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Phase-Field Models for Thin Elastic Structures

Willmore's Energy and Topological Constraints

Thesis presented for the degree of Doctor of Philosophy

to the
Department of Mathematical Sciences

of
Durham University

by
Stephan Jan Wojtowytsch

on
April 11, 2017

Stephan Wojtowytsch – Phase-Field Models for Thin Elastic Membranes

In this dissertation, I develop a phase-field approach to minimising a geometric energy functional in the class of connected structures confined to a small container. The functional under consideration is Willmore's energy, which depends on the mean curvature and area measure of a surface and thus allows for a formulation in terms of varifold geometry. In this setting, I prove existence of a minimiser and a very low level of regularity from simple energy bounds.

In the second part, I describe a phase-field approach to the minimisation problem and provide a sample implementation along with an algorithmic description to demonstrate that the technique can be applied in practice. The diffuse Willmore functional in this setting goes back to De Giorgi and the novel element of my approach is the design of a penalty term which can control a topological quantity of the varifold limit in terms of phase-field functions. Besides the design of this functional, I present new results on the convergence of phase-fields away from a lower-dimensional subset which are needed in the proof, but interesting in their own right for future applications. In particular, they give a quantitative justification for heuristically identifying the zero level set of a phase field with a sharp interface limit, along with a precise description of cases when this may be admissible only up to a small additional set.

The results are optimal in the sense that no further topological quantities can be controlled in this setting, as is also demonstrated. Besides independent geometric interest, the research is motivated by an application to certain biological membranes.

Declaration

The work in this thesis is based on research carried out in the Numerical Analysis Research Group at the Department of Mathematical Sciences, Durham University. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Chapters of this thesis have been previously submitted to research journals for publication. Chapter 3 is based on research for [Woj16]. Chapters 5 and 6 are mainly adapted from [DW15] and [DW17a], while Chapter 7 is taken from [DLW17]. Parts of [DLW17] have also been included in Chapter 5 for easier readability. Chapters 8 and 9 are going to be submitted in forth-coming articles [DW17b, Woj17]. The articles [DW15, DW17a, DW17b] have been written in collaboration with Patrick Dondl and [DLW17] in collaboration with Patrick Dondl and Antoine Lemenant.

Statement of Copyright

The copyright of this thesis rests with the author. No quotation from it should be published without the author's prior written consent and information derived from it should be acknowledged.

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Contents

Chapter 1

Introduction

Geometric energy functionals have been studied extensively by mathematicians at least in the last century, most prominently the area functional (for example as occurring in Plateau's problem as early as 1760) as well as non-local or non-isotropic variations thereof in recent years. While the area functional can be written as the integral of a constant function over the surface in question, physically meaningful energies may also depend on the geometry of the surface itself, for example its normal direction (anisotropic perimeter functionals in crystal grain growth) or its curvatures (elastic membranes). This is expressed through more complicated energy integrands.

In this dissertation, the focus will lie on the case of curvature-dependent energies. The model energy we will investigate is the Willmore functional, which associates to a surface the total integral of its squared mean curvature. Since the total integral of the Gaussian curvature of a surface is determined by its genus, this is arguably the simplest geometric second order energy functional with non-trivial behaviour. Interestingly, this energy is also related to the theory of minimal surfaces in the 3-sphere and to mean curvature flow.

Besides its geometric appeal, Willmore's energy also occurs in the modelling of thin elastic structures as the energy contribution of out-of-plane bending. The energy of a thin structure is usually decomposed into two parts: in-plane stretching and out-of-plane bending. Since the first term in the energy completely dominates the second one for very thin sheets, one can minimise Willmore's energy in a class of isometric embeddings to find good approximations of physical energy minimisers.

An interesting special case of thin elastic structures are so called lipid bilayers which are thin liquid membranes. Being liquid, in-plane stretching does not contribute to the energy at all in this application and the isometry constraint in the minimisation problem turns into an area constraint (since the fluid is virtually incompressible). Thus the objective is to *minimise Willmore's energy among surfaces with prescribed area.* Implicit in this minimisation problem is a selection of a class of surfaces. Generally, the following assumptions are reasonable.

1. All surfaces are embedded in a bounded domain $\Omega \subset \mathbb{R}^3$. If the domain is meant to have no influence, it could be chosen large enough to make the minimisation problem independent of this constraint.

2. All surfaces are connected. Otherwise, we could just consider connected components separately.

3. All surfaces are smooth. Biological membranes typically do not exhibit sharp edges, and the energy would in fact be infinite for a large class of singular behaviours. This condition will be relaxed later to surfaces which can be suitably approximated by smooth surfaces.

4. All surfaces are closed (compact and without boundary) or more generally, they have no boundary inside Ω. This assumption is natural for biological membranes and necessary when using a phase-field approach.

Geometric energies always lead to a technical difficulty in controlling the regularity of parametrisations. Namely, the fact that they depend only on the shape of a space implies an invariance under tangential diffeomorphisms which translates into a lack of compactness for energy minimising sequences of embeddings. In this sense, the liquid membrane problem is mathematically distinct from the corresponding minimisation problem for thin solid sheets.

Due to the tangential invariance, we will pursue an extrinsic approach rather than a parametrised one and use the techniques of geometric measure theory. In this setting, the existence of an energy minimiser becomes easy to establish. On the other hand, the structure of such objects is less obvious, and especially the constraint that surfaces be *connected* is a new feature. In the first part of the thesis, we will demonstrate that the minimisation problem is also well-posed in the class of connected surfaces, and that it is not well-posed in any class of surfaces of fixed genus $g \in \mathbb{N}$.

The second part of the dissertation focuses on an explicit approach to finding minimisers of Willmore's energy under the constraints described above. While existence can be established with the direct method of the calculus of variations, explicitly finding these surfaces is an entirely different matter.

A common approach to finding (local) minimisers of functionals is (numerically) following a gradient flow evolution until it becomes stationary. In the case of Willmore's energy, this leads to several problems.

1. Since the energy is of second order, the corresponding evolution equation is of fourth order and thus numerically difficult to treat.

2. As a fourth order evolution equation, Willmore flow does not allow a maximum principle. In fact, it is known that smooth embedded initial surfaces can be driven to self-intersection in finite time, and this situation is stable under perturbations. They can also be driven out of even convex domains Ω. Thus a gradient flow could potentially take us out of our class of admissible surfaces. A constraint is difficult to implement.

The second observation exposes the structure of the minimisation problem we are dealing with a bit better as a geometric second order double obstacle problem where the obstacles are given by the surface itself through a non-self-penetration constraint and by the boundary of the embedding domain Ω. This structure also rules out the use of Euler-Lagrange equations.

We approach this problem via phase-fields, which goes well together with the extrinsic approach to the existence problem. Namely, instead of solving a highly non-linear geometric problem on a surface, we can solve relatively simple partial differential equations on the domain Ω which then give us some information about a diffuse version of the surface. The price we have to pay for this convenience is solving equations in three dimensions rather than two.

The phase-field approach is – like the varifold approach – by nature extrinsic, and similar difficulties occur. In particular, it is not a priori clear how to understand the topological concept of *connectedness* on the phase-field level and how to enforce it in simulations. This is the key problem of the second part of this dissertation. There, an energy functional is developed which converges to Willmore's energy in a suitable sense, but enforces connectedness of surfaces on a phase-field level. Evidence of the effectiveness of this method is also presented.

Also presented in the second part are technical results on the convergence of phase-fields and their regularity properties near the boundary of Ω. These results are of independent interest for future applications in related problems. In particular, it is shown that a sharp interface surface might not be well approximated by level sets of phase-fields in general, but that this is true if the phase-fields are in addition minimisers of certain energy functionals.

While it would be desirable to write this dissertation from first principles, the scope of the topic does not allow for a complete exposition. In the following, it will be assumed that the reader is familiar with general functional analysis as well as common function spaces and their properties. This includes L^p-spaces, Sobolev spaces $W^{k,p}$ of integer order and their trace and embedding theorems, functions of bounded variation and spaces of continuous and differentiable functions as well as the Riesz-representation theorem and the characterisation of the dual space of continuous function as Radon measures. Further knowledge of measure theory in $\mathbb{R}^n$ is also assumed, as well as knowledge of Calderon-Zygmund regularity theory for elliptic equations of second order and elementary topology. Good sources on these topics are [Bre11, EG92, GT83, Giu84]. Introductions to non-standard topics such as varifolds and phase-fields will be provided.

The thesis is split into two parts. Part I is dedicated to sharp interface models for Willmore's energy, while in Part II diffuse interface models will be studied. Chapters 2 and 4 are used to review known results on sharp and diffuse interface models for Willmore's energy respectively and to introduce the specific problems and notations of the respective part of the dissertation. New results on the sharp interface model for Willmore's energy and the topology of energy minimising sequences are presented in Chapter 3. Original results on phase-field models for Willmore's energy are presented in Chapters 5 through 9.

1.1 Notation

The notation is standard and follows the sources above. Let us fix the following conventions. A *sequence* indexed by $\varepsilon > 0$ can be a countable family indexed by $k \in \mathbb{N}$ and parametrised by an associated sequence $\varepsilon_k \to 0$ or an uncountable family. The results remain the same and we do not distinguish here. The notations D and ∇ will be used equivalently to denote the gradient of a smooth function, the measure-valued gradient of a BV-function will be denoted by D only. D^2 denotes the Hessian of a function and Δ its Laplacian. We abbreviate $B_r = B_r(x)$ if the centre of a ball is clear from the context and $B_r = B_r(0)$ otherwise without comment. Occasionally, we will omit the domain of integration for a measure μ when we integrate over its entire domain. Weak convergence of Radon measures (which is weak* convergence in the dual space of continuous functions if the measures have uniformly bounded supports) will be denoted both by $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$. For convenience we write $A \Subset \mathbb{R}^n$ to mean that $A \subset \mathbb{R}^n$ has compact closure. The letters U, Ω will be used only for open sets, which will be understood implicitly from the notation. We denote

$$|\cdot|_{k,A} = ||\cdot||_{C^k(A)}, \qquad ||\cdot||_{p,\Omega} = ||\cdot||_{L^p(\Omega)}, \qquad ||\cdot||_{k,p,\Omega} = ||\cdot||_{W^{k,p}(\Omega)}.$$

The scalar product of a Hilbert space (in particular, $\mathbb{R}^n$) will be denoted as $\langle \cdot, \cdot \rangle$. As usual, C will denote a constant whose value may change from line to line, but which does not depend on the quantities being investigated (usually a function u and its derivatives, in some instances a variable domain). Further conventions will be introduced when needed.

Part I

Sharp-Interface Models for Thin Elastic Structures

Chapter 2

Background

2.1 Willmore's Energy

2.1.1 Basics

Geometric energy functionals have received attention both in pure and applied mathematics. Such energies arise naturally in the study of geometric problems as well as applications in physics, biology and materials science. A famous example is Willmore's energy

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2 \, \mathrm{d}\mathcal{H}^k \tag{2.1.1}$$

where $\Sigma \subset \mathbb{R}^n$ is an embedded k-dimensional manifold, H denotes its mean curvature and $\mathcal{H}^k$ is the k-dimensional Hausdorff measure. Often, one may reserve the term 'Willmore's energy' for the case of surfaces in 3-space and denote the same energy on plane curves (or occasionally space curves) as Euler's elastica energy. The main goal of this thesis is solving the following problem.

Problem 1. *Minimise Willmore's energy among all connected boundaries $\partial E \in C^2$ of sets $E \Subset \Omega$ such that $\mathcal{H}^2(\partial E) = S$ for some given $\Omega \Subset \mathbb{R}^3$ and $S > 0$.*

This problem is of independent mathematical interest, but can be used to characterise equilibrium shapes of spatially constrained lipid bilayers, for example inner mitochondrial membranes.

The only minimiser of $\mathcal{W}$ in the class of closed immersed C^2-surfaces is the round sphere (although there is a plethora of critical points and local minimisers). As $\mathcal{W}$ is scale-invariant, it becomes apparent that we need both the confinement to Ω and a prescribed area to induce a non-trivial constraint. We imagine S as being very large compared to Ω.

In the next chapter, we will show that this problem admits a solution in a suitably generalised sense. We will further show that it is not possible to prescribe a particular topological type in the minimisation problem. In the second part of this thesis we will develop a method to computationally approximate solutions of the minimisation problem.

Similarly famous is the generalised version of Willmore's energy known as the Canham-Helfrich functional

$$\mathcal{E}^{CH}(\Sigma) = \int_\Sigma \chi_H\,(H-H_0)^2 + \chi_K\,K\,\mathrm{d}\mathcal{H}^2$$

where χ_H, χ_K and H_0 are functions on Σ which may be taken to be constant in the simplest case. Several results in this dissertation will be valid for Canham-Helfrich functionals in a certain regime of χ_H, χ_K and H_0.

We note that for convenience, we consider a normalisation in which the mean curvature is the sum of the principal curvatures rather than their average, as this is more natural in the context of varifolds. In this normalisation, it is customary to consider the functional given by

$$\mathcal{W}(\Sigma) = \frac{1}{4}\int_\Sigma |H|^2\,\mathrm{d}\mathcal{H}^2$$

as Willmore's energy. We will still consider the functional given by (2.1.1), so that our normalisation of Willmore's energy differs from the original functional by a factor of 4. This normalisation simplifies expressions in the second part of the dissertation, where a phase-field approximation is discussed.

2.1.2 Biological Membranes

Both the Willmore and the Canham-Helfrich energy are widely used in the modelling of thin elastic structures. The first mention in that context goes back to Sophie Germain [Ger21]. Later, their importance has been suggested heuristically [Hel73] based on the principle that when a membrane is represented as a graph over its tangent space, only derivatives of at most second degree should occur in at most quadratic expressions at the base point. Another biological motivation in the context of red blood cells is given in [Can70].

The application we have in mind is to inner mitochondrial membranes, which are lipid bilayers of large area confined to a small container. A lipid bilayer is a liquid membrane composed of two layers of molecules with hydrophilic heads and hydrophobic tails held together by electric forces between the tails. A bilayer is typically less than 10nm thick, so its lateral dimensions (several microns) are about 10^3 times larger than its thickness and we can reasonably well idealise it as a surface.

A Helfrich type functional has also been obtained as a macroscopic limit of certain mesoscale models for lipid bilayers [PR09, LPR14]. Here bilayers are modelled using functions $u, v \in BV(\mathbb{R}^n, \{0,1\})$ to express locations of hydrophilic heads and hydrophobic tails of lipid molecules. These functions are coupled through a Monge-Kantorovich distance to be close together, constrained to satisfy $uv \equiv 0$, and the perimeter of $\{u = 1\}$ is penalised. This energy prefers a bilayer structure and suitably rescaled versions Γ-converge to a Helfrich functional with $H_0 = 0$, $\chi_H = 1/2$ and $\chi_K = 1/3$ (for a definition of Γ-convergence, see Definition 4.1.1).

A heuristic way to motivate the occurrence of Willmore's energy in this context comes from thin shell theory. In [FJM02b, FJM02a] Friesecke, James and Müller proved Γ-convergence of non-linear three-dimensional elasticity to geometric bending energies including those of Willmore- or Helfrich-type in the vanishing thickness limit of thin plates. The admissible class here are isometric embeddings of domains in $\mathbb{R}^2$. The restriction to isometries stems from the fact that in-plane stretching energy scales linearly with the thickness of the plate and dominates out of-plane bending, which scales with the third power of the thickness parameter. Thus in the (second order) vanishing thickness limit, we are led to minimise Willmore's energy in a class of isometric immersions. The case of shells (i.e. non-flat structures) has been treated in [FJMM03].

Lipid bilayers differ from shells in that they are liquid, not solid, and thus do not have a reference configuration. Assuming inextensibility (which matches observations), we must therefore also minimise over the space of Riemannian metrics on the bilayer with fixed area and the isometry constraint turns into the fixed area constraint.

2.1.3 A Brief History

Without any claim of completeness, let us give a bit more context of our topic. Willmore's energy is named for T.J. Willmore who studied it in a series of publications [Wil65, Wil71, Wil92, Wil00] and popularised it in his textbook [Wil93]. Independently of Willmore's work, the energy had already been considered as a bending energy for thins plates in [Ger21] and as a conformal invariant of surfaces embedded in $\mathbb{R}^3$ in [BT29, Tho23].

Stationary points (in particular, local minimisers) of the Willmore functional are of interest in models for biological membranes, but they also arise naturally in differential geometry as the stereographic projections of compact minimal surfaces in S^3, see e.g. [PS87], also for examples. A smooth stationary point is a solution of the Euler-Lagrange equation [Tho23, Wei78]

$$\Delta H + (H^2 - 2K)H = 0.$$

The functional also occurs as the energy source of mean curvature flow through

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^{n-1}(M_t) = -\int_{M_t} |H|^2 \,\mathrm{d}\mathcal{H}^{n-1}$$

for a family of surfaces M_t evolving smoothly by mean curvature flow (see e.g. [Hui84]).

From the point of view of the calculus of variations, a natural approach to energies as the ones above is via varifolds [All72], [Hut86b], where existence of minimisers for certain curvature functionals can be proved. The existence of smooth minimising tori was proved by Simon [Sim93], and later generalised to surfaces of arbitrary genus in [BK03].

The long-standing Willmore conjecture that $\mathcal{W}(T) \geq 4\pi^2$ for all tori embedded in $\mathbb{R}^3$ was recently established in [MN14], and the large limit genus of the minimal Willmore energy for closed orientable surfaces in $\mathbb{R}^3$ has been investigated in [KLS10]. The existence of smooth minimising surfaces under isoperimetric constraints has been established in [Sch12]. A good account of the Willmore functional in this context can be found in [KS12].

The case of surfaces constrained to the unit ball was studied in [MR14] and a scaling law for the Willmore energy was found in the regimes of surface area just exceeding 4π and in the large area limit. A parametrised approach to Willmore's energy has been developed in [Riv14] and related papers. In [KMR14] this framework is used to solve the Willmore minimisation problem with prescribed genus and prescribed isoperimetric type. [DGR15] gives a study of the Willmore functional on C^2-graphs and its L^1-lower semi-continuous envelope.

Other avenues of research consider Willmore surfaces in more general ambient spaces [LMS11, LM10, MR13].

In the class of closed surfaces, if χ_K is constant, the second term in the Helfrich functional is of topological nature due to the Gauss-Bonnet theorem. So if the minimisation problem is considered only among surfaces of prescribed topological type it can be neglected. The spontaneous curvature is realistically expected to be non-zero in lipid bilayers due to the inhomogeneity of the bilayer and the presence of different molecules and can have tremendous influence. It should be noted that the full Helfrich energy depends also on the orientation of a surfaces for $H_0 \neq 0$ and not only on its induced (unoriented) varifold. Große-Brauckmann [GB93] gives an example of (non-compact) surfaces M_k of constant mean curvature $H \equiv 1$ converging to a doubly covered plane. This demonstrates that, unlike the Willmore energy, the Helfrich energy need not be lower semi-continuous under varifold convergence for certain parameters.

Recently, existence of minimisers for certain Helfrich-type energies among axially symmetric surfaces under an isoperimetric constraint was proved by Choksi and Veneroni [CV13].

Lower semi-continuity for the Helfrich functional on C^2-boundaries with respect to the L^1-topology of the enclosed sets was established by means of Gauss graphs in [Del97b].

Short time existence for the L^2-gradient flow of the Willmore functional ('Willmore flow') for sufficiently smooth initial data has been shown in [Sim01, KS02] (see also [MS03]) and long time existence for small initial energy and convergence to a round sphere has been demonstrated in [KS01, KS12]. Kuwert and Schätzle's lower bound on existence times in terms of initial curvature concentration in space has been generalised to Willmore flow in Riemannian manifolds of bounded geometry in [Lin13]. It has been shown that Willmore flow can drive smooth initial surfaces to self-intersections in finite time in [MS03]. This issue seems to be prevented by our connectedness functional on the phase-field level, although we do not have a rigorous statement on this. Numerical simulations suggest that singularities can occur in Willmore flow in finite time [MS02]. [DW07, Figure 2] gives a numerical example of a disc pinching off to a torus. A level set approach to Willmore flow is discussed in [DR04].

Willmore flow has been studied numerically for example in [BGN08, Dzi08, DE07] and [BR12], where a two-step time-discretisation is proposed which computes an implicit mean curvature from following a time-step of mean curvature flow. A numerical implementation of the Helfrich functional can be found in [CHM06].

2.1.4 Curves and Euler's Elastica

Let $\gamma : [0, L] \to \mathbb{R}^n$ for $n \geq 2$ be a C^2-closed curve parametrised by arc-length, then the general Willmore functional can be written as

$$\mathcal{W}(\gamma) = \int_\gamma \kappa^2 \, \mathrm{d}\mathcal{H}^1 = \int_0^L |\ddot{\gamma}|^2 \, \mathrm{d}t$$

where κ is just the ordinary geodesic curvature of γ. This energy has been studied as a model for thin elastic rods when $n = 3$ and in image segmentation when $n = 2$. Stationary points of the functional under length constraint solve the Euler-Lagrange equation

$$\kappa'' + \frac{\kappa^3}{2} = \lambda$$

with $\lambda = 0$ if no constraint is posed. Solutions of the unconstrained equation on the whole real line are often called elasticae and have been studied and completely classified in two dimensions already by Euler in [Eul52]. The only stationary points of prescribed length are the circle of that length and a suitable figure eight curve (Bernoulli's lemniscate).

The behaviour of Euler's elastica energy differs from that of Willmore's energy in certain aspects, most importantly through scaling – while $\mathcal{W}(\lambda\cdot\Sigma) = \mathcal{W}(\Sigma)$ for all surfaces Σ and $\lambda \neq 0$, a curve γ satisfies

$$\mathcal{W}(\lambda\cdot\gamma) = \frac{\mathcal{W}(\gamma)}{|\lambda|} \qquad \forall\ \lambda \neq 0. \tag{2.1.2}$$

A family of C^2-curves γ_n which satisfies uniform bounds on length and elastica energy has a limit point $\gamma \in W^{2,2}((0,L),\mathbb{R}^n)$ if its traces are contained in some domain $\Omega \Subset \mathbb{R}^n$. Due to Sobolev embeddings, we see that in particular $\gamma \in C^{1,1/2}([0,L],\overline{\Omega})$. The case of systems of curves has been studied in [BDMP93, BM04, BM07], where the relaxation of the elastica energy on sets with C^2-boundary with respect to the strong L^1-topology is studied. In particular, it is shown that a curve γ which is the $W^{2,2}$-weak limit of embedded C^2-curves γ_n with

$$\limsup_{n\to\infty}\,(L+\mathcal{W})\,(\gamma_n) < \infty$$

can also be approximated by embedded C^2-curves in the strong $W^{2,2}$-sense (L denotes the arc-length of the curve). Since the class of embedded curves is far from being convex, this is not at all immediate and the problem of whether an integral 2-varifold μ arising as the weak limit of embedded C^2-surfaces M_k can be approximated by a different sequence of embedded C^2-surfaces $\widetilde{M_k}$ such that additionally

$$\mathcal{W}(\widetilde{M_k}) \to \mathcal{W}(\mu),$$

is still open. In other contexts, such 'Lavrentiev gap' type phenomena are known to occur.

2.2 A Brief Note on Geometric Measure Theory

2.2.1 Introduction to Varifolds

When proving the existence of minimisers of an energy functional, an invaluable tool is the direct method of the calculus of variations. The method only requires lower semi-continuity and coercivity for a functional in some topology to conclude that a sequence of almost minimisers has an accumulation point which is in fact a minimiser. Integral energies usually do not control the regularity of minimising sequences well, so that weaker topologies with better compactness properties are needed. When weakening the topology, we may need to enlarge the space to obtain a type of closure of the original space in order to obtain a large class of compact sets. The most prominent example of this process is the Sobolev space $W^{1,2}$ for the study of the Dirichlet energy.

Geometric energies like Willmore's energy by definition only depend on the shape of a manifold. This means that the energy associated to two different embeddings of a manifold

M into an ambient Euclidean space

$$f_1, f_2 : M \to \mathbb{R}^n, \qquad f_2 = f_1 \circ \psi, \qquad \psi \in \mathrm{Diffeo}(M, M)$$

is equal. Since tangential reparametrisation is not controlled by the geometric functional, no compactness of a minimising sequence of embeddings can be expected. Indeed, when we denote by $\phi : \overline{\mathbb{C}} \to S^2 \subset \mathbb{R}^3$ the inverse of the stereographic projection and define a family of maps by

$$f_k : \overline{\mathbb{C}} \to S^2 \subset \mathbb{R}^3, \qquad f_k(x) = \phi\left(\frac{x}{k}\right),$$

we see that

$$f_k(\overline{\mathbb{C}}) = S^2 \qquad \forall\ k \in \mathbb{N}, \qquad \lim_{k\to\infty} f_k(x) = \begin{cases} (0,0,1) & x \neq \infty \\ (0,0,-1) & x = \infty \end{cases}$$

pointwise. In this way, geometric energies are not well suited for a parametrised approach. These issues can be solved (using arc-length parametrisation for curves or see [Riv16, Riv13, Riv14] for surfaces), but for us, it will be easier to use the extrinsic approach described below.

Considering energies like Willmore's energy in the class of embedded surfaces leads to the problem of constructing a class of surfaces in which a bound on the area and a suitable integral of mean curvature implies compactness. The most suitable class for this purpose is the class of *integral varifolds*.

Definition 2.2.1. Let $1 \le k \le n$ and $U \subset \mathbb{R}^n$ be open. A k-varifold on U is a Radon measure V on the product space $U \times G(n,k)$ of U with the Grassmannian $G(n,k)$ of unoriented k-planes in $\mathbb{R}^n$.

So a general varifold can be thought of as a measure generalisation of a k-surface which has a location (the projection μ of the measure V onto U) and a direction (the component of the measure which lives on $G(n,k)$, 'slices' of V with respect to μ). For a varifold V, the projected measure

$$\mu = \mu_V = \pi_\sharp V, \qquad \mu(B) = V(\pi^{-1}(B)) = V(B \times G(n,k))$$

under the canonical projection $\pi : U \times G(n,k) \to U$ is called the mass measure of the varifold V. General varifolds still bear too little resemblance to a classical surface to be a satisfactory class for minimisers.

Definition 2.2.2. Let $1 \le k < n$ and $L \subset \mathbb{R}^n$ be a $\mathcal{H}^k$-measurable. Then L is called countably k-rectifiable if there exists a countable collection of Lipschitz maps

$$f_i : \mathbb{R}^k \to \mathbb{R}^n$$

such that

$$\mathcal{H}^k \left(L \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k) \right) = 0.$$

Thus a countably k-rectifiable set can be thought of as a far reaching generalisation of an embedded manifold. Restricting the domain of the Lipschitz maps to the whole of $\mathbb{R}^k$ is not a real restriction due to Kirszbraun's theorem [Fed69, Section 2.10.43] (or [EG92, Theorem 1] for a weaker sufficient statement) and Lipschitz maps could be replaced by C^1-maps in the definition due to Whitney's extension theorem [EG92, Section 6.5]. Making the maps injective, one could even replace $f(\mathbb{R}^m)$ by a C^1-manifolds M_k and a null set M_0:

$$L \subset \bigcup_{k=0}^{\infty} M_k. \tag{2.2.1}$$

Definition 2.2.3. Let $1 \leq k < n$. A measure μ on $\mathbb{R}^n$ is called k-rectifiable if there exists a k-rectifiable set $L \subset \mathbb{R}^n$ and a function $\theta \in L^1_{loc}(\mathcal{H}^k|_L)$ such that

$$\mu = \theta \cdot \mathcal{H}^k|_L. \tag{2.2.2}$$

By the above notation we mean that

$$\mu(B) = \int_{B \cap L} \theta \, \mathrm{d}\mathcal{H}^k$$

which is also sometimes written as $\mu = \mathcal{H}^k|_\theta$ in the literature, assuming that we have set $\theta = 0$ outside L. We will use the notation of (2.2.2) which seems more intuitive. The space $L^1_{loc}(\mathcal{H}^k|_L)$ consists of all functions f such that

$$\int_{K \cap L} |f| \, \mathrm{d}\mathcal{H}^k < \infty$$

for all compact subsets K of $\mathbb{R}^n$. In particular, a rectifiable measure is always a Radon measure.

Note that the support of the measure μ may be significantly larger than the set L which may fail to be closed and even lie dense in $\mathbb{R}^n$. By Rademacher's theorem, Lipschitz maps are differentiable almost everywhere, which generalises to rectifiable sets in the following sense.

Theorem 2.2.4. *[Sim83, Theorems 11.6 and 11.8] A measure μ on $\mathbb{R}^n$ is k-rectifiable if and only if for μ-almost every $x \in \mathbb{R}^n$ there exists a space $S \in G(n,k)$ and a number $\theta(x) \in (0,\infty)$ such that*

$$\lim_{\lambda \to 0} \lambda^{-k} \int f\left(\frac{y-x}{\lambda}\right) \mathrm{d}\mu(y) \;=\; \theta(x) \int_S f \, \mathrm{d}\mathcal{H}^k \qquad \forall\, f \in C_c(\mathbb{R}^n)$$

and such that $\theta \in L^1_{loc}(\mathcal{H}^k|_{\{\theta>0\}})$. The space S is called the weak tangent space to μ at x and denoted by $T_x\mu$.

The function θ is automatically $\mathcal{H}^k$-measurable, and if it is also integrable, then

$$\mu = \theta \cdot \mathcal{H}^k|_{\{\theta>0\}}.$$

The density θ obtained above agrees with the function in (2.2.2). For two measurable functions θ_1, θ_2 on a rectifiable set L, the weak tangent spaces exist and agree $\mathcal{H}^k$-almost everywhere on the set where both functions are positive. In particular for $x \in L$ the tangent space T_xL can be calculated as a classical tangent space when the point x lies in a unique C^1-manifold in the decomposition of rectifiable sets (2.2.1). Approximating the characteristic function $\chi_{B_1(0)}$ from above and below by continuous functions, we have

$$\theta(x) = \Theta_k(x) = \lim_{r\to 0} \frac{\mu(B_r(x))}{\omega_k\, r^k}$$

for μ-almost every $x \in \mathbb{R}^n$ where ω_k is the volume of the unit ball in k dimensions. This could also be seen as an analogue of Lebesgue's differentiation theorem for rectifiable measures or a type of Radon-Nikodym theorem. If $\{\theta > 0\}$ has locally finite $\mathcal{H}^k$-measure, then also

$$\Theta^k(x) = 0 \qquad \text{for } \mathcal{H}^k\text{-almost every } x \in \mathbb{R}^n \setminus \{\theta > 0\}$$

by [EG92, Section 2.3]. Theorem 2.2.4 allows us to define the class of surfaces which we are going to use.

Definition 2.2.5. A varifold V over U is called *rectifiable* if its mass measure μ is rectifiable and if additionally

$$V(f) = \int_U f(x, T_x\mu)\, \theta(x)\, \mathrm{d}\mathcal{H}^k|_L$$

for all $f \in C_c(U \times G(n,k))$. It is called an *integral varifold* if the density θ is $\mathbb{N}$-valued $\mathcal{H}^k|_L$-almost everywhere.

Thus a rectifiable varifold lives on a rectifiable set and the associated direction agrees with the direction given by the tangent space of the rectifiable set. An integral varifold additionally has integer multiplicity.

While generally the varifold V determines the mass measure μ, for a rectifiable varifold the mass measure μ also uniquely determines the varifold V. We will therefore not usually distinguish between μ and V for rectifiable varifolds and also call μ a rectifiable varifold.

We do, however, emphasise that the modes of convergence are different. We say μ_k converges to μ as Radon measures if μ_k converges to μ in the weak* topology in the dual space of continuous functions on $\mathbb{R}^n$, i.e.

$$\int_{\mathbb{R}^n} f \,\mathrm{d}\mu_k \to \int_{\mathbb{R}^n} f \,\mathrm{d}\mu \qquad \forall\ f \in C_c(\mathbb{R}^n)$$

(or $f \in C_c(\overline{U})$ for a suitable $U \subset \mathbb{R}^n$), while we say that μ_k converges to μ as varifolds if the convergence holds in the finer sense of Radon measures on $\mathbb{R}^n \times G(n,k)$ (or $\overline{U} \times G(n,k)$). The modes of convergence can be thought of as analogue to C^0- and C^1-convergence respectively.

Since we are interested in connected surfaces, we make the following definition.

Definition 2.2.6. A Radon measure μ on $\mathbb{R}^n$ is called connected if its support is connected. A varifold is called connected if (the support of) its mass measure is connected.

This convention agrees well with the identification of an integral varifold with its mass measure. The class of connected measures contains surfaces which are only connected in a very weak sense, such as spheres overlapping in only one point

$$\Sigma = \partial B_1(0,0,1) \cup \partial B_1(0,0,-1).$$

Later we will see that this is indeed the strongest concept of connectedness we can guarantee. Our concept of connectedness for measures agrees with that of previous work in [DMR14] as shown in the next Lemma.

Lemma 2.2.7. *A measure μ is disconnected if and only if there are two open sets U_1, U_2 such that $\mu(U_i) > 0$ for $i = 1, 2$, $\mu(\mathbb{R}^n \setminus (U_1 \cup U_2)) = 0$ and $\overline{U_1} \cap \overline{U_2} = \emptyset$.*

Proof. Assume that $\operatorname{spt}(\mu)$ is connected, but there are sets U_1, U_2 with the properties above. Then $\operatorname{spt}(\mu) \cap U_i \neq \emptyset$ for $i = 1, 2$, so

$$\operatorname{spt}(\mu) \cap \big(\mathbb{R}^n \setminus (\overline{U}_1 \cup \overline{U}_2)\big) \neq \emptyset$$

since $\operatorname{spt}(\mu)$ cannot be non-trivially decomposed into two disjoint closed sets $K_i = \operatorname{spt}(\mu) \cap \overline{U}_i$. Since $\overline{U_1} \cup \overline{U_2} = \overline{U_1 \cup U_2}$, there is $x \in \operatorname{spt}(\mu) \setminus \overline{U_1 \cup U_2}$. Taking $r > 0$ such that $B_r(x) \subset$

$\mathbb{R}^n \setminus \overline{U_1 \cup U_2}$, we deduce that

$$\mu\left(\mathbb{R}^n \setminus \overline{U_1 \cup U_2}\right) \geq \mu(B_r(x)) > 0$$

since $x \in \operatorname{spt}(\mu)$, reaching a contradiction.

Now assume that $\operatorname{spt}(\mu)$ is disconnected. Then $\operatorname{spt}(\mu) = K_1 \cup K_2$ has a non-trivial decomposition into relatively closed sets, which are closed also in $\mathbb{R}^n$ since $\operatorname{spt}(\mu)$ is closed by definition. We set

$$U_1 = \left\{x \in \mathbb{R}^n \mid \operatorname{dist}(x, K_1) < \frac{\operatorname{dist}(x, K_2)}{2}\right\}, \quad U_2 = \left\{x \in \mathbb{R}^n \mid \operatorname{dist}(x, K_2) < \frac{\operatorname{dist}(x, K_1)}{2}\right\}$$

and observe that $K_1 \subset U_1$, $K_2 \subset U_2$ since $\operatorname{dist}(x, K_2) > 0$ for all $x \in K_1$ due to the closedness of K_2 and vice versa. Now assume that $x \in \overline{U_1} \cap \overline{U_2}$, then

$$\operatorname{dist}(x, K_1) \leq \frac{\operatorname{dist}(x, K_2)}{2} \leq \frac{\operatorname{dist}(x, K_1)}{4}$$

which is true only if $\operatorname{dist}(x, K_1) = 0$, i.e. $x \in K_1$ since K_1 is closed. The same argument shows that $x \in K_2$, so we have reached a contradiction since $K_1 \cap K_2 = \emptyset$. $\square$

2.2.2 Mean Curvature and Compactness

The set of varifolds with bounded mass is compact by the compactness theorem for Radon measures, or alternatively by the Banach-Alaoglu and Riesz representation theorems. Clearly, the subset of rectifiable or integral varifolds does not have good compactness properties – we can easily imagine a surface of given area in $\mathbb{R}^3$ curling up into a small neighbourhood of a point, so that any point measure (thus in fact any Radon measure on $\mathbb{R}^3$) can be approximated by integral varifolds. Just as control over a gradient term is needed for useful bounds in $W^{1,2}$, we need control over a higher order quantity.

In our geometric setting, the natural quantity is mean curvature. Let M_t be a smooth family of compact C^2-manifolds of dimension k in $\mathbb{R}^n$, parametrised by $t \in (-\varepsilon, \varepsilon)$ via

$$\phi : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^n, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi(t, x) = X \in C^1(M, \mathbb{R}^n).$$

We can compute the variation of area of $M_t = \phi(t, M)$ as

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{H}^k(M_t) = \int_{M_0} \operatorname{div}_{T_xM} X \, \mathrm{d}\mathcal{H}^k = -\int_{M_0} \langle H, X \rangle \, \mathrm{d}\mathcal{H}^k$$

where $\operatorname{div}_{T_xM} X$ is the divergence of the vector field X with respect to the tangent space

T_xM_0 of $M_0 = \phi(0, M)$ at x [Sim83, Section 12]. Here the normalisation of mean curvature may differ from more geometric texts, where H/k is considered as the mean curvature. In our normalisation, the (scalar) mean curvature of the standard $n-1$-sphere is

$$H_{S^{n-1}} = \langle H_{S^{n-1}}, \nu_{S^{n-1}} \rangle \equiv n - 1 > 0.$$

We choose the inner normal for the orientation of a boundary wherever it occurs – note that Willmore's energy depends only on the modulus of the mean curvature vector and not its orientation.

Definition 2.2.8. A rectifiable varifold V in $U \subset \mathbb{R}^n$ is said to have locally finite first variation if there exist a Radon measure δV on U and a δV-measurable vector field $\nu : U \to S^{n-1}$ such that

$$\int \operatorname{div}_{T_x\mu} X \,\mathrm{d}\mu = -\int \langle \nu, X \rangle \,\mathrm{d}\delta V$$

for all $X \in C^1_c(U, \mathbb{R}^n)$. If $\delta V \ll \mu$, then we write

$$\int \operatorname{div}_{T_x\mu} X \,\mathrm{d}\mu = -\int \langle H, X \rangle \,\mathrm{d}\mu \tag{2.2.3}$$

and call H the mean curvature (vector) of V.

The mean curvature of a smooth embedded manifold when expressed in terms of a local parametrisation uses second derivatives, but the variational identity above allows us to define mean curvatures for much less smooth objects. The mean curvature provides us with a useful compactness criterion. The following theorem is a special case of Allard's compactness theorem [All72] which is sufficient for our purposes.

Theorem 2.2.9. *Let $1 \leq k < n$, $U \Subset \mathbb{R}^n$, $1 < p < \infty$ and $C < \infty$. Then the set of integral k-varifolds V with mean curvature $H \in L^p(\mu)$ which satisfy*

$$\operatorname{spt}(\mu) \subset \overline{U}, \qquad \mu(\overline{U}) + ||\, H\, ||^p_{L^p(\mu)} \leq C$$

is compact in the weak topology on the dual space of $C^0(\overline{U} \times G(n,k))$. Since $C^0(\overline{U} \times G(n,k))$ is separable, the weak* topology is locally metrisable and a sequence with these uniform bounds has a sub-sequence which converges in the varifold sense to an integral varifold with p-integrable mean curvature.*

We notice that this is precisely the type of curvature-quantity which we can control by Willmore's energy. Thus we can extend the Willmore functional to integral varifolds by

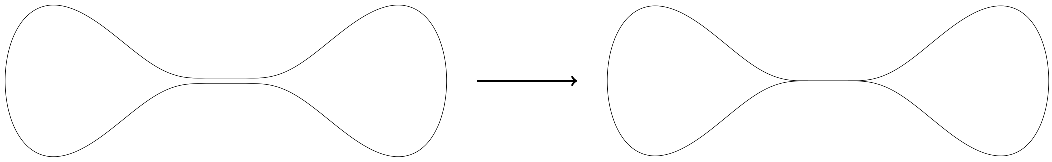

Figure 2.1: Any compact k-manifold which is C^2-immersed into $\mathbb{R}^n$ induces a varifold with mean curvature $H \in L^p(\mu)$ for all p. Points which are covered several times by the immersion have higher multiplicity. Objects with large segments of higher multiplicity can arise as the limits of embedded boundaries.

setting

$$\mathcal{W}(\mu) = \mathcal{W}(V) = \int |H|^2 \,\mathrm{d}\mu.$$

For the convenience of terminology, we make the following definition.

Definition 2.2.10. We call an integral 2-varifold in $\mathbb{R}^3$ with finite mass measure μ and mean curvature $H \in L^2(\mu)$ a Willmore varifold.

We can think of Willmore varifolds as a weak closure of compact surfaces with finite Willmore energy. There are surfaces in this class which are not the limit of a sequence of embedded C^2-surfaces with uniform bounds on area and Willmore energy due to problematic self-crossings – an example is the figure eight space (compare also Section 8.1.2).

2.2.3 Oriented Varifold Hyper-Surfaces

An integral varifold has, by definition, an un-oriented tangent space. When working with embedded hyper-surfaces which bound a compact domain, on the other hand, we can give a natural orientation to the tangent space by specifying an orientation on its normal space (i.e. choosing either the inner or the outer normal as positively oriented). We extend this definition, for simplicity using the dual correspondence between unit vectors and oriented $n-1$-dimensional subspaces of $\mathbb{R}^n$.

Definition 2.2.11. An oriented varifold hyper-surface V^o is a Radon measure on $\mathbb{R}^n \times S^{n-1}$.

Using the canonical projection

$$\pi : S^{n-1} \to G(n,1), \qquad v \mapsto \langle v \rangle = \{\lambda v \mid \lambda \in \mathbb{R}\}$$

and the Grassmannian duality diffeomorphism

$$\delta : G(n,1) \to G(n,n-1), \qquad P \mapsto P^\perp$$

an oriented varifold hypersurface V^o induces a varifold V through the push-forward under the map

$$\mathrm{id}_{\mathbb{R}^n} \times (\delta \circ \pi) : \quad \mathbb{R}^n \times S^{n-1} \to \mathbb{R}^n \times G(n, n-1)$$

or in explicit terms

$$V(f) = \int_{\mathbb{R}^n \times S^{n-1}} f(x, \delta \circ \pi(v)) \, \mathrm{d}V^o(x, v) \qquad \forall\ f \in C_c(\mathbb{R}^n \times G(n, n-1)).$$

The mass measure and mean curvature of V^o are given by those of V. An oriented varifold is called rectifiable if it can be written as

$$V^o(f) = \int_L f(x, \xi_1(T_xL)) \, \theta_1(x) + f(x, \xi_2(T_xL)) \, \theta_2(x) \, \mathrm{d}\mathcal{H}^{n-1}(x)$$

where $\xi_i : G(n, n-1) \to S^{n-1}$ are measurable selection maps for the oriented normal vector to a given space, $\xi_1 = -\xi_2$ pointwise, and L is a rectifiable set, θ_i are functions in $L^1_{loc}(\mathcal{H}^{n-1}|_L)$. The oriented varifold is called integral if both θ_1 and θ_2 are integer-valued almost everywhere. To avoid technicalities, we only give a very partial compactness theorem which is however sufficient for our purposes.

Theorem 2.2.12. *Let $E_k \Subset \Omega \Subset \mathbb{R}^n$ such that $\partial E_k \in C^2$ and*

$$\limsup_{k\to\infty} (\mathcal{W} + \mathcal{H}^{n-1})(\partial E_k) < \infty.$$

Denote by V_k^o the oriented varifold induced by ∂E_k and the outer normal vector. Then there is an oriented integral varifold V^o such that

$$V_k^o \rightharpoonup V.$$

A more general version of the Theorem is proven in [Hut86b]. We only consider the co-dimension 1 case to avoid Grassmannians of orientable sub-spaces and only considered globally oriented surfaces without boundary to avoid currents (see Remark 2.2.13).

2.2.4 The Second Fundamental Form

For simplicity, we restrict ourselves to the case of orientable hyper-surfaces. Here the normal vector ν is uniquely defined up to a choice of sign and the classical second fundamental form II is defined by

$$II : T_xM \to T_xM, \qquad II(v) = \nabla_v \nu$$

where ∇ is the Levi-Civita connection of the ambient space (so for us, the usual derivative in $\mathbb{R}^n$). A different definition A will be given for varifolds.

By using the first variation identity (2.2.3) on vector fields $X(x) = Y(x, T_xM)$ for a suitable Y on smooth manifolds, Hutchinson [Hut86b] introduced a concept of a second fundamental form $A \in L^1(V, \mathbb{R}^{n\times n\times n})$ on varifolds. Namely, by an abuse of notation we identify $P \in G(n,k)$ with the orthogonal projection $P : \mathbb{R}^n \to P \subset \mathbb{R}^n$ and thus embed $G(n,k)$ into the space of linear mappings from $\mathbb{R}^n$ to itself, or equivalently the matrix space $\mathbb{R}^{n\times n}$. For $\phi \in C^1(U \times \mathbb{R}^{n\times n})$ we denote the spatial derivatives of ϕ by $D_j\phi$ and the derivatives in the tangent space directions by $D^*_{jk}\phi$. Then the second fundamental form A in the sense of Hutchinson is defined uniquely by the identity

$$0 = \int \left(P_{ij}D_j\phi + A_{ijk}\, D^*_{jk}\phi + A_{jij}\phi\right) \mathrm{d}V(x,P) \qquad \forall\, i = 1, \dots, n.$$

Varifolds with weak second fundamental form are called *curvature varifolds*. Note that the second fundamental form is defined on $U \times G(n,k)$, not on U like the mean curvature.

The definition of second fundamental form differs slightly from the usual normalisation of the second fundamental form II which measures the oscillation of a normal field ν to the manifold M, while A measures the oscillation of the its unoriented tangent space (or rather, the orthogonal projection onto it) – also see [Hut86b] for a more detailed explanation. The two notions are of course equivalent and the coefficients A_{ijk} of A and the coefficients B_{ijk} of the classic second fundamental form are related by an explicit formula [Hut86b, Proposition 5.2.6].

From the explicit expression, it can be seen that the existence of II and A is equivalent and $A \in L^p(V)$ if and only if $II \in L^p(V)$. If $A \in L^p(V)$, then the mean curvature vector satisfies $H_i(x) = A_{jij}(x, T_x\mu) \in L^p(\mu)$ [Hut86b, Remark 5.2.3]. Furthermore, curvature varifolds with second fundamental form in $L^p(V)$ for $p > k$ are locally graphs of multi-valued $C^{1,1-\frac{k}{p}}$-functions [DLS11, Hut86a] similarly to the regularity given by Morrey's embedding theorem (and Calderon-Zygmund theory) for Sobolev functions u with $\Delta u \in L^p$ for $p > n$. The analogue statement for embedded surfaces is given in [Lan85].

Embedded surfaces in $\mathbb{R}^3$ are orientable for topological reasons. The second fundamental form is a symmetric map $II : T_xM \to T_xM$ with two eigenvalues λ_1, λ_2 (the principal curvatures). The mean curvature is given by

$$H = \frac{1}{2}\,\mathrm{tr}(II) = \frac{\lambda_1 + \lambda_2}{2}$$

as the actual mean of the principal curvatures (geometric normalisation) or

$$H = \mathrm{tr}(II) = \lambda_1 + \lambda_2$$

(analytic normalisation). Because it simplifies the first variation formula (2.2.3), we choose the analytic normalisation of mean curvature. The Gaussian curvature of the surface is

$$K = \det(II) = \lambda_1\lambda_2.$$

Since the principal curvatures are eigenvalues to orthogonal directions $v_1, v_1 \in T_xM$ (or at least can be chosen such, even if $\lambda_1 = \lambda_2$), the Frobenius norm $|\cdot| = ||\cdot||_F$ of the second fundamental form is

$$|\,II\,|^2 = \lambda_1^2 + \lambda_2^2 = (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2 = H^2 - 2K.$$

Using this identity, we can define the Gaussian curvature of an integral varifold with weak second fundamental form $A \in L^2(\mu)$ and thus extend the Canham-Helfrich functional (with parameter $H_0 \equiv 0$) to the class of curvature varifolds with square integrable second fundamental form. To include a parameter $H_0 \neq 0$, we would need *oriented curvature varifolds* since the functional depends on the orientation of the surface in that case.

We remark that the results which we will present in the next chapter do not depend on the specific extension of the Gaussian curvature to curvature varifolds, but only on the fact that the Gaussian curvature of a smooth surface equals that of an integer multiple. This is given for our definition since A agrees for all integer multiples of a given varifold as the density θ cancels out in the defining equation. The property is very sensible and automatically given in a parametrised approach when we consider immersed surfaces which have multiple coinciding segments. Thus the only sensible assumption is that the Gaussian curvature of $\theta \cdot \mu$ agrees with the Gaussian curvature of μ.

Remark 2.2.13. There are further concepts of measure theoretic generalised surfaces which will not be used in this text, most prominently currents [KP08, Sim83]. Currents are constructed in analogy to distributions rather than to measures. They have a simpler algebraic structure and a natural boundary operator, which is lacking for varifolds, but they are prone to extinction phenomena when surfaces with opposite orientations approach each other, making them less suitable for our purposes.

A special case of currents are Gauss graphs [Del96, Del01, Del97a, Del97b] which generalise a point pair (x, ν_x) in $\mathbb{R}^3 \times S^2$ which can be defined for C^1-surfaces in $\mathbb{R}^3$. Currents also play a role in the theory of oriented varifolds [Hut86b] in arbitrary dimension and co-dimension. A notion of boundary for varifolds is discussed in [Man96].

Another short but comprehensive introduction to varifolds can be found in [LM09], along with a locality property for the weak mean curvature of varifolds under certain conditions. A stronger result and further properties of varifolds can be found in [Men09, Men10, Men12, Men13, Men16]. A very brief introduction to varifolds is given in the recent review article [Men17].

Chapter 3

The Topology of Constrained Minimisers

3.1 Introduction

In this chapter, we will show that Problem 1 has a solution in the class of integral varifolds that arise as weak limits of C^2-boundaries of sets $E \Subset \Omega$ with connected boundary ∂E of area $\mathcal{H}^{n-1}(\partial E) = S$. We will further show similar existence and non-existence results for more general Canham-Helfrich functionals and investigate the influence of the parameter χ_K. Finally, we demonstrate that the minimisation problem is not well-posed if in addition to the connectedness of ∂E also its topological type is prescribed. First we prove a structure theorem for Willmore varifolds.

Theorem 3.1.1. *Let μ be a Willmore varifold. Then $\operatorname{spt}(\mu)$ is a rectifiable subset of $\mathbb{R}^3$ and has at most $N \leq \mathcal{W}(\mu)/16\pi$ connected components. Every connected component induces a measure $\mu_1, \dots, \mu_N$ which is a Willmore varifold in itself and the mean curvatures H_{μ_k} and H_μ agree μ_k-almost everywhere. We have*

$$2\sqrt{\frac{\mu_k(\mathbb{R}^n)}{\mathcal{W}(\mu_k)}} \leq \operatorname{diam}(\operatorname{spt}(\mu_k)) \leq \frac{1}{\pi}\sqrt{\mu_k(\mathbb{R}^n)\,\mathcal{W}(\mu_k)} \qquad \forall\ k = 1, \dots, N.$$

Theorem 3.1.1 seems to be known albeit slightly scattered over the current research. We have included its proof due to its relevance in this thesis and since we could not find a suitable single reference.

Note that the theorem is far from obvious and wrong for integral k-varifolds in $\mathbb{R}^n$ with mean curvature in L^p for $p < k$, see Example 3.2.7. Very similar methods can be used to prove a statement on the convergence of Willmore varifolds.

Theorem 3.1.2. *Let $K, M > 0$ and $\Omega \Subset \mathbb{R}^n$ open and μ_k be Willmore varifolds such that*

$$\mathrm{spt}(\mu_k) \subset \overline{\Omega}, \qquad \mu_k(\overline{\Omega}) \leq M, \qquad \mathcal{W}(\mu_k) \leq K \qquad \forall\ k \in \mathbb{N}.$$

Then there exists a Willmore varifold μ such that (up to a subsequence) μ_k converges to μ in the sense of varifolds and

$$\mathrm{spt}(\mu_k) \to \mathrm{spt}(\mu) \cup \{x_1, \ldots, x_N\}$$

in the sense of Hausdorff convergence for a collection of points $x_1, \ldots, x_N \in \overline{\Omega}$. The number N is bounded in terms of $\limsup_{k\to\infty} \mathcal{W}(\mu_k)$ and $N = 0$ if one of the following conditions is met.

1. $\mathrm{spt}(\mu_k)$ *is connected for all $k \in \mathbb{N}$ and $\mu \neq 0$,*
2. $\limsup_{k\to\infty} \mathcal{W}(\mu_k) < \mathcal{W}(\mu) + 16\pi$ *and $\mu \neq 0$,*
3. *the mean curvatures H_k of μ_k satisfy a uniform bound on $||\, H_k||_{L^p(\mu_k)}$ for some $p > 2$.*

The condition $\mu \neq 0$ is satisfied if $\lim_{k\to\infty} \mu_k(\overline{\Omega}) > 0$ along the subsequence which satisfies $\mu_k \stackrel{*}{\rightharpoonup} \mu$. The second condition is met for example if $\mathcal{W}(\mu_k) \to \mathcal{W}(\mu)$. In particular, this means that connectedness is stable in the minimisation problem.

Corollary 3.1.3. *Let $K, M > 0$ and $\Omega \Subset \mathbb{R}^n$ open. The class of integral 2-varifolds V in $\mathbb{R}^n$ satisfying*

1. $\mathrm{spt}(\mu_V) \subset \overline{\Omega}$*, the mass $\mu_V(\overline{\Omega}) \leq M$, $\mathcal{W}(V) \leq K$ and*
2. $\mathrm{spt}(\mu_V)$ *is connected*

is (sequentially) compact under the convergence of varifolds. The same holds for the closure with respect to varifold convergence of connected manifolds which are C^2-embedded into Ω with surface area bounded by M and Willmore energy bounded by K.

This has a direct implication for minimising Willmore's energy in a suitable topological class.

Corollary 3.1.4. *Let $\Omega \Subset \mathbb{R}^n$ be open and $S > 0$. Then there exists a 2-varifold V with mass measure μ_V such that*

(i) $\mathrm{spt}(\mu_V) \subset \overline{\Omega}$ *is connected, $\mu_V(\overline{\Omega}) = S$ and*

(ii) V minimises $\mathcal{W}$ among the varifolds satisfying (i).

The same holds if we add the assumption that V is a varifold limit of connected embedded C^2-surfaces with uniformly bounded Willmore energy and surface area S in (i).

Corollary 3.1.4 follows directly from the Corollary 3.1.3, the definition of varifold convergence and the lower-semicontinuity of Willmore's energy under varifold convergence. A version phrased directly for C^2-boundaries and adapted to phase-field applications can be found in Section 3.2.4.

Now let $g \in \mathbb{N}_0$, $S > 0$ and $\Omega \subset \mathbb{R}^3$ open. Denote by $\mathscr{M}_{g,S,\Omega}$ the space of closed connected orientable genus g surfaces which are C^2-embedded in Ω with surface area S and by $\mathscr{M}_{S,\Omega}$ the union of all $\mathscr{M}_{g,S,\Omega}$ over $g \in \mathbb{N}_0$. The second main result of this chapter is the following.

Theorem 3.1.5. *Let $m \in \mathbb{N}$, $m \geq 2$, $g \in \mathbb{N}_0$ and $\varepsilon > 0$. Then there exists $M \in \mathscr{M}_{g,4\pi m,B_1(0)}$ such that*

$$\mathcal{W}(M) < 4\pi m + \varepsilon.$$

For the proof, we show that we can connect two concentric spheres with almost equal radii by a large number of catenoids. This does not change the area or Willmore's energy much since catenoids are minimal surfaces, but changes the topology to arbitrary genus. A further perturbation with small Willmore energy adds a sufficient amount of area. The argument is similar to [MR14], where two spheres were connected by one catenoid. Our construction is more analytic than geometric and allows for any finite number of catenoids, whereas the construction of [MR14] requires (almost) a whole hemisphere per catenoid.

This has important implications for curvature energies.

Corollary 3.1.6. *Let $g \in \mathbb{N}_0, m \in \mathbb{N}, m \geq 2$ and consider $\Omega = B_1(0)$, $S = 4m\pi$. Then every sequence $M_k \in \mathscr{M}_{g,S,\Omega}$ such that*

$$\mathcal{W}(M_k) \to \inf\{\mathcal{W}(M) \mid M \in \mathscr{M}_{g,S,\Omega}\} = 4\pi m$$

converges to an m-fold covered unit sphere as varifolds, independently of g.

This result differs from the unconstrained case [BK03] or minimisation among C^2-boundaries with prescribed isoperimetric ratio [KMR14]. In both cases, there exists a smooth embedded (i.e. multiplicity 1) surface of genus g which minimises Willmore's energy among all surfaces of genus g (which bound a domain with certain isoperimetric ratio, in the second case).

Corollary 3.1.7. *Denote by $\mathcal{E}$ the Canham-Helfrich energy with constant parameters $\chi_K < 0 < \chi_H$ and $H_0 = 0$. Let $m \in \mathbb{N}, m \geq 2$ and specify $\Omega = B_1(0)$, $S = 4m\pi$. Then every* *sequence $M_k \in \mathscr{M}_{S,\Omega}$ such that*

$$\mathcal{E}(M_k) \to \inf \{\mathcal{E}(M) \mid M \in \mathscr{M}_{S,\Omega}\} = 4\pi\,(4\chi_H m - \chi_K)$$

converges to a higher multiplicity unit sphere $\mu = m \cdot \mathcal{H}^2|_{S^3}$ as varifolds and we have $\mathcal{E}(\mu) <$ $\liminf_{k\to\infty} \mathcal{E}(M_k)$. If $M \in \mathscr{M}_{S',B_1(0)}$ for some $S' > 0$ and

$$\mathcal{E}(M) \leq 4\chi_H S'$$

holds, then M is a topological sphere. Furthermore, if $\chi_K < -4\chi_H$, then for any open $\Omega \Subset \mathbb{R}^3$, $S > 0$, $C > 0$ the functional $\mathcal{E}$ is bounded from below in the class of smooth manifolds $\mathscr{M}_{S,\Omega}$ and in the varifold closure of

$$\mathcal{B}_{S,\Omega,C} := \{M \in \mathscr{M}_{S,\Omega} \mid \mathcal{E}(M) < C\},$$

but not in the union of the closures

$$\bigcup_{k=1}^{\infty} \overline{\mathcal{B}_{S,\Omega,k}}\,.$$

The theorem has some implications for the use of Canham-Helfrich energy in the modelling of lipid bilayers. The multiple covering of a single sphere is unphysical since a biological membrane separating two domains is usually the location of chemical exchange. The higher multiplicity does not increase effective surface area; on the contrary, it would make the transport of any exchanged species more difficult. Obviously, the situation of the corollary is highly idealised, but probably similar phenomena could be observed under more generic conditions.

We also suggest that it might be more appropriate to consider the lower semi-continuous envelope with respect to varifold convergence of the Canham-Helfrich energy in the class of C^2-boundaries rather than its direct extension to curvature varifolds.

The case $\chi_K > 0$ is entirely unphysical. Here we can consider non-constant material parameters. Assume that there are measurable functions χ_H, χ_K and H_0 associated to each surface $M \in \mathscr{M}_{S,\Omega}$.

Corollary 3.1.8. *Let $\Omega \subset \mathbb{R}^3$ open and $r > 0$ such that $\overline{B_r(x)} \subset \Omega$ for some $x \in \mathbb{R}^3, r > 0$. Let $\mathcal{E}$ be the Canham-Helfrich energy with parameters χ_H, χ_K and H_0 satisfying the bounds*

$$||\,\chi_H\,||_{L^\infty(M)} \leq C, \qquad ||\,H_0\,||_{L^2(M)} \leq C, \qquad \delta \leq \chi_K \leq C$$

for some $C, \delta > 0$ independent of $M \in \mathscr{M}_{S,\Omega}$. Assume that $\mu = 4\pi r^2 \cdot \delta_x$ is a point measure *or* $\mu = \mathcal{H}^2|_{\partial B_r(x)}$. *Then there exists a sequence* $M_k \in \mathscr{M}_{4\pi r^2,\Omega}$ *such that* $\mathcal{H}^2|_{M_k} \stackrel{*}{\rightharpoonup} \mu$ *as Radon measures and* $\mathcal{E}(M_k) \to -\infty$. *In the second case, even varifold convergence holds.*

Corollaries 3.1.6, 3.1.7 and 3.1.8 easily follow from Theorem 3.1.5 and the reverse estimate given in Lemma 3.2.3.

3.2 Proofs

3.2.1 Technical Lemmas

In this section, we establish a few geometric properties of varifolds which will be used in the following.

Lemma 3.2.1. *Let* μ *be a finite Radon measure on* $\mathbb{R}^n$ *and* $x \in \mathbb{R}^n$ *a point. Then*

$$\mu(\partial B_r(x)) = 0$$

for all but countably many $r \in \mathbb{R}$.

Proof. Assume that $\mu(\partial B_r(x)) > 0$ for uncountably many $r \in \mathbb{R}$. Since

$$\{r \in \mathbb{R} \mid \mu(\partial B_r(x)) > 0\} = \bigcup_{k=1}^{\infty} \left\{ r \in \mathbb{R} \mid \mu(\partial B_r(x)) > \frac{1}{k} \right\}$$

at least one of the sets on the right hand side must be infinite since otherwise the left hand side could only be countable. But then the measure μ is automatically infinite. □

Recall the following localised Li-Yau inequality originally due to L. Simon [Sim86]. The proof below follows [Top98, Lemma 1] in a formulation adapted to apply to varifolds. In Part II we will give analogue statements for phase-fields in (5.2.4) and (5.2.5).

Lemma 3.2.2. *Let* μ *be a Willmore varifold and* $r > 0$. *Then*

$$\Theta^2(x) := \limsup_{s \to 0} \frac{\mu(B_s(x))}{\pi s^2} \leq \frac{\mu_V(B_r)}{\pi r^2} + \frac{1}{16\pi} \int_{B_r} |H|^2 \, \mathrm{d}\mu \tag{3.2.1}$$

at μ*-almost every point in* $\mathbb{R}^n$. *The classical Li-Yau inequality [LY82]*

$$\Theta^2(x) \leq \frac{1}{16\pi} \mathcal{W}(\mu) \tag{3.2.2}$$

follows by $r \to \infty$.

Note that the equality is usually found with 4π in the place of 16π due to a different normalisation of Willmore's energy or mean curvature. We do not need that the dimension of the embedding space is equal to $n = 3$, while the dimension of the varifold has to be $k = 2$ for the following proof. For a phase-field version of this theorem, see Lemma 5.2.5 and Remark 5.2.6.

