The numerical solution of ordinary differential equation by iteration

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THE NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATION BY ITERATION

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ABSTRACT

This work mainly deals with iterative methods and their rates of convergence, for the solution of non-linear ordinary differential equations.

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INTRODUCTION

In Chapter 1, an explanation of previous work relevant to the study of iterative methods for solving non-linear differential equations in Chebyshev series is given, and an account of methods of assessing and accelerating the convergence of iterative processes.

In Chapter 2 a detailed account is given of various methods: Picards and variations, Runge-Kutta, Newton linearisation and Lie series. Their application to a number of equations and numerical results is also included.

In Chapter 3, analysis of rate of convergence of iterative methods of solution, based on the behaviour of the error functions of each method, is given.

In Chapter 4, numerical and graphical comparisons of theoretical and experimental evaluation of the rate of convergence of iterative methods are given.

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Chapter 1

Introduction

In this chapter previous work relevant to the study of iterative methods for solving nonlinear differential equations is described. Since the methods considered are mostly based on the use of Chebyshev expansions, a brief summary of the properties of Chebyshev polynomials is included as the first section (1.1). The next section (1.2) describes methods which have been suggested for the solution of differential equations and also of integral and integro-differential equations, and in section (1.3) an account is given of methods of assessing and of accelerating the convergence of iterative process.

(1.1.0) Properties of Chebyshev polynomials

(1.1.1) Definitions:

$$\text{(i)} \quad T_n(x) = \cos(n \cos^{-1} X), \quad -1 \le x \le +1$$
$$\text{(ii)} \quad T_n^*(x) = T_n(2x-1), \quad 0 \le x \le 1$$

Since any finite range of values of x can be transformed to any other finite range by a linear change of variables only the first definition is used, i.e. x is taken in the range $-1 \le x \le +1$.

(1.1.2) Recurrence Relation:

$$T_{r+1}(x) - 2x\, T_r(x) + T_{r-1}(x) = 0$$

(1.1.3) Product formula:

$$T_r(x)\, T_s(x) = \tfrac{1}{2}\{T_{r+s}(x) - T_{|r-s|}(x)\}$$

(1.1.4) Integration:

$$\int T_r(x)dx = \begin{cases} T_1(x), & r = 0 \\ \tfrac{1}{4}T_2(x), & r = 1 \\ \tfrac{1}{2}\left\{\dfrac{T_{r+1}(x)}{r+1} - \dfrac{T_{r-1}(x)}{r-1}\right\}, & r > 1 \end{cases}$$

from which

$$\int_{-1}^{x} T_r(x)dx = \begin{cases} T_1(x) + 1, & r = 0 \\ \tfrac{1}{4}(T_2(x) - 1), & r = 1 \\ \tfrac{1}{2}\left\{\dfrac{T_{r+1}(x)}{r+1} - \dfrac{T_{r-1}(x)}{r-1}\right\} + \dfrac{(-1)^{r+1}}{r^2-1}, & r > 1 \end{cases}$$

(1.1.5) Orthogonal properties:

(i) $$\int_{-1}^{1} \frac{T_r(x)\, T_s(x)}{\sqrt{1-x^2}}\, dx = \begin{cases} \pi & \text{for } r = s = 0 \\ \pi/2 & \text{for } r = s \neq 0 \\ 0 & \text{for } r \neq s \end{cases}$$

(ii) for $n > 0$ and $r, s \leq n$

$$\sum_{j=0}^{n}{}'' T_r(x_j)\, T_s(x_j) = \begin{cases} n, & r = s = 0 \text{ or } n \\ \frac{n}{2}, & r = s \neq 0 \text{ or } n \\ 0, & r \neq s \end{cases}$$

where $x_j = \cos \frac{j\pi}{n}$ for $j = 0,1,2, \ldots\ldots, n$

Note: The double prime on summation symbol here and elsewhere indicates that the terms with suffix $j = 0$ and $j = n$ are to be halved. We shall similarly use a single prime on the summation symbol when only the term with suffix $j = 0$ is to be halved.

For example,

(i) $$\sum_{j=0}^{n}{}'' U_j = \tfrac{1}{2}U_o + U_1 + \ldots\ldots + U_{n-1} + \tfrac{1}{2}U_n$$

(ii) $$\sum_{j=0}^{n}{}' U_j = \tfrac{1}{2}U_o + U_1 + \ldots\ldots + U_{n-1} + U_n$$

(1.1.6) Explicit expressions for the first few Chebyshev polynomials:

$$\begin{aligned}
T_o(x) &= 1\\
T_1(x) &= x\\
T_2(x) &= 2x^2 - 1\\
T_3(x) &= 4x^3 - 3x\\
T_4(x) &= 8x^4 - 8x^2 + 1\\
T_5(x) &= 10x^5 - 20x^3 + 5x\\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1
\end{aligned}$$

(1.1.7) Inverse relations giving powers of x in terms of Chebyshev polynomials:

$$\begin{aligned}
1 &= T_o(x)\\
x &= T_1(x)\\
x^2 &= \tfrac{1}{2}(T_o(x) + T_2(x))\\
x^3 &= \tfrac{1}{4}(3T_1(x) + T_3(x))\\
x^4 &= \tfrac{1}{8}(3T_o(x) + 4T_2(x) + T_4(x))\\
x^5 &= \frac{1}{16}(10T_1(x) + 5T_3(x) + T_5(x))\\
x^6 &= \frac{1}{32}(10T_0(x) + 15T_2(x) + 6T_4(x) + T_6(x))
\end{aligned}$$

(1.1.8) Calculation of Chebyshev coefficients:

(i) If f(x) is continuous and of bounded variation in the range (-1, +1), then f(x) can be expressed in the form of an infinite series

$$f(x) = \tfrac{1}{2}A_o T_o(x) + A_1 T_1(x) + \ldots\ldots$$

$$= \sum_{r=0}^{\infty}{}' A_r T_r(x)$$

which is uniformally convergent throughout the range. Using the orthogonal property (1.1.5),

$$A_r = \frac{2}{\pi}\int_{-1}^{+1} \frac{f(x)T_r(x)}{\sqrt{1-x^2}}\,dx$$

$$= \frac{2}{\pi}\int_{o}^{\pi} f(\cos\theta)\cos r\theta\, d\theta$$

This is a familiar representation in the theory of Fourier Series. Here it is used occasionally for conformation purposes.

(ii) The use of the orthogonal property of the summation $\sum_{j=o}^{n}{}'' T_r(x_j)T_s(x_j)$, is of more practical use than integration; its application is carried out as follows.

Define $$C_r = \frac{2}{m}\sum_{j=o}^{m}{}'' f(x_j)T_r(x_j), \quad r = 0,1, \ldots. n$$

$$= \frac{2}{m}\sum_{j=o}^{m}{}'' f(\cos\frac{j\pi}{m})\cos\frac{rj\pi}{m},$$

$$x_j = \cos\frac{j\pi}{m} \quad (j = 0,1, \ldots m)$$

Then

$$P_n(x) = \sum_{r=o}^{n}{}' C_r T_r(x)$$ with $n < m$ is the least square

approximation to $g(\theta) = f(\cos\theta)$ over the $m + 1$ equally spaced points $\theta = \frac{j\pi}{m}$ with the truncated series weights ½ at the beginning and end and 1 elsewhere; and if $m = n$ then

$$P_m(x) = \sum_{r=o}^{m} C_r T_r(x)$$

is the Chebyshev expansion which takes the same values as $f(x)$ at each of the $m + 1$ points $x_j = \cos\frac{j\pi}{m}$, $j = 0,1, \ldots m$.

(1.1.9) Summation by recurrence:

The Chebyshev series

$$f(x) = \sum_{r=o}^{\infty}{}' A_r T_r(x)$$

may be truncated after any term, say the $(n + 1)$th to give an approximation to $f(x)$, an upper bound

$$\sum_{r=n+1}^{\infty} |A_r|$$

for the truncation error being ascertainable at a glance. The approximating finite series may be evaluated in two ways

(i) If the series is first rearranged in the form

$$f(x) = C_0 + C_1 x + \dots\dots\dots + C_n \chi^n$$

It can be evaluated for any given value of χ by the familiar process of nested multiplication. This consists of computing successively the quantities d_n, d_{n-1}, ..., d_o defined by

$$d_r = \chi\, d_{r+1} + C_r, \quad r = m, m-1, \dots\dots 1,0$$

$$d_{n+1} = 0$$

Then $\qquad f(x) = d_o$

(ii) It is possible, however, to evaluate f(x) by recurrence directly from the Chebyshev coefficients A_r. We form successively b_n, b_{n-1}, ..., b_o from

$$b_r = 2\chi\, b_{r+1} - b_{r+2} + A_r, \quad r = n, \dots\dots 1,0$$

$$b_{n+1} = b_{n+2} = 0$$

then $\qquad f(x) = \tfrac{1}{2}(b_o - b_2)$

(1.2.0) <u>The use of Chebyshev polynomials in solution of differential equations</u>:

In considering the use of Chebyshev polynomials in solution of differential equations, it is necessary to record first the effect of differentiating or integrating a Chebyshev series and hence the relations which exist between the coefficients in the series for a function and its derivative or integral.

(1.2.1) Differentiation:

If $$f(x) = \sum_{r=o}^{\infty}{}' A_r T_r(x), \quad f'(x) = \sum_{r=o}^{\infty}{}' C_r T_r(x),$$

then
$$C_{2r} = \sum_{s=r}^{\infty} 2(2s + 1)\, A_{2s+1}, \text{ for } r = 0,1, \ldots.$$

$$C_{2r+1} = \sum_{s=r}^{\infty} 2(2s + 2)\, A_{2s+2}, \text{ for } r = 0,1, \ldots.$$

For the truncated series

$$P(x) = \sum_{r=o}^{n}{}' A_r T_r(x), \quad P'(x) = \sum_{r=o}^{n-1}{}' C_r T_r(x)$$

$$C_{n-1} = 2nA_n$$

$$C_{n-2} = 2(n-1)\, A_{n-1}$$

$$C_{r-1} = 2r\, A_r + C_{r+1} \quad \text{for } r = 1,2, \ldots\ldots, n-2$$

(1.2.2) Integration:

If $$f(x) = \sum_{r=o}^{\infty}{}' A_r T_r(x), \quad \int f(x)\, dx = \sum_{r=1}^{\infty}{}' b_r T_r(x) + b_o$$

where b_o is an arbitary constant, and

$$b_r = \frac{1}{2r}(A_{r-1} - A_{r+1}), \text{ for } r = 1,2, \ldots\ldots$$

For the truncated series

$$P(x) = \sum_{r=o}^{n}{}' A_r T_r(x), \quad \int P(x)dx = \sum_{r=1}^{n+1} b_r T_r(x) + b_o$$

and
$$b_{n+1} = \frac{A_n}{2(n+1)}$$

$$b_n = \frac{A_{n-1}}{2n}$$

$$b_r = \frac{1}{2r}(A_{r-1} - A_{r+1}), \quad r = 1,2,\ldots\ldots n-1$$

For the interpolating series $P(x) = \Sigma'' A_r T_r(x)$ the results are similar with A_n replaced by $\frac{1}{2}A_n$.

(1.2.3) Integration of a function:

Clenshaw and Curtis (1960) [3] suggested as a procedure for integrating a function f(x) defined and well behaved in the range $-1 \le x \le +1$, that, using (1.1.4) and (ii) of (1.1.8).

$$\int_{-1}^{x} f(t)dt = \sum_{r=o}^{m+1} b_r T_r(x)$$

with $f(x) = \sum_{r=o}^{m}{}'' A_r T_r(x)$, $A_r = \frac{2}{m} \sum_{j=o}^{m}{}'' f(x_j)T_r(x_j)$ and

$x_j = \cos\frac{j\pi}{m}$, and then b's and A's are related as in (1.2.2) with

$$b_o = \sum_{\substack{j=o\\ j\neq 1}}^{m}{}'' \frac{(-1)^{j+1}A_j}{j^2-1} - \tfrac{1}{4}A_1$$

Similar expressions hold if f(x) is written as $\sum_{r=o}^{n}{}' A_r T_r(x)$ with $n < m$.

This method expresses the indefinite integral as a Chebyshev series.

Elgendi (1969) [4] suggested a different approach, he connects the values of the integral $\int_{-1}^{x} f(t)dt$ at the points $x_j = -\cos \frac{j\pi}{m}$ $(j = 0,1, \ldots\ldots,m)$ with the values of the function at the same points so that

$$\left[\int_{-1}^{x} f(t)dt\right] = B\left[f\right]$$

where B is a square matrix of order (m+1) and f is the column vector whose elements are $f_j = f(-\cos \frac{j\pi}{m})$. This evaluates the integral at a series of points instead of producing its Chebyshev coefficients. If $f(x_j)$ is calculated at the points $x_j = -\cos \frac{j\pi}{m}$ and represented in the form of a Chebyshev series

$$\sum_{r=o}^{m}{}'' A_r T_r(x),$$

then $\underset{\wedge}{f}$, $\underset{\wedge}{A}$ are connected by

$$\underset{\wedge}{f} = T\,\underset{\wedge}{A}$$

where

$$\underset{\wedge}{f} = \begin{bmatrix} f(x_o) \\ f(x_1) \\ \\ f(x_m) \end{bmatrix} \qquad \underset{\wedge}{A} = \begin{bmatrix} A_o \\ A_1 \\ \\ A_m \end{bmatrix}$$

and

$$T = \begin{bmatrix} \frac{1}{2}T_o(x_o) & T_1(x_o) & \ldots & \frac{1}{2}T_m(x_o) \\ \frac{1}{2}T_o(x_1) & T_1(x_1) & & \frac{1}{2}T_m(x_1) \\ \\ \\ \frac{1}{2}T_o(x_m) & T_1(x_m) & & \frac{1}{2}T_m(x_m) \end{bmatrix}$$

Then if

$$F(x) = \int_{-1}^{x} f(t)dt = \sum_{r=o}^{m+1} b_r T_r(x)$$

$$F(x_j) = \int_{-1}^{x_j} f(t)dt = \sum_{r=o}^{m+1} b_r T_r(x_j)$$

and $\underset{\wedge}{F}, \underset{\wedge}{b}$ are connected by

$$\underset{\wedge}{F} = T'\underset{\wedge}{b} \qquad (1)$$

Where

$$\underset{\wedge}{F} = \begin{bmatrix} F(x_o) \\ F(x_1) \\ \\ F(x_m) \end{bmatrix} \qquad \underset{\wedge}{b} = \begin{bmatrix} b_o \\ b_1 \\ \\ b_{m+1} \end{bmatrix}$$

and

$$T' = \begin{bmatrix} T_o(x_o) & T_1(x_o) & \ldots\ldots & T_{m+1}(x_o) \\ T_o(x_1) & T_1(x_1) & \ldots\ldots & T_{m+1}(x_1) \\ \\ \\ T_o(x_m) & T_1(x_m) & \ldots\ldots & T_{m+1}(x_m) \end{bmatrix}$$

an (m+1) x(m+2) matrix

Also from (1.2.2) the coefficients b_r's and A_r's have the relation

$$\underset{\wedge}{b} = M\underset{\wedge}{A} \qquad (2)$$

where

$$M = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{3} & \frac{1}{8} & \cdots\cdots & \frac{(-1)^{m+1}}{2(m^2-1)} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \cdots\cdots & 0 \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4} & & \vdots \\ \vdots & & \frac{1}{6} & 0 & & \vdots \\ \vdots & & & \frac{1}{8} & 0 \; \cdots & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & & & & & 0 \end{bmatrix}$$

an (m+2) x (m+1) matrix.

The relations (1) and (2) will yield

$$\underset{\wedge}{F} = T'\underset{\wedge}{b} = T'M\underset{\wedge}{A}$$
$$= N\underset{\wedge}{A}$$

where N = T'M is a square matrix of order (m+1) and can be shown to be non-singular.

Hence $$\underset{\wedge}{A} = N^{-1}\underset{\wedge}{F}$$

and $$T\underset{\wedge}{A} = TN^{-1}\underset{\wedge}{F}$$
$$= \underset{\wedge}{f}$$

Therefore $$\underset{\wedge}{F} = (TN^{-1})^{-1}\underset{\wedge}{f}$$
$$= B\underset{\wedge}{f}$$

where $B = (TN^{-1})^{-1}$ is a square matrix of order (m+1)

and

$$\int_{-1}^{x_j} f(t)dt = B\left[\underset{\wedge}{f}\right]$$

The main advantage of this method is that for a certain value of m the elements of the matrix B can be evaluated once and for all independent of the particular function f(x). The method in fact gives alternative quadrature formulae to those obtained by the usual finite difference methods.

(1.2.4) Use of Chebyshev expansions in solving equations:

Any method is designed to produce a series solution of finite degree m;

$$y^{(u)}(x) = \sum_{r=o}^{m}{}' A_r T_r(x) \qquad \text{(i)}$$

which approximates in some sense, the exact solution

$$y(x) = \sum_{r=o}^{\infty}{}' A_r T_r(x) \qquad \text{(ii)}$$

Suggested methods for linear systems are described briefly so as to illustrate the ideas which can then be applied to non-linear systems.

(1.2.5) Direct methods (linear systems):

The idea of these methods is to reduce the solution of the equation to a comparison of Chebyshev expansions. From (ii) above and using the result in (1.1), it is possible to express $y'(x)$, $\int y(x)dx$, and $x^p y(x)$ in similar form; hence any linear differential equation with polynomial coefficients may be reduced to a set of linear equations

$$\sum_{s=o}^{\infty}{}' Q_{rs} A_s = R_r$$

and any linear boundary condition adds a further equation

$$\sum_{r=o}^{\infty}{}' D_r A_r = C$$

Initial or boundary value problems, in which y may be a scalar or vector quantity, all fall into this scheme; the solution is then obtained by solving a restricted set

$$\sum_{s=o}^{m}{}' Q_{rs}A_r = R_r, \qquad \sum_{r=o}^{m}{}' D_rA_r = C$$

The method is described initially by Lanczos (1938) [5] and developed by Clenshaw (1957) [6] and Fox (1962) [7]. In Scraton (1965) [8], it is extended to the case where the coefficients are not polynomials but may be approximated by a (low degree) polynomial, and further extensions to general boundary conditions are treated in Snell (1970) [9]. Knibb and Scraton (1971) [10] apply the same idea to replace a partial differential equation by a set of ordinary differential equations in the A_r. The truncation error as estimated by varying m, is described in [7] and by Phillips (1967) [11].

(1.2.6) Collocation Methods:

In these methods the equation is satisfied exactly at a set of M selected points. It is then not necessary to express each term in the equation in a Chebyshev series, and this enables a wider variety of equations to be tackled. Any pth order linear differential equation

$$q_p(x)\frac{d^py}{dx^p} + q_{p-1}(x)\frac{d^{p-1}y}{dx^{p-1}} \,\ldots\ldots\ldots + q_o(x)y = r(x)$$

reduces, on substituting

$$y = \sum_{o}^{\infty}{}' A_rT_r(x)$$

and using results in (1.1) to

$$\sum_{r=u}^{\infty}{}' Q_r(x)A_r = r(x)$$

Where the $Q_r(x)$ are functions of the $q_r(x)$ and are linear in $T_r(x)$. This is now required to be exact at x_j, $j = 0,1, \ldots\ldots m$ and hence A's satisfy

$$\sum_{r=o}^{\infty}{}' Q_r(x_j)A_r = r(x_j)$$

together with boundary conditions which, as before, produce

$$\sum_{r=o}^{\infty}{}' D_rA_r = C$$

A limited set of these equations are again solved giving $A_o, A_1, \ldots\ldots, A_m$. This method was suggested by Lanczos (1938) [5] and developed by Clenshaw and Norton (1963) [12] and Wright (1964) [13]. The selection of variable points x_j is discussed by Osborne and Watson (1968) [14]. Oliver (1969) [15] gives a discussion of the truncation error. Proposed sets include the extrema of $T_m(x)$ (ie $x_i = \cos \frac{i\pi}{m}$) or the zeros of $T_{m+1}(x)$, i.e. ($x_i = \cos\frac{(2i+1)\pi}{2(m+1)}$).

(1.2.7) Linear integral equations:

Linear integral equations can be treated by techniques which contain elements of both direct and collocation methods. Thus in the Fredholm equation

$$y(x) - \lambda \int_{-1}^{1} K(x,s)\, y(s)\, ds = f(x)$$

Elliot (1963) [20] suggests taking for y the truncated Chebyshev expansion

$\sum_{r=o}^{n}{}' A_r T_r(x)$ and determining the (n+1) A's by satisfying the equation

at the (n+1) points $x_i = \cos \frac{i\pi}{n}$. This gives the equation

$$\sum_{r=o}^{n}{}' A_r T_r(x_i) - \lambda \int_{-1}^{1} \left\{ K(x_i, s) \sum_{r=o}^{n}{}' A_r T_r(s) \right\} ds = f(x_i)$$

for $i = 0, 1, \ldots, n$, $-1 \leq x, s \leq 1$

The Kernel $K(x_i, s)$ may now be approximated by the interpolating polynomial of degree N in the form

$$K(x_i, s) = \sum_{r=o}^{N}{}'' b_r(x_i) T_r(s)$$

Where $$b_r(x_i) = \frac{2}{N} \sum_{j=0}^{N}{}'' K(x_i, \cos \frac{j\pi}{N}) \cos\left(\frac{rj\pi}{N}\right)$$

and so the Fredholm equation is replaced by n+1 equations

$$f(x_i) = \sum_{r=o}^{n}{}' A_r T_r(x_i) - \lambda \int_{-1}^{1} \left\{ \sum_{r=o}^{N}{}'' b_r(x_i) T_r(s) \sum_{p=o}^{n}{}' A_p T_p(s) \right\} ds$$

for $i = 0, 1, \ldots, n$.

Which, using the product formula (1.1.3) and the definite integral formula

$$\int_{-1}^{+1} T_r(u)\,du = \begin{cases} \dfrac{2(-1)^{r+1}}{r^2-1} & \text{for even } r \\ 0 & \text{for odd } r \end{cases}$$

may be reduced to a set of linear equations in the A_r.

Elgendi (1969) [4] suggests a method based on the relation described in (1.2.3) between the integral values and the function values. If $\hat{F}$ is the (m+1)th order column vector of values of

$$\int_{-1}^{x_i} f(u)\,du, \quad x_i = -\cos\frac{i\pi}{m} \quad (i = 0,1,\dots.m) \text{ and } \hat{f}$$

is the vector values of $f(x_i)$, then

$$\hat{F} = B\,\hat{f}$$

In particular

$$F_{m+1} = \int_{-1}^{1} f(u)\,du = \sum_{i=o}^{m}{}' B_{m+1,i+1}\, f(-\cos\frac{i\pi}{m})$$

Thus

$$\int_{-1}^{1} K(x,s)\, y(s)\,ds = \sum_{i=o}^{m}{}' B_{m+1,i+1}\, K(x,x_i)\, y(x_i)$$

which for $x = x_j$,

$$= \sum_{i=o}^{m}{}' B_{m+1,i+1}\, K(x_j,x_i)\, y(x_i)$$

$$= [c^{(j)}]^T\,[y]$$

The Fredholm equation then becomes

$$[y] - \lambda[C^{(j)}]^T[y] = [f]$$

or

$$(I - \lambda C)[y] = [f]$$

and so $[y]$ (the vector of y at the points x_i) may be determined.

(1.2.8) Nonlinear differential equations:

Direct or collocation methods would lead in this case to nonlinear equations in the A's. For the direct method these would result from expressing the nonlinear terms as Chebyshev series, for the collocation method, from expressing the values of the nonlinear terms at the selected points in terms of the Chebyshev coefficients. The solution will thus involve iteration.

An alternative approach which has been adopted, is to use a linear iterative process on the whole solution (i.e. on the vector of A's, if Chebyshev expansions are used); methods suggested are based on Picards iteration (Clenshaw and Norton (1963) [12], Wright (1964) [13]) or a Newton linearisation (Norton (1964) [16]). The Picards idea has also been applied to nonlinear integral equations e.g. by Wolfe (1969) [17]. A full discussion of each of these methods will be given in the next Chapter.

Another method suggested by Weyl (1942) [18] is to linearise the differential equation itself; the Chebyshev collocation method can be applied to this and results are described in Chapter 2.

Recent work on ~~the~~ Lie series generalises these methods and is described in Chapter 3.

The factor of interest in all these methods is the speed of convergence of the iterative process to the truncated Chebyshev series solution. Comparison may also be made between these and other methods which are briefly mentioned.

(1.2.9) Iterative use of Runge-Kutta method for boundary value problems:

The Runge-Kutta method is a standard method for solving initial value problems, by transforming into simultaneous first order equations and using these to integrate from one end of the range of the independent variable to the other. Details of this method are given in many books on numerical analysis such as [19].

The procedure is also adopted for solving boundary value problems. This requires an iterative process in which the equation is solved as an initial value problem and the unknown initial values are successively approximated as functions of the values at the other boundary or boundaries. An example is given in the next Chapter.

(1.3.0) The Order of Convergence

It is natural to consider first the definition of the rate of convergence of an iterative sequence of scalars $x_o, x_1 \ldots. x_i, \ldots.$ If the sequence converges to q, and $1_i = x_i - q$ and if

$\dfrac{(1_{i+1})}{(1_i)^p}$ where p is real tends to a non zero constant C, then p is the order of the sequence and C is the asymptotic error constant (see e.g. Traub p.9 [26]). The information needed can also be quantified, by writing α for the number of new evaluations required per iteration, and then the informational efficiency Eff can be written as a combination of p and α; p/α or $p^{1/\alpha}$ have been suggested.

