Topics in Graph Theory: Extremal Intersecting Systems, Perfect Graphs, and Bireflexive Graphs

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Abstract: In this thesis we investigate three different aspects of graph theory. Firstly, we consider interesecting systems of independent sets in graphs, and the extension of the classical theorem of Erdős, Ko and Rado to graphs. Our main results are a proof of an Erdős-Ko-Rado type theorem for a class of trees, and a class of trees which form counterexamples to a conjecture of Hurlberg and Kamat, in such a way that extends the previous counterexamples given by Baber.

Secondly, we investigate perfect graphs - specifically, edge modification aspects of perfect graphs and their subclasses. We give some alternative characterisations of perfect graphs in terms of edge modification, as well as considering the possible connection of the critically perfect graphs - previously studied by Wagler - to the Strong Perfect Graph Theorem. We prove that the situation where critically perfect graphs arise has no analogue in seven different subclasses of perfect graphs (e.g. chordal, comparability graphs), and consider the connectivity of a bipartite reconfiguration-type graph associated to each of these subclasses.

Thirdly, we consider a graph theoretic structure called a bireflexive graph where every vertex is both adjacent and nonadjacent to itself, and use this to characterise

modular decompositions as the surjective homomorphisms of these structures. We examine some analogues of some graph theoretic notions and define a “dual” version of the reconstruction conjecture.

Declaration

The work in this thesis is based on research carried out in the Department of Computer Science at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification, and the work is all my own work unless specified to the contrary in the text.

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Is it one living being,
Which has separated in itself?
Or are these two, who chose
To be recognized as one?

— from *Gingo Biloba* by Goethe

Dedicated to

Lynn Thomas

and

Stephen Thomas

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Chapter 1

Introduction

A graph is a collection of vertices, which may be adjoined pairwise by *edges*. In everyday language, a graph is a network. Graph theory is a branch of mathematics which studies graphs, which model the real-life networks one sees in various everyday and practical settings: friendship networks, telephone networks, and electrical networks would be some prominent examples. One simple and pervasive example of a graph is the kind that graph theorists call *trees*, so named because they model the branching and vascular structure of trees (and other plants). This type of graph occurs widely: in biological settings such as family trees and phylogeny, in computing settings in searching and storing data, or in communications settings in Morse code (as binary trees). We shall see some problems on trees in Chapter 2.

As with any mathematical subject, graph theory has its ultimate origins in some kind of problem that needed solving. The historical origin of graph theory is thus traced to a seemingly frivolous problem that was considered, and solved, by Euler in 1736 [27].

The locals of Königsberg wondered how to traverse each of the seven bridges of Königsberg exactly once: Euler gave a simple argument that it was not possible as too many of the land masses of Königsberg could be reached from each other in an odd number of ways.

At the time, Euler considered his theorem on the bridges of Konigsberg to be one about the "geometry of position". The phrase "graph" was originally coined by James Joseph Sylvester in 1878 in the paper "Chemistry and Algebra" on molecular diagrams [91]. Indeed there has been a great deal of interaction between the study of graphs and chemistry, as the bonding structure of molecules can be modelled by graphs.

Another problem which is both deeply entwined with the history and developement of graph theory, and also central in modern applications of graph theory, is that of graph colouring. As a first example, consider that there are a list of jobs to be done, and they need to be completed in as short a timeframe as possible, but some of the jobs cannot be done on the same day. How many days can the jobs be completed in?

A *graph colouring* is an assignment of colours to the vertices of a graph so that no pair of vertices linked by an edge receive the same colour. In the scheduling problem, the smallest number of days needed is the same as the smallest number of colours needed in the corresponding graph, whose edges represent two jobs not being able to be done on the same day. This optimum is called the *chromatic number* of the graph.

The historical beginnings of graph colouring are in a different kind of problem, which looks much more like an amusement (much like the problem of Konigsberg's seven bridges) and yet became a very significant area of mathematical research. The *four colour problem* was generated by an observation in 1852 of Francis Guthrie, who, while colouring a map of England's counties, noticed that it was possible to to use four (but not three) colours, so that no counties who were bordering each other shared the same colour. He conjectured that four colours might suffice no matter what the map was. Note that regions sharing a "point" do not count as bordering, so an 8-slice pizza-shaped map requires 2, and not 8 colours. Francis relayed the problem to his brother Frederick Guthrie, who in turn relayed it to his University College London mathematics professor, Augustus De Morgan. De Morgan then became very interested in the problem, and from him it eventually spread widely. The four colour theorem would prove to be an extremely difficult problem. In 1976 - 124 years on - it was finally proved that four colours always suffice by Appel and Haken, and by this point - due to the extreme interest in it - the four colour problem had founded the field of graph colouring and to a certain extent graph theory itself.

By 1976 the scene was very different to 1852. Graph theory was now a major field of mathematics in its own right, and in turn graph colouring was a major area of graph theory. Problems like the reconstruction conjecture of Kelly, the strong perfect graph conjecture of Berge (now solved), and Hadwiger's conjecture took on a status comparable to that previously given to the four colour problem. Many different graph classes were defined by their various properties, often with respect to graph colouring, and there was additional interest - thanks to the rise of computer technology - in algorithms for efficiently computing the chromatic number and other parameters. A prominent example of this was the class of *perfect graphs* defined by Claude Berge - we shall discuss this class at some length in Chapter 3.

In this rest of this chapter we give the basic definitions of graph theory that are needed throughout the thesis, and then give an outline of the content of the chapters of the thesis.

1.1 Basic Definitions

A *binary relation* R on a set X consists of a subset R of the cartesian product $X \times X$. We write aRb if $(a, b) \in R$. If aRa for all $a \in X$ we say R is *reflexive*, if $aRb \Rightarrow bRa$ for all $a, b \in X$ we say R is *symmetric*, and if aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in X$ we say R is *transitive*. A binary relation satisfying all three of these axioms is an *equivalence relation*, and defines a unique partition of X into *equivalence classes*.

A binary relation satisfying the axiom that no element of X is related to itself is called *irreflexive*, and a *graph* is defined to be a pair $G = (V, E)$, made of up a set V

equipped with an irreflexive, symmetric binary relation E. The elements of V are called *vertices* and the elements of E are called *edges*. Another definition of graph is as a subset E of $\binom{V}{2}$, where $\binom{V}{2}$ denotes the set of all 2-element subsets of V; these two definitions are easily seen to be equivalent. These graphs are sometimes called *simple graphs* for clarity as there are other notions of graph-like structures: *digraphs* are obtained from graphs by dropping the symmetry property, and *graphs with loops allowed* are obtained from graphs by dropping the irreflexivity property. We will usually be concerned with simple graphs, however, and any graph is assumed to be a simple graph unless stated otherwise.

We shall not always distinguish between a graph G and its vertices V or its edges E, and we shall say things like "let v be a vertex in G" or let "e be an edge in G" without worrying about this kind of distinction.

The number of vertices of a graph is called its *order* and is typically denoted by n, and the number of edges is its *size* and is typically denoted by m. If v is a vertex of G and $e = \{v, w\}$ is an edge of G, we shall say v is *adjacent* to w, and that w is a *neighbour* of v, and that v and w are *ends* or *endvertices* of the edge e, and are *incident with* e. We also write the shorthand vw for $\{v, w\}$, and write "vw is an edge" or "vw is a nonedge" if $vw \in E(G)$ or $vw \notin E(G)$, respectively.

Two graphs G and H are *isomorphic* if there exists a bijective function $f : V(G) \to V(H)$ such that vw is an edge in $G \Leftrightarrow f(v)f(w)$ is an edge in H, and the function f is a *isomorphism*. If G and H are graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then we say that H is a *subgraph* of G. If H additionally has the property that, for all $v, w \in V(H) : vw$ is an edge in H whenever vw is an edge in G, then we say H is an *induced subgraph* of G, and write $H = G[V(H)]$. If G and H have different vertex sets and H is isomorphic to a subgraph/induced subgraph of G, we will often say simply that H *is* a subgraph/induced subgraph of G, and we will not worry about the distinction. We will also speak of "*the* path on n vertices" and "*the* complete graph on n vertices", and so forth. If $V(G) = V(H)$, then we say H is an *edge subgraph* of G.

If $X \subseteq V(G)$, then $G - X$ is defined as $G[V(G) - X]$. If $X = \{v\}$ we write $G - v$. If X is a subset of $E(G)$ then $G - X$ is the graph obtained by removing the edges of X from the edge set, and $G + X$ is the reverse of this process (for a set of nonedges X). If X is a singleton $\{vw\}$ then we write $G + vw$ (or $G - vw$). Another operation on graphs is *complementation*: The graph $\overline{G} = G - E(G) + (\binom{V}{2} - E(G))$ is referred to as the *complement* of G. Every edge in G is a nonedge in its complement, and vice versa.

If G and H are graphs with disjoint vertex sets, we denote by $G + H$ the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and this is called the *disjoint union* of G and H. A graph that is the disjoint union of two graphs is called *disconnected*, and a graph that is not disconnected is called *connected*. A graph whose complement is disconnected is called *codisconnected*, and a graph that is not codisconnected is called *coconnected*, and a graph that is connected and coconnected is called *biconnected*. A *connected/coconnected component* in G is a connected/coconnected induced subgraph in G which is not contained in any larger connected/coconnected induced subgraph in G. A vertex which forms its own connected component is called an *isolated vertex* or *singleton*, while a vertex which forms its own coconnected component is called a *dominating vertex*.

A graph G is *complete* if every pair of vertices is an edge, and *empty* if every pair of vertices is a nonedge. Every pair of complete graphs on n vertices are isomorphic, and the same goes for empty graphs. We write K_n for the complete graph on n vertices, and E_n for the empty graph on n vertices, and we have $K_n = \overline{E_n}$ and $E_n = \overline{K_n}$. If G is any graph, a complete subgraph of G is called a *clique*, and an empty subgraph is called a *stable set* or *independent set*. The cardinality of the largest clique in G is the *clique number* $\omega(G)$, and the cardinality of the largest stable set in G is the *stability number* or *independence number* $\alpha(G)$.

If v is a vertex, the set of neighbours of v is denoted by $N(v)$ and $d(v) = |N(v)|$ is the *degree* of v. If every vertex has the same degree, G is referred to as a *regular* graph, and k-*regular* if every degree has degree k. A graph G is a cycle if it is connected and 2-regular. A graph with no cycle subgraphs is called a *forest*; if it is connected it is called a *tree*. A graph G is a *path* if it is a tree and every vertex has degree at most 2. Like with complete graphs and empty graphs, the order of paths and cycles determines them up to isomorphism, and we can speak of *the path on n vertices*, or P_n, and also of the *cycle* on n vertices, or C_n. Observe that there is some overlap between all these definitions as $K_2 = P_2$, and $K_3 = C_3$. This latter graph is referred to as a *triangle*. The *length* of a path is referred to by its size, so that the length of P_n is $n - 1$.

Given two vertices v and w in a graph, a *(v, w)-path* is a induced path subgraph P of G such that v and w have degree 1 in P. v and w are the *endpoints* or *ends* of the path, and vertices between v and w on the path are called the *midpoints* of the path. The shortest length of a (v, w)-path is referred to as the *distance* between v and w. Additonally, we say two (v, w)-paths are *disjoint* if they share no vertices or edges besides v and w, and *separated* if they are disjoint and there are no edges between the midpoints of the two paths.

If G is a graph, then a vertex set X such that $G - X$ is disconnected is called a *cutset*. The smallest cardinality of a cutset is the *connectivity* of G, and if this quantity is equal to κ then G is said to be *k-connected* for $k \leq \kappa$. If v is the only member in a singleton cutset we say v is a *cutvertex*. If X is an edge set such that $G - X$ is disconnected we say that X is an *edge-cutset*, and a singleton edge-cutset is a *bridge*. The smallest cardinality of an edge-cutset in G is the *edge-connectivity* of G, and if this quantity is equal to κ we say that G is *k-edge-connected* for $k \leq \kappa$. A graph whose edge connectivity is 1, or equivalently a connected graph in which every edge is a bridge, gives another definition of a tree.

A *graph colouring* of G is a partition of $V(G)$ into stable sets (or more formally, into vertex sets that induce stable sets in G). The smallest number of stable sets required is called the *chromatic number* $\chi(G)$. We could also define it as a function $c : G \to \{1, ..., k\}$ to a set of k "colours" so that if vw is an edge in G then $c(v) \neq c(w)$: up to permutation of the set of colours these notions are equivalent.

A graph having chromatic number 1 or 2 is called *bipartite*. A graph that can be partitioned into k stable sets such that every pair of vertices drawn from distinct stable sets are adjacent is called *complete multipartite*. In the case that $k = 2$ it is called *complete bipartite* and denoted $K_{a,b}$ where a and b are the sizes of the parts. $K_{1,n}$ is called a *claw*.

1.2 Overview of Thesis

In Chapter 2 we consider the graph theoretic analogues of a famous theorem of extremal combinatorics known as the Erdős-Ko-Rado Theorem [25]. A characterisation of the extremal case was also given by Hilton and Milner [51].

EKR Theorem (Erdős, Ko, Rado [25]; Hilton, Milner [51]) *Let n and r be positive integers, $n \geq r$, let S be a set of size n and let $\mathcal{A}$ be a family of subsets of S each of size r that are pairwise intersecting. If $n \geq 2r$, then*

$$|\mathcal{A}| \leq \binom{n-1}{r-1}.$$

Moreover, if $n > 2r$ the upper bound is attained only if the sets in $\mathcal{A}$ contain a fixed element of S.

The EKR Theorem has numerous proofs (see for example [21], [32]). A generalisation of the EKR Theorem to graphs was proposed by Holroyd and Talbot [54]. The EKR Theorem can be interpreted as a property of the empty graph E_n: If the aim is to choose a family of independent r-sets that are pairwise intersecting, the largest way is to choose a *star* - the family of all independent r-sets containing a given vertex v, which is called the *star-centre*. Such an interpretation allows the problem to be generalised to other kinds of graph, where edges may exist. In such a graph, some vertex sets will no longer be independent, making the problem more complicated.

The *r-EKR property* is said to hold for a graph G if the maximum size of an intersecting family of independent r-sets is attained by some star. The *strict r-EKR* *property* is said to hold for a graph G if every maximum-sized intersecting family of independent r-sets in G is a star. The EKR Theorem then states that the graph E_n has the r-EKR property for $r \leq \frac{n}{2}$ and the strict r-EKR property for $r < \frac{n}{2}$. The first r-EKR-type results on graphs were established for disjoint unions of complete graphs, thus generalising the EKR Theorem. More recently results have been established on classes such as paths [53], cycles [92] and chordal graphs [58], and Holroyd and Talbot made a general conjecture [54]. We now state this conjecture (the quantity $\mu(G)$ refers to the smallest size of a maximal independent set in G):

Conjecture (Holroyd, Talbot [54]) *Let r be a positive integer and let G be a graph. Then G is r-EKR if $\mu(G) \geq 2r$ and strictly r-EKR if $\mu(G) > 2r$.*

Firstly, we obtain an EKR result for two classes of trees, thereby providing an initial step towards verifying the above conjecture for trees. The classes are the *depth two claws*, and the *elongated claws with a short limb*, and we refer to Chapter 2 for the precise definitions of these classes.

Theorem *Let G be a tree. If G is a depth-two claw, then G has the strict r-EKR property for $2r \leq \mu(G) - 1$. If G is an elongated claw with n leaves and a short limb, then G is r-EKR if $2r \leq n$.*

Secondly, we investigate a conjecture on cardinalities of star-families in trees, which is related to the EKR conjecture for trees. Generally when proving an EKR-result on graphs it is of use to know how the largest star is obtained; for example for the case of paths, this is the star centred at a *leaf*, a leaf being a vertex of degree 1. Hurlberg and Kamat [58] conjectured that the largest star in any tree would also be centred at a leaf.

Conjecture (Hurlberg, Kamat [58]) *Let r be a positive integer and let T be a tree. Then the largest r-star in T is centred at a leaf.*

This was disproved by Baber by giving an example of a tree where the largest star was a vertex of degree 2 [2]. This still leaves open the question of whether a largest star in a tree can occur at a vertex of *any* degree. We show that this is indeed the case, by giving a class of trees where the largest star has degree n, where n is an arbitrary positive integer.

Theorem *Let n be any positive integer. There exists a tree T and an integer r such that the largest r-star in T is centred at a vertex of degree n.*

Chapter 2 is based on joint published work with Carl Feghali and Matthew Johnson [30].

In Chapter 3 we are interested in perfect graphs and their various subclasses. A graph is *perfect* if every induced subgraph has the property that it has a colouring with as few colours as the size of a largest clique. Perfect graphs are an important class of graphs, due to their favourable algorithmic properties, relations with linear programming, and also simply because they contain lots of basic graph classes as a special case.

Perfect graphs were introduced by Berge [3], and were the subject of the now-famous perfect graph theorems, which both began as conjectures of Berge.

The Perfect Graph Theorems *Let G be a graph. Then the following holds:*

1. *G is perfect if and only if $\overline{G}$ is perfect;*
2. *G is perfect if and only if it has the "Berge property" that neither G nor $\overline{G}$ contain any odd cycles of length 5 or more as induced subgraphs.*

The first part is known as the *perfect graph theorem*, and is due to Lovasz [66], while the second part is known as the *strong perfect graph theorem*, and is due to Chudnovsky, Robertson, Seymour and Thomas [17]. For the sake of brevity in the second part, the cycles that are not triangles are often simply called *holes*, and their complements *antiholes*, so that the forbidden induced subgraphs become the *odd holes*, and *odd antiholes*, respectively. The strong perfect graph theorem proved to be a challenging problem, and its eventual solution in 2002 ran to around 150 pages. The considerable efforts spent to solve these problems played a significant part in the developement of the theory of these graphs.

Our interest in perfect graphs starts with the problem of how we can modify an edge - that is, add or delete an edge - without increasing the clique or stability numbers of the graph. An edge whose deletion does not increase the stability number of the graph is called *free*, while other edges are called *critical.* Likewise, nonedges whose addition to the graph are called free if they do not increase the clique number, and critical otherwise.

Firstly we look at the following problem: we have a graph G and we want to know what kind of graph could we turn G into by a sequence of free edge modifications? Can we disconnect the graph by free edge deletions only, for example? We use this kind of question to provide some characterisations of perfect graphs, in terms of edge modification.

The second part of Chapter 3 relates edge-modification aspects of minimally perfect graphs with the strong perfect graph theorem. This has been investigated by Gasparian, Markossian and Markossian [70], and also by Sebo [88], and was where the critical/free edge concept was originated. These investigations were motivated by a desire to prove the existence of critical edges in minimally imperfect graphs with a view to proving the SPGT.

Sebo's Problem (Sebo [88]) *Let G be a minimally imperfect graph. Show that G contains a critical edge.*

Sebo's problem forms one of the principle problems of the "partitionability" approach to the SPGT. This is probably the second largest body of work devoted to proving the SPGT after the successful structural-decomposition approach which culminated in the proof of Chudnovsky et al. It might possibly be a candidate for a second proof of the SPGT.

Specifically, we observe that in odd holes and odd antiholes, changing an edge increases the clique/stability number if and only if it results in a Berge graph. We prove that something similar can be observed in a hypothetical minimum counterexample to the strong perfect graph theorem, and we speculate that it might be useful for solving Sebo's problem on such a graph.

A *Berge-free edge* in a Berge graph G is an edge xy such that $G - xy$ is also Berge. Non-Berge-free edges are called *Berge-critical*. We show that in a minimum order and minimum size counterexample to the SPGT, every edge is either critical or Berge-critical. We are not able to show also that every edge is either free or Berge-free, which would be a possible next step.

The upshot of this is that in a minimum order/size counterexample to the SPGT either a critical edge must exist, or every edge in the graph must be Berge-critical.

A graph in which every edge is Berge-critical is called *critically Berge*, and this graph class is closely related to the *critically perfect* graphs. A critically perfect graph (as defined by Wagler in [95]) is a perfect graph where the removal of any edge leaves an imperfect graph. By the SPGT the critically Berge graphs and the critically perfect graphs are the same. However from an SPGT-proving point of view the perfection of critically Berge graphs would be potentially of great interest. It would, for example, solve Sebo's problem in a minimum SPGT counterexample.

Theorem *Let G be a minimum counterxample to the SPGT. Then either G contains a critical edge or G is critically Berge.*

Wagler gives several examples of critically perfect (and of course critically Berge) graphs in her PhD thesis [95]. These include graphs G where G and $\overline{G}$ are critically perfect.

Our third main line of results in Chapter 3 moves away from the SPGT and into edge modification in subclasses of the perfect graphs. We investigate *freedom games*. A freedom game consists of a class $\mathcal{C}$ of graphs, which contains both empty and complete graphs, for which the game is to move between the empty graph and complete graph, or vice versa, changing one edge at a time, remaining inside $\mathcal{C}$. Ideally one would also like to be able to start at any graph in the class, and arrive at the complete graph or the empty graph in this way, without "falling out" of the class.

Formally, if every graph $G \in \mathcal{C}$ is either complete or there is a nonedge xy of G such that $G + xy \in \mathcal{C}$, and is either empty or there is an edge xy of G such that $G - xy \in \mathcal{C}$, we say that $\mathcal{C}$ has the *freedom property.* Note that this is not a property of the individual graphs, but rather of $\mathcal{C}$.

As such, the class of perfect graphs does not have the freedom property, due to the existence of critically perfect graphs. Likewise, cographs (because of graphs such as $2K_2$ and C_4) do not have the freedom property. Bipartite graphs and trees cannot be considered because they do not contain complete graphs as a special case.

Nevertheless, we demonstrate several (seven, in total) subclasses of the perfect graphs that *do* have the freedom property, for example the classes of comparability graphs, permutation graphs, and weakly chordal graphs. In some cases, we also show they have an even stronger property. We state this property for split graphs here as an example (but in Chapter 3 we show that it also holds for the classes of interval graphs, chordal graphs, and threshold graphs):

Theorem *Let G be a split graph and let x be a vertex of G. If x is not a dominating vertex, then there exists a non-neighbour y of x such that the graph $G + xy$ is a split graph. Likewise, if x is not an isolated vertex, then there exists a neighbour y of x such that $G - xy$ is a split graph.*

We then apply some of these freedom properties to the critically perfect graphs themselves, and deduce some restrictions on their structure. It was proved by Wagler [95] that they cannot be chordal or co-chordal. We prove additionally that a critically perfect graph cannot be either a comparability or co-comparability graph.

Finally, we show how to associate a bipartite graph to any given class of graphs in a manner somewhat reminiscent of graph reconfiguration, and consider its connectivity.

Chapter 4 concerns a relation between graph homomorphisms and modular decompositions.

A *module* in a graph G is a set of vertices M such that for all $x, y \in M$, $N(x) - M = N(y) - M$. M is a connected component precisely if $N(x) - M = N(y) - M = \emptyset$, so modules generalise the connected components of graphs. The *modular decomposition* of G is then a partitioning of G into modules, and any graph has such a partition. Modular decompositions date back to Gallai in the context of comparability graphs, and are used in the area of perfect graphs to recognise cographs, comparability graphs, permutation graphs, interval graphs, and distance hereditary graphs. The inverse procedure of modular decomposition, *modular substitution*, is used in Lovasz's proof of the perfect graph theorem, and in Chudnovsky et al's proof [17] of the strong perfect graph theorem. The *lexicographic product* of graphs is also a special case of modular decomposition/substitution.

A *graph homomorphism* from a simple graph G to a simple graph H is a function $f : V(G) \to V(H)$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in H. They are useful because they can model a variety of different graph theoretic problems, for example vertex colouring and many other kinds of colouring problem. They can also be composed, which allows algebraic and category-theoretic viewpoints on graphs.

Modular decomposition and graph homomorphisms have been previously linked by Hell and Nesetril: In [49] there is given a notion of "trigraph", a kind of graph structure where in addition to edges and nonedges, "half-edges" are also allowed, as well as loops. The existence of a "trigraph homomorphism" from a simple graph G to a certain 3-vertex trigraph then corresponds to the existence of a module in G.

We consider a different way of doing this: We define a graph structure called a *bireflexive graph*. The idea of a bireflexive graph is that a vertex is adjacent and nonadjacent to itself, and any two distinct vertices are either adjacent or nonadjacent to each other (but not both). At first glance the principle of non-contradiction would not allow this, but there is a way out: defining the relations of adjacency and nonadjacency separately.

We show that the analogue of graph homomorphisms for these bireflexive graphs naturally leads to the modular decomposition. Furthermore by contrasting injective and surjective homomorphisms, we find that the notion of modular decomposition is in a sense "dual" to the notion of the induced subgraph of a graph. We consider that the analogues of perfect graphs and the direct product in this setting are the cographs and the lexicographic product, respectively. Additionally, a "dual" version of the reconstruction conjecture is considered.

Chapter 2

Independent Sets In Trees and the EKR Theorem

2.1 Introduction

In this chapter, we consider graph theoretic versions of the following famous result of Erdős, Ko and Rado:

EKR Theorem (Erdős, Ko, Rado [25]; Hilton, Milner [51]) *Let n and r be positive integers, $n \geq r$, let S be a set of size n and let $\mathcal{A}$ be a family of subsets of S each of size r that are pairwise intersecting. If $n \geq 2r$, then*

$$|\mathcal{A}| \leq \binom{n-1}{r-1}.$$

Moreover, if $n > 2r$ the upper bound is attained only if the sets in $\mathcal{A}$ contain a fixed element of S.

The characterization of the extremal case was provided by Hilton and Milner [51]. The original proof in [25] used the "compression" method, which is now widely used. A second and quite different proof was given by Katona in [59] using what has become known as "Katona's circle method".

There have since been discovered many generalisations, and analogues, of the EKR theorem, for a variety of structures: for example permutations [60], partially ordered sets [26], and vector spaces [33]. Much of this activity has been in extending the theorem to various kinds of graph, which will be our focus.

Throughout this chapter, graphs are simple and undirected. Let K_n denote the complete graph on n vertices, and let $K_{1,n}$ denote a claw. An *independent set* in a graph is a set of pairwise non-adjacent vertices.

Given a graph G and an integer $r \geq 1$, let $\mathcal{I}^{(r)}(G)$ denote the family of independent sets of G of cardinality r. For a vertex v of G, let $\mathcal{I}_v^{(r)}(G)$ be the subset of $\mathcal{I}^{(r)}(G)$ containing all sets that contain v. This is called an *r-star* (or just star) and v is its *centre*. We say that G is r-EKR if no pairwise intersecting family $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$ is larger than the biggest r-star, and strictly r-EKR if every pairwise intersecting family that is not an r-star is smaller than the the largest r-star of $\mathcal{I}^{(r)}(G)$.

The EKR Theorem can be seen as a statement about the maximum size of a family of pairwise intersecting independent sets of size r in the empty graph on n vertices. We quickly obtain another formulation of the EKR Theorem by noting that an independent set of the claw that contains more than one vertex contains only leaves.

Theorem 2.1.1. *Let n and r be positive integers, $n \geq r$. The claw $K_{1,n}$ is r-EKR if $n \geq 2r$ and strictly r-EKR if $n > 2r$.*

There exist EKR results for several graph classes.

One of the first results is due to Berge [5].

Theorem 2.1.2 ([5])**.** *Let $r \geq 1$, $t \geq 2$ and let G be the union of r disjoint copies of K_t. Then G is r-EKR.*

The extremal case of this was characterised by Livingston [64] and two further proofs are found in [43] and [75]. Some generalisations of Theorem 2.1.2 are given by [9, 24, 33]. We state one such generalisation now: it is due to Holroyd, Spencer and Talbot [53].

Theorem 2.1.3 ([53]). *Let G be the disjoint union of $n \geq r$ complete graphs of order at least 2, then G is r-EKR.*

They also obtained the following result.

Theorem 2.1.4 ([53]). *Let G be the disjoint union of $n \geq 2r$ paths, cycles and complete graphs, at least one of which is an isolated vertex. Then G is r-EKR, and it is strictly so if $r < \frac{n}{2}$.*

In light of some of the existing results, Holroyd and Talbot defined the *minimax number* $\mu(G)$ to be the smallest size of a maximal independent set in G. The generalisation of the EKR Theorem below was conjectured by Holroyd and Talbot [54].

Conjecture 2.1.5 (Holroyd, Talbot [54]). *Let r be a positive integer and let G be a graph. Then G is r-EKR if $\mu(G) \geq 2r$ and strictly r-EKR if $\mu(G) > 2r$.*

This conjecture appears difficult to prove or disprove. It is nevertheless known to be true for many graph classes such as the disjoint union of complete graphs each of order at least two, powers of paths [54] and powers of cycles [50, 52, 92]. See [10, 53, 58, 54, 58] for further examples.

For chordal graphs the following result was established by Hurlbert and Kamat [58].

Theorem 2.1.6 ([58]). *Let G be a chordal graph containing a singleton. Then G is r-EKR for $r \leq n/2$ and strictly r-EKR for $r < n/2$.*

For trees and forests this theorem contains the special case that a disjoint union of trees, one of which is a singleton, satisfies the r-EKR property for $r \leq \frac{\mu}{2}$.

A usual technique to prove results of this kind is to find the centre of the largest r-star of a graph and this will prove useful to us. This is easy in the case where a singleton exists, but may be hard in other cases. In this vein, Hurlbert and Kamat [58] conjectured the following for the class of trees.

Conjecture 2.1.7 (Hurlbert, Kamat [58]). *Let n and r be positive integers, $n \geq r$. If T is a tree on n vertices, then there is a largest r-star of T whose centre is a leaf.*

They were able to prove Conjecture 2.1.7 for $1 \leq r \leq 4$ [58]. The conjecture does not, however, hold for any $r \geq 5$. This was shown by Baber [2] who gave counterexamples where the largest r-star in an appropriately defined tree is centred at a vertex whose degree is 2.

2.2 The Main Results

We are interested in the possibility of establishing Conjecture 2.1.5 for trees: The only preexisting result seems to be for paths, which have been established to be r-EKR (for any integer r) in [54].

We consider a subfamily of trees called *elongated claws*. An elongated claw has one vertex that is its *root*. Every other vertex has degree 1 or 2 (it is possible that the root also has degree 1 or 2). A vertex of degree 1 is called a *leaf*. A path from the root to a leaf is a *limb*. A limb is *short* if it contains only one edge. If every leaf is distance 2 from the root (that is, if every limb contains two edges), then the graph is a *depth-two claw*.

We are now ready to state our main results.

Theorem 2.2.1. *Let r be a positive integer and let G be a depth-two claw. Then G is strictly r-EKR if $\mu(G) \geq 2r - 1$.*

Theorem 2.2.1 confirms (and is stronger than) Conjecture 2.1.5 for depth-two claws.

Theorem 2.2.2. *Let n and r be positive integers, $n \geq 2r$, and let G be an elongated claw with n leaves and a short limb. Then G is r-EKR.*

Theorem 2.2.2 does not confirm (but only supports) Conjecture 2.1.5 for the class of elongated claws with short limbs since $\mu(G)$ may be much larger than the number of leaves in G.

We remark that similar EKR results (that is, with weaker bounds than that of Conjecture 2.1.5) were obtained in [53, Theorem 8] and [97, Proposition 4.3]. Satisfying the bound of Conjecture 2.1.5 in Theorem 2.2.2, and in general for elongated claws, is left as an open problem.

In the remaining sections we prove Theorems 2.2.1 and 2.2.2.

Regarding Conjecture 2.1.7, we extend in sections $2.5-2.7$ the counterexamples that were obtained by Baber in various directions: the degree of the largest star centre may be any $n \geq 2$, rather than just 2, and the diameter may be any even $d \geq 6$, rather than just 6.

In the other direction, in Section 2.8 we also establish that Conjecture 2.1.7 holds for caterpillars, and that this is, in some sense, the best possible. Finally, we conclude with a theorem on the number of independent sets in trees, and give a second proof of the EKR property for a disjoint union of complete graphs.

2.3 Depth-two Claws

The following lemma is useful in the proofs of Theorem 2.2.1 and Theorem 2.2.2. Essentially the lemma states that Conjecture 2.1.7 holds for the case where the tree is an elongated claw.

Lemma 2.3.1. *Let r be a positive integer, and let G be an elongated claw. Then there is a largest r-star of G whose centre is a leaf.*

Proof. Let v be a vertex of G that is not a leaf, and let L be the limb of G that contains v (if v is the root, then L can be any limb). Let x be the leaf of L. We find an injection f from $\mathcal{I}_v^{(r)}(G)$ to $\mathcal{I}_x^{(r)}(G)$ which proves that $|\mathcal{I}_x^{(r)}(G)| \geq |\mathcal{I}_v^{(r)}(G)|$ and the lemma immediately follows.

Let w be the unique neighbour of x. Let $A \in \mathcal{I}_v^{(r)}(G)$.

1. If $x \in A$, then let $f(A) = A$.

2. If $x \notin A$ and $w \notin A$, then let $f(A) = A\backslash\{v\} \cup \{x\}$.

3. If $x \notin A$ and $w \in A$, then let $X = \{x = x_1, x_2, \ldots, x_m = v\}$ be the set of vertices in L from x towards v. Let $A \cap X = \{x_{i_1}, \ldots, x_{i_j}\} = Y$ for some $m > j \geq 1$. Let $Z = \{x_{i_1 - 1}, \ldots, x_{i_j - 1}\}$. Observe that $|Y| = |Z|$ and $x \in Z$ since $w \in Y$. Then let $f(A) = (A \cup Z) \backslash Y$.

