A mathematical study of the generation of microseisms by waves at sea

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A MATHEMATICAL STUDY OF THE GENERATION OF MICROSEISMS BY WAVES AT SEA.

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Chapter 1. Introduction.

Sensitve seismographs record oscillations of the surface of the Earth which have an amplitude of a few microns. Many of these are cleary of local origin and are due to wind gusts against the observatory building, traffic, frozen ground, etc. There are however continuous oscillations of periods between 3 and 10 seconds and amplitudes between 0.1 and 20 microns. The word microseism is reserved for this latter type of seismic disturbance.

It has been noticed that the intensity of microseismic disturbance increases simultaneously over large areas of Europe and North America (GUtenberg 1931,1932; Lee 1934). Whipple and Lee (1936) noted that the greatest disturbance is found in coastal regions bordering on a well developed depression, but that equally developed depressions did not necessarily give rise to equal amplitudes. Ramirez (1940) has shewn that microseisms recorded at St. Louis are received from the direction of depressions off the Atlantic coast. Banerji (1930,1935) gave evidence that microseisms were recorded in India as soon as a storm was formed in the mid-Arabian Sea or in the mid-Bay of Bengal, five or six hundred miles away from the coast. He reports that tremendous waves were produced and concludes that these in some way originate the microseisms. He points out that the waves from the storm would take two or three days to arrive at the coast, so that the waves over shallow water could not account for the microseisms which were recorded as soonaas the storm was established.Lee (1935) tabulated the phases of microseisms received at Kew on six occasions when depressions were located over different parts of the Eastern Atlantic and Western Europe. He finds that the phase differences between components confirm the theory that microseisms are Rayleigh waves in the Earth's crust and concludes that microseisms are generated in deep water.

Lee(1934) investigated the effect of the subsoil and of the geological formations under observatories on the amplitude of microseiemic disturbances. Analysis of the motion of a seismograph pillar, and measurements of the earth resistance at Durham and Kew, show that the tilting of the pillars due to microseismic oscillations is negligible; consequently the accuracy with which these oscillations are recorded cannot depend on the subsoil. He found that variations in microseismic amplitudes are due to geographical and geological causes, and that the ratio of horizontal to vertical components at stations on earlier geological formations was in agreement with the theoretical value for Rayleigh waves.

The observations of Lee (1932), Banerji (1935) and other writers indicate that microseisms must be generated in deep water. The fairly obvious source of energy for microseisms,namely sea waves, has for a long time been neglected owing to the inability of the current first order theory of hydrodynamics to account for a pressure variation at depths greater than half a wave--length. Banerji carried out experiments with waves in tanks and found that ''the disturbance starting from a maximum value at the surface diminished up to a certain depth and then increased to another but lower maximum at the bed''. He considered that this phenomenon was a consequence of the compressibility of the water, and in a subsequent analysis he obtained an expression for this pressure. Banerji's results are difficult to explain and it seems likely that his experiments were on the wrong scale. His work has been severely critised by Whipple and Lee (1935) and mistakes in his mathematics have been indicated by Baxter and Archer(1935). Scholte (1943) considered a variable pressure applied to the surface of the sea and found the displacement of the sea bed; shewing that such a variable pressure causes the simultaneous development of gravity and compression waves he uses a first order theory to shew that whilst the gravity waves are attenuated exponentially with depth the compression waves are transmitted unaltered to the sea bed and there produce periodic displacements of the ground. The weakness of this as a theory of the origin of microseisms is that sea waves are not generated by such a pressure variation (Jeffreys 1925).

An earlier theory due originally to Wiechert and until recently supported by Gutenberg (1931) was that microseiems were generated by the surf breaking along the coast line and transferring the energy of the waves to the ground. Undoubtedly some energy will be imparted to the ground by a ''breaker'', but the innumerable breakers along a coast do not break simultaneous so that the surf seems likely to produce a very complex motion. Further most coasts are not of the steep type required by the surf theory.

Bernard (1941) obtained evidence which led him to believe that microseismic oscillations have periods which are half those of the sea waves which give rise to them. He reached this conclusion by comparing the microseisms recorded at Averroes, near Casablanca, with simultaneous observations on the sea waves reaching the coast. The same ratio was noticed by Deacon (1947) between the frequency of microseiems recorded at Kew and sea waves recorded at Perranporth on the north east coast of Cornwall. Deacon's work suggests that the waves entering the coastal region west of the British Isles were responsible for the microseismic activity at Kew. Darbyshire (1950) shows that it is possible to reconcile the two views that microseisms are generated in deep water and in coastal water if we consider that the microseismic activity at any particular observatory is due to more than one source. He selected three occasions when a single depression over the Atlantic was producing large waves whilst the wave activity in the coastal region was small. The records of the Kew Galitzin vertical seismograph were subjected to frequency analysis, and in each case it was possible to identify two bands of frequency. It was possible to identify the microseismic waves which had a two to one frequency ratio with deep water waves in mid-Atlantic and those which had a similar connection with the coastal waves.

Thus a satisfactory theory of the origin of microseisms must be able to explain how surface waves in both deep and coastal waters can influence the sea bed and why the seismic waves have twice the mean frequency of the generating sea wave group. The key to this problem was provided by Miche (1944) who, in a thorough second order investigation of wave motion in an incompressible medium, obtained expressions for the velocity and pressure under progressive and standing waves. He found that under a standing wave there existed a second order pressure variation which was independent of the depth and had a frequency twice that of the surface wave. A shorter proof of the existence of this second order pressure variation under a standing wave was derived by Longuet - Higgins and Ursell (1948). It has also been investigated experimently at Cambridge by Cooper and Longuet-Higgins (1951). This has been used by Longuet-Higgins (1950)[x] to demonstrate that opposite wave groups, that is wave groups of similar characteristics but travelling in opposite directions, originating in a depression or near a coast produce seismic waves of the same order of frequency and amplitude as microseisms.

[x] See also note on p. 224

Chapter 2.

The second order pressure variation on the bed under a train of standing waves in an incompressible fluid.

In the classical study of Hydrodynamics it has been usual to assume that the wave amplitude and derivatives of it are so small compared with the depth that second and higher powers of these quantities may be neglected. In this section the coordinates of the position of a particle and the pressure at a point will be determined to the second order in these small quantities.

We shall use the Langrangian form of coordinates and consider a two dimensional motion; the wave motion being supposed to occur between two vertical parallel planes unit distance apart.

Let (x_0, z_0) be the initial coordinates of any particle of fluid and (x, z) its coordinates at a time t. Then by Lamb § 13 the equations of motion are

$$\left.\begin{aligned} \frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial x_0} + \left(\frac{\partial^2 z}{\partial t^2} - g\right)\frac{\partial z}{\partial x_0} + \frac{1}{\rho}\cdot\frac{\partial p}{\partial x_0} = 0 \\ \frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial z_0} + \left(\frac{\partial^2 z}{\partial t^2} - g\right)\frac{\partial z}{\partial z_0} + \frac{1}{\rho}\cdot\frac{\partial p}{\partial z_0} = 0 \end{aligned}\right\} \qquad (2\cdot 1)$$

where ρ is the density in the neighbourhood of the particle and p is the excess pressure over the atmospheric pressure at time t.

Assuming the liquid to be incompressible, we have as the equation of continuity (Lamb § 14)

$$\frac{\partial(x, z)}{\partial(x_0, z_0)} = 1 \qquad (2\cdot 2),$$

Where the origin is taken in the free surface at rest, z_0 is measured downwards and x_0 is measured horizontally in the direction of the wave motion and perpendicular to the wave crests.

Let h denote the amplitude of the wave motion, then we can write

$$\left.\begin{aligned} x &= x_0 + h\phi_1 + h^2\phi_2 \\ z &= z_0 + h\psi_1 + h^2\psi_2 \\ \frac{p}{\rho} &= g z_0 + h\chi_1 + h^2\chi_2 \end{aligned}\right\} \qquad (2\cdot 3)$$

where ϕ_n, ψ_n, χ_n (n=1,2) are functions of x_0, z_0 and t.

By substituting from equation (2·3) in equations (2·1) and (2·2) and equating to zero the coefficients of h and h^2 we determine the six functions ϕ_n, ψ_n and χ_n (n = 1,2). Thus from equations (2·1)

$$\left(h\frac{\partial^2\phi_1}{\partial t^2} + h^2\frac{\partial^2\phi_2}{\partial t^2}\right)\left(1 + h\frac{\partial\phi_1}{\partial x_0} + h^2\frac{\partial\phi_2}{\partial x_0}\right)$$

$$+\left(h\frac{\partial^2\psi_1}{\partial t^2} + h^2\frac{\partial^2\psi_2}{\partial t^2} - g\right)\left(h\frac{\partial\psi_1}{\partial x_0} + h^2\frac{\partial\psi_2}{\partial x_0}\right) + h\frac{\partial\chi_1}{\partial x_0} + h^2\frac{\partial\chi_2}{\partial x_0} = 0,$$

and

$$\left(h\frac{\partial^2\phi_1}{\partial t^2} + h^2\frac{\partial^2\phi_2}{\partial t^2}\right)\left(h\frac{\partial\phi_1}{\partial z_0} + h^2\frac{\partial\phi_2}{\partial z_0}\right)$$

$$+\left(h\frac{\partial^2\psi_1}{\partial t^2} + h^2\frac{\partial^2\psi_2}{\partial t^2} - g\right)\left(1 + h\frac{\partial\psi_1}{\partial z_0} + h^2\frac{\partial\psi_2}{\partial z_0}\right) + g + h\frac{\partial\chi_1}{\partial z_0} + h^2\frac{\partial\chi_2}{\partial z_0} = 0.$$

From equations (2·2)

$$\left(1 + h\frac{\partial\phi_1}{\partial x_0} + h^2\frac{\partial\phi_2}{\partial x_0}\right)\left(1 + h\frac{\partial\psi_1}{\partial z_0} + h^2\frac{\partial\psi_2}{\partial z_0}\right)$$

$$-\left(h\frac{\partial\phi_1}{\partial z_0} + h^2\frac{\partial\phi_2}{\partial z_0}\right)\left(h\frac{\partial\psi_1}{\partial x_0} + h^2\frac{\partial\psi_2}{\partial x_0}\right) = 1,$$

From the terms in h we have

$$\left.\begin{aligned}
&\frac{\partial\chi_1}{\partial x_0} + \frac{\partial^2\phi_1}{\partial t^2} - g\frac{\partial\psi_1}{\partial x_0} = 0,\\
&\frac{\partial\chi_1}{\partial z_0} + \frac{\partial^2\psi_1}{\partial t^2} - g\frac{\partial\psi_1}{\partial z_0} = 0,\\
&\frac{\partial\phi_1}{\partial x_0} + \frac{\partial\psi_1}{\partial z_0} = 0
\end{aligned}\right\} \qquad (2\cdot4)$$

From the terms in h^2 we have

$$
\left.
\begin{aligned}
&\frac{\partial \chi_2}{\partial x_0} + \frac{\partial^2 \phi_2}{\partial t^2} - g\frac{\partial \psi_2}{\partial x_0} = -\frac{\partial^2 \phi_1}{\partial t^2}\cdot\frac{\partial \phi_1}{\partial x_0} - \frac{\partial^2 \psi_1}{\partial t^2}\cdot\frac{\partial \psi_1}{\partial x_0}\,,\\
&\frac{\partial \chi_2}{\partial z_0} + \frac{\partial^2 \psi_2}{\partial t^2} - g\frac{\partial \psi_2}{\partial z_0} = -\frac{\partial^2 \phi_1}{\partial t^2}\cdot\frac{\partial \phi_1}{\partial z_0} - \frac{\partial^2 \psi_1}{\partial t^2}\cdot\frac{\partial \psi_1}{\partial z_0}\,,\\
&\frac{\partial \phi_2}{\partial x_0} + \frac{\partial \psi_2}{\partial z_0} = -\frac{\partial \phi_1}{\partial x_0}\cdot\frac{\partial \psi_1}{\partial z_0} + \frac{\partial \phi_1}{\partial z_0}\cdot\frac{\partial \psi_1}{\partial x_0}\,.
\end{aligned}
\right\}
\qquad (2\cdot 5)
$$

For a solution of the first order in h it is enough to evaluate ϕ_1 , ψ_1 and χ_1 and to substitute in equations (2·3),

To eliminate χ_1 between equations (2·4), differentiate the first equation with reapect to z_0 and the second equation with respect to x_0 and subtract them ; thus

$$\frac{\partial}{\partial z_0}\cdot\frac{\partial^2 \phi_1}{\partial t^2} - g\frac{\partial^2 \psi_1}{\partial x_0 \partial z_0} - \frac{\partial}{\partial x_0}\cdot\frac{\partial^2 \psi_1}{\partial t^2} + g\frac{\partial^2 \psi_1}{\partial x_0 \partial z_0} = 0,$$

$$\therefore\ \frac{\partial}{\partial z_0}\cdot\frac{\partial^2 \phi_1}{\partial t^2} - \frac{\partial}{\partial x_0}\cdot\frac{\partial^2 \psi_1}{\partial t^2} = 0 \qquad (2\cdot 6)$$

Differentiating the third of equations (2·4) twice with respect to t we have

$$\frac{\partial}{\partial x_0}\cdot\frac{\partial^2 \phi_1}{\partial t^2} + \frac{\partial}{\partial z_0}\cdot\frac{\partial^2 \psi_1}{\partial t^2} = 0 \qquad (2\cdot 7)$$

From equations (2·6) and (2·7) it appears that $\frac{\partial^2 \phi_1}{\partial t^2}$ and $\frac{\partial^2 \psi_1}{\partial t^2}$ are conjugate harmonic functions of x_0 and z_0 . It is therefore convenient to introduce a function $G(x_0, z_0, t)$ defined by

$$\frac{\partial^2 \phi_1}{\partial t^2} = \frac{\partial}{\partial x_0}\cdot\frac{\partial^2 G}{\partial t^2}\,, \qquad \frac{\partial^2 \psi_1}{\partial t^2} = \frac{\partial}{\partial z_0}\cdot\frac{\partial^2 G}{\partial t^2} \qquad (2\cdot 8)$$

Whence, after use of equation (2·7)

$$\frac{\partial^2}{\partial x_0^2}\cdot\frac{\partial^2 G}{\partial t^2} + \frac{\partial^2}{\partial z_0^2}\cdot\frac{\partial^2 G}{\partial t^2} = \nabla^2\frac{\partial^2 G}{\partial t^2} = 0 \qquad (2\cdot 9)$$

If we now integrate equation (2·8) twice with respect to t and use the third of equations (2·4) we find

$$\left.\begin{aligned} \phi_1 &= \frac{\partial G(x_0,z_0,t)}{\partial x_0} + \frac{\partial k(x_0,z_0)}{\partial z_0}.t + \frac{\partial k'(x_0,z_0)}{\partial z_0}, \\ \psi_1 &= \frac{\partial G(x_0,z_0,t)}{\partial z_0} - \frac{\partial k(x_0,z_0)}{\partial x_0}.t - \frac{\partial k'(x_0,z_0)}{\partial x_0}. \end{aligned}\right\} \qquad (2\cdot10)$$

where k and k' are any two arbitrary functions of x_0 and z_0.

If we write $x_0' = x_0 + h\frac{\partial k'}{\partial z_0}$, $z_0' = z_0 - h\frac{\partial k'}{\partial x_0}$ we find that the terms in k' vanish. Thus the function k' is not physically significant as its existence depends on the choice of coordinates. It is thus permissible to neglect the terms in k'.

If we denote by u and w the velocity components of the point (x,z) at time t, we have from equations (2·3)

$$u = h\frac{\partial \phi_1}{\partial t} + h^2\frac{\partial \phi_2}{\partial t},$$

$$w = h\frac{\partial \psi_1}{\partial t} + h^2\frac{\partial \psi_2}{\partial t};$$

or to the first order $u = h\frac{\partial \phi_1}{\partial t}$, $w = h\frac{\partial \psi_1}{\partial t}$.

Hence using the determined values of ϕ_1 and ψ_1 (equations 2·10)

$$u = h\frac{\partial^2 G}{\partial t\,\partial x_0} + h\frac{\partial k(x_0,z_0)}{\partial z_0},$$

$$w = h\frac{\partial^2 G}{\partial t\,\partial z_0} - h\frac{\partial k(x_0,z_0)}{\partial x_0}.$$

Thus it appears that the terms in k [redacted] represent a current independent of time i.e. a motion independent of, and superposed on the periodic motion. There are an infinity of such possible motions. but according to Miche, such currents (courants entraînements) are known to be very feeble in comparison with those following a periodic disturbance. Hence to the first order in h we can neglect k and write

$$\phi_1 = \frac{\partial G}{\partial x_0} \quad \text{and} \quad \psi_1 = \frac{\partial G}{\partial z_0}. \qquad (2\cdot11)$$

On substituting for ϕ_1 and ψ_1 in the third of equations (2·4)

$$\frac{\partial^2 G}{\partial x_0^2} + \frac{\partial^2 G}{\partial z_0^2} = \nabla^2 G = 0,$$

i.e. $G(x, z_0, t)$ is a harmonic function of x_0, z_0.

Substituting for ϕ_1 and ψ_1 in the first two of equations (2·4) we have

$$\frac{\partial \chi_1}{\partial x_0} + \frac{\partial^2}{\partial t^2} \cdot \frac{\partial G}{\partial x_0} - g \frac{\partial^2 G}{\partial x_0 \partial z_0} = 0,$$

and $$\frac{\partial \chi_1}{\partial z_0} + \frac{\partial^2}{\partial t^2} \cdot \frac{\partial G}{\partial z_0} - g \frac{\partial^2 G}{\partial z_0^2} = 0;$$

i.e. $$\frac{\partial}{\partial x_0}\left[\chi_1 + \frac{\partial^2 G}{\partial t^2} - g\frac{\partial G}{\partial z_0}\right] = 0,$$

and $$\frac{\partial}{\partial z_0}\left[\chi_1 + \frac{\partial^2 G}{\partial t^2} - g\frac{\partial G}{\partial z_0}\right] = 0.$$

Hence $$\chi_1 + \frac{\partial^2 G}{\partial t^2} - g\frac{\partial G}{\partial z_0} = f_1(t)$$

or $$\chi_1 = g\psi_1 - \frac{\partial^2 G}{\partial t^2} + f_1(t) \qquad (2{\cdot}12)$$

where $f_1(t)$ is a function of t, whose value is determined by the boundary conditions.

If the motion is considered to be irrotational, we have, in Euler's notation,

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad w = \frac{\partial \phi}{\partial z} \qquad (2{\cdot}13)$$

where ϕ is the velocity potential.

But to the first order we have

$$u = k\frac{\partial \phi_1}{\partial t} \quad \text{and} \quad w = k\frac{\partial \psi_1}{\partial t}.$$

Hence using equation (2·11)

$$u = k\frac{\partial \phi_1}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial G}{\partial t}\right) = \frac{\partial \phi}{\partial x}$$

and $$w = k\frac{\partial \psi_1}{\partial t} = \frac{\partial}{\partial z}\left(k\frac{\partial G}{\partial t}\right) = \frac{\partial \phi}{\partial z}.$$

Hence, suppressing an additive function of t which has no importance for the periodic motion,

$$\phi = k\frac{\partial G(x, y, t)}{\partial t} \qquad (2{\cdot}14)$$

But the current function ψ satisfies

$$u = \frac{\partial\psi}{\partial z} \qquad \text{and} \qquad \omega = -\frac{\partial\psi}{\partial x} \tag{2·15}$$

i.e.. ψ and ϕ are conjugate functions, hence if K is the harmonic conjugate function of G

$$\psi = \frac{\kappa\,\partial K(x,z,t)}{\partial t} \tag{2·16}$$

<u>Determination of ϕ_2, ψ_2 and χ_2.</u>

ϕ_1 and ψ_1 and hence G are periodic functions of time. Suppose their period is 2T, then

$$\left.\begin{aligned} \frac{\partial^2\phi_1}{\partial t^2} &= -\left(\frac{\pi}{T}\right)^2\phi_1, \\ \frac{\partial^2\psi_1}{\partial t^2} &= -\left(\frac{\pi}{T}\right)^2\psi_1, \\ \text{and} \quad \frac{\partial^2 G}{\partial t^2} &= -\left(\frac{\pi}{T}\right)^2 G. \end{aligned}\right\} \tag{2·17}$$

Using equation (2·11), the first of equations (2·5) becomes

$$\frac{\partial\chi_2}{\partial x_0} + \frac{\partial^2\phi_2}{\partial t^2} - g\frac{\partial\psi_2}{\partial x_0} = \left(\frac{\pi}{T}\right)^2\left[\phi_1\frac{\partial\phi_1}{\partial x_0} + \psi_1\frac{\partial\psi_1}{\partial x_0}\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{T}\right)^2\cdot\frac{\partial}{\partial x_0}\left(\phi_1^2 + \psi_1^2\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{T}\right)^2\cdot\frac{\partial}{\partial x_0}\left[\left(\frac{\partial G}{\partial x_0}\right)^2 + \left(\frac{\partial G}{\partial z_0}\right)^2\right].$$

Similarly the second of equations (2 5) becomes

$$\frac{\partial\chi_2}{\partial z_0} + \frac{\partial^2\psi_2}{\partial t^2} - g\frac{\partial\psi_2}{\partial z_0} = \frac{1}{2}\cdot\left(\frac{\pi}{T}\right)^2\cdot\frac{\partial}{\partial z_0}\left[\left(\frac{\partial G}{\partial x_0}\right)^2 + \left(\frac{\partial G}{\partial z_0}\right)^2\right].$$

From these we have, after integration,

$$\chi_2 = g\psi_2 - \frac{\partial^2 F}{\partial t^2} + \frac{1}{2}\left(\frac{\pi}{T}\right)^2\left[\left(\frac{\partial G}{\partial x_0}\right)^2 + \left(\frac{\partial G}{\partial z_0}\right)^2\right] + f_2(t), \tag{2·18}$$

where $$\phi_2 = \frac{\partial F}{\partial x_0} + \nu(z_0).t \qquad (2\cdot19)$$

and $$\psi_2 = \frac{\partial F}{\partial z_0} \qquad (2\cdot20)$$

The earlier determination of ϕ_1 and ψ_1 introduced functions k and k' ; but $k'=0$ by a suitable choice of variables and experience is that ascending currents are usually either nil or very feeble, so that $\frac{\partial k}{\partial x_0}=0$ to higher than the second order. Further $\frac{\partial k}{\partial z_0} = \nu$, a function of z_0 only, so that $\nu(z_0)$ represents a horizontal current independent of the wave motion, variable with depth and of second order in h.

From the third of equations (2·5) we have, on using equations 2·19, 2·20, and 2·11,

$$\frac{\partial^2 F}{\partial x_0^2} + \frac{\partial^2 F}{\partial z_0^2} = \nabla^2 F = -\frac{\partial^2 G}{\partial x_0^2}\cdot\frac{\partial^2 G}{\partial z_0^2} + \frac{\partial^2 G}{\partial x_0 \partial z_0}\cdot\frac{\partial^2 G}{\partial z_0 \partial x_0}$$

i.e. $$\nabla^2 F = -\frac{\partial^2 G}{\partial x_0^2}\cdot\frac{\partial^2 G}{\partial z_0^2} + \left(\frac{\partial^2 G}{\partial x_0 \partial z_0}\right)^2.$$

$$\left.\begin{aligned} &\text{Hence} \quad F = \frac{1}{4}\left[\left(\frac{\partial G}{\partial x_0}\right)^2 + \left(\frac{\partial G}{\partial z_0}\right)^2\right] + G_2 \\ &\text{or} \quad F = \frac{1}{4}\left[\phi_1^2 + \psi_1^2\right] + G_2 \end{aligned}\right\} \qquad (2\cdot21)$$

where G_2 is a harmonic function of x_0, z_0 and periodic in t. The precise value of G_2 depends on that of $f_2(t)$ which in turn depends on the boundary conditions.
Hence equation (2·18) becomes

$$\chi_2 = g\psi_2 - \frac{1}{4}\cdot\frac{\partial^2}{\partial t^2}\left(\phi_1^2+\psi_1^2\right) - \frac{\partial^2 G_2}{\partial t^2} + \frac{1}{4}\left(\frac{\pi}{T}\right)^2\left(\phi_1^2+\psi_1^2\right) + f_2(t) \qquad (2\cdot22)$$

For a progessive wave the classical value of ϕ is (Lamb § 229)

$$\phi = -\frac{L}{T}\cdot\frac{h}{\sinh\frac{\pi H}{L}}\cdot\cosh\frac{\pi}{L}(H-z).\sin\pi\left(\frac{t}{T}-\frac{x}{L}\right)$$

$$= -\frac{bh}{a\sinh aH}\cdot\cosh a(H-z)\sin(bt-ax) \qquad (2\cdot23)$$

where $a = \frac{\pi}{L}$, $b = \frac{\pi}{T}$
and 2T is the period and 2L is the wavelenghh of the progessive wave and H is the depth at of the mass of water and h is the amplitude of the wave.

Two trains of progressive waves with the same characteristics but travelling in the opposite directions interfere to produce standing waves. Suppose the two trains to be defined by equation 2·23 and by

$$\phi = -\frac{bh}{a \sinh aH} \cdot \cosh a(H-z) \cdot \sin(bt+ax),$$

then the velocity potential of the standing wave train is

$$\phi = -\frac{bh \cosh a(H-z)}{a \sinh aH} \cdot \left[\sin(bt-ax) - \sin(bt+ax)\right]$$

$$= \frac{2hb}{a} \cdot \frac{\cosh a(H-z)}{\sinh aH} \cdot \sin ax \cdot \cos bt \qquad (2\cdot24).$$

Hence, after using equation 2·14 we have in Langrangian notation

$$G = \frac{2 \cosh a(H-z_0)}{a \sinh aH} \cdot \sin ax_0 \cdot \sin bt \qquad (2\cdot25)$$

From equation 2·11

$$\left.\begin{aligned} \phi_1 &= \frac{2 \cosh a(H-z_0)}{\sinh aH} \cdot \cos ax_0 \cdot \sin bt \\ \psi_1 &= -\frac{2 \sinh a(H-z_0)}{\sinh aH} \cdot \sin ax_0 \cdot \sin bt \end{aligned}\right\} \qquad (2\cdot26)$$

On substititing for ϕ_1 and ψ_1 in eqaation (2·21)

$$F = \frac{\sin^2 bt}{\sinh^2 aH}\left[\cosh^2 a(H-z_0) \cos^2 ax_0 + \sinh^2 a(H-z_0) \sin^2 ax_0\right] + G_2$$

$$= \frac{\sin^2 bt}{\sinh^2 aH}\left[\left(\frac{\cosh 2a(H-z_0)+1}{2}\right)\left(\frac{\cos 2ax_0+1}{2}\right) + \left(\frac{\cosh 2a(H-z_0)-1}{2}\right)\left(\frac{1-\cos 2ax_0}{2}\right)\right] + G_2$$

$$\therefore F = \frac{\sin^2 bt}{2\sinh^2 aH}\left[\cosh 2a(H-z_0) + \cos 2ax_0\right] + G_2 \qquad (2\cdot27)$$

Hence, after substituting in equations 2·19, 2·20

$$\left.\begin{aligned} \phi_2 &= -\frac{a\sin^2 bt.\sin 2ax_0}{\sinh^2 aH} + \frac{\partial G_2}{\partial x_0} + \nu(z_0).t \\ \psi_2 &= -\frac{a\sin^2 bt.\sinh 2a(H-z_0)}{\sinh^2 aH} + \frac{\partial G_2}{\partial z_0} \end{aligned}\right\} \qquad (2\cdot28)$$

From equation 2·12 we have

$$X_1 = -\frac{2g\sinh a(H-z_0)}{\sinh aH}.\sin ax_0.\sin bt$$

$$+ \frac{2b^2\cosh a(H-z_0)}{a\sinh aH}.\sin ax_0.\sin bt + f_1(t).$$

After reflexion at a barrier the horizontal currents will neutralise each other and for standing waves, near the barrier, we can write $\nu(z_0)=0$. Hence to the first order in h

$$\left.\begin{aligned} x &= x_0 + 2h\frac{\cosh a(H-z_0)}{\sinh aH}.\cos ax_0.\sin bt \\ z &= z_0 - 2h\frac{\sinh a(H-z_0)}{\sinh aH}.\sin ax_0.\sin bt \end{aligned}\right\} \qquad (2\cdot29)$$

$$\frac{p}{\rho g} = z_0 + 2h\frac{\sin ax_0\sin bt}{\sinh aH}\left[-\sinh aH + \frac{b^2}{ag}.\cosh a(H-z_0)\right]$$

$$+ \frac{h}{g}.f_1(t). \qquad (2\cdot30)$$

The pressure must be constant at the surface ($z_0 = 0$)

i.e. when $z_0 = 0$, $p = 0$

$\therefore f_1(t) = 0$ and $\sinh aH = \dfrac{b^2}{ag}\cosh aH$

$$\text{i.e. } \tanh aH = \frac{b^2}{ag} \qquad (2\cdot31)$$

After substituting for $f_1(t)$ and $\frac{b^2}{ag}$ in equation (2·30)

$$\frac{p}{\rho g} = z_0 + 2h\frac{\sin ax_0 \sin bt}{\sinh aH}\left[-\sinh a(H-z_0) + \tanh aH . \cosh a(H-z_0)\right]$$

$$= z_0 + 2h\frac{\sin ax_0 \sin bt}{\sinh aH \cosh aH}\left[-\sinh a(H-z_0)\cosh aH + \sinh aH \cosh a(H-z_0)\right]$$

$$\therefore \frac{p}{\rho g} = z_0 + \frac{2h \sin ax_0 \sin bt}{\sinh aH . \cosh aH} . \sinh az_0 \qquad (2·32)$$

From equations (2·26)

$$\phi_1^2 + \psi_1^2 = \frac{2\sin^2 bt}{\sinh^2 aH}\left[\cosh 2a(H-z_0) + \cos 2ax_0\right]$$

and $$\frac{\partial^2}{\partial t^2}(\phi_1^2+\psi_1^2) = \frac{4b^2 \cos 2bt}{\sinh^2 aH}\left[\cosh 2a(H-z_0) + \cos 2ax_0\right]$$

also $\frac{\pi}{T} = b$. Hence equation (2·22) gives

$$\begin{aligned} \chi_2 = &- \frac{ag \sinh 2a(H-z_0)}{\sinh^2 aH}\sin^2 bt \\ &- \frac{b^2 \cos 2bt}{\sinh^2 aH}\left[\cosh 2a(H-z_0) + \cos 2ax_0\right] \\ &+ \frac{b^2}{\sinh^2 aH}\left[\cosh 2a(H-z_0) + \cos 2ax_0\right]\sin^2 bt \\ &+ f_2(t) + g\frac{\partial G_2}{\partial z_0} - \frac{\partial^2 G_2}{\partial t^2} . \end{aligned} \qquad (2·33)$$

where $\nabla^2 G_2 = 0$ and $p = 0$ at $z_0 = 0$.

It is now necessary to evaluate G_2 and $f_2(t)$, after which equations 2·28 and 2·33 give the complete values of ϕ_2, ψ_2 & χ_2.

Let G_2 be taken as $\cosh 2a(H-z_0) . \cos 2ax_0 . T(bt)$

then
$$\frac{\partial G_2}{\partial z_0} = -2a \sinh 2a(H-z_0) . \cos 2ax_0 . T(bt)$$

and
$$\frac{\partial^2 G_2}{\partial t^2} = \cosh 2a(H-z_0) . \cos 2ax_0 . \frac{\partial^2 T}{\partial (bt)^2} . b^2$$
$$= \cosh 2a(H-z_0) . \cos 2ax_0 . \frac{\partial^2 T}{\partial (bt)^2} . ag \tanh aH$$

after using equation 2·31.

Substitute these values in χ_2 (equation 2·33) and ~~set~~ set $z_0=0$. Then using the fact that $p=0$ (and hence $\chi_2 = 0$) when $z_0=0$, we have

$$-\frac{ag \sinh 2aH}{\sinh^2 aH} \sin^2 bt - \frac{b^2 \cos 2bt}{\sinh^2 aH}(\cosh 2aH + \cos 2ax_0)$$
$$+\frac{b^2}{\sinh^2 aH}(\cosh 2aH + \cos 2ax_0)\sin^2 bt + f_2(t)$$
$$-2ag \sinh 2aH . \cos 2ax_0 . T(bt)$$
$$-\cosh 2aH . \cos 2ax_0 . \frac{\partial^2 T}{\partial (bt)^2} . ag \tanh aH = 0$$

for all values of x_0 and t.

Then since $b^2 = ag \tanh aH$

$$-ag . \frac{2\cosh aH}{\sinh aH} . \sin^2 bt - \frac{ag \cos 2bt}{\sinh aH . \cosh aH}(\cosh 2aH + \cos 2ax_0)$$

$$+\frac{ag}{\sinh aH . \cosh aH}(\cosh 2aH + \cos 2ax_0)\sin^2 bt + f_2(t)$$

$$-2ga \sinh 2aH . \cos 2ax_0 . T(bt) - \cosh 2aH . \cos 2ax_0 \frac{\partial^2 T}{\partial (bt)^2} . ag$$
$$\times \tanh aH = 0.$$

$$\therefore \sin^2 bt\left[-\frac{2\cosh aH}{\sinh aH}+\frac{3(\cosh 2aH+\cos 2ax_0)}{\sinh aH.\cosh aH}\right]$$

$$-\frac{\cosh 2aH+\cos 2ax_0}{\sinh aH \cosh aH}+\frac{f_2(t)}{ag}$$

$$-4\sinh aH \cosh aH.\cos 2ax_0 . T(bt)$$

$$-\cos 2ax_0 \cdot \frac{\partial^2 T}{\partial (bt)^2}\left[\sinh aH.\cosh aH+\frac{\sinh^3 aH}{\cosh aH}\right]=0$$

$$\therefore \sin^2 bt\left[-\frac{2\cosh aH}{\sinh aH}+\frac{3\cosh aH}{\sinh aH}+\frac{3\sinh aH}{\cosh aH}\right.$$

$$\left.+\frac{3\cos 2ax_0}{\sinh aH.\cosh aH}\right]-\frac{\cosh aH}{\sinh aH}-\frac{\sinh aH}{\cosh aH}$$

$$-\frac{\cos 2ax_0}{\sinh aH.\cosh aH}-4\sinh aH.\cosh aH.\cos 2ax_0 . T(bt)$$

$$-\left[\sinh aH.\cosh aH+\frac{\sinh^3 aH}{\cosh aH}\right]\cos 2ax_0 . \frac{\partial^2 T}{\partial (bt)^2}+\frac{f_2(t)}{ag}=0.$$

We now set $\quad T = A\cos 2bt + B$

and $\quad f_2 = ag\,(C\cos 2bt + D)$

where A, B, C and D are constants.

Then writing $\sin^2 bt = \frac{1}{2}(1-\cos 2bt)$,

we have

$$-\frac{1}{2}\cos 2bt\left[\frac{\cosh aH}{\sinh aH}+\frac{3\sinh aH}{\cosh aH}+\frac{3\cos 2ax_0}{\sinh aH\cosh aH}\right]$$

$$+\frac{1}{2}\left[\frac{\cosh aH}{\sinh aH}+\frac{3\sinh aH}{\cosh aH}+\frac{3\cos 2ax_0}{\sinh aH\cosh aH}\right]$$

$$-\frac{\cosh aH}{\sinh aH}-\frac{\sinh aH}{\cosh aH}-\frac{\cos 2ax_0}{\sinh aH\cosh aH}$$

$$-4A\sinh aH.\cosh aH.\cos 2ax_0.\cos 2bt$$

$$-4B\sinh aH.\cosh aH.\cos 2ax_0$$

$$+\left[\sinh aH.\cosh aH+\frac{\sinh^3 aH}{\cosh aH}\right]\cos 2ax_0.4A\cos 2bt$$

$$+C\cos 2bt+D=0$$

for all values of t and all x_0 .

$$\therefore \cos 2bt\left[\left(-\frac{\cosh^2 aH+3\sinh^2 aH}{\sinh aH\cosh aH}+2C\right)\right.$$

$$\left.+\left(8A\frac{\sinh^3 aH}{\cosh aH}-\frac{3}{\sinh aH\cosh aH}\right)\cos 2ax_0\right]$$

$$+\left[\frac{1}{\sinh aH\cosh aH}-8B\sinh aH.\cosh aH\right]\cos 2ax_0$$

$$+\left[\frac{-\cosh^2 aH+\sinh^2 aH}{\sinh aH\cosh aH}+2D\right]=0,$$

for all x_0 and t . Hence

$$A=\frac{3}{8}.\frac{1}{\sinh^4 aH},$$

$$B=\frac{1}{8}.\frac{1}{\sinh^2 aH.\cosh^2 aH}=\frac{1}{2\sinh^2 2aH},$$

$$C = \frac{\cosh^2 aH + 3\sinh^2 aH}{2\sinh aH \cosh aH},$$

$$D = \frac{\cosh^2 aH - \sinh^2 aH}{2\sinh aH \cosh aH} = \frac{1}{\sinh 2aH}.$$

Hence $\dfrac{f_2(t)}{ag} = \dfrac{1}{\sinh 2aH}\left[(\cosh^2 aH + 3\sinh^2 aH)\cos 2bt + 1\right]$

and $T(bt) = \dfrac{3}{8}\cdot\dfrac{\cos 2bt}{\sinh^4 aH} + \dfrac{1}{2\sinh^2 2aH}.$

$$\therefore G_2 = \left[\cosh 2a(H-z_0)\cos 2ax_0\right]\left[\frac{3\cos bt}{8\sinh^4 aH} + \frac{1}{2\sinh^2 2aH}\right]$$

$$\therefore G_2 = \frac{\cosh 2a(H-z_0).\cos 2ax_0}{8\sinh^4 aH}\left\{3\cos 2bt + \tanh^2 aH\right\}. \quad (2\cdot35)$$

Hence after substituting in (2•28) for

$$\phi_2 = -\frac{a\sin^2 bt\sin 2ax_0}{\sinh^2 aH} - \left(\frac{a\cosh 2a(H-z_0)\sin 2ax_0}{4\sinh^4 aH}\right)$$

$$\times\left(3\cos 2bt + \tanh^2 aH\right), \quad (2\cdot36)$$

$$\psi_2 = -\frac{a\sin^2 bt.\sinh 2a(H-z_0)}{\sinh^2 aH}$$

$$-\frac{a\sinh 2a(H-z_0)\cos 2ax_0}{4\sinh^4 aH}\left(3\cos 2bt + \tanh^2 aH\right) \quad (2\cdot36)$$

With the values of G_2 and $f_2(t)$ equation (2•33) gives

$$\chi_2 = -\frac{ag\sinh 2a(H-z_0)\sin^2 bt}{\sinh^2 aH}$$

$$-\frac{b^2\cos 2bt}{\sinh^2 aH}\left[\cosh 2a(H-z_0) + \cos 2ax_0\right]$$

$$+\frac{b^2}{\sinh^2 aH}\left[\cosh 2a(H-z_0) + \cos 2ax_0\right]\sin^2 bt$$

$$+\frac{ag}{\sinh aH}\left[(\cosh^2 aH + 3\sinh^2 aH)\cos 2bt + 1\right]$$

$$-\frac{ag\sinh 2a(H-z_0)\cos 2ax_0}{4\sinh^4 aH}\cdot\left(3\cos 2bt + \tanh^2 aH\right)$$

$$+\frac{3b^2\cosh 2a(H-z_0)\cos 2ax_0}{2\sinh^4 aH}\cdot\cos 2bt\,.$$

But $b^2 = ag\tanh aH,$

$$\therefore \frac{\chi_2}{g} = -\frac{\sinh 2a(H-z_0)\sin^2 bt}{\sinh^2 aH}$$

$$-\frac{\cos 2bt}{\sinh aH\cosh aH}\left[\cosh 2a(H-z_0) + \cos 2ax_0\right]$$

$$+\frac{1}{\sinh aH\cosh aH}\left[\cosh 2a(H-z_0) + \cos 2ax_0\right]\sin^2 bt$$

$$+\frac{1}{\sinh aH\cosh aH}\left[(\cosh^2 aH + 3\sinh^2 aH)\cos 2bt + 1\right]$$

$$-\frac{\sinh 2a(H-z_0)\cos 2ax_0}{4\sinh^4 aH}\cdot\left(3\cos 2bt + \tanh^2 aH\right)$$

$$+\frac{3\cosh 2a(H-z_0)\cdot\cos 2ax_0\cdot\cos 2bt}{2\sinh^3 aH\ \cosh aH}\,. \qquad (2\cdot 37)$$

The complete values of x, z and p to the second order in h are given by the following equations (2·38), (2·39) & (2·40).

$$x = x_0 + 2h\,\frac{\cosh a(H-z_0)}{\sinh aH}\,.\cos a x_0\,.\sin bt$$

$$-\frac{h^2 a}{\sinh^2 aH}\,.\sin 2ax_0\,.\sin^2 bt$$

$$-h^2 a\,.\frac{\cosh 2a(H-z_0)\sin 2ax_0}{4\sinh^4 aH}\left(3\cos 2bt + \tanh^2 aH\right)$$

$$\therefore x = x_0 + 2h\,\frac{\cosh a(H-z_0)\cos a x_0 \sin bt}{\sinh aH}$$

$$-\frac{h^2 a \sin 2ax_0}{\sinh^2 aH}\left[\sin^2 bt + \frac{\cosh 2a(H-z_0)}{4\sinh^2 aH}\left(3\cos 2bt + \tanh^2 aH\right)\right] \quad (2\cdot38).$$

$$z = z_0 - \frac{2h\sinh a(H-z_0)}{\sinh aH}\,.\sin a x_0\,.\sin bt$$

$$-\frac{h^2 a}{\sinh^2 aH}\,.\sinh 2a(H-z_0)\,.\sin^2 bt$$

$$-\frac{h^2 a \sinh 2a(H-z_0)\cos 2ax_0}{4\sinh^4 aH}\,.\left(3\cos 2bt + \tanh^2 aH\right).$$

$$\therefore z = z_0 - \frac{2h\sinh a(H-z_0)\sin a x_0 \sin bt}{\sinh aH}$$

$$-\frac{h^2 a\sinh 2a(H-z_0)}{\sinh^2 aH}\left[\sin^2 bt\right.$$

$$\left. + \frac{\cos 2ax_0}{4\sinh^4 aH}\left(3\cos 2bt + \tanh^2 aH\right)\right] \quad (2\cdot39).$$

and $$\frac{p}{\rho g} = Z_0 + \frac{2h \sin a x_0 \sinh a Z_0 \sin bt}{\sinh aH \ \cosh aH}$$

$$+ \frac{ah^2}{\sinh aH}\Bigg[- \frac{\sinh 2a(H-Z_0)\sin^2 bt}{\sinh aH}$$

$$+ \frac{\{\cosh 2a(H-Z_0) + \cos 2ax_0\}(3\sin^2 bt - 1)}{2\cosh aH}$$

$$- \frac{\sinh 2a(H-Z_0)\cos 2ax_0\,(3\cos 2bt + \tanh^2 aH)}{4\sinh^3 aH}$$

$$+ \frac{3\cosh 2a(H-Z_0)\cos 2ax_0 \,.\, \cos 2bt}{2\sinh^2 aH \,.\, \cosh aH}\Bigg] \tag{2·40}$$

The mean pressure on the bed over a wavelength is denoted by p_m,

where $$\frac{p_m}{\rho g} = \frac{1}{2L}\int_0^{2L}\left[\frac{p}{\rho g}\,dx\right]_{Z_0=H} = \frac{1}{2L}\int_0^{2L}\left[\frac{p}{\rho g}\cdot\frac{\partial x}{\partial x_0}\cdot dx_0\right]_{Z_0=H} \tag{2·41}$$

Putting $Z_0 = H$ in equation (2·40) we have

$$\left[\frac{p}{\rho g}\right]_{Z_0=H} = H + \frac{2h\sin ax_0 \sin bt}{\cosh aH} + \frac{ah^2}{\sinh aH}\Bigg[\frac{(1+\cos 2ax_0)(3\sin^2 bt - 1)}{\cosh aH}$$

$$+ \frac{(\cosh^2 aH + 3\sinh^2 aH)\cos 2bt + 1}{2\cosh aH}$$

$$+ \frac{3\cos 2ax_0 \ \cos 2bt}{2\sinh^2 aH \,.\, \cosh aH}\Bigg]$$

i.e. $$\left[\frac{p}{\rho g}\right]_{z_0=H} = A + B\sin ax_0 + C\cos 2ax_0$$

where

$$A = H + \frac{ah^2}{\sinh aH}\left[\frac{(3\sin^2 bt-1)}{\cosh aH} + \frac{(\cosh^2 aH + 3\sinh^2 aH)\cos 2bt + 1}{2\cosh aH}\right]$$

$$= H + \frac{ah^2\left[1+(2\sinh^2 aH - 1)\cos 2bt\right]}{\sinh aH\ \cosh aH},$$

$$B = \frac{2h\sin bt}{\cosh aH},$$

$$C = \frac{ah^2}{\sinh aH}\left[\frac{3\sin^2 bt - 1}{\cosh aH} + \frac{3\cos 2bt}{2\sinh^2 aH\cosh aH}\right]$$

$$= \frac{ah^2}{2\sinh aH\cosh aH}\left[1 - 3\cos 2bt + \frac{3\cos 2bt}{\sinh^2 aH}\right]$$

$$= \frac{ah^2}{\sinh 2aH}\left[1 + \frac{3(1-\sinh^2 aH)\cos 2bt}{\sinh^2 aH}\right]$$

Putting $Z_0 = H$ in equation 2·38 we obtain

$$x = x_0 + \frac{2h\cos ax_0\sin bt}{\sinh aH}$$

$$- \frac{h^2 a\sin 2ax_0}{\sinh^2 aH}\left[\sin^2 bt + \frac{(3\cos 2bt + \tanh^2 aH)}{4\sinh^2 aH}\right]$$

$$\therefore \left[\frac{\partial x}{\partial x_0}\right]_{z_0=H} = 1 - \frac{2ha\sin bt}{\sinh aH}.\sin ax_0$$

$$- \frac{2h^2 a}{\sinh^2 aH}\left[\sin^2 bt + \frac{3\cos 2bt + \tanh^2 aH}{4\sinh^2 aH}\right]\cos 2ax_0$$

i.e. $$\left[\frac{\partial x}{\partial x_0}\right]_{z_0=H} = 1 + D\sin a x_0 + E\cos 2a x_0$$

$$D = -\frac{2ha\sin bt}{\sinh aH}$$

$$E = -\frac{2ha^2}{\sinh^2 aH}\left[\sin^2 bt + \frac{3\cos 2bt + \tanh^2 aH}{4\sinh^2 aH}\right].$$

Hence $$\left[\frac{p}{\rho g}\cdot\frac{\partial x}{\partial x_0}\right]_{z_0=H}$$

$$= (A + B\sin a x_0 + C\cos 2a x_0)(1 + D\sin a x_0 + E\cos 2a x_0)$$

$$= A + (B + AD)\sin a x_0 + (C + AE)\cos 2a x_0 + BD\sin^2 a x_0$$

$$+ (DC + BE)\sin a x_0\cos 2a x_0 + CE\cos^2 2a x_0$$

$$= A + (B + AD)\sin a x_0 + (C + AE)\cos 2a x_0$$

$$+ \frac{BD}{2}(1 - \cos 2a x_0) + \frac{DC + BE}{2}(\sin 3a x_0 - \sin a x_0)$$

$$+ \frac{CE}{2}(1 + \cos 2a x_0)$$

$$= \left(A + \frac{BD}{2} + \frac{CE}{2}\right) + \left(B + AD - \frac{DC + BE}{2}\right)\sin a x_0$$

$$+ \left(C + AE - \frac{BD}{2} + \frac{CE}{2}\right)\cos 2a x_0 + \frac{DC + BE}{2}\sin 3a x_0.$$

$$\therefore \int_0^{2L}\left[\frac{p}{\rho g}\cdot\frac{\partial x}{\partial x_0}\right]_{z_0=H} dx_0 = \left[\frac{2A + BD + CE}{2}\right]2L,$$ since $a = \frac{\pi}{L}$.

Hence by equation (2·41)

$$\frac{p_m}{\rho g} = \frac{2A + BD + CE}{2}$$

$$BD = -\frac{2h\sin bt}{\cosh aH}\cdot\frac{2ha\sin bt}{\sinh aH} = -\frac{4h^2a\sin^2 bt}{\sinh aH.\cosh aH}$$

$CE = 0$ to the second order in h

$$\therefore \frac{p_m}{\rho g} = H + ah^2\left[1 + \frac{(2\sinh^2 aH - 1)\cos 2bt}{\sinh aH.\cosh aH} - \frac{2\sin^2 bt}{\sinh aH\cosh aH}\right]$$

$$= H + \frac{ah^2}{\sinh aH\cosh aH}\left[1 + (2\sinh^2 aH - 1)\cos 2bt - 1 + \cos 2bt\right]$$

$$\therefore \frac{p_m}{\rho g} = H + \frac{ah^2}{\sinh aH\cosh aH}\left[(2\sinh^2 aH - 1 + 1)\cos 2bt\right]$$

$$= H + \frac{2ah^2\sinh^2 aH}{\sinh aH\cosh aH}.\cos 2bt$$

$$\therefore p_m = \rho g\left[H + 2ah^2\tanh aH.\cos 2bt\right] \qquad (2\cdot42)$$

But $\rho g H$ is the static pressure at the depth H.
Thus we see that the mean pressure on the bed, under a standing wave, has a variable part. Since the frequency of the standing wave is $b/2\pi$ we see that the variation in the mean pressure has a frequency ($\frac{b}{\pi}$), twice that of the standing wave.

Since $b^2 = ag\tanh aH$

$$p_m = \rho g H + 2\rho h^2 b^2\cos 2bt. \qquad (2\cdot43)$$

The amplitude of the pressure variation is proportional to the square of the wave amplitude, and is independent of the depth.

CHAPTER 3.

Evaluation of the mean pressure beneath a given mass of moving fluid.

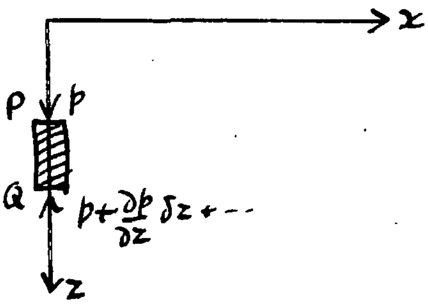
In chapter 2 we considered a periodic, irrotational motion in an incompressible fluid, and demonstrated, by actual evaluation, that there is a variation of the mean pressure beneath a standing wave, and that this variation in the mean pressure arises from the second order terms and is independent of the depth.

In the present chapter, we shall not assume that the motion is either periodic or irrotational. But assuming only that the mass of the fluid remains constant, we shall derive an expression for the pressure at any point. Then by making the motion periodic we shall find the mean pressure over a wavelength.

Consider a quantity of fluid of mass M; the fluid being supposed incompressible. Let us use the Langrangian system of coordinates, x being measured horizontally in the direction of the motion and z vertically downwards.

Then the coordinates of a particular particle of fluid at time t are (x, z), and the coordinates of this particle at an arbitrary time, $t=0$, are (x_0, z_0). The pressure at the point P (x, z) is p and the pressure at $Q(x, z+\delta z)$ is $p+\frac{\partial p}{\partial z}\delta z + \cdots$

The equation of motion of the fluid in the small volume PQ is approximately



$$p-\left(p+\frac{\partial p}{\partial z}\delta z\right)+g\rho\delta z=\rho\delta z.\frac{\partial^2 z}{\partial t^2}$$

$$\therefore\ -\frac{\partial p}{\partial z}\delta z+g\rho\delta z=\rho\delta z.\frac{\partial^2 z}{\partial t^2}$$

Hence the equation of motion of the particle at P is

$$\frac{\partial p}{\partial z}-g\rho=-\rho\frac{\partial^2 z}{\partial t^2} \tag{3.1}$$

The equation of continuity is

$$\rho x dz=\rho_0 dx_0 dz_0, \tag{3.2}$$

where ρ_0 is the density at time $t=0$

$$\therefore \int_M \rho\frac{\partial^2 z}{\partial t^2}dx\,dz=\int_M \rho_0\frac{\partial^2 z}{\partial t^2}dx_0\,dz_0$$

$$=\frac{\partial^2}{\partial t^2}\int_M \rho_0 z\,dx_0\,dz_0$$ since x_0, z_0 are independent of t

i.e. by equation (3·2)

$$\int_M \rho \frac{\partial^2 z}{\partial t^2}\, dx\, dz = \frac{\partial^2}{\partial t^2}\int_M \rho z\, dx\, dz \qquad (3\cdot3)$$

Integrating equation (3·1) over the whole fluid M,

$$\int_M \frac{\partial p}{\partial z}\, dx\, dz - \int_M g\rho\, dx\, dz = -\int_M \rho \frac{\partial^2 z}{\partial t^2}\, dx\, dz$$

$$= -\frac{\partial^2}{\partial t^2}\int_M \rho z\, dx\, dz. \qquad (3\cdot4)$$

In evaluating the integrals of equation (3·4) it is convenient to regard x, z and t as independent variables, rather than x and z as functions of t , and the boundaries of M must in consequence be functions of t.