Proof of Lemma 3.2.2. We will obtain the inequality by inserting a suitable Lipschitz continuous vector field into the first variation inequality. This is justified by approximating the Lipschitz function by C^1-functions. Let $x_0 \in \operatorname{spt}(\mu)$ and $0 < r < R$. Without loss of generality we assume that $x_0 = 0$ and denote $B_\rho := B_\rho(0)$. Set

$$x_r = \max\{|x|, r\}, \qquad X(x) = \left(\frac{1}{x_r^2} - \frac{1}{R^2}\right)_+ x$$

and calculate the tangential divergence of this field to apply the first variation identity

$$\int \operatorname{div}_{T_x\mu} X \,\mathrm{d}\mu = -\int \langle X, H\rangle \,\mathrm{d}\mu.$$

If $|x| < r$, we have

$$\operatorname{div}_{T_x\mu} X(x) = \left(\frac{1}{r^2} - \frac{1}{R^2}\right) \operatorname{div}_{T_x\mu} x = 2\left(\frac{1}{r^2} - \frac{1}{R^2}\right),$$

since the divergence of the field $x \mapsto x$ is 2 on all two-dimensional vector spaces. For $r < |x| < R$

$$\begin{aligned}
\operatorname{div}_{T_x\mu} X(x) &= \langle \nabla_{T_x\mu} |x|^{-2}, x\rangle + \left(\frac{1}{|x|^2} - \frac{1}{R^2}\right) \operatorname{div}_{T_x\mu} x \\
&= \langle (-2)\,|x|^{-4} x^{\|}, x\rangle + 2\left(\frac{1}{|x|^2} - \frac{1}{R^2}\right) \\
&= \frac{-2\,|x^{\|}|^2}{|x|^4} + 2\left(\frac{1}{|x|^2} - \frac{1}{R^2}\right) \\
&= 2\left(\frac{|x^\perp|^2}{|x|^4} - \frac{1}{R^2}\right)
\end{aligned}$$

where $x^{\|} = \pi_{T_x\mu}(x)$ is the tangential component of x (i.e. the orthogonal projection of x onto $T_x\mu$) and $x^\perp = x - x^{\|}$ is the orthogonal component. Finally

$$\operatorname{div}_{T_x\mu} X(x) = 0$$

if $|x| > R$. Thus, assuming that $\mu(\partial B_R) = \mu(\partial B_r) = 0$ (which holds for all but countably many radii due to Lemma 3.2.1) we have

$$0 = \int \operatorname{div}_{T_x\mu} X \,\mathrm{d}\mu + \int \langle X, H\rangle \,\mathrm{d}\mu$$

$$
\begin{aligned}
&= \left(\frac{2}{r^2} - \frac{2}{R^2}\right) \mu(B_r(0)) + \int_{B_R \setminus B_r} \left(\frac{2\,|x^\perp|^2}{|x|^4} - \frac{2}{R^2}\right) \mathrm{d}\mu + \int \langle X, H\rangle \,\mathrm{d}\mu \\
&= \frac{2\,\mu(B_r)}{r^2} + \int_{B_R \setminus B_r} 2\,\frac{|x^\perp|^2}{|x|^4} + \langle X, H\rangle \,\mathrm{d}\mu - \frac{2\,\mu(B_R)}{R^2} + \int_{B_r} \langle X, H\rangle \,\mathrm{d}\mu. \qquad (3.2.3)
\end{aligned}
$$

The last identity follows since $X \equiv 0$ outside B_R. Rearranging, we obtain

$$
\frac{\mu(B_r)}{r^2} = \frac{\mu(B_R)}{R^2} - \frac{1}{2}\int_{B_R \setminus B_r} 2\,\frac{|x^\perp|^2}{|x|^4} + \langle X, H\rangle \,\mathrm{d}\mu_V - \frac{1}{2}\int_{B_r} \langle X, H\rangle \,\mathrm{d}\mu. \qquad (3.2.4)
$$

The last term on the right hand side goes to zero due as $r \to 0$ due to the continuity of measures from above. Due to [Bra78, Chapter 5], the part of the weak mean curvature of an integral varifold μ which is absolutely continuous with respect to μ is perpendicular to its tangent space. This is automatically the case since we assumed $H \in L^2(\mu)$, so we have $\langle H, x\rangle = \langle H, x^\perp\rangle$. For $x \in B_R \setminus B_r$ we have the pointwise estimate

$$
2\,\frac{|x^\perp|^2}{|x|^4} + \langle X, H\rangle \geq -\frac{1}{8}\,|H|^2
$$

for trivial reasons if $\langle x^\perp, H\rangle \geq 0$ and

$$
\begin{aligned}
2\,\frac{|x^\perp|^2}{|x|^4} + \langle X, H\rangle &= 2\,\frac{|x^\perp|^2}{|x|^4} + \left(\frac{1}{|x|^2} - \frac{1}{R^2}\right)\langle x^\perp, H\rangle \\
&= 2\left|\frac{x^\perp}{|x|^2} + \frac{1}{4}H\right|^2 - \frac{1}{R^2}\langle x^\perp, H\rangle - \frac{1}{8}\,|H|^2 \\
&\geq -\frac{1}{8}\,|H|^2
\end{aligned}
$$

if $\langle x^\perp, H\rangle \leq 0$. Finally, we remark that the condition $\mu(\partial B_r) = 0$ can be removed by considering a C^1-approximation of f. Hence, altogether

$$
\Theta^2(x) = \limsup_{r\to 0} \frac{\mu(B_r)}{\pi r^2} \leq \frac{\mu(B_R)}{R^2} + \frac{1}{16}\int_{B_R} |H|^2 \,\mathrm{d}\mu.
$$

□

Let us prove the following result about k-varifolds supported in the unit ball $B = \overline{B_1(0)} \subset \mathbb{R}^n$ which generalises [MR14, Theorem 1] to general $p > 1$.

Lemma 3.2.3. *Let V be an integral k-varifold with weak mean curvature $H \in L^p(\mu)$, $p > 1$ such that $\operatorname{spt}(\mu) \subset B$ and $\mu(B) < \infty$. Then we have*

$$
\int_B |H|^p \,\mathrm{d}\mu \geq k^p\,\mu_V(B). \qquad (3.2.5)
$$

If $k = 2$ and $n = 3$, then equality holds if and only if $\mu = \theta \cdot \mathcal{H}^{n-1}|_{S^{n-1}}$ for an integer $\theta \in \mathbb{N}$.

The last statement could be extended to $k = n-1$ and $n \geq 2$ since all that is needed is that the function θ is constant. A non-constant density however would lead to a singular, tangential part of the first variation. Since the argument below is simpler in this form and covers the case needed for applications, we do not prove the extension.

Proof of Lemma 3.2.3. Use the first variation identity

$$\int \mathrm{div}_{T_x\mu}\, v \,\mathrm{d}\mu = -\int \langle H, v\rangle \,\mathrm{d}\mu$$

with $v(x) = x$. Then $\mathrm{div}_{T_x\mu}\, v \equiv k$ independently of $T_x\mu$ and we have

$$k\,\mu(\mathbb{R}^n) = \int \mathrm{div}_{T_x\mu}\, v \,\mathrm{d}\mu = -\int \langle H, v\rangle \,\mathrm{d}\mu \leq \left(\int |H|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int |x|^{\frac{p}{p-1}} \,\mathrm{d}\mu\right)^{\frac{p-1}{p}} \tag{3.2.6}$$

Using $|x| \leq 1$, we arrive at

$$k\,\mu(\mathbb{R}^n) \leq \left(\int |H|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \mu(\mathbb{R}^n)^{1-\frac{1}{p}} \tag{3.2.7}$$

which is equivalent to the statement of the Lemma. If equality holds, then necessarily

$$\int_B |x|^{\frac{p}{p-1}} \,\mathrm{d}\mu = \mu(B)$$

and thus $|x| = 1$ μ-almost everywhere, so $\mathrm{spt}(\mu) \subset S^{n-1}$. Further, equality holds in Hölder's inequality, so there exist $\alpha, \beta \neq 0$ such that

$$\alpha\,|H|^p = \beta\,|x|^{\frac{p}{p-1}} = \beta,$$

i.e. $|H|$ is constant and in particular bounded. Knowing that $|H|$ is constant, (3.2.7) immediately tells us that $|H| = k$. When $k = 2$, we deduce that μ is a Willmore varifold (even if $1 < p < 2$ initially) and obtain the bound

$$\theta = \Theta^2 \leq \frac{1}{16\pi}\,\mathcal{W}(\mu) = \frac{|H|^2}{16\pi}\mu(\overline{B_1(0)}) = \frac{1}{4\pi}\,\mu(\overline{B_1(0)})$$

at least μ-almost everywhere. When we integrate this inequality over S^2, we see that

$$\theta \equiv \frac{1}{4\pi}\,\mu(\overline{B_1(0)})$$

(at least $\mathcal{H}^2$-almost everywhere) since $\mathcal{H}^2(S^2) = 4\pi$. □

This result is optimal in two ways:

Remark 3.2.4. If $k < n-1$, the uniqueness statement is not true anymore – for example for $n \geq 3$ and $k = 1$, we can take an arbitrary finite union of great circles in S^{n-1} as an example of a 1-varifold satisfying the energy identity. If $n \geq 4$, we can even choose the great circles to be disjoint in different ways and the varifold to be a C^∞-manifold. The most we can hope for is that these superpositions of k-spheres (with multiplicity) are the only varifolds satisfying identity in (3.2.5).

Remark 3.2.5. The inequality (3.2.5) remains true for $p = 1$, or even for the more natural functional

$$\widetilde{\mathcal{W}}_1(V) = \delta V(\mathbb{R}^n),$$

but we cannot apply Hölder's inequality and only reach

$$2\mu(B) = -\int \langle n, x \rangle \, \mathrm{d}\delta V \leq \delta V(\mathbb{R}^n)$$

which only shows that $|x| = 1$ δV-almost everywhere, not μ-almost everywhere. Indeed, the disc

$$D = \{x_3 = 0, x_1^2 + x_2^2 \leq 1\} \subset B_1(0) \subset \mathbb{R}^3$$

induces an integral 2-varifold in $\mathbb{R}^3$ supported in the unit ball whose weak mean curvature is given by the unit normal of the circle lying in the x_1x_2-plane and pointing out of the disc, due to Gauss' theorem. This disc satisfies

$$\widetilde{W}_1(D) = \mathcal{H}^1(\partial D) = 2\pi = 2\,\mathcal{H}^2(D),$$

which corresponds to equality in (3.2.5). Superpositions of discs in different planes retain the same property, so that the uniqueness and regularity assertions break down. Again, the most we can hope for is that the only varifolds satisfying the energy identity are given as superpositions of a unit sphere with higher multiplicity and discs in different planes and with multiplicities.

Finally, we give a simple structure result for compact sets.

Lemma 3.2.6. *[DLW17, Lemma 3.16] Let (X, d) be a metric space and $K \subset X$ is compact. If K is not connected, then there exist two open sets $U_1, U_2 \subset X$ such that*

$$K \subset U_1 \cup U_2, \qquad K \cap U_i \neq \emptyset \quad \textit{for } i = 1,2 \qquad \textit{and} \qquad \operatorname{dist}(U_1, U_2) > 0.$$

Proof. Assume that K is not connected. Then there exist relatively open non-empty sets $W_1, W_2 \subset K$ such that

$$K = W_1 \cup W_2.$$

By definition of the subspace topology, W_1 and W_2 are also relatively closed. Since K is compact, they are even compact, so $\delta := \operatorname{dist}(W_1, W_2) > 0$. We set $U_1 = \{\operatorname{dist}(\cdot, W_1) < \delta/3\}$ and $U_2 = \{\operatorname{dist}(\cdot, W_2) < \delta/3\}$. $\square$

If K is the support of a measure μ, we can pick $x_i \in K \cap U_i$ for $i = 1, 2$ since neither set is empty. As U_i is a neighbourhood of x_i, we see directly from the definition of the support of a measure that $\mu(U_i) > 0$ for $i = 1, 2$.

3.2.2 Structure Theorem

The key idea in the proof of the structure theorem is the scale invariance of Willmore's energy. In that sense the dimension k of a varifold appears to be the point where L^p-integrability of the mean curvature begins to regularise the support. The structure theorem appears to be known, but a concise statement with an elementary proof could not be found.

Proof of Theorem 3.1.1. We know that $\mu = \theta \cdot \mathcal{H}^2|_L$ where L is a 2-rectifiable set (and thus defined up to a set of $\mathcal{H}^2$-measure zero). We fix any representative of θ and pass to

$$L' = \{x \in L \mid \theta(x) \geq 1\}.$$

Since $\theta \geq 1$ $\mathcal{H}^2$-almost everywhere on $\{\theta > 0\}$, we see that $\mu(\mathbb{R}^n \setminus L') = 0$. In particular, for $x \in \mathbb{R}^3 \setminus \operatorname{spt}(\mu)$, there is $r > 0$ such that $\mu(B_r(x)) = 0$, thus $L' \subset \operatorname{spt}(\mu)$. Furthermore, since μ is rectifiable, the limit density

$$\Theta^2(x; \mu) = \lim_{r \to 0} \frac{\mu(B_r(x))}{\pi r^2} = \theta \geq 1$$

exists for $\mathcal{H}^2$-almost every $x \in L'$, see Theorem 2.2.4. Due to the Li-Yau inequality (3.2.2), we have

$$\theta = \Theta^2 \leq \frac{\mathcal{W}(\mu)}{16\pi},$$

so $\mathcal{H}^2|_{L'} \leq \mu \leq \frac{\mathcal{W}(\mu)}{16\pi} \cdot \mathcal{H}^2|_{L'}$. Now, by [EG92, Section 2.3], for $\mathcal{H}^2$-almost every $x \in \mathbb{R}^3 \setminus L'$ we have

$$\Theta^2(x; \mu) = \limsup_{r \to 0} \frac{\mu(B_r(x))}{\pi\, r^2} \leq \frac{\mathcal{W}(\mu)}{16\pi} \limsup_{r \to 0} \frac{\mathcal{H}^2(B_r(x) \cap L')}{\pi r^2} = 0.$$

In summary, we can take a representative of L which is densely contained in $\operatorname{spt}(\mu)$ and satisfies $\Theta^2(\mu, x) \geq 1$ for all $x \in L$. Now take $x \in \operatorname{spt}(\mu)$, $\delta > 0$ and take $r > 0$ so small that $\Theta^2(\mu; x) \geq \frac{\mu(B_r(x))}{\pi r^2} - \delta$. There exists a sequence of radii r for which the upper limit is attained, and we can take slight perturbations of those, so there exists an uncountable family. For technical reasons, we assume that $\mu(\partial B_r(x)) = 0$, which holds for all but countably many $r > 0$ by Lemma 3.2.1. We take a sequence $x_k \in L$ with $\Theta^2(x_k; \mu) \geq 1$. It follows that

$$\begin{aligned}
\Theta^2(\mu; x) &\geq \frac{\mu(B_r(x))}{\pi r^2} - \delta \\
&= \lim_{k\to\infty} \frac{\mu(B_r(x_k))}{\pi r^2} - \delta \\
&\geq \limsup_{k\to\infty} \left(\Theta^2(\mu; x_k) - \frac{1}{16\pi} \int_{B_r(x_k)} |H|^2 \,\mathrm{d}\mu - \delta \right)
\end{aligned}$$

since the translations of the Radon measure μ by $x_k - x$ converge weakly to the original Radon measure. By taking $r, \delta \to 0$ we obtain $\Theta^2(\mu; x) \geq \limsup_{k\to\infty} \Theta^2(\mu; x) \geq 1$. Thus $\operatorname{spt}(\mu) = \{x \mid \Theta^2(x) \geq 1\}$, so $\operatorname{spt}(\mu)$ has finite $\mathcal{H}^2$-measure and is rectifiable.

Now assume that $\operatorname{spt}(\mu)$ is not connected. Then by Lemma 3.2.6, there are two open sets U_1, U_2 such that

$$\operatorname{spt}(\mu) = \big(\operatorname{spt}(\mu) \cap U_1\big) \cup \big(\operatorname{spt}(\mu) \cap U_2\big), \qquad \operatorname{dist}(U_1, U_2) > 0$$

and neither term is empty. It is easy to see that $\mu_i = \mu|_{U_i}$ is an integral varifold for $i = 1, 2$ with mean curvature given by the mean curvature of μ due to the positive separation of the sets. Thus, by the Li-Yau inequality (3.2.2) we have

$$\mathcal{W}(\mu) \geq \mathcal{W}(\mu_1) + \mathcal{W}(\mu_2) \geq 16\pi + 16\pi.$$

If μ_1 and μ_2 are connected, μ has finitely many connected components, otherwise we iterate this procedure. Since Willmore's energy is bounded, there are at most

$$N \leq \frac{\mathcal{W}(\mu)}{16\pi}$$

connected components. Since there are only finitely many connected components, they are relatively closed, hence compact, hence they have a positive distance and they induce Willmore varifolds by themselves. The diameter bounds are due to Simon [Sim86] for immersed surfaces with an explicit constant due to Topping [Top98]. We adapt the proof to varifolds.

The lower diameter bound follows easily from the first variation identity and Hölder's inequality. Abbreviate $\operatorname{diam}(\mu) := \operatorname{diam}(\operatorname{spt}(\mu))$. Now assume that $0 \in \operatorname{spt}(\mu)$ so that

$|x| \leq \operatorname{diam}(\mu)$ for μ-a.e. $x \in \mathbb{R}^n$ and thus

$$\begin{aligned}
2\,\mu(\mathbb{R}^n) &= \int \operatorname{div}_{T_x\mu} x \, \mathrm{d}\mu \\
&= -\int \langle x, H\rangle \, \mathrm{d}\mu \\
&\leq \left(\int |x|^2 \, \mathrm{d}\mu\right)^{\frac{1}{2}} \left(\int |H|^2 \, \mathrm{d}\mu\right)^{\frac{1}{2}} \\
&\leq \operatorname{diam}(\mu) \sqrt{\mu(\mathbb{R}^n)\, \mathcal{W}(\mu)}\,.
\end{aligned}$$

Now we prove the upper diameter bound. By restricting ourselves to one component, we can assume that μ is connected. First assume that $d := \operatorname{diam}(\mu) < \infty$. Then $\operatorname{spt}(\mu)$ is compact and there exist two points $x, y \in \operatorname{spt}(\mu)$ such that $d = |x - y|$. We can assume that $x = 0$ and $y = d \cdot e_1$. Since $\operatorname{spt}(\mu)$ is connected, for any $t \in [0, d]$ there exists an $x \in \operatorname{spt}(\mu)$ such that $x^1 = t$. Fix $r > 0$ and points $x_1, \dots, x_N$ with $x_i^1 = 2i\, r$ for $0 \leq i \leq N$ and take a maximal such collection, i.e.

$$N \leq \frac{\operatorname{diam}(\operatorname{spt}(\mu))}{2r} \leq N + 1.$$

Then the balls $B_r(x_i)$ are disjoint, so

$$\begin{aligned}
\mu(\mathbb{R}^n) &\geq \sum_{i=0}^{N} \mu(B_r(x_i)) \\
&\geq \pi\, r^2 \sum_{i=0}^{N} \left(\Theta^2(x_i) - \frac{1}{16\pi} \int_{B_r(x_i)} |H|^2 \, \mathrm{d}\mu\right) \\
&\geq \pi\, r^2 \left(N + 1 - \frac{1}{16\pi} \mathcal{W}(\mu)\right) \\
&\geq \pi\, r^2 \left(\frac{\operatorname{diam}(\mu)}{2r} - \frac{1}{16\pi} \mathcal{W}(\mu)\right)
\end{aligned}$$

since by the first part of the proof $\Theta^2(\mu, x_i) \geq 1$. This implies that

$$\frac{\pi}{2} \operatorname{diam}(\mu) \leq \frac{\mu(\mathbb{R}^n)}{r} + \frac{\mathcal{W}(\mu)}{16}\, r.$$

The real-valued function $f(r) = \frac{a}{r} + br$ takes its minimum on the positive half-axis at $r_{min} = \sqrt{\frac{a}{b}}$ (for $a, b > 0$) and $f(r_{min}) = 2\sqrt{ab}$, so the optimal bound we can derive with this procedure is

$$\frac{\pi}{2} \operatorname{diam}(\mu) \leq 2 \sqrt{\mu(\mathbb{R}^n)\, \frac{\mathcal{W}(\mu)}{16}}.$$

Thus we have shown the bound on the diameter in the case that already $\operatorname{diam}(\operatorname{spt}(\mu)) < \infty$.

If the diameter is unbounded, we can choose the number N arbitrarily large and $r = 1$. The same estimate as before in the form

$$\mu(\mathbb{R}^n) \geq \pi N - \frac{\mathcal{W}(\mu)}{16}$$

leads to a contradiction with the assumption that $\mu(\mathbb{R}^n) < \infty$. This establishes the upper diameter bound and thus compactness. □

We give a counterexample to the structure theorem for k-varifolds in $\mathbb{R}^n$ which have mean curvatures in L^p for $p < k$.

Example 3.2.7. Let $k \leq n-1$ and $\Sigma = \partial B_1^{\mathbb{R}^{k+1}}(0) \times \{0\}$ be an inclusion of the k-sphere into $\mathbb{R}^n$ and let q_i be a dense sequence in $\mathbb{R}^n$, r_i a sequence of real numbers such that $r_i \to 0$ and

$$\sum_{i=1}^{\infty} \frac{1}{|\log r_i|} < \infty.$$

Then define

$$\mu = \sum_{i=1}^{\infty} \mathcal{H}^k|_{q_i + r_i \cdot \Sigma}.$$

Every truncated sum is a varifold corresponding to an immersed manifold for which we can easily compute volume and mean curvature, so taking the limit and using the lower semi-continuity of the L^p-norm of mean curvature, we get

$$\mu(\mathbb{R}^n) = (k+1)\omega_{k+1} \sum_{i=1}^{\infty} r_i^k < \infty, \qquad \int_{\mathbb{R}^n} |H|^p \,\mathrm{d}\mu \leq (k+1)\omega_{k+1}\, k^p \sum_{i=1}^{\infty} r_i^{k-p} < \infty$$

where ω_d is the volume of the unit ball in d dimensions, so μ is an integral varifold with mean curvature H in $L^p(\mu)$ for all $p < 2$ but $H \notin L^2(\mu)$ since clearly $\mathrm{spt}(\mu) = \mathbb{R}^n$. To see this, take any point $x \in \mathbb{R}^n$ and $r > 0$. It suffices to show that $\mu(B_r(x)) > 0$. All but finitely many r_i satisfy $r_i < r/2$ and there since q_i is a dense sequence in $\mathbb{R}^n$, there is $i \in \mathbb{N}$ such that

$$r_i < r/2, \qquad |x - q_i| < r/2 \qquad \Rightarrow \qquad \mu(B_r(x)) \geq \mathcal{H}^k(q_i + r_i \cdot \Sigma) > 0.$$

Alternatively, if instead we take $q_i = i\, e_1$ and make sure that $r_i < 1/2$ for all $i \in \mathbb{N}$, we obtain an integral varifold μ with mean curvature $H \in L^p(\mu)$ for all $p < 2$ for which $\mathrm{spt}(\mu)$ has infinitely many connected components.

3.2.3 Convergence and Connectedness

Similar methods as in the proof of the structure theorem can be used for the proof of the convergence result. An alternative proof of Theorem 3.1.2 using phase-field methods instead of geometric measure theory can be found in Section 5.3.3.

Proof of Theorem 3.1.2. Since the measures are supported in a bounded domain, a Hausdorff limit $K = \lim_{k\infty} \mathrm{spt}(\mu_k)$ exists up to a subsequence [KP08, Theorem 1.6.6]. Take $x \in \mathrm{spt}(\mu)$. Then for all $r > 0$, we have $\mu(B_r(x)) > 0$ and since

$$0 < \mu(B_r(x)) \leq \liminf_{k\to\infty} \mu_k(B_r(x))$$

by the definition of Radon measure convergence, we find $\mu_k(B_r(x)) > 0$ for all sufficiently large k. This means that $\mathrm{spt}(\mu_k) \cap B_r(x) \neq \emptyset$ and since the property holds for all $r > 0$, there exists a sequence $x_k \in \mathrm{spt}(\mu_k)$ such that $x_k \to x$. This implies that $x \in K$ and thus $\mathrm{spt}(\mu) \subset K$.

Now take $x \in K \setminus \mathrm{spt}(\mu)$. We choose a further subsequence such that the measures

$$\alpha_k(B) = \int_B |H|^2 \,\mathrm{d}\mu_k$$

which localise the Willmore energy of μ_k have a limiting measure α, as the measures are uniformly bounded and we can use the compactness theorem for Radon measures. Now we can take a sequence of points $x_k \in \mathrm{spt}(\mu_k)$ such that $x_k \to x$ and $\Theta^2(\mu_k, x_k) \geq 1$. Then (3.2.1) shows that

$$\begin{aligned} 1 &\leq \frac{\mu_k(B_r(x_k))}{\pi\, r^2} + \frac{1}{16\pi} \int_{B_r(x_k)} |H_k|^2 \,\mathrm{d}\mu_k \\ &\leq \frac{\mu_k(B_{2r}(x))}{\pi r^2} + \frac{1}{16\pi} \alpha_k(B_{2r}(x)). \end{aligned}$$

Choosing r so small that $\mu(B_{3r}(x)) = 0$, and taking r such that $\alpha(\partial B_{2r}(x)) = 0$, we find that the first term on the right vanishes as $k \to \infty$ and are left with

$$\alpha(B_{2r}(x)) = \lim_{k\to\infty} \alpha_k(B_{2r}(x)) \geq 16\pi.$$

We can now take $r \to 0$ and are left with $\alpha(\{x\}) \geq 16\pi$. Since α is a finite Radon measure, there are only finitely many such points and thus $K \setminus \mathrm{spt}(\mu)$ is finite. Let us now consider the three cases in which we claimed that $K = \mathrm{spt}(\mu)$.

If $\mathrm{spt}(\mu_k)$ is connected for all $k \in \mathbb{N}$, then also its Hausdorff limit K is connected. Thus there cannot be any isolated points and thus $\mathrm{spt}(\mu_k) \to \mathrm{spt}(\mu)$ in Hausdorff distance. If we allowed $\mu = 0$, then the Hausdorff limit could be a single point. In the second case, we take

$$U = \{x \in \Omega \mid \operatorname{dist}(x, \operatorname{spt}(\mu)) < \delta\} \cup \bigcup_{i=1}^N B_\delta(x_i)$$

for some $\delta > 0$ so small that all sets in the union are disjoint. Since $\operatorname{spt}(\mu_k) \to K$ in Hausdorff distance, we know that $\operatorname{spt}(\mu_k) \subset U$ for all sufficiently large k. Now we take the modified sequence

$$\widetilde{\mu}_k = \mu_k|_{\{\operatorname{dist}(\cdot, \operatorname{spt}\mu) < \delta\}}$$

which satisfies $\widetilde{\mu}_k \overset{*}{\rightharpoonup} \mu$. The above arguments combined with the lower semi-continuity of Willmore's energy show that

$$\mathcal{W}(\mu_k) \geq \mathcal{W}(\widetilde{\mu}_k) + 16N\pi,$$

thus

$$\mathcal{W}(\mu) + 16\pi > \limsup_{k\to\infty} \mathcal{W}(\mu_k) \geq \liminf_{k\to\infty} \mathcal{W}(\widetilde{\mu}_k) + 16N\pi \geq \mathcal{W}(\mu) + 16N\pi$$

whence $N = 0$. Finally, assume that $H_k \in L^p(\mu_k)$ uniformly in k for some $p > 2$. Then take an isolated point $x \in K \setminus \operatorname{spt}(\mu)$ and calculate as before

$$\begin{aligned} 16\pi &\leq \liminf_{k\to\infty} \int_{B_r(x)} |H_k|^2 \,\mathrm{d}\mu_k \\ &\leq \liminf_{k\to\infty} \left(\int_{B_r(x)} |H_k|^p \,\mathrm{d}\mu_k \right)^{\frac{2}{p}} \mu(B_r(x))^{1-\frac{2}{p}}. \end{aligned}$$

Since the first term is uniformly bounded and the second one goes to zero as $r \to 0$ (since μ does not have atoms), we have reached a contradiction. □

It is easy to give an example $\mu_k \overset{*}{\rightharpoonup} \mu$ with additional points in the Hausdorff limit. Namely, Take $M_k \equiv M \cup \partial B_{r_k}(x)$ for some M, $x \notin M$ and $r_k \to 0$. Then, if μ denotes the varifold induced by M, we have

$$\mu_k \rightharpoonup \mu, \qquad \operatorname{spt}(\mu) = M, \qquad \lim_{k\to\infty} M_k = M \cup \{x\}, \qquad \mathcal{W}(M_k) \equiv \mathcal{W}(M) + 16\pi.$$

A diffuse analogue of this example will be discussed in Example 5.4.2.

Proof of Corollaries 3.1.3 and 3.1.4. The compactness of varifolds satisfying these area and energy bounds with respect to varifold convergence is given by Allard's theorem. The equivalence of compactness and sequential compactness follows from the fact that varifolds can be interpreted as objects in the dual space of continuous functions on $\overline{\Omega} \times G(n,k)$, which is separable. Thus the weak* topology, which is the topology of varifold convergence, is metrisable. The closedness of the class of varifolds with connected support follows from the previous theorem. The existence of a minimiser follows by the direct method of the calculus of variations. The subclasses are closed by definition. □

The Corollaries could easily be extended to $p \geq 2$. The Hausdorff convergence result does not hold for $1 < p < 2$ as the following example shows for general $1 \leq d \leq n-1$ and $p < k$. Let Σ be an inclusion of the d-sphere into $\mathbb{R}^n$ and define

$$\mu_k = \sum_{1 \leq i_1, \dots i_1 \leq k} \mathcal{H}^k|_{(i_1/k, \dots, i_k/k) + r_k \cdot \Sigma}$$

for some $r_k \ll 1/k$. Then we have

$$\mu_k(\mathbb{R}^n) = k^n (d+1)\omega_{d+1}\, r_k^d, \qquad \mathcal{W}(\mu_k) = (d+1)\omega_{d+1} k^n r_k^{d-p}$$

which, for suitably chosen r_k satisfies

$$\mu_k \to 0, \qquad \mathrm{spt}(\mu_k) \to [0,1]^d.$$

However, the non-convergence result is due to small components collapsing away, and it seems that the class of varifolds with connected support should still be closed, even for $k-1 < p < k$ since this would still prevent thin pipes from collapsing away as in the example below, which shows that the class of integral varifolds with connected support is not closed if only a uniform bound on the total variation measure $\delta V(\mathbb{R}^n) \leq C$ is assumed.

Example 3.2.8. For $k = 2$, it is not enough to assume that δV is bounded. We construct a sequence of dumbbell figures with thin middle segments that collapse away. The measure-limit consists of two (almost) spheres with positive spatial separation and has disconnected support while every single approximating manifold was connected. The total mean curvature of the sequence remains bounded.

Take two spheres of radius 1 centred around $(0,0,-2)$ and $(0,0,2)$ in $\mathbb{R}^3$. Make the spheres flat in a C^2-way at the inward points $(0,0,-1)$ and $(0,0,1)$ like in the proof of Lemma 3.2.11. Now pick some manifold M which satisfies the following properties:

1. M is contained in a slice of $\mathbb{R}^3$: $M \subset \mathbb{R}^3 \cap \{0 < x_3 \leq 2\}$,

2. the left half of M is a cylinder:

$$M \cap \{x_3 < 1\} = \{x \in \mathbb{R}^3 \mid 0 < x_3 < 1, \quad x_1^2 + x_2^2 = 1\},$$

3. outside a cylinder, M is a flat plane:

$$M \cap \{x_1^2 + x_2^2 > 2\} = \{x_3 = 2, x_1^2 + x_2^2 > 2\} \qquad \text{and}$$

4. $M \cap \{0 < x_3 \leq 2, x_1^2 + x_2^2 \leq 3\}$ is C^2-smooth and compact.

Now we construct the surfaces Σ_k by connecting the modified spheres $S_\pm$. Consider only k so large that $B_{4/k}(0,0,\pm 1) \cap S_\pm$ is flat. Then we can replace

$$B_{3/k}(0,0,1) \cap S_+ \qquad \text{by} \qquad \frac{1}{k} \cdot (M \cap B_3(0)) + (0,0,1-2/k)$$

and do the same thing on the opposite side. Now we may connect the ends of the two open cylindrical ends using the cylinder

$$Z_k = \{-1 + 2/k \leq x_3 \leq 1 - 2/k, \quad x_1^2 + x_2^2 = 1/k\}.$$

Observe that

$$\int_{\Sigma_k} |H| \,\mathrm{d}\mathcal{H}^2 = 2\int_{S^+} |H| \,\mathrm{d}\mathcal{H}^2 + \frac{2}{k}\int_M |H| \,\mathrm{d}\mathcal{H}^2 + \int_{Z_k} |H| \,\mathrm{d}\mathcal{H}^2.$$

The first term is independent of k, the second one vanishes asymptotically since the mean curvature of the rescaled surface $H_{\frac{1}{k}\cdot M} = k \cdot H_M$ becomes large linearly, but the surface area measure becomes small quadratically in $1/k$ since M has dimension 2:

$$\int_{\frac{1}{k}\cdot M} |H_{\frac{1}{k}\cdot M}| \,\mathrm{d}\mathcal{H}^2 = \frac{1}{k} \cdot \int_M |H| \,\mathrm{d}\mathcal{H}^2.$$

The mean curvature of the cylinder Z_k is the principal curvature k at every point and its area is $2\pi/k \cdot (2 - 2/k)$, so the curvature integral is bounded by 4π and in total, $\int_{\Sigma_k} |H| \,\mathrm{d}\mathcal{H}^2 \leq C$ where C is a constant slightly bigger than 36π (the Willmore integral of a sphere being 16π).

So the total energy along the sequence remains bounded. It is easy to see that

$$\mathcal{H}^2|_{\Sigma_k} \to \mathcal{H}^2_{S_+ \cup S_-}$$

in the sense of Radon measures, so that we approximate a measure with disconnected support by integral varifolds with connected support and uniformly bounded first variations.

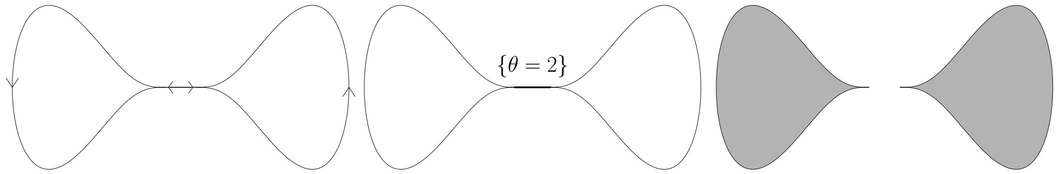

Figure 3.1: Three interpretations of the same picture. Left: An immersed curve γ with only tangential self-contact. Middle: A varifold with a segment of multiplicity two. Right: The boundary of a Caccioppoli set. The segment of higher multiplicity is a ghost interface in the right picture and cannot be seen in the BV-framework.

3.2.4 Limits of C^2-boundaries

To facilitate the transition into a phase-field setting and shift the focus on varifolds which arise as the limit of C^2-boundaries, we give a version of Theorem 3.1.6 which is adapted to this setting. Without the connectedness, it has also been formulated in [DMR14, Proposition 1].

Theorem 3.2.9. *Let $E_k \Subset \Omega \Subset \mathbb{R}^n$ be a sequence of sets with boundaries $\partial E_k \in C^2$ such that*

$$\mathcal{H}^{n-1}(\partial E_k) \to S > 0, \qquad \liminf_{k\to\infty} \mathcal{W}(\partial E_k) < \infty.$$

Denote the characteristic functions χ_{E_k} by u_k and the unit-density varifolds associated to ∂E_k (i.e. the Radon measures $|\nabla \chi_{E_k}|$) by μ_k. Then there exist $u \in BV(\Omega, \{0,1\})$ and an integral $(n-1)$-varifold μ such that up to a subsequence

1. *$u_k \to u$ strongly in $L^1(\Omega)$,*
2. *$\mu_k \rightharpoonup \mu$ as varifolds,*
3. *$|\nabla u| \leq \mu$*
4. *$\operatorname{spt}(\mu) \subset \overline{\Omega}$ and $\mu(\overline{\Omega}) = S$ and*
5. *$\mathcal{W}(\mu) \leq \liminf_{k\to\infty} \mathcal{W}(\mu_k)$.*

If $n = 2, 3$ and the boundaries ∂E_k are connected, then also $\operatorname{spt}(\mu)$ is connected.

The theorem follows directly from Allard's compactness theorem in the disconnected case and Corollary 3.1.3 and the compactness theorem for BV-functions. Since u_k is uniformly bounded in $L^\infty(\Omega)$, convergence actually holds for all $p < \infty$. The connection between $|\nabla u|$ and μ follows by localising the result to any open set $\Omega' \subset \Omega$ and using the properties of BV-functions and Radon measures. In the BV-setting, extinctions can occur between different parts of the gradient when boundaries with opposite orientation meet, while these sum up to multiplicity two in the varifold setting. We establish a stronger connection, which we believe to be new.

Theorem 3.2.10. *Assume the conditions of Theorem 3.2.9. Then $u = \chi_E$ is the characteristic function of a set $E \subset \Omega$ and we have*

$$\partial^* E \approx \{x \in \mathrm{spt}(\mu) \mid \theta(x) \in 2\mathbb{Z} + 1\}.$$

Here $\partial^* E$ denotes the reduced boundary of E [Giu84] and $\approx$ means that we may add a set of $\mathcal{H}^{n-1}$-measure zero on both sides.

Proof. The theorem follows from the compactness theorem for oriented varifold hypersurface. Every boundary ∂E_k induces an oriented varifold V_k^o with the orientation canonically given by the outer normal. Thus there exists an oriented varifold V^o such that $V_k^o \rightharpoonup V^o$. By projection of the convergence, we see that μ is the mass measure of the associated (unoriented) varifold. If we pick an orienting normal vector field ξ on $\mathrm{spt}(\mu)$, we find that

$$V^o(f) = \int f(x,\xi)\,\theta^+ + f(x,-\xi)\,\theta^- \,\mathrm{d}\mathcal{H}^{n-1}$$

for integer valued densities $\theta^\pm$. Inserting functions f for which $f(x,-\xi) = f(x,\xi)$, we observe that the oriented varifold has density $\theta = \theta^+ + \theta^-$ and using test functions f given by

$$f(x,\xi) = \langle v_x, \xi\rangle$$

we observe that $\theta^+ - \theta^-$ is the density associated with the essential boundary, which is either 1 if $x \in \partial^* E$ or 0 otherwise. Since both θ^+, θ^- are integer-valued and either $\theta^+ = \theta^-$ or $\theta^+ - \theta^- \in \{-1, 1\}$, the theorem is proven.

□

The statement that $\theta^+ = \theta^-$ on $\mathrm{spt}(\mu) \setminus \partial^* E$ is actually stronger than what we claimed.

3.2.5 Genus

In this section, we will prove Theorem 3.1.5. First we flatten the unit sphere slightly to have a flat segment on which we can easily glue two surfaces together. We denote by $D_r = B_r(0)$ the disc of radius r around the origin in $\mathbb{R}^2$.

Lemma 3.2.11 ("flattening a sphere")**.** *Let $\varepsilon > 0$. Then there exists $\delta_0 > 0$ such that for* *every* $0 < \delta < \delta_0$ *there exists a convex closed* C^∞*-sphere* $M_\varepsilon \subset B_1(0)$ *in* $\mathbb{R}^3$ *such that*

$$\begin{aligned} M_\varepsilon \cap [D_{2\delta} \times (0,1)] &= \{x_3 = 1 - 3\delta\} \cap [D_{2\delta} \times (0,1)]\,, \\ M_\varepsilon \setminus [D_{4\delta} \times (0,1)] &= S^2 \setminus [D_{4\delta} \times (0,1)] \end{aligned}$$

and

$$\mathcal{W}(M_\varepsilon) < 4\pi + \varepsilon.$$

Proof. Take $f \in C^\infty(-1,1)$, $f(t) = \sqrt{1-t^2}$ and

$$f_\delta : B_1(0) \to \mathbb{R}, \qquad f_\delta(x) = f \circ r_\delta(|x|) = \sqrt{1 - r_\delta^2(|x|)}$$

where $r_\delta \in C^\infty[0,1]$ satisfies

$$r_\delta(t) = \begin{cases} 3\delta & t \le 2\delta \\ t & t \ge 4\delta \end{cases}, \qquad 0 \le r_\delta' \le 1, \qquad 0 \le r_\delta'' \le \frac{4}{\delta}.$$

Then

$$\partial_i f_\delta(x) = (f' \circ r_\delta)\, r_\delta' \,\frac{x_i}{|x|} \tag{3.2.8}$$

$$\partial_{ij}^2 f_\delta(x) = (f'' \circ r_\delta)\,(r_\delta')^2 \,\frac{x_i x_j}{|x|^2} + (f' \circ r_\delta)\, r_\delta'' \,\frac{x_i x_j}{|x|^2} + (f' \circ r_\delta)\, r_\delta' \left[\frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}\right]. \tag{3.2.9}$$

It is easy to see that $D^2 f_\delta$ is negative semi-definite since all three terms in the sum in (3.2.9) are negative semi-definite, so f_δ is concave. Thus

$$M^\delta := \{x \in S^2 \mid x_3 \le 0\} \cup \{(x, f(x)) \mid x \in D_1\}$$

is a convex sphere. The topological type can also be found through the Gaussian curvature integral which coincides for f and f_δ since their boundary values agree (Gauss-Bonnet Theorem).

When we denote $f_0(x) = \sqrt{1 - |x|^2}$, we observe that $|f_\delta - f_0| \le 3\delta$ and

$$\begin{aligned} |\partial_i f_\delta - \partial_i f_0| &= \left[(f' \circ r_\delta)\, r_\delta' - (f' \circ r_0)\right] \frac{x_i}{|x|} \\ |\partial_{ij}^2 f_\delta - \partial_{ij}^2 f_0| &= \left[(f'' \circ r_\delta)\,(r_\delta')^2 - (f'' \circ r_0|)\right] \frac{x_i x_j}{|x|^2} + (f' \circ r_\delta)\, r_\delta'' \,\frac{x_i x_j}{|x|^2} \\ &\quad - \left[(f' \circ r_\delta) r_\delta' - (f' \circ r_0)\right] \left[\frac{x_i x_j}{|x|^3} - \frac{\delta_{ij}}{|x|}\right] \end{aligned}$$

The first term is small since $x_i/|x|$ is bounded and $f'(0) = 0$, so we can choose δ small enough to make f_δ and f_0 close in $C^1(\overline{B_1(0)})$. Curvature prevents us from making them C^2-close, but they are clearly $W^{2,2}$-close since

$$\left|\left|\left[(f''\circ r_\delta)\,(r_\delta')^2-(f''\circ r_0)\right]\frac{x_ix_j}{|x|^2}\right|\right|_{L^2}\leq 2\,||f''||_{L^\infty(-4\delta,4\delta)}\sqrt{\pi\,(4\delta)^2}$$

$$\left|\left|(f'\circ r_\delta)\,r_\delta''\,\frac{x_ix_j}{|x|^2}\right|\right|_{L^2}\leq ||f'||_{L^\infty(-4\delta,4\delta)}\left(\int_{B_{4\delta}(0)}(4/\delta)^2\,\mathrm{d}x\right)^{1/2}$$

$$\left|\left|\left[(f'\circ r_\delta)r_\delta'-(f'\circ r_0)\right]\left[\frac{x_ix_j}{|x|^3}-\frac{\delta_{ij}}{|x|}\right]\right|\right|_{L^2}\leq 2\left(\int_{B_{4\delta}(0)}\left(\frac{2}{|x|}\right)^2\cdot 2\,\left[\,||f''||_{L^\infty(-4\delta,4\delta)}|x|\,\right]^2\,\mathrm{d}x\right)^{1/2}$$

all become small linearly with δ. Since mean curvature H_f, volume element $\mathrm{d}s_f$ and Willmore integrand w_f of the graph

$$\Gamma_f=\{(x,f(x))\mid x\in B_1(0)\subset\mathbb{R}^2\}$$

of f are given by

$$\begin{aligned}H_f&=\frac{(1+f_y^2)\,f_{xx}-2\,f_xf_y\,f_{xy}+(1+f_x^2)\,f_{yy}}{(1+f_x^2+f_y^2)^{3/2}},\\ \mathrm{d}s_f&=\sqrt{1+f_x^2+f_y^2}\qquad\text{and}\end{aligned}$$

$w_f=H_f^2\,\mathrm{d}s_f$, we see that $||w_f-w_g||_{L^1}$ is small if $|f-g|_{C^1}$ and $||f-g||_{W^{2,2}}$ are both small for some $g\in W^{2,2}(D_1)$. So we can chose δ small enough to make this as small as we need for $g=f_\delta$. □

Remark 3.2.12. The radial symmetry of the sphere simplifies the calculations above, but in fact any C^2-surface can be locally flattened around a point when written as a graph over its tangent space. This might be useful for a more general argument when minimising varifolds have double points.

Next we create the handles by which we will connect spheres.

Lemma 3.2.13 ("flattening a catenoid")**.** *Let $R\gg 1$. Then there exists a connected orientable C^∞-manifold $\Sigma\subset\mathbb{R}^3$ such that*

$$\Sigma\setminus Z_R\quad=\quad\big(\{x_3=R+1/2\}\cup\{x_3=-(R+1/2)\}\big)\setminus\overline{Z_R}$$

where Z_R is the cylinder $Z_R=D_{\cosh(R+1)}\times(-R+1/2,R+1/2)$ and furthermore

$$\mathcal{W}(\Sigma)=O(e^{-2R}),\qquad\int_\Sigma K\,\mathrm{d}\mathcal{H}^2=-4\pi$$

where K denotes the Gaussian curvature of Σ.

Proof. Define the surface of revolution

$$\Sigma = \left\{ \begin{pmatrix} f(t)\,\cos\phi \\ f(t)\,\sin\phi \\ g(t) \end{pmatrix} \middle|\; t, \phi \in \mathbb{R} \right\}.$$

If $f(t) = \cosh(t)$ and $g(t) = t$, Σ is the usual catenoid. We consider $f = \cosh$ and an even C^∞-function g satisfying

$$g(t) = \begin{cases} t & |t| \le R \\ R + 1/2 & R \ge R+1 \end{cases}, \qquad 0 < g'(t) \le 1 \text{ for } |t| < R{+}1, \qquad -4 \le g''(t) \le 0 \text{ for } t \ge 0.$$

Then clearly Σ is connected as the continuous image of a connected set and given as the union of two planes outside the cylinder Z_R. The volume element $\mathrm{d}s$ and the mean curvature of Σ are

$$\begin{aligned} \mathrm{d}s &= f\sqrt{(g')^2 + (f')^2}, \\ H &= \frac{ff''g' - ff'g'' - g'(f')^2 - (g')^3}{f\;[(f')^2 + (g')^2]^{3/2}} \\ &= \frac{g'(ff'' - (f')^2 - 1) + g'(1 - (g')^2) - ff'\,g''}{f\;[(f')^2 + (g')^2]^{3/2}} \\ &= \frac{g'(1 - (g')^2) - ff'\,g''}{f\;[(f')^2 + (g')^2]^{3/2}} \end{aligned}$$

since $ff'' - (f')^2 - 1 = 0$ for $f = \cosh$. Thus

$$\begin{aligned} \mathcal{W}(\Sigma) &= 2\pi \int_0^\infty \frac{\left[g'\left(1 - (g')^2\right) - ff'\,g''\right]^2}{f\;[(f')^2 + (g')^2]^{5/2}}\,\mathrm{d}t \\ &\le 4\pi \int_R^{R+1} \frac{(g')^2(1 - (g')^2)^2}{f\;[(f')^2 + (g')^2]^{5/2}} + \frac{f\,(f')^2(g'')^2}{[(f')^2 + (g')^2]^{5/2}}\,\mathrm{d}t \\ &\le 4\pi \int_R^{R+1} \frac{1}{f\,(f')^5}\,\mathrm{d}t + 4\pi \int_R^{R+1} \frac{f\,(g'')^2}{|f'|^3}\,\mathrm{d}t \\ &= O(e^{-2R}). \end{aligned}$$

It remains to show that the total Gaussian curvature is -4π. When we orient Σ by choice of the normal vector

$$\nu = \frac{1}{f\,\sqrt{(f')^2+(g')^2}} \begin{pmatrix} -f\,g'\,\cos\phi \\ -f\,g'\,\sin\phi \\ f\,f' \end{pmatrix}$$

we see that every unit vector $\nu = (\sin\theta)e_\phi + (\cos\theta)e_z \neq (0,0,\pm 1)$ is the normal ν_x at the unique point $x \in \Sigma$ determined by the ϕ-coordinate and t given by

$$\tan\theta = -\frac{g'(t)}{f'(t)}.$$

This is uniquely solvable except for $\tan\theta = 0$ by construction of g. We know that

$$K = \frac{-(g')^2 f'' + f'g'g''}{f\,\left[(f')^2+(g')^2\right]^2} \leq 0 \qquad (\text{since } f'' \geq 0, f'g'' \leq 0)$$

is the determinant of the Gauss map $G : \Sigma \to S^2$, $G(x) = \nu_x$, so

$$4\pi = \mathcal{H}^2(S^2) = \mathcal{H}^2(G(\Sigma)) = \int_\Sigma |K|\,\mathrm{d}\mathcal{H}^2 = -\int_\Sigma K\,\mathrm{d}\mathcal{H}^2.$$

□

Now we are ready to prove this section's main statement.

Proof of Theorem 3.1.5. We first give the proof for $m = 2$. Let $\beta > 0$ to be chosen later depending on $\varepsilon, \delta > 0$. Take M_β constructed like in Lemma 3.2.11, $\delta > 0$ such that M_β coincides with the plane $\{x_3 = 1 - 3\delta\}$ inside the cylinder $D_{2\delta} \times (0,1)$. We may specify δ to be taken sufficiently small later. Take $0 < \rho < \delta/2$ such that there are $g+1$ points $x_1, \dots, x_{g+1}$ in $D_{\delta/2}$ such that the discs $D_\rho(x_i)$ are pairwise disjoint.

Choose $R > 0$ and Σ like in Lemma 3.2.13 such that $\mathcal{W}(\Sigma) = O(e^{-2R}) < \beta$. Then choose $\eta > 0$ such that $\eta\cosh(R+1) < \rho$ and $\eta R < \delta^3$. Finally, define

$$r = \frac{1 - 3\delta - (2R+1)\eta}{1 - 3\delta} < 1, \qquad \tilde{M} = M_\beta \cup r\cdot M_\beta.$$

Since M_β is convex, this is a smooth embedded manifold. By construction, inside the cylinders

$$Z_i := D_\rho(x_i) \times \{0 < z < 1\}, \qquad i = 1, \dots, g+1$$

$\tilde{M}$ is given by the union of the planes $\{z = 1 - 3\delta\}$ and $\{z = 1 - 3\delta - (2R+1)\eta\}$ which have separation $(2R+1)\eta$. Since the Z_i are disjoint, we can replace $\tilde{M}$ inside each cylinder by

$$x_i + \frac{1+r}{2}\,e_z + \eta\cdot\big(\Sigma \cap Z_R\big).$$

We call the resulting manifold M. It is clear that M is a connected surface. Since both the total curvature integral and Willmore's energy are invariant under spatial rescaling and since M is flat on the remaining segments, we have

$$\begin{aligned} \mathcal{W}(M) &= 2\,\mathcal{W}(M_\beta) + (g+1)\,\mathcal{W}(\Sigma) && < (g+3)\,\beta \\ \int_M K\,\mathrm{d}\mathcal{H}^2 &= 2\int_{M_\beta} K\,\mathrm{d}\mathcal{H}^2 + (g+1)\int_\Sigma K\,\mathrm{d}\mathcal{H}^2 = 2\cdot 4\pi + (g+1)\cdot(-4\pi) = 4\pi\,(1-g) \end{aligned}$$

so that M is a closed smoothly embedded orientable genus g surface with small Willmore energy. Unfortunately, $M \not\subset B_1(0)$ and it is not clear after our modifications whether $\mathcal{H}^2(M) = 8\pi$. If $\mathcal{H}^2(M) > 8\pi$, we only need to choose $\beta = \varepsilon/(g+3)$ and set

$$M_g = \sqrt{\frac{8\pi}{\mathcal{H}^2(M)}}\cdot M. \tag{3.2.10}$$

The more complicated case is $\mathcal{H}^2(M) \leq 8\pi$. Then at least

$$\mathcal{H}^2(M) \geq \left(1+r^2\right)\mathcal{H}^2\left(S^2 \setminus [D_{4\delta}\times(0,1)]\right) \geq 8\pi - C\,\delta^2$$

since $\eta R < \delta^3$ and thus $r \geq 1-\delta^2$. Now consider only the inner sphere, which is still spherical around its south pole. Take a function $h \in C_c^\infty(D_r)$ on a small disc such that $h \geq 0$ and $h \not\equiv 0$. Then we may replace a neighbourhood of the south pole of the inner sphere by

$$\tilde\Sigma^t = \left\{\left(x, -\sqrt{r^2-|x|^2} + t\,h\left(\frac{x}{\alpha\sqrt{t}}\right)\right)\,\middle|\, x\in B_r(0)\right\}.$$

The resulting surface is denoted by M^t. Again, this does not change the topological type, but it changes the area and the Willmore functional by

$$\mathcal{H}^2(M^t) \geq \mathcal{H}^2(M) + c\,t^2, \qquad \mathcal{W}(M_t) \leq \mathcal{W}(M) + C\,t$$

as is computed in the proof of [MR14, Proposition 2], at least for suitable spherically symmetric h. Thus we can take $t = O(\delta)$ such that $\mathcal{H}^2(M^t) > 8\pi$ and define M_g again by (3.2.10), this time choosing both β and δ small enough depending on $\varepsilon > 0$.

In the case of $m\in\mathbb{N}$, $m \geq 3$, we simply consider m concentric spheres and connect them by $m+g-1$ catenoids. The modification at the south pole can always be done only for the innermost sphere. To picture that this procedure induces the correct topology, consider first connecting the outer spheres by $g+1$ catenoids. Then we connect the third sphere to the second by one catenoid. This, however, only blows up a small topological disc to a large one since the union of a catenoid and a sphere is homeomorphic to a sphere with a small disc around the north pole removed, i.e. a disc. □

Remark 3.2.14. If we fix a genus g, then we can even find a C^2-smooth map $f : (0,\varepsilon)\times M \to \mathbb{R}^3$ which maps (t, M) to M_t constructed above. In particular, $f(t, M)$ is a C^∞-smooth manifold for all $t \in (0,\varepsilon)$. Clearly, the images converge as varifolds to an m-fold covered sphere as $t \searrow 0$. We can continue the evolution past the m-fold covered sphere in various ways. This describes a singularity in a geometric flow which may occur with decreasing Willmore energy in finite time. It is unclear whether such singularities may appear in the gradient flow of the Willmore functional.

Remark 3.2.15. It is a simple exercise to re-write the calculations above for any $1 < p < \infty$. It is also easy to approximate the varifold induced by

$$\partial B_1(0,0,1) \cup \partial B_1(0,0,-1)$$

by connected surfaces in the same way as above. This shows that no stronger notion of connectedness for μ than topological connectedness of $\mathrm{spt}(\mu)$ can be enforced, as the connection can go through a single point in this case. By Hölder's inequality, we see that connectedness is stable at least for $p \geq 2$, and we have seen in Example 3.2.8 that it is not stable for $p = 1$. The Hausdorff-convergence result in the case $p = 2$ ceases to be valid for $1 < p < 2$ (a counterexample is constructed as in Example 3.2.7), but we conjecture that the stability of connectedness should be true in this range as well.

3.2.6 Application to Curvature Energies

Let us use Theorem 3.1.5 to illustrate phenomena occurring when we minimise curvature energies under area constraint in the unit ball.

Proof of Corollary 3.1.6. By Theorem 3.1.5, there exists a sequence $N_k \in \mathscr{M} := \mathscr{M}_{g,4m\pi,B_1(0)}$ such that $\mathcal{W}(N_k) < 4m\pi + 1/k$. So

$$\inf\left\{\mathcal{W}(M) \,\middle|\, M \in \mathscr{M}\right\} \leq 4m\pi.$$

Now let M_k be a minimising sequence in $\mathscr{M}$. Take a subsequence of M_k. Due to Allard's compactness theorem [All72], there exists an integral varifold V with square integrable mean curvature H such that a further subsequence converges to V as varifolds and

$$\mathcal{W}(V) \leq \limsup_{k\to\infty} \mathcal{W}(M_k) = \inf\left\{\mathcal{W}(M) \,\middle|\, M \in \mathscr{M}\right\} \leq 4m\pi.$$

The convergence of varifolds implies the convergence of their mass measures as Radon measures, so $\mu_V(\overline{B_1(0)}) = 4m\pi$ and $\mu_V(\mathbb{R}^n \setminus \overline{B_1(0)}) = 0$, whence $\mu_V = m \cdot \mathcal{H}^2|_{S^2}$ by Lemma 3.2.3. Since every subsequence has a further subsequence which converges to the same limit and varifold convergence is topological (as a convergence of Radon measures), we see that the whole sequence converges. □

Proof of Corollary 3.1.7. Since $\int_M |H|^2 \,\mathrm{d}\mathcal{H}^2 > 4\mathcal{H}^2(M)$ by Lemma 3.2.3 for manifolds in $B_1(0)$, a manifold M satisfying

$$4\chi_H \mathcal{H}^2(M) \geq \mathcal{E}(M) = \chi_H \int_M |H|^2 \,\mathrm{d}\mathcal{H}^2 + 4\pi\,\chi_K \int_M K \,\mathrm{d}\mathcal{H}^2 \geq 4\chi_H \mathcal{H}^2(M) + 4\pi\chi_K\,(1-g)$$

has genus $g = 0$. As before

$$\inf\left\{\mathcal{E}(M) \mid M \in \mathscr{M}_{4\pi m, B_1(0)}\right\} = 16\pi m\,\chi_H - 4\pi\,|\chi_K|$$

is realised by smooth spheres converging to a multiplicity m sphere. As noted before, a smooth multiplicity m-sphere $V := m \cdot \mathcal{H}^2|_{S^2 \otimes TS^2}$ has total Gaussian curvature $\int K \,\mathrm{d}V = 4\pi m$. Thus

$$\mathcal{E}(V) = 4\pi m\,(4\chi_H - |\chi_K|) < 16\pi m\,\chi_H - 4\pi\,|\chi_K| = \lim_{k\to\infty} \mathcal{E}(M_k).$$

If $\chi_K < -4\chi_H$, then multiplicity m-spheres illustrate that $\mathcal{E}$ is not bounded below on the varifold closure of smooth surfaces, since $m \cdot \mathcal{H}^2|_{S^2}$ can be approximated with finite energy $\mathcal{E}$.

Assume that M_k is a sequence of smooth surfaces with energy $\mathcal{E}$ bounded by C and M_k converges to a varifold V. This implies that their genera and Willmore energies are bounded by

$$g \leq \frac{C}{4\pi\,|\chi_K|} + 1, \qquad \mathcal{W}(M_k) \leq \mathcal{E}(M_k) + 4\pi\,|\chi_K|$$

so

$$\int_{M_k} |A|^2 \,\mathrm{d}\mathcal{H}^2 = \int_{M_k} |H|^2 - 2K \,\mathrm{d}\mathcal{H}^2 \leq 4\,[\,\mathcal{E}(M_k) + 4\pi\,|\chi_K|\,] + 8\pi\left[\frac{C}{4\pi\,|\chi_K|} + 1\right].$$

This is uniformly bounded in k, so V is a curvature varifold [Hut86b]. Clearly

$$\mathcal{E}(V) \geq \chi_H \mathcal{W}(V) - |\chi_K|\,\mathcal{W}(V) \geq -|\chi_K| \int |A|^2 \,\mathrm{d}V$$

is a uniform bound from below in $\overline{\mathcal{B}_{S,\Omega,C}}$. □

The discontinuity is mathematically meaningful. As the catenoid collapses away, two spheres remain in the limit. The Gaussian integral does not see that these spheres happen to coincide.

Proof of Corollary 3.1.8. To approximate a multiplicity one-sphere by manifolds M_k, insert a sphere of radius $1/k$ into a sphere of radius $\approx r$ and connect the two by g_k catenoids, $g_k \to \infty$. Willmore's energy is close to 8π, so the total energy is

$$\begin{aligned}
\mathcal{E}(M_k) &= \int_{M_k} \chi_H^k (H - H_0^k)^2 \,\mathrm{d}\mathcal{H}^2 + \int_{M_k} \chi_K^k \, K \,\mathrm{d}\mathcal{H}^2 \\
&\leq C \int_{M_k} 2\big(|H|^2 + |H_0^k|^2\big) \,\mathrm{d}\mathcal{H}^2 + \delta \int_{M_k \cap \{K<0\}} K \,\mathrm{d}\mathcal{H}^2 + C \int_{M_k \cap \{K>0\}} K \,\mathrm{d}\mathcal{H}^2 \\
&\leq 2C \left(8\pi + 1 + C^2\right) + \delta \int_{M_k} K \,\mathrm{d}\mathcal{H}^2 + \frac{C}{4} \int_{M_k \cap \{K>0\}} H^2 \,\mathrm{d}\mathcal{H}^2 \\
&\leq 2C \left(8\pi + 1 + C^2\right) + 4\pi\delta(1 - g_k) + C^3/4
\end{aligned}$$

since $K = \lambda_1\lambda_2 \leq (\lambda_1 + \lambda_2)^2/4 = H^2/4$ if λ_1 and λ_2 have the same sign. Clearly, this goes to $-\infty$ as $g_k \to \infty$. To approximate a Dirac measure, we approximate a multiplicity m-sphere of radius $r_m = r/\sqrt{m}$ with genus g-manifolds $\tilde{M}_m$, $g \gg m$. $\square$

3.3 Summary and Prospects

We have proven that a solution of Problem 1 exists in a generalised sense (Corollary 3.1.4 or Theorem 3.2.9) and satisfies very mild regularity properties (Theorem 3.1.1). We have furthermore demonstrated that Problem 1 does not generally have a solution in a prescribed topological class, so that the prescription of genus in the minimisation process is not possible (Corollary 3.1.6). We have applied these results also to more general functionals of Canham-Helfrich type and ruled out a certain parameter regime as physically not sensible.

The minimisation problem structurally resembles an obstacle problem with the additional complication that the obstacle is given by the manifold itself and the boundary of the domain Ω, as well as carrying a volume constraint. For this reason, high regularity of minimisers cannot be expected, although the results of Theorem 3.1.1 are not expected to be optimal for minimisers.

The fact that genus cannot be preserved rules out a parametric approach in the minimisation process (or at least poses a significant challenge). The structure as an obstacle problem means that an Euler-Lagrange equation cannot be used to find a minimiser. The non self-penetration constraint and the confinement to Ω pose serious problems as well, since neither is preserved under Willmore flow, which is a fourth order equation (and thus does not support a maximum principle). Furthermore, a numerical implementation even of non-constrained Willmore flow is challenging.

Having solved the existence problem, the second part of this thesis will focus on a phase-field approach to finding minimisers in practice. We remark that the existence of minimisers could also be deduced purely with phase-field methods. We will see phase-field analogues of several sharp interface results below, which can also be used to partly recover the sharp interface versions.

Part II

Phase-Field Models for Thin Elastic Structures

Chapter 4

Preliminaries

4.1 Phase-Fields and Γ-convergence

A phase-field approach can be seen as a method which lifts problems posed for $(n-1)$-dimensional sets which are the boundaries of open subsets of $\mathbb{R}^n$ to a sequence of approximating problems of smooth scalar fields on $\mathbb{R}^n$. It is clear that any problem involving an open set E and its boundary ∂E can equally well be written as a problem involving only its characteristic function χ_E. To simplify notation, we take a step away from notational convention from now on and let the characteristic function be $+1$ inside a set and -1 (not 0) outside.

Often, technical problems arise from considering sets imbued with the L^1-topology of their characteristic functions, which leads to issues in defining the boundary. This is overcome using the concept of the *reduced boundary* $\partial^* E$ and related geometric techniques [Giu84]. Sets where this works well are known as *Caccioppoli sets* or *sets of finite perimeter* and are characterised by $\chi_E \in BV_{loc}(\mathbb{R}^n)$ (or sometimes as appropriately $\chi_E \in BV(\Omega)$ for some $\Omega \Subset \mathbb{R}^n$). They include all sets with Lipschitz boundaries and are the largest class of sets on which we can define a perimeter functional or on which we have a version of Gauss' theorem. For C^1-sets, the boundary and reduced boundary agree.

Assuming that we can re-write problems involving both an open set and its boundary into problems involving functions taking the values ± 1, we can go one step further and introduce a small parameter $\varepsilon > 0$, a length scale for a regularised transition instead of the jump that χ_E makes on ∂E. A phase-field approach rewrites a minimisation problem

$$\mathcal{F}(E) = \min\{F(U) \mid U \in \mathcal{O}\}$$

for some energy $\mathcal{F}$ and a class $\mathcal{O} \subset \mathcal{T}_{\mathbb{R}^n}$ of open sets into a series of regularised minimisation problems

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \min\{\mathcal{F}_\varepsilon(w) \mid w \in X_\varepsilon\}$$

for suitable function spaces X_ε. The idea behind this is that as $\varepsilon \to 0$, u_ε should converge to a function u, which we then hope will contain information about the minimising set E, and $\mathcal{F}_\varepsilon(u_\varepsilon)$ should approach $\mathcal{F}(u)$. Thus we need the minimisers u_ε of the ε-problems to converge to a minimiser of the sharp interface problem in a suitable sense. The topology of choice for this convergence is often the strong L^1-topology, which is the same topology we expect on the class $\mathcal{O}$. In this setting, the appropriate notion of convergence for the functionals $\mathcal{F}_\varepsilon \to \mathcal{F}$ is Γ-convergence.

