k-HARMONIC RIEMANNIAN MANIFOLDS

by

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CHAPTER IV: k-HARMONIC RIEMANNIAN MANIFOLDS

ABSTRACT

In this work we examine n-dimensional Riemannian manifolds with k-harmonic metrics. Ruse's invariant is shown to be a function of one member of a set of two-point invariants; these are the symmetric polynomials of the eigenvalues of an endomorphism of the tangent space at a fixed point (base point) and of the eigenvalues of the inverse endomorphism. These endomorphisms compare the metric tensor at the base point with the pull-back from a variable point via the exponential mapping. If the k-th symmetric polynomial is a function of the two-point invariant distance function alone, the manifold is k-harmonic at the base point. k-harmonic manifolds are k-harmonic at all base points; thus they form a generalisation of harmonic manifolds. We prove for general Riemannian manifolds:

(1) they are harmonic if and only if n-harmonic;

(2) all k-harmonic manifolds are Einstein spaces.

For simply connected Riemannian symmetric spaces we are able to derive the matrix of the required endomorphism explicitly. We investigate whether these spaces are k-harmonic either for all k or else for no k and prove the former if the rank is one. For symmetric spaces of rank greater than one no firm conclusion is reached.

INTRODUCTION

Harmonic Riemannian manifolds have been studied since 1930 when H.S.Ruse investigated the form taken by the "elementary" solution of the generalised Laplace's equation on an analytic Riemannian manifold. If $\Omega(p_0,p)$ is half the square of the geodesic distance from a fixed base point, p_0, to a general neighbouring point, p, a manifold is harmonic at p_0 if the elementary solution is a function of Ω alone. If this is true for every base point then the manifold is defined to be harmonic. An alternative definition requires Ruse's two-point invariant function, $\rho(p_0,p)$ be be a function of Ω alone. A paper published in 1968 by T.J.Willmore [16] shows that Ruse's invariant is a function of one of the elements in a set of n two-point invariants, $\sigma_k(p_0,p)$ $(k = 1, \ldots, n)$. σ_k is the k^{th} symmetric polynomial of the eigenvalues of the matrix, $\omega(p_0,p)$, representing a linear endomorphism of the tangent space at p_0 and σ_k is a function of p; however if it is a function of Ω and otherwise independent of p then the manifold is defined to be k-harmonic at p_0. The manifold is k-harmonic if it is k-harmonic at every base point. In 1970 K. El Hadi in an unpublished thesis [4] defined k-harmonic manifolds in terms of $a(p_0,p)$, the inverse of $\omega(p_0,p)$. We see that Ruse's invariant is one in a set of two-point invariants and hence that the concept of k-harmonic Riemannian manifold is a generalisation of harmonic Riemannian manifold.

This thesis attempts to assess how much is yet known about k-harmonic manifolds.

A summary of basic information regarding general harmonic spaces is given in Chapter I. The chapter starts with a section on affine and Riemannian connections followed by examination of the exponential map, normal coordinates and normal tensors; these are necessary tools for the development of our theory. Alternative definitions for harmonic manifolds are given as well as two infinite sets of necessary and sufficient conditions for a manifold to be harmonic: the Copson and Ruse equations which are expressed in terms of normal tensors and the equations of A.J.Ledger where curvature tensors are used. Properties of harmonic spaces show that they are more general than the spaces of constant curvature yet form a proper subset of the set of Einstein spaces. We conclude the chapter by showing that all decomposable manifolds with positive-definite metric are locally flat.

A.Lichnerowicz has conjectured that all Riemannian manifolds with positive-definite harmonic metric are locally symmetric. Chapter II provides the basic properties of symmetric spaces necessary for the examination of harmonic symmetric spaces; these are established using mainly the approach of S.Helgason [5]. Theorems of A.G.Walker [14] and A.J.Ledger [7] on harmonic symmetric spaces are given in Chapter III. Examination of Jacobi fields leads to the concept of globally harmonic spaces as defined by A.C.Allamigeon [1].

The first three chapters serve as introductory background to the study of k-harmonic Riemannian manifolds. Chapter IV begins with a comparison of the definitions of El Hadi and Willmore; distinction is made between k-harmonic spaces of the positive type and of the negative type. A list of investigations into properties of k-harmonic spaces is given. All k-harmonic spaces are shown to be Einstein. Harmonic spaces are shown to be (-1)-harmonic, i.e., 1-harmonic according to Willmore's definition; the author believes the converse of this result to be false, but no counterexamples have yet been found.

For general Riemannian manifolds computation of the relevant matrices and the symmetric polynomials of their eigenvalues is far from simple. But in the symmetric case use of the Jacobi equations shows that if p is a point on the unit geodesic sphere centre p_0, the eigenvalues are functions of sectional curvatures along plane sections spanned by pairs of elements of a particular form of orthonormal basis of the tangent space at p_0. In the case of symmetric spaces of rank one it is known that the holonomy group is transitive on the unit geodesic sphere centre p_0; from this we deduce that the eigenvalues are independent of p and hence that symmetric spaces of rank one are k-harmonic for all k. A.J.Ledger has proved that no symmetric spaces of rank greater than one are harmonic; it is likely also that these spaces are not k-harmonic for any k, but this is not yet proved. Hence the truth of the conjecture that symmetric spaces are either k-harmonic for all k or else for no k remains open.

CHAPTER I

HARMONIC SPACES

1.1 Affine and Riemannian connections

Let M be a differentiable n-dimensional manifold. By $C^{\infty}(M)$ we denote the set of all real-valued differentiable functions on M. Let $f,g \in C^{\infty}(M)$ and $\lambda \in \mathbb{R}$, $\mathbb{R}$ being the set of real numbers. Defining the operations $f + g$, λg, fg pointwise, it is easy to verify that $C^{\infty}(M)$ is an algebra over $\mathbb{R}$.

A vector field, X, on M is an endomorphism of $C^{\infty}(M)$ which is also a differentiation, that is, X is a map $C^{\infty}(M) \longrightarrow C^{\infty}(M)$ with the properties

(1) $$X(\lambda f + \mu g) = \lambda X(f) + \mu X(g) \text{ for } \lambda,\mu \in \mathbb{R}, \quad f,g \in C^{\infty}(M),$$

(2) $$X(fg) = f(Xg) + (Xf)g \text{ for } f,g \in C^{\infty}(M).$$

By $D^1(M)$ we denote the set of all vector fields on M. We define the three operations:

$$(X,Y) \longrightarrow X + Y \text{ given by } (X + Y)(f) = Xf + Yf,$$

$$(g,X) \longrightarrow gX \quad \text{given by } (gX)(f) = g(Xf), \text{ and}$$

$$(\lambda,X) \longrightarrow \lambda X \quad \text{given by } (\lambda X)(f) = \lambda(Xf).$$

Clearly $D^1(M)$ is a vector space over $\mathbb{R}$. We also define the Lie derivative with respect to X as the endomorphism of $D^1(M)$ given by

$$\theta(X) : Y \longrightarrow [X,Y] \text{ where } [X,Y] \text{ is the vector field}$$

$[X,Y] : f \longrightarrow X(Yf) - Y(Xf)$.

Choose $p \in M$ and $X \in D^1(M)$. The linear mapping $X_p : f \longrightarrow (Xf)(p)$ of $C^\infty(M)$ into $\mathbb{R}$ is a <u>tangent vector</u> to M at p.

$M_p = \{X_p : X \in D^1(M)\}$ is an n-dimensional vector space, the <u>tangent space</u> to M at p. Alternatively we may define a tangent vector to M at p as follows:

A <u>curve</u> is a C^∞ mapping $I \longrightarrow M$ where I is an open interval of $\mathbb{R}$.

Let α be any curve through p, that is $p = \alpha(t_0)$ for some $t_0 \in I$, $\alpha : I \longrightarrow M$. If $f \in C^\infty(M)$, then $(f \circ \alpha) \in C^\infty(I)$.

Define $\alpha_*(t_0)(f) = (f \circ \alpha)'(t_0) = \left.\dfrac{d(f \circ \alpha)}{dt}\right|_{t = t_0}$.

Letting t vary in I, it is easy to verify that $\alpha_*(t)$ satisfies (1) and (2) and hence is a vector field on the submanifold $\alpha(I)$ of M. $\alpha_*(t_0)$ is a tangent vector to M at p. Hence every curve through p defines a member of M_p.

Let M,N be C^∞ manifolds. A mapping $\phi : M \longrightarrow N$ is a <u>diffeomorphism</u> if

(i) ϕ is bijective,

(ii) ϕ and ϕ^{-1} are both differentiable.

Let $\phi : M \longrightarrow N$ be C^∞ and $p \in M$. Given $X_p \in M_p$, the mapping $(\phi_*)_p(X_p) : C^\infty(N) \longrightarrow \mathbb{R}$ given by $(\phi_*)_p X_p : g \longrightarrow X_p(g \circ \phi)$ $(g \in C^\infty(N))$ is a tangent vector to N at $\phi(p)$.

If $X \in D^1(M)$ and $Y \in D^1(\phi(M))$ then X and Y are <u>ϕ-related</u> if $Y_{\phi(p)} = (\phi_*)_p X_p$ for all $p \in M$.

U is the domain of an <u>allowable coordinate system</u>, (x^i), if U is an open subset of M and there exists a C^∞ mapping ϕ

with the properties:

(i) ϕ is a homeomorphism $U \longrightarrow V$, where V is an open subset of $\mathbb{R}^n$, and

(ii) $\phi : q \longrightarrow (x^1(q), \ldots, x^n(q))$ $(q \in N)$.

If $f \in C^\infty(M)$, then $f \circ \phi^{-1}$ is a real valued function with domain V. Let $p \in U$. We denote by $\frac{\partial f}{\partial x^k}(p)$ the real number $\frac{\partial}{\partial x^k}(f \circ \phi^{-1})_{\phi(p)}$.

Then the map $f \longrightarrow \frac{\partial f}{\partial x^k}(p)$ is a tangent vector at p denoted by $\frac{\partial}{\partial x^k}(p)$. It can be shown that $\left\{\frac{\partial}{\partial x^1}(p), \ldots, \frac{\partial}{\partial x^n}(p)\right\}$ is a basis of M_p. If we write $\frac{\partial}{\partial x^k}$ for the mapping $p \longrightarrow \frac{\partial}{\partial x^k}(p)$ then $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$ is a basis for $D^1(U)$.

An <u>affine connection</u> on M is a rule ∇ which assigns to each $X \in D^1(M)$ an endomorphism ∇_X of $D^1(M)$ satisfying

(3) $$\nabla_{fX+gY} = f\nabla_X + g\nabla_Y \text{ and}$$

(4) $$\nabla_X(fY) = f\nabla_X Y + (Xf)Y,$$

where $X, Y \in D^1(M)$ and $f, g \in C^\infty(M)$.

Let U be the domain of an allowable coordinate system (x^i) and $\left\{e_i = \frac{\partial}{\partial x^i}, i=1, \ldots, n\right\}$ be the corresponding basis for $D^1(U)$.

Then there exist functions, Γ^i_{jk} $(i,j,k = 1, \ldots, n)$ on U such that

(5) $$\nabla_{e_i} e_j = \Gamma^k_{ij} e_k.$$

These <u>connection coefficients</u> are not components of any tensor, for if (y^a) is any other allowable coordinate system on U with connection coefficients, Γ'^a_{bc}, it is easy to verify using (3),

(4) and (5) that the transformation equation is

$$(6) \qquad \Gamma'^{a}_{bc} = \frac{\partial y^a}{\partial x^i}\frac{\partial x^j}{\partial y^b}\frac{\partial x^k}{\partial y^c}\Gamma^i_{jk} + \frac{\partial^2 x^k}{\partial y^b \partial y^c}\frac{\partial y^a}{\partial x^k}.$$

Let $V, W \in D^1(M)$. $\nabla_V W$ is the <u>covariant derivative</u> of W with respect to V. If $V = f^i e_i$ and $W = g^j e_j$ then

$$\nabla_V W = \left(f^i \frac{\partial g^k}{\partial x^i} + \Gamma^k_{ij} f^i g^j\right) e_k.$$

In particular, if we substitute $V = \delta^i_1 e_i$, where δ^i_1 is the Kronecker 'delta', we obtain the classical expression for covariant differentiation:

$$g^k{}_{,1} = (\nabla_V W)^k = \frac{\partial g^k}{\partial x^1} + \Gamma^k_{1j} g^j.$$

Let $\alpha : I \longrightarrow M$ be a curve. A vector field $X(t)$ on α is said to be <u>parallel along α</u> if $\nabla_{\alpha_*}(X(t)) = 0$ for all $t \in I$.

Let U be the domain of a coordinate system, (x^i), such that $U \supset \alpha(J)$ where $J \subset I$. Writing $x^i(t) = x^i(\alpha(t))$ and $X(t) = X^i(t) \frac{\partial}{\partial x^i}$ we obtain the differential equations

$$\frac{dX^k}{dt} + \Gamma^k_{ij} X^j \frac{dx^i}{dt} = 0 \quad (k = 1, \ldots, n)$$

as conditions for X to be parallel along α.

α is a <u>geodesic</u> on M if $\nabla_{\alpha_*}\alpha_* = 0$. In a coordinate system this means that

$$\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0.$$

If ∇ is an affine connection on M we define the <u>torsion</u> and curvature tensor fields by

$$T(X,Y) = \nabla_X(Y) - \nabla_Y(X) - [X,Y],$$
$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Clearly $T \in D^1_2(M)$, that is, T is a tensor field of type (1,2). We also have $R \in D^1_3(M)$. If $\{e_i\}$ is a basis of vector fields in a domain U, then we define the components of T and R by

$$T(e_i,e_j) = T^k_{ij}\, e_k \text{ and}$$
$$R(e_i,e_j)e_k = R^h_{kij}\, e_h \,.$$

If $[e_i,e_j] = c^k_{ij}\, e_k$, it is easy to verify that

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} - c^k_{ij}\text{ , and}$$
$$R^h_{kij} = \frac{\partial \Gamma^h_{jk}}{\partial x^i} - \frac{\partial \Gamma^h_{ik}}{\partial x^j} + \Gamma^h_{im}\Gamma^m_{jk} - \Gamma^h_{jm}\Gamma^m_{ik} - c^m_{ij}\Gamma^h_{mk}.$$

Let M be a C^∞ manifold. M is a Riemannian manifold if there exists a tensor field $g \in D^0_2(M)$ satisfying

(i) $g(X,Y) = g(Y,X)$ for all $X,Y \in D^1(M)$.

(ii) g_p is a positive definite form on $M_p \times M_p$ for all $p \in M$. If in (ii) we replace 'positive definite' with 'non-degenerate' we have a pseudo-Riemannian manifold. Generally in this thesis we will understand 'Riemannian' to include 'pseudo-Riemannian'.

The fundamental theorem of Riemannian geometry states that on a Riemannian manifold there exists a unique symmetric connection, ∇, the Riemannian connection, satisfying:

(a) the torsion field is zero,

i.e., $[X,Y] = \nabla_X Y - \nabla_Y X$ for all $X,Y \in D^1(M)$;

(b) g is invariant under parallel translation,

i.e., $\nabla_X g = 0$ for all $X \in D^1(M)$ ([5], p.48)

∇ being symmetric means that $\Gamma^k_{ij} = \Gamma^k_{ji}$ $(i,j,k = 1, \ldots, n)$.
From (a) it follows that all the constants c^k_{ij} are zero.
If (x^i) is an allowable coordinate system on M with domain U, we define the components of g with respect to (x^i) by

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \text{ for } i,j = 1, \ldots, n.$$

Let M be a Riemannian manifold with metric g. The mapping $\phi: M \longrightarrow M$ is an isometry if

(i) ϕ is a diffeomorphism,

(ii) g is invariant under ϕ, $(\phi^* g = g)$

i.e., $g(X,Y) = g(\phi_* X, \phi_* Y)$ for all X,Y in $D^1(M)$.

1.2 The exponential map, normal coordinates and normal tensors

Let M be a C^∞ manifold with affine connection and p_o and arbitrary point of M. W, a neighbourhood of M, is said to be <u>simple convex</u> if for all distinct points p,q of W there exists a unique geodesic (pq) joining them and lying wholly in W. Let (x^i) be an allowable coordinate system on N_{p_o}, a simple convex open neighbourhood of p_o. We say that N_{p_o} is a <u>normal neighbourhood</u> of p_o. If M_{p_o} is the tangent space to M at p_o we define the mapping $Exp_{p_o}: U \longrightarrow N_{p_o}$ (U being an open neighbourhood of $\underline{0}$ in M_{p_o}) as follows:

Let $p \in N_{p_o}$ and α be the unique geodesic parameterised so that $\alpha(0) = p_o$ and $\alpha(1) = p$. If $\alpha_*(0) = X$ then the mapping

$X \longrightarrow p$ is the exponential mapping at p_o and $Exp_{p_o} X = p$.
U is then the domain of Exp (= Exp_{p_o}) withthe required range N_{p_o}.
$Exp : U \longrightarrow N_{p_o}$ is a diffeomorphism. As $\left\{\frac{\partial}{\partial x^i}(p_o)\right\}$ is the basis of M_{p_o} resulting from (x^i) we have $X = y^i \frac{\partial}{\partial x^i}(p_o)$ for all $X \in M_{p_o}$. The mapping $N_{p_o} \longrightarrow \mathbb{R}^n$ given by

$p = Exp(X) \longrightarrow (y^1, \ldots, y^n)$ assigns a normal coordinate system with origin p_o to N_{p_o}. The normal coordinate system (y^i) is derived from the allowable coordinate system (x^i).

Let ϕ be a diffeomorphism of M and $X \in D^1(M)$. We define another vector field X^ϕ by the assignment $p \longrightarrow (\phi_*)_{\phi^{-1}(p)}(X_{\phi^{-1}(p)})$. Suppose that for all vector fields $X, Y \in D^1(M)$ we have

$(\nabla_X Y)^\phi = \nabla_{X^\phi} Y^\phi$. ϕ is said to be an affine transformation of M with respect to ∇ and ∇ is invariant under ϕ.

If α is a geodesic and ϕ an affine transformation then $\nabla_{\alpha_*^\phi} \alpha_*^\phi = (\nabla_{\alpha_*} \alpha_*)^\phi = 0$, that is, $(\phi \circ \alpha)$ is also a geodesic. We deduce that ϕ commutes with Exp,

i.e., $(Exp_{\phi(p_o)} \circ \phi_*)X = (\phi \circ Exp_{p_o})X$ for all $X \in M_{p_o}$.

Further if $X(t)$ is parallel on α, then $(X(t))^\phi$ is parallel on $(\phi \circ \alpha)$.

Let U be a normal neighbourhood of $p_o \in M$ and let $X \neq 0$ be fixed in M_{p_o} so that $Exp\ X \in U$. The mapping $\sigma : I \longrightarrow U$ given by $t \longrightarrow Exp\ tX$ is clearly a geodesic through p_o, where I is an open neighbourhood of $\mathbb{R}$ such that $\sigma(I) \subset U$. If $\{y^i\}$

is a normal coordinate system origin p_o derived from the system $\{x^i\}$ on U and $X = \lambda^i \frac{\partial}{\partial x^i}(p_o)$, we have

$$y^j(\sigma(tX)) = y^j(\mathrm{Exp}\ t\lambda^i \frac{\partial}{\partial x^i}(p_o)) = t\lambda^j.$$ Hence in

normal coordinates geodesics through p_o are of the form

(1) $$y^i(t) = t\lambda^i.$$

Now $\frac{dy^i}{dt}(0) = \lambda^i$, but $\{\lambda^i\}$ being components of X for the basis $\left\{\frac{\partial}{\partial x^i}(p_o)\right\}$ of M_{p_o}, it follows that $\left(\frac{dy^i}{dt}\right)_o = \left(\frac{dx^i}{dt}\right)_o$. Hence entries of the transformation Jacobian of M_{p_o} are

(2) $$\left(\frac{\partial x^i}{\partial y^j}\right)_o = \delta^i_j.$$

Let $X \in M_{p_o}$ be such that $p = \mathrm{Exp}\ X \in N_{p_o}$. Then the <u>differential map</u>, $(d\ \mathrm{Exp})_X : (M_{p_o})_X \longrightarrow M_p$ can be written as a linear map $M_{p_o} \longrightarrow M_p$ by identifying the tangent space at a point of a vector space with the vector space itself. Suppose that t is the matrix of this map with respect to given bases. The following conventional notation will be used to denote the entries of this matrix ([16], p.1052):

Greek suffices will be used for components of elements of M_{p_o}, $^*M_{p_o}$ and tensor products of these spaces (vectors and tensors "at p_o"); Roman suffices will indicate components of vectors and tensors "at p". So with respect to the bases $\left\{\frac{\partial}{\partial x^\alpha}\right\}$ of M_{p_o} and $\left\{\frac{\partial}{\partial x^i}\right\}$ of M_p $(\alpha, i = 1, \ldots, n)$ we write $t = (t^i_\alpha)$.

t is also the matrix of the dual map, $(^*\mathrm{Exp})_X : {}^*M_p \longrightarrow {}^*M_{p_o}$ with respect to corresponding bases $\{dx^i\}$ and $\{dx^\alpha\}$. For if $u = (u^i_\alpha)$

is the matrix of the dual map and we choose $\theta = (\theta^i) \in {}^*M_p$ and $Y = (Y^\alpha) \in M_{p_o}$, the relation

$$(({}^*\mathrm{Exp})_X(\theta))(Y) = \theta((d\,\mathrm{Exp})_X(Y)), \text{ gives us}$$

$u^i_\alpha \theta_i Y^\alpha = \theta_i t^i_\alpha Y^\alpha$. Taking θ and Y as basis elements we obtain $u = t$.

The map $\mathrm{Exp}: U \longrightarrow N_{p_o}$ being a diffeomorphism, it follows from the inverse function theorem for manifolds that $(d\,\mathrm{Exp})_X$ is a linear isomorphism. Hence t has an inverse, $t^{-1} = (t^{-\alpha}_{\ i})$.

Let U be a normal neighbourhood of $p_o \in M$ and (y^i) the normal coordinate system origin p_o and domain U. The affine connection ${}^*\nabla$ has coefficients ${}^*\Gamma^i_{jk}$. From (1) we know that the solution of the differential equations for geodesics

$$(3) \qquad \frac{d^2y^i}{dt^2} + {}^*\Gamma^i_{jk}\frac{dy^j}{dt}\frac{dy^k}{dt} = 0,$$

is $y^i = \lambda^i t$ (λ^i fixed, $i = 1, \ldots, n$). Hence we have at p_o: $({}^*\Gamma^i_{jk})_o \lambda^j \lambda^k = 0$. As this holds for all geodesics through p_o, we deduce

$$(4) \qquad ({}^*\Gamma^i_{jk})_o = 0.$$

Let $T^{ij\ldots}_{pq\ldots}$ be components of any tensor field with respect to an allowable coordinate system (x^i) and ${}^*T^{ij\ldots}_{pq\ldots}$ its components with respect to the derived normal coordinate system (y^i). Then using (2) we have

$$(5) \qquad (T^{ij\ldots}_{pq\ldots})_o = ({}^*T^{ij\ldots}_{pq\ldots})_o.$$

<u>Affine normal tensors</u> are defined at p_o as follows:

$$(6) \qquad (A^i_{jkl\ldots p})_o = (\partial_p \ldots \partial_l {}^*\Gamma^i_{jk})_o, \text{ where } \partial_m = \frac{\partial}{\partial y^m}.$$

Note that the connection coefficients, ${}^*\Gamma^i_{jk}$ are components of a tensor (unlike Γ^i_{jk}). For let (x^i), (x'^i) be allowable coordinate systems with domains U, U' and (y^i), (y'^i) the corresponding normal coordinates origin p_o, where $p_o \in U \cap U'$. On a geodesic through p_o we have $\frac{dy^k}{dt} = (\frac{dx^k}{dt})_o$ and $\frac{dy'^b}{dt} = (\frac{dx'^b}{dt})_o$, whence we have $y^k = (\frac{\partial x^k}{\partial x'^b})_o y'^b$. Hence $\frac{\partial^2 y^k}{\partial y'^b \partial y'^c} = 0$ and from 1.1(6) we obtain

$$(7) \qquad {}^{\circ}\Gamma'^a_{bc} = \frac{\partial y'^a}{\partial y^i}\frac{\partial y^j}{\partial y'^b}\frac{\partial y^k}{\partial y'^c}\,{}^{\circ}\Gamma^i_{jk}.$$

Repeated differentiation of (7) and evaluation at p_o shows that we are justified in asserting that $(A^i_{jkl\ldots p})_o$ is a tensor.