At the time $t = 0$, let us suppose that the mass M of incompressible fluid is contained by the free surface $z = J$, the horizontal plane $z = z'$ and two vertical planes $x = x_1$ and $x = x_2$. At this instant we denote the pressure in the plane $z = z'$ by p_1 and the constant pressure at the free surface by p_s .

Then at this instant $t = 0$

$$\int_M \frac{\partial p}{\partial z}\, dx\, dz = \int_{x_1}^{x_2} (p' - p_s)\, dx$$

$$= (x_2 - x_1) \times (\text{Mean Value of } p' - p_s)$$

$$= (x_2 - x_1)(\bar{p}' - p_s) \qquad (3\cdot5)$$

where $\bar{p}'$ is the mean value of over the interval $x_1 \leq x \leq x_2$

To evaluate the second integral of equation (3·4)

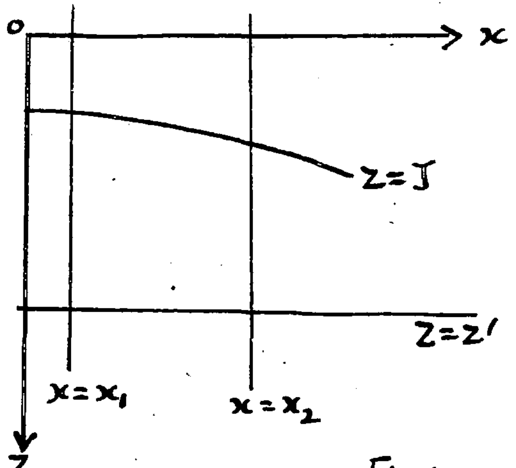
$$\int_M g\rho\, dx\, dz = g\rho\int_M dx\, dz$$

(since the fluid is incompressible)

$$= g\rho\int_{x_1}^{x_2} (z' - J)\, dx$$

$$= g\rho z'\int_{x_1}^{x_2} dx - g\rho\int_{x_1}^{x_2} J\, dx$$

$$= g\rho z'(x_2 - x_1) - g\rho\int_{x_1}^{x_2} J\, dx \qquad (3\cdot6)$$



t=0. Fig. 1

To evaluate the third integral of equation (3·4) it is necessary to find an expression for the integral at times other than $t=0$, and then to set $t=0$ in this expression.

Suppose that at time t the fluid M is bounded by the surfaces

$$z = \mathcal{J}(x,t) \qquad \text{i.e. } A_1 A_2$$
$$z = z' + \mathcal{J}'(x,t) \qquad \text{i.e. } A_1' A_2'$$
$$x = \xi_1(z,t) \qquad \text{i.e. } A_1 A_1'$$
and
$$x = \xi_2(z,t) \qquad \text{i.e. } A_2 A_2'$$

Initially when $t=0$,

$$\mathcal{J}'(x,0) = 0, \quad \xi_1(z,0) = x_1, \quad \xi_2(z,0) = x_2 \tag{3·7}$$

The intersection of $z = \mathcal{J}$, $x = \xi_1(x,t)$ is $A_1(\alpha_1, \gamma_1)$,

The intersection of $z = \mathcal{J}$, $x = \xi_2(x,t)$ is $A_2(\alpha_2, \gamma_2)$,

The interxection of $z = z' + \mathcal{J}'(x,t)$, $x = \xi_1(x,t)$ is $A_1'(\alpha_1', \gamma_1')$,

The intersection of $z = z' + \mathcal{J}'(x,t)$, $x = \xi_2(x,t)$ is $A_2'(\alpha_2', \gamma_2')$.

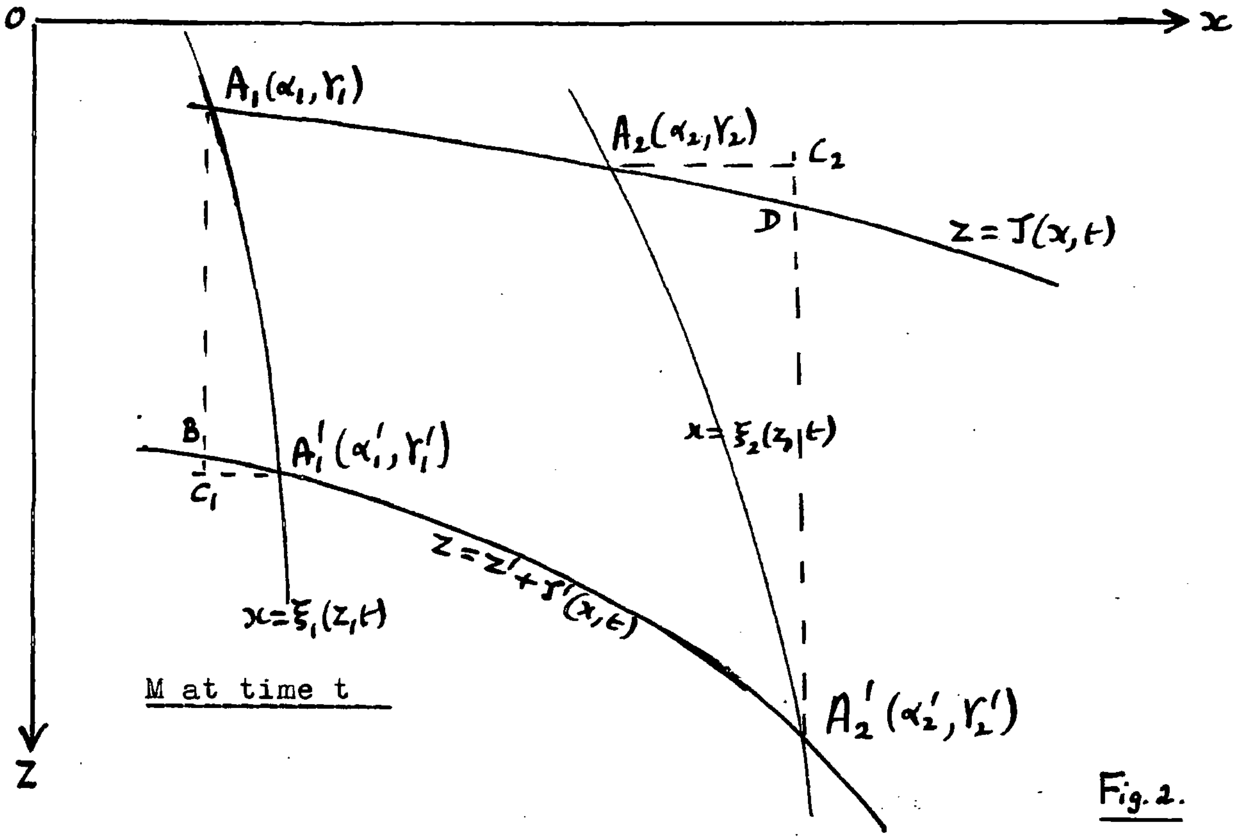


Fig. 2.

Since the mass does not change,

$$\iint_A dx\,dz = A_1 A_1' A_2' A_2 A_1$$

$$= A_1 B A_2' D - C_2 A_2 A_2' C_2 + A_2 D C_2 A_2$$

$$- A_1 C_1 A_1' A_1 + B C_1 A_1' B$$

$$= \int_{\alpha_1}^{\alpha_2'} [(z'+J') - J]\,dx - \int_{\gamma_2}^{\gamma_2'} (\alpha_2' - \xi_2)\,dz$$

$$+ \int_{\alpha_2}^{\alpha_2'} (J - \gamma_2)\,dx - \int_{\gamma_1}^{\gamma_1'} (\xi_1 - \alpha_1)\,dz$$

$$+ \int_{\alpha_1}^{\alpha_1'} \left[\gamma_1' - (z' + J') \right] dx$$

This method of evaluating $\iint$ dx dz over the area A of the mass M at a time t leads to the following expression for $\iint$ z dx dz over the same area:

$$\iint_A z\,dx\,dz = \tfrac{1}{2}\int_{\alpha_1}^{\alpha_2'} \left[(z'+J')^2 - J^2 \right] dx - \int_{\gamma_2}^{\gamma_2'} (\alpha_2' - \xi_2) z\,dz$$

$$+ \tfrac{1}{2}\int_{\alpha_2}^{\alpha_2'} (J^2 - \gamma_2^2)\,dx - \int_{\gamma_1}^{\gamma_1'} (\xi_1 - \alpha_1) z\,dz$$

$$+ \tfrac{1}{2}\int_{\alpha_1}^{\alpha_1'} \left[\gamma_1'^2 - (z' + J')^2 \right] dx$$

$$= \tfrac{1}{2}\int_{\alpha_1'}^{\alpha_2'} (z'+J')^2\,dx - \tfrac{1}{2}\int_{\alpha_1}^{\alpha_2} J^2\,dx + \int_{\gamma_2}^{\gamma_2'} \xi_2 z\,dz$$

$$- \int_{\gamma_1}^{\gamma_1'} \xi_1 z\,dz - \frac{\alpha_2'^2}{2}(\gamma_2'^2 - \gamma_2^2) - \frac{\gamma_2^2}{2}(\alpha_2' - \alpha_2)$$

$$+ \frac{\alpha_1}{2}(\gamma_1'^2 - \gamma_1^2) + \frac{\gamma_1'^2}{2}(\alpha_1' - \alpha_1)$$

$$\iint_A z\,dx\,dz = \tfrac{1}{2}\int_{\alpha_1'}^{\alpha_2'} (z'+\zeta')^2\,dx - \tfrac{1}{2}\int_{\alpha_1}^{\alpha_2} \zeta^2\,dx + \int_{\gamma_2}^{\gamma_2'} \xi_2 z\,dz$$
$$- \int_{\gamma_1}^{\gamma_1'} \xi_1 z\,dz - \tfrac{1}{2}\left[\alpha_1\gamma_1^2 + \alpha_2'\gamma_2'^2 - \alpha_2\gamma_2^2 - \alpha_1\gamma_1'^2\right].$$

$$\therefore \int_M z\,dx\,dz = \int_{\alpha_1'}^{\alpha_2'} \tfrac{1}{2}(z'+\zeta')^2\,dx - \int_{\alpha_1}^{\alpha_2} \tfrac{1}{2}\zeta^2\,dx$$
$$+ \int_{\gamma_2}^{\gamma_2'} \xi_2 z\,dz - \int_{\gamma_1}^{\gamma_1'} \xi_1 z\,dz$$
$$-\tfrac{1}{2}\left[\alpha_2'\gamma_2'^2 - \alpha_1'\gamma_1'^2 - \alpha_2\gamma_2^2 + \alpha_1\gamma_1^2\right] \qquad (3\cdot 8)$$

In order to find $\dfrac{\partial^2}{\partial t^2}\displaystyle\int_M z\,dx\,dz$ it is necessary to apply the fo.lowing theorem twice to each term on the right hand side of equation (3·8) : If $f(x,t)$ is a continuous function of both variables x and t, and x varies between x_0 and x_1 and t between t_0 and t_1, and if x_0 and x_1 are functions of t

$$F(t) = \int_{x_0}^{x_1} f(x,t)\,dx$$

and
$$\frac{\partial F}{\partial t} = \int_{x_0}^{x_1} \frac{\partial f}{\partial t}\,dx + \frac{\partial x_1}{\partial t}\cdot f(x_1,t) - \frac{\partial x_0}{\partial t}\cdot f(x_0,t)$$

(Goursat : Cours d'Analyse section 97)

Hence if a dot denotes a partial differentiation with respect to t :

$$\frac{\partial}{\partial t}\int_M z\,dx\,dz$$

$$= \int_{\alpha_1'}^{\alpha_2'} \frac{\partial}{\partial t}\cdot\tfrac{1}{2}(z'+J')^2 dx + \tfrac{1}{2}\dot{\alpha}_2'\gamma_2'^2 - \tfrac{1}{2}\dot{\alpha}_1'\gamma_1'^2$$

$$-\int_{\alpha_1}^{\alpha_2} \frac{\partial}{\partial t}\cdot\tfrac{1}{2}J^2 dx - \tfrac{1}{2}\dot{\alpha}_2\gamma_2^2 + \tfrac{1}{2}\dot{\alpha}_1\gamma_1^2$$

$$+\int_{\gamma_2}^{\gamma_2'} \frac{\partial\xi_2}{\partial t}\,z\,dz + \dot{\gamma}_2'\gamma_2'\alpha_2' - \dot{\gamma}_2\gamma_2\alpha_2$$ [since z is independent of t]

$$-\int_{\gamma_1}^{\gamma_1'} \frac{\partial\xi_1}{\partial t}\,z\,dz - \dot{\gamma}_1'\gamma_1'\alpha_1' + \dot{\gamma}_1\gamma_1\alpha_1$$

$$-\tfrac{1}{2}\Big[\dot{\alpha}_2'\gamma_2'^2 + 2\alpha_2'\dot{\gamma}_2'\gamma_2' - \dot{\alpha}_1'\gamma_1'^2 - 2\alpha_1'\dot{\gamma}_1'\gamma_1' - \dot{\alpha}_2\gamma_2^2$$
$$- 2\alpha_2\dot{\gamma}_2\gamma_2 + \dot{\alpha}_1\gamma_1^2 + 2\alpha_1\dot{\gamma}_1\gamma_1\Big]$$

$$= \int_{\alpha_1'}^{\alpha_2'} \frac{\partial}{\partial t}\cdot\tfrac{1}{2}(z'+J')^2 dz - \int_{\alpha_1}^{\alpha_2}\frac{\partial}{\partial t}\left(\tfrac{1}{2}J^2\right)dx$$

$$+\int_{\gamma_2}^{\gamma_2'} \dot{\xi}_2\,z\,dz - \int_{\gamma_1}^{\gamma_1'} \dot{\xi}_1\,z\,dz$$

$$\therefore \frac{\partial^2}{\partial t^2}\int_M z\,dx\,dz$$

$$= \int_{\alpha_1'}^{\alpha_2'} \frac{\partial^2}{\partial t^2}\cdot\tfrac{1}{2}(z'+J')^2 dx + \dot{\alpha}_2'\cdot\frac{\partial}{\partial t}\left(\tfrac{1}{2}\gamma_2'^2\right) - \dot{\alpha}_2'\cdot\frac{\partial}{\partial t}\left(\tfrac{1}{2}\gamma_1'^2\right)$$

$$-\int_{\alpha_1}^{\alpha_2} \frac{\partial^2}{\partial t^2}\cdot\tfrac{1}{2}J^2 dx - \dot{\alpha}_2\cdot\frac{\partial}{\partial t}\left(\tfrac{1}{2}\gamma_2^2\right) + \dot{\alpha}_1\frac{\partial}{\partial t}\left(\tfrac{1}{2}\gamma_1^2\right)$$

$$+\int_{\gamma_2}^{\gamma_2'} \ddot{\xi}_2\,z\,dz + \dot{\gamma}_2'\dot{\alpha}_2'\gamma_2' - \dot{\gamma}_2\dot{\alpha}_2\gamma_2' - \int_{\gamma_1}^{\gamma_1'}\ddot{\xi}_1\,z\,dz$$
$$- \dot{\gamma}_1'\dot{\alpha}_1'\gamma_1' + \dot{\gamma}_1\dot{\alpha}_1\gamma_1$$

$$\therefore \frac{\partial^2}{\partial t^2}\int_M z\,dx\,dz$$

$$= \int_{\alpha_1'}^{\alpha_2'} \frac{\partial^2}{\partial t^2}\cdot\tfrac{1}{2}(z'+\mathcal{J}')^2 dx - \int_{\alpha_1}^{\alpha_2}\frac{\partial^2}{\partial t^2}\cdot\tfrac{1}{2}\mathcal{J}^2 dx$$

$$+\int_{\gamma_2}^{\gamma_2'} \ddot{\xi}_2\, z\,dz - \int_{\gamma_1}^{\gamma_1'}\ddot{\xi}_1\, z\,dz$$

$$+\dot{\alpha}_2'\dot{\gamma}_2'\gamma_2' - \dot{\alpha}_1'\dot{\gamma}_1'\gamma_1' - \dot{\alpha}_2\dot{\gamma}_2\gamma_2 + \dot{\alpha}_1\dot{\gamma}_1\gamma_1$$

$$+\dot{\alpha}_2'\dot{\gamma}_2'\gamma_2' - \dot{\alpha}_2\dot{\gamma}_2\gamma_2 - \dot{\alpha}_1'\dot{\gamma}_1'\gamma_1' + \dot{\alpha}_1\dot{\gamma}_1\gamma_1 .$$

$$= \int_{\alpha_1'}^{\alpha_2'} \frac{\partial^2}{\partial t^2}\cdot\tfrac{1}{2}(z'+\mathcal{J}')^2 dx - \int_{\alpha_1}^{\alpha_2}\frac{\partial^2}{\partial t^2}\cdot\tfrac{1}{2}\mathcal{J}^2 dx$$

$$+\int_{\gamma_2}^{\gamma_2'} \ddot{\xi}_2\, z\,dz - \int_{\gamma_1}^{\gamma_1'}\ddot{\xi}_1\, z\,dz$$

$$+2\left[\dot{\alpha}_2'\dot{\gamma}_2'\gamma_2' - \dot{\alpha}_1'\dot{\gamma}_1'\gamma_1' - \dot{\alpha}_2\dot{\gamma}_2\gamma_2 + \dot{\alpha}_1\dot{\gamma}_1\gamma_1\right] \quad (3\cdot9)$$

We need now to return to the evaluation of the third term of equation (3·4) i.e. equation (3·9) to the initial instant.
Initially $t=0$, $\alpha_1=\alpha_1'=x_1$, $\alpha_2=\alpha_2'=x_2$, and $\gamma_1'=\gamma_2'=z'$.
Suppose that when $x=x_1$, $\mathcal{J}=\mathcal{J}_1$
and when $x=x_2$, $\mathcal{J}=\mathcal{J}_2$
Then when $t=0$, equation (3·9) becomes

$$\frac{\partial^2}{\partial t^2}\int_M z\,dx\,dz = \int_{x_1}^{x_2}\left[\frac{\partial^2}{\partial t^2}\cdot\tfrac{1}{2}(z'+\mathcal{J}')^2 - \frac{\partial^2}{\partial t^2}\cdot\tfrac{1}{2}\mathcal{J}^2\right]dx$$

$$+\int_{\mathcal{J}_2}^{z'}\ddot{\xi}_2\, z\,dz - \int_{\mathcal{J}_1}^{z'}\ddot{\xi}_1\, z\,dz$$

$$+2\left[\dot{\alpha}_2'\dot{\gamma}_2'\gamma_2' - \dot{\alpha}_1'\dot{\gamma}_1'\gamma_1' - \dot{\alpha}_2\dot{\gamma}_2\gamma_2 + \dot{\alpha}_1\dot{\gamma}_1\gamma_1\right]$$

$$\frac{\partial^2}{\partial t^2}\int_M z\,dx\,dz = \int_{x_1}^{x_2}\frac{\partial^2}{\partial t^2}\left[\tfrac{1}{2}(z'^2+2z'\zeta'+\zeta'^2-\zeta^2)\right]dx$$

$$+\int_{\varrho_2}^{z'}\ddot{\xi}_2\,z\,dz-\int_{\zeta_1}^{z'}\ddot{\xi}_1\,z\,dz$$

$$+2\left[\dot{\alpha}_2'\dot{\gamma}_2'\gamma_2'-\dot{\alpha}_1'\dot{\gamma}_1'\gamma_1'-\dot{\alpha}_2\dot{\gamma}_2\gamma_2+\dot{\alpha}_1\dot{\gamma}_1\gamma_1\right]$$

But z' is independent of t, hence

$$\frac{\partial^2}{\partial t^2}\int_M z\,dx\,dz$$

$$=\int_{x_1}^{x_2}\left[\frac{\partial^2}{\partial t^2}\cdot\frac{1}{2}\left(\zeta'^2-\zeta^2\right)+z'\ddot{\zeta}'\right]dx+\int_{\varrho_2}^{z'}\ddot{\xi}_2\,z\,dz$$

$$-\int_{\varrho_1}^{z'}\ddot{\xi}_1\,z\,dz+2\left[\dot{\alpha}_2'\dot{\gamma}_2'\gamma_2-\dot{\alpha}_1'\dot{\gamma}_1'\gamma_1'-\dot{\alpha}_2\dot{\gamma}_2\gamma_2+\dot{\alpha}_1\dot{\gamma}_1\gamma_1\right] \quad (3\cdot10)$$

After substituting from equations (3·5), (3·6) and (3·10) equation (3·4) becomes

$$\frac{\overline{p}'-p_s}{\rho}-g z'=$$

$$\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2-\tfrac{1}{2}\zeta'^2\right)-z'\ddot{\zeta}'-g\zeta\right]dx$$

$$-\frac{1}{x_2-x_1}\int_{\varrho_2}^{z'}\ddot{\xi}_2\,z\,dz+\frac{1}{x_2-x_1}\int_{\varrho_1}^{z'}\ddot{\xi}_1\,z\,dz$$

$$-\frac{2}{x_2-x_1}\left[\dot{\alpha}_2'\dot{\gamma}_2'\gamma_2'-\dot{\alpha}_1'\dot{\gamma}_1'\gamma_1'-\dot{\alpha}_2\dot{\gamma}_2\gamma_2+\dot{\alpha}_1\dot{\gamma}_1\gamma_1\right]$$

Hence $\dfrac{\overline{p}' - p_s}{\rho} - gz'$

$$= \frac{1}{x_2 - x_1}\int_{x_1}^{x_2}\left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2 - \tfrac{1}{2}\zeta'^2\right) - z'\ddot{\zeta}' - g\zeta\right]dx$$
$$- \frac{1}{x_2 - x_1}\left[\int_{\zeta_2}^{z'}\ddot{\xi}_2 z\,dz - \int_{\zeta_1}^{z'}\ddot{\xi}_1 z\,dz\right]$$
$$- \frac{2}{x_2 - x_1}\left[\dot{\alpha}_2'\dot{\gamma}_2'\gamma_2' - \dot{\alpha}_1'\dot{\gamma}_1'\gamma_1' - \dot{\alpha}_2\dot{\gamma}_2\gamma_2 + \dot{\alpha}_1\dot{\gamma}_1\gamma_1\right] \quad (3\cdot11)$$

Equation (3·11) expresses the mean pressure on the plane $z = z'$ at the initial instant. It is desirable to transform this into a form which expresses this mean pressure independent of the initial instant chosen.

Let (u', w') denote the velocity components in the plane $z = z'$ at the initial instant,

then $\dfrac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta'^2\right) = \dfrac{\partial}{\partial t}\left(\zeta'\dot{\zeta}'\right) = \zeta'\ddot{\zeta}' + \dot{\zeta}'^2$

When $t = 0$: $\dfrac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta'^2\right) = \omega'^2$ $\quad (3\cdot12)$

Since $z = z' + \zeta'$ is the equation of a surface moving with the fluid we must have

$$\frac{D}{Dt}(z - z' - \zeta') = 0, \quad (3\cdot13)$$

$$\frac{D^2}{Dt^2}(z - z' - \zeta') = 0, \quad (3\cdot14)$$

where $\dfrac{D}{Dt} \equiv \dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x} + \omega\dfrac{\partial}{\partial z}$

$$\frac{D^2}{Dt^2} \equiv \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial z}\right)$$
$$= \frac{\partial^2}{\partial t^2} + u^2\frac{\partial^2}{\partial x^2} + \omega^2\frac{\partial^2}{\partial z^2} + 2u\frac{\partial^2}{\partial x\partial t} + 2\omega\frac{\partial^2}{\partial z\partial t}$$
$$+ 2u\omega\frac{\partial^2}{\partial x\partial z} + \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \omega\frac{\partial u}{\partial z}\right)\frac{\partial}{\partial x}$$
$$+ \left(\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + \omega\frac{\partial \omega}{\partial z}\right)\cdot\frac{\partial}{\partial z}$$

From (3·13) we have
$$\frac{\partial \zeta'}{\partial t} + u\frac{\partial \zeta'}{\partial x} - \omega = 0$$

From (3·14) we have
$$\frac{\partial^2 \zeta'}{\partial t^2} + u^2\frac{\partial^2 \zeta'}{\partial x^2} + 2u\frac{\partial^2 \zeta'}{\partial x \partial t} + \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \omega\frac{\partial u}{\partial z}\right)\frac{\partial \zeta'}{\partial x}$$
$$-\left(\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + \omega\frac{\partial \omega}{\partial z}\right) = 0 \qquad (3\cdot15)$$

At the initial instant $\zeta' = 0$ and $\left[\frac{\partial \zeta'}{\partial x}\right]_{t=0} = 0$

hence $\frac{\partial \zeta'}{\partial t} - \omega' = 0$

and $\therefore \frac{\partial^2 \zeta'}{\partial x \partial t} - \frac{\partial \omega'}{\partial x} = 0$

From (3·15) when $t = 0$
$$\frac{\partial^2 \zeta'}{\partial t^2} + 2u'\frac{\partial^2 \zeta'}{\partial x \partial t} - \left(\frac{\partial \omega'}{\partial t} + u'\frac{\partial \omega'}{\partial x} + \omega'\frac{\partial \omega'}{\partial z}\right) = 0$$

But by the equation of continuity $\frac{\partial u'}{\partial x} + \frac{\partial \omega'}{\partial z} = 0$

$$\therefore \frac{\partial^2 \zeta'}{\partial t^2} + 2u'\frac{\partial \omega'}{\partial x} - \left(\frac{\partial \omega'}{\partial t} + u'\frac{\partial \omega'}{\partial x} - \omega'\frac{\partial u'}{\partial x}\right) = 0$$

since
$$2u'\frac{\partial^2 \zeta}{\partial x \partial t} = 2u'\frac{\partial}{\partial x}\left(\frac{\partial \zeta'}{\partial t}\right) = 2u'\frac{\partial \omega'}{\partial x}$$

Hence
$$\frac{\partial^2 \zeta'}{\partial t^2} + u'\frac{\partial \omega'}{\partial x} + \omega'\frac{\partial u'}{\partial x} - \frac{\partial \omega'}{\partial t} = 0$$

$$\therefore \frac{\partial^2 \zeta'}{\partial t^2} = \frac{\partial \omega'}{\partial t} - \frac{\partial}{\partial x}(u', \omega')$$

or
$$\ddot{\zeta}' = \dot{\omega}' - \frac{\partial}{\partial x}(u', \omega') \qquad (3\cdot16)$$

Since $x = \xi_i$ $(i = 1, 2)$ is the equation of a surface moving with the fluid we must have

$$\frac{D}{Dt}(x - \xi_i) = 0 \quad , \qquad \frac{D^2}{Dt^2}(x - \xi_i) = 0$$

Hence $u - \dfrac{\partial \xi_i}{\partial t} - w \dfrac{\partial \xi_i}{\partial z} = 0,$

and $-\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z}\right) + \dfrac{\partial^2 \xi_i}{\partial t^2} + w^2 \dfrac{\partial^2 \xi_i}{\partial z^2} + 2w\dfrac{\partial^2 \xi_i}{\partial z \partial t}$

$$+ \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right)\frac{\partial \xi_i}{\partial z} = 0$$

Initially $\xi_i = x_i$ $(i = 1, 2)$

$$\therefore \left[\frac{\partial \xi_i}{\partial z}\right]_{t=0} = 0$$

Hence $\quad u_i = \dfrac{\partial \xi_i}{\partial t}$

$$\therefore \frac{\partial u_i}{\partial z} = \frac{\partial^2 \xi_i}{\partial z \partial t}$$

Hence initially, when $t = 0$, we have

$$-\left(\frac{\partial u_i}{\partial t} + u_i\frac{\partial u_i}{\partial x} + w_i\frac{\partial u_i}{\partial z}\right) + \ddot{\xi}_i + 2w_i\frac{\partial u_i}{\partial z} = 0$$

$$\therefore \ddot{\xi}_i = \frac{\partial u_i}{\partial t} + u_i\frac{\partial u_i}{\partial x} - w_i\frac{\partial u_i}{\partial z}$$

But by the equation of continuity $\quad \dfrac{\partial u_i}{\partial x} + \dfrac{\partial w_i}{\partial z} = 0$

Hence $\quad \ddot{\xi}_i = \dot{u}_i - u_i\dfrac{\partial w_i}{\partial z} - w_i\dfrac{\partial u_i}{\partial z}$

$$= \dot{u}_i - \frac{\partial (u_i w_i)}{\partial z}$$

i.e. $\xi_i = \dot{u}_i - \frac{\partial}{\partial z}(u_i \omega_i)$ $\quad (i=1,2) \quad$ (3•17)

Substituting from equations (3•12), (3•16) and (3•17) into equation (3•11) we have

$$\frac{\bar{p}' - p_s}{\rho} - gz'$$

$$= \frac{1}{x_2 - x_1}\int_{x_1}^{x_2}\left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2\right) - \omega'^2 - z'\left\{\dot{\omega}' - \frac{\partial}{\partial x}(u'\omega')\right\} - g\zeta\right]dx$$

$$- \frac{1}{x_2 - x_1}\left[\int_{\zeta_2}^{z'}\left\{\dot{u}_2 - \frac{\partial}{\partial z}(u_2\omega_2)\right\}z\,dz - \int_{\zeta_1}^{z'}\left\{\dot{u}_1 - \frac{\partial}{\partial z}(u_1\omega_1)\right\}z\,dz\right]$$

$$- \frac{2}{x_2 - x_1}\left[\dot{\alpha}_2'\dot{\gamma}_2'\gamma_2' - \dot{\alpha}_1'\dot{\gamma}_1'\gamma_1' - \dot{\alpha}_2\dot{\gamma}_2\gamma_2 + \dot{\alpha}_1\dot{\gamma}_1\gamma_1\right]$$

$$= \frac{1}{x_2 - x_1}\int_{x_1}^{x_2}\left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2\right) - \omega'^2 - \dot{z}\dot{\omega}' - g\zeta\right]dx$$

$$+ \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} z'\frac{\partial}{\partial x}(u'\omega')\,dx$$

$$- \frac{1}{x_2 - x_1}\int_{\zeta}^{z'}\left[z\dot{u} - z\frac{\partial}{\partial z}(u\omega)\right]_{x_1}^{x_2}dz$$

$$- \frac{2}{x_2 - x_1}\left[\dot{\alpha}_2'\dot{\gamma}_2'\gamma_2' - \dot{\alpha}_1'\dot{\gamma}_1'\gamma_1' - \dot{\alpha}_2\dot{\gamma}_2\gamma_2 + \dot{\alpha}_1\dot{\gamma}_1\gamma_1\right]$$

But $$\left[\int_{\zeta}^{z'} z\frac{\partial}{\partial z}(u\omega)\,dz\right]_{x_1}^{x_2} = \left[\left\{z(u\omega)\right\}_{\zeta}^{z_1} - \int_{\zeta}^{z_1} u\omega\,dz\right]_{x_1}^{x_2}$$

HENCE $$\frac{\overline{p_1 - p_s}}{\rho} - g z'$$

$$= \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2\right) - \omega'^2 - z'\dot{\omega}' - g\zeta \right] dx$$

$$- \frac{1}{x_2 - x_1} \left[\int_{\zeta}^{z'} (z\dot{u} + u\omega)\, dz \right]_{x_1}^{x_2} + \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} z' \frac{\partial}{\partial x}(u'\omega')\, dx$$

$$- \frac{1}{x_2 - x_1} \left[(z u \omega)_{\zeta}^{z_1} \right]_{x_1}^{x_2}$$

$$- \frac{2}{x_2 - x_1} \left[\dot{\alpha}_2' \dot{\gamma}_2' \gamma_2' - \dot{\alpha}_1' \dot{\gamma}_1' \gamma_1' - \dot{\alpha}_2 \dot{\gamma}_2 \gamma_2 + \dot{\alpha}_1 \dot{\gamma}_1 \gamma_1 \right]$$

$$= \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2\right) - \omega'^2 - z'\dot{\omega}' - g\zeta \right] dx$$

$$- \frac{1}{x_2 - x_1} \left[\int_{\zeta}^{z'} (z\dot{u} + u\omega)\, dz \right]_{x_1}^{x_2}$$

$$+ \frac{1}{x_2 - x_1} \Big[z'(u_1'\omega_1' - u_2'\omega_2') + u_2\omega_2 z' + (u_2\omega_2\zeta_2)_{\zeta_2}$$
$$- u_1\omega_1 z_1 - (u_1\omega_1\zeta_1)_{\zeta_1} \Big]$$

$$- \frac{2}{x_2 - x_1} \left[\dot{\alpha}_2' \dot{\gamma}_2' \gamma_2' - \dot{\alpha}_1' \dot{\gamma}_1' \gamma_1' - \dot{\alpha}_2 \dot{\gamma}_2 \gamma_2 + \dot{\alpha}_1 \dot{\gamma}_1 \gamma_1 \right]$$

But ($\dot{\alpha}_i, \dot{\gamma}_i$) and ($\dot{\alpha}_i', \dot{\gamma}_i'$) are the velocity components at (x_i, ζ) and (x_i, z'). Hence dropping the dashes we have the exact equation for the mean pressure at time $t = 0$.

$$\frac{\overline{p - p_s}}{\rho} = g z = \int_{x_1}^{x_2} \left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2\right) - \omega^2 - z\dot{\omega} - g\zeta \right]_{x = x_2} dx \tag{3.18}$$
$$- \frac{1}{x_2 - x_1} \left[\int_{\zeta}^{z} (\dot{u}z + u\omega)\, dz - (u\omega z)_{z = \zeta} \right]_{x = x_1}$$

Since the last two terms become

$$\frac{1}{x_2-x_1}\Big[z\,u_2\omega_2 - z\,u_1\omega_1 + u_2\omega_2 z + (u_2\omega_2\zeta_2)_{\zeta_2} - u_1\omega_1 z - (u_1\omega_1\zeta_1)_{\zeta_1}\Big]$$

$$-\frac{2}{x_2-x_1}\Big[u_2\omega_2 z - u_1\omega_1 z - u_2\omega_2\zeta_2 + u_1\omega_1\zeta_1\Big]$$

$$= \frac{1}{x_2-x_1}\Big[(u\omega\zeta)\Big]_{x_1}^{x_2} = \frac{1}{x_2-x_1}\Big[(u\omega z)_{z=\zeta}\Big]_{x_1}^{x_2}$$

Equation (3•18) is true for all values of z and t, and gives the mean pressure on the plane $z=$constant between vertical planes $x=x_1$ and $x=x_2$.

By allowing x_2 to tend to x_1 in this equation we obtain an expression for the pressure at any particular point.

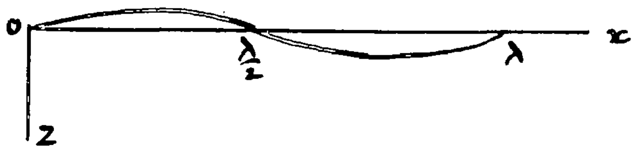
Thus

$$\frac{p-p_s}{\rho} - gz$$

$$= \frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2\right) - \omega^2 - z\dot{\omega} - g\zeta$$

$$-\frac{\partial}{\partial x}\left[\int_\zeta^z (\dot{u}z + u\omega)\,dz - (u\omega z)_{z=\zeta}\right]_{z=\zeta} \qquad (3•19)$$

We now suppose that the motion is periodic in x with wave length λ ; to fix the motion let us set $x_1=0$ and $x_2=\lambda$ and suppose that the origin is in the mean surface level. Then since u and $\dot{u}$ have equal values when $x=0$ and $x=\lambda$



$$\left[\int_\zeta^z (\dot{u}z + u\omega)\,dz - (u\omega z)_{z=\zeta}\right]_{x=0}^{x=\lambda} \qquad (3•20)$$

vanishes identically.

Also $$\int_0^\lambda g\zeta\,dx = 0. \qquad (3•21)$$

Also since the net flow of fluid across the plane z constant is zero over a wave length we have

$$\int_0^\lambda z\dot{\omega}\,dx = z\frac{\partial}{\partial t}\int_0^\lambda \omega\,dx = 0 \tag{3·22}$$

Hence using 3·18, 3·20, 3·21 and 3·22 the mean pressure over a wavelength is given by

$$\frac{\bar{p}-p_s}{\rho} - gz = \frac{1}{\lambda}\cdot\frac{\partial^2}{\partial t^2}\int_0^\lambda \tfrac{1}{2}\delta^2\,dx - \frac{1}{\lambda}\int_0^\lambda \omega^2\,dx \tag{3·23}$$

In water of constant finite depth h , the vertical velocity ω vanishes when $z = h$; so that equation (3·23) indicates that variation in the mean pressure on the bed over one wavelength in water of constant depth depends upon a second order term in the wave amplitude.

CHAPTER 4

The standing wave and the progessive wave.

In this chapter the results of chapter 3 are used to evaluate the mean pressure on the bed in the cases of a standing and a progessive wave.

Suppose that the water is of constant depth h and that the motion is one which to the first approximation consists of two progressive waves of equal lengths λ and period T travelling in opposite directions. Then the equation of the free surface is

$$\zeta = a_1 \cos(kx - \sigma t) + a_2 \cos(kx + \sigma t) + O(a^2 k) \qquad (4\cdot1)$$

$$\left.\begin{aligned} &\text{where} \quad k = \frac{2\pi}{\lambda}, \quad \sigma = \frac{2\pi}{T} \\ &\text{and} \quad \sigma^2 = gk \tanh kh \end{aligned}\right\} \qquad (4\cdot2)$$

[Lamb 1932, page 364]

$O(a^2k)$ is a term of the second and higher orders in a_1 and a_2 the wave amplitudes.

When $z = h$, $\omega = 0$, hence by equation (3•23) the mean pressure $\bar{p}_h$ on the bed ($z = h$) is given by

$$\frac{\bar{p}_h - p_s}{\rho} - gh = \frac{1}{\lambda} \cdot \frac{\partial^2}{\partial t^2} \int_0^\lambda \frac{1}{2}\left[a_1 \cos(kx - \sigma t) + a_2 \cos(kx + \sigma t)\right]^2 dx + O(a^3 \sigma^2 k^2)$$

$$\left[a_1 \cos(kx - \sigma t) + a_2 \cos(kx + \sigma t)\right]^2$$

$$= a_1^2 \cos^2(kx - \sigma t) + a_2^2 \cos^2(kx + \sigma t) + 2a_1 a_2 \cos(kx - \sigma t)\cos(kx + \sigma t)$$

$$= \frac{1}{2}\left[a_1^2 + a_2^2 + a_1^2 \cos(2kx - 2\sigma t) + a_2^2 \cos(2kx + 2\sigma t)\right] + a_1 a_2 (\cos 2kx + \cos 2\sigma t)$$

$$\therefore \int_0^\lambda \left[a_1 \cos(kx-\sigma t) + a_2 \cos(kx+\sigma t)\right]^2 dx$$

$$= \left[(a_1^2+a_2^2)x + \frac{a_1^2}{4k}\sin(2kx-2\sigma t) + \frac{a_2^2}{4k}\sin(2kx+2\sigma t) + \frac{a_1 a_2}{2k}\sin 2kx + a_1 a_2 x \cos 2\sigma t\right]_0^\lambda$$

$$= \frac{\lambda}{2}\left[a_1^2 + a_2^2 + 2a_1 a_2 \cos 2\sigma t\right]$$

Hence $$\frac{\bar{p}_h - p_s}{\rho} - gh = \frac{1}{2}\cdot\frac{\partial^2}{\partial t^2}\left(a_1^2+a_2^2+2a_1a_2\cos 2\sigma t\right) + O(a^3\sigma^2k^2)$$

$$= -2a_1a_2\sigma^2\cos 2\sigma t + O(a^3\sigma^2k^2)$$

Thus to the second order of approximation in a the variation in the mean pressure on the bed is given by

$$\frac{\bar{p}_h - p_s}{\rho} - gh = -2a_1a_2\sigma^2\cos 2\sigma t \tag{4·3}$$

It is apparent that the variationin the mean pressure on the bed ($\bar{p}_h$) is independent of the depth of the water (h), that it is periodic in time with a frequency twice that of the surface waves and in magnitude is proportional to the product of the wave amplitudes.

We may derive the mean pressure variation in the two particular cases of the progressive wave and the standing wave from equation (4·3).

Setting $a_2=0$, and $a_1=a$ we have a progressive wave of amplitude a and period $T=2\pi/\sigma$,and equation (4·3) gives

$$\frac{\bar{p}_h - p_s}{\rho} - gh = 0.$$

Thus to the second order in amplitude the mean pressure on the bed under a progressive wave is constant.

Setting $a_1 = a_2 = \frac{1}{2}a$, we have a standing wave, and the equation of the free surface is

$$\zeta = a \cos kx \cos \sigma t + O(a^2 k) \tag{4·4}$$

From equation (4·3) the fluctuation in the mean pressure at a depth h is given by

$$\frac{\overline{p_h} - p_s}{\rho} - gh = -\frac{1}{2}a^2\sigma^2 \cos 2\sigma t \tag{4·5}$$

From this we see that the mean pressure at a depth h beneath a standing wave has a periodic variation, independent of the depth, with double the frequency of the standing wave and with an amplitude proportional to the square of the wave amplitude.

This conclusion was arrived at in Chapter 2 after evaluating the second order approximation in full.

If in equation (2·43) we write $H = h$, $b = \sigma$ and $h = \frac{a}{2}$ we have

$$p_m = \rho g h + \frac{1}{2}\rho a^2 \sigma^2 \cos 2\sigma t$$

A result in full agreement with equation (4·5)

CHAPTER 5.

The total force over a horizontal plane under a surface wave motion of general form.

Perfectly periodic trains of progressive or standing waves are of very rare occurence on the oceans. To examine the pressure variation on the bed due to the customary surface wave motion on the seas, we consider that the observable surface motion arises from a continuous range or spectrum of wave frequencies.

We measure z vertically downwards from the free surface at rest and x and y horizontally in two perpendicular directions. Let u and v denote the velocities of the point (x,y) in the x and y directions. The symbol A(u,v) denotes the complex wave amplitude and also defines the two dimensional frequency spectrum of the wave motion.

We let $z = \zeta$ denote the equation of the free surface, and we imagine that the fluid is incompressible. After assuming the general conditions necessary for the validity of this work, and in particular the possibility of differentiating under the integral sign, we suppose that the values of ζ and $\partial\zeta/\partial t$ at the initial instant $t = 0$ can be expressed as

$$(\zeta)_{t=0} = \mathcal{R}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A\, e^{i(ukx+vky)}\, du\, dv, \qquad (5{\cdot}1)$$

$$\left(\frac{\partial\zeta}{\partial t}\right)_{t=0} = \mathcal{R}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B\, e^{i(ukx+vky)}\, du\, dv, \qquad (5{\cdot}2)$$

where $\mathcal{R}$ denotes the real part, and B as well as A is a function of (u,v). We further suppose that B is defined by

$$B = i\sigma A \qquad (5{\cdot}3)$$

where $2\pi/\sigma$ is the period of the wave of

$$\text{length} \quad \lambda = \frac{2\pi}{(u^2+v^2)^{1/2}\, k} \qquad (5{\cdot}4)$$

The period equation for waves in water of depth is therefore (Lamb Chapter 9)

$$\sigma^2 = (u^2+v^2)\, g k \tanh\, (u^2+v^2)^{\frac{1}{2}} k h \qquad (5{\cdot}5)$$

The surface wave defined by (5·4) and (5·5) has its crests parallel to the line

$$ux + vy = 0 \qquad (5\cdot 6)$$

We shall now derive a transformation which is to be applied to equations (5·1) and (5·2). A continuous and absolutely integrable function $f(x)$ can be expressed by the exponential form of Fourier's Formula (Titchmarsh : Theory of Fourier Integrals, chapter I) :

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixu}\,du \int_{-\infty}^{\infty} f(t)e^{iut}\,dt.$$

Writing ku for u, this becomes

$$f(x) = \frac{k}{2\pi}\int_{-\infty}^{\infty} e^{-ikxu}\,du \int_{-\infty}^{\infty} f(t)e^{ikut}\,dt.$$

Hence

$$f(x,y) = \frac{k}{2\pi}\int_{-\infty}^{\infty} e^{-ikxu}\,du \int_{-\infty}^{\infty} f(s,y)e^{ikus}\,ds$$

where y is taken as constant.

If $$g(u,y) = \int_{-\infty}^{\infty} f(s,y)e^{ikus}\,ds$$

then $$f(x,y) = \frac{k}{2\pi}\int_{-\infty}^{\infty} e^{-ikxu}\,g(u,y)\,du,$$

where y is constant.

Also $$g(u,y) = \frac{k}{2\pi}\int_{-\infty}^{\infty} F(u,v)e^{-ikyv}\,dv$$

and $$F(u,v) = \int_{-\infty}^{\infty} e^{ikvt}\,g(u,t)\,dt,$$

where u is constant.

Hence

$$f(x,y) = \frac{k}{2\pi}\int_{-\infty}^{\infty} e^{-ikxu}\,du \int_{-\infty}^{\infty} \frac{k}{2\pi} F(u,v)\, e^{-ikyv}\,dv$$

$$= \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(kxu+kyv)} F(u,v)\,du\,dv$$

since u and v are independent,

$$\text{and } F(u,v) = \int_{-\infty}^{\infty} e^{ikvt}\,dt \int_{-\infty}^{\infty} f(s,t)e^{ikus}\,ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(kus+kvt)} f(s,t)\,ds\,dt$$

since s and t are independdnt.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(kux+kvy)} f(x,y)\,dx\,dy.$$

Interchangimg F and f, (u,v) and (x,y) we have

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(kux+kvy)} F(u,v)\,du\,dv$$

where

$$F(u,v) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(kxu+kyv)} f(x,y)\,dx\,dy.$$

Hence we have the transformation for a continuous and absolutely integrable function f(x,y) of two variables :

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(ukx+vky)}\, F(u,v)\, du\, dv, \quad (5.7)$$

where

$$F(u,v) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i(ukx+vky)}\, dx\, dy, \quad (5.8)$$

If F(u,v) denote a complex function of (u,v) , then

$$F_1(u,v) + F_1^*(-u,-v) = R.\,2F_1(u,v),$$

where F_1^* is the conjugate complex function of F_1.
Hence

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}\left[F(u,v) + F^*(-u,-v)\right] e^{i(ukx+vky)}\, du\, dv \quad (5.9)$$

$$= R.\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{i(ukx+vky)}\, du\, dv \quad (5.10)$$

where F(u,v) is a complex function.

where $R.F(u,v) = \frac{1}{2}\left[F(u,v) + F^*(-u,-v)\right]$

$$= \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i(ukx+vky)}\, dx\, dy \quad (5.11).$$

Apply the principle of equations (5.9), (5.10) and (5.11) to equations (5.1) and (5.2), then

$$\frac{1}{2}\left[A(u,v) + A^*(-u,-v)\right] = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\zeta)_{t=0}\, e^{-i(ux+vy)k}\, dx\,dy \tag{5.12}$$

$$\frac{1}{2}\left[B(u,v) + B^*(-u,-v)\right] = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\partial \zeta}{\partial t}\right)_{t=0} e^{-i(ux+vy)k}\, dx\,dy;$$

but from equation (5.3)

$$\frac{1}{2}\left[B(u,v) + B^*(-u,-v)\right] = \frac{1}{2}\left[A(u,v).i\sigma - A^*(-u,-v).i\sigma\right]$$

Hence

$$\frac{1}{2} i\sigma\left[A(u,v) - A^*(-u,-v)\right] = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\partial \zeta}{\partial t}\right)_{t=0} e^{-i(ux+vy)k}\, dx\,dy. \tag{5.13}$$

From (5.12) and (5.13)

$$A(u,v) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\zeta + \frac{1}{i\sigma}\cdot\frac{\partial \zeta}{\partial t}\right)_{t=0} e^{-i(ux+vy)k}\, dx\,dy. \tag{5-14}$$

Now consider the expression

$$\eta = R. \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u,v)\, e^{i(ukx+vky+\sigma t)}\, du\,dv, \tag{5.15}$$

where A(u,v) is given by equation (5.14).

$Z = A(u,v)\, e^{i(ukx+vky+\sigma t)}$ represents a surface wave of amplitude A(u,v), with velocity components (u,v), period $2\pi/\sigma$ in water depth h, of length given by (5.4) and satisfyimg the period equation (5•5) for waves in water of constant depth.

Since $A(u,v)\,e^{i(ukx+vky+\sigma t)}$ satisfies the period equation for waves in water of constant depth, η must also satisfy this equation to the first order of approximation. But from (5•15) we see that

$$(\eta)_{t=0} = R.\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u,v)\,e^{i(ux+vy)k}\,du\,dv$$

$$= (\zeta)_{t=0} \qquad (\text{by } 5\cdot1)\ ;$$

also $$\frac{\partial\eta}{\partial t} = R.\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u,v).\,i\sigma\,e^{i(ukx+vky+\sigma t)}\,du\,dv$$

$$\therefore \left(\frac{\partial\eta}{\partial t}\right)_{t=0} = R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(u,v)\,e^{i(ukx+vky)}\,du\,dv$$

by 5•3

$$= \left(\frac{\partial\zeta}{\partial t}\right)_{t=0} \qquad \text{by } (5\cdot2).$$

Since the initial values of the surface elevation amd its rate of change with respect to time determine the initial potential and kinetic energies of an irrotational motion, then these initial conditions must detrermine a unique irrotational motion. Hence since

$\zeta = \eta$ and $\frac{\partial\zeta}{\partial t} = \frac{\partial\eta}{\partial t}$ for t $=$ 0,

they must also be equal for all values of t ;

that is

$$\mathcal{J} = R.\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u,v)\, e^{i(ukx+vky+\sigma t)}\, du\, dv, \qquad (5\cdot16)$$

for all values of t.

Since uk and vk can take all values then equatiom (5·4) defines all possible wave-lengths and equation (5·6) defines all possible directions. Then the free surface $Z = \mathcal{J}$ can, by equation (5·16), be regarded as created by the sum of a number of superposed wave motions of all possible wavelengths (equation 5·4) and travelling in all possible directions from the origin O. For a given value of k the line OP, where P is the point $(-uk,-vk)$, is perpendicular to the line $ux + vy = 0$ for all pairs of values of (u,v). That is, every line OP is perpendicular to the crest of a wave. So that each vector $\overrightarrow{OP}$ corresponds to a wave component satisfyimg the period equation (5·5). The direction of the vector $\overrightarrow{OP}$ gives the [redacted] direction of propagation of the corresponding wave component. Since $OP^2 = (u^2+v^2)k^2$,

$$OP = \frac{2\pi}{\lambda}, \text{ by equation } (5\cdot4).$$

So that all wave components of the same length correspond to points P lying on the circle centre O and radius $\frac{2\pi}{\lambda}$. Diametrically opposite points correspond to wave components of the same wave-length, with parallel crests but travelling in opposite directions with the same speed. Such pairs of wave components will be called opposite wave components, and will interfere with each other to produce standing waves.

The total energy of the motion.

Before determining the kinetic and potential energies of the free motion of the sea surface we must first extend the Parseval-Plancherel theorem to functions of two variables.

If $F(u) = 2\pi \int_{-\infty}^{\infty} f(t) e^{iut}\, dt$, then $f(x) = \int_{-\infty}^{\infty} F(u) e^{-ixu}\, du$

If $G(u) = 2\pi \int_{-\infty}^{\infty} g(t) e^{iut}\, dt$, then $g(t) = \int_{-\infty}^{\infty} G(u) e^{-itu}\, du$

(Titchmarsh: Theory of Fourier Integrals).

Then
$$\int_{-\infty}^{\infty} F(x) G(x)\, dx = 2\pi \int_{-\infty}^{\infty} G(x)\, dx \int_{-\infty}^{\infty} f(t) e^{ixt}\, dt$$

$$= 2\pi \int_{-\infty}^{\infty} f(t)\, dt \int_{-\infty}^{\infty} G(x) e^{ixt}\, dx$$

$$= 2\pi \int_{-\infty}^{\infty} f(t) g(t)\, dt .$$

Since $G(u) = 2\pi \int_{-\infty}^{\infty} g(t) e^{iut}\, dt$

then $2\pi \int_{-\infty}^{\infty} \bar{g}(-t) e^{-iu(-t)}\, dt = \bar{G}(u)$,

$$\therefore \int_{-\infty}^{\infty} F(x) \bar{G}(x)\, dx = 2\pi \int_{-\infty}^{\infty} f(t) \bar{g}(t)\, dt .$$

If $g = f$ and $G = F$, then

$$\int_{-\infty}^{\infty} |F(x)|^2\, dx = 2\pi \int_{-\infty}^{\infty} |f(t)|^2\, dt,$$

the Parseval- Plancherel Theorem.

By Fourier's Exponential Formula for two variables (see equations 5·7 and 5·8) :

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(kux+kvy)} F(u,v)\,du\,dv,$$

where

$$F(u,v) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(kux+kvy)} f(x,y)\,dx\,dy.$$

Suppose that g and G are similarly related functions.