Definition 4.1.1. Let (X, τ) be a topological space, $\mathcal{F}_\varepsilon, \mathcal{F} : X \to (-\infty, \infty]$ functions, $x \in X$. Then we say that $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}$ at x and write

$$\left[\Gamma - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon\right](x) = \mathcal{F}(x)$$

if the following two conditions are met.

1. For every sequence $x_\varepsilon \to x$, we have $\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) \geq \mathcal{F}(x)$.

2. There exists a sequence $x_\varepsilon \to x$ such that $\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) \leq \mathcal{F}(x)$.

We say that $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}$ if it Γ-converges at every point.

Occasionally, the topology τ on the space X may not be obvious; in that case we may write $\Gamma(\tau)$-convergence. For notational purposes, we remark that by $\Gamma(L^p)$-convergence we mean convergence with respect to the strong (norm) topology of the L^p-space. The sequence in the lim sup-inequality is usually referred to as a recovery sequence.

This notion of convergence is related to the phase-field regularisation of our original sharp interface problem by the following observation.

Lemma 4.1.2. *If* $\Gamma - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon = \mathcal{F}$ *and* $x_\varepsilon \to x_0$ *is a sequence such that*

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0} \inf_{x\in X} \mathcal{F}_\varepsilon(x),$$

then $\mathcal{F}(x_0) = \inf_{x\in X} \mathcal{F}(x)$.

In other words, the limit of (almost) minimisers of $\mathcal{F}_\varepsilon$ is a minimiser of $\mathcal{F}$ (if it exists). To employ this argument, usually the topology τ is chosen in a way that provides a suitable compactness for the sequence x_ε. The sum of Γ-convergent functionals is not necessarily

Γ-convergent (as recovery sequences need not be compatible), but this is true if one of the sequences converges uniformly. For these and more properties of Γ-convergence, see [Bra02].

If we wish to simplify a problem involving a sharp boundary between two phases in $\mathbb{R}^n$ (which is generally hard to solve both analytically and numerically), we expect two ingredients in the energies $\mathcal{F}_\varepsilon$:

1. A strongly penalised term depending on a potential W with exactly two absolute minimisers at ± 1, for example $W(s) = (s^2 - 1)^2$. This is needed for minimisers u_ε to converge to a function taking only these two values.

2. A singular perturbation which regularises the ε-problem but disappears in the limit, often a term involving an integral of a power of the gradient. This ensures that we gain regularity so that the ε-problem becomes easier to solve, but disappears in the limit so that the original singular problem is recovered.

Obviously, phase-fields may be used in much more general contexts. Phase-fields may be vector valued and describe either hyper-surfaces or lower-dimensional objects, depending on the vanishing set of the potential W. The introduction above sketched how a phase-field may be used to approximate a complicated sharp-interface problem by more regular phase-field problems, but often, the process is reversed. In many physical applications, sharp boundaries are an idealisation of transition layers which are small in one dimension compared to the size of the domain, but do have finite extension (thin membranes, fluid mixtures). The phase-field approach to the Willmore problem presented in the first part of this dissertation falls into the first category, while another project of mine [DKW17] (joint work with P. W. Dondl and M. Kurzke) belongs to the second.

This chapter is used to review known results which will either be used in the following or facilitate the understanding of later parts of the thesis. As such, the results of this Chapter are not new. Unless referenced to the contrary, the proofs were written by myself, but the same or similar arguments can be found in the literature referenced throughout the chapter.

4.2 The Example of Modica and Mortola

The simplest conceivable energy satisfying the above conditions is probably

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \frac{\varepsilon}{2}\,|\nabla u|^2 + \frac{1}{\varepsilon}\,W(u)\,\mathrm{d}x$$

for some open set $\Omega \Subset \mathbb{R}^n$ with a function $W \in C^0(\mathbb{R})$ satisfying

$$W(\pm 1) = 0, \qquad W(z) > 0 \quad \forall\, z \in \mathbb{R} \setminus \{-1, 1\}, \qquad \liminf_{|z|\to\infty} W(z) > 0.$$

A prominent example is $W(z) = \frac{(z^2-1)^2}{4}$. The energy $\mathcal{F}_\varepsilon$ is usually known as the Modica-Mortola functional. It contains two competing quantities: While the gradient term favours functions to be essentially constant, the potential term forces them to remain close to the potential wells in most places. Assuming that both terms have the same order of magnitude in the limit, we are lead to conjecture that sequences u_ε along which $\mathcal{F}_\varepsilon(u_\varepsilon)$ remains bounded have large sets where $u_\varepsilon \approx \pm 1$ separated by transition layers of width $\sim \varepsilon$ where $|\nabla u_\varepsilon| = O(\varepsilon^{-1})$.

This observation suggests that an area-segment of this transition should contribute a fixed amount of energy, and that a suitable limit for S_ε would be a multiple of the relative perimeter functional

$$\mathrm{Per}(E, \Omega) = \begin{cases} \frac{1}{2}\int_\Omega \mathrm{d}|D\chi_E| & \chi_E \in BV(\Omega) \\ \infty & \text{else.} \end{cases}$$

Sets $E \subset \Omega$ such that $\chi_E \in BV(\Omega)$ are called sets of finite perimeter or Caccioppoli sets. The perimeter functional is a generalised measure for the size of the boundary ∂E inside Ω, specifically $\mathrm{Per}(E) = \mathcal{H}^{n-1}(\partial^* E)$. Again, $\partial^* E$ denotes the reduced boundary of E [Giu84]. Note that this differs from the usual notation (but not normalisation) by a factor of $1/2$ which is due to the fact that our characteristic functions have a jump of height two at the boundary, not of height one.

This intuition was made rigorous by Modica and Mortola in [Mod87, MM77]. To simplify matters, let us assume that

1. $W \in C^2(\mathbb{R})$ and $W''(\pm 1) > 0$,

2. W is monotone and satisfies the growth-condition $W(s) \geq s^2$ on $(-\infty, -R] \cup [R, \infty)$ for some large $R > 0$ and

3. $$\int_R^\infty \frac{1}{\sqrt{W(s)}} + \frac{1}{\sqrt{W(-s)}}\,\mathrm{d}s < \infty. \tag{4.2.1}$$

In particular, this is obviously true for the potential $W(s) = \frac{1}{4}(s^2-1)^2$ which we will use in the following.

Theorem 4.2.1. *Let $\Omega \Subset \mathbb{R}^n$ and set*

$$\mathcal{F}_\varepsilon : L^1(\Omega) \to \mathbb{R}, \qquad \mathcal{F}_\varepsilon(u) = \begin{cases} \int_\Omega \frac{\varepsilon}{2}\,|\nabla u|^2 + \frac{1}{\varepsilon}\,W(u)\,\mathrm{d}x & u \in W^{1,2}(\Omega) \\ \infty & \textit{else.} \end{cases}$$

Then

$$\Gamma(L^1) - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon = c_0\,\mathrm{Per}(\cdot,\Omega), \qquad c_0 = \int_{-1}^1 \sqrt{2\,W(s)}\,\mathrm{d}s.$$

If u_ε is any sequence such that $\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon)$ is bounded, then there exists $u \in BV(\Omega,\{-1,1\})$ such that $u_\varepsilon \to u$ strongly in $L^p(\Omega)$ for all $p < 2$ (up to a subsequence).

We give a brief proof since ideas from it will be used in the remainder of this dissertation. For the standard potential $W(s) = \frac{(s^2-1)^2}{4}$ we can compute directly $c_0 = \frac{2\sqrt{2}}{3}$.

Proof. **Step 1.** Let us first prove the compactness result. Take any sequence $u_\varepsilon \in W^{1,2}(\Omega)$ such that

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty.$$

Then

$$\begin{aligned} \mathcal{F}_\varepsilon(u_\varepsilon) &= \int_\Omega \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\,W(u_\varepsilon)\,\mathrm{d}x \\ &= \int_\Omega \frac{1}{2\varepsilon}\,\left(\varepsilon\,|\nabla u_\varepsilon| - \sqrt{2\,W(u_\varepsilon)}\right)^2\,\mathrm{d}x + \int_\Omega \sqrt{2\,W(u_\varepsilon)}\,|\nabla u_\varepsilon|\,\mathrm{d}x \\ &\geq \int_\Omega \sqrt{2\,W(u_\varepsilon)}\,|\nabla u_\varepsilon|\,\mathrm{d}x. \end{aligned} \tag{4.2.2}$$

Thus when we take G to be any primitive function of $\sqrt{2\,W}$, we see that the sequence $w_\varepsilon := G(u_\varepsilon)$ satisfies

$$\limsup_{\varepsilon\to 0} \int_\Omega |\nabla w_\varepsilon|\,\mathrm{d}x < \infty.$$

Furthermore

$$\begin{aligned} \limsup_{t\to\infty} \frac{G(t)}{W(t)} &= \limsup_{t\to\infty} \frac{1}{W(t)}\left(C + \int_R^t \sqrt{2\,W(s)}\,\mathrm{d}s\right) \\ &\leq \lim_{t\to\infty} \int_R^t \frac{\sqrt{2\,W(s)}}{W(t)}\,\mathrm{d}s \\ &\leq \sqrt{2}\int_R^\infty \frac{1}{\sqrt{W(s)}}\,\mathrm{d}s < \infty. \end{aligned}$$

The same holds for $t \to -\infty$, so there exists $C > 0$ such that $G \leq C\,(1+W)$. It follows that also

$$\limsup_{\varepsilon\to 0} \int_\Omega |w_\varepsilon|\,\mathrm{d}x \leq \limsup_{\varepsilon\to 0} C\,\left(\mathcal{L}^n(\Omega) + \varepsilon\,\mathcal{F}(u_\varepsilon)\right) < \infty,$$

so in total that w_ε is bounded in $BV(\Omega)$. Using the BV-compactness theorem and the compact embedding into $L^p(\Omega)$ for $1 \le p < n/(n-1)$, we deduce that there exists $w \in BV(\Omega)$ such that (up to a subsequence) $w_\varepsilon \to w$ strongly in $L^p(\Omega)$ for all $p < n/(n-1)$ with

$$|Dw|(\Omega) \le \liminf_{\varepsilon\to 0} \int_\Omega |\nabla w_\varepsilon| \,\mathrm{d}x \le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon). \tag{4.2.3}$$

Since $G(u_\varepsilon) \to w$ in $L^1(\Omega)$, a subsequence converges pointwise almost everywhere. As G is strictly monotone increasing, we can take its inverse function and obtain that $u_\varepsilon \to G^{-1}(w)$ pointwise almost everywhere. Using $W(s) \ge s^2$ for all sufficiently large $|s|$, the bound on $\int_\Omega G(u_\varepsilon)\,\mathrm{d}x$ implies that u_ε is bounded in $L^2(\Omega)$. By a standard result on concentrations and weak compactness (see e.g. [Bre11, Exercise 4.16]) we have that (1) $u_\varepsilon \to G^{-1}(w)$ pointwise and (2) u_ε is bounded in $L^2(\Omega)$ together imply that $u_\varepsilon \to u = G^{-1}(w)$ strongly in $L^p(\Omega)$ for all $1 \le p < 2$.

Step 2. If $u_\varepsilon \to u$ strongly in $L^1(\Omega)$ and $\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty$, then $w = G(u)$ almost everywhere, since L^1-convergence implies convergence pointwise almost everywhere for a subsequence. Clearly u only takes the values ± 1 and w only the values $G(-1), G(+1)$, thus one can easily relate their gradient measures by the difference in the height of the jump:

$$|Du| = \frac{1-(-1)}{G(1)-G(-1)}\,|Dw| = \frac{2}{\int_{-1}^{1} \sqrt{2\,W(s)}\,\mathrm{d}s}\,|Dw|$$

as measures and thus

$$c_0 \operatorname{Per}(\partial\{u=1\}) = \frac{c_0}{2}\,|Du|(\Omega) = |Dw|(\Omega) \le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon)$$

due to (4.2.3). This concludes the proof of the lim inf-inequality.

Step 3. For now we assume $E \Subset \Omega$ such that $\operatorname{Per}(E,\Omega) = \operatorname{Per}(E,\mathbb{R}^n) = \operatorname{Per}(E)$. We imagine the phase-field u_ε as an approximation of the characteristic function of a set E which makes a smoothed out transition on a length-scale ε at the boundary ∂E. Let us for the moment assume that $\partial E \in C^2$. Then we approximate the signed distance function

$$\operatorname{sdist}(x,\partial E) = \begin{cases} \operatorname{dist}(x,\partial E) & x \in E \\ -\operatorname{dist}(x,\partial E) & x \notin E \end{cases}$$

by a function r such that

1. there exists a neighbourhood $U_\delta = \{\operatorname{dist}(x,\partial E) < \delta\}$ of ∂E such that $r(x) = \operatorname{sdist}(x,\partial E)$ for all $x \in U$,

2. $r \geq \delta$ outside U_δ,

3. $r \in C^2(\mathbb{R}^n)$ and

4. $|\nabla r| \leq 1$.

Since $\partial E \in C^2$, sdist is C^2-smooth in a neighbourhood of ∂E and satisfies

$$\nabla \operatorname{sdist}(x) = \nu_{\partial E, \pi(x)}, \qquad \text{in particular } |\nabla \operatorname{sdist}| \equiv 1$$

on U_δ, where the closest point projection $\pi : U_\delta \to \partial E$ is C^2-smooth and uniquely defined (for small $\delta > 0$). We then set

$$u_\varepsilon(x) := q\left(\frac{r(x)}{\varepsilon}\right)$$

where $q : \mathbb{R} \to \mathbb{R}$ is a function satisfying $\lim_{x\to\pm\infty} q(x) = \pm 1$. To get the optimal transition profile which will give us the minimal energy, we choose q as a solution of the optimal profile problem in one dimension, which is a solution of the Euler-Lagrange equation

$$q'' - W'(q) = 0, \qquad \lim_{x\to\pm\infty} q(x) = \pm 1, \qquad q(0) = 0$$

of the functional S_ε in one dimension with length scale $\varepsilon = 1$. The function q will be constructed in detail below in Lemma 4.3.1. We will use two properties in the following:

1. $|q'|^2 = 2\,W(q)$ and

2. $1 - C\,e^{-\alpha x} \leq q(x) \leq 1$ for all $x \in \mathbb{R}$ for some suitable $C, \alpha > 0$.

Now, using the co-area formula [EG92, Section 3.4] and the first identity, we see that for every $0 < \beta < \delta$ we have

$$\begin{aligned}
\int_\Omega \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\,\mathrm{d}x - C\,\varepsilon^{-2}\exp(-\delta/\varepsilon) &\leq \int_{U_\delta} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\,\mathrm{d}x \\
&= \int_{U_\beta} \frac{1}{2\varepsilon}(q')^2\left(\frac{\operatorname{sdist}}{\varepsilon}\right) + \frac{1}{\varepsilon}W\left(q\left(\frac{\operatorname{sdist}}{\varepsilon}\right)\right)\mathrm{d}x \\
&= \int_{U_\beta} \frac{1}{\varepsilon}\sqrt{2\,W\left(q\left(\frac{\operatorname{sdist}}{\varepsilon}\right)\right)}\,q'\left(\frac{\operatorname{sdist}}{\varepsilon}\right)\cdot 1\,\mathrm{d}x \\
&= \int_{-\beta}^{\beta}\left(\int_{\{\operatorname{sdist}=z\}} \frac{1}{\varepsilon}\sqrt{2\,W\left(q\left(\frac{\operatorname{sdist}}{\varepsilon}\right)\right)}\,q'\left(\frac{\operatorname{sdist}}{\varepsilon}\right)\mathrm{d}\mathcal{H}^{n-1}\right)\mathrm{d}z \\
&= \int_{-\beta}^{\beta} \frac{1}{\varepsilon}\sqrt{2\,W\left(\frac{z}{\varepsilon}\right)}q'\left(\frac{z}{\varepsilon}\right)\mathcal{H}^{n-1}\left(\{\operatorname{sdist} = z\}\right)\mathrm{d}z \\
&= \int_{-\beta/\varepsilon}^{\beta/\varepsilon} \sqrt{2\,W(q)}\,q'\,\left(\mathcal{H}^{n-1}(\partial E) + o(1)\right)\mathrm{d}z
\end{aligned}$$

$$\begin{aligned} &\to c_0\, H^{n-1}(\partial E) \\ &= c_0 \mathrm{Per}(E) \end{aligned}$$

where the term $o(1)$ is uniformly small for small β. If E is not compactly contained, then we can use the same construction as before on compact sets $\Omega' \Subset \Omega$ and then let $\Omega' \to \Omega$. This gives precisely the relative perimeter.

Step 4. It remains to show that Caccioppoli sets E contained in Ω can be approximated by sets E_n with C^2-boundaries in the strong L^1-topology such that

$$|D\chi_{E_n}|(\Omega) \to |D\chi_E|(\Omega).$$

Then a diagonal sequence of the recovery sequences for E_n can be used as a recovery sequence for E. Since all our sets will be limits of C^2-boundaries by assumption (with the further property that the approximating surfaces have uniformly bounded Willmore energy), we will skip this part of the proof and refer the interested reader to [Giu84, Theorem 1.24]. □

We note that the proof is entirely analytic, hiding the geometry behind embedding theorems, the co-area formula and, crucially, the chain rule. For this reason, the proof does not generalise to the case when the gradient term is replaced by a fractional Sobolev norm. A different proof is presented in [Bra02] using slicing arguments can be generalised to more general situations, see the remarks at the end of this chapter.

Obviously, potentials growing faster at $\pm\infty$ give better bounds on u_ε. In particular, for $W(u) = \frac{1}{4}\,(u^2-1)^2$, we get strong convergence in $L^p(\Omega)$ for all $p < 4$. From now on, we will consider the normalised functional $S_\varepsilon = \frac{1}{c_0}\mathcal{F}_\varepsilon$, which approximates the perimeter-functional.

Remark 4.2.2. If $E \Subset \Omega$, a slight additional modification shows that we could restrict S_ε to $-1 + W_0^{1,2}(\Omega)$ or even $-1 + W_0^{2,2}(\Omega)$ and obtain the perimeter-functional as the Γ-limit. The $\liminf$-inequality trivially remains in tact, and the recovery sequence can easily be modified like this since q, q' approach $1, 0$ exponentially fast away from ∂E, such that the boundary conditions are almost satisfied anyway. The exponential decay is shown in Lemma 4.3.1 for q and implied by the equality $|q'|^2 = 2\,W(q)$ also for q' since $W(\pm 1) = W'(\pm 1) = 0$ and $W''(\pm 1) > 0$. The equation $q'' = W'(q)$ also implies exponential decay for the second derivative.

Often, the limit u of a finite energy sequence u_ε is not overly instructive since there may be 'ghost interfaces' where two interfaces meet and disappear in the limit. This stems from the fact that the gradient of a BV-function resembles a current, while we work more naturally in a varifold setting. For this reason, it may be more instructive to consider associated measures which are stable in the limit. Thinking of S_ε as an approximation of a perimeter functional, we can localise this to a diffuse surface area Radon measure defined through

$$\mu_\varepsilon^u(U) = \frac{1}{c_0}\int_{\Omega\cap U} \frac{\varepsilon}{2}\,|\nabla u|^2 + \frac{1}{\varepsilon}\,W(u)\,\mathrm{d}x \tag{4.2.4}$$

on open sets $U \subset \mathbb{R}^n$. We call μ_ε^u a diffuse surface measure and note that $S_\varepsilon(u) = \mu_\varepsilon^u(\mathbb{R}^n)$. By the compactness Theorem for Radon measures, if $\limsup_{\varepsilon\to 0} S_\varepsilon(u_\varepsilon) < \infty$, then there is a subsequence $\varepsilon \to 0$ (not relabelled) and a Radon measure μ such that

$$\mu_\varepsilon := \mu_\varepsilon^{u_\varepsilon} \stackrel{*}{\rightharpoonup} \mu$$

as Radon measures (i.e. in the weak* topology when interpreting Radon measures as dual to continuous functions on $\overline{\Omega}$).

4.3 The Stationary Allen-Cahn Equation

In the proof of Theorem 4.2.1, we have used the existence and properties of the optimal profile q, which we will now prove. The profile is governed by the one-dimensional version of the stationary Allen-Cahn equation

$$\Delta u = W'(u). \tag{4.3.1}$$

Lemma 4.3.1. *Let $W \in C^1(\mathbb{R})$ with $W(-1) = W(1) = 0$ and $W > 0$ in $(-1, 1)$. Then there exists a unique solution $q \in C^2(\mathbb{R})$ of the equation*

$$q'' - W'(q) = 0 \tag{4.3.2}$$

satisfying

$$\lim_{x\to-\infty} q(x) = -1, \qquad q(0) = 0, \qquad \lim_{x\to\infty} q(x) = 1.$$

The function q satisfies

$$|q'|^2 = 2\,W(q) \tag{4.3.3}$$

pointwise and if $W \in C^2(\mathbb{R})$ and $W''(1) > 0$, then there exist $C, \alpha > 0$ such that

$$1 - C\,e^{-\alpha x} \le q(x) < 1 \qquad \forall\ x \in \mathbb{R}.$$

If $W(s) = W(-s)$, then $q(-x) = -q(x)$. Equations (4.3.3) *and* (4.3.2) *also imply exponential decay for q' and q'' respectively.*

For the standard example $W(z) = \frac{(z^2-1)^2}{4}$, the optimal profile q is a rescaled version of the hyperbolic tangent, which serves to illustrate the properties in a specific case.

Proof. Assume that q solves (4.3.2), then we also have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{|q'|^2}{2} - W(q)\right) = q'q'' - W'(q)\,q' = (q'' - W'(q))\,q' = 0,$$

so

$$\frac{|q'|^2}{2} - W(q) \equiv c \in \mathbb{R}.$$

We wish to find a transition between -1 and $+1$, so it is clear that c cannot be positive, since otherwise $|q'| \geq \sqrt{c}$ does not have bounded C^1-solutions. If c is negative, on the other hand, $|q'|^2 < 0$ at the zeros of W, so we cannot reach ± 1. Thus only $c = 0$ is admissible. So instead of solving (4.3.2) with conditions at $\pm\infty$ and 0, we solve

$$q_+' = \sqrt{2\,W(q_+)}, \qquad q_+(0) = 0 \tag{4.3.4}$$

forwards in time, for which a unique solution $q \in C^1[0, L_+)$ exists by the Picard-Lindelöff theorem for some maximal $L_+ > 0$ – the Lipschitz continuity of $\sqrt{W}$ follows from the smoothness of W. Similarly, we solve

$$q_-' = -\sqrt{2\,W(q_-)}, \qquad q_-(0) = 0$$

and then define

$$q(t) = \begin{cases} q_+(x) & x > 0 \\ q_-(-x) & x < 0 \end{cases}.$$

Clearly, q is C^1-smooth on the interval (L_-, L_+) and due to the C^1-smoothness and positivity of W also C^2-smooth, except possibly at the origin. By construction, $q' > 0$ and q is monotone increasing. Taking the square of the first order ODE (4.3.4) and differentiating with respect to time, we obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\left((q_+')^2 - 2\,W(q_+)\right) = 2\,\left(q_+'' - W'(q_+)\right)\,q_+'$$

which implies $q_+'' - W'(q_+) = 0$ since $q_+' > 0$. The same holds for q_-, and by continuity, q is C^2-smooth also at the origin and a solution of (4.3.2). Let us now show that $\lim_{x\to\pm\infty} q(x) = \pm 1$.

Since $q' > 0$, the limit $q_\infty := \lim_{x\to L^+} q(x)$ exists (but might be infinite). But since $W(1) = 0$ and $\sqrt{W}$ is Lipschitz-continuous, we immediately see that $q_\infty \leq 1$. If $q_\infty < 1$, then $q' \geq \sqrt{W(q_\infty)} > 0$ and thus $L_+ < \infty$, but then we could continue q by

$$\tilde{q}_+' = \sqrt{2\,W(q)}, \qquad \tilde{q}_+(L_+) = q_\infty,$$

so L_+ would not have been maximal. We deduce that $q_\infty = 1$. Applying the same argument for negative x, we see that q transitions between -1 and 1 on (L_-, L_+). Now assume that W is C^2-smooth and $W''(1) > 0$. When x is large, q is close to 1 and we observe that

$$\begin{aligned}(1-q)' &= -q' = -\sqrt{2\,W(q)}\\ &\approx -\sqrt{2\,W(1) + 2\,W'(1)\,(q-1) + W''(1)\,(q-1)^2} = -\sqrt{W''(1)}\,|q-1|\end{aligned}$$

since $W(1) = W'(1) = 0$ in the potential well. Thus $(1-q)' \leq -C\,(1-q)$ for a slightly smaller constant C and we obtain the exponential decay of $1-q$. □

If W behaves differently around ± 1, we can observe different behaviour of q and transitions between ± 1 in finite time. The proof above illustrates that the optimal profile q is the only interesting solution of the stationary Allen-Cahn equation in one dimension. In n dimensions, we can give a trivial solution by

$$u(x) = q(\langle v, x\rangle + b), \qquad v \in S^{n-1}, b \in \mathbb{R},$$

but other bounded smooth solutions are known to exist. In two dimensions, examples are known which approximate saddle configurations [DFP92, dPKPW10]. The most prominent solution in this class has a zero level set given by the coordinate axes in $\mathbb{R}^2$ and is positive in the first and third quadrants and negative in the second and fourth. Other solutions exist with alternating positive and negative sectors whose borders are asymptotic to a union of lines intersecting in the origin. Solutions asymptotic to minimal surfaces in three dimensional space are also known to exist [dPKW13].

For the solutions described above, clearly $\nabla u(0) = 0$, so the identity $|\nabla u|^2 = W(u)$ which we used to construct the optimal profile in one dimension cannot hold anymore. However, it does hold as an inequality which is often referred to as a Modica-type gradient bound.

Theorem 4.3.2. *[Mod85] Let $W \in C^3(\mathbb{R})$ and $u \in W^{2,2}_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a solution of* (4.3.1)*. Then*

$$|\nabla u|^2 \leq 2\,W(u). \tag{4.3.5}$$

Furthermore, $-1 < u < 1$ or $u \equiv \pm 1$.

In fact, if equality holds everywhere in (4.3.5), then u is automatically the special solution described by extending an optimal profile. The following result is presumably very classical and well-known, but I have not found a reference for it.

Lemma 4.3.3. *Let $n \geq 2$ and $u \in W^{2,2}_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that*

$$-\Delta u + W'(u) = 0 \qquad \text{and} \qquad \frac{|\nabla u|^2}{2} = W(u).$$

Then either $u \equiv \pm 1$ or $u(x) = q(\langle v, x\rangle + b)$ for some $v \in S^{n-1}$, $b \in \mathbb{R}$.

Proof. From the elliptic equation we immediately obtain that $u \in C^\infty(\mathbb{R}^n)$ and due to Theorem 4.3.2, we see further that $|u| < 1$ or $u \equiv \pm 1$. If u is not constant, we can write

$$u = q \circ (q^{-1} \circ u) = q \circ r$$

for a function $r \in C^\infty(\mathbb{R}^n)$. We compute

$$\begin{aligned} \nabla u &= (q' \circ r)\, \nabla r \\ \Delta u &= (q'' \circ r)\, |\nabla r|^2 + (q' \circ r)\, \Delta r \end{aligned}$$

and deduce from

$$0 = \frac{1}{2}\, |\nabla u|^2 - W(u) = \frac{1}{2}\, |q' \circ r|^2\, |\nabla r|^2 - W(q \circ r)$$

that $|\nabla r|^2 \equiv 1$, so

$$0 = \Delta u - W'(u) = (q'' \circ r) + (q' \circ r)\, \Delta r - W'(q \circ r) = (q' \circ r)\, \Delta r.$$

This implies that $\Delta r = 0$ on the whole space since $q' > 0$. So r is a harmonic function with $|\nabla r|^2 \equiv 1$. It follows that $\partial_i r$ is a bounded harmonic function on $\mathbb{R}^n$ for all $i = 1, \dots, n$. Thus $\partial_i r$ is constant by Liouville's theorem and

$$\nabla r \equiv v \qquad \Rightarrow \qquad r(x) = \langle v, x\rangle + t_0.$$

□

In general, solutions to the Allen-Cahn equation $u : \mathbb{R}^n \to (-1, 1)$ can be written in this form and satisfy

$$0 = (q'' \circ r)\, (|\nabla r|^2 - 1) + (q' \circ r)\, \Delta r$$

and $|\nabla r| \leq 1$ due to the Modica gradient bound. The stationary Allen-Cahn equation is the fundamental object for work relating to energies of Modica-Mortola type since it characterises blow-ups of energy minimisers. For simplicity, we only prove the following theorem for the standard potential

$$W(u) = \frac{(u^2-1)^2}{4}$$

which satisfies

$$\big(W'(u)\big)^2 \leq 4\,||u||_\infty^2\, W(u).$$

Theorem 4.3.4. *Let $\Omega \Subset \mathbb{R}^n$, $u_\varepsilon : \Omega \to \mathbb{R}$ a sequence of minimisers of S_ε under the condition that*

$$\frac{1}{|\Omega|}\int_\Omega u_\varepsilon \,\mathrm{d}x = \theta \in (-1,1).$$

Assume additionally that the sequence u_ε is uniformly bounded in $L^\infty(\Omega)$. Then for any $\Omega' \Subset \Omega$ and any sequence $x_\varepsilon \in \Omega'$, the functions

$$\hat{u}_\varepsilon : B_{r/\varepsilon}(0) \to \mathbb{R}, \qquad \hat{u}_\varepsilon(y) = u_\varepsilon(x_\varepsilon + \varepsilon y)$$

converge to a function $\hat{u}$ as $\varepsilon \to 0$ weakly in $W^{2,p}(U)$ for all $p < \infty$ and $U \Subset \mathbb{R}^n$ where $\hat{u}$ is a smooth solution of the stationary Allen-Cahn equation. Here r is such that $\operatorname{dist}(\Omega', \partial\Omega) > r$.

Proof. We can take variations $u_\varepsilon + t\phi$ where $\int_\Omega \phi \,\mathrm{d}x = 0$ to obtain the Euler-Lagrange equation

$$\begin{cases} -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}\,W'(u_\varepsilon) = \lambda_\varepsilon & \text{in } \Omega \\ \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

where λ_ε is not yet determined and arises from the fact that we cannot vary in directions with non-zero integral. However, we can estimate

$$\begin{aligned}
\lambda_\varepsilon &= \frac{1}{|\Omega|}\int_\Omega \lambda_\varepsilon \,\mathrm{d}x \\
&= \frac{1}{|\Omega|}\int_\Omega -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}\,W'(u_\varepsilon)\,\mathrm{d}x \\
&= \frac{1}{|\Omega|\,\sqrt{\varepsilon}}\int_\Omega \frac{1}{\sqrt{\varepsilon}}\,W'(u_\varepsilon)\,\mathrm{d}x \\
&\leq \frac{1}{|\Omega|\,\sqrt{\varepsilon}}\sqrt{|\Omega|}\left(\int_\Omega \frac{1}{\varepsilon}\,W'(u_\varepsilon)^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{|\Omega|\,\varepsilon}}\left(\int_\Omega \frac{1}{\varepsilon}\,4\,||u_\varepsilon||_\infty^2\,W(u_\varepsilon)\,\mathrm{d}x\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \sqrt{\frac{4\,||u_\varepsilon||_\infty^2\,\mu_\varepsilon(\Omega)}{|\Omega|\,\varepsilon}}$$

Thus $\lambda_\varepsilon \leq C\,\varepsilon^{1/2}$ and

$$\begin{aligned}\left(-\Delta\hat{u}_\varepsilon + W'(\hat{u}_\varepsilon)\right)(y) &= \left(-\varepsilon^2\Delta u_\varepsilon + W'(u_\varepsilon)\right)(x_\varepsilon + \varepsilon y)\\ &= \varepsilon\left(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right)(x_\varepsilon + \varepsilon y)\\ &= \varepsilon\,\lambda_\varepsilon.\end{aligned}$$

The right hand side goes to zero at least like $\sqrt{\varepsilon}$. Using that $\hat{u}_\varepsilon$ is uniformly bounded in $L^\infty(U)$, we see that Δu_ε is uniformly bounded in $L^p(U)$ and thus by Calderon-Zygmund theory u_ε is bounded uniformly in $W^{2,p}(U)$ for for all $p < \infty$ and all $U \Subset \mathbb{R}^n$ (passing to some larger U' for the regularity argument). The weak limit $\hat{u}$ satisfies the equation

$$-\Delta\hat{u} + W'(\hat{u}) = 0$$

since the Laplacian converges weakly and the non-linear term converges as $\hat{u}_\varepsilon \to \hat{u}$ strongly in $C^0(\overline{U})$ due to Morrey's embedding theorem. □

In the next chapter, we will see an analogue of this statement also for the Willmore case. The theorem above is somewhat crude and can be improved significantly (see e.g. [LM89]), but it serves to illustrate the importance of the Allen-Cahn equation. As a simple consequence, we see that the L^∞-bound can be improved to

$$\max_{x\in\Omega'} |u_\varepsilon(x)| \to 1.$$

4.4 Equi-Partition of Energy

Since 'good' sequences u_ε are well-described by the Allen-Cahn equation on small scales and solutions to the Allen-Cahn equation satisfy the Modica gradient bound, we can hope that they also satisfy a similar property. We see in (4.2.2) that any recovery sequence for the Modica-Mortola functional also needs to have an asymptotic equipartition of energy in a suitable sense, since the lim sup-property can only hold if

$$\limsup_{\varepsilon\to 0} \int_\Omega \frac{1}{2\varepsilon}\,\left(\varepsilon\,|\nabla u_\varepsilon| - \sqrt{2\,W(u_\varepsilon)}\right)^2\,\mathrm{d}x \leq 0.$$

To measure failure of the equi-partition of energy in the Modica-Mortola functional we introduce the following discrepancy measures:

$$\begin{aligned}
\xi_\varepsilon(B) &= \frac{1}{c_0}\int_B \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}\,W(u_\varepsilon)\,\mathrm{d}x\\
\xi_{\varepsilon,+}(B) &= \frac{1}{c_0}\int_B \left(\frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}\,W(u_\varepsilon)\right)_+\,\mathrm{d}x\\
|\xi_\varepsilon|(B) &= \frac{1}{c_0}\int_B \left|\frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}\,W(u_\varepsilon)\right|\,\mathrm{d}x.
\end{aligned}$$

The equi-partition property needed for the lim sup-construction in the Modica-Mortola functional implies $|\xi_\varepsilon| \to 0$ since

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_\Omega \left|\frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}\,W(u_\varepsilon)\right|\,\mathrm{d}x\\
&\quad= \limsup_{\varepsilon\to 0}\int_\Omega \left|\sqrt{\frac{\varepsilon}{2}}\,|\nabla u_\varepsilon| + \sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}\right|\cdot\left|\sqrt{\frac{\varepsilon}{2}}\,|\nabla u_\varepsilon| - \sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}\right|\,\mathrm{d}x\\
&\quad\le \limsup_{\varepsilon\to 0}\left(\int_\Omega\left(\sqrt{\frac{\varepsilon}{2}}\,|\nabla u_\varepsilon| + \sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}\right)^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega\left(\sqrt{\frac{\varepsilon}{2}}\,|\nabla u_\varepsilon| - \sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}\right)^2\mathrm{d}x\right)\\
&\quad\le \limsup_{\varepsilon\to 0}\sqrt{4\,c_0\,\mu_\varepsilon(\Omega)}\left(\frac{1}{2\varepsilon}\int_\Omega\left(\varepsilon\,|\nabla u_\varepsilon| - \sqrt{2\,W(u_\varepsilon)}\right)^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\
&\quad= 0.
\end{aligned}$$

Note that $\xi_\varepsilon \le 0$ for the recovery sequence of the Modica-Mortola functional since $|\nabla r| \le 1$ for our approximation r of $\mathrm{sdist}(\cdot, \partial E)$ (compare Lemma 4.3.3) and that $|\xi_\varepsilon| \to 0$ exponentially fast in ε. Indeed, it is generally true that $|\xi_\varepsilon| \overset{*}{\rightharpoonup} 0$ as Radon measures in the Willmore case [RS06, Proposition 4.9]. The importance of the discrepancy measures will become apparent in the next chapter, but intuitively it is already clear that $|\xi_\varepsilon|$ is related to how badly blow-ups of u_ε can behave and that $\xi_{\varepsilon,+}$ is somewhat more problematic than $\xi_{\varepsilon,-}$ since $\xi_{\varepsilon,+}$ needs to vanish for behaviour similar to the stationary Allen-Cahn equation, while $\xi_{\varepsilon,-}$ just needs to vanish to ensure that blow ups are in fact optimal profiles.

4.5 Willmore's Energy

It is well known that mean curvature is the L^2-gradient of the area functional – in fact, this variational principle serves to define the mean curvature in the class of varifolds. When we think of the Modica-Mortola functional S_ε as an approximation of the area functional on hyper-surfaces, it is natural to think of its gradient

$$\begin{aligned}\delta S_\varepsilon(u;\phi) &= \int_\Omega \varepsilon\,\langle \nabla u, \nabla\phi\rangle + \frac{1}{\varepsilon}\,W'(u)\phi\,\mathrm{d}x\\ &= \int_\Omega \left(-\varepsilon\,\Delta u_\varepsilon + \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right)\phi\,\mathrm{d}x\end{aligned}$$

as an analogue of mean curvature. The mean curvature of a level set $M_\alpha := \{u = \alpha\}$ of a C^2-function $u : \Omega \to \mathbb{R}$ at $x \in M_\alpha$ is given by

$$H(x) = -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)(x),$$

(see e.g. [ES91]), so $H(x) = \Delta u(x)$ if $|\nabla u| \equiv 1$ in a neighbourhood of x, see also [GT83, Section 14.6]. Thus we see

$$\Delta\,\mathrm{sdist}(x,\partial E) = -H_{\pi(x)} + o(1)$$

in U_δ, also using the C^2-smoothness of sdist (if sdist > 0 inside the compact set E and negative outside for sign convention). Now we can easily compute

$$\begin{aligned}-\varepsilon\,\Delta u_\varepsilon + \frac{1}{\varepsilon}\,W'(u_\varepsilon) &= -\frac{1}{\varepsilon}\,q''\left(\frac{r}{\varepsilon}\right)|\nabla r|^2 - q'\left(\frac{r}{\varepsilon}\right)\Delta r(x) + \frac{1}{\varepsilon}\,W'\left(q\left(\frac{r}{\varepsilon}\right)\right)\\ &= q'\left(\frac{r}{\varepsilon}\right)\left[H_{\pi(x)} + e(x)\right]. \qquad (4.5.1)\end{aligned}$$

inside U_δ where $e(x) \to 0$ as $\mathrm{sdist}(x,\partial E) \to 0$, using the same identities as in Lemma 4.3.3. Outside of U_δ, the optimal profile vanishes exponentially quickly, so the integral is not affected. Again, we could equally well consider $\mathcal{W}_\varepsilon$ to be defined only on $-1 + W_0^{2,2}(\Omega)$ if we are interested in $E \Subset \Omega$. Thus, after dividing by $c_0\varepsilon$ for normalisation purposes, we see that

$$\frac{1}{c_0\varepsilon}\int_\Omega \left(-\varepsilon\,\Delta u_\varepsilon + \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right)^2 \mathrm{d}x \to \mathcal{W}(\partial E)$$

arguing exactly as for the Modica-Mortola functional. This is the lim sup-inequality for the Γ-convergence $\mathcal{W}_\varepsilon \to \mathcal{W}$ for

$$\mathcal{W}_\varepsilon(u) = \frac{1}{c_0\varepsilon}\int_\Omega \left(-\varepsilon\,\Delta u + \frac{1}{\varepsilon}\,W'(u)\right)^2 \mathrm{d}x$$

and has been established in [BP93]. The lim inf-inequality is a lot harder to establish and has been proven in dimension $n = 2$ in [NT07] and in dimensions $n = 2, 3$ in [RS06]. The proof of the lim inf-inequality is quite difficult and we give only a very brief sketch here. Several results of [RS06] are improved in the main text so that the methods are instructive also for our purposes.

Theorem 4.5.1. *[RS06] Let $n = 2, 3$, $\Omega \Subset \mathbb{R}^n$ and $u_\varepsilon \in W^{2,2}(\Omega)$ such that*

$$\limsup_{\varepsilon \to 0} (S_\varepsilon + \mathcal{W}_\varepsilon)(u_\varepsilon) < \infty.$$

Then the associated mass measures μ_ε converge weakly as Radon measures to a measure μ supported in $\overline{\Omega}$ (up to a subsequence). There is an integral varifold V in Ω with square integrable mean curvature H such that μ is its mass measure and

$$\mu(\overline{\Omega}) = \lim_{\varepsilon \to 0} \mu_\varepsilon(\Omega), \qquad \mathcal{W}(\mu) \leq \liminf_{\varepsilon \to 0} \mathcal{W}_\varepsilon(u_\varepsilon).$$

If $\mu \geq |D\chi_E|$ with $\partial E \in C^2$, then $\mathcal{W}(\mu) \geq \mathcal{W}(\partial E)$.

Regularity of μ up to the boundary holds only if boundary conditions are assumed, as will be shown in Chapter 6.

Sketch of Proof:

1. A quantitative estimate for the positive part of the discrepancy measures is established only from bounds on $(S_\varepsilon + \mathcal{W}_\varepsilon)(u_\varepsilon)$ without any assumption of minimality. This estimate also plays an important role for us and can be found in Lemma 5.2.7 in this thesis. This and other technical points (see Lemmas 5.2.2 and 5.2.8 for improved versions as well as Corollary 5.2.3) are the content of the third section of [RS06].

2. A monotonicity formula (Lemma 5.2.4) in suitably estimated form (Lemma 5.2.5) and a result on the local behaviour close to $\mathrm{spt}(\mu)$ (compare Lemmas 7.2.2 and 7.2.4) is used to establish a lower bound on the $n-1$-dimensional density of μ. This implies further $|\xi_\varepsilon| \to 0$ by a Radon-Nikodym argument and that the varifolds

$$V_\varepsilon(\phi) = \int_\Omega \phi\left(x, \nu_\varepsilon(x)\right)\, \mathrm{d}\mu_\varepsilon(x),$$

converge to a rectifiable varifold V with mass measure μ in the varifold sense. Here $\nu_\varepsilon = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$ if $\nabla u_\varepsilon \neq 0$ and $\nu_\varepsilon = 0$ otherwise is the diffuse normal direction. This is the content of the fourth section of [RS06].

3. A blow up argument and a diffuse version of Allard's multi-layer proposition are used together with a result on the behaviour of transition layers (compare Lemma 7.2.2) to establish the integrality of μ. This is the content of the fifth and final section of [RS06].

4. When it is established that μ is an integral varifold with mean curvature $H \in L^2(\mu)$, under the assumption that $u_\varepsilon \to \chi_E$ with $\partial E \in C^2$, a result from [Sch09] implies that

$$\mathcal{W}(\partial E) \leq \mathcal{W}(\mu) \leq \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon).$$

This is explained in the first section of [RS06].

□

As before, we localise $\mathcal{W}_\varepsilon$ using associated Radon-measures. Denote

$$h_\varepsilon = -\varepsilon\,\Delta u_\varepsilon + \frac{1}{\varepsilon}\,W'(u_\varepsilon)$$

and

$$\alpha_\varepsilon(B) := \frac{1}{c_0\varepsilon}\int_B h_\varepsilon^2\,\mathrm{d}x.$$

4.6 Summary, Notation, Assumptions

For a sequence u_ε, we have introduced the following Radon measures: The mass measures

$$\mu_\varepsilon(B) = \frac{1}{c_0}\int_B \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\,W(u_\varepsilon)\,\mathrm{d}x$$

which localise the Modica-Mortola energy, the diffuse Willmore measures

$$\alpha_\varepsilon(B) = \frac{1}{c_0\varepsilon}\int_B \left(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right)^2\,\mathrm{d}x = \frac{1}{c_0\varepsilon}\int_B h_\varepsilon^2\,\mathrm{d}x$$

which localise the diffuse Willmore functional and the discrepancy measures

$$\xi_\varepsilon(B) = \frac{1}{c_0}\int_B \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}\,W(u_\varepsilon)\,\mathrm{d}x.$$

The measures ξ_ε are signed (expected to be non-positive) and we also consider the positive measures $\xi_{\varepsilon,+}$ associated with their Hahn-decomposition and their total variation measures $|\xi_\varepsilon|$.

We will always assume that u_ε is a sequence such that

$$\sup_{\varepsilon>0}(S_\varepsilon + \mathcal{W}_\varepsilon)(u_\varepsilon) < \infty$$

and such that $u \in BV(\Omega)$ and Radon measures μ, α exist which satisfy $u_\varepsilon \to u$ in $L^1(\Omega)$ and $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$, $\alpha_\varepsilon \overset{*}{\rightharpoonup} \alpha$ in the weak* topology of Radon measures. We may later impose more restrictive conditions, and we will usually only consider so small ε that

$$(S_\varepsilon + \mathcal{W}_\varepsilon)(u_\varepsilon) \leq (\mu + \alpha)(\Omega) + 1.$$

This is always possible by choosing an appropriate subsequence. Later we will also consider the functional

$$\mathcal{F}_\varepsilon(u) := \mathcal{W}_\varepsilon(u) + \varepsilon^{-\sigma}\left(S_\varepsilon(u) - S\right)^2 : -1 + W_0^{2,2}(\Omega) \to [0,\infty)$$

for some $\sigma > 0$, which approximates Willmore's energy in the following sense. Note the reduced domain in definition due to modelling assumptions.

Theorem 4.6.1. *Let $E \Subset \Omega \Subset \mathbb{R}^n$ with $n = 2, 3$ such that $\partial E \in C^2$ and $\mathcal{H}^{n-1}(\partial E) = S$. Then there exists a sequence $u_\varepsilon \to \chi_E$ in $L^1(\Omega)$ such that*

$$\mathcal{F}_\varepsilon(u_\varepsilon) \to \mathcal{W}(\partial E).$$

If u_ε is any sequence such that $\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty$*, then there exist limits $u = \lim_{\varepsilon\to 0} u_\varepsilon$ in L^1 and $\mu = \lim_{\varepsilon\to 0} \mu_\varepsilon$ as Radon measures where*

$$|Du| \leq 2\mu, \qquad \operatorname{spt}(\mu) \subset \overline{\Omega}, \qquad \mu(\overline{\Omega}) = S$$

and μ is an integral varifold with square-integrable mean curvature satisfying

$$\mathcal{W}(\mu) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

In particular, this means that

$$\Gamma(L^1) - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon = \mathcal{W}$$

at χ_E with $E \Subset \Omega$, $\partial E \in C^2$ and $\mathcal{H}^{n-1}(\partial E) = S$ when we interpret the Willmore functional of a BV-function as acting on the essential boundary.

Proof. The usual recovery sequence can be used, potentially for a set

$$E_\varepsilon = (1 + \rho_\varepsilon)\, E$$

to fix $S_\varepsilon(u_\varepsilon) \equiv S$ so that the penalisation disappears in the limit, independently of the power σ. A slight modification suffices to ensure $u_\varepsilon \in -1 + W_0^{2,2}(\Omega)$ since $E \Subset \Omega$. By Theorem 4.5.1, μ exists and the relation between u and μ is readily established since it holds for finite

ε and is compatible with the different ways of taking limits for measures and BV-functions. The energy inequality is obvious since $\mathcal{F}_\varepsilon \geq \mathcal{W}_\varepsilon$ and finite energy sequences have uniformly bounded diffuse area S_ε. $\square$

Using the energy bound from [BM10, Theorem 4.1], we could take Bellettini and Mugnai's approximation of the Helfrich energy

$$\mathcal{E}_\varepsilon^{\mathrm{Hel}}(u) = \int_\Omega \frac{2+\chi}{2\varepsilon}\, h_{u,\varepsilon}^2 - \frac{\chi}{2\varepsilon}\left|\varepsilon\, \nabla^2 u - \frac{W'(u)}{\varepsilon}\, \nu_u \otimes \nu_u\right|^2 \,\mathrm{d}x$$

for $\chi \in (-2, 0)$ in place of the diffuse Willmore energy $\mathcal{W}_\varepsilon$. Here $h_{u,\varepsilon}$ is the usual Willmore density associated with u and $\nu_u = \nabla u/|\nabla u|$ is the diffuse normal direction. This extends our results for phase-field approximations of Willmore's energy to certain Canham-Helfrich functionals.

4.7 Concluding Remarks

This introduction has been tailored to include the relevant properties and examples of phase-fields for this thesis, which is dedicated to minimising Willmore's energy. In other research by the author [DKW17], a different application is discussed where the gradient term is replaced by a fractional Sobolev $H^{1/2}$-norm. Functionals of this type arise in physical modelling when Dirichlet's energy is minimised over a half-space given certain boundary values, which can in the stationary (or quasi-stationary) case be treated solely in terms of the boundary values. The limits of functionals of this type can be local or non-local perimeter functionals depending on the power s of the fractional Sobolev space H^s in the functional

$$\mathcal{E}_{s,\varepsilon}(u) = \frac{1}{c_\varepsilon}\left(\frac{1}{2}\,[u_\varepsilon]^2_{H^s(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \frac{1}{\varepsilon}\, W(u)\,\mathrm{d}x\right).$$

In the case $s \leq 1/2$, the optimal profile solution of

$$-(-\Delta)^s u = W'(u), \qquad u' > 0, \qquad \lim_{x\to\pm\infty} u(x) = \pm 1$$

on $\mathbb{R}$ does not have finite energy. In fact, any function on the real line having two different limits at $+\infty$ and $-\infty$ cannot have finite energy due to the non-locality of the norm. The case $s < 1/2$ is thus known as properly non-local, and the operator $-(-\Delta)^s$ bears some resemblance to an integral rather than a differential operator as the singularity at 0 is mild and the decay is slow. A function with finitely many jump discontinuities and compact support $u \in BV((0,1),\{0,1\})$, however, has finite energy. Thus we choose

$$c_\varepsilon = \begin{cases} 1 & s < 1/2 \\ |\log\varepsilon| & s = 1/2 \\ \varepsilon^{(2s-1)/(2s)} & s > 1/2 \end{cases} \cdot$$

For $s \geq 1/2$, (a slightly modified version of the functionals) $\mathcal{E}_{s,\varepsilon}$ converges to the ordinary perimeter functional, for $s \leq 1/2$, the limit is a non-local perimeter functionals introduced in [CRS10] as shown in [SV12]. The non-local perimeter functionals on the other hand Γ-converge to the usual perimeter as $s \to 1/2$ as shown in [ADPM11]. With this normalisation the transition length for $s \leq 1/2$ is proportional to ε and proportional to $\varepsilon^{1/(2s)}$ if $s \geq 1/2$, a more useful normalisation in less unified notation is

$$\mathcal{E}'_{\varepsilon,s}(u) = \frac{\varepsilon^{2s-1}}{2}\,[u]_s^2 + \int_{\mathbb{R}^n} \frac{1}{\varepsilon}\,W(u)\,\mathrm{d}x.$$

For the sake of completeness, we will list a few further properties of phase-fields in this context which will not be relevant in this dissertation, but help place it in the wider context or current research. The review aims only at the illustration of interesting related results and makes no claim of historical or mathematical completeness, which lies far beyond the scope of a brief chapter.

1. Minimisers u_ε of the Modica-Mortola functional on bounded domains with prescribed integral $\frac{1}{|\Omega|}\int_\Omega u_\varepsilon\,\mathrm{d}x \equiv S \in (0,1)$ constraint converge (up to the choice of a subsequence) uniformly to ± 1 away from a hypersurface which is the boundary of a Caccioppoli set which is locally area minimising with prescribed volume [Mod87] and [CC95, Theorem 2] under the assumption that a priori $-1 < u_\varepsilon < 1$. The sequence of Lagrange multipliers remains bounded and plays the rôle of constant mean curvature [LM89].

   This result has been extended in [HT00] to not necessarily minimising stationary points u_ε of the Allen-Cahn functional under total integral constraint without any a priori bound on u_ε, i.e. for solutions of

   $$-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}\,W'(u_\varepsilon) = \lambda_\varepsilon$$

   under the condition that the sequence of Lagrange multipliers λ_ε is uniformly bounded. Along a subsequence, we obtain limits $\lambda_\varepsilon \to \lambda_0$, $\mu_\varepsilon \stackrel{*}{\rightharpoonup} \mu$ where μ is an integral varifold with mean curvature λ_0. A rate of convergence

$$\big||u_\varepsilon| - 1\big| \leq C_{\Omega'}\varepsilon$$

for $\Omega' \Subset \Omega \setminus \operatorname{spt}(\mu)$ has been established. The proof is given only for convergence from outside $[-1, 1]$, but the result holds more generally with an only slightly modified proof.

2. Local minimisers of the Modica-Mortola functional near local minimisers of the perimeter functional in Ω (without integral constraint) have been constructed in [KS89].

3. In [Sav10], a conjecture of de-Giorgi was settled that solutions of the stationary Allen-Cahn equation on $\mathbb{R}^n$ for $n \leq 8$, which are monotone in one direction, are necessarily one-dimensional. More precisely, if

$$\Delta u = W'(u), \quad |u| \leq 1, \quad \partial_{x^1} u > 0 \quad \text{on } \mathbb{R}^n, \quad \text{and} \lim_{x_1 \to \pm\infty} u(x) = \pm 1,$$

then $u(x) = q(\langle v, x\rangle + b)$ for some $v \in S^{n-1}$. Furthermore, a global minimiser of S_1 on $\mathbb{R}^n$ is one-dimensional if $n \leq 7$. Counterexamples in higher dimensions exist [dPKW09], analogous to the change of behaviour in minimal surfaces.

4. Solutions to the Allen-Cahn equation [AC79]

$$\varepsilon u_t = \varepsilon \Delta u - \frac{1}{\varepsilon}\, W'(u),$$

(which is the time-normalised L^2-gradient flow of the Modica-Mortola energy) with well-prepared initial conditions for a surface $M = \partial E$ converge to solutions of mean curvature flow in a suitable sense, which is the L^2-gradient flow of the perimeter functional [Ilm93].

More precisely, an initial condition u_ε^0 is chosen such that $u_\varepsilon^0 \to \chi_E$ in $L^1(\Omega)$, $S_\varepsilon(u_\varepsilon^0) \to \operatorname{Per}(E)$ and such that $|u_\varepsilon^0| \leq 1$, $\xi_\varepsilon^0 \leq 0$, then the associated measures

$$\mu_\varepsilon^t = \left(\frac{\varepsilon}{2}\, |\nabla u_\varepsilon(t,\cdot)|^2 + \frac{1}{\varepsilon}\, W(u_\varepsilon(t,\cdot))\right) \cdot \mathcal{L}^n$$

converge as Radon measures to a motion by mean curvature in the sense of Brakke [Bra78], and for non-fattening initial conditions the zero level sets approach level set mean curvature flow [ESS92]. Non-uniqueness of Brakke flow at four-junctions can be recovered for different well-prepared initial conditions. Results for the related Allen-Cahn action functional or volume-preserving mean curvature flows are also available, see [MR11, MR08] and [Tak15] respectively.

The question whether this holds for multi-phase flows (where for example W has three wells in $\mathbb{R}^2$) is still open and related to the properties of the discrepancy measures. In the scalar case, the non-positivity of the discrepancy measures is propagated in time, while it is not even clear whether an initial condition with non-positive discrepancy measure exists at a triple junction in the vector valued case. In fact, non-positivity fails for solutions of a stationary vector valued Allen-Cahn equation

$$\Delta u = W_u(u)$$

when $u : \mathbb{R}^2 \to \mathbb{R}^2$ and W is chosen to vanish on the unit circle (Ginzburg-Landau model). Similarly, it fails for the fractional case. It is not clear whether it can be attained in the classical case with when W has three zeros in the plane. Multi-phase flows are used for example in the modelling of crystal grain growth and would be a valuable extension of the theory.

5. The ε in front of the time-derivative in the Allen-Cahn equation is included to obtain the right time-normalisation. The asymptotic expansion (4.5.1) shows that

$$\left(\varepsilon\,\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right)(x) \approx H_{\pi(x)}\,q'\left(\frac{\mathrm{sdist}(x,\partial E)}{\varepsilon}\right)$$

if $u_\varepsilon(x) = \phi(\mathrm{sdist}(x,\partial E))$ and $\pi : \Omega \to \partial E$ is the nearest point projection. The interface of u_ε has slope proportional to ε^{-1}, so to have an interface translating in normal time, we need $u_t = O(\varepsilon^{-1})$, which means that we need to rescale time as above – the same is obtained by a formal analysis making the ansatz

$$u_\varepsilon(t,x) = q'\left(\frac{\mathrm{sdist}(x,\partial E(t))}{\varepsilon}\right)$$

where $E(t)$ is a family of sets moving smoothly by mean curvature. The procedure should be complemented by a constant ansatz far away from the interface.

We can consider the terms in the Allen-Cahn equation separately. The first half, $u_t = \Delta u$ is the usual heat equation and describes the diffusion of the interface. It acts on a unit time-scale and wants to 'melt' the steep bump to make u_ε flatter. The heat flow out of the interface is proportional in second order to the mean curvature. The second half of the equation, $u_t = -\frac{1}{\varepsilon^2}W'(u)$ is an ODE which sorts $u \in (0,1)$ into the potential well at 1 and $u \in (-1,0)$ into the potential well at -1 on a very fast time-scale. Splitting these two parts formally into solving the heat equation for a short time and then sorting $u \in [0,1)$ to 1 and $u \in (-1,0)$ to -1 and repeating the procedure is the idea of the thresholding scheme [MBO92]. The thresholding scheme also converges to mean curvature flow, but to level set flow, not Brakke flow [ES91, Eva93].

6. As the Allen-Cahn equation approximates mean curvature flow and interfaces in dimension 1 are collections of points which do not have any curvature, it is clear that solutions of Allen-Cahn equation for well-prepared initial data in dimension $n = 1$ should become stationary in the limit $\varepsilon \to 0$ on the usual time-scale. In fact, more quantitative statements hold. In [CP89a] and [FH89] it is shown that solutions of the Allen-Cahn equation

$$e^{-1/\varepsilon}\varepsilon u_t = \varepsilon u_{xx} - \frac{1}{\varepsilon}\,W'(u), \qquad x \in (0,1), \qquad u_x(0) = u_x(1) = 0$$

with well-prepared initial data for jumps at position h_j^0, $1 \leq j \leq n$ converges to a function u as $\varepsilon \to 0$ which takes only the values $-1, 1$ and jumps at locations $h_j(t)$ governed by an explicit system of ODEs.

A heuristic motivation for this behaviour can be found in [CP89b]. Thus the dynamics of solutions to the Allen-Cahn equation in one dimension are exponentially slow in one dimension – in technical terms, well-prepared initial conditions for transitions at a finite number of points are *dynamically metastable*. Algebraic slowness has also been obtained by energy methods [BK90] (also for Dirichlet boundary values in $\{-1,1\}$) which were extended in [Gra95] to prove exponential slowness, also for the related Cahn-Hillard equation (see below) and its vector-valued version, the Cahn-Morral system. The energy method has been generalised in [OR07] to more generic systems exhibiting dynamic metastability and applied as an example to the Allen-Cahn equation.

Considerations on different time-scales for not well-prepared initial data (phase-separation and formation of meta-stable patterns on short scales), or potentials W with two-wells of different depth can be found for example in [Che92, Che04]. A more extensive review than this can be found in the introduction of [MR16].

The situation is entirely different if the gradient term is replaced by a fractional Sobolev norm. While solutions to the local Allen-Cahn equation in one dimension become exponentially slow as $\varepsilon \to 0$, they are only logarithmically slow if the $H^{1/2}$-seminorm is used instead. This surprisingly fast motion has been described for example in [GM12, PV15, PV16] and plays a key role in the author's work on crystal dislocations [DKW17].

7. While in the case of the Allen-Cahn equation, the limit of solutions to the gradient flow is a solution to the gradient flow of the limit in an appropriate sense, this is not immediate – consider for example the 'wiggly' potentials

$$f_\varepsilon : \mathbb{R} \to \mathbb{R}, \qquad f_\varepsilon(x) = x^2 + 2\varepsilon \sin(x^2/\varepsilon).$$

The sequence f_ε converges uniformly to $f_0(x) = x^2$, thus also in the sense of Γ-convergence, but a solution to the gradient flow of the limit f_0 approaches the absolute minimum at $x = 0$ exponentially fast as $t \to \infty$ independently of its initial condition, while solutions to the gradient flows of f_ε never move more than $\pm\sqrt{2\pi\varepsilon}$ from their initial value. This phenomenon also arises in practical applications, leading to interesting dynamic behaviour which is not captured by energy limits [DKW17].

8. The Modica-Mortola energy also plays a role in two-phase fluids where u represents the concentration of one of the two phases. There the energy is usually rescaled and also known as the Cahn-Hilliard energy

$$\mathcal{E}(u) = \int_\Omega W(u) + \frac{\varepsilon^2}{2}\,|\nabla u|^2\,\mathrm{d}x.$$

The dynamics here are usually described by the Cahn-Hillard equation

$$u_t = \Delta\left(W'(u) - \varepsilon^2 \Delta u\right)$$

which is the H^1-gradient flow of $\mathcal{E}$. The Cahn-Hilliard equation has the advantage of being volume preserving (given suitable boundary conditions) which is physically sensible in this context. For perturbations of a constant or very rough initial conditions, this models phase-separation over time given a small surface tension. Solutions of the Cahn-Hillard equation converge to solutions of the non-local evolution law know as Mullins-Sekerka motion (or two-phase Hele-Shaw flow) [Peg89, ABC94, Che96].

9. The title 'On a modified conjecture of De Giorgi' of [RS06] refers to the fact that the functional proposed by De Giorgi in [DG91] does not integrate h_ε^2 with respect to the suitably normalised Lebesgue measure, but with respect to the diffuse curvature measure μ_ε itself. While this may be conceptually more satisfying, the modern functional $\mathcal{W}_\varepsilon$ has analytic and numerical advantages. In particular, $\mathcal{W}_\varepsilon$ is quadratic in the highest order derivatives, namely the Laplacian. Thus the associated evolution equation is linear in these with constant coefficients, which is significantly more tractable than the highly non-linear coupling that would occur in the original functional. The modified

energy is due to Bellettini and Paolini [BP93].

The simple structure of the functional (especially compared to the highly non-linear Willmore functional) is one of its main advantages. On the other hand, the linearisation also leads to a certain non-convergence phenomenon. Namely, the Γ-limit of $\mathcal{W}_\varepsilon$ is $\mathcal{W}$ at C^2-boundaries and thus by diagonal approximation

$$\Gamma(L^1) - \lim_{\varepsilon\to 0} \mathcal{W}_\varepsilon \leq \widetilde{\mathcal{W}}$$

where $\widetilde{\mathcal{W}}$ is the lower semi-continuous envelope of Willmore's energy with respect to the L^1-topology of open sets. Unfortunately, equality does not hold. For example, strict inequality holds at a figure eight configuration in $\mathbb{R}^2$ where a saddle solution to the stationary Allen-Cahn equation is used around the singular point. On the other hand, the figure eight cannot be approximated by smooth boundaries with uniformly bounded energy since the smoothness at the singularity forces a curvature blow up, see [BP93, BDMP93].

Note, however, that any figure eight given by an immersed smooth curve is an integral varifold with square-integrable mean curvature. Alternative functionals have been proposed which do not have this deficiency by controlling the mean curvatures of the individual level sets, see e.g. [Bel97]. On the other hand, this leads to very non-linear energies which do not lend themselves to numerical implementation.

10. The phase-field approximation of Willmore's energy described above and others as well as their L^2-gradient flows are reviewed in [BMO13]. A convergence result for a diffuse approximation of certain more general Helfrich-type functionals has also been derived by Belletini and Mugnai [BM10].

    Numerical implementations of phase-field models can be found for example in [BKM05, DLRW05, DLW05, DLW06, DW07, DLRW07, DLRW09, Du10, WD07]. The two-step time-stepping algorithm of [BR12] was adapted for phase-field evolutions in [FRW13].