Suppose now that M is an n-dimensional Riemannian manifold and (y^i) a normal coordinate system origin p_o in a normal neighbourhood W. Writing ${}^{\circ}g = ({}^{\circ}g_{ij})$ for the metric tensor, we have $ds^2 = |{}^{\circ}g_{ij}dy^i dy^j|$. The differential equations for geodesics are given by

$$\frac{d^2y^i}{ds^2} + {}^{\circ}\Gamma^i_{jk}\frac{dy^j}{ds}\frac{dy^k}{ds} = 0,$$ where the connection coefficients are Christoffel symbols. It follows that along any geodesic ${}^{\circ}g_{jk}\frac{dy^j}{ds}\frac{dy^k}{ds} = e$, where e is the indicator of the geodesic.

A necessary and sufficient condition that (y^i) be a normal coordinate system is

$$(8) \qquad {}^{\circ}g_{ij}y^j = ({}^{\circ}g_{ij})_o y^j \qquad ([11], \text{pp.}11,12).$$

We define <u>Riemannian normal tensors</u> at p_o by $({}^{\circ}g^{ij}_{\ \ .kl..p})_o = (\partial_p\ldots\partial_l\partial_k{}^{\circ}g^{ij})_o$, where $({}^{\circ}g^{ij})$ is the inverse of g. Using a method outlined by O.Veblen [13] we will obtain relationships between Riemannian and affine normal tensors.

We first note the relation

(9) $A^i_{jkl} = \frac{1}{3}(R^i_{jkl} + R^i_{klj})$, where (R^i_{jkl}) is the curvature tensor. ${}^\circ g^{-1} = ({}^\circ g^{ij})$ being invariant under parallel translation, we have

(10) $${}^\circ g^{ij}{}_{,k} = \partial_k {}^\circ g^{ij} + {}^\circ\Gamma^i_{hk} {}^\circ g^{hj} + {}^\circ\Gamma^j_{hk} {}^\circ g^{ih} = 0.$$

Using (4) and (10) we have

(11) $$({}^\circ g^{ij}_{\circ k})_o = 0.$$

Differentiating (10) partially with respect to y^l, we obtain

(12) $$\partial_l \partial_k {}^\circ g^{ij} + (\partial_l {}^\circ\Gamma^i_{hk}) {}^\circ g^{hj} + {}^\circ\Gamma^i_{hk}(\partial_l {}^\circ g^{hj}) + (\partial_l {}^\circ\Gamma^j_{hk}) {}^\circ g^{ih} + {}^\circ\Gamma^j_{hk}(\partial_l {}^\circ g^{ih}) = 0,$$ and

(13) $$({}^\circ g^{ij}_{\circ kl})_o = -(A^i_{hkl} {}^\circ g^{hj} + A^j_{hkl} {}^\circ g^{ih})_o.$$

Using (9),

(14) $$({}^\circ g_{ij})_o ({}^\circ g^{ij}_{\circ kl})_o = -\tfrac{2}{3}(R_{kl})_o,$$

where $R_{kl} = {}^\circ g^{hm} {}^\circ g_{ih} R^h_{jkl}$ are components of the <u>Ricci tensor</u> of type (0,2). We continue differentiating partially with respect to y^m, y^n, ..., evaluating at p_o, contracting and using relations obtained earlier, thus getting a sequence of relations between Riemannian and affine normal tensors at p_o like (14) above. The next two relations are

(15) $$({}^\circ g_{ij})_o ({}^\circ g^{ij}_{\circ klm})_o = -\tfrac{2}{3}(R_{kl,m})_o,$$ and

(16) $$({}^\circ g_{ij})_o ({}^\circ g^{ij}_{\circ klmn})_o = 4\, S'(A^p_{hkl})_o (A^h_{pmn})_o - 2(A^h_{hklmn})_o + 4\, S'(A^j_{pmn})_o (A^i_{hkl})_o ({}^\circ g_{ij})_o ({}^\circ g^{hp})_o,$$

where S' denotes summation over the three terms: subscript l for the first normal tensor in the product, m,n for second (as above); m for first, l,n for second; n for first, l,m for second.

1.3 <u>The distance function and Ruse's invariant</u>

Let M be a Riemannian manifold and W a normal neighbourhood. Let $p_o, p \in W$. We define the distance function, $\Omega : W \times W \longrightarrow \mathbb{R}$ by

(1) $$\Omega(p_o, p) = \tfrac{1}{2} e r^2,$$

where $r = d(p_o, p)$ is the length of the unique geodesic arc $(p_o p)$ and e is the indicator of the metric. Clearly Ω is a symmetric function and being defined independently of any coordinate system is a <u>two-point invariant function</u>. If $(p_o p)$ is non-null and $p = \mathrm{Exp}_{p_o} sX$, where $X \in M_{p_o}$ is a unit vector, then we have $r = |s|$. Let (y^i) be the coordinates of p with respect to a normal coordinate system centre p_o and $X = X^i \frac{\partial}{\partial y^i}(p_o)$.

Then $y^i = X^i s$ and $({}^{*}g_{ij})_o X^i X^j = e$.

Hence we obtain

(2) $$\Omega = \tfrac{1}{2} ({}^{*}g_{ij})_o y^i y^j.$$

Differentiating, $$\Omega_j \equiv \frac{\partial \Omega}{\partial y^j} = ({}^{*}g_{ij})_o y^i$$
$$= {}^{*}g_{ij}\, y^i, \text{ using 1.2(8).}$$

We deduce,

(3) $${}^{*}g^{ij}\Omega_j = y^i = X^i s,$$

that is, $${}^{*}g^{ij}\Omega_j = s \frac{dy^i}{ds},$$

where d/ds denotes differentiation along the geodesic arc $(p_o p)$.

(3) being a tensor equation, we have

(4) $$g^{ij}\Omega_j = s \frac{dx^i}{ds},$$ where (x^i) is any coordinate system in W. (4) is derived by considering p_o fixed and varying the point with coordinates (x^i), namely p. If however we fix

p and vary the point with coordinates (x^α), namely p_o, the symmetry of Ω gives us

(5) $$g^{\alpha\beta}\Omega_\beta = -s\frac{dx^\alpha}{ds} .$$

The negative sign follows since differentiation along the geodesic arc (pp_o) is $-\frac{d}{ds}$. In the case of normal coordinates origin p_o we have

(6) $$y^\alpha = -{}^{o}g^{\alpha\beta}\,{}^{o}\Omega_\beta = -{}^{o}\Omega_\alpha ,$$

where ${}^{o}\Omega_\beta = \dfrac{\partial\Omega}{\partial y^\beta}$.

(The distinction between coordinates (y^i) and (y^α) must be clarified. In the case of allowable coordinates on W with fixed origin, (x^α) and (x^i) are the coordinates of the variable points p_o and p respectively. However with normal coordinates origin p_o, (y^α) and (y^i) are both coordinates of p, but are used to distinguish the respective cases:

(i) p_o varying and p fixed,

(ii) p_o fixed and p varying).

Now $\Omega(p_o,p)$ being a function of the coordinates (x^α) of p_o and (x^i) of p we can obtain an n x n matrix with entries

$$\Omega_{\alpha i} = \frac{\partial^2\Omega}{\partial x^\alpha \partial x^i} .$$

Define $J = \det(\Omega_{\alpha i})$ and its modulus by $|J|$.

The <u>discriminant function</u> or <u>Ruse's invariant</u> is

(7) $$\rho(p_o,p) = \frac{\sqrt{g g_o}}{|J|} ,$$

where $g = \det(g_{ij})$ and $g_o = \det(g_{\alpha\beta})$. It is immediately evident that $\rho > 0$ ([11], p.18). $\rho(p_o,p)$ is a two-point invariant function symmetric in p_o and p. We obtain a simpler form for ρ in terms of normal coordinates as follows:

$$g^{\alpha\beta}\Omega_{\beta i} = -\frac{\partial y^\alpha}{\partial x^i}, \text{ from (5),}$$

whence

$$J = g_o(-1)^n \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}. \tag{8}$$

But from the transformation formula for tensors,

$$g = {}^*g\left(\frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}\right)^2 \text{ and so}$$

$\rho = \sqrt{\dfrac{gg_o}{J^2}} = \sqrt{\dfrac{{}^*g}{g_o}}$. As ${}^*g = g$ at p_o, we have

$$\rho = \sqrt{\frac{{}^*g}{{}^*g_o}}. \tag{9}$$

<u>Notes</u> (i) Equation (8) shows that $(\Omega_{\alpha i})$ is non-singular, an assumption made in the definition of ρ.

(ii) From (9) is deduced

$$\rho \to 1 \text{ as } p \to p_o. \tag{10}$$

We now derive a connection between the Laplacian $\Delta_2\Omega$ and ρ for fixed p_o and variable p. Using $\Delta_2\Omega = {}^*\Omega^i_{,i}$ and (3) we have

$$(11) \qquad {}^*\Omega^i_{,j} = \delta^i_j + {}^*\Gamma^i_{jk}y^k \text{ and}$$

$$\Delta_2\Omega = n + y^k\frac{\partial}{\partial y^k}\log\sqrt{{}^*g}$$

$$= n + {}^*\Omega^k\frac{\partial}{\partial y^k}\log\rho.$$

In any allowable coordinate system (x^i) this is

$$\Delta_2\Omega = n + \Omega^k\frac{\partial}{\partial x^k}\log\rho. \tag{12}$$

We note that $\Delta_2\Omega \to n$ as $p \to p_o$.

1.4 Definition of harmonic Riemannian manifolds

We give three equivalent definitions for harmonic Riemannian manifolds. Let M be an analytic Riemannian manifold and $p_o \in M$. W is a normal neighbourhood centre p_o. Then M is a harmonic Riemannian manifold if independently of p_o either

(i) there exists a non-constant solution of Laplace's equation, $\Delta_2 u = 0$, in W which is a function of $\Omega = \Omega(p_o,p)$ but otherwise is independent of $p \in W$; or

(ii) $\Delta_2 \Omega$ is a function of $\Omega = \Omega(p_o,p)$ but is otherwise independent of $p \in W$; or

(iii) $\rho = \rho(p_o,p)$ is a function of Ω, but is otherwise independent of p.

If M is harmonic, the solution of Laplace's equation $u = \psi(\Omega)$ is known as the elementary function and the function $\Delta_2\Omega = \chi(\Omega)$ is known as the characteristic function. See [11], pp. 35-40 for the equivalence of the three definitions. In establishing this equivalence, the following relations between the elementary function, the characteristic function and Ruse's invariant are derived for an n-dimensional harmonic space:

(1) $$\psi(\Omega) = A\int_a^{\Omega} \frac{d\omega}{|\omega|^{n/2}\rho(\omega)} + B,$$

where a,A,B are arbitary constants,

(2) $$\chi(\Omega) = n + 2\Omega\frac{d}{d\Omega}\log\rho(\Omega), \text{ and}$$

(3) $$\rho(\Omega) = \exp\int_0^{\Omega}\frac{\chi(\omega) - n}{2\omega}\,d\omega.$$

By virtue of the symmetric property of ρ, namely $\rho(p,p_o) = \rho(p_o,p)$

we deduce that the three functions $\psi(\Omega)$, $\chi(\Omega)$ and $\rho(\Omega)$ are independent of base point, p_o. As an example of harmonic spaces, it can be shown that all Riemannian <u>spaces of constant</u> curvature are harmonic ([11] , pp. 26-30). The characteristic function and Ruse's invariant are given by

(4) $$\chi(\Omega) = 1 + (n-1)\sqrt{2K\Omega}\cot\sqrt{2K\Omega},$$

(5) $$\rho(\Omega) = \left(\frac{\sin^2\sqrt{2K\Omega}}{2K\Omega}\right)^{\frac{n-1}{2}},$$

where K is the curvature of M and n the dimension.

A special case of harmonic manifolds occurs when the characteristic function and hence also Ruse's invariant are constant - the <u>simply harmonic manifolds</u>. Again three equivalent definitions are given. Let M be any n-dimensional harmonic space. Then M is simply harmonic if either

(i) the elementary function $\psi(\Omega)$ is given by

$$\psi(\Omega) = \begin{cases} A/|\Omega|^{\frac{n-1}{2}} + B & (n > 2) \\ A \log\Omega + B & (n = 2), \end{cases}$$

where A,B are arbitrary constants, or

(ii) the characteristic function, $\chi(\Omega)$, is constant, namely

$$\chi(\Omega) = n,$$ or

(iii) Ruse's invariant, $\rho(\Omega)$, is constant, namely

$$\rho(\Omega) = 1.$$

1.5 Conditions for harmonic manifolds

Two sets of conditions have been derived for a Riemannian manifold to be harmonic.

I Copson and Ruse equations ([11] 2.5). These equations were first obtained without proof in 1940, but the proof was given by A. Lichnerowicz in 1944 [8]. Relations between affine normal tensors and metric tensors are derived as follows:

Let M be any analytic Riemannian manifold and $p_o \in M$. N_{p_o} is a mormal neighbourhood of p_o with normal coordinate system (y^i). $({}^*g_{ij})$ is the metric tensor and $({}^*\Gamma^i_{jk})$ the Christoffel symbols. From 1.3(11) we deduce

$$\Delta_2\Omega = \Omega^i_{,i} = n + {}^*\Gamma^i_{ik}y^k.$$

M being analytic we can expand ${}^*\Gamma^i_{ik}$ in a Maclaurin series. Using the definition of affine normal tensors this becomes

$$\Delta_2\Omega = n + \sum_{r=1}^{\infty}\frac{1}{r!}(A^i_{ijk_1k_2\ldots k_r})_o y^j y^{k_1}\ldots y^{k_r},$$

since $(A^i_{ij})_o = 0$. Let $p \in N_{p_o}$ have coordinates (y^i) and $p = \text{Exp } Xs$, where $s = d(p_o,p)$ and $X = (X^i) \in M_{p_o}$ is a unit vector. We have

$$(1) \qquad \Delta_2\Omega = n + \sum_{r=1}^{\infty}\frac{s^r}{r!}(A^i_{ijk_1k_2\ldots k_r})_o X^j X^{k_1}\ldots X^{k_r}.$$

Suppose now that M is harmonic. Then $\Delta_2\Omega = \chi(\Omega)$ can be expanded in a neighbourhood of p_o in the Maclaurin series

$$\Delta_2\Omega = \sum_{t=0}^{\infty}\frac{1}{t!}\chi^{(t)}(0)\Omega^t$$

$$= \sum_{t=0}^{\infty}\frac{s^{2t}e^t}{2^t t!}\chi^{(t)}(0)\ s^{2r} \text{ for } |s| < \delta,$$

where the indicator, e, satisfies $({}^{\circ}g_{ij})_o X^i X^j = e$. Hence,

(2) $$\Delta_2 \Omega = \sum_{t=0}^{\infty} \frac{s^{2t}}{t!} \chi^{(t)}(0) ({}^{\circ}g_{jk_1})_o \ldots ({}^{\circ}g_{k_{2t-2}k_{2t-1}}) X^j \ldots X^{k_{2t-1}}.$$

From the uniqueness of the Maclaurin series we obtain the Copson and Ruse equations valid for all $p_o \in M$:

(3_1) $$A^i_{ijk} = 2\, h_1 {}^{\circ}g_{jk},$$

(3_2) $$A^i_{ijk_1k_2} = 0,$$

....

(3_{2t-1}) $$A^i_{ijk_1 \ldots k_{2t-1}} = h_t\, S({}^{\circ}g_{jk_1} {}^{\circ}g_{k_2k_3} \ldots {}^{\circ}g_{k_{2t-2}k_{2t-1}}),$$

(3_{2t}) $$A^i_{ijk_1 \ldots k_{2t}} = 0.$$

Here S denotes summation is taken over all permutations of the free indices and

(4) $$h_t = \frac{\chi^{(t)}(0)}{2^{t+1} t.t!}.$$

Now $\chi(\Omega)$ being independent of the base point p_o, it follows that $\chi^{(t)}(0)$ and hence all h_t are independent of p_o and are constants on M.

II Ledger's recurrence formula. These conditions first derived in 1954 by A.J.Ledger relate curvature and metric tensors in any allowablecoordinate system. An outline of the derivation of this formula is given below. For details see [11] 2.6.

W is a simple convex neighbourhood of a Riemannian manifold M and (x^i) any allowable coordinate system on W. For $p_o(x^i)_o$ fixed in W we consider the geodesic, σ, joining p_o

to p. For $|s| < \delta$ in W we have corresponding to (1) the Maclaurin series

(5) $$\Delta_2 \Omega = \frac{1}{r!} c_r s^r .$$

Here $c_r = (D^r \Delta_2 \Omega)_{s=0}$, where D is the absolute derivative along the geodesic .

If M is harmonic, comparison of (5) with (2) gives the conditions

(6) $$\partial^{2t}_{jk_1 \dots k_{2t-1}} c_{2t} = 2t(2t)!\, A^i_{ijk_1 \dots k_{2t-1}}$$
$$= 2t(2t)!\, h_t S(g_{jk_1} \dots g_{k_{2t-2}k_{2t-1}}),$$

(7) $$\partial^{2t+1}_{jk_1 \dots k_{2t}} c_{2t+1} = 0,$$

where $\partial^{2t}_{jk_1 \dots k_{2t-1}} = \partial^{2t} / \partial x^j \partial x^{k_1} \dots \partial x^{k_{2t-1}}$ and h_r is given by (4).

The following definitions are made:

Let (T^i_j) be any analytic tensor-field on W. We write $T_m = D^m(T^i_j)$ and $\operatorname{tr} T = T^i_i$. The matrices Λ and Π are defined by $\Lambda = (\Omega^i_{,j})$ and $\Pi = (\Pi^i_j) = (R^i_{klj} x^k x^l)$. Then we have

$$c_r = (D^r \Delta_2 \Omega)_o = (D^r \Omega^i_{,i})_o$$
$$= (D^r \operatorname{tr} \Lambda)_o = \operatorname{tr} (\Lambda_r)_o ,$$

since trace and absolute derivative commute. Now $\operatorname{tr} \Pi = R_{kl} x^k x^l$, where (R_{kl}) is the Ricci tensor. We differentiate the relation $\Omega^k \Omega_{,k} = 2\Omega$ twice covariantly, first with respect to x^i, then with respect to x^j, use the Ricci identity and obtain

$$s\Lambda_1 = s^r\Pi - \Lambda^2 + \Lambda.$$

Applying the operator D^r and using Leibniz's theorem we have

$$s\Lambda_{r+1} + r\Lambda_r = s^2\Pi_r + 2rs\,\Pi_{r-1} + r(r-1)$$
$$- \sum_{q=0}^{r}\binom{r}{q}\Lambda_q\Lambda_{r-q} + \Lambda_r.$$

Evaluation for $s = 0$ and the particular case $r = 1$ gives $(\Lambda_1)_0 = 0$.

Hence <u>Ledger's recurrence formula</u> holds for every point

$p_0 \in M$:

(8) $$(r+1)\Lambda_r = r(r-1)\Pi_{r-2} - \sum_{q=2}^{r-2}\binom{r}{q}\Lambda_q\Lambda_{r-q} \quad (r \geqslant 2).$$

To obtain the curvature conditions for a harmonic manifold we put $r = 2, 3, \ldots$ in (8).

For $r = 2$ in (8) we have $\Lambda_2 = \frac{2}{3}\Pi$, whence

$$c_2 = \operatorname{tr}\Lambda_2 = \tfrac{2}{3}\operatorname{tr}\Pi = \tfrac{2}{3}R_{jk}X^jX^k.$$

Substituting $t = 1$ in (6) gives $\frac{4}{3}R_{jk} = 8h_1 g_{jk}$,

i.e.,

(9_1) $$R_{jk} = k_1 g_{jk}, \text{ where } k_1 = 6h_1.$$

For $r = 3$ in (8) we have $\Lambda_3 = \frac{3}{2}\Pi_1$, whence

$$c_3 = \operatorname{tr}\Lambda_3 = \tfrac{3}{2}\operatorname{tr}\Pi_1 = \tfrac{3}{2}R_{jk_1,k_2}X^jX^{k_1}X^{k_2}.$$

Substituting $t = 1$ in (7) and changing notation gives

(9_2) $$R_{ij,k} + R_{jk,i} + R_{ki,j} = 0.$$

Similarly from $r = 4$ in (8) and $t = 2$ in (6) we have

(9_3) $$S(R^p_{ijq}R^q_{klp}) = k_2\,S(g_{ij}g_{kl}).$$

Also

(9_4) $$S(R_{ij,klm} - R^p_{ijq}R^q_{klp,m}) = 0 \quad (r = 5,\ t = 2 \text{ in}(7))$$ and

(9_5) $$S(32R^p{}_{ijq}\, R^q{}_{klr}\, R^r{}_{mnp} + 9R^p{}_{ijq,m}\, R^q{}_{klp,n})$$
$$= k_3 S(g_{ij} g_{kl} g_{mn}) \quad (r = 6,\ t = 3 \text{ in } (6)).$$

Further conditions, (9_6), (9_7), ... can be derived but the calculation becomes progressively more involved.

1.6 Properties of harmonic spaces

In this section theorems on harmonic spaces will be stated mainly without proof.

The mean value theorems for harmonic functions in harmonic spaces of positive-definite metric were published by T.J.Willmore in 1950.

Theorem 1 Let M be a harmonic space of positive-definite metric and u be a function harmonic in a neighbourhood U. Let $p_o \in U$ and $S(p_o;r)$ be any geodesic sphere centre p_o and radius $r > 0$ such that $S(p_o;r) \subset U$. Then if $\mu(u;p_o;r)$ is the mean value of u over $S = S(p_o;r)$ given by

$$\mu(u;p_o;r) = \int_S u\, dv_{n-1} \Big/ \int_S dv_{n-1},$$

where dv_{n-1} is the volume element in S, we have

$$\mu(u;\ p_o;\ r) = u(p_o).$$

There are two forms of converse of Theorem 1.

Theorem 2 Let M be a harmonic space of positive-definite metric and u be a function of class 2 on a neighbourhood U such that for all $p_o \in U$, $\mu(u;\ p_o;\ r) = u(p_o)$, then u is a harmonic function in U.

<u>Theorem 3</u> Let M be an analytic Riemannian manifold with positive-definite metric. Suppose that for every function u harmonic in any neighbourhood U of M we have $\mu(u;\ p_0;\ r) = u(p_0)$ then M is a harmonic manifold.

For proofs of Theorems 1 to 3 see [11] 2.4.

<u>Theorem 4</u> All harmonic manifolds are Einstein manifolds.
<u>Proof</u> See 1.5 (9_1).

<u>Definitions</u> (1) A Riemannian manifold is <u>conformally flat</u> if it is locally conformal to a flat manifold.

(2) A Riemannian manifold is of <u>normal hyperbolic metric</u> if the signature of its fundamental quadratic form is $\pm (n - 2)$, where n is the dimension of the manifold. (Here <u>signature</u> is defined as the number of positive minus the number of negative terms when the quadratic form is diagonalised).

<u>Theorem 5</u> All harmonic manifolds of dimension 2 or 3 are of constant curvature.