In the case of an iterative solution of a differential equation in a single variable we have successive iterates $y^{(o)}(x)$, $y^{(1)}(x)$ .... $y^{(i)}(x)$... and a true solution $y(x)$. As above, write the error

$$e^{(i)}(x) = y^{(i)}(x) - y(x)$$

Then discussion of the error in this form will require the evaluation of some error norm of which the usual ones are

$$L_1 = \int_a^b |e(x)|dx \bigg/ \int_a^b dx, \quad L_2 = \int_a^b e^2(x)dx \bigg/ \int_a^b dx,$$

or $\quad L_\infty = \max|e(x)|, \; a \le x \le b.$

A relationship may then be obtained as above between the norms of successive iterates.

If the solution of the differential equation is obtained as a Chebyshev series, however, the iterates each give rise to a vector of Chebyshev coefficients, $\hat{A}^{(i)}$ and the relationship becomes,

$$\hat{A}^{(i+1)} = F(\hat{A}^{(i)})$$

(1.3.1) Error estimation of iterative methods:

For the various iterative methods suggested in section (1.2), the solution to the differential system that we hope to achieve is of the form

$$y(x) = \sum_{r=o}^{N}{}' a_r T_r(x)$$

of finite degree N, which approximates the exact solution

$$y^*(x) = \sum_{r=o}^{\infty}{}' A_r T_r(x)$$

to some desired accuracy over the range (-1, +1). The upper bound on the error function $\epsilon(x)$ [15] is given by

$$\begin{aligned}\epsilon(x) &= y(x) - y^*(x) \\ &= \sum_{r=o}^{N}{}' (a_r - A_r)T_r(x) - \sum_{N+1}^{\infty} A_r T_r(x) \\ &= \sum_{r=o}^{N}{}' e_r T_r(x) - \sum_{r=N+1}^{\infty} A_r T_r(x)\end{aligned}$$

At any stage, when selecting N for which

$$\sum_{N+1}^{\infty} A_r T_r(x)$$

is small enough to be neglected, $\epsilon(x)$ can be expressed as

$$\epsilon(x) = \sum_{r=o}^{N} e_r T_r(x)$$

where e_r $(r = 0,1, \ldots, N)$ are the Chebyshev coefficients of the error function $\epsilon(x)$. If after r iterations we define the vectors $\hat{a}^{(r)}$ and $\hat{e}^{(r)}$ such that

$$\hat{a}^{(r)} = \begin{bmatrix} a_o^{(r)} \\ a_1^{(r)} \\ \\ \\ a_N^{(r)} \end{bmatrix} \qquad \hat{e}^{(r)} = \begin{bmatrix} e_o^{(r)} \\ e_1^{(r)} \\ \\ \\ e_N^{(r)} \end{bmatrix}$$

Then ~~for any~~ to first order iterative method, for any we have the representation

$$\underset{\wedge}{a}^{(r+1)} = M\,\underset{\wedge}{a}^{(r)} + \underset{\wedge}{b}$$

Where M (which will in general be a function of h) is a square matrix of order (N+1) and independent of the coefficients a_r $(r = 0,1, \ldots, N)$, and b is a constant vector. If the method is consistent then $\underset{\wedge}{A}$, the vector of the required solution, satisfies $\underset{\wedge}{A} = M\underset{\wedge}{A} + \underset{\wedge}{b}$, so by subtracting these relations we get

$$e^{(r+1)} = M\underset{\wedge}{e}^{(r)}$$

$$= M^r\,\underset{\wedge}{e}^{(o)}$$

where $\underset{\wedge}{e}^{(o)}$ is the error vector in the approximation $\underset{\wedge}{a}^{(o)}$.

(1.3.2) <u>The rate of convergence</u>:

If M (the iteration matrix) is non deficient, then we can show, by expressing $\underset{\wedge}{e}^{(o)}$ in terms of the eigen-vectors of M, that the iterative process converges for any initial approximation $\underset{\wedge}{a}^{(o)}$, iff the spectral radius of M is less than 1, where the spectral radius

$$\rho(M) = \max|\lambda(M)|.$$

In general whether M is deficient or not, we have

$$||\underset{\wedge}{e}^{(r)}|| \leq ||M^r||\ ||\underset{\wedge}{e}^{(o)}||$$

where the norms are defined by

(i) $\quad ||e|| = \{|e_o|^2 + |e_1|^2 + \ldots\ldots + |e_N|^2\}^{\frac{1}{2}}$

(ii) $\quad ||M|| = \max\{\text{eigenvalue of } (M^TM)\}^{\frac{1}{2}}$

(iii) $\quad \rho(M) \leq ||M||$

The average rate of convergence over r iterations $R_r(M)$ is defined by

$$R_r(M) = -\log||M^r||/r.$$

This being the average decrease in $\log||e||$ at each iteration.

Since the Rayleigh quotient $\frac{x^T Mx}{x^T x}$, has a maximum value of $\rho(M)$ for all x, we have that if $e^{(r+1)} = M\, e^{(r)}$ then

$$\frac{e^{(r+1)}\ e^{(r)}}{e^{(r)T}\ e^{(r)}} \leq \rho(M),$$

equality holding only in the case where $e^{(r+1)}$ and $e^{(r)}$ are eigenvectors of M.

For an expansion in orthogonal functions $\phi_r(x)$, such that

$$\int_{-1}^{1} \omega(x)\phi_r(x)\phi_s(x)\,dx = 0,\ r \neq s$$

$$= K_r\ r = s$$

It may be noted (Fox and Parker, p.44 [26] that for any function f we have, writing

$$||f|| = \left\{\int_{-1}^{1} \omega(x)f^2(x)dx\right\}^{\frac{1}{2}},$$

$$||f||^2 = \sum_{r=o}^{N} K_r C_r{}^2 + (f-t_n, f-t_n)\text{ , where}$$

$$t_n = \sum_{r=o}^{N} C_r\phi_r(x)$$

It follows that $||f||^2 \geq \sum_{o}^{N} K_r C_r{}^2$ so that with this definition of norm the function has a larger norm than the vector of its Chebyshev coefficients. In fact, for the error function e(x) and its Chebyshev expansion

$$\sum_{0}^{N} \eta_r T_r(x) \quad \text{we have}$$

$$\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} e^2(x)dx \geq 2\eta_0^2 + \eta_1^2 + \ldots\ldots + \eta_N^2$$

(1.3.3) Acceleration Techniques:

For many problems, however, iterative methods may fail to converge or converge too slowly to be useful. For such cases acceleration procedures of the form $x = f(x_r, x_{r+1}, \ldots\ldots x_{r+s})$ have been proved to be successful, in particular the well known Aitken's δ^2-formula. The terms $x_r, x_{r+1}, \ldots\ldots$ of the sequence of solutions can be either scalar qualities or arrays. The main object of this section is to discuss appropriate procedures to accelerate slowly convergent or non-convergent solutions of both types.

(1.3.4) The ε-Algorithm, (Shanks 1955, Wynn 1956,1962)

Let the sequence of scalars x_r, ($r = 0,1,2,\ldots$) be linked with the sequence $\varepsilon_0^{(r)}$ such that

$$\varepsilon_0^{(r)} = x_r$$

$$\varepsilon_{-1}^{(r)} = 0$$

Then the ε-algorithm defines the quantities $\varepsilon_s^{(r)}$ to satisfy the relation

$$\varepsilon_{s+1}^{(r)} = \varepsilon_{s-1}^{(r+1)} + (\varepsilon_s^{(r+1)} - \varepsilon_s^{(r)})^{-1}, \quad \begin{array}{l} r = 0,1, \ldots\ldots \\ s = 0,1, \ldots\ldots \end{array}$$

If the sequence x_r is slowly convergent then this relation provides a sequence $\epsilon_{2s}^{(r)}$ which when associated with $\epsilon_o^{(r)} = x_r$ can be far more rapid.

Regarding $\epsilon_s^{(r)}$ as scalar quantities, we have

$$\epsilon_1^{(r)} = \epsilon_{-1}^{(r+1)} + \frac{1}{\epsilon_o^{(r+1)} - \epsilon_o^{(r)}}$$

$$= \frac{1}{x_{r+1} - x_r}$$

and

$$\epsilon_2^{(r)} = \epsilon_o^{(r+1)} + \frac{1}{\epsilon_1^{(r+1)} - \epsilon_1^{(r)}}$$

$$= \frac{x_{r+1}^2 - x_r x_{r+2}}{2x_{r+1} - x_r - x_{r+2}}$$

which is in fact Aitken's δ^2-formula. It can be shown, Johnson (1971) [21] that $\epsilon_{2p}^{(r)}$ is found by fitting the hyper plane

$$x_r = \alpha_o + \alpha_1(x_{r+1} - x_r) + \alpha_2(x_{r+2} - x_{r+1}) + \dots + \alpha_p(x_{r+p} - x_{r+p-1})$$

where the α's are determined by the (p+1) lots of values of $\{x_r, x_{r+1}, \dots, x_{r+p}\}$, $r = 0,1,\dots p$. Then α_o, the intersection of this hyper plane with $\delta_s = 0$, $s = 1,2,\dots,p$, is $\epsilon_{2p}^{(r)}$. If the terms x_r are partial sums of a formal power series given by

$$f(Z) = \sum_{i=o}^{\infty} A_i Z^i$$

$$x_j = \sum_{i=o}^{j} A_i Z_o^i \qquad \text{for some } Z_o$$

then it can be shown, Genz (1973) [22] that

$$\varepsilon_{2K}^{(j)} = [K, K+j]_{f(Z)}(Z_o)$$

Where $[K, K+j]_{f(Z)}(Z_o)$ is the Pade's approximants to f(Z) in the form $[K, K+j](Z) = \sum_{0}^{K+j} P_r Z^r / \{1 + \sum_{1}^{K} q_r Z^r\}$ evaluated at $Z = Z_o$.

So the ε-Algorithm can be used to compute Pade approximants for some fixed points.

Also if x_j is a sequence whose terms are given by

$$x_j = x + \sum_{i=1}^{p} b_i \beta_i^j$$

with $|\beta_1| > |\beta_2| \ldots\ldots > |\beta_p|$

then $\varepsilon_{2p}^{(o)} = x$, where x is the limit of the sequence x_j.

(1.3.5) The E-Algorithm

Alternatively let the sequence x_j have the terms $x_r, x_{r+1}, \ldots$ such that

$$x_{r+1} = f(x_r) \qquad (3)$$

If $x = \alpha$ is the solution of (3), and if for $x_r = \alpha + \varepsilon$ we can have the expansion

$$f(\alpha + \varepsilon) = \alpha + C\varepsilon + O(\varepsilon^2)$$

Then it follows ([23]) that either

$$E_n(\alpha + \varepsilon) = \alpha + O(\varepsilon^{n+1}) \text{ for } C \neq 1$$

or

$$E_n(\alpha + \varepsilon) = \alpha + \frac{1}{n+1}\varepsilon + O(\alpha^2) \text{ for } C = 1$$

Where $E_n(x_r)$ satisfies the recurrence relations

$$E_o(x_r) = x_r$$

$$E_n(x_r) = E_{n-1}(x_{r+1}) + \sigma^{-1}$$

$$\sigma^{-1} = \sum_{i=o}^{n-1} \{[E_i(x_{r+n-i+1}) - E_i(x_{r+n-i})]^{-1}\}$$

$$- \sum_{i=o}^{n-1} \{E_i(x_{r+n-i}) - E_i(x_{r+n-i-1})\}^{-1}$$

This is referred to as E-Algorithm which is very much related to ϵ-Algorithm since

$$E_n(x_r) \equiv \epsilon_{2n}^{(r)} \quad \text{for } n = 1,2,\ldots\ldots.$$

In applying this approach to the case in which $\epsilon_s^{(r)}$ and x_r, x_{r+1},.... are sequences of slowly convergent arrays and in particular vectors, the main interest will be focussed on the terms which concern the inverse of those vectors.

P. Wynn (1962) [24] has considered several possibilities regarding the inverse of a vector $\{x_i\}$.

(i) The primitive inverse $\{\frac{1}{x_i}\}$ which deals with each component of $\epsilon_s^{(r)}$ separately, this is equivalent to the simultaneous application of the scalar ϵ-algorithm to each component of $\epsilon_s^{(r)}$.

(ii) The Samelson-inverse of a vector which defines the inverse of a vector $\underset{\wedge}{x}$ by

$$\underset{\wedge}{x}^{-1} = \left\{ \sum_{i=1}^{n} (x_i\bar{x}_i)\right\}^{-1} (\bar{x}_1, \ldots.., \bar{x}_n)$$

where $\underset{\wedge}{x} = (x_1,x_2,\ldots..,x_n)$ and $\bar{x}_r$ is the complex conjugate of x_r.

(1.3.6) The Extrapolation algorithm:

This is an alternative to the ϵ-algorithm, based on fewer evaluations of the original sequence. If the basic iteration is of the form

$$z^{l+1} = G\, z^{l}$$

and if we define a coupled pair of iterative sequences x^{l} and y^{l} such that

$$y^{l} = G\, x^{l}$$

where G is an operator, then Anderson (1965) [25] has established the extrapolation algorithm which defines x^{l+1} as a function of x^{l-1}, x^{l}, y^{l-1} and y^{l}, so that the sequences x^{l} and y^{l} converge more rapidly, than the basic sequence z^{l}.

Define a residual vector r^{l} by

$$r^{l} = y^{l} - x^{l}$$

and

$$u^{l} = x^{l} + \theta^{l}(x^{l-1} - x^{l})$$

$$v^{l} = y^{l} + \theta^{l}(y^{l-1} - y^{l})$$

$$R^{l} = \tfrac{1}{2}(v^{l} - u^{l},\ v^{l} - u^{l})$$

where the inner product of two N-vectors u and v is defined by

$$(u,v) = \sum_{i=1}^{N} u_i v_i w_i$$

w_i being a non-negative weighting factor.

The parameter θ^{l} is chosen so as to minimize the linearized residual R^{l}.

Hence $$\theta^l = (r^l, r^l - r^{l-1})/(r^l - r^{l-1}, r^l - r^{l-1})$$

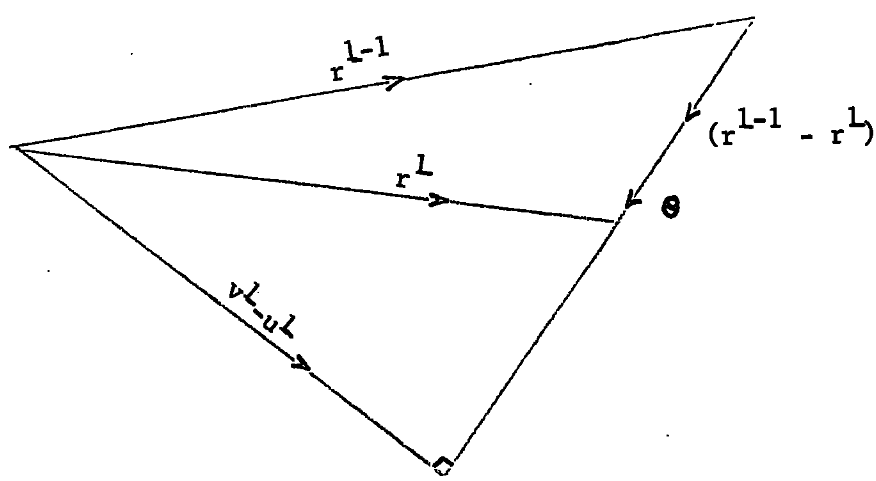
Define $$x^{l+1} = u^l + \beta^l(v^l - u^l), \text{ and if } \beta^l = 1 \text{ then}$$

$$x^{l+1} = v^l$$

$$= y^l + \theta^l(y^{l-1} - y^l)$$

and $$v^l - u^l = r^l + \beta^l(r^{l-1} - r^l)$$

Here the choice of the parameter θ is to make the vector $(v^l - u^l)$, orthogonal to $r^{l-1} - r^l$, and so to minimize it.



This method is suggested as an alternative means of vector extrapolation for many-component vectors, where the components may not be regarded as independent. For such cases the ϵ-algorithm demands an equivalent number of iterates, whereas the above method only uses two.

In the work which follows, the main iteration methods, Picard and Newton and their variants and ~~extremas~~ extensions, are described in detail. They are applied to a number of standard differential equations and the results are analysed. The solutions were all carried out afresh, although some of the same equations and methods have been described in the literature, because (a) the descriptions do not contain enough detail for rates of convergence to be analysed and (b) it was desirable to test all methods under the same computing conditions. Computing was carried out on an IBM 360-67 and all programming was in single-precision so that the results may be considered reliable to say six significant figures.

The rates of convergence of all processes are analysed from the numerical results; estimated rates for the various methods are also obtained in some cases where practicable, by linearising the effects, and also by the usual expansion approach.

Chapter 2

Introduction

In this Chapter a detailed account is given of various iterative methods, Picard and variations, Runge-Kutta, and Newton linearisation. These are applied to a number of equations and the numerical results are given.

(2.0) Picard's Method

Consider the first order differential equation of the form

$$\begin{aligned} y' &= f(x,y) \\ y(\xi) &= \eta \end{aligned} \qquad (2.0.1)$$

Ince (1953) [28] had shown that any differential equation which expresses the derivatives of highest order explicitly in terms of the lower order derivatives and the independent variable, can be expressed by a system of equations and hence any such differential equation can be written as a combination of equations of type (2.0.1). If equation (2.0.1) is replaced by the system

$$y_i'(x) = f\{x, y_{i-1}(x)\} \qquad (2.0.2)$$

for $i = 1,2, \ldots\ldots$

Where $f(x, y_{i-1})$ represents a function of all the dependent variables, an iterative process is produced. Picard's iterative method uses (2.0.2) in the form of the integral equation

$$y_i(x) = \eta + \int_{\xi}^{x} f\{x, y_{i-1}(x)\}dx \qquad (2.0.3)$$

This is the basis of the existence theorem for ordinary initial value differential equations and can be shown to converge under general conditions.

(2.1) Clenshaw-Norton Procedure

C. Clenshaw and H. Norton (1963) [12] have set up an iterative procedure based on the use of Chebyshev series in Picard's iteration, applicable to the solution of both linear and non-linear ordinary differential equations. The first step in constructing the solution of (2.0.1) is then to represent $y_{i-1}(x)$ (the initial approximation to the solution of (2.0.2)), by a truncated Chebyshev series of degree N:

$$y_{i-1}(x) = \sum_{r=o}^{N}{}' A_r T_r(x); \quad y_{i-1}(\xi) = \eta \qquad (2.1.1)$$

Where the coefficients A_r $(r = 0,1,\ldots..N)$ are known. The series may be evaluated at the points

$$x_s = \cos\frac{s\pi}{M}, \; s = 0,1, \;\ldots.., \; M$$

where M is the number of sub-intervals taken in the range $-1 \le x \le +1$, using the recurrence procedure (1.1.9). Now let

$$f\{x,y_{i-1}(x)\} = \sum_{r=o}^{N}{}' A'_r T_r(x) \qquad (2.1.2)$$

then the right hand side of a given differential equation in the form (2.01) represents an algorithm for computing the values of $f(x,y_{i-1})$ for any (x,y) in the region of interest, such that

$$f\{x_s\, y_{i-1}(x_s)\} = C_s$$

$$\text{at } x_s = \cos\frac{s\pi}{M}\,, \quad s = 0,1,\ldots\ldots,M$$

and by the aid of the orthogonal property of summation we get

$$A'_r = \frac{2}{M}\sum_{s=o}^{M}{}'' C_s \cos\frac{sr\pi}{M} \qquad (2.1.3)$$

for $r = 0,1,\ldots\ldots,N$

and hence

$$y'(x) = \sum_{r=o}^{N}{}' A'_r\, T_r(x)$$

The integral formula

$$2rA_r = A'_{r-1} - A'_{r+1} \qquad (2.1.4)$$

for $r = 1,2,\ldots\ldots, N+1$.

will provide the Chebyshev coefficients $A_r^{(i)}$ of the series

$$y_i(x) = \sum_{r=o}^{N+1}{}' A_r^{(i)}\, T_r(x)$$

The integral equation (2.1.4) does not give $A_o^{(i)}$ which is the constant of integration. The boundary condition $y(\xi) = \eta$ will yield $A_o^{(i)}$ such that

$$A_o^{(i)} = 2\eta + 2(A_1^{(i)}T_1(\xi) + A_2^{(i)}T_2(\xi) + \ldots\ldots\ldots + A_{N+1}^{(i)}\, T_{N+1}(\xi))$$

Then the series $\sum_{r=o}^{N+1}{}' A_r^{(i)}\, T_r(x)$ represents an improved approximation to the required solution.

This process may be repeated until each member of the current set of coefficients differs from the corresponding member of the previous set by less than a prescribed amount, that is taken as a measure of accuracy required.

Then Clenshaw-Norton procedure can be outlined as follows:

(i) The Chebyshev series for the initial approximation $y_{i-1}(x)$ is evaluated at the points $x_s = \cos\frac{s\pi}{M}$, $(s = 0,1, \ldots,M)$.

(ii) The values of $f(x,y_{i-1})$ are then computed at the same points $x_s = \cos\frac{s\pi}{M}$.

(iii) The coefficients A'_r $(r = 0,1,\ldots N)$ of the series

$$f(x,y_{i-1}) = \sum_{r=o}^{N}{}' A'_r T_r(x) \text{ are hence calculated.}$$

(iv) New set $A_r^{(i)}$, for $r = 0,1,\ldots, N+1$, are obtained using the integral formula (2.1.4), and the given boundary condition namely $y(\xi) = \eta$.

The solution $y_i(x) = \sum_{r=o}^{N+1}{}' A_r^{(i)} T_r(x)$ is then achieved and this sequence of operations represents one cycle.

To illustrate the procedure, we consider its application to the solution of

Example 1: $\quad y' + y = 0$

$$y(0) = 1$$

which has the solution $\quad y(x) = e^{-x}$

Let the initial approximation $y_o(x) = \sum_{r=o}^{1}{}' A_r T_r(x)$ to be 1-x, so that $y_o(0) = 1$, then

$$y_o(x) = 1-x = \tfrac{1}{2}(2.0)\ T_o(x) - 1.0\ T_1(x)$$

and so

$$A_o = 2.0,\ A_1 = -1.0$$

In this example $f(x,y_o) = -y_o(x)$ provides an algorithm to compute the values of $f(x,y_o) = \Sigma A_r' \ T_r(x)$, where we find that

$$A_o' = 2.0,\ A_1' = 1.0$$

$$A_2' = A_3' = 0$$

and hence using the integral formula, we get new set of coefficients A_r $(r = 0,1,\ldots.\ n+1)$ where

$$A_1 = \frac{A_0' - A_2'}{2} = -1.0$$

$$A_2 = \frac{A_1' - A_3'}{4} = 0.25$$

and

$$A_o = 2(1 + \sum_{r=1}^{K} (-1)^{r-1} A_{2r})$$

$$= 2(1 + A_2 - A_4 + \ldots..)$$

$$= 2.5$$

(where $K = \left[\frac{N+1}{2}\right]$ the max integer $\le \frac{N+1}{2}$).

Therefore the new approximation $y_1(x) = \sum_{r=o}^{2}{}' A_r T_r(x)$ is

$\tfrac{1}{2}(2.5)T_o(x) - T_1(x) + 0.25\ T_2(x)$.

By repeating this process we obtain the 7th Order Chebyshev series approximation to e^{-x} in the range (-1, 1) shown in Table 1.

It is clear from this process that after each iterative cycle the order of the approximation is increased by 1 and so for $i \leq N$ the approximation $y_i(x)$ is merely the truncated Taylor's series for e^{-x} rearranged appropriately. For the general case this would not be so, for if we fix N and continue the process, the coefficients converge to the true values less the truncation error. This truncation error can be reduced by increasing N, e.g.

Since $y_i(x) = \sum_{r=o}^{N}{}' A_r^{(i)} T_r(x)$ is rearrangement of the truncated Taylor series for e^{-x}, it is clear that its maximum error occurs at $x = -1$. If $N = 5$ then this error is given by

$$e - \sum_{r=o}^{5} (-1)^r A_r$$

$$= e - 2.716^{\cdot}$$

$$= 0.00162$$

In contrast comparison of the coefficients $A_r^{(i)}$ with A_r shows that the error of $y_i(x)$ can be reduced; as the number of iterations increased, to

$$\sum_{r=o}^{5} |A_r^{(10)} - A_r| + \sum_{r=6}^{\infty} |A_r|$$

$$= 0.00017$$

where 10 is the number of iterations.

Example 2:

$$y' = y^2$$

$$y(0) = \tfrac{1}{2}$$

Here, let

$$y_o(x) = \tfrac{1}{2}(1 + \tfrac{x}{2})$$

$$= \sum_{r=o}^{N}{}' A_r T_r(x)$$

and N has the values 3,4, ..., 15

The solutions obtained are shown in Table 5, which includes A_r of the

solution $y(x) = \frac{1}{2-x} = \sum_r{}' A_r T_r(x)$.

Example 3:

$$y' = x - y^2$$

$$y(0) = -0.72901$$

$$y_o(x) = -0.72901\ (1 + x)$$

$$= \sum_r^N{}' A_r T_r(x)$$

and let N have the different values as the above example, the solutions are listed in the Table 6. Where the number of iterations required to obtain the approximations A_r is indicated on the superscript on A_r in those tables.

(2.2) Second Order Equations:

Clenshaw-Norton iterative procedure enables us to attack a wide class of differential equations, but at this stage we are restricted to non-linear differential equations of boundary-value type of the form

$$y''(x) = f\{x, y(x), y'(x)\} \qquad (2.2.1)$$

with $y(-1) = \alpha$, $y(+1) = \beta$.