To prove that f is injective we consider distinct $A_1, A_2 \in \mathcal{I}_v^{(r)}(G)$. If $f(A_1)$ and $f(A_2)$ are defined by the same case (of the three above), then it is clear that $f(A_1)$ and $f(A_2)$ are distinct. When they are defined by different cases, we simply note that in the first $f(A)$ always contains v, in the second $f(A)$ contains neither v nor any of its neighbours, and in the third $f(A)$ contains a neighbour of v. $\square$

The property of elongated claws in Lemma 2.3.1 is a much weaker version of the *degree sort property*; a graph has this property if the size of an r-star centred at u is at least the size of an r-star centred at v whenever the degree of u is less than that of v. Hurlbert and Kamat [58] observed that depth-two claws have this property. We note that not all elongated claws possess it. For example, consider an elongated claw with three limbs of lengths 1, 2 and 3. Then the 4-star centred at the neighbour of the root in the limb of length 3 has size 2, but the 4-star centred at the leaf of the limb of length 2 has size 1. It remains to determine which elongated claws — or, more generally, which trees — have the degree sort property. We might also ask which trees have the following weaker property: if $i < j$, then the size of the largest r-star of all those stars centred at vertices of degree i is at least the size of the largest r-star of all those centred at vertices of degree j.

Lemma 2.3.2. *Let n and r be positive integers, $n \geq r$, and let G be a depth-two claw with n leaves. Then the largest r-star of G is centred at a leaf and has size*

$$\binom{n-1}{r-1} 2^{r-1} + \binom{n-1}{r-2}.$$

Proof. By Lemma 2.3.1, there is a largest r-star whose centre is a leaf (and clearly, by symmetry, all leaves are equivalent). So let v be a leaf of G and let c be the root of G. Define a partition: $\mathcal{B} = \{B \in \mathcal{I}_v^{(r)}(G) : c \notin B\}$ and $\mathcal{C} = \{C \in \mathcal{I}_v^{(r)}(G) : c \in C\}$.

Then $|\mathcal{B}| = \binom{n-1}{r-1}2^{r-1}$ since each member of $\mathcal{B}$ intersects $r-1$ of the $n-1$ limbs that do not contain v and can contain either of the 2 vertices (other than the root) of each of those limbs. And $|\mathcal{C}| = \binom{n-1}{r-2}$ since each member of $\mathcal{C}$ contains $r-2$ of the $n-1$ leaves other than v. The proof is complete. □

In order to prove Theorem 2.2.1, we shall need two auxiliary results.

Theorem 2.3.3 (Meyer [72]; Deza and Frankl [21]). *Let n, r and t be positive integers, $n \geq r$, $t \geq 2$, and let G be the disjoint union of n copies of K_t. Then G is r-EKR and strictly r-EKR unless $r = n$ and $t = 2$.*

For a family of sets $\mathcal{A}$ and nonnegative integer s, the s-shadow of $\mathcal{A}$, denoted $\partial_s\mathcal{A}$, is the family $\partial_s\mathcal{A} = \{S : |S| = s, \exists A \in \mathcal{A}, S \subseteq A\}$.

Lemma 2.3.4 (Katona [60]). *Let a and b be nonnegative integers and let $\mathcal{A}$ be a family of sets of size a such that $|A \cap A'| \geq b \geq 0$ for all $A, A' \in \mathcal{A}$. Then $|\mathcal{A}| \leq |\partial_{a-b}\mathcal{A}|$*

The proof of Theorem 2.2.1 is inspired by a proof of the EKR theorem [21]. To the best of our knowledge, the proof is the first to make use of shadows in the context of graphs.

Proof of Theorem 2.2.1. Let c be the root of G and let n be the number of leaves of G. Note that $n = \mu(G)$ so $n \geq 2r - 1$. Let $\mathcal{A} \subseteq I^{(r)}(G)$ be any pairwise intersecting family. Define a partition $\mathcal{B} = \{A \in \mathcal{A} : c \notin A\}$ and $\mathcal{C} = \{A \in \mathcal{A} : c \in A\}$.

Notice that each vertex in each member of $\mathcal{B}$ is either a leaf or the neighbour of a leaf. For $B \in \mathcal{B}$, let M_B be the set of r leaves that each either belongs to B or is adjacent to a vertex in B. We say that M_B *represents* B. Let $\mathcal{M} = \{M_B : B \in \mathcal{B}\}$. Note that each member of $\mathcal{M}$ might represent many different members of $\mathcal{B}$. In fact, consider $M \in \mathcal{M}$. It can represent any independent set that, for each leaf $\ell \in M$, contains either ℓ or its unique neighbour. There are 2^r such sets but they can be partitioned into complementary pairs so, as $\mathcal{B}$ is pairwise intersecting, the number

s_M of members of $\mathcal{B}$ that M represents is at most 2^{r-1}. We also note that $\mathcal{M}$ is pairwise intersecting (since $\mathcal{B}$ is pairwise intersecting).

We have that

$$|\mathcal{B}| = \sum_{M \in \mathcal{M}} s_M \leq \binom{n-1}{r-1} 2^{r-1}, \tag{2.3.1}$$

where the inequality follows from Theorem 2.3.3.

For $B \in \mathcal{B}$, let N_B be the set of $n-r$ leaves that neither belong to B nor are adjacent to a vertex in B. Notice that M_B and N_B partition the set of leaves. Let $\mathcal{N} = \{N_B : B \in \mathcal{B}\}$. For any pair $B_1, B_2 \in \mathcal{B}$, we know that M_{B_1} and M_{B_2} intersect, so $|M_{B_1} \cup M_{B_2}| \leq 2r-1$. The leaves not in this union are members of both N_{B_1} and N_{B_2} and there are at least $n-(2r-1) \geq 0$ of them. Thus we can apply Lemma 2.3.4 to $\mathcal{N}$ with $a = n-r$, $b = n-(2r-1)$ to obtain

$$|\mathcal{N}| \leq |\partial_{r-1}\mathcal{N}|. \tag{2.3.2}$$

Notice that, by definition, $\partial_{r-1}\mathcal{N}$ is a collection of sets of $r-1$ leaves each of which is, for some $B \in \mathcal{B}$, a subset of N_B and is therefore disjoint to M_B and so certainly does not intersect B.

Let us try to bound the size of $\mathcal{C}$. Each $C \in \mathcal{C}$ contains a distinct set of $r-1$ leaves. We know this set must intersect every member of $\mathcal{B}$ so it cannot be a member $\partial_{r-1}\mathcal{N}$. Thus we find

$$|\mathcal{C}| \leq \binom{n}{r-1} - |\partial_{r-1}\mathcal{N}|. \tag{2.3.3}$$

We apply (2.3.2) to (2.3.3) and note that $|\mathcal{N}| = |\mathcal{M}|$ to obtain

$$|\mathcal{C}| \leq \binom{n}{r-1} - |\mathcal{M}|. \tag{2.3.4}$$

Since $s_M \leq 2^{r-1}$ for each $M \in \mathcal{M}$, equality holds in (2.3.1) only if $|\mathcal{M}| \geq \binom{n-1}{r-1}$. Thus combining (2.3.1) and (2.3.4):

$$\begin{aligned} |\mathcal{A}| &= |\mathcal{B}| + |\mathcal{C}| \\ &\leq \sum_{M \in \mathcal{M}} s_M + \binom{n}{r-1} - |\mathcal{M}| \end{aligned}$$

$$
\begin{aligned}
&\leq \binom{n-1}{r-1}2^{r-1} + \binom{n}{r-1} - \binom{n-1}{r-1} \\
&= \binom{n-1}{r-1}2^{r-1} + \binom{n-1}{r-2}. \qquad (2.3.5)
\end{aligned}
$$

Meanwhile, if $|\mathcal{M}| \leq \binom{n-1}{r-1}$ we again get:

$$
\begin{aligned}
|\mathcal{A}| &= |\mathcal{B}| + |\mathcal{C}| \\
&\leq \sum_{M \in \mathcal{M}} s_M + \binom{n}{r-1} - |\mathcal{M}| \\
&\leq \sum_{M \in \mathcal{M}} (2^{r-1} - 1) + \binom{n}{r-1} \\
&\leq (2^{r-1} - 1)\binom{n-1}{r-1} + \binom{n}{r-1} \\
&= \binom{n-1}{r-1}2^{r-1} + \binom{n-1}{r-2}.
\end{aligned}
$$

This proves that G is r-EKR by Lemma 2.3.1. We now show that G is strictly r-EKR. If $r = n$ then $r = 1$ so the result trivially holds. Suppose $r < n$. Then, by Theorem 2.3.3, equality holds in (2.3.1), and therefore in (2.3.5), only if $\mathcal{B}$ is an r-star centred at a leaf x or a neighbour y of a leaf. It follows easily that $\mathcal{C} = \emptyset$ if $\mathcal{A}$ is an r-star centred at y; thus $\mathcal{A} = \mathcal{I}_x^{(r)}(G)$ as desired. $\square$

We demonstrate that if G is a depth-two claw with n leaves, then G is not n-EKR by describing a pairwise intersecting family that is larger than the largest n-star. Let c be the root of G and let $G' = G - c$, a graph containing n copies of K_2 each of which contains one leaf of G. Clearly G' contains 2^n independent sets of size n which can be partitioned into complementary pairs. Let $\mathcal{B}$ be a family of 2^{n-1} independent sets of size n formed by considering each complementary pair and choosing either the one that contains the greater number of leaves of G, or, if they each contain half the leaves, choosing one arbitrarily. Notice that $\mathcal{B}$ is pairwise intersecting but is not a star. Let $\mathcal{C} = \{C \in \mathcal{I}^{(n)}(G) : c \in C\}$. Clearly, $|\mathcal{C}| = \binom{n}{n-1} = n$ and for each pair $C \in \mathcal{C}$, $B \in \mathcal{B}$, we have that $C \cap B \neq \emptyset$. Thus if $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$, then $\mathcal{A}$ is pairwise intersecting, maximal and $|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| = 2^{n-1} + n$. By Lemma 2.3.2, $\mathcal{A}$ has one more element than the largest n-star in G.

The above remark together with Theorem 2.2.1 motivates the following conjecture.

Conjecture 2.3.5. *Let n and r be positive integers, $n > r$ and let G be a depth-two claw with n leaves. Then G is r-EKR.*

2.4 Elongated Claws with Short Limbs

In this section we will prove Theorem 2.2.2. We require some terminology and lemmas. For a vertex v of a graph G, let $G - v$ denote the graph obtained by deleting v and incident edges from G, and let $G \downarrow v$ be the graph obtained from G by deleting the vertex v and all its neighbours and their incident edges.

The following lemma has essentially the same proof as Lemma 2.5 in [58], but we include a proof for completeness.

Lemma 2.4.1. *Let r be a positive integer, and let G be a graph. Let v be a vertex of G and let u be a vertex of $G \downarrow v$. Then*

$$|\mathcal{I}_u^{(r)}(G)| = |\mathcal{I}_u^{(r)}(G - v)| + |\mathcal{I}_u^{(r-1)}(G \downarrow v)|.$$

Proof. Define a partition of $\mathcal{I}_u^{(r)}(G)$: $\mathcal{B} = \{A \in \mathcal{I}_u^r(G) : v \notin A\}$ and $\mathcal{C} = \{A \in \mathcal{I}_u^{(r)}(G) : v \in A\}$. Observe that $|\mathcal{B}| = |\mathcal{I}_u^{(r)}(G - v)|$ and $|\mathcal{C}| = |\mathcal{I}_u^{(r-1)}(G \downarrow v)|$. This implies the lemma. □

Lemma 2.4.2. *Let r be a positive integer and let G be an elongated claw with a short limb with root c. If x is a leaf of G adjacent to c, then x is the centre of a largest r-star of G.*

Proof. Let v be a vertex in G that is not a leaf adjacent to c. We must show that $\mathcal{I}_v^{(r)}(G)$ is no larger than $\mathcal{I}_x^{(r)}(G)$. If $v = c$ this is immediate since $\{A \setminus \{c\} \cup \{x\} : A \in \mathcal{I}_c^{(r)}(G)\}$ has the same cardinality as $\mathcal{I}_c^{(r)}(G)$ and is a subset of $\mathcal{I}_x^{(r)}(G)$.

If $v \neq c$, let L be the limb of G that contains v. To prove the lemma, we find an injection f from $\mathcal{I}_v^{(r)}(G)$ to $\mathcal{I}_x^{(r)}(G)$. Let $A \in \mathcal{I}_v^{(r)}(G)$. We distinguish a number of cases.

1. If $x \in A$, then $f(A) = A$.

2. If $x \notin A$ and $c \notin A$, then $f(A) = A \backslash \{v\} \cup \{x\}$.

3. If $x \notin A$ and $c \in A$, let $X = \{v = x_1, \ldots, x_m\}$ be the set of vertices from v towards the neighbour x_m of c in L. Let $Y = A \cap X = \{x_{i_1}, \ldots, x_{i_j}\}$ for some $m > j \geq 1$. Let $Z = \{x_{i_1+1}, \ldots, x_{i_j+1}\}$ and observe that $|Y| = |Z|$. Then $f(A) = (A \cup Z \cup \{x\}) \backslash (Y \cup \{c\})$.

It can be verified that f is injective as required. □

We now prove Theorem 2.2.2 using an approach based on that of the proof of [58, Theorem 1.22].

Proof of Theorem 2.2.2. Let c be the root of G. Let $\mathcal{A} \subseteq \mathcal{I}^{(r)}(G)$ be any pairwise intersecting family. We must show that $\mathcal{A}$ is no larger than the largest r-star. We use induction on r. If $r = 1$ the result is true so suppose that $r \geq 2$ and that the result is true for smaller values of r.

We now use induction on the number of vertices in G. The base case is that G contains only the root and n leaves; that is, $G = K_{1,n}$ and so the result follows from Theorem 2.1.1. So suppose that the number of vertices is at least $n+2$ and that the result is true for graphs with fewer vertices.

Let x be a leaf adjacent to c. Let v be a leaf that is not adjacent to c. Let w be the unique neighbour of v and let z denote the other neighbour of w.

Define $f : \mathcal{A} \to \mathcal{I}^{(r)}(G)$ such that for each $A \in \mathcal{A}$

$$f(A) = \begin{cases} A \backslash \{v\} \cup \{w\}, & v \in A, z \notin A, A \backslash \{v\} \cup \{w\} \notin \mathcal{A} \\ A, & \text{otherwise.} \end{cases}$$

Define the families:

$$\mathcal{A}' = \{f(A) : A \in \mathcal{A}\},$$

$$\mathcal{B} = \{A \in \mathcal{A}' : v \notin A\},$$

$$\mathcal{C} = \{A \backslash \{v\} : v \in A, A \in \mathcal{A}'\}.$$

Notice that

$$|\mathcal{A}| = |\mathcal{A}'| = |\mathcal{B}| + |\mathcal{C}|. \tag{2.4.1}$$

Claim 1*:* We claim that each of $\mathcal{B}$ and $\mathcal{C}$ is pairwise intersecting.

Proof of Claim 1*:* By the definition of f, we can partition $\mathcal{B}$ into $\mathcal{B}_1 = \{B \in \mathcal{B} : B \in \mathcal{A}\}$ and $\mathcal{B}_2 = \{B \in \mathcal{B} : B \setminus \{w\} \cup \{v\} \in \mathcal{A}\}$. Then $\mathcal{B}_1$ is pairwise intersecting (since $\mathcal{A}$ is intersecting) and $\mathcal{B}_2$ is pairwise intersecting as every member contains w. Next consider $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$. As B_1 and $B_2 \setminus \{w\} \cup \{v\}$ are both in $\mathcal{A}$ they intersect and this intersection does not contain v (since it is not in B_1) so is a superset of $B_1 \cap B_2$. So $\mathcal{B}$ is intersecting.

By definition, if $C \in \mathcal{C}$, then $C \cup \{v\}$ is in $\mathcal{A}'$ and, by the definition of f, also in $\mathcal{A}$. Using the definition of f again, we must have that either z is in C, or $C \cup \{w\}$ is in $\mathcal{A}$. Let C_1 and C_2 be two members of $\mathcal{C}$. Then either they both contain z or if one of them, say C_1, does not, then $C_1 \cup \{w\}$ is in $\mathcal{A}$. As $C_2 \cup \{v\}$ is also in $\mathcal{A}$ and $\mathcal{A}$ is intersecting, we have that $C_1 \cup \{w\}$ and $C_2 \cup \{v\}$ must intersect. By the independence of the two sets, this intersection contains neither v nor w and so C_1 and C_2 must intersect. The claim is proved.

Note that $G - v$ is an elongated claw with a short limb, fewer vertices than G and with n leaves. We also note that each member of $\mathcal{B}$ contains r vertices of $G - v$ and, by Claim 1, $\mathcal{B}$ is pairwise intersecting. By the induction hypothesis, $G - v$ is r-EKR and so the largest intersecting families are r-stars, and, by Lemma 2.4.2, $\mathcal{I}_x^{(r)}(G - v)$ is a largest r-star of $G - v$. Hence

$$|\mathcal{B}| \leq |\mathcal{I}_x^{(r)}(G - v)|. \tag{2.4.2}$$

Note that $G \downarrow v$ is an elongated claw with a short limb, fewer vertices than G and with either n or $n - 1$ leaves. We also note that each member of $\mathcal{C}$ contains $r - 1$ vertices of $G \downarrow v$ and, by Claim 1, $\mathcal{C}$ is pairwise intersecting. By the induction hypothesis,

$G{\downarrow}v$ is $(r-1)$-EKR and so the largest intersecting families are $(r-1)$-stars, and, by Lemma 2.4.2, $\mathcal{I}_x^{(r-1)}(G{\downarrow}v)$ is a largest $(r-1)$-star of $G{\downarrow}v$. Hence

$$|\mathcal{C}| \leq |\mathcal{I}_x^{(r-1)}(G{\downarrow}v)|. \tag{2.4.3}$$

Combining (2.4.1), (2.4.2) and (2.4.3) and applying Lemma 2.4.1:

$$\begin{aligned} |\mathcal{A}| &= |\mathcal{B}| + |\mathcal{C}| \\ &\leq |\mathcal{I}_x^{(r)}(G-v)| + |\mathcal{I}_x^{(r-1)}(G{\downarrow}v)| \\ &= |\mathcal{I}_x^{(r)}(G)| \end{aligned}$$

and the theorem is proved. □

2.5 Star Systems and Counterexamples to Conjecture 2.1.7

A *star system* centred at a vertex v in a graph G, denoted by $\mathcal{I}_v(G)$, is the collection of all independent sets of G containing v. Equivalently, it is the union of all the r-stars centred at v, where r runs over $\{1, ..., \alpha(G)\}$. We shall prove a Theorem about star systems in an appropriately defined tree, and in doing so extend Baber's counterexamples to Conjecture 2.1.7. The strategy is define a tree in a certain way so that we may consider some limiting process involving the sizes of its star systems, and deduce the existence of the required star as a corollary. The largest stars so obtained may be centred at a vertex of arbitrary degree, which extends the counterexample of Baber where this degree is 2.

We shall now describe the tree. An elongated claw where the root has k neighbours and every leaf is distance a from the root is referred to as a (k, a)-*claw*. For all positive integers a, k, n, let C denote a disjoint union of n copies of a (k, a)-claw with roots $c_1, \ldots, c_n$, and let $T = T(n, k, a)$ denote the tree with root x and vertex set $V(C) \cup \{x\}$ and edge set $E(C) \cup \{xc_i : i = 1, \ldots, n\}$.

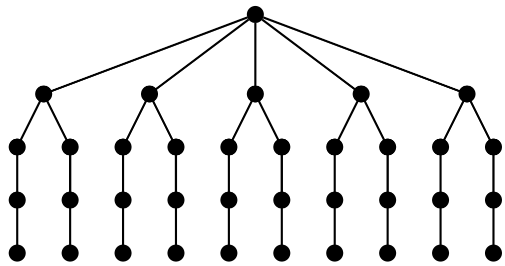

Figure 2.1: $T(5, 2, 3)$

This construction contains the example of Baber as the special case $T(2, n, 2)$. Our counterexamples are contained in the following Theorem.

Theorem 2.5.1. *Suppose $a, n \geq 2$ are fixed. Let $T = T(n, k, a)$ with root x and let $y \in V(T)$ be any leaf of T. Then, for some k_0 and every $k \geq k_0$, there exists r such that $|\mathcal{I}_x^{(r)}(T)| > |\mathcal{I}_y^{(r)}(T)|$.*

We will work with limiting processes, so we establish some notation for that. For any fixed c, let $f(k) \sim cg(k)$ express the fact that $f(k)/g(k) \to c$ as $k \to \infty$. If $c = 1$, we may also say that f is *asymptotic* to g. Also, we will frequently refer to the number of independent sets of a path on n vertices, denoted by $F(n)$ (these numbers are also known as the *Fibonacci* numbers).

Now we are ready to state our theorem on star systems in $T = T(n, k, a)$.

Theorem 2.5.2. *Suppose $a, n \geq 2$ are fixed. Let $T = T(n, k, a)$ with root x and let $y \in V(T)$ be any leaf of T. Then*

$$|\mathcal{I}_x(T)| \sim \frac{F(a-1) + F(a-2)}{F(a-2) + F(a-2)} |\mathcal{I}_y(T)| \text{ as } k \to \infty.$$

Now we show how Theorem 2.5.1 follows from Theorem 2.5.2.

Proof of Theorem 2.5.1. Clearly $F(p) > F(p-1)$ for any $p \geq 1$. Theorem 2.5.2 thus implies that, for large enough k, $|\mathcal{I}_x(T)| > |\mathcal{I}_y(T)|$. Note that, by symmetry, $|\mathcal{I}_y(T)| = |\mathcal{I}_z(T)|$ for any leaf $z \in V(T)$. Therefore, given that $|\mathcal{I}_x(T)| = \sum_{r=1}^{\alpha(T)} |\mathcal{I}_x^{(r)}(T)|$ and $|\mathcal{I}_y(T)| = \sum_{r=1}^{\alpha(T)} |\mathcal{I}_y^{(r)}(T)|$, the conclusion follows. $\square$

We now give the proof of Theorem 2.5.2.

Proof of Theorem 2.5.2. For brevity we write $\mathcal{I}_v$ instead of $|\mathcal{I}_v(T)|$ for any vertex $v \in V(T)$. Recall that the vertices x and y are the root and a leaf, respectively, of T. As the number of independent sets containing any fixed leaf is the same, we only need to show that $|\mathcal{I}_x| \sim \frac{F(a)+F(a-1)}{F(a-1)+F(a-1)}|\mathcal{I}_y|$ as $k \to \infty$. Put $\mathcal{I}'_x = \mathcal{I}_x - (\mathcal{I}_x \cap \mathcal{I}_y)$ and $\mathcal{I}'_x = \mathcal{I}_y - (\mathcal{I}_x \cap \mathcal{I}_y)$.

Let us first approximate $|\mathcal{I}'_x|$ and $|\mathcal{I}'_y|$ as $k \to \infty$. Notice that $|\mathcal{I}'_x|$ and $|\mathcal{I}'_y|$ are equal to the number of independent sets in the graphs $T_x = T \setminus (N(x) \cup \{y\})$ and $T_y = T \setminus (N(y) \cup \{x\})$, respectively.

In order to evaluate $|\mathcal{I}'_x|$, notice that T_x consists of $nk-1$ disjoint copies of P_a and a copy of P_{a-1}. Thus

$$|\mathcal{I}'_x| = F(a)^{nk-1} \cdot F(a-1). \tag{2.5.1}$$

Let us now try to evaluate $|\mathcal{I}'_y|$. Notice that T_y is a disjoint union of $n-1$ copies C of a (k,a)-claw and one elongated claw C' having $k-1$ limbs of length a and one limb of length $a-2$. Let R be the set of roots of C and let $r(C')$ denote the root of C'.

We define a partition of T_y as follows:

- $S_1 = \{A \in \mathcal{I}(T_y) : A \cap (R \cup \{r(C')\}) = \emptyset\}$,
- $S_2 = \{A \in \mathcal{I}(T_y) : r(C') \in A\}$, and
- $S_3 = \{A \in \mathcal{I}(T_y) : r(C') \notin A, A \cap R \neq \emptyset\}$.

In other words, S_1 is the family of sets not containing any of the roots of the elongated claws in T_y; S_2 is the family of sets containing the root of C'; and S_3 is the family of sets containing a root distinct from $r(C')$. In particular, we have $|\mathcal{I}'_y| = |S_1| + |S_2| + |S_3|$ and, moreover, a straightforward argument yields:

$$|S_1| = F(a-2)F(a-1)^{nk-1} \tag{2.5.2}$$

$$|S_2| = F(a-3)F(a-1)^{k-1}\sum_{i=0}^{n-1}\binom{n-1}{i}F(a-1)^{ik}F(a)^{nk-ik-k} \tag{2.5.3}$$

$$|S_3| = F(a-2)F(a)^{k-1}\sum_{i=1}^{n}\binom{n-1}{i}F(a-1)^{ik}F(a)^{nk-ik-k}. \tag{2.5.4}$$

Hence we have

$$\begin{aligned}
|S_2| &= F(a-3)F(a-1)^{k-1}\sum_{i=0}^{n-1}\binom{n-1}{i}F(a-1)^{ik}F(a)^{nk-ik-k}\\
&\leq F(a-1)^{k}\sum_{i=0}^{n-1}\binom{n-1}{\lfloor (n-1)/2\rfloor}F(a-1)^{ik}F(a)^{nk-ik-k}\\
&= \sum_{i=0}^{n-1}\binom{n-1}{\lfloor (n-1)/2\rfloor}F(a-1)^{ik}F(a)^{nk-ik}F(a-1)^{k}F(a)^{-k}\\
&= F(a)^{nk}\sum_{i=0}^{n-1}\binom{n-1}{\lfloor (n-1)/2\rfloor}\left(\frac{F(a-1)}{F(a)}\right)^{ik}\left(\frac{F(a-1)}{F(a)}\right)^{k}\\
&\leq F(a)^{nk}n\binom{n-1}{\lfloor (n-1)/2\rfloor}\left(\frac{F(a-1)}{F(a)}\right)^{k}. \qquad (2.5.5)
\end{aligned}$$

and

$$\begin{aligned}
|S_3| &= F(a-2)F(a)^{k-1}\sum_{i=1}^{n}\binom{n-1}{i}F(a-1)^{ik}F(a)^{nk-ik-k}\\
&\leq F(a)^{k}\sum_{i=1}^{n}\binom{n-1}{\lfloor (n-1)/2\rfloor}F(a-1)^{ik}F(a)^{nk-ik-k}\\
&= \sum_{i=1}^{n}\binom{n-1}{\lfloor (n-1)/2\rfloor}F(a-1)^{ik}F(a)^{nk-ik}\\
&\leq \sum_{i=1}^{n}\binom{n-1}{\lfloor (n-1)/2\rfloor}F(a-1)^{k}F(a)^{nk-k}\\
&= F(a)^{nk}\sum_{i=1}^{n}\binom{n-1}{\lfloor (n-1)/2\rfloor}\left(\frac{F(a-1)}{F(a)}\right)^{ik}\\
&\leq F(a)^{nk}n\binom{n-1}{\lfloor (n-1)/2\rfloor}\left(\frac{F(a-1)}{F(a)}\right)^{k}. \qquad (2.5.6)
\end{aligned}$$

Combining (2.5.1), (2.5.2), (2.5.5) and (2.5.6) we have that as $k \to \infty$:

$$\begin{aligned}
(|\mathcal{I}'_y| + |S_1|)/|\mathcal{I}'_x| &= 2|S_1|/|\mathcal{I}'_x| + |S_2|/|\mathcal{I}'_x| + |S_3|/|\mathcal{I}'_x|\\
&\leq 2|S_1|/|\mathcal{I}'_x| + \frac{F(a)}{F(a-1)}(n+1)\binom{n-1}{\lfloor (n-1)/2\rfloor}\left(\frac{F(a-1)}{F(a)}\right)^{k}\\
&+ \frac{F(a)}{F(a-1)}(n)\binom{n-1}{\lfloor (n-1)/2\rfloor}\left(\frac{F(a-1)}{F(a)}\right)^{k}
\end{aligned}$$

$$\sim \quad 2|S_1|/|\mathcal{I}'_x| = 2F(a-2)/F(a-1). \tag{2.5.7}$$

We also have that

$$|\mathcal{I}'_x \cap \mathcal{I}'_x| = F(a-2)F(a)^{nk-1} = |S_1|. \tag{2.5.8}$$

Thus combining (2.5.1), (2.5.7) and (2.5.8) we obtain:

$$\begin{aligned}
|\mathcal{I}_x|/|\mathcal{I}_y| &= (|\mathcal{I}'_x| + |\mathcal{I}_x \cap \mathcal{I}_y|)/(|\mathcal{I}'_x| + |\mathcal{I}_x \cap \mathcal{I}_y)| \\
&= (|\mathcal{I}'_x| + |S_1|)/(|\mathcal{I}'_y| + |S_1|) \\
&\sim F(a-1)/2F(a-2) + |S_1|/(|\mathcal{I}'_y| + |S_1)|) \\
&\sim F(a-1)/2F(a-2) + |S_1|/(|S_1| + |S_1)|) \\
&= \frac{F(a-1) + F(a-2)}{2F(a-2)},
\end{aligned}$$

as required. □

We end this section by stating an immediate corollary of Theorem 2.5.1.

Corollary 2.5.3. *Suppose $a, n \geq 2$ are fixed. Let $T = T(n,k,a)$ with root x. Then, for some k_0 and every $k \geq k_0$, there exists r such that $|\mathcal{I}^{(r)}_x(T)| > |\mathcal{I}^{(r)}_v(T)|$ for every vertex $v \in V(T) \setminus \{x\}$.*

Proof. By Theorem 2.5.1, it suffices to show that for any leaf $y \in V(T)$ and any internal vertex $w \in V(T) \setminus \{x\}$ we have $|\mathcal{I}^{(r)}_y(T)| \geq |\mathcal{I}^{(r)}_w(T)|$. Let C be the elongated claw of the graph $T \setminus \{x\}$ that contains vertex w and suppose without loss of generality that $y \in V(C)$. Applying Lemma 2.3.1 completes the proof. □

We highlight that the proof of Theorem 2.5.1 does not give exact values of r for which T is a counterexample. Thus, motivated by Conjecture 2.1.5, one may ask if Conjecture 2.1.7 holds whenever $2r \leq \mu(G)$.

2.6 More general counterexamples to Conjecture 2.1.7

Now we describe a more general class of trees than that of Theorem 2.5.1 which can act as a counterexample to Conjecture 2.1.7:

Let T be a tree and let f be a vertex in T.

The rooted tree $G(n,k,T,f)$ is defined by a sequential process by *levels*, where the level of a vertex is one more than its distance from the root. We start with a root vertex x and we say $\mathcal{L}_1 = \{x\}$, and $G_1 = K_1$. We then form the tree G_2 by $V(G_2) = V(G_1) \cup \mathcal{L}_2$, where $|\mathcal{L}_2| = n$, and G_2 has an edge from x to v for all $v \in \mathcal{L}_2$. We then form the tree G_3 by $V(G_3) = V(G_2) \cup \mathcal{L}_3$, where $|\mathcal{L}_3| = nk$, and then adding edges from each vertex in $\mathcal{L}_2$ to k vertices in $\mathcal{L}_3$. Finally, we form $G(n,k,T,f)$ by identifying each vertex in $\mathcal{L}_3$ with a copy of f. If T is a path on a vertices and f is a leaf in T, then we arrive at the tree considered in Theorems 2.5.1 and 2.5.2.

We now state the following Theorem, which is an analogue of Theorem 2.5.2. Note that $F(G)$ refers to the number of independent sets in G (sometimes called the *Fibonacci number* of G).

Theorem 2.6.1. *Let $G(n,k,T,f)$ be defined as above. Then if x is the root and y is a vertex besides x and its neighbours in G, then:*

$$|\mathcal{I}_x(G)| \sim \frac{F(T-y) + F(T \downarrow y)}{F(T \downarrow y) + F(T \downarrow y)} |\mathcal{I}_y(T)| \text{ as } k \to \infty.$$

Proof. Let $\mathcal{I}_x$ denote the number of independent sets in G containing x, and let $\mathcal{I}_y$ denote the number of independent sets in G containing y. We compare $\mathcal{I}'_x = \mathcal{I}_x - (\mathcal{I}_x \cap \mathcal{I}_y)$ and $\mathcal{I}'_y = \mathcal{I}_y - (\mathcal{I}_x \cap \mathcal{I}_y)$, as before.

Let G_x denote the graph $G \downarrow x - y$, and let G_y denote the graph $G \downarrow y - x$. Then $F(G_x) = |I'_x|$ and $F(G_y) = |I'_y|$.

We first count $|I'_x|$:

- $|I'_x| = F(T-y)F(T)^{nk-1}$

Now we partition I'_y and establish some upper bounds on the sizes of the parts. Let $f(y)$ denote the copy of f in the copy of T containing y. Then we partition I'_y into three parts as follows:

- $S_1 = \{A \in \mathcal{I}(T_y) : A \cap (\mathcal{L}_3) = \emptyset\}$,
- $S_2 = \{A \in \mathcal{I}(T_y) : f(y) \in A\}$, and
- $S_3 = \{A \in \mathcal{I}(T_y) : f(y) \notin A, A \cap \mathcal{L}_3 \neq \emptyset\}$.

$$|S_1| = F(T \downarrow y)F(T)^{nk-1}$$

and

$$|S_2| = F(T \downarrow y - f)F(T-f)^{k-1} \sum_{i=0}^{n-1} \binom{n-1}{i} F(T-f)^{ik} f(T)^{nk-ik-k}$$

and

$$|S_3| = F(T \downarrow y)F(T)^{k-1} \sum_{i=1}^{n} \binom{n-1}{i} F(T-f)^{ik} F(T)^{nk-ik-k}$$

.

$$\begin{aligned}
|S_2| &= F(T \downarrow y - f)F(T-f)^{k-1} \sum_{i=0}^{n-1} \binom{n-1}{i} F(T-f)^{ik} F(T)^{nk-ik-k} \\
&\leq F(T-f)^{k} \sum_{i=0}^{n-1} \binom{n-1}{\lfloor (n-1)/2 \rfloor} F(T-f)^{ik} F(T)^{nk-ik-k} \\
&= \sum_{i=0}^{n-1} \binom{n-1}{\lfloor (n-1)/2 \rfloor} F(T-f)^{ik} F(T)^{nk-ik} F(T-f)^{k} F(T)^{-k} \\
&\leq \sum_{i=0}^{n-1} \binom{n-1}{\lfloor (n-1)/2 \rfloor} F(T)^{nk} (\frac{F(T-f)}{F(T)})^{k} \\
&\leq F(T)^{nk}(n) \binom{n-1}{\lfloor (n-1)/2 \rfloor} (\frac{F(T-f)}{F(T)})^{k}
\end{aligned}$$

which when divided by $|I'_x|$ gives:

$$\frac{F(T)}{F(T-f)}(n)\binom{n-1}{\lfloor (n-1)/2 \rfloor}(\frac{F(T-f)}{F(T)})^k$$

And this tends to 0 as k tends to infinity.