Then $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,g(x,y)\,dx\,dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,dx\,dy \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(kux+kvy)} F(u,v)\,du\,dv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\,du\,dv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,e^{i(kux+kvy)}\,dx\,dy$$

$$= \left(\frac{2\pi}{k}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v).\,G(-u,-v)\,du\,dv \qquad (A).$$

But as $\displaystyle g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(kux+kvy)} G(u,v)\,du\,dv$

then $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{G}(-u,-v)\,e^{-i(-ukx-kvy)}\,du\,dv = \bar{g}(x,y).$

Hence in (A) replace $G(u,v)$ by $\bar{G}(-u,-v)$ and $g(x,y)$ by $\bar{g}(x,y)$. Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\bar{g}(x,y)\,dx\,dy = \left(\frac{2\pi}{k}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\,\bar{G}(u,v)\,du\,dv.$$

Putting F G and consequently f g

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2\,dx\,dy = \left(\frac{2\pi}{k}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(u,v)|^2\,du\,dv$$

where $f(x,y)$ and $F(u,v)$ are related by equations (5·7) and (5·8).

Rewrite equation (5.16) as

$$\mathcal{J} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2}\left[A(u,v)e^{i\sigma t} + A^*(-u,-v)e^{-i\sigma t}\right] du\,dv \tag{5.17}$$

and apply the above theoeem :

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{J}^2\,dx\,dy$$

$$= \left(\frac{2\pi}{k}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\tfrac{1}{2}\left\{A(u,v)e^{i\sigma t} + A^*(-u,-v)e^{-i\sigma t}\right\}\right|^2 du\,dv \tag{5.18}$$

$$\left| A(u,v)e^{i\sigma t} + A^*(-u,-v)\,e^{-i\sigma t} \right|^2$$

$$= \left[A(u,v)e^{i\sigma t} + A^*(-u,-v)e^{-i\sigma t}\right]\left[A^*(u,v)\,e^{i\sigma t} + A(-u,-v)\,e^{i\sigma t}\right]$$

$$= A(u,v).\ A^*(u,v) + A(u,v)\,A(-u,-v)\,e^{2i\sigma t}$$

$$+ A^*(u,v)\,A^*(-u,-v)\,e^{-2i\sigma t} + A(-u,-v)\,A^*(-u,-v)$$

$$= 2R.\left[A(u,v)\,A^*(u,v) + A(u,v)\,A(-u,-v)\,e^{2i\sigma t}\right],$$

since $A(u,v)\,A^*(u,v) = A(-u,-v).\,A^*(-u,-v)$ and

$$A(u,v)\,A(-u,-v)\,e^{2i\sigma t} + A^*(u,v)\,A^*(-u,-v)\,e^{-2i\sigma t}$$

$$= 2R.\left[A(u,v)\,A(-u,-v)\,e^{2i\sigma t}\right].$$

Hence equation (5·18) becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \zeta^2\,dx\,dy =$$

$$R.\ 2\left(\frac{\pi}{k}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{A(u,v)\,A^*(u,v) + A(u,v)\,A(-u,-v)\,e^{2i\sigma t}\right\}du\,dv \tag{5-19}$$

The potential energy of the motion is $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2}\rho g\,\zeta^2\,dx\,dy$

$$= R.\,\rho g\left(\frac{\pi}{k}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[A(u,v)\,A^*(u,v) + A(u,v)\,A(-u,-v)\,e^{2i\sigma t}\right]du\,dv \tag{5·20}$$

by equation (5·19).

The Kinetic Energy of the motion is $\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\phi\frac{\partial\phi}{\partial z}\right]_{z=0} dx\,dy$ (5.21)

where z is measured upwards from the disturbed surface (Milne-Thomson § 9.11).

With a surface displacement $A(u,v).\cos\{k(ux+vy)+\sigma t\}$ the complex potential ϖ ($=\phi+i\psi$) is of the form

$$P\sin\left[k\sqrt{u^2+v^2}\left\{\frac{(ux+vy)}{\sqrt{(u^2+v^2)}}+i(z+h)\right\}+\sigma t\right] \qquad (5.22)$$

But $$\left(\frac{\partial\phi}{\partial t}\right)_{z=0} = g\zeta \qquad (5.23)$$

(Milne-Thomson § 14.18)

Then $\dfrac{\phi+i\psi}{P}$

$$= \sin\left[k\sqrt{u^2+v^2}\left\{\frac{ux+vy}{\sqrt{u^2+v^2}}+i(z+h)\right\}\right]\cos\sigma t$$

$$+\cos\left[k\sqrt{u^2+v^2}\left\{\frac{ux+vy}{\sqrt{u^2+v^2}}+i(z+h)\right\}\right]\sin\sigma t$$

$$= \sin k(ux+vy).\cosh(z+h)k\sqrt{u^2+v^2}.\cos\sigma t$$

$$+i\cos k(ux+vy).\sinh(z+h)k\sqrt{u^2+v^2}.\cos\sigma t$$

$$+\cos k(ux+vy).\cosh\left\{k\sqrt{u^2+v^2}.(z+h)\right\}.\sin\sigma t$$

$$-i\sin k(ux+vy).\sinh\left\{k\sqrt{u^2+v^2}.(z+h)\right\}.\sin\sigma t.$$

$$\therefore \phi = P\Big[\sin k(ux+Vy).\ \mathrm{Cosh}\left\{(z+h)k\sqrt{u^2+v^2}\right\}.\ \mathrm{Cos}\,\sigma t$$

$$+ \mathrm{Cos}\,k(ux+Vy).\ \mathrm{Cosh}\left\{(z+h)k\sqrt{u^2+v^2}\right\}.\ \sin\sigma t\Big],$$

hence

$$\frac{\partial\phi}{\partial t} = -P\Big[\sigma\sin k(ux+Vy).\ \mathrm{Cosh}\left\{(z+h)k\sqrt{u^2+v^2}\right\}.\ \sin\sigma t$$

$$- \sigma\,\mathrm{Cos}\,k(ux+Vy).\ \mathrm{Cosh}\left\{(z+h)k\sqrt{u^2+v^2}\right\}.\ \mathrm{Cos}\,\sigma t\Big]$$

$$\therefore \left[\frac{\partial\phi}{\partial t}\right]_{z=0} = P\sigma\,\mathrm{Cosh}\left(hk\sqrt{u^2+v^2}\right).\ \mathrm{Cos}\left\{k(ux+Vy)+\sigma t\right\}.$$

Hence by equation (5·23)

$$P\sigma\,\mathrm{Cosh}\left(hk\sqrt{u^2+v^2}\right).\ \mathrm{Cos}\left\{k(ux+Vy)+\sigma t\right\}$$

$$= g\,A(u,V).\ \mathrm{Cos}\left\{k(ux+Vy)+\sigma t\right\}$$

$$\therefore \quad P = \frac{g\,A(u,V)}{\sigma\,\mathrm{Cosh}\left(hk\sqrt{u^2+v^2}\right)}$$

Hence picking out the real part of ϖ (equation 5·22)

$$\phi = \frac{g\,A(u,V)}{\sigma\,\mathrm{Cosh}\left(hk\sqrt{u^2+v^2}\right)}.\ \mathrm{Cosh}\left\{(z+h)k\sqrt{u^2+v^2}\right\}.\ \sin\left\{k(ux+Vy)+\sigma t\right\}$$

$$= -R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\,g\,A(u,V)\,\mathrm{Cosh}\left\{(z+h)k\sqrt{u^2+v^2}\right\}}{\sigma\,\mathrm{Cosh}\left(hk\sqrt{u^2+v^2}\right)}.\ e^{i(ukx+Vky+\sigma t)}\,du\,dV$$

$$\therefore [\phi]_{z=0} = -R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\,g\,A(u,V)}{\sigma}.\ e^{i(ukx+Vky+\sigma t)}.\ du\,dV.$$

$$\frac{\partial\phi}{\partial z} = \frac{g\,A(u,v).\,k\sqrt{u^2+v^2}.\,\sinh\{k(z+h)\sqrt{u^2+v^2}\}.\,\sin\{k(ux+vy)+\sigma t\}}{\sigma.\,\mathrm{Cosh}\,(hk\sqrt{u^2+v^2})}$$

$$= -R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{igk\,A(u,v).\sqrt{u^2+v^2}\sinh\{k(z+h)\sqrt{u^2+v^2}\}.\,e^{i(ukx+vky+\sigma t)}}{\sigma\,\mathrm{Cosh}\,(hk\sqrt{u^2+v^2})}.\,du\,dv.$$

$$\therefore\left[\frac{\partial\phi}{\partial z}\right]_{z=0} = -R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{igk\,A(u,v).\sqrt{u^2+v^2}}{\sigma}.\,\tanh(hk\sqrt{u^2+v^2})\,e^{i(ukx+vky+\sigma t)}\,du\,dv$$

$$\therefore\left[\phi\frac{\partial\phi}{\partial z}\right]_{z=0} = -R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{A^2(u,v)\,g^2k\sqrt{u^2+v^2}.\,\tanh(hk\sqrt{u^2+v^2})}{\sigma^2}.\,e^{i(ukx+vky+\sigma t)}\,du\,dv.$$

But $\sigma^2 = gk\sqrt{u^2+v^2}.\,\tanh\{(u^2+v^2)^{\frac{1}{2}}kh\}$ from equation (5•5)

$$\therefore\left[\phi\frac{\partial\phi}{\partial z}\right]_{z=0} = -R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\,A^2(u,v).\,e^{i2(ukx+vky+\sigma t)}.\,du\,dv.$$

Hence from (5•21), the Kinetic Energy is

$$\tfrac{1}{2}\rho g\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{i\,A(u,v)\,e^{i(ukx+vky+\sigma t)}\,du\,dv\right\}^2\right]dx\,dy$$

$$= \tfrac{1}{2}\rho g\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left\{R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}B(u,v)\,e^{i(ukx+vky+\sigma t)}\,du\,dv\right\}^2\right]dx\,dy$$

where $B(u,v) = i\,A(u,v)$.

$$R. \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(u,v)\, e^{i(ukx+vky+\sigma t)}\, du\, dv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2}\left[B(u,v)\, e^{i\sigma t} + B^*(-u,-v)\, e^{-i\sigma t}\right] du\, dv.$$

Hence by the same transformation that derives equation (5·18) from equation (5·17), the kinetic energy of the motion is

$$\tfrac{1}{2}\rho g\left(\frac{2\pi}{k}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\tfrac{1}{2}\left\{B(u,v)\, e^{i\sigma t} + B^*(-u,-v)\, e^{-i\sigma t}\right\}\right|^2 du\, dv$$

$$= \rho g\left(\frac{\pi}{k}\right)^2 . R \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[B(u,v)\, B^*(u,v) + B(u,v)\, B(-u,-v)\, e^{2i\sigma t}\right] du\, dv$$

by the same method as equation (5·19) is derived from (5·18). But B(u,v) = iA(u,v) , hence the kinetic energy of the motion is

$$\rho g\left(\frac{\pi}{k}\right)^2 . R \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[A(u,v)\, A^*(u,v) - A(u,v)\, A(-u,-v)\, e^{2i\sigma t}\right] du\, dV \qquad (5{\cdot}24).$$

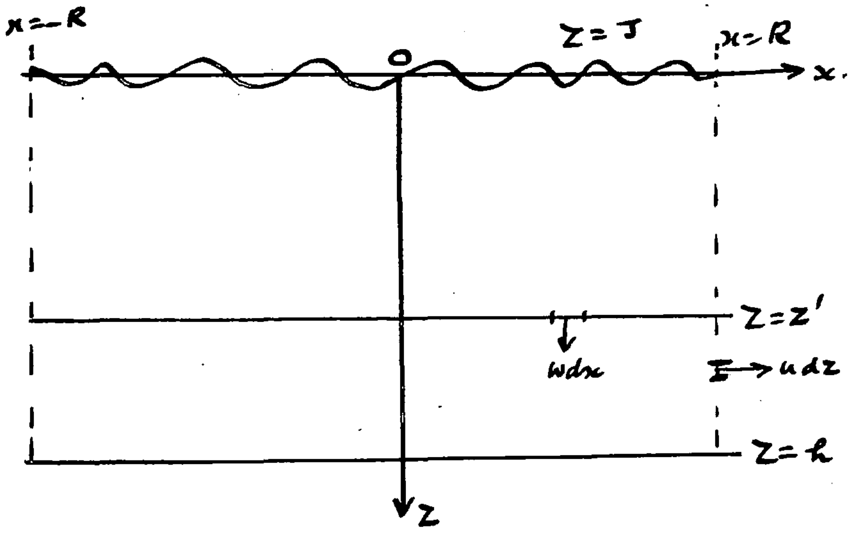
Hence after reference to equations (5·30) and (5·24), the total energy of the motion is

$$2\rho g\left(\frac{\pi}{k}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u,v)\, A^*(u,v)\, du\, dV \qquad (5{\cdot}25)$$

So that the total energy of the general wave motion depends on the square of the modulus of the wave amplitude A(u,v) . We shall return to this result in chapter 9 when we find the displacement of the sea bed due to a wave motion in a finite area.

<u>The force on a given area of the ocean bed.</u>

Consider a region of the water, unit thickness in the y-direction, depth h in the z-direction and bounded by the limits $-R \leqslant x < R$.



$\bar{p}$ denotes the mean pressure on the plane $z = z'$ in this interval.

F denotes the variable part of the total force acting on the plane $z = z'$ in this interval.

p_s is the pressure at the free surface $z = \zeta$.

Then for the equilibrium of the bounded fluid

$$F = 2R\,(\bar{p} - p_s - g\rho z) \tag{5.26}$$

Then from equation (3.18)

$$\frac{F}{\rho} = \int_{-R}^{R}\left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2\right) - \omega^2 - z\dot{\omega} - g\zeta\right]dx \tag{5.27}$$

$$-\left[\int_{\zeta}^{z}(\dot{u}z + u\omega)\,dz - (u\omega z)_{z=\zeta}\right]_{-R}^{R}$$

From the continuity of the flow of the water in the region between $z = h$, $z = z'$ and $x = \pm R$,

$$\int_{-R}^{R} w \, dx = \left[\int_{z}^{h} u \, dz \right]_{-R}^{R}$$

and
$$\int_{-R}^{R} \dot{w} \, dx = \left[\int_{z}^{h} \dot{u} \, dz \right]_{-R}^{R}$$

hence
$$z \int_{-R}^{R} \dot{w} \, dx = \left[z \int_{z}^{h} \dot{u} \, dz \right]_{-R}^{R} \qquad (5{\cdot}28).$$

If the mean level of the free surface $z = \zeta$ is zero at time $t = 0$, consideration of the depression of the free surface and the outflow of water gives,

$$\int_{-R}^{R} \zeta \, dx = \int_{0}^{t} dt \left[\int_{\zeta}^{h} u \, dz \right]_{-R}^{R} \qquad (5{\cdot}29).$$

Extending equation (5·28) to the entire depth of the water, that is putting $z = h$, we have

$$\int_{-R}^{R} z \dot{w} \, dx = \left[h \int_{h}^{h} \dot{u} \, dz \right] = 0 \qquad (5{\cdot}30)$$

Putting $z = h$ and using equations (5·29) and (5·30) for the second term and integrating the third term by parts, equation (5·27) becomes

$$\left[\frac{F}{\rho} \right]_{z=h} = \int_{-R}^{R} \left[\frac{\partial^2}{\partial t^2} \left(\tfrac{1}{2} \zeta^2 \right) - w^2 \right] dx - \left[g \int_{0}^{t} dt \int_{\zeta}^{h} u \, dz \right]_{-R}^{R}$$

$$- \left[z \int_{\zeta}^{h} \dot{u} \, dz - \int_{\zeta}^{h} dz \int_{\zeta}^{z} \dot{u} \, dz + \int_{\zeta}^{h} u w \, dz - (u w z)_{z=\zeta} \right]_{-R}^{R} \qquad (5{\cdot}31).$$

This gives $[F]_{z=h}$ the variable part of the total force, per unit distance in the y-direction, acting on the bed $\{z=h\}$ in the interval $-R < x < R$.

Equation (5·31) is completely exact statement. We have already in this chapter, shown that the motion may be analysed into a frequency spectrum comprising all possible wave frequencies; we suppose that the energy of the motion, given by equation (5·25) is nearly all confined to a narrow band of frequencies; then the motion of the surface will be wave-like. We suppose the mean frequency to be $\sigma/2\pi$ corresponding to a wave-length λ, which is small compared with R.

We now compare the relative sizes of the terms in equation (5·31). In general the relative phase of the motion at two widely separated points on the x-axis will be random. We may, however, suppose that the motion is regular and periodic over any interval of the x-axis less than or equal to $2R_1$ in length. In addition we suppose that initially the motion was confined to an interval $-R_2 < x < R_2$, (where R_2 may be very great compared with R_1), so that the elevation and vertical velocity of the free surface at points outside this interval are initially zero.

We may distinguish three distinct cases :

<u>Case I</u> :When $R \leqslant R_1$, so that the motion is regular over the whole interval $-R < x < R$ which we are considering.

Then $$\int_{-R}^{R}\left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2\right) - \omega^2\right] dx$$

$$\doteq \frac{2R}{\lambda}\int_0^{\lambda}\left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2\right) - \omega^2\right] dx,$$ since the motion is periodic over $-R < x < R$

$$\doteq \frac{2R}{\lambda}\int_0^{\lambda}\left[\zeta\frac{\partial^2\zeta}{\partial t^2} + \left(\frac{\partial\zeta}{\partial t}\right)^2 - \omega^2\right] dx$$

$$= O\left(a^2\sigma^2 R\right),$$ where a is the maximum wave elevation,

if we assume that u and ω are of order $a\sigma$.

For waves in deep water

$$\sigma^2 = gk \tanh kh \quad \text{and} \quad \lambda = \frac{2\pi}{k}$$

$$\therefore \quad \sigma^2 = \frac{2\pi g}{\lambda} \tanh kh$$

that is $g = O(\sigma^2\lambda)$.

Also $\left[\int_{\zeta}^{z} u\,dz\right]_{-R}^{R} = O(a\sigma\lambda)$ for all z.

The remaining terms of (5•31) are of order $a\sigma^2\lambda z$ or $a\sigma^2\lambda^2$; hence if R/λ and R/z are sufficiently lagge we have

$$\left[\frac{F}{\rho}\right]_{z=h} \simeq \int_{-R}^{R} \left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2\right) - \omega^2\right] dx \qquad (5\cdot31)$$

to the first order of small quantities.

In establishing equation (3•18) and hence equation (5•31) we have assumed a constant mass of fluid, hence it must be verified that these second-order pressure variations which are in phase over the whole interval, do not produce any significant motion across the planes $x = \pm R$.

The horizontal and vertical displacements of a particle when there is a standing wave are (Lamb $\int$ 228)

$$X = -Ae^{kz} \sin kx . \sin(2\sigma t + \epsilon),$$

$$Z = Ae^{kz} \cos kx . \sin(2\sigma t + \epsilon).$$

Consider the effect of a pressure distribution

$$\frac{p}{\rho} = \begin{cases} 2a^2\sigma^2 \cos 2\sigma t & (|x| < R), \\ 0 & (|x| > R), \end{cases}$$

acting on the free surface of deep water.

Regarding the pressure as due to ahead of water

$$p = 2a^2\sigma^2\rho \cos 2\sigma t = g\rho A \cos kx \cos 2\sigma t$$

i.e. $$A = \frac{2a^2\sigma^2}{g \cos kx}.$$

The amplitude of the horizontal component of the velocity u is $2\sigma A e^{kz} \sin kx$

$$= \frac{4a^2\sigma^3 e^{kz} \tan kx}{g}$$

$$= O\left(\frac{a^2\sigma}{\lambda}\right),$$

since $g = O(\lambda\sigma^2)$.

Hence the total flow $\left[\int_{\zeta}^{z} u\,dz\right]_{-R}^{R} = O\left(\frac{1}{k}\cdot\frac{a^2\sigma}{\lambda}\right) = O(a^2\sigma)$.

Hence when $\frac{R}{\lambda} \gg 1$ and $R/z \gg 1$

equation (5.31) will be valid. Since ω diminishes rapidly depth and is almost negligible when z $\simeq \frac{1}{2}\lambda$,

then $$\left[\frac{F}{\rho}\right]_{z=\ell} = \frac{\partial^2}{\partial t^2}\int_{-R}^{R} \tfrac{1}{2}\zeta^2\,dx \qquad (5.32)$$

when z is of order λ and $R/\lambda \gg 1$,

<u>Case II:</u> When $R_1 < R \leq R_2$. We suppose that the interval $-R < x < R$ be divided into smaller intervals of length less than or equal to $2R_1$. We assume that the motion in each of these sub-intervals is regular and periodic but that there are random phase differences between successive intervals.

Since, $$\left|\left\{(\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_n)^2 + i\,(\sin\theta_1 + \sin\theta_2 + \cdots + \sin\theta_n)^2\right\}^{\frac{1}{2}}\right|$$

(where the θ's are random)

$$= \sqrt{1^2 + 1^2 + \cdots \text{ to } n \text{ terms}}$$

$$= \sqrt{n},$$

since the products $2\cos(\theta_p-\theta_q)$ have zero sum because of the random values of $(\theta_p-\theta_q)$, the sum of n vectors of comparable modulus in random phase relationship with one another increases like $\sqrt{n}$ times the mean modulus.

Hence $$\int_{-R}^{R}\left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\delta^2\right)-\omega^2\right]dx$$

$$=\sum_{-R}^{R}\int_{-R_1}^{R_1}\left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\delta^2\right)-\omega^2\right]dx$$

$$=O\left[\sqrt{n}\int_{-R_1}^{R_1}\left\{\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\delta^2\right)-\omega^2\right\}dx\right]$$

$$=O\left[\left(\frac{R}{R_1}\right)^{\frac{1}{2}}\cdot a^2\sigma^2\cdot R_1\right]$$

$$=O\left[a^2\sigma^2(RR_1)^{\frac{1}{2}}\right]$$

after assuming u and w to be of order $a\sigma$. The remaining terms of equation (5·31) are of order $a\sigma^2\lambda z$ or $a\sigma^2\lambda^2$; hence if R/λ and R/z are sufficiently large equation (5·31) is still valid. ω decreases exponentially with increase of z, hence if z is of order λ and $\sigma^2(RR_1)^{1/2}$ or $(RR_1)^{1/2}/\lambda$ is very much greater than unity, equation (5·32) remains valid.

<u>Case III :</u> When $R > R_2$.

By allowing R to tend to infinity an exact expression for the total force $[F]_{z=h}$ over the whole plane $z = $ constant may be obtained. After reference to Lamb (1932, § 238) we see that a standing wave with a surface elevation

$\eta = \cos\sigma t\cos kx$ on deep water arises from a velocity potential $\phi = \frac{g}{\sigma}\sin\sigma t.\, e^{-kz}\cos kx$ where $\sigma^2 = gk$.

Generalising this by Fourier's double-integral theorem, an elevation $$\eta = \frac{1}{\pi}\int_0^\infty \cos\sigma t\, dk \int_{-\infty}^{\infty} f(\alpha)\cos k(x-\alpha)\, d\alpha$$

arises from a velocity potential

$$\phi = \frac{1}{\pi}\int_0^\infty \frac{g}{\sigma}\sin\sigma t \,.\, e^{-kz} dk \int_{-\infty}^{\infty} f(\alpha)\cos k(x-\alpha)\, d\alpha,$$

where the initial conditions are $\eta = f(x)$, $\phi_0 = 0$, where the zero suffix indicates the surface- value ($z = 0$). If the initial elevation be confined to the immediate neighbourhood of the origin, so that $f(\alpha)$ vanishes for all but infinitesimal values of α , we have,

$$\phi = \frac{g}{\pi}\int_0^\infty \frac{\sin\sigma t}{\sigma} \,.\, e^{-kz} \cos kx\, dk$$

after assuming $$\int_{-\infty}^{\infty} f(\alpha)\, d\alpha = 1 \,.$$

This value of ϕ may be expanded in the form

$$\phi = \frac{gt}{\pi}\int_0^\infty \left\{1 - \frac{\sigma^2 t^2}{\underline{|3}} + \frac{\sigma^4 t^4}{\underline{|5}} - \cdots \right\} e^{-kz} \cos kx\, dk$$

$$= \frac{gt}{\pi}\int_0^\infty \left\{1 - \frac{gt^2 k}{\underline{|3}} + \frac{(gt^2)^2}{\underline{|5}} k^2 - \cdots \right\} e^{-kz} \cos kx\, dk,$$

after using $\sigma^2 = gk$.

Now $$\int_0^\infty e^{ky} \operatorname{Cos} kx \, . \, k^n \, dk$$

$$= R. \int_0^\infty e^{k(y+ix)} . \, k^n \, dk$$

$$= R. \int_0^\infty \frac{1}{y+ix} . \, d\left(e^{k(y+ix)}\right) k^n \, dk$$

$$= R . \frac{1}{y+ix} \left[e^{k(y+ix)} . k^n - n \int k^{n-1} . \, e^{k(y+ix)} dk \right]_0^\infty .$$

Putting $y = -z = -r \operatorname{Cos}\theta$, $x = r \sin\theta$

$$\int_0^\infty e^{-kz} \operatorname{Cos} kx . \, k^n dk =$$

$$R . \left[\frac{\underline{|n} . (-)^n \, e^{-kr \operatorname{Cos}\theta}}{r^n . \, e^{-ni\theta} . \, r e^{-i\theta}} \left\{ \operatorname{Cos} (kr \sin\theta) + i \sin (kr \operatorname{Sin}\theta) \right\} \right]_0^\infty$$

$$= \frac{\underline{|n}}{r^{n+1}} . \quad \operatorname{Cos} (n+1) . \theta .$$

Hence $$\phi = \frac{gt}{\pi} \left\{ \frac{\operatorname{Cos}\theta}{r} - \frac{\underline{|1}}{\underline{|3}} . \frac{gt^2 \operatorname{Cos} 2\theta}{r^2} + \frac{\underline{|2}}{\underline{|5}} (gt^2)^2 \frac{\operatorname{Cos} 3\theta}{r^3} + \cdots \right\}$$

$$= \frac{gt}{\pi} \left\{ \frac{z}{x^2+z^2} - \frac{\underline{|1}}{\underline{|3}} . \, gt^2 . \frac{(z^2 - x^2)}{(z^2+x^2)^2} + \cdots \right\}$$

$$\simeq \frac{gt}{\pi} . \frac{z^2}{x^2+z^2} , \quad \text{when } \frac{gt^2}{(x^2+z^2)^2} \text{ is small.}$$

Thus we have that the velocity potential of the motion due to an initial elevation of the free surface concentrated in the line $x = z = 0$ is proportional to $gtz(x^2+z^2)^{-1}$, when $gt^2(x^2+z^2)^{-2}$ is small. A similar result will hold when the initial disturbance is distributed over a finite interval of the x-axis. Hence for very large R the velocities across the planes $z = \pm R$ will initially be proportional to R^{-2}, and the total flow $\left[\int_{\zeta}^{z} u\, dz\right]_{-R}^{R}$ will be proportional to R^{-1}.

The terms of the equation (5·31) to be evaluated at the planes $z = \pm R$ therefore tend to zero. But since the total potential energy is finite, we may assume that the first integral of (5·31) converges. Hence the total force $[F]_{z=h}$ over the whole plane is given by

$$\left[\frac{F}{\rho}\right]_{z=h} = \int_{-\infty}^{\infty}\left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2\right) - \omega^2\right] dx \tag{5·32}$$

Since ω decreases exponentially with increase of depth

$$\left[\frac{F}{\rho}\right]_{z=h} = \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\tfrac{1}{2}\zeta^2\, dx \quad \text{approximately} \tag{5·33}$$

These results may be extended to motion in three dimensions. Let S be a square given by $-R < x < R$, $-R < y < R$ on the $z=0$ plane. Suppose that the motion in S is wave-like with a mean wave-length λ. Then if z is comparable with λ, and R/λ and $(RR_1)^{1/2}/\lambda$ are both large compared with unity, where $2R_1$ is the side of the largest square over which the second-order pressure variations are effectively in phase, the variable part of the total force acting on the bed inside the square S is $[F]_{z=h}$ where

$$\left[\frac{F}{\rho}\right]_{z=h} = \int_{-R}^{R}\int_{-R}^{R}\left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2\right) - \omega^2\right] dx\, dy \tag{5·34}$$

Since ω diminishes rapidly with depth,

$$\left[\frac{F}{\rho}\right]_{z=h} = \frac{\partial^2}{\partial t^2}\int_{-R}^{R}\int_{-R}^{R}\tfrac{1}{2}\zeta^2\,dx\,dy \qquad (5\cdot35)$$

if $z > \tfrac{1}{2}\lambda$.

If it supposed that the motion is initially confined to a finite region of the (x,y) plane, then the motion produces a total force $\mathbf{F}$ over the whole bed, given by

$$\frac{\mathbf{F}}{\rho} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\partial^2}{\partial t^2}\left(\tfrac{1}{2}\zeta^2\right) - \omega^2\right]dx\,dy$$

$$\simeq \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2}\zeta^2\,dx\,dy \qquad (5\cdot36)$$

if $z > \tfrac{1}{2}\lambda$.

Pressure variation at the sea-bed in terms of the frequency spectrum.

By reference to equations (5•19) and (5•36) we sea that the variable part of the total force acting on the entire area of the sea-bed, that is the whole area of the xy-plane is given by

$$\frac{F}{\rho} = R.\left(\frac{\pi}{k}\right)^2 \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{A(u,v)A^*(u,v) + A(u,v)A(-u,-v)e^{2i\sigma t}\right\}du\,dv$$

$$= -R.\,4\left(\frac{\pi}{k}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u,v)A(-u,-v)\sigma^2 e^{2i\sigma t}\,du\,dv \qquad (5•37)$$

Now $A(u,v)$ and $A(-u,-v)$ are the complex amplitudes of opposite wave-components in the frequency spectrum. So that three conclusions may be drawn from equation (5•37), viz.:

(I) The variations in the total force on the sea bed arise only from opposite pairs of wavescomponents in the frequency spectrum into which the sea motion may be analysed.

(2) The contribution to F from any opposite pair of wave components is of twice their frequency and proportional to the product of their amplitudes.

(3) The total force F is the integrated sum of the contributions from all opposite pairs of wave components.

A wave group is a complicated wave motion, the component simple waves all travelling in the same direction; such a motion may be defined as one which most of the energy is confined to a small region of the (u,v) plane, excluding the origin. A single wave group will not possess opposite pairs of wave components and so cannot cause variations in the force on the sea-bed. For appreciable variations in the total force on the bed the surface motion must possess at least two wave-groups which are opposite, im the sense that some wave-components of the first group are opposite to some wave components of the second.

We now determine the total force over a finite area of the sea-bed. We take as the area a square S, symmetrically situated with respect to the origin and the axes of x and y and defined by $-R < x < R$, $-R < y < R$.

Let us now define a hypothetical motion of the sea surface, where the equation of the surface at any instant is $z = \zeta'$, such that at any time :

$$\left.\begin{aligned} \zeta' &= \zeta \\ \text{and} \quad \frac{\partial \zeta'}{\partial t} &= \frac{\partial \zeta}{\partial t} \end{aligned}\right\} \text{within the square S,} \qquad (5{\cdot}38)$$

$$\text{and} \quad \zeta' = \frac{\partial \zeta'}{\partial t} = 0, \text{ outside the square S.} \qquad (5{\cdot}39)$$

This motion will not satisfy the equations of motion, especially near the boundaries of S, but it enables us to replace integrals between the limits $-R$ and R by those with limits $\pm\infty$. Thus equation (5·36) yields :

$$\frac{F}{\rho} = \frac{\partial^2}{\partial t^2} \int_{-R}^{R}\int_{-R}^{R} \tfrac{1}{2}\zeta^2 \, dx\, dy = \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2}\zeta'^2 \, dx\, dy \qquad (5{\cdot}40)$$

We also define $A'(u,v;t)$ by the equations

$$\left.\begin{aligned} \zeta' &= R.\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A'(u,v;t)\, e^{i(ukx+vky+\sigma t)}\, du\, dv, \\ \frac{\partial \zeta'}{\partial t} &= R.\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\sigma A'(u,v;t)\, e^{i(ukx+vky+\sigma t)}\, du\, dv. \end{aligned}\right\} \qquad (5{\cdot}41)$$

by analogy with equation (5•16) and other equations defining ζ in terms of A(u,v). Hence

$$\frac{1}{2}\left[A'(u,v;t)+A'^{*}(-u,-v;t)\right]e^{i\sigma t}=\left(\frac{k}{2\pi}\right)^{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta' e^{-i(ux+vy)k}\,dx\,dy,$$

$$\frac{1}{2}i\sigma\left[A'(u,v;t)-A'^{*}(-u,-v;t)\right]e^{i\sigma t}=\left(\frac{k}{2\pi}\right)^{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial\zeta'}{\partial t}\cdot e^{-i(ux+vy)k}\,dx\,dy;$$

hence

$$A'(u,v;t)e^{i\sigma t}=\left(\frac{k}{2\pi}\right)^{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\zeta'+\frac{1}{i\sigma}\cdot\frac{\partial\zeta'}{\partial t}\right)e^{-i(ux+vy)k}\,dx\,dy \qquad (5\cdot42)$$

The actual motion of the sea surface is taken to be defined by equation (5•16).

Because of (5•38) and (5•39)

$$\left.\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta'\,dx\,dy &= \int_{-R}^{R}\int_{-R}^{R}\zeta\,dx\,dy,\\ \text{and}\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial\zeta'}{\partial t}\,dx\,dy &= \int_{-R}^{R}\int_{-R}^{R}\frac{\partial\zeta}{\partial t}\,dx\,dy.\end{aligned}\right\} \qquad (5\cdot43)$$

Then equation (5•42) becomes

$$A'(u,v;t)e^{i\sigma t}=\left(\frac{k}{2\pi}\right)^{2}\int_{-R}^{R}\int_{-R}^{R}\left(\zeta+\frac{1}{i\sigma}\cdot\frac{\partial\zeta}{\partial t}\right)e^{-i(ux+vy)k}\,dx\,dy \qquad (5\cdot44)$$

But from equation (5·16), where $A(u_1,v_1)$ is a neighbouring wave component,

$$\zeta = R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u_1,v_1)e^{i(u_1kx+v_1ky+\sigma_1 t)}\,du_1\,dv_1$$

$$= R.\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u_1,v_1)\,e^{i\sigma_1 t}\cdot e^{i(u_1kx+v_1ky)}\,du_1\,dv_1$$

where $\sigma_1 = \sigma(u_1,v_1)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2}\left[A(u_1,v_1)e^{i\sigma_1 t} + A^{*}(-u_1,-v_1)e^{-i\sigma_1 t}\right]e^{i(u_1kx+v_1ky)}\,du_1\,dv_1$$

Similarly

$$\frac{\partial\zeta}{\partial t} = R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\,\sigma_1\,A(u_1,v_1)e^{i\sigma_1 t}\cdot e^{i(u_1kx+v_1ky)}\,du_1\,dv_1$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2}\left[i\sigma_1\,A(u_1,v_1)\,e^{i\sigma_1 t} - i\sigma_1 A^{*}(-u_1,-v_1)e^{-i\sigma_1 t}\right]e^{i(u_1kx+v_1ky)}\,du_1\,dv_1$$

Hence $\zeta + \frac{1}{i\sigma}\cdot\frac{\partial\zeta}{\partial t}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2}\Big[A(u_1,v_1)\,e^{i\sigma_1 t} + A^{*}(-u_1,-v_1)e^{-i\sigma_1 t} + \frac{\sigma_1}{\sigma}A(u_1,v_1)e^{i\sigma_1 t}$$

$$- \frac{\sigma_1}{\sigma}\cdot A^{*}(-u_1,-v_1)e^{-i\sigma_1 t}\Big]e^{i(u_1kx+v_1ky)}\,du_1\,dv_1$$

That is $\quad \zeta + \frac{1}{i\sigma}\cdot\frac{\partial\zeta}{\partial t}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2}\left[\left(1+\frac{\sigma_1}{\sigma}\right)A(u_1v_1)e^{i\sigma_1 t} + \left(1-\frac{\sigma_1}{\sigma}\right)A^*(-u_1-v_1)e^{-i\sigma_1 t}\right]e^{i(u_1kx+v_1ky)}\,du_1dv_1$$

Hence equation (5•44) becomes $\quad A'(u,v;t) =$

$$= \left(\frac{k}{2\pi}\right)^2\int_{-R}^{R}\int_{-R}^{R}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2}\left\{\left(1+\frac{\sigma_1}{\sigma}\right)A(u_1v_1)e^{i\sigma_1 t} + \left(1-\frac{\sigma_1}{\sigma}\right)A^*(-u_1-v_1)e^{-i\sigma_1 t}\right\}\right.$$

$$\left. e^{i(u_1kx+v_1ky)}\,du_1dv_1\right]\cdot e^{-i(ukx+vky+\sigma t)}\,dx\,dy$$

$$= \left(\frac{k}{2\pi}\right)^2\int_{-R}^{R}\int_{-R}^{R}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2}\left\{A(u_1v_1).\left(1+\frac{\sigma_1}{\sigma}\right)e^{-i[(u-u_1)kx+(v-v_1)ky+(\sigma-\sigma_1)t]}\right.\right.$$

$$\left.\left. + A^*(-u_1-v_1)\left(1-\frac{\sigma_1}{\sigma}\right)e^{-i[(u-u_1)kx+(v-v_1)ky+(\sigma+\sigma_1)t]}\,du_1\,dv_1\right]dx\,dy$$

But $\displaystyle \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A^*(-u_1-v_1)\,e^{-i\{(u-u_1)kx+(v-v_1)ky+(\sigma+\sigma_1)t\}}\,du_1dv_1$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A^*(u_2v_2)\,e^{-i\{(u+u_2)kx+(v+v_2)ky+(\sigma+\sigma_1)t\}}\,du_2dv_2$$

where $\quad u_1 = -u_2\,,\quad v_1 = -v_2$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A^*(u_1,v_1)\,e^{-i\{(u+u_1)kx+(v+v_1)ky+(\sigma+\sigma_1)t\}}\,du_1\,dv_1$$

Hence $A'(u,v;t) =$

$$= \left(\frac{k}{2\pi}\right)^2 \int_{-R}^{R}\int_{-R}^{R} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}\left\{ A(u_1 v_1)\left(1+\frac{\sigma_1}{\sigma}\right) e^{-i\{(u-u_1)kx+(v-v_1)ky+(\sigma-\sigma_1)t\}} \right.\right. \qquad (5\ 45)$$

$$\left.\left. + A^*(u_1 v_1).\left(1-\frac{\sigma_1}{\sigma}\right) e^{-i\{(u+u_1)kx+(v+v_1)ky+(\sigma+\sigma_1)t\}} \right\} du_1\, dv_1 \right] dx\, dy$$

Since k is as yet unspecified we can write, for convenience,

$$k = \frac{\pi}{R} \qquad (5\text{-}46).$$

$$\int_{-R}^{R}\int_{-R}^{R} e^{-i(u-u_1)kx} . e^{-i(v-v_1)ky} \, dx\, dy$$

$$= \int_{-R}^{R} \left[\frac{e^{-i(u-u_1)kx}}{-i\ (u-u_1)\ k} . e^{-i(v-v_1)ky} \right]_{-R}^{R} dy$$

$$= \int_{-R}^{R} \left[\frac{e^{i(u-u_1)\pi} - e^{-i(u-u_1)\pi}}{i(u-u_1)} \right] . e^{-i(v-v_1)ky} dy$$

$$= \frac{2\sin(u-u_1)\pi}{(u-u_1)k} \int_{-R}^{R} e^{-i(v-v_1)ky} dy$$

$$= \frac{2\sin(u-u_1)\pi}{(u-u_1)k(-i)(v-v_1)} \left[e^{-i(v-v_1)\pi} - e^{i(v-v_1)\pi} \right]$$

$$= \frac{4\sin(u-u_1)\pi . \sin(v-v_1)\pi}{(u-u_1).(v-v_1)\ k^2} .$$

Similarly $$\int_{-R}^{R}\int_{-R}^{R} e^{-i(u+u_1)kx-} \cdot e^{-i(V+V_1)ky} \cdot dx\,dy$$

$$= \frac{4\sin(u+u_1)\pi \cdot \sin(V-V_1)\pi}{(u+u_1)(V+V_1)\cdot k^2} .$$

Hence equation (5•45) becomes

$$A'(u,v;t) =$$
$$\frac{k^2}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Bigg[\left\{A(u_1V_1)\left(1+\frac{\sigma_1}{\sigma}\right)e^{-i(\sigma-\sigma_1)t} \cdot \frac{\sin(u-u_1)\pi \cdot \sin(V-V_1)\pi}{(u-u_1)(V-V_1)k^2}\right\}$$
$$+\left\{A^*(u_1V_1)\left(1-\frac{\sigma_1}{\sigma}\right)e^{-i(\sigma+\sigma_1)t} \cdot \frac{\sin(u+u_1)\pi \cdot \sin(V+V_1)\pi}{(u+u_1)(V+V_1)k^2}\right\}\Bigg]du_1 dv_1$$

That is

$$A'(u,v;t)$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u_1V_1)\left(1+\frac{\sigma_1}{\sigma}\right) \cdot \frac{\sin(u-u_1)\pi}{(u-u_1)\pi} \cdot \frac{\sin(V-V_1)\pi}{(V-V_1)\pi} \cdot e^{-i(\sigma-\sigma_1)t}\,du_1 dV_1$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A^*(u_1V_1)\left(1-\frac{\sigma_1}{\sigma}\right) \cdot \frac{\sin(u+u_1)\pi}{(u+u_1)\pi} \cdot \frac{\sin(V+V_1)\pi}{(V+V_1)\pi} \cdot e^{-i(\sigma+\sigma_1)t}\,du_1 dV_1$$

$$= I_1 + I_2 \qquad (5\text{•}47)$$

In the derivation of equation (5·32) it has already been stipulated that λ is very much less than R, so the frequency spectrum of the motion ζ consists of waves whose length, given equation (5·4) is small compared with 2R. The factors in the denominators of I_1 and I_2 make the integrands small except when $(u_1, v_1) \simeq (u,v)$ in I_1 and $(u_1, v_1) \simeq (-u,-v)$ in I_2. In either case $\sigma_1 \simeq \sigma$. So that in either case $1 - \frac{\sigma_1}{\sigma} \simeq 0$ and $1 + \frac{\sigma_1}{\sigma} \simeq 2$, that is

$$A'(u,v;t) \simeq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u_1,v_1) . \frac{\sin(u-u_1)\pi}{(u-u_1)\pi} . \frac{\sin(v-v_1)\pi}{(v-v_1)\pi} . e^{-i(\sigma-\sigma_1)t} \, du_1 dv_1 . \qquad (5·48).$$

Although $A(u,v;t)$ is dependent on t the integrals for $\frac{\partial A'}{\partial t}$, $\frac{\partial^2 A'}{\partial t^2}$, etc. contain factors $(\sigma-\sigma_1)$, $(\sigma-\sigma_1)^2$, etc. which are small over the critical range of integration near (u,v), where the two wave components are nearly alike. These expressions are therefore small, and so $A'(u,v;t)$ is only a slowly varying quantity, in time. Hence we may use $A'(u,v)$ for $A'(u,v;t)$.

From equation (5·40) and (5·41)

$$\left[\frac{F}{\rho}\right]_{z=h} = R.\left(\frac{\pi}{k}\right)^2 . \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[A'(u,v). A'^{*}(u,v) + A'(u,v). A'(-u,-v) e^{2i\sigma t}\right] du\,dv$$

$$= R.\left(\frac{\pi}{k}\right)^2 . \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2i\sigma\, A'(u,v). A'(-u,-v). e^{2i\sigma t} du\,dv$$

$$\simeq -R. 4\left(\frac{\pi}{k}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma^2 A'(u,v). A'(-u,-v). e^{2i\sigma t} du\,dv. \qquad (5·49)$$

Comparison of equations (5·37) and (5·49) shows that the expression for $\mathbf{F}'$ the force over a finite area (equation5·49) is similar to that over the whole plane (equation 5·37) except that the original spectrum A of the actual motion is replaced by a new spectrum A'.
Equation (5·48) shows that A' is the weighted mean of neighbouring wave components, $A(u,v)$ and $A(u_1, v_1)$, of the original spectrum. That is each wave component in the new spectrum is a blend of neighbouring wave components in the original spectrum, and further each wave component in the original spectrum contributes to neighbouring components in the new spectrum.
From equations (5·4) and (5·46) $\frac{2R}{\lambda} = (u^2+v^2)^{1/2}$ so that the number of wave lengths of any wave component intercepted on the x-axis inside the square region S is u, and the corresponding number on the y-axis is v. Neighbouring wave components of the new spectrum are those such that the number of wave-lengths intercepted on any diameter of S does not differ by 2 or 3 from the corresponding number for the original wave component.

Thus in order to calculate the total force on the sea-bed under a limited region of an actual motion, we may obtain a close approximation to the required result, by calculating the total force over the entire plane for a hypothetical motion. This new motion being such that over the finite region the elevation of the surface and its rate of change in time are the same as in the original motion, but outside the region they are zero. If the dimensions of the region are much greater than the mean wave-length of the original motion, then the new motion will have within the region a frequency spectrum, which differs only slightly from that of the original motion. The contribution of each wave component of the old spectrum to several neighbouring components of the new spectrum results in the new spectrum being a 'blurred' edition of the old 'sharp' spectrum. This blurring may be regarded as being due to an inability to define the spectrum exactly from a knowledge of conditions over only a limited area. The amount of blurring is not enough to prevent satisfactory results being obtained by the new spectrum A'.

Since $A'(u,v)$ is the frequency spectrum of the hypothetical free motion in which at time $t=t_1$, ζ and $\frac{\partial \zeta}{\partial t}$ take their actual values within the square S defined by $-R < x < R$, $-R < y < R$ but are zero outside the square. Then when $t=t_1$ all the potential energy and nearly all the kinetic energy of the motion is contained in the square S. Hence, after reference to equation (5•25), the total energy of the square is very nearly

$$2\rho g\left(\frac{\pi}{k}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A'(u,v).\,A'^*(u,v)\,du\,dv \qquad (5\cdot 50).$$

CHAPTER 6.

Wave motion in a heavy compressible fluid.

In the preceding chapters the fluid has been regarded as incompressible. This assumption is only valid so long as the time taken for a disturbance to be propagated to the bed is small compared with the period of the waves, that is

$$\frac{h}{c} \ll T \qquad \text{or} \qquad h \ll cT \tag{6.0}$$

For ocean waves h may be of the order of several kilometers, c is about 1·4 km/sec, and T lies between about 5 and 20 secs. With these values of c and T we see that cT lies in the range

$$7{\cdot}0 \text{ km.} \leqslant cT \leqslant 28{\cdot}0 \text{ km.}$$

That is, condition (6·0) is not satisfied.
It follows that a satisfactory theory must therefore take account of the compressibility of the sea water.

In this chapter we shall develope to the second order the theory of the wave motion, in a compressible medium, which to the first order of amplitudes is a standing wave of the gravity type.

We shall first of all build up the general equations and then solve them by successive approximations.

General Equations.

Take rectangular axes Ox, Oy and Oz with the origin in the free surface at rest, the z axis vertically downwards and the y axis parallel to the wave crests.

The motion is taken to be periodic in the x-direction with wavelength λ .

Let $z = h$ be the equation of the sea bed (assumed rigid) and $z = \zeta$ the equation of the free surface.

Let $\underline{u}$ be the velocity, p the pressure, ρ the density of the fluid, and let p_s and ρ_s denote the values of p and ρ respectively at the free surface.

Assuming that the viscosity is negligible and that the motion is irrotational,

$$\underline{u} = -\operatorname{grad} \phi \tag{6.1}$$

where ϕ is the velocity potential.

Assuming ρ to be a function of p only,

$$\frac{\partial \phi}{\partial t} - \tfrac{1}{2}\underline{u}^2 + gz - \int_{p_s}^{p} \frac{dp}{\rho} = 0 \tag{6.2}$$

(Milne-Thomson: 'Theoretical Hydrodynamics page 82)
where ϕ contains an arbitrary function of time t.

Set $$P = \int_{\rho_s}^{\rho} \frac{dp}{\rho} \qquad (6\cdot3)$$

Assume that the relation between p and ρ is

$$\frac{dp}{d\rho} = c^2 = \text{constant} \qquad (6\cdot4)$$

that is, that the velocity of sound in the fluid, c, is constant.

Then from (6·3) and (6·4)

$$P = \int_{\rho_s}^{\rho} c^2 \frac{d\rho}{\rho} = c^2 \log\left(\frac{\rho}{\rho_s}\right) \qquad (6\cdot5)$$

The equation of continuity (Milne-Thomson page 68) is

$$\frac{D\rho}{Dt} - \rho\nabla^2\phi = 0 \qquad (6\cdot6)$$

where $\frac{D}{Dt}$ denotes (as in chapter 3) the differentiation following the motion.

Hence $$\frac{D\rho}{Dt} = \rho\,\nabla^2\phi$$

$$\therefore \nabla^2\phi = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}\right)$$

$$= \frac{1}{\rho}\left(\frac{\partial\rho}{\partial t} + w\frac{\partial\rho}{\partial z}\right)$$

$$= \frac{\partial(\log\rho)}{\partial t} + u\frac{\partial}{\partial x}(\log\rho) + v\frac{\partial}{\partial y}(\log\rho) + w\frac{\partial}{\partial z}(\log\rho)$$

$$= \frac{D}{Dt}(\log\rho)$$

$$= \frac{D}{Dt}(\log\rho - \log\rho_s) \quad \text{since } \rho_s = \text{constant}$$

$$= \frac{D}{Dt}\left(\log\frac{\rho}{\rho_s}\right)$$

$$= \frac{D}{Dt}\left(\frac{P}{c^2}\right) \quad \text{by } (6\cdot5)$$

Hence $$\nabla^2\phi = \frac{1}{c^2}\cdot\frac{DP}{Dt} \tag{6·7}$$

Eliminating P between equations (6·2), (6·3) and (6·7)

$$c^2\nabla\phi = \frac{D}{Dt}\left(\frac{\partial\phi}{\partial t} - \frac{1}{2}\underline{u}^2 + gz\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial t}\right) + w\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right)$$

$$-\frac{\partial}{\partial t}\left(\frac{1}{2}\underline{u}^2\right) - u\frac{\partial}{\partial x}\left(\frac{1}{2}\underline{u}^2\right) - v\frac{\partial}{\partial y}\left(\frac{1}{2}\underline{u}^2\right) - w\frac{\partial}{\partial z}\left(\frac{1}{2}\underline{u}^2\right)$$

$$+g\left(\frac{\partial z}{\partial t} + u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} + w\frac{\partial z}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial t^2} + \underline{u}\,\nabla\left(\frac{\partial\phi}{\partial t}\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\underline{u}^2\right) - \underline{u}\,\nabla\left(\frac{1}{2}\underline{u}^2\right) + gw$$

$$\left[\text{since } \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0\right]$$

$$= \frac{\partial^2\phi}{\partial t^2} - \frac{\partial}{\partial t}\left(\frac{1}{2}\underline{u}^2\right) + \underline{u}\,\nabla\left(\frac{\partial\phi}{\partial t}\right) - \underline{u}\,\nabla\left(\frac{1}{2}\underline{u}^2\right) - g\frac{\partial\phi}{\partial z} \tag{6·8}$$

since $\frac{\partial\phi}{\partial z} = -w$, the motion being irrotational.

But $\underline{u} = -\nabla\phi$, hence

$$\underline{u}\cdot\nabla\left(\frac{\partial\phi}{\partial t}\right) = \underline{u}\cdot\frac{\partial}{\partial t}(\nabla\phi) = \underline{u}\cdot\frac{\partial}{\partial t}(-\underline{u}) = -\frac{1}{2}\cdot\frac{\partial\underline{u}^2}{\partial t}$$

Hence equation (6·8) becomes

$$\frac{\partial^2\phi}{\partial t^2} - c^2\nabla^2\phi - g\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial t}\left(\frac{1}{2}\underline{u}^2\right) - \underline{u}\cdot\nabla\left(\frac{1}{2}\underline{u}^2\right) = 0 \tag{6·9}$$

This is our differential equation for ϕ for which we now find a solution by successive approximation.

The first thing is to express the boundary conditions which a solution must satisfy.