<u>Proof</u> Let M be a two-dimensional harmonic manifold. Then the curvature is given by

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}g_{21}} = \frac{R_{1212}}{g} = -\frac{R_{ij}}{g_{ij}},$$

since $g\,R_{ij} = -g_{ij}R_{1212}$. M being an Einstein manifold we deduce immediately that K is constant on M

It can also be shown that all three-dimensional Einstein spaces and all n-dimensional conformally flat Einstein spaces are of constant curvature. This will complete the proof of Theorem 5 and give:

Theorem 6 All conformally flat harmonic manifolds are of constant curvature.

Theorem 7 All harmonic manifolds with normal hyperbolic metric are manifolds of constant curvature.

This theorem was first proved by A.Lichnerowicz and A.G.Walker in 1945; the proof can be found in [11] pp. 68-71.

As a corollary to Theorems 5,6 and 7 we have:

Theorem 8 If a simply harmonic manifold has one of the following properties, then it is locally flat:

(i) it has dimension 2 or 3, or

(ii) it is conformally flat, or

(iii) it is of normal hyperbolic metric.

Theorem 9 Every simply harmonic manifold of positive-definite metric is locally flat.

Proof See [11] p.71.

The set of harmonic manifolds contains the set of manifolds of constant curvature as a subset. It is itself a proper subset of the set of Einstein manifolds. That it is a proper subset can be shown by the following example.

Example Let M be the 4-dimensional Riemannian manifold with metric

$$ds^2 = 2\,du\,dv + 2\,R_e(e^{2iku}z^2)du^2 + |dz|^2,$$

where $z = x + iy$, k is a constant and $R_e(\ )$ means 'real part of'. Writing $x^1 = x$, $x^2 = y$, $x^3 = u$, $x^4 = v$, the metric tensor has matrix

$$(g_{ij}) = \begin{pmatrix} I_2 & O_2 \\ O_2 & A \end{pmatrix},$$ where I_2, O_2 are the identity

and zero matrices of order 2 respectively and $A = \begin{pmatrix} 2f & 1 \\ 1 & 0 \end{pmatrix}$,

where $f = f(x^1,x^2,x^3) = ((x^1)^2 - (x^2)^2)\cos 2kx^3 - 2\,x^1x^2 \sin 2kx^3$.

M is of normal hyperbolic metric; however it is not of constant curvature. (For example:

$$R_{2313} = 2 \sin 2kx^3, \quad g_{21}g_{33} - g_{23}g_{31} = 0, \text{ but}$$
$$R_{2323} = 2 \cos 2kx^3, \quad g_{22}g_{33} - g_{23}g_{32} = 2\,f(x^1,x^2,x^3).)$$

From Theorem 7 we deduce that M is not harmonic.

It is easy to verify that the Ricci tensor is identically zero and hence M is an Einstein manifold which not harmonic.

1.7 Decomposable harmonic spaces

Let M_1 and M_2 be Riemannian manifolds of dimension m and m' respectively. Let $n = m + m'$ and consider the product space $M = M_1 \times M_2$. This is given a Riemannian structure as follows:

(i) Let $p = (q,r) \in M$. If N_q, N_r are neighbourhoods of q,r respectively in M_1, M_2 respectively, then $N_p = N_q \times N_r$ is a neighbourhood of p in M.

(ii) For each point p in the topological space M, a coordinate system (x^i) can be given which can be considered as the product of two systems of coordinates (x^{α}) in M_1 and $(x^{\alpha'})$ in M_2. Let $p = (q,r) \in M$. Then we use the following notational convention for coordinates:

p has coordinates (x^i) $\quad i = 1,2, \ldots, n;$

q has coordinates (x^{α}) $\quad \alpha = 1,2, \ldots, m;$

r has coordinates $(x^{\alpha'})$ $\quad \alpha' = 1,2, \ldots, m'$

or $\alpha' = m+1, m+2, \ldots, n$.

The coordinate system (x^i) on M is said to be decomposable.

(iii) M is given the metric

$$ds^2 = g_{ij}dx^i dx^j,$$

that is,

$$ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta} + g_{\alpha'\beta'}dx^{\alpha'}dx^{\beta'}, \tag{1}$$

where the two sums on the right of (1) give the metrics on M_1 and M_2 respectively.

Definition A Riemannian manifold is decomposable if it is locally isometric to the product of two Riemannian manifolds.

Let $M = M_1 \times M_2$ be a decomposable Riemannian manifold and $T = (T^{ij\ldots k}_{rs\ldots t})$ be a tensor field on M. For any given component $T^{ij\ldots k}_{rs\ldots t}$ we can substitute a Greek letter which is either unprimed or primed. For example we can substitute α for i if $1 \leq i \leq m$ or α' for i if $m + 1 \leq i \leq n$. Hence the components of T can be partitioned into three classes:

(a) The first class of T is the set of components for which every substitution is unprimed.

(b) The second class of T is the set of components for which every substitution is primed.

(c) The <u>mixed class of T</u> is the set of components for which some substitutions are unprimed and some are primed. Clearly the three classes are invariant under a decomposable coordinate transformation. In particular if one of the three classes is empty in one decomposable coordinate system, it is empty in all decomposable coordinate systems.

<u>Definition</u> A tensor field T on $M = M_1 \times M_2$ is <u>seperable</u> if in a decomposable coordinate system the mixed class of T is empty. T is <u>decomposable</u> if it is seperable and its first class depends only on variables of M_1 and its second class only on variables of M_2.

If T is decomposable each non-empty class defines a tensor field on M_1 and M_2. The sum, contraction and contracted product of decomposable tensor fields are decomposable. In particular the metric, curvature, Ricci and affine normal tensor fields are decomposable.

<u>Theorem 10</u> Every decomposable harmonic manifold is simply harmonic.

<u>Proof</u> This is a modification of the proof of A.Lichnerowicz [8].

Let $M = M_1 \times M_2$ be a decomposable harmonic manifold. Let (y^i) be a normal coordinate system origin p_o, which is decomposable into the normal coordinate systems, (y^α) origin q_o and $(y^{\alpha'})$ origin r_o. M being harmonic the Copson and Ruse equations are valid and using the notation of 1.5 we have for $t \geqslant 1$

(2) $$A^i_{ijk_1 \ldots k_{2t-1}} = h_t \, S({}^*g_{jk_1} {}^*g_{k_2 k_3} \ldots {}^*g_{k_{2t-2} k_{2t-1}}).$$

Now let $t \geqslant 2$ and suppose that we substitute a positive even number of both unprimed and primed Greek letters for the 2t letters $j, k_1, \ldots, k_{2t-1}$. Normal tensors being decomposable, the component on the left of (2) is zero, but the factor of h_t on the right is not identically zero. For example, if $t = 2$,

$$A^i_{i\alpha\beta\alpha'\beta'} = 4h_2 \left({}^\circ g_{\alpha\beta}{}^\circ g_{\alpha'\beta'} + {}^\circ g_{\alpha\alpha'}{}^\circ g_{\beta\beta'} + {}^\circ g_{\alpha\beta'}{}^\circ g_{\beta\alpha'}\right),$$

which implies that $h_2 \, {}^\circ g_{\alpha\beta}{}^\circ g_{\alpha'\beta'} = 0 \qquad \alpha, \beta = 1, \ldots, m$

$$\alpha', \beta' = m+1, \ldots, n.$$

We deduce that $h_t = 0$ for $t \geqslant 2$ and hence that

(3) $\qquad \chi^{(t)}(0) = 0 \quad (t \geqslant 2)$, where $\chi(\Omega)$ is the characteristic function of M.

Lichnerowicz derived an (incorrect) inequality relating $\chi^{(1)}(0)$ and $\chi^{(2)}(0)$. The correct version due to T.J.Willmore is

(4) $$(\chi^{(1)}(0))^2 \leqslant -\tfrac{5}{2}\chi^{(2)}(0)(n-1).$$

From (3) and (4) we deduce that $\chi^{(t)}(0) = 0$ for $t \geqslant 1$ and hence that $\chi(\Omega) = n$.

Q.e.d.

An alternative argument (see [11] pp. 214-216) cna be summarised as follows:

Consider a decomposable normal coordinate system on $M = M_1 \times M_2$ origin $p_o = (q_o, r_o)$ and let $p = (q, r)$ be in the corresponding normal neighbourhood of p_o. Let Ω, Ω_1, Ω_2 and ρ, ρ_1, ρ_2 denote the distance and discriminant functions in M, M_1 and M_2 respectively.

It is easy to verify

(5) $$\Omega = \Omega_1 + \Omega_2, \text{ and}$$

(6) $$\rho = \rho_1\rho_2.$$

Suppose now that M is harmonic so that ρ is essentially a function of Ω alone. From equations (5) and (6) it follows that this is possible if and only if

$$\rho_1 = \text{constant}, \rho_2 = \text{constant}.$$

An objection to the argument above runs as follows:

Suppose that there exists an m-dimensional Riemannian manifold, M^{*}_1, whose characteristic and discriminant functions are expressed in terms of the distance function, Ω_1, by

$$\chi_1(\Omega_1) = m + 2\,\Omega_1 \text{ and}$$
$$\rho_1(\Omega_1) = e^{\Omega_1}.$$

Suppose further that there exists an m'-dimensional Riemannian manifold, M^{*}_2, such that

$$\chi_2(\Omega_2) = m' + 2\,\Omega_2 \text{ and}$$
$$\rho_2(\Omega_2) = e^{\Omega_2}.$$

Clearly M^{*}_1 and M^{*}_2 are harmonic and so is their product, M^{*}. Further relations (5) and (6) are satisfied, but M^{*} is a decomposable harmonic manifold which is nor simply harmonic. It is not easy to see <u>'a priori'</u> why manifolds M^{*}_1 and M^{*}_2 with these characteristic and discriminant functions cannot exist.

CHAPTER II

SYMMETRIC SPACES

2.1 Isometry groups of Riemannian manifolds

Let M be any analytic Riemannian manifold. I(M), the set of all isometries of M, is a group under composition of mappings, known as the isometry group of M. We give I(M) a topology, $\mathcal{J}$, the compact-open topology, as follows ([5], p.167):

Let C and U be respectively compact and open subsets of M and let

$$W(C, U) = \{g: g(C) \subset U,\ g \in I(M)\}.$$

$\mathcal{J}$ is the smallest topology containing all the sets W(C, U); it has a countable basis, Ω, consisting of all finite intersections of sets of the form $W(\overline{O}_i, O_j)$, where $\{O_i\}$ is a countable basis of the topology of M, each O_i having compact closure. ([5], p.167 Lemma 2.1).

There are four fundamental properties of I(M):

(i) The group multiplication, I(M) X I(M) $\longrightarrow$ I(M) is continuous.

(ii) The inverse mapping, I(M) $\longrightarrow$ I(M) is continuous.

(iii) The group action, I(M) X M $\longrightarrow$ M is continuous.

(iv) I(M) is locally compact.

From (i) and (ii) we deduce that I(M) with the compact-open topology is a topological group ([3], p. 26). Properties (i) to (iv) can be proved using sequences (see [5], pp. 167-169). We will give alternative proofs for (i) and (iii).

<u>Proof of (i)</u> Let B be any member of Ω and define

$$B^* = \{(g,h): gh \in B, (g,h) \in I(M) \times I(M)\}.$$

Clearly B^* is non-empty. We show that B^* is open.

Choose $(g_o, h_o) \in B^*$. Now B being a basis element, there exists an integer, N, and sequences $\{i_1, \ldots, i_N\}$, $\{j_1, \ldots, j_N\}$ such that

$$B = \bigcap_{r=1}^{N} W(\overline{O}_{i_r}, O_{j_r}).$$

For each r, we have

$$h_o\overline{O}_{i_r} \subset g_o^{-1}O_{j_r}, \text{ i.e., } h_o\overline{O}_{i_r} \text{ and } (g_o^{-1}O_{j_r})' \text{ are}$$

disjoint closed sets. Now M being a metric space is normal. Hence there exist sequences of non-empty open sets H_r and K_r such that $\overline{H}_r \cap \overline{K}_r = \emptyset$, $H_r \supset h_o\overline{O}_{i_r}$ and $K_r \supset (g_o^{-1}O_{j_r})'$, $(r = 1, \ldots, N)$ (see, for example [6], p.41, Theorem 2-6).

$\overline{O}_{j_r}$ being compact and g_o a homeomorphism, it follows that $C_r = \overline{H}_r$ is compact.

Thus $h_o\overline{O}_{i_r} \subset H_r \subset C_r \subset g_o^{-1}O_{j_r}$ $(r = 1, \ldots, N)$.

Now define

$$U_1 = \bigcap_{r=1}^{N} W(C_r, O_{j_r}), \quad U_2 = \bigcap_{r=1}^{N} W(\overline{O}_{i_r}, H_r).$$

Clearly $(g_o, h_o) \in U_1 \times U_2$ and $U_1 \times U_2$ is open in the product topology. Further let $(g, h) \in U_1 \times U_2$. Then

$$gh(\overline{O}_{i_r}) \subset gH_r \subset gC_r \subset O_{j_r} \quad (r = 1, \ldots, N)$$

i.e.,
$$gh \in \bigcap_{r=1}^{N} W(\overline{O}_{i_r}, O_{j_r}) = B.$$

Hence, $U_1 \times U_2$ is an open subset of B^* and B^* is open.

Proof of (iii) We will show that the mapping $f:(g,p) \longrightarrow g(p)$ of $I(M) \times M$ into M is continuous.
Let U be open in M and choose any (g_o, p_o) in $f^{-1}(U)$.
Denoting by $N(p; s)$ the open sphere centre p and radius s, then there exists $r > 0$ such that $N(g_o(p_o); r) \subset U$. The sets

$$O_{g_o} = W(\{p_o\}, N(g_o(p_o); r/2)) \text{ and } N_{p_o} = N(p_o; r/2)$$

are open in $I(M)$ and M respectively.
Hence $O_{g_o} \times N_{p_o}$ is an open neighbourhood of (g_o, p_o) in $I(M) \times M$ and a subset of $f^{-1}(U)$, which proves the continuity of f.

To summarise: $I(M)$ is a locally compact topological transformation group of M. In general $I(M)$ is disconnected. We will denote by G the identity component of $I(M)$, that is, the component containing $e = id_M$.

2.2 Symmetric Spaces

Let M be a C^∞ manifold with affine connection ∇. Let $p_o \in M$ and N_{p_o} be a normal neighbourhood of p_o. For $p \in N_{p_o}$ let γ_p be the unique geodesic parameterised so that $\gamma_p(0) = p_o$ and $\gamma_p(1) = p$. Then if $q = \gamma_p(-1)$, s_{p_o} is the mapping $N_{p_o} \longrightarrow N_{p_o}$ given by $s_{p_o}(p) = q$. Alternatively s_{p_o}, the geodesic symmetry at p_o, is the mapping $Exp_{p_o}(X) \longrightarrow Exp_{p_o}(-X)$ for $X \in M_{p_o}$. Hence if (y^i) is a normal coordinate system with origin p_o,

$$s_{p_o} :(y^1, \ldots, y^n) \longrightarrow (-y^1, \ldots, -y^n).$$

Note that s_{p_o} is <u>involutive</u>, i.e., $s_{p_o}^2 = id_{N_{p_o}}$ but $s_{p_o} \neq id_{N_{p_o}}$.
We also have $(ds_{p_o})_{p_o} = - id_{M_{p_o}}$.

<u>Definition (1)</u> A C^∞ manifold with affine connection, M, is an <u>affine locally symmetric manifold</u> if for every point p_o of M, the geodesic symmetry, s_{p_o}, is an affine transformation of a normal neighbourhood of p_o.

Let M be an affine locally symmetric manifold and F be a tensor field of odd degree, i.e., F is of type (r,t), where (r + t) is of odd parity. Let $p_o \in M$ and let $X_1, \ldots, X_r \in M_{p_o}$ $\omega_1, \ldots, \omega_t \in {}^*M_{p_o}$ be arbitrary contravariant and covariant vectors. We have

$$F_{p_o}(X_1, \ldots, X_r, \omega_1, \ldots, \omega_t) = F_{p_o}(X_1^{s_{p_o}}, \ldots, X_r^{s_{p_o}}, \omega_1^{s_{p_o}}, \ldots, \omega_t^{s_{p_o}})$$
$$= F_{p_o}(-X_1, \ldots, -X_r, -\omega_1, \ldots, -\omega_t)$$
$$= (-1)^{r+t} F_{p_o}(X_1, \ldots, X_r, \omega_1, \ldots, \omega_t).$$

Hence F = 0 on M. In particular if T and R are the torsion and curvature tensor fields respectively, T and ∇R are of degree 3 and 5 respectively and hence T = 0 and $\nabla R = 0$.
The converse statement is also true ([5], pp.164-165).
Hence:

A C^∞ manifold with an affine connection is affine locally symmetric if and only if T = 0 and $\nabla R = 0$.

<u>Definition (2)</u> A C^∞ manifold with affine connection, M, is a (globally) <u>affine symmetric manifold</u> if for every point p_o of M, the geodesic symmetry, s_{p_o}, is an affine transformation of M.

These definitions extend naturally to Riemannian manifolds.

Definition (3) A Riemannian manifold, M, is a Riemannian locally symmetric manifold if for every point p_o of M, the geodesic symmetry, s_{p_o}, is an isometry of a normal neighbourhood of p_o.

All Riemannian locally symmetric spaces are affine locally symmetric, since isometries are affine transformations.

Definition (4) A Riemannian manifold, M, is a Riemannian globally symmetric manifold if every point p_o of M is an isolated fixed point of an involutive isometry of M.

Let M be a Riemannian globally symmetric manifold and σ_{p_o} be an involutive isometry with isolated fixed point p_o. We will show that $\sigma_{p_o} = s_{p_o}$ and hence that every Riemannian globally symmetric space is locally symmetric. This will also show that we could have equivalently have defined Riemannian globally symmetric spaces to be Riemannian manifolds in which all geodesic symmetries are global isometries of M, i.e., $s_{p_o} \in I(M)$ for all $p_o \in M$.

Proof that $\sigma_{p_o} = s_{p_o}$ Since p_o is an isolated fixed point of σ_{p_o}, there exists a normal neighbourhood, N_{p_o} of p_o, in which $\sigma_{p_o}(p) \neq p$ if $p \neq p_o$. Choose $p \neq p_o$ in N_{p_o} and suppose that $p = \mathrm{Exp}_{p_o} X$ $(X \in M_{p_o})$. Writing T for the linear endomorphism $(d\sigma_{p_o})_{p_o}$ of M_{p_o}, we see that $M_{p_o} = V_1 + V_2$ (direct sum), where $X_1 = \frac{1}{2}(X + T(X)) \in V_1$, $X_2 = \frac{1}{2}(X - T(X)) \in V_2$, and $X = X_1 + X_2$.

Suppose that $V_1 \neq 0$ and contains $X_1 \neq 0$. Then there exists an open neighbourhood (t_1, t_2) of 0 in $\mathbb{R}$ such that $\gamma_{X_1} = \{q : q = \mathrm{Exp}_{p_o}\, t\, X_1,\ t \in (t_1, t_2)\}$ is a subset of N_{p_o}. γ_{X_1} is a geodesic arc in N_{p_o} all of whose points are invariant under σ_{p_o} since $T(X_1) = X_1$, which is contrary to hypothesis. Hence $V_1 = 0$, $T(X) = -X$ and σ_{p_o} is the geodesic symmetry.

The converse of the result above is not generally true. However a Riemannian locally symmetric space is globally symmetric provided it is complete and simply connected ([5], p. 187). In the case of a Riemannian globally symmetric space, M, we have seen in 2.1 that the isometry group, I(M), is a locally compact topological transformation group of M. It can be shown further that I(M) can be given an analytic structure compatible with the open-compact topology in which it is a <u>Lie transformation group</u> ([5], p.171). It is also known that A(M), the group of affine transformations of a C^∞ manifold with affine connection, is a Lie transformation group ([5], p.229).

2.3 Isotropy subgroups and involutive automorphisms

Let M be an affine symmetric manifold and A(M) be the Lie group of affine transformations of M. A(M) is transitive on M. For let $p, q \in M$ and m be the mid-point of the unique geodesic arc (pq), i.e., if we parameterise (pq) so that $\gamma(0) = p$, $\gamma(1) = q$, then $\gamma(\frac{1}{2}) = m$. We have $s_m(p) = q$.

Now let G be the identity component of I(M). s_m is not necessarily in G, but we show that G is transitive on M. For, with m defined as above, let $T_m = s_m s_p$. Then $T_m(p) = q$. Letting q tend to p on the geodesic arc, we see that T_m and $e = id_M$ lie on a continuous arc in A(M) and hence that $T_m \in G$. The affine transformation, T_m, is called the <u>transvection with base (pq)</u>. Clearly the set of all transvections is a transitive subgroup of G.

Fix p_o in M. We define the <u>isotropy subgroups of A(M) and G at p_o</u>, $\bar{H}$ and H respectively as the subgroups of transformations leaving p_o invariant. The choice of p_o is immaterial. For let $p \in M$, $p \neq p_o$, and let $\bar{H}_p$ and H_p be the respective subgroups at p. The mapping $h \longrightarrow s_m h s_m$ establishes the isomorphisms $\bar{H}_{(p_o)} \cong \bar{H}_p$ and $H_{(p_o)} \cong H_p$, where m is the mid-point of $(p_o p)$. Clearly $\bar{H}$ and H are closed subgroups of A(M) and G respectively.

G being transitive on M and H a closed subgroup of G, we can write M = G/H by identifying the element $g(p_o)$ with the left coset gH in G/H. Hence all symmetric spaces are <u>homogeneous spaces</u>.

Let M = G/H be a symmetric space. Writing s for the geodesic symmetry, s_{p_o}, we consider the mapping $\sigma : A(M) \longrightarrow A(M)$ given by $\sigma(g) = sgs$. Clearly σ is an involutive (inner) automorphism of A(M). But by considering the curve $(\sigma \circ \gamma)$, where γ is a curve in G joining $g \in G$ to e, we deduce that σ is an automorphism of G. Let $G_\sigma = \{g : \sigma(g) = g, g \in G\}$

and $(G_\sigma)_e$ be its identity component. We have:

(a) $H \subset G_\sigma$ Given $h \in H$ and p in a normal neighbourhood of p_0 in M, since geodesics through p_0 are mapped into geodesics through p_0 (see p. 7), we obtain the relation

$$sh(p) = hs(p). \quad \text{Hence } h \in G_\sigma.$$

(b) $(G_\sigma)_e \subset H$ Given $g \in (G_\sigma)_e$, there exists a continuous curve $\gamma : [0,1] \longrightarrow G_\sigma$ parametrised so that $\gamma(0) = e$ and $\gamma(1) = g$. If τ is the action of G on M, $\Gamma = (\tau \circ \gamma)$ is a continuous curve in M joining p_0 to $p = g(p_0)$. If $t \in [0,1]$ we have

$$(s\gamma(t))(p_0) = (\gamma(t)s)(p_0), \text{ since } \gamma(t) \in G_\sigma$$
$$= (\gamma(t))(p_0).$$

Hence Γ is invariant under s. p_0 being an isolated fixed point of s implies that $g \in H$.

Combining (a) and (b) we have the result that H lies between G_σ and the identity component of G_σ. This motivates the following definition.

Definition Let G be a connected Lie group and H a closed subgroup of G. (G,H) is a symmetric pair if there exists an involutive automorphism, σ, such that $(G_\sigma)_e \subset H \subset G_\sigma$, where G_σ is the subgroup of G invariant under σ and $(G_\sigma)_e$ is the identity component of G_σ.

2.4 The Lie algebra of G

Let M = G/H be a blobally symmetric space and $e = id_M$. We make an algebra from the vector (tangent) space, G_e, as follows:

Let $X, Y \in G_e$. The left-invariant vector fields $\widetilde{X}$, $\widetilde{Y}$

on G are formed by left-translations: if $g \in G$, $\tilde{X}_g = dL_g(X)$, where $L_g : G \longrightarrow G$ is given by $L_g(a) = ga$. If $\tilde{Y}_g = dL_g$, then we define $[X,Y] = [\tilde{X},\tilde{Y}]_e = (\tilde{X}\tilde{Y} - \tilde{Y}\tilde{X})_e$. G_e with the bracket operation $(X,Y) \longrightarrow [X,Y]$ we denote by $\mathfrak{g}$, the Lie algebra of G. Alternatively, $\mathfrak{g}$ is the vector space of all left-invariant vector fields with the bracket given by $(\tilde{X},\tilde{Y}) \longrightarrow [\tilde{X},\tilde{Y}] = [\tilde{X},\tilde{Y}]_e$ (these vector fields are uniquely determined by the tangent vectors $\tilde{X}_e$, $\tilde{Y}_e$).