And

$$\begin{aligned}
|S_3| &= F(T\downarrow y)F(T)^{k-1}\sum_{i=1}^{n-1}\binom{n-1}{i}F(T-f)^{ik}F(T)^{nk-ik-k}\\
&\leq F(T)^k\sum_{i=1}^{n-1}\binom{n-1}{\lfloor (n-1)/2 \rfloor}F(T-f)^{ik}F(T)^{nk-ik-k}\\
&= \sum_{i=1}^{n-1}\binom{n-1}{\lfloor (n-1)/2 \rfloor}F(T-f)^{k}F(T)^{nk-k}\\
&\leq \sum_{i=1}^{n-1}\binom{n-1}{\lfloor (n-1)/2 \rfloor}F(T)^{nk}(\frac{F(T-f)}{F(T)})^k\\
&\leq F(T)^{nk}(n-1)\binom{n-1}{\lfloor (n-1)/2 \rfloor}(\frac{F(T-f)}{F(T)})^k
\end{aligned}$$

which when divided by $|I'_x|$ gives:

$$\frac{F(T)}{F(T-f)}(n-1)\binom{n-1}{\lfloor (n-1)/2 \rfloor}(\frac{F(T-f)}{F(T)})^k$$

And this tends to 0 as k tends to infinity.

Thus by the algebra of limits,

$$\frac{|I'_y|}{|I'_x|}\sim\frac{|S_1|+|S_2|+|S_3|}{|I'_x|}\sim\frac{|S_1|+0+0}{|I'_x|}\sim\frac{|S_1|}{|I'_x|}\sim\frac{F(T\downarrow y)}{F(T-y)}$$

Which, recombining with the remaining independent sets from $I_x \cap I_y$ gives the limit stated in the theorem. □

Further counterexamples to Conjecture 2.1.7 can be deduced by letting the tree T be any tree where the leaves are all symmetrical, and applying the symmetry of the leaves.

2.7 Trees with Multiple Large Non-Leaf Centred Stars

In Theorem 2.5.1 we saw that for any $d \geq 2$, there exists a tree, in which there exists a vertex of degree d, such that this vertex is contained in more independent r-sets than any leaf, for some $r \geq 5$. It is possible for there to be arbitrarily many non-leaf vertices with this property (e.g. 100 vertices of degree 100), and they can even be of varying degrees: this is what we now give a proof outline for (as before in Sections 2.5 and 2.6, by studying the limiting behaviour of a family of trees, rather than directly constructing one). Once again we start with a rooted tree and organise vertices by their levels, but this time the construction is a bit more complicated.

Let $S = \{k_1, k_2, ..., k_n\}$ be a multiset of positive integers. Then there is a tree $T = T(S, a, t)$, constructed by levels, as follows:

Level 1: Start with a root vertex x. Define $\mathcal{L}_1 = \{x\}$.

Level 2: Add n vertices as neighbours to x and denote these by v_i ($i \in \{1, ..., n\}$). The set of such vertices is called $\mathcal{L}_2$.

Level 3: Add k_i new vertices as neighbours to each vertex v_i added in Level 2. These vertices are called $\mathcal{L}_3$.

Level 4: Let K_i be equal to $\frac{1}{k_i}\prod_{j=1}^{n} k_j$, and then add tK_i new vertices as neighbours to each of the vertices defined in Level 3. These vertices are called $\mathcal{L}_4$.

Levels 5+: Identify one leaf of an a-vertex path with each of the vertices in $\mathcal{L}_4$ (i.e. one a-vertex path for each vertex). The new vertices are called members of $\mathcal{L}_{d+1}$ according as they have distance d from x.

For example, if $S = \{2, 2\}$ and $a = 2$, $t = 1$, we get the 23-vertex tree resulting from appending a single pendant vertex to each leaf of a full binary tree on 15 vertices. This is the simplest case of the construction. We prove the existence of the trees we seek by making t very large in a limiting process.

We will also require the following useful Lemma.

Lemma 2.7.1. *Let F be a forest. Then if l is a leaf and v is a vertex connected to l via a path P all of whose midpoints (i.e. vertices besides l and v) have degree 2, and such that v is not a leaf, then l is contained in strictly more independent sets than every other vertex of P (including v).*

Proof. We argue by induction. The theorem is easily seen to be true in forests of order 4 or less, where the leaf centred star systems are strictly larger than all the others. Now let P be the path from l to v and let l_2 be a leaf of F besides l, which must exist since every tree (or forest) with an edge has at least two leaves.

We consider the vertices on P and count how many independent sets they belong to in $T \downarrow l_2$ and $T - l_2$; these represent the independent sets containing l_2 or not containing l_2, respectively. Both these forests have lesser order than T and so we can apply the induction hypothesis to these. We just have to be careful in one case.

If we are comparing l to the neighbour w (in P of v), which becomes a leaf in $F \downarrow l_2$ if l_2 is adjacent to v then we cannot apply the induction hypothesis to w (and w is *not* contained in *strictly* fewer independent sets than l in this graph). To deal with this case, we simply observe that in $G - l_2$, w is not a leaf and so by applying the induction hypothesis in this graph we get strict inequality in T overall.

Otherwise, we can apply the induction hypothesis, and we are done. □

Theorem 2.7.2. *Let S be a multiset of positive integers of cardinality n, where $n \geq 2$. Then there is a positive integer t such that in the tree $T(S, 2, t)$, the n largest star system centres are precisely the $\mathcal{L}_2$-vertices, and all of them are strictly larger than any non-$\mathcal{L}_2$-centred star system.*

Proof. For an arbitrary vertex $v \in V(T(S, 2, t))$, let $\mathcal{I}_v$ denote the number of independent sets containing the vertex v in $T(S, 2, t)$. In what follows parameters such as S and n shall remain fixed whilst t is increased, so that $\mathcal{I}_v$ is a function of t.

Claim 1: $|\mathcal{I}_l| > |\mathcal{I}_v|$ for any leaf l and any vertex in $\mathcal{L}_3$ or $\mathcal{L}_4$ lying on the path from l to x.

Claim 2: $|\mathcal{I}_{v_i}| \sim |\mathcal{I}_{v_j}|$ for all $v_i, v_j \in \mathcal{L}_2$.

Claim 3: $|\mathcal{I}_{v_i}| \sim c|\mathcal{I}_y|$ where $c < 1$, for any $v_i \in \mathcal{L}_2$ and any leaf y.

Claim 4: $|\mathcal{I}_{v_i}| \sim c|\mathcal{I}_x|$ where $c < 1$, for any $v_i \in \mathcal{L}_2$.

To see that these four claims prove the theorem, let us assume they are true, and compare any vertex v_i in $\mathcal{L}_2$ with any other vertex v outside $\mathcal{L}_2$. If $v = x$ we are done by Claim 4. If v is a leaf at distance 3 from v_i we are done by Claim 3. If v is some other leaf we use Claim 2 and Claim 3, and if v is in $\mathcal{L}_3$ or $\mathcal{L}_4$ we use Claim 1, Claim 2 and Claim 3. And so, we are done.

Now we go through the claims, giving a proof of the first two. For the third and fourth claims we sketch an outline, as the details are similar to claim 2 but are somewhat tedious. We write $T = T(S, 2, t)$.

Proof of Claim 1: This is achieved by applying Lemma 2.7.1 to T, with v being the level 3 vertex.

Proof of Claim 2: To achieve this step we look at the vertex v_1 (without loss of generality) and prove that as t tends to infinity, the fraction of independent sets in $T(S, 2, t)$ that contain v_1 and a vertex in $\mathcal{L}_3$ tends to 0.

The number of independent sets containing v_1 is equal to $F(G_1)$, where $G_1 = H_1 + H_2 + ...H_n$, where H_1 consists of tL copies of P_2, and H_i consists of v_i joined to k_i $(tK_i, 2)$-claws for $i \geq 2$. Here we use the phrase "$(tK_i, 2)$-claw": it means an elongated claw with tK_i limbs each of length 2.

We prove that $F(G_1) \sim F(G_2)$, where G_2 is obtained from G_1 by deleting every $\mathcal{L}_3$-vertex: each neighbour of each v_i for $2 \leq i \leq n$. To see this, we count the independent sets in G_1 that use an $\mathcal{L}_3$-vertex and count the independent sets that don't (i.e. the independent sets of G_2), and divide the first one by the second.

The second one - the Fibonacci number of G_2 - is easy to calculate: $F(G_2) = 2^{n-1}3^{ntL}$, where L is the product of the k_i.

The number of independent sets in G_1 using at least one $\mathcal{L}_3$-vertex is equal to:

$$\sum_{\lambda} 3^{tL} \prod_{i=2}^{n} 2^{\lambda_i t K_i} 3^{(k_i-\lambda_i)tK_i} 2^{c(\lambda)}$$

Here $\lambda = (\lambda_2, ..., \lambda_n)$ is a vector expressing the number of neighbours of v_i as λ_i, for $2 \leq i \leq n$. $c(\lambda)$ refers to the number of the λ_i that are nonzero. Thus, the whole expression amounts to a sum of products of powers of 2 and 3 that depend on t. The number of ways to choose λ is finite and does not increase with increasing t. Thus, to show that the whole expression when divided by $F(G_2) = 2^{n-1}3^{ntL}$ tends to zero (as t tends to infinity) it is enough to show that the largest term in the sum when divided by $2^{n-1}3^{ntL}$ tends to zero.

So let us choose the largest. We assume without loss of generality that k_2 is the largest of all the k_i (or else just reorder the vertices). Therefore, the largest term is going to be the one with the fewest twos and the most threes; that is, when K_i is the smallest (meaning k_i needs to be the largest), and the independent sets contain precisely one neighbour of v_2 (and no neighbours of any other v_i). Therefore the expression for this term is

$$3^{tL} 2^{tK_2} 3^{(k_2-1)tK_2} \prod_{i=3}^{n} 3^{k_i t K_i} 2^{n-2}$$

which when divided by $2^{n-1}3^{ntL}$ gives

$$\frac{1}{2}(\frac{2}{3})^{tK_2}$$

which clearly tends to 0 as t tends to infinity. Thus $|\mathcal{I}_{v_1}| \sim F(G_2) = 2^{n-1}3^{ntL}$, and the same argument as given above can repeated for any other v_i (if $i = 2$, we would choose the second largest of the k_i instead of k_2). So this proves Claim 2.

Sketch of Claim 3: The strategy for Claim 3 consists of breaking down the family of independent sets containing y, where y is a leaf, as well as at least one $\mathcal{L}_3$-vertex, into 5 parts, and proving that the quantity of each of these parts tends to 0 when divided by $2^{n-1}3^{ntL}$ (the approximate number of independent sets containing v_1).

Thus, the portion of the star-system centred at y and using at least one $\mathcal{L}_3$-vertex is somehow negligible. One is then left with $\frac{2^n+1}{3}3^{ntL}$ independent sets containing y and no $\mathcal{L}_3$-vertex. This then produces the limit $\frac{2^n+1}{2^{n-1}3}$, which takes values lying between 2/3 and 5/6, depending on the value of n. The 5 parts require 5 cases, and we omit the details as they are each similar to the proof of Claim 2.

Sketch of Claim 4: This again uses the idea that "almost none" of the independent sets containing the root x contain any $\mathcal{L}_3$-vertex, being of a quantity which tends to 0 when divided by $2^{n-1}3^{ntL}$. The analogous limit to Claim 3 is much smaller, being equal as it is to $\frac{1}{2^{n-1}}$.

$\square$

Some Remarks: Thus, for example, there is a tree where the 100 largest star system centres have degree 100 (or any 100 positive integers you like; just make them all at least 2). Just setting S to be 100 copies of 99 is all that is required, before letting t tend to infinity. By symmetry, this also applies to the r-star centres too, so there exists a tree T where the largest 100 r-star centres all have degree 100 (although, we cannot specify the value of r beyond it being at least 5).

We also remark that the construction we gave was for $T(S, a, t)$; i.e. a need not be 2. As it happens the above proof goes through also if a is larger, with the one exception. If $a = 3$ and $n = 2$, then the limiting argument for the leaf versus the level 2 vertex becomes hamstrung by the fact that $\frac{2^2+1}{2^{2-1}} = \frac{F(3)}{F(1)} = 5/2$, so we cannot conclude that the $\mathcal{L}_2$-vertices are larger star system centres than the leaves. This problem happens because the smallest (bigger:smaller) ratio between two successive Fibonacci numbers occurs when the first one is 5 and the second one is 2. After that, it tends to the golden ratio squared which is about 2.618.

2.8 Conjecture 2.1.7 is true for Caterpillars

A *caterpillar* is a tree T where every vertex is either on or adjacent to a vertex on some underlying path P. More generally, a t-*caterpillar* is a tree where every vertex is at distance at most t from some underlying path P. The trees $T(2,k,t)$ in Theorem 2.5.1 are t-caterpillars, which means Conjecture 2.1.7 does not hold for t-caterpillars for $t \geq 2$.

Here we show that Conjecture 2.1.7 does hold for caterpillars (t-caterpillars with $t = 1$), which is therefore the best possible result in a sense. On the other hand, even for caterpillars it is *not* true that every leaf-star is larger than every non-leaf star: see the elongated claw with limbs of length 1, 2, and 3.

Theorem 2.8.1. *Let T be a caterpillar, and let $r \geq 1$. Then there exists a largest star in T whose centre is a leaf.*

We make a few definitions in order to prove Theorem 2.8.1.

Terminology: If T is a caterpillar, and x is a vertex that is not a leaf of T, we say x is a *path-vertex*. if two leaves are at distance 2 from each other we call them *siblings*. An *intermediate* leaf is a leaf that has two path-vertices at distance 2 from it. Otherwise a leaf is said to be an *end-leaf*.

Definition: Let T be a caterpillar and let x be a path-vertex. Then there exist two leaves l_1 and l_2 such that x lies on the path between l_1 and l_2, and this path is of minimal length with this property. These leaves are unique up to isomorphism (i.e. they might have siblings). We say that x is *in between* l_1 and l_2.

Definition: Let P_n denote the path on n vertices. Let $P_n(a,b)$ denote the graph obtained from P_n by adding $a-1$ siblings to the leaf at one end, and $b-1$ siblings to the leaf at the other end. So $P_n = P_n(1,1)$.

We will build up our caterpillar from P_n to start, then to $P_n(a,1)$, and from there to $P_n(a,b)$, before moving on to general caterpillars by adding intermediate leaves to $P_n(a,b)$.

Lemma 2.8.2. *Let T be the path on n vertices, P_n. Let $r \leq \alpha(T)$. Then the largest r-star in T is centred at a leaf.*

Proof. Let l be a leaf and x be a path-vertex. We find an injection from I_x^r (the star at x) into I_l^r (the star at l) by looking for independent sets that contain x but not l.

Let $I(x,l)$ denote the family of sets in I_x^r that do not contain l. Let $I(l,x)$ denote the family of sets in I_l^r that do not contain x. $I(x,l)$ is of equal cardinality to the collection of all $(r-1)$-sets in the graph $T \downarrow x - l$, where x and its two neighbours have been removed, and l has been removed. $I(l,x)$ is of equal cardinality to the collection of all $(r-1)$-sets in the graph $T \downarrow l - x$, where l and its one neighbour has been removed, and x has been removed.

$T \downarrow x - l$ is an induced subgraph of $T \downarrow l - x$. If the subgraph induced by the path from l to x is P_k, then each graph will have two components, one of which is P_{k-3}.

The other component will be a path in each case, but in $T \downarrow l - x$ it will have one more vertex, being P_{n-k} as opposed to P_{n-k-1}. As P_{n-k-1} is an induced subgraph of P_{n-k}, we have $T \downarrow x - l$ being an induced subgraph of $T \downarrow l - x$.

Thus we have an injection from the independent $(r-1)$-subsets of $T \downarrow x - l$ into the independent $(r-1)$-subsets of $T \downarrow l - x$. So we have an injection g from $I(x,l)$ into $I(l,x)$.

Let f be a function from I_x^r to I_l^r, defined by $f(A) = A$ if $l \in A$ and $f(A) = g(A)$ otherwise. Then f is an injection. □

Lemma 2.8.3. *Let $T = P_n(a,1)$. Let $r \leq \alpha(T)$. Then (a) the largest star is centred at a leaf having $a-1$ siblings, and (b) the leaf at the other end of the path is a larger star centre than any path-vertex.*

Proof. This uses the same "subgraph" method as the previous lemma. We start by proving (a).

Starting with a leaf l_1 that has $a-1$ silblings, and l_2 that has no siblings, we remove the independent r-sets containing both l_1 and l_2, and then look to make up $r-1$-sets in $T \downarrow l_1 - l_2$ and $T \downarrow l_2 - l_1$.

Since l_1 and l_2 have equal degree, neither of these graphs can be an induced subgraph of the other as before. Also, the larger star is now associated with the subgraph rather than the supergraph.

We have that $T \downarrow l_1 - l_2$ is a subgraph of $T \downarrow l_2 - l_1$, as the $a-1$ siblings are now disconnected vertices in the former, while the latter is a tree.

However a subgraph of a graph, that has the same number of vertices, will have more independent r-sets, and hence $|I^r_{l_1}| \geq |I^r_{l_2}|$.

(b): Suppose that the subgraph induced by the path from the path-vertex x to the leaf l is P_k. In this case star centred at the leaf l is associated with the graph $T \downarrow l - x$ which consists of P_{k-3} together with all the vertices that do not lie on a path from x to l. In the case of the vertex x we have $T \downarrow x - l$ which consists of P_{k-3} together with all the vertices that do not lie on a path from x to l, except those that are neighbours of x.

Therefore $T \downarrow x - l$ is an induced subgraph of $T \downarrow l - x$, will have fewer independent $(r-1)$-sets, and we get that $I(x, l)$ injects into $I(l, x)$.

We reach the conclusion that I^r_x injects into I^r_l.

Therefore the leaves in this graph are larger star-centres than all the path-vertices.

□

Now we move onto $P_n(a, b)$ for $b > 1$:

Lemma 2.8.4. *Let T be a graph and let v and w be vertices in T such that v is a leaf and $|I^r_v(T)| \geq |I^r_w(T)|$ for all r. Let T' be the graph obtained by adding a leaf u at distance 2 from v. Then $|I^r_v(T')| \geq |I^r_w(T')|$ for all r.*

Proof. Let r be arbitrary. Then $|I^r_v(T')| = |I^r_v(T)| + |I^{r-1}_v(T)|$, as the first and second summand counts the sets that do not or do contain u respectively.

Meanwhile $|I_w^r(T')| \leq |I_w^r(T)| + |I_w^{r-1}(T)|$. The inequality is because not every set in $I_w^{r-1}(T)$ gives rise to a r-set in $I_w^r(T')$, as some sets will already contain the neigbour of v and u.

□

Corollary 2.8.5. *Consider the graph $T = P_n(a, b)$. Then if l is a leaf with $b-1$ neighbours, and x is a path-vertex, $I_x^r(T) \leq I_l^r(T)$.*

Proof. Follows from $b-1$ applications of Lemma 2.8.4 to the graph $T = P_n(a, 1)$ and the vertex l.

□

Lemma 2.8.6. *Let $r \geq 1$. The leaf-centred r-stars in $P_n(a, b)$ are larger than the path-vertex-centred stars.*

Now we move onto caterpillars.

Theorem 2.8.7. *Let T be a caterpillar, let $r \geq 1$ and let x be a path-vertex such that x is between l_1 and l_2. Then $|I_{l_1}^r| \geq |I_x^r| \leq |I_{l_2}^r|$.*

Proof. Let T be a caterpillar and assume that the statement in the theorem holds for all caterpillars having fewer vertices than T. Let x be a path-vertex between the leaves l_1 and l_2. If T has no intermediate leaves then $T = P_n(a, b)$ for some $a, b \geq 1$. In that case we are done by Lemma 2.8.6. So assume T has at least one intermediate leaf, and let l_3 be a leaf not a sibling of l_1 or l_2, which must now exist.

The star $I_{l_1}^r(T)$ may be partitioned into two sets, those which do not contain l_3 and those that do contain l_3. Thus:

$$|I_{l_1}^r(T)| = |I_{l_1}^r(T - l_3)| + |I_{l_1}^{r-1}(T \downarrow l_3)|$$

We can do the same for x and l_2, therefore the inequalities:

$$|I_{l_1}^r(T - l_3)| \quad \geq \quad |I_x^r(T - l_3)| \quad \leq \quad |I_{l_2}^r(T - l_3)|, \tag{2.8.1}$$

$$|I_{l_1}^{r-1}(T \downarrow l_3)| \;\; \geq \;\; |I_x^{r-1}(T \downarrow l_3)| \;\; \leq \;\; |I_{l_2}^{r-1}(T \downarrow l_3)|, \tag{2.8.2}$$

will give the theorem.

First we prove inequality (2.8.1). $T - l_3$ is a caterpillar in which x is a path-vertex between l_1 and l_2 the leaves. Also, $T - l_3$ is of a smaller order than T. Therefore $|I_{l_1}^r(T - l_3)| \geq |I_x^r(T - l_3)| \leq |I_{l_2}^r(T - l_3)|$ from the inductive hypothesis.

To prove (2.8.2), observe that if l_3 has siblings in T, then $T \downarrow l_3$ is not a caterpillar.

However, let T' be the caterpillar obtained by removing l_3, its neighbourhood and its siblings from T. Then the set of all independent $(r-1)$-sets in $T \downarrow l_3$ containing x and exactly 1 of the siblings of l_3 has cardinality equal to $k|I_x^{r-2}(T')|$, where k is the number of siblings of l_3 in T.

Likewise the set of all independent $(r-1)$-sets in $T \downarrow l_3$ containing x and exactly 2 of the siblings of l_3 has cardinality equal to $\binom{k}{2}|I_x^{r-3}(T')|$.

Partitioning we get

$$|I_x^{r-1}(T \downarrow l_3)| = k|I_x^{r-2}(T')| + \binom{k}{2}|I_x^{r-3}(T')| + \binom{k}{3}|I_x^{r-4}(T')| + \ldots + \binom{k}{k}|I_x^{r-(k+1)}(T')|$$

Now we can do the same for l_1 and l_2 as they are vertices in T'. So we have

$$|I_{l_1}^{r-1}(T \downarrow l_3)| = k|I_{l_1}^{r-2}(T')| + \binom{k}{2}|I_{l_1}^{r-3}(T')| + \binom{k}{3}|I_{l_1}^{r-4}(T')| + \ldots + \binom{k}{k}|I_{l_1}^{r-(k+1)}(T')|$$

and the same for l_2.

Now if l_1 and l_2 are end-leaves in T', we have that $|I_{l_1}^r(T')| \geq |I_x^r(T')|$ for all r. Hence

$$|I_{l_1}^{r-1}(T \downarrow l_3)| \geq |I_x^{r-1}(T \downarrow l_3)|$$

and the same for l_2.

If l_1 and l_2 are not both end-leaves in T', then since $|T'| < |T|$, we have the inductive hypothesis yielding that

$$|I_{l_1}^{r-1}(T \downarrow l_3)| \geq |I_x^{r-1}(T \downarrow l_3)|$$

and the same for l_2.

Therefore we get the inequality (2.8.2). $\square$

Theorem 2.8.7 also implies:

Theorem 2.8.8. *Let T be a caterpillar, and let x be a path-vertex such that x is between l_1 and l_2. Then l_1 and l_2 are contained in (non-strictly) more independent sets than x in T.*

Proof. Let T be a caterpillar. Then adding up the cases of Theorem 2.8.7 for $r = 1, 2, 3, ..., \alpha(T)$, gives the result. $\square$

Theorems 2.8.7 and 2.8.8 together imply Theorem 2.8.1.

2.9 Remark on the number of independent r-sets in paths

Let us mention a property of trees with respect to independent r-sets. Let T be a tree and $r \geq 1$. How many independent r-sets are there in T? It turns out that the minimum value is obtained on two cases: paths (for all r) and 2-subdivisions of trees (for large r). We state this as a theorem:

Theorem 2.9.1. *Let T be a tree on n vertices, and $r \geq 1$ be a positive integer. Assume also that T possesses the minimum possible number of independent r-sets amongst the trees on n vertices. Then either T is a path, or T is a 2-subdivision of a tree and $r = \alpha(T)$.*

Observation: A tree is not a 2-subdivision of a tree if and only if it contains two vertices not of degree 2 that are at odd distance from each other, which we will call *nodal vertices*.

Using this observation, Lemma 2.4.1 and induction, we prove Theorem 2.9.1.

Proof. The theorem is true for $n \leq 5$. Suppose the theorem holds for all numbers strictly less than n, and let T be a tree on n vertices. Assume that T is not P_n or a 2-subdivision of a tree.

Then T contains at least one pair of nodal vertices that are at odd distance from each other. Amongst these vertices let a and b be at minimal odd distance from each other. Then on the path between a and b there are not any nodal vertices. Also note that vertices a and b cannot both be leaves since T is not a path.

Case 1: $r > 3$, and either a or b is a leaf

Let a be a leaf. If $d(a, b) = 1$, then $T \downarrow v$ is a disconnected forest on $n - 2$ vertices, and so will have strictly more independent $r - 1$-sets than P_{n-2}. If $d(a, b) > 1$, then $T \downarrow a$ is a forest on $n - 2$ vertices having a pair of nodal vertices at odd distance from each other. As $r > 3$, we can apply the inductive hypothesis to this graph and get that $I_{r-1}(T \downarrow a) > I_{r-1}(P_{n-2})$.

Case 2: $r > 3$, and a and b are not leaves

Let v be any leaf in T. Then $T - v$ is a tree having a pair of nodal vertices at odd distance from each other (a and b). Thus we have $I_r(T - v) > I_r(P_{n-1})$ by induction.

Case 3: $r = 3$, and either a or b is a leaf

If $r = 3$, then assuming a is a leaf, $T \downarrow a$ might be a tree, and the 2-independent sets will be the same as in P_{n-2}.

Consider any leaf not equal to a, say, v. Then $T - v$ is still a forest (a tree, actually) having a pair of nodal vertices (a and b) at odd distance from each other. Also $3 \leq \alpha(T - v)$ since $T - v$ has at least 5 vertices. Thus we have $I_r(T - v) > I_r(P_{n-1})$ by induction.

Case 4: $r = 3$, and a and b are not leaves

This is the same as the previous case, except v can be any leaf of T.

In every case we have that $I_r(T - v) > I_r(P_{n-1})$ or $I_{r-1}(T \downarrow a) > I_{r-1}(P_{n-2})$ and so we are done by Lemma 2.4.1. $\square$

2.10 Remark on Disjoint Unions Of Complete Graphs

We also remark upon the following method of deducing an Erdős-Ko-Rado theorem for disjoint unions of complete graphs, directly from the Erdős-Ko-Rado theorem, which to the best of our knowledge is new.

Lemma 2.10.1. *Let G be the disjoint union of r copies of K_t. Then any intersecting family of t-independent sets in G has cardinality at most $t^r/t = t^{r-1}$.*

Proof. Let the vertices of G be numbered $v_{i,j}$, where vertex $v_{i,j}$ is the jth vertex of the ith copy of K_t (so $1 \leq i \leq r$ and $1 \leq j \leq t$).

The vertex set of G can be partitioned into t classes according to the value of j associated to that vertex. The result follows from noting that independent r-sets using vertices from different parts of the partition are disjoint. □

Theorem 2.10.2. *Let G be a disjoint union of n copies of the complete graph K_t. Then G is r-EKR for $r \leq n/2$, and strictly so if $r < n/2$.*

Proof. In order for two independent r-sets to intersect in some vertex, they must first intersect in some connected component. By the Erdős-Ko-Rado Theorem, there are at most $\binom{n-1}{r-1}$ ways for this to happen.

Therefore, the independent sets must each come from one of at most $\binom{n-1}{r-1}$ classes, each class determined by the components it uses. Within each class, there can be at most $t^r/t = t^{r-1}$ independent sets, by Lemma 2.10.1.

Multiplying together, we get that the size of an intersecting family is at most $\binom{n-1}{r-1}t^{r-1}$. But this is the size of a star centred at some vertex of G. The strictness component of the theorem follows from applying the strictness component of the Erdős-Ko-Rado Theorem to the quantity $\binom{n-1}{r-1}$. □

Chapter 3

Edge Modification in Perfect Graphs

3.1 Introduction

In this chapter we are interested in perfect graphs and their various subclasses. A graph G is *perfect* if every induced subgraph has the property that it has a vertex colouring with as few colours as the size of a largest clique in that induced subgraph. Clearly this is the best possible, as any clique of ω vertices will require ω distinct colours. Thus, the perfect graphs represent the solutions to a kind of extremal problem in vertex colouring.

Perfect graphs are an important class of graphs and have been a pillar of the developement of graph theory. Firstly, they hold great appeal for their nice duality properties. Secondly, many of the most basic graph classes in graph theory are perfect: Complete graphs, empty graphs, paths, trees, forests and bipartite graphs (but crucially not cycles) are perfect. Slightly more advanced, but well-known classes such as chordal graphs, comparability graphs, cographs, interval graphs, split graphs, and threshold graphs are also included. Furthermore, perfect graphs have deep connections with distinct areas such as structural graph theory [16, 17], Shannon capacity [3, 4, 7], linear programming [34], and graph algorithms [15, 44].

Perfect graphs were first introduced by Berge at a conference in Halle in 1960. Berge then gave a series of talks [3, 7] in the early 1960s, in which he expanded upon the concept. It was during these talks that Berge posed two conjectures about perfect graphs, which were later to become known as the perfect graph theorems:

The Perfect Graph Conjectures (Berge [3]) *Let G be a graph. Then the following is true:*

1. *G is perfect if and only if $\overline{G}$ is perfect;*

2. *G is perfect if and only if it G does not contain any odd cycles of length* 5 *or more ("odd holes") or the complement of an odd cycle of length* 5 *or more ("odd antiholes") as induced subgraphs.*

The first statement became known as *The Perfect Graph Conjecture.* The second statement, which clearly implies the first one, became known as *The Strong Perfect Graph Conjecture.* These two conjectures and the successful effort to turn them into theorems would play a important role in encouraging the developement of the theory of perfect graphs. Used in the second statement are the *holes* and *antiholes*: These are the cycles which are not triangles, and respectively their complements.

A sequence of proofs of perfection for various subclasses of perfect graphs in the 1960s began to provide evidence for Berge's conjectures. The first graph class to be explicitly recognised as having the property we now call perfection were the bipartite graphs and their complements: Gallai [36] had observed this as a corollary of Kőnig's theorem on bipartite graphs [62]. Hajnal and Suranyi [45] then proved that chordal graphs are perfect. In [6] Berge gives proofs of perfection of chordal graphs, unimodular graphs, interval graphs, comparability graphs and the line graphs of bipartite graphs. This provided evidence for and interest in Berge's conjectures. In [4] Berge himself gives a detailed account of how he came to develop his theories and the two conjectures.

During the 60s the problem of proving The Perfect Graph Conjecture had become a very great question in graph theory. Finally its truth was demonstrated by László

Lovász [65, 66], who in 1972 gave two proofs of this conjecture, whereby it became known as *The Perfect Graph Theorem*.

The Perfect Graph Theorem (Lovász [66])

The complement of a perfect graph is perfect.

Fulkerson in [34] found his own proof almost immediately after Lovász. Fulkerson had reduced the problem to the important lemma now known as the "Substitution Lemma", but had been unable to prove this lemma. After he learned of Lovász's success, he proved the lemma in a matter of hours. Later other proofs were found by Padberg [76] and Gasparian [38] that used linear algebra.

The proofs of The Perfect Graph Theorem essentially divide into three types: Graph-theoretic (Lovász), Linear Algebra (Padberg and Gasparian), and Polyhedral/Linear Programming (Fulkerson). It should also be pointed out that Lovász and Fulkerson both used the Substitution Lemma, while Padberg and Gasparian did not. Thus the proofs of The Perfect Graph Theorem also divide into two types: those using the Substitution Lemma (Lovász and Fulkerson) and those using Linear Algebra (Padberg and Gasparian). To the best of our knowledge, *all* the known proofs of The Perfect Graph Theorem fall into one of these types.

After the proof of The Perfect Graph Theorem, the second - and harder - of Berge's conjectures still remained open. It became standard to name the graphs having no odd holes or odd antiholes as *Berge graphs*. A perfect graph must clearly be Berge, by the fact that an odd hole has chromatic number 3 and clique number 2, and an odd hole of length $2k + 1$ has chromatic number $k + 1$ and clique number k. This allowed a more succinct statement of the conjecture.

The Strong Perfect Graph Conjecture (Version 2)

Every Berge graph is perfect.

Several approaches to The Strong Perfect Graph Conjecture where developed. These were roughly divided into three depending on the use of purely graph-theoretic methods, linear algebraic methods, or polyhedral methods [82]. In this respect, it resembled the situation for the proof of The Perfect Graph Theorem.

Eventually in 2002 the problem was solved - in the positive - with a proof of some considerable complexity. This proof is due to Chudnovsky, Robertson, Seymour and Thomas [17]. It arrived only shortly before the passing of Claude Berge. To date, this is still the only proof. The methods used were of the purely graph-theoretic kind ("structural graph theory"), the Strong Perfect Graph Theorem itself being deduced from the ability of any Berge graph to be decomposed by certain perfection-preserving operations into smaller graphs which are each either: a bipartite graph or the complement of a bipartite graph, a line graph of a bipartite graph or the complement of such a graph, or a doubled split graph (self complementary class) (see [17] for details).

The Strong Perfect Graph Theorem (Chudnovsky, Robertson, Seymour and Thomas [17])

Every Berge graph is perfect.