At the bed, $z = h$ and $-\omega = \left(\frac{\partial\phi}{\partial z}\right)_{z=h} = 0$ (6·10)

At the free surface $z = \zeta$ and $p = p_s$

and by (6·5) $P_{z=\zeta} = c^2 \log \rho_s/\rho_s = c^2 \log 1 = 0$ (6·11)

and $\therefore$ by (6·2) $\left[\frac{\partial\phi}{\partial t} - \frac{1}{2}\underline{u}^2 + gz\right]_{z=\zeta} = 0$ (6·12)

From (6·11) $\left(\frac{DP}{Dt}\right)_{z=\zeta} = 0$ (6·13)

and hence $\left(\nabla^2\phi\right)_{z=\zeta} = \left(\frac{DP}{Dt}\right)_{z=\zeta} = 0$ (6·14)

Equations 6·12, 6·13 and 6·14 express the conditions to be satisfied at the free surface $z = \zeta$. It is, however, more convenient to have conditions satisfied at the bed $z = 0$. These can be obtained by expanding the equations by Taylor's Theorem:

$$\left[\frac{\partial\phi}{\partial t} - \frac{1}{2}\underline{u}^2 + g(z+\zeta)\right]_{z=0} = \left[\frac{\partial\phi}{\partial t} - \frac{1}{2}\underline{u}^2 + g\zeta\right]_{z=\zeta} = 0 \quad \text{by (6·12)}$$

and
$$\frac{\partial\phi}{\partial t} - \frac{1}{2}\underline{u}^2 + g(z+\zeta)$$
$$= \left(\frac{\partial\phi}{\partial t} - \frac{1}{2}\underline{u}^2 + gz\right) + \zeta.\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t} - \frac{1}{2}\underline{u}^2 + gz\right)$$
$$+ \frac{\zeta^2}{\underline{|2}}\cdot\frac{\partial^2}{\partial z^2}\left(\frac{\partial\phi}{\partial t} - \frac{1}{2}\underline{u}^2 + gz\right) + \text{------}$$
$$= \frac{\partial\phi}{\partial t} - \frac{1}{2}\underline{u}^2 + gz + \zeta\left(\frac{\partial^2\phi}{\partial t\,\partial z} - \bar{u}\frac{\partial\bar{u}}{\partial z} + g\right)$$
$$+ \frac{\zeta^2}{\underline{|2}}\left[\frac{\partial^3\phi}{\partial t\,\partial z^2} - \underline{u}\frac{\partial^2\underline{u}}{\partial z^2} - \left(\frac{\partial\underline{u}}{\partial z}\right)^2\right] + \text{----}$$

$$\therefore \left[\frac{\partial\phi}{\partial t} - \frac{1}{2}\underline{u}^2 + g(z+\zeta)\right]_{z=0} = \left(\frac{\partial\phi}{\partial t} - \frac{1}{2}\underline{u}^2 + gz\right)_{z=0}$$
$$+ \zeta\left(\frac{\partial^2\phi}{\partial t\,\partial z} - \underline{u}\frac{\partial\underline{u}}{\partial z} + g\right)_{z=0} + \frac{\zeta^2}{\underline{|2}}\left[\frac{\partial^3\phi}{\partial t\,\partial z^2} - \underline{u}\frac{\partial^2\underline{u}}{\partial z^2} - \left(\frac{\partial\underline{u}}{\partial z}\right)^2\right]_{z=0}$$
$$+ \text{--------}$$

That is

$$\left(\frac{\partial\phi}{\partial t} - \frac{1}{2}\underline{u}^2\right)_{z=0} + \zeta\left(\frac{\partial^2\phi}{\partial t\,\partial z} - \underline{u}\cdot\frac{\partial\underline{u}}{\partial z} + g\right)_{z=0}$$

$$+ \frac{\zeta^2}{\underline{|2}}\left[\frac{\partial^3\phi}{\partial t\,\partial z^2} - \underline{u}\cdot\frac{\partial^2\underline{u}}{\partial z^2} - \left(\frac{\partial\underline{u}}{\partial z}\right)^2\right]_{z=0} + \cdots = 0 \qquad (6{\cdot}15)$$

By Taylor's theorem

$$F(z+\zeta) = F(z) + \zeta F'(z) + \frac{\zeta^2}{\underline{|2}}F''(z) + \cdots$$

and $\therefore \left[F(z+\zeta)\right]_{z=0} = \left[F(z)\right]_{z=\zeta} = F(0) + \zeta F'(0) + \frac{\zeta^2}{\underline{|2}}F''(0) + \cdots$

Applying this to equation (6•14)

$$\left[\nabla^2\phi\right]_{z=\zeta} = 0 = \left[\nabla^2\phi\right]_{z=0} + \zeta\left[\frac{\partial}{\partial z}\nabla^2\phi\right]_{z=0} + \frac{\zeta^2}{\underline{|2}}\left[\frac{\partial^2}{\partial z^2}\nabla^2\phi\right]_{z=0} + \cdots \qquad (6{\cdot}16)$$

In order to define the solution completely it is necessary to add a further condition expressing the assumption that the origin is in the undisturbed free surface Since the mass contained below the free surface is the same as in the undisturbed state we have

$$\int_0^\lambda dx \int_\zeta^h \rho\, dz = \int_0^\lambda dx \int_0^h \rho_0\, dz \qquad (6{\cdot}17)$$

where the suffix o denotes the value in the undisturbed state.
Equation (6•17) can be rewritten

$$\int_0^\lambda dx \int_0^h (\rho - \rho_0)\, dz - \int_0^\lambda dx \int_0^\zeta \rho\, dz = 0 \qquad (6{\cdot}18)$$

Set $\displaystyle\int_0^\zeta \rho\, dz = F(\zeta) + = \left[F(z+\zeta)\right]_{z=0}$

$$= \left[F(z) + \zeta\cdot\frac{\partial F(z)}{\partial z} + \frac{\zeta^2}{\underline{|2}}\cdot\frac{\partial^2 F(z)}{\partial z^2} + \cdots\right]_{z=0}$$

Hence since $F(z) = \int_0^z \rho\, dz$, $F(0) = 0$

and $\frac{\partial F}{\partial z} = \rho$ $\therefore$ $F'(0) = [\rho]_{z=0}$

and $\frac{\partial^2 F}{\partial z^2} = \frac{\partial \rho}{\partial z}$ $\therefore \left(\frac{\partial^2 F}{\partial z^2}\right)_{z=0} = \left(\frac{\partial \rho}{\partial z}\right)_{z=0}$

etc.

$$\therefore \int_0^{\zeta} \rho\, dz = \zeta[\rho]_{z=0} + \frac{\zeta^2}{L^2}\left[\frac{\partial \rho}{\partial z}\right]_{z=0} + \text{-------}$$

Hence equation (6•18) becomes

$$\int_0^{\lambda} dx \int_0^{h} (\rho - \rho_0)\, dz -$$

$$- \int_0^{\lambda} dx \left[\zeta (\rho)_{z=0} + \tfrac{1}{2}\zeta^2 \left(\frac{\partial \rho}{\partial z}\right)_{z=0} + \text{---}\right] = 0 \quad (6{\cdot}19)$$

From equations (6•2), (6•3) and (6•5)

$$\frac{\rho}{\rho_s} = e^{\frac{P}{c^2}} = e^{\frac{1}{c^2}\left(\frac{\partial \phi}{\partial t} - \frac{1}{2}\underline{u}^2 + gz\right)} \quad (6{\cdot}20)$$

so that $\frac{\rho_0}{\rho_s} = e^{\frac{gz}{c^2}}$ (6•21)

since $\underline{u} = 0$ in the undisturbed state.

From equation (6•4)

$$\left[p\right]_{p_s}^{p} = \left[\rho c^2\right]_{\rho_s}^{\rho}$$

$$\therefore\ p - p_s = (\rho - \rho_s)c^2$$

$$\therefore\ p_0 - p_s = (\rho_0 - \rho_s)c^2 \quad (6{\cdot}22)$$

$$= \left(\rho_s e^{\frac{gz}{c^2}} - \rho_s\right)c^2 \quad \text{by } (6{\cdot}21)$$

$$\therefore \; p_0 - p_s = c^2 \rho_s \left(e^{gz/c^2} - 1 \right)$$

To find solutions for equation (6•9) we write

$$\left.\begin{aligned}
\phi &= \epsilon \phi_1 + \epsilon^2 \phi_2 + --- \\
\underline{u} &= \epsilon \underline{u}_1 + \epsilon^2 \underline{u}_2 + --- \\
J &= \epsilon J_1 + \epsilon^2 J_2 + --- \\
p - p_0 &= \epsilon p_1 + \epsilon^2 p_2 + --- \\
\rho - \rho_0 &= \epsilon \rho_1 + \epsilon^2 \rho_2 + ---
\end{aligned}\right\} \qquad (6{\cdot}24)$$

where ϵ is a small parameter.
We can substitute from equation (6•24) in equations (6•9), (6•10) and (6•16) :

$$\frac{\partial^2}{\partial t^2}(\epsilon \phi_1 + \epsilon^2 \phi_2 + --) - c^2 \nabla^2 (\epsilon \phi_1 + \epsilon^2 \phi_2)$$

$$- g \frac{\partial}{\partial z}(\epsilon \phi_1 + \epsilon^2 \phi_2 + -) - \frac{\partial}{\partial t}\left\{ \tfrac{1}{2} (\epsilon \underline{u}_1 + \epsilon^2 \underline{u}_2 + --) \right\}$$

$$- (\epsilon \underline{u}_1 + \epsilon^2 \underline{u}_2 + --) \operatorname{grad} \left\{ \tfrac{1}{2} (\epsilon \underline{u}_1 + \epsilon^2 \underline{u}_2 + --)^2 \right\} = 0;$$

$$\left\{ \frac{\partial}{\partial z} (\epsilon \phi_1 + \epsilon^2 \phi_2 + --) \right\}_{z=h} = 0; \qquad \text{and}$$

$$\left\{ \nabla^2 (\epsilon \phi_1 + \epsilon^2 \phi_2 + --) \right\}_{z=0}$$

$$+ (\epsilon J_1 + \epsilon^2 J_2 + --) \left\{ \frac{\partial}{\partial z} \nabla^2 (\epsilon \phi_1 + \epsilon^2 \phi_2 + --) \right\}_{z=0}$$

$$+ -------- = 0.$$

For a first approximation to the values of ϕ , $\underline{u}$, etc. we can neglect powers of ϵ above the first, and for a second approximation we neglect powers of ϵ above the square.
By equating to zero the coefficients of ϵ and ϵ^2 we may thus obtain equations by means of which ϕ_1 and ϕ_2 may be determined.
For the first approximation we have

$$\frac{\partial^2 \phi_1}{\partial t^2} - c^2 \nabla^2 \phi_1 - g \frac{\partial \phi_1}{\partial z} = 0, \qquad (6\cdot 25)$$

$$\left(\frac{\partial \phi_1}{\partial z} \right)_{z=h} = 0, \qquad (6\cdot 26)$$

$$\left(\nabla^2 \phi_1 \right)_{z=0} = 0. \qquad (6\cdot 27)$$

From equations (6·1), (6·15) and (6·22) for the first approximation we have

$$\underline{u}_1 = -\operatorname{grad} \phi_1 \qquad (6\cdot 28)$$

$$\left(\frac{\partial \phi_1}{\partial t} \right)_{z=0} + g \zeta_1 = 0 \qquad (6\cdot 29)$$

and $$\left(p_0 + \epsilon p_1 + \epsilon^2 p_2 + \cdots - p_s \right) = c^2 \left(\rho_0 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \cdots - \rho_s \right)$$

Equating the coefficient of ϵ to zero

$$p_1 = c^2 \rho_1$$

After substituting in equation (6·20)

$$\frac{\rho_0 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \cdots}{\rho_s} = e^{\frac{1}{c^2}\left[\frac{\partial}{\partial t}\left(\epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots \right) - \frac{1}{2}\left(\epsilon \underline{u}_1 + \epsilon^2 \underline{u}_2 + \cdots \right)^2 + gz \right]}$$

$$= e^{gz/c^2} \left[1 + \frac{1}{c^2} \left\{ \frac{\partial}{\partial t}\left(\epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots \right) - \frac{1}{2}\left(\epsilon \underline{u}_1 + \epsilon^2 \underline{u}_2 + \cdots \right)^2 \right\} \right.$$

$$+ \frac{1}{c^4} \cdot \frac{1}{\underline{|2}} \left\{ \frac{\partial}{\partial t}\left(\epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots \right) - \frac{1}{2}\left(\epsilon \underline{u}_1 + \epsilon^2 \underline{u}_2 + \cdots \right)^2 \right\}^2$$

$$+ \cdots\cdots\cdots$$

Equating to zero the coefficient of ϵ , we have

$$\frac{\rho_1}{\rho_s} = \frac{e^{\frac{gz}{c^2}}}{c^2} \cdot \frac{\partial \phi_1}{\partial t}$$

Hence $$\frac{p_1}{\rho_s} = c^2 \frac{\rho_1}{\rho_s} = \frac{\partial \phi_1}{\partial t} \cdot e^{2\gamma z} \qquad (6\cdot30)$$

where $2\gamma = g/c^2$ (6·31)

For the second approximation we obtain :

from (6·9) $$\frac{\partial^2 \phi_2}{\partial t^2} - c^2 \nabla^2 \phi_2 - g \frac{\partial \phi_2}{\partial z} = \frac{\partial}{\partial t}(\underline{u}_1^2) \qquad (6\cdot32)$$

from(6·10) $$\left(\frac{\partial \phi_2}{\partial z}\right)_{z=h} = 0 \qquad (6\cdot33)$$

from (6·16) $$\left(\nabla^2 \phi_2\right)_{z=0} = - \mathcal{J}_1 \left(\frac{\partial}{\partial z} \nabla^2 \phi_1\right)_{z=0} \qquad (6\cdot34)$$

from (6·1) $$\underline{u}_2 = - \operatorname{grad} \phi_2 \qquad (6\cdot35)$$

from (6·15) $$g \mathcal{J}_2 = -\left(\frac{\partial \phi_2}{\partial t} - \frac{1}{2} \underline{u}_1^2\right) - \mathcal{J}_1 \left(\frac{\partial^2 \phi_1}{\partial t \partial z}\right)_{z=0} \qquad (6\cdot36)$$

from (6·22) $$p_2 = c^2 \rho_2$$

and after substituting in equation (6·20)

$$\frac{p_2}{\rho_s} = c^2 \frac{\rho_2}{\rho_s} = \left[\frac{\partial \phi_2}{\partial t} - \frac{\underline{u}_1^2}{2} + \frac{1}{2c^2} \cdot \left(\frac{\partial \phi_1}{\partial t}\right)^2\right] \cdot e^{2\gamma z} \qquad (6\cdot37)$$

On substituting for ρ and $\mathcal{J}$ in equation (6·19) we obtain

$$\int_0^\lambda dx \int_0^h (\epsilon \rho_1 + \epsilon^2 \rho_2 + \dots)\, dz$$

$$- \int_0^\lambda dx \Big[(\epsilon_1 \mathcal{J}_1 + \epsilon_2 \mathcal{J}^2 + \dots)(\rho_0 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \dots)_{z=0}$$

$$+ \frac{1}{2} (\epsilon \mathcal{J}_1 + \epsilon^2 \mathcal{J}_2 + \dots)^2 \left\{ \frac{\partial}{\partial z} (\rho_0 + \epsilon \rho_1 + \dots)_{z=0} \right\} + \dots \Big] = 0.$$

From (6·21) $$\frac{\rho_0}{\rho_s} = e^{\frac{2gz}{c^2}} = e^{2\gamma z}$$

$$\therefore \quad (\rho_0)_{z=0} = \rho_s \cdot e^0 = \rho_s .$$

Equating the coefficient of ϵ we have

$$\int_0^\lambda dx \int_0^h \rho_1 \, dz - \int_0^\lambda dz . \, \zeta_1 (\rho_0)_{z=0} = 0$$

which is $$\int_0^\lambda dx \int_0^h \rho_1 \, dz - \rho_s \int_0^\lambda \zeta_1 \, dz = 0$$

$$\therefore \quad \int_0^\lambda dx \int_0^h \frac{\rho_1}{\rho_s} dz - \int_0^\lambda \zeta_1 \, dz = 0.$$

Substituting from (6·29) and (6·30)

$$\int_0^\lambda dx \int_0^h \frac{1}{c^2} \cdot \frac{\partial \phi_1}{\partial t} . \, e^{2\gamma z} dz + \frac{1}{g} \int_0^\lambda \left(\frac{\partial \phi_1}{\partial t} \right)_{z=0} dx = 0$$

$$\therefore \quad \frac{g}{c^2} \int_0^\lambda dx \int_0^h \frac{\partial \phi_1}{\partial t} . e^{2\gamma z} dz + \int_0^\lambda \left(\frac{\partial \phi_1}{\partial t} \right)_{z=0} dx = 0$$

$$\therefore \quad 2\gamma \int_0^\lambda dx \int_0^h \frac{\partial \phi_1}{\partial t} \, e^{2\gamma z} dz + \int_0^\lambda \left(\frac{\partial \phi_1}{\partial t} \right)_{z=0} dx = 0 \qquad (6·38)$$

Equating the coefficients of ϵ^2 we have

$$\int_0^\lambda dx \int_0^h \rho_2 \, dz - \int_0^\lambda dx \left[\zeta_1 \rho_1 + \rho_0 \zeta_2 + \frac{1}{2} \zeta_1^2 \cdot \frac{\partial \rho_0}{\partial z} \right]_{z=0} = 0$$

From equations (6·36) and (6·29)

$$\zeta_2 = -\frac{1}{g} \left(\frac{\partial \phi_2}{\partial t} - \frac{1}{2} u_1^2 \right)_{z=0} + \frac{1}{g^2} \left(\frac{\partial \phi_1}{\partial t} \cdot \frac{\partial^2 \phi_1}{\partial z \partial t} \right)_{z=0}$$

From (6·21) $$\frac{\partial \rho_0}{\partial z} = \frac{g}{c^2} \cdot \rho_s \cdot e^{\frac{gz}{c^2}} = \frac{g}{c^2} \cdot \rho_s e^{2\gamma z}$$

Substituting these results and from equations (6·37), (6·30) and (6·29) we have

$$\int_0^\lambda dx \int_0^h \frac{\rho_s}{c^2}\left[\frac{\partial\phi_2}{\partial t} - \frac{1}{2}\underline{u}_1^2 + \frac{1}{2c^2}\left(\frac{\partial\phi_1}{\partial t}\right)^2\right] e^{2\gamma z}\,dz$$

$$-\int_0^\lambda \left[-\frac{\rho_s}{gc^2}\cdot\left(\frac{\partial\phi_1}{\partial t}\right)^2 e^{2\gamma z} - \rho_s\cdot\frac{e^{2\gamma z}}{g}\cdot\left\{\frac{\partial\phi_2}{\partial t} - \frac{1}{2}\underline{u}_1^2 - \frac{\partial^2\phi_1}{\partial z\partial t}\cdot\frac{1}{g}\cdot\frac{\partial\phi_1}{\partial t}\right\}\right.$$

$$\left. + \frac{1}{2}\cdot\frac{1}{g^2}\cdot\left(\frac{\partial\phi_1}{\partial t}\right)^2\cdot\frac{g}{c^2}\cdot\rho_s\cdot e^{2\gamma z}\right]_{z=0} dx = 0.$$

$$\therefore \frac{1}{c^2}\int_0^\lambda dx \int_0^h \left[\frac{\partial\phi_2}{\partial t} - \frac{1}{2}\underline{u}_1^2 + \frac{1}{2c^2}\left(\frac{\partial\phi_1}{\partial t}\right)^2\right] e^{2\gamma z}\,dz$$

$$+\int_0^\lambda \left[\frac{1}{gc^2}\cdot\left(\frac{\partial\phi_1}{\partial t}\right)^2 + \frac{1}{g}\left(\frac{\partial\phi_2}{\partial t} - \frac{1}{2}\underline{u}_1^2 - \frac{1}{g}\cdot\frac{\partial^2\phi_1}{\partial z\partial t}\cdot\frac{\partial\phi_1}{\partial t}\right)\right.$$

$$\left. - \frac{1}{2gc^2}\cdot\left(\frac{\partial\phi_1}{\partial t}\right)^2\right]_{z=0} dx = 0.$$

But $\quad 2\gamma = \dfrac{g}{c^2}$

$$\therefore 2\gamma\int_0^\lambda dx \int_0^h \left[\frac{\partial\phi_2}{\partial t} - \frac{1}{2}\underline{u}_1^2 + \frac{1}{2c^2}\left(\frac{\partial\phi_1}{\partial t}\right)^2\right] e^{2\gamma z}\,dz$$

$$+\int_0^\lambda \left[\frac{\partial\phi_2}{\partial t} - \frac{1}{2}\underline{u}_1^2 - \frac{1}{g}\cdot\frac{\partial^2\phi_1}{\partial z\partial t}\cdot\frac{\partial\phi_1}{\partial t} + \frac{1}{2c^2}\cdot\left(\frac{\partial\phi_1}{\partial t}\right)^2\right]_{z=0} dx = 0.$$

$$\therefore 2\gamma\int_0^\lambda dx \int_0^h \frac{\partial\phi_2}{\partial t}\cdot e^{2\gamma z}\,dz + \int_0^\lambda \left(\frac{\partial\phi_2}{\partial t}\right)_{z=0} dx$$

$$= 2\gamma\int_0^\lambda dx \int_0^h \left[\frac{1}{2}\underline{u}_1^2 - \frac{1}{2c^2}\left(\frac{\partial\phi_1}{\partial t}\right)^2\right] e^{2\gamma z}\,dz$$

$$+\int_0^\lambda \left[\frac{1}{2}\underline{u}_1^2 + \frac{1}{g}\cdot\frac{\partial^2\phi_1}{\partial z\partial t} - \frac{1}{2c^2}\cdot\left(\frac{\partial\phi_1}{\partial t}\right)^2\right]_{z=0} dx \qquad (6\cdot39)$$

Suppose that ϕ and $\mathfrak{J}$ are periodic functions satisfying equations 6·9, 6·10, 6·15 and 6·16. If P and ρ are defined by equation 6·20 then these equations imply 6·6, 6·11 and 6·13. Providing that grad P is not identically zero, equations 6·11 and 6·13 show that $z = \mathfrak{J}$ is a surface moving with the fluid. But since the equation of continuity is satisfied (6·6), it follows that the left-hand side of 6·17, 6·18, or 6·19 is at most a constant. Hence any periodic solution $\phi = \phi_1^*$ of equations 6·25, 6·26 and 6·27 must make the left hand side of equation 6·39 a constant, say C_1^*.

Then a solution of (6·39) is given by

$$\phi_1 = \phi_1^* - \frac{C_1^*}{\lambda} e^{-2\gamma h} t \qquad (6\cdot40)$$

Since

$$2\gamma\int_0^\lambda dx \int_0^h \frac{\partial \phi_1}{\partial t} e^{2\gamma z} dz + \int_0^\lambda \left(\frac{\partial \phi_1}{\partial t}\right)_{z=0} dx$$

$$= 2\gamma\int_0^\lambda dx \int_0^h \frac{\partial \phi_1^*}{\partial t} e^{2\gamma z} dz + \int_0^\lambda \left(\frac{\partial \phi_1^*}{\partial t}\right)_{z=0} dx$$

$$-\frac{1}{\lambda}\int_0^\lambda dx \left[C_1^* e^{-2\gamma h} e^{2\gamma z}\right]_0^h - \frac{1}{\lambda}\int_0^\lambda C_1^* e^{-2\gamma h} dx$$

$$= C_1^* - C_1^*\left(1 - e^{-2\gamma h}\right) - C_1^* e^{-2\gamma h}$$

$$= 0.$$

Hence if ϕ_1^* is any periodic solution of (6·25), (6·26) and (6·27), a solution of these equations and (6·39) is found by adding to ϕ_1^* a constant multiple of t. Similarly if ϕ_2^* is any periodic solution of (6·32), (6·33) and (6·34) a solution of these equations and equation (6·39) is to be found by adding a constant multiple of t to ϕ_2^*.

Determination of the first approximation ϕ_1 .

We assume that ϕ_1 is a simple progressive wave of the form

$$\phi_1 = Z(z)\, e^{i(kx+\sigma t)} \tag{6·41}$$

where $k = 2\pi/\lambda$, $\sigma = 2\pi/T$ and Z is a function of z only.

Writing

$$Z = e^{-\gamma z}\, Z_1(z) \tag{6·42}$$

$$\phi_1 = e^{-\gamma z}\, Z_1(z)\, .\, e^{i(kx+\sigma t)}$$

$$\frac{\partial^2 \phi_1}{\partial t^2} = -\sigma^2 \phi_1,$$

$$\frac{\partial^2 \phi_1}{\partial x^2} = -k^2 \phi_1$$

$$\frac{\partial \phi_1}{\partial z} = e^{i(kx+\sigma t)} \left[e^{-\gamma z} \frac{dZ_1}{dz} - \gamma e^{-\gamma z} Z_1 \right]$$

$$\frac{\partial^2 \phi_1}{\partial z^2} = e^{i(kx+\sigma t)} . e^{-\gamma z} \left(\frac{d^2 Z_1}{dz^2} - \gamma \frac{dZ_1}{dz} \right)$$

$$- e^{i(kx+\sigma t)} . e^{-\gamma z} \left(\frac{dZ_1}{dz} - \gamma Z_1 \right)$$

$$= e^{i(kx+\sigma t)} . e^{-\gamma z} \left(\frac{d^2 Z_1}{dz^2} - 2\gamma \frac{dZ_1}{dz} + \gamma^2 Z_1 \right)$$

Hence $\nabla^2 \phi_1 = e^{i(kx+\sigma t)} . e^{-\gamma z} \left(\frac{d^2 Z_1}{dz^2} - 2\gamma \frac{dZ_1}{dz} + \gamma^2 Z_1 - k^2 Z_1 \right)$

Hence equation (6·25) gives

$$-\sigma^2 \phi_1 - c^2 . e^{i(kx+\sigma t)} . e^{-\gamma z} \left[\frac{d^2 Z_1}{dz^2} - 2\gamma \frac{dZ_1}{dz} + \gamma^2 Z_1 - k^2 Z_1 \right]$$

$$- g . e^{i(kx+\sigma t)} . e^{-\gamma z} . \left(\frac{dZ_1}{dz} - \gamma Z_1 \right) = 0$$

$$\therefore c^2 \frac{d^2 Z_1}{dz^2} + (g - 2c^2 \gamma) \frac{dZ_1}{dz} - g(\gamma + c^2 k^2 - c^2 \gamma^2 - \sigma^2) Z_1 = 0 .$$

But $2\gamma = \frac{g}{c^2}$ (equation 6·31) and we write

$$\alpha^2 = k^2 - \frac{\sigma^2}{c^2} + \gamma^2 \tag{6·43}$$

So that
$$\frac{d^2 Z_1}{dz^2} - \alpha^2 Z_1 = 0 \tag{6·44}$$

Assuming that $\alpha \neq 0$, this has a solution

$$Z_1 = Ae^{\alpha z} + Be^{-\alpha z} \tag{6·45}$$

where A and B are constants, hence from (6·42)

$$\phi_1 = \left[Ae^{\alpha z} + Be^{-\alpha z}\right] . e^{-\gamma z} \, e^{i(kx+\sigma t)}$$

$$\therefore \; \phi_1 = \left[Ae^{-(\gamma-\alpha)z} + Be^{-(\gamma+\alpha)z}\right] . e^{i(kx+\sigma t)} \tag{6·46}$$

$$\therefore \; \frac{\partial \phi_1}{\partial z} = \left[-A(\gamma-\alpha)e^{-(\gamma-\alpha)z} - B(\gamma+\alpha)e^{-(\gamma+\alpha)z}\right] . e^{i(kx+\sigma t)}$$

Hence equation (6·26) becomes

$$-A(\gamma-\alpha)e^{-(\gamma-\alpha)h} - B(\gamma+\alpha)e^{-(\gamma+\alpha)h} = 0 \tag{6·47}$$

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \left[A(\gamma-\alpha)^2 e^{-(\gamma-\alpha)z} + B(\gamma+\alpha)^2 e^{-(\gamma+\alpha)z}\right] . e^{i(kx+\sigma t)}$$
$$- k^2\left[Ae^{-(\gamma-\alpha)z} + Be^{-(\gamma+\alpha)z}\right] . e^{i(kx+\sigma t)}$$

$$\therefore \left[\nabla^2(\phi_1)\right]_{z=0} = \left[A(\gamma-\alpha)^2 + B(\gamma+\alpha)^2 - k^2(A+B)\right] . e^{i(kx+\sigma t)}$$

Hence equation (6·27) becomes

$$A\left\{(\gamma-\alpha)^2 - k^2\right\} + B\left\{(\gamma+\alpha)^2 - k^2\right\} = 0 \tag{6·48}$$

The eliminant of A and B between equations (6·47) and (6·48) is

$$\Delta(\sigma_1, k) = \begin{vmatrix} -(\gamma-\alpha)e^{-(\gamma-\alpha)h} , & -(\gamma+\alpha)e^{-(\gamma+\alpha)h} \\ (\gamma-\alpha)^2-k^2 , & (\gamma+\alpha)^2-k^2 \end{vmatrix}$$

$$= -e^{-\gamma h}\Big[e^{\alpha h}(\gamma^3+\gamma^2\alpha-\gamma k^2-\gamma\alpha^2-\alpha^3+\alpha k^2)$$
$$- e^{-\alpha h}(\gamma^3-\gamma^2\alpha-\gamma k^2-\gamma\alpha^2+\alpha^3-\alpha k^2)\Big]$$

$$= -e^{-\gamma h}\Big[(\gamma^3-\gamma\alpha^2-k^2\gamma)(e^{\alpha h}-e^{-\alpha h})$$
$$+ (\alpha\gamma^2-\alpha^3+\alpha k^2)(e^{\alpha h}+e^{-\alpha h})\Big]$$

$$= -e^{-\gamma h}\Big[2\gamma(\gamma^2-\alpha^2-k^2)\sinh\alpha h + 2\alpha(\gamma^2-\alpha^2+k^2)\operatorname{Cosh}\alpha h\Big] \quad (6{\cdot}49)$$

In order that equations (6·47) and (6·48) may possess non-zero solutions $\Delta(\sigma_1, k) = 0$

(Ferrar: Higher Algebra I p.173)

i.e. $\gamma(\gamma^2-\alpha^2-k^2)\sinh\alpha h + \alpha(\gamma^2-\alpha^2+k^2)\operatorname{Cosh}\alpha h = 0$

$$\gamma(\gamma^2-\alpha^2-k^2) + \alpha(\gamma^2-\alpha^2+k^2)\operatorname{Coth}\alpha h = 0$$

$$\alpha h \operatorname{Coth}\alpha h + \frac{\gamma(\gamma^2-\alpha^2-k^2)h}{\gamma^2-\alpha^2+k^2} = 0$$

but $\quad k^2 = \alpha^2 + \dfrac{\sigma^2}{c^2} - \gamma^2$

$$\therefore \alpha h \operatorname{Coth}\alpha h + \frac{\gamma(\gamma^2-\alpha^2-\alpha^2-\frac{\sigma^2}{c^2}+\gamma^2)h}{\gamma^2-\alpha^2+\alpha^2+\frac{\sigma^2}{c^2}-\gamma^2} = 0$$

but $\quad c^2 = \dfrac{g}{2\gamma}$

$$\therefore \alpha h \operatorname{Coth} \alpha h + \gamma\left(2\gamma^2 - 2\alpha^2 - \frac{2\sigma^2\gamma}{g}\right) h \cdot \frac{g}{2\sigma^2 g} = 0$$

$$\therefore \alpha h \operatorname{Coth} \alpha h + \frac{gh}{\sigma^2}\left(\gamma^2 - \alpha^2 - \frac{\sigma^2\gamma}{g}\right) = 0$$

$$\therefore \alpha h \operatorname{Coth} \alpha h - \frac{g}{h\sigma^2}(\alpha h)^2 - \gamma h\left(1 - \gamma h \cdot \frac{g}{h\sigma^2}\right) = 0$$

$$\text{or } f(\alpha, h) \equiv \alpha h \operatorname{Coth} \alpha h - P(\alpha h)^2 - Q = 0 \qquad (6{\cdot}50)$$

where $$P = \frac{g}{h\sigma^2}\,, \qquad Q = \gamma h(1 - P\gamma h) \qquad (6{\cdot}51)$$

If the depth and the period of the progressive wave are known, that is h and σ are given, then equation (6.50) determines α, and hence since

$$k^2 = \alpha^2 + \frac{\sigma^2}{c^2} - \gamma^2 = \alpha^2 + \frac{\sigma^2}{c^2} - \frac{g^2}{4c^4}\,,$$

we have k and hence λ, since $\lambda = 2\pi/k$; where c is a constant, the velocity of sound in the water.

Now $\displaystyle \operatorname*{Lt}_{\theta \to 0} \frac{\theta}{\tanh\theta} = 1$, hence as αh tends to zero, $f(\alpha, h)$ tends to $(1 - Q)$, which is assumed positive.

But $\quad Q = \gamma h(1 - P\gamma h) = \gamma h\left[1 - \dfrac{g\gamma}{\sigma^2}\right] < 1$

When αh is large and positive, $f(\alpha h)$ is negative, since $\displaystyle \operatorname*{Lt}_{\theta \to \infty} \frac{\theta}{\tanh\theta} = \infty$

since $\tanh\theta \to 1$ as $\theta \to \infty$

Write $\eta = \alpha^2 h^2$, then

$$f \equiv \eta^{\frac{1}{2}} \operatorname{Coth} \eta^{\frac{1}{2}} - P\eta - Q$$

$$\therefore \frac{\partial f}{\partial \eta} \equiv \frac{1}{2}\eta^{-\frac{1}{2}} \operatorname{Coth} \eta^{\frac{1}{2}} - \frac{1}{2} \operatorname{Cosech}^2 \eta^{\frac{1}{2}} - P$$

$$\therefore \frac{\partial^2 f}{\partial \eta^2} \equiv -\frac{1}{4}\eta^{-3/2} \operatorname{Coth} \eta^{\frac{1}{2}} - \frac{1}{4}\eta^{-1} \operatorname{Cosech}^2 \eta^{\frac{1}{2}} + \frac{1}{2}\eta^{-\frac{1}{2}} \operatorname{Cosech}^2 \eta^{\frac{1}{2}} \cdot \operatorname{Coth} \eta^{\frac{1}{2}}$$

If we put $\theta = \eta^{\frac{1}{2}}$

$$\frac{\partial^2 f}{\partial \eta^2} = -\frac{1}{4\theta^3}\cdot \mathrm{Coth}\,\theta - \frac{1}{4\theta^2}\cdot \mathrm{Cosech}^2\theta + \frac{1}{2\theta}\cdot \mathrm{Cosech}^2\theta . \mathrm{Coth}\,\theta$$

$$= -\frac{\mathrm{Coth}\,\theta}{4\theta^3} - \left(\frac{\mathrm{Cosech}\,\theta}{2\theta}\right)^2 \cdot \left(1 - \frac{2\theta}{\tanh\theta}\right)$$

$$< 0$$

Hence $\frac{\partial^2 f}{\partial \eta^2}$ is always negative.

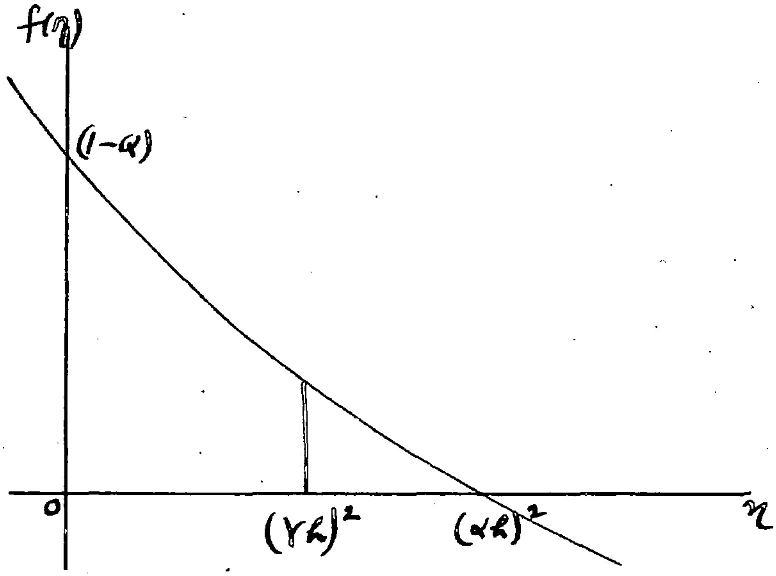
Thus we have (1) $f(\alpha h) \to (1-Q) > 0$ as $\alpha h \to 0$

(2) $f(\alpha h) < 0$ when $\alpha h \to +\infty$

(3) $\frac{\partial^2 f}{\partial \eta^2} < 0$ i.e. $f = 0$ has only one real zero.

This real zero corresponds to a gravity type wave. There are an infinity of imaginary zeroes each corresponding to a compression type wave (Whipple & Lee 1935)

We now assume that α is the positive real root of equation (6 50).



Now $f(\gamma h) = \gamma h\,\mathrm{Coth}\,\gamma h - \frac{g}{h\sigma^2}\cdot\gamma^2 h^2 - \gamma h\left(1 - \frac{g}{h\sigma^2}\cdot\gamma h\right)$

$$= \gamma h\,\mathrm{Coth}\,\gamma h - \gamma h$$

$$= \gamma h\,(\mathrm{Coth}\,\gamma h - 1)$$

> 0 since $|\mathrm{Coth}\,\theta| > 1$.

Hence $(\gamma h)^2 < (\alpha h)^2$

or $\gamma^2 < \alpha^2$ (6•52)

Hence, by (6•43) $\alpha^2 - \gamma^2 = k^2 - \frac{\sigma^2}{c^2} > 0$

$$\therefore \quad k^2 > \frac{\sigma^2}{c^2} > 0. \tag{6•53}$$

Thus corresponding to α which satisfies equation (6•50) there is a real value of k, that is to say a real wave motion, since $k = 2\pi/\lambda$.

From equations (6•47) and (6•48)

$$\frac{A}{(\gamma+\alpha)e^{-(\gamma+\alpha)h}} = -\frac{B}{(\gamma-\alpha)e^{-(\gamma-\alpha)h}}$$

$$\therefore \quad \frac{A}{(\gamma+\alpha)e^{-\alpha h}} = -\frac{B}{(\gamma-\alpha)e^{\alpha h}} .$$

Hence after substituting in equation (6•46) the first approximation for ϕ is

$$\phi_1 = \left[(\gamma+\alpha)e^{-\alpha h}.e^{-(\gamma-\alpha)z} - (\gamma-\alpha)e^{\alpha h}.e^{-(\gamma+\alpha)z}\right].e^{i(kx+\sigma t)}$$

$$\therefore \quad \phi_1 = \left[(\gamma+\alpha)e^{-\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha)e^{\alpha h-(\gamma+\alpha)z}\right].e^{i(kx+\sigma t)} \tag{6•54}$$

This value should satisfy equation (6•38).
It will now be shewn that it does in fact do so :-
From (6•54)

$$\frac{\partial\phi_1}{\partial t} = i\sigma\left[(\gamma+\alpha)e^{-\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha)e^{\alpha h-(\gamma+\alpha)z}\right]e^{i(kx+\sigma t)}$$

$$\therefore \int_0^{\lambda}\left(\frac{\partial\phi_1}{\partial t}\right)_{z=0} dx = i\sigma\left[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\right]\int_0^{\lambda} e^{i(kx+\sigma t)}\,dx$$

$$= \frac{i\sigma}{ik}\left[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\right]\left[e^{i(kx+\sigma t)}\right]_0^{\lambda}$$

$$= \frac{\sigma}{k}\left[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\right]e^{i\sigma t}\left(e^{ik\lambda}-1\right)$$

$$= \frac{\sigma}{k}\left[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\right]e^{i\sigma t}\left(e^{i2\pi}-1\right)$$

$$= 0. \qquad \left(\text{since} \quad 2\pi = k\lambda\right).$$

$$\int_0^{h}\frac{\partial\phi_1}{\partial t}.e^{2\gamma z}dz = i\sigma\int_0^{h}\left[(\gamma+\alpha)e^{-\alpha h+(\gamma+\alpha)z} - (\gamma-\alpha)e^{\alpha h+(\gamma-\alpha)z}\right]e^{i(kx+\sigma t)}.dz$$

$$= i\sigma e^{i(kx+\sigma t)}\left[e^{-\alpha h+(\gamma+\alpha)z} - e^{\alpha h+(\gamma-\alpha)z}\right]_0^{h}$$

$$= i\sigma e^{i(kx+\sigma t)}.\left(e^{\alpha h} - e^{-\alpha h}\right).$$

$$\therefore \int_0^{\lambda}dx\int_0^{h}\frac{\partial\phi_1}{\partial t}.e^{2\gamma z}dz = \frac{i\sigma}{ik}\left(e^{\alpha h}-e^{-\alpha h}\right)\left[e^{i(kx+\sigma t)}\right]_0^{\lambda}$$

$$= \frac{\sigma}{k}\left(e^{\alpha h}-e^{-\alpha h}\right)e^{i\sigma t}.\left(e^{i2\pi}-1\right)$$

$$= 0.$$

That is

$$2\gamma\int_0^{\lambda}dx\int_0^{h}\frac{\partial\phi_1}{\partial t}\,e^{2\gamma z}dz + \int_0^{\lambda}\left(\frac{\partial\phi_1}{\partial t}\right)_{z=0}dx = 0.$$

Thus ϕ_1 given by (6·54) does satisfy equation (6·38). Since the equations (6·25), (6·26), (6·27), (6·28), (6·29) and (6·30), which determine the first approximation are all linear, the sum of any number of solutions is also a solution. Therefore we may take as the first approximation

$$\phi_1 = \left[(\gamma+\alpha)\, e^{\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha)\, e^{\alpha h-(\gamma+\alpha)z} \right] \times \left[b_1 \sin(kx-\sigma t) + b_2 \sin(kx+\sigma t) \right] \quad (6{\cdot}55)$$

representing two waves of the same wavelength travelling in opposite directions.

Determination of the second approximation ϕ_2.

We now substitute the value of ϕ_1 given by equation (6·55) into equations (6·32), (6·33), (6·34) and (6·39) and solve the resulting equations for ϕ_2.

From equation (6·28)

$$-\underline{u}_1 = \operatorname{grad}\phi_1 = \underline{i}\,\frac{\partial\phi_1}{\partial x} + \underline{j}\,\frac{\partial\phi_1}{\partial z}$$

where $\underline{i}$ and $\underline{j}$ are unit vectors in the directions Ox, Oz respectively.

Then $\underline{u}_1^2 = \left(\underline{i}\,\frac{\partial\phi_1}{\partial x} + \underline{j}\,\frac{\partial\phi_1}{\partial z}\right)^2 = \left(\frac{\partial\phi_1}{\partial x}\right)^2 + \left(\frac{\partial\phi_1}{\partial z}\right)^2$

From (6·55)

$$\frac{\partial\phi_1}{\partial x} = \left[(\gamma+\alpha)\, e^{\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha)\, e^{\alpha h-(\gamma+\alpha)z} \right] k \times \left[b_1 \cos(kx-\sigma t) + b_2 \cos(kx+\sigma t) \right]$$

$$\frac{\partial\phi_1}{\partial z} = \left[-(\gamma^2-\alpha^2)\, e^{\alpha h-(\gamma-\alpha)z} + (\gamma^2-\alpha^2)\, e^{\alpha h-(\gamma+\alpha)z} \right] \times \left[b_1 \sin(kx-\sigma t) + b_2 \sin(kx+\sigma t) \right]$$

$$\therefore \overline{u_1^2} = k^2 \left[(\gamma+\alpha) e^{-\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha) e^{\alpha h-(\gamma+\alpha)z} \right]^2$$

$$\times \left[b_1^2 \cos^2(kx-\sigma t) + b_2^2 \cos^2(kx+\sigma t) + 2b_1 b_2 \cos(kx-\sigma t)\cos(kx+\sigma t) \right]$$

$$+ \left[(\gamma^2-\alpha^2) e^{-\alpha h-(\gamma-\alpha)z} - (\gamma^2-\alpha^2) e^{\alpha h-(\gamma+\alpha)z} \right]^2$$

$$\times \left[b_1^2 \sin^2(kx-\sigma t) + b_2^2 \sin^2(kx+\sigma t) + 2b_1 b_2 \sin(kx-\sigma t)\sin(kx+\sigma t) \right]$$

$$= k^2 \left[(\gamma+\alpha) e^{-\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha) e^{\alpha h-(\gamma+\alpha)z} \right]^2$$

$$\times \left[\tfrac{1}{2} b_1^2 \{1+\cos 2(kx-\sigma t)\} + \tfrac{1}{2} b_2^2 \{1+\cos 2(kx+\sigma t)\} + b_1 b_2 (\cos 2kx + \cos 2\sigma t) \right]$$

$$+ \left[(\gamma^2-\alpha^2) e^{-\alpha h-(\gamma-\alpha)z} - (\gamma^2-\alpha^2) e^{\alpha h-(\gamma+\alpha)z} \right]^2$$

$$\times \left[\tfrac{1}{2} b_1^2 \{1-\cos 2(kx-\sigma t)\} + \tfrac{1}{2} b_2^2 \{1-\cos 2(kx+\sigma t)\} + b_1 b_2 (-\cos 2kx + \cos 2\sigma t) \right].$$

$$\therefore \frac{\partial}{\partial t}\left(\overline{u_1^2}\right) =$$

$$\sigma \left[k^2 \left\{ (\gamma+\alpha) e^{-\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha) e^{\alpha h-(\gamma+\alpha)z} \right\}^2 - \left\{ (\gamma^2-\alpha^2) e^{-\alpha h-(\gamma-\alpha)z} - (\gamma^2-\alpha^2) e^{\alpha h-(\gamma+\alpha)z} \right\}^2 \right]$$

$$\times \left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t) \right]$$

$$-\sigma \left[k^2 \left\{ (\gamma+\alpha) e^{-\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha) e^{\alpha h-(\gamma+\alpha)z} \right\}^2 + (\gamma^2-\alpha^2)^2 \left\{ e^{-\alpha h-(\gamma-\alpha)z} - e^{\alpha h-(\gamma+\alpha)z} \right\}^2 \right] 2b_1 b_2 \sin 2\sigma t.$$

The coefficient of $\sigma\left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right]$ is

$$k^2\left[(\gamma+\alpha)^2 e^{-2\alpha h-2(\gamma-\alpha)z} - 2(\gamma^2-\alpha^2) e^{-2\gamma z} + (\gamma-\alpha)^2 e^{2\alpha h-2(\gamma+\alpha)z}\right]$$

$$-(\gamma^2-\alpha^2)^2\left[e^{-2\alpha h-2(\gamma-\alpha)z} - 2e^{-2\gamma z} + e^{2\alpha h-2(\gamma+\alpha)z}\right].$$

$$= (\gamma+\alpha)^2\left\{k^2-(\gamma-\alpha)^2\right\} e^{2\alpha h-2(\gamma-\alpha)z}$$

$$+(\gamma-\alpha)^2\left\{k^2-(\gamma+\alpha)^2\right\} e^{2\alpha h-2(\gamma+\alpha)z}$$

$$+2(\gamma^2-\alpha^2)(\gamma^2-\alpha^2-k^2)e^{-2\gamma z}$$

$$= \frac{C^{①}}{\sigma} e^{-2(\gamma-\alpha)z} + \frac{C^{②}}{\sigma} e^{-2(\gamma+\alpha)z} - 2\frac{C^{③}}{\sigma} e^{-2\gamma z}$$

$$\left.\begin{aligned} C^{①} &= -\sigma\left\{(\gamma-\alpha)^2-k^2\right\}(\gamma+\alpha)^2 e^{-2\alpha h} \\ C^{②} &= -\sigma\left\{(\gamma+\alpha)^2-k^2\right\}(\gamma-\alpha)^2 e^{2\alpha h} \\ C^{③} &= -\sigma\left\{\gamma^2-\alpha^2-k^2\right\}(\gamma^2-\alpha^2) \end{aligned}\right\} \quad (6\cdot56)$$

The coefficient of $2b_1 b_2 \sin 2\sigma t$ is

$$-\sigma\left[k^2(\gamma+\alpha)^2 e^{-2\alpha h-2(\gamma-\alpha)z} - 2k^2(\gamma^2-\alpha^2)e^{-2\gamma z}\right.$$

$$+k^2(\gamma-\alpha)^2 e^{2\alpha h-2(\gamma+\alpha)z}$$

$$+(\gamma^2-\alpha^2)^2 e^{-2\alpha h-2(\gamma-\alpha)z}$$

$$\left.+2(\gamma^2-\alpha^2)^2 e^{-2\gamma z} + (\gamma^2-\alpha^2)^2 e^{2\alpha h-2(\gamma+\alpha)z}\right]$$

The coefficient of $2b_1 b_2 \sin 2\sigma t$ is

$$-\sigma\Big[(\gamma+\alpha)^2\{k^2+(\gamma-\alpha)^2\}e^{-2\alpha h-2(\gamma-\alpha)z}$$

$$-\sigma(\gamma-\alpha)^2\{k^2+(\gamma+\alpha)^2\}e^{2\alpha h-2(\gamma+\alpha)z}$$

$$+2\sigma(\gamma^2-\alpha^2)\{k^2+(\gamma^2-\alpha^2)\}e^{-2\gamma z}$$

$$= C^{④}e^{-2(\gamma-\alpha)z}+C^{⑤}e^{-2(\gamma+\alpha)z}-2C^{⑥}e^{-2\gamma z}$$

where
$$\left.\begin{aligned}
C^{④} &= -\sigma\{(\gamma-\alpha)^2+k^2\}(\gamma+\alpha)^2e^{-2\alpha h}\\
C^{⑤} &= -\sigma\{(\gamma+\alpha)^2+k^2\}(\gamma-\alpha)^2e^{2\alpha h}\\
C^{⑥} &= -\sigma\{\gamma^2-\alpha^2+k^2\}(\gamma^2-\alpha^2)
\end{aligned}\right\} \quad (6\cdot57)$$

Hence

$$\frac{\partial u_1^2}{\partial t}=\Big[C^{①}e^{-2(\gamma-\alpha)z}+C^{②}e^{-2(\gamma+\alpha)z}-2C^{③}e^{-2\gamma z}\Big]$$

$$\times\Big[b_1^2\sin 2(kx-\sigma t)-b_2^2\sin 2(kx+\sigma t)\Big]$$

$$+\Big[C^{④}e^{-2(\gamma-\alpha)z}+C^{⑤}e^{-2(\gamma+\alpha)z}-2C^{⑥}e^{-2\gamma z}\Big]2b_1b_2\sin 2\sigma t.$$

Thus equation (6·32) becomes

$$\frac{\partial^2\phi_2}{\partial t^2}-c^2\nabla^2\phi_2-g\frac{\partial\phi_2}{\partial z}=$$

$$\Big[C^{①}e^{-2(\gamma-\alpha)z}+C^{②}e^{-2(\gamma+\alpha)z}-2C^{③}e^{-2\gamma z}\Big]$$

$$\times\Big[b_1^2\sin 2(kx-\sigma t)-b_2^2\sin 2(kx+\sigma t)\Big]$$

$$+\Big[C^{④}e^{-2(\gamma-\alpha)z}+C^{⑤}e^{-2(\gamma+\alpha)z}-2C^{⑥}e^{-2\gamma z}\Big]2b_1b_2\sin 2\sigma t. \quad (6\cdot58)$$

From equations (6·29) and (6·34)

$$\left(\nabla^2\phi_2\right)_{z=0} = \frac{1}{g}\cdot\left(\frac{\partial\phi_1}{\partial t}\right)_{z=0}\cdot\left(\frac{\partial}{\partial z}\cdot\nabla^2\phi_1\right)_{z=0}$$

From equation (6·55)

$$\frac{\partial\phi_1}{\partial t} = \left[(\gamma+\alpha)e^{-\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha)e^{\alpha h-(\gamma+\alpha)z}\right]\sigma\left[-b_1\cos(kx-\sigma t) + b_2\cos(kx+\sigma t)\right]$$

$$\therefore\left(\frac{\partial\phi_1}{\partial t}\right)_{z=0} = \sigma\left[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\right]\left[-b_1\cos(kx-\sigma t)+b_2\cos(kx+\sigma t)\right].$$

$$\frac{\partial^2\phi_1}{\partial x^2} = -\left[(\gamma+\alpha)e^{-\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha)e^{\alpha h-(\gamma+\alpha)z}\right]k^2 \times\left[b_1\sin(kx-\sigma t)+b_2\sin(kx+\sigma t)\right].$$

$$\frac{\partial^2\phi_1}{\partial z^2} = (\gamma^2-\alpha^2)\left[(\gamma-\alpha)e^{-\alpha h-(\gamma-\alpha)z} - (\gamma+\alpha)e^{\alpha h-(\gamma+\alpha)z}\right] \times\left[b_1\sin(kx-\sigma t)+b_2\sin(kx+\sigma t)\right]$$

$$\therefore \nabla^2\phi_1 = \Big[(\gamma^2-\alpha^2)\left\{(\gamma-\alpha)e^{-\alpha h-(\gamma-\alpha)z} - (\gamma+\alpha)e^{\alpha h-(\gamma+\alpha)z}\right\}$$
$$-k^2\left\{(\gamma+\alpha)e^{-\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha)e^{\alpha h-(\gamma+\alpha)z}\right\}\Big]$$
$$\times\left[b_1\sin(kx-\sigma t)+b_2\sin(kx+\sigma t)\right]$$

$$\therefore\left[\frac{\partial\nabla^2\phi_1}{\partial z}\right]_{z=0} = \Big[(\gamma^2-\alpha^2)\left\{-(\gamma-\alpha)^2e^{-\alpha h} + (\gamma+\alpha)^2e^{\alpha h}\right\}$$
$$-k^2\left\{-(\gamma^2-\alpha^2)e^{-\alpha h} + (\gamma^2-\alpha^2)e^{\alpha h}\right\}\Big]$$
$$\times\left[b_1\sin(kx-\sigma t)+b_2\sin(kx-\sigma t)\right]$$

$$\therefore \left[\frac{\partial \nabla^2\phi_1}{\partial z}\right]_{z=0} = (\gamma^2-\alpha^2)\left[\{-(\gamma-\alpha)^2+k^2\}e^{-\alpha h} - \{-(\gamma+\alpha)^2+k^2\}e^{\alpha h}\right]$$
$$\times\left[b_1 \sin(kx-\sigma t) + b_2 \sin(kx+\sigma t)\right].$$