It should be noted that convergence does not necessarily occur for boundary-value problems but must be investigated for each problem.

Method of Solution:

Let the initial approximation to the solution of (2.2.1) be represented by

$$y_{i-1}(x) = \sum_{r=o}^{N}{}' A_r^{(i-1)} T_r(x)$$

in the range $-1 \le x \le +1$. Such that

$$y_{i-1}(-1) = \alpha, \; y_{i-1}(+1) = \beta$$

Also let

$$y'_{i-1}(x) = \sum_{r=o}^{N}{}' A_r'^{(i-1)} T_r(x)$$

where the coefficients $A_r^{(i-1)}$, $A_r'^{(i-1)}$ $(r = 0,1,\ldots.,N)$ are of known values.

Then the series $y_{i-1}(x) = \Sigma' A_r^{(i-1)} T_r(x)$ and $y'_{i-1}(x) = \Sigma' A_r'^{(i-1)} T_r(x)$ can be evaluated at the points $x_s = \cos\frac{s\pi}{M}$, $(s = 0,1,\ldots.., M)$ by the aid of a recturrence procedure similar to (1.1.9). Hence the values of

$f\{x, y_{i-1}(x), y'_{i-1}(x)\}$ are computed by a given algorithm, for any values of (x,y,y') in the required region, and thus the coefficients A''_r of the series

$$y'' = f\{x, y_{i-1}(x), y'_{i-1}(x)\} = \sum_{r=o}^{N}{}' A''_r T_r(x)$$

can be computed directly using the summation formula

$$A''_r = \frac{2}{M} \sum_{s=o}^{M}{}'' f_s \cos\frac{rs\pi}{M}$$

for $r = 0,1,\ldots,N$ and where $f_s = f\{x_s, y_{i-1}(x_s)\ y'_{i-1}(x_s)\}$.

The relations

$$2r\ A'_r = A''_{r-1} - A''_{r+1}; \qquad r = 1,2,\ldots,N$$

$$2r\ A_r = A'_{r-1} - A'_{r+1}; \qquad r = 2,3,\ldots,N$$

and the given boundary conditions will enable us to produce the new sets of coefficients $A_r^{(i)}$ and $A_r'^{(i)}$ and hence

$$y_i(x) = \sum_{r=o}^{N}{}' A_r^{(i)} T_r(x)$$

$$y'_i(x) = \sum_{r=o}^{N}{}' A_r'^{(i)} T_r(x)$$

These new series for $y_i(x)$ and $y'_i(x)$ could be used to start another cycle of the iterative procedure. To illustrate this, we consider Van der Pol's equation as an example

<u>Example 4</u> $$y'' = \tfrac{1}{4}(1-y^2)y' - \frac{1}{16}y$$

$$y(-1) = 0, \; y(+1) = 2.0$$

Here let

$$y_o(x) = \sum_{r=o}^{N}{}' A_r^{(o)} T_r(x)$$

$$= \tfrac{1}{2}(A_o^{(o)})T_o(x) + A_1^{(o)}T_1(x)$$

Now $$y_o(-1) = \tfrac{1}{2}A_o^{(o)} - A_1^{(o)} = 0$$

$$y_o(+1) = \tfrac{1}{2}A_o^{(o)} + A_1^{(o)} = 2$$

and from these we get $A_o^{(o)} = 2.0$, $A_1^{(o)} = 1.0$ and $A_2^{(o)} = A_3^{(o)} = \ldots\ldots = A_N^{(o)} = 0$.

<u>i.e.</u> $$y_o(x) = \tfrac{1}{2}(2.0)T_o(x) + T_1(x)$$

$$= 1 + x$$

and hence let $y_o'(x) = 1$, so that $A_o'^{(o)} = 2.0$, $A_1'^{(o)} = 0.0$, $A_2'^{(o)} = A_3'^{(o)} = \ldots. = A_N'^{(o)} = 0.0$.

Then Picard's method will produce new sets of coefficients $A_r^{(1)}$ and $A_r'^{(1)}$ $(r = 0,1,\ldots.,N)$ where $A_o^{(1)}$, $A_1^{(1)}$ and $A_o'^{(1)}$ are deduced from the boundary conditions such that

$$A_o^{(1)} = 2-2(A_2^{(1)} + A_4^{(1)} + A_6^{(1)} + \ldots\ldots..)$$

$$A_1^{(1)} = 1 - (A_3^{(1)} + A_5^{(1)} + A_7^{(1)} + \ldots\ldots.)$$

$$A_o'^{(1)} = 2\sum_{r=o}^{N}(2r+1)\,A_{2r+1}^{(1)}$$

The process has settled to a reasonable approximation after 10 iterations, and the coefficients A_r of the approximation

$$y(x) = \sum_{r=o}^{N}{}' A_r T_r(x)$$

Where N was taken to be equal to 17 are shown in Table 7. Here the truncation error in y(x) is so small as to be ineffective, while a significant build of round-off error is expected due to the repeated evaluation of the functions y(x), y'(x) and f{x,y,y'}.

Example 5:

Consider the problem

$$y'' + \lambda^2 y = 0$$

$$y(-1) = 0, \; y(+1) = 1$$

It has the solution $y(x) = \sin \lambda(1+x)/\sin 2\lambda$ in the range $-1 \leq x \leq 1$ for $\lambda \neq \frac{\pi}{2}$.

$$\text{let } y_o(x) = \tfrac{1}{2}(1 + x)$$

$$= \sum_{r=o}^{N}{}' A_r T_r(x)$$

(where $A_o = 1$, $A_1 = \frac{1}{2}$, $A_2 = \ldots\ldots\ldots\ldots = A_N = 0$).

be our initial approximation to the required solution, this being the simplest polynomial which satisfies the given boundary conditions. Taking $\lambda = 1.25 < \frac{\pi}{2}$ the method of Picard iteration gave the values which are listed in Table 2 which includes for comparison the leading coefficients A_r in the finite Chebyshev series for y(x).

Example 6: As in example 5, with

$$\lambda > \frac{\pi}{2}$$

The results for this are given in the Appendix. The solution diverges; explanation and treatment via acceleration procedures are discussed in the next Chapter.

In this example if we consider the case $\lambda = 2$, the solutions obtained by Picard's are given in Table 3 which also includes A_r of the solution

$$y(x) = \sum_{r=o}^{N}{}' A_r T_r(x)$$

The process was terminated after 10 iterations, and the degree of the approximations was fixed to be ($N = 9$). In this case, however Picard's method was diverging and hence it failed to get a reasonable approximation to the solution $y(x)$. Clearly this indicates that convergence is not secured for a wide class of problems when Picard's method is used.

(2.3) Application to Integral equations:

A similar approach can be used for problems originally formulated as integral equations. Thus Wolfe (1969) [17] used the truncated Chebyshev series

$$y(x) = \sum_{0}^{N}{}' A_r T_r(x)$$

The coefficients A_r ($r = 0,1,\ldots.,N$) are in this case determined iteratively by

$$y_{s+1}(x_i) = y_s(x_i) + \lambda \int_{-1}^{+1} K(x_i,t) y_s(t)\, dt \qquad (2.3.1.)$$

where $$y_s(x) = \sum_{r=o}^{N}{}' A_r^{(s)} T_r(x)$$

$$f(x_i) = y_o(x_i)$$

For each $x_i = \cos \frac{i\pi}{N}$, $(i = 0,1,\ldots,N)$, $K(x_i,t)$ is approximated by a polynomial in t of degree M of the form

$$K(x_i,t) = \sum_{r=o}^{M}{}'' b_r(x_i)\, T_r(t)$$

$$b_r(x_i) = \frac{2}{M} \sum_{s=o}^{M}{}'' K(x_i, \cos \frac{s\pi}{M}) \cos \frac{rs\pi}{M}$$

for $(r = 0,1,\ldots,M)$

Hence equation (2.3.1) becomes

$$y_{s+1}(x_i) = y(x_i) + \lambda \int_{-1}^{1} \left\{ \sum_{r=o}^{M}{}'' b_r T_r(t) \sum_{r=o}{}' A_r^{(s)} T_r(t) \right\} dt$$

the series $\Sigma'' b_r T_r(t)$ and $\Sigma' A_r^{(s)} T_r(t)$ are multiplied together and the integration can then be carried out using (1.1.3) and (1.1.4). Therefore for each $x_i = \cos \frac{i\pi}{N}$, $y_{s+1}(x_i)$ can be determined, and hence the Chebyshev expansion taking these values can be found as

$$\sum_{r=o}^{N}{}' A_r^{(s+1)} T_r(x)$$

using the property of summation (1.1.6), where

$$A_r^{(s+1)} = \frac{2}{N} \sum_{i=o}^{N}{}'' y(x_i)\, T_r(x_i), \quad r = 0,1,\ldots,N$$

and these are used in (2.3.1) to continue the iteration until convergence is reached.

This method can be easily extended to the solution of Volterra equation

$$y(x) = f(x) + \lambda \int_{-1}^{x} K(x,t)y(t)\, dt$$

$$y(-1) = Y$$

No further work has been done on this method in this thesis and it is included here only for completeness.

(2.4) Weyls Method:

Weyl (1942) [18] suggested an iterative method for solution of equations of the general type

$$y^{(r+1)}(x) + f(x,y,\ldots\ldots,y^{(r-1)})y^{(r)} = 0 \qquad (2.4.1)$$

or $$y^{(r+1)}(x) + f(x,y,\ldots\ldots,y^{(r-1)})y^{(r)} = g(x,y,\ldots\ldots,y^{(r-1)}) \qquad (2.4.2)$$

where $y^{(r)}(x)$ denotes the rth derivative of $y(x)$.

If the problem is an initial value one, for example if $y(0)$, $y'(0),\ldots$, $y^{(r)}(0)$ are known, solving the equation as a linear differential equation in $y^{(r)}$, using approximate values for $y,y',\ldots\ldots$ in f and g, produces an iterative procedure which is known to converge, namely

$$y_{i+1}^{(r)}(x) = y^{(r)}(0)\, e^{-\int_0^x f_i\, dx} \qquad (2.4.3)$$

or $$y_{i+1}^{(r)}(x) = \left\{y^{(r)}(0) + \int_0^x g_i\, e^{\int_0^x f_i\, dx}\, dx\right\} e^{-\int_0^x f_i\, dx} \qquad (2.4.4)$$

where y_i is the ith iterate and $f_i \equiv f(x,y_i,y_i',\ldots\ldots,y_i^{(r-1)})$ and similarly g_i.

The values of $y_{i+1}^{(r-1)}$, $y_{i+1}^{(r-2)}$, ..., y_{i+1} are obtained by successive ~~iter~~integration. The procedure may also be used for boundary value problems, but here convergence is not guaranteed.

The method may be applied to any equation which is linear in two derivatives. For example the equation to which it was first applied.

$$y''' + y\, y'' = 0$$

$$\text{with} \quad y(0) = y'(0) = 0,\ y''(0) = 1$$

produces iterates

$$y_0 = 0$$

$$y_1'' = 1 \text{ giving } y_1 = \tfrac{1}{2} x^2$$

$$y_2'' = \exp(-\tfrac{1}{6} x^3), \text{ and so on.}$$

It can be seen that, this method is, when $g \equiv 0$, simply a variant of Picard's method produced by a change of variable.

If

$$\log y^{(r)} = U$$

is substituted into

$$y^{(r+1)} + f\acute{y}^{(r)} = 0$$

it becomes

$$U' + f = 0$$

and Picard's method applied to this would produce, with U(0) known ($= \log y^{(r)}(0)$)

$$U_{i+1} = U(0) - \int_0^x f_i\, dx$$

which gives

$$y_{i+1}^{(r)} = y^{(r)}(0)\, e^{-\int_0^x f_i\, dx} \quad \text{as in (2.4.3) above.}$$

When $g \neq 0$ a further approximation is introduced. Picard's method now would give, with the same substitution

$$U_{i+1} = U(0) - \int_0^x f_i dx + \int_0^x g_i/y^{(r)} dx \tag{2.4.6}$$

or
$$y_{i+1}^{(r)} = y^{(r)}(0)\, e^{-\int_0^x f_i dx}\, e^{\int_0^x g_i/y^{(r)} dx}$$

$$= e^{-\int_0^x f_i dx}\, y^{(r)}(0)\left\{1 + \int_0^x g_i\, e^{-U} dx + \ldots\ldots\right\} \tag{2.4.7}$$

and from (2.4.6)

$$e^{-U} = e^{-U(0)} . e^{\int_0^x f_i dx} . e^{-\int_0^x g_i e^{-U} dx}$$

$$= \frac{1}{y^{(r)}(0)}\, e^{\int_0^x f_i dx} + \ldots\ldots\ldots\ldots$$

substituting this approximation in (2.4.7) gives

$$y_{i+1}^{(r)} = e^{-\int_0^x f_i dx}\left\{y^{(r)}(0) + \int_0^x g_i\, e^{\int_0^x f_i dx} dx\right\}$$

which is (2.4.4).

<u>The Use of Chebyshev Series in Weyls Method</u>:

We now consider the details involved in carrying out the iterative procedure while representing the functions which are used in (2.4.4) by a polynomial approximation in the form of the following series:

$$y_i(x) = \sum_{k=o}^{N}{}' A_k T_k(x)$$

$$y_i'(x) = \sum_{k=o}^{N}{}' A_k' T_k(x)$$

$$y_i''(x) = \sum_{k=o}^{N}{}' A_k'' T_k(x)$$

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$$y_i^{(r-1)}(x) = \sum_{k=o}^{N}{}' A_k^{(r-1)} T_k(x)$$

$$y_i^{(r)}(x) = \sum_{k=o}^{N}{}' A_k^{(r)} T_k(x)$$

$$f(x, y_i, y_i', \ldots, y_i^{(r-1)}) = \sum_{k=o}^{N}{}' b_k T_k(x)$$

$$g(x, y_i, y_i', \ldots y_i^{(r-1)}) = \sum_{k=o}^{N}{}' C_k T_k(x)$$

Given the Chebyshev series for $y_i^{(r-1)}$, $y_i^{(r-2)}$, ..., y_i', y_i, we calculate the Chebyshev series for f_i and g_i. The values of $y^{(s)}$ ($s = 0,1, \ldots, r-1$) can be computed at the points $x_j = \cos \frac{j\pi}{N}$, ($j = 0,1,\ldots,N$) using the recurrence formula (1.1.9) and hence the values of f_i and g_i are evaluated at each of the (N+1) points.

Using the orthogonal property of summation (1.1.8) we obtain the Chebyshev coefficients b_k, C_k ($k = 0,1,\ldots,N$) such that

$$b_k = \frac{2}{N} \sum_{j=o}^{N}{}'' f(x_j, y_i(x_j), \ldots, y_i^{(r-1)}(x_j)) T_k(x_j)$$

and

$$C_k = \frac{2}{N} \sum_{j=o}^{N}{}'' g(x_j, y_i(x_j) \ldots . y_i^{(r-1)}(x_j))\ T_k(x_j)$$

for $k = 0,1,\ldots.,N$.

Now let:

$$\text{(i)} \qquad F(x) = \int_o^x f_i\ dx$$

$$= \sum_{k=o}^{N} b_k \int_o^x T_k(x)\ dx$$

$$= \sum_{k=o}^{N+1}{}' d_k\ T_k(x)$$

where

$$d_0 = 2(b_2 - b_4 + b_6 \ldots\ldots\ldots (-1)^{r+1}\ b_{2r})\ r > 0$$

$$2rd_r = b_{r-1} - b_{r+1},\ r = 1,2,\ldots.., N-1$$

$$d_N = \frac{b_{N-1}}{2N}$$

$$d_{N+1} = \frac{b_N}{2(N+1)}$$

$$\text{(ii)} \qquad E(x) = EXP(F(x))$$

$$= \sum_{k=o}^{N}{}' e_k\ T_k(x)$$

$$\text{(iii)} \qquad H(x) = \sum_{k=o}^{N}{}' h_k\ T_k(x)$$

$$= +\ EXP(-F(x))$$

Thus evaluating $F(x_j)$ at the points $x_j = \cos\frac{j\pi}{N}$ $(j = 0,1,\ldots,N)$ will enable us to calculate the Chebyshev series for $E(x)$ and $H(x)$

(iv) $$K(x) = E(x)\, g_i$$
$$= \sum_{k=o}^{N}{}' K_k\, T_k(x)$$

Where this series may be approximated by one of the following methods:

(a) Multiplying the series $E(x) = \sum_k{}' e_k\, T_k(x)$ by each term of the series $g_i = \sum_k{}' C_k\, T_k(x)$. This is carried out using the relation

$$T_m(x)\, T_n(x) = \tfrac{1}{2}\,\{T_{m+n}(x) + T_{|m-n|}(x)\}$$

and then equate coefficients of $T_k(x)$ of both sides of (iv) to calculate the coefficients K_k $(K = 0, 1, \ldots, N)$.

(b) Evaluating $K(x_j) = E(x_j)\, g\{x_j, y_i(x_j), \ldots\ldots, y_i^{(r-1)}(x_j)\}$ at the point $x_j = \cos\frac{j\pi}{N}$ $(j = 0,1, \ldots, N)$ and use the summation formula (ii) of (1.1.8) to obtain K_k.

(v) $$G(x) = \int_o^x K(x)\, dx$$
$$= \sum_{k=0}^{N}{}' K_k \int_o^x T_k(x)\, dx$$
$$= \sum_{k=0}^{N+1}{}' 1_k T_k(x)$$

Where

$$l_0 = 2\,(k_2 - k_4 + k_6 - \ldots\ldots.(-1)^{r+1}\; K_{2r}),\ r > 0$$

$$l_r = \frac{K_{r-1} - K_{r+1}}{2r}, \qquad r = 1,2, \ldots.., N-1$$

$$l_N = \frac{K_{N-1}}{2N}$$

$$l_{N+1} = \frac{K_N}{2(N+1)}$$

$$\text{(vi)} \qquad Q(x) = \sum_{k=0}^{N}{}' q_k T_k(x) = H(x).G(x)$$

Where this series is approximated by either method (a) or (b).

Hence

$$y_{i+1}^{(r)}(x) = y^{(r)}(0)\,H(x) + Q(x)$$

$$= \sum_{k=0}^{N}{}' a_k^{(r)}\; T_k(x)$$

is known, and successive integration would provide the solution y(x) to the original problem.

To summarize the procedure described here, we note that the right hand side of equation (2.4.3) or (2.4.4) is reduced to a truncated Chebysher series of known coefficients. This series is then integrated using (1.2:2), where a set of (N+1) simultaneous equations is formed in the (N+1) unknown coefficients. The solution of these equations gives an improved approximation $y_{i+1}(x) = \sum_k{}' a_k T_k(x)$ to the solution of the differential equation and where this may be used to start another iterative cycle.

The essence of this method is demonstrated by considering its application to the following examples;

Example 2: (Norton 64)

$$y' = y^2$$

$$y(0) = \tfrac{1}{2}$$

Which has the solution $y(x) = {}^{1}/(2-x)$ in the range $-1 \leq x \leq +1$.

We reduce this problem to the form

$$y' + f(x,y)y = g(x,y)$$

Where $f(x,y) = -y, \quad g(x,y) = 0$

The iterative process will have the form

$$y_{i+1}(x) = y_{(0)} \, e^{-\int_0^x f_i \, dx}$$

$$= \tfrac{1}{2} \, e^{\int_0^x y_i(x) dx}$$

Taking the initial approximation

$$y_o(x) = \sum_{k=o}^{N}{}' a_k T_k(x)$$

$$= \tfrac{1}{2} (1 + \tfrac{x}{2})$$

and N has the values 3, 5, 7. the process converged to a reasonable solution after only 10 iterations. The solutions in the coefficients

A_k (k = 0,1, ...N) are shown in table (5). Compared with the coefficients A_r of the solution $y(x) = \sum_{r=o}^{N}{}' A_r T_r(x) = {}^1/(2-x)$, obtained by

$$A_r = \frac{2}{N} \sum_{j=o}^{N}{}'' f(x_j)\, T_r(x_j)$$

$$= \frac{2}{N} \sum_{j=o}^{N}{}'' \frac{1}{(2-x_j)}\, T_r(x_j)$$

Example 3: (Norton 1964)

$$y' = x-y^2$$

$$y(o) = -0.72901$$

This can be rearranged in the form

$$y'_{i+1}(x) + f(x,y_i)\; y_{i+1} = g(x,y_i)$$

where $f(x,y_i) = y_i(x)$ and $g(x,y_i) = x$

The iterative process will be

$$y_{i+1}(x) = e^{-\int_o^x y_i dx} \{ \int_o^x x\, e^{\int_o^x y_i dx} dx - 0.72901 \}$$

Taking $y_o(x) = -0.72901\,(1 + \frac{x}{2}) = \sum_{r=o}^{N}{}' A_r T_r(x)$, we get the results shown in table (6), compared with the results obtained by Norton (64).

<u>Example 4:</u> Van der Pol's equation

$$y'' = \frac{1}{4}(1-y^2)y' - \frac{1}{16} y$$

$$y(-1) = 0, \quad y(+1) = 2$$

This can be arranged in the form

$$y''_{i+1} + f(x,y_i)y'_{i+1} = g(x,y_i)$$

where $\quad f(x,y_i) = -\frac{1}{4}(1-y^2),$

and $\quad g(x,y_i) = -\frac{1}{16} y_i$

If we take $\quad y_o(x) = 1 + x$

$$= \sum_{r=o}^{N}{}' A_r T_r(x)$$

Then

$$f(x,y_i) = \frac{1}{4}(1-(1+x)^2) = \frac{x}{2} + \frac{x^2}{4}.$$

$$= \sum_{r=o}^{N}{}' b_r T_r(x)$$

$$= \tfrac{1}{2}(1/4)T_0(x) + \tfrac{1}{2} T_1(x) + \frac{1}{8} T_2(x)$$

$$b_0 = \frac{1}{4}, \; b_1 = \frac{1}{2}, \; b_2 = \frac{1}{8}, \; b_3 = \ldots\ldots = b_N = 0.$$

and $\quad g(x,y_i) = -\frac{1}{16} y_i(x)$

$$= -\frac{1}{16}(1+x)$$

$$= \sum_{r=0}^{N}{}' c_r T_r(x)$$

$$= \tfrac{1}{2}(-\tfrac{1}{8})\, T_0 - \frac{1}{16} T_1(x)$$

$$C_0 = -\frac{1}{8},\; C_1 = -\frac{1}{16},\; C_2 = \ldots\ldots = C_N = 0.$$

Thus Weyls process can be started from these initial values and the 17th Order Chebyshev series approximation obtained is shown in table (7).

<u>Example 5</u>:

$$y'' + \lambda^2 y = 0$$

$$y(-1) = 0, \quad y(+1) = 1$$

This has the form

$$y'' + f(x,y)y' = g(x,y), \quad f(x,y) = 0, \; g(x,y) = -\lambda^2 y$$

<u>i.e.</u>

$$y'_{i+1}(x) = e^{-\int_0^x f_i dx} \{ \int_0^x g_i \, e^{\int_0^x f_i dx} dx + y'(0) \}$$

$$= e^{-\int_0^x 0\, dx} \{ \int g_i \, e^{\int_0^x 0\, dx} dx + y'(0) \}$$

$$= \int_0^x -\lambda^2 y_i dx + y'(0)$$

or $$y_{i+1}(x) = \int_0^x \{ -\int_0^x \lambda^2 y \, dx + y'(0) \} dx + y(0)$$

Which is Picards method for the solution of the above.

(2.5) Use of Runge-Kutta method for boundary value problems:

Consider the solution of Falkner-~~Stern~~ Skan equation of the form,

$$y'''(x) + y(x)\,y''(x) + \beta(1-y'(x)^2) = 0$$

with the boundary conditions $y(0) = 0$, $y'(0) = 0$, $y'(\infty) = 1$.

and. β = constant (0.01) in the range $0 \le x \le \infty$.

In order to use Runge-Kutta procedure, $y''(0)$ is also required. Therefore the iterative process employed here is to improve approximate values of $y''(0)$ until $y'(\infty) = 1$.

(i) Let $y''(0) = V_r$

(ii) Apply R-K using initial values $y(0) = 0$, $y'(0) = 0$, and $y''(0) = V_r$ and record $y'(\infty)$, say U_r

(iii) let again $y''(0) = V_{r+1}$ such that $V_{r+1} \simeq V_r$

(iv) Apply R-K, using the initial values $y(0) = 0$, $y'(0) = 0$, and $y''(0) = V_{r+1}$ and record $y'(\infty)$, say U_{r+1}

By linear interpolation

$$V_{r+2} = \frac{V_{r+1}(1-U_r) - V_r(1-U_{r+1})}{U_{r+1} - U_r}$$

and for $r = 1$,

(v) Calculate V_{r+2} using the above relation

(vi) Apply R-K using $y(0) = 0$, $y'(0) = 0$, $y''(0) = V_{r+2}$ and record $y'(\infty)$

(vii) if $y'(\infty) = 1$, then solution is achieved, else set $r = r + 1$ and go back to step (v).

Unfortunately the efficiency of this method depends entirely, in this case, on the initial gussed - values of $y''(0)$, and so convergence is considerably slower if the initial values of $y''(0)$ are not anywhere near the right solution.

For the solution of the above equation, the initial values of $y''(0)$ were taken as $V_0 = -.95$, $V_1 = 1.05$. The interval of integration was $h = 0.1$ over the range $0 \leq x \leq 20$. Only six iterations were needed to obtain the solution to 5 d.p. shown in table (8b).

2.6 Newtons Method

For the system $y' = f(x,y)$, we assume $f(x,y)$ to be a function of y regular in a region which includes the solution and our approximation to it for every value of x in the range (-1, +1).