The proof ran to 179 pages; it was shortened to 131 pages in [15]. Paul Seymour has given an overview of the search for the proof in [89]. Additional survey articles on the strong perfect graph theorem (SPGT) are available (see [82],[93]).

For general references on perfect graphs, there are books dedicated to them by Golumbic [42], Berge and Chvátal [8] and Alfonsin and Reed [1]. There is a large collection of some 120 subclasses of perfect graphs described by Hougardy [56], with exhaustive information on the various inclusions between subclasses. The survey of graph classes by Brandstadt, Le and Spinrad [13] has a large section devoted to perfect graphs, and contains a lot of useful information.

An important notion that comes up time and again in the theory of perfect graphs (for example, in the proofs of both the weak and strong perfect graph theorems) is that of a *minimally imperfect graph.* A minimally imperfect graph is an imperfect graph with no imperfect proper induced subgraphs. By descending induction, any imperfect graph contains a minimally imperfect induced subgraph. Conversely, any graph containing a minimally imperfect induced subgraph is imperfect. Hence, a graph is perfect if and only if it contains no minimally imperfect graph as an induced subgraph.

The Perfect Graph Theorems for Minimally Imperfect Graphs

1. *The Perfect Graph Theorem: The complement of a minimally perfect graph is a minimally imperfect graph.*

2. *The Strong Perfect Graph Theorem: The minimally imperfect graphs are precisely the odd holes and the odd antiholes.*

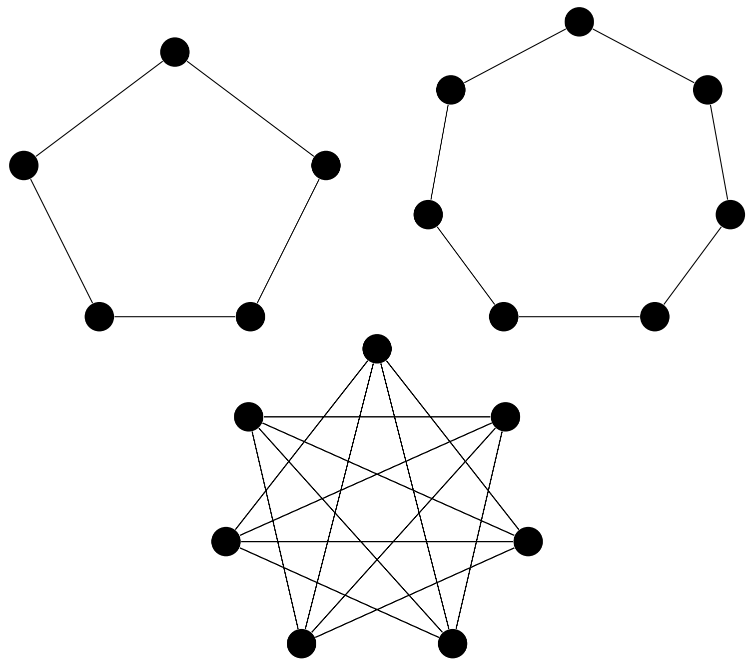

Figure 3.1: The first three examples of minimally imperfect graphs are given by C_5, C_7 and $\overline{C_7}$.

The first few odd holes and odd antiholes are illustrated in Figure 3.1, with the first one being C_5. Let us take a moment to note *why* they are imperfect.

A minimum partition into cliques has size 3: It consists of two of the edges and the "leftover" vertex. On the other hand, if we choose any 3 vertices $\{v_1, v_2, v_3\}$, numbered according as they are positioned cyclically clockwise from v_1, then if v_2 is not adjacent to v_1 it must be at least distance 2 from it, and if v_3 is not adjacent to v_2 it must be another 2 vertices away, in which case v_3 is adjacent to v_1 and $\{v_1, v_2, v_3\}$ is not a stable set. Thus $2 = \alpha(C_5)$, and $2 < 3$.

On the other hand, $\omega(C_5) = 2$ clearly, while the graph is not bipartite as if we colour vertices red, blue, red, blue, red... alternately, the oddness of the length of our sequence leaves us finishing with the first and fifth vertices both labeled red. Thus $\chi(C_5) = 3$, and $2 < 3$.

These two arguments, although given for C_5 as an example, readily generalise to the rest of the odd holes and antiholes.

3.1.1 Subclasses of the perfect graphs and algorithmic properties

As we said, one reason the perfect graphs are so important is the fact that they contain within them many of the basic and naturally occurring graph classes of graph theory; as if "by accident". We now pause to give a description of a few of these classes; we will need refer to them again later in the chapter. We also mention a few of their algorithmic properties.

Basic graph types: Many of the graph types that typically appear at the beginning of textbooks on graph theory are perfect. *Complete graphs, empty graphs, complete multipartite graphs, triangles, paths, trees, forests, and bipartite graphs*; all these are perfect. The notable absentee from this list is the class of *cycles*: The clarification of the relationship between the class of perfect graphs and the class of cycles is found in the strong perfect graph theorem.

Chordal Graphs: A *chordal graph* is a graph containing no induced cycles besides triangles. Several proofs exist that chordal graphs are perfect, see for example [23, 22]. Equivalently, they are the graphs that admit a vertex ordering with the property that any vertex, together with its neighbours which precede it in the ordering, form a clique. Yet another characterisation of chordal graphs is as the intersection graphs of the subtrees of a tree [39]. Chordal graphs are also closely connected to the notion of *treewidth* [81].

Comparability Graphs: A *comparability graph* is a graph whose edges can be oriented such that the resulting binary relation forms a partially ordered set. These contain the bipartite graphs as a subclass. The perfection of comparability graphs is equivalent to Mirsky's Theorem [86], and the perfection of the complements of comparability graphs is equivalent to Dilworth's Theorem [86]. Early investigations of these graphs were by Gallai [37] and Ghouila-Houri [40].

Cographs: A graph is a *cograph* if it can be built up from complete graphs via the operations of complementation and disjoint union. The perfect graph theorem together with the perfection of complete graphs and the fact that a disjoint union of perfect graphs is itself perfect implies the perfection of this class. There are many characterisations of this graph class [42]: They are the graphs where every induced path either has length 1 or length 2 [42]; they are the special case of comparability graphs where the orientation is required to be a partial order of order dimension 2 [42].

Split Graphs: A graph is a *split graph* if and only if it has a vertex set bipartition into a clique and a stable set. Equivalently, they are the chordal graphs whose complements are also chordal - this fact was proved in [31] by Foldes and Hammer, where split graphs were first defined. By this and the perfection of chordal graphs, split graphs are perfect. Another elegant characterisation is that they are the intersection graphs of the subtrees of a tree, restricted to the special case where all the trees are stars [13].

Threshold Graphs: A graph is said to be a *threshold graph* if it possesses a *threshold elimination ordering*, which is an ordering of the vertex set such that every vertex is either adjacent to everything coming before it in the ordering, or else is nonadjacent to everything coming before it in the ordering. Equivalently, they are the graphs that are both split graphs and cographs, thereby implying that they are perfect. They were introduced by Chvatal and Hammer in [19], and are extensively studied in the textbook "Threshold Graphs and Related Topics" [68].

Interval Graphs: A graph G is said to be an *interval graph* if there is a one to one correspondence between the vertices of G and a set of intervals of the real line such that two vertices are adjacent precisely if their corresponding intervals intersect. They are perfect by virtue of being a subclass of the chordal graphs: in fact they are equivalent to the chordal graphs whose complement is a comparability graph [42]. Also, the relation of interval graphs to the notion of *pathwidth* is the same as for chordal graphs and treewidth [81].

Permutation Graphs: A graph G is a *permutation graph* if there is a pair of linear orderings of $V(G)$ such that xy is an edge in G if and only if x is less than y in one of the orderings but x is more than y in the other ordering. Equivalently, the vertices of G are the elements of a permutation and the edges of G represent the pairs of elements reversed in this permutation: Hence the term "Permutation Graphs". They are the comparability graphs which are also the complements of comparability graphs [42], and are hence perfect. The cographs are the special case of permutation graphs, restricted to the case where the permutation representing G is separable [12].

Weakly Chordal Graphs: A graph G is said to be *weakly chordal* if it has no holes or antiholes of length at least 5. They were first introduced by Hayward [79]. They have nice structural decomposition properties which are somewhat similar to the decomposition of Berge graphs used to prove the Strong Perfect Graph Theorem (see [93] for details) and share some (but not all) of the properties of the chordal graphs, of which they are a generalisation [79].

Trivially Perfect Graphs: A partially ordered set P is called a *tree* if every suborder of P obtained by taking all the elements less than or equal to some element x is a linear ordering. These objects are closely related to the trees of graph theory: The Hasse diagram of an order-theoretic tree is a graph-theoretic tree. The comparability graphs of order-theoretic trees are called the *trivially perfect graphs.* As comparability graphs, they are perfect. Like for the relationship between chordal graphs and treewidth, and the relationship between interval graphs and pathwidth, there is a relationship between trivially perfect graphs and *treedepth* [49].

The perfect graphs themselves can be recognised in polynomial time [16], thanks to the SPGT which says they are equivalent to the Berge graphs, and the 2005 achievement of Chudnovsky, Cornuéjols, Liu, Seymour and Vušković, who gave an $O(n^9)$-time algorithm for recognising Berge graphs. By this point the problems of finding a maximum clique and maximum stable set in a perfect graph, and a minimum vertex colouring and minimum clique covering in a perfect graph, had all been solved with the finding of polynomial time algorithms [44].

For some special subclasses of perfect graphs the maximum clique and minimum colouring problems can be solved in $O(n+m)$-time, for example comparability graphs [71] and their subclasses such as cographs and permutation graphs.

We shall have recourse to mention each of these subclasses defined above again, later in the chapter.

3.1.2 Main results

Our interest in perfect graphs focuses on their properties under edge modification, and this will be split into three main parts: Sections 3.3, 3.4 and 3.5. These follow Section 3.2, in which we give a proof of the Perfect Graph Theorem by Gasparian.

In Section 3.3 we introduce the critical edges/nonedges of a graph, and use them to characterise perfect graphs. If G is a graph with stability number $\alpha(G)$ and clique number $\omega(G)$, an edge xy is critical if $\alpha(G - xy) > \alpha(G)$, and a nonedge xy critical if $\omega(G + xy) > \omega(G)$, and any other edges/nonedges are free.

For example: suppose we have a graph G and we wish to modify G via successive edge deletions in order to give it some specified property? Can we disconnect G by deleting only free edges from G, for example? Can we turn G into a cograph? We use this kind of question to provide some characterisations of perfect graphs, in terms of edge modification. We also look at free nonedges, where the addition of the nonedge to the edge set of G does not increase the clique number (and a similar notion of critical edge applies here as well).

We also consider edge deletions that decrease the clique number, and edge additions that decrease the stability number, as this can also be used to characterise perfect graphs (this uses the strong perfect graph theorem).

In Section 3.4 we look at edge modification in minimally imperfect graphs. For some time now, the critical edges and nonedges of minimally perfect graphs have been seen to be related to the the strong perfect graph theorem. This has been investigated by Gasparian, Markossian and Markossian [70], who introduced the concept of critical edges/nonedges, and it was then developed further by Sebo [88]. These investigations were motivated by a desire to prove the existence of critical edges in minimally imperfect graphs with a view to proving the SPGT.

Sebo's Problem (Sebo [88]) *Let G be a minimally imperfect graph. Show that G contains a critical edge.*

Sebo's problem forms one of the principle problems of the linear algebraic approach to the SPGT. The linear algebraic method essentially begins with the proofs of The Perfect Graph Theorem by Padberg [76] and Gasparian [38]. Expanding upon this method could possibly be a candidate approach for a second proof of the SPGT [82, 70, 88].

Our investigation in this direction begins with an easy observation on a property shared by the odd holes and the odd antiholes, which we call the *dichotomy property*.

Dichotomy Property (Theorem 3.4.5) *Let G be an odd hole or odd antihole. Then the following holds:*

1. *If xy is an edge in G then xy either is a critical edge and $G - xy$ is Berge, or xy is free edge and $G - xy$ is not Berge.*

2. *If xy is a nonedge in G then xy either is a critical nonedge and $G + xy$ is Berge, or xy is a free nonedge and $G + xy$ is not Berge.*

The point of the dichotomy property is that the Berge structure of the odd holes and odd antiholes is related to the parameters α and ω. Our aim is to try and replicate this relationship not just in odd holes and odd antiholes, but in hypothetical counterexamples to the SPGT.

We make the following definitions, which apply to any graph G (whether G is Berge or not). A *Berge-free edge* in a graph G is an edge e such that $G - e$ is not Berge. Non-Berge-free edges are called *Berge-critical.*

We can therefore restate the dichotomy property for odd holes and odd antiholes.

Dichotomy Property (Theorem 3.4.5) *Let G be an odd hole or odd antihole. Then every edge/nonedge xy is critical if and only if it is Berge-free (and free if and only if it is Berge-critical).*

Now, recalling the idea of a minimum imperfect Berge graph, we define a *smallest imperfect Berge graph* to be a minimum imperfect Berge graph with the smallest possible size. The nonexistence of such a graph is equivalent to the SGPT.

In Section 3.4 we will prove the following theorem for smallest imperfect Berge graphs, which is a weaker version of the dichotomy property.

Theorem 3.4.7 *Let G be a smallest imperfect Berge graph. Then for every edge xy in G, either xy is critical or xy is Berge-critical.*

A key property of the smallest imperfect Berge graphs - which is not readily deducible in a minimum imperfect Berge graph - is that deleting an edge never produces another minimally imperfect graph. The idea that you never get two “adjacent pairs” of minimally imperfect graphs - one differing from another in a single edge change - seems intuitive enough, especially in light of the actual structure of the odd holes and odd antiholes, but it is seemingly difficult to prove. The notion of smallest imperfect Berge graph effectively serves as a kind of “way out” of this problem.

A corollary of Theorem 3.4.7 is that in a smallest imperfect Berge graph either a critical edge must exist, or every edge in the graph must be Berge-critical. We call a Berge graph with all edges Berge-critical *critically Berge.*

Critically Berge graphs have been studied by Wagler, in the form of *critically imperfect graphs*. A critically perfect graph, as defined by Wagler in [95, 96], is a perfect graph where the removal of any edge leaves an imperfect graph. By the SPGT the critically Berge graphs and the critically perfect graphs are the same.

Wagler gives several examples of these graphs in her PhD thesis [95]. These include graphs G where G and $\overline{G}$ are critically Berge. The knowledge that critically Berge are perfect, or at any rate not minimally imperfect, would imply the existence of critical edges in smallest imperfect Berge graphs. Thus, greater knowledge about the properties of critically Berge graphs may provide a route to a solution of Sebo's Problem.

In Section 3.5 of this chapter the focus is on edge modification in specific subclasses of the perfect graphs. We investigate *freedom games*. A freedom game consists of a class $\mathcal{C}$ of graphs, which contains both empty and complete graphs, for which the game is to move between the empty graph and complete graph, or vice versa, changing one edge at a time, remaining inside $\mathcal{C}$. Ideally one would also like to be able to start at any graph in the class, and arrive at the complete graph or the empty graph in this way, without "falling out" of the class.

Formally, if every graph $G \in \mathcal{C}$ is either complete or there is a nonedge xy of G such that $G + xy \in \mathcal{C}$, and is either empty or there is an edge xy of G such that $G - xy \in \mathcal{C}$, we say that $\mathcal{C}$ has the *freedom property*. Note that this is not a property of the individual graphs, but rather $\mathcal{C}$.

The class of perfect graphs does not have the freedom property, due to the existence of critically perfect graphs. Likewise, cographs (because of $2K_2$ and C_4) do not have the freedom property. Bipartite graphs and forests cannot be considered because they do not contain complete graphs as a special case (although, they do have the edge subtraction part of the property).

Nevertheless, we demonstrate several subclasses of the perfect graphs that *do* have the freedom property. For example, we will prove following theorem on split graphs:

Theorem 3.5.1 *Let G be a split graph and let x be a vertex of G. If x is not a dominating vertex, then there exists a non-neighbour y of x such that the graph $G + xy$ is a split graph. Likewise, if x is not an isolated vertex, then there exists a neighbour y of x such that $G - xy$ is a split graph.*

This property is actually stronger than the freedom property. We call this *local freedom*, where the edge to be added can be chosen from the set of non-neighbours of any vertex, and the edge to be deleted can be chosen from the set of neighbours of any vertex, while the assumption of not being an isolated vertex (respectively dominating vertex) replaces that of the graph not being empty (respectively complete).

We demonstrate that this local freedom property holds for not only the class of split graphs, but also for the classes of interval, threshold, and chordal graphs. For the classes of permutation graphs and comparability graphs, we demonstrate that the weaker property of freedom holds. We then discuss some ramifications of these results, including for the critically perfect graphs of Wagler, and deduce some restrictions on the structure of a critically perfect graph.

Finally, we show how one can take an arbitrary class of graphs $\mathcal{C}$, and an arbitrary positive integer n and generate a so-called $(\mathcal{C}, n)$*-reconfiguration graph*, whose vertices are adjacent if and only if the corresponding n-vertex graphs in the class differ by a single edge change. We prove a connectivity result for when $\mathcal{C}$ is the class of cographs, which in some sense shows that the cographs as a graph class are in some sense "less well connected" than (for example) the class of chordal graphs, but "more well connected" than the class of perfect graphs.

3.2 Some Definitions and a proof of the Perfect Graph Theorem

We shall now give a proof of The Perfect Graph Theorem. Every proof of The Perfect Graph Theorem falls into one out of of two types. The first type are those that use the notion of graph substitution: This includes Lovász's original proof [66], and that of Fulkerson [34]. The second method is to use linear algebra, specifically, multiplying together the "clique matrix" and the "stable set matrix" of the graph, and then using the dimension theorem of linear algebra. The proof we give here uses the matrix method: it is due to Gasparian [70]. Gasparian's proof is an elegant simplification of the methods of Padberg [76], to whom the first proof of The Perfect Graph Theorem by the matrix method is due.

We actually prove that a minimally imperfect graph has the following five properties, of which only the first is needed for deducing The Perfect Graph Theorem. A clique (or stable set) is called *big* if it is of the maximum size amongst the cliques (or stable sets) in the graph.

Theorem 3.2.1 (Padberg 1974 [76]; Gasparian 1996 [70]).

Let G be a minimally imperfect graph on n vertices. Then G satisfies the following:

1. $n = 1 + \alpha(G)\omega(G)$
2. *G has n big stable sets $S_1, ..., S_n$ and n big cliques $K_1, ..., K_n$.*
3. *The orderings of the big stable sets and big cliques can be chosen such that the pairs (S_i, K_i) satisfy $|S_i \cap K_i| = 0$, and $|S_i \cap K_j| = 1$ if $i \neq j$. The pairs (S_i, K_i) are then referred to as "mates".*
4. *Every vertex of G is contained in α big stable sets and ω big cliques.*
5. *Every vertex v of G has the property that $G - v$ is uniquely ω-colourable, and uniquely α-coverable.*

Graphs satisfying conditions $1 - 5$ are called *partitionable.* For this reason, the linear algebra approach to the perfect graph theorems is often referred to as the "partitionability approach". Before proving Theorem 3.2.1, we need the following lemma:

Lemma 3.2.2. *A graph G is perfect if and only if for every induced subgraph H of G, there exists a stable set S in H that intersects every big clique in G.*

Proof. For the "if" direction, assume G is a graph with clique number ω and for every induced subgraph H of G, there exists a stable set S in H that intersects every big clique in G. By induction we can assume that all the proper induced subgraphs of G are perfect. Therefore we need only to prove that G can be coloured with ω colours. Since $\omega(G-S) = \omega(G)-1$, and $G-S$ is perfect, we can colour $G-S$ with $\omega - 1$ colours. Replacing S into G, we can augment this colouring to an ω-colouring of G.

Conversely, for the "only if" direction, if we assume G is perfect, then any colour class of G will intersect every big clique in G. □

Proof of Theorem 3.2.1

Let G be a minimally imperfect graph, having vertex set $\{1, ..., n\}$, $\alpha(G) = \alpha$ and $\omega(G) = \omega$. Let $S_0 = \{v_1, ..., v_\alpha\}$ be a maximum stable set in G. For every $v_i \in S_0$, there is a partition of $G - v_i$ into ω colour classes. Define a sequence of $N = 1 + \alpha\omega$ stable sets by taking S_0, then the colour classes of $G - v_1$, then the colour classes of $G - v_2$, and so on. This defines a sequence $\mathcal{S} = S_0, ..., S_N$ of stable sets.

The set of colour classes of each $G - v_i$ will be called a *band*. The sequence $\mathcal{S}$ thus consists of S_0, followed by α consecutive bands.

Let u be any vertex of G. If $u \in S_0$ then u occurs in S_0, and all but one of the bands, giving α occurrences in $\mathcal{S}$ in total. If u is not a member of S_0, it will appear once in each band, again giving a total of α occurrences. Thus we can say, every vertex of G occurs in precisely α of the stable sets in $\mathcal{S}$. This will be enough to prove (4) later on.

Now let K be a big clique in G. If K does not intersect S_0, then it forms an ω-clique in every $G - v_i$, and hence intersects everything in $\mathcal{S}$ except for S_0. If K intersects S_0 in one element v_j, then K is an ω-clique in all $G - v_i$ except when $i = j$, when it is an $\omega - 1$ clique. Thus it intersects everything in all bands besides the jth band, where it intersects everything except one stable set. Thus, any big clique K is disjoint from precisely one element of $\mathcal{S}$.

By Lemma 3.2.2, in a minimally imperfect graph, every stable set has some big clique disjoint from it. Therefore we may generate a sequence of big cliques $\mathcal{K} = K_0, ..., K_N$, where S_i is disjoint from K_i for each i, and K_i intersects every S_j such that $i \neq j$.

Consider the matrix A whose rows are the characteristic vectors of the elements of $\mathcal{S}$, indexed over the vertex set of G. Similarly, let B be the matrix whose columns are the characteristic vectors over the vertex set of G. By the above, $AB = J - I$, which is an $N \times N$ matrix of full rank. Thus $n \geq N$. Also, because the vertex deleted subgraphs of G are perfect, $n - 1 \leq \alpha\omega = N - 1$, so $n \leq N$. Therefore, $n = N = \alpha\omega + 1$. This proves (1) (and thus The Perfect Graph Theorem). We also know that every stable set in $\mathcal{S}$ is big, since $n - 1 = \alpha\omega$.

Let c be the characteristic vector of a big stable set in G. We show that it is a row of A by considering also the matrix equation $\underline{t}A = \underline{c}$.

We have $AB = J - I$ which implies $I = J - AB = \omega^{-1}AJ - AB = A(\omega^{-1}J - B)$. Hence $A^{-1} = \omega^{-1}J - B$.

Now, since A has full rank, the equation $\underline{t}A = \underline{c}$ has a solution. We examine its properties.

$\underline{t}A = \underline{c}$ implies $\underline{t} = \underline{c}A^{-1} = \omega^{-1}\underline{c}J - \underline{c}B^T = \omega^{-1}(\omega\underline{1}) - \underline{c}B^T = \underline{1} - \underline{c}B^T$.

Therefore $\underline{t}$ is a $(0, 1)$-vector, and it has a 0 in every place except one, implying that $\underline{c}$ is a row of A. So $\mathcal{S}$ consists of *all* the big stable sets of G. A similar calculation gives the same information of about $\mathcal{K}$ and the big cliques of G. This proves (2) and (3).

We saw earlier that every vertex u occurs in α elements of $\mathcal{S}$. Applying this argument to $\overline{G}$ we get that u occurs in ω stable sets. This is (4). To see (5), note however we generate $\mathcal{S}$, we generate every big stable set exactly once. Once we have generated

$\mathcal{S}$, changing any of the bands would cause some big stable set to appear more than once, which is impossible. $\square$

Proof of The Perfect Graph Theorem

Let G be a minimally imperfect graph, with order n and clique number ω and stability number α. Since G has more vertices than $\alpha\omega$, the same then applies to $\overline{G}$, which then cannot be coloured with ω colours, because the number of vertices is too large. So $\overline{G}$ is imperfect (and thus minimally imperfect). $\square$

Theorem 3.2.1 implies the following criterion for perfection, which is originally due to Lovász [65]. We shall refer to it as *Lovász's Criterion*.

Theorem 3.2.3 (Lovász' Criterion)**.** *A graph G is perfect if and only if it satisfies $\alpha(H)\omega(H) \geq |H|$ for induced subgraphs H of G.*

Proof. If G is perfect, then any $H \preceq G$ can be partitioned into $\omega(H)$ cliques, each of size at most $\alpha(H)$. Hence $\alpha(H)\omega(H) \geq |H|$ for all $H \preceq G$.

If G is not perfect, then it has a minimally imperfect induced subgraph H. Then $\alpha(H)\omega(H) > |H|$ by part (1) of Theorem 3.2.1. $\square$

Another consequence of Theorem 3.2.1 is a different statement of the SPGT. We define the graph invariant π to be $\min(\alpha, \omega)$. Let G be a minimally imperfect graph. Theorem 3.2.1 immediately implies that any minimally imperfect graph with $\pi(G) = 2$ is either an odd hole or an odd antithole. To see this, assuming $\omega = 2$, Theorem 3.2.1 says that G is 2-regular and therefore a union of cycles. This means G is an odd hole, since triangles and even cycles are perfect and disconnected graphs are not minimally imperfect. The case $\alpha = 2$ is much the same, and gives the following theorem:

Theorem 3.2.4. *Let G be a graph. Then the following are equivalent:*

1. *G is a minimally imperfect Berge graph*

2. *G is a minimally imperfect graph with $\pi(G) \neq 2$*

Theorem 3.2.5. *The following are equivalent:*

1. *The Strong Perfect Graph Theorem: Every minimally imperfect graph is an odd cycle of length* 5 *or more (an odd hole) or the complement of one (an odd antihole);*

2. *Every minimally imperfect graph G satisfies the equation $\pi(G) = 2$.*

Theorem 3.2.1 contains several facts about minimally imperfect graphs which are not obtained from Lovász's original proof. However, it does not contain the "substitution lemma" of Lovász's proof [66]. Recall that a *module* in a graph G is a vertex set $M \subseteq V(G)$ such that $N(x) - M = N(y) - M$ for all $x, y \in M$. We now state the substitution lemma:

Theorem 3.2.6 (Lovász' Substitution Lemma)**.** *Let G be a minimally imperfect graph. Then G contains no modules besides $V(G)$ and the singletons $\{x\}$, where $x \in V(G)$.*

3.3 Edge Modification and the Parameters α and ω

Let G be a graph with n vertices. A vertex colouring of a graph by k colours is a partition of $V(G)$ into k stable sets. We could equivalently define it as a complete multipartite graph G' on n vertices with clique number k, together with an embedding of G into G'.

The chromatic number measures how few colours one can use in a vertex colouring. Equivalently, it measures the smallest number k for which there is a complete k-partite graph into which G can be embedded. We can then consider the graph colouring problem as an edge addition problem where we try to turn our original graph into a complete multipartite graph with clique number as small as possible.

Having considered this problem we can change it: suppose we insist on the clique number of the output graph being $\omega(G)$, and ask how close to a complete multipartite graph we can make it by adding edges. If it is not a complete multipartite graph, can we at least have it be a cograph, or a graph that is either disconnected or its complement is, or some other property generalising complete multipartite graphs?

A graph is perfect if and only if every induced subgraph can be modified into a complete multipartite graph without increasing ω. We shall see that the graphs which can be modified without increasing ω into a cograph, or a graph that is disconnected or codisconnected, are actually the same class of graphs. First we need some definitions.

Let G be a graph with stability number α and clique number ω. Recall that an edge xy is critical if $\alpha(G - xy) > \alpha(G)$, and a nonedge xy critical if $\omega(G + xy) > \omega(G)$, and any other edges/nonedges are free.

More generally if X is a collection of edges of G, write $G - X$ for the graph obtained by removing X from the edge set of G. If X is an edge set of G then we say X is *critical* if $\alpha(G - X) > \alpha(G)$ and *free* if $\alpha(G - X) = \alpha(G)$.

If X is a collection of nonedges of G, write $G + X$ for the graph obtained by adding X to the edge set of G. If X is a nonedge set of G then we say X is *critical* if $\omega(G + X) > \omega(G)$ and *free* if $\omega(G - X) = \omega(G)$.

A graph G_n obtained from $G_0 = G$ by successive edge modifications via the sequence $G_0, G_1, ..., G_n$, where G_i is obtained from G_{i-1} by adding or removing a free edge in G_{i-1} is called a *free graph* of G. If it is obtained entirely via free nonedge additions it is called an *upper graph* of G, and if it is obtained entirely via free edge deletions it is called a *lower graph* of G.

Intuitively, an upper or lower graph of a graph G is in some sense “close” G. What we will see is that perfect graphs are “close” to certain other perfect graphs, but never to imperfect graphs.

Theorem 3.3.1. *Let G be a graph. A graph is ω-colourable if and only if G has an upper complete ω-partite graph. A graph is α-coverable if and only if it has a lower graph which is a disjoint union of α cliques. A graph is perfect if and only if all of its induced subgraphs have both of these properties.*

Proof. Given an ω-colouring, adding edges between all the different coloured vertices gives the required graph. Conversely, given an upper ω-partite graph, the parts are colour classes in an ω-colouring.

Likewise, given a covering by α cliques remove all edges between the cliques, and given a lower graph which is a disjoint union of α cliques, those cliques will form the appropriate clique cover. □

What if we generalise the class of complete multipartite/disjoint union graphs? Effectively, we are generalising the notion of a graph colouring. We shall see that, by Lovász' Criterion (Theorem 3.2.3), the class of graphs that this produces is the same as the perfect graphs.

Theorem 3.3.2. *Let G be a graph. Then the following conditions, when satisfied by every induced subgraph H of G, are equivalent:*

1. *H is ω-colourable (G is perfect)*
2. *H has an upper graph H' which is perfect*
3. *H has a lower graph H' which is perfect*
4. *H has a free graph H' which is perfect*

Proof. ($1 \Rightarrow 2$): If H is ω-colourable, then any ω-colouring of G gives us a complete multipartite graph which is an upper graph of G. Since complete multipartite graphs are perfect, we have what we need.

($2 \Rightarrow 3$): If H' is a perfect upper graph of H, then $\overline{H'}$ is a lower graph of H, and it is perfect by the Perfect Graph Theorem.

$(3 \Rightarrow 4)$: Define the sequence H_1, H_2 by $H_1 = H, H_2 = H'$. This sequence satisfies the definition of H' being a free graph of H.

$(4 \Rightarrow 1)$: Given a sequence $H_1, ..., H_n = H'$ whereby H' is a free graph of H, consider H_i and H_{i+1} for some i. If H_{i+1} is a lower graph of H_i, then $\alpha(H_{i+1}) = \alpha(H_i)$, and since removing edges cannot increase the clique number, we have $\omega(H_{i+1}) \leq \omega(H_i)$. If H_{i+1} is an upper graph of H_i, then $\omega(H_{i+1}) = \omega(H_i)$, and since adding edges cannot increase the stability number, we have $\alpha(H_{i+1}) \leq \alpha(H_i)$. Applying this inductively we get $\alpha(H)\omega(H) \geq \alpha(H_r)\omega(H_r)$.

Since H_r is perfect, it satisfies Lovász' criterion and $\alpha(H_r)\omega(H_r) \geq |H_r|$. But $|H_r| = |H|$, and therefore $\alpha(H)\omega(H) \geq |H|$, which implies that H is perfect by Lovász' criterion, and is therefore $\omega(H)$-colourable. $\square$

Example: "Unfolded Tetrahedron"

Although an upper graph of G will not have larger clique number than G, it may have smaller independence number. In terms of colour classes, this means some perfect graphs have no ω-colourings where there is a colour class which is an α-stable set, and we now give an example of this which we call an *unfolded tetrahedron*.

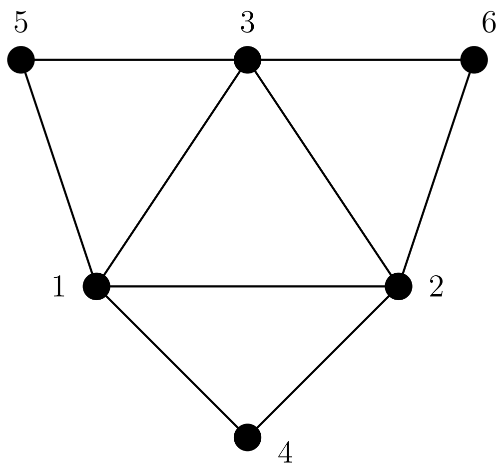

Figure 3.2: The unfolded tetrahedron.

Let G be the graph in Figure 3.2 on vertex set $\{1, 2, 3, 4, 5, 6\}$ where $1, 2, 3$ induce a triangle, $4, 5, 6$ an antitriangle, and $(1, 4), (2, 4), (1, 5), (3, 5), (2, 6), (3, 6)$ are the only other edges. Then $4, 5, 6$ is not a colour class in any ω-colouring. Correspondingly, the complete ω-partite graph representing the colouring has stability number 2. Note that G *does* have an upper cograph with $\alpha = 3$.

Now we consider two classes of graphs $\mathcal{X}$ and $\mathcal{Y}$. $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ denote the set of all graphs G such that every induced subgraph of G has a lower graph in $\mathcal{X}$ and an upper graph in $\mathcal{Y}$.