Hence $g(\nabla^2\phi_2)_{z=0}$

$$= \sigma(\gamma^2-\alpha^2)\left[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\right]\left[\{k^2-(\gamma-\alpha)^2\}e^{-\alpha h} + \{k^2-(\gamma+\alpha)^2\}e^{\alpha h}\right]$$
$$\times\left[-b_1\cos(kx-\sigma t) + b_2\cos(kx+\sigma t)\right]\left[b_1\sin(kx-\sigma t) + b_2\sin(kx+\sigma t)\right].$$

$$= \tfrac{1}{2}\sigma(\gamma^2-\alpha^2)\Big[(\gamma+\alpha)\{k^2-(\gamma-\alpha)^2\}e^{-2\alpha h} - (\gamma+\alpha)\{k^2-(\gamma+\alpha)^2\}$$
$$-(\gamma-\alpha)\{k^2-(\gamma-\alpha)^2\} + (\gamma-\alpha)\{k^2-(\gamma+\alpha)^2\}e^{2\alpha h}\Big]$$
$$\times\left[-b_1^2\sin 2(kx-\sigma t) + b_2^2\sin 2(kx+\sigma t) - 2b_1b_2\sin 2\sigma t\right].$$

That is

$$(\nabla^2\phi_2)_{z=0} = D\left[b_1^2\sin 2(kx-\sigma t) - b_2^2\sin 2(kx+\sigma t) + 2b_1b_2\sin 2\sigma t\right] \tag{6.59}$$

where $D = -\dfrac{\sigma(\gamma^2-\alpha^2)}{2g}\Big[(\gamma+\alpha)\{-(\gamma-\alpha)^2+k^2\}e^{-2\alpha h}$

$$-(\gamma+\alpha)\{k^2-(\gamma+\alpha)^2\} - (\gamma-\alpha)\{k^2-(\gamma-\alpha)^2\} + (\gamma-\alpha)\{k^2-(\gamma+\alpha)^2\}e^{2\alpha h}\Big]$$

$$= \frac{-\sigma(\gamma^2-\alpha^2)}{2g}\Big[(\gamma+\alpha)\{k^2-(\gamma-\alpha)^2\}e^{-2\alpha h} + (\gamma-\alpha)\{k^2-(\gamma+\alpha)^2\}e^{2\alpha h}$$
$$+ 2\gamma(3\alpha^2+\gamma^2-k^2)\Big].$$

But $\Delta(\sigma_1 k)=0$, hence by equations (6·48) and (6·47)

$$(\gamma-\alpha)\{(\gamma+\alpha)^2-k^2\}e^{\alpha h}=(\gamma+\alpha)\{(\gamma-\alpha)^2-k^2\}e^{-\alpha h}$$

$$\therefore\ (\gamma-\alpha)\{(\gamma+\alpha)^2-k^2\}e^{2\alpha h}=(\gamma+\alpha)\{(\gamma-\alpha)^2-k^2\}$$

$$\text{and } (\gamma+\alpha)\{(\gamma-\alpha)^2-k^2\}e^{-2\alpha h}=(\gamma-\alpha)\{(\gamma+\alpha)^2-k^2\}$$

Hence $$D=-\frac{\sigma(\gamma^2-\alpha^2)}{2g}\Big[-(\gamma-\alpha)\{(\gamma+\alpha)^2-k^2\}-(\gamma+\alpha)\{(\gamma-\alpha)^2-k^2\}$$
$$+(\gamma+\alpha)\{(\gamma+\alpha)^2-k^2\}+(\gamma-\alpha)\{(\gamma-\alpha)^2-k^2\}\Big].$$

$$=-\frac{\sigma(\gamma^2-\alpha^2)}{2g}\Big[2\alpha\{(\gamma+\alpha)^2-k^2\}-2\alpha\{(\gamma-\alpha)^2-k^2\}\Big]$$

$$=\frac{\sigma\alpha(\gamma^2-\alpha^2)}{g}\Big[(\gamma+\alpha)^2-(\gamma-\alpha)^2\Big]$$

$$\therefore\ D=-\frac{4\sigma\alpha^2\gamma}{g}.\ (\gamma^2-\alpha^2). \tag{6·60}$$

We now substitute for ϕ_1 in equation (6·39).
We have already shewn that

$$\underline{u}_1^2=k^2\Big[(\gamma+\alpha)e^{-\alpha h-(\gamma-\alpha)z}-(\gamma-\alpha)e^{\alpha h-(\gamma+\alpha)z}\Big]^2$$
$$\times\Big[\tfrac{1}{2}b_1^2\{1+\cos 2(kx-\sigma t)\}+\tfrac{1}{2}b_2^2\{1+\cos 2(kx+\sigma t)\}+b_1b_2(\cos 2kx+\cos 2\sigma t)\Big]$$
$$+(\gamma^2-\alpha^2)^2.\Big[e^{-\alpha h-(\gamma-\alpha)z}-e^{\alpha h-(\gamma+\alpha)z}\Big]^2$$
$$\times\Big[\tfrac{1}{2}b_1^2\{1-\cos 2(k-\sigma t)\}+\tfrac{1}{2}b_2^2\{1-\cos 2(kx+\sigma t)\}$$
$$+b_1b_2(-\cos 2kx+\cos 2\sigma t)\Big]$$

$$\int_0^h k^2\left[(\gamma+\alpha)^2 e^{-2\alpha h+2\alpha z} - 2(\gamma^2-\alpha^2) + (\gamma-\alpha)^2 e^{2\alpha h-2\alpha z}\right] dz$$

$$= k^2\left[\frac{(\gamma+\alpha)^2}{2\alpha}.e^{-2\alpha h+2\alpha z} - 2(\gamma^2-\alpha^2)z - \frac{(\gamma-\alpha)^2}{2\alpha}.e^{2\alpha h-2\alpha z}\right]_0^h$$

$$= \frac{k^2}{2\alpha}\left[4\gamma\alpha - 4(\gamma^2-\alpha^2)\alpha h - (\gamma+\alpha)^2 e^{-2\alpha h} + (\gamma-\alpha)^2 e^{2\alpha h}\right].$$

$$\int_0^h \left[e^{-2\alpha h+2\alpha z} - 2 + e^{2\alpha h-2\alpha z}\right] dz$$

$$= \frac{1}{2\alpha}\left[e^{-2\alpha h+2\alpha z} - 4\alpha z - e^{2\alpha h-2\alpha z}\right]_0^h$$

$$= \frac{1}{2\alpha}\left[-4\alpha h + e^{2\alpha h} - e^{-2\alpha h}\right].$$

Hence $\quad 2\gamma\int_0^\lambda dx \int_0^h \frac{1}{2}\underline{u}_1^2\, dz$

$$= \frac{\gamma k^2}{2\alpha}\left[4\gamma\alpha - 4(\gamma^2-\alpha^2)\alpha h - (\gamma+\alpha)^2 e^{-2\alpha h} + (\gamma-\alpha)^2 e^{2\alpha h}\right]$$
$$\times\left[\tfrac{1}{2}(b_1^2+b_2^2)\lambda + b_1 b_2 \operatorname{Cos} 2\sigma t.\ \lambda\right]$$

$$+\frac{\gamma(\gamma^2-\alpha^2)^2}{2\alpha}.\left(-4\alpha h + e^{2\alpha h} - e^{-2\alpha h}\right)$$
$$\times\left[\tfrac{1}{2}(b_1^2+b_2^2)\lambda + b_1 b_2 \lambda \operatorname{Cos} 2\sigma t\right] \qquad (6.61)$$

$$\frac{\partial\phi_1}{\partial t} = \sigma\left[(\gamma+\alpha)e^{-\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha)e^{\alpha h-(\gamma+\alpha)z}\right] \times \left[-b_1 \cos(kx-\sigma t) + b_2 \cos(kx+\sigma t)\right].$$

$$\therefore \left(\frac{\partial\phi_1}{\partial t}\right)^2 = \sigma^2 e^{-2\gamma z}\left[(\gamma+\alpha)^2 e^{-2\alpha h+2\alpha z} - 2(\gamma^2-\alpha^2) + (\gamma-\alpha)^2 e^{2\alpha h-2\alpha z}\right]$$

$$\times\left[\tfrac{1}{2}b_1^2\{1+\cos 2(kx-\sigma t)\} + \tfrac{1}{2}b_2^2\{1+\cos 2(kx+\sigma t)\} - b_1 b_2(\cos 2kx + \cos 2\sigma t)\right]$$

$$\therefore \int_0^h \left(\frac{\partial\phi_1}{\partial t}\right)^2 e^{2\gamma z}\,dz$$

$$= \frac{\sigma^2}{4\alpha}\left[4\gamma\alpha - 4(\gamma^2-\alpha^2)\alpha h - (\gamma+\alpha)^2 e^{-2\alpha h} + (\gamma-\alpha)^2 e^{2\alpha h}\right]$$

$$\times\left[\tfrac{1}{2}b_1^2\{1+\cos 2(kx-\sigma t)\} + \tfrac{1}{2}b_2^2\{1+\cos 2(kx+\sigma t)\} - b_1 b_2(\cos 2kx + \cos 2\sigma t)\right]$$

$$\therefore \frac{\gamma}{c^2}\int_0^\lambda dx \int_0^h \left(\frac{\partial\phi_1}{\partial t}\right)^2 e^{2\gamma z}\,dz$$

$$= \frac{\gamma\sigma^2}{2c^2\alpha}\left[4\gamma\alpha - 4(\gamma^2-\alpha^2)\alpha h - (\gamma+\alpha)^2 e^{-2\alpha h} + (\gamma-\alpha)^2 e^{2\alpha h}\right]$$

$$\times\left[\tfrac{1}{2}(b_1^2+b_2^2)\lambda - b_1 b_2 \lambda.\cos 2\sigma t\right] \quad (6{\cdot}62)$$

Using the value of $\underline{u}_1^2$ determined earlier

$$\tfrac{1}{2}(\underline{u}_1^2)_{z=0} = \frac{k^2}{2}\left[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\right]^2$$

$$\times\left[\tfrac{1}{2}b_1^2\{1+\cos 2(kx-\sigma t)\} + \tfrac{1}{2}b_2^2\{1+\cos 2(kx+\sigma t)\} + b_1 b_2(\cos 2kx + \cos 2\sigma t)\right]$$

$$+ \frac{(\gamma^2-\alpha^2)^2}{2}.\left[e^{-\alpha h} - e^{\alpha h}\right]^2$$

$$\times\left[\tfrac{1}{2}b_1^2\{1-\cos 2(kx-\sigma t)\} + \tfrac{1}{2}b_2^2\{1-\cos 2(kx+\sigma t)\} + b_1 b_2(\cos 2\sigma t - \cos 2kx)\right].$$

$$\therefore \int_0^\lambda \frac{1}{2}\left(\underline{u}_1^2\right)_{z=0} dx =$$

$$\frac{1}{2}k^2\left[(\gamma+\alpha)e^{-\alpha h}-(\gamma-\alpha)e^{\alpha h}\right]^2\cdot\left[\frac{1}{2}(b_1^2+b_2^2)\lambda+b_1b_2\lambda\cos 2\sigma t\right]$$

$$+\frac{1}{2}(\gamma^2-\alpha^2)^2\cdot\left[e^{-\alpha h}-e^{\alpha h}\right]^2\cdot\left[\frac{1}{2}(b_1^2+b_2^2)\lambda+b_1b_2\lambda\cos 2\sigma t\right]. \tag{6.63}$$

$$\frac{\partial^2\phi_1}{\partial z\partial t}=\sigma\left[-(\gamma^2-\alpha^2)e^{-\alpha h-(\gamma-\alpha)z}+(\gamma^2-\alpha^2)e^{\alpha h-(\gamma+\alpha)z}\right]$$
$$\times\left[-b_1\cos(kx-\sigma t)+b_2\cos(kx+\sigma t)\right]$$

$$\therefore\left(\frac{\partial\phi_1}{\partial t}\cdot\frac{\partial^2\phi_1}{\partial z\partial t}\right)_{z=0}=$$

$$\sigma^2(\gamma^2-\alpha^2)(-e^{-\alpha h}+e^{\alpha h})\left[(\gamma+\alpha)e^{-\alpha h}-(\gamma-\alpha)e^{\alpha h}\right]$$
$$\times\left[b_1\cos(kx-\sigma t)-b_2\cos(kx+\sigma t)\right]^2$$

$$=\sigma^2(\gamma^2-\alpha^2)(e^{\alpha h}-e^{-\alpha h})\left[(\gamma+\alpha)e^{-\alpha h}-(\gamma-\alpha)e^{\alpha h}\right]$$
$$\times\left[\frac{1}{2}b_1^2\{1+\cos 2(kx-\sigma t)\}+\frac{1}{2}b_2^2\{1+\cos 2(kx+\sigma t)\}\right.$$
$$\left.-b_1b_2(\cos 2kx+\cos 2\sigma t)\right].$$

$$\therefore\int_0^\lambda\frac{1}{g}\left(\frac{\partial\phi_1}{\partial t}\cdot\frac{\partial^2\phi}{\partial z\partial t}\right)_{z=0}dx$$

$$=\frac{\sigma^2(\gamma^2-\alpha^2)}{g}(e^{\alpha h}-e^{-\alpha h})\left[(\gamma+\alpha)e^{-\alpha h}-(\gamma-\alpha)e^{\alpha h}\right]$$
$$\times\left[\frac{1}{2}(b_1^2+b_2^2)\lambda-b_1b_2\lambda\cos 2\sigma t\right]. \tag{6.64}$$

$$\left(\frac{\partial\phi_1}{\partial t}\right)_{z=0} = \sigma^2\left[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\right]^2 \times\left[b_1 \cos(kx-\sigma t) - b_2\cos(kx+\sigma t)\right]^2$$

$$\therefore \frac{1}{2c^2}\int_0^\lambda \left(\frac{\partial\phi_1}{\partial t}\right)_{z=0} dx =$$

$$\frac{\sigma^2}{2c^2}\left[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\right]^2 \times\left[\tfrac{1}{2}(b_1^2+b_2^2)\lambda - b_1 b_2\lambda\cos 2\sigma t\right] \tag{6.65}$$

Adding together equations (6·61), (6·62), (6·63), (6·64) and (6·65) we have from equation (6·39)

$$2\gamma\int_0^\lambda dx\int_0^h \frac{\partial\phi_2}{\partial t}e^{2\gamma z}dz + \int_0^\lambda\left(\frac{\partial\phi_2}{\partial t}\right)_{z=0}dx$$

$$= E^{①}(b_1^2+b_2^2) + E^{②}\,2b_1b_2\cos 2\sigma t$$

$$\begin{aligned}
E^{①} = {} & \frac{\gamma k^2}{2\alpha}\left[4\gamma\alpha - 4(\gamma^2-\alpha^2)\alpha h - (\gamma+\alpha)^2e^{-2\alpha h} + (\gamma-\alpha)^2e^{2\alpha h}\right]\frac{\lambda}{2} \\
& + \frac{\gamma(\gamma^2-\alpha^2)^2}{2\alpha}\left(-4\alpha h + e^{2\alpha h} - e^{-2\alpha h}\right)\frac{\lambda}{2} \\
& - \frac{\gamma\sigma^2}{2c^2\alpha}\left[4\gamma\alpha - 4(\gamma^2-\alpha^2)\alpha h - (\gamma+\alpha)^2e^{-2\alpha h} + (\gamma-\alpha)^2e^{2\alpha h}\right]\frac{\lambda}{2} \\
& + \frac{k^2}{2}\left[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\right]^2\frac{\lambda}{2} + \frac{(\gamma^2-\alpha^2)^2}{2}\left(e^{\alpha h} - e^{-\alpha h}\right)^2\cdot\frac{\lambda}{2} \\
& + \frac{\sigma^2(\gamma^2-\alpha^2)}{g}\cdot\left(e^{\alpha h} - e^{-\alpha h}\right)\cdot\left[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\right]\frac{\lambda}{2} \\
& - \frac{\sigma^2}{2c^2}\left[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\right]^2\cdot\frac{\lambda}{2}.
\end{aligned}$$

$$\therefore E^{(1)} = \frac{\lambda\gamma}{4\alpha}(\alpha^2-\gamma^2)\Big[4\gamma\alpha - 4(\gamma^2-\alpha^2)\alpha h - (\gamma+\alpha)^2 e^{-2\alpha h} + (\gamma-\alpha)^2 e^{2\alpha h}$$

$$- 4(\alpha^2-\gamma^2)\alpha h + (\alpha^2-\gamma^2)e^{2\alpha h} - (\alpha^2-\gamma^2)e^{-2\alpha h}\Big]$$

$$+ \frac{\lambda}{4}(\alpha^2-\gamma^2)\Big[\{(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\}^2 + (\alpha^2-\gamma^2)(e^{\alpha h}-e^{-\alpha h})^2\Big]$$

$$- \frac{\sigma^2(\gamma^2-\alpha^2)}{g}\Big[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\Big]\frac{\lambda}{2}(e^{\alpha h}-e^{-\alpha h}).$$

$$\therefore \frac{2E^{(1)}}{\lambda(\alpha^2-\gamma^2)} = \frac{\gamma}{2\alpha}\Big[4\gamma\alpha - (\gamma+\alpha)^2 e^{-2\alpha h} + (\gamma-\alpha)^2 e^{2\alpha h}$$

$$+ (\alpha^2-\gamma^2)e^{2\alpha h} - (\alpha^2-\gamma^2)e^{-2\alpha h}\Big]$$

$$+\frac{1}{2}\Big[\{(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\}^2 + (\alpha^2-\gamma^2)(e^{\alpha h}-e^{-\alpha h})^2\Big]$$

$$-\frac{\sigma^2}{g}(e^{\alpha h}-e^{-\alpha h})\Big[(\gamma+\alpha)e^{-\alpha h} - (\gamma-\alpha)e^{\alpha h}\Big]$$

$$= 2\gamma^2 - (\gamma^2+\gamma\alpha)e^{-2\alpha h} + (\alpha\gamma-\gamma^2)e^{2\alpha h} + \frac{1}{2}(\gamma+\alpha)^2 e^{-2\alpha h}$$

$$-(\gamma^2-\alpha^2) + \frac{1}{2}(\gamma-\alpha)^2 e^{2\alpha h} + \frac{1}{2}(\alpha^2-\gamma^2)e^{2\alpha h} + \frac{1}{2}(\alpha^2-\gamma^2)e^{-2\alpha h}$$

$$-(\alpha^2-\gamma^2) - \frac{2\gamma\sigma^2}{g} + \frac{(\gamma-\alpha)\sigma^2}{g}e^{2\alpha h} + \frac{(\gamma+\alpha)\sigma^2}{g}e^{-2\alpha h}$$

$$= 4\gamma\left(\gamma-\frac{\sigma^2}{g}\right) + 2\Big[\alpha^2-\gamma^2-(\gamma+\alpha)\frac{\sigma^2}{g}\Big]e^{-2\alpha h}$$

$$+2\Big[\alpha^2-\gamma^2+2(\gamma-\alpha)\frac{\sigma^2}{g}\Big]e^{2\alpha h}.$$

$$= 4\gamma\left(\gamma-\frac{\sigma^2}{g}\right) + 2(\alpha+\gamma)\left(\alpha-\gamma+\frac{\sigma^2}{g}\right)e^{-2\alpha h}$$

$$+2(\alpha-\gamma)\left(\alpha+\gamma-\frac{\sigma^2}{g}\right)e^{2\alpha h}$$

$$\left[\text{Since } \frac{\sigma^2}{c^2} = k^2-\alpha^2+\gamma^2 \text{ and } 2\gamma = \frac{g}{c^2} \ \therefore \frac{\sigma^2}{g} = \frac{k^2-\alpha^2+\gamma^2}{2\gamma}\right]$$

$$\therefore \frac{2E^{(1)}}{\lambda(\alpha^2-\gamma^2)} =$$

$$4\gamma\left(\gamma - \frac{k^2-\alpha^2-\gamma^2}{2\gamma}\right) + 2(\alpha+\gamma)\left(\alpha-\gamma+\frac{k^2-\alpha^2+\gamma^2}{2\gamma}\right)e^{-2\alpha h}$$
$$+ 2(\alpha-\gamma)\left(\alpha+\gamma-\frac{k^2-\alpha^2+\gamma^2}{2\gamma}\right)e^{2\alpha h}$$

$$= 2(\gamma^2+\alpha^2-k^2) + \frac{(\alpha+\gamma)(k^2+2\alpha\gamma-\alpha^2-\gamma^2)e^{-2\alpha h}}{\gamma}$$
$$+ \frac{(\alpha-\gamma)(-k^2+2\alpha\gamma+\alpha^2+\gamma^2)e^{2\alpha h}}{\gamma}$$

$$= 2(\gamma^2+\alpha^2-k^2) - \frac{(\gamma-\alpha)\{(\gamma+\alpha)^2-k^2\} + (\gamma+\alpha)\{(\gamma-\alpha)^2-k^2\}}{\gamma}$$

[Since $(\gamma-\alpha)\{(\gamma+\alpha)^2-k^2\}e^{\alpha h} = (\gamma+\alpha)\{(\gamma-\alpha)^2-k^2\}e^{-\alpha h}$

$$\therefore (\gamma+\alpha)\{(\gamma-\alpha)^2-k^2\}e^{-2\alpha h} = (\gamma-\alpha)\{(\gamma+\alpha)^2-k^2\}$$

and $(\gamma-\alpha)\{(\gamma+\alpha)^2-k^2\}e^{2\alpha h} = (\gamma+\alpha)\{(\gamma-\alpha)^2-k^2\}$]

$$\therefore \frac{4E^{(1)}}{\lambda(\alpha^2-\gamma^2)} = 2(\alpha^2+\gamma^2-k^2) - $$
$$- \frac{(\gamma-\alpha)(\gamma+\alpha)^2 - k^2(\gamma-\alpha) + (\gamma+\alpha)(\gamma-\alpha)^2 - k^2(\gamma+\alpha)}{\gamma}$$

$$= 2(\alpha^2+\gamma^2-k^2) - \frac{2\gamma(\gamma^2-\alpha^2) - 2\gamma k^2}{\gamma}$$

$$= 2\gamma^2+2\alpha^2-2k^2-2\gamma^2+2\alpha^2+2k^2$$

$$= 4\alpha^2.$$

$$\therefore \quad E^{(1)} = \alpha^2(\alpha^2 - \gamma^2)\lambda. \tag{6.67}$$

$E^{(2)}$ does not enter into the subsequent work and so need not be simplified. We now have four equations (6·58), (6·33), (6·59) and (6·66) from which to determine .
Guided by the form of equation (6·58) we set

$$\phi_2 = \left[F^{(1)}e^{-2(\gamma-\alpha)z} + F^{(2)}e^{-2(\gamma+\alpha)z} - 2F^{(3)}e^{-2\gamma z}\right]$$
$$\times\left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right]$$
$$+\left[F^{(4)}e^{-2(\gamma-\alpha)z} + F^{(5)}e^{-2(\gamma+\alpha)z} - 2F^{(6)}e^{-2\gamma z}\right]$$
$$\times 2b_1b_2 \sin 2\sigma t + \phi_2' \tag{6.68}$$

where $F^{(1)}, F^{(2)}, F^{(3)}, F^{(4)}, F^{(5)}$ and $F^{(6)}$ are

functions of σ, α, γ, k and h, that is, of the physical properties of the motion and the medium, and ϕ_2' is a function of the variables x, z and t.
We now substitute for ϕ_2 from (6·68) in (6·58) :

$$\frac{\partial\phi_2}{\partial z} = -2\left[F^{(1)}(\gamma-\alpha)e^{-2(\gamma-\alpha)z} + F^{(2)}(\gamma+\alpha)e^{-2(\gamma+\alpha)z}\right.$$
$$\left.-\gamma F^{(3)}e^{-2\gamma z}\right]\left[b_1^2\sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right]$$
$$-2\left[F^{(4)}(\gamma-\alpha)e^{-2(\gamma-\alpha)z} + F^{(5)}(\gamma+\alpha)e^{-2(\gamma+\alpha)z} - 2\gamma F^{(6)}e^{-2\gamma z}\right]$$
$$\times 2b_1b_2 \sin 2\sigma t + \frac{\partial\phi_2'}{\partial z}.$$

$$\frac{\partial^2\phi_2}{\partial z^2} = 4\Big[(\gamma-\alpha)^2 F^{①} e^{-2(\gamma-\alpha)z} + (\gamma+\alpha)^2 F^{②} e^{-2(\gamma+\alpha)z}$$

$$- 2\gamma^2 F^{③} e^{-2\gamma z}\Big]\Big[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\Big]$$

$$+ 4\Big[(\gamma-\alpha)^2 F^{④} e^{-2(\gamma-\alpha)z} + (\gamma+\alpha)^2 F^{⑤} e^{-2(\gamma+\alpha)z}$$

$$- 2\gamma^2 F^{⑥} e^{-2\gamma z}\Big].2 b_1 b_2 \sin 2\sigma t \; + \frac{\partial^2\phi_2'}{\partial z^2} .$$

$$\frac{\partial^2\phi_2}{\partial x^2} = -\Big[F^{①} e^{-2(\gamma-\alpha)z} + F^{②} e^{-2(\gamma+\alpha)z} - 2F^{③} e^{-2\gamma z}\Big] 4k^2$$

$$\times \Big[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\Big] + \frac{\partial^2\phi_2'}{\partial x^2} .$$

$$\nabla^2\phi_2 = 4\Big[\{(\gamma-\alpha)^2-k^2\} F^{①} e^{-2(\gamma-\alpha)z} + \{(\gamma+\alpha)^2-k^2\} F^{②} e^{-2(\gamma+\alpha)z}$$

$$-2(\gamma^2-k^2) F^{③} e^{-2\gamma z}\Big]\Big[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\Big]$$

$$+ 4\Big[(\gamma-\alpha)^2 F^{④} e^{-2(\gamma-\alpha)z} + (\gamma+\alpha)^2 F^{⑤} e^{-2(\gamma+\alpha)z}$$

$$- 2\gamma^2 F^{⑥} e^{-2\gamma z}\Big].2 b_1 b_2 \sin 2\sigma t \; + \nabla^2\phi_2' .$$

$$\frac{\partial^2\phi_2}{\partial t^2} = -4\sigma^2\Big[F^{①} e^{-2(\gamma-\alpha)z} + F^{②} e^{-2(\gamma+\alpha)z} - 2F^{③} e^{-2\gamma z}\Big]$$

$$\times \Big[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\Big]$$

$$-\Big[F^{④} e^{-2(\gamma-\alpha)z} + F^{⑤} e^{-2(\gamma+\alpha)z} - 2F^{⑥} e^{-2\gamma z}\Big] 8 b_1 b_2 \sigma^2 \sin 2\sigma t$$

$$+ \frac{\partial^2\phi_2'}{\partial t^2} .$$

Equation (6.58) becomes

$$\frac{\partial^2 \phi_2'}{\partial t^2} - c^2 \nabla^2 \phi_2' - g\frac{\partial \phi_2'}{\partial z} = A^{①}\left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right] + A^{②}.\, 2b_1 b_2 \sin 2\sigma t.$$

Now select $F^{①}, F^{②}, F^{③}, F^{④}, F^{⑤}$ and $F^{⑥}$ so that the right-hand side of this equation vanishes identically. Then

$$\begin{aligned} &F^{①} e^{-2(\gamma-\alpha)z}\left[-4\sigma^2 - 4c^2\{(\gamma-\alpha)^2-k^2\} + 2g(\gamma-\alpha)\right] \\ &+F^{②} e^{-2(\gamma+\alpha)z}\left[-4\sigma^2 - 4c^2\{(\gamma+\alpha)^2-k^2\} + 2g(\gamma+\alpha)\right] \\ &-2F^{③} e^{-2\gamma z}\left[-4\sigma^2 - 4c^2(\gamma^2-k^2) + 2\gamma g\right] \\ &\equiv C^{①} e^{-2(\gamma-\alpha)z} + C^{②} e^{-2(\gamma+\alpha)z} - 2C^{③} e^{-2\gamma z}; \end{aligned}$$

and

$$\begin{aligned} &F^{④} e^{-2(\gamma-\alpha)z}\left[-4\sigma^2 - 4c^2(\gamma-\alpha)^2 + 2g(\gamma-\alpha)\right] \\ &+F^{⑤} e^{-2(\gamma+\alpha)z}\left[-4\sigma^2 - 4c^2(\gamma+\alpha)^2 + 2g(\gamma+\alpha)\right] \\ &-2F^{⑥} e^{-2\gamma z}\left[-4\sigma^2 - 4c^2\gamma^2 + 2g\gamma\right] \\ &\equiv C^{④} e^{-2(\gamma-\alpha)z} + C^{⑤} e^{-2(\gamma+\alpha)z} - 2C^{⑥} e^{-2\gamma z}. \end{aligned}$$

Hence

$$\left.\begin{aligned} F^{①} &= \frac{C^{①}}{-4\sigma^2 - 4c^2\{(\gamma-\alpha)^2-k^2\} + 2g(\gamma-\alpha)} \\ F^{②} &= \frac{C^{②}}{-4\sigma^2 - 4c^2\{(\gamma+\alpha)^2-k^2\} + 2g(\gamma+\alpha)} \\ F^{③} &= \frac{C^{③}}{-4\sigma^2 - 4c^2(\gamma^2-k^2) + 2g\gamma} \end{aligned}\right\} \quad (6.69)$$

$$\left.\begin{aligned} F^{④} &= \frac{c^{④}}{-4\sigma^2 - 4c^2(\gamma-\alpha)^2 + 2g(\gamma-\alpha)} \\ F^{⑤} &= \frac{c^{⑤}}{-4\sigma^2 - 4c^2(\gamma+\alpha)^2 + 2g(\gamma+\alpha)} \\ F^{⑥} &= \frac{c^{⑥}}{-4\sigma^2 - 4c^2\gamma + 2g\gamma} \end{aligned}\right\} \quad (6\cdot69)$$

And $$\frac{\partial^2 \phi_2'}{\partial t^2} - c^2 \nabla^2 \phi_2' - g\frac{\partial \phi_2'}{\partial z} = 0. \quad (6\cdot70)$$

$$\begin{aligned} \left(\frac{\partial \phi_2}{\partial z}\right)_{z=h} = &-2\left[F^{①}(\gamma-\alpha)e^{-2(\gamma-\alpha)h} + F^{②}(\gamma+\alpha)e^{-2(\gamma+\alpha)h} - 2F^{③}\gamma e^{-2\gamma h}\right] \\ &\times\left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right] \\ &-2\left[F^{④}(\gamma-\alpha)e^{-2(\gamma-\alpha)h} + F^{⑤}(\gamma+\alpha)e^{-2(\gamma+\alpha)h} - 2F^{⑥}\gamma e^{-2\gamma h}\right] \\ &\times 2b_1 b_2 \sin 2\sigma t \quad + \left(\frac{\partial \phi_2'}{\partial z}\right)_{z=h} \end{aligned}$$

Hence equation (6·33) becomes

$$\begin{aligned} \left(\frac{\partial \phi_2'}{\partial z}\right)_{z=h} = &\, G^{①}\left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right] \\ &+ G^{②}\, 2b_1 b_2 \sin 2\sigma t \end{aligned} \quad (6\cdot71)$$

where $$\left.\begin{aligned} G^{①} &= 2(\gamma-\alpha)e^{-2(\gamma-\alpha)h}F^{①} + 2(\gamma+\alpha)e^{-2(\gamma+\alpha)h}F^{②} - 4\gamma e^{-2\gamma h}F^{③}, \\ \text{and } G^{②} &= 2(\gamma-\alpha)e^{-2(\gamma-\alpha)h}F^{④} + 2(\gamma+\alpha)e^{-2(\gamma+\alpha)h}F^{⑤} - 4\gamma e^{-2\gamma h}F^{⑥} \end{aligned}\right\} \quad (6\cdot72)$$

Also
$$\left(\nabla^2\phi_2\right)_{z=0}$$
$$= 4\left[\{(\gamma-\alpha)^2-k^2\}F^{(1)}+\{(\gamma+\alpha)^2-k^2\}F^{(2)}-2(\gamma^2-\alpha^2)F^{(3)}\right]$$
$$\times\left[b_1^2\sin 2(kx-\sigma t)-b_2^2\sin 2(kx+\sigma t)\right]$$
$$+4\left[(\gamma-\alpha)^2F^{(4)}+(\gamma+\alpha)^2F^{(5)}-2\gamma^2F^{(6)}\right]2b_1b_2\sin 2\sigma t$$
$$+\left(\nabla^2\phi_2'\right)_{z=0}.$$

Hence equation (6·59) becomes

$$\left(\nabla^2\phi_2'\right)_{z=0}=\left[D-4\{(\gamma-\alpha)^2-k^2\}F^{(1)}-4\{(\gamma+\alpha)^2-k^2\}F^{(2)}\right.$$
$$\left.+8(\gamma^2-k^2)F^{(3)}\right]\left[b_1^2\sin 2(kx-\sigma t)-b_2^2\sin 2(kx+\sigma t)\right]$$
$$+\left[D-4(\gamma-\alpha)^2F^{(4)}-4(\gamma+\alpha)^2F^{(5)}+8\gamma^2F^{(6)}\right]2b_1b_2\sin 2\sigma t.$$

That is
$$\left(\nabla^2\phi_2'\right)_{z=0}=(D+H^{(1)})\left[b_1^2\sin 2(kx-\sigma t)-b_2^2\sin 2(kx+\sigma t)\right]$$
$$+(D+H^{(2)}).\ 2b_1b_2\sin 2\sigma t \qquad (6\cdot73)$$

where
$$\left.\begin{aligned}H^{(1)}&=-4\{(\gamma-\alpha)^2-k^2\}F^{(1)}-4\{(\gamma+\alpha)^2-k^2\}F^{(2)}+8(\gamma^2-\alpha^2)F^{(3)}\\ \text{and}\quad H^{(2)}&=-4(\gamma-\alpha)^2F^{(4)}-4(\gamma+\alpha)^2F^{(5)}+8\gamma^2F^{(6)}\end{aligned}\right\}(6\cdot74)$$

We now substitute in equation (6·66)

$$\left(\frac{\partial\phi_2}{\partial t}\right)_{z=0}=\left(F^{(1)}+F^{(2)}-2F^{(3)}\right)2\sigma\left[-b_1^2\cos 2(kx-\sigma t)-b_2^2\cos 2(kx+\sigma t)\right]$$
$$+\left(F^{(4)}+F^{(5)}-2F^{(6)}\right)4b_1b_2\cos 2\sigma t+\left(\frac{\partial\phi_2'}{\partial t}\right)_{z=0}.$$

$$\therefore \int_0^\lambda \left(\frac{\partial \phi_2}{\partial t}\right)_{z=0} dx = -\frac{\sigma}{k}\left(F^{(1)} + F^{(2)} - 2F^{(3)}\right)$$

$$\times \left[b_1^2 \sin 2(kx - \sigma t) + b_2^2 \sin 2(kx + \sigma t)\right]_0^\lambda$$

$$+ \left(F^{(4)} + F^{(5)} - 2F^{(6)}\right) 4 b_1 b_2 \left[x \cos 2\sigma t\right]_0^\lambda + \int_0^\lambda \left(\frac{\partial \phi_2'}{\partial t}\right)_{z=0} dx$$

$$= \int_0^\lambda \left(\frac{\partial \phi_2'}{\partial t}\right)_{z=0} dx + \left(F^{(4)} + F^{(5)} - 2F^{(6)}\right) 4 b_1 b_2 \cdot \frac{2\pi}{k} \cos 2\sigma t$$

since $k\lambda = 2\pi$.

$$\frac{\partial \phi_2}{\partial t} \cdot e^{2\gamma z} = \left[F^{(1)} e^{\alpha z} + F^{(2)} e^{-2\alpha z} - 2F^{(3)}\right] 2\sigma$$

$$\times \left[-b_1^2 \cos 2(kx - \sigma t) - b_2^2 \cos 2(kx + \sigma t)\right]$$

$$+ \left[F^{(4)} e^{2\alpha z} + F^{(5)} e^{-2\alpha z} - 2F^{(6)}\right] 4 b_1 b_2 \cos 2\sigma t + e^{2\gamma z} \cdot \frac{\partial \phi_2'}{\partial t} .$$

$$\therefore \int_0^\lambda dx \int_0^h \frac{\partial \phi_2}{\partial t} \cdot e^{2\gamma z} dz$$

$$= -\frac{\sigma}{k}\left[\frac{F^{(1)} e^{2\alpha z}}{2\alpha} - \frac{F^{(2)} e^{-2\alpha z}}{2\alpha} - 2F^{(3)} z\right]_0^h$$

$$\times \left[b_1^2 \sin 2(kx - \sigma t) + b_2^2 \sin 2(kx + \sigma t)\right]_0^\lambda$$

$$+ \left[\frac{F^{(4)} e^{2\alpha z}}{2\alpha} - \frac{F^{(5)} e^{-2\alpha z}}{2\alpha} - 2F^{(6)} z\right]_0^h \cdot \left[4 b_1 b_2 x \cos 2\sigma t\right]_0^\lambda$$

$$+ \int_0^\lambda dx \int_0^h e^{2\gamma z} \cdot \frac{\partial \phi_2'}{\partial t} \cdot dz .$$

$$\therefore \quad \int_0^\lambda dx \int_0^h \frac{\partial \phi_2}{\partial t} . e^{2\gamma z} dz$$

$$= \left[F^{④} e^{2\alpha h} - F^{⑤} e^{-2\alpha h} - 4F^{⑥} \alpha h - F^{④} + F^{⑤} \right] \frac{4 b_1 b_2 \pi}{\alpha k} . \cos 2\sigma t$$

$$+ \int_0^\lambda dx \int_0^h e^{2\gamma z} . \frac{\partial \phi_2'}{\partial t} . dz \quad .$$

Hence equation (6·66) becomes

$$2\gamma \int_0^\lambda dx \int_0^h e^{2\gamma z} . \frac{\partial \phi_2'}{\partial t} dz + \int_0^\lambda \left(\frac{\partial \phi_2'}{\partial t} \right)_{z=0} dx$$

$$= E^{①} (b_1^2 + b_2^2) + E^{②} 2 b_1 b_2 \cos 2\sigma t$$

$$- \left\{ F^{④} e^{2\alpha h} - F^{⑤} e^{-2\alpha h} - 4F^{⑥} \alpha h - F^{④} + F^{⑤} \right\} \frac{8\pi b_1 b_2 \gamma}{\alpha k} \cos 2\sigma t$$

$$- \left(F^{④} + F^{⑤} - 2F^{⑥} \right) \frac{8 b_1 b_2 \pi}{k} . \cos 2\sigma t \, .$$

That is

$$2\gamma \int_0^\lambda dx \int_0^h e^{2\gamma z} \frac{\partial \phi_2'}{\partial t} dz + \int_0^\lambda \left(\frac{\partial \phi_2'}{\partial t} \right)_{z=0} dx$$

$$= E^{①} (b_1^2 + b_2^2) + \left(E^{②} + I \right) 2 b_1 b_2 \cos 2\sigma t \qquad (6·75)$$

Where

$$I = - \frac{4\pi\gamma}{\alpha k} \left[F^{④} e^{2\alpha h} - F^{⑤} e^{-2\alpha h} - 4F^{⑥} \alpha h - F^{④} + F^{⑤} \right]$$

$$- \frac{4\pi}{k} \left[F^{④} + F^{⑤} - 2F^{⑥} \right] \qquad (6·76)$$

We now have ϕ_2 given by equation (6·68) involving ϕ_2' which is determined by equations (6·70), (6·71), (6·73) and (6·75).
We now seek to reduce the right hand members of equations (6·71) and (6·73) to zero by writing

$$\phi_2' = \left[J^{①} e^{-(\gamma-\alpha')z} + J^{②} e^{-(\gamma+\alpha')z} \right] \times \left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t) \right] + \left[J^{③} e^{-(\gamma-\alpha'')z} + J^{④} e^{-(\gamma+\alpha'')z} \right] 2b_1 b_2 \sin 2\sigma t + \phi_2'' \quad (6·77)$$

where $J^{①}$, $J^{②}$, $J^{③}$ and $J^{④}$ are functions of the constants of the motion and the medium, and

$$\left.\begin{aligned} \alpha'^2 &= 4k^2 - \frac{4\sigma^2}{c^2} + \gamma^2 \\ \text{and} \quad \alpha''^2 &= -\frac{4\sigma^2}{c^2} + \gamma^2 \end{aligned}\right\} \quad (6·78)$$

Substitute for ϕ_2' in equation (6·71).

$$\frac{\partial \phi_2'}{\partial z} = -\left[J^{①}(\gamma-\alpha') e^{-(\gamma-\alpha')z} + J^{②}(\gamma+\alpha') e^{-(\gamma+\alpha')z} \right] \times \left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t) \right] - \left[J^{③}(\gamma-\alpha'') e^{-(\gamma-\alpha'')z} + J^{④}(\gamma+\alpha'') e^{-(\gamma+\alpha'')z} \right] 2b_1 b_2 \sin 2\sigma t + \frac{\partial \phi_2''}{\partial z}$$

Hence by equation (6·71)

$$\left(\frac{\partial \phi_2''}{\partial z}\right)_{z=h} = \left[G^{①} + J^{①}(\gamma-\alpha')e^{-(\gamma-\alpha')h} + J^{②}(\gamma+\alpha')e^{-(\gamma+\alpha')h}\right]$$
$$\times\left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right]$$
$$+\left[G^{②} + J^{③}(\gamma-\alpha'')e^{-(\gamma-\alpha'')h} + J^{④}(\gamma+\alpha'')e^{-(\gamma+\alpha'')h}\right]$$
$$\times 2 b_1 b_2 \sin 2\sigma t.$$

$$\therefore \left(\frac{\partial \phi_2''}{\partial z}\right)_{z=h} = 0 \qquad (6·79)$$

if $G^{①} + J^{①}(\gamma-\alpha')e^{-(\gamma-\alpha')h} + J^{②}(\gamma+\alpha')e^{-(\gamma+\alpha')h} = 0,$ (6·80)

and $G^{②} + J^{③}(\gamma-\alpha'')e^{-(\gamma-\alpha'')h} + J^{④}(\gamma+\alpha'')e^{-(\gamma+\alpha'')h} = 0.$ (6·81)

From equation (6·77)

$$\frac{\partial^2 \phi_2'}{\partial z^2} = \left[J^{①}(\gamma-\alpha')^2 e^{-(\gamma-\alpha')z} + J^{②}(\gamma+\alpha')^2 e^{-(\gamma+\alpha')z}\right]$$
$$\times\left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right]$$
$$+\left[J^{③}(\gamma-\alpha'')^2 e^{-(\gamma-\alpha'')z} + J^{④}(\gamma+\alpha'')^2 e^{-(\gamma+\alpha'')z}\right]$$
$$\times 2 b_1 b_2 \sin 2\sigma t + \frac{\partial^2 \phi_2''}{\partial z^2},$$

$$\frac{\partial^2 \phi_2'}{\partial x^2} = -4k^2\left[J^{①}e^{-(\gamma-\alpha')z} + J^{②}e^{-(\gamma+\alpha')z}\right]$$
$$\times\left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right] + \frac{\partial^2 \phi_2''}{\partial x^2}.$$

$$\therefore \left[\nabla^2\phi_2'\right]_{z=0}$$

$$= \left[J^{①}(\gamma-\alpha')^2 + J^{②}(\gamma+\alpha')^2 - 4k^2 J^{①} - 4k^2 J^{②}\right]$$

$$\times \left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right]$$

$$+ \left[J^{③}(\gamma-\alpha'')^2 + J^{④}(\gamma+\alpha'')^2\right] 2b_1 b_2 \sin 2\sigma t$$

$$+ \left[\nabla^2\phi_2''\right]_{z=0}.$$

$$\left[\nabla^2\phi_2''\right]_{z=0} = \left[D + H^{①} - J^{①}\{(\gamma-\alpha')^2 - 4k^2\} - J^{②}\{(\gamma+\alpha')^2 - 4k^2\}\right]$$

$$\times \left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right]$$

$$+ \left[D + H^{②} - J^{③}(\gamma-\alpha'')^2 - J^{④}(\gamma+\alpha'')^2\right] 2b_1 b_2 \sin 2\sigma t.$$

That is $\left[\nabla^2\phi_2''\right]_{z=0} = 0.$ (6•82)

if $D + H^{①} = J^{①}\{(\gamma-\alpha')^2 - 4k^2\} + J^{②}\{(\gamma+\alpha')^2 - 4k^2\}$ (6•83)

and $D + H^{②} = J^{③}(\gamma-\alpha'')^2 + J^{④}(\gamma+\alpha'')^2$ (6•84)

In order to determine $J^{①}$ and $J^{②}$ we cross multiply equations (6•80) and (6•83), thus

$$\frac{J^{(1)}}{-(\gamma+\alpha')e^{-(\gamma+\alpha')h}.(D+H^{(1)})-\{(\gamma+\alpha')^2-4k^2\}.G^{(1)}}$$

$$=\frac{J^{(2)}}{(\gamma-\alpha')e^{-(\gamma-\alpha')h}.(D+H^{(1)})+\{(\gamma-\alpha')^2-4k^2\}.G^{(1)}}$$

$$=\frac{1}{(\gamma-\alpha')e^{-(\gamma-\alpha')h}\{(\gamma+\alpha')^2-4k^2\}-(\gamma+\alpha')e^{-(\gamma+\alpha')h}.\{(\gamma-\alpha')^2-4k^2\}}$$

But

$$\{(\gamma-\alpha')^2-4k^2\}(\gamma+\alpha')e^{-(\gamma+\alpha')h}-\{(\gamma+\alpha')-4k^2\}(\gamma-\alpha')e^{-(\gamma-\alpha')h}$$

$$=e^{-\gamma h}\left[(\gamma+\alpha')(\gamma^2-2\gamma\alpha'+\alpha'^2-4k^2)e^{-\alpha'h}-(\gamma-\alpha')(\gamma^2+2\gamma\alpha'+\alpha'^2-4k^2)e^{\alpha'h}\right]$$

$$=-e^{-\gamma h}\left[\gamma(\gamma^2-\alpha'^2-4k^2)(e^{\alpha'h}-e^{-\alpha'h})+\alpha'(\gamma^2-\alpha'^2+4k^2)(e^{\alpha'h}+e^{-\alpha'h})\right]$$

$$=-2e^{-\gamma h}\left[\gamma(\gamma^2-\alpha'^2-4k^2)\sinh\alpha'h+\alpha'(\gamma^2-\alpha'^2+4k^2)\cosh\alpha'h\right]$$

$$=\Delta(2\sigma,2k) \tag{6.85}$$

Hence

$$J^{(1)}=\frac{\{(\gamma+\alpha')^2-4k^2\}G^{(1)}+(\gamma+\alpha')e^{-(\gamma+\alpha')h}.(D+H^{(1)})}{\Delta(2\sigma,2k)} \tag{6.86}$$

$$J^{(2)}=-\frac{\{(\gamma-\alpha')^2-4k^2\}G^{(1)}+(\gamma-\alpha')e^{-(\gamma-\alpha')h}.(D+H^{(1)})}{\Delta(2\sigma,2k)} \tag{6.87}$$

providing

$$\Delta(2\sigma,2k)\equiv-2e^{-\gamma h}\left[\gamma(\gamma^2-\alpha'^2-4k^2)\sinh\alpha'h+\alpha'(\gamma^2-\alpha'^2+4k^2)\cosh\alpha'h\right]$$

$$\neq 0.$$

In order to determine $J^{③}$ and $J^{④}$ we cross multiply equations (6·81) and (6·84), thus

$$\frac{J^{③}}{-(\gamma+\alpha'')e^{-(\gamma+\alpha'')h}.(D+H^{②}) - (\gamma+\alpha'')^2.G^{②}}$$

$$= \frac{J^{④}}{(\gamma-\alpha'')^2 G^{②} + (\gamma-\alpha'')e^{-(\gamma-\alpha'')h}.(D+H^{②})}$$

$$= \frac{1}{(\gamma-\alpha'')(\gamma+\alpha'')^2 e^{-(\gamma-\alpha'')h} - (\gamma-\alpha'')^2(\gamma+\alpha'')e^{-(\gamma+\alpha'')h}}$$

when $k=0$ $\alpha'^2 = \alpha''^2$, hence

$$(\gamma-\alpha'')(\gamma+\alpha'')^2 e^{-(\gamma-\alpha'')h} - (\gamma-\alpha'')^2(\gamma+\alpha'')e^{-(\gamma+\alpha'')h} = -\Delta(2\sigma,0)$$

Hence

$$J^{③} = \frac{(\gamma+\alpha'')^2 G^{②} + (\gamma+\alpha'')e^{-(\gamma+\alpha'')h}.(D+H^{②})}{\Delta(2\sigma,0)} \quad (6{\cdot}88)$$

$$J^{④} = -\frac{(\gamma-\alpha'')^2 G^{②} + (\gamma-\alpha'')e^{-(\gamma-\alpha'')h}.(D+H^{②})}{\Delta(2\sigma,0)} \quad (6{\cdot}89)$$

provided that

$$\Delta(2\sigma,0) \equiv -2e^{-\gamma h}\left[\gamma(\gamma^2-\alpha''^2)\sinh\alpha''h + \alpha''(\gamma^2-\alpha''^2)\cosh\alpha''h\right] \neq 0 \quad (6{\cdot}90)$$

We now substitute for ϕ_2' in equation (6·75)

$$\frac{\partial\phi_2'}{\partial t} = 2\sigma\left[J^{①}e^{-(\gamma-\alpha')z} + J^{②}e^{-(\gamma+\alpha')z}\right] \times \left[-b_1^2\cos 2(kx-\sigma t) - b_2^2\cos 2(kx+\sigma t)\right]$$

$$+ 2\sigma\left[J^{③}e^{-(\gamma-\alpha'')z} + J^{④}e^{-(\gamma+\alpha'')z}\right]2b_1b_2\cos 2\sigma t + \frac{\partial\phi_2''}{\partial t}.$$

$$\therefore \int_0^\lambda \left(\frac{\partial \phi_2'}{\partial t}\right)_{z=0} dx =$$

$$2\sigma\left(J^{①}+J^{②}\right)\left[\frac{-b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)}{2k}\right]_0^\lambda$$

$$+ 2\sigma\left(J^{③}+J^{④}\right)\left[2x\, b_1 b_2 \cos 2\sigma t\right]_0^\lambda + \int_0^\lambda \left(\frac{\partial \phi_2''}{\partial t}\right)_{z=0} dx$$

$$= 2\sigma\left(J^{③}+J^{④}\right) 2\lambda\, b_1 b_2 \cos 2\sigma t + \int_0^\lambda \left(\frac{\partial \phi_2''}{\partial t}\right)_{z=0} dx \, .$$

$$\int_0^h \frac{\partial \phi_2'}{\partial t} e^{2\gamma z} dz = 2\sigma\left[\frac{J^{①} e^{(\gamma+\alpha')z}}{\gamma+\alpha'} + \frac{J^{②} e^{(\gamma-\alpha')z}}{\gamma-\alpha'}\right]_0^h$$

$$\times\left[-b_1^2 \cos 2(kx-\sigma t) - b_2^2 \cos 2(kx+\sigma t)\right]$$

$$+ 2\sigma\left[\frac{J^{③} e^{(\gamma+\alpha'')z}}{\gamma+\alpha''} + \frac{J^{④} e^{(\gamma-\alpha'')z}}{\gamma-\alpha''}\right]_0^h 2\, b_1 b_2 \cos 2\sigma t$$

$$+ \int_0^h \frac{\partial \phi_2''}{\partial t} e^{2\gamma z} dz$$

$$\therefore \int_0^\lambda dx \int_0^h \frac{\partial \phi_2'}{\partial t} e^{2\gamma z} dz$$

$$= 2\sigma\left[\frac{J^{③}\left(e^{(\gamma+\alpha'')h}-1\right)}{\gamma+\alpha''} + \frac{J^{④}\left(e^{(\gamma-\alpha'')h}-1\right)}{\gamma-\alpha''}\right] 2\lambda\, b_1 b_2 \cos 2\sigma t$$

$$+ \int_0^\lambda dx \int_0^h e^{2\gamma z} \frac{\partial \phi_2''}{\partial t} dz \, .$$

Thus equation (6·75) becomes

$$4\gamma\sigma\left[\frac{J^{③}\{e^{(\gamma+\alpha'')h}-1\}}{\gamma+\alpha''}+\frac{J^{④}\{e^{(\gamma-\alpha'')h}-1\}}{\gamma-\alpha''}\right]2\lambda b_1 b_2 \cos 2\sigma t$$

$$+2\gamma\int_0^\lambda dx\int_0^h e^{2\gamma z}\frac{\partial\phi_2''}{\partial t}\,dz$$

$$+2\sigma\left(J^{③}+J^{④}\right)2\lambda b_1 b_2 \cos 2\sigma t+\int_0^\lambda\left(\frac{\partial\phi_2''}{\partial t}\right)_{z=0}dx$$

$$=E^{①}(b_1^2+b_2^2)+\left(E^{②}+I\right)2b_1 b_2\cos 2\sigma t.$$

That is

$$2\gamma\int_0^\lambda dx\int_0^h e^{2\gamma z}\frac{\partial\phi_2''}{\partial t}\,dz+\int_0^\lambda\left(\frac{\partial\phi_2''}{\partial t}\right)_{z=0}dx$$

$$=E^{①}(b_1^2+b_2^2)+\left(E^{②}+I+K\right)2b_1 b_2\cos 2\sigma t \qquad (6\cdot 91)$$

where

$$K=-2\sigma\left(J^{③}+J^{④}\right)\lambda-4\gamma\sigma\lambda\left[\frac{J^{③}\{e^{(\gamma+\alpha'')h}-1\}}{\gamma+\alpha''}+\frac{J^{④}\{e^{(\gamma-\alpha'')h}-1\}}{\gamma-\alpha''}\right].$$

It now remains to express equation (6·70) in terms of ϕ_2''. We have

$$\frac{\partial^2\phi_2'}{\partial t^2}=\left[J^{①}e^{-(\gamma-\alpha')z}+J^{②}e^{-(\gamma+\alpha')z}\right]4\sigma^2$$

$$\times\left[-b_1^2\sin 2(kx-\sigma t)+b_2^2\sin 2(kx+\sigma t)\right]$$

$$-\left[J^{③}e^{-(\gamma-\alpha'')z}+J^{④}e^{-(\gamma+\alpha'')z}\right]8\sigma^2 b_1 b_2\sin 2\sigma t$$

$$+\frac{\partial^2\phi_2''}{\partial t^2}.$$

$$\nabla^2\phi_2' = -4k^2\left[J^{①}e^{-(\gamma-\alpha')z} + J^{②}e^{-(\gamma+\alpha')z}\right]$$

$$\times\left[b_1^2\sin 2(kx-\sigma t) - b_2^2\sin 2(kx+\sigma t)\right]$$

$$+\left[J^{①}(\gamma-\alpha')^2e^{-(\gamma-\alpha')z} + J^{②}(\gamma+\alpha')^2e^{-(\gamma+\alpha')z}\right]$$

$$\times\left[b_1^2\sin 2(kx-\sigma t) - b_2^2\sin 2(kx+\sigma t)\right]$$

$$+\left[J^{③}(\gamma-\alpha'')^2e^{-(\gamma-\alpha'')z} + J^{④}(\gamma+\alpha'')^2e^{-(\gamma+\alpha'')z}\right]2b_1b_2\sin 2\sigma t$$

$$+\ \nabla^2\phi_2''.$$

and

$$\frac{\partial\phi_2'}{\partial z} = -\left[J^{①}(\gamma-\alpha')e^{-(\gamma-\alpha')z} + J^{②}(\gamma+\alpha')e^{-(\gamma+\alpha')z}\right]$$

$$\times\left[b_1^2\sin 2(kx-\sigma t) - b_2^2\sin 2(kx+\sigma t)\right]$$

$$-\left[J^{③}(\gamma-\alpha'')e^{-(\gamma-\alpha'')z} + J^{④}(\gamma+\alpha'')e^{-(\gamma+\alpha'')z}\right]2b_1b_2\sin 2\sigma t$$

$$+\ \frac{\partial\phi_2''}{\partial z}.$$

Hence, in $\frac{\partial^2\phi_2'}{\partial t^2} - c^2\nabla^2\phi_2' - g\frac{\partial\phi_2'}{\partial z}$, the coefficient of $J^{①}e^{-(\gamma-\alpha')z}\left[b_1^2\sin 2(kx-\sigma t) - b_2^2\sin 2(kx+\sigma t)\right]$ is

$$-4\sigma^2 + 4c^2k^2 - c^2(\gamma-\alpha')^2 + g(\gamma-\alpha')$$

$$= -4\sigma^2 + 4c^2k^2 - c^2(\gamma^2 - 2\gamma\alpha' + \alpha'^2) + 2c^2\gamma(\gamma-\alpha')$$

$$= -4\sigma^2 + 4c^2k^2 - c^2(\gamma^2 - 2\gamma\alpha' + \alpha'^2 - 2\gamma^2 + 2\gamma\alpha')$$

$$= -4\sigma^2 + 4c^2k^2 - c^2\left(-\gamma^2 + 4k^2 - \frac{4\sigma^2}{c^2} + \gamma^2\right) \text{ by (6.78)}$$

$$= 0.$$

The coefficient of

$$J^{②} e^{-(\gamma+\alpha')z}\left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right] \text{ is}$$

$$-4\sigma^2+4c^2k^2 - c^2(\gamma+\alpha')^2 + g(\gamma+\alpha')$$

$$= -4\sigma^2+4c^2k^2 - c^2(\gamma^2+2\gamma\alpha'+\alpha'^2)+2c^2\gamma(\gamma+\alpha')$$

$$= -4\sigma^2+4c^2k^2 - c^2(\gamma^2+2\gamma\alpha'+\alpha'^2-2\gamma^2-2\gamma\alpha')$$

$$= -4\sigma^2+4c^2k^2 - c^2\left(4k^2-\frac{4\sigma^2}{c^2}\right) \qquad \text{by } (6\cdot78)$$

$$= 0.$$

Hence the coefficient of $\left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right]$ is zero.