Let H be the isotropy subgroup of $\tilde{p}_0 \in M$. Then H is a Lie subgroup of G since it is a closed subgroup of G ([3], p.135). Further $\mathfrak{h}$, the Lie algebra of H, is a Lie subalgebra of ([5], p.102).

Let $X \in \mathfrak{g}$ be fixed $(X \neq 0)$. The set $\{tX : t \in \mathbb{R}\}$ is a one-parameter subalgebra of $\mathfrak{g}$ and is the tangent space at e to a one-dimensional submanifold of G; this submanifold is a curve $\gamma_X : \mathbb{R} \longrightarrow G$ and is a subgroup of G by the Lie subgroup-subalgebra correspondence. We parameterise so that γ_X is a group monomorphism, that is, $\gamma_X(t + t') = \gamma_X(t)\,\gamma_X(t')$ $(t,t' \in \mathbb{R})$, $\gamma_X(0) = e$. We define $d\gamma_X(0) = X$. The mapping $\exp : \mathfrak{g} \longrightarrow G$ given by $tX \longrightarrow \gamma_X(t)$ is the exponential map of $\mathfrak{g}$ in G. The subgroup $\{\exp tX : t \in \mathbb{R}\}$ is called the one-parameter subgroup corresponding to X.

Now let $\phi : K_1 \longrightarrow K_2$ be a homomorphism of Lie subgroups, K_1 and K_2, of G. A general theorem of Lie group theory states that if $\mathfrak{k}_1$, $\mathfrak{k}_2$ are the Lie algebras of K_1, K_2 then the linear map $(d\phi)_e : \mathfrak{k}_1 \longrightarrow \mathfrak{k}_2$ is a Lie algebra homomorphism ([3], p.113). Let $X \in \mathfrak{k}_1$ and let γ_X be the corresponding one-parameter subgroup $(X \neq 0)$. We have $\phi\gamma_X = \gamma_{(d\phi)_e X}$. Hence it follows:

$$\exp\,(d\phi)_e X = \gamma_{(d\phi)_e X}(1) = \phi\gamma_X(1) = \phi(\exp X).$$

We have shown that the diagram below is commutative.

$$(1) \qquad \begin{array}{ccc} \mathfrak{k}_1 & \longrightarrow & \mathfrak{k}_2 \\ \exp \downarrow & (d\phi)_e & \downarrow \exp \\ K_1 & \xrightarrow[\phi]{} & K_2 \end{array}$$

If A is any automorphism of G, then from (1) we have $A(\exp X) = \exp((dA)_e X)$ $(X \in \mathfrak{g})$ and $(dA)_e$ is an automorphism of $\mathfrak{g}$. For $x \in G$, the mapping $I(x): g \longrightarrow xgx^{-1}$ is an inner automorphism; the derivative, $Ad(x) = (dI(x))_e$ is thus an automorphism of $\mathfrak{g}$ and hence non-singular. The homomorphism $Ad: G \longrightarrow GL(\mathfrak{g})$ is the <u>adjoint representation of G</u>. $Ad_G(G)$, the range of the homomorphism, is clearly a subgroup of $GL(\mathfrak{g})$ known as the <u>adjoint group</u>. If $X \in \mathfrak{g}$ we have $\exp(Ad(x)X) = x \exp X\, x^{-1}$. Defining the <u>adjoint representation of $\mathfrak{g}$</u>, by $ad = (dAd)_e : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}) = (GL(\mathfrak{g}))_{identity}$, it can be shown that $ad(X)$ is the linear endomorphism of $\mathfrak{g}$ given by $ad(X)(Y) = [X,Y]$ ([3], p.123). $GL(\mathfrak{g})$ being a matrix Lie group and $\mathfrak{gl}(\mathfrak{g})$ its Lie algebra, let e be the matrix exponential. Application of the exponential to both sides of $ad = (d\,Ad)_e$ gives

$$(2) \qquad e^{ad\,X} = Ad(\exp X) \qquad (X \in \mathfrak{g}).$$

Let (G,H) be the symmetric pair corresponding to the globally symmetric space, M. If H is the isotropy subgroup at $p_o \in M$, then $H \subset G_\sigma$, where G_σ is the subgroup of G invariant under the involutive automorphism, σ. From (1) we have the relation $\exp((d\sigma)_e X) = \sigma(\exp X)$ and hence $\mathfrak{h} = \{X \in \mathfrak{g} : (d\sigma)_e X = X\}$. The identity $X = \frac{1}{2}(X + (d\sigma)_e X) + \frac{1}{2}(X - (d\sigma)_e X)$ enables us to write

(3) $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sum), where $\mathfrak{m} = \{X \in \mathfrak{g} : (d\sigma)_e X = -X\}$.

$\mathfrak{m}$ is a subspace of $\mathfrak{g}$, but not a subalgebra.

Indeed from $(d\sigma)_e [X,Y] = [(d\sigma)_e X, (d\sigma)_e Y]$ we have

(4) if $X, Y \in \mathfrak{m}$, then $[X,Y] \in \mathfrak{h}$, and

(5) if $X \in \mathfrak{m}$, $Y \in \mathfrak{h}$, then $[X,Y] \in \mathfrak{m}$.

The statements, "$\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$", (4) and (5) can be summarised by

(6) $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, $[\mathfrak{m},\mathfrak{h}] \subset \mathfrak{m}$ and $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$

2.5 Action of one-parameter subgroups of G

Let (G,H) be the symmetric pair corresponding to a globally symmetric space, M. We decompose the Lie algebra, $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H, the isotropy subgroup at $p_o \in M$. We will show that $\mathfrak{m}$ and M_{p_o} are isomorphic. Let π be the canonical projection $G \longrightarrow M$, given by $\pi(g) = gH$ (left coset). Its derivative, $(d\pi)_e$ maps $\mathfrak{g}$ into M_{p_o}.

(i) $\mathfrak{h}$ is the kernel of this linear mapping. For, let $X \in \ker(d\pi)_e$ and $f \in C^\infty(M)$. Then if $t \in \mathbb{R}$, $tX(f \circ \pi) = ((d\pi)_e(tX))(f) = 0$, which implies that $(f \circ \pi)$ is constant on the one-parameter subgroup of G corresponding to X. Now choosing f so that $f(p) \neq f(p_o)$ for all $p \neq p_o$ in a normal neighbourhood N_{p_o}, we have $\pi(\exp tX) = p_o$ and hence $X \in \mathfrak{h}$. Conversely let $X \in \mathfrak{h}$ and $f \in C^\infty(M)$. As $(\exp tX)(p_o) = p_o$, we have

$((d\pi)_e X)(f) \;=\; \left[\frac{d}{dt} f((\exp tX)(p_o))\right]_{t=0} \;=\; 0.$ Hence $X \in \ker (d\pi)_e$.

(ii) Let $X \in M_{p_o}$, $X \neq 0$. X defines a unique geodesic $\gamma_X : \mathbb{R} \longrightarrow M$ parameterised so that $\gamma_X(0) = p_o$ and $d\gamma_X(0) = X$. If $\gamma_X(u) = p_u$ $(u \in \mathbb{R})$, for each $t \in \mathbb{R}$ we have the transvection, $T_t = s_{p_t} s_{p_o}$. The curve $\Gamma : t \longrightarrow T_t$ is a one-parameter subgroup of G and defines a unique $\overline{X} \in \mathfrak{g}$ such that $\overline{X} = d\Gamma(0)$ and $T_t = \exp(t\overline{X})$. Now $(\sigma \circ \Gamma)(t) = \sigma T_t = s_{p_o} s_{p_t} = - T_t$. Hence $(d\sigma)_e \overline{X} = d(\sigma \circ \Gamma)(0) = - \overline{X}$ and so $\overline{X} \in \mathfrak{m}$.

(i) and (ii) establish the required linear isomorphism.

$\gamma_X(t)$, the image of $\exp t\overline{X}$ under π, is also the image of tX under $\mathrm{Exp} = \mathrm{Exp}_{p_o}$. Hence (altering notation):

(1) $$\pi(\exp tX) = \mathrm{Exp}(t(d\pi)_e X) \qquad (X \in \mathfrak{m}).$$

Now let $Z \in M_{p_o}$ and let $Z(t)$ be the vector field formed by parallel translation of Z along γ_X $(Z(t) \in M_{p_t})$. Fix $t \neq 0$ and define the parallel translations along γ by $\tau : M_{p_o} \longrightarrow M_{p_t}$ and $\tau' : M_{p_{t/2}} \longrightarrow M_{p_t}$. We thus have

(2) $$\tau(Z) \;=\; Z(t) \;=\; \tau' Z(t/2).$$

Now $s_{p_{t/2}}$ being an affine transformation, parallelism is preserved under $(ds_{p_{t/2}})$, i.e., $(ds_{p_{t/2}})_{p_o}(Z)$ $(\in M_{p_{t/2}})$ and $(ds_{p_{t/2}})_{p_{t/2}}(Z(t/2))$ are parallel. Hence,

(3) $$\tau'((ds_{p_{t/2}})(Z(t/2))) \;=\; (ds_{p_{t/2}})(Z).$$

But $s_{p_{t/2}}$ being the geodesic symmetry at $p_{t/2}$ we have

(4) $$(ds_{p_{t/2}})(Z(t/2)) \;=\; - Z(t/2).$$

It follows from (2), (3) and (4) that $(ds_{p_{t/2}})(Z) = -\tau(Z)$.

Hence, $(dT_t)(Z) = d(s_{p_{t/2}} s_{p_o})(Z) = \tau Z$,

that is,

(5) $\qquad (dT_t) = \tau.$

Equations (1) and (5) show that the affine connection on a symmetric space necessarily has the two following properties:

(i) geodesics through the base point p_o are orbits under action of one-parameter subgroups of G,

(ii) given a one-parameter subgroup of G, the action of its differentials on vectors in M_{p_o} is parallel translation along the geodesic which is the orbit of p_o under the given subgroup, i.e., $(d \exp tX)(Z) = \tau Z \quad (X \in \mathfrak{m}, Z \in M_{p_o})$.

<u>Definition</u> Let (G,H) be a symmetric pair. For $g \in G$, the diffeomorphism $\tau(g) : G/H \longrightarrow G/H$ is given by $\tau(g)g'H = gg'H$. The <u>linear isotropy group</u>, H^{ρ}, is the group of linear transformations $(d\tau(h)) : M_{p_o} \longrightarrow M_{p_o}$ $(h \in H)$, where $M = G/H$ and p_o is the coset H.

We now consider $Ad_G(H)$, a subgroup of the adjoint group. Let $h \in H$ and $X \in \mathfrak{m}$. Then is s_{p_o} is the geodesic symmetry at p_o and σ is the involutive automorphism of (G,H), then $\sigma = I(s_{p_o})$ and we have

$$\begin{aligned}(d\sigma)_e(\mathrm{Ad}\, h(X)) &= d(I(s_{p_o} h))_e(X)\\ &= d(I(hs_{p_o}))_e(X), \text{ since } H \subset G_\sigma\\ &= \mathrm{Ad}\, h((d\sigma)_e(X))\\ &= -\mathrm{Ad}\, h\,(X), \text{ since } X \in \mathfrak{m}.\end{aligned}$$

Hence $\mathrm{Ad}\, h(X) \in \mathfrak{m}$, i.e., $\mathrm{Ad}\, h|_{\mathfrak{m}}$ is a linear endomorphism of $\mathfrak{m}$. This we express by

(6) $\mathrm{Ad}_G(H)(\mathfrak{m}) \subset \mathfrak{m}$.

Further, for $h \in H$ we have $\tau(h): g(p_o) = \pi(g) \longrightarrow hg(p_o) = \pi(hgh^{-1})$. Taking differentials we obtain

(7) $(d\tau(h))_{p_o} \circ (d\pi)_e = (d\pi)_e \circ \mathrm{Ad}\, h$,

where both sides denote linear mappings of $\mathfrak{m}$. It is easy to verify that the groups H^* and $\mathrm{Ad}_G(H)$ are isomorphic.

2.6 Connections and Metrics on Symmetric Spaces

Definition Let G/H be a homogeneous space with G a connected Lie group. G/H is *reductive* if

(i) there exists a subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sum), where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively;

(ii) $\mathrm{Ad}_G(H)(\mathfrak{m}) \subset \mathfrak{m}$.

K. Nomizu [9] has examined the *G-invariant affine connections* on reductive homogeneous spaces, that is, affine connections on G/H invariant under left action of G acting on G/H as a Lie transformation group. He showed that there is a one-to-one correspondence between the set of G-invariant affine connections and the set of *connection functions*, that is, the set of bilinear functions, $\alpha: \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathfrak{m}$ which are invariant by $\mathrm{Ad}(h)$ $(h \in H)$. The correspondence is derived as follows:

There exists a neighbourhood U of e in G with the properties:

(i) $U = N \times K$ (topological product);

(ii) $\dim N = \dim \mathfrak{m}$, $\dim K = \dim \mathfrak{h}$;

(iii) $K \subset H_o$, the identity component of H (see [3], p.110)

Under the canonical projection, π, N is diffeomorphic to a neighbourhood N^* of p_o. Given $X \in \mathfrak{m}$, the vector field X^* on N^* is defined by

$$p = \pi(c) \longrightarrow X^*_p = (d(\tau(c))_{p_o} \circ (d\pi)_e)X$$
$$= ((d\pi)_e \circ \dot{A}d(c))X, \text{ from 2.5(7).}$$

The correspondence between G-invariant affine connections and connection functions is given by

$$\alpha(X,Y) = (\nabla_{X^*}Y^*)_{p_o} \qquad (X,Y \in \mathfrak{m}).$$

Invariant affine connections can have two properties:

(A1): Let $X \in \mathfrak{m}$ and $x(s) = \exp sX$ be the one-parameter subgroup of G generated by X. If $x^*(s) = \pi(x(s))$, then $x^*(s)$ is a geodesic of G/H through p_o.

(A2): Let $x(s)$ and $x^*(s)$ be defined as above and $Y \in \mathfrak{m}$. Then parallel translation of $(d\pi)_e Y$ at p_o along $x^*(s)$ is the same as left-translation of Y by $x(s)$.

If G/H has property (A1), there exists a unique invariant affine connection with trivial torsion, the <u>canonical affine connection of the first kind</u>. The connection function is

(1) $$\alpha(X,Y) = \tfrac{1}{2}[X,Y] \qquad ([9], \text{ p.48}).$$

If G/H has property (A2), there exists a unique invariant affine connection, the <u>canonical affine connection of the second kind</u>. The connection function is given by

(2) $$\alpha(X,Y) = 0.$$

If R and T are the curvature and torsion tensor fields for this connection we have for all $X,Y,Z \in \mathfrak{m}$

(3) $$R(X,Y)Z = -[[X,Y]_{\mathfrak{h}},Z],$$

(4) $\qquad T(X,Y) = -[X,Y]_{m}$, and

(5) $\qquad \nabla R = \nabla T = 0 \qquad$ ([9], p.49).

Now let (G,H) be a symmetric pair. Clearly from 2.4(3) and 2.5(6) G/H is a reductive homogeneous space. Further the properties (i) and (ii) of p.41 are precisely the properties (A1) and (A2). As $[m,m] \subset h$ the canonical affine connections of the first and second kind are identical. Hence on symmetric homogeneous spaces there exists a unique canonical invariant affine connection given by the connection function $\alpha(X,Y) = 0$. Further this is the ohly affine connection of G/H which is invariant by the geodesic symmetry at each point. Given $X,Y,Z \in m$, the curvature and torsion tensor fields satisfy:

(6) $\qquad R(X,Y)Z = -[[X,Y],Z]$,

(7) $\qquad T(X,Y) = 0$, and

(8) $\qquad \nabla R = 0$.

(Equations (7) and (8) have been obtained in 2.2 by consideration of the geodesic symmetry).

Symmetric spaces often admit Riemannian metrics. To establish this, we first assume that G/H has a metric, g. Now g_{p_o}, its value at $p_o = \{H\}$, being invariant by the geodesic symmetry, s_{p_o}, induces the canonical affine connection. We write $Q = \pi^* g_{p_o}$, where π is the canonical projection. The relation between Ad(h) and $(d\tau(h))_{p_o}$, namely 2.5(7), shows that Q is a metric on m if and only if Q is adjoint-invariant. Conversely, any non-degenerate bilinear form on $m \times m$ which is adjoint invariant induces a metric at p_o and hence globally on M by parallel translation.

Now consider the Killing form, $B: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ given by $B(X,Y) = tr(ad\ X \circ ad\ Y)$. Let α be any automorphism of $\mathfrak{g}$. Then $ad\ \alpha X = \alpha \circ ad\ X \circ \alpha^{-1}$ and $tr(AC) = tr(CA)$ (A,C any endomorphisms) imply

$$\begin{aligned} B(\alpha X, \alpha Y) &= tr(ad\ \alpha X \circ ad\ \alpha Y) \\ &= tr(\alpha \circ ad\ X \circ ad\ Y \circ \alpha^{-1}) \\ &= tr(ad\ X \circ ad\ Y) \\ &= B(X,Y). \end{aligned}$$

In particular, B is adjoint-invariant. However B is non-degenerate if and only if $X \neq 0$ implies $ad\ X \neq 0$, that is, if and only if the centre of $\mathfrak{g}$ is 0.

Definitions A Lie algebra, $\mathfrak{g}$, is simple if it is non-abelian and its only ideals are $\mathfrak{g}$ and $\{0\}$. $\mathfrak{g}$ is semi-simple if $\{0\}$ is its only abelian ideal, or equivalently if its centre is $\{0\}$. A Lie group is simple or semi-simple if its Lie algebra is simple or semi-simple.

Hence B is a non-degenerate form on $\mathfrak{g} \times \mathfrak{g}$ if and only if G is semi-simple. It is easy to verify that if $X \in \mathfrak{h}$ and $Y \in \mathfrak{m}$ then $B(X,Y) = 0$. Hence $B|_{\mathfrak{m}}$ is a non-degenerate form on $\mathfrak{m} \times \mathfrak{m}$ if and only if G is semi-simple; it induces a G-invariant metric on G/H known as the Cartan metric. Now given a Riemannian globally symmetric space $M = G/H$, it is known that H is compact in the compact-open topology ([5], p.173). Hence $Ad_G(H)$ is a compact subgroup of the adjoint group. This result motivates the following definition.

<u>Definition</u> Let (G,H) be a symmetric pair. Then (G,H) is a <u>Riemannian symmetric pair</u> if the group $Ad_G(H)$ is compact.

On pp. 38-39 we have seen that a Riemannian globally symmetric space gives rise to the involutive automorphism $s = (d\sigma)_e$ of $\mathfrak{g}$ with eigenspaces $\mathfrak{h}$ and $\mathfrak{m}$. The symmetry condition can be expressed in terms of Lie algebras rather than groups.

<u>Definition</u> An <u>orthogonal symmetric Lie algebra</u> is a pair $(\mathfrak{g}, s)$ such that

(i) $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$;

(ii) s is an involutive automorphism of $\mathfrak{g}$;

(iii) $\mathfrak{h}$, the set of fixed points of s, is a compactly imbedded subalgebra of $\mathfrak{g}$.

Further, $(\mathfrak{g}, s)$ is <u>effective</u> if $\mathfrak{h} \cap z = \{0\}$, where z is the centre of $\mathfrak{g}$. For example, $(\mathfrak{g}, s)$ is effective if $\mathfrak{g}$ is semi-simple. Let $(\mathfrak{g}, s)$ be an orthogonal symmetric Lie algebra, G a connected Lie group with Lie algebra $\mathfrak{g}$ and H a Lie subgroup of G with Lie algebra $\mathfrak{h}$. Then (G,H) is a symmetric pair <u>associated</u> with $(\mathfrak{g}, s)$. Further if G is simply connected and H connected, (G,H) is a Riemannian symmetric pair ([5], p.178).

Symmetric spaces can be classified as follows:

Let (G,H) be a symmetric pair associated with the orthogonal symmetric Lie algebra, $(\mathfrak{g}, s)$.

<u>Definitions</u> 1. (G,H) is <u>compact</u> if $\mathfrak{g}$ is semi-simple and compact.

2. (G,H) is <u>non-compact</u> if $\mathfrak{g}$ is semi-simple and non-compact.

3. (G,H) is <u>Euclidean</u> if in the canonical decomposition, $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, $\mathfrak{m}$ is an abelian ideal of $\mathfrak{g}$.

The three classes are characterised by sectional curvature. Let (G,H) be a Riemannian symmetric pair associated with $(\mathfrak{g}, s)$, where $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Let S be any two-dimensional subspace of $\mathfrak{m}$ and K(S) its sectional curvature.

(i) If (G,H) is compact, then $K(S) \geqslant 0$.

(ii) If (G,H) is non-compact, then $K(S) \leqslant 0$.

(iii) If (G,H) is Euclidean, then $K(S) = 0$. ([5], p.205)

Finally we make the following definition.

<u>Definition</u> Let (G,H) be a symmetric pair associated with the orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$. The eigenspaces of s are $\mathfrak{h}$ and $\mathfrak{m}$. (G,H) is <u>irreducible</u> if

(i) $\mathfrak{g}$ is semi-simple and $\mathfrak{h}$ contains no non-zero ideals of $\mathfrak{g}$;

(ii) the algebra $ad_{\mathfrak{g}}(\mathfrak{h})$ acts irreducibly on $\mathfrak{m}$, that is, if $\mathfrak{k}$ is a subspace of $\mathfrak{m}$ such that $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$, then either $\mathfrak{k} = \mathfrak{m}$ or $\mathfrak{k} = \{0\}$.

CHAPTER III

HARMONIC SYMMETRIC SPACES

3.1 Harmonic Symmetric Spaces

We first examine how Ledger's curvature conditions for harmonic spaces can be applied to symmetric spaces.

Let M be an n-dimensional globally symmetric Riemannian manifold and p_o any point of M. Let (x^i) be a coordinate system valid in a neighbourhood W of p_o, so that for any $X \in M_{p_o}$, $X = X^i \frac{\partial}{\partial x^i}(p_o)$ and $v = v(X) = g_{p_o}(X,X)$, where g_{p_o} is the metric tensor at p_o. Consider the endomorphism $Y \longrightarrow R(X,Y)X$ of M_{p_o}. (Here X is fixed and R is the curvature tensor). This endomorphism has matrix $\Pi(X) = (\Pi^i_j) = (R^i{}_{klj}X^kX^l)$. The following theorem was proved by A.G.Walker in 1946 [14].

<u>Theorem 1</u> M is harmonic if and only if the eigenvalues of $\Pi(X)$ are of the form $\alpha_1 v, \alpha_2 v, \ldots, \alpha_n v$, where the coefficients $\alpha_1, \ldots, \alpha_n$ are independent of X.

<u>Proof</u> We recall notation used in Section 1.5.
$\Lambda = (\Lambda^i_j) = (\Omega^i{}_{,j})$. If $T = (T^i_j)$ is any tensor field on W, $T_r = D^r(T^i_j)$. The vanishing of the covariant derivatives of the curvature tensor field for a symmetric space yields $\Pi_r = 0$ for $r \geqslant 1$.