A small change δy in y gives formally

$$\begin{aligned} \frac{d}{dx}(y + \delta y) &= f(x,y + \delta y) \\ &= f(x,y) + \delta y \, \frac{\partial f}{\partial y}(x,y) + O(\delta^2 y) \qquad 2.6.1 \end{aligned}$$

We define a sequence $\{Y_i\}$ of approximations to the solution $y(x)$ by considering the leading terms in Taylor-series expansion for $f(x,y)$, suggesting the Newtons iteration formula

$$Y_i'(x) = f(x,Y_{i-1}) + (Y_i - Y_{i-1}) \frac{\partial f}{\partial y}(x,Y_{i-1}) \qquad 2.6.2$$

i.e.

$$Y_i'(x) - Y_i \frac{\partial f}{\partial y}(x,Y_{i-1}) = f(x,Y_{i-1}) - Y_{i-1} \frac{\partial f}{\partial y}(x,Y_{i-1}) \qquad 2.6.3$$

For each iterative cycle, a particular solution $\bar{y}_i(x) = v(x)$ of the inhomogeneous linear equation (2.6.3) may be calculated, to which is to be added a multiple $\mu\, U(x)$ of the solution $U(x)$ of the homogeneous equation,

$$U'_i(x) - U_i \frac{\partial f}{\partial y}(x, \bar{y}_{i-1}) = 0$$

The factor μ is chosen so that the resulting iterate $y_i(x) = V(x) + \mu\, U(x)$ satisfies the given boundary condition.

Norton Procedure:

Norton (1964)[16] has made use of Chebyshev series in Newtons iteration (2.6.3), simply by representing the functions which are used in (2.6.3) and hence deriving relations between the coefficients in Chebyshev series

$$y_i(x) = \sum_r{}' A_r^{(i)}\, T_r(x)$$

$$y_{i-1}(x) = \sum_r{}' A_r^{(i-1)} T_r(x)$$

$$y'_i(x) = \sum_r{}' A_r'^{(i)}\, T_r(x)$$

$$f(x, y_{i-1}) = \sum_r{}' b_r\, T_r(x)$$

$$\frac{\partial f}{\partial y}(x, y_{i-1}) = \sum_r{}' C_r\, T_r(x)$$

By substituting these expressions in equation (2.63), we have

$$\sum_r{}' A'^{(i)}_r T_r(x) - \left(\sum_r{}' C_r T_r(x)\right)\left(\sum_r{}' A_r^{(i)} T_r(x)\right)$$

$$= \sum_r{}' b_r T_r(x) - \left(\sum_r{}' C_r T_r(x)\right)\left(\sum_r{}' A_r^{(i-1)} T_r(x)\right) \qquad 2.6.4$$

Products of Chebyshev polynomials $T_r(x)\ T_s(x)$ may be dealt with in the usual way, however Norton has found that the simplification produced by taking only the first term of the series $\sum_r{}' C_r T_r(x)$ gives satisfactory results. We may then simply equate the coefficients of $T_{r-1}(x)$ and $T_{r+1}(x)$ in the right and left members of (2.64) to obtain the formulae

$$A'^{(i)}_{r-1} = b_{r-1} + \tfrac{1}{2}\,\mathrm{Co}\ \left(A^{(i)}_{r-1} - A^{(i-1)}_{r-1}\right)$$

$$A'^{(i)}_{r+1} = b_{r+1} + \tfrac{1}{2}\,\mathrm{Co}\ \left(A^{(i)}_{r+1} - A^{(i-1)}_{r+1}\right)$$

where Co is the first term of $\sum_r{}' C_r T_r(x)$.

Also using the relation

$$2r\, A_r = A'_{r-1} - A'_{r+1}, \qquad r = 1,2, \ldots\ldots N. \qquad 2.65$$

If on the right hand side of (2.65) we substitute the expressions for $A'^{(i)}_{r-1}$ and $A'^{(i)}_{r+1}$, we derive a set of N linear algebraic equations in $A_r^{(i)}$. These equations may be written in the form

$$\tfrac{1}{2} \, Co \, (A^{(i)}_{r-1} - A^{(i)}_{r+1}) - 2r \, A_r^{(i)} = P_r \qquad 2.6.6$$

where $$P_r = (b_{r+1} - b_{r-1}) + \tfrac{1}{2} \, Co \, (A^{(i-1)}_{r-1} - A^{(i-1)}_{r+1}), \quad r = 1,2, \ldots\ldots N.$$

Quantities such as $A_r^{(i)}$, $A_r^{(i-1)}$ for $r > N$ which occur in the above relation are assumed to be zero.

Then Norton iterative process can now be outlined as follows;

(i) Given the coefficients $A_r^{(i-1)}$, $(r = 0, 1, \ldots., N)$ in the series $y_{i-1}(x) = \sum_{r=0}^{N}{}' A_r^{(i-1)} T_r(x)$, we evaluate $y_{i-1}(x_s)$ at the points $x_s = \cos \frac{s\pi}{M}$ $(s = 0, 1, \ldots., M)$ by using a formula similar to (1.1.9) of Chapter 1.

(ii) At the same points we compute the values of $f(x_s, y_{i-1})$ and $\frac{\partial f}{\partial y}(x_s, y_{i-1})$ for $(s = 0, 1, \ldots., M)$.

(iii) The coefficients b_r $(r = 0,1, \ldots\ldots,N)$ in expression $f(x,y_{i-1}) = \sum{}' b_r T_r(x)$ are now derived using the formulae

$$b_r = \sum_{s=0}^{M}{}' \beta_s T_r(x_s) \; ; \qquad r = 0,1, \ldots., N-1$$

$$b_N = \tfrac{1}{2} \sum_{s=0}^{M}{}' \beta_s T_N(x_s)$$

where

$$\beta_s = \frac{2}{M} \, f(x_s, \, y_{i-1}(x_s)) \qquad s = 0,1, \ldots\ldots, M-1$$

and

$$\beta_M = \frac{1}{M} \, f(x_M, \, y_{i-1}(x_M)) = \frac{1}{M} \, f(-1, \, y_{i-1}(-1))$$

(iv) The coefficient $\mathcal{C}_o$ of $\frac{\partial f}{\partial y}(x, y_{i-1}) = {\sum_{r=o}^{N}}' \mathcal{C}_r T_r(x)$ is given by

$$\mathcal{C}_o = {\sum_{s=0}^{M}}' C'_s \, T_o(x_s)$$

where

$$C'_s = \frac{2}{M} \frac{\partial f}{\partial y} (x_s, y_{i-1}(x_s)); \qquad s = 0, 1, \ldots\ldots, M-1$$

and

$$C'_M = \frac{1}{M} \frac{\partial f}{\partial y} (x_M, y_{i-1}(x_M)) = \frac{1}{M} \frac{\partial f}{\partial y} (-1, y_{i-1}(-1)).$$

Equations (2.6.6) produce one solution of inhomogeneous starting with zero coefficients, and one of homogeneous starting with unit coefficients. These two solutions are combined to determine μ such that the boundary condition is satisfied.

It should be clear from the description given that the Newton process is exact when applied to a linear equation, if $\frac{\partial f}{\partial y}$ is retained as a function of x. To illustrate the Norton iterative procedure we consider the example 2;

$$y' = y^2, \qquad y(0) = \tfrac{1}{2}$$

Where this has the solution $y(x) = \frac{1}{(2-x)}$ in the region $-1 \le x \le 1$. We take N = 5 and consider the Chebyshev series to the initial approximation $y_o(x)$ is of the form

$$y_o(x) = {\sum_{r=1}^{N}}' A_r^{(0)} T_r(x)$$

$$= \tfrac{1}{2} \left(1 + \frac{x}{2}\right)$$

then $A_0^{(0)} = 1.0, \quad A_1^{(0)} = 0.25, \quad A_2^{(0)} = \ldots . = A_s^{(0)} = 0$

and hence

$$f(x,y_o) = y_o^2 = \sum_r{}' b_r T_r(x)$$

where $b_0 = \frac{9}{16}, \quad b_1 = 0.25, \quad b_2 = \frac{1}{32}$ and $b_3 = b_4 \ldots . = b_5 = 0.$

and

$$\frac{\partial f}{\partial y}(x,y_o) = y_o = \sum_{r=o}^{N}{}' \mathcal{E}_r T_r(x)$$

with $C_0 = 2.0, \quad C_1 = 0.5, \quad C_2 = \ldots . = C_5 = 0.$

only C_0 in this case is considered for $\frac{\partial f}{\partial y}(x,y_o)$.

Then the process can be started from

$$A_0^{(0)} = 1, \quad A_1^{(0)} = 0.25, \quad A_2^{(0)} = \ldots .. = A_5^{(0)} = 0$$

$$b_0 = \frac{9}{16}, \quad b_1 = \frac{1}{4}, \quad b_2 = \frac{1}{32}, \quad b_3 = \ldots .. = b_5 = 0$$

and $C_0 = 2.$

Hence $P_r = (b_{r+1} - b_{r-1}) + \frac{1}{2} C_o (A_{r-1}^{(0)} - A_{r+1}^{(0)}), \quad (r = 1, \ldots . 5)$

can be calculated, and so we have

$$P_1 = \frac{15}{32}, \quad P_2 = 0, \quad P_3 = \frac{1}{32}, \quad P_4 = P_5 = 0$$

Equation (2.66) can be rearranged in the form

$$A_{r-1} = A_{r+1} - 2r\, A_r + P_r, \quad r = 1, \ldots .., N$$

Where the solution E_r of this nonhomogeneous equation can be generated by recursive solution starting with $E_{N+1} = E_{N+2} = 0$, and $E_0, E_1, \ldots, E_{N-1}$ can be calculated in succession from

$$E_{r-1} = E_{r+1} + 2r\, E_r + P_r, \quad r = 1,2, \ldots . N$$

such that

$$E_0 = \frac{3}{16}$$
$$E_1 = \frac{1}{8}$$
$$E_2 = \frac{-1}{32}$$

$$E_3 = E_4 = \ldots = E_{N+2} = 0$$

Similarly starting with $F_{N+1} = 0$, $F_N = 1$, and employing the corresponding homogenous relation;

$$F_{r-1} = F_{r+1} + 2r\, F_r, \quad r = N, N-1, \ldots\ldots, 1$$

we derive a sequence $\{F_r\}$ such that, for N = 5

$$F_{N+1} = 0$$
$$F_N = 1$$
$$F_{N-1} = 10, \; F_{N-2} = 81 \; \ldots., \; F_0 = 4626$$

We may now construct a solution

$$A_r^{(1)} = E_r + \mu F_r \qquad r = 0,1,\ldots\ldots,N$$

where $\mu = \dfrac{1 - E_0 + 2E_2 - 2E_4}{F_0 - 2F_2 + 2F_4}$ is determined so that the function $y_1(x) = \sum_{r=o}^{N}{}' A_r^{(1)} T_r(x)$ satisfies the boundary condition $y(0) = \frac{1}{2}$.

* The result of this example is shown in table (5).

2.7 Second Order Equations:

The extension of Newtons method for the equation

$$y''(x) = f(x,y,y') \qquad 2.7.1$$

is given by

$$y''_{i+1} - g(x)y'_{i+1} - h(x)y_{i+1} = f(x,y_i,y'_i) - g(x)\, y'_i - h(x)y_i$$

where $g(x) = \frac{\partial f}{\partial y'}(x,y_i,y'_i)$

and $h(x) = \frac{\partial f}{\partial y}(x,y_i,y'_i)$.

Norton procedure for this system is as follows:

Let the functions occuring in the above be represented by these series

$$y_i(x) = \sum_{r=o}^{N}{}' A_r T_r(x)$$

$$y_{i-1}(x) = \sum_{r}{}' a_r T_r(x)$$

$$y'_{i-1}(x) = \sum_{r}{}' a'_r T_r(x)$$

$$y'_i(x) = \sum_{r}{}' A'_r T_r(x)$$

$$y''_i(x) = {\sum_r}' A''_r \, T_r(x)$$

$$f(x, y_{i-1}, y'_{i-1}) = {\sum_r}' b_r T_r(x)$$

$$h(x) = {\sum_r}'' C_r T_r(x)$$

$$g(x) = {\sum_r}' C'_r \, T_r(x)$$

For simplicity only the first constant term in each of the Chebyshev expansions of h(x) and g(x) will be considered.
Substituting these expressions in (2.7.1) and equate coefficients of $T_r(x)$ we obtain the relation,

$$A''_r - \tfrac{1}{2} C_o' \, A_r' - \tfrac{1}{2} C_o A_r = P_r$$ [redacted]

where

$$P_r = b_r - \tfrac{1}{2} C_o' a_r' - \tfrac{1}{2} C_o a_r, \qquad r = 0, 1, \ldots .N$$

using the relations

$$2r \, A'_r = A''_{r-1} - A''_{r+1}, \; r = 1, \ldots\ldots, N$$

$$2r \, A_r = A'_{r-1} - A'_{r+1}, \; r = 2, \ldots\ldots, N$$

the above equation may be written as the following system of recurrence equations

$$C_o A_{r-1} = C_o A_{r+1} + 2r \, (2A_r' - C_o' \, A_r) + 2 \, (P_{r+1} - P_{r-1})$$

$$A'_{r-1} = A'_{r+1} + 2r \, A_r, \qquad r = N, N-1, \ldots\ldots\ldots, 1$$ 2.7.2

and every solution of this system may be expressed as the sum of one particular solution E_r, E'_r for $r = 0, 1, \ldots., N$ and linear combination of the independent solutions of the corresponding homogeneous system

$$C_o A_{r-1} = C_o A_{r+1} + 2r\,(2A'_r - C_o' A_r) \qquad 2.73$$

$$A'_{r-1} = A'_{r+1} + 2r A_r$$

Here we need to construct two solutions of this system, F_r, F'_r and G_r, G'_r (say) which tend to zero as r tends to infinity. We then determine the constants μ and ν in the expression

$$A_r = E_r + \mu F_r + \nu G_r$$

to ensure that the iterate $y_i(x) = \sum_r{}' A_r T_r(x)$ satisfies the prescribed boundary conditions.

Example 4 Van der Pols' equation

$$y''(x) = \frac{1}{4}(1-y^2)y' - \frac{1}{16}y$$

$$y(-1) = 0, \; y(+1) = 2$$

Initially let

$$y_o(x) = \sum_r{}' a_r T_r(x)$$

$$= 1 + x$$

$$= \tfrac{1}{2}(2)\, T_o(x) + T_1(x)$$

and

$$y'_o(x) = 1$$

Then $$f(x,y_o,y'_o) = \frac{1}{4}(1-y^2_o)y'_o - \frac{1}{16}y_o$$

$$= \sum_r{}' b_r T_r(x)$$

$$= \tfrac{1}{2}(-\tfrac{3}{8})\,T_o(x) - \frac{9}{16}T_1(x) - \frac{1}{8}T_r(x)$$

$$h(x) = \frac{\partial f}{\partial y}(x,y_o,y'_o) = -\tfrac{1}{2}y_o y'_o - \frac{1}{16}$$

$$= \sum_r{}' C_r T_r(x)$$

$$= \tfrac{1}{2}(-\tfrac{9}{8})T_o - \tfrac{1}{2}T_1(x)$$

$$g(x) = \frac{\partial f}{\partial y'}(x,y_o,y'_o) = \frac{1}{4}(1-y_o^2)$$

$$= \sum_r{}' C'_r T_r(x)$$

$$= \tfrac{1}{2}(-\tfrac{1}{4})\,T_o - \tfrac{1}{2}T_1 - \frac{1}{8}T_2$$

Hence the process can be started from

$$a_o = 2,\quad a_1 = 1,\quad a_2 = \dots = a_N = 0$$
$$a'_o = 2,\quad a'_1 = 0,\quad a'_2 = \dots = a'_N = 0$$
$$b_o = -\frac{3}{8},\ b_1 = -\frac{9}{16},\ b_2 = -\frac{1}{8},\ b_3 = \dots\dots = b_N = 0$$
$$C_o = -\frac{9}{8},\ C'_o = -\frac{1}{4}$$

and then $P_r = b_r - \tfrac{1}{2}C'_o a'_r - \tfrac{1}{2}C_o a_r$

can be computed for $(r = 0, 1, \dots\dots, N)$.

Now the recurrence relations (2.7.2) can be solved easily by letting

$$E_{N+1} = E'_{N+1} = 0, \quad E_N = E'_N = 1.$$

Similarly equations (2.7.3) are solved, when starting with $F_{N+1} =$ $F'_{N+1} = G_{N+1} = G'_{N+1} = 0$, $F_N = F'_N = G_N = 1$ and $G'_N = -1$.

The values of A_r given by $A_r = E_r + \mu F_r + \nu G_r$ for $r = 0,1, \ldots, N$ are determined so that the constants μ and ν are chosen to ensure that the approximation $y_i(x) = \sum_r{}' A_r T_r(x)$ satisfies the given boundary conditions.

From the set of coefficients A_r $(r = 0,1,\ldots N)$ we can compute the coefficients A'_r $(r = 0,1,\ldots N)$ by letting $A'_{N+1} = A'_N = 0$ and use the relation $A'_{r-1} = A_{r+1} + 2r\,A_r$ $(r = N, N-1, \ldots, 1)$. and hence we use the new sets A_r, A'_r to start another iterative cycle. The solution we have obtained for Van der Pols equation is given in the tables (7a), with various values of N.

2.8 Recursive Procedures when C_o is small:

When C_o is small, and more-over when N (the degree of the wanted approximation) is large, both solutions of the nonhomogeneous relation, E_r and F_r and subsequently F'_r, G_r and G'_r become very large and hence a build up of errors can swamp the desired solution.

The following modification due to G.F. Miller [16] overcomes this difficulty.

(a) For first order equations the sequence $\{F_r\}$ may be computed as before. In place of $\{E_r\}$, however we compute for $K = N, N-1, \ldots, 1$, sequences $E_r^{(k)}$ $(r = K-1, K, \ldots, N+1)$ satisfying the relation

$$C_o E_{k-1} = C_o E_{k+1} + 4K E_K + 2(P_{k+1} - P_{k-1})$$

such that $\qquad E_k^{(k)} = 0, \quad E_{N+1}^{(k)} = 0$

Given the sequence $\{E_r^{(k)}\}$ we compute the quantity $E_{k-1}^{(k)}$ and hence the new sequence $\{E_r^{(k-1)}\}$ from the relations

$$\left.\begin{aligned} E_{k-1}^{(k)} &= E_{k+1}^{(k)} + \frac{2}{C_o}(P_{k+1} - P_{k-1}) \\ E_r^{(k-1)} &= E_r^{(k)} - \alpha_k F_r \end{aligned}\right\} , \quad r = K, K+1, \ldots, N$$

where

$$P_k = b_k - \tfrac{1}{2} C_o a_k$$

and $\qquad \alpha_k = E_{k-1}^{(k)} / F_{k-1}$

Thus each sequence is obtained from its predecessor by subtracting a multiple of $\{F_r\}$. We finally obtain a solution $\{E_r^{(1)}\}$ with the desired property that it is not dominated by $\{F_r\}$. Finally the solution with the desired property is given by $E_r = E_r^{(1)}$ and hence we may construct a solution

$$A_r = E_r + \mu F_r, \qquad r = 0, 1, \ldots\ldots, N$$

(b) For second order equations, as in the case of first order equations there may be cancellation consequent upon the use of $A_r = E_r + \mu F_r + \nu G_r$ to obtain A_r when C_o is small. Here the sequences $\{F_r\}$ and $\{F_r'\}$ are computed by straightforward application of the recurrence relations

$$C_o \; B_{r-1} = C_o B_{r+1} \; + \; 2r \; (2B'_r - C'_o \, B_r)$$
$$B'_{r-1} = B'_{r+1} \; + \; 2r \; B_r$$

for $r = N, N-1, \ldots\ldots, 1$

Starting with $F_N = F'_N = 1$ and $F_{N+1} = F'_{n+1} = 0$. To obtain a second solution $\{G_r\}$, $\{G'_r\}$ which is essentially distinct from $\{F_r\}$, $\{F'_r\}$ we compute sequences $\{G_r\}$, $\{G'_r\}$ for $r = K-1, K, \ldots\ldots, N$ and $K = N, N-1, \ldots\ldots, 1$ from the relations

$$\begin{aligned} G_{N+1}^{(N)} &= G_N^{(N)} = 0 \\ G_N^{'(N)} &= 1, \; G_{N+1}^{'(N)} = 0 \\ C_o \; G_{k-1}^{(k)} &= C_o \; G_{k+1}^{(k)} + 4K \; G_k^{'(k)} \\ G_{k-1}^{'(k)} &= G_{k+1}^{'(k)} \end{aligned}$$

$$\left.\begin{aligned} G_r^{(k-1)} &= G_r^{(k)} - \gamma_k \, F_r \\ G_r^{'(k-1)} &= G_r^{'(k)} - \gamma_k \, F_r' \end{aligned}\right\} \quad r = K-1, K, \ldots\ldots, N$$

where $\gamma_k = G_{k-1}^{k} \, / \, F_{k-1}$ is chosen so that $G_{k-1}^{(k-1)} = 0$

Finally we take $G_r = G_r^{(1)}$, and $G_r' = G_r^{'(1)}$

To obtain a solution $\{E_r\}$, $\{E'_r\}$ of the nonhomogeneous system

$$C_o \; A_{r-1} = C_o \; A_{r+1} + 2r \; (2A'_r - C_o' \; A_r) + 2 \; (P_{r+1} - P_{r-1})$$

$$A'_{r-1} = A'_{r+1} + 2r\,A_r$$

$$\text{for } r = N,\ N-1, \ldots\ldots, 1$$

which is not dominated by $\{F_r\}$, $\{F'_r\}$ or by $\{G_r\}$, $\{G'_r\}$, we calculate sequences $\{E_r^{(k)}\}$, $\{E'^{(k)}_r\}$, $(r = K-1, K, \ldots\ldots N)$ and $K = N, N-1, \ldots\ldots 1$, concurrently with sequences $\{G_r^{(k)}\}$, $\{G'^{(k)}_r\}$ from the ~~table~~ above relation

$$E_{N+1}^{(N)} = E_N^{(N)} = E_{N+1}^{'(N)} = E_N^{'(N)} = 0$$

$$C_o\,E_{k-1}^{(k)} = C_o\,E^{(k)}_{k+1} + 2\,(P_{k+1} - P_{k-1})$$

$$E_{k-1}^{'(k)} = E'^{(k)}_{k+1}$$

$$\left.\begin{aligned} E_r^{(k-1)} &= E_r^{(k)} - \alpha_k F_r - \beta_k\,G_r^{(k-1)} \\ E'^{(k-1)}_r &= E_r^{'(k)} - \alpha_k F_r' - \beta_k G_r^{'(k-1)} \end{aligned}\right\}\ r = K, K+1, \ldots\ldots, N$$

where α_k and β_k are determined so that

$$E_{k-1}^{(k-1)} = E^{'(k-1)}_{k-1} = 0$$

and since $G_{k-1}^{(k-1)} = 0$, we have

$$\alpha_k = E_{k-1}^{(k)} \,/\, F_{k-1}$$

$$\beta_k = (E_{k-1}^{'(k)} - \alpha_k F'_{k-1}) \,/\, G_{k-1}^{'(k-1)}$$

Finally the solution with the desired property is given by

$$E_r = E_r^{(1)} \text{ and } E'_r = E_r^{'(1)}.$$

Example 3:

The problem $y' = y^2$

$$y(0) = \tfrac{1}{2}$$

The solutions obtained by taking $y_o(x) = \frac{1}{2}(1 + \frac{x}{2})$ for various values of N and applying the Miller's modifications, are listed in table (5a) compared with the solutions of the same problem without the modifications.

2.9 Lie series:

Recently H. Knapp and G. Wanner (1968) [29] have published a report in which they have established a general iterative process based on a perturbation method, making use of the theory of Lie series. For the numerical solution of ordinary differential equations of the form

$$y'_i(x) = f_i\{x, y_i(x), \ldots\ldots, y_n(x)\} \tag{2.9.1}$$

for $i = 1, 2, \ldots, n$

and $y_i(x_o) = y_{io}$

An exact formula is

$$y_i(x) = \hat{y}_i(x) + \sum_{\alpha=0}^{s} \int_{x_o}^{x} \frac{(x-\xi)^\alpha}{\alpha!} \{D_2 D^\alpha y_i\}_{\xi, \hat{y}(\xi)} d\xi + R_{is}(x) \tag{2.9.2}$$

$R_{is}(x)$ being a remainder term given by

$$R_{is}(x) = \int_{x_0}^{x} \frac{(x-\xi)^s}{s!} \{[D^{s+1}y_i]_{\xi,y(\xi)} - [D^{s+1}y_i]_{\xi,\hat{y}(\xi)}\} \, d\xi \qquad 2.9.3$$

where $\hat{y}_i(x)$ is an approximation to $y(x)$ such that

$$\hat{y}'_i(x) = \hat{f}_i(x,\hat{y}_1,\ldots,\hat{y}_n), \quad i = 1,\ldots,n \qquad 2.9.4$$

$$[D]_{x,y} = \frac{\partial}{\partial x} + f(x,y)\frac{\partial}{\partial y}$$

$$[D_1]_{x,y} = \frac{\partial}{\partial x} + \hat{f}(x,y)\frac{\partial}{\partial y} \qquad 2.9.5$$

and

$$D_2 = (D_1 - D).$$

Treating for simplicity the case when y is a scalar and the equation is $y'(x) = f(x,y)$, the sth order iterative process derived from Lie series is then

$$y_{r+1}(x) = y_r(x) + \sum_{\alpha=0}^{s} \int_{x_0}^{x} \frac{(x-\xi)^\alpha}{\alpha!} \{D_2 D^\alpha y\}_{\xi,y_r(\xi)} \, d\xi \qquad 2.9.6$$

where y_r, y_{r+1} are the rth, r+1th iterates respectively. Details are given in [29] of the application of this method to various differential equations including example 2. In the applications quoted, Chebyshev series were not used, a Taylor series expansion was taken as the first approximation to the solution and this was improved once. The integrations in the Lie series expansion were carried out using Gauss quadrature, and the solution was developed step by step using a controlled step size. An additional facility is the use in boundary value problems of the connection matrix which calculates the derivatives of the functions at the far boundary with respect to the chosen initial values. The far boundary conditions can then be satisfied by solving for the initial values using Newton's method rather than the linear secant method in the Ruge Kutta section.