The coefficient of $J^{③} e^{-(\gamma-\alpha'')z} \cdot 2b_1b_2 \sin 2\sigma t$ is

$$-4\sigma^2 - c^2(\gamma-\alpha'')^2 + g(\gamma-\alpha'')$$

$$= -4\sigma^2 - c^2(\gamma^2-2\gamma\alpha''+\alpha''^2) + 2c^2\gamma(\gamma-\alpha'')$$

$$= -4\sigma^2 - c^2(\alpha''^2-\gamma^2) \qquad \text{by } (6.78)$$

$$= 0.$$

The coefficient of $J^{④} e^{-(\gamma+\alpha'')z} \cdot 2b_1b_2 \sin 2\sigma t$ is

$$4\sigma^2 - c^2(\gamma+\alpha'')^2 + g(\gamma+\alpha'')$$

$$= 4\sigma^2 - c^2(\gamma^2+2\gamma\alpha''+\alpha''^2) + c^2\gamma(\gamma+\alpha'')$$

$$= 4\sigma^2 - c^2(-\gamma^2+\alpha''^2)$$

$$= 0.$$

Hence the coefficient of $2b_1 b_2 \sin 2\sigma t$ is zero, and so equation (6•70) transforms into

$$\frac{\partial^2 \phi_2''}{\partial t^2} - c^2 \nabla^2 \phi_2'' - g \frac{\partial \phi_2''}{\partial z} = 0 \qquad (6\text{•}92).$$

We thus have four equations from which to determine ϕ_2'', namely :

$$\frac{\partial^2 \phi_2''}{\partial t^2} - c^2 \nabla^2 \phi_2'' - g \frac{\partial \phi_2''}{\partial z} = 0, \qquad (6\text{•}92)$$

$$\left(\frac{\partial \phi_2''}{\partial z}\right)_{z=h} = 0, \qquad (6\text{•}79)$$

$$\left(\nabla^2 \phi_2''\right)_{z=0} = 0, \qquad (6\text{•}82)$$

$$2\gamma \int_0^\lambda dx \int_0^h e^{2\gamma z} \frac{\partial \phi_2''}{\partial t}\, dz + \int_0^\lambda \left(\frac{\partial \phi_2''}{\partial t}\right)_{z=0} dx$$

$$= E^{(1)}(b_1^2 + b_2^2) + (E^{(2)} + I + K)\, 2b_1 b_2 \cos 2\sigma t, \qquad (6\text{•}91)$$

It has already been shewn that a solution of the four equations 6•25, 6•26 , 6•27 and 6•39 is to be found by adding a constant multiple of t to a solution of the equations 6•25 , 6•26 and 6•27. Also equations 6•92 , 6•79, 6•82 and 6•91 are derived from equations 6•25 , 6•26 , 6•27 and 6•39 respectively by the same changes of variable. Hence a solution of equations 6•92 , 6•79 , 6•82 amd 6•91 is to be found by adding a constant multiple of t to a solution of 6•92 , 6•79 , and 6•82,

Now $\phi_2'' = 0$ satisfies these last three equations. Hence a solution of all four equations for ϕ_2'' is

$$\phi_2'' = c'' t \qquad (6\text{•}93)$$

On substituting from (6·92) into (6·91) we have

$$2\gamma \int_0^\lambda dx \int_0^h c'' e^{2\gamma z}\, dz + \int_0^\lambda c''\, dx$$

$$= 2\gamma \cdot \frac{c''}{2\gamma}\left(e^{2\gamma h} - 1\right) \cdot \lambda + c'' \lambda$$

$$= c'' \lambda e^{2\gamma h} = E^{(1)}(b_1^2 + b_2^2) + \left\{E^{(2)} + I + K\right\} 2 b_1 b_2 \cos 2\sigma t$$

Hence $$c'' \lambda e^{2\gamma h} = E^{(1)}(b_1^2 + b_2^2) \qquad (6{\cdot}94)$$

and $$E^{(2)} + I + K = 0 .$$

Hence $$\phi_2'' = \frac{E^{(1)}(b_1^2 + b_2^2)}{\lambda e^{2\gamma h}} \cdot t ,$$

From equation (6·77)

$$\phi_2' = \left[J^{(1)} e^{-(\gamma - \alpha')z} + J^{(2)} e^{-(\gamma + \alpha')z}\right]\left[b_1^2 \sin 2(kx - \sigma t) - b_2^2 \sin 2(kx + \sigma t)\right]$$

$$+ \left[J^{(3)} e^{-(\gamma - \alpha'')z} + J^{(4)} e^{-(\gamma + \alpha'')z}\right] 2 b_1 b_2 \sin 2\sigma t$$

$$+ \frac{E^{(1)}(b_1^2 + b_2^2)}{\lambda e^{2\gamma h}} \cdot t .$$

Then from equation (6·68)

$$\phi_2 = \left[F^{(1)} e^{2\alpha z} + F^{(2)} e^{-2\alpha z} - 2F^{(3)}\right] e^{-2\gamma z}\left[b_1^2 \sin 2(kx - \sigma t) - b_2^2 \sin 2(kx + \sigma t)\right]$$

$$+ \left[F^{(4)} e^{2\alpha z} + F^{(5)} e^{-2\alpha z} - 2F^{(6)}\right] e^{-2\gamma z} \cdot 2 b_1 b_2 \sin 2\sigma t$$

$$+ \left[J^{(1)} e^{\alpha' z} + J^{(2)} e^{-\alpha' z}\right] e^{-\gamma z} \cdot \left[b_1^2 \sin 2(kx - \sigma t) - b_2^2 \sin 2(kx + \sigma t)\right]$$

$$+ \left[J^{(3)} e^{\alpha'' z} + J^{(4)} e^{-\alpha'' z}\right] e^{-\gamma z} \cdot 2 b_1 b_2 \sin 2\sigma t$$

$$+ E^{(1)}(b_1^2 + b_2^2)\, \lambda^{-1} e^{-2\gamma h} \cdot t \qquad (6{\cdot}95)$$

We thus have an expression for ϕ_2 involving a number of quantities

$$F^{①}, F^{②}, F^{③}, F^{④}, F^{⑤}, F^{⑥}, J^{①}, J^{②}, J^{③}, J^{④}$$

and $E^{①}$.

ϕ_2 will be finite provided that $J^{①}$, $J^{②}$, $J^{③}$ and $J^{④}$ remain finite.

For this, it has already been stipulated that neither $\Delta(2\sigma, 2k)$ nor $\Delta(2\sigma, 0)$ must vanish. Hence before proceeding with ϕ_2 we must examine these functions further.

By definition

$$\Delta(2\sigma,2k) = -2e^{-\gamma h}\left[\gamma(\gamma^2-\alpha'^2-4k^2)\sinh\alpha' h + \alpha'(\gamma^2-\alpha'^2+4k^2)\cosh\alpha' h\right].$$

Since $\alpha'^2 = 4k^2 - \dfrac{4c^2}{\sigma^2} + \gamma^2$,

$$\begin{aligned}\Delta(2\sigma,2k) &= -2e^{-\gamma h}\Big[\gamma\left(\frac{4\sigma^2}{c^2}-8k^2\right)\sinh\left(4k^2-\frac{4c^2}{\sigma^2}+\gamma^2\right)^{\frac{1}{2}}h \\ &\quad + \left(4k^2-\frac{4c^2}{\sigma^2}+\gamma^2\right)^{\frac{1}{2}}\cdot\frac{4\sigma^2}{c^2}\cdot\cosh\left(4k^2-\frac{4\sigma^2}{c^2}+\gamma^2\right)^{\frac{1}{2}}h\Big].\end{aligned}$$

Hence $\Delta(\theta\sigma, \theta k) =$

$$-2e^{-\gamma h}\Big[\gamma\left(\frac{\theta^2\sigma^2}{c^2}-2\theta^2k^2\right)\sinh\left(\theta^2k^2-\frac{\theta^2\sigma^2}{c^2}+\gamma^2\right)^{\frac{1}{2}}h$$

$$+\left(\theta^2k^2-\frac{\theta^2\sigma^2}{c^2}+\gamma^2\right)^{\frac{1}{2}}\cdot\frac{\theta^2\sigma^2}{c^2}\cdot\cosh\left(\theta^2k^2-\frac{\theta^2\sigma^2}{c^2}+\gamma^2\right)^{\frac{1}{2}}h\Big]$$

Putting $\beta^2 = \theta^2k^2 - \dfrac{\theta^2\sigma^2}{c^2} + \gamma^2$

$$\Delta(\theta\sigma,\theta k) = -2e^{-\gamma h}\left[\gamma\frac{\theta^2\sigma^2}{c^2}\left(1-\frac{2k^2c^2}{\sigma^2}\right)\sinh\beta h + \frac{\beta\theta^2\sigma^2}{c^2}\cosh\beta h\right]$$

$\therefore \Delta(\theta\sigma, \theta k)$

$$= -2e^{-\gamma h} \cdot \frac{\theta^2\sigma^2}{c^2} \sinh \beta h \left[\gamma\left(1 - \frac{2k^2c^2}{\sigma^2}\right) + \beta \coth \beta h\right] \qquad (6\cdot96)$$

From (6•53) $\quad k^2 - \frac{\sigma^2}{c^2} > 0$, hence

$\beta^2 = \theta^2\left(k^2 - \frac{\sigma^2}{c^2}\right) + \gamma^2$ is a positive increasing function of θ^2 . But $\beta \coth \beta h$ is an increasing function of β^2 when $\beta^2 > 0$ and hence is an increasing function of θ^2 . Hence equation (6•96) shows that $\Delta(\theta\sigma,\theta k)$ can only vanish for one positive value of θ .
But $\Delta(\sigma, k)$ must vanish in order that ϕ_1 shall exist. Hence $\Delta(2\sigma, 2k)$ cannot vanish.
It is however still possible that $\Delta(2\sigma, 0)$ should vanish. This will be considered in the next chapter; for the time being it is assumed that $\Delta(2\sigma, 0)$ is not zero.

CHAPTER 7.

Application of the results of chapter 6 to ocean waves.

For ocean waves we may take

$$\left.\begin{aligned} &\sigma = 0.5 \text{ sec}^{-1} \quad , \; g = 0{\cdot}98 \times 10^3 \text{ cm/ sec}^2 \\ &h = 10^6 \text{ cm} \quad , \text{ velocity of sound in water} \\ &\qquad = c \doteq 1{\cdot}4 \times 10^5 \text{ cm/ sec} \end{aligned}\right\} \quad (7{\cdot}1)$$

By equation (6•51)

$$P\gamma h = \frac{\gamma g}{\sigma^2} = \frac{g}{2c^2}\cdot\frac{g}{\sigma^2} = \frac{9{\cdot}8^2 \times 10^4}{2 \times 1{\cdot}4^2 \times 10^{10} \times 0{\cdot}5^2} = 0{\cdot}98 \times 10^{-4}$$

$$\therefore P\gamma h \doteq 1{\cdot}0 \times 10^{-4} \quad (7{\cdot}2)$$

$$\gamma h < \frac{9{\cdot}8 \times 10^2 \times 10^6}{2 \times 1{\cdot}4^2 \times 10^{10}} = \frac{1}{0{\cdot}4} \times 10^{-2}$$

$$\therefore \gamma h < 2{\cdot}5 \times 10^{-2} \quad (7{\cdot}3)$$

By equation (6 51)

$$Q = \gamma h(1 - P\gamma h) < 2{\cdot}5 \times 10^{-2}(1 - 10^{-4})$$

$$\therefore Q < 2{\cdot}5 \times 10^{-2} \quad (7{\cdot}4)$$

Hence, since $\alpha h \operatorname{Coth} \alpha h \geqslant 1$ for real αh,
equation (6•50) shews that

$$\alpha h \operatorname{Coth} \alpha h - P(\alpha h)^2 \simeq 0$$

i.e. $P\alpha h$ is of the same order as $\operatorname{Coth} \alpha h$

Then $\dfrac{\gamma}{\alpha} = \dfrac{P\gamma h}{P\alpha h} \simeq \dfrac{P\gamma h}{\operatorname{Coth} \alpha h} \simeq \dfrac{1{\cdot}0 \times 10^{-4}}{1}$

$$\therefore \frac{\gamma}{\alpha} \simeq 10^{-4} \quad (7{\cdot}5)$$

From equation (6•50)

$$\coth \alpha h = P\alpha h + \frac{Q}{\alpha h}$$

$$< P\alpha h + \frac{2\cdot 5\times 10^{-2}}{\alpha h} \quad \text{by (7•4)}$$

$$\coth \alpha h < P\alpha h\left[1 + \frac{2\cdot 5\times 10^{-2}}{P\alpha^2 h^2}\right]$$

$$< P\alpha h\left[1 + \frac{2\cdot 5\times 10^{-2}}{P(\gamma h)^2}\cdot\left(\frac{\gamma}{\alpha}\right)^2\right]$$

$$< P\alpha h\left[1 + \frac{2\cdot 5\times 10^{-2}\times 10^{-8}}{2\cdot 5\times 10^{-2}\times 1\cdot 0\times 10^{-4}}\right]$$

by (7•2) and (7•3)

That is $\coth \alpha h \simeq P\alpha h\left[1 + 10^{-10}\right]$

or $\coth \alpha h = P\alpha h\left[1 + O\left(\frac{\gamma}{\alpha}\right)\right]$ (7•7)

Now $\frac{\sigma^2}{c^2} = \frac{g}{Ph}\cdot\frac{2\gamma}{g} = \frac{2\gamma}{Ph} = \frac{2\gamma\alpha}{P\alpha h}$

$$= 2\gamma\alpha\,.\,\tanh \alpha h\,.\left[1 + O\left(\frac{\gamma}{\alpha}\right)\right]$$

By (6•43) $k^2 = \alpha^2 + \frac{\sigma^2}{c^2} - \gamma^2$

$$\therefore\ k^2 = \alpha^2\left[1 + \frac{1}{\alpha^2}\cdot 2\gamma\alpha\,.\,\tanh \alpha h\left\{1 + O\left(\frac{\gamma}{\alpha}\right)\right\} - \left(\frac{\gamma}{\alpha}\right)^2\right]$$

$$\therefore\ k^2 = \alpha^2\left[1 + 2\left(\frac{\gamma}{\alpha}\right)\tanh \alpha h + O\left(\frac{\gamma}{\alpha}\right)^2\right] \quad (7•8)$$

We now determine the functions ϕ_1 and ϕ_2 after expanding the coefficients $F^{①}$, $F^{②}$, etc. , and neglecting powers of $\frac{\gamma}{\alpha}$ after making use of equations (7•7) and (7•8).

From equations (6·56) and (6·69)

$$F^{①} = \frac{-\sigma\{(\gamma-\alpha)^2-k^2\}\,(\gamma+\alpha)^2 e^{-2\alpha h}}{-4\sigma^2-4c^2\{(\gamma-\alpha)^2-k^2\}+2g(\gamma-\alpha)}$$

$$= \frac{-\sigma\left[\frac{\gamma^2}{\alpha^2}-\frac{2\gamma}{\alpha}+1-1-\frac{2\gamma}{\alpha}\tanh\alpha h-O\left(\frac{\gamma}{\alpha}\right)^2\right]\left[\frac{\gamma^2}{\alpha^2}+\frac{2\gamma}{\alpha}+1\right]e^{-2\alpha h}}{-\frac{4\sigma^2}{\alpha^4}-\frac{4c^2}{\alpha^2}\left[\frac{\gamma^2}{\alpha^2}-\frac{2\gamma}{\alpha}+1-1-\frac{2\gamma}{\alpha}\tanh\alpha h-O\left(\frac{\gamma}{\alpha}\right)^2\right]+\frac{2g}{\alpha^3}\left(\frac{\gamma}{\alpha}-1\right)}$$

$$\simeq \frac{\frac{2\gamma}{\alpha}.\sigma(\tanh\alpha h+1)\,e^{-2\alpha h}}{-\frac{4\sigma^2}{\alpha^4}+\frac{4\sigma^2}{\alpha^2}.\frac{1}{2\gamma\alpha}.\operatorname{Coth}\alpha h\left[1-O\left(\frac{\gamma}{\alpha}\right)\right]\frac{2\gamma}{\alpha}(1+\tanh\alpha h)-\frac{2g}{\alpha^3}}$$

$$= \frac{\frac{2\gamma}{\alpha}.\sigma(1+\tanh\alpha h)\,e^{-2\alpha h}}{-\frac{4\sigma^2}{\alpha^4}\left[1-\operatorname{Coth}\alpha h-1+O\left(\frac{\gamma}{\alpha}\right)\operatorname{Coth}\alpha h+O\left(\frac{\gamma}{\alpha}\right)\right]-\frac{2g}{\alpha^3}}$$

$$\simeq \frac{2\gamma\alpha^3}{4\sigma}.\frac{1+\tanh\alpha h}{\operatorname{Coth}\alpha h}.e^{-2\alpha h}$$

$$= \frac{\gamma\alpha^3}{2\sigma}.\tanh\alpha h.\frac{2e^{-\alpha h}}{e^{\alpha h}+e^{-\alpha h}}$$

$$= \frac{\gamma\alpha^3}{2\sigma}.\tanh\alpha h.\frac{e^{-\alpha h}}{\cosh\alpha h}$$

$$\therefore F^{①} \simeq \frac{\gamma\alpha^3}{2\sigma}.\frac{e^{-\alpha h}\sinh\alpha h}{\cosh^2\alpha h} \qquad (7·9)$$

$$F^{②} = \frac{-\sigma\{(\gamma+\alpha)^2-k^2\}(\gamma-\alpha)^2 e^{2\alpha h}}{-4\sigma^2-4c^2\{(\gamma+\alpha)^2-k^2\}+2g(\gamma+\alpha)}$$

$$= \frac{-\sigma\left\{\frac{\gamma^2}{\alpha^2}+\frac{2\gamma}{\alpha}+1-1-2\frac{\gamma}{\alpha}\tanh\alpha h-O\left(\frac{\gamma}{\alpha}\right)^2\right\}\left\{\left(\frac{\gamma}{\alpha}\right)^2-2\frac{\gamma}{\alpha}+1\right\}e^{2\alpha h}}{-\frac{4\sigma^2}{\alpha^2}-\frac{4c^2}{\alpha^2}\left\{\frac{\gamma^2}{\alpha^2}+\frac{2\gamma}{\alpha}+1-1-\frac{2\gamma}{\alpha}\tanh\alpha h-O\left(\frac{\gamma}{\alpha}\right)^2\right\}+\frac{2g}{\alpha^3}\left(\frac{\gamma}{\alpha}+1\right)}$$

$$\simeq \frac{-2\sigma\frac{\gamma}{\alpha}(1-\tanh\alpha h)e^{2\alpha h}}{-\frac{4\sigma^2}{\alpha^4}-\frac{4}{\alpha^2}\cdot\frac{\sigma^2}{2\gamma\alpha}\coth\alpha h\left[1-O\left(\frac{\gamma}{\alpha}\right)\right]\cdot\frac{2\gamma}{\alpha}\cdot(1-\tanh\alpha h)+\frac{2g}{\alpha^3}\left(\frac{\gamma}{\alpha}+1\right)}$$

$$\simeq \frac{-\sigma\, 2\gamma\alpha^3(1-\tanh\alpha h)e^{2\alpha h}}{-4\sigma^2(1+\coth\alpha h-1)+2g\alpha}$$

$$\simeq \frac{\gamma\alpha^3(1-\tanh\alpha h)e^{2\alpha h}}{2\sigma\coth\alpha h}$$

$$= \frac{\gamma\alpha^3 \cdot \sinh\alpha h\cdot 2e^{-\alpha h}\cdot e^{2\alpha h}}{2\sigma\cdot\cosh\alpha h\cdot(e^{\alpha h}+e^{-\alpha h})}$$

$$= \frac{\gamma\alpha^3}{\sigma}\cdot\frac{\sinh\alpha h}{\cosh\alpha h}\cdot\frac{e^{\alpha h}}{e^{\alpha h}+e^{-\alpha h}}$$

$$\therefore F^{②} \simeq \frac{\gamma\alpha^3 e^{\alpha h}}{2\sigma}\cdot\frac{\sinh\alpha h}{\cosh\alpha h} \qquad (7\cdot10)$$

$$F^{③} = \frac{-\sigma(\gamma^2 - a^2 - k^2)(\gamma^2 - \alpha^2)}{-4\sigma^2 - 4c^2(\gamma^2 - k^2) + 2g\gamma}$$

$$\simeq \frac{-\sigma\left(\frac{\gamma^2}{\alpha^2} - 1 - 1 - \frac{2\gamma}{\alpha}\tanh\alpha h\right)\left(\frac{\gamma^2}{\alpha^2} - 1\right)}{-\frac{4\sigma^2}{\alpha^4} - \frac{4c^2}{\alpha^2}\left(\frac{\gamma^2}{\alpha^2} - 1 - \frac{2\gamma}{\alpha}\tanh\alpha h\right) + \frac{2g\gamma}{\alpha^4}}$$

$$\simeq \frac{-2\sigma\left(1 + \frac{\gamma}{\alpha}\tanh\alpha h\right)}{-\frac{4\sigma^2}{\alpha^4} + \frac{4}{\alpha^2}\cdot\frac{\sigma^2}{2\gamma\alpha}\cdot\coth\alpha h\left[1 - O\left(\frac{\gamma}{\alpha}\right)\right]\left(1 + \frac{2\gamma}{\alpha}\tanh\alpha h\right) + \frac{2g\gamma}{\alpha^4}}$$

$$\simeq \frac{-\left(1 + \frac{\gamma}{\alpha}\right)\tanh\alpha h}{-\frac{2\sigma}{\alpha^4} + \frac{\sigma}{\gamma\alpha^3}\left(\coth\alpha h + \frac{2\gamma}{\alpha}\right) + \frac{2g\gamma}{\sigma\alpha^4}}$$

$$\simeq -\frac{\alpha^3\gamma}{\sigma\coth\alpha k}$$

$$F^{③} \simeq -\frac{\alpha^3\gamma}{\sigma}\cdot\tanh\alpha h \qquad (7\cdot11)$$

$$F^{④} = \frac{-\sigma\{(\gamma-\alpha)^2 + k^2\}(\gamma+\alpha)^2 e^{-2\alpha h}}{-4\sigma^2 - 4c^2(\gamma-\alpha)^2 + 2g(\gamma-\alpha)}$$

$$\simeq \frac{-\sigma\left\{\frac{\gamma^2}{\alpha^2} - \frac{2\gamma}{\alpha} + 1 + 1 + \frac{2\gamma}{\alpha}\tanh\alpha h\right\}\left(\frac{\gamma^2}{\alpha^2} + \frac{2\gamma}{\alpha} + 1\right)e^{-2\alpha h}}{-\frac{4\sigma^2}{\alpha^4} - \frac{4c^2}{\alpha^2}\left(\frac{\gamma^2}{\alpha^2} - \frac{2\gamma}{\alpha} + 1\right) + \frac{2g}{\alpha^3}\left(\frac{\gamma}{\alpha} - 1\right)}$$

$$\simeq \frac{-\sigma\left(2 - \frac{2\gamma}{\alpha} + \frac{2\gamma}{\alpha}\tanh\alpha h\right)e^{-2\alpha h}}{-\frac{4\sigma^2}{\alpha^4} - \frac{4}{\alpha^2}\cdot\frac{\sigma^2}{2\gamma\alpha}\cdot\coth\alpha h\left[1 - O\left(\frac{\gamma}{\alpha}\right)\right]\left(1 - \frac{2\gamma}{\alpha}\right) + \frac{2g}{\alpha^3}\left(\frac{\gamma}{\alpha} - 1\right)}$$

$$\simeq \frac{\alpha^3\gamma e^{-2\alpha h}}{\sigma\coth\alpha h}$$

$$\therefore F^{④} \simeq \frac{\alpha^3\gamma e^{-2\alpha h}}{\sigma}\cdot\tanh\alpha h. \qquad (7\cdot12)$$

$$F^{⑤} = \frac{-\sigma\{(\gamma+\alpha)^2+k^2\}(\gamma-\alpha)^2 e^{2\alpha h}}{-4\sigma^2-4c^2(\gamma+\alpha)^2+2g(\gamma-\alpha)}$$

$$\doteq \frac{-\sigma\{\frac{\gamma^2}{\alpha^2}+\frac{2\gamma}{\alpha}+1+1+\frac{2\gamma}{\alpha}\tanh\alpha h\}(\frac{\gamma^2}{\alpha^2}-\frac{2\gamma}{\alpha}+1)e^{2\alpha h}}{-\frac{4\sigma^2}{\alpha^4}-\frac{4c^2}{\alpha^2}\left(\frac{\gamma^2}{\alpha^2}+\frac{2\gamma}{\alpha}+1\right)+\frac{2g}{\alpha^3}\left(\frac{\gamma}{\alpha}-1\right)}$$

$$\doteq \frac{-2\sigma\left(\frac{\gamma}{\alpha}+1+\frac{\gamma}{\alpha}\tanh\alpha h\right)e^{2\alpha h}}{-\frac{4\sigma^2}{\alpha^4}-\frac{4}{\alpha^2}\cdot\frac{\sigma^2}{2\gamma a}\coth\alpha h\left[1-O(\frac{\gamma}{\alpha})\right](1+\frac{2\gamma}{\alpha})+\frac{2g}{\alpha^3}\left(\frac{\gamma}{\alpha}-1\right)}$$

$$\doteq \frac{\alpha^3\gamma e^{2\alpha h}}{\sigma.\ \coth\alpha h}$$

$$\therefore\ F^{⑤} \doteq \frac{\alpha^3\gamma e^{2\alpha h}}{\sigma}.\ \tanh\alpha h. \qquad (7\cdot13)$$

$$F^{⑥} = \frac{-\sigma(\gamma^2-\alpha^2+k^2)\ (\gamma^2-\alpha^2)}{-4\sigma^2-4c^2\gamma+2g\gamma}$$

$$\doteq \frac{-\sigma\left(\frac{\gamma^2}{\alpha^2}-1+1+\frac{2\gamma}{\alpha}\tanh\alpha h\right)\left(\frac{\gamma^2}{\alpha^2}-1\right)}{-4\frac{\sigma^2}{\alpha^4}-\frac{4\gamma^2}{\alpha^4\sigma^2}\cdot\frac{\coth\alpha h}{2\alpha\gamma}\left[1-O(\frac{\gamma}{\alpha})\right]+\frac{2g\gamma}{\alpha^4}}$$

$$\doteq \frac{\frac{2\gamma}{\alpha}.\ \sigma.\ \tanh\frac{\alpha}{h}}{-\frac{4\sigma^2}{\alpha^4}}$$

$$\therefore\ F^{⑥} \doteq -\frac{\alpha^3\gamma}{2\sigma}.\ \tanh\alpha h. \qquad (7\cdot14)$$

From equation (6·60) $\quad \mathcal{D} \doteq \frac{4\sigma}{9}\gamma\alpha^4$ (7·15)

From equations (6·72), since $F^{(1)}, F^{(2)}, F^{(3)}, F^{(4)}, F^{(5)}$ and $F^{(6)}$ are all of order $\frac{\gamma\alpha^3}{\sigma}$, we have

$$G^{(1)} \doteq -2\alpha e^{2\alpha h} F^{(1)} + 2\alpha e^{-2\alpha h} F^{(2)}$$

$$\doteq -\frac{\gamma\alpha^4}{\sigma} \cdot \frac{e^{\alpha h}\sinh\alpha h}{\cosh^2\alpha h} + \frac{\gamma\alpha^4}{\sigma} \cdot \frac{e^{-\alpha h}\sinh\alpha h}{\cosh^2\alpha h}$$

(by equations 7·9 and 7·10) ;

that is $\quad G^{(1)} \doteq -\frac{2\gamma\alpha^4}{\sigma} \cdot \frac{\sinh^2\alpha h}{\cosh^2\alpha h}$ (7·16)

Similarly $\quad G^{(2)} \doteq -2\alpha e^{2\alpha h} F^{(4)} + 2\alpha e^{-2\alpha h} F^{(5)}$

$$\doteq -\frac{2\gamma\alpha^4}{\sigma} \cdot \tanh\alpha h + \frac{2\gamma\alpha^4}{\sigma} \cdot \tanh\alpha h$$

(by equations 7·12 and 7·13)

That is $\quad G^{(2)} = O\left(\frac{\gamma^2\alpha^3}{\sigma}\right)$ at most (7·17)

From equations (6 74)

$$H^{(1)} \doteq -4(\alpha^2 - k^2) F^{(1)} - 4(\alpha^2 - k^2) F^{(2)} - 8k^2 F^{(3)}$$

$$\doteq -8k^2 F^{(3)} + O(\alpha\gamma F^{(1)}) \quad \text{(by equation 7·8)}$$

$$\doteq \frac{8\alpha^5\gamma}{\sigma} \cdot \tanh\alpha h \quad \text{(by equation 7·11)} \qquad (7·18)$$

and $\quad H^{(2)} \doteq -4\alpha^2 F^{(4)} - 4\alpha^2 F^{(5)}$

$$\doteq \frac{4\gamma\alpha^5}{\sigma} \cdot e^{-2\alpha h} \tanh\alpha h - \frac{4\gamma\alpha^5}{\sigma} \tanh\alpha h \cdot e^{2\alpha h}$$

(by equations 7·12 and 7·13)

that is $H^{(2)} \simeq -\dfrac{8\gamma\alpha^5}{\sigma} . \cosh 2\alpha h . \tanh \alpha h$ (7·19)

From equation (6·50)

$$\alpha h \coth \alpha h - \frac{g(\alpha h)^2}{h\sigma^2} \simeq 0$$

or $$\frac{\sigma}{g} \simeq \frac{\alpha \tanh \alpha h}{\sigma}$$

and so equation (7·15) may be written

$$D \simeq \frac{4\gamma\alpha^5}{\sigma} . \tanh \alpha h .$$

Therefore

$$D + H^{(1)} \simeq \frac{12\gamma\alpha^5}{\sigma} \tanh \alpha h \qquad (7·20)$$

and

$$D + H^{(2)} \simeq \frac{4\gamma\alpha^5}{\sigma} \tanh \alpha h \left\{1 - 2\cosh 2\alpha h\right\}$$

$$= \frac{4\gamma\alpha^5}{\sigma} \frac{\tanh \alpha h}{\cosh \alpha h} . \left(\cosh \alpha h - 2\cosh \alpha h . \cosh 2\alpha h\right)$$

$$= -\frac{4\gamma\alpha^5}{\sigma} . \frac{\tanh \alpha h}{\cosh \alpha h} . \cosh 3\alpha h \qquad (7·21)$$

From equations (6·78)

$$\alpha'^2 = 4\alpha^2\left[1 + \frac{2\gamma}{\alpha}\tanh \alpha h + O\left(\frac{\gamma}{\alpha}\right)^2\right]$$

$$- 8\gamma\alpha \tanh \alpha h \left[1 + O\left(\frac{\gamma}{\alpha}\right)\right] + \gamma^2$$

$$\simeq 4\alpha^2 \qquad (7·22)$$

and $\alpha''^2 \simeq 0.$ (7·23)

Also from equation (6·67)

$$E^{(1)} \simeq \lambda\alpha^4 . \qquad (7·24)$$

By definitiion

$$\Delta(2\sigma, 2k) = -2e^{-\gamma h}\Big[\gamma(\gamma^2 - \alpha'^2 - 4k^2)\sinh\alpha' h + \alpha'(\gamma^2 - \alpha'^2 + 4k^2)\cosh\alpha' h\Big]$$

Hence, by equations (7·8) and (7·22),

$$\Delta(2\sigma, 2k) \doteq -2e^{-\gamma h}\Big[-8\gamma^2 \sinh 2\alpha h + 16\alpha^2\gamma \tanh\alpha h \cosh 2\alpha h\Big]$$

$$= -16\gamma\alpha^2 e^{-\gamma h}\Big[-2\sinh\alpha h\cosh\alpha h + 2\tanh\alpha h(\cosh^2\alpha h + \sinh^2\alpha h)\Big]$$

$$= -32\gamma\alpha^2 e^{-\gamma h}.\tanh\alpha h.\sinh^2\alpha h \qquad (7{\cdot}25)$$

Also equation (6·90) gives

$$\Delta(2\sigma, 0) = -2e^{-\gamma h}\Big[\gamma(\gamma^2 - \alpha''^2)\sinh\alpha'' h + \alpha''(\gamma^2 - \alpha''^2)\cosh\alpha'' h\Big]$$

$$\therefore \Delta(2\sigma, 0) = -2e^{-\gamma h}(\gamma^2 - \alpha''^2)(\gamma\sinh\alpha'' h + \alpha''\cosh\alpha'' h)$$

But $\qquad \alpha''^2 \doteq -\dfrac{4\sigma^2}{c^2} = -8\gamma\alpha\tanh\alpha h.\Big[1 + O\Big(\dfrac{\gamma}{\alpha}\Big)\Big],$

hence

$$\Delta(2\sigma, 0) \doteq -16e^{-\gamma h}\gamma\alpha\tanh\alpha h(\gamma\sinh\alpha'' h + \alpha''\cosh\alpha'' h)$$

$$\doteq -16e^{-\gamma h}\gamma\alpha\alpha''\tanh\alpha h.\cosh\alpha'' h \qquad (7{\cdot}26)$$

Hence from equation (6·86)

$$J^{(1)} \doteq \frac{\alpha'!\, e^{-(\gamma+\alpha')h}.(D + H^{(1)})}{\Delta(2\sigma, 2k)}$$

That is

$$J^{(1)} \doteq -\frac{\alpha'!\, e^{-(\gamma+\alpha')h}.12\gamma\alpha^5\tanh\alpha h}{\sigma.32\gamma\alpha^2 e^{-\gamma h}.\tanh\alpha h.\sinh^2\alpha h}$$

by (7·20)

$$\therefore \quad J^{(1)} \simeq -\frac{3}{4} \cdot \frac{\alpha^4}{\sigma} \cdot \frac{e^{-\alpha' h}}{\sinh^2 \alpha h} \tag{7·27}$$

From equation (6·87)

$$J^{(2)} \simeq -\frac{-\alpha' . e^{-(\gamma-\alpha')h} . (D+H^{(1)})}{\Delta(2\sigma, 2k)}$$

$$= -\frac{\alpha' . e^{-(\gamma-\alpha')h} . 12\gamma\alpha^5 \tanh \alpha h}{\sigma . 32\gamma\alpha^2 e^{-\delta h} . \tanh \alpha h . \sinh^2 \alpha h}$$

by 7·20 and 7·25

$$\therefore J^{(2)} \simeq -\frac{3\alpha^4}{4\sigma} . \frac{e^{\alpha' h}}{\sinh^2 \alpha h} \tag{7·28}$$

From equation (6·88)

$$J^{(3)} \simeq \frac{\alpha'' e^{-(\gamma+\alpha'')h} . (D+H^{(2)})}{\Delta(2\sigma, 0)}$$

$$= \frac{\alpha'' . e^{-(\gamma+\alpha'')h} . 4\gamma\alpha^5 . \tanh \alpha h . \cosh 3\alpha h}{\sigma . \cosh \alpha h . 16 e^{-\delta h} . \gamma\alpha\alpha' \tanh \alpha h . \cosh \alpha'' h}$$

by equations 7·21 and 7·26

$$\therefore J^{(3)} \simeq \frac{\alpha^4}{4\sigma} . \frac{e^{-\alpha'' h}}{\cosh \alpha'' h} . \frac{\cosh 3\alpha h}{\cosh \alpha h} \tag{7·29}$$

From equation (6·89)

$$J^{(4)} \simeq \frac{\alpha'' e^{-(\gamma-\alpha'')h} . (D+H^{(2)})}{\Delta(2\sigma, 0)}$$

That is

$$J^{④} \simeq \frac{\alpha'' e^{-(\gamma-\alpha'')h} . 4\gamma\alpha^5 . \tanh\alpha h . \cosh 3\alpha h}{\sigma \cosh\alpha h . 16 e^{-\gamma h} . \gamma\alpha\alpha'' \tanh\alpha h . \cosh\alpha'' h}$$

$$\therefore J^{④} \simeq \frac{\alpha^4}{4\sigma} . \frac{e^{\alpha'' h}}{\cosh\alpha'' h} . \frac{\cosh 3\alpha h}{\cosh\alpha h} \qquad (7\cdot30)$$

By equation (7·8)

$$k^2 = \alpha^2\left[1 + 2\left(\frac{\gamma}{\alpha}\right)\tanh\alpha h + O\left(\frac{\gamma}{\alpha}\right)^2\right],$$

$$= \alpha^2\left[1 + 2\gamma h . \left(\frac{\tanh\alpha h}{\alpha h}\right) + O\left(\frac{\gamma}{\alpha}\right)^2\right],$$

but $\dfrac{\alpha h}{\tanh\alpha h} \geqslant 1$, hence neglecting, not only terms of order $\dfrac{\gamma}{h}$ but also of order γh ,

$$k^2 \simeq \alpha^2 \qquad \therefore \quad k \simeq \alpha .$$

Hence, from equations (6·78)

$$\alpha' \simeq 2\alpha \quad \text{and} \quad \alpha'' \simeq 2i\frac{\sigma}{c} .$$

Hence we may replace αh, $\alpha' h$, $\alpha'' h$ and $e^{\gamma h}$ by

kh, $2kh$, $2i\sigma\frac{h}{c}$ and $e^0 = 1$, respectively.

But $$\alpha^2 = k^2 - \frac{\sigma^2}{c^2} + \gamma^2$$

$$\therefore \quad \frac{\sigma^2}{c^2} \simeq k^2 - \alpha^2$$

$$\simeq \alpha^2 + 2\alpha\gamma\tanh\alpha h - \alpha^2 \text{ by } (7\ 8)$$

$$\therefore \quad \frac{\sigma^2}{c^2} \simeq 2\alpha\gamma\tanh\alpha h$$

$$\text{or} \quad \sigma^2 \simeq 2c^2\gamma k\tanh\alpha h$$

By equations (7.1) and (7.3)

$$2c^2\gamma \simeq \frac{2 \times 1.4^2 \times 10^{10} \times 2.5 \times 10^{-2}}{10^6}$$

$$\simeq 980$$

$$\simeq g$$

Hence $$\sigma^2 \simeq gk \tanh kh \qquad (7.31)$$

We now find thw values of ϕ_1 and ϕ_2 to the order of greatness used so far in this chapter.
From equation (6.55) ,

$$\phi_1 = \left[(\gamma+\alpha)e^{-\alpha h-(\gamma-\alpha)z} - (\gamma-\alpha)e^{\alpha h-(\gamma+\alpha)z}\right]$$

$$\times\left[b_1 \sin(kx-\sigma t) + b_2 \sin(kx+\sigma t)\right]$$

$$= e^{-\gamma z}\left[(\gamma+\alpha)e^{-\alpha h+\alpha z} - (\gamma-\alpha)e^{\alpha h-\alpha z}\right]\frac{\sigma}{2k^2\sinh kh}$$

$$\times\left[\frac{b_1 2k^2\sinh kh}{\sigma}\sin(kx-\sigma t) + \frac{b_2 2k^2\sinh kh}{\sigma}\sin(kx+\sigma t)\right]$$

That is

$$\phi_1 = \frac{\sigma e^{-\gamma z}}{2k^2\sinh kh}\left[(\gamma+\alpha)e^{\alpha(z-h)} - (\gamma-\alpha)e^{-\alpha(z-h)}\right]$$

$$\times\left[a_1 \sin(kx-\sigma t) - a_2 \sin(kx+\sigma t)\right],$$

where
$$\left.\begin{aligned} a_1 &= \frac{2k^2\sinh kh}{\sigma}\cdot b_1 \\ a_2 &= -\frac{2k^2\sinh kh}{\sigma}\cdot b_2 \end{aligned}\right\} \qquad (7.32)$$

$$\therefore \phi_1 = \frac{\sigma e^{-\gamma z}}{2k^2 \sinh kh}\left[\gamma\left\{e^{\alpha(z-h)} - e^{-\alpha(z-h)}\right\} + \alpha\left\{e^{\alpha(z-h)} + e^{-\alpha(z-h)}\right\}\right]$$

$$\times\left[a_1 \sin(kx-\sigma t) - a_2 \sin(kx+\sigma t)\right]$$

$$= \frac{\sigma e^{-\gamma z}}{k^2 \sinh kh}\left[\gamma \sinh\alpha(z-h) + \alpha \cosh\alpha(z-h)\right]$$

$$\times\left[a_1 \sin(kx-\sigma t) - a_2 \sin(kx+\sigma t)\right].$$

Neglecting terms of order of γh, and setting $\alpha = k$, $e^{-\gamma z} = 1$, we have

$$\phi_1 \simeq \frac{\sigma}{k^2 \sinh kh} \cdot k \cosh k(z-h) \cdot \left[a_1 \sin(kx-\sigma t) - a_2 \sin(kx+\sigma t)\right]$$

That is

$$\phi_1 = \frac{\sigma}{k} \cdot \frac{\cosh k(z-h)}{\sinh kh} \cdot \left[a_1 \sin(kx-\sigma t) - a_2 \sin(kx+\sigma t)\right] \qquad (7\cdot 33)$$

Using equations (7·9), (7·10), (7·11), (7·27) and (7·28) the coefficient of $\left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right]$ in ϕ_2, given by equation (6·95), is

$$e^{-2\gamma z} \cdot \frac{\gamma \alpha^3}{\sigma} \cdot \tanh \alpha h \left\{\frac{e^{-\alpha h + 2\alpha z}}{2\cosh \alpha h} + \frac{e^{\alpha h - 2\alpha z}}{2\cosh \alpha h} - 2\right\}$$

$$- \frac{3\alpha^4}{4\sigma \sinh^2 \alpha h} \cdot \left\{e^{\alpha' h + \alpha z} + e^{\alpha' h - \alpha z}\right\} e^{-\gamma z};$$

on neglecting terms of order γh this becomes

$$\frac{-3k^4}{4\sigma \sinh^2 kh}\left\{e^{k(2h-z)} + e^{-k(2h-z)}\right\}$$

$$= \frac{-3k^4 \cosh k(2h-z)}{2\sigma \sinh^2 kh}.$$

Hence the term involving $\left[b_1^2 \sin 2(kx-\sigma t) - b_2^2 \sin 2(kx+\sigma t)\right]$ becomes

$$-\frac{3k^4 \cdot \cosh k(2h-z)}{2\sigma \sinh^2 kh} \cdot \frac{\sigma^2}{4k^4 \sinh^2 kh}\left[a_1^2 \sin 2(kx-\sigma t) - a_2^2 \sin 2(kx+\sigma t)\right]$$

$$= -\frac{3\sigma}{8} \cdot \frac{\cosh k(2h-z)}{\sinh^4 kh}\left[a_1^2 \sin 2(kx-\sigma t) - a_2^2 \sin 2(kx+\sigma t)\right]$$

Where a_1 and a_2 are defined by equations (7·32).
Using equations (7·12), (7·13), (7·14), (7·29) and (7·30)
the coefficient of $2b_1 b_2 \sin 2\sigma t$ is

$$e^{-2\gamma z} \cdot \frac{\gamma\alpha^3}{\sigma} \cdot \tanh \alpha h \left\{ e^{2\alpha z - 2\alpha h} + e^{-2\alpha z + 2\alpha h} + \right\}$$

$$+ e^{-\gamma z} \cdot \frac{\alpha^4}{4\sigma} \cdot \frac{\cosh 3\alpha h}{\cosh \alpha'' h \cdot \cosh \alpha h}\left\{ e^{-\alpha'' h + \alpha'' z} + e^{\alpha'' h - \alpha'' z}\right\} \quad ;$$

On neglecting terms of order γh and putting $\alpha = k$, this becomes

$$\frac{k^4}{4\sigma} \cdot \frac{\cosh 3kh}{\cosh 2i\frac{\sigma}{c}h \cdot \cosh kh} \cdot \left\{ e^{i \cdot \frac{2\sigma}{c}(z-h)} + e^{-i \cdot \frac{2\sigma}{c}(z-h)}\right\}$$

$$= \frac{k^4}{4\sigma} \cdot \frac{\cosh 3kh}{\cos \frac{2\sigma h}{c} \cdot \cosh kh} \cdot 2 \cos \frac{2\sigma(z-h)}{c} \quad .$$

Hence, after substituting for b_1 and b_2 the term involving $b_1 b_2 \sin 2\sigma t$ becomes

$$-\frac{k^4}{2\sigma} \cdot \frac{\cosh 3kh \cdot \cos \frac{2\sigma(z-h)}{c}}{\cos \frac{2\sigma h}{c} \cdot \cosh kh} \cdot \frac{2\sigma^2}{4k^4 \sinh^2 kh} \cdot a_1 a_2 \sin 2\sigma t$$

$$= -\frac{\sigma}{8} \cdot \frac{\cosh 3kh}{\sinh^2 kh \cdot \cosh kh} \cdot \frac{\cos \frac{2\sigma(z-h)}{c}}{\cos \frac{2\sigma h}{c}} \cdot 2a_1 a_2 \sin 2\sigma t.$$

The remaining term in equation (6·95) becomes

$$\lambda k^4 . \lambda^{-1} . \frac{\sigma^2}{4k^4 \sinh^2 kh} . (a_1^2 + a_2^2) t$$

$$= \frac{\sigma}{4} . \frac{a_1^2 + a_2^2}{\sinh^2 kh} . \sigma t .$$

Thus the value of ϕ_2 is

$$\phi_2 = -\frac{3\sigma}{8} . \frac{\cosh k(z-h)}{\sinh^4 kh} . \left[a_1^2 \sin 2(kx - \sigma t) - a_2^2 \sin 2(kx + \sigma t)\right]$$

$$- \frac{\sigma}{8} . \frac{\cosh 3kh}{\sinh^2 kh . \cosh kh} . \frac{\cos 2\sigma(z-h)/c}{\cos 2\sigma h/c} . 2a_1 a_2 \sin 2\sigma t$$

$$+ \frac{\sigma}{4} . \frac{a_1^2 + a_2^2}{\sinh^2 kh} . \sigma t \qquad (7{\cdot}34)$$

We now use equations (7·33) and (7·34) to investigate the changes with depth which occur in the second order pressure term, p_2, given by equation (6·37).
Let λ_g and λ_c be the lengths of a gravity wave and a compression wave;

then $\lambda_g = \frac{2\pi}{k}$ and $\lambda_c = \frac{2\pi c}{\sigma}$ (7·35)

$$\left(\frac{\lambda_g}{\lambda_c}\right)^2 = \frac{1}{k^2} . \frac{\sigma^2}{c^2}$$

neglecting terms of order $\frac{\gamma}{\alpha}$ and γh ,

$$\left(\frac{\lambda_g}{\lambda_c}\right)^2 = \frac{1}{\alpha^2} . 2\gamma\alpha \tanh \alpha h$$

$$\therefore \left(\frac{\lambda_g}{\lambda_c}\right) = 2^{\frac{1}{2}} . \left(\frac{\gamma}{\alpha}\right)^{\frac{1}{2}} \tanh^{\frac{1}{2}} \alpha h \qquad (7{\cdot}36)$$

$$\simeq 2^{\frac{1}{2}} . 10^{-2} . \tanh^{\frac{1}{2}} \alpha h$$ by equation (7·5)

$$\therefore \lambda_g > \frac{\sqrt{2}}{100} . \lambda_c \qquad (7{\cdot}37)$$

Case (1) : When the depth is less than half the length of a gravity wave.

$$z < \tfrac{1}{2}\lambda_g \quad \text{i.e.} \quad z < \frac{\pi}{k}$$

Also by (7·37) and (7·35)

$$z < \frac{1}{100\sqrt{2}} \cdot \lambda_c = \frac{2\pi c}{100\sqrt{2}\,.\,\sigma}$$

$$\therefore \frac{2\sigma z}{c} < \frac{2\pi\sqrt{2}}{100}$$

$$\frac{\cos 2\sigma(z-h)/c}{\cos 2\sigma h/c} = \cos\frac{2\sigma z}{c} + \sin\frac{2\sigma z}{c} \cdot \tan\frac{2\sigma h}{c}$$

$$< \cos\frac{2\pi\sqrt{2}}{100} + \sin\frac{2\pi\sqrt{2}}{100} \cdot \tan\left(\frac{10}{1\cdot 4}\right)$$

by 7·1

$$\simeq 1.$$

Hence from (7·34)

$$\phi_2 \simeq -\frac{3\sigma}{8} \cdot \frac{\cosh 2k(z-h)}{\sinh^4 kh} \cdot \left[a_1^2 \sin 2(kx-\sigma t) - a_2^2 \sin 2(kx+\sigma t)\right]$$

$$-\frac{\sigma}{8} \cdot \frac{\cosh 3kh}{\sinh^2 kh \cdot \cosh kh} \cdot 2a_1 a_2 \sin 2\sigma t$$

$$+\frac{\sigma}{4} \cdot \frac{a_1^2 + a_2^2}{\sinh^2 kh} \cdot \sigma t \qquad (7\cdot 38)$$

which is independent of c.
That is, the motion is unaffected by the compressibility of the water, since ϕ_1 is also independent of c.