The first four values of Λ_r at p_o are $\Lambda_o = I$, $\Lambda_1 = 0$, $\Lambda_2 = \frac{2}{3}\Pi$, $\Lambda_3 = 0$ and Ledger's recurrence formula reduces to

(1) $$(r + 1)\Lambda_r = - \sum_{q=2}^{r-2}\binom{r}{q}\Lambda_q\Lambda_{r-q} \quad (r \geqslant 4) \text{ (cf. 1.5(8)).}$$

It is easy to establish by induction using (1) that values of Λ_r at p_o are

(2) $\qquad \Lambda_{2r+1} = 0 \qquad\qquad (r = 1, 2, \ldots)$

(3) $\qquad \Lambda_{2r} = (-1)^{r+1}\beta_r \Pi^r, \ \beta_r > 0 \quad (r = 1, 2, \ldots).$

From 1.5(2) and 1.5(5) we deduce that

$$c_r = tr(\Lambda_{2r})_o = h_r(g_{k_1 l_1})\ldots(g_{k_r l_r})X^{k_1}\ldots X^{k_r}X^{l_1}\ldots X^{l_r},$$

that is,

(4) $\qquad tr(\Lambda_{2r}) = h_r v^r,$

where h_r is independent of X for all r if and only if M is harmonic. Hence, from (3), the necessary and sufficient conditions for M to be harmonic are

$$tr(\Pi^r) = \gamma_r v^r \quad (r = 1, 2, \ldots),$$ where the coefficients γ_r are independent of X. Now if $a_1, \ldots, a_n$ are the eigenvalues of Π, then $tr(\Pi^r) = a_1{}^r + \ldots + a_n{}^r$. We deduce that $a_i = \alpha_i v$ $(i = 1, \ldots, n)$ and all eigenvalues are constant multiples of v(X).

Q.e.d.

For a simply harmonic manifold, all the constants, h_r, in (4) are zero. Hence,

<u>Corollary</u> A globally symmetric Riemannian manifold is simply harmonic if and only if the eigenvalues of $\Pi(X)$ are all zero.

Note that the point of application , p_o, for theorem 1 is immaterial since if the conditions hold at one point of application, action of a member of the transitive isometry group will establish the validity at any other point of the manifold.

<u>Definition</u> Let M be any Riemannian manifold and S a connected submanifold of M. S is a <u>totally geodesic submanifold</u> of M if, for every $p \in S$, each M-geodesic which is tangent to S at p is a curve in S. Further if S is a totally geodesic submanifold of M, all S-geodesics through p are also M-geodesics ([5], p.79, Lemma 14.3).

Thus S-geodesic symmetries are derived from M-geodesic symmetries by restriction. We deduce that all totally geodesic submanifolds of a locally symmetric space are themselves locally symmetric.

<u>Definition</u> A subspace, $\mathfrak{s}$, of a Lie algebra, $\mathfrak{g}$, is a <u>Lie triple system</u> if $X, Y, Z \in \mathfrak{s}$ implies $[X,[Y,Z]] \in \mathfrak{s}$. For example, if $(\mathfrak{g}, s)$ is an orthogonal symmetric Lie algebra, the eigenspace, $\mathfrak{m}$ with eigenvalue -1, is a Lie triple system.

Lie triple systems give rise to totally geodesic submanifolds of symmetric spaces. For if $M = G/H$ is a globally symmetric space, suppose that we identify p_o with the coset $\{H\}$ and the tangent space M_{p_o} with the subspace $\mathfrak{m}$ of the Lie algebra of $I(M)$. Let $\mathfrak{s}$ be a subspace of $\mathfrak{m}$ such that $\mathfrak{s}$ is a Lie triple system. Then $S = \mathrm{Exp}_{p_o} \mathfrak{s}$ has a natural differentiable structure in which it is a totally geodesic submanifold of M satisfying $S_{p_o} = \mathfrak{s}$ ([5], p.189).

<u>Definition</u> Let M be a globally symmetric space. The <u>rank</u> of M is the maximal dimension of a flat, totally geodesic submanifold of M.

The rank of a symmetric space could alternatively be defined in terms of the dimension of a Lie subalgebra. For a totally geodesic submanifold is flat if and only if the Lie triple system from which it is mapped is abelian. (2.6(6))

__Definition__ Let $\mathfrak{g}$ be a Lie algebra. A subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is a __Cartan subalgebra__ if

(i) $\mathfrak{s}$ is a maximal abelian subalgebra of $\mathfrak{g}$;

(ii) for each $X \in \mathfrak{s}$, the endomorphism ad X of $\mathfrak{g}$ is semi-simple, that is, each subspace of $\mathfrak{g}$ invariant under ad X has an invariant complementary subspace.

If $\mathfrak{g}$ is semi-simple then it has a Cartan subalgebra ([5], p.140). Further any two Cartan subalgebras are isomorphic under an automorphism of $\mathfrak{g}$ and hence all Cartan subalgebras have the same dimension ([5], p. 213). We deduce that if $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is the canonical decomposition of the orthogonal symmetric Lie algebra of the symmetric space M = G/H, then the rank of M equals the dimension of a Cartan subalgebra of $\mathfrak{m}$.

Harmonic symmetric spaces are characterised by their rank. We know from Chapter I Theorems 9 and 10 that all decomposable harmonic spaces are locally flat, so we confine our attention to symmetric manifolds with indecomposable metrics, i.e., non-Euclidean manifolds.

__Theorem 2__ Let M be a harmonic globally symmetric manifold with positive definite indecomposable metric. Then M has rank equal to one.

<u>Proof</u> This theorem was first proved by A.J.Ledger [7] in 1957.

Writing M = G/H and let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the canonical decomposition of the Lie algebra. Let N be the subset of G which maps under the canonical projection, π, onto N^*, a neighbourhood of $p_0 = \{H\}$ (see pp. 42-43). If U is the topological product N X K, N and K have Lie algebras $\mathfrak{m}$ and $\mathfrak{h}$ respectively. If g is the metric on N^*, $(\pi^* g)$ is a metric on U equal to the metric on U induced as a submanifold of G. Further $\exp(\mathfrak{m})$ = N, we have seen above that $\mathfrak{m}$ is a Lie triple system and so N is a totally geodesic submanifold of G.

Let $\{e_a\}$ (a = 1, ..., m) be a basis for $\mathfrak{g}$ such that

$\{e_i\}$ (i = 1, ..., n) is a basis for $\mathfrak{m}$ and

$\{e_{i'}\}$ (i' = n+1, ..., m) is a basis for $\mathfrak{h}$.

Defining the constants of structure, C^c_{ab}, of $\mathfrak{g}$ by

(5) $[e_a, e_b] = C^c{}_{ab}\, e_c$, we have from 2.4(6)

(6) $C^k{}_{ij} = C^k{}_{i'j'} = C^{k'}{}_{ij'} = 0.$

Now the Cartan metric on a Lie group is obtained from the torsion-free <u>(0)-connection</u> with connection function $\alpha(X,Y) = \frac{1}{2}[X,Y]$. This connection is G-invariant and satisfies condition (A1) on p. 43. The curvature transformation is given by $R(X,Y) = -\frac{1}{4}\,\mathrm{ad}([X,Y])$ ([9], p.49).

N being totally geodesic we can express the curvature tensor on it by components:

$$R^h{}_{ijk} = \tfrac{1}{4} C^h{}_{ia} C^a{}_{jk} \qquad (h,i,j,k = 1, \ldots, n \quad a = 1, \ldots, m).$$

Let $X = X^i e_i \in \mathfrak{m}$. Then the endomorphism, $\frac{1}{2}$ ad X, of $\mathfrak{g}$ has matrix $\quad A(X) = (A^a_b) = (\frac{1}{2} C^a{}_{ib} X^i)$.

We have, $\quad A^2 = \frac{1}{4}(C^a{}_{ib} C^b{}_{jc} X^i X^j) = (R^a{}_{ijc} X^i X^j)$,

that is,

$$(7) \qquad A^2 = \begin{pmatrix} (R^h{}_{ijk} X^i X^j) & 0 \\ 0 & (R^{h'}{}_{ijk'} X^i X^j) \end{pmatrix}$$

Here we have used the relations (6).

Let the rank of M be 1 and $\mathfrak{s}$ be a Cartan subalgebra of $\mathfrak{m}$. We choose a basis $\{e_a\}$ of $\mathfrak{g}$ so that $\{e_1, \ldots, e_l\}$ is a basis of $\mathfrak{s}$. Let $X = (X^i) \in \mathfrak{s}$, $X \neq 0$, i.e., $X^i = 0$ if $i > l$. Then a result of E. Cartan states that the eigenvalues of $A(X)$ are fixed linear combinations of the components, X^i. We deduce that A^2 has eigenvalues of the form

$$(8) \qquad \Big(\sum_{i=1}^{l} A^b{}_i X^i\Big)^2 \quad (b = 1, \ldots, m),$$ where the coefficients, $A^b{}_i$ are independent of X. Now consider the matrix $\Pi(X) = (R^h{}_{ijk} X^i X^j)$. Clearly all eigenvalues of $\Pi(X)$ are eigenvalues of $(A(X))^2$ and are of form (8).

We now use the hypothesis that M is harmonic. By Theorem 1 we know that all eigenvalues of $\Pi(X)$ $(X \neq 0)$ are of form $\alpha^j v(X)$, where each α^j is independent of X and $v(X) = g_{p_0}(X,X) > 0$, since the metric is positive definite. M being indecomposable cannot be flat, so $\alpha^{j_0} > 0$ for some j_0, $1 \leq j_0 \leq n$. Hence the expression $\Big(\sum_{i=1}^{l} A^{j_0}{}_i X^i\Big)^2$ is positive definite for all $X = (X^i) \in \mathfrak{s}$. This is only possible if the summation is taken over one term, i.e., the rank of M is one.

Q.e.d.

3.2 Jacobi fields

Let M be a complete, simply connected Riemannian manifold and $p_0 \in M$. Let I,J be open neighbourhoods of 0 in $\mathbb{R}$ and $\gamma: I \longrightarrow M$ be a geodesic through p_0 parameterised by arc distance, i.e., $d(\gamma(s), p_0) = |s|$, $s \in I$. A variation of γ is a C mapping, $G: I \times J \longrightarrow M$ such that

(i) for all ϵ in J the curve $\gamma_\epsilon : s \longrightarrow G(s,\epsilon)$ is a unit-speed geodesic;

(ii) $\gamma_0 = \gamma$.

A Jacobi field on γ is a vector field X on γ such that there exists a variation G of γ for which $X(s) = \frac{\partial G}{\partial \epsilon}(s,0)$. For each $s \in I$ we can define an arc $\sigma_s : J \longrightarrow M$ by $\sigma_s(\epsilon) = \gamma_\epsilon(s) = G(s,\epsilon)$. Clearly, $X(s) = \dfrac{d\sigma_s(\epsilon)}{d\epsilon}(0)$, that is, the Jacobi field is given by tangents to the transversals of γ. We show that X satisfies the variational equations

$$\frac{D^2X}{ds^2} + R(X,\gamma_*)\gamma_* = 0, \tag{1}$$

where $\frac{D}{ds}$ is the intrinsic derivative on γ, R is the curvature tensor and $\gamma_*(s) = \frac{\partial G}{\partial s}(s,0)$. Let (x^i) be a local coordinate system valid over a neighbourhood of p_0 containing $\gamma(I)$ so that on γ, $X(s) = X^i(s)\frac{\partial}{\partial x^i}(s)$ and $\gamma_*(s) = \lambda^i(s)\frac{\partial}{\partial x^i}(s)$. Now on γ we have $\frac{DX^i}{ds} = \frac{dx^i}{ds} + \Gamma^i_{jk}\lambda^j X^k$, where (Γ^i_{jk}) are connection coefficients. However,

$$\frac{dX(s)}{ds} = \frac{d}{ds}\frac{d\sigma_s(\epsilon)}{d\epsilon}(0) = \frac{\partial^2}{\partial s\,\partial\epsilon}G(s,0) = \frac{d}{d\epsilon}\frac{d\gamma_\epsilon(s)}{ds}\bigg|_{\epsilon=0},$$

that is,

$$\frac{dX(s)}{ds} = \frac{d(\gamma_\epsilon)_*}{d\epsilon}\bigg|_{\epsilon=0}. \tag{2}$$

Using the Ricci identity ([15], p. 215) we have

$$\lambda^i_{,lk}\lambda^k\chi^l - \lambda^i_{,kl}\lambda^k\chi^l = R^i_{hkl}\lambda^h\lambda^k\chi^l.$$

Hence,

(3) $$\left(\frac{D^2\gamma_*}{ds\,d\epsilon} - \frac{D^2\gamma_*}{d\epsilon\,ds}\right)\frac{\partial}{\partial x^i} = -R(X,\gamma_*)\gamma_*.$$

From (2) we deduce that on , $\frac{DX}{ds} = \frac{D(\gamma_\epsilon)_*}{d\epsilon}$ and hence $\frac{D^2X}{ds^2} = \frac{D^2(\gamma_\epsilon)_*}{ds\,d\epsilon}$. We have from (3)

$$\frac{D^2X}{ds^2} - \frac{D^2(\gamma)_*}{d\epsilon\,ds} + R(X,\gamma_*)\gamma_* = 0.$$

The variational equations (1) follow since γ is a geodesic.

Let γ be any geodesic on M. By $\mathcal{E}(\gamma)$ we denote the vector space of Jacobi fields on γ. The dimension of $\mathcal{E}(\gamma)$ is 2n. If $X, Y \in \mathcal{E}(\gamma)$ it is easy to verify using the variational equations and the symmetry and antisymmetry properties of the Riemannian curvature tensor that

$$\frac{D}{ds}\, g_{\gamma(s)}\left(\frac{DX}{ds}, \gamma_*\right) = \frac{D}{ds}\left(g_{\gamma(s)}\left(\frac{DX}{ds}, Y\right) - g_{\gamma(s)}\left(\frac{DY}{ds}, X\right)\right)$$
$$= 0,$$

where $g_{\gamma(s)}$ is the metric tensor at $\gamma(s)$. Hence there exist constants C, C' such that

(4) $$g_{\gamma(s)}\left(\frac{DX}{ds}, \gamma_*\right) = C,$$

(5) $$g_{\gamma(s)}\left(\frac{DX}{ds}, Y\right) - g_{\gamma(s)}\left(\frac{DY}{ds}, X\right) = C'.$$

We define subspaces of $\mathcal{E}(\gamma)$ as follows:

$\bar{\mathcal{E}}(\gamma)$ is the subspace of Jacobi fields on γ for which C = 0 in (4). $\mathrm{Dim}(\bar{\mathcal{E}}(\gamma)) = 2n - 1$.

$\mathcal{E}_0(\gamma)$ is the subspace of Jacobi fields on γ for which $X(0) = 0$. $\mathrm{Dim}(\mathcal{E}_0(\gamma)) = n$.

$\overline{\mathcal{E}}_0(\gamma) = \mathcal{E}_0(\gamma) \cap \overline{\mathcal{E}}(\gamma)$. $\text{Dim}(\overline{\mathcal{E}}_0(\gamma)) = n - 1$.

If $X \in \overline{\mathcal{E}}_0(\gamma)$ then $X(0) = 0$ and for all $s \in I$ we have

(6) $$g_{\gamma(s)}(X(s), \gamma_*(s)) = 0.$$

Conversely, $X(0) = 0$ and for some $s_0 \neq 0$

(7) $$g_{\gamma(s_0)}(X(s_0), \gamma_*(s_0)) = 0 \text{ implies } X \in \overline{\mathcal{E}}_0(\gamma).$$

For details of the results above, see [1], p.106.

Now let $Y \in \overline{\mathcal{E}}_0(\gamma)$. Suppose that $B_{\gamma(0)} = \{X_1, \ldots, X_n = \gamma_*(0)\}$ is an orthonormal basis of $M_{\gamma(0)}$. Let $B_{\gamma(s)} = \{X_1(s), \ldots, X_n(s) = \gamma_*(s)\}$ be the orthonormal frame at $\gamma(s)$ formed by parallel translation of $B_{\gamma(0)}$ on γ. We have $\frac{DX_a(s)}{ds} = 0$ $(a = 1, \ldots, n)$. Defining the coordinates $y^a(s)$ of $Y(s)$ by $y^a(s) = g_{\gamma(s)}(Y(s), X_a(s))$, from (6) we have $y^n(s) = 0$. Writing $Y_a(s) = y^a(s)X_a(s)$ (a not summed) we see that $Y(s)$ can be written as the sum of $(n - 1)$ mutually orthogonal Jacobi fields on :

$$Y(s) = Y_1(s) + \ldots + Y_{n-1}(s) = y^a(s)X_a(s) \quad \text{(a summed)}.$$

Let (x^i) be a local coordinate system valid in a neighbourhood of p_0 containing $\gamma(I)$ so that each $X_a(s) \in B_{\gamma(s)}$ and $Y_a \in \overline{\mathcal{E}}_0(\gamma)$ have components $(X^i{}_a(s))$ and $(Y^i{}_a(s))$ respectively.

Denoting $\frac{d}{ds}$ by ' we have for each a, $1 \leq a \leq n - 1$,

$$\frac{DY^i{}_a}{ds} = (y^a X^i{}_a)' + \Gamma^i_{jk} y^a X^j{}_n X^k{}_a$$
$$= y^{a\prime} X^i{}_a + y^a \frac{DX^i{}_a}{ds} = y^{a\prime} X^i{}_a.$$

Hence $\frac{D^2 Y_a}{ds^2} = y^{a\prime\prime} X_a$ and using the variational equations we have

$$y^{a\prime\prime} X_a + y^b R(X_b, \gamma_*)\gamma_* = 0.$$

Thus we deduce the <u>Jacobi equations</u>

(8) $$y^{a''} + K_{ab}y^b = 0 \quad (a = 1, \ldots, n-1),$$

where writing R for the Riemannian curvature tensor, the sectional curvatures, K_{ab}, are given by

$$K_{ab} = K_{ab}(s) = R(X_a(s), \gamma_*(s), X_b(s), \gamma_*(s)).$$

We recall that M is a complete, simply connected Riemannian manifold. Let $p_o, p \in M$ and γ be a unit-speed geodesic joining p_o to p parameterised so that $\gamma(s_o) = p_o$, $\gamma(s_1) = p$.
γ is <u>minimal</u> if $L(\gamma) = |s_o - s_1| = d(p_o, p)$ (the metric distance). M being complete, for all $p_o, p \in M$ there exists a minimal geodesic arc which joins them. Further all geodesics are defined with domain $\mathbb{R}$, that is, Exp_{p_o} has domain M_{p_o} and is surjective.

Let γ be the geodesic joining $p_o = \gamma(s_o)$ to $p = \gamma(s_1)$. Then we define

$$\mathcal{E}_{s_o,s_1}(\gamma) = \mathcal{E}_{s_o}(\gamma) \cap \mathcal{E}_{s_1}(\gamma)$$
= vector space of Jacobi fields on γ

which vanish at p_o and p. From (7) we have

(9) $$\mathcal{E}_{s_o,s_1}(\gamma) \subset \overline{\mathcal{E}}_{s_o}(\gamma).$$

The <u>index of the pair (s_o, s_1)</u> is the number
$\lambda(\gamma; s_o, s_1) = \dim(\mathcal{E}_{s_o,s_1}(\gamma))$. Clearly from (9), $(\gamma; s_o, s_1) \leq n-1$.
p_o and p are <u>conjugate points on γ</u> if $(\gamma; s_o, s_1) \neq 0$, $p \neq p_o$.
If $\gamma|_{[s_o, s_1]}$ is minimal, there are no conjugate points on the open geodesic arc $\gamma|_{(s_o, s_1)}$.

Let σ be a unit-speed geodesic ray from p_o with domain

$[0,\infty)$ $(\sigma(0) = p_o)$. Suppose that $p_1 = \sigma(s_1)$ is a conjugate point of p_o on σ and is such that $\sigma|_{[0,s]}$ is minimal for all $s < s_1$. p_1 is the <u>first conjugate point of p_o</u> on σ. The <u>index of σ</u>, $\lambda(\sigma)$ is defined by $\lambda(\sigma) = \lambda(\sigma;0, s_1)$. If p_o has no conjugate points on σ we define $\lambda(\sigma) = 0$. The locus of the first conjugate point of p_o for all geodesic rays through p_o is called the <u>residual locus of p_o</u> and denoted by R_{p_o}.

Let $p_o \in M$ and $Z \in M_{p_o}$. For the geodesic $\gamma^Z: t \to \operatorname{Exp}_{p_o} tZ$, we define $\lambda(Z) = \lambda(\gamma^Z;0, 1)$. Let $u \in (M_{p_o})_Z$. Then if $\bar{\gamma}^Z$ is the path $t \to tZ$ on M_{p_o}, the assignment $x^u(t) = tu$ at tZ is a <u>linear homogeneous vector field</u> on $\bar{\gamma}^Z$. Let w be the identification of the vector space M_{p_o} with the tangent space at any of its points. We write

$$X^u(t) = ((d\operatorname{Exp}_{p_o})_{tZ})x^u(t). \quad \text{Then } X^u \in \mathcal{E}_o(\gamma^Z).$$

Let (x^i) be a local coordinate system and $X^u(t) = \lambda^i{}_u \frac{\partial}{\partial x^i}(t)$.
Then we have,

$$\frac{D\lambda^i{}_u}{dt}(0) = \left(\frac{d\lambda^i{}_u}{dt} + \Gamma^i_{jk}\lambda^j{}_u \frac{dx^k}{dt}\right)_{t=0}$$

$$= \frac{d\lambda^i{}_u}{dt}(0), \text{ since } \lambda^j{}_u(0) = 0.$$

Hence, $$\frac{DX^u}{dt}(0) = d\operatorname{Exp}_{p_o}\left(\frac{d}{dt}(X^u(t))_{t=0}\right) = d\operatorname{Exp}_{p_o} u, \text{ i.e.,}$$

(10) $$\frac{DX^u}{dt}(0) = w(u).$$

We deduce that the mapping $\phi:(M_{p_o})_Z \to \mathcal{E}_o(\gamma^Z)$ given by $\phi(u) = X^u$ is linear and bijective. If K is the kernel of the linear mapping $(d\operatorname{Exp}_{p_o})_Z$, then ϕ maps K onto $\mathcal{E}_{0,1}(\gamma^Z)$.

Let $M^1_{p_o}(Z)$ be the subspace of M_{p_o} equal to $w(K)$. Then,

$$M^1_{p_o}(Z) = \left\{\frac{DX}{dt}(0) : X \in \mathcal{E}_{0,1}(\gamma^Z)\right\}.$$

If $\{X_1(t), \ldots, X_n(t)\}$ is a basis of $\mathcal{E}_0(\gamma^Z)$, we have

(11) $\left\{\frac{DX_1}{dt}(0), \ldots, \frac{DX_\lambda}{dt}(0)\right\}$ is a basis of $M^1_{p_o}(Z)$.

Let $M^2_{p_o}(Z)$ be the orthogonal complement of $M^1_{p_o}(Z)$. It generates vector fields which vanish at $\gamma^Z(1)$ but not at p_o,

i.e., $M^2_{p_o}(Z) = \{X(0) : X \in \mathcal{E}_1(\gamma^Z)\}$.

(12) $\{X_{\lambda+1}(0), \ldots, X_n(0)\}$ is a basis of $M^2_{p_o}$.

3.3 Globally harmonic spaces

Harmonic spaces have been defined globally by A.C.Allamigeon [1]. The definition of Ruse's invariant can be extended to manifolds with affine connection as well as Riemannian manifolds.

Let M be a C^∞ simply connected manifold with affine connection. Let τ be the volume element; this is an n-form invariant under parallel translations. Let $p_o \in M$ and $p \in W$, a normal neighbourhood of p_o. Define $X \in M_{p_o}$ such that $\mathrm{Exp}_{p_o} X = p$. Let U be the domain of Exp_{p_o} with range W. $\mathrm{Exp}^*_{p_o} \tau_p$ is the pull-back of the volume element at p to p_o and is identified with an n-form at p_o. The "ratio" of the two n-forms at p_o we denote by $R(X)$,

that is,

(1) $$\mathrm{Exp}_{p_o}^*(\tau_p) = R(X)\tau_{p_o}.$$

Allanigeon defined Ruse's invariant for a geodesic arc in terms of the function $R: U \longrightarrow \mathbb{R}$.