The equations treated were

(a) $y' = 1 - e^{-y}(\mathrm{Sin}\ x - \mathrm{Cos}\ x), \quad y(o) = 0$

(b) $y_1' = y_2, \quad y_1(0) = 0$

$y'_2 = y_1, \quad y_2(0) = 1$

(c) $y' = y^2$, $y(0) = 1$ (Example 2)

(d) $y' = -xy^3 \quad y(x_o) = y_o$

(e) A restricted three body problem

(f) A boundary value problem

$y'_1 = y_2$, $y'_2 = \exp(y_1)$, $y_1(0) = y_1(1) = 0$, with correction via Newtons method.

Since iterative methods were not used the results are not applicable to consideration of rates of convergence; information is available on the improvement arising from one step, but only in the solution at particular values of x, obtained using different step lengths. In example 2, using a Lie series of order 3, an error of 3.14 in y(0.9) from the Taylors series is reduced to an error of 3×10^{-15} i.e. a reduction factor of 10^{-15}, but this is based on a step by step approach using 23 steps from $x = 0$.

It does appear that Lie series could be used as the basis of higher order iteration processes instead of as a once for all correction. The given system is based on an elaborate computorised recursive generation of derivatives of the functions based on a set of standard elementary functions in order to evaluate the terms $D_y{}^{\nu}$ and $D_2 D_y{}^{\nu}$; an approach using Chebyshev series might be more economical. Thus the Lie series may be capable of development to yield Chebyshev iteration methods of higher order than Newton. However no further work has been done on these lines in this thesis.

The rates of convergence of these processes will be considered in the next chapter and the numerical results obtained above will be analysed in the light of this in Chapter 4.

Chapter 3

(3.0) Analysis of Rate of Convergence of Iterative Methods of Solution

(3.1) Behaviour of the error function:

Given an iterative process which is described by $y_{r+1}(x) = F\{y_r(x)\}$, where F is some operator, and a true solution y(x) and error functions $e_r(x)$, $e_{r+1}(x)$ so that $y_r(x) = y(x) + e_r(x)$, then

$$\begin{aligned} y_{r+1}(x) &= y(x) + e_{r+1}(x) \\ &= F\{y_r(x)\} \\ &= F\{y(x) + e_r(x)\} \end{aligned} \qquad (3.1.1)$$

and if the right hand side of (3.1.1) is expandible in the form

$$F(y+e_r) = F(y) + G(y,e_r)$$

and since $y(x) = F\{y(x)\}$, then

$$e_{r+1}(x) = G\{y(x),\ e_r(x)\} \qquad (3.1.2)$$

This is the general relation governing the rate of convergence of the iterative method. If the $e_r(x)$ for $r = 0,1,\ldots.$ are expressed as a truncated Chebyshev series then the relation (3.1.2) will be expressible in the form

$$A^{(r+1)} = H\ A^{(r)}$$

where $A^{(r)}$ is the vector of Chebyshev coefficients and the behaviour of the process will depend on the matrix H (which will in general be a function of A). For the solution of the first order ordinary differential equation $y' = f(x,y)$ with $y = y_o$ at $x = x_o$, the following methods may be analysed

(a) The Zeroth order Lie Series, is

$$y_{r+1}(x) = y_r(x) + \int_{xo}^{x} (D_2 y(\xi))_{\xi, y_r(\xi)}\, d\xi$$

$$= y_r(x) + \int_{xo}^{x} \{f(\xi,y_r) - \hat{f}(\xi,y_r)\} d\xi$$

$$= y_r(x) - \int_{xo}^{x} \hat{f}(\xi,y_r) d\xi + \int_{xo}^{x} f(\xi,y_r) d\xi$$

$$= y_o + \int_{xo}^{x} f(\xi,y_r) d\xi$$

where $y_r(x) = y_o + \int_{xo}^{x} \hat{f}(\xi,y_r) d\xi$ by definition and therefore

$$y_{r+1}(x) = y(x_o) + \int_{xo}^{x} f(\xi,y_r) d\xi$$

(which is Picard's iterative process for initial value problem $y' = f(x,y)$).

i.e.

$$y(x) + e_{r+1}(x) = y(x_o) + \int_{xo}^{x} f\{\xi,\ y+e_r\} d\xi$$

$$= y_o + \int_{xo}^{x} \left\{ f(\xi,y) + e_r \frac{\partial f}{\partial y} (\xi,y) + \ldots \right\} d\xi$$

$$e_{r+1}(x) = \int_{xo}^{x} e_r(\xi) \frac{\partial f}{\partial y} (\xi,y) d\xi + \ldots$$

If $\frac{\partial f}{\partial y}(x,y)$ is bounded, then it is possible to ensure $||e_{r+1}|| < || e_r ||$, that the convergence of the process can be guaranteed, by taking $(x-x_o)$ sufficiently small. The remainder term in the Lie series formulation gives the same estimate of the error in an alternative form.

(b) The first order Lie series:

The iterative method based on first order Lie series is

$$y_{r+1}(x) = y_o + \int_{xo}^{x} f(\xi,y_r)d\xi + \int_{xo}^{x}(x-\xi)[f(\xi,y_r) - y_r'(\xi)]\,\frac{\partial f}{\partial y}\,d\xi$$

and the remainder term is known to be

$$\int_{x_o}^{x}(x-\xi)\left\{\frac{\partial f}{\partial x}(\xi,y) + y'\frac{\partial f}{\partial y}(\xi,y) - \frac{\partial f}{\partial x}(\xi,y_r)-y_r'\frac{\partial f}{\partial y}(\xi,y_r)\right\} d\xi$$

Hence

$$y_{r+1}' \text{ is estimated as } f(x,y_r) + \int_{xo}^{x}\left\{f(\xi,y_r)-y_r'\right\}\frac{\partial f}{\partial y}(\xi,y_r)d\xi$$

and the error term in y' is, writing $y_r = y + e_r$ and expanding

$$e_{r+1}' = -\int_{xo}^{x}\Big\{f_x + ff_y - (f_x + e_r f_{xy} + \tfrac{1}{2}e_r^2 f_{xyy} + \ldots..)$$

$$- (f + e_r f_y + \tfrac{1}{2}e_r^2 f_{yy} + \ldots.)(f_y + e_r f_y + \ldots..)\Big\} d\xi$$

$$= \int_{xo}^{x} e_r(\xi)\left\{f_{xy} + f_y^2 + f_{yy}\right\}d\xi - \tfrac{1}{2}\int_{xo}^{x} e_r^2(\xi)\Big\{3f_y f_{yy} +$$

$$ff_{yyy} + f_{xyy}\Big\}\, d\xi$$

(c) Newton's Method:

The error in the Newton iteration formula may be determined directly. Since

$$y_{r+1}' = f(x,y_r) + y_{r+1}\frac{\partial f}{\partial y}(x,y_r) - y_r\frac{\partial f}{\partial y}(x,y_r), \text{ we have}$$

on expanding

$$e'_{r+1} = \left\{ e_r f_y + \frac{e_r^2}{2} f_{yy} + \frac{e_r^3}{6} f_{yyy} + \ldots\ldots \right\}$$

$$+ (e_{r+1} - e_r)\left\{ f_y + e_r f_{yy} + \frac{e_r^2}{2} f_{yyy} + \ldots\ldots\ldots \right\}$$

i.e. $$e'_{r+1} - e_{r+1}F = -\tfrac{1}{2} e_r^2 f_{yy} - \frac{1}{3} e_r^3 f_{yyy} + \ldots\ldots.$$

where $F = \frac{\partial f}{\partial y} (x, y_r)$, and since $e_r = e_{r+1} = 0$ at $x = x_o$

this gives

$$e_{r+1} = \int_{xo}^{x} \left\{ -\tfrac{1}{2} e_r^2 f_{yy} - \frac{1}{3} e_r^3 f_{yyy} \right\} d\xi + \int_{xo}^{x} F(\xi) d\xi \int_{xo}^{\xi} \left\{ -\tfrac{1}{2} e_r^2 f_{yy} - \frac{1}{3} e_r^3 f_{yyy} \right\} d\xi' + \int_{xo}^{x} F(\xi) d\xi \int_{xo}^{\xi} F(\xi') d\xi' \int_{xo}^{\xi'} \left\{ -\tfrac{1}{2} e_r^2 f_{yyy} - \frac{1}{3} e_r^3 f_{yyy} \right\} d\xi'' + \ldots\ldots\ldots\ldots$$

or $$e'_{r+1} = -\tfrac{1}{2} e_r^2 f_{yyy} - \frac{1}{3} e_r^3 f_{yyy} + \frac{\partial f}{\partial y} (x, y_r) \int_{xo}^{x} \left\{ -\tfrac{1}{2} e_r^2 f_{yy} - \frac{1}{3} e_r^3 f_{yyy} \right\} d\xi + \frac{\partial f}{\partial y} (x, y_r) \int_{xo}^{x} \frac{\partial f}{\partial y} (\xi, y_r) d\xi \int_{xo}^{\xi} \left\{ -\tfrac{1}{2} e_r^2 f_{yy} - \frac{1}{3} e_r^3 f_{yyy} \right\} d\xi'$$

with a leading term in e'_{r+1} of $-\tfrac{1}{2} e_r^2 f_{yy}$.

If however the Newton formula is used in the form

$$y'_{r+1} = f(x, y_r) + \mu (y_{r+1} - y_r)$$

where μ is a constant $(= \frac{\partial f}{\partial y} (x_o, y_o))$, then

$$e'_{r+1} = \mu e_{r+1} + e_r f_y + \frac{e_r^2}{2} f_{yy} + \ldots\ldots\ldots - \mu e_r$$

$$e'_{r+1} - \mu e_{r+1} = e_r \int_{xo}^{x} \frac{d}{d\xi} \left\{ \frac{\partial f}{\partial y} (\xi, y) \right\} d\xi + \frac{e_r^2}{2} f_{yy}$$

giving

$$e'_{r+1} = e_r \int_{xo}^{x} \frac{d}{d\xi} \left\{ \frac{\partial f}{\partial y} (\xi, y) \right\} d\xi \text{ as a leading term}$$

$$= e_r \int_{xo}^{x} (f_{xy} + f f_{yy}) \, d\xi$$

(d) Weyl's method:

The iterative process for the homogeneous equation $y' = yF(x,y)$, $y = y_o$ at $x = x_o$ is of the form

$$y_{r+1}(x) = y_o \, e^{\int_{x_o}^{x} F(x, y_r) dx}$$

where

$$y(x) = y_o \, e^{\int_{x_o}^{x} F(x,y) dx}$$

then

$$\log y_{r+1}(x) = \log y_o + \int_{xo}^{x} F(x, y_r) dx$$

so that

$$\frac{e_{r+1}}{y} = \int_{xo}^{x} e_r \frac{\partial F}{\partial y} (x,y) dx$$

and

$$e_{r+1} = y(x) \int_{xo}^{x} e_r(\xi) \frac{\partial F}{\partial y} (\xi, y) d\xi$$

This can be compared with Picard's method for the same problem $y' = yF(x,y) = f(x,y)$, for which the error has been found to be

$$e_{r+1} = \int_{xo}^{x} e_r \frac{\partial f}{\partial y}\, dx$$

$$= \int_{xo}^{x} e_r(\xi) \left\{F(\xi,y) + y(\xi) \frac{\partial F}{\partial y}(\xi,y)\right\} d\xi.$$

For example, consider the equation $y' = y^2$, $y = 1$ at $x = 0$ which has the solution $y(x) = 1/(1-x)$. Then $\frac{\partial F}{\partial y} = 2/(1-x)$ and with initial solution $y_r(x) = 1 + x$ where $e_r(x) = -x^2$

(i) In Picard's iteration $y_{r+1}(x) = y_o + \int_{xo}^{x} f(x,y_r)dx$

$$y_{r+1}(x) = 1 + \int_{o}^{x} (1+x)^2 dx$$

$$= x + x^2 + \frac{1}{3} x^3$$

therefore the error function will be $e_{r+1}(x) = -\frac{2}{3}x^3$.

Here we have $\frac{\partial f}{\partial y} = 2y$ so that the estimated error is

$$e_{r+1}(x) = \int_{o}^{x} e_r \frac{\partial f}{\partial y}\, dx$$

$$= \int_{o}^{x} -2x^2(1+x)dx$$

$$= -\frac{2}{3} x^3 + \ldots\ldots..$$

(ii) Weyl's iteration

$$y_{r+1}(x) = y_o\, e^{\int_{o}^{x} f(x,y_r)dx}$$

$$f(x,y) = y \text{ and so}$$

$$y_{r+1}(x) = e^{\int_{o}^{x} (1+x)dx}$$

$$= e^{(x + \frac{x^2}{2})}$$

$$= 1 + x + x^2 + \frac{2}{3} x^3 + \ldots\ldots\ldots$$

Therefore $e_{r+1}(x) = -\frac{1}{3}x^3$ and also the estimated error function will be

$$e_{r+1}(x) = y(x)\int_0^x e_r(x)\,\frac{\partial f}{\partial y}(x,y)dx$$

$$= \frac{1}{(1-x)}\int_0^x -x^2 dx$$

$$= \frac{-x^3}{3(1-x)} = -\frac{x^3}{3}(1 + x + x^2 + \ldots\ldots\ldots)$$

$$= -\frac{x^3}{3} - \ldots\ldots$$

(iii) Newton's formulation gives,

$$y'_{r+1} - 2(1+x)\,y_{r+1} = -(1+x)^2$$

which with $y_{r+1} = 1$ at $x = 0$, gives

$$y_{r+1}(x) = 1 + x + x^2 + x^3 + x^4 + \frac{4}{5}x^5$$

so $$e_{r+1} = -\frac{1}{5}x^5$$

$$e'_{r+1} = -x^4$$

This corresponds to the expression already obtained in (c)

$$e'_{r+1} = -\tfrac{1}{2}e_r^2 f_{yy} = -x^4$$

(iv) Modified Newton formulation gives,

$$y'_{r+1} = 2y_{r+1} = (1+x)^2 - 2(1+x)$$

$$= x^2 - 1$$

for which the solution is,

$$y_{r+1}(x) = 1 + x + x^2 + x^3 + \tfrac{1}{2}x^4$$

and so $$e_{r+1}(x) = -\tfrac{1}{2}x^4$$

$$e'_{r+1}(x) = -2x^3$$

which is derived from the leading term

$$-x^2 \int_0^x (f_{xy} + ff_{yy})d\xi$$

$$= -2x^3$$

(v) Lie series gives,

$$y_{r+1}(x) = 1 + \int_0^x (1+\xi)^2 d\xi + 2\int_0^x (x-\xi)\left\{(1+\xi)^2-1\right\}(1+\xi)d\xi$$

$$= 1 + x + x^2 + x^3 + \tfrac{1}{2}x^4 + \ldots\ldots\ldots$$

and so $$e_{r+1}(x) = -\tfrac{1}{2}x^4$$

$$e'_{r+1}(x) = -2x^3$$

Here $$f_y = \frac{\partial f}{\partial y} = 2y, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2$$

since $e_r(x) = -x^2 + \ldots\ldots$, the expression for the error of (b) gives,

$$e'_{r+1} = \int_0^x (-\xi^2)\left\{4y^2 + 2y^2\right\}d\xi + \ldots\ldots \text{ with a}$$

leading term $-2x^3$.

In general it would be expected that the exact Newton formulation with error $E_{en} = -\tfrac{1}{2}e_r^2 f_{yy}$, would be more accurate than Lie series, with an error

$$E_\ell = \int_{xo}^x e_r(\xi)\left\{f_{xy} + ff_{yy} + f_y^2\right\}d\xi$$

or modified Newton with error

$$E_{mn} = e_r(x)\int_{xo}^{x}\left\{f_{xy} + ff_{yy}\right\}d\xi$$

since

$$E_\ell = e_r(x)\int_{xo}^{x}\left\{f_{xy} + ff_{yy} + f_y^2\right\}d\xi$$

$$-\int_{xo}^{x}\left\{f_y + \int_{xo}^{\xi'} f_y^2\, d\xi\right\} e_r'(\xi')\, d\xi'$$

$$= E_{mn} + e_r(x)\int_{xo}^{x} f_y^2\, d\xi - \int_{xo}^{x} f_y\, e_r'(\xi)d\xi$$

$$-\int_{xo}^{x} e_r'(\xi')\, d\xi' . \int_{xo}^{\xi'} f_y^2\, d\xi$$

Table 3.01

Method	Picard's	Weyl's	Modified Newton	Exact Newton	Lie Series
$e_{r+1}(x)$	$-\frac{2}{3}x^3$	$\frac{-x^3}{3}$	$-\frac{1}{2}x^4$	$-\frac{1}{5}x^5$	$-\frac{1}{2}x^4$
$e'_{r+1}(x)$	$-2x^2$	$-x^2$	$-2x^3$	$-x^4$	$-2x^3$

(3.2) Relation between Newton's Method and Lie Series:

The expression for iteration via first order Lie series

$$y'_{r+1}(x) = f(x,y_r) + \int_{xo}^{x}\left\{f(\xi,y_r) - y_r'(\xi)\right\}\frac{\partial f}{\partial y}(\xi,y_r)d\xi$$

may be transformed to give the Newton iteration form with extra terms; we have

$$y'_{r+1}(x) = f(x,y_r) + \int_{xo}^{x} f(\xi,y_r)\frac{\partial f}{\partial y}(\xi,y_r)d\xi - \left[\frac{\partial f}{\partial y}(\xi,y_r)\right]_{xo}^{x}$$

$$+ \int_{xo}^{x} y_r(\xi)\frac{d}{d\xi}\left\{\frac{\partial f}{\partial y}(\xi,y_r)\right\}d\xi$$

$$= f(x,y_r) + \int_{xo}^{x} y'_{r+1} \frac{\partial f}{\partial y} (\xi,y_r)\, d\xi + \int_{xo}^{x} \left\{f(\xi,y_r)-y'_{r+1}(\xi)\right\} \frac{\partial f}{\partial y}\, d\xi$$

$$+ \int_{xo}^{x} y_r(\xi) \frac{d}{d\xi} \left\{\frac{\partial f}{\partial y} (\xi,y_r)\right\} d\xi - \frac{\partial f}{\partial y}(\xi,y_r) y_r(x) + y_o \frac{\partial f}{\partial y} (x_o,y_o)$$

$$= f(x,y_r) + y_{r+1} \frac{\partial f}{\partial y} (x,y_r) - y_r \frac{\partial f}{\partial y} (x,y_r) - \int_{xo}^{x} y_{r+1}(\xi) \frac{d}{d\xi}$$

$$\left(\frac{\partial f}{\partial y}\right) d\xi + \int_{xo}^{x} y_r(\xi)\left\{\frac{\partial f}{\partial y}\right\} d\xi + \int_{xo}^{x} \left\{f(\xi,y_r) - y'_{r+1}(\xi)\right\} \frac{\partial f}{\partial y}\, d\xi$$

The first three terms give the Newton iteration formula. The additional terms vanish under the two assumptions;

(i) $y_{r+1}(\xi) = f(\xi,y_r(\xi))$, the Picard approximation

(ii) $\frac{\partial f}{\partial y} = \text{constant}$.

(3.3) <u>Runge-Kutta Method, boundary value:</u>

An equation solved with one initial condition missing and determined from the corresponding value gives rise to a functional relation from ξ, the initial value, to η, the boundary value

$$\eta = f(\xi)$$

and then the solution of $\eta = f(\xi) = \eta_o$ can be carried out by iterative methods. If the derivative $f'(\xi)$ is known or can be estimated as in the Lie series development discussed, Newton's method can be used, otherwise the easiest thing is to use the secant method and this is known to have order 1.62 and asymptotic constant depending on $\left\{f''/f'\right\}^{0.62}$. An analysis of the numerical results of Runge-Kutta calculation is given in the next Chapter.

(3.4) Divergent sequences made convergent:

For a particular example (example 6) the Picard method is found to produce a divergent sequence (see Table 3). The explanation may be given simply,

In solving $y'' + \lambda^2 y = 0$, using the boundary conditions $y(-1) = 0$, $y(1) = 1$, the following sequence of approximations are obtained

$$y_o(x) = \tfrac{1}{2}(1+x)$$

$$y_1(x) = (\tfrac{1}{2} + \frac{\lambda^2}{3})(1 + x) - \frac{\lambda^2}{12}(1 + x)^3$$

$$y_2(x) = (\tfrac{1}{2} + \frac{\lambda^2}{3} + \frac{14\lambda^4}{45})(1 + x) - \frac{\lambda^2}{6}(\tfrac{1}{2} + \frac{\lambda^2}{3}).$$
$$(1 + x)^3 + \frac{\lambda^4}{240}(1 + x)^5$$

The coefficient of the first term in $(1 + x)$ thus produces increasing numbers of terms of the power series expansion of λ cosec 2λ, which is divergent for $\lambda > \frac{\pi}{2}$. The ϵ- Algorithm can now be used to provide a sum for this expansion.

Let $\epsilon^{(m)}_{o,r} = a^{(m)}_r$, $r = 0,1,2,\ldots\ldots,N$

Applying the ϵ-Algorithm

$$\epsilon^{(m)}_{s+1} = \epsilon^{(m+1)}_{s-1} + \frac{1}{\epsilon^{(m+1)}_s - \epsilon^{(m)}_s} \qquad \begin{array}{l} s = 0,1,\ldots\ldots, \\ m = 0,1,\ldots\ldots, \end{array}$$

$$\epsilon^{(m)}_{-1} = 0$$

where, $\epsilon^{(m)}_{2s}$ is found to converge to the right solution A_r of $y(x) = \Sigma' A_r T_r(x)$.

(3.5) Convergence of iterative methods, matrix analysis:

When the function is being described by a vector of Chebyshev coefficients $\underline{A}$, iterative methods may be reduced to the form

$$\hat{A}^{(r+1)} = M\,\hat{A}^{(r)} + \hat{b}, \text{ or alternatively}$$

$$\underline{e}^{(r+1)} = M\,\underline{e}^{(r)}$$

Our objectives by using this, are to construct an iteration matrix M independent of the elements of the vectors $\hat{A}^{(r)}$, $\hat{A}^{(r+1)}$ or $\hat{e}^{(r+1)}$, $\hat{e}^{(r)}$ of the Chebyshev coefficients of two consecutive approximations obtained by the iterative process. Then one can show that the iterative method converges for any initial approximation $\hat{A}^{(o)}$ if and only if $\rho(M)$ (the spectral radius of M) is less than 1, where $\rho(M) = \max |\lambda(M)|$, and hence the rate of convergence $R_n(M)$ which is defined by the equation

$$R_n(M) = -\log_e ||M||, \quad ||M|| \geq |\lambda|$$

Such a matrix can be formed exactly for linear equations; for non-linear equations an approximation is obtained by linearising. Details of the matrix for general first order equation for the various methods are derived below and then eigenvalues are investigated for the particular equations discussed.

(3.6) Derivation of iteration matrix:

(1) Picard's Method:

(a) Exact analysis, linear equations:

If $y_r(x) = \sum_{j=o}^{N}{}' A_j^{(r)} T_j(x)$ and $f(x,y) = P(x)y + Q(x)$;

then $$f(x,y_r) = P(x) \sum_{j=o}^{N}{}' A_j^{(r)} T_j(x) + Q(x)$$

If P(x), Q(x) are polynomials of low order or can be fitted by such polynomials, this reduces to

$$f(x,y_r) = \sum_{o}^{N}{}' B_j^{(r)} T_j(x)$$ directly; in any case this form

can be derived, if necessary by using collocation to expand $P(x)\, T_j(x)$ for each j.

The $B_j^{(r)}$ are linear in the $A_j^{(r)}$ so that $\hat{B}^{(r)} = R\,\hat{A}^{(r)}$, where R is independent of r.

Then Picard's method

$$y_{r+1}(x) = \int_o^x f(x,y_r)dx + y(o)$$

corresponds to

$$\hat{A}^{(r+1)} = S\,\hat{B}^{(r)} + \underline{b}$$

where

$$A_i^{(r+1)} = \frac{1}{2i}\left\{B_{i+1}^{(r)} - B_{i-1}^{(r)}\right\},\quad i = 1, \ldots\ldots, N$$

$$A_o^{(r+1)} = y(o) + A_2^{(r+1)} - A_4^{(r+1)} \quad \ldots\ldots\ldots$$

so that $Z^T \hat{A}^{(r+1)} = y(o)$ where Z^T is $(\tfrac{1}{2}\ 0\ -1\ 0\ +1\ \ldots\ldots\ldots)$

Then

$$S = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & \frac{3}{4} & 0 & \frac{5}{12} & \cdots\cdots & \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots\cdots & 0 \\ 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & -\frac{1}{6} & 0 & \frac{1}{6} & 0 & & \\ \vdots & 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{8} & \cdots\cdots & 0 \\ & & & & & \ddots & \ddots & \end{bmatrix}$$

The first row of S, S_{1j} is given by

$$S_{1j} = \frac{(-1)^{[j/2]}(j-1)\{(-1)^j+1\}}{j(j-2)}$$

for $j > 2$ and $\underset{\wedge}{b}$ is the vector $b_o = 2y(o)$, $b_i = 0$ for $i \neq 0$. Hence the Picard process corresponds to the iterative procedure

$$\underset{\wedge}{A}^{(r+1)} = M \underset{\wedge}{A}^{(r)} + \underset{\wedge}{b}$$

where M, the iteration matrix is SR.