By Theorem 3.3.2, we have the following:

Theorem 3.3.3. *Let $\mathcal{P}$ be the class of perfect graphs. Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of perfect graphs such that $\mathcal{X}$ contains the class $\mathcal{DU}$ of disjoint union graphs and $\mathcal{Y}$ contains the class $\mathcal{CM}$ of complete multipartite graphs. Then $\mathcal{P} = \mathcal{B}(\mathcal{X}, \mathcal{Y})$.*

Proof. Apply Theorem 3.3.2 to the class of perfect graphs, in light of the fact that $\mathcal{P} = \mathcal{B}(\mathcal{DU}, \mathcal{CM})$. □

Theorem 3.3.4. *Let $\mathcal{P}$ be the class of perfect graphs. Let $\mathcal{CO}$, $\mathcal{CP}$, and $\mathcal{PM}$ denote the classes of Cographs, Comparability Graphs, and Permutation Graphs, respectively. Then $\mathcal{P} = \mathcal{B}(\mathcal{CO}, \mathcal{CO}) = \mathcal{B}(\mathcal{CP}, \mathcal{CP}) = \mathcal{B}(\mathcal{PM}, \mathcal{PM}) = B(\mathcal{P}, \mathcal{P})$.*

Proof. This follows from the fact that disjoint union graphs and complete multipartite graphs are cographs, comparability graphs, and permutation graphs, and the fact that cographs, comparability graphs, and permutation graphs are all perfect. □

3.3.1 Connectivity

Here we show that the perfect graphs are, in a sense, characterisable as the graphs that are hereditarily close to the class of *half-connected* graphs, that is, graphs who are either disconnected or codisconnected.

Theorem 3.3.5. *Let G be a graph. Then the following are equivalent:*

1. *G is perfect.*

2. *$\forall H \preceq G$: H has a disconnected lower graph or is a clique.*

3. *$\forall H \preceq G$: H has a codisconnected upper graph or is a stable set.*

Proof. ($1 \Rightarrow 2$): Assume G is perfect, and let $H \preceq G$. If H is not a clique, then $\alpha(H) \geq 2$, and so the disjoint union lower graph (guaranteed by perfection of G) of H is disconnected.

($1 \Leftarrow 2$) : Assume that G is not a clique, so $\alpha \geq 2$. Let us assume inductively that all the proper induced subgraphs satisfy the implication we are trying to prove, so that G is subperfect.

Let G be a disconnected lower graph of G, with $G = G_1 \uplus G_2$, where $\alpha(G_1) = \alpha_1$ and $\alpha(G_2) = \alpha_2$, and $\alpha_1 + \alpha_2 = \alpha(G)$. Since G_1, G_2 are induced subgraphs of G, they are perfect (since G is subperfect). Therefore, G_1 has a lower graph which is a disjoint union of α_1 cliques, and G_2 has a lower graph which is a disjoint union of α_2 cliques, which can be combined to give a lower graph of G which is a disjoint union of α cliques. Therefore, G is perfect.

($2 \Leftrightarrow 3$) follows from the Perfect Graph Theorem. □

Up until now, we have focused on operations that do not increase α or ω.

We now deviate to give another characterisation of perfect graphs, focusing instead on operations which *do decrease* α and ω, and this can give us another connectivity-based characterisation of perfect graphs. The Strong Perfect Graph Theorem is essential for the proof.

First, we introduce the following notions.

An edge subgraph G' of a graph G is called a 2-*cut* if G' is obtained by removing all edges between two halves of a bipartition of G. A 2-*join* of G is a supergraph obtained by adding all edges between two halves of a bipartition of G. An edge subgraph of G is called *reducing* if it has strictly smaller clique number than G, and an edge supergraph is called reducing if it has strictly smaller stability number than G.

Theorem 3.3.6. *Let G be a graph. Then the following are equivalent:*

1. *G is perfect.*
2. *For all $H \preceq G$: H has a reducing 2-cut or is a stable set, and has a reducing 2-join or is a clique.*
3. *For all $H \preceq G$: H has a free 2-join or is a stable set.*
4. *For all $H \preceq G$: H has a free 2-cut or is a clique.*

Proof. ($1 \Rightarrow 2$): Assume G is perfect, and let $H \preceq G$. If H is not a stable set or clique, then $\alpha(H) \geq 2$, and (by perfection of G) we take a partition of G into α-cliques. Dividing this collection of cliques into two nonempty collections A and B (we can do this since α is at least two), and adding edges between everything in A and everything in B, we get a reducing 2-cocut of G.

Since $\omega(G) \geq 2$, performing the dual operation with an ω-colouring gives us a reducing 2-cut of G.

($2 \Rightarrow 1$): As usual, by induction we can assume the subperfection of G. So, we need to prove that G is not minimally imperfect.

If $\omega(G) = 2$, then a reducing 2-cut halves G into two parts A and B, each with clique number strictly less than two. But this means that they are stable sets, so G is bipartite, and hence perfect. Likewise if $\alpha(G) = 2$, then a reducing 2-cocut halves G into two parts A and B, each with coclique number strictly less than two. But this means that they are cliques, so G is the complement of a bipartite graph, and is hence perfect.

Therefore we may assume that $\omega(G), \alpha(G) \geq 3$. But by the Strong Perfect Graph Theorem, this implies that G is not minimally imperfect, since the only minimally imperfect graphs are the odd holes and odd antiholes. Therefore G must be perfect.

($3 \Leftrightarrow 1$): Any upper complete multipartite graph of G may be arrived at by a sequence of free joins, and any sequence of free joins will eventually generate an upper graph which is complete multipartite. Therefore this criterion is equivalent to perfection.

$(4 \Leftrightarrow 1)$: Any lower disjoint union graph of G may be arrived at by a sequence of free cuts, and any sequence of free cuts will eventually generate a lower graph which is a disjoint union of cliques. Therefore this criterion is equivalent to perfection by Theorem 3.3.1. □

It is apparent that if G is a bipartite graph, and H is an induced subgraph of G, then any 2-colouring of H defines a 2-cut of H, unless H has no edges. As such we called a graph all of whose induced subgraphs either have 2-cuts or are stable sets *weakly bipartite*. Therefore, part (2) of Theorem 3.3.6 states that a graph is perfect if and only if both it and its complement are weakly bipartite.

We note that part (2) of Theorem 3.3.6 and the notion of weak bipartiteness is not new as it has appeared previously in [55] under the name of "2-divisible graphs".

3.3.2 Pancritical Graphs

The simplest special case of an edge modification of a graph is when there is only one edge change - recall that an nonedge is critical if adding it increases the clique number and that an edge is critical if removing it increases the stability number. This leads to the following definition.

If G is a graph with an edge e, then we say G is a *pancritical* graph if every edge and every nonedge is critical.

Firstly, no perfect graph can be pancritical, besides the complete graphs and empty graphs.

Theorem 3.3.7. *Let G be a perfect graph of order n. Then if G is pancritical, $G = K_n$ or $G = E_n$.*

Proof. Let G be a perfect graph that is not a clique or stable set. Since G has an upper complete multipartite graph, it has at least one free nonedge, unless it is complete multipartite. In the case where G is complete multipartite, it cannot be a disjoint union of cliques for it would then have to be a clique or stable. Therefore, since G has a lower disjoint union of cliques, it has at least one free edge. □

We shall refer to the complete and empty graphs as *trivial* pancritical graphs. The Strong Perfect Graph Theorem, together with Theorem 3.3.7, implies that any pancritical graph contains an odd hole or odd antihole.

Theorem 3.3.8. *Let G be a nontrivial pancritical graph. Then G contains an odd hole or odd antihole as an induced subgraph.*

Proof. This is immediate from the preceding theorem and the Strong Perfect Graph Theorem. □

Example (The cycle on 5 *vertices):* The first imperfect graph C_5 also gives the first example of a pancritical graph (see fig 3.3). Adding any edge to C_5 gives a triangle, and removing any edge gives the complement of a triangle. Meanwhile, the clique number and stability number of C_5 are both 2.

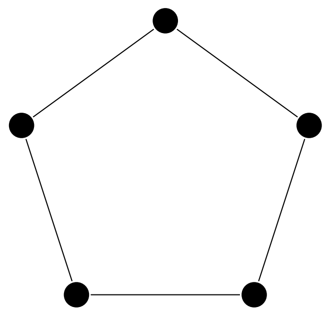

Figure 3.3: The cycle on 5 vertices

The notion of pancritical graphs gives us reason to mention *Paley graphs.* A Paley graph is the graph defined on q vertices, where q is the power of an odd prime congruent to 1 mod 4, and vertex i is adjacent to vertex j if and only if $i - j$ is a square in mod q arithmetic. They are denoted $P(q)$.

Example (The Paley graph on 13 vertices): The Paley graph $P(13)$ on 13 vertices (see Fig. 3.4) provides an additional example of a pancritical graph. Adding any edge creates a K_4, whereas the graph itself has clique number 3. Self-duality (a property held by all Paley graphs) of the graph then implies that it is pancritical.

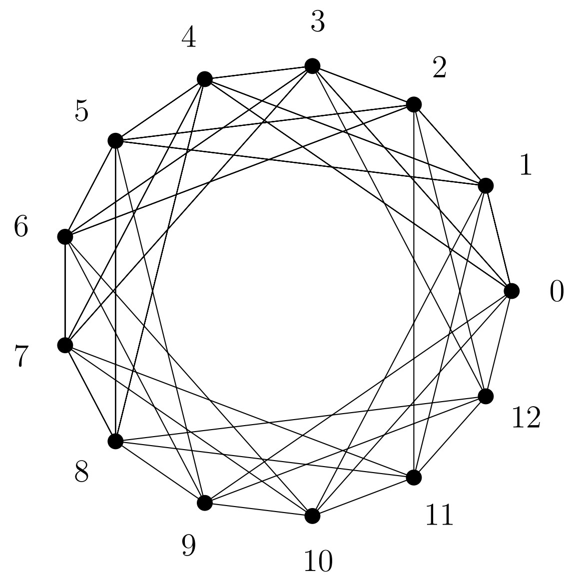

Figure 3.4: The Paley graph $P(13)$

Observe that for example $\{1, 2, 3, 7, 10\}$ forms a C_5 in $P(13)$. This, gives rise to the following conjecture.

Conjecture 3.3.9. *Every nontrivial pancritical graph contains C_5 as an induced subgraph.*

3.4 Edge Modification and The SPGT

The partitionability approach for the SPGT retains some significance for two reasons: firstly, a second proof of such a major result would be interesting (none exists presently); secondly, the proof could potentially be quite short in comparison to the proof of Chudnovsky et al. Recall that a critical edge/nonedge is one whose deletion/addition increases the stability/clique number. Several results have been obtained in the direction of partitionability and critical edges. They are mostly summed up by Sebo in [87, 88].

One of the most prominent obstacles to this approach to the SPGT is the difficult task of establishing of the existence of critical edges/nonedges in a given minimally imperfect graph. At present nobody has established the existence of a critical edge in an arbitrary minimally imperfect graph without going through the entire proof of the SPGT as given in [17].

In the absence of existence results on critical edges in minimally imperfect graphs from partitionability methods, several nonexistence results have been obtained, stating that "not too many" critical edges/nonedges can occur in minimally imperfect Berge graphs. Initial results in this direction were obtained by Gasparian, Markossian and Markossian [70], and then improved upon by Sebo [88, 87]. Perhaps the most interesting result of Sebo is the following:

Theorem 3.4.1 (Sebo [87])**.** *Let G be a minimally imperfect Berge graph. Then any vertex of G incident with both a critical edge and a critical nonedge, is incident with precisely one critical edge and precisely one critical nonedge.*

Theorem 3.4.1 implies, for example, that if every vertex is incident with both a critical edge and a critical nonedge, the graph must have even order (because it implies the existence of a matching in the graph). It also contrasts with the situation in odd holes/antiholes, where every vertex is incident with 2 critical edges and with 2 critical nonedges: Theorem 3.4.1 implies that in a minimally imperfect Berge graph, *no* vertex can be incident with 2 critical edges and with 2 critical nonedges.

Sebo has also obtained the following results on critical edges in minimally imperfect Berge graphs (again of the nonexistence kind). Note that the theorems are vacuous if (as stated in the SPGT) no minimally imperfect Berge graphs exist. Therefore, assuming for a contradiction that minimally imperfect graphs do exist, one may also speak of a *minimum imperfect Berge graph:* This is an imperfect Berge graph of the smallest possible order, which can be a useful additional assumption (and is indeed used in [17]) because some things that are easy to prove for minimum imperfect Berge graphs are hard to prove for minimally imperfect Berge graphs.

Theorem 3.4.2 (Sebo [87])**.** *Let G be a minimally imperfect Berge graph. Let $\{v_1, ..., v_k\}$ be a set of vertices such that for all $i < j$ there is a path connecting v_i*

with v_j consisting of critical edges in G. Then $G[v_1, ..., v_k]$ is a clique.

Theorem 3.4.3 (Sebo [87])**.** *Let G be a minimally imperfect Berge graph. Let $v_1, ..., v_k$ be a sequence of vertices such that $v_i v_{i+1}$ is a critical edge in G for all $1 \leq i \leq k-1$. Then $k < \omega(G)$.*

Theorem 3.4.4 (Sebo [88])**.** *Let G be a minimum imperfect Berge graph. Then every vertex has degree at least $2\omega - 1$ and codegree at least $2\alpha - 1$.*

Our aim in this section is to expose a connection between critical edges and another kind of edge called a "Berge-critical edge". Roughly speaking, the idea - which we call "dichotomy" - is that *any* edge must be either critical, or else Berge-critical. The potential benefit of this is that by demonstrating the nonexistence of Berge-critical edges, one may demonstrate the existence of critical edges.

3.4.1 Dichotomy in Odd Holes and Odd Antiholes

Now we define what Berge-critical edges are, and what the dichotomy property is.

Let G be a graph. We define a pair xy in a graph G to be *Berge-critical* if the graph obtained by switching the status of xy is not Berge, i.e. contains an odd hole or antihole as an induced subgraph. Otherwise, xy is *Berge-free*. Importantly, note that the graph G itself does not have to be Berge for this definition to apply!

The dichotomy property consists of an observation about the structure of odd holes and antiholes.

Let x and y be vertices in an odd hole. If xy is an edge and we delete it, then $G - xy$ is Berge, and $G - xy$ has stability number 1 more than G. If xy is a nonedge and we add it, then $G + xy$ is Berge if and only if xy is a triangular chord in $G + xy$: if it isn't we get a shorter odd hole in $G + xy$. On the other hand, $G + xy$ has increased clique number compared to G if and only if xy is a triangular chord in $G + xy$. So to sum it up, the Bergeness of the graph changes if and only if the parameters α/ω stay the same. Equivalently the Bergeness stays the same if and only if the parameters α/ω change. This is called *the dichotomy property.*

The dichotomy property of odd holes and antiholes may be summed up in the following theorem:

Theorem 3.4.5 (Dichotomy property)**.** *Let G be an odd hole or odd antihole. Let xy be an edge or nonedge of G. Then xy is critical if and only if it is Berge-free. Equivalently, xy is free if and only if it is Berge-critical.*

We are interested in whether a similar relationship holds in a minimally imperfect Berge graph. We shall see that it does, but to further realise this possibility, we introduce a slightly different, but equivalent, form of the SPGT.

3.4.2 Minimum Imperfect Berge Graphs and Dichotomy

Now instead of odd holes and odd antiholes, we look at putative counterexamples to the SPGT: Minimally imperfect Berge graphs, and how the dichotomy property can be applied in this context. Before we do this though, it will be necessary for us to look at something a bit more specific than the notion of a minimally imperfect Berge graph.

First, we say that a *minimum imperfect Berge graph* is an imperfect Berge graph with the smallest possible number of vertices. Secondly, we say a *small imperfect Berge graph* is a minimum imperfect Berge graph with the smallest possible number of edges. Finally, we say that a minimum imperfect Berge graph G is *normal* if none of the graphs obtained by deleting a single edge of G are a minimum imperfect Berge graph. If $\overline{G}$ is normal we say G is conormal. A normal and conormal minimum imperfect Berge graph is called *binormal.*

In attempting to prove the SPGT, the assumptions of minimum imperfect, small, and normal are all *natural* assumptions - the nonexistence of minimally imperfect Berge graphs is equivalent to the nonexistence of these special kinds of minimally imperfect Berge graphs. We note in passing that, minimum imperfect Berge graphs were used in the proof of the SPGT [17].

Theorem 3.4.6. *The following are equivalent:*

1. *The Strong Perfect Graph Theorem: There is no minimally imperfect Berge graph.*
2. *There is no minimum imperfect Berge graph.*
3. *There is no small imperfect Berge graph.*
4. *There is no normal minimum imperfect Berge graph.*

Proof. The first statement clearly implies statements two, three and four, while the second implies the first (by induction) and the third implies the second (by induction). The second clearly implies the fourth. The fourth implies the third, as a small imperfect Berge graph not being normal would contradict the minimality of its size. □

We emphasise the property of normality, because it is exactly what we need to get a dichotomy-like relationship akin to what we found in odd holes and odd antiholes, which we call *weak dichotomy.*

Theorem 3.4.7 (weak dichotomy property)**.** *Let G be a normal minimum imperfect Berge graph. Let xy be an edge in G. Then xy is either critical or Berge-critical. If G is conormal then every nonedge is either critical or Berge-critical.*

Proof. Let G be a normal minimum imperfect Berge graph. Let xy be an edge of G, and assume that it is Berge-free. Since the order of $G - xy$ is minimum, all of its proper induced subgraphs are perfect if they are Berge. But this follows from Bergeness of $G - xy$: therefore $G - xy$ is subperfect.

Clearly $G - xy$ is subperfect, and the normality of G implies that it cannot be minimally imperfect, so therefore $G - xy$ must be perfect. This implies that $G - xy$

satisfies the Lovász bound: $\alpha(G-xy)\omega(G-xy) \geq n$. On the other hand, $\alpha(G)\omega(G) < n$ by Theorem 3.2.1. Therefore $\alpha(G-xy)\omega(G-xy) > \alpha(G)\omega(G)$. Meanwhile, $\omega(G-xy) \leq \omega(G)$. Together these inequalities imply that $\alpha(G-xy) > \alpha(G)$, i.e. that xy is critical. $\square$

Three immediate questions arise:

1. Can we improve upon the weak dichotomy in a minimum imperfect Berge graph, to get the full dichotomy property?
2. How can we justify the stronger assumption of binormality, in place of normality?
3. How can we find Berge-free edges in a Berge graph?

For question 1, the dichotomy property states that the critical and Berge-critical edges form a bipartition of the edge set. The weak dichotomy only states that the union of the critical and Berge-critical edges gives the edge set. It would obviously be good to realise more of this structure that exists in the odd holes and antiholes, in the Berge minimally imperfect graphs.

For question 2 let us define a *close pair* to be a pair of minimally imperfect graphs, each of whom is obtained from the other by a single edge change. Close pairs certainly seem counterintuitive, but it is as yet not clear how to prove them contradictory. A proof of their nonexistence would allow us to strengthen weak dichotomy to include both edges and nonedges of the same graph.

For question 3, the only investigations into this second issue are due to Wagler in her PhD thesis [95, 96]. From the investigations of Wagler we know that there do exist *critically Berge graphs*: Berge graphs where all the edges are Berge-critical edges, and even Berge graphs where there are only Berge-critical edges and nonedges. Therefore, we cannot conclude from weak dichotomy that critical edges exist: we might find ourselves in a critically Berge graph.

Interestingly Wagler notes that all known examples of such graphs appear to either be line graphs of bipartite graphs, or obtained from line graphs of bipartite graphs via three different operations.

In any case the conclusion is that a minimum imperfect graph satisfying the assumption of normality, must either have a critical edge or else be of the kind considered by Wagler (critically Berge).

3.4.3 Existence Of Berge-Critical Edges

Now we apply Theorem 3.4.7 (the weak dichotomy property) together with Sebo's Theorems in order to obtain the existence of Berge-critical edges in normal minimum imperfect Berge graphs.

Sebo's Theorems are nonexistence results for critical edges in minimally imperfect graphs: they say a minimally imperfect graph cannot have "too many" critical edges, in some sense. In a minimally imperfect graph with the weak dichotomy property, the flipside of this is that we can deduce the existence of Berge-critical edges. We now give a few of these results.

Theorem 3.4.8. *Let G be a binormal minimum imperfect Berge graph. Let v be a vertex of G. Then one of the following must occur:*

1. *v has entirely Berge-critical edges with its neighbours*
2. *v has entirely Berge-critical nonedges with its non-neighbours*
3. *v has Berge-critical edges with all of its neighbours except one, and Berge-critical nonedges with all of its non-neighbours except one.*

Proof. Since G is binormal, every edge is critical or Berge-critical (by Theorem 3.4.7). If v has no critical edges with its neighbours we get the first condition, and if v has no critical nonedges with its non-neighbours we get the second condition, and the only remaining possibility is the second condition, which follows by Theorem 3.4.1 which says v cannot have multiple critical edges and multiple critical nonedges. □

Theorem 3.4.9. *Let G be a binormal minimum imperfect Berge graph. Then every path of ω edges contains a Berge-critical edge, and every antipath of α nonedges contains a Berge-critical nonedge.*

Proof. Follows from Theorem 3.4.3 and weak dichotomy.

□

Theorem 3.4.10. *Let G be a normal minimum imperfect Berge graph. Let P be an induced path of length l in G. Then P has at least one Berge-critical edge for every two consecutive edges: the number of Berge-critical edges is at least $\frac{l}{2}$ if l is even and is at least $\frac{l-1}{2}$ if l is odd.*

Proof. Since G is normal, any free edge is Berge-critical. Let e be a critical edge of G in P. Any edges to either side of e must be free, or else we would get an induced P_3 consisting of critical edges, and contradicting Theorem 3.4.3. Therefore any edges either side of e must be Berge-critical, by weak dichotomy. □

3.4.4 Berge-Critical Edges Contained in a P4 and Berge-Critical Edges Incident with a Vertex

In this section we look at the containment of Berge-critical edges within induced P_4s in minimum imperfect Berge graphs, and then the existence of Berge-critical edges within the neighbourhood of an arbitrary vertex. Again, we use Theorem 3.4.7 (the weak dichotomy property) and Sebo's Theorems.

For now, we need a few definitions. Let G be a graph with an induced P_4, P. If $P = v_1v_2v_3v_4$, the edges v_1v_2 and v_3v_4 are called the *wings* and the edge v_2v_3 is called the *midpoints*. The nonedge v_1v_4 are called the *endpoints* and the nonedges v_1v_3 and v_2v_4 are called the *diagonals*.

Let G be a graph. Observe that a nonedge $e = xy$ in a graph G is critical iff there exists at least one $(\omega - 1)$-clique K such that x and y are each adjacent to everything in K. This clique will be called a *blocker* of e. Similarly the $(\alpha - 1)$-coclique nonadjacent to both ends of a critical edge is called a *coblocker*.

Recall Theorem 3.2.1 that any minimally imperfect graph G is also *partitionable*: In particular, for every big clique there is precisely one big stable set disjoint from it, and vice versa.

Lemma 3.4.11. *Let G be a minimally imperfect graph. Let ab be a critical nonedge with blocker $K = aKb$ and let cd be an critical edge with coblocker $S = cSd$. Then there are exactly four possibilities:*

1. $|\{a, b\} \cap \{c, d\}| = 1$ *and* $K \cap S = \emptyset$.
2. $\{a, b\} \cap \{c, d\} = \emptyset$ *and either:*
   (a) $K \cap S = \{x\}$
   (b) $K \cap S = \emptyset$ *and* $\{a, b\} \subseteq S$
   (c) $K \cap S = \emptyset$ *and* $\{c, d\} \subseteq K$.

Proof. If the first scenario happens, let x be the common vertex. Nothing can be in a forcer *and* an coforcer of x, so the intersection of these must be empty.

If the second scenario happens, suppose $K \cap S$ is empty. Then we have four intersections of big cliques vs big stable sets to consider:

1. $[K \cup a] \cap [S \cup c]$
2. $[K \cup a] \cap [S \cup d]$
3. $[K \cup b] \cap [S \cup c]$
4. $[K \cup b] \cap [S \cup d]$

Assume without loss of generality that $a \notin S$, and $b \in S$. Then the bottom two intersections have cardinality 1, and c and d are not in K (or else the bottom two intersections would be of cardinality two). Now the top two intersections have cardinality 0. This contradicts the uniqueness of mates (partitionability, Theorem 3.2.1 part (3)). Thus either a and b are in S, or neither are, and either c and d are in K, or neither are. If a and b are both not in S and c and d are both not in K, then all the intersections will be empty, again contradicting the uniqueness of mates.

□

Theorem 3.4.12. *Let G be a minimally imperfect graph. Let $v_1v_2v_3v_4$ be an induced P_4 in G such that the nonedge $v1v4$ is critical and the edge $v2v3$ is critical. Then G is C_5.*

Proof. By Lemma 3.4.11, we take blocker v_1Kv_4 and coblocker v_2Sv_3, and look to see which of $2a/2b/2c$ holds. If $K \cap S = \{x\}$, then $v_1v_2v_3v_4xv_1$ is an induced C_5, and since G is minimally imperfect, this must be the whole graph.

Otherwise, $\{1, 4\} \subseteq S$ (contradicted by the edges between v_1 and v_2 or v_4 and v_3) or $\{2, 3\} \subseteq K$ (contradicted by the nonedges between v_1 and v_3 or between v_2 and v_4). Thus $2a$ holds and the graph being C_5 is the only possibility remaining. □

Theorem 3.4.13. *Let G be a minimally imperfect Berge graph. Then if P is a P_4 in G, either the endpoints or the midpoints are free.*

Theorem 3.4.14. *Let G be a binormal minimum imperfect Berge graph. Then if P is a P_4 in G, either the endpoints or the midpoints are Berge-critical.*

Therefore given a P_4, in a binormal minimum imperfect Berge graph G, either a wing and the midpoints, or both wings, are Berge-critical, and either a diagonal and the endpoints, or both diagonals, are Berge-critical, and furthermore, at least one of the endpoints/midpoints are Berge-critical.

We can use this to show that every vertex in a normal minimum imperfect Berge graph is incident with some Berge-critical edge. First we need the following result of Olariu [20]:

Theorem 3.4.15 (Olariu [20]). *Let G be a minimally imperfect graph with vertex v. Then there exist at least two P_4s containing v as a midpoint and at least two P_4s containing v as an endpoint.*

Theorem 3.4.16. *Let G be a normal minimum imperfect Berge graph and let v be a vertex of G. Then v is incident with a Berge-critical edge.*

Proof. By Theorem 3.4.15 v is the midpoint of some P_4, P. By Theorem 3.4.10, either the wing of P incident with v or edge between the midpoints incident with v is Berge-critical. □

Another way to get Berge-critical edges incident with each vertex is to use Theorem 3.4.2 and Theorem 3.4.4.

Theorem 3.4.17. *Let G be a binormal minimum imperfect Berge graph. Let v be a vertex of G. Then v is incident with at least ω Berge-critical edges and at least α Berge-critical nonedges. Furthermore, at least one of these quantities is greater still: v is either incident with $2\omega - 2$ Berge-critical edges or $2\alpha - 2$ Berge-critical nonedges.*

Proof. By Theorem 3.4.2 and Theorem 3.4.4, at most $\omega - 1$ of the neighbours of v can be critical (or else the critical neighbours of v would form a clique with strictly more than ω vertices, which is impossible). Therefore by Theorem 3.4.4, at least ω of the $\geq 2\omega - 1$ neighbours of v must induce free edges with v. These edges and nonedges are then Berge-critical by normality of G.

For the second part, by Theorem 3.4.1 either v's neighbourhood has all but one free edge, which by Theorem 3.4.4 constitutes at least $2\omega - 2$ edges, or v's coneighbourhood has all but one free nonedge, which by Theorem 3.4.4 constitutes at least $2\alpha - 2$ nonedges. By binormality, these edges (or nonedges) are then Berge-critical. □

3.4.5 Edges that are Critical and Berge-Critical

Having obtained some guarantees on the existence of Berge-critical edges/nonedges in normal/conormal graphs, we look at what happens when an edge is both critical and Berge-critical; unlike free and Berge-free edges, these are not ruled out by weak dichotomy. As usual we assume G is a minimum imperfect Berge graph.

If e is a Berge-critical edge and $G - e$ contains an odd hole, then in G, e must be a triangular chord connecting two vertices in a cycle of length $n \geq 5$. It must be a triangular chord, or else G would be containing an odd hole. This structure is called an *odd hat*. The endpoints of e, and e itself, are called the *sides* of the hat.

If e is a Berge-critical edge and $G - e$ contains an odd antihole, then e must be the endpoints of an induced even-length path in $\overline{G}$: we call this an *even length antipath*.

Note that it is possible that an edge can be Berge-critical in both the above ways. Also, a nonedge being Berge-critical means it is contained in an odd hat in $\overline{G}$ (*odd antihat*) or an even length path in G. Both these possibilities can happen for a single nonedge, too.

Now we show the existence of a critical Berge-critical edge leads to the existence of a sequence of 3 Berge-critical edges, provided G is a normal minimum imperfect Berge graph.

Theorem 3.4.18. *Let G be a normal minimum imperfect Berge graph. Let e be a critical Berge-critical edge. Then e is the middle edge of a weak P_4-subgraph of Berge-critical edges. The P_4 is either induced or consists of three edges of a C_4 in G.*

Proof. Let e be the critical Berge-critical edge. If e is the sides of an odd hat, or the ends of an even length antipath, then it lies on an even hole. The edges on this hole to either side of e must be free or else G would contain a critical P_3, which is not allowed. Since these are free and G is normal, they must be Berge-critical. If the structure was an odd hat on 7 or more vertices, the Berge-critical edges form an induced P_4 in G. If it was a 5-hat or an even antipath, they are three edges of an induced C_4 in G. □

3.4.6 Triangles and Berge-Critical edges

Recall that Theorem 3.2.1 immediately implies that any minimally imperfect graph with $\pi(G) = 2$ is either an odd hole or an odd antithole. Therefore, any minimally imperfect Berge graph is not triangle free, and neither is its complement.

Theorem 3.4.19. *Let G be a normal minimum imperfect Berge graph. If e is a Berge-critical edge in G then e is contained in a triangle.*

Proof. This is immediate from the existence of triangles in an odd hat, and from the existence of triangles in odd antipaths. □

In particular, if G is a normal minimum imperfect Berge graph then any edge not contained in a triangle is critical. If G is also conormal, the same holds for nonedges and antitriangles.

3.4.7 Distance 2 and Berge-Critical Edges

In this section we look at distance and Critical/Berge-critical, with a special emphasis on distance 2.

It is easy to observe, in any graph G having clique and stability number both greater than or equal to 2 (i.e. G is neither a clique nor a stable set), and a pair of nonadjacent vertices x and y, that if $\omega(G+xy) > \omega(G)$ - i.e. xy is a critical nonedge - then x and y must have a common neighbour. Since x and y are nonadjacent, this means they are at distance 2 from each other in G. Similarly, in a critical edge xy, we will have $d_{\overline{G}}(x, y) = 2$.

Interestingly, this necessary condition is also sufficient for odd holes and odd antiholes. In an odd hole, every edge xy is critical and the vertex on the opposite side of the hole from xy forms the midpoint of a P_3 joining x to y. Equally, every pair xy such that $d(x, y) = 2$ forms a critical nonedge, as adding it forms a triangle. Note that this is a generalisation of the "pancritical" property we saw earlier.

This observation that nonedges (respectively edges) that are further than 2 apart in G (respectively $\overline{G}$) cannot be critical, when combined with the weak dichotomy property, provides a potentially quite large number of critical nonedges (respectively edges) in G.

Theorem 3.4.20. *Let G be a conormal minimum imperfect Berge graph. Then if x and y are nonadjacent vertices at distance at least 3 in G, then xy is a Berge-critical nonedge. If G is normal, then the same is true for vertices at distance 3 or more in $\overline{G}$.*

Proof. If xy were not Berge critical, then it would be critical by weak dichotomy. But this is impossible, since x and y have no common neighbours. □

Theorem 3.4.21. *Let G be a conormal minimum imperfect Berge graph. If x and y are nonadjacent vertices such that xy is a nonedge that is not Berge-critical, then x and y are at distance 2, and every induced cycle in G containing x and y is a square.*

Proof. Let G be a normal minimum imperfect Berge graph and let x and y be nonadjacent vertices such that xy is a nonedge that is not Berge-critical. By Theorem 3.4.20 they are at distance 2 from each other.

Now, x and y must not be lying on any even hole of length $n \geq 6$, or else adding xy to G would create an odd hole, contradicting that xy is not Berge-critical. Also, since they are nonadjacent, they are clearly not contained in a triangle, and the only remaining possibility is a C_4. □

3.4.8 Distance 2, Holes, and Star-Cutsets

The results on induced paths of length 2 in minimally imperfect graphs can be combined with a result of Chvatal called the "Star-Cutset Lemma". Together, they can be used to improve Theorem 3.4.21 slightly. First we make some definitions. A cutset C in G is called a *star-cutset* if $G[C]$ has a star as a spanning tree - or equivalently, if $G[C]$ has a dominating vertex. In 1985 Chvatal [18] proved that a minimally imperfect graph could not contain such a cutset. It is equivalent to having a star-cutset in G, that you have two vertices x and y such that every path between x and y passes through the closed neighbourhood of some vertex v.

Theorem 3.4.22 (Chvatal [18])**.** *Let G be a minimally imperfect graph. Then G does not contain a star cutset.*

We can use Theorem 3.4.22 to establish the following fact: In a minimally imperfect graph, every P_3 is contained in a hole.

Theorem 3.4.23. *Let G be a minimally imperfect graph. Every P_3 is contained in a hole of G as an induced subgraph, and every $\overline{P_3}$ is contained in an antihole of G as an induced subgraph.*

Proof. We prove the first part, since the second is equivalent to it by duality (i.e. the perfect graph theorem). Let x and y be nonadjacent vertices at distance 2 from each other in G. Let z be a common neighbour of x and y. Let $P_0 = xzy$, and enumerate by $P_1, ..., P_n$ the set of induced paths between x and y not using z.

Let $N(P_i)$ denote the midpoints of P_i (the points besides x and y on the path) which are adjacent to z. If some P_i had empty $N(P_i)$, then xP_iyzx is a hole, so we assume they are all nonempty. Then $z + \bigcup_{i=1}^{n} N(P_i)$ is a star-cutset. □

This in turn implies that all edges lie on holes.

Corollary 3.4.24. *Let G be a minimally imperfect graph. Then every edge is contained in a hole, and every nonedge is contained in an antihole.*

Proof. We just prove every edge is contained in a hole. The second part follows by duality.