<u>Definition</u> Let g be an orientated geodesic arc. <u>Ruse's invariant of g</u> is the number

(2) $$\rho(g) = R(\frac{dg}{dt}(0)).$$

For all $p_o, p \in W$ we can define a geodesic arc $\gamma: [0,1] \longrightarrow M$ parameterised so that $\gamma(0) = p_o$, $\gamma(1) = p$. If $X = \dot{\gamma}_o(0)$ then $\rho(p_o,p) = \rho(\gamma) = R(X)$.

If M is a Riemannian manifold then for a normal coordinate system origin p_o on W, (y^i), the volume element can be expressed by $\tau = \sqrt{(\det({}^o g_{ij}))}\, dy^1 \wedge dy^2 \wedge \ldots \wedge dy^n$, where $({}^o g_{ij})$ is the metric tensor. It is easy to verify using (1) that $\rho(p_o,p)$ has the properties of Ruse's invariant as defined in Section 1.3 (see [1], pp. 99-101).

<u>Definition</u> Let M be a Riemannian manifold. M is <u>globally harmonic</u> if for every geodesic arc, g, Ruse's invariant $\rho(g)$ is a C^∞ function of L(g), the length of g.

Let M be a C^∞ manifold with affine connection and possessing a local volume element, τ. Let $p_o \in M$ and $\gamma: I \longrightarrow M$ be a geodesic parameterised so that $\gamma(0) = p_o$ (I is an open neighbourhood of 0 in $\mathbb{R}$). We will derive an expression for τ applied to a basis of $\mathcal{E}_o(\gamma)$. Let $\{X_1, \ldots, X_n\}$ be such a basis. Let $p = \gamma(t)$ be any point on γ. We define the geodesic arc $(p_o p)$ by $g_t: [0,1] \longrightarrow M$ given by $g_t(s) = \gamma(st)$. Again let $\dot{\gamma}_o(0) = Z$ and $\{u_1, \ldots, u_n\}$ be a basis of $(M_{p_o})_Z$. We have n linearly independent linear homogeneous vector fields by assigning

(3) $\qquad Y_i(t) = t\mathring{u}_i$ to $tZ \in \tilde{\gamma}^Z$ (see p.58). $\mathcal{E}_o(\gamma)$ has basis $\{X_1(t), \ldots, X_n(t)\}$ where

(4) $\qquad X_i(t) = (d\,\mathrm{Exp}_{p_o})_Z\, Y_i(t).$

As with 3.2(10) we have

(5) $\qquad \dfrac{DX_i}{dt}(0) = w(u_i)$, where w is the identification $(M_{p_o})_Z \longrightarrow M_{p_o}$. We now have

$$\tau_{\gamma(t)}(X_1(t), \ldots, X_n(t)) = (\mathrm{Exp}^*_{p_o}\tau_{\gamma(t)})(wY_1, \ldots, wY_n), \text{ by (4)}$$
$$= t^n R(tZ)\tau_{p_o}(wu_1, \ldots wu_n), \text{ by (1)}.$$

Hence we have

(6) $\qquad \tau(X_1(t), \ldots, X_n(t)) = t^n\rho(g_t)\tau(\dfrac{DX_1}{dt}(0), \ldots, \dfrac{DX_n}{dt}(0)).$

<u>Theorem 3</u> Let M be a complete, globally harmonic manifold. Then there exists a real number $L \geqslant 0$ (possibly infinite) and a non-negative integer λ such that for all geodesic rays the distance to the first conjugate point is always L and the index is always λ.

<u>Proof</u> (Allamigeon [1]). Let $p_o \in M$ and σ be any unit-speed geodesic ray from p_o. g_t being defined as at the bottom of p.60, we have $\rho(g_t) = \rho^*(t)$, where $\rho = \rho^* \circ L$ and (6) reduces to

(7) $\qquad \tau(X_1(t), \ldots, X_n(t)) = t^n\rho^*(t)\tau(\dfrac{DX_1}{dt}(0), \ldots, \dfrac{DX_n}{dt}(0))$

for all geodesic rays from p_o. The left hand side of (7) vanishes if and only if $t = 0$ or $\sigma(t)$ and $\sigma(0)$ are conjugate. Define L to be the least positive root of the equation $\rho^*(x) = 0$, if such a root exists. If on the other hand $\rho^*(x) > 0$ for all $x \geqslant 0$, p_o has no conjugate points and we define $L = \infty$ and $\lambda = 0$. Suppose however that $L < \infty$. Then it equals the distance along each geodesic ray to the first conjugate point. Let σ be any geodesic ray. We must show that $\lambda = \lambda(\sigma;0, L)$ is in fact independent of σ. Let $\{X_1(t), \ldots, X_n(t)\}$ be a basis of $\mathcal{E}_0(\sigma)$ such that $\{X_1(t), \ldots, X_\lambda(t)\}$ is a basis of $\mathcal{E}_{0,L}(\sigma)$. Let $\bar{X}_i(t)$ be the vector obtained by parallel translation of $X_i(t)$ along $\sigma|[t,L]$ $(i = 1, \ldots, n)$.

Then from the formula

$$\frac{DX_i}{dt}(L) = \lim_{t \to L} \frac{1}{t - L}(\bar{X}_i(t) - X_i(t))$$

(see for example, [5], p.41) we have

$$\bar{X}_i(t) = \begin{cases} (t - L)\frac{DX_i}{dt}(L) + \epsilon_i(t - L) & (1 \leq i \leq \lambda) \\ X_i(L) + \epsilon_i(t - L) & (\lambda + 1 \leq i \leq n), \end{cases}$$

where $\epsilon_i : \mathbb{R} \to M_{\sigma(L)}$ are C^∞ vector-valued functions such that $\epsilon_i(s) \to 0$ as $s \to 0$ $(i = 1, \ldots, n)$.

We now have

$$\begin{aligned} \tau(X_1(t), \ldots, X_n(t)) &= \tau(\bar{X}_1(t), \ldots, \bar{X}_n(t)) \\ &= (t - L)^\lambda \tau\left(\frac{DX_1}{dt}(L), \ldots, \frac{DX_\lambda}{dt}(L), X_{\lambda+1}(L), \ldots, X_n(L)\right) + o(t - L)^\lambda. \end{aligned}$$

Using (7) we have

$$\rho^\circ(t) = A_\sigma(t - L)^\lambda + o(t - L)^\lambda,$$

where

$$A_\sigma = \frac{\tau_{\sigma(L)}\left(\frac{DX_1}{dt}(L), \ldots, \frac{DX_\lambda}{dt}(L), X_{\lambda+1}(L), \ldots, X_n(L)\right)}{L^n \tau_{\sigma(0)}\left(\frac{DX_1}{dt}(0), \ldots, \frac{DX_n}{dt}(0)\right)}.$$

(Here we use $\frac{K}{t^n} = \frac{K}{L^n} + o(t - L)$).

Now λ is independent of σ if and only if $0 < \rho^\circ(t) < \infty$ for $0 \leq t < L$, that is, if and only if A_σ has no singularities or zeros for any basis of $\mathcal{E}_0(\sigma)$ on any geodesic ray .

(i) As $X_i(0) = 0$ $(i = 1, \ldots, n)$ we have

$$\frac{DX_i}{dt}(0) = \lim_{t \to 0} \frac{1}{t} \phi_t^{-1} X_i(t),$$ where ϕ_t is the parallel translation from $\sigma(0)$ to $\sigma(t)$. $\{X_i(t), i = 1, \ldots, n\}$ being a linearly independent set for $0 \leqslant t \leqslant L$, we have $\left\{\frac{DX_i}{dt}(0), i = 1, \ldots, n\right\}$ is linearly independent. Hence A_σ has no singularities.

(ii) Put $Z = -L\,\sigma_*(L)$ and clearly $\left\{\frac{DX_1}{dt}(L), \ldots, \frac{DX_\lambda}{dt}\right\}$ is a basis of $M^1_{\sigma(L)}(Z)$. $\{X_{\lambda+1}(L), \ldots, X_n(L)\}$ is a basis of $M^2_{\sigma(L)}(Z)$ (see 3.2(11), (12)).

$\left\{\frac{DX_1}{dt}(L), \ldots, \frac{DX}{dt}(L), X_{\lambda+1}(L), \ldots, X_n(L)\right\}$ is hence a basis of $M_{\sigma(L)}$ and so A_σ has no zeros.

Q.e.d.

From Theorem 3 we can deduce that for a compact, complete, globally harmonic manifold all geodesics are closed and of length 2L.

For let σ be a unit-speed geodesic ray from $p_0 \in M$ with $\sigma(L) = p$. p_0 and p being conjugate let σ' be another unit-speed geodesic ray from p_0 such that $\sigma'(L) = p$ and $\sigma'_*(L) = -\sigma_*(L)$. We define $\gamma: [0, 2L] \longrightarrow M$ by

$$\gamma(s) = \begin{cases} \sigma(s) & \text{if } 0 \leqslant s \leqslant L \\ \sigma'(2L - s) & \text{if } L \leqslant s \leqslant 2L. \end{cases}$$

Then γ is a closed curve from p_0. We must show that $\gamma_*(0) = -\gamma_*(2L)$. Let $0 < a < L$. Then $\gamma|_{[a, a+L]}$ is a geodesic arc of length L and hence joins conjugate points.

The arc $\gamma' : [0, L] \longrightarrow M$ given by

$$\gamma'(s) = \begin{cases} \gamma(a - s) & \text{if } 0 \leqslant s \leqslant a \\ \gamma(2L + a - s) & \text{if } a \leqslant s \leqslant L \end{cases}$$

is continuous, of length L, joins $\gamma(a)$ to $\gamma(a + L)$ and hence is a geodesic arc. γ' is C^{∞} and the result follows.

Harmonic manifolds of dimension 2 or 3 with positive-definite metric were shown in Chapter I Theorem 5 to be necessarily of constant curvature and hence locally symmetric. Further it is known that harmonic manifolds of dimension 4 with positive-definite metric are locally symmetric ([11], pp.142-150). A.Lichnerowicz has conjectured that all harmonic manifolds are necessarily locally symmetric. A.Avez [2] considered the case of compact, simply connected, globally harmonic manifolds in Theorem 4 below.

<u>Theorem 4</u> Let M be a compact, simply connected, globally harmonic Riemannian manifold with positive-definite metric. Then M is locally symmetric.

<u>Proof</u> M being compact we can define for $f, g \in C^2(M)$ the global scalar product

$$\langle f, g \rangle = \int_M f(p)\, g(p)\, \tau_p$$

where τ_p is the volume element at $p \in M$.

Let $p_0 \in M$. Then we have seen as a consequence of Theorem 3 above that all geodesics through p_0 are closed and of length 2L; geodesic arcs of length less than L have no conjugate points. The geodesic symmetry, s_{p_0}, can be defined globally on M as follows: if $p \in R_{p_o}$, the residual locus of p_o, we have $s_{p_o}(p) = p$; if $p \notin R_{p_o}$, $s_{p_o}(p) = q$, where p_o is the mid-point of the geodesic arc (pq).

Let f be an eigenfunction of the Laplace-Beltrami operator Δ_2, that is

(8) $$(\Delta_2 f)(p) = \lambda f(p) \qquad (p \in M).$$

Fix $p_o \in M$ and let $N(p, p_o)$ be the elementary solution of $\Delta_2 N(p, p_o) = \delta_p(p_o)$, where $\delta_p(p_o)$ is the Dirac 'δ'-function. $N(p, p_o)$ can be regarded as the potential due to a unit charge at p_o. We have by definition of 'δ':

$$f(p_o) = \int_M f(p)\, \delta_p(p_o)\, \tau_p$$
$$= \langle \Delta_2 N(p, p_o), f(p) \rangle.$$

Now Δ_2 being a symmetric differential operator ([5], p.387) we have

(9) $$\langle N(p, p_o), (\Delta_2 f)(p) \rangle = f(p_o).$$

(8) and (9) imply

(10) $$\lambda \langle N(p, p_o), f(p) \rangle = f(p_o).$$

We write $p = s_{p_o}(q)$. Lichnerowicz [8] has shown that $\tau_{s_{p_o}}(q) = \tau_q$ and Allamigeon that $N(p, p_o)$ is a function of $d(p, p_o)$ alone, i.e.,

$$N(s_{p_o}(q), p_o) = N(q, p_o).$$

Hence,

$$f(p_o) = \lambda \int_M N(s_{p_o}(q), p_o)\, f(s_{p_o}(q))\, \tau_{s_{p_o}}(q)$$
$$= \lambda \int_M N(q, p_o)\, f(s_{p_o}(q))\, \tau_q,$$

that is,

(11) $\lambda < N(p, p_o), f(s_{p_o}(p)) > \ = \ f(p_o).$

Now let $h(p)$ be another eigenfunction of Δ_2 with eigenvalue μ different from λ, i.e.,

$$(\Delta_2 h)(p) = \mu h(p).$$

From the symmetry of Δ_2 we have

$$< \Delta_2 f, h > = < f, \Delta_2 h >,$$

whence, $< \lambda f, h > = < f, \mu h >,$

that is, $(\lambda - \mu) < f, h > = 0.$

Thus $\lambda \neq \mu$ implies $< f, h > = 0.$

From (11) we have

$$<< N(p, p_o), f(s_{p_o}(p)) >, h(p_o) > = < f, h > = 0.$$

Let $s = d(p, p_o)$. In a neighbourhood of p_o, $N(p, p_o) = O(s^{2-n})$. The integral $< N(p, p_o), f(s_{p_o}(p)) >$ is therefore uniformly convergent in p_o on M and Fubini's theorem gives

$$<< N(p, p_o), f(s_{p_o}(p)) >, h(p_o) > = << N(p, p_o), h(p_o) >, f(s_{p_o}(p)) >$$
$$= 0.$$

But using the symmetry property, $N(p, p_o) = N(p_o, p)$, we have from (10)

$$\mu < N(p, p_o), h(p_o) > = h(p),$$

whence, $< h(p), f(s_{p_o}(p)) > = 0.$

The function $(f \circ s_{p_o})$ is therefore orthogonal to E, the set of eigenfunctions of Δ_2 whose eigenvalues are not equal to λ. We know that the set $E \cup \{f : \Delta_2 f = \lambda f, f \in C^2(M)\}$ is dense in C(M). Consequently, $f(s_{p_o}(p)) \in \{f : \Delta_2 f = \lambda f, f \in C^2(M)\}$ and $(\Delta_2 f)(s_{p_o}(p)) = \lambda f(s_{p_o}(p))$. We put $\bar{g}_p = g_{s_{p_o}(p)}$, where g is the given positive-definite metric on M and write $\bar{\Delta}_2$ for the Laplacian of $\bar{g}$. Hence $(\bar{\Delta}_2 f)(p) = \lambda f(p)$ and using (8) we have $(\bar{\Delta}_2 - \Delta_2)(f) = 0$.

A result of Kolmogoroff has shown that corresponding to $F \in C^2(M)$ there exists a sequence of finite linear combinations of eigenfunctions of Δ_2 converging uniformly to F. The same property holds for all partial derivatives of F of order less than or equal to 2.

Hence, if $F \in C^2(M)$,

$$(\bar{\Delta}_2 - \Delta_2)(F) = 0. \tag{12}$$

Fix $p_o \in M$ and let p be an arbitrary point of M. Let (y^i) be a normal coordinate system origin p derived from the local coordinates, that is, the metric $^*g = {}^*(g_{ij})$ and the given metric satisfy $^*g_p = g_p$. Now M being a compact metric space is complete ([6], p.81). Hence the coordinate homeomorphism

$$\phi : q \longrightarrow (y^1(q), \ldots, y^n(q)) \in \mathbb{R}^n \text{ has domain}$$

$M - R_p$. Let $X = X^i \frac{\partial}{\partial y^i}(p) \quad M_p$ and define

$$F_q = F_X(q) = \tfrac{1}{2} X_i X_j y^i(q) y^j(q), \text{ where } q \in M - R_p$$

and $X_i = g_{ik}X^k$. We will write ∂_i for $\frac{\partial}{\partial y^i}$.

As $(\partial_i F)(p) = 0$ $(i = 1, \ldots, n)$ we have

$$\begin{aligned}(\Delta_2 F)(p) &= ((\det {}^*g)^{-\frac{1}{2}}\, \partial_j({}^*g^{ij}(\det {}^*g)^{\frac{1}{2}}\partial_i F))(p)\\ &= ({}^*g^{ij}\partial_j\, \partial_i F)(p)\\ &= ({}^*g^{ij})_p X_i\, X_j\\ &= g_p(X, X).\end{aligned}$$

From (11) we have

$$\bar{g}_p(X, X) = g_p(X, X).$$

As p and X are arbitrary, $\bar{g} = g$. Hence M is locally symmetric. Since M is complete and simply connected, it is also globally symmetric (see p. 34).

<u>Remark</u> The function F defined near the bottom of p. 67 is not necessarily globally defined on M and hence does not belong to $C^2(M)$.

For, let $Z = (Z^i) \in M_p$ be any unit vector orthogonal to X, i.e., $X_i Z^i = 0$. Let $Y = (g_p(X,X))^{-\frac{1}{2}} X$.

Define $\sigma_a : [0,L] \longrightarrow M$ $(a = 1, 2)$ by $\sigma_1(t) = \mathrm{Exp}_p tY$, $\sigma_2(t) = \mathrm{Exp}_p tZ$. Then σ_1 and σ_2 are geodesic rays joining p to points of R_p. We have

$$F(\sigma_1(t)) \longrightarrow \tfrac{1}{2}L^2\, g_p(X,X) \text{ as } t \longrightarrow L, \text{ but}$$

$$F(\sigma_2(t)) \longrightarrow 0.$$

In the case where R_p is a single point locus, for example, on S^2, there is a singularity at this point.

CHAPTER IV

k-HARMONIC RIEMANNIAN MANIFOLDS

4.1 Definitions of k-harmonic Riemannian manifolds

k-harmonic manifolds were first defined by T.J.Willmore in 1966 [16]. Let M be any Riemannian manifold and $p_0 \in M$ be the origin of a normal coordinate system with domain W. We will use the notational convention of Section 1.2:

Let $p \in W$. $D(p_0)$ and $D(p)$ are the vector spaces of tensors at p_0 and p respectively. Greek suffices will be used for components of members of $D(p_0)$ and Roman suffices for components of members of $D(p)$. We denote by

$$D(p_0, p) = D(p_0) \otimes D(p)$$

the space of bi-tensors over (p_0, p) (see for example, J.L.Synge, "Relativity: The General Theory," North-Holland, 1964, pp.48-50).

An example of a bi-tensor is the tensor with components

$$\Omega_{\alpha i} = \frac{\partial^2 \Omega}{\partial x^\alpha \partial x^i}, \tag{1}$$

where $\Omega(p_0, p)$ is the distance function. We also define the bi-tensors with components $\Omega_i^\alpha = g^{\alpha\beta}\Omega_{\beta i}$, $\Omega_\alpha^i = g^{ij}\Omega_{\alpha j}$, and we note from (1) that $\Omega_{\alpha i} = \Omega_{i\alpha}$.

The "pure" bi-tensors (ω_j^i), (ω_β^α) are defined by

$$\omega_j^i = \Omega_\alpha^i \Omega_j^\alpha, \tag{2}$$

$$\omega_\beta^\alpha = \Omega_i^\alpha \Omega_\beta^i. \tag{3}$$

These are two-point invariant functions and we have

$$\det(\omega^i_j) = \det(\omega^\alpha_\beta) = \frac{J^2}{g g_0},$$

where $J = \det(\Omega_{\alpha i})$, $g = \det(g_{ij})$, $g_0 = \det(g_{\alpha\beta})$.

Clearly from 1.3(7) we have

$$(4) \qquad \det(\omega^i_j) = \det(\omega^\alpha_\beta) = \frac{1}{\rho^2},$$

where $\rho(p_0, p)$ is Ruse's invariant.

Let $\sigma_k = \sigma_k(p_0,p)$ denote the k^{th} symmetric polynomial of the eigenvalues of the matrix (ω^i_j). The following is Willmore's definition:

<u>Definition (1)</u> A Riemannian manifold, M, is <u>k-harmonic at p_0</u> if p_0 is the origin of a normal neighbourhood W, such that, if $p \in W$ and Ω is the distance function, then $\sigma_k(p_0,p)$ is a function which depends only upon Ω and not otherwise upon p. M is <u>k-harmonic</u> if it is k-harmonic at p_0 for all $p_0 \in M$.

The matrices (ω^i_j) and (ω^α_β) can be interpreted geometrically as follows:

Let $p_0 \in M$ and $p \in W$, where W is a normal neighbourhood origin p_0. Then there exists $Y \in M_{p_0}$ such that $\mathrm{Exp}_{p_0} Y = p$. The mapping $(d\,\mathrm{Exp}_{p_0})_Y : (M_{p_0})_Y \longrightarrow M_p$ is linear. Let w be the identification $(M_{p_0})_Y \longrightarrow M_{p_0}$. Then w is a natural linear isomorphism and the mapping $(d\,\mathrm{Exp}_{p_0})_Y \circ w^{-1} : M_{p_0} \longrightarrow M_p$ is linear. Let $t = (t^i_\alpha)$ be its matrix. We have shown on pp. 8-9 that t is also the matrix of the dual map $w \circ (\mathrm{Exp}^*)_Y : {}^*M_p \longrightarrow {}^*M_{p_0}$. The inverse matrix $t^{-1} = (t^{-\alpha}_{\ i})$

is the matrix of the linear maps $M_p \longrightarrow M_{p_0}$ and ${}^*M_{p_0} \longrightarrow {}^*M_p$.

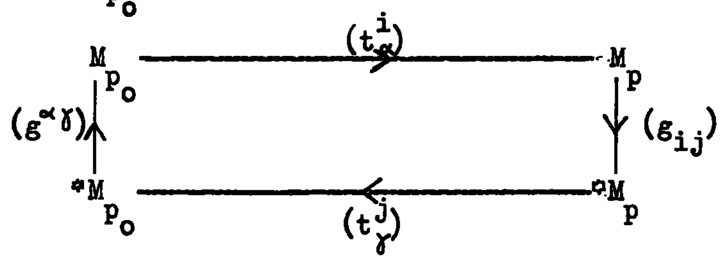
Now let $X = (X^\beta) \in M_{p_0}$ and consider the composition of linear maps

$$X^\beta \longrightarrow t^i_\beta X^\beta \longrightarrow g_{ij} t^i_\beta X^\beta \longrightarrow t^j_\gamma g_{ij} t^i_\beta X^\beta \longrightarrow g^{\alpha\gamma} t^j_\gamma g_{ij} t^i_\beta X^\beta.$$

Let

(5) $$a^\alpha_\beta = g^{\alpha\gamma} t^j_\gamma g_{ij} t^i_\beta = g^{\alpha\gamma} h_{\gamma\beta},$$

where $h_{\gamma\beta} = t^j_\gamma t^i_\beta g_{ij}$. (a^α_β) is the matrix of a linear endomorphism of M_{p_0} as can be seen from the diagram below.



This diagram also defines a linear endomorphism of M_p with matrix (a^i_j) given by

(6) $$a^i_j = t^i_\alpha g^{\alpha\gamma} t^k_\gamma g_{kj} = h^{ik} g_{kj},$$

where $h^{ik} = t^i_\alpha t^k_\gamma g^{\alpha\gamma}$. Using inverses we obtain the inverse endomorphisms of M_{p_0} and M_p with matrices (b^α_β) and (b^i_j) respectively given by

(7) $$b^\alpha_\beta = (t^{-\alpha}_{\ i}\, t^{-\gamma}_{\ j}\, g^{ij}) g_{\gamma\beta} = h^{\alpha\gamma} g_{\gamma\beta}, \text{ and}$$

(8) $$b^i_j = g^{ik}(t^{-\alpha}_{\ k}\, t^{-\beta}_{\ j}\, g_{\alpha\beta}) = g^{ik} h_{kj}.$$

Clearly $(a^i_j)^{-1} = (b^i_j)$, $(a^\alpha_\beta)^{-1} = (b^\alpha_\beta)$ and also $(h_{\alpha\beta})^{-1} = (h^{\alpha\beta})$, $(h_{ij})^{-1} = (h^{ij})$.