Case (ii). When the depth is of the order of one gravity wave length.

$$Z \simeq \frac{\sqrt{2}}{100}\lambda_c = \frac{\sqrt{2}}{100} \cdot \frac{2\pi\sigma}{c}$$

$$\therefore \frac{2\sigma Z}{c} \simeq 4\pi \cdot \frac{\sqrt{2}}{100} \quad \text{and} \quad \frac{2\sigma h}{c} < \frac{10}{1.4}$$

Hence, again, $$\frac{\cos 2\sigma(z-h)/c}{\cos 2\sigma h/c} \simeq 1.$$

Also $$e^{-kz} \simeq e^{-k\lambda_g} = e^{-k.\frac{2\pi}{k}} = e^{-2\pi} < 0.002.$$

Hence $$e^{-kh} < e^{-kz} < 0.002$$

and $$e^{kh} > e^{kz} > 400.$$

Hence $$\frac{\cosh k(z-h)}{\sinh kh} = \frac{e^{kz}.e^{-kh} + e^{-kz}.e^{kh}}{e^{kh} - e^{-kh}}$$

$$\simeq \frac{2 \times 400 \times 0.002}{400 - 0.002}$$

$$\simeq 0.004$$

hence, from equation (7·33)

$$\frac{\partial \phi_1}{\partial t} \simeq 0 \quad ; \quad \nabla^2 \phi_1 \simeq 0.$$

also $$\frac{\cosh 3kh}{\sinh^2 kh . \cosh kh} = \frac{4(e^{3kh} + e^{-3kh})}{(e^{kh} - e^{-kh})^2 . (e^{kh} + e^{-kh})}$$

$$\simeq \frac{400^3 \times 4}{400^2 \times 400}$$

$$\simeq 4.$$

and

$$\frac{\cos 2k(z-h)}{\sinh^4 kh} = \frac{(e^{2kz}.e^{-2kh}+e^{-2kz}.e^{2kh})8}{(e^{kh}-e^{-kh})^4}$$

$$\simeq \frac{(400 \times 0.002)^2 \times 4 \times 8}{400^4}$$

$$\simeq 0.$$

also $$\frac{1}{\sinh^2 kh} = \frac{4}{(e^{kh}-e^{-kh})^2} \simeq \frac{4}{400^2} \simeq 0.$$

Hence $$\phi_2 \simeq -\frac{\sigma}{8}.4.2a_1a_2 \sin 2\sigma t$$

$$\therefore \frac{\partial \phi_2}{\partial t} \simeq -2a_1a_2\sigma^2 \cos 2\sigma t.$$

Then equation (6.37) gives

$$\frac{p_2}{\rho_s} \simeq -2a_1a_2\sigma^2 \cos 2\sigma t. \qquad (7.39)$$

$[\text{Since } e^{2\gamma z} < e^{2\gamma h} < e^{5\times 10^{-2}} = e^{0.02} \simeq 1]$

That is at the depth equal to the length of a gravity wave there is a second order pressure variation with twice the frequency of the gravity wave.

<u>Case (iii). When the depth is compar able to the length of of a compression wave.</u>

$$z \simeq \lambda_c \leq \frac{100}{\sqrt{2}}.\lambda_g$$

$$e^{-kz} \simeq e^{-k.\frac{100}{\sqrt{2}}.\frac{2\pi}{k}} = e^{-\frac{200\pi}{\sqrt{2}}} \simeq 0.$$

$$\therefore \phi_1 \simeq 0.$$

AND $$\phi_2 \simeq -\frac{\sigma}{8}.4.\frac{\cos 2\sigma(z-h)/c}{\cos 2\sigma h/c}.2a_1a_2 \sin 2\sigma t.$$

$$\therefore \phi_2 \doteq -\sigma . \frac{\cos 2\sigma(z-h)/c}{\cos 2\sigma h/c} . a_1 a_2 \sin 2\sigma t \qquad (7\cdot40)$$

That is, the motion reduces to a compression wave at depth of the order of the length of a compression wave.

Equation (6·9) is the general differential equation for a wave motion in a heavy compressible fluid. It is interesting to see which terms in this equation dominate for the ocean waves of this section.

If we take the value of ϕ_1, given by equation (7·33), for ϕ we find that

$$\frac{\partial^2\phi}{\partial x^2} = -\frac{k\sigma \cosh k(z-h)}{\sinh kh} . \left[a_1 \sin(kx-\sigma t) - a_2 \sin(kx+\sigma t)\right]$$

$$\frac{\partial^2\phi}{\partial z^2} = \frac{k\sigma \cosh k(z-h)}{\sinh kh} . \left[a_1 \sin(kx-\sigma t) - a_2 \sin(kx+\sigma t)\right]$$

that is $\nabla^2\phi = 0$.

So that no compression term appears.

$$\frac{\partial\phi}{\partial z} = \frac{\sigma \sinh k(z-h)}{\sinh kh} . \left[a_1 \sin(kx-\sigma t) - a_2 \sin(kx+\sigma t)\right]$$

$$\rightarrow 0 \quad \text{as} \quad z \rightarrow h.$$

These results are in full agreement with the usual first order theory.

If we now take the value of ϕ_2, given by equation (7·38) for ϕ we find that

$$\frac{\partial\phi}{\partial z} = -\frac{3\sigma k}{4} . \frac{\sinh 2k(z-h)}{\sinh^4 kh} . \left[a_1^2 \sin 2(kx-\sigma t) - a_2^2 \sin 2(kx+\sigma t)\right]$$

$$+ \frac{\sigma^2}{4c} . \frac{\cosh 3kh}{\sinh^2 kh . \cosh kh} . \frac{\sin 2\sigma(z-h)/c}{\cos 2\sigma h/c} . 2a_1 a_2 \sin 2\sigma t.$$

$$\frac{\partial^2\phi}{\partial z^2} = -\frac{3\sigma k^2}{2}\cdot\frac{\cosh 2k(z-h)}{\sinh^4 kh}$$

$$\times\left[a_1^2\sin 2(kx-\sigma t) - a_2^2 \sin 2(kx+\sigma t)\right]$$

$$+\frac{\sigma^3}{2c^2}\cdot\frac{\cosh 3kh}{\sinh^2 kh.\cosh kh}\cdot\frac{\cos 2\sigma(z-h)/c}{\cos 2\sigma h/c}\cdot 2a_1a_2\sin 2\sigma t,$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{3\sigma^2k^2}{2}\cdot\frac{\cosh 2k(z-h)}{\sinh^4 kh}\cdot\left[a_1^2\sin 2(kx-\sigma t) - a_2^2\sin 2(kx+\sigma t)\right],$$

That is

$$\nabla^2\phi = \frac{\sigma^3}{2c^2}\cdot\frac{\cosh 3kh}{\sinh^2 kh.\cosh kh}\cdot\frac{\cos 2\sigma(z-h)/c}{\cos 2\sigma h/c}\cdot 2a_1a_2\sin 2\sigma t.$$

As $\quad z \to 0 \qquad c^2\nabla^2\phi \to 0$

and $\quad g\dfrac{\partial\phi}{\partial z}\quad$ remains finite for large h.

As $\quad z \to h \qquad \dfrac{\partial\phi}{\partial z} \to 0$

and

$$c^2\nabla^2\phi \to \frac{\sigma^3}{2}\cdot\frac{\cosh 3kh}{\sinh^2 kh.\cosh kh}\cdot\sec\frac{2\sigma h}{c}$$

$$\times\, 2a_1a_2\sin 2\sigma t.$$

It thus appears that the ocean may be considered as comprising two layers: a surface layer in which the gravity term of equation (6·9) dominates the motion and the remainder of the ocean, below a depth equal to the length of a gravity wave, in which the compression term of equation (6·9) determines the motion. In the surface layer the motion is similar to that to be expected in a heavy non-compressible fluid (see chapter 2) and produces a

But $$\frac{2\sigma}{c\gamma} = \frac{2\sigma}{cg} \cdot 2c^2 = \frac{4\sigma c}{g}$$

$$= \frac{4 \times 0.5 \times 1.4 \times 10^5}{9.8 \times 10^2} \text{ by equation (7.1)}$$

$$\simeq 2.8 \times 10^2 \ .$$

That is $$\frac{2\sigma h}{c} \simeq n\pi + \frac{\pi}{2} \ .$$

This is the same result as is obtained by putting $\cos 2\sigma h/c$ equal to zero in equation (7.40); which is to be expected since equation (7.40) is derived from equation (6.95). This condition, when $\frac{2\sigma h}{c} = n\pi + \frac{\pi}{2}$, is one of resonance between the surface and the sea bed.

From equation (7.40) we see that the length of a compression wave is $\frac{\pi c}{\sigma}$, so that resonance occurs when

$$h = \left(\frac{n}{2} + \frac{1}{4}\right)\frac{\pi c}{\sigma}, \qquad (7.41)$$

that is when the depth is about $\left(\frac{1}{2}n + \frac{1}{4}\right)$ times the length of a compression wave.

<u>Summary:</u> In an incompressible fluid, there is a second order pressure variation under a standing wave. This pressure variation exists at all depths and has a frequency twice that of the standing wave and an amplitude proportional to the square of the mean amplitude of the surface wave (see chapters 2 and 3).

In a compressible fluid, the elasticity of the liquid has little effect on a surface layer of depth equal to a wave length of the gravity wave, and the liquid in this layer has a motion very much as would be expected if the whole liquid were inelastic. In the lower layer the motion is small and is controlled by the elasticity.

The compression wave in the lower layer has the same frequency as the pressure variation, and may be regarded as a consequence of the pressure variation at the interface of the two layers.

The fluid being regarded as incompressible, a pressure variation applied to the free surface, may be used to estimate the displacement at the bed, due to a standing wave which produces a similar pressure variation at a depth of the length of a gravitational wave.

CHAPTER 8.

The displacement of the sea bed due to an oscillatory force applied at the free surface.

In this chapter we shall make a first order investigation similar to that made by Stoneley(1926) and by Scholte (1943).

The origin is taken on the surface of the ground, the z-axis is measured vertically downwards and the x and y axes are taken in the horizontal plane and are perpendicular.

The superficial water is of depth h, so that in the undisturbed state the equations of the sea bed and the surface of the sea are $z = 0$ and $z = -h$ respectively.

We shall neglect the effect of the water's viscosity.

Let u', v', w', denote the displacement components of the water in the x, y and z directions respectively.

p_0 and ρ_0 are the pressure and the density of the water in the undisturbed state; p_1 and ρ_1 are the changes in the pressure and density respectively. Then the actual pressure p and the actual density ρ are given by

$$\left.\begin{aligned} p &= p_0 + p_1 \\ \rho &= \rho_0 + \rho_1 \end{aligned}\right\} \qquad (8{\cdot}1)$$

In the Eulerian system the equations of motion are

$$\left.\begin{aligned} \rho_0 \frac{\partial^2 u'}{\partial t^2} &= -\frac{\partial p_1}{\partial x}, \\ \rho_0 \frac{\partial^2 v'}{\partial t^2} &= -\frac{\partial p_1}{\partial y}, \\ \rho_0 \frac{\partial^2 w'}{\partial t^2} &= -\frac{\partial p_1}{\partial z}. \end{aligned}\right\} \qquad (8{\cdot}2)$$

The continuity equation for a compressible fluid (Lamb : chapter I, section 7) is

$$\frac{D\rho}{Dt} + \rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \quad , \qquad (8\cdot3)$$

where $\dfrac{D}{Dt} \equiv \dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x} + v\dfrac{\partial}{\partial y} + \omega\dfrac{\partial}{\partial z}$,

and u , v , ω are the velocity components in the x, y and z directions respectively.

Also the changes in the pressure and density are connected by the relation

$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt} , \qquad (8\cdot4)$$

where $c^2 = \left(\dfrac{dp}{d\rho} \right)_{\rho=\rho_0}$.

$$\frac{Dp}{Dt} = \frac{\partial p_1}{\partial t} + \frac{\partial(z+\omega')}{\partial t} \cdot \frac{\partial p}{\partial(z+\omega')} \simeq \frac{\partial p_1}{\partial t} + \frac{\partial p_0}{\partial z} \cdot \frac{\partial \omega'}{\partial t} \qquad (8\cdot5)$$

Hence by equations (8•4) and (8•5)

$$c^2 \frac{D\rho}{Dt} = \frac{\partial p_1}{\partial t} + \frac{\partial p_0}{\partial z} \cdot \frac{\partial \omega'}{\partial t}$$

$$= \frac{\partial p_1}{\partial t} + \rho_0 g \frac{\partial \omega'}{\partial t} , \quad \text{since} \quad p_0 = \rho_0 g z .$$

Hence by equation (8•3)

$$\frac{\partial p_1}{\partial t} + \rho_0 g \frac{\partial \omega'}{\partial t} + c^2 \rho_0 \frac{\partial}{\partial t} \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial \omega'}{\partial z} \right) = 0 . \qquad (8\cdot6)$$

From the first of equations (8·2)

$$\rho_0 \frac{\partial^3 u'}{\partial t^3} = - \frac{\partial}{\partial x}\left(\frac{\partial p_1}{\partial t}\right).$$

Hence from (8 2) and (8 6)

$$\left.\begin{aligned}
\rho_0 \frac{\partial^3 u'}{\partial t^3} &= \frac{\partial}{\partial x}\left\{\rho_0 g \frac{\partial w'}{\partial t} + c^2 \rho_0 \frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)\right\}, \\
\rho_0 \frac{\partial^3 v'}{\partial t^3} &= \frac{\partial}{\partial y}\left\{\rho_0 g \frac{\partial w'}{\partial t} + c^2 \rho_0 \frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)\right\}, \\
\rho_0 \frac{\partial^3 w'}{\partial t^3} &= \frac{\partial}{\partial z}\left\{\rho_0 g \frac{\partial w'}{\partial t} + c^2 \rho_0 \frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)\right\}.
\end{aligned}\right\} \quad (8{\cdot}7)$$

If the motion is assumed to be irrotational, we can write

$$\left.\begin{aligned}
u &= \frac{\partial u'}{\partial t} = - \frac{\partial \phi}{\partial x}, \\
v &= \frac{\partial v'}{\partial t} = - \frac{\partial \phi}{\partial y}, \\
w &= \frac{\partial w'}{\partial t} = - \frac{\partial \phi}{\partial z},
\end{aligned}\right\} \quad (8{\cdot}8)$$

where ϕ is the velocity potential.

Then $$\frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v'}{\partial t}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w'}{\partial t}\right)$$

$$= -\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)$$

$$= -\nabla^2 \phi .$$

Also $$\frac{\partial^3 u'}{\partial t^3} = \frac{\partial^2}{\partial t^2}\left(\frac{\partial u'}{\partial t}\right) = -\frac{\partial^2}{\partial t^2}\cdot\frac{\partial \phi}{\partial x}\,, \text{ etc.}$$

Hence the first of equations (8·7) becomes

$$-\rho_0 \frac{\partial^2}{\partial t^2}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}\left\{-\rho_0 g \frac{\partial \phi}{\partial z} - c^2\rho_0\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)\right\},$$

that is

$$\frac{\partial^2}{\partial t^2}\cdot\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(c^2\nabla^2\phi + g\frac{\partial \phi}{\partial z}\right).$$

Similarly the second and third of (8·7) become

$$\frac{\partial^2}{\partial t^2}\cdot\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(c^2\nabla^2\phi + g\frac{\partial \phi}{\partial z}\right),$$

$$\frac{\partial^2}{\partial t^2}\cdot\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}\left(c^2\nabla^2\phi + g\frac{\partial \phi}{\partial z}\right).$$

Hence ϕ must satisfy

$$\frac{\partial^2 \phi}{\partial t^2} = c^2\nabla^2\phi + g\frac{\partial \phi}{\partial z} \qquad (8{\cdot}9)$$

In what follows we shall assume that the depth of the ocean remains constant and that the density of the suboceanic layer of the earth is uniform, and equal to ρ_2. We shall also assume the density of the water to be constant and equal to ρ_1.

We suppose that ϕ varies according as

$$\phi = e^{i(\sigma t - \xi x - \beta z)} \qquad (8{\cdot}10)$$

then substituting for ϕ in (8·9)

$$\frac{\partial^2\phi}{\partial t^2} - c^2\nabla^2\phi - g\frac{\partial\phi}{\partial z}$$

$$= -\sigma^2\phi - c^2(\xi^2+\beta^2)\phi + ig\beta\phi$$

$$= 0$$

$$c^2\beta^2 + ig\beta + c^2\xi^2 - \sigma^2 = 0 \qquad (8\cdot 11)$$

$$\therefore \beta = -\frac{ig}{2c^2} \pm \sqrt{\frac{\sigma^2}{c^2} - \xi^2 - \frac{g^2}{4c^2}} \qquad (8\cdot 12)$$

$$= -\frac{ig}{2c^2} \pm \varsigma$$

where $$\varsigma = \sqrt{\frac{\sigma^2}{c^2} - \xi^2 - \frac{g^2}{4c^2}} \qquad (8\cdot 13).$$

Hence by 8·10, 8·12 and 8·13,

$$\phi = e^{i(\sigma t - \xi x) - \frac{gz}{2c^2}} \left[Ae^{-i\varsigma z} + Be^{i\varsigma z} \right], \qquad (8\cdot 14)$$

where A and B are determined by the boundary conditions. Also

$$u = -\frac{\partial\phi}{\partial x} = i\xi e^{i(\sigma t - \xi x) - \frac{gz}{2c^2}} \left[Ae^{-i\varsigma z} + Be^{i\varsigma z} \right] \qquad (8\cdot 15)$$

and

$$w = -\frac{\partial\phi}{\partial z}$$

$$= e^{i(\sigma t - \xi x) - \frac{gz}{2c^2}} \left[\left(i\varsigma + \frac{g}{2c^2}\right) Ae^{-i\varsigma z} + \left(-i\varsigma + \frac{g}{2c^2}\right) Be^{i\varsigma z} \right] \qquad (8\cdot 16)$$

By Lamb section 40

$$\frac{p}{\rho_1} = \frac{\partial\phi}{\partial t} - \Omega - \tfrac{1}{2}q^2 + F(t)$$

Where Ω is the potential function of the body forces.

Hence $$\frac{p}{\rho_1} = \frac{\partial\phi}{\partial t} + gz - \tfrac{1}{2}q^2 + F(t).$$

At the surface

$$0 = -gh - \tfrac{1}{2}q^2 + F(t)$$ at the same instant.

Hence $$\frac{p}{\rho_1} = \frac{\partial\phi}{\partial t} + g(z+d)$$

or $$p = \rho_1\frac{\partial\phi}{\partial t} + g\rho_1(z+h) \qquad (8{\cdot}17)$$

For the displacements in the material of the sea bed Bullen (page 21) gives

$$\rho_2\frac{\partial^2 u_i}{\partial t^2} = (\lambda+\mu)\frac{\partial\theta}{\partial x_i} + \mu\nabla^2 u_i$$

after neglecting external forces, where λ and μ are Lamé 's elastic coefficients and

$$\theta = \frac{\partial u_i}{\partial x_i} = div\,(u)$$

$$\therefore \frac{\partial\theta}{\partial x_i} = \frac{\partial\theta}{\partial x_1} + \frac{\partial\theta}{\partial x_2} + \frac{\partial\theta}{\partial x_3} = grad\,\theta = \nabla\theta$$

$$= grad\,\{div\,u\} = \nabla(div\,u)$$

Hence $\rho \dfrac{\partial^2 u_i}{\partial t^2} = (\lambda+\mu)\nabla(\mathrm{div}\, u_i) + \mu \nabla^2 u_i$

for a displacement vector u_i ($i = 1,2,3$).
Let U and W denote the components of the displacement of the sea bed in the x and z directions respectively.
Following Bullen (page 86) we set

$$\left.\begin{aligned} U &= \frac{\partial \phi'}{\partial x} + \frac{\partial \psi'}{\partial z} \,, \\ W &= \frac{\partial \phi'}{\partial z} - \frac{\partial \psi'}{\partial x} \,. \end{aligned}\right\} \tag{8·18}$$

A displacement of the sea bed satisfies

$$\rho_2 \frac{\partial^2 r}{\partial t^2} = (\lambda+\mu)\nabla.\mathrm{div}\, r + \mu \nabla^2 r, \tag{8·19}$$

$$r = \underline{i}\, U + \underline{k}\, W$$

$\underline{i}$ and $\underline{k}$ being unit vectors in the x and z directions respectively. Equation (8 19) becomes

$$\rho_2 \frac{\partial^2}{\partial t^2}(\underline{i}\,U + \underline{k}\,W) = (\lambda+\mu)\nabla.\mathrm{div}(\underline{i}\,U + \underline{k}\,W) + \mu\nabla^2(\underline{i}\,U + \underline{k}\,W)$$

$$= (\lambda+\mu)\nabla\left(\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z}\right) + \mu\left[\underline{i}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2}\right) + \underline{k}\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2}\right)\right]$$

$$= (\lambda+\mu)\left[\underline{i}\,\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z}\right) + \underline{k}\,\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z}\right)\right]$$

$$+ \mu\left[\underline{i}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2}\right) + \underline{k}\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2}\right)\right]$$

Hence

$$\rho_2 \frac{\partial^2 U}{\partial t^2} = (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}+\frac{\partial W}{\partial z}\right)+\mu\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial z^2}\right),$$

and

$$\rho_2 \frac{\partial^2 W}{\partial t^2} = (\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial x}+\frac{\partial W}{\partial z}\right)+\mu\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial z^2}\right).$$

But

$$\frac{\partial U}{\partial x}+\frac{\partial W}{\partial z} = \frac{\partial^2\phi'}{\partial x^2}+\frac{\partial^2\psi'}{\partial x\partial z}+\frac{\partial^2\phi'}{\partial z^2}-\frac{\partial^2\psi'}{\partial x\partial z} = \nabla^2\phi',$$

and

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial z^2} = \frac{\partial^3\phi'}{\partial x^3}+\frac{\partial^3\psi'}{\partial x^2\partial z}+\frac{\partial^3\phi'}{\partial x\partial z^2}+\frac{\partial^3\psi'}{\partial z^3}$$

$$= \frac{\partial}{\partial x}\nabla^2\phi'+\frac{\partial}{\partial z}\nabla^2\psi'.$$

Hence

$$\rho_2\frac{\partial^2 U}{\partial t^2} = (\lambda+\mu)\frac{\partial}{\partial x}\nabla^2\phi'+\mu\frac{\partial}{\partial x}\nabla^2\phi'+\mu\frac{\partial}{\partial z}\nabla^2\psi'$$

$$\therefore \rho_2\frac{\partial^2 U}{\partial t^2} = (\lambda+2\mu)\frac{\partial}{\partial x}\nabla^2\phi'+\mu\frac{\partial}{\partial z}\nabla^2\psi' \qquad (8\cdot20)$$

Also

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial z^2} = \frac{\partial^3\phi'}{\partial x^2\partial z}-\frac{\partial^3\psi'}{\partial x^3}+\frac{\partial^3\phi'}{\partial z^3}-\frac{\partial^3\psi'}{\partial x\partial z^2}$$

$$= \frac{\partial}{\partial z}\nabla^2\phi'-\frac{\partial}{\partial x}\nabla^2\psi'$$

Hence

$$\rho_2\frac{\partial^2 W}{\partial t^2} = (\lambda+2\mu)\frac{\partial}{\partial z}\nabla^2\phi'-\mu\frac{\partial}{\partial x}\nabla^2\psi' \qquad (8\cdot21)$$

Substituting for U and W from (8·18) into (8·20) and (8·21)

$$\rho_2\left[\frac{\partial}{\partial x}\cdot\frac{\partial^2\phi'}{\partial t^2}+\frac{\partial}{\partial z}\cdot\frac{\partial^2\psi'}{\partial t^2}\right]=(\lambda+2\mu)\frac{\partial}{\partial x}\nabla^2\phi'+\mu\frac{\partial}{\partial z}\nabla^2\psi'$$

and

$$\rho_2\left[\frac{\partial}{\partial z}\cdot\frac{\partial^2\phi'}{\partial t^2}-\frac{\partial}{\partial x}\cdot\frac{\partial^2\psi'}{\partial t^2}\right]=(\lambda+2\mu)\frac{\partial}{\partial z}\nabla^2\phi'-\mu\frac{\partial}{\partial z}\nabla^2\psi'.$$

Hence

$$\left.\begin{aligned}\rho_2\frac{\partial^2\phi'}{\partial t^2}&=(\lambda+2\mu)\nabla^2\phi',\\ \text{and}\quad \rho_2\frac{\partial^2\psi'}{\partial t^2}&=\mu\nabla^2\psi'.\end{aligned}\right\}\tag{8·22}$$

Take $\phi'=\dfrac{C}{i\sigma}\cdot e^{i(\sigma t-\xi x-pz)}$ (8·23)

and $\psi'=\dfrac{D}{i\sigma}\cdot e^{i(\sigma t-\xi x-qz)}$ (8·24).

In order that (8·23) should satisfy the first of (8·22)

$$\rho_2 C i\sigma=-(\lambda+2\mu)\frac{C}{i\sigma}(\xi^2+p^2)$$

$$\therefore\rho_2\sigma^2=(\lambda+2\mu)(\xi^2+p^2)$$

$$\left.\begin{aligned}\therefore\quad p^2&=\frac{\rho_2\sigma^2}{\lambda+2\mu}-\xi^2\\ &=\frac{\sigma^2}{\alpha_2^2}-\xi^2\end{aligned}\right\}\tag{8·25}$$

where $$\alpha_2 = \sqrt{\frac{\lambda + 2\mu}{\rho_2}} \qquad (8\cdot26)$$

that is α_2 is the velocity of the compression (dilatational or irrotational) wave. (Bullen section 4·1).

In order that (8·24) should satisfy the second of (8·22)

$$\rho_2 \, i\sigma = - \frac{\mu}{i\sigma} \left(\xi^2 + q^2 \right)$$

$$\therefore \quad \rho_2 \sigma^2 = \mu \left(\xi^2 + q^2 \right)$$

that is $$\left. \begin{aligned} q^2 &= \frac{\rho_2 \sigma^2}{\mu} - \xi^2 \\ &= \frac{\sigma^2}{\beta_2^2} - \xi^2 , \end{aligned} \right\} \qquad (8\cdot27)$$

where $$\beta_2 = \sqrt{\frac{\mu}{\rho_2}} \qquad (8\cdot28)$$

that is β_2 is the velocity of the distortional (rotational or equivoluminal or shake) wave. (Bullen 4·1)

The velocity components of a solid particle are

$$u_2 = \frac{\partial U}{\partial x} \qquad \text{and} \qquad \omega_2 = \frac{\partial W}{\partial z} \qquad (8\cdot29)$$

From equations (8·18), (8·23) and (8·24)

$$U = -\frac{1}{\sigma}\left[C\,\xi^{-ipz} + D q e^{-iqz} \right] . \, e^{i(\sigma t - \xi x)} ,$$

and

$$W = -\frac{1}{\sigma}\left[C p e^{-ipz} - D \xi e^{-iqz} \right] e^{i(\sigma t - \xi x)} ;$$

hence by equation (8·29)

$$\left.\begin{aligned} u_2 &= -i\left(C\xi e^{-ipz} + Dq e^{-iqz}\right).\, e^{i(\sigma t - \xi x)} \\ w_2 &= -i\left(Cp e^{-ipz} - D\xi e^{-iqz}\right).\, e^{i(\sigma t - \xi x)} \end{aligned}\right\} \quad (8\cdot30)$$

The stresses across the XOY plane parallel to OX and OZ respectively are p_{zx} and p_{zz}, where by Bullen (§ 5·1)

$$\left.\begin{aligned} p_{zx} &= \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ p_{zz} &= \lambda \operatorname{div} \mathbf{r} + 2\mu \frac{\partial w}{\partial z} \end{aligned}\right\} \quad (8\cdot31).$$

Hence

$$p_{zx} = \mu i \frac{\xi}{\sigma}\left(Cp e^{-ipz} - D\xi e^{-iqz}\right).\, e^{i(\sigma t - \xi x)}$$

$$+ \mu \frac{i}{\sigma}\left(Cp\xi e^{-ipz} + Dq^2 e^{-iqz}\right) e^{i(\sigma t - \xi x)}$$

$$= \mu \cdot \frac{i}{\sigma}\left[2p\xi C e^{-ipz} + D(q^2 - \xi^2) e^{-iqz}\right] e^{i(\sigma t - \xi x)}$$

$$\therefore\ p_{zx} = i\mu k^2\left[\frac{2\xi p}{k^2}.\, C e^{-ipz} + \left(\frac{q^2}{k^2} - \frac{\xi^2}{k^2}\right) D e^{-iqz}\right] e^{i(\sigma t - \xi x)}$$

where $$k^2 = \frac{\sigma^2}{\beta_2^2} = \frac{\sigma^2 \rho_2}{\mu} \quad (8\cdot32)$$

That is

$$p_{zx} = i\rho_2\sigma\left[\frac{2\xi p}{k^2}.\, C e^{-ipz} + \left(\frac{q^2}{k^2} - \frac{\xi^2}{k^2}\right) D e^{-iqz}\right] e^{i(\sigma t - \xi x)} \quad (8\cdot33)$$

$$p_{zz} = \lambda \operatorname{div} r + 2\mu \frac{\partial W}{\partial z}$$

$$= \lambda \frac{\partial U}{\partial x} + (\lambda + 2\mu)\frac{\partial W}{\partial z}$$

$$= \frac{\lambda \xi i}{\sigma}\left[C\xi e^{-ipz} + Dq e^{-iqz}\right] e^{i(\sigma t - \xi x)}$$

$$+ (\lambda + 2\mu)\frac{i}{\sigma}\left[Cp^2 e^{-ipz} - D\xi q e^{-iqz}\right] e^{i(\sigma t - \xi x)}$$

$$= \frac{i}{\sigma}\Big[\left\{C\lambda\xi^2 + (\lambda+2\mu)Cp^2\right\} e^{-ipz}$$

$$+ D\left\{\lambda\xi q - (\lambda+2\mu)\xi q\right\} e^{-iqz}\Big] e^{i(\sigma t - \xi x)}$$

$$= \frac{i}{\sigma}\left[C\left\{\lambda\xi^2 + (\lambda+2\mu)p^2\right\} e^{-ipz} - D.\, 2\mu\xi q e^{-iqz}\right] e^{i(\sigma t - \xi x)}$$

But by equation (8•25)

$$(\lambda + 2\mu)p^2 = \rho_2\sigma^2 - (\lambda+2\mu)\xi^2$$

$$\therefore p_{zz} = \frac{i}{\sigma}\left[C\left\{\rho_2\sigma^2 - 2\mu\xi^2\right\} e^{-ipz} - D.\, 2\mu\xi q e^{-iqz}\right] e^{i(\sigma t - \xi x)}$$

$$= \frac{i k^2}{\sigma}\left[C\left(\frac{\rho_2\sigma^2}{k^2} - \frac{2\mu\xi^2}{k^2}\right) e^{-ipz} - D.\frac{2\mu\xi q}{k^2} e^{-iqz}\right] e^{i(\sigma t - \xi x)}$$

Hence using equation (8•32)

$$p_{zz} = i\rho_2\sigma\left[C\left(1 - \frac{2\xi^2}{k^2}\right) e^{-ipz} - D.\frac{2\xi q}{k^2} e^{-iqz}\right].\, e^{i(\sigma t - \xi x)} \qquad (8{\cdot}34)$$

Continuity of velocity and stress at the ocean bed.

Equations (8·15), (8·16) and (8·17) give the components of velocity and pressure at a point (x,z) in the ocean.

Equations (8·30), (8·33) and (8·34) give the components of velocity and of stress at the point (x,z) of the sea bed.

At the surface of the ocean-bed the normal components of velocity and stress must be continuous.

The continuity of normal stress gives :

$$\left[\rho_1 \frac{\partial \phi}{\partial t} + g\rho_1 (z+h)\right]_{z=0} + g\rho_1 (\text{ Elevation of lower surface of sea })$$

$$= \left[p_{zz}\right]_{z=0} + g\rho_1 h - g\rho_2 (\text{ Elevation of surface of sea-bed})$$

$$\therefore \rho_1 \left[\frac{\partial \phi}{\partial t}\right]_{z=0} + g\rho_1 \left[\int -\frac{\partial \phi}{\partial z}\, dt\right]_{z=0} = \left[p_{zz}\right]_{z=0} - g\rho\left[W\right]_{z=0}$$

That is

$$\rho_1 i\sigma\, e^{i(\sigma t - \xi x)} (A+B) +$$

$$\frac{g\rho_1}{i\sigma} \cdot e^{i(\sigma t - \xi x)} \cdot \left[A\left(i\varsigma + \frac{g}{2c^2}\right) + B\left(-i\varsigma + \frac{g}{2c^2}\right)\right]$$

$$= i\rho_2 \sigma \left[C\left(1 - \frac{2\xi^2}{k^2}\right) - D \cdot \frac{2\xi q}{k^2}\right] e^{i(\sigma t - \xi x)}$$

$$+ \frac{g\rho_2}{\sigma}\left[Cp - D\xi\right] e^{i(\sigma t - \xi x)}$$

That is

$$\frac{\rho_1}{\rho_2}\left[A\left(1-\frac{ig\varsigma}{\sigma^2}-\frac{g^2}{2c^2\sigma^2}\right)+B\left(1+\frac{ig\varsigma}{\sigma^2}-\frac{g^2}{2c^2\sigma^2}\right)\right]$$

$$=C\left(1-\frac{2\xi^2}{k^2}-\frac{igp}{\sigma^2}\right)-D\left(\frac{2\xi q}{k^2}-\frac{ig\xi}{\sigma^2}\right)$$

The continuity of normal velocity at the sea-bed gives :

$$\left[e^{i\left(\sigma t-\xi x-\frac{gz}{c^2}\right)}\left\{A\left(i\varsigma+\frac{g}{2c^2}\right)e^{-i\varsigma z}\right.\right.$$

$$\left.\left.+B\left(-i\varsigma+\frac{g}{2c^2}\right)e^{i\varsigma z}\right\}\right]_{z=0}$$

$$=\left[-iCpe^{-pz}+iD\xi e^{-qz}\right]_{z=0}.\,e^{i(\sigma t-\xi x)}$$

$$\therefore\ A\left(i\varsigma+\frac{g}{2c^2}\right)+B\left(-i\varsigma+\frac{g}{2c^2}\right)=-ipC+i\xi D \qquad (8\cdot36)$$

Since we assume that the viscosity of the water may be neglected it follows that the horizontal movements of the ocean-bed and the ocean are independent and that there is no stress in the plane $z = 0$,

that is $\left[p_{zx}\right]=0$

Hence from equation (8·33)

$$2\frac{\xi p}{k^2}.\,C+\left(\frac{q^2}{k^2}-\frac{\xi^2}{k^2}\right)D=0,$$

But $$q^2=\frac{\sigma^2}{\beta_2^2}-\xi^2 \qquad \text{(equation 8·27)}$$

$$=k^2-\xi^2 \qquad \text{(equation 8·32)}$$

Hence $$\frac{2\xi p}{k^2}\cdot C + \left(1-\frac{2\xi^2}{k^2}\right)D = 0 \qquad (8\cdot37)$$

To determine A,B,C and D we need a fourth equation in addition to (8·35), (8·36) and (8·37). This extra equation can be derived from the conditions at the free surface of the ocean. Let us suppose that the pressure at the free surface is the real part of $P e^{i(\sigma t-\xi x)+gh/c^2}$,

then

$$\left[\rho_1 \frac{\partial\phi}{\partial t}\right]_{z=-h} + g\rho_1\left[\text{elevation of the free surface above the plane } z=-h\right]$$

$$= P e^{i(\sigma t-\xi x)+gh/c^2}.$$

That is

$$\rho_1 i\sigma e^{i(\sigma t-\xi x)+\frac{gh}{c^2}}\cdot\left[Ae^{i\zeta h}+Be^{-i\zeta h}\right]$$

$$+\frac{g\rho_1}{i\sigma}\cdot e^{i(\sigma t-\xi x)+\frac{gh}{c^2}}\cdot\left[\left(i\zeta+\frac{g}{2c^2}\right)Ae^{i\zeta h}+B\left(-i\zeta+\frac{g}{2c^2}\right)e^{-i\zeta h}\right]$$

$$= P e^{i(\sigma t-\xi x)+\frac{gh}{c^2}}$$

$$\therefore\quad i\sigma\rho_1\left[A\left(1-\frac{ig\zeta}{\sigma^2}-\frac{g^2}{2\sigma^2c^2}\right)e^{i\zeta h}\right.$$

$$\left.+B\left(1+\frac{ig\zeta}{\sigma^2}-\frac{g^2}{2c^2\sigma^2}\right)e^{-i\zeta h}\right]$$

$$= P \qquad (8\cdot38)$$

We know by chapters 6 and 7 that a periodic disturbance of the free surface of the sea sets up both gravitational and compression waves and that the former are attenuated exponentially with increase of depth. In any case we wish to investigate the effect of the compression waves at depth h, so we neglect the gravitational terms in the four boundary equations.

It is convenient to write

$$\eta = \frac{\xi}{k} = \frac{\xi\beta_2}{\sigma} \qquad (8\cdot40)$$

Equations (8·35), (8·36), (8·37) and (8·38) then give :

$$\frac{\rho_1}{\rho_2}(A+B) = C(1-2\eta^2) - 2Dq\frac{\eta^2}{\xi}, \qquad (8\cdot41)$$

$$(A-B)s = -pC + \xi D \quad , \qquad (8\cdot42)$$

$$\frac{2\eta^2 p}{\xi}\cdot C + (1-2\eta^2)D = 0 \quad . \qquad (8\cdot43)$$

$$i\sigma\rho_1\left[Ae^{ish} + Be^{-ish}\right] = P \quad . \qquad (8\cdot44)$$

From (8·41) and (8·42)

$$2A = \left\{\frac{\rho_2}{\rho_1}(1-2\eta^2) - \frac{p}{s}\right\}C - \left\{2\frac{\rho_2}{\rho_1}\cdot\frac{q\eta^2}{\xi} - \frac{\xi}{s}\right\}D.$$

$$2B = \left\{\frac{\rho_2}{\rho_1}(1-2\eta^2) + \frac{p}{s}\right\}C - \left\{2\frac{\rho_2}{\rho_1}\cdot\frac{q\eta^2}{\xi} + \frac{\xi}{s}\right\}D$$

Substituting for A and B in equation (8·44)

$$2P = i\sigma\rho_1\left[\left\{\frac{\rho_2}{\rho_1}(1-2\eta^2)-\frac{p}{\varsigma}\right\}Ce^{i\varsigma h}-\left(\frac{2\rho_2}{\rho_1}\cdot\frac{q\eta^2}{\xi}-\frac{\xi}{\varsigma}\right)De^{i\varsigma h}\right.$$

$$\left.+\left\{\frac{\rho_2}{\rho_1}(1-2\eta^2)+\frac{p}{\varsigma}\right\}Ce^{-i\varsigma h}-\left(\frac{2\rho_2}{\rho_1}\cdot\frac{q\eta^2}{\xi}+\frac{\xi}{\varsigma}\right)De^{-i\varsigma h}\right]$$

$$= i\sigma\rho_1\left[C\cdot\frac{\rho_2}{\rho_1}(1-2\eta^2)(e^{i\varsigma h}+e^{-i\varsigma h})-\frac{p}{\varsigma}C(e^{i\varsigma h}-e^{-i\varsigma h})\right.$$

$$\left.-\frac{2\rho_2}{\rho_1}\cdot\frac{q\eta^2}{\xi}(e^{i\varsigma h}+e^{-i\varsigma h})+\frac{\xi}{\varsigma}C(e^{i\varsigma h}-e^{-i\varsigma h})\right]$$

$$\therefore \frac{P}{i\sigma\rho_1} = C\left[\frac{\rho_2}{\rho_1}(1-2\eta^2)\cos\varsigma h - i\frac{p}{\varsigma}\sin\varsigma h\right]$$

$$-D\left[\frac{2\rho_2}{\rho_1}\cdot\frac{q\eta^2}{\xi}\cdot\cos\varsigma h - i\frac{\xi}{\varsigma}\sin\varsigma h\right]$$

But by (8·43) $\qquad C = \dfrac{(2\eta^2-1)\xi D}{2\eta^2 p}$

$$\therefore \frac{P}{i\sigma\rho_1} = D\left[\frac{(2\eta^2-1)\xi}{2\eta^2 p}\left\{\frac{\rho_2}{\rho_1}(1-2\eta^2)\cos\varsigma h - \frac{ip}{\varsigma}\sin\varsigma h\right\}\right.$$

$$\left.-\frac{2\rho_2}{\rho_1}\cdot\frac{q\eta^2}{\xi}\cos\varsigma h + \frac{i\xi}{\varsigma}\sin\varsigma h\right]$$

$$\therefore \frac{P}{i\sigma\rho_1} = D\left[-\frac{\rho_2}{\rho_1}\left\{\frac{(2\eta^2-1)^2\xi}{2\eta^2 p} + \frac{2q\eta^2}{\xi}\right\}\cosh i\zeta h + \frac{1}{2\eta^2}\cdot\frac{\xi}{\zeta}\cdot\sinh i\zeta h\right]$$

$$\therefore D = \frac{P}{i\sigma\rho_1\left[\frac{\rho_1}{\rho_2}\left\{\frac{(2\eta^2-1)^2\xi}{2\eta^2 p} + \frac{2q\eta^2}{\xi}\right\}\cosh i\zeta h - \frac{1}{2\eta^2}\cdot\frac{\xi}{\zeta}\sinh i\zeta h\right]} \tag{8·45}$$

We let W_0 denote the vertical displacement of the ocean bed at the point (x, 0, 0) ; then after putting z=0 in the expression for W ,

$$W_0 = -\frac{1}{\sigma}\,(Cp - D\xi)\,e^{i(\sigma t-\xi x)}$$

$$= \frac{D}{\sigma}\left[\frac{(2\eta^2-1)\xi}{2\eta^2} - \xi\right]e^{i(\sigma t-\xi x)}$$

by equation (8·43)

$$= \frac{-\xi}{\sigma}\cdot\frac{D}{2\eta^2}\cdot e^{i(\sigma t-\xi x)}$$

After using equation (8·45)

$$W_0 = \frac{-\xi P e^{i(\sigma t-\xi x)}}{i\sigma^2\rho_1 2\eta^2\left[\frac{\rho_2}{\rho_1}\left\{\frac{(2\eta^2-1)^2\xi}{2\eta^2 p} + \frac{2q\eta^2}{\xi}\right\}\cosh i\zeta h - \frac{1}{2\eta^2}\cdot\frac{1}{\zeta}\cdot\sinh i\zeta h\right]}$$

$$= \frac{-P e^{i(\sigma t-\xi x)}}{i\sigma^2\rho_2\left[\left\{\frac{(2\eta^2-1)^2}{p} + \frac{4q\eta^4}{\xi^2}\right\}\cosh i\zeta h - \frac{\rho_1}{\rho_2}\cdot\frac{1}{\zeta}\cdot\sinh i\zeta h\right]}$$

That is $$W_0 = \frac{-P\, e^{i(\sigma t - \xi x)}}{\sigma^2 \rho_2\; G(\xi)}\,, \qquad (8\cdot46)$$

where

$$G(\xi) = \left\{\frac{(2\eta^2-1)^2}{p} + \frac{4q\eta^4}{\xi^2}\right\} \cosh i\zeta h - \frac{\rho_1}{\rho_2}\cdot\frac{1}{\zeta}\cdot \sinh i\zeta h.$$

But $$\eta = \frac{\xi}{k} = \frac{\xi\beta_2}{\sigma} \quad \text{by equation } (8\cdot40)$$

$$p = \left(\frac{\sigma^2}{\alpha_2^2} - \xi^2\right)^{\frac{1}{2}} = i\left(\xi^2 - \frac{\sigma^2}{\alpha_2^2}\right)^{\frac{1}{2}} \quad \text{by } (8\cdot25)$$

$$q = \left(\frac{\sigma^2}{\beta_2^2} - \xi^2\right)^{\frac{1}{2}} = i\left(\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}} \quad \text{by } (8\cdot27)$$

and $$\zeta = \left(\frac{\sigma^2}{c^2} - \xi^2\right)^{\frac{1}{2}} = i\left(\xi^2 - \frac{\sigma^2}{c^2}\right)^{\frac{1}{2}} \quad \text{by } (8\cdot13)$$

Hence $G(\xi) =$

$$i\left[\left(\frac{2\xi^2\beta_2^2}{\sigma^2} - 1\right)^2 \cdot \frac{\left(\xi^2 - \frac{\sigma^2}{\alpha_2^2}\right)^{-\frac{1}{2}}}{i} + 4i\left(\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)\cdot\frac{\xi^4\beta_2^4}{\sigma^2\xi^2}\right] \cosh i\zeta h$$

$$- \frac{\rho_1}{\rho_2}\cdot\left(\xi^2 - \frac{\sigma^2}{c^2}\right)^{-\frac{1}{2}} \cdot \sinh i\zeta h$$

$$= \left[\left(\frac{\beta_2}{\sigma}\right)^4 \left\{\left(2\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^2\left(\xi^2 - \frac{\sigma^2}{\alpha_2^2}\right)^{-\frac{1}{2}} - 4\left(\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}}\xi^2\right\}\right] \cosh i\zeta h$$

$$- \frac{\rho_1}{\rho_2}\left(\xi^2 - \frac{\sigma^2}{c^2}\right)^{-\frac{1}{2}} \sinh i\zeta h.$$

but $\operatorname{Cosh} i\,\delta h = \operatorname{Cosh}\left[-\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h\right] = \operatorname{Cosh}\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h$

and $\operatorname{Sinh} i\,\delta h = \operatorname{Sinh}\left[-\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h\right] = -\operatorname{Sinh}\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h$

$$\therefore\ G(\xi) =$$

$$\left(\frac{\beta_2}{\sigma}\right)^4\left\{\left(2\xi^2-\frac{\sigma^2}{\beta_2^2}\right)^2\left(\xi^2-\frac{\sigma^2}{\alpha_2^2}\right)^{-\frac{1}{2}} - 4\left(\xi^2-\frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}}\xi^2\right\}\cdot\operatorname{Cosh}\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h$$

$$+\frac{\rho_1}{\rho_2}\cdot\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{-\frac{1}{2}}\sinh\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h. \qquad (8\cdot47)$$

With the aid of equation (8•46) we may now determine the vertical displacemeht of a point of the sea bed caused by a force at the surface which is represented by a more general function. The new function of force selected is $P\,e^{i\sigma t-\frac{r}{a}+\frac{gh}{c^2}}$ applied to the free surface, where r is the distance of the point on the sea surface from the point (0,0,-d) and a is the distance at which the force is e times that at the point where r = 0. The maximum force is at (0,0,-d) that is vertically above the origin, and the force decreases radially symmetrically. With this pressure system we have only to deal with a limited area within which the waves are generated.

Denote the displacements U and W of the sea bed, produced by the pressure variation $P\,e^{i(\sigma t-\xi x)+gh/c^2}$, by $F e^{-i\xi x}$ and $G_1 e^{-i\xi x}$ reapectively.

Then the displacements due to a pressure variation

$$P e^{i(\sigma t - \xi x \cos\gamma - \xi y \sin\gamma) + \frac{gh}{c^2}}$$ will be

$$U' = F\cos\gamma \cdot e^{-i(\xi x \cos\gamma + \xi y \sin\gamma)}$$ parallel to Ox,

$$V' = F\sin\gamma \cdot e^{-i(\xi x \cos\gamma + \xi y \sin\gamma)}$$ parallel to Oy,

and $$W' = G_1 e^{-i(\xi x \cos\gamma + \xi y \sin\gamma)}$$ parallel to Oz.

By allowing γ all values from 0 to 2π we can find the average value of these functions for all γ. The result, which will be independent of the azimuth γ, will be radially symmetrical.

The average force $= \frac{1}{2\pi}\int_0^{2\pi} P e^{i(\sigma t - \xi x \cos\gamma - \xi y \sin\gamma)} d\gamma;$

on writing $x = r\cos\theta$, $y = r\sin\theta$, where $r^2 = +\sqrt{x^2+y^2}$,

this becomes $$\frac{Pe^{i\sigma t}}{2\pi}\int_\alpha^{2\pi+\alpha} e^{-i\xi r\cos(\gamma-\theta)} \cdot d(\gamma-\theta).$$

That is, the average force

$$= \frac{Pe^{i\sigma t}}{2\pi}\int_0^{2\pi}\left\{\cos(\xi r\cos\phi) - i\sin(\xi r\sin\phi)\right\}d\phi$$

since α may be anything.

Since $\cos\phi = -\cos(\pi-\phi)$ and $\cos(2\pi-\phi) = -\cos(\pi+\phi)$,

the imaginary part is zero,

the average force $= \dfrac{P e^{i\sigma t}}{2\pi} \displaystyle\int_0^{2\pi} \cos(\xi r \cos\phi)\, d\phi$

$$= P e^{i\sigma t} \cdot J_0(\xi r) \qquad (8{\cdot}48),$$

where J_0 denotes Bessel's function of the first kind and zero order.

The average value of W' is

$$\frac{1}{2\pi}\int_0^{2\pi} G_1 \cdot e^{-i(\xi x \cos\gamma + \xi y \sin\gamma)} \cdot d\gamma$$

$$= G_1 \cdot J_0(\xi r) \qquad (8{\cdot}49),$$

After reference to equations (8•46) and (8•48) we see that the value of W_0 caused by the pressure

$$P e^{i\sigma t} \cdot J_0(\xi r)$$

is $W_0 = - \dfrac{P e^{i\sigma t}}{\sigma^2 \rho_2\, G(\xi)} \cdot J_0(\xi r)$ (8•50)

We now seek to transform the function

$P e^{i\sigma t} \cdot J_0(\xi r)$ into $P e^{i\sigma t - \frac{r}{a}}$.

According to Titchmarsh (''Theory of Fourier Integrals'' section 8•1)

$$f(x) = \int_0^{\infty} k(x,u)\, du \int_0^{\infty} k(u,y) f(y)\, dy$$

for an arbitrary function $f(x)$.

With $k(x,u) \equiv J_0(x,u)$ we have

$$e^{-\frac{r}{a}} = \int_0^\infty J_0(\xi r)\,\xi\, d\xi \int_0^\infty J_0(s\xi)\, s e^{-\frac{s}{a}} ds \quad (8\cdot51)$$

According to Watson (''Theory of Bessel's Functions'' chapter XIII section 13·2)

$$\int_0^\infty e^{-at} J_\nu(bt)\, t^{\nu+1} dt = \frac{2a.(2b)^\nu.\Gamma(\nu+\frac{3}{2})}{(a^2+b^2)^{\nu+3/2}.\sqrt{\pi}}$$

$$= \frac{2a.(2b)^\nu}{(a^2+b^2)^{\nu+3/2}} \cdot \frac{\lfloor 2\nu+1 \,.\, \Gamma(\frac{1}{2})}{2^{2\nu+1}.\lfloor\nu.\Gamma(\frac{1}{2})}$$

$$= \frac{a\, b^\nu}{2^\nu (a^2+b^2)^{\nu+3/2}} \cdot \frac{\lfloor 2\nu+1}{\lfloor \nu}$$

Setting $\nu = 0$

$$\int_0^\infty e^{-at} . J_0(bt)\, t\, dt = \frac{a}{(a^2+b^2)^{3/2}} .$$

Replacing a by $\frac{1}{a}$, t by s, b by ξ and ν by μ we have that

$$\int_0^\infty e^{-\frac{s}{a}} . J_0(s\xi) . s\, ds = \frac{1}{a} \cdot \frac{1}{\left(\frac{1}{a^2} + \xi^2\right)^{3/2}} \quad (8\cdot52)$$

Hence using equations (8·51) and (8·52)

$$e^{-\frac{r}{a}} = \int_0^\infty J_0(\xi r)\,\xi\,.\,\frac{1}{a}\,.\,\frac{1}{\left(\frac{1}{a^2}+\xi^2\right)^{3/2}}\,.\,d\xi \qquad (8\cdot53)$$

By considering the force on an annulus of radius r and centre (o,o,-h) we see that the total force exerted on the free surface z + h = o is

$$\int_0^\infty P e^{i\sigma t-\frac{r}{a}}\,.\,2\pi r\,dr$$

$$= 2\pi P e^{i\sigma t}\int_0^\infty e^{-\frac{r}{a}}\,.\,r\,dr$$

$$= 2\pi P e^{i\sigma t}\left[-rae^{-\frac{r}{a}}\right]_0^\infty + 2\pi P e^{i\sigma t}\int_0^\infty ae^{-\frac{r}{a}}dr$$

$$= 2\pi a^2 P e^{i\sigma t} \qquad (8\cdot54)$$

We now suppose that a tends to zero in such a way that $2\pi P a^2$ keeps throughout the value K. Then the force over the surface is equivalent to a force concentrated at a point distance r from (o,o, -h) of value

$$\underset{a\to 0}{Lt}\, P e^{i\sigma t-\frac{r}{a}} = \underset{a\to 0}{Lt}\, P e^{i\sigma t}\int_0^\infty J_0(\xi r)\frac{\frac{1}{a}}{\left(\xi^2+\frac{1}{a^2}\right)^{3/2}}\,.\,d\xi$$

(by 8·53)

$$= \underset{a\to 0}{Lt}\, P a^2 e^{i\sigma t}\int_0^\infty \frac{J_0(\xi r)\,\xi\,d\xi}{\left(1+a^2\xi^2\right)^{3/2}}$$

$$= \frac{K}{2\pi}\,e^{i\sigma t}\int_0^\infty J_0(\xi r)\,\xi\,d\xi\,. \qquad (8\cdot55)$$

Application of the operator $\int_0^\infty \frac{a^2 \xi \, d\xi}{(1+a^2\xi^2)^{3/2}}$ ...

to the function $P e^{i\sigma t} J_0(\xi r)$ has given the function $P e^{i\sigma t - r/a}$, so that applying the same operator to equation (8.50) we see that a pressure variation $P e^{i\sigma t - r/a}$ applied to the free surface produces a vertical displacement W_0' of the bed at a point distance r from the origin, where

$$W_0' = -\int_0^\infty \frac{a^2 \xi}{(1+a^2\xi^2)^{3/2}} \cdot \frac{P e^{i\sigma t}}{\sigma^2 \rho_2 \, G(\xi)} \cdot J_0(\xi r) \, d\xi .$$

$$= -\frac{K e^{i\sigma t}}{2\pi \sigma^2 \rho_2} \int_0^\infty \frac{J_0(\xi r) . \xi \, d\xi}{(1+a^2\xi^2)^{3/2} . G(\xi)} .$$

Putting $a = 0$, we have that a point force

$\frac{K}{2\pi} e^{i\sigma t} \int_0^\infty J_0(\xi r) \xi \, d\xi$ applied to the free surface produces a vertical deflection

$$-\frac{K e^{i\sigma t}}{2\pi \sigma^2 \rho_2} \int_0^\infty \frac{J_0(\xi r) \xi \, d\xi}{G(\xi)}$$

at a distance r from the origin.