Chapter 5

On the Uniform Convergence of Phase-Fields

5.1 Introduction

This chapter is dedicated to the study of how phase-fields u_ε for Willmore's energy approach their limit u away from the set $\operatorname{spt}(\mu)$ where u_ε makes a fast transition. It is clear that L^1-convergence holds; in fact, it is easy to show that L^p-convergence holds for all $p < \infty$. We will show that L^∞-convergence does not hold in three dimensions, but will give a strong substitute which we name *essentially uniform convergence.* The chapter focuses on technical properties of phase-fields, which will be needed in the applications in Chapter 7. In this chapter, we will always make the following non-restrictive assumptions:

1. The sequence u_ε has finite energy, i.e. $\limsup_{\varepsilon\to\infty} \mathcal{E}_\varepsilon(u_\varepsilon) := (\mathcal{W}_\varepsilon + S_\varepsilon)(u_\varepsilon) < \infty$,
2. all quantities have a limit, i.e. $u_\varepsilon \to u$ in $L^1(\Omega)$, $\mu_\varepsilon \rightharpoonup \mu$ and $\alpha_\varepsilon \rightharpoonup \alpha$ and
3. ε is small enough for the phase-fields to resemble the limit in the sense that we assume that $\mu(\mathbb{R}^n) = \bar\mu$, $\alpha(\mathbb{R}^n) = \bar\alpha$ and $\mu_\varepsilon(\mathbb{R}^n) \leq \bar\mu + 1$, $\alpha_\varepsilon(\mathbb{R}^n) \leq \bar\alpha + 1$.

We can take a continuous representative of $u_\varepsilon \in W^{2,2}(\Omega) \hookrightarrow C^{0,1/2}(\overline{\Omega})$ if Ω is regular and $u_\varepsilon \in C^{0,1/2}_{loc}(\Omega)$ else. For u we take the representative that is constant ± 1 on $\Omega \setminus \operatorname{spt}(\mu)$ (which exists since $|Du| \leq 2\mu$). Then the following hold.

Theorem 5.1.1. *1. Let $\Omega' \Subset \Omega$. Then there exists $C > 0$ such that $|u_\varepsilon| \leq C$ on Ω' for all $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)^2$ and $u_\varepsilon \in C^{0,1/2}(B_\varepsilon(x))$ for all $x \in \Omega'$ with*

$$|u_\varepsilon(y) - u_\varepsilon(z)| \leq \frac{C}{\varepsilon^{1/2}}\, |y - z|^{1/2} \qquad \forall\, y, z \in B_\varepsilon(x).$$

2. *Let $\Omega' \Subset \Omega$. Then $u_\varepsilon \to u$ in $L^p(\Omega')$ for all $1 \leq p < \infty$.*

3. *Let $n = 2$, $\Omega' \Subset \Omega$. Then there exist $\bar\varepsilon > 0$, $C > 0$ such that*
$$\sup_{x\in\Omega'} |u_\varepsilon(x)| \leq 1 + C\,\varepsilon^{1/2} \qquad \forall\ \varepsilon < \bar\varepsilon.$$

4. *Let $n = 2$, $\Omega' \Subset \Omega \setminus \operatorname{spt}(\mu)$. Then there exist $\bar\varepsilon > 0$, $C > 0$ such that*
$$\sup_{\Omega'} |u_\varepsilon - u| \leq C\,\varepsilon^{1/2} \qquad \forall\ \varepsilon < \bar\varepsilon$$

5. *Let $n = 2$, $I \Subset (-1, 1)$ not empty. Then there exists a compact set $K \subset \overline{\Omega}$ and a subsequence $\varepsilon \to 0$ (not relabelled) such that $u_\varepsilon^{-1}(I) \to K$ in Hausdorff distance. K satisfies*
$$K \cap \Omega = \operatorname{spt}(\mu) \cap \Omega.$$

6. *Let $n = 3$, $\tau > 0$. Then there are only finitely many points $x \in \Omega$ with the following property:*
$$\exists\ x_\varepsilon \to x \quad \textit{such that } \limsup_{\varepsilon\to 0} |u_\varepsilon(x_\varepsilon)| \geq 1 + \tau.$$
The number of points can be bounded in terms of $\bar\mu$, $\bar\alpha$ and τ.

7. *Let $n = 3$, $\tau > 0$. Then there are only finitely many points $x \in \Omega \setminus \operatorname{spt}(\mu)$ with the following property:*
$$\exists\ x_\varepsilon \to x \quad \textit{such that } \limsup_{\varepsilon\to 0} \big|\, u_\varepsilon(x_\varepsilon) - u(x)\big| \geq \tau.$$
The number of such points can be bounded in terms of $\bar\mu$, $\bar\alpha$ and τ.

8. *Let $n = 3$, $\Omega' \Subset \Omega \setminus \operatorname{spt}(\mu)$. If α has no atoms in $\overline{\Omega'}$, then $u_\varepsilon \to u$ uniformly on Ω'. In particular, if V is an integral varifold supported in Ω with mass measure μ such that $\mu_\varepsilon \to \mu$ and additionally $\alpha_\varepsilon(\Omega) \to \mathcal{W}(\mu)$, then u_ε converges to u uniformly on all $\Omega' \Subset \Omega \setminus \operatorname{spt}(\mu)$.*

9. *Let $n = 3$, $I \Subset (-1, 1)$. Then there exists a compact set $K \subset \overline{\Omega}$ and a subsequence $\varepsilon \to 0$ (not relabelled) such that $u_\varepsilon^{-1}(I) \to K$ in Hausdorff distance. K satisfies*
$$K \cap \Omega = (\operatorname{spt}(\mu) \cap \Omega) \cup \{x_1, \dots, x_N\}$$
for finitely many points $x_1, \dots, x_N \in \Omega$. The number N can be bounded in terms of $\bar\mu$, $\bar\alpha$ and I. If α has no atoms outside $\operatorname{spt}(\mu)$, then $K \cap \Omega = \operatorname{spt}(\mu) \cap \Omega$.

10. *There exists a countable set* $\Delta \subset \Omega \setminus \operatorname{spt}(\mu)$, *such that* $u_\varepsilon \to u$ *pointwise everywhere on* $\Omega \setminus (\operatorname{spt}(\mu) \cup \Delta)$. *In particular, for* $C \Subset \Omega \setminus \operatorname{spt}(\mu)$, $s > 0$ *such that* $\mathcal{H}^s(C) < \infty$ *we have that* $u_\varepsilon \to u$ $\mathcal{H}^s|_C$*-almost everywhere. Since* u_ε *is uniformly bounded in* $L^\infty(C)$, *furthermore* $u_\varepsilon \to u$ *in* $L^p(\mathcal{H}^s|_C)$ *for all* $p < \infty$.

11. *If* $\Omega_1 \Subset \Omega_2 \Subset \Omega$ *and* $|u_\varepsilon| \geq 1/\sqrt{2}$ *on* Ω_2, *then there exists* $C > 0$ *such that*

$$\mu_\varepsilon(\Omega_1) \leq C\,\varepsilon^2.$$

The condition $|u_\varepsilon| \geq 1/\sqrt{2}$ *is always satisfied for small enough* ε *if either* $n = 2$ *or if* $n = 3$ *and all atoms of* α *are sufficiently small.*

The statement and the proof are split over Corollaries 5.2.3, 5.2.23, Lemma 5.2.8, Theorems 5.2.12, 5.2.13, 5.2.18, 5.2.21 and 5.2.27.

Under the same assumptions, Nagase and Tonegawa [NT07] proved (3) and uniform convergence in two dimensions. For the sake of completeness, we repeat their argument in the proof of Theorem 5.2.12 here and apply their techniques to establish the rate of convergence, which was only partly established there.

The differences between the cases $n = 2$ and $n = 3$ arise from the sharp interface problem, not the phase-field approximation. Namely, due to the fact that Willmore's energy is scale invariant, the sequence of manifolds

$$M_k = \partial B_1(0) \cup \partial B_{1/k}(0)$$

has Willmore energy $\mathcal{W}(M_k) \equiv 32\pi$ in $n = 3$ dimensions. It satisfies $M_k \to \partial B_1(0)$ in the measure sense, but $M_k \to \partial B_1(0) \cup \{0\}$ in Hausdorff distance. Such a sequence can be used to show that uniform convergence cannot hold for the phase-field problem. The analogue of Willmore's energy on curves (Euler's elastica energy) is not scale invariant since the exponent of the mean curvature $p = 2$ is higher than the dimension $n - 1 = 1$ of the manifold.

It is an important feature of our analysis that we only assume that $\mathcal{E}_\varepsilon(u_\varepsilon)$ is bounded and not necessarily that u_ε is a local minimiser or stationary point of a related functional under suitable side conditions. This is of central importance for applications in biology, where Willmore's energy is usually not the only term contributing to the total energy in a model.

We will give an example of a sequence of functions demonstrating that $\beta = 1/2$ is the optimal rate of convergence. Also in three dimensions, our result is sharp. While the formulation is new, it is geometrically intuitive. Namely, the sets

$$\Delta_\tau \coloneqq \{x \in \Omega \setminus \operatorname{spt}(\mu) \mid \exists\, x_\varepsilon \to x \text{ such that } \limsup_{\varepsilon\to 0} |u_\varepsilon(x_\varepsilon) - u(x)| \geq \tau\}$$

and $\Delta \coloneqq \bigcup_{\tau>0} \Delta_\tau = \bigcup_{k=1}^\infty \Delta_{1/k}$ encode how far u_ε is from converging uniformly to u. Since u is locally constant on $\Omega\setminus\operatorname{spt}(\mu)$, it is easy to see that $u_\varepsilon \to u$ locally uniformly on $\Omega\setminus\operatorname{spt}(\mu)$ if and only if $\Delta = \emptyset$. We show that the τ-distant sets Δ_τ are finite for all $\tau > 0$, but may be non-empty. So while uniform convergence cannot be achieved in general, the set where it fails by any given positive amount is as small as can be.

This is still a strong statement, and we shall call such functions converging essentially uniformly on $\Omega\setminus\operatorname{spt}(\mu)$. Essentially uniform convergence is especially suited for investigating functionals that depend on individual level sets and can be used to deduce uniform convergence for certain minimising sequences, see Section 5.3. The new technique is particularly useful in fourth order problems where energy competitors cannot be constructed as easily as in generalised Modica-Mortola functionals.

The chapter is organised as follows. In Section 5.2.1, we collect a few helpful results that will us allow to deal with the boundedness and L^p-convergence of u_ε in Section 5.2.2, uniform convergence in two and three dimensions in Sections 5.2.3 and 5.2.4 respectively and Hausdorff convergence of the level sets of u_ε to $\operatorname{spt}(\mu)$ in Section 5.2.5. Applications to uniform convergence for minimisers, the stationary Allen-Cahn equation and varifold geometry in three dimensions will be discussed in Section 5.3. We conclude the chapter with examples demonstrating that our results are sharp in Section 5.4.

5.2 Proofs

5.2.1 Auxiliary Estimates

In this section, we will collect a few improved estimates. The first Lemma is essentially obvious from the energy estimates, but important in controlling the Sobolev norms of u_ε from the control over $\mathcal{E}_\varepsilon(u_\varepsilon)$.

Lemma 5.2.1. *Let $u_\varepsilon \in W^{2,2}(\Omega)$. Then there is a constant C depending on $\mathcal{E}_\varepsilon(u_\varepsilon)$ and Ω such that*

$$||u_\varepsilon||_{2,\Omega} \leq C, \qquad ||\nabla u_\varepsilon||_{2,\Omega} \leq \frac{C}{\sqrt{\varepsilon}}, \qquad ||\Delta u_\varepsilon||_{2,\Omega} \leq \frac{C}{\varepsilon^{7/2}}.$$

If $\partial\Omega \in C^{0,1}$, we may use the Sobolev embeddings to see that $u_\varepsilon \in C^{0,1/2}(\overline{\Omega})$. On irregular sets, u_ε is still regular in the interior.

Proof. The first two estimates follow directly from the bound on S_ε as above. The bound on Δu_ε follows from an application of Young's inequality to obtain

$$\frac{\varepsilon}{2}\int_\Omega (\Delta u_\varepsilon)^2\,\mathrm{d}x \le \alpha_\varepsilon(\Omega) + \int_\Omega \frac{1}{\varepsilon^3}\,W'(u_\varepsilon)^2\,\mathrm{d}x$$

together with the estimate

$$\frac{1}{\varepsilon^3}\int_\Omega W'(u_\varepsilon)^2\,\mathrm{d}x \le \frac{C}{\varepsilon^3}\,\left(1+||u_\varepsilon||_{6,\Omega}^6\right) \le \frac{C}{\varepsilon^3}\,\left(1+||u_\varepsilon||_{1,2,\Omega}^6\right) \le \frac{C}{\varepsilon^{3+6/2}}. \tag{5.2.1}$$

□

Obviously, the Laplacian estimate is far from being optimal. When we have uniform L^∞-bounds over a set Ω', the integral can be dominated by $4\,||u_\varepsilon||_\infty^2\int_{\Omega'} W(u_\varepsilon)\,\mathrm{d}x$ instead, so $||\Delta u_\varepsilon||_{2,\Omega'} = O(\varepsilon^{-3/2})$. This is indeed the growth rate for optimal interfaces.

The next Lemma is a sharpened version of [RS06, Propositions 3.4 and 3.5] concerning how much mass the measures μ_ε can create while the phase-fields remain close to ± 1 or even outside $[-1,1]$.

Lemma 5.2.2. *Let $\Omega_0 \Subset \Omega_\infty \subset \Omega$, and $\delta := \operatorname{dist}(\Omega_0, \partial\Omega_\infty)$. Then for any $N \ge 1$ we have*

$$\begin{aligned}\int_{\Omega_0\cap\{|u_\varepsilon|>1\}} & 2\varepsilon\,|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon}\,W'(u_\varepsilon)^2\,\mathrm{d}x \\ &\le \frac{1-\left(\frac{(N+2)\,\varepsilon}{2\,\delta}\right)^N}{1-\frac{(N+2)\,\varepsilon}{2\,\delta}}\,\frac{\varepsilon^2}{2}\,\alpha_\varepsilon(\Omega_\infty\cap\{|u_\varepsilon|>1\}) \\ &\quad + \left(\frac{(N+2)\,\varepsilon}{2\,\delta}\right)^N \int_{\Omega_\infty\cap\{|u_\varepsilon|>1\}} 2\varepsilon\,|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon}\,W'(u_\varepsilon)^2\,\mathrm{d}x\end{aligned}$$

and for $0<\tau<1-1/\sqrt{2}$ we have

$$\begin{aligned}&\mu_\varepsilon(\Omega_0\cap\{1-\tau\le|u_\varepsilon|\le 1\}) \\ &\le \left(4\tau + \frac{4\,(N+2)\,\varepsilon}{\delta}\right)\left(\frac{1-\left(\frac{2\,(N+2)\,\varepsilon}{\delta}\right)^N}{1-\frac{2\,(N+2)\,\varepsilon}{\delta}}\right)\mu_\varepsilon(\Omega_\infty\cap\{|u_\varepsilon|\le 1-\tau\}) \\ &\quad + \frac{\varepsilon^2}{2}\left(\frac{1-\left(\frac{2\,(N+2)\,\varepsilon}{\delta}\right)^N}{1-\frac{2\,(N+2)\,\varepsilon}{\delta}}\right)\alpha_\varepsilon(\Omega_\infty\cap\{|u_\varepsilon|<1\}) \\ &\quad + \left(\frac{2\,(N+2)\,\varepsilon}{\delta}\right)^N \mu_\varepsilon(\Omega_\infty\cap\{1-\tau\le|u_\varepsilon|\le 1\}).\end{aligned}$$

To understand these complicated estimates better, let us first deduce a few easy consequences in the limit $\varepsilon \to 0$.

Corollary 5.2.3. *1. Assume that $r > 0$ and $B_r(x) \Subset \Omega$. Then*

$$\limsup_{\varepsilon\to 0} \frac{4\,\mu_\varepsilon(B_r \cap \{|u_\varepsilon| > 1\})}{\varepsilon^2} \leq \alpha(\overline{B_r}).$$

2. Assume that $|u_\varepsilon| \geq 1/\sqrt{2}$ on $B_{r+\delta}$ for some $r, \delta > 0$ and all sufficiently small $\varepsilon > 0$. Then

$$\limsup_{\varepsilon\to 0} \frac{2\,\mu_\varepsilon(B_r \cap \{|u_\varepsilon| < 1\})}{\varepsilon^2} \leq \alpha(\overline{B_r}).$$

3. Assume that $\Omega' \Subset \Omega''$ and that $|u_\varepsilon| \geq 1/\sqrt{2}$ on Ω''. Then there exist $\varepsilon_0, C > 0$ such that

$$\mu_\varepsilon(\Omega') \leq C\,\varepsilon^2 \qquad \forall\ \varepsilon < \varepsilon_0.$$

4. Assume that $\Omega' \Subset \Omega''$. Then there exist $\varepsilon_0, C > 0$ such that

$$\mu_\varepsilon\left(\Omega' \cap \{|u_\varepsilon| > 1\}\right) \leq C\,\varepsilon^2 \qquad \forall\ \varepsilon < \varepsilon_0.$$

5. Let $x \in \mathbb{R}^n$, $r > 0$, $0 < \tau < 1 - 1/\sqrt{2}$. Then

$$\limsup_{\varepsilon\to 0} \mu_\varepsilon\big(\{u_\varepsilon \geq 1-\tau\} \cap B_r(x)\big) \leq 4\,\tau\,\mu\left(\overline{B_r(x)}\right).$$

Proof. The statement is essentially obvious from Lemma 5.2.2 with the complicated terms all vanishing as $\varepsilon \to 0$. For the first point, we use Lemma 5.2.2 with $N \geq 10$ and $B_r, B_{r+\delta}$ for small $\delta > 0$ in conjunction with

$$\int_{B_{r+\delta}} \frac{1}{\varepsilon}\,W'(u_\varepsilon)^2\,\mathrm{d}x \leq C_{\bar\mu,\bar\alpha,r,n}\,\varepsilon^{-4}$$

from (5.2.1) to bound $\int_{\{|u_\varepsilon|>1\}} \frac{1}{\varepsilon}\,W'(u_\varepsilon)^2\,\mathrm{d}x$ over $B_{r+\delta}$. Note that $W'(u)^2 \geq 4W(u)$ for $u \geq 1$ and take first $\varepsilon \to 0$ and subsequently $\delta \to 0$.

For the second point, $N = 3$ suffices. Here the key feature is that $\mu_\varepsilon(B \cap \{|u_\varepsilon| < 1/\sqrt{2}\}) = 0$ for all $B \subset B_{r+\delta}(x)$ since $|u_\varepsilon| \geq 1/\sqrt{2}$. The third and fourth points follows very similarly.

The fifth point is proven by considering balls $B_r(x)$ and $B_{r+\delta}(x)$ first and then taking $\varepsilon \to 0$ and subsequently $\delta \to 0$. □

Proof of Lemma 5.2.2. Take $g \in C^{0,1}(\mathbb{R})$ and $\eta \in C^{0,1}_c(\Omega)$ and calculate

$$\begin{aligned}\int_\Omega h_\varepsilon\, g(u_\varepsilon)\,\eta\,\mathrm{d}x &= \int_\Omega \left(-\varepsilon\,\Delta u_\varepsilon + \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right) g(u_\varepsilon)\,\eta\,\mathrm{d}x\\ &= \int_\Omega \varepsilon\, g'(u_\varepsilon)\,|\nabla u_\varepsilon|^2\,\eta + \varepsilon\, g(u_\varepsilon)\,\langle \nabla u_\varepsilon, \nabla\eta\rangle + \frac{1}{\varepsilon}\,W'(u_\varepsilon)\,g(u_\varepsilon)\,\eta\,\mathrm{d}x.\end{aligned}$$

We specify either

$$g(u) = \begin{cases} W'(u) & |u| \geq 1-\tau \\ \frac{W'(1-\tau)}{1-\tau}\,u & |u| \leq 1-\tau \end{cases} \quad \text{and } A_\tau := \{1-\tau \leq |u_\varepsilon|\}, \quad B_\tau := \{|u_\varepsilon| < 1-\tau\}$$

where $|u_\varepsilon|$ lies above (respectively below) $1-\tau$ or

$$g(u) = \begin{cases} \frac{W'(1-\tau)}{1-\tau}\,u & |u| \leq 1-\tau \\ W'(u) & 1-\tau \leq |u| \leq 1 \\ 0 & |u| \geq 1 \end{cases} \quad \text{and } A_\tau := \{1-\tau \leq |u_\varepsilon| \leq 1\}, \quad B_\tau := \{|u_\varepsilon| < 1-\tau\}.$$

In both cases we can write

$$\begin{aligned}\int_\Omega h_\varepsilon\, g(u_\varepsilon)\,\eta\,\mathrm{d}x &= \int_{A_\tau} W''(u_\varepsilon)\,\varepsilon\,|\nabla u_\varepsilon|^2\,\eta + \frac{1}{\varepsilon}\,W'(u_\varepsilon)^2\,\eta\,\mathrm{d}x\\ &\qquad + \frac{W'(1-\tau)}{1-\tau}\int_{B_\tau} \varepsilon\,|\nabla u_\varepsilon|^2\,\eta + \frac{1}{\varepsilon}\,W'(u_\varepsilon)\,u_\varepsilon\,\eta\,\mathrm{d}x\\ &\qquad + \int_\Omega \varepsilon\, g(u_\varepsilon)\,\langle\nabla\eta, \nabla u_\varepsilon\rangle\,\mathrm{d}x\\ &\geq \int_{A_\tau} (3\,(1-\tau)^2 - 1)\,\varepsilon\,|\nabla u_\varepsilon|^2\,\eta + \frac{1}{\varepsilon}\,W'(u_\varepsilon)^2\,\eta\,\mathrm{d}x\\ &\qquad - \tau\,(1+\tau)\int_{B_\tau} \varepsilon\,|\nabla u_\varepsilon|^2\,\eta + \frac{1}{\varepsilon}\,(u_\varepsilon^2 - 1)\,u_\varepsilon^2\,\eta\,\mathrm{d}x\\ &\qquad + \int_\Omega \varepsilon\, g(u_\varepsilon)\,\langle\nabla\eta, \nabla u_\varepsilon\rangle\,\mathrm{d}x.\end{aligned}$$

This can be rearranged to

$$\begin{aligned}&\int_{A_\tau} (3\,(1-\tau)^2 - 1)\,\varepsilon\,|\nabla u_\varepsilon|^2\,\eta + \frac{1}{\varepsilon}\,W'(u_\varepsilon)^2\,\eta\,\mathrm{d}x + \tau\,(1+\tau)\int_{B_\tau} \frac{1}{\varepsilon}\,(1-u_\varepsilon^2)\,u_\varepsilon^2\,\eta\,\mathrm{d}x\\ &\qquad \leq \tau\,(1+\tau)\int_{B_\tau} \varepsilon\,|\nabla u_\varepsilon|^2\,\eta\,\mathrm{d}x + \int_\Omega h_\varepsilon\, g(u_\varepsilon)\,\eta\,\mathrm{d}x - \int_\Omega \varepsilon\, g(u_\varepsilon)\,\langle\nabla\eta, \nabla u_\varepsilon\rangle\,\mathrm{d}x.\end{aligned}$$

We will further estimate the last two terms on the right hand side. Observe that

$$\begin{aligned}\left|\int_\Omega h_\varepsilon\, g(u_\varepsilon)\,\eta\,\mathrm{d}x\right| &\leq \int_\Omega \left(\frac{1}{2\varepsilon}\, g(u_\varepsilon)^2 + \frac{\varepsilon}{2}\, h_\varepsilon^2\, \chi_{\{g\neq 0\}}\right)\eta\,\mathrm{d}x\\ &= \int_{A_\tau} \frac{1}{2\varepsilon} W'(u_\varepsilon)^2\,\mathrm{d}x + \int_{B_\tau} \frac{1}{2\varepsilon}\,\tau^2(1+\tau)^2\, u_\varepsilon^2\,\mathrm{d}x + \frac{\varepsilon}{2}\int_{\{g\neq 0\}} h_\varepsilon^2\,\eta\,\mathrm{d}x.\end{aligned}$$

Inserted in the previous inequality (with two terms on the left hand side), this gives

$$\begin{aligned}&\int_{A_\tau} (3\,(1-\tau)^2 - 1)\,\varepsilon\,|\nabla u_\varepsilon|^2\,\eta + \left(\frac{1}{\varepsilon} - \frac{1}{2\varepsilon}\right) W'(u_\varepsilon)^2\,\eta\,\mathrm{d}x\\ &\qquad + \frac{\tau\,(1+\tau)}{\varepsilon}\int_{B_\tau}\left(1 - u_\varepsilon^2 - \frac{\tau\,(1+\tau)}{2}\right) u_\varepsilon^2\,\eta\,\mathrm{d}x \qquad\qquad (5.2.2)\\ &\qquad \leq \tau\,(1+\tau)\int_{B_\tau} \varepsilon\,|\nabla u_\varepsilon|^2\,\eta\,\mathrm{d}x + \frac{\varepsilon^2}{2}\int_{\{g\neq 0\}} \frac{1}{\varepsilon}\, h_\varepsilon^2\,\eta\,\mathrm{d}x + \int_\Omega \varepsilon\, g(u_\varepsilon)\,\langle\nabla\eta, \nabla u_\varepsilon\rangle\,\mathrm{d}x.\end{aligned}$$

Finally, we consider the term involving the gradient of η, which we now specify. First, we choose a sequence of sets $\Omega_0 \Subset \Omega_1 \Subset \ldots \Subset \Omega_{N-1} \Subset \Omega_\infty$ such that $\mathrm{dist}(\partial\Omega_k, \partial\Omega_{k+1}) \geq \delta/(N+1)$ and $\mathrm{dist}(\partial\Omega_{N-1}, \partial\Omega_\infty) \geq \delta/(N+1)$. Now, we take a cut-off function $0 \leq \eta \leq 1$ satisfying $\eta \equiv 1$ on Ω_0, $\eta \equiv 0$ outside Ω_1 and $|\nabla\eta| \leq (N+2)/\delta$. First consider

$$g(u) = W'(u)\cdot\chi_{\{|u|>1\}}$$

which corresponds to the first type of function g for $\tau = 0$. Then (5.2.2) simplifies to

$$\int_{\{|u_\varepsilon|>1\}}\left(2\varepsilon\,|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon}\, W'(u_\varepsilon)^2\right)\eta\,\mathrm{d}x \leq \frac{\varepsilon^2}{2}\int_{\{|u_\varepsilon|>1\}} \frac{1}{\varepsilon}\, h_\varepsilon^2\,\eta\,\mathrm{d}x + \int_\Omega \varepsilon\, g(u_\varepsilon)\,\langle\nabla\eta, \nabla u_\varepsilon\rangle\,\mathrm{d}x$$

which implies the weaker estimate

$$\begin{aligned}&\int_{\Omega_0\cap\{|u_\varepsilon|>1\}} 2\varepsilon\,|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon}\, W'(u_\varepsilon)^2\,\mathrm{d}x\\ &\qquad \leq \frac{\varepsilon^2}{2}\,\alpha_\varepsilon(\Omega_1\cap\{|u_\varepsilon|>1\}) + \frac{(N+2)\,\varepsilon}{2\,\delta}\int_{\Omega_1\cap\{|u_\varepsilon|>1\}} 2\varepsilon\,|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon}\, W'(u_\varepsilon)^2\,\mathrm{d}x.\end{aligned}$$

We perform this estimate iteratively for pairs Ω_k, Ω_{k+1} and $\Omega_{N-1}, \Omega_\infty$ to obtain

$$\begin{aligned}&\int_{\Omega_0\cap\{|u_\varepsilon|>1\}} 2\varepsilon\,|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon}\, W'(u_\varepsilon)^2\,\mathrm{d}x\\ &\qquad \leq \frac{\varepsilon^2}{2}\left(\sum_{k=0}^{N-1}\left(\frac{(N+2)\,\varepsilon}{2\,\delta}\right)^k\right)\alpha_\varepsilon(\Omega_\infty\cap\{|u_\varepsilon|>1\})\\ &\qquad\quad + \left(\frac{(N+2)\,\varepsilon}{2\,\delta}\right)^N\int_{\Omega_\infty\cap\{|u_\varepsilon|>1\}} 2\varepsilon\,|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon}\, W'(u_\varepsilon)^2\,\mathrm{d}x.\end{aligned}$$

Simplifying the sum by a geometric series gives the correct formula. Now we may focus on the case $\tau \in (0, 1-1/\sqrt{2})$. In this situation, take g for general τ with $g \equiv 0$ above 1. Then clearly the following inequalities hold and may be used to simplify (5.2.2):

1. $3\,(1-\tau)^2 - 1 \geq 3\,\left(1/\sqrt{2}\right)^2 - 1 = 3/2 - 1 = 1/2$,

2. $\frac{1}{2}W'(u_\varepsilon)^2 = \frac{1}{2}\,u_\varepsilon^2\,(u_\varepsilon^2-1)^2 \geq \frac{1}{4}\,(u_\varepsilon^2-1)^2 = W(u_\varepsilon)$ for $u_\varepsilon \geq 1-\tau \geq 1/\sqrt{2}$,

3. $\tau^2\,u_\varepsilon^2 \leq \tau^2(1-\tau)^2 \leq \tau^2(2-\tau)^2 = 4\,W(1-\tau) \leq 4\,W(u_\varepsilon)$ for $|u_\varepsilon| \leq 1-\tau$ and

4. $1 - u_\varepsilon^2 - \tau(1+\tau)/2 \geq 1-(1-\tau)^2 - \tau(1+\tau)/2 = 3\tau(1-\tau)/2 \geq 3\tau/8$ for $|u_\varepsilon| \leq 1-\tau$.

Thus, when we simplify the constants, (5.2.2) implies that

$$\begin{aligned}\int_{\{1-\tau<|u_\varepsilon|<1\}} &\left(\frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\,W(u_\varepsilon)\right)\eta\,\mathrm{d}x + \frac{3\,\tau^2}{8}\int_{\{|u_\varepsilon|\leq 1-\tau\}} \frac{1}{\varepsilon}\,u_\varepsilon^2\,\eta\,\mathrm{d}x\\ &\leq 2\tau\int_{\{|u_\varepsilon|\leq 1-\tau\}} \varepsilon\,|\nabla u_\varepsilon|^2\,\eta\,\mathrm{d}x + \frac{\varepsilon^2}{2}\int_{\{|u_\varepsilon|<1\}}\frac{1}{\varepsilon}\,h_\varepsilon^2\,\eta\,\mathrm{d}x + \int_\Omega \varepsilon\,g(u_\varepsilon)\,\langle\nabla\eta,\nabla u_\varepsilon\rangle\,\mathrm{d}x.\end{aligned}$$

Again we use Young's inequality to deal with the boundary integral.

$$\begin{aligned}\int_\Omega \varepsilon\,g(u_\varepsilon)\,\langle\nabla\eta,\nabla u_\varepsilon\rangle\,\mathrm{d}x &\leq \int_{\{1-\tau\leq|u_\varepsilon|\leq 1\}} \varepsilon\,|W'(u_\varepsilon)|\,|\nabla u_\varepsilon|\,|\nabla\eta|\,\mathrm{d}x\\ &\qquad + \int_{\{|u_\varepsilon|\leq 1-\tau\}} \varepsilon\,\tau\,(1+\tau)\,|u_\varepsilon|\,|\nabla u_\varepsilon|\,|\nabla\eta|\,\mathrm{d}x\\ &\leq \frac{(N+2)\,\varepsilon}{\delta}\int_{\{1-\tau<|u_\varepsilon|\leq 1\}\cap\Omega_1} \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon}W'(u_\varepsilon)^2\,\mathrm{d}x\\ &\qquad + \frac{(1+\tau)\,(N+2)\,\varepsilon}{\delta}\int_{\{|u_\varepsilon|\leq 1-\tau\}\cap\Omega_1} \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon}\,\tau^2 u_\varepsilon^2\,\mathrm{d}x\\ &\leq \frac{2\,(N+2)\,\varepsilon}{\delta}\,\mu_\varepsilon(\Omega_1\cap\{1-\tau\leq|u_\varepsilon|<1\})\\ &\qquad + \frac{2\,(1+\tau)\,(N+2)\,\varepsilon}{\delta}\,\mu_\varepsilon(\Omega_1\cap\,\{|u_\varepsilon|\leq 1-\tau\}).\end{aligned}$$

Thus overall

$$\begin{aligned}\mu_\varepsilon(\Omega_0\cap\{1-\tau\leq|u_\varepsilon|\leq 1\}) \leq &\left(4\tau + \frac{4\,(N+2)\,\varepsilon}{\delta}\right)\mu_\varepsilon\,(\Omega_1\cap\{|u_\varepsilon|\leq 1-\tau\})\\ &+ \frac{\varepsilon^2}{2}\,\alpha_\varepsilon(\Omega_1\cap\{|u_\varepsilon|\leq 1\})\\ &+ \frac{2\,(N+2)\,\varepsilon}{\delta}\,\mu_\varepsilon(\Omega_1\cap\{1-\tau\leq|u_\varepsilon|\leq 1\}).\end{aligned}$$

Again we can iterate the estimate to find that

$$\begin{aligned}
&\mu_\varepsilon(\Omega_0 \cap \{1-\tau \le |u_\varepsilon| \le 1\}) \\
&\quad \le \left(4\tau + \frac{4\,(N+2)\,\varepsilon}{\delta}\right)\left(\sum_{k=0}^{N-1}\left(\frac{2\,(N+2)\,\varepsilon}{\delta}\right)^k\right)\mu_\varepsilon(\Omega_\infty \cap \{|u_\varepsilon| \le 1-\tau\}) \\
&\quad + \frac{\varepsilon^2}{2}\left(\sum_{k=0}^{N-1}\left(\frac{2\,(N+2)\,\varepsilon}{\delta}\right)^k\right)\alpha_\varepsilon(\Omega_\infty \cap \{|u_\varepsilon| < 1\}) \\
&\quad + \left(\frac{2\,(N+2)\,\varepsilon}{\delta}\right)^N \mu_\varepsilon(\Omega_\infty \cap \{1-\tau \le |u_\varepsilon| \le 1\}).
\end{aligned}$$

□

Before we move on, let us recall two results. A key tool in our argument is a simplified monotonicity formula. We will first give the exact version, which is a phase-field analogue of the varifold monotonicity formula

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^{-k}\mu(B_\rho(x))\right) = \frac{\mathrm{d}}{\mathrm{d}\rho}\int_{B_\rho(x)} \frac{|D^\perp r|^2}{r^n}\,\mathrm{d}\mu + \rho^{-(k+1)}\int_{B_\rho(x)} \langle y-x, H\nu\rangle\,\mathrm{d}\mu(y)$$

which holds for k-varifolds in $\mathbb{R}^n$ in the distributional sense [Sim83, Chapter 17] (where $D^\perp$ is the gradient orthogonal to $T_x\mu$ and $r(y) = |x-y|$). Note that this reduces to

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^{-k}\mu(B_\rho(x))\right) = \int_{\partial B_\rho(x)} \frac{|D^\perp r|^2}{r^n}\,\mathrm{d}\mu + \rho^{-(k+1)}\int_{B_\rho(x)} \langle y-x, H\rangle\,\mathrm{d}\mu(y)$$

for almost all radii ρ (namely all $\rho > 0$ such that $\mu(\partial B_\rho(x)) = 0$), compare also (3.2.4).

Lemma 5.2.4. *[RS06, Lemma 4.2] For $x \in \Omega$ we have*

$$\begin{aligned}
\frac{d}{d\rho}\left(\rho^{1-n}\,\mu_\varepsilon(B_\rho(x))\right) = &-\frac{\xi_\varepsilon(B_\rho(x))}{\rho^n} + \frac{1}{c_0\,\rho^{n+1}}\int_{\partial B_\rho(x)} \varepsilon\,\langle y-x, \nabla u_\varepsilon\rangle^2\,\mathrm{d}\mathcal{H}^{n-1}(y) \\
&+ \frac{1}{c_0\,\rho^n}\int_{B_\rho(x)} h_\varepsilon\,\langle y-x, \nabla u_\varepsilon\rangle\,\mathrm{d}y.
\end{aligned}$$

Proof. We assume $x = 0$ and write $B_\rho := B_\rho(0)$ for $\rho > 0$. Then for $h > 0$ we introduce the cut-off function

$$\eta : [0,\infty) \to \mathbb{R}\,, \qquad \eta(r) = \begin{cases} 1 & r \le \rho \\ 1 - (r-\rho)/h & r \in (\rho, \rho+h) \\ 0 & r \ge \rho + h \end{cases}$$

and for later use the vector field $V(x) := \eta(|x|)\cdot x$. Note that $\eta = \eta_h$ does depend on the small parameter. As usual, we abbreviate $r := |x|$. This means

$$\begin{aligned}
\frac{\mu_\varepsilon(B_{\rho+h}) - \mu_\varepsilon(B_\rho)}{h} &= \int_{B_{\rho+h}\setminus B_\rho} \frac{1}{h}\,\mathrm{d}\mu_\varepsilon \\
&= -\int_{\mathbb{R}^n} \eta'(r)\,\mathrm{d}\mu_\varepsilon \\
&= -\frac{1}{\rho}\int_{\mathbb{R}^n} \rho\,\eta'(r)\,\mathrm{d}\mu_\varepsilon \\
&= \frac{1}{\rho}\int_{\mathbb{R}^n} -(r\,\eta' + n\,\eta) + (r-\rho)\,\eta' + n\,\eta\,\mathrm{d}\mu_\varepsilon \\
&= \frac{1}{\rho}\int_{\mathbb{R}^n} -\operatorname{div}(V) + (r-\rho)\,\eta' + n\,\eta\,\mathrm{d}\mu_\varepsilon\,.
\end{aligned}$$

When we take $h \to 0$ later, the second term drops out because the integrand is bounded and μ_ε is absolutely continuous with respect to $\mathcal{L}^n$. The first of the three terms is computed as follows.

$$\begin{aligned}
-\int_{\mathbb{R}^n} \operatorname{div}(V)\,\mathrm{d}\mu_\varepsilon &= -\int_{\mathbb{R}^n} \operatorname{div}(V)\,\left(\frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\,W(u_\varepsilon)\right)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \frac{\varepsilon}{2}\,\langle \nabla\,|\nabla u_\varepsilon|^2, V\rangle + \frac{1}{\varepsilon}\,W'(u_\varepsilon)\,\langle \nabla u_\varepsilon, V\rangle\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \varepsilon\,V^i\,(\partial_i\partial_j u_\varepsilon)\,\partial_j u_\varepsilon + \frac{1}{\varepsilon}\,W'(u_\varepsilon)\,\langle \nabla u_\varepsilon, V\rangle\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} -\varepsilon\,\partial_j V^i\,\partial_i u_\varepsilon\,\partial_j u_\varepsilon - \varepsilon\,V^i\,\partial_i u_\varepsilon\,\partial_j\partial_j u_\varepsilon + \frac{1}{\varepsilon}\,W'(u_\varepsilon)\,\langle \nabla u_\varepsilon, V\rangle\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} -\varepsilon\,\left(\eta'\,\frac{x^i\,x^j}{|x|} + \eta\,\delta^i_j\right)\,\partial_i u_\varepsilon\,\partial_j u_\varepsilon + \left(-\varepsilon\,\Delta u_\varepsilon + \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right)\,\langle \nabla u_\varepsilon, V\rangle\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} -\varepsilon\,\eta'(r)\,\frac{\langle x, \nabla u_\varepsilon\rangle^2}{|x|} - \eta\,\varepsilon\,|\nabla u_\varepsilon|^2 + h_\varepsilon\,\langle \nabla u_\varepsilon, x\rangle\,\eta\,\mathrm{d}x\,.
\end{aligned}$$

Now we can take $h \to 0$ and obtain (with $\eta = \eta_h$)

$$\begin{aligned}
&\lim_{h\searrow 0} \frac{\mu_\varepsilon(B_{\rho+h}) - \mu_\varepsilon(B_\rho)}{h} \\
&\quad= \frac{1}{\rho^2}\int_{\partial B_\rho} \varepsilon\,\langle x, \nabla u_\varepsilon\rangle^2\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{\rho}\int_{B_\rho} (-2)\,\frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + h_\varepsilon\,\langle \nabla u_\varepsilon, x\rangle\,\mathrm{d}x \\
&\qquad\quad + \frac{n}{\rho}\int_{B_\rho} \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\,W(u_\varepsilon)\,\mathrm{d}x \\
&\quad= \frac{1}{\rho^2}\int_{\partial B_\rho} \varepsilon\,\langle \nabla u_\varepsilon, x\rangle^2\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{\rho}\int_{B_\rho} h_\varepsilon\,\langle \nabla u_\varepsilon, x\rangle\,\mathrm{d}x + \frac{1}{\rho}\int_{B_\rho} \frac{1}{\varepsilon}\,W(u_\varepsilon) - \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2\,\mathrm{d}x \\
&\qquad\quad + \frac{n-1}{\rho}\int_{B_\rho} \frac{1}{\varepsilon}\,W(u_\varepsilon) + \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2\,\mathrm{d}x \\
&\quad= \frac{1}{\rho^2}\int_{\partial B_\rho} \varepsilon\,\langle \nabla u_\varepsilon, x\rangle^2\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{\rho}\int_{B_\rho} h_\varepsilon\,\langle \nabla u_\varepsilon, x\rangle\,\mathrm{d}x + \frac{n-1}{\rho}\mu_\varepsilon(B_\rho) - \frac{1}{\rho}\,\xi_\varepsilon(B_\rho)\,.
\end{aligned}$$

A similar computation can of course be done for $h < 0$ so the function $f(\rho) := \mu_\varepsilon(B_\rho)$ is differentiable with the derivative given above. Furthermore

$$\begin{aligned}
\frac{d}{d\rho}\left(\rho^{1-n} f(\rho)\right) &= (1-n)\,\rho^{-n} f(\rho) + \rho^{1-n} f'(\rho) \\
&= \frac{1-n}{\rho^n} f(\rho) - \frac{\xi_\varepsilon(B_\rho)}{\rho^n} + \frac{n-1}{\rho^n}\, f(\rho) + \frac{1}{\rho^{n+1}} \int_{\partial B_\rho} \varepsilon\, \langle \nabla u_\varepsilon, x\rangle^2 \,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad + \frac{1}{\rho^n} \int_{B_\rho} h_\varepsilon \,\langle \nabla u_\varepsilon, x\rangle \,\mathrm{d}x
\end{aligned}$$

Cancelling out the two equal terms proves the result. $\square$

In low dimensions $n = 2, 3$, the second and third term in the monotonicity formula can easily be estimated after integration to a localised Li-Yau type formula (3.2.1). The proof is a slightly corrected version of that of [RS06, Proposition 4.5].

Lemma 5.2.5. *[RS06, Proposition 4.5] Let $0 < r < R < \infty$ if $n = 3$ and $0 < r < R \leq 1$ if $n = 2$, then*

$$\begin{aligned}
r^{1-n} \mu_\varepsilon(B_r(x)) \leq 3\, R^{1-n} \mu_\varepsilon(B_R(x)) + 2 \int_r^R \frac{\xi_{\varepsilon,+}(B_\rho(x))}{\rho^n}\,\mathrm{d}\rho \\
+ \frac{1}{2\,(n-1)^2} \alpha_\varepsilon(B_R(x)) + \frac{r^{3-n}}{(n-1)^2}\, \alpha_\varepsilon(B_r(x)) \\
+ \frac{R_0^2\, R^{1-n}}{(n-1)^2}\, \alpha_\varepsilon(B_R(x))
\end{aligned} \tag{5.2.3}$$

where $R_0 := \min\{R, R_\Omega\}$ and R_Ω is a radius such that $\Omega \subset B(0, R_\Omega/2)$

Proof. Without loss of generality we may assume that $x = 0$ and write $B_\rho := B(0,\rho)$, $f(\rho) = \rho^{1-n}\mu_\varepsilon(B_\rho)$. Observe that for any function $g : B_R \to \mathbb{R}$ we have

$$\begin{aligned}
\int_r^R \rho^{-n} \int_{B_\rho} g(x)\,\mathrm{d}x\,\mathrm{d}\rho &= \int_{B_R} g(x) \int_{\max\{|x|,r\}}^R \rho^{-n}\,\mathrm{d}\rho\,\mathrm{d}x \\
&= \frac{1}{n-1} \int_{B_R} g(x) \left(\frac{1}{\max\{|x|,r\}^{n-1}} - \frac{1}{R^{n-1}}\right) \mathrm{d}x
\end{aligned}$$

and

$$\int_r^R \rho^{-(n+1)} \int_{\partial B_\rho} g(x)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}\rho = \int_{B_R \setminus B_r} \frac{g(x)}{|x|^{n+1}}\,\mathrm{d}x.$$

Using this to integrate the derivative and using Young's inequality with $\lambda \in (0,1)$ we obtain

$$
\begin{aligned}
f(R) - f(r) &= \int_r^R f'(\rho)\,\mathrm{d}\rho \\
&= \int_r^R \frac{-\,\xi_\varepsilon(B_\rho)}{\rho^n}\,\mathrm{d}\rho + \frac{1}{c_0}\int_{B_R\setminus B_r} \frac{\varepsilon\,\langle \nabla u_\varepsilon, y\rangle^2}{|y|^{n+1}} + \frac{1}{n-1}\,\frac{h_\varepsilon\,\langle y, \nabla u_\varepsilon\rangle}{|y|^{n-1}}\,\mathrm{d}y \\
&\quad + \frac{1}{(n-1)\,c_0\,r^{n-1}}\int_{B_r} h_\varepsilon\,\langle y, \nabla u_\varepsilon\rangle\,\mathrm{d}y - \frac{1}{(n-1)\,c_0\,R^{n-1}}\int_{B_R} h_\varepsilon\,\langle y, \nabla u_\varepsilon\rangle\,\mathrm{d}y \\
&\geq \int_r^R \frac{-\,\xi_{\varepsilon,+}(B_\rho)}{\rho^n}\,\mathrm{d}\rho \\
&\quad + \frac{1}{c_0}\int_{B_R\setminus B_r} \frac{\varepsilon\,\langle \nabla u_\varepsilon, y\rangle^2}{|y|^{n+1}} - \frac{1}{n-1}\left((n-1)\,\varepsilon\,\frac{\langle y, \nabla u_\varepsilon\rangle^2}{|y|^{2(n-1)}} + \frac{1}{4\,(n-1)\,\varepsilon}\,h_\varepsilon^2\right)\mathrm{d}y \\
&\quad - \frac{1}{c_0\,r^{n-1}}\int_{B_r} \lambda\,\frac{\varepsilon\,\langle y, \nabla u_\varepsilon\rangle^2}{2\,|y|^2} + \frac{1}{2\lambda}\,\frac{|y|^2\,h_\varepsilon^2}{(n-1)^2\,\varepsilon}\,\mathrm{d}y \\
&\quad - \frac{1}{c_0\,R^{n-1}}\int_{B_R} \lambda\,\frac{\varepsilon\,\langle y, \nabla u_\varepsilon\rangle^2}{2\,|y|^2} + \frac{1}{2\lambda}\,\frac{|y|^2\,h_\varepsilon^2}{(n-1)^2\,\varepsilon}\,\mathrm{d}y \\
&\geq \int_r^R \frac{-\,\xi_{\varepsilon,+}(B_\rho)}{\rho^n}\,\mathrm{d}\rho - \frac{1}{4\,(n-1)^2}\int_{B_R\setminus B_r} \frac{1}{\varepsilon}\,h_\varepsilon^2\,\mathrm{d}y \\
&\quad - \lambda\,f(r) - \frac{1}{2\lambda}\,\frac{r^2}{(n-1)^2\,r^{n-1}}\int_{B_r} \frac{h_\varepsilon^2}{\varepsilon}\,\mathrm{d}y \\
&\quad - \lambda\,f(R) - \frac{1}{2\lambda}\,\frac{R_0^2}{(n-1)^2\,R^{n-1}}\int_{B_R} \frac{h_\varepsilon^2}{\varepsilon}\,\mathrm{d}y.
\end{aligned}
$$

In the second inequality, we used that $|y| \leq R_0$ wherever $h_0 \neq 0$ and that $2(n-1) \leq n+1$ in dimensions $n = 2, 3$, so that $|y|^{n+1} \leq |y|^{2(n-1)}$ for all $|y|$ if $n = 3$ and for $|y| \leq 1$ if $n = 2$. This allows us to cancel the singular integrals containing $\langle \nabla u_\varepsilon, y\rangle$ which we cannot control. When we bring all the relevant terms to the other side, this shows that

$$
\begin{aligned}
(1+\lambda)\,f(R) - (1-\lambda)\,f(r) \geq -\int_r^R \frac{\xi_{\varepsilon,+}(B_\rho)}{\rho^n}\,\mathrm{d}\rho - \frac{1}{4\,(n-1)^2}\,\alpha_\varepsilon(B_R\setminus B_r) \\
- \frac{r^{3-n}}{2\,\lambda\,(n-1)^2}\,\alpha_\varepsilon(B_r) - \frac{R_0^2}{2\lambda\,(n-1)^2\,R^{n-1}}\,\alpha_\varepsilon(B_R).
\end{aligned}
$$

Setting $\lambda = 1/2$ and multiplying by two proves the Lemma. □

Remark 5.2.6. If $n = 3$, we may let $R \to \infty$ and subsequently $\varepsilon \to 0$, $r \to 0$ and finally $\lambda \to 0$ in the proof so that we have

$$
\limsup_{r\to 0} r^{1-n}\mu(B_r(x)) \leq \frac{1}{4\,(n-1)^2}\,\alpha(\overline{\Omega})
$$

at every point $x \in \mathbb{R}^3$ such that $\alpha(\{x\}) = 0$ (i.e. when $\lim_{r\to 0}\alpha(B_r) = 0$) since $|\xi_\varepsilon| \to 0$. Using the results of [RS06], μ is an integral varifold, so this yields a Li-Yau-type [LY82] inequality

$$\theta^*(\mu, x) = \limsup_{r\to 0} \frac{\mu(B_r(x))}{\pi\, r^2} \leq \frac{1}{16\,\pi}\alpha(\overline{\Omega}) \tag{5.2.4}$$

which we had obtained in the sharp interface limit directly with $\mathcal{W}(\mu)$ in place of α in (3.2.2).

In $n = 2$ dimensions, we had to assume $R \leq 1$. Indeed, an inequality of this type cannot hold since circles with large enough radii have arbitrarily small elastic energy. Still, setting $R = 1$, a similar bound on the multiplicity in terms of $\bar\alpha$ and $\overline{\mu}$ can be obtained.

The version of the monotonicity formula (5.2.3) which we will use is the simplified expression

$$r^{1-n}\mu_\varepsilon(B_r(x)) \leq 3\, R^{1-n}\mu_\varepsilon(B_R(x)) + 3\,\alpha_\varepsilon(B_R(x)) + 2\int_r^R \frac{\xi_{\varepsilon,+}(B_\rho(x))}{\rho^n}\, d\rho. \tag{5.2.5}$$

This holds generally if $n = 3$, and when $R \leq 1$ if $n = 2$. We can think of this as a diffuse analogue of the localised Li-Yau inequality (3.2.1) with error terms stemming from the phase-field level.

Furthermore, we have the following estimate for the positive part of the discrepancy measures. It is a precise quantitative refinement of the classic statement that smooth solutions of the stationary Allen-Cahn equation $-\Delta u + W'(u) = 0$ on $\mathbb{R}^n$ satisfy $|\nabla u|^2 \leq 2\, W(u)$ [Mod85].

Lemma 5.2.7. *[RS06, Lemma 3.1] Let $n = 2, 3$. Then there are $\delta_0 > 0, M \in \mathbb{N}$ such that for all $0 < \delta \leq \delta_0$, $0 < \varepsilon \leq \rho$ and*

$$\rho_0 := \max\{2, 1 + \delta^{-M}\varepsilon\}\,\rho$$

we have

$$\begin{aligned}\rho^{1-n}\xi_{\varepsilon,+}(B_\rho(x)) \leq C\,\delta\,\rho^{1-n}\,\mu_\varepsilon(B(x,2\rho)) + C\,\delta^{-M}\varepsilon^2\,\rho^{1-n}\int_{B(x,\rho_0)} \frac{1}{\varepsilon}\,h_\varepsilon^2\,\mathrm{d}x \\ + C\,\delta^{-M}\varepsilon^2\,\rho^{1-n}\int_{B(x,\rho_0)\cap\{|u_\varepsilon|>1\}} \frac{1}{\varepsilon^3}W'(u_\varepsilon)^2\,\mathrm{d}x + \frac{C\,\varepsilon\,\delta}{\rho}.\end{aligned}$$

5.2.2 L^p-regularity

Now we are ready to prove the first major result. Denote

$$\Omega_\varepsilon^\beta := \{x \in \Omega_\varepsilon \mid B_{\varepsilon^\beta}(x) \subset \Omega\}, \qquad \Omega_\varepsilon := \Omega_\varepsilon^{1/2}.$$

Lemma 5.2.8. *Assume that $\beta < 1$ and ε is so small that $\varepsilon \leq \varepsilon^\beta/4$. Then there is $C_{\bar\alpha,\bar\mu,n,\beta} >$*

0 *such that*

$$||u_\varepsilon||_{\infty,\Omega_\varepsilon^\beta} \leq C_{\bar\alpha,\bar\mu,n,\beta}.$$

Take $x \in \Omega_\varepsilon^\beta$ *and set* $B_\varepsilon \coloneqq B_\varepsilon(x)$. *Then* u_ε *is Hölder-continuous on* B_ε *with*

$$|u_\varepsilon(y) - u_\varepsilon(z)| \leq \frac{C_{\bar\alpha,\bar\mu,n,\beta,\gamma}}{\varepsilon^\gamma}\,|y-z|^\gamma$$

for all $y, z \in B_\varepsilon$ *and* $\gamma \leq 1/2$ *if* $n = 3$, $\gamma < 1$ *if* $n = 2$.

Optimal interfaces have precisely these Hölder-coefficients, so they cannot be improved.

Proof. **Step 1.** In a first step, we will prove that for sufficiently small $\varepsilon > 0$ and $x \in \Omega_\varepsilon^\beta$ we have a bound

$$\int_{B_{2\varepsilon}(x)\cap\{|u_\varepsilon|>1\}} \frac{1}{\varepsilon^3} W'(u_\varepsilon)^2 \,\mathrm{d}x' \leq C_{\bar\alpha,\bar\mu,n,\beta}.$$

First, we observe that due to Sobolev embeddings scaled to small balls, we have

$$||u_\varepsilon||_{\infty,B_{\varepsilon^\beta}(x)} \leq C_n\,\varepsilon^{-\beta n/2}\,||u_\varepsilon||_{2,2,B_{\varepsilon^\beta}(x)} \leq C_n\,\varepsilon^{-(n\beta+7)/2}.$$

Now, we consider Lemma 5.2.2 for $N = N_\beta$ such that $N_\beta\,(1-\beta) \geq 9 + n\beta$. Using $\varepsilon^\beta - 2\varepsilon \geq \varepsilon^\beta/2$, this tells us that

$$\begin{aligned}\int_{B_{2\varepsilon}\cap\{|u_\varepsilon|>1\}} \frac{1}{\varepsilon}\,W'(u_\varepsilon)^2\,\mathrm{d}x' &\leq (1 + c_{N_\beta,\varepsilon^{1-\beta}})\,\frac{\varepsilon^2}{2}\,\alpha_\varepsilon(B_{\varepsilon^\beta})\\ &\quad + \varepsilon^2\,2^{N_\beta}\,(N_\beta+2)^{N_\beta}\,\varepsilon^{2\,(n\beta+7)/2}\,(1 + ||u_\varepsilon||^2_{\infty,B_{\varepsilon^\beta}})\,4\,\mu_\varepsilon(B_{\varepsilon^\beta})\\ &\leq C_{\bar\alpha,\bar\mu,\beta,n}\,\varepsilon^2.\end{aligned}$$

Note that the terms depending on β are uniformly bounded and vanish as $\varepsilon \to 0$.

Step 2. Defining the blow up $\tilde u_\varepsilon \colon B_2(0) \to \mathbb{R}$ by $\tilde u_\varepsilon(y') = u_\varepsilon(x + \varepsilon y')$ we observe that

$$\int_{B_2(0)} (W'(\tilde u_\varepsilon))^2\,\mathrm{d}y' \leq C_{\bar\alpha,\bar\mu,n,\beta}.$$

after rescaling the previous estimate and that hence

$$\begin{aligned}
\int_{B_2(0)} \tilde{u}_\varepsilon^2 \,\mathrm{d}x' &= \int_{B_2(0)} \left(|\tilde{u}_\varepsilon| - 1 + 1\right)^2 \,\mathrm{d}y' \\
&\leq \int_{B_2(0)} \left((|\tilde{u}_\varepsilon| - 1)_+ + 1\right)^2 \,\mathrm{d}y' \\
&\leq 2 \int_{B_2(0)} \left(|\tilde{u}_\varepsilon| - 1\right)_+^2 + 1 \,\mathrm{d}y' \\
&\leq \varepsilon^{3-n} \int_{\{|u_\varepsilon|>1\}\cap B_{2\varepsilon}(x)} \frac{1}{\varepsilon^3} W'(u_\varepsilon)^2 \,\mathrm{d}y' + 2^{n+1}\,\omega_n \\
&\leq C_{\bar{\alpha},\bar{\mu},n,\beta}.
\end{aligned}$$

As usual, ω_n denotes the volume of the n-dimensional unit ball.

Step 3. Now a direct calculation shows that

$$\begin{aligned}
\int_{B_2(0)} \left(\Delta\tilde{u}_\varepsilon - W'(\tilde{u}_\varepsilon)\right)^2 \,\mathrm{d}y' &= \int_{B_2(0)} (\varepsilon^2 \Delta u_\varepsilon - W'(u_\varepsilon))^2\,(x+\varepsilon y)\,\mathrm{d}y' \\
&= c_0\,\varepsilon^{3-n}\,\alpha_\varepsilon(B_{2\varepsilon}(x)),
\end{aligned}$$

thus

$$||\,\Delta\tilde{u}_\varepsilon\,||_{2,B_2(0)} \leq ||\,\Delta\tilde{u}_\varepsilon - W'(\tilde{u}_\varepsilon)\,||_{2,B_2(0)} + ||\,W'(\tilde{u}_\varepsilon)\,||_{2,B_2(0)} \leq C_{\bar{\alpha},\bar{\mu},n,\beta}.$$

In total, we see that

$$||\,\tilde{u}_\varepsilon\,||_{2,B_2(0)} + ||\,\Delta\tilde{u}_\varepsilon\,||_{2,B_2(0)} \leq C_{\bar{\alpha},\bar{\mu},n,\beta}.$$

Therefore, the elliptic estimate [GT83, Theorem 9.11] implies that

$$||\tilde{u}_\varepsilon||_{2,2,B_1(0)} \leq C_{\bar{\alpha},\bar{\mu},n,\beta}.$$

Using the Sobolev embeddings

$$W^{2,2}(B_1(0)) \hookrightarrow W^{1,p}(B_1(0)) \hookrightarrow C^{0,\gamma}(\overline{B_1(0)})$$

for $p \leq 6$, $\gamma \leq 1/2$ if $n = 3$ and $p < \infty$, $\gamma < 1$ if $n = 2$, we deduce that

$$|\tilde{u}_\varepsilon|_{0,\gamma,B_1(0)} \;\leq\; C_{\bar{\alpha},\bar{\mu},n,\beta,\gamma}.$$

In particular, this shows that

$$||\tilde{u}_\varepsilon||_{\infty,B_1(0)} \leq C_{\bar{\alpha},\bar{\mu},n,\beta}.$$

Since this holds for all balls $B_\varepsilon(x)$ with $x \in \Omega_\varepsilon^\beta$, we can deduce that

$$|| \, u_\varepsilon \, ||_{\infty, \Omega_\varepsilon^\beta} \leq C_{\bar{\alpha}, \bar{\mu}, n, \beta}.$$

Furthermore, for $x \in \Omega_\varepsilon^\beta$ and $y, z \in B_\varepsilon(x)$, we deduce

$$\begin{aligned} |u_\varepsilon(z) - u_\varepsilon(y)| &= |\tilde{u}_\varepsilon((z-x)/\varepsilon) - \tilde{u}_\varepsilon(\,(y-x)/\varepsilon)| \\ &\leq C_{\bar{\alpha}, \bar{\mu}, n, \beta, \gamma} \, |\,(y-x)/\varepsilon - (z-x)/\varepsilon\,|^\gamma \\ &= \frac{C_{\bar{\alpha}, \bar{\mu}, n, \beta, \gamma}}{\varepsilon^\gamma} \, |z-y|^\gamma. \end{aligned}$$

□

Remark 5.2.9. Note that Ω_ε is growing as $\varepsilon \to 0$, so that the local boundedness and Hölder continuity hold on every set $\Omega' \Subset \Omega$ with constants independent of Ω', at least for small enough $\varepsilon > 0$. We shall make use of this in the following. The proof shows further more that the dependence on β vanishes as $\varepsilon \to 0$.

If we have information on the boundary values of u_ε, the previous Lemma can be sharpened and the proof be simplified. This will be discussed in detail in Chapter 6.

Remark 5.2.10. We can use blow up sequences $\tilde{u}_\varepsilon(y) = u_\varepsilon(x_\varepsilon + \varepsilon y)$ along a sequence $x_\varepsilon \in \Omega$. If x_ε has a limit $x \in \Omega$, then $B_r(x) \subset \Omega$ for some $r > 0$, and thus we may define $\tilde{u}_\varepsilon$ on $B_{r/(2\varepsilon)}$ for all $\varepsilon > 0$ so small that $x_\varepsilon \in B_{r/2}(x)$. Like above $||\tilde{u}_\varepsilon||_{2,2,U} \leq C$ for all $U \Subset \mathbb{R}^n$, so there is a function $\tilde{u} \in W^{2,2}_{loc}(\mathbb{R}^n)$ such that $\tilde{u}_\varepsilon \rightharpoonup \tilde{u} \in W^{2,2}_{loc}(\mathbb{R}^n)$.

Then in particular $-\Delta\tilde{u}_\varepsilon + W'(\tilde{u}_\varepsilon) \rightharpoonup -\Delta\tilde{u} + W'(\tilde{u})$ (using compact embeddings on the non-linear term), and we obtain that

$$|| -\Delta\tilde{u} + W'(\tilde{u})||^2_{2,U} \leq \liminf_{\varepsilon \to 0} || -\Delta\tilde{u}_\varepsilon + W'(\tilde{u}_\varepsilon)||^2_{2,U} = \liminf_{\varepsilon \to 0} c_0 \, \varepsilon^{3-n} \, \alpha_\varepsilon(x_\varepsilon + \varepsilon U).$$

Thus $-\Delta\tilde{u} + W'(\tilde{u}) = 0$ if $n = 2$ or if $n = 3$ and x is not an atom of α. Elliptic regularity shows that $\tilde{u} \in C^\infty(\mathbb{R}^n)$, so $\tilde{u}$ is an entire solution of the stationary Allen-Cahn-Equation. In this way, Lemma 5.2.8 implies an analogue of Theorem 4.3.4 for the Willmore case.

Corollary 5.2.11. *Let $1 \leq p < \infty$. Then $u_\varepsilon \to u$ in $L^p(\Omega')$ for all $\Omega' \Subset \Omega$.*

Proof. We know that $u_\varepsilon \to u$ in $L^2(\Omega)$ and that the sequence u_ε is bounded uniformly in $L^\infty(\Omega')$. Hölder's inequality does the rest. □

5.2.3 Convergence in Two Dimensions

In this section, we shall consider $n = 2$. Our first result resembles Lemma 6.2.4 on the convergence of phase-fields from outside $[-1, 1]$ also at the interface. This version does not require boundary values.

Theorem 5.2.12. *Take Ω_ε^β as in Lemma 5.2.8. Then there exists $C > 0$ depending only on $\bar\alpha, \bar\mu, \beta$ such that*

$$\sup_{x\in\Omega_\varepsilon^\beta} |u_\varepsilon(x)| \leq 1 + C\varepsilon^{1/2}.$$

The result is given in [NT07, Lemma 3.2]. We repeat the proof here for the reader's convenience, also since we use it to establish the rate of convergence also from inside $[-1, 1]$.