If xy is an edge, then there must exist some vertex z such that z is adjacent to precisely one out of x and y. Otherwise, x and y have the same neighbours in

$V(G) - \{x, y\}$, contradicting Theorem 3.2.6 (substitution lemma). Taking x, y, z we obtain a P_3 in G. Since the P_3 lies on a hole by Theorem 3.4.23, xy must lie on this same hole. □

Note that Theorem 3.4.21 could now be slightly modified to specify that the two vertices in the Theorem are contained in at least one C_4.

Theorem 3.4.25. *Let G be a conormal minimum imperfect Berge graph. If x and y are nonadjacent vertices such that xy is a nonedge that is not Berge-critical, then x and y are at distance 2, and there is some square containing x and y, and every induced cycle in G containing x and y is a square.*

3.4.9 A Note on Distance 2 and a Property of Odd Holes

We stop to note a curious property of the odd holes.

Let G be a graph. The 2-graph of G, which we denote by $2(G)$, is the graph with vertex set $V(G)$ and for $x, y \in V(G)$, x and y are adjacent in $2(G)$ if and only if they are at distance 2 in G.

The question we then pose is: For which graphs does the equation $G \simeq 2(G)$ hold? For cycles, we have the following theorem:

Theorem 3.4.26. *The cycle C_n is isomorphic to its own 2-graph if and only if $n \geq 5$ and n is odd: That is, if it is an odd hole.*

Proof. Since there are no links in a triangle, we have $2(C_3) = \overline{C_3}$. If n is even, then no vertex can reach another vertex at odd distance from it, while it can reach every vertex at even distance from it, so $2(C_n) = C_{\frac{n}{2}} \uplus C_{\frac{n}{2}}$ in this case. Meanwhile, if n is odd and at least 5, and the vertices are ordered $v_1, v_2, ..., v_n$, then the ordering $v_1, v_3, ..., v_n, v_2, v_4, ..., v_{n-1}, v_1$ forms a Hamiltonian cycle C in $2(C_n)$. Furthermore, since every vertex is linked to exactly 2 vertices in C_n, $2(C_n)$ is 2-regular and therefore the edges of C must be all the edges of $2(C_n)$, making it an n-cycle. □

Problem: Which graphs besides the odd holes have this property? It would be nice if this result characterised odd holes, but adding a single triangular chord to an even cycle of length 6 or more shows there are other graphs with this property.

3.4.10 Concluding Remarks to Section 3.4

In this section we have shown how critical and Berge-critical edges are directly related via the dichotomy property in the odd holes and the odd antiholes, and how some of this connection - weak dichotomy - can be preserved in a minimum counterexample to The SPGT. We have shown certain hypotheses, which combined with the weak dichotomy Property may lead to the existence of critical edges in a minimum counterexample.

In order for this approach to the SPGT were to be developed further it would require progress on three outstanding problems.

Firstly, the problem of proving the existence of Berge-free edges in a minimum imperfect Berge graph. For this, perhaps some insight would be gained by more investigations to the critically Berge and critically perfect graphs studied by Wagler [95].

Secondly, it would be greatly useful to be able to solve the closeness problem: That no minimally imperfect graph differs from another minimally imperfect graph in a single edge change. Doing so would allow one to upgrade the weak dichotomy property in our minimum counterexample to the SPGT to apply to G and $\overline{G}$, rather than just G itself.

Thirdly, it would be useful to disprove (in a minimum counterexample to the SPGT) the existence of an edge that is both critical and Berge-critical.

Specifically, if the second problem could be solved, and the first problem could be solved by showing a Berge-free edge incident with every vertex of our minimum counterexample, it would show that no minimum counterexample of odd order to the SPGT can exist. Meanwhile, if the second and third problems could be solved, it would guarantee the full dichotomy property in our minimum counterexample.

3.5 Freedom Games

We have considered previously the concepts of edge freedom and edge Berge-freedom. A nonedge xy in a graph G has freedom if $\omega(G + xy) = \omega(G)$, and Berge-freedom if $G + xy$ is Berge. An edge xy has freedom if $\alpha(G - xy) = \alpha(G)$, and Berge-freedom if $G - xy$ is Berge. So we can speak also of free and Berge-free edges, and we can speak about graphs having freedom in the sense of having free edges and Berge-freedom in the sense of having Berge-free edges.

In this section we consider a way of speaking of entire graph classes as having freedom (or not).

First we introduce some terminology: We start with a graph $G \in \mathcal{C}$, where $\mathcal{C}$ is some graph class which is required to contain K_n for all n. If x and y are nonadjacent vertices in G, then the pair x and y is a *$\mathcal{C}$-free* nonedge if $G+xy$ is in $\mathcal{C}$, and *$\mathcal{C}$-critical* otherwise.

For example, the critically perfect graphs from earlier in this chapter, are just the perfect graphs with all edges perfect-critical. Dually, the cocritically perfect graphs are the perfect graphs with all nonedges perfect-critical.

Now we define the game. It starts with a graph $G \in \mathcal{C}$, and the objective of the game is to choose $\mathcal{C}$-free edge additions one by one, until eventually the complete graph is reached. If we do this, we say we have won the *positive $\mathcal{C}$-freedom game* for the graph G. If one can always win the game, no matter how $G \in \mathcal{C}$ is chosen, we say $\mathcal{C}$ has *positive freedom*. Additionally, we say if G has a $\mathcal{C}$-free nonedge that G has positive $\mathcal{C}$-freedom, so that a class $\mathcal{C}$ has positive freedom if and only if all of its members have positive $\mathcal{C}$-freedom.

We also have a dual version: It starts with a graph $G \in \mathcal{C}$, where $\mathcal{C}$ is some graph class which is required to contain the empty graph E_n for all n. If x and y are adjacent vertices in G, then the pair x and y is *$\mathcal{C}$-free* if $G - xy$ is in $\mathcal{C}$, and *$\mathcal{C}$-critical* otherwise. Is there a sequence of "moves" - of $\mathcal{C}$-free edge deletions - which leads us to the empty graph? If so, we say we have won the *negative $\mathcal{C}$-freedom game*, and we say $\mathcal{C}$ has *negative freedom* if this is always possible. Additionally, we say if G has a $\mathcal{C}$-free nonedge that G has negative $\mathcal{C}$-freedom, so that a class $\mathcal{C}$ has negative freedom if and only if all of its members have negative $\mathcal{C}$-freedom.

A graph class $\mathcal{C}$ possessing both positive and negative $\mathcal{C}$-freedom has *freedom*.

If $\mathcal{C}$ is the class of graphs with clique number at most ω, the $\mathcal{C}$-free/critical edges are the free/critical edges we have seen previously: the freedom game is a generalisation of that concept.

Taking examples, the k-colourable graphs have negative freedom (because they are closed under taking subgraphs) but not positive freedom. Likewise the k-connected graphs have positive freedom (being closed under taking supergraphs) but not negative freedom. It would be nice to have some interesting classes possessing both positive freedom and negative freedom. Of course one could take the class of *all* graphs, but that hardly counts as interesting. We shall see that seven nontrivial graph classes exist with the freedom property, and they are all subclasses of the class of perfect graphs.

3.5.1 Freedom and Perfect Graphs

The first and most important point to emphasise: The perfect graphs themselves, as a graph class, have neither positive freedom nor negative freedom. We see this by choosing any of the critically perfect/cocritically perfect graphs discovered by Wagler [95].

Another subclass of perfect graphs that has neither positive freedom nor negative freedom is is the class of cographs. If one chooses $2K_2$ in the positive game, or C_4

in the negative game, there is no legitimate move as whatever we do a P_4 is created (see Fig 3.5).

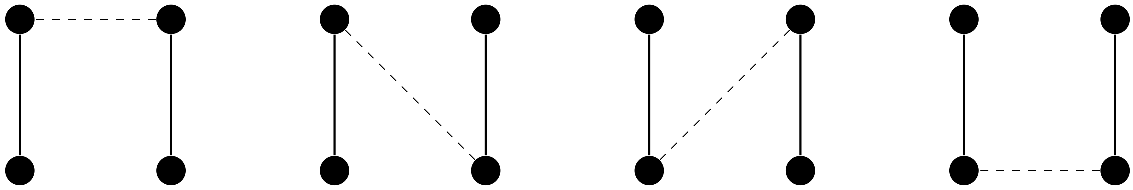

Figure 3.5: No way to add an edge to $2K_2$ without getting a P_4

We introduce in this section six classes, all subclasses of the perfect graphs, which do have the freedom property. The first subclass of the perfect graphs which we demonstrate does have the freedom property is the *split graphs*, and this is also the class for which this fact is easiest to prove. Recall that a split graph is a graph that can have its vertex set bipartitioned into two halves, one inducing a clique and the other a stable set. It is this characterisation which we shall make use of.

Following our terminology outlined above, we say an edge or nonedge is *split-critical* if switching its status produces a non-split graph, and we say it is *split-free* otherwise. In fact for split graphs, a stronger property than freedom is satisfied: every non-dominating vertex is incident with a split-free nonedge, and every non-isolated vertex is incident with a split-free edge.

Theorem 3.5.1. *Let G be a split graph, and let v be a vertex of G. If v is not dominating, there exists a non-neighbour y of v such that $G + xy$ is split. If v is not isolated, there exists a neighbour z of v such that $G - xz$ is split.*

Proof. Let G be a split graph and v be a vertex of G. Let S and K be the clique and stable set respectively, partitioning G into two. If either S or K is empty then the theorem is easily seen to hold, so we will assume they are both nonempty from here on. We show that we can add an edge to v and get a split graph, unless it is dominating. Applying this to G and additionally $\overline{G}$, we get the theorem.

Case 1: Assume $v \in K$. If v has a non-neighbour in S, then add this edge to get another split graph. If v is neighbours with everything in S, then it must be dominating, so we are done.

Case 2: Assume $v \in S$. If v has a non-neighbour in K, we add the edge to this vertex and we are done. Assume then that v has no non-neighbours in K. Then $K + v$ is a clique, and $(S - v, K + v)$ is a split partition of G. Now we apply the analysis of Case 1 again, and we are done. □

Inspired by this property of split graphs we shall make a new game. Given a graph G, which is a member of a class $\mathcal{C}$, we will play an adversarial 2-player game in which Player 1 chooses a vertex x in G, which is not allowed to be a dominating vertex, and Player 2 chooses a nonedge xy incident with x and adds this edge to get another graph $G + xy \in \mathcal{C}$. Player 1 wins if Player 2 has no legitimate moves, and Player 2 wins if a complete graph is reached. Note that in this *local positive freedom* game, "dominating vertex" has replaced "complete graph", and "isolated vertex" will replace "empty graph" in the dual version, *local negative freedom.*

Vertices which are either dominating or where adding an edge is possible have *positive $\mathcal{C}$-freedom.* Graphs where every vertex has positive $\mathcal{C}$-freedom have *local positive $\mathcal{C}$-freedom.* Classes where every member of the class has local positive $\mathcal{C}$-freedom have the *local positive freedom* property. Local negative freedom is defined analagously but with edge deletion replacing edge addition and isolated vertices replacing dominating vertices. Additionally, *local freedom* refers to the combination of local positive freedom and local negative freedom.

We saw in Theorem 3.5.1 that the class of split graphs satisfies not only the freedom property, but also the stronger property of local freedom (positive and negative). As it happens, this property holds true for four of our six classes.

Our second graph class is another subclass of the perfect graphs, and extends the split graphs in a natural way. The *chordal graphs* are the graphs not containing any hole as an induced subgraph. Therefore, a graph is split if and only if it and its complement are chordal. We will show that as for the split graphs, the chordal graphs have the local freedom property. Equivalently, Player 2 can always win the local freedom game.

Observe that even though split graphs are a subclass of the chordal graphs, the freedom property for the class of split graphs is *not* a special case of the freedom property for the class of chordal graphs. This is because the freedom property is a property of the entire class, rather than the individual graphs which are members of the class.

We begin by defining the terms *chordal-critical* edge/nonedge and *chordal-free* edge/nonedge as we did for the other classes: an edge/nonedge is chordal-critical if switching its status takes us out of the class. Also, if x and y are a pair of vertices in G then we say that they are a 2-*pair* if all induced paths between them have length 2.

First, we prove that the class of chordal graphs has local positive freedom. We prove existence of chordal-free nonedges incident with any non-dominating vertex. To do this, we must first see the equivalence of 2-pairs and chordal nonedges.

Lemma 3.5.2. *Let G be a chordal graph and $e = xy$ a nonedge in G. Then xy is chordal if and only if x and y form a 2-pair.*

Proof. First we assume that x and y form a 2-pair. If $G + xy$ contained an induced hole C, then $C - xy$ would be an induced path of length 3 or more between x and y, and that is a contradiction.

Conversely, assume that $G + xy$ is chordal. If x and y was not a 2-pair, then there must have been some induced path P of length 3 or more between x and y in G. But adding xy to get $P + xy$ gives us a hole in $G = xy$, again a contradiction. □

Using this theorem, we can prove the existence of a chordal-free edge incident with a non-dominating vertex in a chordal graph. It will be convenient for us to refer to an induced path between nonneighbours of length 3 or more as *long*.

Theorem 3.5.3. *The class of chordal graphs has local positive freedom.*

Proof. Let x be a vertex. We seek a proof that either x is dominating or is a member of a 2-pair, and then appeal to Lemma 3.5.2. If x is not dominating, it must have a vertex at distance 2 from it, say y. Let v be the midpoint of an induced path of length 2 from x to y. If there are no long induced paths between x and y we are done (since x and y form a 2-pair). Assume then that we have a long induced path P_1 between x and y, whose middle vertices will be denoted $p_{1,1}, p_{1,2}, \ldots$ etc.

If v were not adjacent to every vertex of P_1, then we would get a hole in G, because the paths xvy and P_1 are induced paths. Therefore $vp_{1,i}$ are edges for all i. Since x is not adjacent to $p_{1,2}$, $xvp_{1,2}$ is an induced path of length 3, and so is $xp_{1,1}p_{1,2}$: they are long paths.

Now we repeat the previous argument but with $p_{1,2}$ playing the role of y, and v and $p_{1,1}$ both playing the role of v.

If there is no long induced path between x and $p_{1,2}$ then we are done (since x and $p_{1,2}$ form a 2-pair), so let P_2 be a long induced path between x and $p_{1,2}$, with middle vertices denoted by $p_{2,1}, p_{2,2}, \ldots$ etc, and consider the induced paths $xvp_{1,2}$, $xp_{1,1}p_{1,2}$ and P_2. Applying the same argument we had before, there are edges from v and $p_{1,1}$ to all the vertices of P_2. In particular, neither is $p_{2,1}$ (since $p_{2,1}$ is not adjacent to $p_{1,2}$) but both are adjacent to $p_{2,2}$, so both $xvp_{2,2}$ and $xp_{1,1}p_{2,2}$ are both induced paths of length 2 from x to $p_{2,2}$.

Now if $p_{2,2}$ forms an even pair with x we are done. If not, we get another induced long path between x and $p_{2,2}$, P_3, so that $v, p_{1,1}$ and $p_{2,1}$ are adjacent to all of its vertices. In particular since v and $p_{1,1}$ and $p_{2,1}$ are adjacent to both endpoints of P_3, neither of these can be $p_{3,1}$, since that would imply P_3 is of length 2.

Continuing in this way, if we never get a 2-pair involving x, this makes the sequence $v, p_{1,1}, p_{2,1}, \ldots$ of distinct vertices continue indefinitely, which implies that the graph must be infinite. This is a contradiction (to our usual unstated assumption that all graphs must be finite). By Lemma 3.5.2, the proof is complete. □

The proof of the negative freedom for chordal graphs takes a different tack, using the elimination characterisation of the notion of a chordal graph.

A *simplicial elimination ordering* in a graph G is an ordering of the vertex set $v_1, ..., v_N$ such that for all k such that $k < j$ and $k < i$, $v_k v_j$ and $v_k v_i$ implies that $v_j v_i$ is an edge. A graph has a simplicial elimination ordering precisely if it is chordal (see [23]).

Theorem 3.5.4. *The class of chordal graphs has local negative freedom.*

Proof. Let G be a chordal graph on n vertices and fix a vertex x and a simplicial elimination ordering so that $x = v_i$ in the ordering $v_1, ..., v_N$. Assume v_i is not an isolated vertex. If v_i has no neighbours before it in the ordering, we can remove any edge incident with v_i and keep this simplicial elimination ordering as proof of chordality.

Assume then that v_i does have neighbours occurring before it in the ordering. Choose the earliest, v_j. Now deleting the edge $v_j v_i$ still keeps the same elimination ordering, since everything before v_j in the ordering is not adjacent to v_i, and everything else comes after v_j in the ordering, so doesn't care whether v_j has an edge deleted from it. □

Our third class arises again as a relative of the split graphs. A *threshold graph* is a graph G which is both a split graph and a cograph. Equivalently - and this is our *chosen* definition - a threshold graph is a graph G which can be whittled down to the null graph by the successive deletion of vertices which are either dominating or isolated. Formally, an ordering $v_1, ..., v_N$ of the vertices of G such that v_i either is dominating or is isolated in $G[v_1, ..., v_i]$ is called a *threshold elimination ordering* (TEO), and a threshold graph is a graph that possesses at least one such ordering. Threshold graphs were first studied by Chvátal and Hammer in [19], where the original and quite different definition using linear programming is given (which gives rise to the name "threshold graph"). Their paper also contains proofs of the equivalence of the definition of Chvátal and Hammer and the definition in terms of TEOs.

Observe that the threshold graphs are a self-complementary class. This is obvious from either the fact that they are split cographs, or from the self-duality of the notion of a TEO (but interestingly, this is not obvious from the linear programming definition given by Chvátal and Hammer). This fact means that local positive freedom, local negative freedom, and local freedom are all equivalent.

We also make the terminology that given a threshold graph equipped with a TEO, a vertex is either an *isovertex* if it is nonadjacent to everything preceding it in the TEO, or a *domvertex* if it is adjacent to everything preceding it in the TEO. We make repeated use of the fact that interchanging the order of a pair of vertices v_i, v_{i+1} produces another TEO, provided that v_i and v_{i+1} are either both domvertices or both isovertices.

Theorem 3.5.5. *The class of threshold graphs has local freedom.*

Proof. Let G be a threshold graph, and let there be a threshold elimination ordering $v_1, ..., v_N$. We first take an arbitrary vertex $x = v_i$ and assume it is a domvertex. We assume that $v_i \neq v_1$, and note that every vertex in every threshold graph can have an ordering where it doesn't come first (swap it with the 2nd vertex if necessary).

To add an edge to v_i, take the earliest isovertex coming after v_i in the ordering, say v_j. Such a vertex must exist, or else everything after v_i is a domvertex which makes v_i dominating in G. Since every vertex v_k strictly between v_i and v_j is a domvertex, moving v_i "to the right" to be immediately before v_j in the ordering gives another TEO. Then, if we swap v_i and v_j in that ordering, and add the edge $v_j v_i$, we have a TEO for $G + v_i v_j$.

To remove an edge from v_i, take the latest isovertex coming before v_i in the ordering, say v_j. If this vertex does not exist, then we can move v_i "to the left" be 2nd in the ordering, and it is still a TEO, and then this TEO is also a TEO of $G - v_1 v_i$.

So let us assume we do have this vertex v_j. Since every vertex v_k strictly between v_j and v_i in the ordering is a domvertex, we can move v_i "to the left" to be immediately after v_j in the ordering. Then swap v_j and v_i, and we have a TEO of $G - v_i v_j$. □

Our fourth class is the class of *interval graphs*. A graph G is an interval graph if there exists a collection of intervals $I_1, ..., I_N$ on the real line such that vertices $v_1, ..., v_N$ satisfy $v_i v_j \in E(G)$ if and only if $I_i \cap I_j \neq \emptyset$. Note that while the intervals may be closed or open in the above definition, restricting them to be open makes no difference to the graphs one gets, as one can simply change closed intervals to open intervals and slide them fractionally to get the same pattern of intersections. We now state this well-known fact as a lemma:

Lemma 3.5.6. *Let G be an interval graph with vertex set $v_1, ..., v_N$. Then it has a representation by open intervals, all of whose endpoints are distinct.*

Theorem 3.5.7. *The class of interval graphs has local freedom.*

Proof. Let G be an interval graph with vertex set $v_1, ..., v_N$ and interval representation $I_1, ..., I_N$, where all the endpoints are distinct by Lemma 4.2.2. Let v_i be a vertex of G. If v_i is not dominating, then I_i has an interval both of whose endpoints are to the left or both of whose endpoints are to the right. If I_j and I_k are the first and second closest such intervals to I_i, without loss of generality being positioned to the right, then slide the right endpoint of I_i right until it is precisely halfway between the left endpoint of I_j and the left endpoint of I_k. Since all the endpoints are distinct, we can do this without overlapping two intervals at once. Therefore, we have added one edge to v_i. Note that if the second one I_k does not exist and there is no second interval to the right of I_i, then just slide the right endpoint of I_i to be halfway between the left and right endpoints of I_j and it is no problem.

If v_i is not isolated, we can remove an edge from v_i by following the same sort of process. Let I_j and I_j be the intervals intersecting I_i (and without loss of generality having their right endpoint to the right of the right endpoint of I_i) whose left endpoints are the first and second (respectively) closest to the right endpoint of I_i. Then slide the right endpoint of I_i to the left until it is halfway between the respective left endpoints of I_j and I_k. As before it is no problem if there is only one interval I_j intersecting I_i with its right endpoint to the right of $I - i$, as one simply slides the right endpoint of $I - i$ to halfway between the left endpoint of I_j and whatever the next endpoint (be it left or right) positioned to the left of the left endpoint of I_j is. □

A fifth class of graphs is the class of *permutation graphs*. A permutation graph is a graph G such that $V(G)$ is able to be linearly ordered in two ways - $<_1$ and $<_2$ - such that xy is an edge in G if and only if: ($x <_1 y$ and $x >_2 y$) or ($x >_1 y$ and $x <_2 y$). So in a sense, a permutation graph represents to what extent two linear orderings fail to be isomorphic: the empty graph is obtained if they are isomorphic, while the complete graph is obtained if they are the duals of each other. Permutation graphs are perfect: for example they are the graphs that are both comparability and incomparability graphs [42].

We show that permutation graphs have the freedom property. First we need a lemma. If $<$ is a linear order on a set X, the *predecessor* of $x \in X$ is the maximum of the linear suborder of $<$ obtained by removing x and all elements greater than x. The *successor* of $x \in X$ is the minimum of the linear suborder of $<$ obtained by removing x and all elements lesser than x.

Lemma 3.5.8. *Let G be a permutation graph represented by the linear orders $<_1$ and $<_2$. If y is the predecessor of x in $<_1$, then swapping the positions of x and y in $<_1$ gives a pair of linear orders representating either $G - xy$ or $G + xy$, according as $x <_2 y$ or $x >_2 Y$, respectively.*

Proof. In the order $<_1$ the relative positions of all pairs besides xy are remaining unchanged after interchanging x and y - this is because y is the predecessor of x. Therefore no edge (or nonedge) besides xy will be affected. On the other hand, the status of xy will clearly be reversed. □

Theorem 3.5.9. *Let G be a permutation graph on n vertices, which is not the complete graph K_n. Then there exists a pair xy of nonadjacent vertices in G such that $G + xy$ is a permutation graph. Furthermore if G is not the empty graph E_n, there exists a pair xy of adjacent vertices such that $G - xy$ is a permutation graph.*

Proof. Let $<_1$ and $<_2$ be the two linear orderings of G. We show that if we cannot find some pair of nonadjacent vertices xy such that $G + xy$ is a permutation graph, then $G = K_n$. To this end, let x_1 be a vertex in G. Assume without loss of generality that it is not the minimum element of $<_1$. Let x_2 be the predecessor of x_1 in $<_1$. If $x_2 <_2 x_1$, then interchanging x_1 and x_2 in $<_1$ gives a pair of linear orders representing $G + x_1x_2$, by Lemma 3.5.8. Otherwise x_1x_2 is already an edge in G. In this case, we move on to the predecessor x_3 of x_2. If x_3 is less than x_2 in order $<_2$ as well, we interchange the order of x_3 and x_2 in $<_1$ to obtain a pair of linear orders representing $G + x_2x_3$, by Lemma 3.5.8. If on the other hand $x_2 <_2 x_3$, then we have $x_1 <_2 x_2 <_2 x_3$ and $x_3 <_1 x_2 <_1 x_1$, meaning that x_1, x_2, x_3 all form a clique in G.

Continuing this way taking predecessors, we either get the opportunity to add an edge preserving the permutation-graphness of the resulting graph, or we eventually reach the minimum element x_k of the order $<_1$, and $\{x_1, ..., x_k\}$ form a clique in G. If that happens, switching back to x_1, we instead take successors and repeat the same argument, and eventually if we cannot add an edge to make a permutation graph we get to the maximum element x_j under $<_1$, and we have that $\{x_k, ..., x_1, ..., x_j\}$ is a clique so the whole graph is a clique. □

We can also prove a local freedom property for permutation graphs (weaker than local freedom itself): there does not exist a permutation graph in which there exists a vertex which is incident with only permutation-critical edges and nonedges.

Theorem 3.5.10. *Let G be a permutation graph with a vertex x. Then there exists either an nonedge xy such that $G + xy$ is a permutation graph, or an edge xy such that $G - xy$ is a permutation graph.*

Proof. Let y be the vertex of G that immediately precedes x in the order $<_1$ (without loss of generality). Since $y <_1 x$, we know that reversing the order of x and y under $<_1$ will reverse the status of the edge xy: if $x <_2 y$ it will change xy from an edge to a nonedge, while if $x >_2 y$ it will change xy from a nonedge to an edge. Furthermore, no other edges are altered. By Lemma 3.5.8 we are done. □

For our sixth class of graphs, we turn to the *comparability graphs*. These are the graphs whose edge set can represent a partially ordered set, if given appropriate orientations. We will show that this graph class satisfies both negative and positive freedom. In the interests of clarity we will prove a theorem for partially ordered sets: That in any partially ordered set, it is possible to remove an arc and still have a partially ordered set (unless the partial ordering is the empty ordering), and it is possible to add an arc and still have a partially ordered set (unless the partial ordering is a total order). Freedom for comparability graphs follows directly from this.

Theorem 3.5.11. *Let P be a nonempty partially ordered set with underlying set V and (strict) ordering relation $<$. Then there exist $x, y \in V, x < y$, such that $P - (x < y)$ is a partially ordered set.*

Proof. Since P is not empty, there exist $x, y \in V$ with $x < y$. Let x be a minimal element in $<$ with this property, and then let y be a minimal element in the partial order $P[v \in V(G)$ such that $x < v]$ generated by the things which are greater than x in $<$. Clearly, there can be no element z such that $x < z$ and $z < y$. Therefore, deleting the arc from y to x does not break the transitivity axiom (which is the only one which could be broken) and therefore results in a partial order. □

The second assertion for partially ordered sets has a slightly more complicated proof which requires a case analysis.

Theorem 3.5.12. *Let P be a partially ordered set (which is assumed not to be a total order) with underlying set V and (strict) ordering relation $<$. Then there exist $x, y \in V$, x and y being incomparable, such that $P + (x < y)$ is a partially ordered set.*

Proof. Since P is not a total ordering, we have the existence of a vertex y, for which there exists a vertex x, such that y is incomparable to x. We shall choose these in such a way that we can apply a minimality argument. Specifically, we first choose some element y that has elements incomparable to it, and secondly we choose x to be *minimal* amongst those things which are incomparable to y.

Suppose we try and add an arc from y to x to get $P + (x < y)$ and see if this is a partial ordering. It could be, providing the transitivity axiom is not broken. As in the previous proof, the antisymmetry axiom cannot be broken (since x is incomparable to y). We now divide the rest of the proof into a case analysis.

Suppose that z is a vertex such that we have added and arc from y to x to the original partial order, and we want to know whether transitivity is broken by the triple x, y, z.

Case 1: If $z < y$ in P, then after adding an arc from y to x, y will be greater than both x and z, implying that x, y, z is not a counterexample to transitivity in $P + (x < y)$.

Case 2: If y and z are incomparable in P, then $z < x$ contradicts the minimality of x (with respect to incomparability to y). Therefore either z and x are incomparable (and all three of x, y, z are pairwise incomparable) or $x < z$, in which case adding an arc from y to x would make x less than both y and z (and transitivity would be satisfied). So this case is fine.

Case 3: If $y < z$ in P, then $z < x$ contradicts the incomparability of x and y, so the only possibilities are either $x < z$, or x and z are incomparable. The first out of these two results in no contradiction to transitivity upon adding the arc $x < y$, so it's fine. The tricky case is the second one.

Let us now assume that $y < z$ in P, and x and z are incomparable in P. We claim that as well as x being minimal with respect to incomparability to y in P, x is also minimal with respect to incomparability to z in P.

To see this let x' be a fourth element of P such that $x' < x$ and x' is incomparable to z. Since x is minimal with respect to incomparability to y, either $x' < y$ or $y < x'$. In the first case, we get by transitivity that $x < z$, and this is a contradiction (to x

being incomparable to z), while in the second case we get by transitivity that $y < x$, which is also a contradiction (to x being incomparable to y).

But now we have the precise same situation we started with, except with z instead of y. Therefore we can repeat our previous argument, using the minimality of x with respect to incomparability to z in P. Then two things can happen: Either we are able to add the arc from z to x and obtain a new partial ordering $P+(x<z)$, or we generate a new element z' for which we repeat the argument again, and so forth. It is easy to see that this leads to an infinite ascending chain, which is a contradiction as there are only finitely many elements in P. □

We now deduce freedom - both positive and negative - for comparability graphs.

Theorem 3.5.13. *Let G be a comparability graph. Then G has both positive freedom and negative freedom.*

Proof. Let G be a comparability graph, and choose any orientation of its edges which renders a partially ordered set. Applying Theorems 3.5.11 and 3.5.12 to this orientation, we get negative freedom and positive freedom, respectively. □

Our seventh and final class of graphs is that of the *weakly chordal graphs*. These are the graphs with no holes or antiholes of length at least 5 as induced subgraphs. Before we state the freedom theorem for the class of weakly chordal graphs, we state an important equivalent characterisation due to Hayward, Hoang and Maffray [79].

Lemma 3.5.14 (Hayward, Hoang and Maffray [79])**.** *Let G be a graph. Then G is weakly chordal if and only if every induced subgraph of G either possesses a* 2*-pair or else is a clique.*

We now show the positive and negative freedom properties both hold for the class of weakly chordal graphs.

Theorem 3.5.15. *The class of weakly chordal graphs has both positive and negative freedom.*

Proof. Let G be a chordal graph which is not complete. If we show that G has a nonedge xy such that $G + xy$ is weakly chordal then this is enough, because it would establish positive freedom and in turn negative freedom, since the class of weakly chordal graphs has self-complementarity.

By Lemma 3.5.14, G has a 2-pair; we call it xy. Adding xy as an edge will not create any new holes in $G + xy$, as a new hole would require xy to be joined by an induced path in G of length 3 or more, which is not the case. But any hole or antihole of length at least 5 must either be a hole of length 5 or more, or else contain a C_4 by being an antihole of length 6 or more. Therefore no hole or antihole of length at least 5 can be created in $G + xy$, which means that $G + xy$ is weakly chordal because G is. □

Now we present an application of some of the results from Section 3.5 in the area of critically and cocritically perfect graphs.

3.5.2 An Application to Critically and Cocritically Perfect Graphs

Recall that previously in this chapter (in Section 3.4) we have looked at Berge-critical edges/nonedges, and critically/cocritically Berge graphs where all edges/nonedges are Berge-critical. By the SPGT these are equivalent to the critically/cocritically perfect graphs, where the deletion of any edge renders the graph imperfect. These graphs were the subject of the PhD thesis of Wagler [95] (who called cocritically perfect graphs “anticritically perfect graphs”).

Some of the results obtained by Wagler concerned the structures which cannot exist in critically/cocritically perfect graphs, and the properties which cannot hold in critically/cocritically perfect graphs. For example, Wagler showed that no critically/cocritically perfect graph can be a weakly chordal graph, which is quite a good result as the class of weakly chordal graphs is “fairly large”, in some sense (e.g. about 87 percent of graphs on ≤ 10 vertices; see [56]).

We remark that it is possible to deduce, from our results for any of the classes above for which we proved freedom properties in the previous section, that no critically/cocritically perfect graph is contained in any of them. For example, suppose G was a critically perfect comparability graph. Freedom of comparability graphs, together with the critical perfection of G, implies that there is a pair of imperfect comparability graphs, which of course cannot be true.

We now state this as a theorem. We also remark that the class of graphs that are either comparability graphs or the complement of one may be thought of as "somewhat large" (e.g. about 52 percent of graphs on ≤ 10 vertices; see [56]).

Theorem 3.5.16. *Let G be graph. If either G or $\overline{G}$ is a comparability graph, then G is not a critically perfect graph (or a cocritically perfect graph).*

This is new as far as we know. We could state a similar theorem for, say, chordal graphs or permutation graphs, but these would already be contained in Theorem 3.5.16 or the result of Wagler [95] on weakly chordal graphs (which our methods also prove).

We could also state something stronger than just "a chordal graph is not a critically/cocritically perfect graph". Since we know every (non-dominating/non-isolated respectively) vertex in a chordal graph can have an edge added so that the graph remains chordal, and have an edge deleted so that the graph remains chordal, we know it can have an edge added/deleted so that the graph remains perfect.

We define a *perfect-critical vertex* to be a non-isolated vertex in a perfect graph where deleting any of the edges incident to it makes the graph imperfect, and we define a *perfect-cocritical vertex* to be a non-dominating vertex in a perfect graph where adding any edge to be incident with it makes the graph imperfect. We can state the following stronger theorem for chordal graphs, which is thereby no longer a special case of the results of Wagler:

Theorem 3.5.17. *Let G be a chordal graph. Then G does not contain any perfect-critical vertices or any perfect-cocritical vertices.*

The similar theorems for split, threshold, or interval graphs are contained in Theorem 3.5.17 as a special case.

In the next section we define a "reconfiguration"-type graph, which is associated with a given graph class, and connect it to the freedom properties for the various graph classes we have considered.