We note
$$a^\alpha_\beta = t^{-\alpha}_{\ i}\, a^i_j\, t^j_\beta, \text{ and}$$
$$b^\alpha_\beta = t^{-\alpha}_{\ i}\, b^i_j\, t^j_\beta.$$

Hence (a^α_β) and (a^i_j) have equal eigenvalues; likewise (b^α_β) and (b^i_j).

Suppose now that we have a normal coordinate system origin p_o. Then $t^i_j = \delta^i_j$. From 1.3(6) we have $y^\alpha = -{}^*\Omega^\alpha$ and so from (2)

$$\omega^i_j = {}^*\Omega^i_\alpha {}^*\Omega^\alpha_j = {}^*g^{ik}{}^*g_{\alpha\beta}{}^*\Omega^\beta_k {}^*\Omega^\alpha_j$$

$$= {}^*g^{ik}{}^*g_{\alpha\beta}\, \delta^\beta_k\, \delta^\alpha_j.$$

But from (8),

$$b^i_j = {}^*g^{ik}{}^*g_{\alpha\beta}\, \delta^\beta_k\, \delta^\alpha_j.$$

Hence $(\omega^i_j) = (b^i_j)$ and similarly $(\omega^\alpha_\beta) = (b^\alpha_\beta)$.

We can also derive the expressions

(9) $$a^\alpha_\beta = \Omega^{-\alpha}_i\, \Omega^{-i}_\beta,$$

(10) $$a^i_j = \Omega^{-i}_\alpha\, \Omega^{-\alpha}_j,$$

where $(\Omega^{-\alpha}_i) = (\Omega^i_\alpha)^{-1}$ and $(\Omega^{-i}_\alpha) = (\Omega^\alpha_i)^{-1}$.

K. El Hadi has defined k-harmonic spaces somewhat differently ([4], pp. 88-91).

Let M be a complete Riemannian manifold and $p_o \in M$. Let $u \in M_{p_o}$ be a unit vector. Then for $0 \leq t < \infty$ we define

$$f_{tu} = (d\,\mathrm{Exp}_{p_o})_{tu} \circ (w(tu))^{-1},$$

where $w(tu)$ is the identification $(M_{p_o})_{tu} \longrightarrow M_{p_o}$.

Let $p = \mathrm{Exp}_{p_o} tu$. $f_{tu} : M_{p_o} \longrightarrow M_p$ is linear.

We define h_{p_o} as the pull-back of g via f_{tu},

i.e., $$h_{p_o}(X,Y) = (f^*_{tu}\, g_p)(X,Y) \text{ for all } X,Y \in M_{p_o}.$$

We now have two symmetric bilinear forms at p_o, both non-degenerate. Hence there exists an endomorphism ϕ_{tu} of M_{p_o}

such that $g_{p_o}({}_{tu}X,Y) = h(X,Y)$ for all $X,Y \in M_{p_o}$.
In fact given a local coordinate system (x^i) we have

(11) $$(\phi_{tu})^\alpha_\beta = g^{\alpha\gamma}h_{\gamma\beta}.$$

By $\sigma_k(\phi_{tu})$ we denote the k^{th} symmetric polynomial of the eigenvalues of (ϕ_{tu}). El Hadi makes the following definition.

<u>Definition (2)</u> A Riemannian manifold M is <u>k-harmonic at p_o</u> if $\sigma_k(\phi_{tu})$ is a function of Ω alone.

Comparison of (5) and (11) shows that $(\phi_{tu})^\alpha_\beta = a^\alpha_\beta$ and hence (ϕ_{tu}) and ω are inverse matrices. The definitions (1) and (2) are not equivalent but complementary; each defines n formally distinct sets of Riemannian manifolds. We will distinguish between the two types by referring to k-harmonic manifolds of the positive type or k-harmonic manifolds of the negative type arising from symmetric polynomials of the eigenvalues of (a^α_β) (i.e., ϕ_{tu}) or (b^α_β) (i.e., ω) respectively.

<u>Definition (3)</u> Let M be a Riemannian manifold and $p_o \in M$. Let $p \in W$, where W is a normal neighbourhood of p_o. If $p = \operatorname{Exp}_{p_o} X$, let $h = h_{p_o} = (w \circ (\operatorname{Exp}_{p_o}{}^{*})_X)\, g_p$, where w is the identification defined on p.70. Let $a = (a^\alpha_\beta)$ be the matrix of the endomorphism $g_{p_o} h^{-1}$ and $b = (b^\alpha_\beta) = a^{-1}$.

(i) M is <u>k-harmonic of the positive type at p_o</u> (or (+k)-harmonic at p_o) if $\sigma_k = \sigma_k(a(p_o,p))$, the k^{th} symmetric polynomial of the eigenvalues of a is a function of Ω alone $(1 \leq k \leq n)$.

(ii) M is <u>k-harmonic of the negative type at p_o</u> (or $(-k)$-harmonic at p_o) if $\sigma_{-k} = \sigma_{-k}(b(p_o,p))$, the k^{th} symmetric polynomial of the eigenvalues of b is a function of Ω alone $(1 \leq k \leq n)$.

M is <u>(+k)-harmonic</u> (resp. <u>(-k)-harmonic</u>) if it is (+k)-harmonic (resp. (-k)-harmonic) at all points p_o on M.

<u>Definition (4)</u> (i) M is <u>simply (+k)-harmonic</u> if σ_k is constant.

(ii) M is <u>simply (-k)-harmonic</u> if σ_{-k} is constant.

4.2 <u>Properties of k-harmonic manifolds</u>

From 4.1(4) it is clear that an n-dimensional Riemannian manifold is classically harmonic if and only if it is n-harmonic and if and only if it is (-n)-harmonic. Hence (+k)- and (-k)-harmonic manifolds can be regarded as generalisations of harmonic manifolds. It can be conjectured that the $(2n - 2)$ sets of k-harmonic manifolds $(k = \pm 1, \ldots, \pm(n - 1))$ are all precisely the set of harmonic manifolds. In this case we would have shown that harmonic manifolds possess $(2n - 2)$ two-point invariants distinct from Ruse's invariant which are all functions of Ω alone. The conjecture may be false.

We will adopt the following notation:

Let H(n) be the set of n-dimensional harmonic manifolds; H(k; n), the set of k-harmonic manifolds of the positive type and H(-k; n), the set of k-harmonic manifolds of the negative type. We can list various fields of enquiry.

A. What subset or equality relations exist between the sets $H(n)$ and $H(\pm k;\ n)$ $(1 \leqslant k \leqslant n)$?

B. Can simply $\pm k$-harmonic manifolds be characterised?

C. Can necessary and sufficient conditions for a manifold to belong to $H(\pm k;\ n)$ be given either in terms of affine normal tensors (cf. Copson & Ruse equations) or in terms of curvature tensors (cf. Ledger's equations)?

D. Are k-harmonic manifolds necessarily Einstein spaces?

E. If M is a symmetric manifold can simple conditions that $M \in H(k;\ n)$ be obtained? What is the rank of k-harmonic symmetric manifolds? If $M \in H(k;\ n)$ for one value k, does this imply that $M \in H(k;\ n)$ for all $k \in \{\pm 1, \ldots, \pm n\}$?

Regarding Problem A two results can be immediately stated.

<u>Theorem 1</u> Let M be an n-dimensional Riemannian manifold. The following statements are equivalent:

(i) M is harmonic.

(ii) M is n-harmonic.

(iii) M is (-n)-harmonic.

<u>Proof</u> We use 4.1(4).

<u>Theorem 2</u> Let M be a harmonic Riemannian manifold. Let $1 \leqslant k \leqslant n$.

(a) If M is k-harmonic, then M is -(n-k)-harmonic.

(b) If M is (-k)-harmonic, then M is (n-k)-harmonic.

(Symbolically: $H(-(n-k);\ n) \subset H(k;\ n) \cap H(n)$

and $H(n-k;\ n) \subset H(-k;\ n) \cap H(n)$.)

<u>Proof</u> We compare characteristic equations of (a^{α}_{β}) and (b^{α}_{β}).

The following result of T.J.Willmore is less trivial ([16], p.1056).

<u>Theorem 3</u> All harmonic Riemannian manifolds are (-1)-harmonic.

<u>Proof</u> Let $p_0 \in M$ and $p \in W$, where W is a normal neighbourhood of p_0. In the case of normal coordinates origin p_0 we have using 1.3(6) the expression for Beltrami's first differential parameter

$$\Delta_1 {}^*\Omega = {}^*g^{\alpha\beta} \frac{\partial^*\Omega}{\partial y^\alpha} \frac{\partial^*\Omega}{\partial y^\beta} = {}^*g_{\alpha\beta} y^\alpha y^\beta = 2^*\Omega.$$

Hence for any allowable coordinate system on W with p_0 and p having coordinates (x^α) and (x^i) respectively, we have

$$(1) \qquad \Omega^\alpha \Omega_\alpha = 2\,\Omega,$$

where $\Omega^\alpha = g^{\alpha\beta}\Omega_\beta = g^{\alpha\beta}\dfrac{\partial\Omega}{\partial x^\beta}$. We differentiate (1) covariantly with respect to x^k, apply the transformation $g_p^{-1} = (g^{ik}) : {}^*M_p \longrightarrow M_p$ and obtain from the i^{th} component:

$$(2) \qquad \Omega^i_\alpha \Omega^\alpha = \Omega^i \quad (i = 1, \ldots, n).$$

(2) is differentiated covariantly on both sides with respect to x^j and we obtain

$$\Omega^i_\alpha \Omega^\alpha_j + \Omega^i_{\alpha,j}\Omega^\alpha = \Omega^i_j ,$$

whence,

$$b^i_j = \Omega^i_j - \Omega^i_{j,\alpha}\Omega^\alpha,$$

where in the last term we have interchanged the order of covariant differentiation. Hence

$$\sigma_{-1} = \operatorname{tr}(b^i_j) = \Delta_2\Omega - (\Delta_2\Omega)_{,\alpha}\Omega^\alpha.$$

Now M being harmonic we have $\Delta_2\Omega = \chi(\Omega)$, where χ is the characteristic function. Thus

$$\sigma_{-1} = \chi(\Omega) - \chi'(\Omega)\Omega^{\alpha}\Omega_{\alpha},$$

that is

$$\sigma_{-1} = \chi(\Omega) - 2\chi'(\Omega)\Omega \text{ from } (1).$$

Q.e.d.

<u>Corollary 1</u> All harmonic Riemannian manifolds are (n-1)-harmonic.

<u>Proof</u> We use Theorem 2.

<u>Corollary 2</u> All simply harmonic Riemannian manifolds are simply (-1)-harmonic and simply (n-1)-harmonic.

<u>Proof</u> $\chi(\Omega) = n$ implies $\sigma_{-1} = n$, using (3).

The converse of Theorem 3 has not been proved and may indeed be false. However El Hadi has proved the converse of Corollary 2.

<u>Theorem 4</u> All simply (-1)-harmonic Riemannian manifolds are simply harmonic.

<u>Proof</u> (Outline) $\sigma_{-1} = n$ implies

$$\Delta_2\Omega - \Omega^{\alpha}(\Delta_2\Omega)_{\alpha} = n.$$ Writing

$f = \Delta_2\Omega - n$, it can be shown that the differential equation

$$f + y^{\alpha}\frac{\partial f}{\partial y^{\alpha}} = 0$$ with boundary conditions

$f \to 0$ as $y^{\alpha} \to 0$ $(\alpha = 1, \ldots, n)$ has unique solution $f \equiv 0$ on W.

For details see [4] pp. 99-102.

It is highly plausible that all harmonic Riemannian manifolds are k-harmonic for all $k \in \{\pm 1, \ldots, \pm n\}$. We will show in 4.6 that compact, simply connected, symmetric harmonic spaces are k-harmonic for all k. The converse of Theorem 3 may be false but no counterexamples have yet been found.

4.3 k-harmonic and Einstein spaces

With regard to Problem D of Section 4.2 we are able to reach a definite conclusion. Use is made of the expressions connecting affine and Riemannian normal tensors which were derived in Section 1.2.

Theorem 5 All (-1)-harmonic Riemannian manifolds are Einstein spaces.

Proof Let M be a (-1)-harmonic Riemannian manifold. Let $p_o \in M$ and W be a normal neighbourhood of p_o. M being analytic, there exists a subset U of W which is a neighbourhood of p_o in which the n^2 functions $({}^{\circ}g^{ij})_p$ $(p \in U)$ can be expanded as a Maclaurin series in terms of $({}^{\circ}g^{ij})_{p_o} = ({}^{\circ}g^{ij})_o$. Let p have normal coordinates (y^i). Then $y^i = X^i s$, where s is the geodesic distance, $d(p_o,p)$ and $X = (X^i)$ is the unit tangent vector at p_o defining the geodesic arc $(p_o p)$. Now

$$({}^{\circ}g^{ij})_p = ({}^{\circ}g^{ij})_o + (\partial_k{}^{\circ}g^{ij})_o y^k + (\partial_l\partial_k{}^{\circ}g^{ij})_o \frac{y^k y^l}{2!} + \dots$$

$$= ({}^{\circ}g^{ij})_o + ({}^{\circ}g^{ij}_{.k})_o X^k s + ({}^{\circ}g^{ij}_{.kl})_o X^k X^l \frac{s^2}{2!} + O(s^3).$$

Hence,

$$\sigma_{-1}(p_o,p) = (b^i_i)$$

$$= n + ({}^{\circ}g_{ij})_o({}^{\circ}g^{ij}_{.k})_o X^k s + ({}^{\circ}g_{ij})_o({}^{\circ}g^{ij}_{.kl})_o X^k X^l \frac{s^2}{2!} + O(s^3).$$

We now use the hypothesis that M is (-1)-harmonic.

$$\sigma_{-1}(p_o,p) = f(\Omega) = a_o + a_1\, s^2/2 + a_2\, s^4/2^2.2! + O(s^6),$$

where the coefficients a_k are given by

$$a_k = \left(\frac{d^k f(\Omega)}{d\Omega^k}\right)_o.$$

We now compare coefficients of the two Maclaurin series to get

$$a_o = n;$$

$$({}^{\circ}g_{ij})_o({}^{\circ}g^{ij}_{.k})_o X^k = 0 \quad \text{(cf. 1.2(11))};$$

(1) $$({}^{\circ}g_{ij})_o({}^{\circ}g^{ij}_{.kl})_o X^k X^l = a_1;$$

(2) $$({}^{\circ}g_{ij})_o({}^{\circ}g^{ij}_{.klm})_o X^k X^l X^m = 0;$$

(3) $$({}^{\circ}g_{ij})_o({}^{\circ}g^{ij}_{.klmn})_o X^k X^l X^m X^n = 3\, a_2, \text{ and so on.}$$

Using equations (1) and 1.2(14), the independence of a_1 and X implies that $-\frac{2}{3}(R_{kl})_o X^k X^l$ is also independent of X. But X is a unit vector and so

$$(R_{kl})_o = k_1\, (g_{kl})_o, \text{ where } k_1 = -3/2\, a_1.$$

p_o being an arbitrary point of M, it follows that M is an Einstein space.

Q.e.d.

Equations (2) and (3) are necessary conditions for M to be (-1)-harmonic and merit further examination. The method is similar to that used to derive the Copson and Ruse equations. Equations (2) and 1.2(15) yield at p_o (and hence on M):

$$R_{lm,k} + R_{mk,l} + R_{kl,m} = 0,$$

which is an immediate consequence of the Einstein condition. However, using (3) and 1.2(16) we obtain at p_o:

(4) $$A^i_{iklmn} - 2\, S'(A^i_{hkl})(A^h_{imn}) - 2\, S'(A^i_{hkl})(A^j_{pmn})({}^{\circ}g_{ij})({}^{\circ}g^{hp}) = k_2\, S({}^{\circ}g_{kl}{}^{\circ}g_{mn}).$$

We define

$$B_{klmn} = S'(A^{i}_{hkl})(A^{h}_{imn}) + S'(A^{i}_{hkl})(A^{j}_{pmn})({}^{*}g_{ij})({}^{*}g^{hp})$$

and use Theorem 5 and the Copson and Ruse equation

(5) $$A^{i}_{iklmn} = h_2 S({}^{*}g_{kl}{}^{*}g_{mn}) \quad (1.5(3_3))$$

to deduce that in a <u>harmonic space</u>

(6) $$B_{klmn} = L\, S({}^{*}g_{kl}{}^{*}g_{mn}),$$

where L is a constant which may well be zero. A direct verification of (6) is by no means obvious. The converse result, namely that (4) implies (5) and (6) independently seems very implausible and strengthens belief that the converse of Theorem 3 is false.

We now prove the general theorem.

<u>Theorem 6</u> All (-k)-harmonic manifolds are Einstein spaces for all k, $1 \leq k \leq n$.

<u>Proof</u> We need an expression for σ_{-k} similar to that for σ_{-1} on the bottom of p.78. Let

$$\lambda^n + \sum_{k=1}^{n} (-1)^k \sigma_{-k} \lambda^{n-k} = 0$$

be the characteristic equation of (b^{α}_{β}) with real eigenvalues $\lambda_1, \ldots, \lambda_n$. Writing $s_r = (\lambda_1)^r + \ldots + (\lambda_n)^r$ and $a_k = (-1)^k \sigma_{-k}$ we have Newton's formulae:

$$\left.\begin{array}{l} a_1 = -s_1 \\ s_1 a_1 + 2a_2 = -s_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ s_{k-1}a_1 + s_{k-2}a_2 + \ldots + s_1 a_{k-1} + k a_k = -s_k \end{array}\right\}$$

These solve to give $a_k = \frac{(-1)^k}{k!} b_k$, i.e., $\sigma_{-k} = \frac{1}{k!} b_k$, where

$$b_k = \begin{vmatrix} s_1 & 1 & & & & 0 \\ s_2 & s_1 & 2 & & & \\ s_3 & s_2 & s_1 & 3 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ s_{k-1} & s_{k-2} & \cdots & & & (k-1) \\ s_k & s_{k-1} & \cdots & & & s_1 \end{vmatrix}.$$

Expanding, $b_k = s_1 b_{k-1} - (k-1) \begin{vmatrix} & & 0 \\ & & 0 \\ A_{k-1} & & \vdots \\ & & (k-2) \\ \hline s_k & s_{k-1} \cdots & s_2 \end{vmatrix}$,

where the $(k-1) \times (k-1)$ matrix, A_{k-1} is such that det $A_{k-1} = b_{k-2}$.

Hence we obtain the reduction formula

$$(7) \quad b_k = s_1 b_{k-1} - (k-1)s_2 b_{k-2} + (k-1)(k-2)s_3 b_{k-3} + \cdots + (-1)^{k-1} k! s_{k-1} b_1 + (-1)^k k! s_k.$$

The term

$$c\, s_{p_1}^{q_1} s_{p_2}^{q_2} \cdots s_{p_t}^{q_t}, \text{ where } c \text{ is constant and}$$

p_i and q_i are positive integers for $i \in \{1, \ldots, t\}$, is said to be of <u>weight w</u> if $w = \sum_{i=1}^{t} p_i q_i$. A polynomial is said to be <u>isobaric of weight w</u> if all of its terms are of weight w. From (7) an easy induction argument shows that b_k and hence σ_{-k} are both isobaric of weight k. Hence we can write

$$(8) \qquad \sigma_{-k} = \sum_{p,q,t} c_{p_1,\ldots,p_t}^{q_1,\ldots,q_t} s_{p_1}^{q_1} \cdots s_{p_t}^{q_t},$$

where summation is taken over all $t, p_1, \ldots, p_t, q_1, \ldots, q_t$ satisfying

(i) $t \in \{1, \ldots, k\}$;

(ii) $1 \leq p_1 < p_2 < \ldots < p_t \leq k$;

(iii) $q_1, \ldots, q_t$ are positive integers satisfying $\sum_{i=1}^{k} p_i q_i = k$ and $c^{q_1, \ldots, q_t}_{p_1, \ldots, p_t}$ are rational numbers depending on the combinations $\{p_1, \ldots, p_t\}$ and $\{q_1, \ldots, q_t\}$.

Now
$$\begin{aligned} s_p{}^q &= (\lambda_1{}^p + \lambda_2{}^p + \ldots + \lambda_n{}^p)^q \quad \text{(p,q positive integers)} \\ &= (\mathrm{tr}(b^i_j)^p)^q \\ &= (b^i_l\, b^l_m \ldots b^r_i)^q \quad \text{(product of p terms)} \end{aligned}$$

that is,
$$\begin{aligned} s_p{}^q &= \Big[(\delta^i_l + ({}^*g_{lj})_o({}^*g^{ij}_{.uv})_o X^u X^v \Omega + o(\Omega^2)) \ldots \\ &\quad \ldots (\delta^r_i + ({}^*g_{ij})_o({}^*g^{rj}_{.uv})_o X^u X^v \Omega + o(\Omega^2))\Big]^q \\ &= \left[n + p({}^*g_{ij})_o({}^*g^{ij}_{.uv})_o X^u X^v \Omega + o(\Omega^2)\right]^q \\ &= n^q - \tfrac{2}{3}\, pq\, n^{q-1}(R_{uv})_o X^u X^v \Omega + o(\Omega^2), \end{aligned}$$

where use has been made of 1.2(14).

If we now define
$$\begin{aligned} B_{p,q} &= -\tfrac{2}{3}\, pq\, n^{q-1}, \\ A_k &= \sum_{p,q,t} c^{q_1, \ldots, q_t}_{p_1, \ldots, p_t}\, n^{q_1 + \ldots + q_t}, \\ B_k &= \sum_{p,q,t} c^{q_1 \ldots, q_t}_{p_1, \ldots, p_t} \sum_{j=1}^{t} n^{r_j} B_{p_j, q_j}, \end{aligned}$$

where $r_j = q_1 + \ldots + q_t - q_j$, then clearly A_k, B_k are constants for all $k \in \{1, \ldots, n\}$. Using (8) we have

(9) $$\sigma_{-k} = A_k + B_k\,(R_{uv})_o X^u X^v \Omega + o(\Omega^2),$$

noting that $A_1 = n$ and $B_1 = \frac{2}{3}$.

Theorem 6 is now completed on the same lines as Theorem 5.

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We now consider the case of k-harmonic spaces of the positive type.

<u>Theorem 7</u> All k-harmonic Riemannian manifolds are Einstein spaces $(k > 0)$.

<u>Proof</u> (Outline) Covariant Riemannian normal tensors (instead of contravariant type) are derived in terms of affine normal tensors and hence curvature tensors.

We have

$$a^i_j = \delta^i_j + \tfrac{1}{3}(R^i_{ujv} + R^i_{vju})X^u X^v \Omega + o(\Omega^2).$$

The proof proceeds as in Theorems 6 and 5. For details see [4] pp. 94-98 where El Hadi uses a neater method than that on pp. 81-82 to show that the expressions A_k and B_k in (9) are constants. Indeed, he explicitly evaluates these coefficients and the following expression is obtained for σ_k:

$$\sigma_k = (-1)^n \binom{n-1}{k-1}\left[\frac{n}{k} + \tfrac{2}{3}(R_{uv})_o X^u X^v \Omega\right] + o(\Omega^2).$$

4.4 <u>The differential of the exponential mapping for symmetric spaces</u>

In the case of symmetric Riemannian spaces we will be able to obtain expressions for σ_k $(k = \pm 1, \ldots, \pm n)$ in terms of sectional curvatures. This is because the differential of the exponential mapping takes a simple form. This we derive in two ways: first using the Jacobi equations (3.2(8)) and secondly outlining a method using an expression of S. Helgason.