Proof. We proved the estimates

$$\frac{1}{\varepsilon^3}\int_{\{|\tilde u_\varepsilon|>1\}} W'(\tilde u_\varepsilon)^2\,\mathrm{d}x \leq C, \qquad \int_{B_2(0)} \tilde h_\varepsilon^2\,\mathrm{d}x \leq \alpha_\varepsilon(\Omega)\,\varepsilon.$$

Take a sequence of monotone increasing convex functions $g_k \in C^2(\mathbb{R})$ such that

$$g_k \to \max\{0, \cdot\}$$

uniformly on $\mathbb{R}$ and set $u_\varepsilon^k = g_k(\tilde u_\varepsilon - 1)$ where $\tilde u_\varepsilon$ is the blow up as in Lemma 5.2.8. Then

$$\begin{aligned}
\nabla u_\varepsilon^k &= g_k'\,\nabla\tilde u_\varepsilon\\
\Delta u_k &= g_k''\,|\nabla\tilde u_\varepsilon|^2 + g_k'\,\Delta\tilde u_\varepsilon\\
&\geq g_k'\,\left(\Delta\tilde u_\varepsilon - W'(\tilde u_\varepsilon)\right) + g_k'\,W'(\tilde u_\varepsilon)\\
&\geq g_k'\,\left(\,\Delta\tilde u_\varepsilon - W'(\tilde u_\varepsilon)\right).
\end{aligned}$$

By [GT83, Theorem 8.17] we obtain

$$\sup_{x\in B_1(0)} \tilde u_\varepsilon^k(x) \leq C\,\left(||\tilde u_\varepsilon^k||_{2,B_2(0)} + ||\,g_k'\,\tilde h_\varepsilon\,||_{2,B_2(0)}\right)$$

and taking $k \to \infty$, we get

$$\sup_{x\in B_1(0)} \max\{\tilde u_\varepsilon - 1, 0\} \leq C\,\left(||\,(u_\varepsilon - 1)_+||_{2,B_2(0)} + ||\,\tilde h_\varepsilon\,||_{2,B_2(0)}\right) \leq C\,\varepsilon^{1/2}.$$

Since this holds for all balls of size ε centred in Ω_ε^β with uniform constants, the theorem is proven. For the other direction, substitute u_ε by $-u_\varepsilon$. $\square$

Away from $\mathrm{spt}(\mu)$, we get convergence from inside $(-1, 1)$ as well.

Theorem 5.2.13. *Let $\Omega' \Subset \Omega \setminus \operatorname{spt}(\mu)$. Then there exist $\bar{\varepsilon}, C > 0$ such that*

$$\max_{x\in\Omega'} \big| u_\varepsilon(x) - u(x)\big| \leq C\,\varepsilon^{1/2}$$

for all $\varepsilon < \bar{\varepsilon}$. While $\bar{\varepsilon}$ cannot be estimated in terms of Ω' and energy values, C depends only on $\bar{\alpha}, \bar{\mu}$.

Proof. In a first step, it is necessary to show that $u_\varepsilon \to u$ uniformly on Ω'. This has been done for example in [NT07, Proposition 4.2] or [DLW17, Theorem 2.1] – the proof is similar to the one of Theorem 5.2.21 and will not be given here, but see Remark 5.2.24.

The second step resembles the proof of Theorem 5.2.12 when take g_k to be an approximation of $\max\{0, -z\}$ instead by smooth convex and monotone decreasing functions. The key estimate is $\mu_\varepsilon(\Omega') \leq C\,\varepsilon^2$ from Corollary 5.2.3, which is applicable since $|u_\varepsilon| \geq 1/\sqrt{2}$ due to uniform convergence.

When taking an approximating sequence of phase-fields $u_\varepsilon \to u$, the convergence can be slow in ε despite $\mathcal{E}_\varepsilon(u_\varepsilon) \approx \mathcal{E}(u)$. When a recovery-type sequence u_ε for sets $E_\varepsilon \to E$ is chosen (e.g. in C^2-topology), then $\bar{\varepsilon}$ depends also on the speed of convergence $E_\varepsilon \to E$ which cannot be estimated by energy bounds. $\square$

Unlike its three-dimensional counterpart, this result does not require the existence of α but only a uniform bound on $\mathcal{W}_\varepsilon(u_\varepsilon)$ and infinitesimal Hölder continuity. The proof above implicitly used the following result in the second step.

Corollary 5.2.14. *Let $\Omega' \Subset \Omega \setminus \operatorname{spt}(\mu)$. Then there exists $C > 0$ depending only on $\bar{\alpha}, \bar{\mu}$ and $\operatorname{dist}(\Omega', \partial\Omega \cup \operatorname{spt}(\mu))$ such that*

$$\mu_\varepsilon(\Omega') \leq C\,\varepsilon^2$$

for all sufficiently small $\varepsilon > 0$.

Remark 5.2.15. While it is not possible to obtain a convergence rate better than $\varepsilon^{1/2}$ (see Example 5.4.1), there are only few points where the convergence becomes this slow. For $\beta < 2/3$, and $B_r(x) \Subset \Omega \setminus \operatorname{spt}(\mu)$, set

$$D_\varepsilon := \{s \in (0, r) \mid \min_{\partial B_s(x)} |u_\varepsilon| \leq 1 - \varepsilon^\beta\}.$$

Then for all $\sigma < 2 - 3\beta$ there exists $C > 0$ which depends on the Hölder continuity of u_ε on ε-balls and $\bar{\alpha}$ such that

$$\mathcal{L}^1(D_\varepsilon) \leq C\,\varepsilon^\sigma$$

for all small ε. Assume the contrary. Note that due to $C^{0,\gamma}$-Hölder continuity for all $\gamma < 1$ on ε-balls, we know that

$$|u_\varepsilon(x)| \leq 1 - \varepsilon^\beta \qquad \Rightarrow \qquad |u_\varepsilon(y)| \leq 1 - \frac{\varepsilon^\beta}{2} \quad \forall\, y \in B_{c\,\varepsilon^{1+\beta/\gamma}}(x)$$

for some small positive c depending on γ. We take γ so close to one that $\sigma + 2\beta + \beta/\gamma < 2$. Due to our assumption, there are $N_\varepsilon = O(\varepsilon^{\sigma-(1+\beta/\gamma)}) \gg 1$ radii $s_{1,\varepsilon}, \dots, s_{N_\varepsilon,\varepsilon}$ radii in D_ε such that $|s_{i,\varepsilon} - s_{j,\varepsilon}| \geq c\varepsilon^{1+\beta/\gamma}$ for all $1 \leq i \neq j \leq N_\varepsilon$. Now we take points

$$x_{i,\varepsilon} \in \partial B_{s_{i,\varepsilon}}(x), \qquad |u_\varepsilon(x_{i,\varepsilon})| \leq 1 - \varepsilon^\beta$$

and compute

$$\begin{aligned}
\mu_\varepsilon(B_r(x)) &\geq \sum_{i=1}^{N_\varepsilon} \mu_\varepsilon\left(B_{c\varepsilon^{1+\beta/\gamma}}(x_{i,\varepsilon})\right) \\
&\geq N_\varepsilon \left[\pi(c\varepsilon^{1+\beta/\gamma})^2\right] \frac{1}{\varepsilon} W\left(1 - \frac{\varepsilon^\beta}{2}\right) \\
&= O\left(\varepsilon^{\sigma-(1+\beta/\gamma)}\varepsilon^{2(1+\beta/\gamma)}\,\varepsilon^{-1}\,\varepsilon^{2\beta}\right) \\
&= O\left(\varepsilon^{\sigma+2\beta+\beta/\gamma}\right).
\end{aligned}$$

Due to Corollary 5.2.14, this is also $O(\varepsilon^2)$, but our γ is close enough to 1 to show that

$$\liminf_{\varepsilon\to 0} \varepsilon^{-2}\mu_\varepsilon(B_r(x)) \to \infty.$$

This suggests that on most of $\Omega' \Subset \Omega \setminus \operatorname{spt}(\mu)$, the convergence should have a better rate than $\sqrt{\varepsilon}$.

5.2.4 Convergence in Three Dimensions

In this section, we will investigate the convergence of u_ε in $n = 3$ dimensions. As we shall see in Example 5.4.1, uniform convergence away from the interface does not hold in this case. Therefore, we are forced to introduce a new notion of convergence which is better adapted to phase-field problems.

Definition 5.2.16. Let $U \subset \mathbb{R}^n$, $f_\varepsilon, f\colon U \to \mathbb{R}$ continuous functions. Then we say that $f_\varepsilon \to f$ *essentially uniformly (e.u.)* if the sets

$$\Delta_\tau := \{x \in U \mid \exists\, x_\varepsilon \to x \text{ such that } \limsup_{\varepsilon\to 0} |f_\varepsilon(x_\varepsilon) - f(x)| \geq \tau\}$$

are finite for all $\tau > 0$.

Since we assume f to be continuous, locally uniform convergence corresponds to $\Delta_\tau = \emptyset$ for all $\tau > 0$ and implies essentially uniform convergence. Even without the assumption of continuity, e.u. convergence implies convergence pointwise everywhere on the complement of a countable set. With this definition, our results on convergence in three dimensions can be summarised as

$$u_\varepsilon \to u \text{ e.u. on } \Omega \setminus \operatorname{spt}(\mu) \qquad \text{and} \qquad (|u_\varepsilon| - 1)_+ \to 0 \text{ e.u. on } \Omega.$$

Remark 5.2.17. Essentially uniform convergence is a powerful tool for our purposes, but still quite far from uniform convergence. The following properties are easy to establish.

1. Assume that $f_\varepsilon \to f$ e.u. on U. Then $\Delta = \bigcup_{\tau>0} \Delta\tau = \bigcup_{k=1}^\infty \Delta_{1/k}$ is countable and $f_\varepsilon(x) \to f(x)$ for all $x \in U \setminus \Delta$.

2. Let $K \Subset U \setminus \Delta$. Then $f_\varepsilon \to f$ uniformly on K.

3. Δ is countable and may lie dense in U, in which case the previous point is vacuous. In particular, it may happen that $f_\varepsilon \to f$ e.u. but there exists no open set $U' \subset U$ such that $f_\varepsilon \to f$ uniformly on U'. We shall see in Example 5.4.1 that this may happen in our case of finite energy sequences u_ε.

In one space dimension, the same kind of convergence was used by Dal Maso and Iurlano for phase-fields governed by a Modica-Mortola energy [DMI13, Proof of Proposition 1]. In one dimension, the Modica-Mortola functional controls functions well enough to show essentially uniform convergence. In Remark 5.2.26 we discuss under what assumptions our techniques can be adapted to prove essentially uniform convergence in higher dimensions.

As in the two dimensional case, we begin by proving convergence from outside $[-1, 1]$, also at $\operatorname{spt}(\mu)$.

Theorem 5.2.18. *Let $\tau > 0$ and $x \in \Omega$ a point for which there exists a sequence $x_\varepsilon \to x$ such that $\limsup_{\varepsilon\to 0} |u_\varepsilon(x_\varepsilon)| \geq 1 + \tau$. Then there exists $\bar\theta > 0$ depending only on $\bar\alpha, \bar\mu$ and τ such that $\alpha(\{x\}) \geq \bar\theta$. In particular, there are only finitely many such points.*

Proof. Passing to a subsequence (not relabelled) and replacing τ by $\tau/2$, we may assume that $|u_\varepsilon(x_\varepsilon)| \geq 1 + \tau$ for all ε. Since Ω is open, there exists $r > 0$ such that $B_{4r}(x) \subset \Omega$. Thus $B_{3r}(x) \subset \Omega_\varepsilon$ for all sufficiently small ε, so we may use Lemma 5.2.8 with uniform constants. Since $x_\varepsilon \to x$, for all sufficiently small $\varepsilon > 0$ we have $B_\varepsilon(x_\varepsilon) \subset B_r(x)$, and by

Hölder-continuity of u_ε, there is $0 < c < 1$ such that

$$|u_\varepsilon| \geq 1 + \frac{\tau}{2} \quad \text{on } B_{c\varepsilon}(x_\varepsilon)$$

which implies that

$$\mu_\varepsilon(B_r(x)) \geq \mu_\varepsilon(B_{c\varepsilon}(x_\varepsilon)) \geq \omega_n\,(c\varepsilon)^n\,\frac{W(1+\tau/2)}{\varepsilon}.$$

Using Corollary 5.2.3, we find that $\alpha(B_{2r}) \geq \omega_n\,c^n\,W(1+\tau/2)$ where c only depends on the Hölder constant of u_ε on $B_{c\varepsilon}(x_\varepsilon)$ and thus only on the energy bounds. Taking $r \to 0$, we see that

$$\alpha(\{x\}) \geq C_{\bar\alpha,\bar\mu,\tau}.$$

A point with the properties of x is therefore an atom of α with a minimal size depending on $\bar\alpha, \bar\mu$ and τ. In particular, since $\bar\alpha < \infty$, there are only finitely many such points. □

Note that we had to use the limiting measure α. Its existence may always be achieved by taking a subsequence $\varepsilon \to 0$. On the other hand, if we add bumps as Example 5.4.1 based at points along a dense sequence in some $\Omega' \Subset \Omega \setminus \operatorname{spt}(\mu)$, we see that all points $x \in \Omega'$ are limits of bad sequences. Thus the existence of α is of critical importance for the argument above.

We slightly abuse notation and denote by $\mathcal{W}_\varepsilon, S_\varepsilon, \mathcal{E}_\varepsilon$ also the functionals given by the same formulae as above on the function space $L^1(B_1(0))$ instead of $L^1(\Omega)$.

Lemma 5.2.19. *Let $n = 2, 3$, $B = B_1(0) \subset \mathbb{R}^n$, $\theta \in [0, 1)$ and*

$$X_\theta := \{u \in W^{2,2}(B) \mid |u(0)| \leq \theta\}.$$

Then the function

$$e\colon [0,1) \to \mathbb{R}, \quad e(\theta) := \liminf_{\varepsilon \to 0} \inf_{u \in X_\theta} \mathcal{E}_\varepsilon(u)$$

is strictly positive.

Proof. For a contradiction, assume that there is $\theta \in [0, 1)$ and a sequence $u_\varepsilon \in X_\theta$ such that $\mathcal{E}_\varepsilon(u_\varepsilon) \to 0$. As usual, denote $B_\rho := B_\rho(0)$ and the diffuse mass and Willmore measures by μ_ε and α_ε, respectively, despite the change of domain. Consider the densities

$$f_\varepsilon(\rho) := \rho^{1-n}\mu_\varepsilon(B_\rho)$$

for $\rho \in [\varepsilon, 1]$. By the Hölder continuity on $B_{1/2}$ from Lemma 5.2.8, we get $f_\varepsilon(\varepsilon) = \varepsilon^{1-n}\mu_\varepsilon(B_\varepsilon) \geq \bar c > 0$ for a uniform constant depending only on θ (since $\bar\mu = \bar\alpha = 0$ by assumption). In the next step, we will apply Lemma 5.2.7 with $\delta = \eta_\varepsilon\,(\varepsilon/\rho)^\beta$ for some $0 < \beta < 1/M$ and $\eta_\varepsilon \to 0$ so slowly that

1. $\eta_\varepsilon^{-M}\alpha_\varepsilon(B) \to 0$ and
2. $\eta_\varepsilon^{-M}\varepsilon^{1-M\beta} \leq 1$.

Note that the second condition also implies that $\delta^{-M}\varepsilon = (\varepsilon/\rho)^{-M\beta}\,\eta_\varepsilon^{-M}\varepsilon \leq 1$ for $\rho \geq \varepsilon$. In particular, $\delta < \delta_0$ independently of $\rho \geq \varepsilon$ for all small enough $\varepsilon > 0$. Using the estimated monotonicity formula from Lemma 5.2.5 for $\varepsilon = r < R = 1/3$ together with the estimates for

- $\xi_{\varepsilon,+}$ from Lemma 5.2.7 for the δ given above, for
- $||u_\varepsilon||_{\infty,B_{2/3}}$ from Lemma 5.2.8 and for
- $\int_{B_{2/3}\cap\{|u_\varepsilon|>1\}} \frac{1}{\varepsilon^3} W'(u_\varepsilon)^2\,\mathrm{d}x$ from Lemma 5.2.2 with $N = 3$,

we obtain

$$\begin{aligned}
f_\varepsilon(\varepsilon) &\leq 3\,R^{1-n}\,\mu_\varepsilon(B_R) + 3\,\alpha_\varepsilon(B_R) + 2\int_r^R \frac{\xi_{\varepsilon,+}(B_\rho)}{\rho^n}\,\mathrm{d}\rho\\
&\leq 3\,R^{1-n}\,\mu_\varepsilon(B_R) + 3\,\alpha_\varepsilon(B_R) + 2\,C\int_r^R \eta_\varepsilon\,\frac{\varepsilon^\beta}{\rho^{1+\beta}}\,\rho^{1-n}\,\mu_\varepsilon(B_{2\rho})\,\mathrm{d}\rho\\
&\quad + \int_r^R \frac{\varepsilon^{2-M\beta}}{\rho^{n-M\beta}}\,\eta_\varepsilon^{-M}\left(\alpha_\varepsilon(B_{2\rho}) + \int_{B_{2\rho}\cap\{|u_\varepsilon|>1\}}\frac{1}{\varepsilon^3}W'(u_\varepsilon)^2\,\mathrm{d}x\right) + \frac{\varepsilon^{1+\beta}\,\eta_\varepsilon}{\rho^{2+\beta}}\,\mathrm{d}\rho\\
&\leq 3\,R^{1-n}\,\mu_\varepsilon(B_R) + 3\,\alpha_\varepsilon(B_R) + \int_r^R \frac{2\,C\,\eta_\varepsilon\,\varepsilon^\beta}{\rho^{1+\beta}}\,f_\varepsilon(2\rho)\,\mathrm{d}\rho\\
&\quad + \frac{C}{1+\beta}\,\varepsilon^{1+\beta}\left[r^{-(1+\beta)} - R^{-(1+\beta)}\right]\eta_\varepsilon\\
&\quad + \frac{C}{n-1-M\beta}\,\varepsilon^{2-M\beta}\left\{r^{1-n+M\beta} - R^{1-n+M\beta}\right\}\eta_\varepsilon^{-M}\\
&\qquad\cdot\left\{\alpha_\varepsilon(B_{2R}) + \frac{1-\left(\frac{5\varepsilon}{R}\right)^3}{1-\frac{5\,\varepsilon}{R}}\,\alpha_\varepsilon(B_{2R}) + \frac{1}{\varepsilon^2}\left(\frac{5\,\varepsilon}{R}\right)^3 ||u_\varepsilon||_{\infty,B_{2R}}\cdot 4\,\mu_\varepsilon(B_{2R})\right\}\\
&\leq \gamma_\varepsilon + \int_r^{2R}\frac{2\,C\,\eta_\varepsilon\,\varepsilon^\beta}{\rho^{1+\beta}}\,f_\varepsilon(\rho)\,\mathrm{d}\rho
\end{aligned}$$

with $\gamma_\varepsilon \to 0$ as $\varepsilon \to 0$. We may now use Grönwall's inequality backwards in time to deduce that

$$f_\varepsilon(\varepsilon) \leq \gamma_\varepsilon\,\exp\left(\int_\varepsilon^{2/3}\frac{C\,\eta_\varepsilon\,\varepsilon^\beta}{\rho^{1+\beta}}\,\mathrm{d}\rho\right) \leq C\,\gamma_\varepsilon.$$

This is a contradiction since $\gamma_\varepsilon \to 0$, but on the other hand $f_\varepsilon(\varepsilon) \geq \bar{c} > 0$ due to Hölder continuity. $\square$

In the next step of our program, we will reduce the problem of uniform convergence to this minimisation problem. The central tool in doing so is the following rescaling result, compare e.g. the proof of [RS06, Theorem 5.1].

Lemma 5.2.20. *Let $u_\varepsilon \colon B_r(x) \to \mathbb{R}$, $\lambda > 0$ and $\hat{u}_\varepsilon \colon B(0, r/\lambda) \to \mathbb{R}$ with*

$$\hat{u}_\varepsilon(y) = u_\varepsilon(x + \lambda y).$$

Set $\hat{r} := r/\lambda$, $\hat{\varepsilon} := \varepsilon/\lambda$,

$$\hat{\mu}_\varepsilon := \frac{1}{c_0}\left(\frac{\hat{\varepsilon}}{2}\,|\nabla \hat{u}_\varepsilon|^2 + \frac{1}{\hat{\varepsilon}}\,W(\hat{u}_\varepsilon)\right)\mathcal{L}^n, \qquad \hat{\alpha}_\varepsilon := \frac{1}{c_0\,\hat{\varepsilon}}\left(\hat{\varepsilon}\,\Delta \hat{u}_\varepsilon - \frac{1}{\hat{\varepsilon}}\,W'(\hat{u}_\varepsilon)\right)^2\mathcal{L}^n.$$

Then

$$\hat{r}^{1-n}\,\hat{\mu}_\varepsilon(B(0,\hat{r})) = r^{1-n}\,\mu_\varepsilon(B_r(x)), \qquad \hat{r}^{3-n}\,\hat{\alpha}_\varepsilon(B(0,\hat{r})) = r^{3-n}\,\alpha_\varepsilon(B_r(x)).$$

With this in mind, we proceed to our main result on convergence away from $\mathrm{spt}(\mu)$ in three dimensions.

Theorem 5.2.21. *Let $\tau > 0$ and $x \in \Omega \setminus \mathrm{spt}(\mu)$ such that there exists a sequence $x_\varepsilon \to x$ with the property that*

$$\limsup_{\varepsilon\to 0} |u_\varepsilon(x_\varepsilon) - u(x)| \geq \tau.$$

Then there exists $\bar{\theta} > 0$ depending only on τ such that $\alpha(\{x\}) \geq \bar{\theta}$. In particular, there are only finitely many such points.

Proof. In a first step, we reduce the argument to proving the atom property for points x that admit a sequence $x_\varepsilon \to x$ such that

$$\liminf_{\varepsilon\to 0} |u_\varepsilon(x_\varepsilon)| \leq 1 - \tau.$$

Without loss of generality, we may assume that $u(x) = 1$. Assume that there is a subsequence $x_\varepsilon \to x$ such that $u_\varepsilon(x_\varepsilon) < 0$. Since $u_\varepsilon \to u$ in $L^1(\Omega)$ (so pointwise almost everywhere, up to a subsequence), and u is locally constant, there is also a sequence $\tilde{x}_\varepsilon \to x$ such that $u_\varepsilon(\tilde{x}_\varepsilon) \geq 1 - \tau/2$. Using the continuity of u_ε, we obtain a sequence $x'_\varepsilon \to x$ such that $|u_\varepsilon(x'_\varepsilon)| \leq 1 - \tau$. Passing to a subsequence in ε, we may assume that this holds for all ε.

So assume that $x_\varepsilon \to x \in \Omega$ and $|u_\varepsilon(x_\varepsilon)| \leq 1-\tau$. Since $\Omega \setminus \operatorname{spt}(\mu)$ is open, there is $r > 0$ such that $B(x, 3r) \subset \Omega \setminus \operatorname{spt}(\mu)$. As $x_\varepsilon \to x$, $B(x_\varepsilon, r) \subset B(x, 2r)$ for almost all $\varepsilon > 0$. We have $\mu(B(x, 3r)) = 0$, so (using the terminology of Lemmas 5.2.19 and 5.2.20)

$$\begin{aligned}
\alpha(B_{3r}(x)) &\geq \alpha(\overline{B_{2r}(x)})\\
&\geq \limsup_{\varepsilon\to 0} \left(\alpha_\varepsilon(B_{2r}(x) + r^{1-n}\mu_\varepsilon(B_{2r}(x))\right)\\
&\geq \limsup_{\varepsilon\to 0} \left(\alpha_\varepsilon(B_r(x_\varepsilon)) + r^{1-n}\mu_\varepsilon(B_r(x_\varepsilon))\right)\\
&= \limsup_{\hat\varepsilon\to 0} \left(\hat\alpha_\varepsilon(B_1(0)) + \hat\mu_\varepsilon(B_1(0))\right)\\
&\geq \limsup_{\hat\varepsilon\to 0} \inf_{u\in X_{1-\tau}} \left(\mathcal{W}_{\hat\varepsilon} + S_{\hat\varepsilon}\right)(u)\\
&\geq \bar\theta
\end{aligned}$$

with $\hat u_\varepsilon(y) = u_\varepsilon(x_\varepsilon + ry)$ and $\hat\varepsilon = \varepsilon/r$. Letting $r \to 0$, we establish that

$$\alpha(\{x\}) \geq \bar\theta$$

where $\bar\theta$ only depends on τ. Again, x is an atom of a fixed minimal size, so there are only finitely many such points. □

Corollary 5.2.22. *Assume that $\Omega' \Subset \Omega \setminus \operatorname{spt}(\mu)$. Then the following hold true.*

1. *For all $\tau > 0$ there exists $\bar c_\tau > 0$ such that if α has no atoms of size at least $\bar c$ in $\overline{\Omega'}$, then $\big|\,|u_\varepsilon| - 1\big| < \tau$ on Ω'.*
2. *If $\bar c$ is small enough and all atoms of α in Ω' are smaller than $\bar c$, then for every $\Omega'' \Subset \Omega'$ there exists $C > 0$ such that $\mu_\varepsilon(\Omega'') \leq C\,\varepsilon^2$ for all sufficiently small $\varepsilon > 0$.*
3. *If α has no atoms in $\overline{\Omega'}$ at all, then $u_\varepsilon \to u$ uniformly on Ω'.*
4. *If μ is the mass measure of a varifold V and $\alpha(\overline{\Omega}) = \mathcal{W}(V)$ (i.e. u_ε is a recovery sequence for its limit), then $u_\varepsilon \to u$ locally uniformly in $\Omega \setminus \operatorname{spt}(\mu)$.*

Proof. All but the last point are obvious. Clearly, it suffices to show that α has no atoms outside $\operatorname{spt}(\mu)$. For a contradiction, assume that $x_0 \notin \operatorname{spt}(\mu)$ is an atom of α and choose $\Omega' = \Omega \setminus \overline{B_r(x_0)}$ such that $B_r(x_0) \Subset \Omega \setminus \operatorname{spt}(\mu)$. Then consider the sequence $\bar u_\varepsilon = u_\varepsilon$ pointwise. Clearly still $\bar\mu_\varepsilon \rightharpoonup \mu$, but $\liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon(\bar u_\varepsilon) < \mathcal{W}(V)$ contradicting the $\Gamma - \liminf$ inequality from [RS06]. □

The following is an easy corollary once essentially uniform convergence is established. We state it here in order to illustrate the properties of this mode of convergence.

Corollary 5.2.23. *There exists a countable set* $\Delta \subset \Omega \setminus \operatorname{spt}(\mu)$*, such that* $u_\varepsilon \to u$ *pointwise everywhere on* $\Omega \setminus (\operatorname{spt}(\mu) \cup \Delta)$*. In particular, for* $C \Subset \Omega \setminus \operatorname{spt}(\mu)$*,* $s > 0$ *such that* $\mathcal{H}^s(C) < \infty$ *we have that* $u_\varepsilon \to u$ $\mathcal{H}^s|_C$*-almost everywhere.*

Proof. The statement follows from Remark 5.2.17 point (1), which is evident from the definition of essentially uniform convergence. □

A few remarks are in order.

Remark 5.2.24. The only difference to the case $n = 2$ lies in the different rescaling properties of α_ε in two and three dimensions. There, we could deduce that $\alpha(B_{3r}(x)) \geq \bar{\theta}/r$, which gives a contradiction as $r \to 0$ and establishes uniform convergence of $|u_\varepsilon| \to 1$ on sets $\Omega' \Subset \Omega \setminus \operatorname{spt}(\mu)$.

Remark 5.2.25. As pointed out, if $\Omega' \Subset \Omega$ and $\mu(\overline{\Omega'}) = \alpha(\overline{\Omega'}) = 0$, then $|u_\varepsilon| \to 1$ uniformly on every $\Omega'' \Subset \Omega'$. However, the convergence has no *a priori* rate in ε in $n = 3$ dimensions. Functions like

$$u_\varepsilon = 1 + f(\varepsilon)\, g(\,(x - x_0)/\varepsilon)$$

will not lead to atoms of α if $g \in C_c^\infty(\mathbb{R}^n)$ and $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. For similar considerations, see Example 5.4.1.

Remark 5.2.26. The argument presented above can clearly be adapted to other situations with the following ingredients:

1. a sequence of functions u_ε converging to a function u which induces two sequences of Radon measures $\mu_\varepsilon \rightharpoonup \mu, \alpha_\varepsilon \rightharpoonup \alpha$ uniformly bounded on compact subsets,
2. an infinitesimal generation of mass property like

$$|u_\varepsilon(x) - u(x)| \geq \theta \qquad \Rightarrow \qquad \varepsilon^{1-n}\mu_\varepsilon(B_\varepsilon) \geq \bar{c}_\theta,$$

3. a monotonicity formula resembling

$$R^{1-n}\mu_\varepsilon(B_R) \geq c_1\, r^{1-n}\mu_\varepsilon(B_r) - c_2\, \alpha_\varepsilon(B_R) + \Xi_\varepsilon, \qquad c_1, c_2 > 0,$$

   for μ_ε which involves only $\mu_\varepsilon, \alpha_\varepsilon$ and an error term Ξ_ε which goes to zero and
4. a critical or sub-critical rescaling property for α_ε.

Then we can re-write the problem of uniform convergence into a minimisation problem and employ the same arguments as above. Depending on the nature of the rescaling property, we may be able to obtain uniform convergence this way (as for $n = 2$) or essentially uniform convergence (as for $n = 3$).

5.2.5 Hausdorff Convergence

In applications, we like to think of $\mathrm{spt}(\mu)$ as being approximated by the set $\{u_\varepsilon = 0\}$. This is rigorously justified in the next theorem, which can be thought of as a diffuse version of Theorem 3.1.2. In fact, we will show that Theorem 3.1.2 follows from Theorem 5.2.27 in Lemma 5.3.4 in the case that the approximating varifolds are smooth boundaries.

Theorem 5.2.27. *Let $I \Subset (-1,1)$ be non-empty, not necessarily open. Then, up to a subsequence, $u_\varepsilon^{-1}(I)$ converges to a compact set $K \subset \overline{\Omega}$ in Hausdorff distance such that*

1. *$K \cap \Omega = \mathrm{spt}(\mu) \cap \Omega$ if $n = 2$ or $n = 3$ and α has no atoms in $\Omega \setminus \mathrm{spt}(\mu)$,*
2. *$K \cap \Omega = (\mathrm{spt}(\mu) \cap \Omega) \cup \bigcup_{k=1}^N \{x_k\}$ for finitely many points $x_k \in \Omega$ if $n = 3$. The number N of points can be bounded in terms of I and $\limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon)$.*

Proof. In accordance with convention, we may replace $u_\varepsilon^{-1}(I)$ with its closure without affecting the limit. Since $u_\varepsilon^{-1}(I) \subset \Omega$ is bounded, there is a compact set $K \subset \overline{\Omega}$ and a subsequence (not relabelled) such that

$$u_\varepsilon^{-1}(I) \to K$$

in Hausdorff distance. K can be calculated as the Kuratowski limit

$$K = \{x \in \overline{\Omega} \mid \exists\, x_\varepsilon \in u_\varepsilon^{-1}(I) \text{ such that } x_\varepsilon \to x\}.$$

We will show that $\mathrm{spt}(\mu) \subset K$ anticipating the results of Chapter 7. Take $x \in \mathrm{spt}(\mu)$. By the fifth point of Corollary 5.2.3 and the fact that $\xi_{\varepsilon,+} \to 0$, we see that for small enough $\tau > 0$ we have

$$\liminf_{\varepsilon \to 0} \frac{\mathcal{L}^n\left(B_r(x) \cap \{|u_\varepsilon| \geq 1 - \tau\}\right)}{\varepsilon} > 0,$$

see also Corollary 5.2.3. Thus there exists $y \in B_r(x)$ such that $u_\varepsilon(y) \leq 1 - \tau$. In fact there exists a large set of such y as witnessed by the fact that the Lebesgue measure of this set is greater than $c\varepsilon$ for some $c > 0$. We can use this as in the proof of Lemma 7.2.4 to deduce that there exists a point $y \in B_{r/2}(x)$ to which we can apply Lemma 7.2.2 which tells us that on a disc $B_{L\varepsilon}(y)$ around y, u_ε is C^0-close to an optimal profile type transition. We can choose $L > 0$ arbitrarily large here, so in particular we see that there exists $y' \in B_{L\varepsilon}(y) \subset B_r(x)$ such that $u_\varepsilon(y') \in I$. The statements presented below are sharper, so we do not present the proof in greater detail.

For the inverse inclusion, assume that $x \in \Omega \cap K \setminus \mathrm{spt}(\mu)$. Take $r > 0$ such that $B_r(x) \subset \Omega \setminus \mathrm{spt}(\mu)$. If $n = 2$ or $n = 3$ and α has no atoms in $\overline{B_r(x)}$, we see that $|u_\varepsilon| \to 1$ uniformly on $\overline{B_r(x)}$, which leads to a contradiction. If $n = 3$ in general, then x must be an atom of α with a minimal size depending only on

$$\sup_{\theta \in I} |\theta| < 1.$$

Since there can only be finitely many such points, the theorem is proven. □

Remark 5.2.28. In the case where we have $u_\varepsilon \in -1 + W_0^{2,2}(\Omega)$, we can extend u_ε to a larger domain as a constant function outside Ω. Thus we obtain the stronger result that $u_\varepsilon^{-1}(I) \to \operatorname{spt}(\mu)$ if $n = 2$ (or if $n = 3$ and α does not have atoms outside $\operatorname{spt}(\mu)$) and $u_\varepsilon^{-1}(I) \to \operatorname{spt}(\mu) \cup \{x_1, \dots, x_N\}$ (up to a subsequence) if $n = 3$ for a finite collection of points $x_i \in \overline{\Omega}$. If $n = 2$ or $n = 3$ and α has no atoms, the uniqueness of the limit implies that actually the whole sequence $u_\varepsilon^{-1}(I)$ converges to $\operatorname{spt}(\mu)$. The same holds for periodic boundary conditions.

Without boundary conditions, the relationship of $K \cap \partial\Omega$ and $\operatorname{spt}(\mu) \cap \partial\Omega$ is more complicated. If $\partial\Omega \in C^2$, we may consider an optimal interface transition for $\partial\Omega$ such that only the positive part of the transition lies inside Ω. This induces the measure $\mu = 1/2\, \mathcal{H}^{n-1}|_{\partial\Omega}$. So μ may well fail to be an integral varifold at the boundary, and the inclusion $\operatorname{spt}(\mu) \cap \partial\Omega \not\subset K \cap \partial\Omega$ need not hold (take $I \Subset (-1, 0)$). Further details are given in Section 6.3.

5.3 Applications

5.3.1 Minimising Sequences Converge Uniformly

In the first application, we demonstrate how essentially uniform convergence can be used to obtain uniform convergence under additional assumptions. It formalises the intuition that phase-fields have no energetic incentive not to converge uniformly in three dimensions.

Lemma 5.3.1. *Let $X = -1 + W_0^{2,2}(\Omega)$, $S, V \in \mathbb{R}$, $\lambda > 0$, $\chi \geq 0$ and*

$$\mathcal{E}_\varepsilon \colon X \to [0, \infty), \qquad \mathcal{E}_\varepsilon(u) = \mathcal{W}_\varepsilon(u) + \lambda\, (S_\varepsilon(u) - S)^2 + \chi \left(\frac{1}{2} \int_\Omega (u + 1)\, \mathrm{d}x - V \right)^2$$

an associated energy functional. Furthermore, assume that $u_\varepsilon \in X$ and $u \in BV(\Omega)$ are such that

$$\mathcal{E}_\varepsilon(u_\varepsilon) = \min_{v \in X} \mathcal{E}_\varepsilon(v), \qquad u_\varepsilon \to u \text{ in } L^1(\Omega).$$

As usual, let $\mu_\varepsilon \rightharpoonup \mu$, $\alpha_\varepsilon \rightharpoonup \alpha$. Then $\operatorname{spt}(\alpha) \subset \operatorname{spt}(\mu)$. In particular, $u_\varepsilon \to u$ uniformly on compact sets $K \subset \overline{\Omega} \setminus \operatorname{spt}(\mu)$.

The parameter S and V play the roles of a preferred surface area and enclosed volume and λ, χ express the strength of the preference.

The existence of minimisers of $\mathcal{E}_\varepsilon$ follows from the direct method of the calculus of variations and Sobolev embedding theorems, compare Lemma 5.2.1. A similar statement holds if X is $W^{2,2}(\Omega)$ or the subspace of $W^{2,2}(\Omega)$ with vanishing normal derivatives.

Proof. Note that the sequence $\bar{u}_\varepsilon \equiv -1$ keeps $\mathcal{E}_\varepsilon$ uniformly bounded, so $\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) < \infty$. Let us consider a subsequence $\varepsilon \to 0$ such that all three terms in the energy have a limit.

By Corollary 5.2.22, $u_\varepsilon \to u$ locally uniformly in $\overline{\Omega} \setminus \mathrm{spt}(\mu)$ if α has no atoms outside $\mathrm{spt}(\mu)$, so it suffices to show the inclusion $\mathrm{spt}(\alpha) \subset \mathrm{spt}(\mu)$. By extending u_ε to a slightly larger domain Ω' as a constant function, we only need to consider the case that $\mathrm{spt}(\alpha) \Subset \Omega$. Recall that the support of a measure is the collection of all points, such that any neighbourhood of the point has positive measure. Thus for a contradiction, we may assume that there exists a ball $B_{2r}(x) \subset \Omega \setminus \mathrm{spt}(\mu)$ such that $\alpha(B_r(x)) > 0$.

Since there are only finitely many points $x \in \overline{\Omega} \setminus \mathrm{spt}(\mu)$ such that there exists a sequence $x_\varepsilon \to x$ with the property that $\limsup_{\varepsilon \to 0} |\,|u_\varepsilon(x_\varepsilon)| - 1| \geq \tau$ for any given $\tau > 0$, we can choose two radii $r < r_1 < r_2 < 2r$ and the ring domain

$$R := \{y \in \Omega \mid r_1 < |x - y| < r_2\}$$

such that $|u_\varepsilon| \geq 1/\sqrt{2}$ on $\overline{R'}$ for all sufficiently small ε and a slightly larger set R' such that $R \Subset R'$. Since by Corollary 5.2.3 we know that

$$\int_R \frac{1}{\varepsilon} W'(u_\varepsilon)^2 \,\mathrm{d}x \leq C\,\varepsilon^2,$$

we can pick a ring

$$R_\varepsilon = \{y \in \Omega \mid r_\varepsilon < |x - y| < r_\varepsilon + |\log(\varepsilon)|^{-1}\} \Subset R$$

such that

$$\varepsilon^{-2}\,\mu_\varepsilon(R_\varepsilon) + \int_{R_\varepsilon} \frac{1}{\varepsilon^3} W'(u_\varepsilon)^2 \,\mathrm{d}x \leq C\,|\log(\varepsilon)|^{-1}.$$

Then we choose a cut-off function η such that $\eta \equiv 1$ inside $B_{r_\varepsilon}(x)$, $\eta \equiv 0$ outside $B_{r_\varepsilon + |\log\varepsilon|^{-1}}(x)$, $|\nabla\eta| \leq 2\,|\log\varepsilon|$, $|\Delta\varepsilon| \leq C\,|\log\varepsilon|^2$ and define

$$\hat{u}_\varepsilon = (1 - \eta)\,u_\varepsilon + \eta.$$

Since $|u_\varepsilon| \geq 1/\sqrt{2}$ on R, we can suppose that without loss of generality $u_\varepsilon \geq 1/\sqrt{2}$. It follows directly from $\hat{u}_\varepsilon \geq u_\varepsilon > 0$ that

$$\int_R \frac{1}{\varepsilon}\, W(\hat{u}_\varepsilon)\,\mathrm{d}x \leq \int_R \frac{1}{\varepsilon}\, W(u_\varepsilon)\,\mathrm{d}x$$

and furthermore $0 = \hat{\mu}_\varepsilon(B_{r_1}(x)) \leq \mu_\varepsilon(B_{r_1}(x))$. Finally, we note that

$$\int_R \frac{\varepsilon}{2}\, |\nabla \hat{u}_\varepsilon|^2\,\mathrm{d}x \leq \varepsilon \int_R |\nabla u_\varepsilon|^2\,(1-\eta)^2 + (u_\varepsilon - 1)^2\,|\nabla\eta|^2\,\mathrm{d}x = O(\varepsilon^2\,|\log\varepsilon|^2)$$

due to Corollary 5.2.3, so in particular that $\hat{\mu}_\varepsilon \rightharpoonup \mu$ and $\lim_{\varepsilon\to 0} S_\varepsilon(\hat{u}_\varepsilon) = \lim_{\varepsilon\to 0} S_\varepsilon(u_\varepsilon)$. Since $u_\varepsilon \to 1$ in $L^1(B_{2r}(x))$ already before the modification, we do not change the limiting integral either:

$$\lim_{\varepsilon\to 0}\int_\Omega \hat{u}_\varepsilon\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon\,\mathrm{d}x.$$

Hence the last two terms in $\mathcal{E}_\varepsilon$ converge to the same limits as before. Thus it suffices to show that $\liminf_{\varepsilon\to 0}\mathcal{W}_\varepsilon(\hat{u}_\varepsilon) < \liminf_{\varepsilon\to 0}\mathcal{W}_\varepsilon(u_\varepsilon)$ to see that

$$\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(\hat{u}_\varepsilon) < \liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon),$$

which means that u_ε cannot be a minimiser of $\mathcal{E}_\varepsilon$ for some small $\varepsilon > 0$. This is the contradiction we are looking for. So calculate

$$\begin{aligned}
&\hat{\alpha}_\varepsilon(B_{r_\varepsilon + |\log\varepsilon|^{-1}}(x)) = \hat{\alpha}_\varepsilon(R_\varepsilon)\\
&\quad = \frac{1}{c_0\,\varepsilon}\int_{R_\varepsilon}\left(\varepsilon\,\Delta\hat{u}_\varepsilon - \frac{1}{\varepsilon}\,W'(\hat{u}_\varepsilon)\right)^2\,\mathrm{d}x\\
&\quad \leq \frac{1+\delta}{c_0\,\varepsilon}\int_{R_\varepsilon}\left(\varepsilon\,\Delta u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right)^2\,(1-\eta)^2\,\mathrm{d}x\\
&\qquad\quad + \left(1+\frac{1}{\delta}\right)\frac{1}{c_0\,\varepsilon}\int_{R_\varepsilon}\left(-2\varepsilon\,\langle\nabla\eta,\nabla u_\varepsilon\rangle + \varepsilon\,(1-u_\varepsilon)\,\Delta\eta + \frac{1}{\varepsilon}[W'(u_\varepsilon)(1-\eta) - W'(\hat{u}_\varepsilon)]\right)^2\,\mathrm{d}x\\
&\quad \leq \frac{1+\delta}{c_0\,\varepsilon}\int_{R_\varepsilon}\left(\varepsilon\,\Delta u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right)^2\,\eta^2\,\mathrm{d}x + \left(1+\frac{1}{\delta}\right)\frac{3}{c_0\,\varepsilon^3}\int_{R_\varepsilon}[W'(u_\varepsilon)(1-\eta) - W'(\hat{u}_\varepsilon)]^2\,\mathrm{d}x\\
&\qquad\qquad + \left(1+\frac{1}{\delta}\right)\frac{3\,\varepsilon}{c_0}\int_{R_\varepsilon} 4\,\langle\nabla\eta,\nabla u_\varepsilon\rangle^2 + (u_\varepsilon-1)^2\,(\Delta\eta)^2\,\mathrm{d}x\\
&\quad \leq (1+\delta)\,\alpha_\varepsilon(R_\varepsilon) + \left(1+\frac{1}{\delta}\right)\frac{3}{c_0\,\varepsilon^3}\int_{R_\varepsilon} W'(u_\varepsilon)^2\,\mathrm{d}x + \left(1+\frac{1}{\delta}\right)\,C\,\left(\varepsilon^2\,|\log\varepsilon|^2 + \varepsilon^4\,|\log\varepsilon|^4\right)\\
&\quad \leq (1+\delta)\,\alpha_\varepsilon(R_\varepsilon) + \left(1+\frac{1}{\delta}\right)\frac{3}{c_0\,|\log\varepsilon|} + \left(1+\frac{1}{\delta}\right)\,C\,\left(\varepsilon^2\,|\log\varepsilon|^2 + \varepsilon^4\,|\log\varepsilon|^4\right)
\end{aligned}$$

for $\delta > 0$. Here we used that $\hat{u}_\varepsilon$ is a convex combination of u_ε and 1 pointwise, so that the estimate in the middle integral works. Taking first $\varepsilon \to 0$ and then $\delta \to 0$, it follows that

$$\hat{\alpha}(B_{2r}) = \alpha(B_{2r} \setminus B_{r_0}) < \alpha(B_{2r})$$

where $r_0 = \lim_{\varepsilon \to 0} r_\varepsilon > r$. This implies the contradiction and concludes the proof. $\square$

Cases of independent geometric interest are the formal limits $\lambda = \chi = \infty$ and $\lambda = \infty$, $\chi = 0$. The problem becomes more complex, and in the first case, solutions can only exist if $V < \mathcal{L}^n(\Omega)$ and $S > c_0$ for some c_0 depending on V through the isoperimetric inequality relative to Ω. These limits can be expressed in phase-field models for example in the choice of admissible functions

$$X_\varepsilon(\Omega) = \{u \in -1 + W_0^{2,2}(\Omega) \mid S_\varepsilon(u) = S\}$$

as a (non-linear) sub-manifold of $W_0^{2,2}$ or simply by choosing $\lambda = \lambda_\varepsilon = \varepsilon^{-1}$, and similarly in the first case. Our simple modification clearly does not go through in either scenario, but we believe that the same result should still hold.

We will see that uniform convergence still holds for a penalised functional when we add a version of the topological term discussed in Chapter 7. For simplicity, we restrict ourselves to the case discussed there, but a total integral term could easily be included. The following Lemma should be read after Chapter 7 and is included here because of a better fit by topic despite the fact that it comes logically later.

Assume that $\mathcal{C}_\varepsilon^1$ is associated to a function $\phi_1 \in C_c(1/\sqrt{2}, 1)$ and $\mathcal{C}_\varepsilon^2$ to a $\phi_2(s) = \phi_1(-s)$ and suitable F_1, F_2.

Lemma 5.3.2. *Let $X = -1 + W_0^{2,2}(\Omega)$, $\sigma > 0$, $\kappa > 2$ and*

$$\mathcal{E}_\varepsilon \colon X \to [0, \infty), \qquad \mathcal{E}_\varepsilon(u) = \mathcal{W}_\varepsilon(u) + \varepsilon^{-\sigma} \left(S_\varepsilon(u) - S\right)^2 + \varepsilon^{-\kappa} \left(\mathcal{C}_\varepsilon^1 + \mathcal{C}_\varepsilon^2\right)(u).$$

Assume that $u_\varepsilon \in X$ are minimisers of $\mathcal{E}_\varepsilon$ u, α, μ as usual. Then $\operatorname{spt}(\alpha) \subset \operatorname{spt}(\mu)$ and $u_\varepsilon \to u$ locally uniformly in $\overline{\Omega} \setminus \operatorname{spt}(\mu)$.

Sketch of Proof: The proof proceeds in two steps. In the first one, we assume that if $x \in \operatorname{spt}(\mu)$ and $y \in \Omega$ such that $y_\varepsilon \to y$ and $u_\varepsilon(y_\varepsilon) \in \operatorname{supp}(\phi_1)$. Then we deduce that

$$\liminf_{\varepsilon \to 0} \left(\frac{1}{\varepsilon} \int_{B_r(x)} \phi_1(u_\varepsilon(x))\,\mathrm{d}x\right) \left(\frac{1}{\varepsilon^3} \int_{B_r(y)} \phi_1(u_\varepsilon(y))\,\mathrm{d}y\right) > 0$$

for all $r > 0$ using Lemma 7.2.4 on the first term and infinitesimal Hölder regularity on the second, reaching a contradiction. Having excluded the situation of Example 5.4.2, we use the same modification as in Lemma 5.3.1 on u_ε in the second step of the proof. Since we know that $|u_\varepsilon| \geq 1/\sqrt{2}$ on the whole ball $B_{r_2}(x)$ rather than just the ring domain R, the difference in the diffuse area functional can be controlled to be $o(\varepsilon^\sigma)$ for all $\sigma < 2$. Thus the same argument as above goes through. $\square$

5.3.2 Global Solutions of the Stationary Allen-Cahn Equation

Our next application is to the stationary Allen-Cahn equation in low dimensions. While usually the blow up is used to obtain information about the geometric object, here we can go the other way around. Namely, we exclude the existence of solutions making a 'bump' in $\mathbb{R}^n$ for $n = 2, 3$, but no proper transition.

Lemma 5.3.3. *Let $n = 2, 3$. Then there is no global solution of the stationary Allen-Cahn equation $-\Delta u + W'(u) = 0$ satisfying*

$$\lim_{|x|\to\infty} \sqrt{|x|}\,|u(x) - 1| = 0, \qquad u(0) \neq 1.$$

Proof. Assume that there is such a function. Due to the condition at infinity, u is bounded, so the Allen-Cahn equation forces that $u \in [-1, 1]$ via an easy contradiction. Thus $|u(0)| \leq \theta$ for some $\theta < 1$. Furthermore, we know that $|\nabla u|^2 \leq 2\,W(u)$ from a well known gradient estimate by Modica [Mod85].

Let $u_\varepsilon \in W^{2,2}(B_1(0))$, $u_\varepsilon(x) = u(x/\varepsilon)$. These functions satisfy $|u_\varepsilon(0)| \leq \theta$, $\mathcal{W}_\varepsilon(u_\varepsilon) \equiv 0$ and

$$\begin{aligned}
S_\varepsilon(u_\varepsilon) &= \int_{B_1} \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\,W(u_\varepsilon)\,\mathrm{d}x \\
&= \varepsilon^{n-1}\left(\int_{B_{R_\varepsilon}} \frac{1}{2}\,|\nabla u|^2 + W(u)\,\mathrm{d}x + \int_{B_{1/\varepsilon}\setminus B_{R_\varepsilon}} \frac{1}{2}\,|\nabla u|^2 + W(u)\,\mathrm{d}x\right) \\
&= \varepsilon^{n-1}\left(C_u\,R_\varepsilon^n + 2\int_{B_{1/\varepsilon}\setminus B_{R_\varepsilon}} W(u)\,\mathrm{d}x\right) \\
&\leq \varepsilon^{n-1}\left(C_u\,R_\varepsilon^n + \int_{B_{1/\varepsilon}\setminus B_{R_\varepsilon}} (u-1)^2\,\mathrm{d}x\right) \\
&= \varepsilon^{n-1}\left(C_u\,R_\varepsilon^n + \int_{R_\varepsilon}^{1/\varepsilon} \left(\frac{\eta_\varepsilon}{\sqrt{r}}\right)^2 r^{n-1}\,\mathrm{d}r\right) \\
&= C_u\,R_\varepsilon^n \varepsilon^{n-1} + \frac{\eta_\varepsilon^2\,\varepsilon^{n-1}}{n-1}\left(\varepsilon^{1-n} - R_\varepsilon^{n-1}\right)
\end{aligned}$$

where $R_\varepsilon \to \infty$ slower than $\varepsilon^{(1-n)/n}$ so that the first term disappears in the limit. We can choose

$$\eta_\varepsilon := \sup_{|x|\geq R_\varepsilon} \sqrt{|x|}\,|u(x)-1| \to 0$$

due to the assumption that $|u(x)-1|/|x|^{1/2} \to 0$. Thus $u_\varepsilon \in W^{2,2}(B_1(0))$ satisfies $|u_\varepsilon(0)| \leq \theta$ and

$$\lim_{\varepsilon\to 0} (\mathcal{W}_\varepsilon + S_\varepsilon)(u_\varepsilon) = 0$$

in contradiction to Lemma 5.2.19. □

5.3.3 Hausdorff-Convergence of Manifolds with Bounded Energy

Last but not least we show how our results can be used to obtain results on the interplay between varifold and Hausdorff convergence using only our PDE techniques. This is a proof of Theorem 3.1.2 for smooth boundaries using only phase-field arguments.

Lemma 5.3.4. *Let M_k be a sequence of compact C^2-surfaces and μ_k their induced (multiplicity 1) varifolds. Assume that μ is an integral varifold such that $\mu_k \rightharpoonup \mu$ as varifolds and*

$$\limsup_{k\to\infty} \left[\mathcal{W}(M_k) + \mathcal{H}^2(M_k)\right] < \infty.$$

Then $\lim_{k\to\infty} M_k = \mathrm{spt}(\mu)\cup\{x_1,\dots,x_N\}$ for a finite collection of points x_i, $i=1,\dots,N$ in the sense of convergence in Hausdorff distance for every subsequence along which the limit exists. Moreover, if M_k is connected for all $k\in\mathbb{N}$ or $\lim_{k\to\infty}\mathcal{W}(V_k) = \mathcal{W}(V)$, then

$$\mathrm{spt}(\mu) = \lim_{k\to\infty} M_k.$$

Proof. A simple contradiction shows that $\mathrm{spt}(\mu)\subset \lim_{k\to\infty} M_k$, so only the inverse direction is difficult. This concerns the uniform or essentially uniform convergence of phase-fields away from u_ε which we established using exclusively PDE techniques and no geometric measure theory at all.

As M_k is compact, orientable and embedded, it is the boundary of a set $E_k\subset\mathbb{R}^3$. Since furthermore $M_k\in C^2$, there is a sequence $\varepsilon_k\to 0$ such that the signed distance function $\mathrm{sdist}(\cdot, M_k)$ is C^2-smooth on

$$U_k = \{x\in\mathbb{R}^n \mid \mathrm{dist}(x,M_k) < \sqrt{\varepsilon_k}\} \Subset \Omega$$

and we can consider the sequence

$$u_k\colon \Omega\to\mathbb{R}, \qquad u_k(x) = q(r_k(x)/\varepsilon_k)$$

where r_k is a smooth approximation of $\mathrm{sdist}(x, M_k)$ as in the proof of Theorem 4.2.1. A further slight modification gives us $u_k \in -1 + W_0^{2,2}(\Omega)$ and $M_k \equiv \{u_k = 0\}$. If we choose ε_k sufficiently small, it becomes obvious that

$$\lim_{k\to\infty} \mu_k = \mu, \qquad \lim_{k\to\infty} \alpha_k(\mathbb{R}^3) = \lim_{k\to\infty} \mathcal{W}(M_k).$$

We can therefore invoke Corollary 5.2.27 (with boundary values) to see that

$$\lim_{k\to\infty} M_k = \lim_{k\to\infty} \{u_k = 0\} = \mathrm{spt}(\mu) \cup \{x_1, \dots, x_N\}$$

for a finite collection of points $x_1, \dots, x_N \in \overline{B_R(0)}$. Now assume additionally that M_k is connected for all $k \in \mathbb{N}$. Then, by standard results on Hausdorff convergence, $\lim_{k\to\infty} M_k$ is connected, so the finite collection of points must be empty.

Last, assume that we have a recovery sequence, i.e. $\lim_{k\to\infty} \mathcal{W}(M_k) = \mathcal{W}(V)$. If we choose ε_k sufficiently small also $\mathcal{W}_{\varepsilon_k}(u_k) \to \mathcal{W}(V)$, thus

$$\lim_{k\to\infty} M_k = \lim_{k\to\infty} \{u_k = 0\} = \mathrm{spt}(||V||)$$

as explained in Corollary 5.2.22. □

Corollary 5.3.5. *If M_k is connected for all $k \in \mathbb{N}$, then also* $\mathrm{spt}(\mu)$ *is connected.*

In particular, we have shown with phase-field techniques that the problem of minimising Willmore's energy in the class of connected surfaces arising as the limits of boundaries is well posed in three dimensions (Corollary 3.1.4).

5.4 Counterexamples

In this section, we give examples showing that our results are optimal. All constructions are simple perturbations of an optimal interface recovery sequence.

Example 5.4.1 (Simple Example in $\mathbb{R}^2$ and $\mathbb{R}^3$). Let $E \Subset \Omega$ with $\partial E \in C^2$ and denote by $d(x) = \mathrm{sdist}(x, \partial E)$ the signed distance function from ∂E. We take an optimal interface transition $u^\varepsilon(x) = q(d(x)/\varepsilon)$ where d is a smooth approximation of the signed distance function from ∂E and d is constant for $\mathrm{dist}(x, \partial E) > \delta$ for some $\delta > 0$. Now take $x_0 \notin \partial E$, $g \in C_c^\infty(\mathbb{R}^n)$ and set

$$u_g^\varepsilon(x) := u^\varepsilon(x) + \varepsilon^\beta\, g(\varepsilon^{-\gamma}(x - x_0)).$$

For small enough $\varepsilon > 0$, we know that $u^\varepsilon \equiv \pm u_\varepsilon(d(\delta)/\varepsilon) \approx 1 - e^{-\delta/\varepsilon}$ close to x_0, which simplifies the energies of the modified functions. Up to a small modification, we may assume that $u^\varepsilon \equiv 1$ around x_0. If $\gamma > 0$ (or $\gamma = 0$ and g has sufficiently small support), this implies the following identities.

$$S_\varepsilon(u^\varepsilon_g) = S_\varepsilon(u^\varepsilon) + \varepsilon^{1+2\beta+(n-2)\gamma}\int_{\mathbb{R}^n}\frac{1}{2}\,|\nabla g|^2\,\mathrm{d}x + \varepsilon^{2\beta+n\gamma-1}\int_{\mathbb{R}^n} g^2\,\frac{(2+\varepsilon^\beta g)^2}{4}\,\mathrm{d}x,$$
$$\mathcal{W}_\varepsilon(u^\varepsilon_g) = \mathcal{W}_\varepsilon(u^\varepsilon) + \int_{\mathbb{R}^n}\frac{1}{\varepsilon}\left(\varepsilon^{1+\beta-2\gamma}\,\Delta g - \varepsilon^{\beta-1}\,g\,(1+\varepsilon^\beta\,g)\,(2+\varepsilon^\beta\,g)\right)^2(\varepsilon^{-\gamma}(x-x_0))\,\mathrm{d}x.$$

In the bending energy, both terms scale differently unless

$$1+\beta-2\gamma = \beta-1 \qquad \Leftrightarrow \qquad \gamma = 1.$$

In this situation, we can simplify the integral to give

$$\mathcal{W}_\varepsilon(u^\varepsilon_\gamma) \approx \mathcal{W}_\varepsilon(u^\varepsilon) + \varepsilon^{2\beta-3+n}\int_{\mathbb{R}^n}(\Delta g - 2\,g)^2\,\mathrm{d}x$$

under the assumption that $\beta > 0$. For a compactly supported non-zero function g the last term cannot be zero, so we have the heuristic condition

$$2\beta - 3 + n \geq 0 \quad \Leftrightarrow \quad \beta \geq \frac{3-n}{2}$$

for the energy to remain finite. Conversely, it is easy to see that the energy does remain finite in these cases in both $n = 2$ and $n = 3$ dimensions. Setting $\beta = 0$ shows that u_ε need not converge uniformly to ± 1 away from the interface in three dimensions. In two dimensions, setting $\beta = 1/2$ shows that we cannot obtain a convergence rate better than $\sqrt{\varepsilon}$.

Note in particular that we have $\mu^\varepsilon_g(B_r(x)) = O(\varepsilon^2)$ if α_g has an atom at x and $\mu^\varepsilon_g(B_r(x)) = o(\varepsilon^2)$ otherwise, both in two and three dimensions. In three dimensions, we can consider the function

$$f\colon [0,1) \to (0,\infty), \quad f(\theta) := \inf\left\{[\mathcal{W}_1 + S_1](u) \mid u \in 1 + W^{2,2}_0(B_1(0)),\ u(0) = \theta\right\}.$$

It is continuous and satisfies $\lim_{\theta\to 1} f(\theta) = 0$, so we can take a sequence $\theta_n \to 1$ such that

$$\sum_{n=1}^\infty f(\theta_n) < \infty.$$

Then we take corresponding minimisers g_n, a dense subsequence x_n in $\Omega \setminus \partial E$ and define

$$\varepsilon_n = \min\left\{\min_{1\leq i\neq j\leq n}\frac{|x_i - x_j|}{2},\ \min_{1\leq i\leq n}\frac{\mathrm{dist}(x_i,\partial E)}{2},\ \frac{1}{2^n}\right\}$$

and

$$u_n(x) = \begin{cases} \pm g_n((x-x_i)/\varepsilon_n) & \text{in } B_{\varepsilon_n}(x_n) \\ u(x) & \text{else} \end{cases}$$

with the choice of sign for g_n such that the function is continuous. Then $\varepsilon_n \to 0$, $u_n \to u$ in $L^1(\Omega)$ and essentially uniformly, but there exists no open set $\Omega' \Subset \Omega \setminus \partial E$ such that $u_n \to u$ uniformly on Ω'.

The next example gives a different modification in three dimensions only. It shows that in $\mathbb{R}^3$, uniform convergence away from the interface may fail even if the discrepancy measures $|\xi_\varepsilon|$ vanish exponentially fast in ε and $\xi_\varepsilon \leq 0$ for all $\varepsilon > 0$. Another implication is that there is no guaranteed rate of convergence for $\mu_\varepsilon(\Omega') \to 0$ for $\Omega' \Subset \Omega \setminus \operatorname{spt}(\mu)$.

Example 5.4.2 (Second Example in $\mathbb{R}^3$). Consider a set $\Omega' \Subset \Omega \setminus \operatorname{spt}(\mu)$ and $x_0 \in \Omega'$. Assume that u^ε is an optimal profile type recovery sequence, or at least that u^ε is constant on Ω'. Let $r > 0$ such that $B_r(x_0) \Subset \Omega'$ and $\varepsilon^{3/4} < r_\varepsilon < r/2$. Then the functions

$$\bar u_\varepsilon(x) = \begin{cases} u^\varepsilon(x) & x \notin B_r(x_0) \\ \pm q(\widetilde{\operatorname{sdist}}(x, \partial B_{r_\varepsilon}(x_0))) & x \in B_r(x_0) \end{cases}$$

are C^2-smooth (if the sign of the optimal profile is chosen correctly). Here $\widetilde{\operatorname{sdist}}$ is an appropriate approximation of the signed distance function modified to give the correct constant for a continuous matchup. This is a recovery-type sequence for ∂E with an additional interface at spheres $\partial B_{r_\varepsilon}(x_0)$ and can easily be seen to satisfy

$$\bar\mu_\varepsilon \rightharpoonup \mu, \qquad \bar\mu_\varepsilon(\Omega') \approx \frac{4\pi\, r_\varepsilon^2}{3}, \qquad \alpha_\varepsilon(\Omega') \approx 16\pi$$

since spheres of any radius have Willmore energy 16π in three dimensions with our normalisation of the Willmore functional. As r_ε may go to zero arbitrarily slowly, so can $\mu_\varepsilon(\Omega')$.

This shows that no penalisation of the discrepancy measures can enforce uniform convergence away from the interface in three dimensions. In two dimensions, this does not work since small circles have large elastica energy while the Willmore functional on surfaces in $\mathbb{R}^3$ is scaling invariant.

Chapter 6

On the Boundary Regularity of Phase-Fields

6.1 Introduction

In Chapter 7 we will assume that u_ε is not only $W^{2,2}$-regular but even

$$u_\varepsilon \in -1 + W_0^{2,2}(\Omega).$$

This can be expressed equivalently as

$$u_\varepsilon \in W^{2,2}_{loc}(\mathbb{R}^n) \text{ and } u_\varepsilon \equiv -1 \text{ outside } \Omega \tag{6.1.1}$$

due to the Sobolev extension theorems. If $\partial\Omega \in C^2$, it is equivalent to the modelling assumption $u_\varepsilon \equiv -1$ on $\partial\Omega$ and $\partial_\nu \equiv 0$ on $\partial\Omega$ which expresses that surfaces are contained in Ω by $u_\varepsilon = -1$ on $\partial\Omega$ and that they may only touch $\partial\Omega$ tangentially by $\partial_\nu u_\varepsilon = 0$ on $\partial\Omega$. If we assume boundary conditions (6.1.1) or periodic boundary conditions, the results of Theorem 5.1.1 may be sharpened as follows:

1. The sequence u_ε is bounded in $L^\infty(\Omega)$ and $u_\varepsilon \to u$ in $L^p(\Omega)$ for all $p < \infty$,

2. a function u_ε is Hölder continuous with constants as above on every ball $B_\varepsilon(x)\cap\Omega$ for $x \in \overline{\Omega}$,

3. we may replace $\Omega' \Subset \Omega$ by $\overline{\Omega}$ and $\Omega' \Subset \Omega \setminus \mathrm{spt}(\mu)$ by $\Omega' \Subset \mathbb{R}^n \setminus \mathrm{spt}(\mu)$ in Theorem 5.1.1 for $n = 2$,

4. we may replace "finitely many points in Ω (or $\Omega \setminus \mathrm{spt}(\mu)$)" with "finitely many points in $\overline{\Omega}$ (or $\overline{\Omega} \setminus \mathrm{spt}(\mu)$)" if $n = 3$,

5. the Hausdorff limit is $K = \mathrm{spt}(\mu)$ if $n = 2$ or $n = 3$ and α has no atoms, and $K = \mathrm{spt}(\mu) \cup \{x_1, \dots, x_N\}$ in general if $n = 3$ with $x_1, \dots, x_N \in \overline{\Omega}$, and

6. μ is the mass measure of an integral varifold.