3.5.3 A Bipartite Reconfiguration-type Graph

Let $\mathcal{C}$ be a class of graphs, and let n be a positive integer. We define the $(\mathcal{C}, n)$-*reconfiguration graph* as the graph whose vertex set is the set of all n-vertex graphs who are members of $\mathcal{C}$, and whose vertices are adjacent if and only if the corresponding pair of graphs can be obtained from one another by adding/deleting a single edge. We can note, quickly, that the graph must have only even cycles and is therefore bipartite. The question we wish to ask is, when is this graph connected?

Clearly if $\mathcal{C}$ possesses either positive or negative freedom then the $(\mathcal{C}, n)$-reconfiguration graph is connected for all n, since every vertex has a path to the vertex corresponding to the complete or empty graph, respectively.

Meanwhile, if $\mathcal{C} = \mathcal{P}$, the class of perfect graphs, then the graphs in [95] which are both critically perfect and cocritically perfect, will be isolated vertices. So in this case the $(\mathcal{P}, n)$-reconfiguration graph will be disconnected for any of the infinitely many values of n for which such graphs exist.

There is a third situation, where a graph class has neither positive nor negative freedom, and yet the $(\mathcal{C}, n)$-reconfiguration graph is still connected. For example, if $\mathcal{C}$ is the class of cographs (denoted $\mathcal{CO}$), and $n = 4$, then although we cannot add any edges to $2K_2$ and get a cograph, we can delete one edge to get $K_2 + 2K_1$, and add edges to one of the ends of the sole remaining K_2 to make it dominating, and add edges however we like from there to reach K_4, all the while remaining a cograph. Continuing along these lines one can easily see that the $(\mathcal{CO}, n)$-reconfiguration graph for $n = 4$ is connected.

In fact, the $(\mathcal{CO}, n)$-reconfiguration graph is connected for any n, which we now prove.

Theorem 3.5.18. *Let n be a positive integer. Then the ($\mathcal{CO}$,n)-reconfiguration graph is connected.*

Proof. Before we proceed with an inductive proof that $\mathcal{G}$ is connected, we make a claim.

Claim 1: If G is a cograph on n vertices that has a path in the ($\mathcal{CO}$,n)-reconfiguration graph to the (vertex of) the empty graph, then it has a path in the ($\mathcal{CO}$,n)-reconfiguration graph to the (vertex of) the complete graph. The converse is also true.

Proof of Claim 1: By freedom for threshold graphs (see Theorem 3.5.5), one can pass freely between the complete and empty graph (and vice versa) whilst remaining in this class at all times, and the class of threshold graphs is a subclass of the class of cographs. Thus the claim is true.

Now we make our inductive hypothesis: Any cograph G has a path in the ($\mathcal{CO}$,n)-reconfiguration graph ending at the complete graph, and also a path in the ($\mathcal{CO}$,n)-reconfiguration graph ending at the empty graph.

Note for the base case that the claim is trivially true if $n \leq 2$. Suppose we have established the inductive hypothesis for every positive integer less than some fixed positive integer n.

Letting G be a cograph on n vertices, either G is disconnected or codisconnected. If G is disconnected, we have that G is the disjoint union of G_1 and G_2, and following the paths in the ($\mathcal{CO}$,$|G_1|$)-reconfiguration graph from G_1 to $E_{|G_1|}$ and in the ($\mathcal{CO}$,$|G_2|$)-reconfiguration graph from G_2 to $E_{|G_2|}$ gives us a path from G to E_n. From E_n we can get to K_n by Claim 1.

If G is codisconnected, we have to do the same thing, but this time we follow the paths that arrive at the complete graphs, and go from G to K_n. From there we can also get to E_n by Claim 1. Thus the proof is complete as we have showed that the vertices of the complete and empty graphs are each connected to everything via paths. □

In light of Theorem 3.5.18 we note that connectivity of the $\mathcal{C}$-reconfiguration graph is a plausible weakening of the properties of positive and negative freedom, and that cographs provide a "middle example" to this effect.

We also note that one could consider the nonexistence of close pairs of minimally imperfect ($\mathcal{MI}$) graphs (see Section 3.4) - which is a corollary of the SPGT - could be expressed as the $(\mathcal{MI}, n)$-reconfiguration graph being empty or null for all n.

3.5.4 Concluding Remarks to Section 3.5

We have seen that despite the fact that the freedom property fails to hold for the class of perfect graphs, it holds for many of their subclasses: Split graphs, chordal graphs, interval graphs, threshold graphs, permutation graphs, and comparability graphs. Also, except for comparability graphs, an even stronger property is satisfied: Local freedom, for split, chordal, interval and threshold graphs; and a weaker version of local freedom for permutation graphs.

Further directions would consist of two aspects. Firstly, it would consist of trying to prove local freedom properties for permutation graphs and comparability graphs, or possibly finding counterexamples that show they do not have it. Secondly, one is inclined to consider what, if any, other classes of perfect graphs satisfy a freedom property. Some candidates might include perfectly orderable graphs, unimodular graphs, totally unimodular graphs, parity graphs, and quasi-parity graphs. In some cases there would be consequences for the structure which of the critically/cocritically perfect graphs.

For the $\mathcal{C}$-reconfiguration graphs, we saw that the cographs can be seen to occupy a space in between the perfect graphs and the classes which had freedom properties.

For future directions in this area, one could consider that the $\mathcal{C}$-reconfiguration graphs are bipartite, and ask whether it is possible that *every* bipartite graph arises in such a way.

Chapter 4

Bireflexive Graphs, Homomorphisms, and Modular Decomposition

What is the "dual" notion to that of the induced subgraph? In this chapter we argue that it is the modular decomposition, a notion of graph theory which came to prominence in 1972 when László Lovász [66] used it to prove the Perfect Graph Theorem, having earlier been introduced in 1967 by Lovász's supervisor, Tibor Gallai [36]. The modular decomposition of a graph is an important concept in graph theory with connections to diverse areas such as connectivity, perfect graphs, graph reconstruction and graph algorithms.

The duality we seek will arise by considering both induced subgraphs and modular decompositions in terms of graph homomorphisms.

To this end, we will first introduce the basic aspects of the modular decomposition, and the important notion of a "prime" graph. Then we introduce homomorphisms of simple graphs, and the concepts of weak subgraph as an injective homomorphism, and H-colouring as a surjective homomorphism.

We then show how to link these concepts by a novel definition of a graph-like structure we call a "bireflexive graph" and show that this naturally gives rise to the induced subgraph as an injective homomorphism, and modular decomposition as a surjective homomorphism, so that these are dual concepts.

Finally we investigate some aspects of this definition, and consider the idea that some problems stated in terms of induced subgraphs may have dual versions: We consider the "dual" of the vertex reconstruction conjecture based on the idea of modular decomposition being the dual notion of that of induced subgraph. We consider whether a weakened version of the reconstruction conjecture may be true.

4.1 Introduction

Modular decompositions were first described by Gallai [36] in his work on partially ordered sets and the related comparability graphs. Roughly speaking, the modular decomposition functions as a way of extending the notions of connectedness of graphs. It does this by introducing the concept of a module, which generalises the concept of a connected component of a graph. As such, it differs completely from other connectedness related notions (for example k-connectedness, or t-toughness).

Since its introduction by Gallai the modular decomposition has had a tendency to reappear in various different contexts in graph theory, one example being in the theory of perfect graphs. For example, the original proof by László Lovász [66] of the Perfect Graph Theorem uses the modular decomposition, in the form of Lovász's Susbtitution Lemma (Lemma 3.2.3). Secondly, some important subclasses of the class of perfect graphs, for example the class of cographs, the class of comparability graphs, and the class of permutation graphs satisfy similar properties to Lemma 3.2.3. Also, modular decompositions in perfect graphs are also closely related to skew partitions - an important notion in the theory of perfect graphs - by a theorem of Reed [80].

Part of the interest in the concept of a module is that it allows one to think of *any* graph as if it were a disconnected graph, or the complement of a disconnected graph;

the modules of a graph replicate some of the properties of the connected components of a disconnected graph. This has led to the modular decomposition of graphs being applied to the reconstruction conjecture of Kelly [47, 61], a notoriously difficult conjecture that is nevertheless relatively easy to prove true on disconnected graphs (and their complements). The paper by Brignall, Georgiou and Waters [14] represents a rare partial progress on this difficult problem. More generally any problem which can be solved easily on disconnected graphs but not connected graphs is potentially suited to a modular approach.

The modular decomposition has been studied from an algorithmic point of view, as it can be evaluated in linear time in arbitrary graphs [71], and is applied in the recognition of various subclasses of the perfect graphs: cographs [71], permutation graphs [71], comparability graphs [73], interval graphs [57, 73], and distance hereditary graphs [42, 90], which seems to provide further evidence of its relationship to graphs that are perfect. It also has increasingly found application in efficient algorithms for representing graphs visually in the plane - the area known as *graph drawing* [77, 90].

Graph homomorphisms are, rougly speaking, structure-preserving maps between graphs. They arose as a natural consequence of the spreading of the concept of homomorphism from group theory where it originated in the late 1800s to the rest of mathematics. Graph homomorphisms first became prominent in the 1960s in the work of Sabidussi [83] and Hedrlin and Pultr [48] and around the same time the related concept of an automorphism group [84] of a graph was introduced. Since then the area has rapidly expanded, and is the subject of hundreds of papers as well as a textbook by Hell and Nešetřil which is dedicated to the area [49]. The two most famous applications of graph homomorphisms are:

1. Generalising graph colouring. For example graph homomorphisms generalise the notion of graph colouring, whereby a graph is k-colourable if and only if it has a homomorphism to the complete graph on k vertices. This led to a

homomorphism from G to H being dubbed an "H-colouring of G". Additionally homomorphisms to more general classes of graphs than the complete graphs can express some generalisations of graph colouring arising independently of the concept of a graph homomorphism: For example, fractional colouring [85] and circular colouring [41] occur by generalising the class of complete graphs to the classes of Kneser Graphs and Circular Complete Graphs respectively.

2. Constraint satisfaction problems [28, 29, 74]: The so-called constraint satisfaction problems (CSPs) are equivalent to problems of determining the existence of a graph homomorphism to a corresponding digraph.

We will give precise definitions of homomorphism and their various types in Section 4.3.

4.2 Modular Decomposition: Modules, Skeletons and Primality

Now we define modules and modular decompositions of graphs, and prove some of their basic properties. The treatment roughly follows that found in the lecture notes of Lozin [67].

Let G be a graph. A *module* in a graph G is a vertex set $M \subseteq V(G)$ such that $N(x) - M = N(y) - M$ for all $x, y \in M$.

Modules are guaranteed to exist. Regardless of the structure of the graph G, all sets $\{v\}$ for $v \in V(G)$ are modules, as is $V(G)$ itself. These are then referred to as *trivial* modules. Also, this gives two ways to partition any graph on at least two vertices into modules (if $G = K_1$ then the two ways coincide).

Once we have a partition of $V(G)$ into modules $M_1, ...M_k$, any pair of modules either have all possible edges passing between them, or only nonedges passing between them. Therefore, if we generate a subset $S \subseteq V(G)$ by choosing exactly one vertex from each module, then $G[S]$ preserves all the information about the adjacencies between the modules. A vertex set S so generated is said to be a *skeleton* of G, and it's structure/isomorphism type $G[S]$ is determined by the choice of modules. A *modular decomposition* of G consists of a partition of G into modules, together with the vertices $v_1, ...v_k$ chosen from the modules, and the induced subgraph $G[v_1, ..., v_k]$ of G.

If we have a modular decomposition of G, with skeleton S and modules $M_1, ..., M_k$, we may speak of the reverse process, where G is obtained from S via the *modular substitution* of $M_1, ..., M_k$ for the vertices of G.

Note that for arbitrary graphs G and H, we make no fuss over the technical distinction between whether H *is* a skeleton of G or whether it is *isomorphic to* a skeleton of G, and likewise for modules. In this way the situation is similar for other graph relationships such as subgraphs and induced subgraphs.

Observe also that if we choose all modules to be singletons, G itself is our skeleton, and if we choose only the one module $V(G)$, then K_1 is our skeleton. These are therefore called the *trivial skeletons* of G.

Connected components of graphs give immediate examples of nontrivial modules and nontrivial skeletons. The connected components of a disconnected graph, as well as the coconnected components of a graph whose complement is disconnected (codisconnected graph), are nontrivial modules. In a disconnected graph $\overline{K_2}$ will be a nontrivial skeleton (unless $G = \overline{K_2}$), and similarly K_2 is a nontrivial skeleton of codisconnected graphs (unless $G = K_2$).

Modules, therefore, generalise connected/coconnected components. In Figure 4.1 we give an example of a module which does not arise as a connected/coconnected component.

A graph has nontrivial modules precisely if it has nontrivial skeletons, which we state here formally.

Proposition 4.2.1. *Let G be a graph. Then the following are equivalent:*

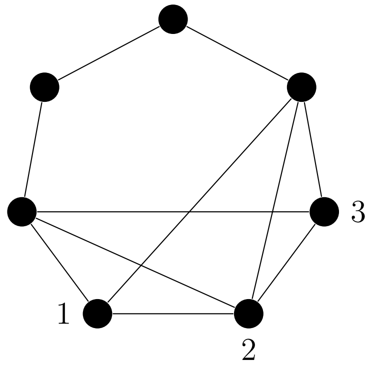

Figure 4.1: A biconnected graph with a module formed by the vertices labelled $\{1, 2, 3\}$.

1. *G has no nontrivial modules.*

2. *G has no nontrivial skeletons.*

A graph satisfying either, and hence both, of these conditions is called *prime*. A graph is said to be *decomposable* if it is not prime.

Examples: Graphs that are not biconnected are all decomposable if and only if they have at least three vertices. This includes as special cases complete graphs, empty graphs, and more generally complete multipartite graphs and their complements. Paths are prime provided they have at least four vertices. Cycles are prime when they have at least five vertices. Trees are prime if no leaves have siblings, and decomposable otherwise; a skeleton of a tree is obtained by merging leaves with their siblings.

Prime graphs are integral to the theory of modular decomposition. The reason for this is that any decomposable graph is associated with a unique prime graph, in a special way. Before describing this we state a couple of lemmas:

Lemma 4.2.2. *Let G be a graph and let M_1 and M_2 be intersecting modules of G. Then $M_1 \cap M_2$ is a module of G.*

Proof. If $|M_1 \cap M_2 = 1|$ then we are done since any singleton is a module. Therefore, let $v_1.v_2$ be distinct vertices in $M_1 \cap M_2$, and let w be a vertex in G. If w lies outside $M_1 \cap M_2$, then either it lies outside M_1, in which case v_1 and v_2 have the same relation to w by modularity of M_1, or it lies outside M_2 in which case v_1 and v_2 have the same relation to w by the modularity of M_2. □

Lemma 4.2.3. *Let G be a graph and let M_1 and M_2 be intersecting modules of G. Then $M_1 \cup M_2$ is a module.*

Proof. Let v_1, v_2 be distinct vertices of $M_1 \cup M_2$, and let w be a vertex of G which lies outside $M_1 \cup M_2$. If v_1 and v_2 both lie in M_1 or both lie in M_2 we are done, so assume $v_1 \in M_1$ and $v_2 \in M_2$. Observe that w is outside the intersection of M_1 and M_2, since it lies outside even their union.

Everything in $M_1 \cap M_2$ bears to w the same relation, by the fact that $M_1 \cap M_2$ is a module (see Lemma 4.2.2). This must be shared by anything in M_1, and everything in M_2, which then makes $M_1 \cup M_2$ a module. □

We define a *principal skeleton* of G to be a skeleton S of G such that $G[S]$ is a prime graph. A module is called a principal module of G if it is part of a modular decomposition of G whose skeleton is a principal skeleton of G.

It is easy to see that if G is a disconnected graph of n components, the principal skeleton is isomorphic to $\overline{K_2}$. Likewise if G is codisconnected, the principal skeleton is uniquely determined and is isomorphic to K_2. Note that the assignment of principal modules with vertices of the principal skeleton is not unique unless $n = 2$.

What if G is neither disconnected nor codisconnected? We say such a graph is *biconnected*, and in this case the set of principal modules partitions $V(G)$. This implies that all principal skeletons of G are isomorphic.

Theorem 4.2.4. *Let G be a biconnected graph. Then the principal modules of G are uniquely determined and partition $V(G)$, and the principal skeleton is uniquely determined, and given a modular decomposition of G, the modules of this decomposition are principal if and only if the skeleton is principal.*

Proof. Let M_1 and M_2 be two maximal modules of G which are not G itself. If M_1 and M_2 are not disjoint, then $M_1 \cup M_2$ forms a larger module of G by Lemma 4.2.3.

If $M_1 \cup M_2$ is not to be $V(G)$ itself, then this contradicts the maximality of M_1 and M_2. Therefore $M_1 \cup M_2 = V(G)$.

$M_1 \setminus M_2$ and $M_2 \setminus M_1$ must both be nonempty, by maximality of M_1 and M_2, so let $x \in M_1 \setminus M_2$ and let $y \in M_2 \setminus M_1$. If x and y are adjacent, then by modularity of M_1, everything in M_1 is adjacent to y. Repeating this argument, everything in M_1 is adjacent to everything in $M_2 \setminus M_1$, and $\overline{G}$ is disconnected. If x and y are nonadjacent, a similar argument shows that G must be disconnected.

Therefore, we have established that M_1 and M_2 must be disjoint. Also, since any vertex of G must occur in some module, by induction it must occur in a principal module of G (by finiteness of G, the iteration of taking larger modules must terminate at some principal module). Therefore, the principal modules of G form a partition of $V(G)$ into modules.

Finally, the skeleton S of G generated by this partition is prime. If S had some vertices forming a nontrivial module in S, then the union of the principal modules corresponding to these vertices would be contained in some larger nontrivial module in G, contradicting that those modules are principal.

$\square$

In particular we note that Theorem 4.2.4 implies the following:

Corollary 4.2.5. *Let G be a graph. Then G only has one principal skeleton, up to isomorphism.*

Theorem 4.2.4 forms - for the biconnected graphs - an analogue to the theorem in number theory which says any number factors uniquely into primes. There are, in fact, two ways in number theory of defining prime number. Firstly, p is prime when it is only divisible by 1 and p. Secondly, p is prime if p dividing ab implies p divides at least one out of a and b.

If we take as analogues to "n divides m" the statements "H is a module of G" or "H is a skeleton of G", it seems our definition of "prime" is analagous to the first of these: A graph G is prime iff it has no "divisors" besides K_1 and itself (where "divisor" can refer to either modules or skeletons).

What if we instead said a graph P is prime if and only if whenever P is a module of a graph G, that has modular decomposition with skeleton S and modules $M_1, ..., M_n$, that P must be a module of either S or M_i for some $1 \leq i \leq n$? Likewise, we could say a graph P is prime if and only if whenever P is a skeleton of a graph G, that has modular decomposition with skeleton S and modules $M_1, ..., M_n$, that P must be a skeleton of either S or M_i for some $1 \leq i \leq n$?

We shall see that these are equivalent to our initial definiton. First an easy lemma:

Lemma 4.2.6. *Let G be a graph. Then if M is a module in G, and $M \subseteq X$, then M is also a module in the induced subgraph $G[X]$ of G.*

Theorem 4.2.7. *Let P be a graph. Then the following are equivalent:*

1. *P is prime.*
2. *For all graphs G such that P is a module of G, with G having modular decomposition with skeleton S and modules $M_1, ..., M_n$, then P must be a module of either S or M_i for some $1 \leq i \leq n$.*
3. *For all graphs G such that P is a skeleton of G, with G having modular decomposition with skeleton S and modules $M_1, ..., M_n$, then P must be a skeleton of either S or M_i for some $1 \leq i \leq n$.*

Proof. ($1 \Rightarrow 2$) First we identify P with the set of vertices of G which generate P as a module. If $P \subseteq M_i$ for some $1 \leq i \leq n$, then P is a module of M_i and we are done.

Therefore, we assume that P contains vertices from multiple modules. If $P \cap M_i \geq 2$ for some i, then by Lemma 4.2.2 and Lemma 4.2.6, $P \cap M_i$ is a submodule of P, and is a nontrivial module in P. This contradicts the fact that P is prime.

Thus, P contains either one or no vertices from each of the M_i. But since P is an induced subgraph of the skeleton S of G and is a module in G, it must be a module in S.

($1 \Rightarrow 3$) If P is a skeleton of G, then since P is prime, it must be the principal skeleton of any skeleton of G besides K_1, by Corollary 2.1.5. Therefore, P is a skeleton of S unless $S \simeq K_1$. Meanwhile, if $S \simeq K_1$, then the only module in the modular decomposition is G itself, and P is a skeleton of G.

($2 \Rightarrow 1$) If P satisfies (2) but is not prime, then P must have a nontrivial decomposition, for which the skeleton and modules are all proper induced subgraphs of P. Since P cannot be a module of a proper induced subgraph of itself, we get a contradiction.

($3 \Rightarrow 1$) If P satisfies (3) but is not prime, then P must have a nontrivial decomposition, for which the skeleton and modules are all proper induced subgraphs of P. Since P cannot be a skeleton of a proper induced subgraph of itself, we get a contradiction. □

4.3 Graph Homomorphisms

Now we define graph homomorphisms and give some of their most basic properties (our treatment roughly follows [49]). In the setting of graph homomorphisms it is common to encounter different notions of graphs - graphs with loops, directed edges and so forth - and changing the type of graph considered is often useful for representing different concepts. In the introduction to this thesis we defined a graph by a set V of vertices equipped with an irreflexive symmetric relation on V. Such a structure is sometimes called a *simple* graph, typically when different notions of graphs are under consideration and there may be risk of confusion.

Let G and H be graphs. A graph homomorphism is a mapping which preserves all the adjacencies of the first graph in the second. More precisely, a *graph homomorphism* from G to H is defined as a function $f : V(G) \to V(H)$ such that whenever vw is an edge in G, $f(v)f(w)$ is an edge in H. If f is respectively injective, surjective, or has an inverse, then f is said to be, respectively, a *monomorphism*, an *epimorphism*, or an *isomorphism*.

A few basic properties of graphs can be expressed in terms of homomorphisms.

Proposition 4.3.1. *Let G and H be graphs. Then there exists a monomorphism from G to H iff G is isomorphic to a subgraph of H.*

Proof. If f is a monomorphism from G to H, then taking H and deleting the vertices which are not of the form $f(v) : v \in V(G)$ and the edges which are not of the form $f(v)f(w) : v, w \in V(G)$ gives the subgraph G' of H to which $f : G \to G'$ is an isomorphism.

Conversely, if G is isomorphic via f to a subgraph G' of H, and $i : G' \to H$ is the inclusion map from G' to H, then $i \circ f$ is a monomorphism from G to H. $\square$

Proposition 4.3.2. *Let G be a graph. Then there exists an epimorphism from G to K_n iff G has a graph colouring using n colours.*

Proof. ($\Leftarrow$) First consider a graph colouring with n colours as a mapping $c : V(G) \to \{1, ..., n\}$ such that adjacent vertices receive different colours. Let K_n be the complete graph with vertex set $\{v_1, ..., v_n\}$. We claim that defining $f(v)$ to be the vertex of K_n with the same index as the colour received by v in c is an epimorphism.

Assume v and w are neighbours in G. Then since $c(v) \neq c(w)$, $f(v) \neq f(w)$ and since K_n is complete, all distinct vertices are adjacent. So $f(v)f(w)$ in K_n.

($\Rightarrow$) Assume that $f : G \to K_n$ is an epimorphism. Define c by colouring each vertex v in G by the index of the vertex of K_n it got mapped to by f. Since f is an epimorphism, and K_n has no loops, it maps adjacent vertices to different vertices. Therefore they have received different colours. $\square$

What Proposition 4.3.2 says is that graph epimorphisms generalise graph colouring: the generalisation is that when the target graph is not complete, adjacent vertices must not only receive different colours, but there are some "illegal" colour pairings which also cannot be received by adjacent vertices. These illegal pairings are represented by deleting edges from K_n.

This more general problem is called "H-colouring". In light of Theorem 4.3.1, the notions of subgraph and H-colouring are dual notions - one being represented by monomorphisms and the other being represented by epimorphisms.

When graphs are defined as reflexive symmetric binary relations (i.e. reflexive graphs) epimorphisms to complete graphs (or empty graphs) do not describe graph colourings or any related graph theoretical notion. For this reason the irreflexive definition of graphs has the advantage.

Another important property of graph homomorphisms is their "mono-epi" composition property, which we now explain. Let $f : G \to H$ be a graph homomorphism. Define the *quotient* graph $Q = Q(f)$ of f as follows: The vertex set of Q is the vertex set of H induced by $f(G)$. Two distinct vertices v and w of Q are adjacent in Q if and only if their inverse images $f^{-1}(v)$ and $f^{-1}(w)$ have an edge between them in G.

Theorem 4.3.3. *Let G and H be graphs and $f : G \to H$ be a graph homomorphism. Then there exists a graph G' such there exists an epimorphism $g : G \to G'$ and there exists a monomorphism $h : G' \to H$ such that $f = h \circ g$.*

Proof. Let G' be the quotient of f. Since the vertex set of G' is just $f(G)$, the map $g : G \to G'$ defined by $g(v) = f(v)$ is a surjective function from G' to H, and note that by the way G' is defined, $g : G \to G'$ is a homomorphism. Then letting $i : G' \to H$ be the inclusion map of G' into H (which is a monomorphism) gives $f = i \circ g$. □

This allows us to establish that graph homomorphisms have the following remarkable property: A graph can be characterised up to isomorphism by the number of homomorphisms it has to every possible auxiliary graph. To this end, let $\mathrm{Hom}(X, G)$ and $\mathrm{Hom}(X, H)$ denote the sets of all homomorphisms from X to G and H respectively.

Theorem 4.3.4. *Let G and H be graphs. Then G and H are isomorphic if and only if for every auxiliary graph X : $|Hom(X, G)| = |Hom(X, H)|$.*

Proof. Let G and H be graphs and assume that $\mathrm{Hom}(X, G)$ and $\mathrm{Hom}(X, H)$ have the same cardinality for any auxiliary graph X.

Let $\mathrm{Mono}(X, G)$ and $\mathrm{Mono}(X, H)$ denote the respective numbers of monomorphisms from X to G and H. We show that $\mathrm{Mono}(X, G) = \mathrm{Mono}(X, H)$ for all X by induction.

Firstly, if X has order 1, then any homomorphism (any map, in fact) from X to G or H is a monomorphism. Therefore, $\mathrm{Mono}(X, G) = \mathrm{Mono}(X, H)$ (and G and H have the same order).

Therefore we assume X has order at least 2 and that the equality we seek to show holds for all smaller auxiliary graphs.

Now we use the mono-epi composition to divide up the homomorphisms based on the quotient graphs of X to obtain the sum:

$$Hom(X, G) = Mono(X, G) + \sum_{Q \neq X} Mono(Q, G) \tag{4.3.1}$$

$$Hom(X, H) = Mono(X, H) + \sum_{Q \neq X} Mono(Q, H) \tag{4.3.2}$$

The left hand sides of (1) and (2) are equal by assumption, and the rightmost terms on the right hand sides of (1) and (2) are equal by induction, since the quotients Q have fewer vertices than X. Therefore, $\mathrm{Mono}(X, G)$=$\mathrm{Mono}(X, H)$.

Now $\mathrm{Mono}(H, H)$=$\mathrm{Mono}(G, H)$=$\mathrm{Mono}(G, G)$, which implies the existence of a monomorphism both from G to H and from H to G. Therefore, G and H are isomorphic. $\square$

4.3.1 Homomorphic Equivalence and Cores

If G and H are graphs they are said to be *homomorphically equivalent* if there exists a graph homomorphism from G to H and also a homomorphism from H to G. We write $G \sim H$ for this, and in fact $\sim$ is an equivalence relation on graphs: It is clearly reflexive (identity map is a homomorphism), symmetric (the statement of the property is symmetric in G and H), and transitive (compositions of homomorphisms are homomorphisms).

In fact, there is a canonical representative of each equivalence class under the relation $\sim$. A *core* is a graph not homomorphic to any of its proper subgraphs.

Lemma 4.3.5. *Let G and H be homomorphically equivalent cores. Then $G \simeq H$.*

Proof. Let $f : G \to H$ and $g : H \to G$ be homomorphisms. If f were not surjective then $g \circ f$ would be a homomorphism from G to a proper subgraph of itself, a contradiction. Likewise g must be surjective, or else $f \circ g$ would be a homomorphism from H to a proper subgraph of itself. Therefore, G and H are isomorphic. □

Theorem 4.3.6. *Let G and H be graphs. Then G and H have unique cores G' and H' respectively, and $G \sim H$ if and only if $G' \simeq H'$.*

Proof. Firstly, G must have a core: If G is not a core itself, repeatedly applying homomorphisms to proper subgraphs will eventually arrive at one. Secondly, if G' and G'' were cores, with $f' : G \to G'$ and $f'' : G \to G''$, then f'' restricted to $V(G')$ is a homomorphism from G' to G'' and f' restricted to $V(G'')$ is a homomorphism from G'' to G', implying $G' \simeq G''$ by Lemma 4.3.5.

For the second part, if $G \sim H$, then we have homomorphisms from G' to G, from G to H, and from H to H', and vice versa, meaning G' and H' are isomorphic by Lemma 4.3.5. Meanwhile, if G' and H' are isomorphic, then $G \sim H$ by the fact that $\sim$ is an equivalence relation. □

Since any homomorphism from a complete graph must be injective, some obvious examples of cores are the complete graphs. Additionally, complete graphs as cores can be used to define perfect graphs:

Theorem 4.3.7. *Let G be a graph. Then G is perfect if and only if every induced subgraph H of G has a complete core.*

Proof. H has an $\omega(H)$-clique if and only if it has a homomorphism from $K_{\omega(H)}$, and has an $\omega(H)$-colouring if and only if it has a homomorphism to $K_{\omega(H)}$. □

4.3.2 The Direct Product

Let G and H be graphs. What does it mean to multiply the two together? There are several different ways of defining graph multiplication, leading to several different concepts [46]. We will now define one of these. The *direct product* $G \times H$ has vertex set $V(G) \times V(H)$ and (v, w) is adjacent to (v', w') if and only if vv' and ww'.

Proposition 4.3.8. *The direct product $\times$ satisfies the following properties:*

1. *$\times$ is associative: $(G_1 \times G_2) \times G_3 = G_1 \times (G_2 \times G_3)$ for all graphs G_1, G_2, G_3*
2. *$\times$ is commutative: $G \times H = H \times G$ for all graphs G and H.*

The direct product is the most natural product in the setting of graph homomorphisms, as it has the property that homomorphisms from some graph X to G and H produce a homomorphism from X to $G \times H$.

Theorem 4.3.9. *Let G and H be graphs. If X is an auxiliary graph with homomorphisms $f_1 : X \to G$ and $f_2 : X \to H$, then the function $f : X \to G \times H$ defined by $f(x) = (f_1(x), f_2(x))$ is a homomorphism.*

This property leads to the direct product often being called the "categorical" product of graphs.

4.4 Bireflexive Graphs

In this section we give a new definition of graphs which is different to the usual ones, and that relates homomorphisms to the modular decomposition. A *bireflexive graph* G consists of a set V of vertices together with symmetric binary relations E and N on V satisfying the axioms:

1. For all $x, y \in V(G) : xEy$ or xNy

2. For all $x, y \in V(G) : xEy$ and xNy if and only if $x = y$

The members of E are called edges, and are said to be adjacent, while the members of N are called nonedges, and are said to be nonadjacent. Thus, every vertex is both adjacent and nonadjacent to itself, and to every other vertex it is either only adjacent, or only nonadjacent. Additionally, to say H is an *induced subgraph* of G is to say that $V(H) \subseteq V(G)$, and for all $x, y \in V(H) : xEy$ in $H \Rightarrow xEy$ in G and xNy in $H \Rightarrow xNy$ in G.

There is a natural $1-1$ correspondence between bireflexive graphs and simple graphs. If $G = (V, E, N)$ is a bireflexive graph, the set E of edges between *distinct* vertices defines a simple graph G' on V. Also, the set N of nonedges between distinct vertices defines the complement of G', $\overline{G'}$.

Despite this correspondence, the definition of bireflexive graphs gives them different algebraic properties to simple graphs, as they have different homomorphisms.

If G and H are bireflexive graphs, a *bireflexive homomorphism* is a function $f : G \to H$ such that $xEy \Rightarrow f(x)Ef(y)$ and $xNy \Rightarrow f(x)Nf(y)$. A *bireflexive monomorphism* is an injective bireflexive homomorphism and a *bireflexive epimorphism* is a surjective bireflexive homomorphism. A *bireflexive isomorphism* is defined as a bijective bireflexive homomorphism, and a *bireflexive antimorphism* is defined as a function $f : G \to H$ such that $xEy \Rightarrow f(x)Nf(y)$ and $xNy \Rightarrow f(x)Ef(y)$.

Bireflexive isomorphisms and antimorphisms correspond to the isomorphisms and complements of simple graphs. Bireflexive monomorphisms and epimorphisms are quite different, however.

Theorem 4.4.1. *Let G and H be bireflexive graphs. Then there exists a monomorphism from G to H iff G is isomorphic to an induced subgraph of H.*

Proof. This follows from the definitions and the fact that any injective function is a bijection from its domain to its image. □

Theorem 4.4.2. *Let G and H be bireflexive graphs. Let f be an epimorphism from G to H. Then H is isomorphic to a skeleton of G, and the fibres of f together with this skeleton form a modular decomposition of G.*

Conversely, given a modular decomposition of G with H as the skeleton and $M_1, ..., M_n$ as the modules, there exists an epimorphism $f : G \to H$ whose fibres are equal to $M_1, ..., M_n$.

Proof. ($\Rightarrow$) Let $f : G \to H$ be an epimorphism. Let $f^{-1}(x)$ and $f^{-1}(y)$ be fibres of f with $x \neq y$. Let $x_1, x_2 \in f^{-1}(x)$, and $y_1 \in f^{-1}(y)$. If x_1y_1 is an edge in G, xy must be an edge in H (since f is a homomorphism). Since $x \neq y$, xy cannot be a nonedge in H, which prohibits x_2y_1 from being a nonedge in G. Therefore x_2y_1 must be an edge in G.