(b) Non-linear equation:

The procedure may be applied to non-linear equations by linearising the derivative in the neighbourhood of the true solution. S remains as above.

$B_i^{(r)} = B_i(\underset{\wedge}{A}^{(r)})$, a functional relationship depending on f, however writing $A_i^{(r)} = \alpha_i + e_i^{(r)}$, α_i being the true solution value, we may put

$$B_i^{(r)} = B_i(\alpha) + \sum_j \frac{\partial B_i}{\partial A_j} \Big|_\alpha e_j^{(r)}$$

or

$$B^{(r)} = B(\alpha) + Q \underset{\wedge}{e}^{(r)}$$

Then the Picard process corresponds to

$$A^{(r+1)} = S\ B\ (\alpha) + S\ Q\ e^{(r)} + \underset{\wedge}{b}$$

and the error relationship is

$$e^{(r+1)} = S\ Q\ e^{(r)} \quad \text{so that M is now SQ}$$

where

$$q_{ij} = \frac{\partial B_i}{\partial A_j} \Big|_{\underset{\wedge}{A} = \underline{\alpha}}$$

The iteration matrices for various N for the first order equations dealt with have been analysed, and then maximum eigenvalues determined. The following Table 3.02 shows the results. The iteration matrix M of example 3 ($y' = x - y^2$) has the elements m_{ij} where

$$m_{ij} = -q_{ij} = \frac{\partial B_i}{\partial A_j}\bigg|_{\underset{\wedge}{A} = \underline{\alpha}}$$

and hence the eigen values have the same absolute values as shown in the Table 3.02.

Table 3.02

N	Example 1 $\max\lvert\lambda\rvert$	Example 2 $\max\lvert\lambda\rvert$	Example 3 $\max\lvert\lambda\rvert$
4	0.4564	0.4233	0.4233
5	0.4510	0.3709	0.3709
6	0.4496	0.3260	0.3260
7	0.4497	0.3033	0.3033
8	0.4497	0.2492	0.2492
9	.	0.2367	0.2367
10	.	0.2134	0.2134
11	.	0.2134	.
12	.	.	.
13	0.4497	0.2134	0.2134

Second order equations can be treated in a precisely similar way.

(2) The same sort of analysis can be carried out for Newton's method. Since this is exact for a linear equation, nonlinear equations only will be considered.

The problem $y' = y^2$, $y(o) = \frac{1}{2}$ has the solution $y(x) = 1/(2-x)$ in the range $-1 \leq x \leq 1$. Hence for any N

$$f(x,y_r) = y_r^2$$

$$= \sum_{r}^{N}{}' b_r \, T_r(x)$$

and

$$b_o = \tfrac{1}{2}A_o^2 + A_1^2 + \ldots\ldots. + A_N^2$$

$$b_1 = A_oA_1 + A_1A_2 + \ldots..+ A_{N-1} A_N$$

$$b_2 = \tfrac{1}{2}A_1^2 + A_oA_2 + \ldots...+ A_{N-2} A_N$$

$$\vdots$$

so that $$P_r = b_r - \tfrac{1}{2}C_oA_r, \; r = 0,1,\ldots..,N$$

$$= b_r - A_r, \text{ since } c_o = 2.0$$

and if $\underset{\wedge}{A} = \underline{\alpha} + \underline{S}$, then linearising

$$\underset{\wedge}{P} = R \underset{\wedge}{S} + \underline{K} + O(\delta^2)$$

where $R_{ij} = \dfrac{\partial P_{i-1}}{\partial A_{j-1}}\Big|_{\underset{\wedge}{A} = \underline{\alpha}}$ which gives as above

$$R = \begin{bmatrix} \alpha_0-1 & 2\alpha_1 & 2\alpha_2 & \cdot & \cdot & \cdot \\ \alpha_1 & \alpha_0+\alpha_2-1 & \alpha_1+\alpha_3 & \cdot & \cdot & \cdot \\ \alpha_2 & \alpha_1+\alpha_3 & \alpha_0+\alpha_4-1 & \cdot & \cdot & \cdot \\ \alpha_3 & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & & & & & \end{bmatrix}$$

and $R_{ij} = \alpha_{i+j-2} + \alpha_{|i-j|}$, $R_{ii} = \alpha_0 + \alpha_{2i} - 1$

Newton's process will be,

$$A_{r-1} - 2r\,A_r - A_{r+1} = P_{r+1} - P_{r-1}$$

and so the solution vector E satisfies

$$B\,\underset{\wedge}{E} = L\,\underset{\wedge}{P}$$

where

$$B = \begin{bmatrix} 1 & -2 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & -4 & -1 & 0 & & \vdots \\ 0 & 0 & 1 & -6 & -1 & & \vdots \\ \vdots & & & 1 & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & \vdots \\ \vdots & & & & & \ddots & -2(N-1) \\ 0 & \cdots & \cdots & \cdots & \cdots & & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & & \\ 0 & 0 & -1 & 0 & 1 & \\ \vdots & & 0 & -1 & 0 & \\ \vdots & & & & \ddots & \ddots \\ \vdots & & & & & \ddots \end{bmatrix}$$

and similarly the homogeneous solution vector $\underset{\wedge}{F}$ satisfies $F_{r-1} - 2rF_r + F_{r+1} = 0$, $r = 1, \ldots, N-1$

$F_N = 1$

$B \underset{\wedge}{F} = \underset{\wedge}{S}$ where $\underset{\wedge}{S}$ is the vector $S_i = 0$, $i < N$, $S_N = 1$

The boundary condition $y(o) = \frac{1}{2}$ now means $\underset{\wedge}{A} = \underline{E} + \mu \underset{\wedge}{F}$ where μ must satisfy

$$\tfrac{1}{2}E_0 - E_2 + E_4 - \ldots. + \mu(\tfrac{1}{2}F_0 - F_2 + F_4 \ldots.) = \tfrac{1}{2}$$

$\underline{i}.\underline{e}.$ $Z^T \underset{\wedge}{E} + \mu Z^T \underset{\wedge}{F} = \frac{1}{2}$

where Z^T is $(\frac{1}{2}, 0, -1, 0, 1, 0, -1, \ldots\ldots\ldots)$

Thus $\mu = (\frac{1}{2} - Z^T\underset{\wedge}{E}) / (Z^T\underset{\wedge}{F})$, and substituting for μ, we have,

$$\underset{\wedge}{A} = \underset{\wedge}{E} + \frac{\underset{\wedge}{F}(\frac{1}{2} - Z^T\underset{\wedge}{E})}{(Z^T \underset{\wedge}{F})}$$

$$= B^{-1} L \underset{\wedge}{P} + \frac{B^{-1}\underset{\wedge}{S}}{(Z^T B^{-1} \underset{\wedge}{S})} \left\{\tfrac{1}{2} - Z^T B^{-1} L \underset{\sim}{P}\right\}$$

$$= \left\{B^{-1} L - B^{-1} \underset{\wedge}{S} \underset{\wedge}{Z}^T B^{-1} L\right\} P + \text{terms independent of } P.$$

and so the iteration matrix is found, substituting for $\underset{\wedge}{P}$, to be $M = QR$ where R is given above. It is the eigenvalues of this matrix which determine whether and how rapidly the iteration process converges.

The matrix Q determined from the above analysis has been calculated for various N. Examples for N = 4, N = 8, N = 11, are given (see print out). These are independent of the function f, which only determines the matrix $\underset{\wedge}{P}$. It is clear that Q becomes very ill-conditioned for high order N, the values decreasing in magnitude very significantly from the first row onward. It might be conjectured that the use of such a matrix would lead to unsatisfactory performance. The eigenvalues of $M = QR$ are as listed for the example 2 ($y' = y^2$, $y(o) = \frac{1}{2}$), in Table 3.03.

Table 3.03

N	$\max\|\lambda\|$
4	0.1619
5	0.1164
6	0.0980
7	0.0910
8	0.1050
9	0.1889
10	0.3085
11	0.5235
12	1.1905
13	3.7060
14	-
15	-
16	-

(3) Modified Newton's Method:

The modified Newton's method was devised so as to overcome the disadvantage of the Newton's process. It is awkward to write down the matrix Q formally for this case, but it can be developed automatically and comparison values are given for N = 4, 8, 11.

A table of eigen values of $M = QR$, is given below for the same equation ($y' = y^2$, $y(o) = \frac{1}{2}$).

Table 3.04

N	max$\|\lambda\|$
4	0.1160
5	0.0607
6	0.06117
7	0.06671
8	0.06939
9	0.07059
10	0.07107
11	0.07126
12	0.07131
13	0.07133
14	0.07134
15	0.07134
16	.
17	.
18	.

The results obtained here on the relative performance of the various methods are checked in the next Chapter by direct comparison with numerical results.

Chapter 4

(4.0) Numerical Results on Rates of Convergence:

As a numerical check on the previous analysis, we take the ratio $\frac{e_i^{(m)}}{e_i^{(m-1)}}$ as an estimate of rate of convergence, when the process has converged sufficiently for the components to be uncoupled from each other.

By taking the average of $\frac{e_o^{(m)}}{e_o^{(m-1)}}, \frac{e_1^{(m)}}{e_1^{(m-1)}}, \ldots\ldots\ldots\ldots, \frac{e_n^{(m)}}{e_n^{(m-1)}}$

over a set of m's and subsequently the mean value of this average, we can have better estimates.

In practice it may be observed that a computer subroutine for calculating the ratios

$$\frac{e_r^{(m)}}{e_r^{(m-1)}} \quad \text{for} \quad \begin{array}{l} r = 0,1,2,\ldots\ldots \\ m = 0,1,2,\ldots\ldots \end{array}$$

fails at the stages where $e_r^{(m)}$ tends to zero. Hence one should treat this with caution. Omitting few terms of both ends of the sequence $\{e_r^{(m)}/e_r^{(m-1)}\}$ one would expect a reasonable approximation.

For example we choose the problem $y' = y^2$, $y(0) = \frac{1}{2}$. The previously obtained maximum eigenvalues of the iteration matrices corresponding to Picard's and Newton's methods are listed in Table 4.0.1 below, and compared with convergence ratio obtained for both methods as described above.

In the table, N represents the degree of Chebyshev approximation used.

Table for the comparison of estimates of maximum eigenvalues for the solution of $y' = y^2$, $y(0) = \frac{1}{2}$.

Table 4.0.1

Picard's Method			Newton's Method	
max|e-value|	Av. Ratio|	N	max |e-value|	Av. Ratio
0.4233	-	4	0.1619	0.1520
0.3709	0.3282	5	0.1164	0.0830
0.3260	0.2690	6	0.0980	0.0710
0.3033	0.2642	7	0.0910	0.0980
0.2492	0.2555	8	0.1050	0.0840
0.2367	0.2567	9	0.1890	0.0890
0.2134	0.2566	10	0.3090	0.2480
0.2134	0.2566	11	0.5240	0.5300
⋮	⋮	12	⋮	⋮
		13		
		14		
		15		

(4.1) Examples and Figures

In what follows, rates of convergence (i.e. - log(|average ratio|) or -log(max|e.value|)) are given for the examples already discussed. Where possible, theoretical and numerical estimates are compared.

Example 1:

$$y' + y = 0, \; y(0) = 1$$

Table 4.1.1

N	max e-value	R_p
4	0.4564	0.7844
5	0.4510	0.7963
6	0.4496	0.7994
7	0.4497	0.7992
8	0.4497	0.7992
9		
10		

R_p - The rate of convergence of Picard's method estimated by theoretical approach (reduction).

(in this case the theoretical estimate is exact).

Example 2:

$$y' = y^2, \; y(0) = \tfrac{1}{2}$$

(a) The rates of convergence estimates of iterative methods by the theoretical approach.

Table 4.1.2a

N	R_p	R_N	R_{MN}
4	0.8597	1.8208	2.1542
5	0.9918	2.1507	2.8018
6	1.1208	2.3228	2.7941
7	1.1930	2.3969	2.7073
8	1.3895	2.2538	2.6680
9	1.4409	1.6660	2.6508
10	1.5446	1.1744	2.6440
11	1.5446	0.6463	2.6415
12	.	–	2.6407
13	.	–	2.6404
14	.	–	2.6403
15	.	.	2.6403
16	.	.	.
17	.	.	.

R_p - Rate of convergence of Picard's method

R_N - Rate of convergence of Newton's method

R_{MN} - Rate of convergence of Modified Newton's method

(b) The rates of convergence estimates of iterative methods, by the average ratio technique

Table 4.1.2b

N	R_P	R_W	R_N
4			1.8837
5	1.1282	1.5552	2.4920
6	1.3094	1.9215	2.6513
7	1.3310	1.9104	2.3250
8	1.3645	1.8887	2.4808
9	1.3626	1.8905	2.4250
10	1.3599	1.8897	*1.3964
11	1.3601	1.8910	*0.1201
12	1.3604	1.8913	*0.6340
13	1.3602	1.8913	
14	1.3602	.	
15	1.3604	.	
16	.	.	
17	.		

R_P - Rate of convergence of Picard's method

R_W - Rate of convergence of Weyl's method

R_N - Rate of convergence of Newton's method

Notes

(i) In Picard's and Weyl's methods, rates of convergence for N = 4 were not obtainable, because solutions by both methods are not stable in the early stages. The rate of convergence of Picard's for N = 5 was calculated by desk machine.

(ii) * the rates of convergence of Newton's method were not available for N > 11 because solutions were then not possible.

Example 3:

$$y' = x - y^2, \quad y(0) = -0.72901$$

Table 4.1.3

R_W	N	R_{P1}	R_{P2}
1.0904	4	0.8597	0.6157
1.1582	5	0.9918	0.8449
1.1057	6	1.1208	1.1325
1.1056	7	1.1930	1.1648
1.1058	8	1.3895	1.2067
1.1071	9	1.4410	1.2062
1.1068	10	1.5446	1.2078
1.1067	11	1.5466	1.2057
1.1068	12	1.5446	1.2094
1.1068	13	1.5446	1.2076
1.1068	14	1.5446	1.2071

R_W - The rate of convergence of Weyl's method estimated by average ratio technique.

R_{P1} - The rate of convergence of Picard's method estimated by the theoretical approach.

R_{P2} - The rate of convergence of Picard's method estimated by average ratio technique.

Example 4:

The Van der Pol's equation

$$y'' = \tfrac{1}{4}(1-y^2)y' - \frac{1}{16}y$$

$$y(-1) = 0, \quad y(+1) = 2$$

Table 4.1.4

R_P	N	R_{MN}
2.3667	5	4.5330
1.5232	6	4.7560
2.1459	7	5.5240
2.2452	8	4.7870
2.2574	9	4.8220
2.2428	10	4.7790
2.2455	11	4.8250
2.2515	12	4.9380
2.2450	13	4.7960
2.2431	14	4.8470
2.2450	15	4.8790
2.2450	16	4.7740
2.2450	17	4.7210
2.2429	18	4.7680

R_P - The rate of convergence of Picard's method estimated by the average ratio technique

R_{MN} - The rate of convergence of Modified Newton's method by the same technique

Example 5:

$$y'' + \lambda^2 y = 0, \quad y(-1) = 0, \quad y(+1) = 1$$

$$\text{for } \lambda = \quad 1.25 < \frac{\pi}{2}$$

Table 4.1.5

N	Average Ratio	R_P
5	0.0320	3.4052
6	0.0332	3.4055
7	0.0457	3.0850
8	0.0457	3.0849
9	0.0416	3.1797
10	0.0416	3.1797
11	0.0413	3.1860
12	0.0413	3.1860
.	.	.
.	.	.
.	.	.
.	.	.

R_P - The rate of convergence of Picard's method estimated by experimental technique (average ratio).

Example 6:

$$y'' + \lambda^2 y = 0, \; y(-1) = 0, \; y(+1) = 1$$
$$\text{for } \lambda = 2 > \frac{\pi}{2}$$

Table 4.1.6

N	Average ratio	R_ϵ
4	0.5643	0.5722
5	0.3543	1.0376
6	0.2875	1.2465
7	0.2049	1.5852
8	0.1763	1.7356
9	0.1781	1.7254
10	0.1779	1.7261
.	.	.
.	.	.
.	.	.

R_ϵ - the average rate of convergence of ϵ-algorithm estimated experimentally (average ratio).

This example shows the average rates of convergence of the ϵ-algorithm applied to a divergent sequence of solutions in Chebyshev coefficients, that obtained by Picard's method (see Table 3 of appendix). In this case one would expect a higher rate of convergence, but since the first ϵ which is (Aitken's δ^2-formula) gave a good approximation to the answer (see Table 4), the rates of convergence of the rest of ϵ's then became very low. However, investigation for rates of convergence of each ϵ could have been carried out, but this was not thought be worthwhile since the errors at later stages were so small.

(4.2) Comparison between iterative methods:

It is of interest to compare the applications of the present iterative methods to a simple initial value problem. Accordingly the comparison was made in the case of the first order equation $y' = y^2$, $y(0) = {}^1/2$.

The Picards program gave the solution in Chebyshev series of degree 10 to this problem in the range $-1 \leq x \leq 1$ to 6 decimal places, in 10 cycles. This implies a total of about 5,000 multiplications. Weyls iteration in Chebyshev series requires a total of about 4,800 multiplications over 10 cycles to secure the desired accuracy.

Norton (1964) [16] gave an estimation of 20,000 multiplications in 12 cycles for Newtons method's solution of degree 25 to the above problem, and 2,560 multiplications for Runge-Kutta method. Our estimate of multiplications required for Newtons method is 5,600, and for the modified Newtons method is 6,500 in 10 cycles. It should also be noted that in both modified Newton and Newtons methods, evaluation of an extra function ($\frac{\partial f}{\partial y}$) is required.

Table 4.2.1

Method	Rate of convergence = R	no. of multiplications per cycle = n	Efficiency = R/n
Picard's	1.5446	500	3.1×10^{-3}
Weyl's *	1.8897	480	3.9×10^{-3}
Newton's	1.1744	560	2.1×10^{-3}
M. Newton's	2.6440	650	4.1×10^{-3}
Runge-Kutta	-	total multiplications = 2,560	-

* Rate of convergence of Weyl's method was estimated by experimental technique (average ratio). The others were calculated theoretically (Reduction).

To summarise the rate of convergence of all iterative methods, for the examples considered, we take the degree of the approximate solutions N to be 10 and construct a general Table 4.2.2 below, which contains the theoretical and the experimental approximations of the rates of convergence (R).

Table 4.2.2

Method	Picard's		Weyl's		Newton's		Modified Newton's	
Examples	R_{P1}	R_{P2}	R_{W1}	R_{W2}	R_{N1}	R_{N2}	R_{MN1}	R_{MN2}
Example 1	0.80	-	-	-	-		-	-
Example 2	1.54	1.36	-	1.89	1.17	1.40	2.64	-
Example 3	1.54	1.21	-	1.11	-	-	-	-
Example 4	-	2.24	-	-	-	-	-	4.78
Example 5	-	3.18	-	-	-	-	-	-
Example 6	-		-	-	-	-	-	-

(4.3) The order of Convergence of Runge-Kutta method:

The error functions $e_r(x)$, r = 0,1,2, ..., in the solution of Falkner-~~Stern~~ Skan equation

$$y''' + y y'' + \beta(1- y'^2) = 0$$

$$y(0) = 0, \; y'(0) = 0, \; y'(\infty) = 1, \quad \beta = 0.01$$

by Runge-Kutta, are calculated for x = 0, 1, 2, ... The estimated values of α (the order of convergence) defined by the relations

$$e_{r+1} = K e_r^{\alpha}$$

or $$\log |e_{r+1}| = \log |K| + \alpha \log |e_r|$$

where K is a constant, are calculated by plotting graphs of log $|e_{r+1}|$ against log $|e_r|$ for r = 0,1,2,...., which are tabulated below.

x	1	2	3	4	5
$\log\|e_o\|$, $\log\|e_1\|$	(-1.5, -1.3)	(-0.2, 0)	(0.3, 0.5)	(0.7, 0.9)	(1.0, 1.2)
$\log\|e_1\|$, $\log\|e_2\|$	(-1.3, -3.4)	(0, -2.1)	(0.5, -1.5)	(0.9, -1.1)	(1.2, -0.9)
$\log\|e_2\|$, $\log\|e_3\|$	(-3.4, -5.1)	(-2.1, -3.8)	(-1.5, -3.2)	(-1.1, -3.0)	(-0.9, -2.6)
$\log\|e_3\|$, $\log\|e_4\|$	(-5.1, -8.7)	(-3.8, -7.4)	(-3.2, -6.8)	(-3.0, -6.4)	(-2.6, -6.2)
α	1.72	1.80	1.92	1.73	1.95

The average order of convergence of Runge-Kutta is approximated by (α = 1.82).

This compares well with the theoretical estimate of order of convergence of the Secant method [26] which is $\alpha = 1.62$.

Conclusions

Methods of solution of nonlinear differential equations have been compared, both numerically and theoretically, in their performance on a set of particular equations.

Picards method is the simplest and the most well-balanced method, and it is rather efficient due to the fact that the process involves less evaluation of the functions y(x), f(x,y) that cuts down a considerable build up of round-off errors through the computation. Efficiency factor~~s~~ ~~range from~~ is $(3.1 \times 10^{-3})\lambda$ for example 2. Convergence is guaranteed for initial value problems. Boundary-value problems, for example in the case $y'' + \lambda^2 y = 0$ for $\lambda > \frac{\pi}{2}$ may not converge. The failure of convergence in this case can be easily rectified using the ϵ - algorithm technique.

Weyls method is noted for its significant success in obtaining solutions for initial value problems of the form

$$y_{r+1}^{(n)} - y_r \dot{y}_{r+1}^{(n-1)} = 0, \quad y(0) = y'(0) = \dots = y(0)^{(n-2)} = 0, y(0)^{(n-1)} = 1.$$

For boundary value problems convergence of this method is not assured. It is a variant of Picard's method obtained by a transformation of the variable. It has an efficiency of 3.9×10^{-3} for example 2.

Newtons method, theoretically, is the most efficient method of all, but due to involvement of a great number of multiplications and evaluation of considerable number of functions, a build-up of round-off errors effects the final outcome of this process; a modification improves this but requires further multiplications. If the degree of the required approximations are large enough, Newtons method converges faster than Picards when convergence holds, and often provides the most powerful technique to secure convergence in a wider class of problems. The efficiency of modified Newtons is 4.1×10^{-3} for example 2.

The Lie series formulation has been shown to generalise all these methods and to provide a family of iterative methods of all orders.

A method of analysis based on evaluating a linear approximation to the iteration matrix connecting successive vectors of Chebyshev coefficients has been tested and its numerical results are found to compare reasonably with the results provided by carrying out the iteration. It is hoped that this idea might be developed to give information about the behaviour of these iteration methods on general classes of equations.

REFERENCES

1. Tables of Chebyshev polynomials Sn(x) and Cn(x). U.S. department of Commerce (N.B.S.) (Applied Maths. Series 9).
2. Clenshaw, C.W. (1962). Chebyshev Series for mathematical functions. (N.P.L.), Maths. tables vol. 5.
3. Clenshaw, C.W., and Curtis A.R. (1960). A method for Numerical integration on an automatic computer. Numerical Math: Vol 2, pp 197-205.
4. Elgendi, S.E. (1969). Chebyshev solution of differential, integral and integro-differential equations. Comp. J. 12 pp282-287.
5. Lanczos, C. (1938). Applied Analysis, London, Pitman.
6. Clenshaw, C.W.(1957). The numerical solution of linear differential equations in Chebyshev series Proc. Cant. Phil. Soc. Vol. 53, p.134.
7. Fox, L. (1962). Chebyshev methods for ordinary differential equations. The Comp. J. Vol 4 p.338.
8. Scraton, R.E. (1965). The solution of linear differential equations in Chebyshev series. The Comp. J. Vol 8, p.57.
9. Snell, J. (1970). The solution in Chebyshev series of systems of linear differential equations with general boundary conditions. Comp. J. 13 pp103-106.
10. Knibb, D. and Scraton, R.E. (1971). On solution of Parabolic partial differential equations in Chebyshev series. Comp. J. 14, pp428-432.
11. Phillips, W. (1967). An estimation of error estimates for two Chebyshev series methods of solving ordinary differential equations. M.Sc. Dissertation, University of Newcastle upon Tyne.

12. Clenshaw, C.W. and Norton, H.J. (1963). The solution of non-linear ordinary differential equations in Chebyshev series. Comp. J. 6 p.88.

13. Wright, K. (1964). Chebyshev collocation methods for ordinary differential equations. Comp. J. 6 p.358.

14. Osborne, M.R. and Watson, G.A. (1968). Note on two methods of solving ordinary linear differential equations. Comp. J. 10 p.383.

15. Oliver, J. (1969). An error estimation technique for the solution of ordinary differential equations in Chebyshev series. Comp. J. 12 pp.57-62.

16. Norton, H.J. (1964). The iterative solution of non-linear ordinary differential equations in Chebyshev series. Comp. J. 7 p.76.

17. Wolfe, M.A. (1969). The Numerical solution of non-singular integral and integro-differential equations by iteration with Chebyshev series. Comp. J. 12, pp. 193-196.