On the other hand, not requiring boundary values can lead to much simpler proofs as in Theorem 5.2.21 or Corollary 5.2.22 where we would otherwise have to go through lengthy additional arguments.

This chapter further illuminates the behaviour of u_ε and μ_ε at $\partial\Omega$. Partial regularity results for uniformly bounded boundary values and phase-fields whose level sets meet $\partial\Omega$ at a right angle will be discussed in Lemmas 6.2.1, 6.2.2 and 6.2.4. On the other hand, regularity of u_ε and μ may fail at $\partial\Omega$ even if the boundary is smooth as we will demonstrate in a series of examples. The case of free boundary values is important for example for the proof of essentially uniform convergence via the minimisation problem as given above. Due to the results listed above for the case that boundary conditions (6.1.1) are given, the chapter can be skipped by a reader only interested in this situation.

Theorem 6.1.1. *Let $\partial\Omega \in C^2$. Then the following hold true.*

1. *There exists a sequence $u_\varepsilon \in W^{2,2}(\Omega)$ such that $(\mathcal{W}_\varepsilon + S_\varepsilon)(u_\varepsilon) \to 0$, but u_ε is not bounded in $L^\infty(\Omega)$.*

2. *There exists a sequence u_ε such that such that $\alpha = 0$, $\mu = 0$ but K contains an open subset of $\partial\Omega$. Similar constructions give $K = \{x_0\}$ or $K = \gamma$ for a point $x_0 \in \partial\Omega$ and a closed curve $\gamma \subset \partial\Omega$.*

3. *Let $S > 0$. Then there exists a point $x_0 \in \partial\Omega$ and a sequence $u_\varepsilon \in W^{2,2}(\Omega)$ such that $|u_\varepsilon| \leq 1$ in $\overline{\Omega}$, $\mathcal{W}_\varepsilon(u_\varepsilon) \equiv 0$, $\mu_\varepsilon(\Omega) \equiv S$, $K = \emptyset$ and $\mu = S \cdot \delta_{x_0}$.*

If Ω is convex, any point x_0 or closed curve γ in $\partial\Omega$ can be chosen and u_ε may be such that it is not uniformly bounded in $\Omega \cap U$ for all open sets U with $U \cap \partial\Omega \neq \emptyset$.

This shows that for example the minimisation problem for

$$\mathcal{F}_\varepsilon = \mathcal{W}_\varepsilon + \varepsilon^{-\sigma}(S_\varepsilon - S)^2$$

is not physically meaningful without boundary conditions or with partly free boundary conditions $u_\varepsilon \equiv +1$ on Γ_+, $u_\varepsilon \equiv -1$ on Γ_- and u_ε left free on $\partial\Omega \setminus (\Gamma_+ \cup \Gamma_-)$.

6.2 Partial Regularity at the Boundary

In this chapter, we describe partial regularity results for weakly controlled boundary values.

Lemma 6.2.1. *Assume that u_ε is continuous on $\overline{\Omega}$ and there is $\theta \geq 1$ such that $|u_\varepsilon| \leq \theta$ on $\partial\Omega$ for all $\varepsilon > 0$. Then the following hold true.*

1. *There exists $C > 0$ such that $\mu_\varepsilon(\{|u_\varepsilon| \geq \theta\}) \leq C\,\varepsilon^2$.*
2. *For the set $\tilde\Omega_\varepsilon = \{x \in \Omega \,|\, B_{2\varepsilon}(x) \subset \Omega\}$ we can show that there exists C depending only on $\bar\alpha, \gamma$ and θ such that*

$$||u_\varepsilon||_{\infty,\tilde\Omega_\varepsilon} \leq C, \qquad |u_\varepsilon(y) - u_\varepsilon(z)| \leq \frac{C_{\bar\alpha,\theta,\gamma}}{\varepsilon^\gamma}\,|y-z|^\gamma$$

if there is $x \in \tilde\Omega_\varepsilon$ such that $y, z \in B_\varepsilon(x)$ and $\gamma \leq 1/2$ if $n = 3$, $\gamma < 1$ if $n = 2$.

Proof. This proof is an adaptation of the proof of Lemma 5.2.8 using a modified argument in the first step of the proof. We observe that for the proof of Lemma 5.2.8 to work, we needed that $B_{2\varepsilon}(x) \subset \Omega$ to employ the elliptic inequality

$$||\tilde u_\varepsilon||_{2,2,B_1(0)} \leq C\,\left(||\tilde u_\varepsilon||_{2,B_2(0)} + ||\Delta \tilde u_\varepsilon||_{2,B_2(0)}\right)$$

and an estimate of $\int_{B_{2\varepsilon}(x)} \frac{1}{\varepsilon^n} W'(u_\varepsilon)^2\,\mathrm{d}x$. The first one we are given directly by the choice of Ω_ε^β or $\tilde\Omega_\varepsilon$, for the second one we needed the separation from $\partial\Omega$ to employ Lemma 5.2.2 above. Here, we can obtain it through integration by parts

$$\begin{aligned}
c_0\,\alpha_\varepsilon(\{|u_\varepsilon| > \theta'\}) &= \int_{\{|u_\varepsilon|>\theta'\}} \frac{1}{\varepsilon}\left(\varepsilon\,\Delta u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right)^2\,\mathrm{d}x \\
&= -\frac{2}{\varepsilon}\int_{\partial\{|u_\varepsilon|>\theta'\}} W'(u_\varepsilon)\,\partial_\nu u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad + \int_{\{|u_\varepsilon|>\theta'\}} \varepsilon\,(\Delta u_\varepsilon)^2 + \frac{2}{\varepsilon}\,W''(u_\varepsilon)\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^3}\,W'(u_\varepsilon)^2\,\mathrm{d}x \\
&\geq \int_{\{|u_\varepsilon|>\theta'\}} \varepsilon\,(\Delta u_\varepsilon)^2 + \frac{4}{\varepsilon}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^3}\,W'(u_\varepsilon)^2\,\mathrm{d}x
\end{aligned}$$

for $\theta' > \theta$ when $\{|u_\varepsilon| > \theta'\}$ is a Caccioppoli set (i.e. for almost all $\theta' > \theta$). If $|u_\varepsilon| < \theta'$ on $\partial\Omega$, the set $\{u_\varepsilon > \theta'\}$ does not touch the boundary $\partial\Omega$, so $\partial\{u_\varepsilon > \theta'\} \subset \{u_\varepsilon = \theta'\} \subset \Omega$. Because $W'(\theta) > 0$ and $\partial_\nu u_\varepsilon$ is inward pointing on $\partial\{u_\varepsilon > \theta\}$, the boundary integral is non-positive. The rest of the argument goes through as before. Additionally, taking $\theta' \to \theta$ establishes the first claim. $\square$

Another situation with a similar improvement is that of prescribed Neumann boundary data.

Lemma 6.2.2. *Assume that Ω is a Caccioppoli set and $\partial_\nu u_\varepsilon = 0$ almost everywhere on $\partial\Omega$ with respect to the boundary measure $|D\chi_\Omega|$. Then the following hold true.*

1. *There exists $C > 0$ such that $\mu_\varepsilon(\{|u_\varepsilon| \geq 1\}) \leq C\,\varepsilon^2$.*

2. *For the set $\tilde\Omega_\varepsilon = \{x \in \Omega \,|\, B_{2\varepsilon}(x) \subset \Omega\}$ we can show that there exists C depending only on $\bar\alpha$ and γ such that*

$$||u_\varepsilon||_{\infty,\tilde\Omega_\varepsilon} \leq C, \qquad |u_\varepsilon(y) - u_\varepsilon(z)| \leq \frac{C}{\varepsilon^\gamma}\,|y-z|^\gamma$$

if there is $x \in \tilde\Omega_\varepsilon$ such that $y, z \in B_\varepsilon(x)$. Here $\gamma \leq 1/2$ if $n = 3$, $\gamma < 1$ if $n = 2$.

If $\partial\Omega \in C^2$ and $\partial_\nu u_\varepsilon = 0$ almost everywhere on $\partial\Omega$, then the second statement can be sharpened as follows:

2'. *For all $x \in \overline{\Omega}$ there exists a constant C depending only on $\bar\alpha, \bar\mu, \gamma$ and $\partial\Omega$ such that*

$$|u_\varepsilon(x)| \leq C, \qquad |u_\varepsilon(y) - u_\varepsilon(z)| \leq \frac{C}{\varepsilon^\gamma}\,|x-y|^\gamma \qquad \forall\ y, z \in B_\varepsilon(x) \cap \overline{\Omega}.$$

The dependence of C on $\partial\Omega$ vanishes in the limit $\varepsilon \to 0$.

In particular, for regular boundaries, the Neumann condition implies the boundedness of solutions (in particular also on the boundary).

Proof. As before, we obtain

$$\begin{aligned}
\alpha_\varepsilon(\{|u_\varepsilon| > \theta'\}) &= \int_{\{|u_\varepsilon|>\theta'\}} \frac{1}{\varepsilon}\left(\varepsilon\,\Delta u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right)^2 \mathrm{d}x \\
&= -\frac{2}{\varepsilon}\int_{\partial\Omega\cap\partial\{|u_\varepsilon|>\theta'\}} W'(u_\varepsilon)\,\partial_\nu u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1} - \frac{2}{\varepsilon}\int_{\partial\{|u_\varepsilon|>\theta'\}\cap\Omega} W'(u_\varepsilon)\,\partial_\nu u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1} \\
&\quad + \int_{\{|u_\varepsilon|>\theta'\}} \varepsilon\,(\Delta u_\varepsilon)^2 + \frac{2}{\varepsilon}\,W''(u_\varepsilon)\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^3}\,W'(u_\varepsilon)^2\,\mathrm{d}x \\
&\geq \int_{\{|u_\varepsilon|>\theta'\}} \varepsilon\,(\Delta u_\varepsilon)^2 + \frac{4}{\varepsilon}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^3}\,W'(u_\varepsilon)^2\,\mathrm{d}x
\end{aligned}$$

for any $\theta' > 1$ such that $\{|u_\varepsilon| > \theta'\}$ is a Caccioppoli set. Here the boundary integral can be split into two parts, one of which has a sign, while the other one vanishes due to the Neumann condition. This implies the boundedness on $\tilde\Omega_\varepsilon$ and the bound on the mass measures $\mu_\varepsilon(\{|u_\varepsilon| > \theta'\})$ as before. We can take $\theta' \to 1$ to prove the first part of the Lemma.

Now assume that $\partial\Omega \in C^2$ and pick $x \in \partial\Omega$. The rest of the argument is a fairly standard 'straightening the boundary' argument with the feature that the boundary becomes flatter as $\varepsilon \to 0$. Without loss of generality, we assume that $x = 0$. We may now blow up to

$$\tilde{u}_\varepsilon : B_2(0) \cap (\Omega/\varepsilon) \to \mathbb{R}, \qquad \tilde{u}_\varepsilon(y) = u_\varepsilon(\varepsilon y).$$

We pick a C^2-diffeomorphism $\phi_\varepsilon : B_2(0) \to B_2(0)$ such that

1. $\phi_\varepsilon(\Omega/\varepsilon \cap B_2(0)) = B_2^+(0)$,
2. $\phi_\varepsilon \to \mathrm{id}_{B_2(0)}$ in $C^2(B_2(0), B_2(0))$ as the domain becomes increasingly flat,
3. under ϕ_ε, the normal to $\partial\Omega/\varepsilon$ gets mapped to e_n on the boundary, i.e. the orthogonality condition is preserved.

With this we obtain a function

$$\tilde{w}_\varepsilon : B_2^+(0) \to \mathbb{R}, \qquad \tilde{w}_\varepsilon(y) = u_\varepsilon(\phi_\varepsilon^{-1}(y))$$

in flattened coordinates. Since ϕ_ε is C^2-smooth, it preserves $W^{2,2}$-functions and it is easy to calculate

$$\begin{aligned} \partial_i \tilde{u}_\varepsilon &= \partial_i (\tilde{w}_\varepsilon \circ \phi_\varepsilon) \\ &= \partial_i (\phi_\varepsilon)_j \; ((\partial_j \tilde{w}_\varepsilon) \circ \phi_\varepsilon) \\ \partial_{ij}\, \tilde{u}_\varepsilon &= \partial_{ij}(\phi_\varepsilon)_k \; ((\partial_k \tilde{w}_\varepsilon) \circ \phi_\varepsilon) + \partial_i(\phi_\varepsilon)_k \, \partial_j(\phi_\varepsilon)_l \; ((\partial_{kl}\tilde{w}_\varepsilon) \circ \phi_\varepsilon)\,. \end{aligned}$$

In shorter notation, this means that

$$\nabla \tilde{u}_\varepsilon = D\phi \cdot \nabla \tilde{w}_\varepsilon, \qquad \Delta \tilde{u}_\varepsilon = a_\varepsilon^{ij}\, \partial_{ij} \tilde{w}_\varepsilon + \langle \Delta\phi_\varepsilon, \nabla \tilde{w}_\varepsilon \rangle$$

with

$$a_\varepsilon^{ij} = \langle \partial_i \phi_\varepsilon, \partial_j \phi_\varepsilon \rangle.$$

The coefficients are C^1-differentiable – so the associated operator A_ε can be equivalently written in divergence form – and C^1-close to δ_{ij}. We observe that

$$(\Delta \tilde{u}_\varepsilon - W'(\tilde{u}_\varepsilon))\,(\phi(y)) = \left(\partial_i \left(a_\varepsilon^{ij}\, \partial_j \tilde{w}_\varepsilon\right) - (\partial_i\, a_\varepsilon^{ij}) \partial_j \tilde{w}_\varepsilon + \langle \Delta\phi_\varepsilon, \nabla \tilde{w}_\varepsilon \rangle - W'(\tilde{w}_\varepsilon)\right)(y).$$

We extend $\tilde{w}_\varepsilon$ by even reflection to the whole ball $B_2(0)$, which preserves the $W^{2,2}$-smoothness since we preserved the property that $\partial_\nu \tilde{u}_\varepsilon = 0$ on the boundary when straightening the boundary. We observe that

$$\partial_i \,(a_\varepsilon^{ij} \partial_j \tilde{w}_\varepsilon) - \langle \operatorname{div} A_\varepsilon - \Delta\phi_\varepsilon, \nabla \tilde{w}_\varepsilon \rangle =: f_\varepsilon \in L^2(B_2(0))$$

since

$$\begin{aligned}\int_{B_2(0)} W'(\tilde{w}_\varepsilon)^2 \,\mathrm{d}y &= 2\int_{B_2^+(0)} W'(\tilde{w}_\varepsilon)^2(y)\,\mathrm{d}y \\ &= 2\int_{\Omega/\varepsilon\cap B_2(0)} W'(\tilde{u}_\varepsilon((z)))\,\det(D\phi_\varepsilon^{-1})(z)\,\mathrm{d}z \\ &\leq 2(1+c_\varepsilon)\int_{B_{2\varepsilon}(x)} \frac{1}{\varepsilon^n} W'(\tilde{u}_\varepsilon)\,\mathrm{d}z \\ &\leq C\end{aligned}$$

as shown above. The constants c_ε vanish as $\varepsilon \to 0$ and $\phi_\varepsilon \to \mathrm{id}$. The coefficients a_{ij} are uniformly elliptic and approach δ_{ij} uniformly as $\varepsilon \to 0$, so we can employ the elliptic estimate

$$\begin{aligned}||\nabla\tilde{w}_\varepsilon||_{L^2(B_{3/2})} &\leq C\left\{||\tilde{w}_\varepsilon||_{L^2(B_2)} + ||f_\varepsilon + \langle \operatorname{div} A_\varepsilon - \Delta\phi_\varepsilon, \nabla\tilde{w}_\varepsilon\rangle||_{L^2(B_2)}\right\} \\ &\leq C\left\{||\tilde{w}_\varepsilon||_{L^2(B_2)} + ||f_\varepsilon||_{L^2(B_2)} + ||\operatorname{div} A_\varepsilon - \Delta\phi_\varepsilon||_{L^\infty(B_2)}\,||\nabla\tilde{w}_\varepsilon||_{L^2(B_2)}\right\}.\end{aligned}$$

The constant is uniform in ε and $||\operatorname{div} A_\varepsilon - \Delta\phi_\varepsilon||_{L^\infty(B_2)} \to 0$ as $\varepsilon \to 0$, so we can bring the term to the other side and obtain a uniform $W^{1,2}$-bound for all sufficiently small ε, where the necessary smallness depends only on $\mathcal{W}_\varepsilon(u_\varepsilon)$ and $\partial\Omega$. In a second step, this gives us a uniform bound on $||\tilde{w}_\varepsilon||_{W^{2,2}(B_1(0))}$, which gives us a uniform bound on $||\tilde{u}_\varepsilon||_{W^{2,2}(B_{3/2}(0)\cap\Omega/\varepsilon)}$ after transforming back. The rest follows as before. □

Remark 6.2.3. The case that Ω has finite perimeter and $\partial_\nu u_\varepsilon = 0$ almost everywhere on the reduced boundary is a generalisation of the situation in which $\partial\Omega \in C^2$ and the level sets of u_ε meet $\partial\Omega$ at a ninety degrees angle. Such conditions arise naturally when we search for surfaces of minimal perimeter bounding a prescribed volume and may be useful also for models containing Willmore's energy [AK14].

We give an improvement of the L^∞-bound up to the boundary which implies L^p-convergence for all finite p.

Lemma 6.2.4. *Assume that there is $\theta \geq 1$ such that $|u_\varepsilon| \leq \theta$ on $\partial\Omega$ for all $\varepsilon > 0$. Then the following hold true.*

1. *If $n = 2$, $\partial\Omega \in C^{1,1}$ and $\theta > 1$, then for every $\beta < 1$ there exists a constant C depending only on $\bar{\alpha}, \theta, \Omega$ and β such that $\sup_{x\in\Omega} |u_\varepsilon(x)| \leq \theta + C\varepsilon^\beta$ for all $\varepsilon > 0$.*

   If $\theta = 1$, then for every $\beta < 1/2$ there exists a constant C depending only on $\bar{\alpha}, \Omega$ and β such that $\sup_{x\in\Omega} |u_\varepsilon(x)| \leq 1 + C\varepsilon^\beta$ for all $\varepsilon > 0$.

2. *If $n = 3$ and $\partial\Omega \in C^{1,1}$, then for every $p < \infty$ there exists C depending only on $\bar\mu, \bar\alpha, \theta, p$ and Ω such that $||u_\varepsilon||_{p,\Omega} \leq C$. Furthermore, for every $\sigma > 0$ there exists C depending only on $\bar\alpha, \theta, \Omega$ and σ such that $||u_\varepsilon||_{\infty,\Omega} \leq C\,\varepsilon^{-\sigma}$.*

We assume that also in three dimensions, uniformly bounded boundary values lead to uniform interior bounds.

Proof. The proof is a modified version of that of [RS06, Proposition 3.6]. We follow that proof closely, but use a different maximum principle.

Let $\theta' > \theta \geq 1$ such that $\{|u_\varepsilon| > \theta'\}$ has finite perimeter and define $w_\varepsilon := (u_\varepsilon - \theta')_+$. Then $w_\varepsilon \in W_0^{1,2}(\Omega)$ and from the same integration by parts as before we obtain that

$$||w_\varepsilon||_{1,2,\Omega}^2 \leq \int_{\{u_\varepsilon > \theta'\}} W'(u_\varepsilon)^2 + |\nabla u_\varepsilon|^2 \leq \alpha_\varepsilon(\Omega)\,\varepsilon.$$

The function satisfies

$$\begin{aligned}
\int_\Omega w_\varepsilon\,(-\Delta\phi)\,\mathrm{d}x &= \int_{\{u_\varepsilon > \theta'\}} (u_\varepsilon - \theta')\,(-\Delta\phi)\,\mathrm{d}x \\
&= -\int_{\partial\{u_\varepsilon > \theta'\}} (u_\varepsilon - \theta')\,\partial_\nu\phi\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\{u_\varepsilon > \theta'\}} \langle\nabla\phi, \nabla u_\varepsilon\rangle\,\mathrm{d}x \\
&= \int_{\partial\{u_\varepsilon > \theta'\}} \phi\,\partial_\nu u_\varepsilon - (u_\varepsilon - \theta')\,\partial_\nu\phi\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\{u_\varepsilon > \theta'\}} \phi\,(-\Delta u_\varepsilon)\,\mathrm{d}x \\
&\leq \int_{\{u_\varepsilon > \theta'\}} \phi\,(-\Delta u_\varepsilon)\,\mathrm{d}x
\end{aligned}$$

for $\phi \geq 0$. Again, this holds true because $\partial\{u_\varepsilon > \theta'\} \subset \{u_\varepsilon = \theta'\}$. Obviously

$$\begin{aligned}
\int_{\{u_\varepsilon > \theta'\}} \phi\,(-\Delta u_\varepsilon)\,\mathrm{d}x &= \int_{\{u_\varepsilon > \theta'\}} \left(-\Delta u_\varepsilon + \frac{1}{\varepsilon^2}W'(u_\varepsilon) - \frac{1}{\varepsilon^2}\,W'(u_\varepsilon)\right)\phi\,\mathrm{d}x \\
&\leq \int_{\{u_\varepsilon > \theta'\}} \frac{1}{\varepsilon}\left(h_\varepsilon - \frac{1}{\varepsilon}\,W'(\theta')\right)_+ \phi\,\mathrm{d}x,
\end{aligned}$$

so $-\Delta w_\varepsilon \leq \frac{1}{\varepsilon}\,\chi_{\{u_\varepsilon > \theta'\}}\left(h_\varepsilon - \frac{1}{\varepsilon}\,W'(\theta')\right)_+$ in the distributional sense. When we consider the solution $\psi_\varepsilon \in W_0^{1,2}(\Omega)$ of the problem

$$-\Delta\psi_\varepsilon = \frac{1}{\varepsilon}\left(h_\varepsilon - \frac{1}{\varepsilon}\,W'(\theta')\right)_+ \chi_{\{u_\varepsilon > \theta'\}},$$

the weak maximum principle [GT83, Theorem 8.1] applied to $w_\varepsilon - \psi_\varepsilon$ implies that

$$u_\varepsilon \leq \theta + w_\varepsilon \leq \theta + \psi_\varepsilon. \tag{6.2.1}$$

We proceed to estimate

$$\begin{aligned}
||\Delta\psi_\varepsilon||_{q,\Omega}^q &= \varepsilon^{-q}\int_{\{u_\varepsilon>\theta'\}}\left(h_\varepsilon - \frac{1}{\varepsilon}\,W'(\theta')\right)_+^q \mathrm{d}x \\
&\le \varepsilon^{-q}\left(\int_{\{u_\varepsilon>\theta'\}} 1\,\mathrm{d}x\right)^{1-q/2}\left(\int_\Omega h_\varepsilon^2\,\mathrm{d}x\right)^{q/2} \\
&\le \varepsilon^{-q}\left(\frac{\varepsilon^3}{W'(\theta')^2}\int_{\{u_\varepsilon>\theta'\}}\frac{1}{\varepsilon^3}\,W'(u_\varepsilon)^2\,\mathrm{d}x\right)^{1-q/2}\left(\varepsilon\int_\Omega \frac{1}{\varepsilon}\,h_\varepsilon^2\,\mathrm{d}x\right)^{q/2} \\
&\le c_{\bar\alpha,q}\,(W'(\theta'))^{q-2}\,\varepsilon^{-q+3(1-q/2)+q/2} \\
&= c_{\bar\alpha,q}\,(W'(\theta'))^{q-2}\,\varepsilon^{3-2q}
\end{aligned}$$

for $1 \le q < 2$. Thus $||\Delta\psi_\varepsilon||_{q,\Omega} \le C_{\bar\alpha,q}\,(W'(\theta'))^{1-2/q}\,\varepsilon^{3/q-2}$, and by the elliptic estimate [GT83, Lemma 9.17], we have

$$||\psi_\varepsilon||_{2,q,\Omega} \le c_{\Omega,\bar\alpha,q}\,(W'(\theta'))^{1-2/q}\,\varepsilon^{3/q-2}.$$

Let us insert this estimate into (6.2.1). If $n = 3$, we take $q = 3/2$ and use that $W^{2,3/2}(\Omega)$ embeds into $L^p(\Omega)$ for all finite p. Thus (taking some $\theta' > 1$ if $\theta = 1$), we see that $u_\varepsilon \le \theta' + \psi_\varepsilon$ where ψ_ε is uniformly bounded in $L^p(\Omega)$. We may use the same argument on the negative part of u_ε, so in total u_ε is uniformly bounded in $L^p(\Omega)$ for all $1 \le p < \infty$ by domination through ψ_ε. Taking $q = 3/(2-\sigma) > 3/2$ proves the L^∞-estimate by the same comparison.

If $n = 2$, we have a Sobolev embedding $W^{2,q}(\Omega) \to L^\infty(\Omega)$ for all $q > 1$. Assuming that $\theta > 1$ and $\beta < 1$ we take $\theta' \to \theta$ to obtain

$$u_\varepsilon \le \theta + w_\varepsilon \le \theta + \psi_\varepsilon \le \theta + C_{\Omega,\bar\alpha,q}\,(W'(\theta))^{1-2/q}\,\varepsilon^{3/q-2}.$$

For $q = 3/(2+\beta)$, this gives $u_\varepsilon \le 1 + C\,\varepsilon^\beta$. Here $q \in (1,2)$ is admissible since $\beta \in (0,1)$. If $\theta = 1$, we may take $0 < \beta < 1/2$, $q = (3-2\beta)/2 \in (1,2)$ and $1+\varepsilon^\beta \le \theta' \le 1 + 2\varepsilon^\beta$ to obtain

$$|u_\varepsilon| \le 1 + C_{\Omega,\bar\alpha,q}\varepsilon^{\beta(1-2/q)+(3/q-2)} = 1 + C_{\Omega,\bar\alpha,q}\varepsilon^\beta$$

with the approximation $W'(\theta') = O(\varepsilon^\beta)$. □

Corollary 6.2.5. *If either*

1. $u_\varepsilon \in C^0(\overline{\Omega})$ *and there exists* $\theta \ge 1$ *such that* $|u_\varepsilon| \le \theta$ *on* $\partial\Omega$ *for all* $\varepsilon > 0$ *or*

2. $\partial\Omega \in C^2$ *and* $\partial_\nu u_\varepsilon = 0$ *a.e. on* $\partial\Omega$,

then $u_\varepsilon \to u$ *in* $L^p(\Omega)$.

Proof. The sequence u_ε converges to u in $L^1(\Omega)$ and is bounded in $L^q(\Omega)$ for all $q < \infty$ (or even $L^\infty(\Omega)$). Hölder's inequality implies L^p-convergence. □

Remark 6.2.6. If $n = 2$, $\beta < 1/2$ and $|u_\varepsilon| \leq 1 + \varepsilon^\beta$ on $\partial\Omega$, then the proof still shows that

$$\sup_\Omega |u_\varepsilon| \leq 1 + C\,\varepsilon^\beta$$

for this particular β. The case $\beta = 1/2$ is still open at the boundary.

For a counterexample to uniform boundedness on Ω without boundary conditions, see Example 6.3.1. Even with boundary values satisfying $|u_\varepsilon| \leq 1$ on $\partial\Omega \in C^2$, we shall construct a sequence u_ε for which uniform Hölder continuity fails at the boundary in Example 6.3.3.

6.3 Counterexamples to Boundary Regularity

The idea here is simple: namely, the energy $\mathcal{W}_\varepsilon$ can be seen to control the $W^{2,2}$-norm of blow ups of phase-fields onto ε-scale since those are asymptotic to bounded entire solutions of the stationary Allen-Cahn equation $-\Delta\tilde u + W'(\tilde u) = 0$ at (almost all) points away from the boundary. At the boundary on the other hand, the asymptotic behaviour corresponds to solutions of the same equation on half-space, whose behaviour is essentially governed by their boundary values. To make this precise, take $h \in C_c^\infty(\mathbb{R}^n)$ and $H := \{x_n > 0\}$. The energy

$$\mathcal{F}\colon W^{1,2}_{loc}(H) \to \mathbb{R}\cup\{\infty\}, \quad \mathcal{F}(u) = \int_H \frac{1}{2}\,|\nabla u|^2 + W(u)\,\mathrm{d}x$$

has a minimiser $\tilde u$ in the affine space $(1+h)+W^{1,2}_0(H)$ by the direct method of the calculus of variations. Namely, take a sequence u_k such that $\lim_{k\to\infty}\mathcal{F}(u_k) = \inf \mathcal{F}(u) \leq \mathcal{F}(h+1) < \infty$. Then

$$||\nabla u_k||_{L^2(H)} \leq C, \qquad \text{and} \quad (u_k-1)^2(x) \leq (u_k-1)^2(u_k+1)^2(x) = 4\,W(u_k(x))$$

at all points $x \in H$ such that $u_k(x) \geq 0$. Using the boundary values, also the negative part of u_k is uniformly controlled in $L^2(H)$ by the H^1-semi norm. Thus the sequence u_k is bounded in $W^{1,2}(H)$ and there exists $\tilde u$ such that $u_k \rightharpoonup \tilde u$ (up to a subsequence). Since the affine space is convex and strongly closed, it is weakly = weakly* closed and $\tilde u \in 1 + h + W^{1,2}_0(H)$. For any $R > 0$, we can use the compact embedding $W^{1,2}(B_R^+) \to L^4(B_R^+)$ to deduce that

$$\int_{B_R^+} \frac{1}{2}\,|\nabla\tilde u|^2 + W(\tilde u)\,\mathrm{d}x \leq \liminf_{k\to\infty}\int_{B_R^+}\frac{1}{2}\,|\nabla u_k|^2 + W(u_k)\,\mathrm{d}x \leq \liminf_{k\to\infty}\int_H \frac{1}{2}\,|\nabla u_k|^2 + W(u_k)\,\mathrm{d}x.$$

Letting $R \to \infty$ shows that $\tilde{u}$ is in fact a minimiser of $\mathcal{F}$. If $h \geq 0$, then

$$1 + (\tilde{u} - 1)_+ \in 1 + h + W_0^{1,2}(H), \qquad \mathcal{F}\left(1 + (\tilde{u} - 1)_+\right) \leq \mathcal{F}(\tilde{u})$$

with strict inequality unless $\tilde{u} = 1 + (\tilde{u} - 1)_+$. Since we assume $\tilde{u}$ to be a minimiser, we find that $\tilde{u} \geq 1$ almost everywhere. The same argument shows that $\tilde{u} \leq 1 + ||h||_\infty$ almost everywhere. Calculating the Euler-Lagrange equation of $\mathcal{F}$, we see that $\tilde{u}$ is a weak solution of

$$-\Delta \tilde{u} + W'(\tilde{u}) = 0.$$

On the convex set

$$C_h := \{u \in W^{1,2}(H) \mid u = 1 + h \text{ on } \partial H, u \geq 1\}$$

the operator

$$A\colon C_h \to W^{-1,2}(H), \quad A(u) = -\Delta u + W'(u)$$

is well-defined (since $n \leq 3$ and W' has cubical growth) and strongly monotone, so the equation $Au = 0$ has a unique solution $\tilde{u} \in C_h$ which coincides with the minimiser $\tilde{u}$ of $\mathcal{F}$ in $1 + h + W_0^{1,2}(H)$). A bootstrapping argument via elliptic regularity theory shows that $\tilde{u} \in C^\infty_{loc}(\overline{H})$. By trace theory we have that

$$||h||^2_{2,\partial H} \;=\; ||\tilde{u} - 1||^2_{2,\partial H} \;\leq\; ||\tilde{u} - 1||^2_{1,2,H}/2 \;\leq\; \mathcal{F}(\tilde{u}) \;\leq\; \mathcal{F}(1 + h).$$

In this way, we can fully control the mass density $\tilde{\mu} = \frac{1}{2}\,|\nabla \tilde{u}|^2 + W(\tilde{u})$ created by $\tilde{u}$ in terms of its boundary values. For later purposes, we have to obtain suitable decay estimates for the functions $\tilde{u}$ depending on h. In a first step, we show that the limit $\lim_{|x|\to\infty} \tilde{u}(x) = 1$ exists. Assume the contrary. Then there exist $\theta > 1$ and a sequence $x_k \in H$ such that

$$|x_k| \to \infty, \qquad \tilde{u}(x_k) \geq \theta.$$

Taking a suitable subsequence, we may assume that the balls $B_1(x_k)$ are disjoint and $|x_k| \geq R + 2$ is so large that h is supported in $B_R(0)$. If $B_2(x_k) \subset H$, we may proceed as in Lemma 5.2.8 to deduce uniform Hölder continuity on the balls $B_1(x_k)$ from the L^∞-bound to $\tilde{u}$ and the fact that $\tilde{u}$ solves $\Delta \tilde{u} = W'(\tilde{u})$. This means that there exists $r > 0$ such that $\tilde{u} \geq (1 + \theta)/2$ on $B_r(x_k)$. Otherwise, the same argument still goes through after extending $\tilde{u}$ by a standard reflection principle and the fact that the boundary values are constant on

$\partial H \cap B_2(x_k)$. The geometry of H gives us $\mathcal{L}^n(B_r(x_k) \cap H) \geq \omega_n\, r^n/2$. So we deduce that

$$\mathcal{F}(\tilde{u}) \geq \sum_{k=0}^{\infty} \int_{B_r(x_k)} W(\,(1+\theta)/2)\,\mathrm{d}x \geq \sum_{k=0}^{\infty} W((1+\theta)/2)\,\omega_n\, r^n/2 = \infty$$

in contradiction to the definition of $\tilde{u}$. Now we can estimate the decay of $\tilde{u}$ in a more precise fashion. Since $h \in C_c(\partial H)$, there is $C_h > 0$ such that $h \leq C_h\, e^{-|x|}$ on ∂H. To simplify the following calculations, we assume that $C_h = 1$. Then we claim that $1 \leq u \leq 1 + e^{-|x|}$ for all $x \in \mathbb{R}^n$. Assume the contrary and observe that $\psi(x) = 1 + e^{-|x|}$ satisfies

$$\Delta\psi(x) = \left(1 + \frac{1-n}{|x|}\right) e^{-|x|}, \qquad W'(\psi(x)) = \left(2 + 3\,e^{-|x|} + e^{-2\,|x|}\right) e^{-|x|},$$

so in particular $\Delta\psi(x) \leq W'(\psi(x))$ for all $x \in \mathbb{R}^n$. Since $\tilde{u} = h \leq \psi$ on ∂H by assumption and $\lim_{|x|\to\infty} \tilde{u}(x) = 1$, there must be a point $x_0 \in H$ such that

$$(\psi - u)(x_0) = \min_H (\psi - u) < 0,$$

but then

$$\Delta(\psi - u)(x_0) \leq W'(\psi(x_0)) - W'(u(x_0)) < 0$$

so $\psi - u$ cannot be minimal at x_0. This proves the claim. It follows that

$$\int_{H\setminus B_R^+} W(\tilde{u})\,\mathrm{d}x \leq 2 \int_R^{\infty} e^{-2r}\, r^{n-1}\,\mathrm{d}r = P_n(R)\, e^{-2R}$$

where P_n is a polynomial of degree n depending on the dimension. To estimate the second part of the energy functional, we use the gradient bound

$$|\nabla u(x)| \leq n\,\sqrt{n}\, \sup_{\partial Q} |u| + \frac{1}{2} \sup_Q |\Delta u|$$

from [GT83, Section 3.4] where Q is a cube of side length $d = 1$ with a corner at x. Applied to our problem, for $x \in \partial B_R^+$ we can find a cube Q satisfying $\bar{Q} \cap \bar{B_R^+} = \{x\}$ such that

$$|\nabla \tilde{u}(x)| \;=\; |\nabla(\tilde{u} - 1)|(x) \leq n\,\sqrt{n}\, \sup_{\partial Q} |\tilde{u} - 1| + \frac{1}{2} \sup_Q |W'(\tilde{u})| \;\leq\; (n\,\sqrt{n} + 5/2)\, e^{-|x|}.$$

Thus we also have

$$\int_{H\setminus B_R^+} \frac{1}{2}\, |\nabla \tilde{u}|^2\,\mathrm{d}x \;\leq\; \left(n\,\sqrt{n} + 5/2\right)^2\; P_n(R)\, e^{-2R}$$

Finally, we remark that the same type of estimate obviously holds for $\Delta\tilde{u} = W'(\tilde{u}) \in L^2(H)$. Having given the general construction for suitable functions of zero $\mathcal{W}_1$ curvature energy, we are finally ready to apply these results to obtain counterexamples. For simplicity, we construct the counterexamples first on the half space H and transfer them to bounded Ω later on.

Example 6.3.1 (Counterexample to Boundedness). Fix $h \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq h \leq e^{-|x|}$, $h \not\equiv 0$ and set $h_\theta = \theta\, h$. Every function of this type induces a minimiser $\tilde{u}_\theta$. We may take a sequence $\theta_\varepsilon \to \infty$ such that $\varepsilon^{n-1}/\theta_\varepsilon^4 \to 0$ and set $u_\varepsilon(x) = \tilde{u}_{\theta_\varepsilon}(x/\varepsilon)$. Clearly, u_ε becomes unbounded as $\varepsilon \to 0$, but

1. $\mathcal{W}_\varepsilon(u_\varepsilon) \equiv 0$ and

2. $S_\varepsilon(u_\varepsilon) = \varepsilon^{n-1}\,\mathcal{F}(\tilde{u}_{\theta_\varepsilon}) \leq C\,\varepsilon^{n-1}\,\mathcal{F}(h_{\theta_\varepsilon}) \to 0$.

So the sequence u_ε induces limiting measures $\mu = \alpha = 0$, but fails to be uniformly bounded.

The next example is a technically more demanding version of this one where the energy scaling is chosen so that we create an atom of size $S > 0$ at the origin.

Example 6.3.2 (Counterexample to Boundary Regularity of μ). Take $h_\theta, \tilde{u}_\theta$ as above. Then the map

$$f\colon [0,\infty) \to \mathbb{R}, \quad f(\theta) = \mathcal{F}(\tilde{u}_\theta) = \inf\{\mathcal{F}(u) \mid u \in 1 + h_\theta + W_0^{1,2}(H)\}$$

is continuous. To see this, take pairs θ_1, θ_2 and the corresponding minimisers $\tilde{u}_1$, $\tilde{u}_2$ and observe that

$$\tilde{u}_{1,2} = \frac{\theta_2}{\theta_1}\,[\tilde{u}_1 - 1] + 1 \;\in\; 1 + h_{\theta_2} + W_0^{1,2}(H).$$

Since

$$W(1+\alpha u) = ((1+\alpha u)^2 - 1)^2/4 = (2\alpha u + \alpha^2 u^2)^2/4 \leq \max\{\alpha^2, \alpha^4\} W(1+u)$$

we have

$$f(\theta_2) = \mathcal{F}(\tilde{u}_2) \leq \mathcal{F}(\tilde{u}_{1,2}) \leq \max\left\{\left(\frac{\theta_2}{\theta_1}\right)^2, \left(\frac{\theta_2}{\theta_1}\right)^4\right\}\mathcal{F}(\tilde{u}_1) = \max\left\{\left(\frac{\theta_2}{\theta_1}\right)^2, \left(\frac{\theta_2}{\theta_1}\right)^4\right\} f(\theta_1).$$

Reversing the roles of θ_1 and θ_2 shows that f is continuous. Now let $S > 0$. Due to the continuity of f in θ and the trace inequality

$$\theta^2 ||h||_{2,\partial H}^2 = ||h_\theta||_{2,\partial H}^2 \leq \mathcal{F}(\tilde{u}_\theta)$$

we can pick a sequence $\theta_\varepsilon \to \infty$ at most polynomially in $1/\varepsilon$ such that $\mathcal{F}(\tilde u_{\theta_\varepsilon}) = S\,\varepsilon^{1-n}$. As before, set $u_\varepsilon(x) = \tilde u_{\theta_\varepsilon}(x/\varepsilon)$ and observe that $\mathcal{W}_\varepsilon(u_\varepsilon) \equiv 0$, $S_\varepsilon(u_\varepsilon) \equiv S$. It remains to show that $\mu = S\,\delta_0$, i.e. that the limiting measure is concentrated in one point. The functions $\tilde u_\theta$ actually tend to shift more of their mass towards the origin as $\theta \to \infty$ since the steepness (and overall height) is best concentrated on a ball of small radius for a low energy.

The same application of the maximum principle as before shows that $\tilde u_\theta \le \tilde w_\theta := 1 + \theta(\tilde u_1 - 1)$ since

$$\Delta(\tilde w_\theta - \tilde u_\theta) = \theta\,\Delta\tilde u_1 - \Delta\tilde u_\theta = \theta\,W'(\tilde u_1) - W'(\tilde u_\theta) \le W'(\tilde w_\theta) - W'(\tilde u_\theta)$$

is monotone in $\tilde w_\theta$, $\tilde u_\theta$ and the boundary values satisfy $\tilde u_\theta = \tilde w_\theta$ on ∂H and $\lim_{|x|\to\infty} \tilde u_\theta = \lim_{|x|\to\infty} \tilde w_\theta = 1$. Like above, we now obtain that

$$\int_{H\setminus B_R^+} \frac{1}{2}\,|\nabla\tilde u_\varepsilon|^2 + W(\tilde u_\varepsilon)\,\mathrm{d}x \le \max\{\theta_\varepsilon^2, \theta_\varepsilon^4\}\,P_n(R)\,e^{-2R}.$$

Thus we can choose a sequence $R_\varepsilon \to \infty$ such that $\theta_\varepsilon^4\,P_n(R_\varepsilon)\,e^{-2R_\varepsilon} \to 0$ and $\varepsilon\,R_\varepsilon \to 0$ since θ_ε grows only polynomially in $1/\varepsilon$ and the exponential term dominates (take e.g. $R_\varepsilon = \varepsilon^{-1/2}$). Thus for all $R > 0$

$$\mu_\varepsilon(B_R(0)) = \varepsilon^{1-n}\int_{B^+_{R/\varepsilon}} |\nabla\tilde u_{\theta_\varepsilon}|^2 + W(\tilde u_{\theta_\varepsilon})\,\mathrm{d}x \ge \varepsilon^{1-n}\int_{B^+_{R_\varepsilon}} |\nabla\tilde u_{\theta_\varepsilon}|^2 + W(\tilde u_{\theta_\varepsilon})\,\mathrm{d}x \to S$$

and hence $\mu(B_R(0)) \ge S$. Taking $R \to 0$ shows that $\mu(\{0\}) = \mu(\overline{H}) = S$, i.e. $\mu = S\,\delta_0$.

Functions as described above can appear as minimisers of functionals like $\mathcal{W}_\varepsilon + \varepsilon^{-1}\,(S_\varepsilon - S)^2$ which are used to search for minimisers of Willmore's energy with prescribed surface area – even as functions with energy zero. The same is true for functionals including the topological penalisation term discussed below.

By construction, the previous example shows that the inclusion $\mathrm{spt}(\mu) \subset \lim_{\varepsilon\to 0} u_\varepsilon^{-1}(I)$ need not be true for any $I \Subset (-1,1)$ since $u_\varepsilon \ge 1$ and thus $K = \emptyset$. We use a similar construction to demonstrate that the reverse inclusion need not hold, either.

Example 6.3.3 (Counterexample to Hausdorff Convergence). Using the same arguments as above, if $0 \le h \le 2$, we can find a solution $\tilde u \in (1-h) + W_0^{1,2}(H) \cap C^\infty_{loc}(\overline{H})$ of

$$-\Delta\tilde u + W'(\tilde u) = 0 \quad \text{in } H, \qquad \bar u = 1 - h \quad \text{on } \partial H$$

satisfying $-1 \le \tilde u \le 1$, $\lim_{|x|\to\infty} \tilde u(x) = 1$ and $\mathcal{F}(\tilde u) \le \mathcal{F}(1+h) < \infty$. Decay estimates are harder to obtain here since W' is not monotone inside $[-1,1]$, but we will not need them, either. If we take h such that $h(0) = 2$, $h \in C_c^\infty(B_1)$, we can use continuity up to the boundary to deduce that $\tilde{u}^{-1}(\rho)\cap B_1^+ \neq \emptyset$ for all $\rho \in (-1,1)$. So when we set $u_\varepsilon(x) = \tilde{u}(x/\varepsilon)$, we see that

1. $\mu_\varepsilon(H) = \varepsilon^{n-1}\,\tilde{\mu}(H) = \varepsilon^{n-1}\,\mathcal{F}(\tilde{u}) \to 0$,
2. $\mathcal{W}_\varepsilon(u_\varepsilon) \equiv 0$ and
3. $0 \in \lim_{\varepsilon\to 0} u_\varepsilon^{-1}(I)$ in the Hausdorff sense for all $\emptyset \neq I \Subset (-1,1)$.

Example 6.3.4 (Counterexample to Uniform Hölder Continuity). If we take h like in the previous example and replace it by $h^\omega(x) = h(\omega x)$ we observe that the associated minimisers satisfy

$$\mathcal{F}(\tilde{u}^\omega) \leq \mathcal{F}(h^\omega) \leq \mathcal{F}(h)$$

for all $\omega \geq 1$ since the gradient term stays invariant in two dimensions and decreases in three, while the integral of the double well potential decreases in both cases for any fixed h. Thus, if we take any sequence $\omega_\varepsilon \to \infty$ and define $u_\varepsilon(x) = \tilde{u}^{\omega_\varepsilon}(x/\varepsilon)$, we get the same results as before. As the function becomes steeper and steeper on the boundary faster than ε, uniform Hölder continuity up to the boundary cannot hold, even for uniformly bounded boundary values.

Example 6.3.5 (Counterexample to Boundary Regularity of μ with $-1 < u_\varepsilon < 1$). We can refine the examples to show that growth of u_ε on $\partial\Omega$ is not the only reason that μ might develop atoms on $\partial\Omega$, but that this is in fact possible with $|u_\varepsilon| \leq 1$. This happens when we prescribe highly oscillating boundary values on ∂H. Let $h \in C_c^\infty(\partial H)$, then for any $u \in H^1(H)$ with $u|_{\partial H} = g$ we have

$$\int_H |\nabla u|^2\,\mathrm{d}x \geq [h]^2_{H^{1/2}(\partial H)} = c_{n-1}\int_{\partial H\times\partial H} \frac{|h(x)-h(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}x\,\mathrm{d}y.$$

for a constant depending on the dimension $n-1 \in \{1,2\}$. For any $S' > 0$ and $\delta > 0$ we can construct $h \in C^\infty(H)$ such that

1. $0 \leq h \leq \delta$,
2. $\mathrm{supp}(h) \subset B_1(0)$ and
3. $[h]^2_{H^{1/2}} \geq S'$.

We construct a solution of the stationary Allen-Cahn equation with the boundary values $1-h$ as before, but for a modified potential

$$\overline{W}(s) = \begin{cases} W(1-2\delta) & s \leq 1-2\delta \\ W(s) & s \geq 1-2\delta \end{cases}.$$

An energy minimiser will never dip below $1 - 2\delta$ then, and consequently never below $1 - \delta$ by the maximum principle if δ is chosen so small that W' is monotone on $[1 - 2\delta, \infty)$. The rest of the proof goes through as before with suitable scaling of h to get the right energy since W' behaves correctly just below 1, as it does slightly above 1. We will not repeat the details.

The boundary values need to be constructed with slightly more care since we cannot just have vertical growth and the $H^{1/2}$-norm behaves badly under spacial scaling. This is compensated in the boundary construction by having a larger number of faster oscillations. When we have constructed h with a large enough half-norm, we can always reduce it by scaling with a constant < 1.

For the sake of simplicity, we chose to construct the examples on half space due to its scaling invariance. Let us sketch how they can be transferred to C^2-domains. If $\Omega \Subset \mathbb{R}^n$ and $\partial\Omega \in C^2$ there exists $x_0 \in \partial\Omega$ such that $|x_0| = \max_{x\in\partial\Omega} |x|$. At x_0, both principal curvatures of $\partial\Omega$ are strictly positive, so in a ball around x_0, up to a rigid motion we may write

$$\Omega \cap B_r(x_0) = \{x \in B_r(x_0) \mid x_n > \phi(\hat{x})\}$$

where $\hat{x} = (x^1, \ldots, x^{n-1})$ and ϕ is a strictly convex C^2-function satisfying $\phi(0) = 0$, $\nabla\phi(0) = 0$ and $\Omega \subset H$. If Ω is convex in the first place, this is possible at every point $x_0 \in \partial\Omega$.

Thus, the function $u_\varepsilon(x) = \tilde{u}(x/\varepsilon)$ is well-defined on Ω for any of the functions $\tilde{u}$ constructed above. If ε is chosen small enough, the difference between H and Ω/ε becomes negligible for any given $\tilde{u}$ and we can still construct counterexamples to boundedness, local Hölder-continuity, relationship between $\mathrm{spt}(\mu)$ and the Hausdorff limit of the level sets and to the regularity of μ this way.

Using the exponential decay (or modifying functions to become constant for larger arguments) it is also possible to create singular behaviour for example along curves in the convex portion of the boundary by placing singular solutions of the stationary Allen-Cahn equation at an increasing number of points distributed along the curve.

We restricted our analysis to convex boundary points since then $u_\varepsilon = \tilde{u}_\theta(x/\varepsilon)$ is well-defined for all small $\varepsilon > 0$, whereas at other points, half space does not provide enough information to fill an entire neighbourhood of x_0. We believe that the same pathologies can arise at general boundary points.

Chapter 7

Thin Elastic Structures with Constraint

7.1 Introduction

In this chapter, we will finally provide a partial phase-field solution to Problem 1 and a computationally feasible method of finding minimisers. The solution is partial in that it only controls *connectedness* and does not guarantee that a limiting surface can be approximated by C^2-boundaries with bounded Willmore energy. A contribution to this field will be given in Chapter 8.

An often cited advantage of phase-fields is that they are capable of changing their topology; in that sense our endeavour is non-standard. It should be noted that our phase-fields may still change their topology (at least in three dimensions), only connectedness is enforced.

7.1.1 Topology and Phase-Fields

Examples of topological changes and loss of connectedness in simulations for biological problems governed by bending energies or our type are given in [DW07, Du10].

In [BLS15], a geodesic distance function has been used to minimise the length of a connected set K containing a prescribed set of points $x_1, \dots, x_N$ in two dimensions (Steiner's problem). Our setting is different in two ways: 1. Steiner's problem has a finite number of a priori known points which need to be contained in K while the transition layer of the phase-field has no special points and 2. the phase-field approximation of Steiner's problem works in dimension $n = 2$, while we work in ambient space of dimension $n = 2, 3$ where the curves used in the definition of the distance function have codimension 2.

Previous work in [DMR11] provides a first attempt at an implementation of a topological constraint in a phase-field model for elastic strings modelled by the one-dimensional version of the Willmore energy, Euler's elastica. This technique prevents transitions in simulations for simple situations, but may fail in more complex cases, see Section 9.3. Similar numerical approaches to tracking the topology using a diffuse Euler number are discussed in [DLW05, DLRW07].

This approach was complemented by a method put forth in [DMR14], which relies on a second phase-field subject to an auxiliary minimisation problem used to identify connected components of the transition layer. While our functional can be seen as using a diffuse measure of path-connectedness, the functional in [DMR14] generalises more directly the notion of connectedness. For this model, a Γ-convergence result was obtained, showing that limits of bounded-energy sequences must describe a connected structure. Unfortunately, the complicated nested minimisation problem makes it unsuitable for computation.

Approaches of regularising limit interfaces have been developed by Bellettini in [Bel97] and investigated analytically and numerically in [ERR14]. The approaches work by introducing non-linear terms of the phase-field in order to control the Willmore energies of the level sets individually and exclude transversal crossings (which phase-fields for De Giorgi's functional can develop). These regularisations may prevent loss of connectedness along a gradient flow in practice, but do not lead to a variational statement via Γ-convergence.

Furthermore, we would like to emphasise that we can easily describe a weakly* continuous evolution of varifolds along which connectedness is lost. Except at one singular time, the varifolds are embedded C^2-manifolds and the evolution is C^2-smooth – see Remark 3.2.14.

It thus is not clear whether the approach of [Bel97] does prevent topological transitions, in particular, the loss of connectedness, in three ambient space dimensions. At least, it is more difficult to implement due to the highly non-linear term including the Willmore energies of level sets.

As we have seen in Chapter 3, topological genus is not continuous under varifold convergence and minimising sequences of a constrained minimisation problem with fixed genus may change topological type in the limit.

At this point, we thus know of no other model which can control the topology of phase-field limits. Furthermore, our results are optimal since they allow us to control as much of the topology as can be controlled even for a sharp interface and they allow for efficient implementation.

7.1.2 Connectedness

In order to ensure that the support of the limiting measure μ is connected we include an auxiliary term $\mathcal{C}_\varepsilon$ in the energy functional. The heuristic idea behind this is that if the support of the limiting measure $\mathrm{spt}(\mu)$ is connected, then so should the set $\{\rho_1 < u_\varepsilon < \rho_2\}$ for $-1 < \rho_1 < \rho_2 < 1$. These level sets away from ± 1 can be heuristically viewed as approximations of $\mathrm{spt}(\mu)$ as supported by Lemma 5.3.4.

Our concept is to introduce a quantitative notion of path-connectedness and penalise the measured disconnectedness. Take a weight function $F \in C^0(\mathbb{R})$ such that

$$F \geq 0, \qquad F \equiv 0 \text{ on } [\rho_1, \rho_2], \qquad F(-1),\ F(1) > 0.$$

and the associated geodesic distance

$$d^{F(u)}(x, y) = \inf \left\{ \int_K F(u) \, \mathrm{d}\mathcal{H}^1 \;\middle|\; K \text{ connected, } x, y \in K, \mathcal{H}^1(K) \leq \omega(\varepsilon) \right\},$$

where $\omega(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In particular, $\omega(\varepsilon) \equiv \infty$ is not excluded. If $\{\rho_1 < u_\varepsilon < \rho_2\}$ is connected, we can connect any two points $x, y \in u_\varepsilon^{-1}(\rho_1, \rho_2)$ by a curve of length zero. If it is not, then $d^{F(u)}(x, y)$ gives a quantitative notion of how badly path-connectedness fails between these two points. To obtain a global notion, we take a second weight function $\phi \in C_c(-1, 1)$ resembling a bump, i.e.,

$$\phi \geq 0, \qquad \{\phi > 0\} = (\rho_1, \rho_2) \Subset (-1, 1), \qquad \int_{-1}^{1} \phi(u) \, \mathrm{d}u > 0$$

and take the double integral

$$\mathcal{C}_\varepsilon(u) = \frac{1}{\varepsilon^2} \int_\Omega \int_\Omega \phi(u(x)) \, \phi(u(y)) \, d^{F(u)}(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

As connectedness is a non-local concept, the non-local nature of the functional is not surprising. So, if $\{\phi(u_\varepsilon) > 0\} = \{\rho_1 < u_\varepsilon < \rho_2\}$ is connected, we can connect any two points $x, y \in \Omega$ such that $\phi(u_\varepsilon(x)) \, \phi(u_\varepsilon(y)) > 0$ with a curve of length zero, hence $d^{F(u_\varepsilon)}(x, y) = 0$ and both the integrand and the double integral vanish.

If on the other hand $\mathrm{spt}(\mu)$ is disconnected, then we expect that $d^{F(u_\varepsilon)}$ should be able to discern different connected components such that $\liminf_{\varepsilon \to 0} \mathcal{C}_\varepsilon(u_\varepsilon) > 0$. The core part of our proof is concerned with precisely that. We need to show that ϕ detects components of the interface and that $d^{F(u_\varepsilon)}$ distinguishes them. For the first result, we need to understand the structure of the interfaces converging to μ and make sure they cannot be so steep in $u_\varepsilon^{-1}(\rho_1, \rho_2)$ that the double integral does not see them in the limit. For the second part, the challenge is to understand how phase-fields converge away from the interface which has been treated above. Precisely, we need convergence on curves, i.e. on objects of co-dimension two.

7.1.3 Main Results

For our application to connectedness, we define the total energy of an ε-phase-field as

$$\mathcal{E}_\varepsilon(u) = \begin{cases} \mathcal{W}_\varepsilon(u) + \varepsilon^{-\sigma}\,(S_\varepsilon(u) - S)^2 + \varepsilon^{-\kappa}\,\mathcal{C}_\varepsilon(u) & u \in -1 + W_0^{2,2}(\Omega) \\ +\infty & \text{else} \end{cases} \tag{7.1.1}$$

for $\sigma, \kappa > 0$.

Remark 7.1.1. If $\omega(\varepsilon) < \infty$, existence of mininimisers for the functional $\mathcal{E}_\varepsilon$ is a simple exercise in the direct method of the calculus of variations, since uniform convergence of a minimising sequence $u_{\varepsilon,k} \in W^{2,2}(\Omega) \to C^0(\overline{\Omega})$ for fixed ε guarantees convergence of the distance term.

Theorem 7.1.2. *Let $u_\varepsilon \in X$ be a sequence such that $(\mathcal{W}_\varepsilon + S_\varepsilon)(u_\varepsilon) \leq C$ for some $C > 0$ and μ, α Radon measures such that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$, $\alpha_\varepsilon \overset{*}{\rightharpoonup} \alpha$. If $\operatorname{spt}(\mu)$ is disconnected, then*

$$\liminf_{\varepsilon\to 0} \mathcal{C}_\varepsilon(u_\varepsilon) > 0.$$

Corollary 7.1.3. *Let $n = 2, 3$ and $u_\varepsilon \in X$ a sequence such that $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) < \infty$. Then the diffuse mass measures μ_ε converge weakly* to a measure μ with connected support $\operatorname{spt}(\mu) \subset \overline{\Omega}$ and area $\mu(\overline{\Omega}) = S$.*

The main result of [RS06] can be applied to deduce Γ-convergence of our functionals in the following sense:

Corollary 7.1.4. *Let $n = 2, 3$, $S > 0$, $\Omega \Subset \mathbb{R}^n$ and $E \Subset \Omega$, with smooth boundary $\partial E \in C^2$ with area $\mathcal{H}^{n-1}(\partial E) = S$. Then*

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\chi_E - \chi_{E^c}) = \begin{cases} \mathcal{W}(\partial E) & \partial E \textit{ is connected} \\ +\infty & \textit{otherwise} \end{cases}$$

7.2 Proofs

The proofs are organised in the following way. We begin with technical Lemmata (Section 7.2.1) and give the proofs of Theorems 7.1.2, 7.1.3 and 7.1.4 in Section 7.2.2.

7.2.1 Auxiliary Results

In this section, we will derive technical results concerning how phase-field approximations interact with the function ϕ as needed for the functional $\mathcal{C}_\varepsilon$ to impose connectedness.

The following arguments rely on the rectifiable structure of the measure μ that we are approximating. Specifically, we introduce the diffuse normal direction by

$$\nu_\varepsilon := \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$$

when $\nabla u_\varepsilon \neq 0$ and 0 else.

Lemma 7.2.1. *[RS06, Propositions 4.1, 5.1] Define the $n-1$-varifold $V_\varepsilon := \mu_\varepsilon \otimes \nu_\varepsilon$ by*

$$V_\varepsilon(f) = \int_{\mathbb{R}^n} f(x, \langle \nu_\varepsilon \rangle^\perp) \,\mathrm{d}\mu_\varepsilon \qquad \forall\ f \in C_c(\mathbb{R}^n \times G(n, n-1)).$$

Then there is an integral varifold V such that $V_\varepsilon \to V$ weakly as Radon measures on $\mathbb{R}^n \times G(n, n-1)$ (varifold convergence). The limit satisfies

$$\mu_V = \mu, \qquad H_\mu^2\, \mu \leq \alpha$$

where μ_V is the mass measure of V and H_μ denotes the generalised mean curvature of μ. In particular, $\mathcal{W}(\mu) \leq \bar\alpha$.

The following result is a suitably adapted version of [RS06, Proposition 5.5] for our purposes. It shows that given small discrepancy measures and small oscillation of the gradient, a bounded energy sequence looks very much like an optimal interface in small balls. Using our improved bounds from Lemma 5.2.8, we can drop most of their technical assumptions.

Lemma 7.2.2. *Let $\delta, \tau > 0$ and denote $\nu_{\varepsilon,n} = \langle \nu_\varepsilon, e_n \rangle$. Then there exist $0 < L < \infty$ depending on δ and τ only and $\gamma > 0$ depending on $\bar\alpha, \delta$ and τ such that the following holds for all $x_0 \in \mathbb{R}^n$. If*

1. *$|u_\varepsilon(x_0)| \leq 1 - \tau$ and*
2. *$|\xi_\varepsilon|(B_{4L\varepsilon}(x_0)) + \int_{B_{4L\varepsilon}(x_0)} 1 - \nu_{\varepsilon,n}^2 \,\mathrm{d}\mu_\varepsilon \leq \gamma\, (4L\varepsilon)^{n-1}$*

then also the following two properties hold:

- *The blow up* $\tilde u_\varepsilon(y) = u_\varepsilon(x_0+\varepsilon y)$ *is* $C^{0,1/4}$*-close to an optimal profile* q *on* $B_{3L}(0)$*:*

$$|\pm\tilde u_\varepsilon - q(\pi_n(\cdot)-t_1)|_{0,1/4,B_{3L}(0)} < \delta.$$

The optimal profile q *is the function from the* $\limsup$*-construction,* $t_1 = q^{-1}(u_\varepsilon(x))$ *and* π_n *denotes the projection onto the* n*-th coordinate.*

- $|u_\varepsilon(\hat x, x_n+t)| \geq 1-\tau/2$ *for all* $L\varepsilon \leq |t| \leq 3L\varepsilon$*, where* $\hat x = (x_1,\dots,x_{n-1})$ *and* u *changes sign in between.*

Proof. Without loss of generality, we may assume that $x_0 = 0$ and write $B_r := B_r(0)$. Recall that $q'(t) = \sqrt{2\,W(q(t))}$ and $\lim_{t\to\pm\infty} q(t) = \pm 1$. Thus we can pick $L>0$ such that $|q(t)| \geq 1-\tau/4$ for all $t > L$.