A similar argument applies if x_1y_1 is a nonedge in G, leading to x_2y_1 being a nonedge in G. Therefore, x_1 and x_2 have the same relation to everything outside $f^{-1}(x)$, and applying this to all pairs in $f^{-1}(x)$ gives proof of it being a module. Since the adjacency/nonadjacency between these modules is determined according to the structure of H, H is isomorphic to a skeleton of this modular decomposition (the skeleton is the whole of H because f is surjective).

($\Leftarrow$) Let G be partitioned into modules $M_1, ..., M_n$ with a skeleton $G[x_1, ..., x_n]$, where $x_i \in M_i$. Define $f : G \to G[x_1, ..., x_n]$ by $f(x) = x_i$ such that $x \in M_i$. Let xEy in G with $f(x) = x_i, f(y) = x_j$. If $i \neq j$, then everything in M_i is adjacent to everything in M_j, and x_i is adjacent to x_j. If $i = j$, then then $f(x) = f(y)$ is self-adjacent. In either case, $xEy \Rightarrow f(x)f(y)$. A similar argument applies for when xNy in G.

□

4.4.1 Bireflexive Graphs and Lexicographic Products

If we have bireflexive graphs G and H, there is a difficulty in defining the product of G and H in the same way as we did for the direct product of simple graphs. We could say $(x, y)E(x', y')$ if and only if xEx' and yEy', and $(x, y)N(x', y')$ if and only if xNx' and yNy'. But then if xNx' and $xyEy'$, or xEx' and yNy', and these vertices are distinct, there is no way to assign $((x, y),(x', y'))$ to either relation, so the product will not be a bireflexive graph.

However, there is a kind of graph product that does not have this difficulty and which extend to bireflexive graphs, called the lexicographic product.

The idea of a lexicographic ordering is that one looks for the leftmost "coordinate" in which two items differ, and this coordinate "decides" which item comes first in the ordering. The lexicographic product of graphs is similar: Given a vector representing one vertex from each graph, its adjacency with other vectors is obtained by looking at the leftmost coordinate in which they differ.

We formally define the *lexicographic product* $G \circ H$ of two bireflexive graphs G and H as follows. Let $(x, y),(x', y')$ be ordered pairs such that $x, x' \in V(G)$ and $y, y' \in V(H)$. If $x \neq x'$, then $xEx' \Rightarrow (x, y)E(x', y')$ and $xNx' \Rightarrow (x, y)N(x'y')$. If $x = x'$, then $yEy' \Rightarrow (x, y)E(x', y')$ and $yNy' \Rightarrow (x, y)N(x'y')$.

It might also be pointed out that the lexicographic product is equivalent to the special case of the modular decomposition, where every module is the same.

We now give a few basic properties of the lexicographic product of bireflexive graphs:

Theorem 4.4.3. *The lexicographic product satisfies the following properties:*

1. *The lexicographic product is well defined: $G \circ H$ defines a bireflexive graph.*

2. The lexicographic product is associative: $(G_1 \circ G_2) \circ G_3 \simeq G_1 \circ (G_2 \circ G_3)$

3. The lexicographic product is not commutative.

Proof. 1. If $(x, y) \neq (x', y')$, then either $x \neq x'$ or $y \neq y'$. In both these cases, exactly one of $(x, y)E(x', y')$ and $(x, y)N(x', y')$ holds. Meanwhile, if $(x, y) = (x', y')$, xEx' and yNy' so $(x, y)E(x', y')$ and $(x, y)N(x', y')$.

2. Comparing $(((x, y), z), ((x', y'), z'))$ and $(((x, (y, z), (x', (y', z')))$, if $x \neq x'$ then the status of the right hand side is determined by the relation between x and x', while the status of the left hand side is determined by the relation between (x, y) and (x', y') which is determined by the relation between x and x'.

   If $x = x'$, but $y \neq y'$, then the then right hand side is determined by the relation between y and y'. Also $(x, y) \neq (x', y')$ and so the left hand side is determined by the relation between (x, y) and (x', y') which is determined by the relation between y and y'.

   If $x = x'$ and $y = y'$, then the relation between z and z' determines everything, so in all three cases $(((x, y), z), ((x', y'), z'))$ and $(((x, (y, z), (x', (y', z')))$ have the same relation.

3. If G is connected and H is disconnected then $G \circ H$ is connected but $H \circ G$ is disconnected. The simplest example would be $K_2 \circ E_2$ which is C_4 (which is connected) and $E_2 \circ K_2$ which is $2K_2$ (which is disconnected).

□

We show next that given an auxiliary bireflexive graph X, with bireflexive homomorphisms to G and H, we can define "coordinate-wise" a bireflexive homomorphism to $G \circ H$. In this respect, the lexicographic product plays a role for bireflexive graphs akin to that of the direct product for graphs.

Theorem 4.4.4. *Let G and H be bireflexive graphs. If X is an auxiliary bireflexive graph with bireflexive homomorphisms $f_1 : X \to G$ and $f_2 : X \to H$, then the function $f : X \to G \circ H$ defined by $f(x) = (f_1(x), f_2(x))$ is a bireflexive homomorphism.*

Proof. Let $x, y \in E(X)$, and define f as above, so that $f(x) = (f_1(x), f_2(x))$, and $f(y) = (f_1(y), f_2(y))$. We need to check that $xEy \Rightarrow f(x)Ef(y)$ and $xNy \Rightarrow f(x)Nf(y)$.

If $f_1(x) = f_1(y)$, then $xEy \Rightarrow f_2(x)Ef_2(Y)$, since f_2 is a bireflexive homomorphism. Then by definition of the lexicographic product, $f_2(x)Ef_2(y) \Rightarrow f(x)f(y)$. Similarly $xNy \Rightarrow f_2(x)Nf_2(Y) \Rightarrow f(x)Nf(y)$.

Meanwhile, if $f_1(x) \neq f_1(y)$, f_1 is a bireflexive homomorphism and $xEy \Rightarrow f_1(x)Ef_1(y) \Rightarrow f(x)Ef(y)$, and $xNy \Rightarrow f_1(x)Nf_1(y) \Rightarrow f(x)Nf(y)$.

In either case, f is a bireflexive homomorphism. $\square$

4.4.2 Analogues of Theorems 4.3.3, 4.3.4 and 4.3.6

The analogue of Theorem 4.3.3 holds also for bireflexive graphs:

Theorem 4.4.5. *Let G and H be bireflexive graphs and $f : G \to H$ be a bireflexive homomorphism. Then there exists a graph G' such there exists an bireflexive epimorphism $g : G \to G'$ and there exists a bireflexive monomorphism $h : G' \to H$ such that $f = h \circ g$.*

Proof. By the definition of bireflexive homomorphism and Theorem 4.4.3, f defines a modular decomposition with skeleton S which is isomorphic to the induced subgraph $H[f(G)]$ of H, which then has an inclusion map into H. $\square$

Using the decomposition in Theorem 4.4.5 and the same proof as before for Theorem 4.3.6 we obtain a similar result for bireflexive homomorphisms.

Theorem 4.4.6. *Let G and H be bireflexive graphs. Let $BHom(X, G)$ and $BHom(X, H)$ denote the sets of all bireflexive homomorphisms from X to G and H respectively. Then G and H are isomorphic if and only if for every auxiliary graph $X : |BHom(X, G)| = |BHom(X, H)|$.*

The bireflexive core of a graph G is the same thing as its principal skeleton. This is always an induced subgraph of G, which is unlike the case of homomorphisms between simple graphs: for example any odd hole (an odd cycle of length at least 5) has core K_3 and yet does not contain a triangle as a subgraph. The analogue of Theorem 4.3.6, which has the same proof, can be stated as:

Theorem 4.4.7. *Let G and H be bireflexive graphs. Then G and H are (bireflexively) homomorphically equivalent if and only if they have the same principal skeleton.*

4.5 Analogues Of Colouring and Perfect Graphs

A graph colouring of a simple graph G may be characterised as a homomorphism $f : G \to H$, where H is a complete graph, and the number of colours used is $|f(G)|$. Equivalently a graph colouring can be defined as a graph homomorphism $f : G \to H$, where $f^{-1}(x)$ is an empty graph for all $x \in V(H)$ (and the number of colours used is $|f(G)|$). Also equivalently, we could define a graph colouring as a homomorphism having *both* these properties.

Realised for bireflexive graphs, these are not equivalent. The first leads to a modular decomposition with a complete skeleton, the second leads to a modular decomposition with empty modules, and the third leads to a partition of the graph making it a complete multipartite graph.

We shall take the third property as the "right" definition of a bireflexive colouring, so that a bireflexive graph has a *bireflexive graph colouring* with k-colours if and only if it is complete k-partite.

A perfect graph is a graph G where every induced subgraph H of G has a graph colouring with $\omega(H)$ colours. If we were to define a perfect bireflexive graph according to this definition, since complete multipartite graphs are closed under induced subgraphs we would just get that the complete multipartite graphs as the perfect bireflexive graphs. This would be undesirable however, since the class of perfect bireflexive graphs would be not be closed under taking complements. To this end, let us recall the *Substitution Lemma* of Lovász [66]:

Theorem 4.5.1. *Let G be a perfect graph on N vertices $\{v_1, ..., v_N\}$, and let $H_1, ...H_N$ be perfect graphs. The graph G' obtained by the modular substitution of $H_1, ...H_N$ for $\{v_1, ..., v_N\}$ is perfect.*

If $\mathcal{C}$ is a graph class, we define the *monic closure* of $\mathcal{C}$ to be the set of graphs G in $\mathcal{C}$ all of whose induced subgraphs are in $\mathcal{C}$. We define the *epic closure* of C to be the set of graphs G in $\mathcal{C}$ such the modular substitution of graphs in $\mathcal{C}$ for vertices of G makes another graph $G' \in \mathcal{C}$.

According to Theorem 4.5.1, we may equivalently define the class $\mathcal{P}$ of perfect graphs as formed by taking the ω-colourable graphs, then taking the monic closure, and then taking the epic closure.

Now adapt this definition to bireflexive graphs. We take our class $\mathcal{P}'$ the epic closure of the monic closure of the complete multipartite graphs. This time the monic closure doesn't do anything, but the epic closure does.

We define the class $\mathcal{P}'$ of *bireflexively perfect* graphs as follows:

1. $K_1, K_2, E_2 \in \mathcal{P}'$
2. If G and H are in $\mathcal{P}'$, and G' is obtained by substituting H for a vertex of G, then G' is in $\mathcal{P}'$

Note that this is the same class we defined before, as any complete multipartite graph can be generated from K_2 and E_2 by repeated substitution.

The *cographs* are a graph class $\mathcal{CO}$ defined as follows:

1. $K_1 \in \mathcal{CO}$
2. $G \in \mathcal{CO} \Rightarrow \overline{G} \in \mathcal{CO}$

3. $G, H \in \mathcal{CO} \Rightarrow G + H \in \mathcal{CO}$

Lemma 4.5.2. *Let G be a graph. Then G is a cograph if and only if it is P_4-free.*

We now prove the following theorem:

Theorem 4.5.3. $\mathcal{CO} = \mathcal{P}'$.

Proof. ($\mathcal{CO} \subseteq \mathcal{P}'$): To see that every cograph is in $\mathcal{P}'$, observe that $K_1 \in \mathcal{P}'$, and that if $G, H \in \mathcal{P}'$, substituting G and H for the vertices of E_2 implies $G + H \in \mathcal{P}'$.

Secondly, we see that $\mathcal{P}'$ is closed under complementation: Let $\mathcal{G}_0$ be the class containing just K_1, K_2 and E_2. Let G_i be defined as the class of graphs obtained from graphs in G_{i-1} via a single modular substitution. Then since substituting $\overline{H}$ for a vertex of $\overline{G}$ gives the complement of the graph obtained by substituting H for the same vertex of G, we have that G_i is closed under complementation for all i by induction. Therefore, $\mathcal{P}'$ is closed under complementation.

Since $\mathcal{P}'$ contains all the initial graphs of $\mathcal{CO}$ and is closed under the same operations, it contains $\mathcal{CO}$.

($\mathcal{P}' \subseteq \mathcal{CO}$): Since any graph in P' is obtained by repeated modular substitution started from $\{K_1, K_2, E_2\}$, and the graph P_4 is prime with respect to the modular substitution/decomposition, no graph in $\mathcal{P}'$ contains an induced P_4. Therefore, every graph in $\mathcal{P}'$ is a cograph, by Lemma 4.5.2. $\square$

As a corollary of this, we find that our bireflexive perfect graphs are closed under the taking of complements.

We also note that perfect graphs and cographs have the interesting property that they are closed under all three bireflexive operations: monomorphisms (monic closure), epimorphisms (epic closure) and antimorphisms (taking complements). Another existing graph class is the class of permutation graphs.

4.6 The Dual Of The Reconstruction Conjecture

Let G be a graph on n vertices. The following conjecture is due to Kelly [61] and Ulam [94]. Before stating it, we introduce the now-standard "deck of cards" terminology, which is due to Frank Harary [47] who played an important role in popularising the conjecture.

Let G and H be graphs on a set $V = \{v_1, ..., v_n\}$ of vertices, where n is at least 3, and define the *deck* of each graph, $D(G)$ and $D(H)$, by $D(G) = \{G[V - v_i]\}, 1 \leq i \leq n$ and $D(H) = \{H[V - v_i]\}, 1 \leq i \leq n$. Note that the decks are considered as multisets, as some graphs may occur multiple times. The members of the deck are called *cards*.

Conjecture 4.6.1 (The Reconstruction Conjecture)**.** *Let G and H be graphs on n vertices, where n is at least 3. If the cards in each of their decks can be paired up into isomorphic pairs, then $G \simeq H$.*

The provision that n be at least 3 is crucial, as $D(K_2) = D(E_2) = \{K_1, K_1\}$.

The form stated above is the most common form of the reconstruction conjecture. However, there is another formulation using the *expanded deck*, which consists of *all* the proper induced subgraphs, not just those on $n - 1$ vertices (and again, counting multiple occurrences multiply). Indeed, the expanded deck is used by Kelly in proving the reconstruction conjecture for trees [61]. The reconstructibility of the expanded deck from the deck of $(n - 1)$-cards is a result known as *Kelly's Lemma*, and may be established by a simple counting argument.

Lemma 4.6.2 (Kelly's Lemma[47],[61])**.** *Let G and H be graphs on n vertices which have the same deck (of $(n-1)$-cards). Then G and H must have the same expanded deck (of $(< n)$-cards).*

According to Kelly's Lemma, we may state a second version of the reconstruction conjecture. Also, we define a *proper* bireflexive monomorphism to be one that is not a epimorphism.

Conjecture 4.6.3 (The Reconstruction Conjecture, alternative version)**.** *Let G and H be graphs on n vertices, where n is at least 3, and assume that for any auxiliary graph X, the number of proper bireflexive monomorphisms from X to G and from X to H are equal. Then G and H are isomorphic.*

Given that bireflexive monomorphisms are dual to bireflexive epimorphisms, we could make a similar statement for epimorphisms from larger graphs than G (and perhaps conjecture its truth).

Some Notation: Let $\{X \twoheadrightarrow G\}$ denote the set of epimorphisms from X to G. Let $\{X \hookrightarrow G\}$ denote the set of monomorphisms from X to G. For the cardinalities of these sets we use $|X \twoheadrightarrow G|$ and $|X \hookrightarrow G|$ respectively.

Further, $\underline{\twoheadrightarrow}$ denotes an epimorphism that is a *proper* epimorphism (i.e. not a monomorphism). Similarly for $\underline{\hookrightarrow}$.

Thus the reconstruction conjecture states that for G a graph on $n > 2$ vertices:

$|X \underline{\hookrightarrow} G| = |X \underline{\hookrightarrow} H| \ \forall X \Rightarrow G \simeq H.$

The "dual" statement states that for a graph G on n vertices:

$|X \underline{\twoheadrightarrow} G| = |X \underline{\twoheadrightarrow} H| \ \forall X \Rightarrow G \simeq H.$

A graph satisfying these conjectures is called *reconstructible* (or *monic-reconstructible*) and *epic-reconstructible*, respectively. The attendant decks are called the *monic deck* and *epic deck*.

Note that the assumption that $n > 2$ can be removed; this follows from the fact that no graph can be both disconnected and codisconnected. $|X \underline{\twoheadrightarrow} K_n| > 0$ implies that $|X \underline{\twoheadrightarrow} E_n| = 0$, unless $n = 1$, and likewise $|X \underline{\twoheadrightarrow} E_2| > 0$ implies that $|X \underline{\twoheadrightarrow} K_2| = 0$, unless n=1. We state this as a theorem.

Theorem 4.6.4. *E_n and K_n are both epic-reconstructible.*

Similar to monic-reconstruction, we have the following theorem for epic-reconstruction:

Theorem 4.6.5. *If G is epic-reconstructible then so is $\overline{G}$.*

Proof. Let G an epic-reconstructible graph. Let H be a graph such that $\overline{G}$ and $\overline{H}$ have the same epic deck. Then $|X \twoheadrightarrow \overline{G}| = |\overline{X} \twoheadrightarrow G|$ and $|X \twoheadrightarrow \overline{H}| = |\overline{X} \twoheadrightarrow H|$. Also as $\overline{G}$ and $\overline{H}$ have the same epic deck: $|X \twoheadrightarrow \overline{G}| = |X \twoheadrightarrow \overline{H}|$.

Hence G and H are co-epic, and $G \simeq H$. Therefore $\overline{G} \simeq \overline{H}$. □

Theorem 4.6.6. *If G is a prime graph then G is epic-reconstructible.*

Proof. Let X be a graph such that $|X \twoheadrightarrow G| > 0$. Then G is the core (i.e. principal skeleton) of X. Any graph with the same epic deck as G must also be the core of X. The epic-reconstructibility follows from the uniqueness of cores. □

We note that the paper of Brignall et. al. [14] has made some progress towards proving the reconstruction conjecture for graphs having a nontrivial modular decomposition. If such a result were proved, it would imply in conjunction with Theorem 4.6.6 that a weaker form of the reconstruction conjecture is true, where one combines the standard (monic) deck with the epic deck, and that this would be enough to reconstruct any graph. We state it now as a conjecture.

Conjecture 4.6.7 (Monic-Epic Reconstruction Conjecture)**.** *Let G and H be graphs on n vertices. If G and H have the same monic deck, and the same epic deck, then G and H are isomorphic.*

Glossary of Terms

2-cut/2-join (70): If G is a graph then a 2-*cut* is the operation of adding all possible edges between the sets A and B, where A and B are two sets that partition $V(G)$. The opposite operation of removing all possible edges is called a 2-*join*.

Antihole (49): The complement of a non-triangular cycle.

Antimorphism (126): A function $f : G \to H$ between bireflexive graphs G and H such that $xEy \Rightarrow f(x)Nf(y)$ and $xNy \Rightarrow f(x)Ef(y)$. Equivalent to the taking of complements.

Antipath (81): The complement of a path.

Antihat (85): The complement of a hat, a hat being a graph obtained by adding a single edge between two vertices at distance 2 in a hole (i.e. a *triangular chord*).

Asymptotic Functions (28): Two functions f and g of a real variable t are said to be *asymptotic* to one another if $\frac{f(t)}{g(t)}$ tends to 1 as t tends to infinity. We will also use the notation $f \sim g$.

Berge-critical edge/nonedge (11,58): An edge (resp. nonedge) xy such that $G - xy$ (resp. $G + xy$) is not Berge. Only used in the context of Berge graphs or odd holes or odd antiholes.

Berge-free Edge/nonedge (11,58): An edge/nonedge that is not Berge-critical.

Berge Graph (9,50): A graph with no induced odd holes or induced odd antiholes.

Biconnected (5): A graph (or bireflexive graph) is biconnected if there is a path between any two vertices consisting entirely of edges, and also a path between any two vertices consisting entirely of nonedges.

Big Clique/Stable Set (61): A Clique/Stable Set of maximum cardinality.

Bireflexive Graph (125): A set V together with two binary relations E and N such that $E \cap N = \{(v, v) : v \in V\}$ and $E \cup N = V$.

Bireflexive Homomorphism (126): A function $f : G \to H$ between bireflexive graphs G and H such that $xEy \Rightarrow f(x)Ef(y)$ and $xNy \Rightarrow f(x)Nf(y)$.

Bireflexive Epimorphism (126): A surjective function $f : G \to H$ between bireflexive graphs G and H such that $xEy \Rightarrow f(x)Ef(y)$ and $xNy \Rightarrow f(x)Nf(y)$. Equivalent to the modular decomposition, where G decomposes to H.

Bireflexive Monomorphism (126): An injective function $f : G \to H$ between bireflexive graphs G and H such that $xEy \Rightarrow f(x)Ef(y)$ and $xNy \Rightarrow f(x)Nf(y)$. Equivalent to the induced subgraph, where G is an induced subgraph of H.

Blocker (82): If G is a graph with clique number ω and xy is a critical nonedge, then a *blocker* of xy is a clique K of cardinality $\omega - 1$ such that x and y are both adjacent to every vertex in K.

(C,n)-reconfiguration graph (107): If $\mathcal{C}$ is a class of graphs and n is a positive integer, then the $(\mathcal{C}, n)-$ reconfiguration graph is the graph whose vertices are the graphs in $\mathcal{C}$ of order n and the vertices are adjacent if one can be obtained from the other by deleting or adding a single edge.

Card (134): An $(n-1)$-vertex induced subgraph of an n-vertex graph G is called a *card* of G.

Caterpillar, t-caterpillar (40): A *caterpillar* tree is defined to be a tree where every vertex is adjacent to some induced path subgraph of the tree. A *t-caterpillar* is where the vertices are allowed to be at distance at most t from this path.

Chordal Graph (53): A graph with no induced holes.

Claw (7): The complete bipartite graph $K_{1,n}$ is often called a *claw*.

Close Pair (79): A pair of minimally imperfect graphs such that each can be obtained from the other by adding or deleting a single edge.

Cograph (54): A graph with no induced P_4 subgraphs.

Comparability Graph (54): A graph whose edges can be oriented to make a (strict) partially ordered set.

Core (124,130): If G is a simple graph, the *core* of G is the unique graph G' to which G admits a homomorphism but such that any homomorphism $f : G' \to G'$

is an isomorphism. For a bireflexive graph G the definition is analogous, with the difference that one has to specify that a core has no *nontrivial* homomorphisms onto an induced subgraph of itself, as all bireflexive graphs admit a bireflexive homomorphism to K_1. The bireflexive core is equivalent to the pricipal skeleton.

Critically Berge Graph (11,58): A Berge graph where deleting any edge makes the graph non-Berge. If adding any edge makes the graph non-Berge we say it is *cocritically Berge*.

Critical Edge/nonedge (10,66): An edge (respectively nonedge) xy such that $\alpha(G-xy) > \alpha(G)$ (resp. $\omega(G+xy) > \omega(G)$).

Critically Perfect Graph (11,59): A perfect graph where deleting any edge makes the graph imperfect. If adding any edge makes the graph imperfect we say it is *cocritically perfect*.

Deck (134): The multiset of all cards of a graph G is the *deck* of G.

Decomposable Graph (116): A graph which has a nontrivial module, or equivalently a nontrivial skeleton.

Degree Sort Property (20): The degree sort property (for some integer r) is satisfied by a graph if v having fewer neighbours than w implies that w is not the centre of a larger r-star than v.

Depth Two Claw (18): A *depth two claw* is an elongated claw that is a $(k,2)$-claw for some positive integer k.

Diagonals (81): The nonedges v_1v_3 and v_2v_4 in an induced path $v_1v_2v_3v_4$ on 4 vertices.

Dichotomy Property/Weak Dichotomy Property (58,78): A graph satisfies the *dichotomy property* if every edge/nonedge is either critical and Berge-free, or free and Berge-critical. *Weak dichotomy* is where every edge is either critical or Berge-critical.

Direct Product (125): Given simple graphs G and H, the *direct product* is the graph $G \times H$ with vertex set $V(G) \times V(H)$ and such that (v, w) is adjacent to (v', w') if and only if v is adjacent to v' and w is adjacent to w'.

Elongated Claw (18): A tree T is an *elongated claw* if it has one or fewer vertices with degree strictly more than 2. If such a vertex x exists it is called the *root* or *centre* of the claw. If x has degree k and the leaves are all at distance a from x then we say T is a (k, a)-*claw*. The paths induced by going from the root to the leaves are called the *limbs*. A limb of length 1 is called *short*.

End-leaf (40): In a caterpillar with a path P as its underlying path, the leaves of this path and their siblings are called *end-leaves*.

Endpoints (of a path on 4 vertices)(81): The nonedge v_1v_4 in an induced path $v_1v_2v_3v_4$ on 4 vertices.

Epic Closure (132): Given a graph class $\mathcal{C}$ the *epic closure* of $\mathcal{C}$ is the class of all graphs G reachable from a graph in $\mathcal{C}$ via a chain of epimorphisms whose preimages (i.e. skeletons) and fibers (i.e. modules) are all in $\mathcal{C}$.

Epic Deck (135): The *epic deck* of a bireflexive graph G is the multiset (there may be repetitions) of all graphs H admitting a proper bireflexive epimorphism to G. (And an analagous such deck may be defined in terms of nontrivial modular decompositions for simple graphs).

Epic Reconstructible (135): A bireflexive graph is *epic-reconstructible* if it can be identified up to isomorphism from its epic deck (an analogue using modular decomposition can be defined for simple graphs).

Expanded Deck (134): The *expanded deck* of a graph G consists of the multiset of proper induced subgraphs of G (repetitions may occur). Determined from the deck by Kelly's Lemma.

Freedom (59,91): Let $\mathcal{C}$ be a class of graphs. $\mathcal{C}$ has the property of *positive freedom* if for any non-complete graph G in $\mathcal{C}$, there exists a nonedge xy, such that $G + xy$ is in $\mathcal{C}$. If for any non-empty G we may find an edge $x'y'$ such that $G - x'y'$ is in $\mathcal{C}$, we call it *negative freedom*. The conjunction of these two is called *freedom*.

Hat (85): A *k-hat* is a k-vertex cycle with an additional edge added between two vertices at distance 2 on the cycle (a *triangular chord*).

Hole (9): A cycle which is not a triangle.

Independent Set (5): If G is a graph, a set I of vertices is an *independent set* if no two of its vertices are adjacent to one another.

Intermediate Leaf (40): Any leaf of a caterpillar that is not an end-leaf.

Interval Graph (55): A graph whose vertices can be represented by intervals on the real line in such a way that the edges correspond to the intersections of these intervals.

Isomorphism (121,126): A homomorphism which admits an inverse homomorphism is an *isomorphism.* Similarly a bireflexive isomorphism is a bireflexive homomorphism which admits an inverse bireflexive homomorphism.

Leaf (8): A vertex of degree 1, typically in a tree or forest.

Level (32): In a tree with a distinguished root vertex x, the *level* of a vertex v is $d+1$, where d is the distance from v to x.

Lexicographic Product (128): The *lexicographic product* of G and H is the unique graph $G \circ H$ having a modular decomposition with skeleton G and all modules equal to H.

Local Freedom (60,92): As for freedom, except that we now require it be possible to add an edge to (respectively delete an edge from) every non-dominating (respectively non-isolated) vertex in G.

Mates (61): In a partitionable graph, every big clique has a unique big stable set to which it is disjoint, and vice versa. A pair consisting of a big clique and a big stable set that are disjoint is called a pair of *mates.*

Midpoints (of a path on 4 vertices)(81): The edges v_2v_3 in an induced path $v_1v_2v_3v_4$ on 4 vertices.

Minimally Imperfect Graph (51): An graph that is *not* perfect, but all of whose proper induced subgraphs *are* perfect.

Minimum Imperfect Berge Graph (75): An imperfect Berge graph of the minimum possible otder. This makes sense, when one is assuming that imperfect Berge graphs exist (typically for a contradiction).

Minimax Number (17): The minimum size of a maximal independent set.

Module (65,114): A *module* in a graph G is a subset M of $V(G)$ such that for all x, y in M and all v in $V(G) - M$, xv and yv are either both edges or both nonedges. A module that is a singleton or G itself is called *trivial*.

Modular Decomposition, Skeleton (115): If $M_1, ..., M_k$ is a partition of a graph G into modules, and S is the graph describing the adjacencies between distinct modules, then we say G has *modular decomposition* with *skeleton* S and modules $M_1, ..., M_k$.

Modular Substitution (115): The reverse of modular decomposition. If G has a modular decomposition with skeleton S and modules $M_1, ..., M_k$ then we say G is a *modular substitution* of S by the modules $M_1, ..., M_k$.

Monic Closure (132): Given a graph class $\mathcal{C}$ the *monic closure* is the class of graphs in $\mathcal{C}$ all of whose induced subgraphs are also in $\mathcal{C}$.

Monic Deck (135): Given a bireflexive graph G, the *monic deck* of G is the multiset of all graphs with proper monomorphisms to G, with multiple occurrences if the image of the monomorphism occurs multiple times in G. It is equivalent to the expanded deck of G.

Monic Reconstructible (135): A bireflexive graph G is *monic-reconstructible* if it is determined up to isomorphism by its monic deck. It is equivalent to the usual vertex-reconstuctibility.

Nodal Vertex (45): If T is a tree and v and w are vertices, they are a *nodal pair* if they are both not of degree 2, and are at odd distance from each other.

Normal (77): A minimally imperfect graph G is said to be *normal* if no edge can be deleted from G to form another minimally imperfect graph. If no edge is able to be added to G to form another minimally imperfect graph, we say G is *conormal*. If G is both normal and conormal we say it is *binormal*.

Paley Graph (73): If $q = p^k \equiv 1 \pmod 4$ is a power of an odd prime p, then the *Paley Graph* on q vertices is formed by taking the elements of the finite field F_q and declaring an edge between elements of F_q, a and b, if and only if $a - b$ is a square under multiplication in F_q.

Pancritical Graph (72): A graph where every edge and every nonedge is critical.

Partitionable Graph (61): A graph G where $n = \alpha(G)\omega(G) + 1$, and G has n big stable sets and n big cliques, which can be paired into mates.

Path-vertex (40): A non-leaf in a caterpillar.

Perfect Graph (9,48): A graph G such that $\chi(G) = \omega(G)$, and the same equation holds for all its induced subgraphs (with their own values of χ and ω).

Permutation Graph (55): A graph G that admits a pair of linear orderings of its vertex set, in such a way that two vertices are adjacent if and only the two linear orderings disagree on which vertex is greater than the other.

Prime Graph (116): A graph with no modular decomposition where the modules (or equivalently, the skeleton) are not all trivial.

Principal Skeleton (117): The *principal skeleton* is the skeleton of a graph which is prime. It is the bireflexive core.

Proper Monomorphism/Epimorphism (135): A monomorphism (resp. epimorphism) that is not an epimorphism (resp. monomorphism) is called *proper*.

Reconstructible Graph (135): A graph is *reconstructible* if it is determined up to isomorphism by its deck. Equivalent to monic reconstructibility, by the equivalence of the monic deck to the expanded deck and by Kelly's Lemma.

Siblings (40): Two leaves in a tree are *siblings* if they are at distance 2 from each other (i.e. they have a common neigbhour).

Simple Graph, or Graph (4,120): A *simple graph*, or *graph*, is a set V together with an irreflexive symmetric binary relation on V.

(Simple) Graph Homomorphsim (120,121): A *simple graph homomorphism*, or *graph homomorphism* between two (simple) graphs G and H is a function $f : G \to H$ such that if xy is an edge in G then $f(x)f(y)$ is an edge in H.

(Simple) Graph Epimorphism (120,121): A *simple graph epimorphism*, or *graph epimorphism* between two (simple) graphs G and H is a surjective function $f : G \to H$ such that if xy is an edge in G then $f(x)f(y)$ is an edge in H.

(Simple) Graph Monomorphism (120,121): A *simple graph monomorphism*, or *graph monomorphism* between two (simple) graphs G and H is an injective function $f : G \to H$ such that if xy is an edge in G then $f(x)f(y)$ is an edge in H.

Split Graph (54): A graph G is *split* if it has a partition of its vertex set into a stable set and a clique.

Stable Set (5): If G is a graph, a set S of vertices is a *stable set* if no two of the vertices in S are adjacent to one another. Same meaning as *independent set.*

Star-Cutset (87): A cutset where there is at least one vertex adjacent to every other vertex in the cutset.

Stars, Star Systems, and their Centres (7,27): An *r-star with centre v* is the intersecting family consisting precisely of the independent r-sets containing a single vertex v. *Star systems centred at v* are defined similarly, just without the stipulation about the cardinality of the independent sets.

Subdivision (45): A *k-subdivision* of an edge uv in a graph G is the process of taking the disjoint union of G and a path P on $k-1$ vertices, deleting the edge uv, and adding an edge from one leaf of the path P to u and from the other leaf to v (or adding an edge from the only vertex of P to both u and v if P is a singleton). A k-subdivision of a graph G is the graph G' obtained by k-subdividing every edge of G.

Threshold Graph (54): A graph obtainable recursively by starting with a singleton and successively adding one vertex that is either isolated or dominating.

Trivial Module/Trivial Skeleton (114,115): If G is a graph then the modules/skeletons of G which are either singletons or the entire graph G itself are called *trivial modules/skeletons*.

Trivially Perfect Graph (55): An *ordered tree* is a partially ordered set P where the set $\{y : y < x\}$ induces a linear ordering for every vertex $x \in P$. A *trivially perfect graph* is a graph whose edges can be oriented to form an ordered tree.

Upper/Lower Graph (66): If G is a graph of order n then the supergraphs (of order n) of G with the same clique number are called its *upper graphs*, while the subgraphs (of order n) with the same stability number are called its *lower graphs*.

Weakly Chordal Graph (55): A graph G having no induced holes or induced antiholes on 5 or more vertices.

Wings (81): The edges v_1v_2 and v_3v_4 in an induced path $v_1v_2v_3v_4$ on 4 vertices.

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