18. Moore, F.K. (Ed.), Theory of Lammar Flows (High speed Aerodynamics and Jet Propulsion Series v. 4), London, Oxford University Press, 1964.

19. Modern Computing Methods (1961), (N.P.L.). Notes on applied science, No. 16 (H.M.S.O.)

20. Elliot, D. (1963), Acchebyshev series for the numerical solution of Fredholm integral equations. The Comp. J. Vol 6 No. 1, pp.102-110.

21. Johnson, R. (1971), New theory of Pade' approximants. Maths. department. Durham University.

22. Genz, A.C. (1973). Application of the ε - Algorithm to Quadrature problems. (Mathematical Institute, University of Kent, England).

23. Johnson, R. (1971). Fixed points and Pade' approximants. Maths. department, Durham University.

24. Wynn, P. (1962). Acceleration techniques for iterated vector and matrix problems. Math. Comput. 16 pp. 301-322.

25. Anderson, D.G. (1965). Iterative procedure for non-linear integral equations. Journal of the Assoc. for computing Machinery 12, No. 4, pp.547-560.

26. Traub, J.F. Iterative methods for the solution of equations. Prentice-Hall, Englewood Cliffs, N.J. 1964.

27. Fox, L. and Parker (1968), Chebyshev polys. in numerical analysis, O.U.P.

28. Ince, E.L. (1953). Ordinary differential equations. London: Constable, New York: Dover.

29. Knapp, H. and Wanner, G. (1968). The numerical solution of ordinary differential equations by Grobener's method of Lie series. MRC technical summary report 880.

30. Shanks, D. (1955). Non-linear transformations of divergent and slowly convergent sequences. J. Math. Phys., v. 34 p.1.

31. Wynn, P. (1956). On a device for computing the $e_m(S_n)$ transformation. MATC, v.10 p.91.

APPENDIX

Table 1

Picards Solution of y' = -y, y(0) = 1

r	$A_r^{(10)}$	$A_r^{(10)}$	$A_r^{(10)}$	A_r
0	2.531240	2.53142	2.532143	2.532132
1	-1.130220	-1.130336	-1.130337	-1.130318
2	0.270830	0.271496	0.271497	0.271495
3	-0.044260	-0.044323	-0.044322	-0.044337
4	0.0052077	0.005471	0.005471	0.005474
5	-0.000525	-0.000551	-0.000551	-0.000543
6		0.000046	0.000046	0.000045
7	.	-0.000002	-0.000002	-0.000003
8	.	0.000000	0.000000	0.000000
9	.	.	-0.000000	-0.000000
10		.	0.000000	0.000000
11	.		-0.000000	-0.000000
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
.	.			.

Table 2

Solution of $y'' + \lambda^2 y = 0$, $y(-1) = 0$, $y(1) = 1$, $\lambda = 1.25 < \frac{\pi}{2}$

r	$A_r^{(20)}$ Picards	A_r
0	2.048282	2.048396
1	0.538073	0.538073
2	-0.542529	-0.542591
3	-0.038850	-0.038850
4	0.018635	0.018637
5	0.000784	0.000784
6	-0.000248	0.000248
7	-0.000007	-0.000008
8	0.000002	0.000001
9	0.000000	0.000000
10	.	.
.	.	.
.	.	.
.	.	.
.		
.		

$$\text{Solution } y(x) = \sin \lambda\,(1+x)/\sin 2\lambda$$

$$= \sum_{r=o}^{N}{}' A_r T_r(x)$$

$$A_r = \frac{2}{N} \sum_{i=o}^{N}{}'' y(x_i)\, T_r(x_i),\quad r = 0,1,\ldots\ldots,N$$

Table 3

Solution of $y'' + \lambda^2 y = 0$, $y(-1) = 0$, $y(+1) = 1$, and $\lambda = 2 > \frac{\pi}{2}$

r	$A_r^{(0)}$	$A_r^{(1)}$	$A_r^{(2)}$	$A_r^{(3)}$	$A_r^{(4)}$	$A_r^{(5)}$	$A_r^{(6)}$	$A_r^{(7)}$	$A_r^{(8)}$	$A_r^{(9)}$	A_r
0	1.000000	1.999998	3.583326	6.144426	10.295337	17.024353	27.932968	45.617310	74.286041	120.761993	-0.538010
1	0.500000	0.583333	0.613888	0.626024	0.630920	0.632902	0.633703	0.634028	0.634156	0.634266	0.634252
2	0.0	-0.499995	-1.333330	-2.687497	-4.883308	-8.443119	-14.214073	-23.569595	-38.736145	-63.323196	0.847859
3		0.0	-0.118054	-0.132175	-0.137901	-0.140222	0.141161	-0.141541	-0.141691	-0.141750	-0.141805
4			0.041666	0.110666	0.239582	0.439082	0.762544	1.286929	2.137027	3.515159	-0.081691
5			0.0	0.006250	0.007133	0.007495	0.007641	0.007700	0.007724	0.007735	0.007742
6				-0.001389	0.001530	-0.008213	-0.015107	-0.026285	-0.044404	-0.737809	0.002890
7				-0.000099	0.000024	-0.000176	-0.000186	0.000190	-0.000192	-0.000194	-0.000192
8				0.0	0.000014	0.000072	0.000148	0.000274	0.000477	0.000807	-0.000053
9	0.0	0.0	0.0	0.0	0.0	0.000002	0.000002	0.000002	0.000003	0.000003	0.000003

Table 4

ϵ- Algorithm for the solution of $y'' + \lambda^2 y = 0$, $y(-1) = 0$,

$y(1) = 1$, $\lambda = 2 > \frac{\pi}{2}$.

r	$\epsilon_{2r}^{(o)}$	$\epsilon_{4r}^{(o)}$	$\epsilon_{6r}^{(o)}$	$\epsilon_{8r}^{(o)}$	A_r
0	-0.714288	-0.538744	-0.538003	-0.538006	-0.538010
1	0.631577	0.634247	0.634250	0.634247	0.634252
2	0.750002	0.847370	0.847851	0.847834	0.847859
3	-0.142856	-0.141805	-0.141806	-0.141822	-0.141805
4	0.000000	-0.081698	-0.081699	-0.081689	-0.081691
5	.	0.007743	0.007740	0.007740	0.007742
6	.	0.000000	0.002892	0.002889	0.002890
7	.	.	-0.000193	-0.000193	-0.000192
8	.	.	-0.000000	-0.000058	-0.000053
9	0.000000	0.000000	-0.000000	0.000003	0.000003
					0.000000

Solution $y(x) = \sin \lambda(1+x)/\sin 2\lambda$

$$A_r = \frac{2}{N} \sum_{i=o}^{N}{}'' y(x_i)T_r(x_i), \quad r = 0,1,\dots,N$$

Table 5a

Solution of $y' = y^2$, $y(0) = \frac{1}{2}$

	N = 8		N = 10	
r	Newton $A_r^{(10)}$	M.Newton $A_r^{(10)}$	Newton $A_r^{(10)}$	M.Newton $A_r^{(10)}$
0	1.153809	1.154306	1.164063	1.154975
1	0.308837	0.309206	0.310059	0.309540
2	0.082596	0.082827	0.083496	0.082957
3	0.022141	0.022186	0.022476	0.022233
4	0.005924	0.005936	0.006027	0.005958
5	0.001554	0.001557	0.001611	0.001596
6	0.000261	0.000262	0.000406	0.000423
7	0.001881	0.001881	-0.000008	0.000093
8	0.000160	0.000160	0.001596	-0.000065
9			0.000110	0.001391
10			0.000000	0.000082
11				
12				

Newton's method gave unstable solutions for this problem when N > 12

Table 5

Solution of $y' = y^2$, $y(0) = \frac{1}{2}$

r	A_r Picard	A_r Weyl	A_r M.Newton	A_r
0	1.154687	1.154713	1.154305	1.154696
1	0.309395	0.309408	0.309206	0.309395
2	0.082901	0.082905	0.082827	0.082883
3	0.022213	0.022215	0.022186	0.022214
4	0.005952	0.005952	0.005936	0.005975
5	0.001595	0.001596	0.001557	0.001596
6	0.000427	0.000429	0.000423	0.000408
7	0.000114	0.000116	0.000093	0.000114
8	0.000032	0.000037	-0.000065	0.000049
9	0.000009	0.000008	0.001391	0.000008
10	0.000000	0.000000	0.000082	0.000000
11	0.000000	0.000000	0.000000	0.000000

The solution $y(x) = \frac{1}{(2-x)}$

$$= \sum_{r=o}^{N}{}' A_r T_r(x)$$

$$A_r = \frac{2}{N} \sum_{i=o}^{N}{}'' \frac{1}{(2-x_i)} T_r(x_i), \quad r = 0,1,\dots,N$$

Table 6

Solution of $y' = x-y^2$, $y(0) = -0.72901$

r	$A_r^{(13)}$ Picard's	$A_r^{(11)}$ Weyl's	$A_r^{(11)}$ Newton's (Norton 64)
0	-1.331811	-1.331804	-1.331820
1	-0.565767	-0.565764	-0.565775
2	0.065562	0.065562	0.065558
3	-0.012310	-0.012310	-0.012312
4	0.002575	0.002575	0.002575
5	-0.000560	-0.000560	-0.000560
6	0.000124	0.000124	0.000124
7	-0.000028	-0.000027	-0.000028
8	0.000006	0.000006	0.000006
9	-0.000001	-0.000001	-0.000001
10	0.000000	0.000000	0.000000
11	-0.000000	.	.
12	0.000000	.	.
13	0.000000	.	.
14	0.000000	0.000000	0.000000

This problem has the formal solution, (Norton 1964),

$$y(x) = A_i'(x)/A_i(x)$$

where $A_i(x)$ is the Airy integral given by

$$A_i(x) = \frac{1}{\pi}\int_0^\infty \cos(\tfrac{1}{3}t^3 + xt)dt$$

Table 7a

Solution of $y'' = \frac{1}{4}(1-y^2)y' - \frac{1}{16}y$, $y(-1) = 0$, $y(+1) = 2$

	N = 3		N = 10	
r	Newton $A_r^{(10)}$	M.Newton $A_r^{(10)}$	Newton $A_r^{(10)}$	M.Newton $A_r^{(10)}$
0	2.066008	2.066482	2.050637	2.068076
1	1.023438	1.022152	1.027443	1.023983
2	-0.033004	-0.033241	-0.031566	-0.032795
3	-0.0234386	-0.022152	-0.025146	-0.024856
4			-0.001544	-0.001367
5			0.000914	0.000901
6			0.000148	0.000137
7			-0.000025	-0.000026
8			-0.000009	-0.000009
9			0.000000	0.000000
10			0.000000	0.000000
11				
12				
13				
14				
15				
16				

Table 7

Solution of $y'' = \frac{1}{4}(1-y^2)y' - \frac{1}{16}y$, $y(-1) = 0$, $y(1) = 2$

r	$A_r^{(10)}$ Picard's	$A_r^{(10)}$ Weyl's	$A_r^{(10)}$ Mod. Newton's
0	2.068066	2.068054	2.068076
1	1.023980	1.023978	1.023982
2	-0.032794	-0.032795	-0.032795
3	-0.024856	-0.024858	-0.024856
4	-0.001367	-0.001369	-0.001367
5	0.000901	0.000897	0.000901
6	0.000136	0.000132	0.000137
7	-0.000026	-0.000015	-0.000026
8	-0.000009	-0.000006	-0.000009
9	0.000000	-0.000005	0.000000
10	0.000000	-0.000003	0.000000
11	.	-0.000004	.
12		-0.000006	.
13	.	-0.000005	.
14		-0.000005	
15	.	-0.000004	.
16		-0.000003	
17	.	-0.000001	.
18	0.000000	-0.000000	0.000000

Table 8

Solution of $y''' + yy'' + \beta(1-y'^2) = 0$

$y(0) = y'(0) = 0, \; y'(\infty) = 1, \; \beta = 0.01$

(a)

r	0	1	2	3	4	5
V_r	0.950	1.050	0.49437	0.49437	0.48277	0.48244
U_r	1.62147	1.73797	0.89664	1.01811	1.00050	1.00000

(b)

r	1st iteration			6th iteration		
	$y(x_r)$	$y'(x_r)$	$y''(x_r)$	$y(x_r)$	$y'(x_r)$	$y''(x_r)$
0	0.0	0.0	0.95	0.0	0.0	0.48244
1	0.466230	0.91018	0.80595	0.23771	0.46832	0.43679
2	1.699440	1.46716	0.28967	0.89889	0.82254	0.25101
3	3.250250	1.59183	0.02948	1.81121	0.97065	0.06483
4	4.849380	1.60272	0.00393	2.80020	0.99994	0.00643
5	6.453740	1.60578	0.00255	3.79960	1.00000	0.00024
6	8.060690	1.60804	0.00202	4.79959	1.00000	0.00000
7	9.669680	1.60988	0.00168	4.79958	1.00000	0.00000
8	11.280350	1.61143	0.00143	6.79958	1.00000	0.00000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
19	29.006310	1.62034	0.00056	17.79957	1.00000	-0.00000
20	30.687510	1.62147	0.00054	18.79957	1.00000	-0.00000

PRINT OUT

Q_1 - represents the matrix Q of Newton's method

Q_2 - represents the matrix Q of Mod. Newton's method

Q_1 of order 5 (N=4)

0.2659E 00	0.5319E 00	-0.4044E 00	-0.8310E 00	0.1382E 00
0.5651E 00	0.1302E 00	-0.6094E 00	-0.2659E 00	0.4433E-01
0.1357E 00	0.2715E 00	-0.1856E 00	-0.2992E 00	0.4982E-01
0.2216E-01	0.4432E-01	0.1330E 00	-0.6925E-01	-0.1551E 00
0.2770E-02	0.5540E-02	0.1662E-01	0.1163E 00	-0.1939E-01

Q_2 of order 5 (N=4)

-0.4993E-02	-0.2631E 00	-0.1710E-02	0.2514E 00	0.7118E-02
0.0	0.0	0.0	0.0	0.0
-0.4993E-02	0.2409E-01	0.3445E-02	-0.1522E-01	-0.6431E-01
-0.2240E-02	0.5237E-01	0.6840E-02	-0.5408E-02	-0.2847E-01
0.2497E-02	0.1765E-01	0.3973E-01	0.1871E-01	0.3513E-01

Q_1 of order 9 (N=8)

0.2661E 00	0.5321E 00	-0.4036E 00	-0.8253E 00	0.1873E 00
0.4961E 00	-0.7422E-01	-0.8750E 00	-0.9000E 01	
0.5652E 00	0.1303E 00	-0.6090E 00	-0.2633E 00	0.6621E-01
0.2224E 00	-0.7031E-01	-0.5000E 00	-0.8000E 01	
0.1357E 00	0.2715E 00	-0.1855E 00	-0.2986E 00	0.5464E-01
0.4932E-01	-0.7812E-02	0.0	-0.1000E 01	
0.2217E-01	0.4434E-01	0.1330E 00	-0.6893E-01	-0.1524E 00
0.2797E-01	-0.5371E-02	-0.7812E-02	-0.3750E 00	
0.2737E-02	0.5474E-02	0.1642E-01	0.1150E 00	-0.3106E-01
-0.1190E 00	0.1172E-01	-0.2686E-02	-0.3906E-01	
0.2715E-03	0.5429E-03	0.1629E-02	0.1140E-01	0.9610E-01
-0.2002E-01	-0.9743E-01	0.7980E-02	-0.2930E-02	
0.2249E-04	0.4498E-04	0.1349E-03	0.9445E-03	0.7961E-02
0.8118E-01	-0.1396E-01	-0.8181E-01	0.5783E-02	
0.1599E-05	0.3198E-05	0.9595E-05	0.6717E-04	0.5661E-03
0.5773E-02	0.7012E-01	-0.1026E-01	-0.7069E-01	
0.9995E-07	0.1999E-06	0.5997E-06	0.4198E-05	0.3538E-04
0.3608E-03	0.4382E-02	0.6186E-01	-0.4418E-02	

Q_2 of order 9 (N=8)

-0.4398E-08	-0.2631E 00	-0.1874E-02	0.2508E 00	0.5579E-02
0.1167E-01	0.1863E-02	0.6234E-03	0.1580E-03	
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	
-0.4398E-08	0.240E-01	0.3281E-02	-0.1576E-01	-0.6585E-01
-0.7473E-02	-0.3266E-02	-0.7496E-03	-0.2114E-03	
0.1759E-07	0.5257E-01	0.7494E-02	-0.3226E-02	-0.2232E-01
-0.4666E-01	-0.7448E-02	-0.2500E-02	-0.6284E-03	
-0.1099E-06	0.1641E-01	0.3564E-01	0.5074E-02	-0.3336E-02
-0.1457E-01	-0.3550E-01	-0.6365E-02	-0.1966E-02	
0.8971E-06	0.3772E-02	0.1104E-01	0.2838E-01	0.4477E-02
-0.1870E-02	-0.1061E-01	-0.2853E-01	-0.5521E-02	
-0.9081E-05	0.7940E-03	0.2599E-02	0.8609E-02	0.2372E-01
0.4132E-02	-0.1188E-02	-0.8254E-02	-0.2371E-01	
-0.3046E-05	0.1674E-03	0.5727E-03	0.2052E-02	0.7060E-02
0.2035E-01	0.3646E-02	-0.1275E-02	-0.8167E-02	
0.6867E-05	0.3863E-04	0.1357E-03	0.5001E-03	0.1830E-02
0.6453E-02	0.1957E-01	0.9589E-02	0.1883E-01	

Q_1 of order 12 (N=11)

0.2661E 00	0.5321E 00	-0.4036E 00	-0.8253E 00	0.1865E 00
0.4961E 00	-0.6641E-01	-0.2500E 00	0.2000E 01	0.1600E 02
-0.4096E 04	-0.6554E 05			
0.5652E 00	0.1303E 00	-0.6090E 00	-0.2633E 00	0.6644E-01
0.2244E 00	-0.3125E-01	-0.2500E 00	0.0	0.0
-0.2304E 04	-0.2458E 05			
0.1357E 00	0.2715E 00	-0.1855E 00	-0.2986E 00	0.5466E-01
0.4907E-01	-0.1172E-01	0.0	0.1000E 01	0.0
-0.1024E 04	-0.2048E 05			
0.2217E-01	0.4434E-01	0.1330E 00	-0.6893E-01	-0.1524E 00
0.2802E-01	-0.4150E-02	-0.1562E-01	-0.6250E-01	-0.2000E 01
-0.1120E 03	-0.2048E 04			
0.2737E-02	0.5474E-02	0.1642E-01	0.1150E 00	-0.3106E-01
-0.1190E 00	0.1186E-01	-0.1465E-02	0.1172E-01	0.6250E-01
-0.9000E 01	-0.1280E 03			
0.2715E-03	0.5429E-03	0.1629E-02	0.1140E-01	0.9610E-01
-0.2002E-01	-0.9742E-01	0.8057E-02	0.4883E-03	0.1953E-01
-0.1000E 01	-0.1600E 02			
0.2249E-04	0.4498E-04	0.1349E-03	0.9445E-03	0.7961E-02
0.8118E-01	-0.1296E-01	-0.8181E-01	0.599tE-02	0.1709E-02
-0.6250E-01	-0.2000E 01			
0.1599E-05	0.3198E-05	0.9595E-05	0.6717E-04	0.5661E-03
0.5773E-02	0.7012E-01	-0.1025E-01	-0.7043E-01	0.4639E-02
-0.7812E-02	-0.3750E 00			
0.9961E-07	0.1992E-06	0.5976E-06	0.4183E-05	0.3526E-04
0.3596E-03	0.4367E-02	0.6165E-01	-0.7838E-02	-0.6181E-01
0.3174E-02	-0.2344E-01			
0.5518E-08	0.1104E-07	0.3311E-07	0.2318E-06	0.1954E-05
00.1992E-04	0.2420E-03	0.3415E-02	0.5497E-01	-0.6189E-02
-0.5508E-01	0.1953E-02			
0.2753E-09	0.5506E-09	0.1652E-08	0.1156E-07	0.9745E-07
0.9938E-06	0.1207E-04	0.1704E-03	0.2742E-02	0.4958E-01
-0.5017E-02	-0.4977E-01			

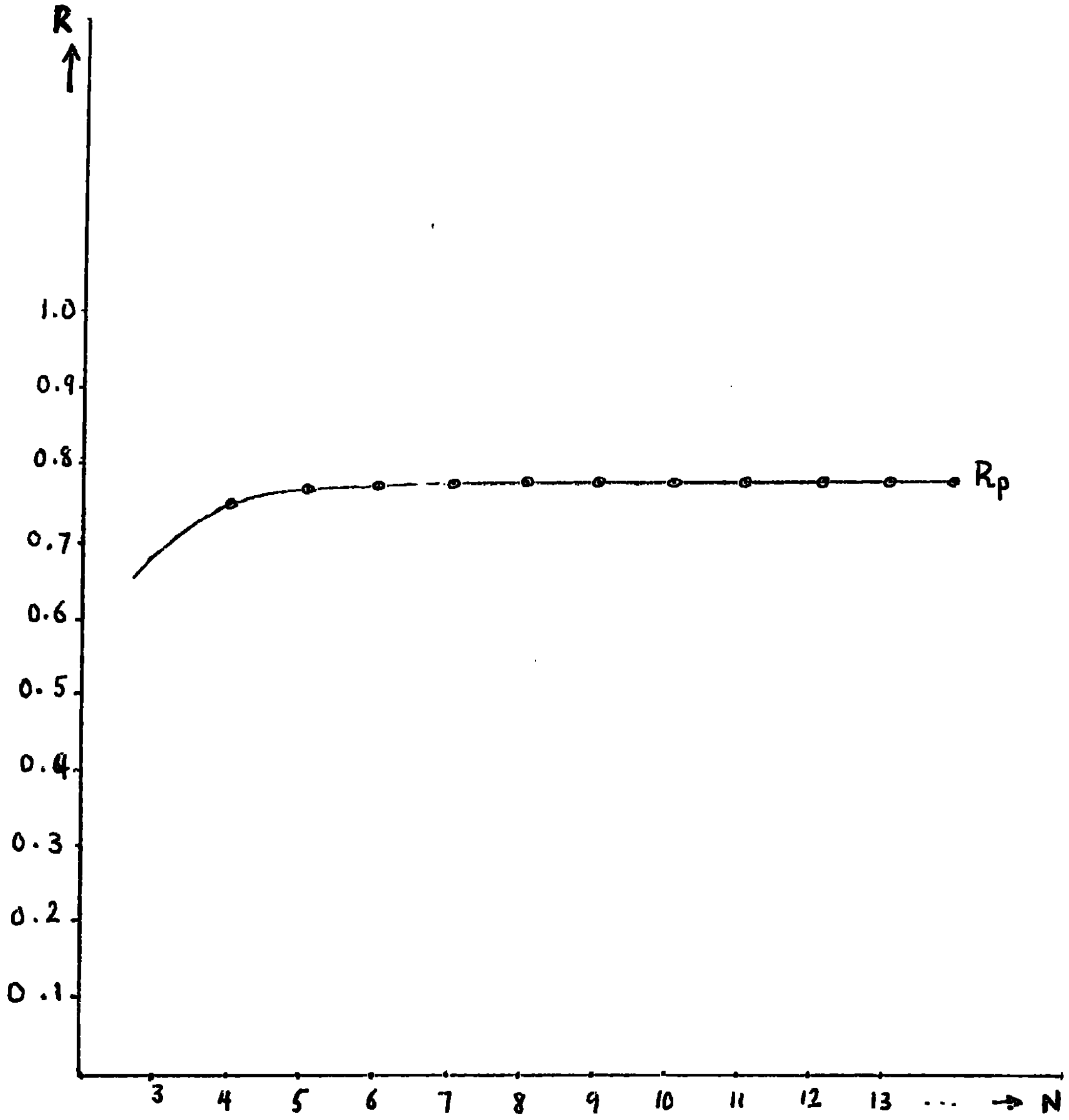
Q_1 of order 12 (N=11) contd.