Assume for a contradiction that there is no constant $\gamma > 0$ such that the results of the Lemma hold. Then for $\gamma^j \to 0$, there must be a sequence u_ε^j such that $|u_\varepsilon^j(0)| \leq 1-\tau$, $\mathcal{W}_\varepsilon(u_\varepsilon^j) \leq \bar\alpha + 1$ and

$$|\xi_\varepsilon^j|(B_{4L\varepsilon}) + \int_{B_{4L\varepsilon}} 1-(\nu_{\varepsilon,n}^j)^2\,\mathrm{d}\mu_\varepsilon \leq \gamma^j\,(4L\varepsilon)^{n-1},$$

but the conclusions of the Lemma do not hold. Considering the blow ups $\tilde u^j : B_{4L}\to\mathbb{R}$ with $\tilde u^j(y) = u_\varepsilon^j(\varepsilon y)$ we obtain

$$||\tilde u^j||_{2,2,B_{3L}} \leq C_{\bar\alpha,n,L}$$

like in Lemma 5.2.8. Hence there is $\tilde u \in W^{2,2}(B_{3L})$ such that

$$\tilde u^j \rightharpoonup \tilde u \quad \text{in } W^{2,2}(B_{3L}).$$

Since $W^{2,2}$ embeds compactly into $W^{1,2}$ and L^4, we see that

$$\begin{aligned}\int_{B_{3L}} \big|\,|\nabla\tilde u|^2/2 - W(\tilde u)\,\big|\,\mathrm{d}x &= \lim_{j\to\infty}\int_{B_{3L}} \big|\,|\nabla\tilde u^j|^2/2 - W(\tilde u^j)\,\big|\,\mathrm{d}x\\ &\leq \lim_{j\to\infty} \varepsilon^{1-n}|\xi_\varepsilon^j|(B_{4L\varepsilon})\\ &\leq \liminf_{j\to\infty}\,(4L)^{n-1}\gamma^j\\ &= 0\end{aligned}$$

and when we set $\hat\nabla u = (\partial_1 u, \dots, \partial_{n-1}u)$, we get

$$\begin{aligned}
\int_{B_{3L}} |\hat\nabla\tilde u|\,\mathrm{d}x &= \lim_{j\to\infty}\int_{B_{3L}} |\hat\nabla\tilde u^j|\,\mathrm{d}x\\
&= \lim_{j\to\infty}\int_{B_{3L}} \sqrt{|\nabla\tilde u_j|^2 - |\partial_n\tilde u^j|^2}\,\mathrm{d}x\\
&\leq \liminf_{j\to\infty}\int_{B_{4L}} |\nabla\tilde u^j|\,\sqrt{1-\left(\tilde\nu_n^j\right)^2}\,\mathrm{d}x\\
&\leq \liminf_{j\to\infty}\left(\omega_n\,(4L)^n\right)^{1/2}\left(\int_{B_{4L}} |\nabla\tilde u^j|^2\,\left(1-(\tilde\nu_n^j)^2\right)\,\mathrm{d}x\right)^{\frac12}\\
&\leq \liminf_{j\to\infty}\sqrt{8L\,\omega_n}\left(\varepsilon^{1-n}\int_{B_{4L\varepsilon}} 1-(\nu_n^j)^2\,\mathrm{d}\mu_\varepsilon\right)^{\frac12}\\
&\leq \liminf_{j\to\infty}\sqrt{8L\,\omega_n\,\gamma^j}\\
&= 0.
\end{aligned}$$

Thus we can see that

$$|\nabla\tilde u|^2 = 2\,W(\tilde u), \qquad \nabla\tilde u = (0,\dots,0,\partial_n\tilde u).$$

Clearly, this means that $\tilde u(y) = p(y_n)$ for a function p with $p' = \pm\sqrt{2\,W(p)}$. Using that $|\tilde u(0)| \leq 1-\tau$ and the Picard-Lindelöff theorem on the uniqueness of the solutions to ODEs, we see that $p(y_n) = \pm q(y_n - \bar y)$ for some $\bar y \in \mathbb{R}$ which can easily be fixed by the initial condition for $p(0)$.

Since weak $W^{2,2}$-convergence implies strong $C^{0,1/4}$-convergence in $n = 2,3$ dimensions, we see that there is $j \in \mathbb{N}$ such that the claim of the Lemma holds for u_ε^j contradicting our assumption. Thus the Lemma is proven. □

To deal with the rectifiable sets in the next section more easily we prove a structure result for rectifiable sets. The result seems standard, but we have been unable to find a reference for it. As usual, we call a function on a closed set differentiable if it admits a differentiable extension to a larger open set.

Lemma 7.2.3. *Let M be a countably k-rectifiable set in $\mathbb{R}^n$. Denote by B the closed unit ball in k dimensions. Then there exist injective C^1-functions $f_i : B \to \mathbb{R}^n$ with $\nabla f_i \neq 0$ on B such that*

$$\mathcal{H}^k\left(M \setminus \bigcup_{i=1}^\infty f_i(B)\right) = 0$$

and such that $f_i(B)\cap f_j(B) = \emptyset$ for all $i \neq j$.

Proof. According to [KP08, Lemma 5.4.2] or [Sim83, Lemma 11.1] there is a countable collection of C^1-maps $g_i : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$M \subset N \cup \bigcup_{i=1}^{\infty} g_i\left(\mathbb{R}^k\right)$$

where $\mathcal{H}^k(N) = 0$. Without loss of generality, N is assumed to be disjoint from the other sets. First we need to make the individual maps g_i one-to-one. To do that, we define the set where injectivity fails in a bad way:

$$A_i := \left\{x \in \mathbb{R}^k \mid \forall\ r > 0\ \exists\ y \in B_r(x) \text{ such that } g_i(x) = g_i(y)\ \right\}.$$

Due to the failure of local injectivity, we see that the Jacobian $J_{g_i}(x)$ vanishes on A_i. Since g_i is a C^1-function, the set $D_i := J_{g_i}^{-1}(0)$ is closed and by the Morse-Sard Lemma [Fed69, 3.4.3]

$$\mathcal{H}^k\left(g_i(D_i)\right) = 0.$$

Set $U_i := \mathbb{R}^k \setminus D_i$. Now as in [EG92, Chapter 1.5, Corollary 2] we can use Vitali's covering theorem [EG92, Chapter 1.5, Theorem 1] to obtain a countable selection of closed balls B_i^j such that f_i is injective with non-vanishing gradient on B_i^j for all $j \in \mathbb{N}$ and

$$\mathcal{L}^k\left(U_i \setminus \bigcup_{j=1}^{\infty} B_i^j\right) = 0.$$

Since the boundary of a k-ball has Hausdorff dimension $k-1$, we could equally well take open balls. Since C^1-functions map sets of $\mathcal{L}^k$-measure zero to sets of $\mathcal{H}^k$-measure zero, we have shown that we can write

$$M \subset \tilde{N} \cup \bigcup_{j=1}^{\infty} \tilde{g}_i(B^\circ)$$

where $\mathcal{H}^k(\tilde{N}) = 0$, $\tilde{g}_i : B \to \mathbb{R}^n$ is one-to-one, C^1, and has a non-vanishing gradient everywhere on the closed ball B. The functions $\tilde{g}_m$ are obtained by rescaling suitable restrictions of g_i from B_i^j to the unit ball. Finally, we have to cut out the sets that get hit by more than one function $\tilde{g}_m$. Inductively, we define

$$\tilde{U}_m := B^\circ \setminus \tilde{g}_m^{-1}\left(\bigcup_{l=1}^{m-1} \tilde{g}_l(B)\right).$$

Finally, we use Vitali's Lemma again to pick collections of closed balls $\tilde{B}_m^l$ such that

$$\mathcal{L}^k\left(\tilde{U}_m \setminus \bigcup_{l=1}^{\infty} \tilde{B}_m^l\right) = 0.$$

Rescaling the restricted functions from these balls and translating to the unit ball gives us the result. □

The proof of the following Lemma resembles that of the integrality of μ in [RS06, Lemma 4.2]. It faces different challenges: while we do not need to prove multi-layeredness, we cannot zoom in on the tangent space since we need a macroscopic measure contribution to the double integral. Thus we need Lemma 7.2.3 to approximate macroscopically the structure of μ.

Lemma 7.2.4. *Let $\phi \in C^0(\mathbb{R})$ such that $\phi \geq 0$ and $\int_{-1}^{1} \phi(u)\,\mathrm{d}u > 0$. If $x \in \operatorname{spt}(\mu)$, then*

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{B_r(x)} \phi(u_\varepsilon)\,\mathrm{d}z > 0$$

for all $r > 0$.

Proof. **Step 1.** As usual, we assume that $x = 0$, $\mu(\partial B_{r/2}(x)) = 0$ and denote $B = B_{r/2}(x)$. This means that all the ε-balls of positive integral we are going to find will actually lie in $B_r(x)$ and is a purely technical condition. Let ζ be a small constant to be specified later. For further use, denote by $\hat{B}$ the closed unit ball in $\mathbb{R}^{n-1}$.

As μ is an integral varifold, we know that $\operatorname{spt}(\mu)$ is rectifiable. This means by Lemma 7.2.3 that there are countably many C^1-functions $f_i : \hat{B} \to \mathbb{R}^n$ such that

$$\operatorname{spt}(\mu) \subset M_0 \cup \bigcup_{i=1}^{\infty} f_i(\hat{B}), \quad \mathcal{H}^{n-1}(M_0) = 0 \quad f_i(\hat{B}) \cap f_j(\hat{B}) = \emptyset$$

for $i \neq j$. Since μ has second integrable mean curvature $H_\mu^2 \cdot \mu \leq \alpha$, we can further use the Li-Yau inequality (5.2.4) to bound the maximum multiplicity of μ uniformly by

$$\theta_{\max} \leq \frac{\alpha(\overline{\Omega})}{16\,\pi},$$

at least $\mathcal{H}^{n-1}$-almost everywhere. Now since $\mathcal{H}^{n-1}(\operatorname{spt}(\mu)) < +\infty$ we can find $N \in \mathbb{N}$ such that

$$\mathcal{H}^{n-1}\left(\big(\operatorname{spt}(\mu) \cap B\big) \setminus \bigcup_{i=1}^{N} f_i(\hat{B}) \right) < \frac{\zeta}{\theta_{\max}}.$$

Since f_i is injective and has non-vanishing tangent maps everywhere, $M := \bigcup_{i=1}^{N} f_i\left(\hat{B}^\circ\right)$ is a C^1-manifold. We observe that

$$\mathcal{H}^{n-1}(\operatorname{spt}(\mu) \cap B \setminus M) < \frac{\zeta}{\theta_{\max}}$$

and hence

$$\mu\,(B \setminus M) < \zeta.$$

Since the maps in question are smooth and the unit discs are orientable, for every i we can pick a continuous unit normal field to $f_i(\hat{B})$ (e.g. using cross products). Since the discs are compact and disjoint (thus a positive distance apart), the fields defined on each disc separately induce a continuous unit vector field on the union of their closures.

Now we use the Tietze-Urysohn extension theorem to obtain a vector field X on B such that $X = \nu_M$ on M and projecting on the unit ball we ensure $|X| \leq 1$. After an easy modification, we may assume that $|X| = 1$ on a neighbourhood of M. We then define

$$G : \mathbb{R}^n \times G(n, n-1) \to \mathbb{R}, \qquad G(x, S) = \langle X_x, \nu_S \rangle^2$$

where ν_S is one of the unit normals to S. Note that G is continuous since X is. Using the non-negativity of G and the fact that $T_x\mu = T_xM$ for $\mathcal{H}^{n-1}$-almost every $x \in M \cap \operatorname{spt}(\mu)$ we interpret μ as dual to $C^0(\mathbb{R}^n \times G(n, n-1))$ and observe

$$\begin{aligned}
\langle \mu, G \rangle &= \int_{\operatorname{spt}(\mu)} \theta(x)\, G(x, T_x\mu)\, \mathrm{d}\mathcal{H}^{n-1} \\
&\geq \int_{\operatorname{spt}(\mu)\cap M} \theta(x)\, G(x, T_xM)\, \mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\operatorname{spt}(\mu)\cap M} \theta(x)\, \mathrm{d}\mathcal{H}^{n-1} \\
&= \mu(M) \\
&\geq \mu(B) - \zeta.
\end{aligned}$$

Step 2. By varifold convergence, we know that $\lim_{\varepsilon\to 0}\langle \mu_\varepsilon, G\rangle = \langle \mu, G\rangle \geq \mu(B) - \zeta$, and $|X|, |\nu_\varepsilon| \leq 1$ so

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \int_B \left|1 - \langle \nu_\varepsilon, X\rangle^2\right| \mathrm{d}\mu_\varepsilon &= \limsup_{\varepsilon\to 0} \int_B 1 - \langle \nu_\varepsilon, X\rangle^2\, \mathrm{d}\mu_\varepsilon \\
&\leq \limsup_{\varepsilon\to 0} \left(\mu_\varepsilon(B) - \langle \mu_\varepsilon, G\rangle\right) \\
&\leq \zeta.
\end{aligned}$$

For $\gamma, \varepsilon, L > 0$ we define the set

$$U_{\varepsilon,\gamma,L} := \left\{ x \in B \;\middle|\; \frac{1}{(4L\varepsilon)^{n-1}} \int_{B_{4L\varepsilon}(x)} \left|1 - \langle \nu_\varepsilon, X\rangle^2\right| \mathrm{d}\mu_\varepsilon > \gamma/4 \right\}.$$

Let $x_1, \dots, x_K$ be points in $U_{\varepsilon,\gamma,L}$ being maximal for the property that the balls $B_{4L\varepsilon}(x_i)$

are disjoint. Then by definition

$$\zeta \geq \int_B \left|1-\langle\nu_\varepsilon, X\rangle^2\right| \mathrm{d}\mu_\varepsilon \geq \sum_{i=1}^K \int_{B_{4L\varepsilon}(x_i)} \left|1-\langle\nu_\varepsilon, X\rangle^2\right| \mathrm{d}\mu_\varepsilon \geq K\,(4L\varepsilon)^{n-1}\,\gamma/4.$$

At the same time, we know that the balls $B(x_i, 8L\varepsilon)$ cover $U_{\varepsilon,\gamma,L}$ because otherwise we could bring in more disjoint balls, therefore

$$\begin{aligned}\frac{\mathcal{L}^n(U_{\varepsilon,\gamma,L})}{\varepsilon} &\leq K\,\frac{\omega_n\,(8L\varepsilon)^n}{\varepsilon}\\ &\leq \frac{4\,\zeta}{\gamma\,(4L\varepsilon)^{n-1}}\,\frac{\omega_n\,(8L\varepsilon)^n}{\varepsilon}\\ &= 2^{n+4}\,\omega_n\,L\,\zeta/\gamma.\end{aligned}$$

For a given γ, we choose $\zeta = \zeta(\gamma)$ such that this is $\leq \mu(B)/4$.

Step 3. Knowing that $|\xi_\varepsilon|(B) \to 0$, we can use the same argument as in the second step to show for

$$V_{\varepsilon,\gamma,L} := \left\{ x \in B \;\middle|\; \frac{|\xi_\varepsilon|\,(B_{4L\varepsilon}(x))}{(4L\varepsilon)^{n-1}} > \gamma/2 \right\}$$

the estimate

$$\frac{\mathcal{L}^n(V_{\varepsilon,\gamma,L})}{\varepsilon} \leq \mu(B)/4$$

for all sufficiently small $\varepsilon > 0$.

Step 4. Now choose U as a neighbourhood of M on which $|X| = 1$ and $\tau > 0$ like in Corollary 5.2.3 satisfying

$$\liminf_{\varepsilon\to 0} \mu_\varepsilon\left(U \cap \{|u_\varepsilon| \leq 1-\tau\}\right) \geq \frac{3\,\mu(B)}{4}.$$

This is easily achieved when $\mu(M) > 3\,\mu(B)/4$. Furthermore we take $\delta \ll 1$ suitably small for small deviations of the optimal interface to behave similarly enough, L and γ as in Lemma 7.2.2 and $\zeta = \zeta(\gamma)$. Using steps one through three, we see that

$$\begin{aligned}&\liminf_{\varepsilon\to 0} \frac{\mathcal{L}^n\left(\{|u_\varepsilon| \leq 1-\tau\} \cap U \setminus (U_{\varepsilon,\gamma,L} \cup V_{\varepsilon,\gamma,L})\right)}{\varepsilon}\\ &\quad\geq \liminf_{\varepsilon\to 0} \frac{\mathcal{L}^n\left(\{|u_\varepsilon| \leq 1-\tau\} \cap U\right)}{\varepsilon} - \frac{\mathcal{L}^n\left(U_{\varepsilon,\gamma,L}\right)}{\varepsilon} - \frac{\mathcal{L}^n\left(V_{\varepsilon,\gamma,L}\right)}{\varepsilon}\\ &\quad\geq 3\,\mu(B)\,/4 - \mu(B)/4 - \mu(B)/4\\ &\quad= \mu(B)/4.\end{aligned}$$

Using the reverse argument of step 2, we can see that there are at least K points $x_1, \ldots, x_K$

in $\{|u_\varepsilon| \leq 1-\tau\} \cap U \setminus (U_{\varepsilon,\gamma,L} \cup V_{\varepsilon,\gamma,L})$ such that the balls $B_{4L\varepsilon}(x_i)$ are disjoint with

$$K \geq \frac{\mu(B)}{8^{n+1}\, L^n\, \varepsilon^{n-1}}.$$

Step 5. To apply Lemma 7.2.2, we must "freeze" the coefficients of the vector field X to a single unit vector. We compute

$$\begin{aligned}
&\frac{1}{(4L\varepsilon)^{n-1}}\left|\int_{B_{4L\varepsilon}(x_i)} \left(1-\langle \nu_\varepsilon, X\rangle^2\right) - \left(1-\langle \nu_\varepsilon, X_i\rangle^2\right)\, d\mu_\varepsilon\right| \\
&\qquad = \frac{1}{(4L\varepsilon)^{n-1}}\left|\int_{B_{4L\varepsilon}(x_i)} \langle \nu_\varepsilon, X_i\rangle^2 - \langle \nu_\varepsilon, X\rangle^2\, d\mu_\varepsilon\right| \\
&\qquad = \frac{1}{(4L\varepsilon)^{n-1}}\left|\int_{B_{4L\varepsilon}(x_i)} \langle \nu_\varepsilon, X_i - X\rangle\, \langle \nu_\varepsilon, X_i + X\rangle\, d\mu_\varepsilon\right| \\
&\qquad \leq |X_i + X|_{C^0(B_{4L\varepsilon}(x_i))} \cdot |X_i - X|_{C^0(B_{4L\varepsilon}(x_i))}\, \frac{1}{(4L\varepsilon)^{n-1}} \cdot \int_{B_{4L\varepsilon}(x_i)} d\mu_\varepsilon \\
&\qquad \leq 2\, C_{\bar\alpha,L,n}\, |X_i - X|_{C^0(B_{4L\varepsilon}(x_i))}
\end{aligned}$$

for all X_i such that $|X_i| \leq 1$. When we set $X_i = X(x_i)$, the last term converges to zero – so eventually it is smaller than $\gamma/4$ and

$$\frac{1}{(4L\varepsilon)^{n-1}}\int_{B_{4L\varepsilon}(x_i)} 1 - \langle \nu_\varepsilon, X_i\rangle^2\, d\mu_\varepsilon < \gamma/2.$$

Since $x_i \in U$, we finally see that $|X_i| = 1$ and Lemma 7.2.2 can be applied.

Step 6. Since u_ε is $C^{0,1/4}$-close to a one-dimensional optimal profile on $B_{3L\varepsilon}(x_i)$ which transitions from -1 to 1, we see that for each $s \in (-(1-\tau), (1-\tau))$ there must be a point $y_i \in B_{3L\varepsilon}(x_i)$ such that $u_\varepsilon(y_i) = s$. By Hölder continuity, we deduce that

$$\int_{B_{3L\varepsilon}(x_i)} \phi(u_\varepsilon)\, dx \geq \bar\theta\, \varepsilon^n$$

for a constant $\bar\theta$ depending on the support of ϕ and on $\bar\alpha, n$ for the Hölder constant. Since the balls are disjoint by construction, we can add this up to

$$\begin{aligned}
\frac{1}{\varepsilon}\int_B \phi(u_\varepsilon)\, dx &\geq \frac{1}{\varepsilon}\sum_{j=1}^{M}\int_{B_{3L\varepsilon}(x_i)} \phi(u_\varepsilon)\, dx \\
&\geq \frac{1}{\varepsilon}\, M\, \bar\theta\, \varepsilon^n \\
&\geq \frac{\mu(B)\, \bar\theta}{8^{n+1}\, L^n} \\
&> 0.
\end{aligned}$$

This concludes the proof. □

At this point, also the reverse inclusion for the Hausdorff limits that was claimed in Lemma 5.3.4 can been proven – compare the sketch of the proof there and the proof above.

7.2.2 Proof of the Main Results

Having dealt with the necessary auxiliary results, we can proceed to prove our main results. We begin with the main statement about connectedness.

Proof of Theorem 7.1.2. The proof is structured as follows. First, we show that we can find neighbourhoods of connected components which have positive distance with respect to the usual metric on $\mathbb{R}^n$. Then we need to show that they also have positive distance with respect to the pseudometric $d^{F(u_\varepsilon)}$. Intuitively, this makes sense since any connecting curve should have to leave the interfacial layer between the two sets. This is simple if $n = 2$ and slightly more technical if $n = 3$.

Without loss of generality, we may assume that there are $-1 < \theta_1 < \theta_2 < 1$ such that $\{\phi > 0\} \subset (\theta_1, \theta_2)$ and $F \geq 1$ outside (θ_1, θ_2). This is only a minor assumption and could easily be removed, but simplifies the proof.

Step 1. Assume that $\mathrm{spt}(\mu)$ is not connected. Since $\mathrm{spt}(\mu)$ is compact, according to Lemma 3.2.6 there are disjoint open sets U_1, U_2 such that

$$\mathrm{spt}(\mu) \subset U_1 \cup U_2, \qquad \mu(U_i) > 0, i = 1, 2, \qquad \delta := \mathrm{dist}(U_1, U_2) > 0.$$

Now

$$\begin{aligned}\liminf_{\varepsilon\to 0} \mathcal{C}_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon\to 0} \int_{U_1} \phi(u_\varepsilon(x))\,\mathrm{d}x \cdot \liminf_{\varepsilon\to 0} \int_{U_2} \phi(u_\varepsilon(y))\,\mathrm{d}y \\ \cdot \liminf_{\varepsilon\to 0} \mathrm{dist}^{F(u_\varepsilon)}(U_1, U_2).\end{aligned}$$

Since the first two factors are strictly positive according to Lemma 7.2.4, it suffices to show that $\liminf_{\varepsilon\to 0} \mathrm{dist}^{F(u_\varepsilon)}(U_1, U_2) > 0$.

Step 2. For a contradiction, assume that $\mathrm{dist}^{F(u_\varepsilon)}(U_1, U_2) \to 0$. Pick a sequence c_ε such that $c_\varepsilon \to 0$ but still $\frac{\mathrm{dist}^{F(u_\varepsilon)}(U_1,U_2)}{c_\varepsilon} \to 0$. Then there exist a connected set K_ε and points $x_\varepsilon, y_\varepsilon \in \overline{\Omega}$ such that

$$x_\varepsilon \in K_\varepsilon \cap \partial U_1, \qquad y_\varepsilon \in K_\varepsilon \cap \partial U_2, \qquad \int_{K_\varepsilon} F(u_\varepsilon)\,\mathrm{d}\mathcal{H}^1 \leq c_\varepsilon.$$

If $n = 2$, we know that $|u_\varepsilon| \to 1$ uniformly on $\Omega \setminus (U_1 \cup U_2)$, so in particular $u_\varepsilon \notin [\theta_1, \theta_2]$ on

$K_\varepsilon \subset \Omega \setminus (U_1 \cup U_2)$ and

$$\int_{K_\varepsilon} F(u_\varepsilon)\,\mathrm{d}\mathcal{H}^1 \geq \mathcal{H}^1(K_\varepsilon \setminus (U_1 \cup U_2)) \geq \delta > 0$$

since K_ε connects U_1 to U_2. This is a contradiction to our assumption. In the case $n = 3$ we need a further argument.

Step 3. In this step, we will use the competition between the distance function driving u_ε away from ± 1 along K_ε and the energy bounds in three dimensions.

Take a subsequence realising the lim inf. The 1-Lipschitz map $\pi(x) := \operatorname{dist}(x, U_1)$ maps K_ε to a connected set containing $0 = \pi(x_\varepsilon)$ and $\delta = \pi(y_\varepsilon)$, so $[0, \delta] \subset \pi(K_\varepsilon)$. Furthermore, $\pi^{-1}(0, \delta) \subset \mathbb{R}^2 \setminus (U_1 \cup U_2)$ since $\operatorname{dist}(U_1, U_2) = \delta$. Take the set

$$K'_\varepsilon := \{t \in [0, \delta] \,:\, \exists\, x \in K_\varepsilon \text{ such that } t = \pi(x) \text{ and } u_\varepsilon(x) \notin [\theta_1, \theta_2]\}$$

of points whose pre-image contributes a lot to the weighted length of K_ε. Then

$$\begin{aligned}\mathcal{H}^1(K'_\varepsilon) &\leq \mathcal{H}^1\left(K_\varepsilon \cap \{u_\varepsilon \notin [\theta_1, \theta_2]\}\right)\\ &\leq \int_{K_\varepsilon} F(u_\varepsilon)\,\mathrm{d}\mathcal{H}^1\\ &\leq c_\varepsilon.\end{aligned}$$

Pick M intervals

$$I_k = \left[\frac{2k-1}{2M}\,\delta,\ \frac{k}{M}\,\delta\right]$$

inside $[0, \delta]$. Fix $1 \leq k \leq M$. When ε is so small that $c_\varepsilon < \frac{\delta}{4M}$, we deduce that

$$\mathcal{H}^1(I_k \setminus K'_\varepsilon) \geq \mathcal{H}^1(I_k) - \mathcal{H}^1(K'_\varepsilon) \geq \frac{\delta}{2M} - \frac{\delta}{4M} = \frac{\delta}{4M}. \tag{7.2.1}$$

In particular, there exist points $x_{i,\varepsilon} \in \pi^{-1}(I_k \setminus K'_\varepsilon)$ and (up to a subsequence) $x_{i,\varepsilon} \to x_i \in \Omega \setminus \operatorname{spt}(\mu)$ for $i = 1, \ldots, M$. By construction, x_i is in the δ-distant set A_δ for $\delta = \min\{|1+\theta_1|, |1-\theta_2|\}$. Letting $M \to \infty$, we show that there is a countable collection of such points, contradicting the essentially uniform convergence of $|u_\varepsilon| \to 1$ in $\mathbb{R}^n \setminus \operatorname{spt}(\mu)$.

□

Now Corollary 7.1.3 is an obvious consequence of Theorem 7.1.2.

Proof of Corollary 7.1.3: Let u_ε be a sequence such that $\mathcal{E}_\varepsilon(u_\varepsilon)$ is bounded. Then in particular $|\mu_\varepsilon(\Omega) - S| \leq \varepsilon^{\sigma/2}$, so $\mu_\varepsilon(\mathbb{R}^n) = \mu_\varepsilon(\Omega)$ is bounded and $\mu_\varepsilon \rightharpoonup \mu$ for some Radon measure

μ – for this and other properties see [EG92, Chapter 1]. Clearly

$$\mu(\overline{\Omega}) \geq \limsup_{\varepsilon\to 0} \mu_\varepsilon(\overline{\Omega}) = S$$

and on the other hand

$$\mu(\overline{\Omega}) \leq \mu(\mathbb{R}^n) \leq \liminf_{\varepsilon\to 0} \mu_\varepsilon(\mathbb{R}^n) = S$$

so $\mu(\overline{\Omega}) = S$. If $U = \mathbb{R}^n \setminus \overline{\Omega}$, we have

$$\mu(U) \leq \liminf_{\varepsilon\to 0} \mu_\varepsilon(U) = 0,$$

so $\operatorname{spt}(\mu) = \bigcap_{U \text{ open}, \mu(U)=0} U^c \subset \overline{\Omega}$. Since $\mathcal{E}_\varepsilon(u_\varepsilon)$ is bounded, we have $\mathcal{C}_\varepsilon(u_\varepsilon) \to 0$, so due to Theorem 7.1.2, $\operatorname{spt}(\mu)$ is connected. □

We now proceed to prove Corollary 7.1.4.

Proof of the $\liminf$*-inequality:* It follows from Theorem 7.1.3 that $\mathcal{E}_\varepsilon(u_\varepsilon) \to \infty$ if ∂E is disconnected. If ∂E is connected, the main part of this inequality is to show that if $u_\varepsilon \to \chi_E - \chi_{E^c}$ in $L^1(\Omega)$ and $\mu_\varepsilon(\Omega) \leq S+1$, then $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{W}(\partial E)$. Since $\mathcal{E}_\varepsilon \geq \mathcal{W}_\varepsilon$ and enforces the surface area estimate, we obtain with [RS06] that

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{W}(\partial E).$$

□

Proof of the $\limsup$*-inequality:* We may restrict our analysis to the case of connected boundaries with area $\mathcal{H}^{n-1}(\partial E) = S$. Since $E \Subset \Omega$, $U_\delta := \{\operatorname{dist}(\cdot, E) < \delta\} \subset \Omega$ for all sufficiently small δ, and since $\partial E \in C^2$ is embedded, there is $\delta > 0$ such that

$$\psi : \partial E \times (-\delta, \delta) \to U_\delta, \quad \psi(x,t) = x + t\,\nu_x$$

is a diffeomorphism. Considering Chapter 4, it only remains to show that $\lim_{\varepsilon\to 0} \varepsilon^{-\kappa}\mathcal{C}_\varepsilon(u^\varepsilon) = 0$. We will show that even $\mathcal{C}_\varepsilon(u^\varepsilon) \equiv 0$ along this sequence. Since ∂E is connected and ψ is a diffeomorphism, all the level sets

$$\{u^\varepsilon = \rho\} = \psi(\partial E, \varepsilon\, q_\varepsilon^{-1}(\rho))$$

are connected manifolds for $\rho \in (-1,1)$. We know that

$$\{\phi(u^\varepsilon) > 0\} = \{\rho_1 < u^\varepsilon < \rho_2\}$$

and pick any $\rho \in (\rho_1, \rho_2)$. Now let $x, y \in \Omega$, $\phi(u^\varepsilon(x)), \phi(u^\varepsilon(y)) > 0$. We can construct a curve from x to y by setting piecewise

$$\gamma_1 : [0, d(x)]] \to \Omega\,, \quad \gamma_1(t) = \pi(x) + t\,\nu_{\pi(x)},$$
$$\gamma_3 : [0, d(y)] \to \Omega\,, \quad \gamma_3(t) = \pi(y) + t\,\nu_{\pi(y)}$$

and γ_2 any curve connecting $\pi(x)$ to $\pi(y)$ in $\{u^\varepsilon = \rho\}$. This curve exists since connected manifolds are path-connected. The curve $\gamma = \gamma_3 \oplus \gamma_2 \oplus \gamma_1^{-1}$ connects x and y and satisfies by construction $\phi(\gamma(t)) > 0$, so $F(\gamma(t)) \equiv 0$. Therefore we deduce

$$d^{F(u^\varepsilon)}(x, y) = 0$$

if $\phi(u^\varepsilon(x)), \phi(u^\varepsilon(y)) \neq 0$ since the connecting curves have uniformly bounded length and $\omega(\varepsilon) \to \infty$. Thus in particular

$$\frac{1}{\varepsilon^2} \int_{\Omega\times\Omega} \phi(u^\varepsilon(x))\, \phi(u^\varepsilon(y))\, d^{F(u^\varepsilon)}(x, y)\, \mathrm{d}x\, \mathrm{d}y \equiv 0.$$

Like in Remark 4.2.2, we can satisfy the boundary conditions $u_\varepsilon \in -1 + W_0^{2,2}(\Omega)$ by a slight modification of the usual recovery sequence. □

7.3 Extensions

In this chapter, we have developed a strategy to enforce connectedness of diffuse interfaces. Below we shall see that the strategy fares well in applications and can efficiently be implemented and seems to be more generally applicable to a wider class of problems. Our results can be extended to the following situations.

- We can include a soft volume constraint like

  $$F\left(\frac{1}{2}\int_\Omega u_\varepsilon + 1\,\mathrm{d}x\right)$$

  for continuous functions $F \geq 0$. We could also include a hard volume constraint under the assumption that the sharp interface limit supports the hard volume constraint, in particular we have to prescribe a volume smaller than that of Ω and compatible with the area constraint through an isoperimetric inequality in Ω.

- Another popular constraint compatible with our functional and results is minimising a distance from a given configuration as

$$A_\varepsilon(u) = \int_\Omega |u - g| \,\mathrm{d}\lambda$$

where λ is a finite Radon measure on Ω and $g \in L^1(\lambda)$. This functional originates in problems in image segmentation, but in our context it can be understood as prescribing certain points to lie inside or outside the membrane according to experimental data.

- We can use the same modelling techniques for a finite collection of membranes given by $u_\varepsilon^1, \dots, u_\varepsilon^N$ inside an elastic container given by U_ε. The governing energy could be composed of a sum of the individual elastic energies $\mathcal{E}_\varepsilon$ and interaction energies I_ε like

$$I_\varepsilon(u_\varepsilon^i, u_\varepsilon^j) = \frac{1}{\varepsilon}\int_\Omega (u_\varepsilon^i + 1)^2\,(u_\varepsilon^j + 1)^2\,\mathrm{d}x$$

which prevent penetration of the phases $u_\varepsilon^i \approx 1$ and $u_\varepsilon^j \approx 1$ or, in a slight variation, enforce confinement of $u_\varepsilon^i \approx 1$ to $U_\varepsilon \approx 1$.

- As mentioned above, we can use the phase-field approximation of Helfrich's energy [BM10] in place of the diffuse Willmore functional.

Chapter 8

The Role of Blow-Ups

8.1 Introduction

We have demonstrated above how we can force the limiting surface μ to be connected through an appropriate penalty term. In the first part of this chapter, we instead concentrate on the approximability of μ by smooth boundaries. In the second we will demonstrate that this method controls the topology of phase-fields on a finer level than before along a continuous time-evolution and can in particular be used to preserve connectedness.

Characterising the Γ-limit of the functionals $\mathcal{W}_\varepsilon$ at sets E which do not have a C^2-boundary is an open problem. A natural candidate for the Γ-limit is the L^1-lower semi-continuous envelope

$$\widetilde{\mathcal{W}}(E) = \inf\left\{ \liminf_{k\to\infty} \mathcal{W}(\partial E_k) \;\middle|\; \chi_{E_k} \xrightarrow{L^1} \chi_E, \quad \partial E_k \in C^2 \right\}$$

of the functional $\mathcal{W}$ defined on C^2-sets, which picks the best approximation of E by C^2-sets E_k and returns the limit energy of the approximating sets. Indeed, $\widetilde{\mathcal{W}}$ is an upper bound for $\Gamma - \lim_{\varepsilon\to 0} \mathcal{W}_\varepsilon$ by a diagonal sequence argument. Nonetheless, the two functionals do not agree.

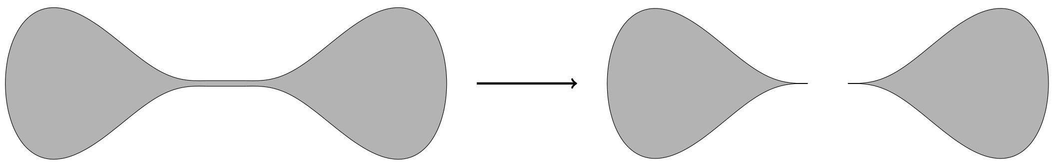

Figure 8.1: The lower semi-continuous envelope is calculated by the energy of approximating sets with C^2-boundaries.

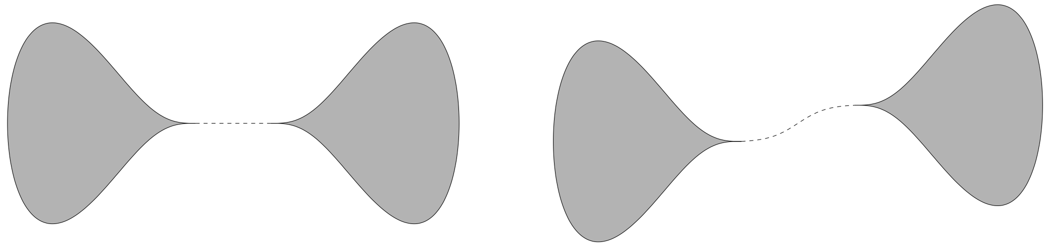

Figure 8.2: The lower semi-continuous envelope of Willmore's energy is not an integral functional. Both grey sets sketched above have the same elastica integral quantities, but approximation by C^2-boundaries leads to multiplicity two ghost interfaces which are straight for the left set (not contributing to $\widetilde{\mathcal{W}}$) but need to be curved for the right one (positive contribution). For a more rigorous argument, see [BM04].

8.1.1 The Relaxed Willmore Functional

For technical reasons, we may rather consider the relaxation of $\mathcal{W} + \mathrm{Per}$ instead of the relaxation of $\mathcal{W}$ alone. Note that the lower-semi continuous envelope $\mathcal{W} + \mathrm{Per}$ depends on the class of admissible sets. As we wish to prescribe boundary conditions $u_\varepsilon \in -1 + W_0^{2,2}(\Omega)$ for the phase-fields, we take the lower semi-continuous envelope with respect to Ω

$$(\widetilde{\mathcal{W} + \mathrm{Per}})_\Omega(E) = \inf\left\{\liminf_{k\to\infty}(\mathcal{W}+\mathrm{Per})(\partial E_k)\;\middle|\; \chi_{E_k} \xrightarrow{L^1} \chi_E,\ \partial E_k \in C^2,\ E_k \Subset \Omega\right\}. \quad (8.1.1)$$

This differs from the lower semi-continuous envelope of $\mathcal{W} + \mathrm{Per}$ with respect to $\mathbb{R}^n$ and the lower semi-continuous envelope if general C^2-boundaries relative to Ω are permitted, see also Figure 8.3. From the opening question posed in Problem 1, we see that the definition given in (8.1.1) is the correct one for us. We will identify $\widetilde{\mathcal{W} + \mathrm{Per}} = (\widetilde{\mathcal{W} + \mathrm{Per}})_\Omega$ in the following.

Lemma 8.1.1. *The following are true.*

1. *If* $\widetilde{\mathcal{W}}_\Omega(E) < \infty$ *and* $E_k \Subset \Omega$ *with* $\partial E_k \in C^2$ *such that* $E_k \to E$ *in* L^1 *with*

$$\lim_{k\to\infty} \mathcal{W}(\partial E_k) = \widetilde{\mathcal{W}}(E),$$

   then $\limsup_{k\to\infty} \mathrm{Per}(E_k) < \infty$.

2. *In general, the lower semi-continuous envelope* $(\widetilde{\mathcal{W} + \mathrm{Per}})_\Omega$ *of the sum of Willmore's energy and the perimeter functional does not agree with the sum* $\widetilde{\mathcal{W}}_\Omega + \mathrm{Per}$ *of the perimeter functional and the lower semi-continuous envelope of Willmore's energy, even when evaluated at sets* $E \Subset \Omega$.

3. $\lim_{\lambda\to 0}(\widetilde{\mathcal{W} + \lambda\mathrm{Per}})_\Omega = \widetilde{\mathcal{W}}_\Omega$ *uniformly on* $BV(\Omega, \{-1, 1\})$.

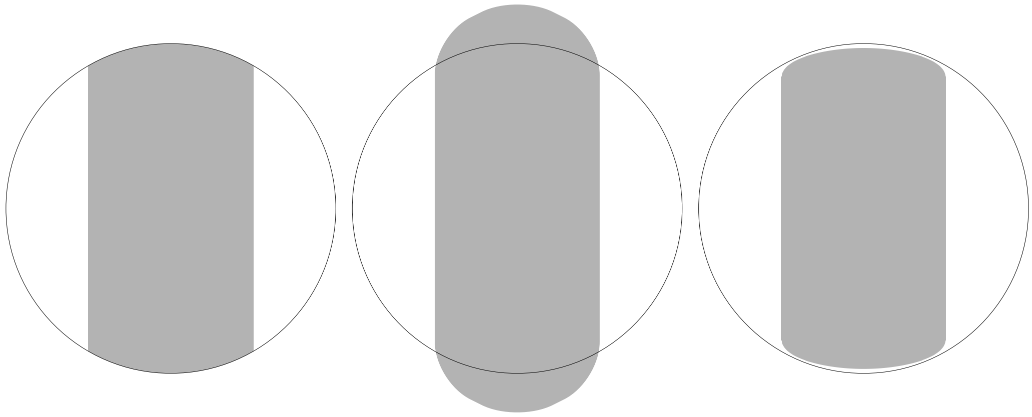

Figure 8.3: The lower semicontinuous envelope of $\mathcal{W}$ (or $\mathcal{W}+\mathrm{Per}$) at the set of the left hand side inside the domain $\Omega = B_1(0)$ depends on the domain of definition for $\mathcal{W}$ (or $\mathcal{W}+\mathrm{Per}$ respectively). If general sets $E \Subset \mathbb{R}^n$ with C^2-boundary ∂E (not necessarily contained in Ω) are permitted and the convergence is taken in $L^1(\Omega)$, both functionals are finite, whereas if the sets have to be compactly contained in Ω, the sharp bend at the boundary forces a blow up of curvature energy in the relaxation process. Depending on the situation, either functional can be more meaningful, but we will always take the right hand version of the relaxation.

The Lemma is presumably classic, but we have not found a proof in the literature, so we will proceed to prove it here.

Proof. **1.** Since $\Omega \Subset \mathbb{R}^n$, we have $\mathrm{diam}(\Omega) < \infty$ and we have $\Omega \subset B_{2\,\mathrm{diam}(\Omega)}(x)$ for all $x \in \Omega$. Without loss of generality, $0 \in \Omega$ and we use Lemma 3.2.3 to compute

$$\mathcal{W}(\partial E_k) = \mathcal{W}\left(\partial \frac{E_k}{2\,\mathrm{diam}(\Omega)}\right) \geq 4\,\mathcal{H}^2\left(\partial \frac{\mathcal{E}_k}{2\,\mathrm{diam}(\Omega)}\right) = \frac{1}{\mathrm{diam}(\Omega)^2}\,\mathrm{Per}(E_k).$$

This gives us the uniform perimeter bound in $n = 3$ dimensions, the same argument goes through in dimension $n = 2$ with slightly different scaling.

2. Any set which requires ghost interfaces in the approximation violates equality. Examples in two dimensions can be seen in Figures 8.1 and 8.2.

3. From the first point, we obtain that

$$\widetilde{\mathcal{W}}_\Omega \leq (\widetilde{\mathcal{W}+\lambda\mathrm{Per}})_\Omega \leq \left(1+\lambda\,\mathrm{diam}(\Omega)^2\right)\widetilde{\mathcal{W}}_\Omega.$$

□

Remark 8.1.2. Any functional $\mathcal{F} : X \to [0,\infty)$ on a metric space (X, d) can be relaxed in the way described above with respect to the metric topology. The relaxed function $\widetilde{\mathcal{F}}$ is always lower semi-continuous with respect to the metric topology.

Generalisations for topological spaces are also available. Here the relaxation of $\mathcal{F}$ is defined to be the largest lower semi-continuous functional $\mathcal{G}$ such that $\mathcal{G} \leq \mathcal{F}$. Formulae like (8.1.1) which allow a direct computation are only valid in first countable spaces, of course.

8.1.2 The Figure Eight

The figure eight space in $\mathbb{R}^2$ is the prime example of a set for which the L^1-lower semi-continuous envelope of $\mathcal{W}$ (Euler's elastica energy) does not agree with the $\Gamma(L^1)$-limit of $\mathcal{W}_\varepsilon$. The exotic saddle-solutions to the stationary Allen-Cahn equation described in Section 4.3 can be used to create a singular transition which can be matched to the usual optimal profile construction already a distance for example $\sim \sqrt{\varepsilon}$ away from the singular point. For such a sequence of phase-fields u_ε, one can compute that $\mathcal{W}_\varepsilon(u_\varepsilon) \to \mathcal{W}(\gamma)$, where $\mathcal{W}(\gamma)$ is the Euler elastica energy of the figure eight space, viewed as an immersed parametrised curve in two dimensions (and thus in particular finite). By the converse estimate from the lim inf-construction in [RS06] and the locality of the mean curvature of 1-varifolds established in [LM09], we see that

$$\left[\Gamma - \lim_{\varepsilon\to 0}(\mathcal{W}_\varepsilon + S_\varepsilon)\right](E) = (\mathcal{W} + \mathcal{H}^1)(\gamma)$$

if γ is a figure eight curve and E its enclosed set. For a more detailed account of this process, see [BP93].

However, $\widetilde{\mathcal{W}}(E)$ is infinite since any approximation of the figure eight by embedded curves must approximate the self-crossing by two sharp bends, leading to asymptotically infinite energy. A simple proof of this fact goes as follows, if we consider the sum $\mathcal{W} + \mathrm{Per}$ instead of $\mathcal{W}$ only.

1. Assume that $\widetilde{\mathcal{W}}(E)$ is finite and take a sequence of a sets E_k realising the lower limit. Then the boundaries of the sets E_k are compact embedded one-dimensional manifolds, so they are given by a finite union of smooth C^2-curves γ_k^l, $1 \leq l \leq N_k$, and the energy bound on $(\mathcal{W} + \mathcal{H}^1)(\partial E_k)$ implies a uniform bound on the number N_k of curves since for any $\gamma = \gamma_k^l$ we have

$$\begin{aligned}
\mathcal{W}(\gamma) &= \int_\gamma \kappa^2 \,\mathrm{d}\mathcal{H}^1 \\
&= \frac{1}{\mathcal{H}^1(\gamma)} \int_\gamma \kappa^2 \,\mathrm{d}\mathcal{H}^1 \int_\gamma 1 \,\mathrm{d}\mathcal{H}^1 \\
&\geq \frac{1}{\mathcal{H}^1(\gamma)} \left(\int_\gamma |\kappa| \,\mathrm{d}\mathcal{H}^1\right)^2 \\
&\geq \frac{4\pi^2}{\mathcal{H}^1(\gamma)}
\end{aligned}$$

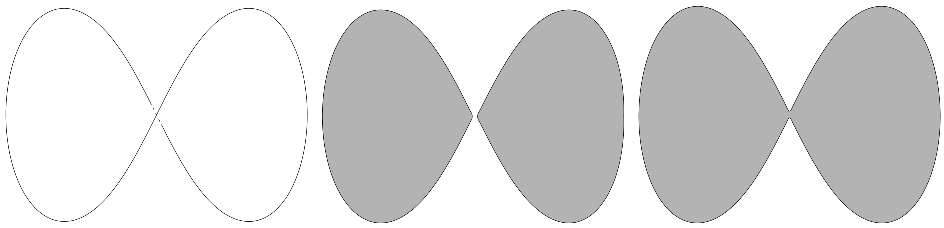

Figure 8.4: Two different interpretations of the figure eight in two dimensions: As an immersed smooth curve on the left and approximated by smooth embedded boundaries in two different ways on the right.

by Hölder's inequality and the fact that the integral of the curvature κ of an embedded closed curve is $\pm 2\pi$, depending on its orientation.

2. Now we choose a subsequence of approximating sets which always have the same number of curves. The uniform bounds on length and curvature integral imply bounds on the $W^{2,2}$-norm of an arclength-parametrisation since the lengths of the curves γ_k^l are bounded from below for energetic reasons and $|\kappa| = |\ddot{\gamma}|$. The $W^{2,2}$-bounds induce $C^{1,1/2}$-bounds, and for example the Arzela-Ascoli theorem yields the existence of a C^1-converging subsequence of the parametrised boundary curves. Clearly, the limiting family of curves $\{\gamma^l\}_{l=1}^N$ need not be embedded anymore. We assume that $\gamma^{l_1}(t_1) = \gamma^{l_2}(t_2)$ where l_1 and l_2 need not be distinct.

   It is easy to show that if $\dot{\gamma}^{l_2}(t_2) \neq \pm\dot{\gamma}^{l_1}(t_1)$, then the curves cross, and uniform convergence shows that this is also true for some large enough $k \in \mathbb{N}$, contradicting the embeddedness of $\{\gamma_k^l\}_{l=1}^N$.

3. On the other hand, ∂E must be contained in this limit (although the two can well be distinct – see Section 3.2.4), which means that there must be a double point with non-tangential contact. This gives us the desired contradiction.

This demonstrates how exotic solutions to the stationary Allen-Cahn equation can lead to strict inequality

$$\left[\Gamma - \lim_{\varepsilon\to 0}(\mathcal{W}_\varepsilon + S_\varepsilon)\right](\chi_E) < \left(\widetilde{\mathcal{W} + \mathrm{Per}}\right)(E) = \infty$$

at sets $E \subset \mathbb{R}^2$ whose boundary is not C^2-embedded. The figure eight is, however, connected, and thus not a priori excluded by the functionals $\mathcal{C}_\varepsilon$ described in the previous chapter. To the author's knowledge, the question whether strict inequality can occur at sets with $(\widetilde{\mathcal{W} + \mathrm{Per}})(E) < \infty$ is open.

It seems that the fundamental object for the functionals $S_\varepsilon, \mathcal{W}_\varepsilon$ are rather the measures

μ_ε and not the functions u_ε – this is supported by the fact that we obtain the Willmore energy of the figure eight varifold μ as the Γ-limit at this point, rather than the lower semicontinuous envelope of Willmore's energy.

In this chapter we will develop a functional which does not control the connectedness of a limiting varifold, but its approximability by C^2-boundaries.

8.2 Approximating the Relaxed Willmore Functional

For slender structures, we must think of a non-embedded configuration as the limit of embedded surfaces, so the fundamental object is $\widetilde{W}$ rather than the Willmore energy of a varifold interpretation of a limiting measure. Therefore we are interested in approximating the lower semicontinuous envelope of Willmore's energy.

Heuristically, it seems that an exotic solution must occur at such singular points and that sequences which have a recovery sequence structure with optimal profiles cannot exhibit such behaviour. Recall that we established the existence of a $W^{2,2}_{loc}$-weak limit $\tilde{u}$ of the blow-ups

$$\tilde{u}_\varepsilon(y) = u_\varepsilon(x_\varepsilon + \varepsilon y)$$

of a sequence of phase-fields u_ε along a sequence of points $x_\varepsilon \in \Omega' \Subset \Omega$. In two dimensions, we further saw that

$$\Delta\tilde{u}_\varepsilon - W'(\tilde{u}_\varepsilon) \quad \begin{array}{ll} \to 0 & \text{strongly in } L^2_{loc}(\mathbb{R}^2) \\ \rightharpoonup \Delta\tilde{u} - W'(\tilde{u}) & \text{weakly in } L^2_{loc}(\mathbb{R}^2) \end{array},$$

so that $\tilde{u}$ is a global solution of the stationary Allen-Cahn equation. Furthermore

$$\Delta\tilde{u}_\varepsilon = (\Delta\tilde{u}_\varepsilon - W'(\tilde{u}_\varepsilon)) + W'(\tilde{u}_\varepsilon) \to W'(\tilde{u}) = \Delta\tilde{u}$$

strongly in $L^2_{loc}(\mathbb{R}^2)$, so $\tilde{u}_\varepsilon \to \tilde{u}$ even strongly in $W^{2,2}_{loc}(\mathbb{R}^2)$ by the elliptic estimate

$$||\, D^2 u\,||_{L^2(B_R)} \leq C\, \left\{||u||_{L^2(B_{2R})} + ||\, \Delta u||_{L^2(B_R)}\right\}.$$

Definition 8.2.1. Let $\Omega \subset \mathbb{R}^n$ be open. We say that a sequence of phase-fields $u_\varepsilon \in W^{2,2}_{loc}(\Omega)$ has the *blow-up property* if for all compact $\Omega' \Subset \Omega$ and all sequences $x_\varepsilon \in \Omega'$ the blow-up sequence $\tilde{u}_\varepsilon$ has a subsequence $\varepsilon \to 0$ such that $\tilde{u}_\varepsilon$ has a $W^{2,2}_{loc}(\mathbb{R}^n)$-strong limit $\tilde{u}$ and either $\tilde{u} = \pm 1$ or

$$\tilde{u}(y) = q(\langle v, y\rangle + b)$$

for some $v \in S^{n-1}$ and $b \in \mathbb{R}$.

Remark 8.2.2. A simple contradiction argument shows that the constant functions and optimal profiles are in fact the only blow-up limits which can occur along sequences with the blow-up property. In two dimensions, the existence of the limit follows automatically.

Conjecture 8.2.3. *Let $\Omega \Subset \mathbb{R}^n$ for $n = 2, 3$ and $u_\varepsilon \in -1 + W_0^{2,2}(\Omega)$ a sequence of phase-fields with the blow-up property such that $u_\varepsilon \to \chi_E$ strongly in $L^1(\Omega)$ for some $E \Subset \Omega$. Then*

$$\liminf_{\varepsilon\to 0} (\mathcal{W}_\varepsilon + S_\varepsilon)(u_\varepsilon) \geq (\widetilde{\mathcal{W} + \mathrm{Per}})_\Omega (E).$$

The conjecture would of course determine the Γ-limit of certain extended functionals if we introduce a penalisation which vanishes at recovery sequences, but enforces the blow-up property.

Corollary 8.2.4. *Assume that Conjecture 8.2.3 is true. Let $\Omega \Subset \mathbb{R}^n$ for $n = 2, 3$ and $\mathcal{G}_\varepsilon : W^{2,2}(\Omega) \to \mathbb{R}$ a functional such that*

(1) Sequences $u_\varepsilon \in -1 + W_0^{2,2}(\Omega)$ such that

$$\limsup_{\varepsilon\to 0} (\mathcal{W}_\varepsilon + S_\varepsilon + \mathcal{G}_\varepsilon)(u_\varepsilon) < \infty$$

have the blow-up property, and

(2) at sets with $\partial E \in C^2$ we have

$$\left[\Gamma - \lim_{\varepsilon\to 0} (\mathcal{W}_\varepsilon + S_\varepsilon + \mathcal{G}_\varepsilon)\right] (\chi_E) = \mathcal{W}(\partial E) + \mathrm{Per}(E).$$

Then

$$\Gamma - \lim_{\varepsilon\to 0} (\mathcal{W}_\varepsilon + S_\varepsilon + \mathcal{G}_\varepsilon) = \widetilde{\mathcal{W} + \mathrm{Per}}_\Omega$$

at all $u \in BV(\Omega, \{-1, 1\})$.

Similar results could be established for functionals involving penalties like $\varepsilon^{-\sigma}(S_\varepsilon - S)^2$. We only give examples of functionals $\mathcal{G}_\varepsilon$ for which the Corollary holds.

8.2.1 Blow-Up Controlling Functionals

In this section, we describe various examples of functionals $\mathcal{G}_\varepsilon$ that satisfy the conditions of Conjecture 8.2.3. The idea is to use Lemma 4.3.3 and suitable penalisations which force blow-ups into a geometrically rigid situation where only small perturbations of optimal profiles are admissible.

In a slight abuse of notation, we denote the density of the discrepancy measures also by $\xi_\varepsilon = \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}\,W(u_\varepsilon)$. In two dimensions, we consider the functionals

$$\begin{aligned}
\mathcal{G}^{(2)}_{\varepsilon,1}(u) &= \frac{\varepsilon^{-(n-1+\sigma)}}{c_0}\int_\Omega |\xi_\varepsilon|\,\mathrm{d}x\\
\mathcal{G}^{(2)}_{\varepsilon,2}(u) &= \varepsilon^{-(n-2+\sigma)}\int_\Omega |\xi_\varepsilon|^2\,\mathrm{d}x\\
\mathcal{G}^{(2)}_{\varepsilon,3}(u) &= \varepsilon^{-(3+\sigma)}\int_\Omega |\xi_\varepsilon|^2\,\mathrm{d}x + \varepsilon^{1-\sigma}\int_\Omega |\nabla\xi_\varepsilon|^2\,\mathrm{d}x
\end{aligned}$$

for some $\sigma > 0$. We will see, that in three dimensions an additional penalty is needed. Consider the energy

$$\mathcal{W}_{\varepsilon,p}(u) = \frac{1}{c_0\,\varepsilon}\int_\Omega \left|-\varepsilon\,\Delta u + \frac{1}{\varepsilon}\,W'(u)\right|^p\,\mathrm{d}x.$$

We can think of $\mathcal{W}_{\varepsilon,p}$ as an approximation of the energy functional $\mathcal{W}_p(M) = \int_M |H|^p\,\mathrm{d}\mathcal{H}^2$ (although no proof of Γ-convergence has been given for $p \neq 2$). In three dimensions, we define

$$\mathcal{G}^{(3)}_{\varepsilon,k} = \mathcal{G}^{(2)}_{\varepsilon,k} + \varepsilon^\lambda\,\mathcal{W}_{\varepsilon,p}$$

for $k \in \{1,2,3\}$ and some $p > 2$ and $0 < \lambda < p-2$. These functionals are sufficiently regularising to exclude saddle configurations and thus also limiting varifolds which are not smoothly approximable.

Due to the Sobolev embedding theorems, all functionals are well-defined on $W^{2,2}(\Omega)$ if $n = 2$ and $W^{2,p}(\Omega)$ if $n = 3$. We will show that $\mathcal{W}_\varepsilon + S_\varepsilon + \mathcal{G}^{(n)}_{\varepsilon,k}$ are blow-up controlling functionals, by which we mean that they force finite energy sequences to have the blow-up property.

Lemma 8.2.5. *Let $\Omega \Subset \mathbb{R}^n$ for $n = 2,3$ and $u_\varepsilon \in -1 + W^{2,2}_0(\Omega)$ be a sequence such that*

$$\sup_{\varepsilon>0}\left(\mathcal{W}_\varepsilon + S_\varepsilon + \mathcal{G}^{(n)}_{\varepsilon,k}\right)(u_\varepsilon) < \infty$$

for $k \in \{1,2,3\}$. Take any sequence $x_\varepsilon \in \mathbb{R}^n$ and consider the blow up functions

$$\tilde u_\varepsilon(y) = u_\varepsilon(x_\varepsilon + \varepsilon y).$$

Then there exists a subsequence $\varepsilon \to 0$ and a function $\tilde u \in C^\infty(\mathbb{R}^n)$ such that $\tilde u_\varepsilon \to \tilde u$ strongly in $W^{2,2}(U)$ for all $U \Subset \mathbb{R}^n$ and $W^{2,p}_{loc}(\mathbb{R}^n)$ in three dimensions or if the penalty term $\varepsilon^\lambda\mathcal{W}_{\varepsilon,p}$ is included also in two dimensions. The function $\tilde u$ satisfies

$$\tilde u \equiv \pm 1 \qquad \textit{or} \qquad \exists\; v \in S^{n-1}, b \in \mathbb{R} \quad \textit{such that} \quad u = q(\langle v, \cdot\rangle + b).$$

Proof. We only focus on the case $n = 3$ as the two-dimensional case is a simpler application of the same argument. Since $\mathcal{G}_{\varepsilon,3} \geq \mathcal{G}_{\varepsilon,2}$, it suffices to consider $\mathcal{G}_{\varepsilon,1}, \mathcal{G}_{\varepsilon,2}$. As in Lemma 5.2.8, we compute that

$$\begin{aligned}
||-\Delta\tilde{u}_\varepsilon + W'(\tilde{u}_\varepsilon)\,||^p_{L^p(B_R(0))} &= \varepsilon^{p+1-n}\int_{B_{R\varepsilon}(x_\varepsilon)}\left|-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right|^p \mathrm{d}x\\
&\leq \varepsilon^{p+1-n}\,\mathcal{W}_{\varepsilon,p}(u_\varepsilon)\\
&\leq C\varepsilon^{p-2-\lambda},\\
\int_{B_R}\left|\frac{1}{2}\,|\nabla\tilde{u}_\varepsilon|^2 - W(\tilde{u}_\varepsilon)\right| \mathrm{d}x &= \varepsilon^{1-n}\,|\xi_\varepsilon|(B_{R\varepsilon}(x_\varepsilon))\\
&\leq C\varepsilon^{\sigma},\\
\int_{B_R}\left|\frac{1}{2}\,|\nabla\tilde{u}_\varepsilon|^2 - W(\tilde{u}_\varepsilon)\right|^2 \mathrm{d}x &= \varepsilon^{2-n}\int_{B_{R\varepsilon}(x_\varepsilon)}\left|\frac{\varepsilon}{2}\,|\nabla u|^2 - \frac{1}{\varepsilon}\,W(u)\right|^2 \mathrm{d}x\\
&\leq C\varepsilon^{\sigma}.
\end{aligned}$$

We have the bound $||u_\varepsilon||_{L^\infty(\mathbb{R}^n)} \leq C$ from Lemma 5.2.8 and Calderon-Zygmund theory shows that

$$\tilde{u}_\varepsilon \rightharpoonup \tilde{u}^R$$

weakly in $W^{2,p}(B_R)$ for a subsequence in ε for any $R > 0$. A diagonal sequence argument shows that $\tilde{u}^R$ can in fact be chose as the restriction of a single function $\tilde{u} \in W^{2,p}_{loc}(\mathbb{R}^n)$ onto B_R. By the lower semi-continuity of the norm under weak convergence we deduce

$$-\Delta\tilde{u} + W'(\tilde{u}) = 0, \qquad |\nabla\tilde{u}|^2 = 2\,W(\tilde{u}). \tag{8.2.1}$$

By Sobolev embeddings, $\tilde{u}_\varepsilon \to \tilde{u}$ strongly in $C^0(\overline{B_R})$ and hence $\tilde{u}$ is bounded on B_R by the L^∞-bound on u_ε. This also shows that $W'(\tilde{u}_\varepsilon) \to W'(\tilde{u})$ converges strongly in $L^p(B_R)$ and thus

$$-\Delta\tilde{u}_\varepsilon = (-\Delta\tilde{u}_\varepsilon + W'(\tilde{u}_\varepsilon)) - W'(\tilde{u}_\varepsilon) \to -W'(\tilde{u}) = -\Delta\tilde{u}$$

converges strongly in $L^p(B_R)$ for all $R > 0$. A usual elliptic argument then shows that

$$\tilde{u}_\varepsilon \to \tilde{u}$$

strongly in $W^{2,p}(B_R)$ for all $R > 0$. By Hölder's inequality, the convergence also holds in $W^{2,2}(B_R)$, which is optimal in the two-dimensional case without penalisation. Due to Lemma 4.3.3 and (8.2.1), the function $\tilde{u}$ is either an optimal profile or a constant function as in the statement of the Lemma. $\square$

Note that a penalisation only of the discrepancy would be insufficient in three space dimensions, since for example functions like

$$\widetilde{u}(y) = q\left(|x| + c_0\right), \qquad c_0 \in \mathbb{R}$$

could arise as blow-up limits of finite energy sequences and only create finite-sized atoms of $\mathcal{W}_\varepsilon$ – this illustrates the necessity of using a small multiple of $\mathcal{W}_{\varepsilon,p}$ for $p > 2$.

Lemma 8.2.6. *Let $\Omega \Subset \mathbb{R}^n$ for $n = 2, 3$. Then*

$$\left[\Gamma(L^1) - \lim_{\varepsilon\to 0}\left(\mathcal{W}_\varepsilon + S_\varepsilon + \mathcal{G}^{(d)}_{\varepsilon,k}\right)\right](\chi_E) = \left(\mathcal{W} + \mathcal{H}^{n-1}\right)(\partial E)$$

if $E \Subset \Omega$ and $\partial E \in C^2$ and $k \in \{1, 2, 3\}$, $d \in \{2, 3\}$.

Here we set the functionals to $+\infty$ if $u \notin -1 + W^{2,2}_0(\Omega)$. In applications, we will of course assume that $d = n$.

Proof. Write $\mathcal{G}_\varepsilon = \mathcal{G}^{(d)}_{\varepsilon,k}$. We trivially have

$$\begin{aligned}\liminf_{\varepsilon\to 0}\left(\mathcal{W}_\varepsilon + S_\varepsilon + \mathcal{G}_\varepsilon\right)(u_\varepsilon) &\geq \lim_{\varepsilon\to 0}(\mathcal{W}_\varepsilon + S_\varepsilon)(u_\varepsilon)\\ &\geq \left(\mathcal{W} + \mathcal{H}^{n-1}\right)(\partial E)\end{aligned}$$

if $u_\varepsilon \to \chi_E$ strongly in $L^1(\Omega)$ due to Theorem 4.5.1, so the lim inf-inequality holds trivially. For the usual recovery sequence

$$u_\varepsilon(x) = q\left(\frac{\mathrm{sdist}(x, \partial E)}{\varepsilon}\right)$$

the discrepancy term $\frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}\,W(u_\varepsilon)$ vanishes identically at the interface. It does not vanish away from the interface, since we need to smooth the distance function a little bit and satisfy boundary conditions, but since $q(z) \to \pm 1$, $q'(z), q''(z) \to 0$ exponentially fast as $z \to \pm\infty$, the penalisation $\mathcal{G}_\varepsilon$ vanishes as $\varepsilon \to 0$.

If $n = 3$, observe additionally that the recovery sequence satisfies

$$\mathcal{W}_{\varepsilon,p}(u_\varepsilon) \to \int_{\partial E} |H|^p \,\mathrm{d}\mathcal{H}^{n-1} < \infty$$

by the same proof as for the usual Willmore functional. Thus also this penalisation vanishes at C^2-boundaries. $\square$

8.2.2 An Approximation of $\widetilde{\mathcal{W}}$

We finally prove a special case of the conjecture. We denote

$$\mathcal{G}_\varepsilon(u) = \mathcal{G}^{(3)}_{\varepsilon,3}(u) = \varepsilon^{-(3+\sigma)} \int_\Omega |\xi_\varepsilon|^2 \,\mathrm{d}x + \varepsilon^{1-\sigma} \int_\Omega |\nabla \xi_\varepsilon|^2 \,\mathrm{d}x + \varepsilon^\lambda \,\mathcal{W}_{\varepsilon,p}(u)$$

with $p > n$, $0 < \lambda < p-2$ and $\sigma > 0$. Note that we use the three-dimensional penalty functional with the curvature-dependent term $\mathcal{W}_{\varepsilon,p}$ also in two dimensions.