Let M be a compact, complete, simply connected, locally symmetric Riemannian manifold. We recall that for Riemannian manifolds, not necessarily locally symmetric, if $\gamma : I \longrightarrow M$ is a geodesic arc through p_o, $\{X_1(s), \ldots, X_n(s)\}$ is any orthonormal basis of $\bar{\mathcal{E}}_o(\gamma)$ and $Y(s) = y^a(s)X_a(s)$ is any member of $\bar{\mathcal{E}}_o(\gamma)$, then the Jacobi equations are

$$(1) \qquad y^{a''} + K_{ab}y^b = 0 \quad (a = 1, \ldots, n-1)$$

where $K_{ab} = K_{ab}(s) = R_{(s)}(X_a(s), \dot{\gamma}(s), X_b(s), \dot{\gamma}(s))$.
Now M is locally symmetric and γ is a geodesic. Hence on γ

$$(2) \qquad \frac{DK_{ab}(s)}{ds} = (\nabla_{\gamma_*} R)(X_a(s), \dot{\gamma}(s), X_b(s), \dot{\gamma}(s)) = 0,$$

which implies that the coefficients K_{ab} are constants on γ.
Now $K_{ba} = K_{ab}$ by the symmetry of R and so the matrix $K = (K_{ab})$ being real symmetric has real eigenvalues $K_1, \ldots, K_{n-1}$. Let these be arranged so that $K_1 \geqslant K_2 \geqslant \ldots \geqslant K_{n-1}$. Then K is orthogonally equivalent to $L = \mathrm{diag}(K_1, \ldots, K_{n-1})$. Let P be the orthogonal matrix such that $L = PKP^{-1}$. Under the orthogonal transformation $X_a(s) \longrightarrow P_a^b X_b(s)$ of bases of $\bar{\mathcal{E}}_o(\gamma)$ the Jacobi equations take the form

$$(3) \qquad y^{a''} + K_a y^a = 0 \quad (a = 1, \ldots, n-1, \text{ not summed}).$$

Clearly K_a is the sectional curvature of the plane section, $S_a(s)$, spanned by $X_a(s)$ and $\dot{\gamma}(s)$; this sectional curvature is invariant under parallel translation. In order to specify a basis of $\bar{\mathcal{E}}_o(\gamma)$ under which the Jacobi equations take the simple form (3) we make the following definition.

<u>Definition</u> Let $p_o \in M$ and $Z \in M_{p_o}$ be a unit vector. Let γ^Z be the unit-speed geodesic ray from p_o with initial vector Z. A <u>diagonal basis (or D-basis) of $\bar{\mathcal{E}}_o(\gamma^Z)$</u> is an orthonormal basis, $D_Z = \{X_1(s), \ldots, X_{n-1}(s)\}$ of $\bar{\mathcal{E}}_o(\gamma^Z)$ such that for $a \neq b$, $1 \leqslant a,b \leqslant n-1$ and $s \geqslant 0$ we have

$$R_{\gamma^Z(s)}(X_a(s), Z(s), X_b(s), Z(s)) = 0,$$

where $Z(s)$ is the parallel translate of Z on γ^Z.

A D-basis is <u>ordered</u> if the sectional curvatures $K_a = K(S_a)$ satisfy $K_1 \geqslant K_2 \geqslant \ldots \geqslant K_{n-1}$.

Note that on the hypothesis that M is compact we have $K_a \geqslant 0$ $(a = 1, \ldots, n-1)$ (see p.47).

We will now solve the Jacobi equations (3) for an ordered D-basis of $\bar{\mathcal{E}}_o(\gamma^Z)$ given the boundary conditions $y^a(0) = 0$ $(a = 1, \ldots, n-1)$.

(i) <u>Suppose $K_a = 0$.</u> We have $y^a(s) = c(a)\, s$, where each constant $c(a)$ is independent of s. Hence if $Y_a(s) = y^a(s)X_a(s)$ (unsummed), we have $Y_a(s) = s\, c(a)\, X_a(s)$ and $Y_a'(0) = c(a)\, X_a(0)$. In view of equation 3.2(2) we may choose the parameter of the variation of γ^Z defining Y so that $g_{p_o}(Y_a'(0), Y_a'(0)) = 1$ $(a = 1, \ldots, n-1)$.

Hence

(4) $$Y_a(s) = s\, X_a(s).$$

(ii) <u>Suppose $K_a > 0$.</u> Using the boundary conditions as in (i) we obtain

(5) $$Y_a(s) = \frac{\sin(\sqrt{K_a}\, s)}{\sqrt{K_a}}\, X_a(s)$$

Note that the points $\gamma^Z(\frac{n\pi}{\sqrt{K_a}})$ $(n = 1, 2, \ldots)$ are all conjugate. If

(6) $$s_o = \frac{\pi}{\sqrt{K_1}},$$

then $\gamma^Z(s_o)$ belongs to the residual locus of p_o.

Let Q be the rectangle $[0,s_o] \times [0,\epsilon_o)$ where $\epsilon_o > 0$. For each $a = 1, \ldots, n-1$ we define the variation $G_a : Q \longrightarrow S_a$ satisfying

(i) $G_a(s,0) = \tilde{\gamma}^Z(s)$, where $\tilde{\gamma}^Z : \mathbb{R} \longrightarrow S_a \subset M_{p_o}$ is the ray $s \to sZ$,

(ii) If $\frac{\partial G_a}{\partial \epsilon}(s,0) = U_a(s) \in (M_{p_o})_{sZ}$, we require $(w(sZ))(U_a(s)) = sX_a(0)$, where $w(sZ)$ is the identification $(M_{p_o})_{sZ} \longrightarrow M_{p_o}$.

Each U_a is therefore a linear homogeneous vector field on $\tilde{\gamma}^Z$, i.e., a Jacobi field in S_a. We define further the variation $H_a = \text{Exp} \circ G_a : Q \longrightarrow M$. Then each Jacobi field $Y_a(s)$ on $\gamma^Z = \text{Exp} \circ \tilde{\gamma}^Z$ given by

$$Y_a(s) = \frac{\partial H_a}{\partial \epsilon}(s,0) = (d\,\text{Exp}_{p_o})_{sZ} \circ \frac{\partial G_a}{\partial \epsilon}(s,0)$$

is related to $U_a(s)$ by

$$\begin{aligned} Y_a(s) &= (d\,\text{Exp}_{p_o})_{sZ}(U_a(s)) \\ &= s((d\,\text{Exp}_{p_o})_{sZ} \circ w(sZ)^{-1})((X_a(0)). \end{aligned}$$

We will write f_{sZ} for $(d\,\text{Exp}_{p_o})_{sZ} \circ w(sZ)^{-1}$. Using (4) and (5) we have

(7) $$(f_{sZ})(X_a(0)) = \begin{cases} X_a(s), & \text{if } K_a = 0 \\ \dfrac{\sin(\sqrt{K_a}\, s)}{\sqrt{K_a}\, s} X_a(s), & \text{if } K_a > 0. \end{cases}$$

We also have

$$(8) \qquad (f_{sZ})(Z) = Z(s).$$

An alternative method of obtaining (7) and (8) is to use a group-theoretical expression of S. Helgason for the differential of the exponential mapping in Riemannian globally symmetric spaces ([5], p.180 Theorem 4.1). Let π be the canonical projection $G \longrightarrow M$ and $\bar{Y},\bar{Z}$ be the images of Y,Z under $(d\pi)^{-1} = (d\pi)_e\Big|_{\mathfrak{m}}^{-1}$. Then

$$(9) \qquad (d\,\mathrm{Exp})_Z(Y) = d(\tau(\exp \bar{Z})) \circ (d\pi) \circ \sum_{n=0}^{\infty} \frac{(T_{\bar{Z}})^n(\bar{Y})}{(2n+1)!},$$

where $(T_{\bar{Z}})(\bar{Y}) = (\mathrm{ad}\,\bar{Z})^2\Big|_{\mathfrak{m}}(\bar{Y})$. As $(\exp s\bar{Z})$ is a transvection its differential maps vectors in M_{p_o} into vectors in M_p, where $p = \gamma^Z(s)$, by parallel translation and (8) is immediate.

Let $X_a(s) \in D_Z$ and $\bar{X}_a = (d\pi)^{-1}(X_a)(0)$. Using 2.6(6) we have

$$(T_{s\bar{Z}})(\bar{X}_a) = [s\bar{Z},[s\bar{Z},\bar{X}_a]] = (d\pi)^{-1}(s^2R(Z,X_a)Z),$$

where $X_a = X_a(0)$. Hence

$$\begin{aligned} R(Z,X_a)Z &= \sum_{b=1}^{n-1} g_{p_o}(R(Z,X_a)Z,X_a)X_b + g_{p_o}(R(Z,X_a)Z,Z)Z \\ &= -\sum_{b=1}^{n-1} R(Z,X_a,Z,X_b)X_b - R(Z,X_a,Z,Z)Z \\ &= -R(Z,X_a,Z,X_a)X_a, \text{ since } D_Z \text{ is a D-basis} \\ &= -K_aX_a. \end{aligned}$$

Hence we can show by induction that

$$(T_{s\bar{Z}})^n(\bar{X}_a) = (d\pi)^{-1}((-K_a)^n s^{2n} X_a).$$

For $K_a \geqslant 0$ the following infinite series uniformly convergent for all $s \geqslant 0$ is obtained

$$(f_{sZ})(X_a) = (d\exp s\bar{Z})\left\{\sum_{n=0}^{\infty} \frac{(-K_a)^n s^{2n}}{(2n+1)!} X_a(0)\right\}$$

and (7) is easily deduced.

4.5 <u>The action of holonomy transformations on Jacobi fields</u>

Let (G,H) be a Riemannian symmetric pair and let $M = G/H$. The linear group of transformations of M_{p_o} induced by parallel translation of frames about all closed curves through p_o is the <u>holonomy group</u>, K. Now if M is irreducible and simply connected it is known that K is connected and $K = H^o$, where H^o is the identity component of the linear isotropy group ([5], p.337).

Let γ^X be a geodesic ray from p_o with $X \in M_{p_o}$ a unit vector. Let V be a variation of γ^X so that $\{\gamma_\epsilon\}$ is a one-parameter family of unit-speed geodesic rays such that $\gamma_o = \gamma^X$. We will write $X_\epsilon(s)$ for $\gamma_{\epsilon *}(s)$. Let $T \in K$. Then there is an element h of H such that $T = (d\tau(h))_{p_o}$. Define $X'_\epsilon = T(X_\epsilon(0))$ (in M_{p_o}), $\bar{X} = (d\pi)^{-1}(X_\epsilon)$ (in $\mathfrak{m}$) and $\overline{X'_\epsilon} = (d\pi)^{-1}(X'_\epsilon)$ (in $\mathfrak{m}$), where as in 4.4 $(d\pi)^{-1} = (d\pi)_e\Big|_{\mathfrak{m}}^{-1}$. Let $D_X = \{X_1(s), \ldots, X_{n-1}(s)\}$ be an ordered D-basis of $\mathcal{E}_o(\gamma^X)$. Using 2.5(7) we have $\overline{X'_\epsilon} = (\mathrm{Ad}\ h)(\bar{X}_\epsilon)$. Let $\gamma_{X'_\epsilon}$ be the geodesic ray $s \to \mathrm{Exp}\ s\, X'_\epsilon(0)$.

$$\begin{aligned}\gamma_{X'_\epsilon}(s) &= \pi(\exp s\,\overline{X'}_\epsilon) &&= \pi(\exp \mathrm{Ad}(s\bar{X}_\epsilon))\\ &= \pi(h \exp s\bar{X}\ h^{-1}) &&= (\tau(h) \circ \pi)(\exp s\bar{X})\\ &= (\tau(h) \circ \gamma_{X_\epsilon(0)})(s).\end{aligned}$$

Writing Γ_ϵ for $\tau(h) \circ \gamma$, the family $\{\Gamma_\epsilon\}$ of geodesic rays is the image of $\{\gamma_\epsilon\}$ under the isometry $\tau(h)$ and we correspond to each Jacobi field Y on $\gamma = \gamma^X$ a Jacobi field Y' on $\Gamma = \tau(h)(\gamma)$. Let $X' = T(X)$ and $X'_a = T(X_a(0))$ $(a = 1, \ldots, n-1)$. Then $B_{X'} = \{X'_1, \ldots, X'_{n-1}, X'\}$ is an orthonormal basis of M_{p_o}. Now $\tau(h)$ maps transversals, σ_s of $\{\gamma_\epsilon\}$ onto transversals Σ_s of $\{\Gamma_\epsilon\}$, so we have for $s > 0$

$$Y'(s) = \frac{d\Sigma_s(\epsilon)}{d\epsilon}(0) = (d\tau(h))_{\gamma(s)}\frac{d\sigma_s(\epsilon)}{d\epsilon}(0) = T_{\gamma(s)}Y(s),$$

where $T_{\gamma(s)} = (d\tau(h))_{\gamma(s)} : M_{\gamma(s)} \longrightarrow M_{\Gamma(s)}$ is an isometry induced by T. Further if $X'_a(s)$ is the parallel translate of X'_a on Γ we have $T_{\gamma(s)}X_a(s) = X'(s)$. Hence,

$$g_{\Gamma(s)}(Y'(s), X'_a(s)) = g_{\Gamma(s)}(T_{\gamma(s)}Y(s), T_{\gamma(s)}X_a(s))$$
$$= g_{\gamma(s)}(Y(s), X_a(s)).$$

If $Y(s) = y^a(s)X_a(s)$ and $Y'(s) = z^a(s)X'_a(s)$, we have shown that $y^a(s) = z^a(s)$. Further, $D_{X'} = \{X'_1(s), \ldots, X'_{n-1}(s)\}$ is a D-basis of $\mathcal{E}_o(\gamma^{X'})$. The components y^a and z^a must satisfy an identical Jacobi equation of form 4.4(3). We deduce that the sectional curvatures, K_a, are invariant under T.

Conversely, let $D_X = \{X_1(s), \ldots, X_{n-1}(s)\}$ and $D_{X'} = \{X'_1(s), \ldots, X'_{n-1}(s)\}$ be ordered D-bases of $\mathcal{E}_o(\gamma^X)$ and $\mathcal{E}_o(\gamma^{X'})$, where $X, X' \in M_{p_o}$. Let T be the transformation of M_{p_o} such that $T(X_a(0)) = X'_a(0)$ $(a = 1, \ldots, n-1)$ and $T(X) = X'$. Suppose further that $K(S_a) = K(S'_a)$, where S_a and S'_a are the plane sections spanned by X, X_a and X', X'_a respectively. T then induces an isometry of Jacobi fields in a neighbourhood of p_o in M and is therefore itself an isometry. Hence $T \in K$.

We have proved that if $X, X' \in M_{p_o}$, D_X and $D_{X'}$ are ordered D-bases of $\mathcal{E}_o(\gamma^X)$ and $\mathcal{E}_o(\gamma^{X'})$, $K_a = K(S_a)$ and $K'_a = K(S'_a)$ and T is the transformation given by $T(X_a(0)) = X'_a(0)$, $T(X) = X'$, then $K_a = K'_a$ $(a = 1, \ldots, n-1)$ if and only if $T \in K$, the holonomy group.

4.6 k-harmonic symmetric spaces

In this final section we investigate Problem E of p. 75 and attempt to derive conditions for symmetric spaces to be k-harmonic. Let M be a compact, simply connected, irreducible, n-dimensional Riemannian globally symmetric space. Let $p_0 \in M$ and $p \in W$, where W is a normal neighbourhood of p_0. We will derive an explicit formula for $\sigma_k(p_0,p)$, the symmetric polynomials of the matrix (a^α_β). Let γ be the unit-speed geodesic ray from p_0 through p. Let s be the arc length along γ and $X = \gamma_*(0)$. Then $p = \mathrm{Exp}_{p_0} sX$. Let D_X be an ordered D-basis of $\mathcal{E}_0(\gamma)$. If the sectional curvatures are $K_1 \geqslant K_2 \geqslant \dots \geqslant K_{n-1} \geqslant 0$ with $K_1 > 0$, suppose that $K_{n-r} > 0$ but $K_{n-r+1} = \dots = K_{n-1} = 0$. Then $r \leqslant$ rank of M. In particular, if M is of rank one, r = 1, i.e., all sectional curvatures are positive. We evaluate $a^\alpha_\beta = g^{\alpha\gamma}h_{\gamma\beta}$, where $h = (h_{\gamma\beta})$ is the pull-back of $(g_{ij})_p$ under $\mathrm{Exp}^*_{p_0}$.

Let $Y,Z \in M_{p_0}$. Then
$$h(Y,Z) = ((\mathrm{Exp}_{p_0}^*)g_p)(Y,Z) = g_p(f_{sX}(Y),f_{sX}(Z)),$$
where f_{sX} is as defined on p. 86.

Using 4.4(7) and 4.4(8) we have

(1) $$h(X,X_a(0)) = h(X_a(0),X_b(0)) = 0, \text{ if } a \neq b,$$

since the D-basis is orthonormal.

(2) $$\text{If } K_a > 0,\ h(X_a(0),X_a(0)) = \frac{\sin^2(\sqrt{K_a}\, s)}{K_a\, s^2}.$$

We note that $h(X_a(0),X_a(0)) \longrightarrow 1$ as $s \longrightarrow 0$.

(3) $$\text{If } K_a = 0,\ h(X_a(0),X_a(0)) = 1.$$

$$h(X,X) = 1. \tag{4}$$

We thus have the n x n diagonal matrix

$$h = (h_{\alpha\beta}) = \mathrm{diag}\left(\frac{\sin^2(\sqrt{K_1}\; s)}{K_1\; s^2}, \ldots , \frac{\sin^2(\sqrt{K_{n-r}}\; s)}{K_{n-r}\; s^2}, \underbrace{1, \ldots, 1}_{(r \text{ times})}\right).$$

But as $\{X_1(0), \ldots, X_{n-1}(0), X_n(0) = X\}$ is an orthonormal basis of M_{p_o}, we have

$$g_{p_o}^{-1}(X_a(0), X_b(0)) = \delta^{ab}.$$

Hence,

$$(a^{\alpha}_{\beta}) = \mathrm{diag}\left(\frac{\sin^2(\sqrt{K_1}\; s)}{K_1\; s^2}, \ldots, \frac{\sin^2(\sqrt{K_{n-r}}\; s)}{K_{n-r}\; s^2}, 1, \ldots, 1\right).$$

$(b^{\alpha}_{\beta}) = (a^{\alpha}_{\beta})^{-1}$ is the diagonal matrix of reciprocals.

The symmetric polynomials can now be written down.

$$\sigma_1(p_o,p) = \sigma_1(s,X) = \frac{\sin^2(\sqrt{K_1}\; s)}{K_1\; s^2} + \ldots + \frac{\sin^2(\sqrt{K_{n-r}}\; s)}{K_{n-r}\; s^2} + r. \tag{5_1}$$

. . . .

$$\sigma_n(p_o,p) = \sigma_n(s,X) = \frac{(\sin^2(\sqrt{K_1}\; s) \ldots (\sin^2(\sqrt{K_{n-r}}\; s)}{K_1 \ldots K_{n-r}\; s^{2(n-r)}}. \tag{5_n}$$

. . . .

$$\sigma_{-n}(p_o,p) = \sigma_{-n}(s,X) = \frac{K_1 \ldots K_{n-r}\; s^{2(n-r)}}{(\sin^2(\sqrt{K_1}\; s) \ldots (\sin^2(\sqrt{K_{n-r}}\; s)}. \tag{5_{-n}}$$

We can now prove Theorem 8.

Theorem 8 All compact, simply connected, irreducible, Riemannian globally symmetric spaces of rank one are k-harmonic for all $k \in \{\pm 1, \ldots, \pm n\}$.

Proof Let M be a manifold satisfying the conditions of the theorem. Choose $p_o \in M$ and $p \in W$, where W is a normal neighbourhood of p_o. We use a result of M.Berger (see J.Simons [12]) that the holonomy group at p_o, H, is transitive on S_{n-1}, the unit sphere centre 0 in M_{p_0} if and only if M is of rank one. Thus given X, $X' \in S_{n-1}$, there exists $T \in H$ such that $T(X) = X'$. If D_X is an ordered D-basis of $\mathcal{E}_0(\gamma^X)$ then $D_{X'} = T(D_X)$ is an ordered D-basis of $\mathcal{E}_0(\gamma^{X'})$ and the sectional curvatures $K_1, \ldots, K_{n-1}$ are invariant under T (see 4.5). H being transitive on S_{n-1}, it follows that $K_1, \ldots, K_{n-1}$ are independent of X and hence using the equations (5) we deduce that $\sigma_k(p_0,p) = \sigma_k(s)$ and so M is k-harmonic for all k.

Q.e.d.

__Corollary__ Let M be a compact, simply connected, irreducible, Riemannian globally symmetric manifold with positive-definite harmonic metric. Then M is k-harmonic for all k.

__Proof__ We use Theorem 2 of Chapter 3 (p. 51).

We will obtain a simpler proof of Theorem 8 after proving Theorem 9.

__Definition__ A Riemannian manifold, M, is said to be __two-point homogeneous__ if for any two point pairs $p_1, p_2 \in M$, $q_1, q_2 \in M$ satisfying $d(p_1,p_2) = d(q_1,q_2)$, there exists an isometry $g \in I(M)$ such that $g(p_1) = q_1$ and $g(p_2) = q_2$.

__Theorem 9__ A two-point homogeneous manifold, M, with positive-definite metric is k-harmonic for all k.

__Proof__ (T.J.Willmore [16]) Let $p_0 \in M$ and take $p_2 = q_2 = p_0$ in the definition above. Then there exists an isometry of M which maps any point on $S(p_0;s)$, the geodesic sphere centre

p_o radius s ($s > 0$), onto any otherpoint p' on $S(p_o;s)$.

In terms of normal coordinates origin p_o, we have

$$a^i_j = (g^{ik})_{p_o} (g_{kj})_p.$$

Hence (a^i_j) has the same eigenvalues for all points p on $S(p_o;s)$. Thus $\sigma_k(p_o,p)$ is a function of Ω alone and M is k-harmonic for all k.

Q.e.d.

Theorem 8 can now be proved using Theorem 9 and the following result:

Let M be a Riemannian globally symmetric space of rank one. Then M is a two-point homogeneous space. ([5], p.355).

Note that this alternative proof does not require all the given conditions for Theorem 8. Indeed if M is non-compact, then $K_a > 0$ for some a, and for such a sectional curvature the solution of the Jacobi equation is

$$(f_{sX})(X_a(0)) = \frac{\sinh(\sqrt{-K_a}\, s)}{\sqrt{-K_a}\, s} \quad \text{(cf. 4.4(7)).}$$

We have

$$(a\ ) = \operatorname{diag}\left(\frac{\sinh^2(\sqrt{-K_1}\, s)}{-K_1\, s^2}, \ldots, \frac{\sinh^2(\sqrt{-K_{n-r}}\, s)}{-K_{n-r}\, s^2}, \ldots, 1\right).$$

Symmetric polynomials are easily obtained and the proof given on pp. 91-92 is still valid. We conclude that compactness is not a necessary condition for a symmetric to be k-harmonic for all k.

It is highly probable that the converse of Theorem 8 is true, namely that symmetric spaces of rank greater than one cannot be k-harmonic for any k. Yet examination of the equations (5) suggests that the sectional curvatures may vary in such a way that even though σ_n and σ_{-n} are not functions of s alone, other symmetric polynomials may be independent of choice of X. But this is highly unlikely.

If the conjecture that Riemannian symmetric spaces are k-harmonic for all k if and only if the rank is one, be true, where can we find an example of a space which is k-harmonic for $k \neq \pm n$ but not classically harmonic? We would have to examine compact, reductive homogeneous, non-symmetric Einstein manifolds. For if Avez' Theorem (Chapter III Theorem 4) is correct these spaces cannot be harmonic. But here it is unlikely that σ_k will reduce to the simple form we have in the symmetric case. And even there we were unable to come to any firm conclusion.

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