0.1251E-10	0.2503E-10	0.7508E-10	0.5256E-09	0.4430E-08
0.4517E-07	0.5487E-06	0.7744E-05	0.1246E-03	0.2254E-02
0.4523E-01	-0.2262E-02			

Q_2 of order 12 (N=11)

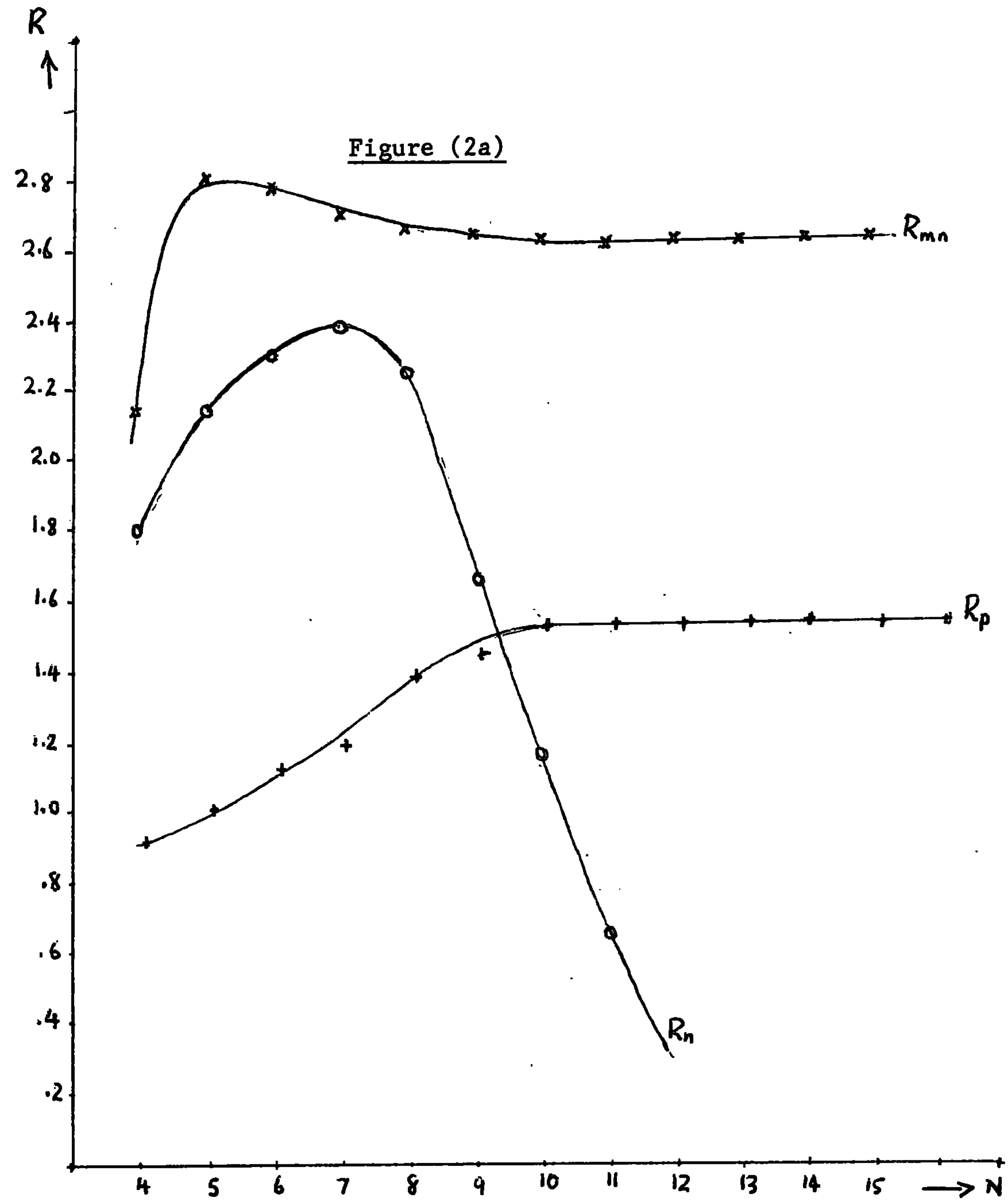
-0.1095E-13	-0.2631E 00	-0.1874E-02	0.2508E 00	0.5579E-02
0.1167E-01	0.1863E-02	0.6234E-03	0.1580E-03	0.4493E-04
0.1175E-04	0.6413E-06			
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0			
-0.1095E-13	0.2404E-01	0.3281E-02	-0.1576E-01	-0.6585E-01
-0.7473E-02	-0.3266E-02	-0.7496E-03	-0.2115E-03	-0.5628E-04
-0.1497E-04	-0.3826E-05			
0.4381E-13	0.5257E-01	0.7494E-02	-0.3226E-02	-0.2232E-01
-0.4666E-01	-0.7448E-02	-0.2500E-02	-0.6282E-03	-0.1711E-03
-0.4579E-04	-0.1245E-04			
-0.2738E-12	0.1641E-01	0.3564E-01	0.5074E-02	-0.3336E-02
-0.1457E-01	-0.3550E-01	-0.6366E-02	-0.1967E-02	-0.5086E-03
-0.1375E-03	-0.3693E-04			
0.2234E-11	0.3772E-02	0.1104E-01	0.2838E-01	0.4477E-02
-0.1869E-02	-0.1061E-01	-0.2853E-01	-0.5509E-02	-0.1628E-02
-0.4270E-03	-0.1150E-03			
-0.2262E-10	0.7939E-03	0.2599E-02	0.8608E-02	0.2372E-01
0.4129E-02	-0.1198E-02	-0.8290E-02	-0.2383E-01	-0.4844E-02
-0.1294E-02	-0.3685E-03			
0.2737E-09	0.1675E-03	0.5733E-03	0.2055E-02	0.7069E-02
-0.2038E-01	0.3767E-02	-0.8380E-03	-0.6786E-02	-0.2045E-01
-0.4317E-02	-0.1221E-02			
-0.3854E-08	0.3653E-04	0.1270E-03	0.4672E-03	0.1706E-02
0.5992E-02	0.1786E-01	0.3442E-02	-0.6217E-03	-0.5736E-02
-0.1791E-01	-0.3893E-02			
0.6194E-07	0.8247E-05	0.2881E-04	0.1069E-03	0.3966E-03
0.1459E-02	0.5197E-02	0.1589E-01	0.3160E-02	-0.4774E-03
-0.4950E-02	-0.1589E-01			
0.4830E-07	0.1849E-05	0.6835E-05	0.2491E-04	0.9267E-04
0.3448E-03	0.1274E-02	0.4581E-02	0.1430E-01	0.2856E-02
-0.5978E-03	-0.5076E-02			
-0.5085E-07	0.2919E-06	0.1419E-05	0.6144E-05	0.2338E-04
0.8799E-04	0.3283E-03	0.1218E-02	0.4418E-02	0.1419E-01
0.7005E-02	0.1383E-01			

Figure 1



R_p - theoretical rate of convergence of Picards method for the solution of

$$y' + y = 0, \quad y(0) = 1$$



R_p - theoretical rate of convergence of Picards method

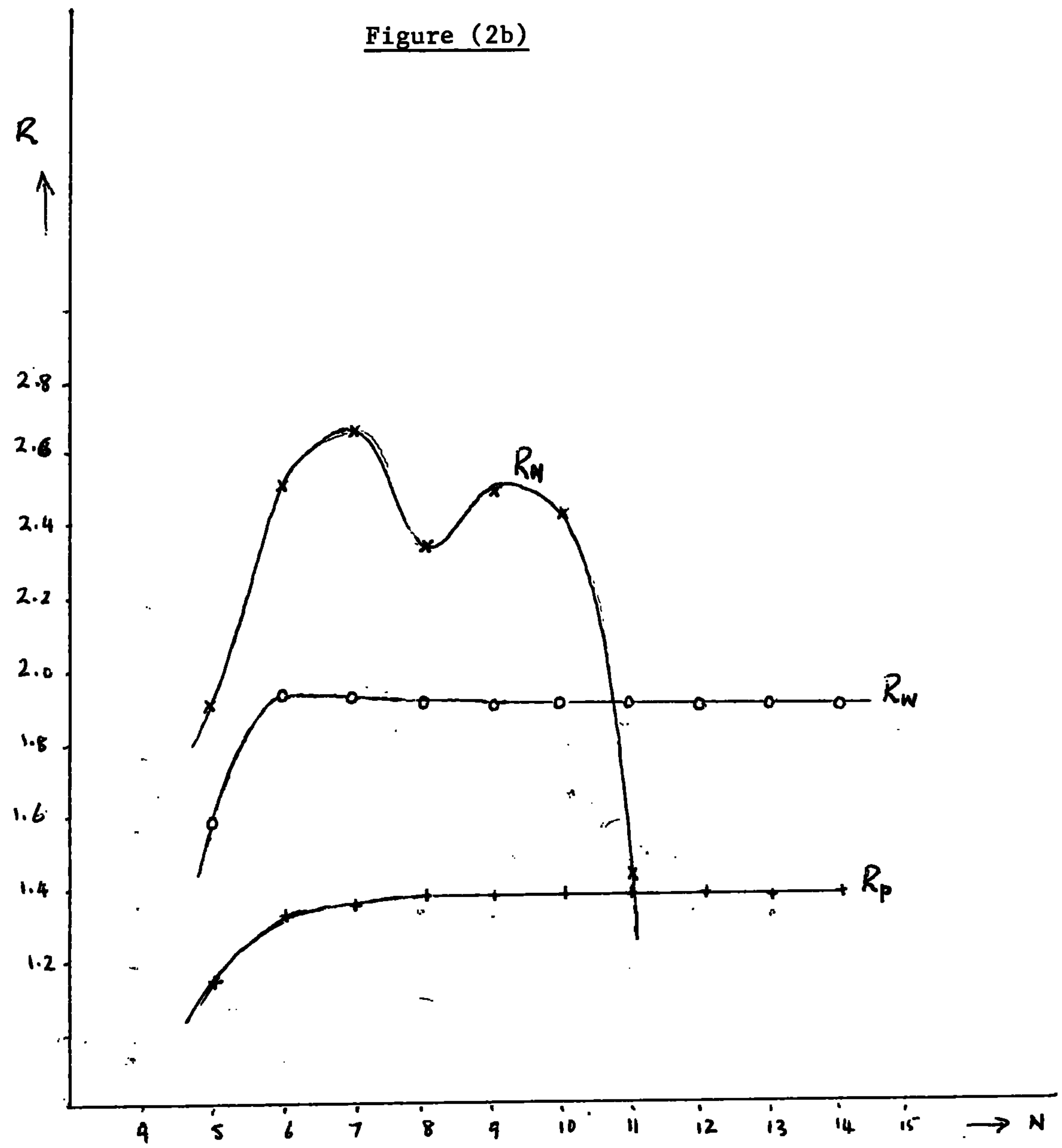
R_n - " " " " Newtons method

R_{mn} - " " " " Modified Newtons method

for the solution of

$$y' = y^2, \; y(0) = {}^1/2$$

Figure (2b)



R_p - experimental rate of convergence of Picards method

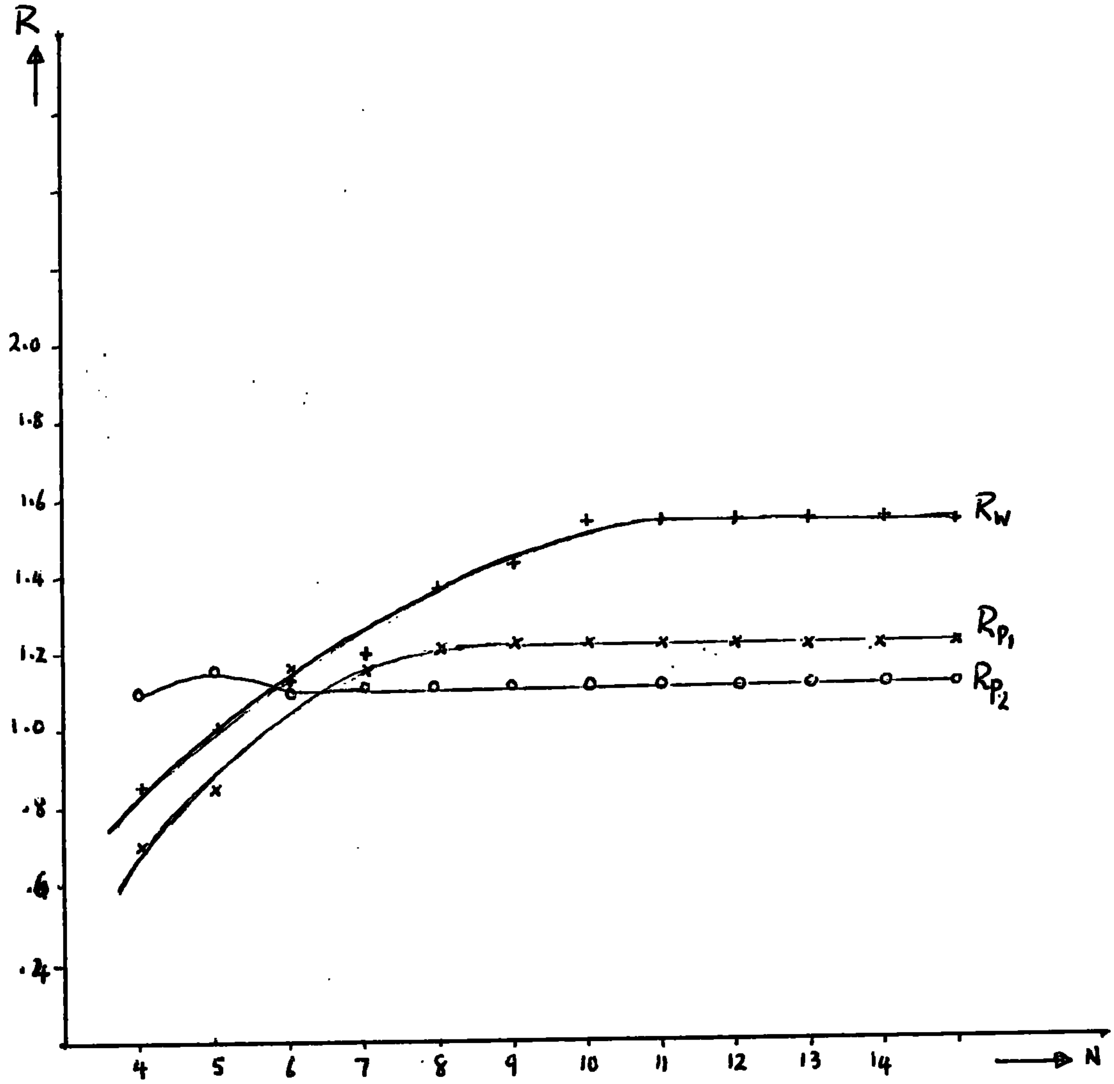
R_w - " " " " Weyls "

R_N - " " " " Newtons "

for the solution of

$$y' = y^2, \; y(0) = {}^1/2 .$$

Figure (3)



R_{p1} - theoretical rate of convergence of Picards method

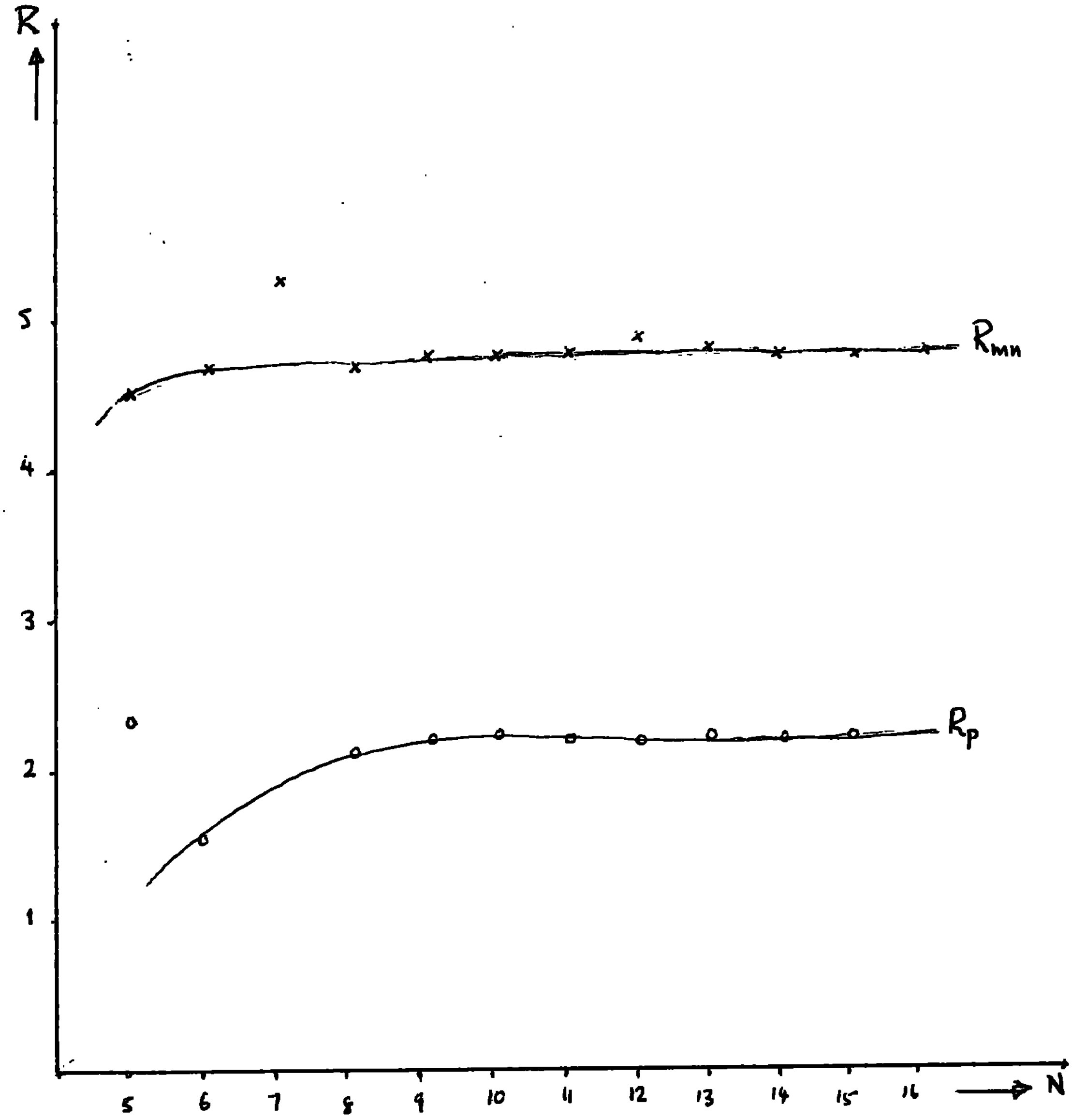
R_{p2} - experimental rate of convergence of Picards method

R_w - experimental rate of convergence of Weyls method

for the solution of

$$y' = x-y^2, \; y(0) = -0.72901$$

Figure (4)



R_p - Experimental rate of convergence of Picards method

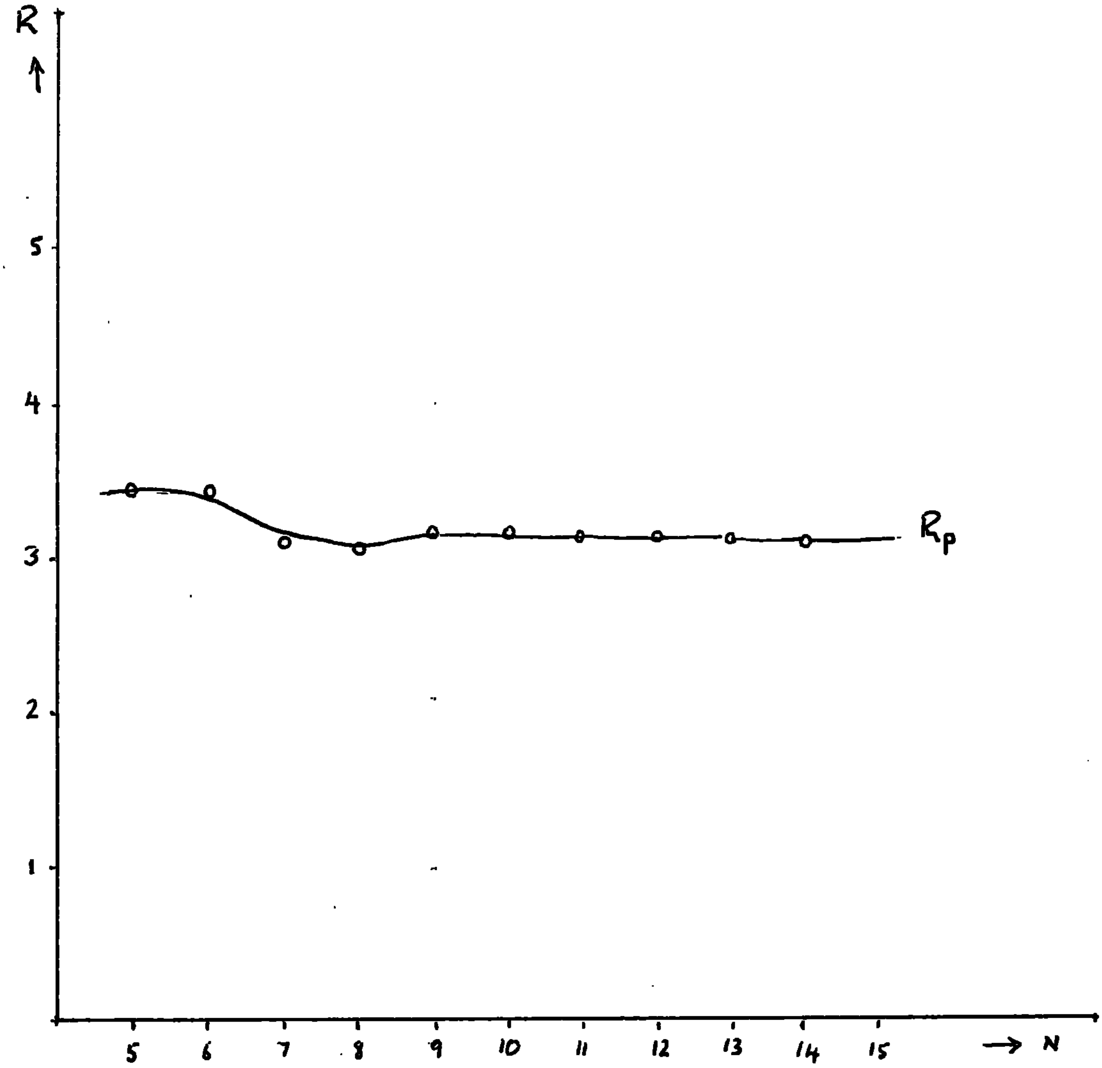
R_{mn} - Experimental rate of convergence of modified Newtons method

for the solution of

$$y'' = \frac{1}{4}(1-y^2)\ y' - \frac{1}{16}y$$

$$y(-1) = 0,\ y(1) = 2$$

Figure (5)

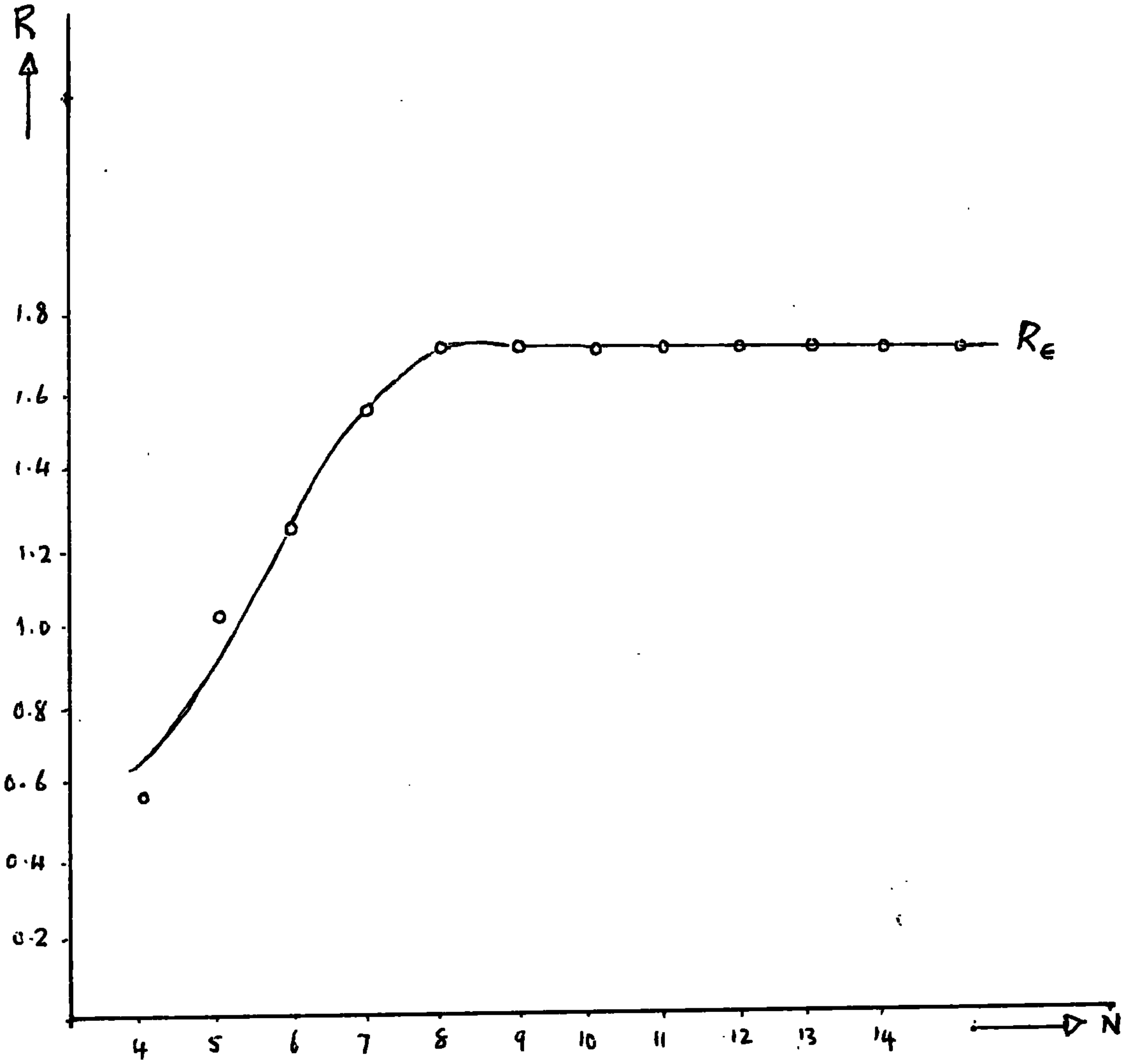


R_p - Experimental rate of convergence of Picards method for the solution of

$$y'' + \lambda^2 y = 0, \quad y(-1) = 0, \; y(+1) = 1$$

$$\text{and} \quad \lambda = 1.25 < \frac{\pi}{2}$$

Figure (6)



R_ϵ - the average rate of convergence of the ϵ - Algorithm estimated experimentally (average ratio).