Theorem 8.2.7. *Let $\Omega \Subset \mathbb{R}^n$ for $n = 2,3$ and $u_\varepsilon \in -1 + W^{2,2}_0(\Omega)$ a sequence of phase-fields such that $u_\varepsilon \to \chi_E$ strongly in $L^1(\Omega)$ for some $E \Subset \Omega$. Then*

$$\liminf_{\varepsilon\to 0}\, (\mathcal{W}_\varepsilon + S_\varepsilon + \mathcal{G}_\varepsilon)(u_\varepsilon) \geq (\widetilde{\mathcal{W} + \mathrm{Per}})_\Omega\,(E).$$

This easily implies the following Γ-convergence.

Corollary 8.2.8. *We have*

$$\Gamma - \lim_{\varepsilon\to 0} (\mathcal{W}_\varepsilon + S_\varepsilon + \mathcal{G}_\varepsilon) = \widetilde{\mathcal{W} + \mathrm{Per}}$$

at all functions $u \in BV(\Omega, \{-1,1\})$.

Analogous statements can be made for functionals like

$$\widetilde{\mathcal{E}}_\varepsilon = \mathcal{W}_\varepsilon + \varepsilon^{-\sigma}(S_\varepsilon - S)^2 + \mathcal{G}_\varepsilon.$$

Proof of Theorem 8.2.7. In an abuse of notation, we identify the measures $\mu_\varepsilon, \xi_\varepsilon$ with their densities. On $\Omega_\varepsilon = \{|u_\varepsilon| < 1\}$ we can define $r_\varepsilon = \varepsilon\, q^{-1}(u_\varepsilon)$ such that $u_\varepsilon = q\left(\frac{r_\varepsilon}{\varepsilon}\right)$ and thus

$$\begin{aligned}
\mu_\varepsilon &= \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\,W(u_\varepsilon)\\
&= \frac{1}{\varepsilon}\,W\left(q\left(\frac{r_\varepsilon}{\varepsilon}\right)\right)\,\left[|\nabla r_\varepsilon|^2 + 1\right] && (8.2.2)\\
\xi_\varepsilon &= \frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}\,W(u_\varepsilon)\\
&= \frac{1}{\varepsilon}\,W\left(q\left(\frac{r_\varepsilon}{\varepsilon}\right)\right)\,\left[|\nabla r_\varepsilon|^2 - 1\right] && (8.2.3)\\
\nabla\xi_\varepsilon &= \frac{1}{\varepsilon}\,W'\left(q\left(\frac{r_\varepsilon}{\varepsilon}\right)\right) q'\left(\frac{r_\varepsilon}{\varepsilon}\right)\left[|\nabla r_\varepsilon|^2 - 1\right]\frac{\nabla r_\varepsilon}{\varepsilon} + \frac{1}{\varepsilon}\,W\left(q\left(\frac{r_\varepsilon}{\varepsilon}\right)\right)\,\nabla\left(|\nabla r_\varepsilon|^2\right) && (8.2.4)\\
h_\varepsilon &= \varepsilon\,\Delta u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon)\\
&= q'\left(\frac{r_\varepsilon}{\varepsilon}\right)\,\Delta r_\varepsilon + \frac{1}{\varepsilon}\,W\left(q\left(\frac{r_\varepsilon}{\varepsilon}\right)\right)\,\left[|\nabla r_\varepsilon|^2 - 1\right]. && (8.2.5)
\end{aligned}$$

To simplify expressions, we will in the following leave out the arguments of q, q', q'' and always assume that the functions are evaluated at $\frac{r_\varepsilon}{\varepsilon}$.

Since $\mathcal{W}_\varepsilon + S_\varepsilon + \mathcal{G}_\varepsilon$ is blow-up controlling, we see that $\tilde u_\varepsilon(y) := u_\varepsilon(x+\varepsilon y)$ is $W^{2,p}$-close to an optimal profile transition $q_v = q(\langle\cdot, v\rangle)$ in some direction $v \in S^{n-1}$ on a ball B_R for $x \in \{u_\varepsilon = 0\}$ and small enough ε. In particular, $\tilde u_\varepsilon$ is $C^{1,\alpha}$-close to q_v since we took $p > n$ and thus

$$q(-2R) < u_\varepsilon < q(2R) \quad \text{and} \quad \nabla \tilde u_\varepsilon \neq 0$$

on B_R. It follows that

$$\nabla \tilde u_\varepsilon \neq 0 \text{ on } \quad B_{R\varepsilon}(N_\varepsilon) := \{y \in \mathbb{R}^n \mid \operatorname{dist}(y, N_\varepsilon) < R\varepsilon\}$$

where $N_\varepsilon := \{u_\varepsilon = 0\}$ and by the same argument

$$\{q(-R/2) < u_\varepsilon < q(R/2)\} \subset B_{R\varepsilon}(N_\varepsilon).$$

Thus we see that for $\alpha \in (q(-R/2), q(R/2))$ the sets $\{u_\varepsilon > \alpha\}$ satisfy

1. $\partial\{u_\varepsilon > \alpha\} \in C^2$,

2. $d^H(\partial\{u_\varepsilon > \alpha\}, N_\varepsilon) \leq R\varepsilon$ and consequently

3. $\chi_{\{u_\varepsilon > \alpha\}} \to \chi_E$ as $\varepsilon \to 0$.

In particular,

$$\liminf_{\varepsilon\to 0} \mathcal{W}(\partial\{u_\varepsilon > \alpha\}) \geq \widetilde{\mathcal{W}}(E)$$

due to the definition of $\widetilde{\mathcal{W}}$. We compute

$$\begin{aligned}
\alpha_\varepsilon(\Omega) &\geq \frac{1}{c_0\,\varepsilon}\int_{B_{R\varepsilon}(N_\varepsilon)} \left(\varepsilon\,\Delta u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right)^2 \mathrm{d}x\\
&= \frac{1}{c_0\,\varepsilon}\int_{B_{R\varepsilon}(N_\varepsilon)} \left(q'\,\Delta r_\varepsilon + \frac{1}{\varepsilon}\,W'(q)\,\left[|\nabla r_\varepsilon|^2 - 1\right]\right)^2 \mathrm{d}x\\
&\geq \frac{1}{c_0\,\varepsilon}\int_{B_{R\varepsilon}(N_\varepsilon)} (1-\delta)\,(q')^2\,(\Delta r_\varepsilon)^2 - \frac{1}{4\delta}\,\frac{1}{\varepsilon^2} W'\,(q)^2\,\left[|\nabla r_\varepsilon|^2 - 1\right]^2 \mathrm{d}x
\end{aligned}$$

for all $\delta > 0$. Since $q(-2R) < q(r_\varepsilon/\varepsilon) < q(2R)$ on $B_{R\varepsilon}(N_\varepsilon)$ due to the local uniform continuity, we see that the second term goes to zero as $\varepsilon \to 0$ since

$$\frac{1}{\varepsilon}\int_{B_{R\varepsilon}(N_\varepsilon)} \frac{1}{\varepsilon^2} W'\,(q)^2\,\left[|\nabla r_\varepsilon|^2 - 1\right]^2 \mathrm{d}x \leq \frac{4}{W(q(2R))\,\varepsilon^3}\int_\Omega W(u_\varepsilon)^2\,\left[|\nabla r_\varepsilon|^2 - 1\right]^2 \mathrm{d}x$$

since $(W')^2 \leq 4W$ on $(-1, 1)$. The right hand side vanishes due to the penalisation of the quadratic discrepancy density. Now we observe that the level sets of u_ε agree with the level sets of r_ε and thus have mean curvatures (see e.g. [ES91])

$$H = \operatorname{div}\left(\frac{\nabla r_\varepsilon}{|\nabla r_\varepsilon|}\right) = \frac{1}{|\nabla r_\varepsilon|}\left(\Delta r_\varepsilon - \frac{\left\langle \nabla r_\varepsilon, \nabla\left(|\nabla r_\varepsilon|^2\right)\right\rangle}{2\,|\nabla r_\varepsilon|^2}\right).$$

We continue the computation with the first term

$$\begin{aligned}
&\frac{1}{c_0\varepsilon}\int_{B_{R\varepsilon}(N_\varepsilon)} (q')^2\,(\Delta r_\varepsilon)^2\,\mathrm{d}x \\
&\quad = \frac{1}{c_0\varepsilon}\int_{B_{R\varepsilon}(N_\varepsilon)} (q')^2\left(\operatorname{div}\left(\frac{\nabla r_\varepsilon}{|\nabla r_\varepsilon|}\right) + \frac{\left\langle \nabla r_\varepsilon, \nabla\left(|\nabla r_\varepsilon|^2\right)\right\rangle}{2\,|\nabla r_\varepsilon|^3}\right)^2 |\nabla r_\varepsilon|^2\,\mathrm{d}x \\
&\quad \geq \frac{1}{c_0\varepsilon}\int_{B_{R\varepsilon}(N_\varepsilon)} (1-\delta)\,(q')^2\left[\operatorname{div}\left(\frac{\nabla r_\varepsilon}{|\nabla r_\varepsilon|}\right)\right]^2 |\nabla r_\varepsilon|^2\,\mathrm{d}x \\
&\qquad\qquad - \frac{1}{c_0\varepsilon}\int_{B_{R\varepsilon}(N_\varepsilon)} \frac{1}{4\delta}\,(q')^2\left(\frac{\left\langle \nabla r_\varepsilon, \nabla\left(|\nabla r_\varepsilon|^2\right)\right\rangle}{2\,|\nabla r_\varepsilon|^3}\right)^2 |\nabla r_\varepsilon|^2\,\mathrm{d}x.
\end{aligned}$$

Note that $|\nabla r_\varepsilon| \geq \beta$ for some β close to 1. Again, the second term vanishes as $\varepsilon \to 0$ since

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{B_{R\varepsilon}(N_\varepsilon)} (q')^2\left(\frac{\left\langle \nabla r_\varepsilon, \nabla\left(|\nabla r_\varepsilon|^2\right)\right\rangle}{2\,|\nabla r_\varepsilon|^3}\right)^2 |\nabla r_\varepsilon|^2\,\mathrm{d}x \\
&\quad \leq \frac{1}{4\beta^2\varepsilon}\int_{B_{R\varepsilon}(N_\varepsilon)} W(q)\,\left|\nabla\left(|\nabla r_\varepsilon|^2\right)\right|^2\,\mathrm{d}x \\
&\quad \leq \frac{1}{4\beta^2\varepsilon}\int_{B_{R\varepsilon}(N_\varepsilon)} \frac{W(q)^2}{W(q(2R))}\,\left|\nabla\left(|\nabla r_\varepsilon|^2\right)\right|^2\,\mathrm{d}x \\
&\quad = \frac{1}{4\beta^2\,W(q(2R))\,\varepsilon}\int_{B_{R\varepsilon}(N_\varepsilon)} \varepsilon^2\left|\nabla\xi_\varepsilon - \frac{1}{\varepsilon^2}\,W'(q)\,q'\,\left[|\nabla r_\varepsilon|^2 - 1\right]\,\nabla r_\varepsilon\right|^2\,\mathrm{d}x \\
&\quad \leq \frac{1}{2\beta^2\,W(q(2R))}\int_\Omega \varepsilon\,|\nabla\xi_\varepsilon|^2 + \frac{4\,W(q)^2}{\varepsilon^3}\,\left[|\nabla r_\varepsilon|^2 - 1\right]^2\,\mathrm{d}x \\
&\quad \leq \frac{\varepsilon}{2\beta^2\,W(q(2R))}\int_\Omega |\nabla\xi_\varepsilon|^2\,\mathrm{d}x + \frac{1}{2\,W(q(2R))\,\beta^2\,\varepsilon^3}\int_\Omega |\xi_\varepsilon|^2\,\mathrm{d}x
\end{aligned}$$

vanishes due to our penalisation. Finally, we calculate the remaining term.

$$\begin{aligned}
&\frac{1}{c_0\varepsilon}\int_{B_{R\varepsilon}(N_\varepsilon)} (q')^2\left[\operatorname{div}\left(\frac{\nabla r_\varepsilon}{|\nabla r_\varepsilon|}\right)\right]^2 |\nabla r_\varepsilon|^2\,\mathrm{d}x \\
&\quad \geq \frac{\beta}{c_0}\int_{B_{R\varepsilon}(N_\varepsilon)} (q')^2\left[\operatorname{div}\left(\frac{\nabla r_\varepsilon}{|\nabla r_\varepsilon|}\right)\right]^2 \frac{|\nabla r_\varepsilon|}{\varepsilon}\,\mathrm{d}x \\
&\quad = \frac{\beta}{c_0}\int_{-R/2}^{R/2}\left(\int_{\{r_\varepsilon = z\}} H^2\,\mathrm{d}\mathcal{H}^{n-1}\right)(q')^2(z)\,\mathrm{d}z \\
&\quad \geq \left[\widetilde{\mathcal{W}}(E) - o(1)\right]\,\frac{\beta}{c_0}\int_{-R/2}^{R/2} (q')^2\,\mathrm{d}z
\end{aligned}$$

so finally

$$\liminf_{\varepsilon\to 0}\mathcal{W}_\varepsilon(u_\varepsilon) \geq \beta\,(1-\delta)^2\,\frac{\int_{q(-R/2)}^{q(R/2)}\sqrt{2\,W(s)}\,\mathrm{d}s}{c_0}\,\widetilde{\mathcal{W}}(E)$$

since the $o(1)$ error term vanishes automatically as $\varepsilon \to 0$. We may take $\delta \to 0$ and $R \to \infty$ now to obtain

$$\liminf_{\varepsilon\to 0}\mathcal{W}_\varepsilon(u_\varepsilon) \geq \beta\,\widetilde{\mathcal{W}}(E).$$

Now, since the blow-ups converge to optimal profiles as $\varepsilon \to 0$, we can choose β arbitrarily close to 1 for small enough ε, thus in total

$$\liminf_{\varepsilon\to 0}\mathcal{W}_\varepsilon(u_\varepsilon) \geq \widetilde{\mathcal{W}}(E).$$

A simpler argument establishes the same result for S_ε, so the proof is complete. □

Remark 8.2.9. Despite the lengthy calculations, the functional $\mathcal{G}_\varepsilon$ was chosen specifically to allow a simple proof. We believe that the same should be true under a lot milder penalisations (or even general phase-fields with the blow-up property) and will pursue this in the future.

We will see below that we can say a lot more about phase-fields with a blow-up property on the topological level. We believe that some of the techniques could be extended to the smooth setting, but we have been unable to establish the quantitative estimates needed for this purpose so far.

8.2.3 Comparison with Existing Methods

Other phase-field approximations of $\widetilde{\mathcal{W}}$ have been proposed, for example the functionals

$$\begin{aligned}\mathcal{W}_\varepsilon^{\mathrm{Bel}}(u) &= \int_\Omega \left[\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)\right]^2 \mathrm{d}\mu_\varepsilon\\ &= \int_\Omega \left[\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)\right]^2 \left(\frac{\varepsilon}{2}\,|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right)\mathrm{d}x\end{aligned}$$

introduced by Bellettini in [Bel97], which converge to $\widetilde{\mathcal{W}}$ in the $\Gamma(L^1)$-sense in any dimension $n \geq 2$. The advantage of $\mathcal{W}_\varepsilon^{\mathrm{Bel}}$ over $\mathcal{W}_\varepsilon$ is that the integrand with respect to the diffuse surface measures is given precisely by the mean curvature of the level sets of u_ε, so that the lower semicontinuous envelope is automatically controlled by diffuse quantities and the Modica-Mortola functional.

The disadvantage of $\mathcal{W}_\varepsilon^{\mathrm{Bel}}$ with respect to an implementation is the high degree of nonlinearity in the highest order term. The term div $\left(\frac{\nabla u}{|\nabla u|}\right)$ is discussed in detail in [ES91], where it is shown that it is uniformly elliptic along level sets and totally degenerate in the normal direction. The numerical implementation is challenging at best even for the operator, let alone a gradient flow for an energy functional which contains the square of this term integrated with respect to a non-trivial measure.

Other functionals with similar approaches and deficiencies have been proposed for example in [ERR14] and [Mug13], see also the overview article [BMO13].

By comparison, the philosophy put forth in this chapter is different: Instead of introducing a new, directly geometric term into the energy, we introduce a term which only acts on the phase-field level without geometric meaning, which forces the phase-field to adopt geometrically meaningful behaviour. This may be less philosophically satisfying (or not, depending on one's taste), but has clear advantages with respect to an implementation. In two dimensions, the highest order terms in the energy

$$\widetilde{\mathcal{E}}_\varepsilon = \mathcal{W}_\varepsilon + \varepsilon^{-2}(S_\varepsilon - S)^2 + \mathcal{G}^{(2)}_{\varepsilon,\,2}$$

are simply given by

$$\varepsilon\,(\Delta u)^2,$$

leading to a semi-linear evolution equation with constant coefficients in time. This means that matrices can be assembled once at the very beginning of a simulation rather than in every time-step, which speeds up simulations significantly and allows for example for direct solvers based on factorisations rather than iterative solvers (if desired).

While the three-dimensional counterpart of $\widetilde{\mathcal{E}}_\varepsilon$ does not enjoy this feature anymore, the choice $p = 4$, $\lambda = 1.5$ and $\sigma = 1$ would lead to a functional

$$\frac{1}{c_0\varepsilon}\int_\Omega \left(\varepsilon\Delta u - \frac{W'(u)}{\varepsilon}\right)^2 + \varepsilon^{3/2}\left(\varepsilon\Delta u - \frac{W'(u)}{\varepsilon}\right)^4 + \varepsilon^{-4}\left(\frac{\varepsilon}{2}\,|\nabla u|^2 - \frac{W(u)}{\varepsilon}\right)^2 \mathrm{d}x$$

with leading order contribution

$$\varepsilon\,(\Delta u)^2 + \varepsilon^{7/2}(\Delta u)^4$$

which is convex and relatively 'tame' compared to functionals like $\mathcal{W}_\varepsilon^{\mathrm{Bel}}$. We also avoid potential problems associated to points where $\nabla u = 0$. Even with the mild penalisation of $\nabla \xi_\varepsilon$ which we needed in the proof of Theorem 8.2.7, the functionals are relatively well-behaved, although the elimination of the second-order penalty term in the energy will be the focus of future work.

It should be noted that also the topological functionals $\mathcal{C}_\varepsilon$ have a regularising effect in simulations – compare Figures 9.1 and 9.3 in Chapter 9. Without the topological penalty, we observe self-crossings along saddle-solutions of the stationary Allen-Cahn equation, which do not occur with the penalty term.

This is assumed to depend on the specific implementation – we chose α, β close to 1 and $\kappa = 1$ and a second associated functional corresponding to α', β' close to -1 and $\kappa' = 1$. If $0 \in (\alpha, \beta)$, then at least certain saddle solutions are expected to be permitted, as a level set close to zero is connected. If $\kappa < 1$, then the penalty should not be regularising enough for small ε, since the level-sets are only disconnected on a length scale $\sim \varepsilon$. In both cases, we expect to see saddle solutions as blow-ups, while they seem to be prevented in two-dimensional simulations for suitably chosen topological penalties.

Remark 8.2.10. The existence problem for the gradient flow of $\mathcal{W}_\varepsilon^{\mathrm{Bel}}$ is open, but formal asymptotic expansions suggest that the gradient flows of $\mathcal{W}_\varepsilon^{\mathrm{Bel}}$ approach Willmore flow as $\varepsilon \to 0$ for an appropriate scaling of the time-parameter. This result is only valid as long as formal asymptotic expansions hold, and in particular not when self-intersections occur in Willmore flow. Self-intersections are a stable property of a class of initial conditions [MS03], but non-tangential self-intersections should heuristically lead to infinite energy in $\widetilde{\mathcal{W}}$.

Thus we are lead to conjecture that the gradient flows of $\mathcal{W}_\varepsilon^{\mathrm{Bel}}$ or more generally any functional $\widetilde{\mathcal{W}}^\varepsilon$ which approximates $\widetilde{\mathcal{W}}$ fail to approach the gradient flow of $\mathcal{W}$ in singular situations. The motion could be compared to a version of Willmore flow which has been modified to satisfy a maximum principle and has not been described yet. This idea will be pursued further in the following section.

Numerical simulations for the gradient flow of $\mathcal{W}_\varepsilon$ on the other hand suggest convergence to Willmore flow past the critical time [BMO13]. Again, this seems to suggest that the fundamental object for the model based on $\mathcal{W}_\varepsilon$ is the diffuse surface μ_ε (associated to $\mathcal{W}$) rather than the function u_ε (associated to $\widetilde{\mathcal{W}}$).

8.3 Topology-preserving Time-evolution

8.3.1 Intuition and Heuristics

As pointed out above, the topological concept of connectedness is a non-local invariant of a space, and it is thus clear that our topological functional $\mathcal{C}_\varepsilon$ has to be non-local to capture the notion. The change of topology (in particular, loss of connectedness) in a surface evolution on the other hand happens locally, so entirely local functionals are suited for preventing a loss of connectedness (among other things) in a continuous time evolution.

Since level sets $\{u_\varepsilon = \theta\}$ and even approximate level sets $\{\alpha < u_\varepsilon < \beta\}$ are highly unstable under perturbations, we introduce a more stable notion of topology for a phase-field in this chapter. For a set $A \subset \mathbb{R}^n$, denote

$$B_r(A) := \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, A) < r\}$$

and again

$$N_\varepsilon := \{x \in \Omega \mid u_\varepsilon(x) = 0\}.$$

Lemma 8.3.1. *Let $E \Subset \Omega$, $\partial E \in C^2$ and u_ε the usual recovery sequence for χ_E. Then ∂E is a deformation retract of $B_{\lambda\varepsilon}(\{u_\varepsilon = 0\}) = \{q(-\lambda) < u_\varepsilon < q(\lambda)\}$ for all $\lambda > 0$ and all small enough $\varepsilon > 0$.*

The required smallness of ε may of course depend on λ and E.

Proof. For some small $r > 0$, the map

$$\psi : B_r(\partial E) \to \partial E \times (-r, r), \qquad x \mapsto (\pi(x), \operatorname{sdist}(x, \partial E))$$

composed of the closest point projection and the signed distance function is a diffeomorphism. For any $\varepsilon > 0$, the nearest point projection

$$\pi : B_{\lambda\varepsilon}(\partial E) \to \partial E$$

is a retraction. The map $\pi : B_{\lambda\varepsilon}(\partial E) \to B_{\lambda\varepsilon}(\partial E)$ is homotopic to the identity on $B_{\lambda\varepsilon}(\partial E)$ relative to ∂E by

$$h(t, x) = \pi(x) + t \operatorname{sdist}(x, \partial E)\, \nu_{\pi(x)}.$$

$\square$

In particular, the fattened zero level set captures not only the number of connected components of the zero level set, but also the cohomology groups (in this smooth case). We will show that the topology of the fattened zero-level set is stable under small perturbations. For the proof, we need a discrete version of the blow-up property.

Definition 8.3.2. We say that a function $u \in -1 + W_0^{2,2}(\Omega)$ satisfies an ε-blow-up criterion at level (R, δ) if for all points $x \in \overline{\Omega}$ the blow-up function

$$\tilde{u}(y) = u(x + \varepsilon y)$$

satisfies either

$$||\, \tilde{u} - q(\langle \cdot, v\rangle + b)||_{W^{2,2}(B_R(x))} < \delta$$

for $b = q^{-1}(u(x))$ and appropriately chosen $v \in S^{n-1}$ or

$$||\tilde{u} - (\pm 1)||_{W^{2,2}(B_R(x))} < \delta$$

where ± 1 denotes the constant function of that value. For the purpose of the blow up, we use constant continuations of u to $\mathbb{R}^n$ to avoid dealing with boundary behaviour separately.

8.3.2 On the Approximation of Level Sets by Manifolds

Before we come to the main results of this section, we need a Lemma about the approximation of there zero level sets of phase-fields u_ε with the blow-up property by smooth manifolds. We blow up to an ε-scale, where we simply consider the approximation of linear spaces. Let $R \gg 1$ and $\eta \in C_c^\infty(B_1)$ be a standard mollifier, i.e. $\eta \geq 0$ and $\int_{\mathbb{R}^n} \eta \,\mathrm{d}x = 1$. Additionally, we assume that η is radially symmetric, so that $\eta * f = f$ for all linear functions f. Then for $r \in L^1(B_R)$ we define $\hat{r} \in L^1(B_{R-1})$ by the convolution $\hat{r} = r * \eta$.

Lemma 8.3.3. *Let $R \gg 1$ and denote $\pi_1 : B_R \to \mathbb{R}$, $\pi_1(x) = x_1$. Then for $k = 1, 2$ and $\delta > 0$, there exists a $\beta > 0$ such that if*

$$||\, r - \pi_1 \,||_{W^{k,2}(B_R)} < \beta$$

then $\{\hat{r} = 0\}$ is a C^∞-graph over $\{x_1 = 0\}$ of a function $\phi : \{x_1 = 0\} \cap B_{R/2} \to \mathbb{R}$ and $|\phi|_{C^k} < \delta$.

Proof. Assume that there is a sequence of functions u_n such that $r_n \to \pi_1$ in $W^{k,2}(B_R)$, and note that by construction $\hat{r}_n - \pi_1 = \widehat{r_n - \pi_1}$ since the second function is linear and η is radially symmetric. Since $r_n - \pi_1 \to 0$ in $L^1(B_R)$, standard analysis shows that $\widehat{r_n - \pi_1} \to 0$ in $C^m(B_{R-2})$ for any $m \in \mathbb{N}$ since the mollifier is not rescaled as in other applications.

In particular, $\nabla \hat{r}_n \neq 0$ on B_{R-2} for all sufficiently large n and all level sets of $\hat{u}_n$ are embedded C^∞-manifolds. It is immediately obvious that $\{\hat{r}_n = 0\}$ and $\{x_1 = 0\}$ are close in Hausdorff-distance, and careful examination of the proof of the regular value theorem via the inverse function theorem shows that they are close also in C^k-parametrisation.

□

8.3.3 On the Fine Topology of Phase-Fields

We will apply the result of the previous section to $r = \varepsilon\, q^{-1}(u)$ and set $\hat{u} = q\left(\frac{\hat{r}}{\varepsilon}\right)$. Note that if u, w are $W^{k,p}$-close and $|u|, |w| \leq 1 - \bar{\delta}$, then the associated functions r_u, r_w are also $W^{k,p}$-close (but with large constants for small $\bar{\delta} > 0$).

Lemma 8.3.4. *Assume that u is a phase-field with the blow-up property at the level (R, δ). For all sufficiently small $\delta > 0$, there exist $0 < \lambda_{R,\delta} < \lambda^{R,\delta} < R/2$ such that for all $\lambda_{R,\delta} < \lambda, \mu < \lambda^{R,\delta}$, the sets $B_{\lambda\varepsilon}(\{u_\varepsilon = 0\})$ and $B_{\mu\varepsilon}(\{u_\varepsilon = 0\})$ are homeomorphic.*

The lower bound $\lambda_{R,\delta}$ is needed since for too small λ, the fattened level sets would be just as unstable under perturbations as the level sets themselves. The upper bound is needed since for very large $\lambda \gg R$, the fattened set might develop points of self-contact not corresponding to the behaviour of the phase-field.

Proof of Lemma 8.3.4. **Step 1.** We demonstrate that the boundary $[\partial B_\lambda(\{\tilde{u} = 0\})] \cap B_R$ is the union of the graphs of two continuous functions over a linear space, at least inside $B_{R/2}$.

Take any $x \in \{u = 0\}$, then on the ball B_R the blow up of u around x is δ-close to an optimal profile in some direction v. Without loss of generality, $v = e_n$. For $\hat{x} = (x_1, \dots, x_{n-1})$ we define

$$\Gamma_{\hat{x}} = \{x_n \mid (\hat{x}, x_n) \in B_\lambda(\{\tilde{u} = 0\}) \cap B_R\}.$$

Let us show that for all $\hat{x} \in \{x_n = 0\} \cap B_{R/2}$, the slice $\Gamma_{\hat{x}}$ is an open interval. Due to the continuous embedding $W^{2,2}(B_R) \to C^0(B_R)$, we see that in Hausdorff distance

$$d^H(\{\tilde{u} = 0\} \cap B_{R/2},\ \{x_n = 0\} \cap B_{R/2}) \leq C\delta$$

since $\tilde{u}$ must cross between positive and negative values to stay close to an optimal profile. Thus in particular

$$\left\{-\frac{\lambda}{2} < x_n < \frac{\lambda}{2}\right\} \cap B_{R/2} \ \subset\ B_\lambda(\{\tilde{u} = 0\}) \cap B_{R/2}.$$

for all small enough $\delta > 0$. Now assume that there exist $\lambda/2 \leq s < t$ such that $s \notin \Gamma_{\hat{x}}$ but $t \in \Gamma_{\hat{x}}$ for some $\hat{x} \in B_{R/2}$. Then there exists

$$y \in \{u = 0\} \cap B_{3R/4} \subset B_{C\delta}(\{x_n = 0\}) \cap B_{3R/4}$$

such that

$$|(\hat{x}, t) - y| < \lambda, \quad |(\hat{x}, s) - y| > \lambda \qquad \Rightarrow \qquad |y - (\hat{x}, s)| > |y - (\hat{x}, t)|$$

so we have reached a contradiction. Thus $[\partial B_\lambda(\{\tilde{u} = 0\})] \cap B_R$ can be written as the union of the graphs of two functions $g_\pm$, without any statement about the continuity of these functions so far. It is, however, clear from the Hausdorff-distance estimate that

$$-\lambda - C\delta \leq g_- \leq -\lambda + C\delta, \qquad \lambda - C\delta \leq g_+ \leq \lambda + C\delta.$$

Pick any $y \in [\partial B_\lambda(\{\tilde{u} = 0\})] \cap B_R$. Then $\mathrm{dist}(y, \{\tilde{u} = 0\}) = \lambda$ and since $\tilde{u}$ is continuous, there exists $x \in \{\tilde{u} = 0\}$ such that $|y - x| = \lambda$. By definition, $B_\lambda(x) \subset B_\lambda(\{\tilde{u} = 0\})$, and we calculate

$$\lambda^2 = |x-y|^2 = |\hat{x} - \hat{y}|^2 + |x_n - y_n|^2 \geq |\hat{x} - \hat{y}|^2 + |\lambda - 2C\delta|^2 \quad \Rightarrow \quad |\hat{x} - \hat{y}|^2 \leq 4C\lambda\delta.$$

In particular, if δ is small enough, we can write $g_+ \geq f$ around $\hat{x}$, where f is the graph representation of $\partial B_\lambda(\hat{y})$. So at every point $\hat{x}$, there is a continuous function $f_{\hat{x}}$ on a small ball $B_r(\hat{x})$ in $\{x_n = 0\}$ for some r independent of $\hat{x}$ such that

1. $f_{\hat{x}}(\hat{x}) = g_+(\hat{x})$ and
2. $g_+ \geq f_{\hat{x}}$ on $B_r(\hat{x})$.

It follows that g_+ is continuous, and the same is true for g_-.

Step 2. Like in the previous section, we observe that the set $\{\hat{u} = 0\}$ is C^2-close to a linear space in a ball $B_{R\varepsilon}(x)$ if δ is small enough. Thus we can write the linear space as a graph over the smooth manifold $\{\hat{u} = 0\}$, and successively also the boundaries of the set $B_{\lambda\varepsilon}(\{u = 0\})$. There is a canonical choice of g_+ and g_- by choosing g_+ always on the side where the optimal profile approximated by the local blow up is positive. Thus we have

$$\partial B_{\lambda\varepsilon}(\{u = 0\}) = \{y + \hat{g}_{\lambda,+}(y)\,\nu_y \mid y \in \{\hat{u} = 0\}\} \cup \{y + \hat{g}_{\lambda,-}(y)\,\nu_y \mid y \in \{\hat{u} = 0\}\}$$

and

$$B_{\lambda\varepsilon}(\{u = 0\}) = \{y + t\,\nu_y \mid y \in \{\hat{u} = 0\}, \quad g_{\lambda,-}(y) < t < g_{\lambda,+}\}\,.$$

Therefore we have a homeomorphism

$$\phi : \{\hat{u} = 0\} \times (0,1) \to B_{\lambda\varepsilon}(\{u = 0\}), \qquad \phi(y,t) = y + [t\,g_{\lambda,-}(y) + (1-t)\,g_{\lambda,+}(y)]\;\nu_y.$$

The same is true for μ in the same regime as λ, and thus the two fattenings are homeomorphic. □

So the choice of topology of a phase-field does not depend on the fattening parameter $\lambda > 0$ in a sensible regime. The dependence on R is only relevant for being able to choose λ large enough, the relevant control is about the C^0-norm of u close to the zero level.

Lemma 8.3.5. *Assume that u, w are phase-fields with the blow-up property at the level (R, δ). For $\lambda \ll R$ and small enough δ, we have the following property: There exists $\beta > 0$ such that if*

$$\varepsilon^{-n} \int_\Omega (u - w)^2 \,\mathrm{d}x < \beta,$$

then the sets $B_{\lambda\varepsilon}(\{u = 0\})$ and $B_{\lambda\varepsilon}(\{w = 0\})$ are homotopy equivalent.

The condition is likely not optimal and could be improved, but it implies a sufficient stability for our application to gradient flow evolutions.

Proof. The proof of this Lemma is similar to the previous one with slight modifications as we let u change this time, not λ. When we blow up u and w around the same point $x \in \{\tilde{u} = 0\}$, we observe that

$$\begin{aligned}
||\,\tilde{u} - \tilde{w}\,||_{2,2,B_R} &\leq ||\,\tilde{u} - \phi_{v_1,b_1}\,||_{2,2,B_R} + ||\,\phi_{v_1,b_1} - \phi_{v_2,b_2}\,||_{2,2,B_R} + ||\,\tilde{w} - \phi_{v_2,b_2}\,||_{2,2,B_R} \\
&\leq ||\,\tilde{u} - \phi_{v_1,b_1}\,||_{2,2,B_R} + C\,||\,\phi_{v_1,b_1} - \phi_{v_2,b_2}\,||_{2,B_R} + ||\,\tilde{w} - \phi_{v_2,b_2}\,||_{2,2,B_R} \\
&\leq ||\,\tilde{u} - \phi_{v_1,b_1}\,||_{2,2,B_R} - \phi_{v_2,b_2}\,||_{2,B_R} + ||\,\tilde{w} - \phi_{v_2,b_2}\,||_{2,2,B_R} \\
&\qquad + C\,\{||\,\tilde{u} - \phi_{v_1,b_1}\,||_{2,B_R} + ||\tilde{u} - \tilde{v}\,||_{2,B_R} + ||\,\tilde{w} - \phi_{v_2,b_2}\,||_{2,B_R}\} \\
&\leq C\beta + 2(C+1)\,\delta
\end{aligned}$$

since on the finite dimensional space of optimal profiles $\phi_{v,b} = q(\langle\cdot, v\rangle + b)$ parametrised by $(v, b) \in \mathbb{R}^n \times \mathbb{R}$, the norms induced by $W^{2,2}(B_R)$ and $L^2(B_R)$ are equivalent. Thus the blow-ups $\tilde{u}, \tilde{w}$ are $W^{2,2}$-close on B_R.

This implies that the smoothed functions $\hat{u} = u * \eta, \hat{w} = w * \eta$ are C^2-close after blowing up, and a close examination of the proof of the regular value theorem via the implicit function theorem shows that $\{\hat{w} = 0\}$ and $\{\hat{u} = 0\}$ are both C^2-graphs over the same linear space in small enough neighbourhoods. It follows that we can write $\{\hat{w} = 0\}$ as a graph over $\{\hat{u} = 0\}$ locally, and thus also globally, since $\{\hat{u} = 0\}$ has a smooth choice of normal vector. Consequently the sets $\{\hat{u} = 0\}$ and $\{\hat{w} = 0\}$ are homeomorphic.

We have seen above that the fattenings $B_\varepsilon(\{u = 0\}), B_\varepsilon(\{w = 0\})$ are homotopy equivalent to the zero level sets of the smoothed function $\{\hat{u} = 0\}, \{\hat{w} = 0\}$. We now use the transitivity of the homotopy equivalence of spaces to conclude the proof. □

Thus if we use blow-up controlling functionals to approximate Willmore's energy, the topology of the fattened zero level set is a meaningful concept for phase-fields and does not depend on the choice of the fattening parameter $\lambda > 0$ except in possibly requiring smaller ε for larger or too small λ.

8.3.4 Gradient-Flows

We will show that all solutions of a gradient flow associated to an energy which enforces an approximate blow-up property for positive ε have a topology-preserving property. The natural space for a gradient flow solution is the Bochner space

$$V = \left\{ u \in L^2\left([0,T],\, -1 + W_0^{2,2}(\Omega)\right) \;\middle|\; \frac{\mathrm{d}u}{\mathrm{d}t} \in L^2\left([0,T], W^{-2,2}(\Omega)\right) \right\}.$$

It is well known that this space embeds into

$$C^0\left([0,T], L^2(\Omega)\right),$$

see for example [Eva10, Section 5.9.2], where the proof is given in the case of functions with values in H^1 instead of H^2. However, the proof goes through in the exact same way in our case. The previous Lemma immediately implies the following.

Corollary 8.3.6. *Let $u \in C^0([0,T], L^2(\Omega))$ be a function such that $u(t,\cdot)$ satisfies the blow-up criterion at the level (R,δ) for some small positive $\varepsilon > 0$ with constants independent of the time. If δ is small enough, the fattened level-sets $B_{\lambda\varepsilon}(\{u(t,\cdot) = 0\})$ are homotopy equivalent for all times $t \in [0,T]$ and $\lambda \in [1/2, 2]$ (for small enough ε).*

So we have shown that phase-fields satisfying the positive ε blow-up criterion preserve the topological type of the fattened zero level set in a continuous time-evolution (for example, the L^2-gradient flow). Of course, loss of genus phenomena like the one described in Remark 3.2.14 can also occur in the limit $\varepsilon \to 0$ as smaller and smaller catenoidal connections can be captured by phase-field approximations as $\varepsilon \to 0$.

Thus we have a stability of genus phenomenon for phase-field evolutions for positive ε but not for the singular limit if we enforce 'good' behaviour of the phase-field at the transition layer. Topological changes can only occur on the phase-field level when passing through an exotic solution of the stationary Allen-Cahn equation in two dimensions or, possibly, an entirely different function in three dimensions (leading to a high concentration of curvature on a small ball). Excluding those 'bad' blow-up behaviours leads to a higher rigidity in the phase-fields and seems to induce a minimal bending scale or a type of interior and exterior sphere condition. The length-scale of this minimal bending seems to be larger $R\varepsilon$ for all $R > 0$, but since q approaches pure phase ± 1 exponentially fast on $\mathbb{R}$, a recovery sequence for a sequence of C^2-surfaces with interior and exterior spheres of radius proportional to d_ε can be constructed provided that

$$d_\varepsilon \gg \varepsilon\, |\log \varepsilon|.$$

We can call the idea of following the gradient flow of a suitable blow-up controlling approximation of Willmore's energy the *blow-up control method* and compare it to the topology controlling functionals of the last chapter, which we could dub the *distance function method*. Let us compare the two methods. The following table comprises the most important properties.

Distance function method	**Blow-up control method**
• Unclear Γ-limit at immersed curves	• Can approximate $\widetilde{\mathcal{W}}$
• Only controls connectedness, allows topological transitions	• Controls topological type and 'smoothness' through $\widetilde{\mathcal{W}}$
• Admits a variational statement	• Admits a dynamical statement
• Non-local functional	• Local functional
• Requires computation of a geodesic distance function	• Destroys quasi-linearity

So in particular, the blow-up control method allows us to begin with a given number of connected components at time $t = 0$ of a continuous time-evolution and will preserve the number of components. The components may come into contact and changes of topological type may occur in the singular limit $\varepsilon \to 0$ (but not for fixed $\varepsilon > 0$). The description of the limit of such gradient flow dynamics is entirely open at the moment.

It seems that more could be said for example for initial conditions with knotted tori for the blow-up control method, but we shall not investigate such questions here.

Chapter 9

An Implementation of the Topological Constraint

9.1 Introduction

At first glance, the energy $\mathcal{E}_\varepsilon$ looks dreadful from an implementation point of view since in every time-step, we have to find the geodesic distance $d^{F(u_\varepsilon)}(x, y)$ for all points $x, y \in \Omega$. This can be significantly simplified to allow for an efficient implementation. For convenience, we take $\omega_\varepsilon \equiv \infty$.

1. The distance only needs to computed between points x, y in the diffuse surface

$$\Sigma_\varepsilon = \{x \in \Omega \mid \phi(u_\varepsilon(x)) > 0\}.$$

2. If x_0, x_1 and y_0, y_1 lie in the same path-component C_x, C_y of $\overline{\Sigma_\varepsilon}$ respectively, then

$$d^{F(u_\varepsilon)}(x_1, y_1) = d^{F(u_\varepsilon)}(x_0, y_0)$$

since x_0 and x_1 (or y_0 and y_1) can be connected by a curve γ lying entirely in $\overline{\Sigma_\varepsilon}$ which has length zero due to the fact that $F \equiv 0$ on $[\alpha, \beta]$. This means that (provided the connected components of Σ_ε have been found) the distance only has to be computed between connected components.

In simulations, it has also proven favourable to use two topological functionals $\mathcal{C}^1_\varepsilon, \mathcal{C}^2_\varepsilon$ associated to functions ϕ_1, F_1 and ϕ_2, F_2 respectively such that ϕ_1 has support close to $+1$ and ϕ_2 has support close to -1. By keeping a diffuse level set close to $+1$ and one close to

-1 connected, we create two barriers against a disintegrating interface. It is clear that this case is also covered by the results of Chapter 7.

We will now describe an efficient implementation of phase-field Willmore flow with topological constraint and area penalisation.

9.2 The Algorithm

We use a variant of Dijkstra's algorithm similar to the one of [BCPS10] to compute the geodesic distance function used in the topological term of our energy functional.

9.2.1 The Distance Function on a Graph

Let Γ be a finite connected (undirected) graph with vertices v and edges e that have weights $w_e \geq 0$. The distance of two vertices v, v' is defined as the length of the shortest path connecting v and v'. Here the length of a path is the sum of the weights of all the edges along the path and continuity is expressed via the condition that consecutive edges share a node. Precisely, we have

$$d(v, v') = \inf \left\{ \sum_{i=1}^{n} w_{e_i} \mid v = v_0, v' = v_n, \quad v_{i-1}, v_i \in e_i, \quad \forall\, 1 \leq i \leq n \in \mathbb{N} \right\}$$

where the infimum goes over $n \in \mathbb{N}$ and over all paths of length n connecting $v = v_0$ to $v' = v_n$. Assume that we are given a sequence of graphs Γ_h associated to a sequence of triangulations with a spacial grid scale h for $h \to 0$ in the sense that a vertex of Γ_h corresponds bijectively to a triangle and that the weight of the edge e is computed as a convex combination of the values a continuous function $f \geq 0$ assumes on e.

The triangulations may force us to walk zig-zagging to connect two points, so the distance on the graph may not approximate the distance function

$$d^f(x, y) = \inf \left\{ \int_\gamma f \,\mathrm{d}\mathcal{H}^1 \,\middle|\, \gamma \text{ curve from } x \text{ to } y \right\},$$

but assuming that triangulations do not degenerate, it approximates a function which is related to d^f in a bi-Lipschitz sense uniformly in h:

$$c\, d^{\Gamma_h}(v, v') \leq d^f(x_v, x_{v'}) \leq C\, d^{\Gamma_h}(v, v')$$

where the points $x_v, x_{v'}$ are the centres of mass of their triangles, v and v' do not lie in the same triangle, and the constants $c, C > 0$ are uniform in h. Note that if there exists a unique shortest curve $\bar\gamma$ between x and y then

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} d^{F(u+ts)}(x,y) = \int_{\bar\gamma} F'(u)\, s\, \mathrm{d}\mathcal{H}^1 \qquad \forall\ s \in C^\infty(\mathbb{R}^n).$$

This identity will be postulated heuristically for the procedure below.

9.2.2 Dijkstra's Algorithm

Dijkstra's algorithm describes a procedure to calculate the distance function $v \mapsto d(v, \bar v)$ for a given $\bar v \in \Gamma$. Our version is a simplified version of the one that was proposed in [BCPS10] for the fast marching method.

The idea is to find the shortest path connecting two elements by marking elements as known when we are sure from the algorithm that a shorter path cannot exist and checking whether they give a shorter path to their neighbours than has been found before.

To keep things simple, in the description we assume that to every vertex we associate a data structure which includes a distance D and a predecessor vertex pointer P. In the set-up of the algorithm, set $D_{\bar v} = 0$ for the given vertex $\bar v$, $D_v = \infty$ for all $v \neq \bar v$, $P_v = \mathrm{NIL}$ for all v. Here NIL is the pointer equivalent of an empty set and a convenient abstraction, but could be replaced by any given value. Create two lists K and U of known and unknown vertices and set $K = \emptyset$, $U = \Gamma$.

1. Take an element $v \in U$ such that $D_v = \min\{D_{v'} \mid v' \in U\}$. (In the first step, this is $\bar v$.) Move v from U to K. For all elements u connected to u by an edge e_{vu}, check if

   $$D_v + w_{uv} < D_u.$$

   If so, replace D_u by $D_v + w_{uv}$ and set $P_u = v$.

2. Repeat step 1 until $K = \Gamma$ and $U = \emptyset$. If all elements in U have distance ∞, the graph is disconnected. In this case, the algorithm can be aborted (and all remaining predecessor pointers be set to some common value, for example, all distances left at ∞).

In the following, we will always assume that our graphs are connected. The algorithm can be terminated prematurely according to certain criteria, e.g. when the last vertex out of a list of nodes we are interested in is marked as known. This will be used below.

The algorithm could easily be adapted for asymmetric graphs. If only the distance function is needed and its derivative is not, we need not remember the predecessor pointers.

By following predecessor pointers back from an element v through the predecessor pointers (until the pointer becomes $NULL$) we obtain a shortest path between $\bar{v}$ and v.

9.2.3 Treating the Topological Term

Now, we will describe how to include the topological term in an explicit fashion in given finite element code.

The description is given in the two-dimensional case assuming that the finite element space corresponds to a triangulation of Ω with grid length scale h. Dimension three and more general basis element shapes can obviously be treated by the same method.

In the set up of the simulation, create a graph Γ such that

1. every node v of Γ corresponds to a triangle $\Delta = \Delta_v$ in the triangulation of the domain Ω associated to our finite element space and vice versa and
2. two vertices v_1, v_2 are connected by an edge e if and only if the triangles Δ_1, Δ_2 share a side.

This can, of course, be done implicitly. It is also advantageous if an element knows its volume $|\Delta|$ and potentially diameter $\mathrm{diam}(\Delta)$. Given a Galerkin space function $u = u^k$ in time step k, do the following.

1. For all triangles Δ in the triangulation, compute the average integral
$$u_\Delta = \frac{1}{|\Delta|}\int_\Delta u\,\mathrm{d}x.$$
2. For the edge e between two triangles Δ, Δ' define the weight of the edge by
$$w_e = \frac{F(u_\Delta) + F(u_{\Delta'})}{2}\,h.$$
The constant h is included as an approximation of the distance between the midpoints of Δ and Δ' up to bounded scalar factor.
3. Create a list I of all interface elements, i.e. all elements such that
$$u_\Delta \in [\alpha, \beta].$$
Remember the length $|I|$ of the list.
4. Create a new list T whose components will be lists C of triangles Δ. We think of the C's as connected components of the interface and T as expressing the topology.

5. Take an arbitrary element $\Delta \in I$ and create a new list C containing only the element Δ and remove Δ from I. Run Dijkstra's Algorithm to compute the distance function $d(\cdot, \Delta)$ on the graph Γ. When you encounter an interface element $\Delta' \in I$ such that $d(\Delta, \Delta') = 0$, transfer Δ' from I to C.

   Abort the algorithm when you encounter the first element $\Delta' \in \Gamma$ such that $d(\Delta, \Delta') > 0$. Do not add Δ' to C.

6. Repeat step 5 until I is empty. Now we have lists $C_1, \dots, C_n$ of equivalence classes of I inside Γ. If there is only one list $C = C_1$, the interface is connected and $\mathcal{C}_\varepsilon = \delta\mathcal{C}_\varepsilon = 0$. In this case, abort the algorithm.

7. Iterate through the list T over the components C_i and create a list Φ of the integrals

$$\Phi_i = \frac{1}{\varepsilon} \int_{C_i} \phi(u) \,\mathrm{d}x$$

   for later use.

8. If there are at least two components, create a symmetric array G whose elements G_{ij}, $1 \leq i, j \leq n$ are lists of vertices. We will store the shortest curve (geodesic) between the components C_i and C_j in G_{ij}.

   Also create a symmetric array d_{ij} in which to store the distance $\mathrm{dist}(C_i, C_j) = d(\Delta_i, \Delta_j)$ for arbitrary triangles $\Delta_i \in C_i$, $\Delta_j \in C_j$.

9. Take the component C_i and run Dijkstra's algorithm from an arbitrary element $\Delta \in C_i$. Use a counter to abort the algorithm when you have found $d(\Delta, \Delta')$ for the remaining $|I| - 1$ interface elements Δ'.

   Take $j = i + 1$ and a triangle $\Delta' \in C_j$. Set $d_{ij} = d(\Delta, \Delta')$. Then, use the predecessor pointer from Dijkstra's algorithm to find the element Δ'' before Δ'. If $\Delta'' \in C_j$, replace $\Delta' = \Delta''$ and repeat. If $\Delta'' \notin C_j$, add Δ'' to the list G_{ij}. Take the predecessor element Δ''' of Δ''. If $\Delta''' \notin C_i$, add it to G_{ij}, otherwise stop and move on to the next component $j' = j + 1 < n$.

10. Repeat step 9 for $i = 1, \dots, n$. Now we know all connected components C_i of the interface, their distances d_{ij} and shortest connections G_{ij} in the graph.

11. Compute the value of the topological functional

$$\mathcal{C}_\varepsilon(u) = \frac{2}{\varepsilon^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d_{ij} \,\Phi_i \,\Phi_j .$$

12. We do a three-fold nested iteration: Iterate over the components C_i, over the elements $\Delta \in C_i$ and over the basis functions s whose support overlaps with Δ to compute the component contributions to the force

$$[\delta \mathcal{C}_\varepsilon]^{C_i}(u; s) = \left(2 \sum_{j \neq i} d_{ij} \Phi_j \right) \frac{1}{\varepsilon} \int_{C_i} \phi'(u)\, s \,\mathrm{d}x.$$

13. We do a three-fold nested iteration: Iterate over the geodesics G_{ij}, over the elements $\Delta \in G_{ij}$ and over the basis functions s whose support overlaps with Δ to compute the geodesic contributions to the force

$$[\delta C_\varepsilon]^{G_{ij}}(u; s) = \Phi_i\, \Phi_j \int_{G_{ij}} F'(u)\, s \,\mathrm{d}\mathcal{H}^1.$$

The line integral can be approximated by taking the integral over the element Δ and multiplying by

$$c_\Delta = \frac{\mathrm{diam}(\Delta)}{|\Delta|}$$

to account for the fact that we integrate with respect to a different measure. The quantity can be approximated globally if the elements of the triangulation are sufficiently similar. In particular, for regular sequences of triangulations, this can be chosen to simply be $c_\Delta \equiv h^{1-n}$ where h is the spacial grid scale.

This algorithm can easily be implemented fully nested in a given implementation with explicit or implicit time-stepping, but only leads to explicit treatment of the topological term. An implicit implementation has been found to be less efficient due to the high instability of the topological term.

Clearly, steps 1 – 3 can be parallelised. Dijkstra's algorithm is not suitable for parallelisation, but it has to be called only a small number of times and can be aborted after running through only a small number of elements before the interface has been understood completely in real simulations.

In this way it is easy to include the topological term in given finite elements code for diffuse Willmore flow. Variations with respect to the structure of the graph are easy, like taking integration points as vertices or connecting triangles by an edge in the graph if they share a corner.

9.3 Simulations in Two Dimensions

In our simulation, we follow a finite element version of the L^2-gradient flow of

$$\mathcal{E}_\varepsilon = \mathcal{W}_\varepsilon + \varepsilon^{-\sigma}(S_\varepsilon - S)^2 + \varepsilon^{-\kappa}\left(\mathcal{C}_\varepsilon^{(1)} + \mathcal{C}_\varepsilon^{(2)}\right)$$

for $\sigma = 2$, $\kappa = 1$ and $\varepsilon = 1.5 \cdot 10^{-2}$. The domain of the phase-fields is chosen to be the unit disc in two dimensions which is triangulated by a mesh of triangular elements of diameter approximately $h = 6 \cdot 10^{-3}$. The basis functions are approximately 250.000 subdivision surfaces supported in the two-ring around a node and in particular H^2-conforming, using a finite element backend by P. W. Dondl also described in [DMR11, DHR16]. This allows for a direct implementation of the weak formulation of the gradient flow equation since the highest order term is the parabolic bi-Laplace evolution equation, i.e. of fourth order.

Time stepping is done with a hybrid implicit/explicit Euler method, namely the leading fourth order term $\varepsilon u_t = -\varepsilon\,\Delta^2 u$ is discretised implicitly and the lower order non-linear terms are discretised explicitly in time. The time step size is $\hat{\tau} = \varepsilon\tau = 10^{-5}$ (in rescaled time). In this semi-implicit formulation, the system matrix for time-stepping is constant in time and has to be assembled only once. For this reason, a direct QR-factorisation solver from the C++ library CHOLMOD [Dav08] was used. Matrices were implemented using the Armadillo library. This appeared to have a comparable performance to iterative solvers, but proved to be slightly faster and more user-friendly.

We also attempted a fully implicit time-stepping scheme with iterative solvers, but it seems that the coarse discretisation of the geodesic distance function does not support this well. The geodesic contributions to the force are concentrated along curves or chains of elements which are much thinner than an interface, so a small time-step has to be chosen for the sake of numerical stability and the second derivatives of the geodesic distance function (which are not even guaranteed to exist in a satisfactory theoretical sense) are not well approximated.

The functions ϕ_1, ϕ_2 and F_1, F_2 were chosen via $\phi_2(z) = \phi_1(-z)$ and $F_2(z) = F_1(-z)$ where ϕ_1 and F_1 are piecewise C^∞- and globally $C^{1,1}$-functions created with fourth order polynomials

$$\phi_1(z) = \begin{cases} \frac{30}{(\beta-\alpha)^5}\,(z-\alpha)^2(\beta-z)^2 & z \in [\alpha,\beta] \\ 0 & \text{else} \end{cases}$$

$$F_1(z) = \begin{cases} \frac{4}{(1+\alpha)^2}(z-\alpha)^2 & z \in (-\infty, \alpha] \\ 0 & z \in [\alpha, \beta] \\ \frac{4}{(1-\beta)^2}(z-\beta)^2 & z \in [\beta, \infty) \end{cases}$$

where the normalising constants insure that $\int_{-1}^{1} \phi_1(z)\,\mathrm{d}z = 1$ and $F_1(\pm 1) = 4$. In our simulation, $\alpha = 0.85$ and $\beta = 0.95$.

We see in Figure 9.1 that without the inclusion of the topological term, the transition layer disintegrates into several connected components along the gradient flow of $\mathcal{W}_\varepsilon + \varepsilon^{-\sigma}(S_\varepsilon - S)^2$.

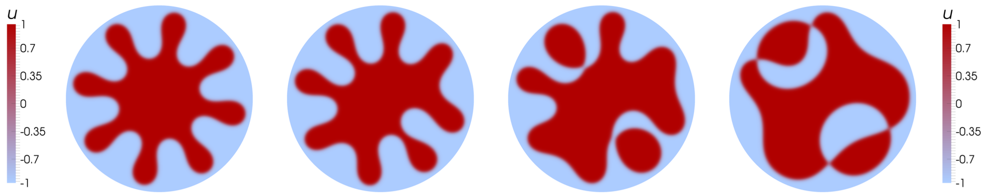

Figure 9.1: Gradient flow of $\mathcal{W}_\varepsilon + \varepsilon^{-\sigma}(S_\varepsilon - S)^2$. From left to right: Phase-field u for approximately $t = 7.5 \cdot 10^{-5}$, $t = 3 \cdot 10^{-4}$, $t = 7.5 \cdot 10^{-4}$ and $t = 1.8 \cdot 10^{-3}$.

To compare implementations of topological side conditions, we include the topological term suggested in [DMR11], which penalises a deviation of a diffuse signed curvature integral from 2π in the simulation. This term prevents the initial pinch-off, but at a later time, the interface will pinch off in a more complicated way which keeps the diffuse winding number close to 2π. The phenomenon is a simultaneous pinch off at several points as seen in Figure 9.2. The far right plot in Figure 9.2 illustrates the diffuse curvature density as distributed along the curve at pinch off time. We can observe the formation of a circle with negative total curvature $\approx -2\pi$ (due to the phase-field switching in the other direction from $+1$ to -1), and two components with total curvature $\approx 2\pi$ so that the total curvature of the whole interface stays close to 2π.

Unfortunately we have been unable to implement the topology controlling term $\mathcal{A}_{u,\varepsilon}$ from [DMR14] and the associated gradient flow in practice due to the complicated nested minimisation procedure of the energy functional. The need to find in each time-step an absolute minimiser of $\mathcal{A}_{u,\varepsilon}$ has prevented us from giving a practical implementation and convinced us to develop the simpler functional $\mathcal{C}_\varepsilon$ instead. For this reason, we do not have an implementation for comparison.

In Figure 9.3, a flow for $\mathcal{E}_\varepsilon$ with the additional term of $\mathcal{C}_\varepsilon$ on the other hand can be seen

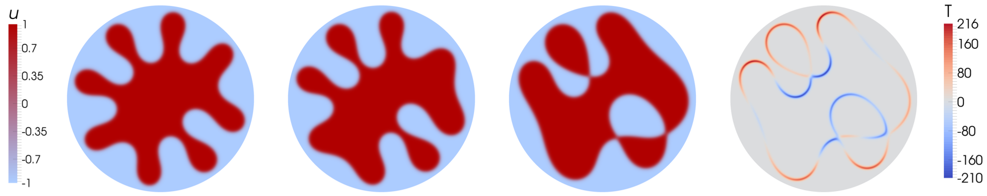

Figure 9.2: Gradient flow with penalty on a diffuse winding number as suggested in [DMR11]. From left to right: Phase-field u for approximately $t = 3{\cdot}10^{-4}$, $t = 7.5{\cdot}10^{-4}$ and $t = 1.8{\cdot}10^{-3}$, then a plot of the diffuse winding number density denoted T at time $t = 1.8 \cdot 10^{-3}$.

to stably flow past those singular situations.

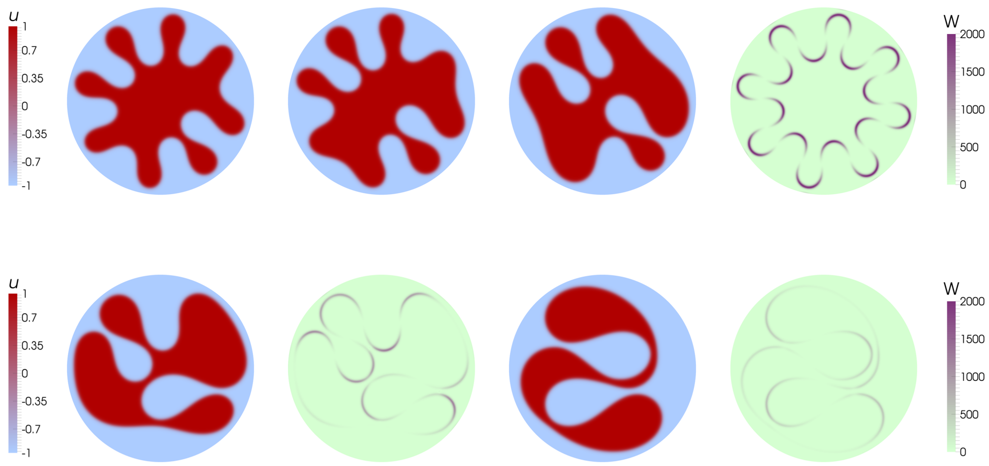

Figure 9.3: Evolution including our new topological penalty term $\mathcal{C}_\varepsilon$. Top line, from left to right: phase-field u for approximately $t = 3{\cdot}10^{-4}$, $t = 7.5{\cdot}10^{-4}$ and $t = 1.8{\cdot}10^{-3}$, then a plot of the diffuse Willmore energy density (denoted W here) of the initial condition. Bottom line, left to right: Phase-field u and diffuse Willmore energy density first for approximately $t = 6.6 \cdot 10^{-3}$ and then for approximately $t = 3.6 \cdot 10^{-2}$.

Comparing the three scenarios above, we observe that there is virtually no difference in the plots at time $3 \cdot 10^{-4}$ and that the plots for both modified (penalised using either the old or the new method) functionals at time $7.5 \cdot 10^{-4}$ still look very similar. It can thus be argued that the topological condition does not affect the shape of the curve in a major way except when it has to in order to prevent loss of connectedness.

In Figure 9.3, we see non-trivial geometric changes along the gradient flow for later times. This demonstrates the necessity of continuing the flow beyond the critical times. It should be emphasised that our focus is not on implementing a scheme to approximate Willmore flow using phase-fields but on finding minimisers of the diffuse interface problem using a gradient flow. Existence of Willmore flow for long time and topological changes along it are still an open field of research.

Chapter 10

Summary

The problem we had set for ourselves in the beginning of this dissertation was to *minimise Willmore's energy among connected structures with large surface area confined to a small container* and in the subclass of such *structures which are weakly approximable by C^2-boundaries*. We have demonstrated that a solution to both problems exists in a weak sense in Corollary 3.1.4 and gave elementary properties of such minimisers in Theorem 3.1.1. In the proof we employed the direct method of the calculus of variations, complemented with a result on the relationship between varifold convergence and Hausdorff convergence of the support of the associated mass measures, which we established in Theorem 3.1.2. We further demonstrated in Theorem 3.1.6 that connectedness is indeed the only topological quantity which can be controlled in terms of Willmore's energy, even for minimising sequences. Associated results for the more general Helfrich functional were given in Corollaries 3.1.7 and 3.1.8 and for more general Willmore-type functionals with exponent $p \neq 2$ in Remark 3.2.15. For use in the second part of the dissertation, we also established a link to the theory of Caccioppoli sets in Theorem 3.2.9.

In the second part of this dissertation, we designed a phase field approach which allows for a numerical approximation of the minimisation problem to find approximate minimisers numerically. To this end, we introduced the new notion of essentially uniform convergence in Definition 5.2.16 and proved new sharp results on the L^∞-boundedness of phase-fields in Theorem 5.1.1. Among others, we gave a precise description of the Hausdorff convergence of level sets of phase-fields in Theorem 5.2.27 which provides a partial justification of the common identification of the zero level set with a sharp interface limiting surface. These results were then used in Theorem 7.1.2 and Corollaries 7.1.3 and 7.1.4 to demonstrate that a sequence of topological penalty functionals designed in Section 7.1.2 enforce connectedness of the limit of diffuse surface measures. In Chapter 8, we described a different penalisation method which can be used to obtain a Γ-approximation of the relaxed function $\widetilde{\mathcal{W} + \mathrm{Per}}$ (Theorem 8.2.7) and showed that the gradient-flow evolutions of a wide class of functionals with similar penalty terms have a topology-preserving property (Corollary 8.3.6) which is distinct from both the gradient flow of $\mathcal{W}_\varepsilon$ and continuous surface evolutions of finite energy, potentially even the gradient flow of $\mathcal{W}$ in non-smooth situations. In Chapter 9 we described an efficient numerical implementation of a gradient flow of the diffuse Willmore-functionals and presented numerical proof that the penalty term is successful in finite element simulations. Further results, especially of a more technical nature and on the boundary behaviour of phase fields, can be found in the text.

We have thus given an analytic solution to the original problem and provided a numerical method of explicitly finding energy minimisers. Remaining open questions, especially as outlined in Conjecture 8.2.3 and Remark 8.2.10, will be the focus of future research.

Chapter 11

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