Hence a concentrated force $K e^{i\sigma t}$ applied at a point (o,o -h) produces a vertical deflection of the bed equal to

$-\frac{K e^{i\sigma t}}{2\pi \sigma^2 \rho_2} \int_0^\infty \frac{J_0(\xi r) \xi \, d\xi}{G(\xi)}$ at a distance r from the origin.

It is convenient to let $W(\sigma,r)e^{i\sigma t}$ denote the vertical displacement of the sea bed at a distance r from the origin due to force $e^{i\sigma t}$ applied to the sea surface immediately above the origin, then

$$W(\sigma,r)e^{i\sigma t} = -\frac{1}{2\pi}\int_0^\infty \frac{J_0(\xi r)\xi e^{i\sigma t}}{\rho_2\,\sigma^2 G(\xi)}\cdot d\xi \qquad (8\cdot 56)$$

We notice that as h tends to zero

$$G(\xi) \to \left(\frac{\beta_2}{\sigma}\right)^4\left[\left(2\xi^2-\frac{\sigma^2}{\beta_2^2}\right)^2\left(\xi^2-\frac{\sigma^2}{\alpha_2^2}\right)^{-\frac{1}{2}} - 4\xi^2\left(\xi^2-\frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}}\right]$$

then $G(\xi)=0$ becomes

$$\left(2\xi^2-\frac{\sigma^2}{\beta_2^2}\right)^2\left(\xi^2-\frac{\sigma^2}{\alpha_2^2}\right)^{-\frac{1}{2}} - 4\xi^2\left(\xi^2-\frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}} = 0,$$

which is the Rayleigh wave equation; it is in fact Stoneley's equation (24). For this to be so it is essential that the square roots $(\xi^2-\sigma^2/\alpha_2^2)^{\frac{1}{2}}$ and $(\xi^2-\sigma^2/\beta_2^2)^{\frac{1}{2}}$ be taken positive or zero. Since $\cosh\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h$ and $\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{-\frac{1}{2}}\sinh\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h$ are singled valued functions of ξ, the choice of sign for $\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}$ is immaterial.

To evaluate the right hand side of equation (8·56) we take ξ to be a complex variable, so that for $G(\xi)=0$, when h tends to zero, to be the Rayleigh wave equation the real parts of $\left(\xi^2-\frac{\sigma^2}{\alpha_2^2}\right)^{\frac{1}{2}}$ and $\left(\xi^2-\sigma^2/\beta_2^2\right)^{\frac{1}{2}}$ must be positive or zero. Thus the field of integration is restricted to one sheet of the Riemann surface (see Osgood; Functions of of a complex variable) bounded by the cuts

$$R\left(\xi^2-\frac{\sigma^2}{\alpha_2^2}\right)^{\frac{1}{2}}=0, \quad R\left(\xi^2-\frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}}=0 \qquad (8\cdot 57)$$

where σ is also taken as complex.

These cuts are rectangular hyperbolas :

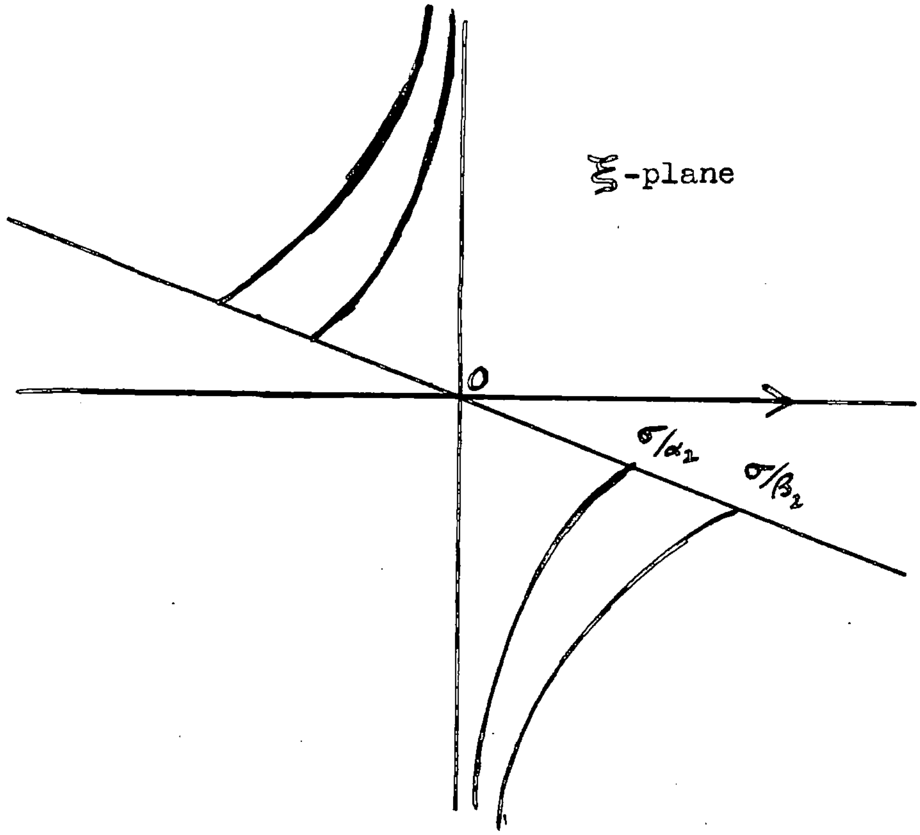


Figure 7.

Before proceeding further it is necessary to consider more fully the function $G(\xi)$. Since $G(\xi)$ vanishes at certain points on the real axis if σ is real, we take σ as ~~imaginary~~ complex and allow argσ to tend to zero. $W(\sigma, r)e^{i\sigma t}$ will then contain converging or diverging waves as argσ tends to zero through positive or negative values. Since we require diverging waves we allow argσ to tend to zero through negative values.

When σ is real.

It is assumed throughout the following that we are restricted to that part of the ξ -plane in which

$$-\frac{\pi}{2} \leq \arg\left(\xi^2-\frac{\sigma^2}{\alpha_2^2}\right)^{\frac{1}{2}} \leq \frac{\pi}{2} \; ; \quad -\frac{\pi}{2} \leq \arg\left(\xi^2-\frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}} \leq \frac{\pi}{2} \qquad (8\cdot58).$$

When $\operatorname{Cosh}\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h \neq 0$, we write

$$G(\xi) = 4\operatorname{Cosh}\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h \,.\, G_1(\xi), \qquad (8\cdot59)$$

where $$G_1(\xi) = \left(\frac{\beta_2}{\sigma}\right)^4\cdot\left[\left(\xi^2-\frac{\sigma^2}{2\beta_2^2}\right)^2\left(\xi^2-\frac{\sigma^2}{\alpha_2^2}\right)^{-\frac{1}{2}} - \xi^2\left(\xi^2-\frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}}\right]$$

$$+\frac{1}{4}\left(\frac{\rho_1}{\rho_2}\right)\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{-\frac{1}{2}}\cdot\tanh\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h \,. \qquad (8\cdot60)$$

$$= G_{11}(\xi) + G_{12}(\xi), \text{ say.}$$

When $\operatorname{Cosh}\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h = 0$, we have

$$G(\xi) = \frac{\rho_1}{\rho_2}\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{-\frac{1}{2}}\cdot\sinh\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h$$

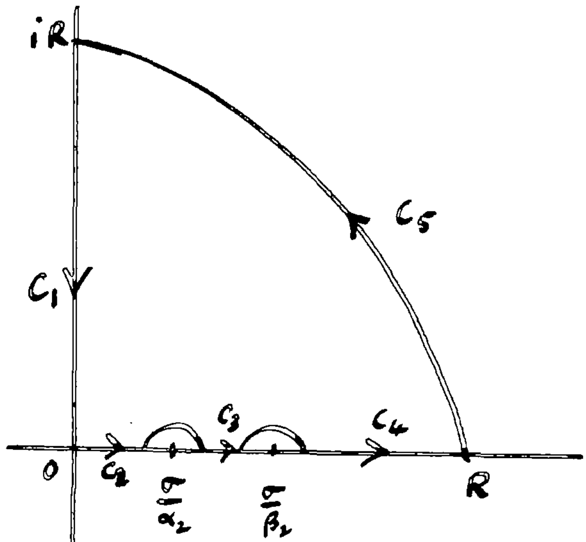
$$= \frac{\rho_1}{\rho_2}\cdot\frac{2(-1)^n h}{(2n+1)\pi}, \qquad (8\cdot61)$$

where n is an integer.

By (8·59) every zero of $G_1(\xi)$ is also a zero of $G(\xi)$. But since (8·60) never vanishes it follows that $G(\xi)$ cannot vanish unless $\operatorname{Cosh}\left(\xi^2-\frac{\sigma^2}{c^2}\right)^{\frac{1}{2}}h \neq 0$.

Hence, by (8·59), every zero of $G(\xi)$ is also a zero of $G_1(\xi)$ and the zeroes of $G(\xi)$ and $G_1(\xi)$ are identical.

In the following we shall find it more convenient to deal with $G_1(\xi)$ than with $G(\xi)$.



We shall first suppose that σ is real.
Let $C = c_1 + c_2 + c_3 + c_4 + c_5$, be closed contour in the ξ-plane.
Where c_1 is the imaginary axis from iR to O; c_2, c_3 and c_4 are the real axis from O to σ/α_2, and from σ/β_2 to R respectively; and c_5 is an arc of a large circle of radius R in the first quadrant.
c_2 and c_3 are taken along the upper side of the cuts along the real axis defined by equations (8·56), and the contour is indented inwards at $\xi = \sigma/\alpha_2$ (where in general G_{11} has an infinity) at $\xi = \sigma/\beta_2$, and at any zero of G_{11} on the real or imaginary axis.
By considering the variation of G_{11} round this contour we shall prove that, when

$$0 < \arg \xi < \frac{\pi}{2} \text{, then } -\pi < \arg G_{11} < 0 \, .$$

On c_1 and c_2, ξ^4 is real and non-negative, so that

$$\left(\xi^2 - \frac{\sigma^2}{\alpha_2^2}\right)^{\frac{1}{2}} G_{11} = \left(\beta_2 \frac{\xi}{\sigma}\right)^4 \left[\left(1 - \frac{\sigma^2}{2\beta_2^2 \xi^2}\right)^2 - \left(1 - \frac{\sigma^2}{\alpha_2^2 \xi^2}\right)^{\frac{1}{2}} \left(1 - \frac{\sigma^2}{\beta_2^2 \xi^2}\right)^{\frac{1}{2}}\right]$$

$$\geqslant \left(\frac{\beta_2 \xi}{\sigma}\right)^4 \cdot \left[1 - \frac{\sigma^2}{\beta_2^2 \xi^2} - \left(1 - \frac{\sigma^2}{\alpha_2^2 \xi^2}\right)^{\frac{1}{2}} \left(1 - \frac{\sigma^2}{\beta_2^2 \xi^2}\right)^{\frac{1}{2}}\right]$$

$$= \left(\frac{\beta_2 \xi}{\sigma}\right)^4 \cdot \left(1 - \frac{\sigma^2}{\beta_2^2 \xi^2}\right)^{\frac{1}{2}} \cdot \left[\left(1 - \frac{\sigma^2}{\beta_2^2 \xi^2}\right)^{\frac{1}{2}} - \left(1 - \frac{\sigma^2}{\alpha_2^2 \xi^2}\right)^{\frac{1}{2}}\right], \quad (8\cdot 62)$$

$$> 0 \text{ , since } \alpha_2 > \beta_2 \, .$$

On c_1, $\xi^2 = -\eta^2$,say, where η is positive.

Let $1 + \frac{\sigma^2}{\alpha_2^2 \eta} = p^2$, $1 + \frac{\sigma^2}{\beta_2^2 \eta} = q^2$, then $p < q$,

and
$$4 i \eta p G_{11} = \left(\frac{\beta_2^4 \eta^4}{\sigma^4}\right) \cdot \left[\left(1 + \frac{\sigma^2}{2\beta_2^2 \eta^2}\right)^2 - pq\right]$$

$$> \left(\frac{\beta_2^4 \eta^4}{\sigma^4}\right) \cdot \left[1 + \frac{\sigma^2}{\beta_2^2 \eta^2} - pq\right]$$

$$= \left(\frac{\beta_2^4 \eta^4}{\sigma^4}\right) \cdot \left(q^2 - pq\right)$$

$$= \left(\frac{\beta_2 \eta^4}{\sigma^4}\right) \cdot q(q - p)$$

$$> 0$$

$$\therefore \; i G_{11} > 0$$

that is G_{11} is of the form (-i) (poisitive quatity)

so that $\arg G_{11} = -\frac{\pi}{2}$, say on c_1 .

On c_2 ; let $\frac{\sigma^2}{\alpha_2^2 \xi^2} - 1 = P^2$ and $\frac{\sigma^2}{\beta_2^2 \xi^2} - 1 = Q^2$.

Then
$$\xi (i P) G_{11} = \left(\frac{\beta_2 \xi}{\sigma}\right)^4 \cdot \left[\left(\frac{\sigma^2}{2\beta_2^2 \xi^2} - 1\right)^2 - (i P)(i Q)\right],$$

which is positive.

Hence G_{11} is of the fotm (-i).(positive quantity)

$\therefore$ $\arg G_{11} = -\frac{\pi}{2}$, say on c_2 .

So that on c_1 and c_2 (excluding the point $\frac{\sigma}{\alpha_2}$)

$$\arg G_{11} = -\frac{\pi}{2} \text{ , say.} \qquad (8\cdot 63).$$

In the neighbourhood of σ/α_2

$$G_{11} \to \frac{\left(\frac{\beta_2}{\alpha_2}\right)^4 \cdot \left[1 - \frac{\alpha_2^2}{2\beta_2^2}\right]}{\left[\left(\frac{\sigma}{\alpha_2} + \rho e^{i\theta}\right)^2 - \frac{\sigma^2}{\alpha_2^2}\right]^{\frac{1}{2}}} \simeq \frac{\text{positive quantity}}{\sqrt{\frac{2\sigma}{\alpha_2}\rho} \; . \; e^{i\theta/2}}$$

arg G lies between 0 and $-\frac{\pi}{2}$.

In the special case when $\alpha_2^2 = 2\beta_2^2$ the first term of G_{11} becomes negligible compared with the second. Hence we have in any case $-\frac{\pi}{2} - \epsilon \leqslant \arg G_{11} \leqslant \epsilon$ (8·64)

where ϵ is arbitrarily small.

On c_3, $\left(\xi^2 - \frac{\sigma^2}{2\beta_2^2}\right)^2\left(\xi^2 - \frac{\sigma^2}{\alpha_2^2}\right)^{\frac{1}{2}}$ is real and $\xi^2\left(\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}}$ is imaginary and non-vanishing, except at σ/β_2, where the former term is positive. Hence G_{11} is non-vanishing and

$$-\frac{\pi}{2} < \arg G_{11} < 0 \qquad (8{\cdot}65)$$

On c_4, G_{11} is real i.e. arg G_{11} is zero, and for large positive ξ we have

$$G_{11} = \left(\frac{\beta_2}{\sigma}\right)^4 \xi^3\left[\left(1 - \frac{\sigma^2}{2\beta_2^2\xi^2}\right)^2\left(1 - \frac{\sigma^2}{\alpha_2^2\xi^2}\right)^{-\frac{1}{2}} - \left(1 - \frac{\sigma^2}{\beta_2^2\xi^2}\right)^{\frac{1}{2}}\right]$$

$$\simeq \left(\frac{\beta_2}{\sigma}\right)^4 \xi^3\left[\left(1 - \frac{\sigma^2}{\beta_2^2\xi^2}\right)\left(1 + \frac{\sigma^2}{2\alpha_2^2\xi^2}\right) - \left(1 - \frac{\sigma^2}{2\beta_2^2\xi^2}\right)\right]$$

$$= \left(\frac{\beta_2}{\sigma}\right)^4 \xi^3\left[\frac{\sigma^2}{2\alpha_2^2} - \frac{\sigma^2}{2\beta_2^2}\right]\cdot\frac{1}{\xi^2} \qquad (8{\cdot}66)$$

$$< 0 \quad \text{since} \quad \frac{\sigma}{\alpha_2} < \frac{\sigma}{\beta_2}. \qquad (8{\cdot}67)$$

Since $G_{11} > 0$ when $\xi = \frac{\sigma}{\beta_2}$, G_{11} has an odd number of zeroes, say $2N+1$, on c_4. At each zero arg G_{11} is diminished by π (since we travel round it in a clockwise direction). Thus on the real axis arg G_{11} takes successively the values :

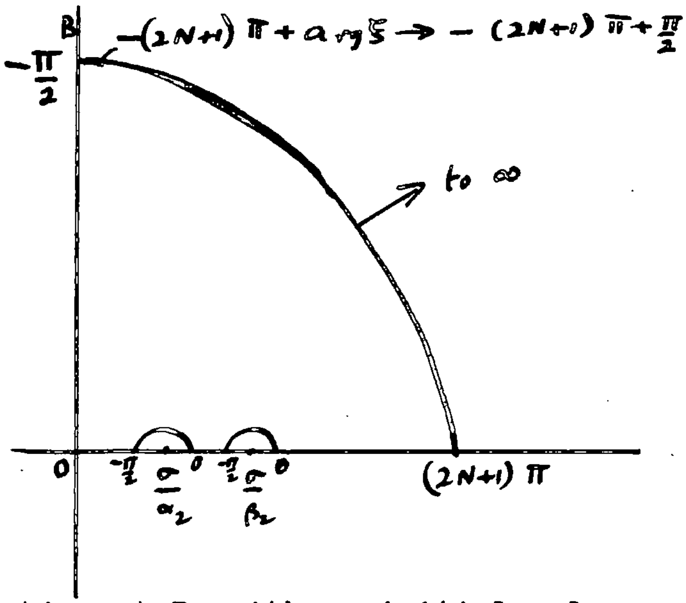
$$\arg G = 0, -\pi, -2\pi, \; \cdots\cdots\cdots, -(2N+1)\pi. \qquad (8{\cdot}68).$$

On c_5 equation (8·64) is still valid, so that when R is large

$$\arg G_{11} \sim -(2N+1)\pi + \arg\xi \qquad (8{\cdot}69)$$

Hence the final value of arg G_{11} on completing the circuit C is $-(2N+1)\pi + \frac{\pi}{2}$.

The values of arg G_{11} round the contour C are as shewn :



Thus starting at B, with an initial value arg $G_{11} = -\frac{\pi}{2}$, and completing the circuit C in an anticlockwise direction the final value of arg G_{11} is $-(2N+1)+\frac{\pi}{2}$. Hence the increase in arg G_{11} in describing the circuit once equals

$$-(2N+1)\pi + \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = -2N\pi.$$

Now, since G_{11} is regular and has no poles inside C, the increase in arg G_{11} in describing the circuit once equals (by Cauchy's Integral Theorem) $2\pi n$, where n is the number of zeroes in the interior of C. Hence we have

$$-2N\pi = 2n\pi$$

$$\text{or} \quad -N = n \qquad (8\cdot70).$$

Since N and n are both essentially non negative we must have $N = n = 0$ (8·71).

In other words <u>G_{11} has no zeroes inside C, and has just one zero on the positive real axis</u>.

Now if f(z) is any function of z (not a constant), regular and non-vanishing within a simple closed contour C, then it is possible to define without ambiguity a function $f_1(z)$ given by $f_1 = e^{-i\log f}$

Since $|f_1| = e^{\arg f}$ it follows from the maximum modulus theorem (Titchmarsh : Theory of Functions, 1932, chap. V; If f(z) be an analytic function, regular in a region D and on its boundary C, then $|f(z)|$ reaches its maximum on the boundary C and not at an interior point) that $e^{\arg f}$, and hence arg f, takes its greatest and least values only on C itself.

Applying this result to G_{11}, we see that from equations (8·63), (8·64), (8·65), (8·68), (8·69) and (8·70) that, at all interior points of the first quadrant,

$$-\pi - \epsilon < \arg G_{11} < \epsilon \qquad (8\cdot72)$$

where ϵ is as small as we please. Hence

$$-\pi \leq \arg G_{11} \leq 0 \qquad (8\cdot73).$$

But any interior point of the quadrant may be surrounded by a contour consisting entirely of interior points, at each of which (8·73) holds good. Therefore in (8·73) the inequality signs may be replaced by strict inequalities; that is, when

$$\left.\begin{array}{ll} \text{when} & 0 < \arg \xi < \frac{\pi}{2}, \\ \text{then} & -\pi < \arg G_{11} < 0 \end{array}\right\} \qquad (8\cdot74).$$

Also when $0 < \arg z < \frac{\pi}{2}$, we can shew that

$$-\frac{\pi}{2} < \arg \tanh z < \frac{\pi}{2}.$$

Thus

$$\arg \tanh z = \arg \frac{\sinh x \cos y + i \cosh x \sin y}{\cosh x \cos y + i \sinh x \sin y}$$

where both x and y are positive, since $0 < \arg z < \frac{\pi}{2}$.

$$\therefore \arg \tanh z = \arg(\sinh x \cos y + i \cosh x \sin y) \times (\cosh x \cos y - i \sinh x \sin y)$$

$\therefore$ the imaginary part of arg tanh z

$$= \cosh^2 x \sin y \cos y - \sinh^2 x \sin y \cos y$$

$$= \tfrac{1}{2} \sin 2y$$

and the real part of arg tanh z

$$= \cos^2 y \sinh x \cosh x + \sin^2 y \sinh x \cosh x$$

$$= \tfrac{1}{2} \sinh 2x > 0$$

since $x > 0$.

Hence, since the real part of $\arg \tanh Z$ is positive, tanh(z) lies in the first or fourth quadrant, and so

$$\left.\begin{aligned} -\frac{\pi}{2} < \arg \tanh Z < \frac{\pi}{2} \\ \text{when} \quad 0 < \arg Z < \frac{\pi}{2} \end{aligned}\right\} \tag{8.75}$$

Also when $0 < \arg Z < \frac{\pi}{2}$,

$$\arg\left\{\frac{\tanh Z}{Z}\right\} = \arg\left\{\frac{\sinh(x+iy)}{\cosh(x+iy)} \cdot \frac{1}{(x+iy)}\right\}$$

$$= \arg\left\{\frac{\sinh x \cos y + i \cosh x \sin y}{\cosh x \cos y + i \sinh x \sin y} \cdot \frac{1}{x+iy}\right\}$$

$$= \arg\left\{(\sinh x \cos y + i \cosh x \sin y). (\cosh x \cos y - i \sinh x \sin y). (x-iy)\right\}$$

$$= \arg \tfrac{1}{2}\left\{(\sinh 2x + i \sin 2y)(x-iy)\right\} .$$

Imaginary part of $\arg\left\{\frac{\tanh Z}{Z}\right\}$

$$= \tfrac{1}{2}(x \sin 2y - y \sinh 2x)$$

$$< 0,$$

since $\dfrac{\sinh 2x}{x} > \dfrac{\sin 2y}{y}$,

since these are both unity when $x = y = 0$ and thereafter $\dfrac{\sinh 2x}{x}$ increases while $\left|\dfrac{\sin 2y}{y}\right| < 1$.

Hence we have that, when

$$\left.\begin{aligned} &0 < \arg z < \frac{\pi}{2} \\ \text{then}\quad &-\pi < \arg\left\{z^{-1}\tanh z\right\} < 0 \end{aligned}\right\} \qquad (8{\cdot}76)$$

We shall now apply the results (8·74), (8·75) and (8·76) to the function G_1.

At all points in the interior of the first quadrant we have

$$0 < \arg\left(\xi^2 - \frac{\sigma^2}{c^2}\right)^{\frac{1}{2}} < \frac{\pi}{2},$$

and thereafter, by (8·76),

$$-\pi < \arg G_{12} < 0 \qquad (8{\cdot}77)$$

Also by (8·74) G_{11} is non vanishing and

$$-\pi < \arg G_{11} < 0$$

$\therefore$ G_1, $= G_{11} + G_{12}$, is non-vanishing in the interior of the first quadrant and

$$-\pi \quad \arg G_1 < 0 \qquad (8{\cdot}78).$$

Since, when σ is real,

$$G_1(-\xi) = G(\xi) \quad ; \quad G_1(\xi^*) = G_1^*(\xi) \qquad (8{\cdot}79)$$

(a star denoting a conjugate complex),

G_1 is a non-vanishing in all the other three quadrants. <u>Thus G_1 has no complex zeroes</u>. But every zero of $G(\xi)$ is a zero of $G_1(\xi)$. <u>Hence $G(\xi)$ has no complex zeroes when σ is real</u>.

On the imaginary axis, and on the real axis when $|\xi| < \dfrac{\sigma}{\beta_2}$ we have seen that G_{11} is non-vanishing and

$$-\frac{\pi}{2} \leqslant \arg G_{11} \leqslant 0 .$$

On the other hand G_{12} is real on either of the axes and so <u>G_1 has no zeroes on the imaginary axis or on this part of the real axis.</u>

On the positive real axis, when $\xi \geqslant \frac{\sigma}{\beta_2}$, G_1 is real, and therefore, if the axis is approached from the interior of the first quadrant we have from (8·76)

$$\arg G_1 = -\pi \text{ or } 0 . \qquad (8\cdot78).$$

But, if we travel along the real axis in the direction of ξ increasing, passing above the poles and zeroes of G_1, at each pole arg G_1 is increased by π and at each zero it is diminished by π. Therefore, the poles and zeroes of G_1 must occur alternately. Further, for large ξ, G_1 is ultimately negative and $\arg G_1 = -\pi$.
There is, therefore, at least one zero in the interval $\xi > \frac{\sigma}{\beta_2}$. For, either there is a pole in this interval or not. If there is a pole, arg G_1 must at some point be changed back from 0 to $-\pi$. If there is no pole, $G_{12}(\xi)$ is always positive and $G_1\left(\frac{\sigma}{\beta_2}\right) > G_{11}\left(\frac{\sigma}{\beta_2}\right) > 0$.

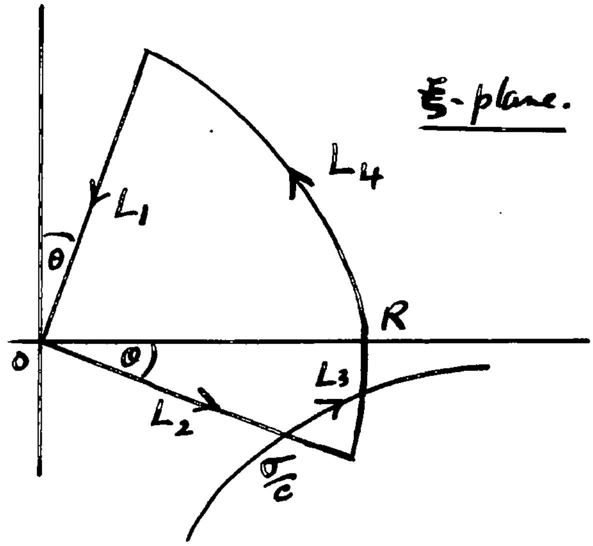
Therefore continuity requires that G_1 should vanish at some point. In this latter case, however, there is only one positive zero, since if there were two zeroes there would be a pole separating them. But the zeroes of $G(\xi)$ and $G_1(\xi)$ are identical and every zero of $G(\xi)$ is also a zero of $G_1(\xi)$, also the function $G(\xi)$ has zeroes only when $\cosh\left(\xi^2 - \frac{\sigma^2}{c^2}\right)^{\frac{1}{2}} l$ is not zero, so we conclude that when σ is real, the positive zeroes of $G(\xi)$ are all $\geqslant \sigma/\beta_2$ and are separated alternately on the real axis by the zeroes of

$$\cosh\left(\xi^2 - \frac{\sigma^2}{c^2}\right)^{\frac{1}{2}} l ,$$

We shall now suppose <u>σ to be complex</u> .

Suppose $\arg \sigma = -\theta$, where $0 < \theta < \frac{\pi}{2}$, since we wish $\arg \sigma$ to approach zero through negative values.

Let $L, = L_1 + L_2 + L_3 + L_4$, be a closed contour in the ξ -plane



L_1 is part of the line

$\arg \xi = \frac{\pi}{2} - \theta$, $0 \leq |\xi| \leq R$;

L_2 is part of the line

$\arg \xi = -\theta$, $0 \leq |\xi| \leq \left|\frac{\sigma}{\alpha_1}\right|$;

L_3 is part of the rectangular hyperbola

$$\arg \left(\xi^2 - \frac{\sigma^2}{c^2}\right)^{\frac{1}{2}} = 0 ,$$

L_4 is an arc of the circle $|\xi| = R$.

We shall now consider the variation of G_1 round L.

We can write

$$G_{11} = \left(\frac{\beta_2}{\sigma}\right)^4 \left[\left(\frac{\xi^2}{\alpha^2} - \frac{1}{2\beta_2^2}\right)^2 \left(\frac{\xi^2}{\sigma^2} - \frac{1}{\alpha_2^2}\right)^{-\frac{1}{2}} - \frac{\xi^2}{\alpha^2}\left(\frac{\xi^2}{\alpha^2} - \frac{1}{\beta_2^2}\right)^{\frac{1}{2}} \right] .$$

On L_1 , since $\arg\left(\frac{\xi}{\sigma}\right) = \frac{\pi}{2}$, we have from (8·60)

$$\arg G_{11} = \theta - \frac{\pi}{2}$$

also, since $0 < \arg \left(\xi^2 - \frac{\sigma^2}{c^2}\right)^{\frac{1}{2}} < \frac{\pi}{2}$, wehave by (8·76)

$$-\pi < \arg G_{12} < 0,$$

$$\therefore \quad -\pi < \arg G_1 < 0.$$

When $-\theta < \arg \xi < \frac{\pi}{2} - \theta$ it follows from (8·74) that

$$\theta - \pi < \arg G_{11} < \theta .$$

So that on L_2 we have, taking the limit as $\arg \xi \to -\theta$,

$$\theta - \pi \leq \arg G_{11} \leq \theta .$$

But, on L_2 , $\arg \left(\xi^2 - \frac{\sigma^2}{c^2}\right)^{\frac{1}{2}} = \theta - \frac{\pi}{2}$, and so using (8·75) we find, except possibly at $\xi = \sigma/c$,

$$\theta - \pi < \arg G_{12} < \theta .$$

At $\xi = \frac{\sigma}{c}$, we have $\arg G_{12} = 0$.
Hence at all points on L_2 we have

$$\theta - \pi < \arg G_1 < \theta .$$

On L_3 (excluding the point σ/c)

$$\theta - \pi \leq \arg G_{11} < \theta , \text{ and } \arg G_{12} = 0,$$

so that $\theta - \pi < \arg G_1 < \theta$.

On L_4, if R is sufficiently large, G_{12} is small compared with G_{11} and we have

$$\arg G_1 \sim \arg G_{11} ,$$

so that $\theta - \pi - \epsilon < \arg G_1 < \theta + \epsilon$,
where ϵ is arbitrarily small.

$\therefore$ G_1 has no zeroes inside L. (Since it returns to its initial value after describing L)

Thus, when $-\frac{\pi}{2} < \arg \sigma < 0$, G_1 has no zeroes on the positive real axis, and so, as $\arg \sigma$ tends to zero, the zeroes cannot approach the positive real axis from above. But the zeroes are continuous functions of σ, hence it follows that they must approach the positive real axis from below.

We now return to the evaluation of the right hand side of equation (8•56).. The cuts in the ξ-plane given by equations (8•57) and shewn in figure 7 approach the positive real axis from below as $\arg \sigma$ tends to zero. So that we are restricted to a single sheet of a Riemann's surface bounded by a cut along the negative imaginary axis from the origin to $-i\infty$ and along the real positive axis from the origin to σ/β_2. Also, as seen above, the zeroes of $G(\xi)$ approach the positive real axis from below as $\arg \sigma$ tends to zero through negative values. Hence the path of integration must be taken along the upper side of the cuts from 0 to σ/α_2 and from 0 to σ/β_2 and must be indented above the real axis near the zeroes of $G(\xi)$.

The path of integration is shewn in figure 8; where ξ_1 , ξ_2 , ------ are the real zeroes of $G(\xi)$:

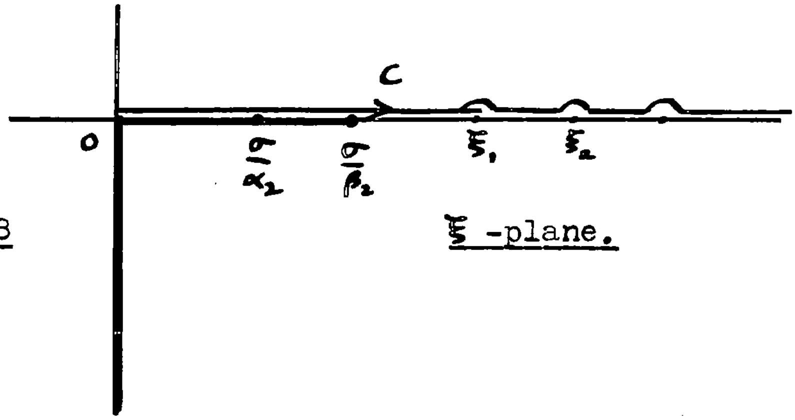


Fig.8 ξ -plane.

Now Bessel's function of the first kind and zero order can be expressed as the sum of two Hankel's functions (Watson; Theory of Bessel Functions § 7·22) :

$$J_0(\xi r) = \frac{1}{2}\left[H_0^{(1)}(\xi r) + H_0^{(2)}(\xi r)\right] \tag{8·79}$$

Let $$I = \int_0^\infty \frac{J_0(\xi r)\,\xi\,d\xi}{G(\xi)}$$

$$= \frac{1}{2}\int_0^\infty \frac{H_0^{(1)}(\xi r)\,\xi\,d\xi}{G(\xi)} + \frac{1}{2}\int_0^\infty \frac{H_0^{(2)}(\xi r)\,\xi\,d\xi}{G(\xi)}$$

$$= I_1 + I_2 \tag{8·80}$$

For I_1 the path of integration is transformed into $C_1 + C_2$, where C_1 is the positive imaginary axis from 0 to $i\infty$ and C_2 is the quadrant of the circle $|\xi| = R$ from iR to R and R is made to tend to infinity. (see figure 9):

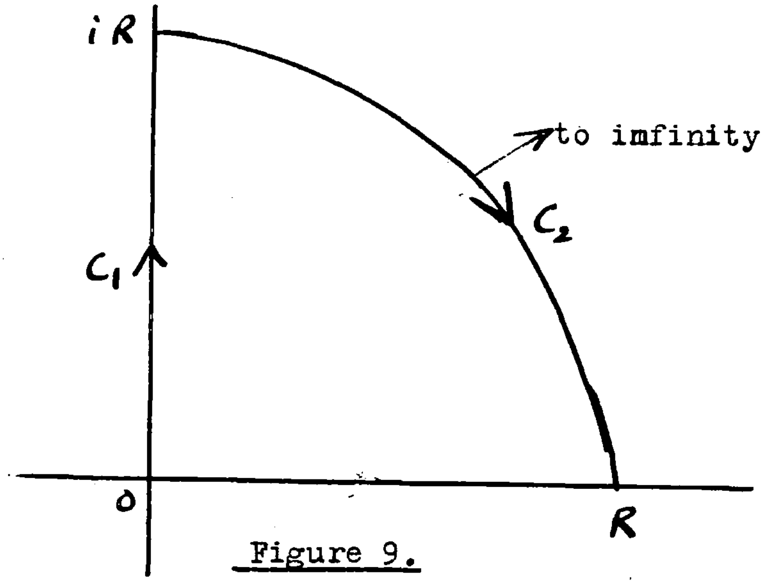


Figure 9.

So that

$$I_1 = \frac{1}{2}\int_{C_1} \frac{H_0^{(1)}(\xi r)\,\xi\,d\xi}{G(\xi)} + \frac{1}{2}\int_{C_2} \frac{H_0^{(2)}(\xi r)\,\xi\,d\xi}{G(\xi)}$$

$$= I_1^{(1)} + I_1^{(2)} \qquad (8{\cdot}81).$$

The integral I_2 is evaluated along a contour specially selected so that the integral over part of it will cancel out. This part of the contour is C_3, which is taken along the left hand side of the imaginary axis cut from 0 to $-i\infty$.

Along C_1, $\xi = Re^{i\frac{\pi}{2}}$, and along C_3, $\xi = Re^{-i\frac{\pi}{2}}$ also $G(\xi) = G(-\xi)$ and $H_0^{(2)}(te^{-i\frac{\pi}{2}}) = -H_0^{(1)}(te^{i\frac{\pi}{2}})$.

Hence

$$\int_{C_1} \frac{H_0^{(1)}(\xi r)\,\xi\,d\xi}{G(\xi)} = -\int_{C_3} \frac{H_0^{(2)}(\xi r)\,\xi\,d\xi}{G(\xi)}$$

or

$$I_1^{(1)} = -I_2^{(1)} \qquad (8{\cdot}82)$$

where $$I_2^{(1)} = \frac{1}{2}\int_{C_3} \frac{H_0^{(2)}(\xi r)\,\xi\,d\xi}{G(\xi)}$$

To extend the contour to infinity we use C_5 which is a quadrant of the circle $|\xi| = R$ from $-iR$ to R.
Now C_5 starts on the right hand side of the imaginary axis cut from 0 to $-i\infty$, so that a contour C_4 is necessary to link up C_3 with C_5. This contour C_4 surrounds the cuts in the ξ-plane. The contour C_5 sweeps over the zeroes of $G(\xi)$ (approached from below), so that these must be compensated for by integration round small circles C_6', C_6'', etc. round the zeroes ξ_1, ξ_2, etc.
The complete contour for I_2 is shewn by figure 10. So that

$$I_2 = \frac{1}{2}\int_{C_3} \frac{H_0^{(2)}(\xi r)\,\xi\,d\xi}{G(\xi)} + \frac{1}{2}\int_{C_4} \frac{H_0^{(2)}(\xi r)\,\xi\,d\xi}{G(\xi)}$$

$$+ \frac{1}{2}\int_{C_5} \frac{H_0^{(2)}(\xi r)\,\xi\,d\xi}{G(\xi)} + \frac{1}{2}\int_{C_6} \frac{H_0^{(2)}(\xi r)\,\xi\,d\xi}{G(\xi)}$$

$$= I_2^{(1)} + I_2^{(2)} + I_2^{(3)} + I_2^{(4)} \qquad (8\cdot83)$$

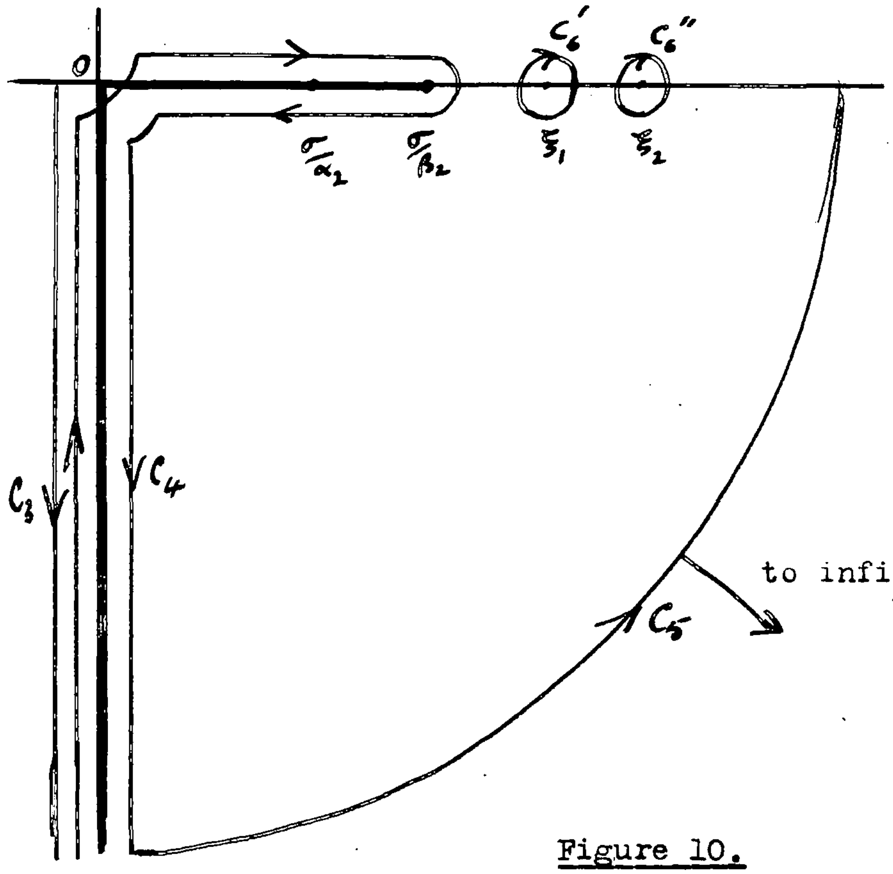


Figure 10.

According to Watson (Theory of Bessel Functions) :
when $|z|$ is large and $-\pi + \epsilon \leq \arg z \leq 2\pi - \epsilon$

$$H_0^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cdot e^{i(z - \frac{\pi}{4})} \quad ;$$

when $|z|$ is large and $-2\pi + \epsilon \leq \arg z \leq \pi - \epsilon$

$$H_0^{(2)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cdot e^{-i(z - \frac{\pi}{4})} \quad .$$

Hence when $|\xi|$ is large and

$$-\pi + \epsilon \leq \arg \xi \leq \pi - \epsilon$$

$$\left.\begin{aligned} \tfrac{1}{2} H_0^{(1)}(\xi r) &\sim \frac{\xi^{-\frac{1}{2}} \cdot e^{i(\xi r - \frac{\pi}{4})}}{(2\pi r)^{\frac{1}{2}}} \\ \tfrac{1}{2} H_0^{(2)}(\xi r) &\sim \frac{\xi^{-\frac{1}{2}} \cdot e^{-i(\xi r - \frac{\pi}{4})}}{(2\pi r)^{\frac{1}{2}}} \end{aligned}\right\} \qquad (8\cdot84)$$

The B.A. Mathematical Tables VI give the value of $J_0(25)$ as $0\cdot0962667833$.

The asymptotic series for $J_0(25)$ is

$0\cdot11338892\text{s}\,\text{Sin}\,25^c + 0\cdot11226159\,\text{Cos}\,25^c$

$= 0\cdot11338892\,\text{Sin}\,1432\cdot39448783^\circ + 0\cdot11226159\,\text{Cos}\,1432\cdot39448783^\circ$

$= 0\cdot11338892\,\text{Sin}(1440-7\cdot60551217)$

$\qquad + 0\cdot11226159\,\text{Cos}(1440-7\cdot60551217)$

$= 0\cdot11338892\,\text{Sin}\{8\pi-(7^\circ\ 36\cdot331')\}$

$\qquad 0\cdot11226159\,\text{Cos}\{8\pi-(7^\circ\ 36\cdot331')\}$

$= -0\cdot11338892\,\text{Sin}(7^\circ\ 36\cdot331') + 0\cdot11226159\,\text{Cos}(7^\circ\ 36\cdot331')$

$= -0\cdot015007 + 0\cdot111274$

$= 0\cdot096267.$

So that to six decimal places, the limits of working with seven figure tables, the asymptotic value of $J_0(25)$ agrees with the values of $J_0(25)$.

Now in chapter 10 it is shewn (see 10·4) that a likely value of ξ_1 is $5\cdot29125 \times 10^{-6}$, so that when $r = 2 \times 10^8$ cm. , $\xi_1 r = 10\cdot6 \times 10^2$.

Hence it is justifiable to use the asymptotic values of $H_0^{(1)}(\xi r)$ and $H_0^{(2)}(\xi r)$ in the evaluation of $I_1^{(2)}$ and $I_2^{(3)}$, $I_2^{(4)}$ respectively.

Using the asympto-tic value for $H_0^{(2)}(\xi r)$ the contribution to I_2 from the neighbourhood of the zeroes of G(ξ) is, by Cauchy's Residue Theorem, ($-2\pi i$) x residues of the integrand at ξ_m (m =1, 2, ---, N) That is

$$I_2^{(4)} = -2\pi i \cdot \frac{e^{i\frac{\pi}{4}}}{(2\pi r)^{1/2}} \cdot \sum_{m=1}^{N} \frac{1}{\frac{d}{d\xi}\left\{\xi^{-\frac{1}{2}} \cdot e^{i\xi r} \cdot G(\xi)\right\}_{\xi_m}} .$$

But $\frac{d}{d\xi}\left\{\xi^{-\frac{1}{2}} \cdot e^{i\xi r} \cdot G(\xi)\right\}$

$$= -\tfrac{1}{2}\,\xi^{-3/2} \cdot e^{i\xi r} \cdot G(\xi) + i r \xi^{-\frac{1}{2}} \cdot e^{i\xi r} \cdot G(\xi)$$

$$+ \xi^{-\frac{1}{2}} \cdot e^{i\xi r} \cdot \frac{dG}{d\xi} .$$

But $\left[G(\xi_m)\right]_{m=1,2,\dots N} = 0$

$$\therefore I_2^{(4)} = -\frac{2\pi i \cdot e^{i\frac{\pi}{4}}}{(2\pi r)^{1/2}} \cdot \sum_{m=1}^{N} \frac{\xi_m^{\frac{1}{2}} \cdot e^{-i\xi_m r}}{\left(\frac{dG}{d\xi}\right)_{\xi_m}} \qquad (8.85)$$

When ξ is large, as it is along C_2 and C_5,

$$G(\xi) = \frac{\beta_2^4}{\sigma^4}\cdot 4\xi^3\left[\left(1-\frac{\sigma^2}{\xi^2\beta_2^2}\right)\left(1+\frac{\sigma^2}{2\xi^2\alpha_2^2}\right)-\left(1-\frac{\sigma^2}{2\xi^2\beta_2^2}\right)\right] \times \operatorname{Cosh}\xi h\left(1-\frac{\sigma^2}{2c^2\xi^2}\right)$$

$$+\frac{\rho_1}{\rho_2}\cdot\frac{1}{\xi}\cdot\left(1+\frac{\sigma^2}{2\xi^2c^2}\right)\sinh\xi h\left(1-\frac{\sigma^2}{2c^2\xi^2}\right)$$

$$= \frac{\beta_2^4}{\sigma^4}\cdot 4\xi^3\left[1+\frac{\sigma^2}{2\xi^2\alpha_2^2}-\frac{\sigma^2}{\xi^2\beta_2^2}-1+\frac{\sigma^2}{2\xi^2\beta_2^2}\right] \times \operatorname{Cosh}\xi h\left(1-\frac{\sigma^2}{2c^2\xi^2}\right)$$

$$+\frac{\rho_1}{\rho_2}\cdot\frac{1}{\xi}\left(1+\frac{\sigma^2}{2c^2\xi^2}\right)\sinh\xi h\left(1-\frac{\sigma^2}{2c^2\xi^2}\right)$$

$$= \frac{\beta_2^4}{\sigma^2}\cdot 2\xi\left(\frac{1}{\alpha_2^2}-\frac{1}{\beta_2^2}\right)\operatorname{Cosh}\xi h\left(1-\frac{\sigma^2}{2c^2\xi^2}\right)$$

$$+\frac{\rho_1}{\rho_2\xi}\cdot\left(1+\frac{\sigma^2}{2c^2\xi^2}\right)\sinh\left(1-\frac{\sigma^2}{2c^2\xi^2}\right)$$

$$\simeq \frac{\beta_2^2}{\sigma^2}\cdot 2\xi\left(\frac{1}{\alpha_2^2}-\frac{1}{\beta_2^2}\right)\operatorname{Cosh}\xi h$$

$$\simeq \frac{\beta_2^2\left(\beta_2^2-\alpha_2^2\right)}{\sigma^2\alpha_2^2}\cdot\xi e^{\xi h}.$$

Hence by (8·81) and (8·84)

$$I_1^{②} \simeq \frac{e^{-\frac{i\pi}{4}} \cdot \sigma^2 \alpha_2^2}{\beta_2^2 (\beta_2^2 - \alpha_2^2) \cdot (2\pi r)^{\frac{1}{2}}} \int_{C_2} \xi^{-\frac{1}{2}} \cdot e^{i\xi r} \cdot e^{-\xi h} \cdot d\xi$$

Then putting $\xi = Re^{i\theta}$,

$$I_1^{②} \simeq - \frac{e^{-\frac{i\pi}{4}} \cdot \sigma^2 \alpha_2^2 R^{\frac{1}{2}}}{(2\pi r)^{\frac{1}{2}} \cdot \beta_2^2 (\beta_2^2 - \alpha_2^2)} \int_0^{\frac{\pi}{4}} e^{rR(i\cos\theta - \sin\theta) - hR(\cos\theta + i\sin\theta)} \cdot i e^{i\theta/2} \cdot d\theta$$

$$\therefore |I_1^{②}| < - \frac{\sigma^2 \alpha_2^2 R^{\frac{1}{2}}}{(2\pi r)^{\frac{1}{2}} \beta_2^2 (\beta_2^2 - \alpha_2^2)} \int_0^{\frac{\pi}{2}} e^{-rR\sin\theta - hR\cos\theta} \cdot d\theta$$

$$= - \frac{\sigma^2 \alpha_2^2 R^{\frac{1}{2}}}{(2\pi r)^{\frac{1}{2}} \beta_2^2 (\beta_2^2 - \alpha_2^2)} \int_0^{\frac{\pi}{2}} e^{-R\sin(\theta + \phi)} \cdot d\theta$$

where $\tan\phi = \frac{h}{r}$

$$< \frac{\sigma^2 \alpha_2^2 R^{\frac{1}{2}}}{(2\pi r)^{\frac{1}{2}} \beta_2^2 (\beta_2^2 - \alpha_2^2)} \int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}(\theta + \phi)} \cdot d\theta$$

$$= \frac{\sigma^2 \alpha_2^2 R^{\frac{1}{2}}}{(2\pi r)^{\frac{1}{2}} \beta_2^2 (\beta_2^2 - \alpha_2^2)} \cdot \frac{\pi}{2R} \left[e^{-\frac{2R}{\pi}(\theta + \phi)} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\sigma^2 \alpha_2^2 \pi}{2 (2\pi r)^{\frac{1}{2}} \cdot \beta_2^2 (\beta_2^2 - \alpha_2^2)} \cdot \frac{1}{R^{\frac{1}{2}}} \left[e^{-R - \frac{2R\phi}{\pi}} - e^{-\frac{2R\phi}{\pi}} \right]$$

$\rightarrow 0$ as $R \rightarrow \infty$.

Hence $I_1^{②} \to 0$ as $R \to \infty$. (8·86)

Similarly from (8·82) and (8·84)

$$I_2^{③} \simeq \frac{e^{i\frac{\pi}{4}} \cdot \sigma^2 \cdot \alpha_2^2}{(2\pi r)^{\frac{1}{2}} \cdot \beta_2^2 (\beta_2^2 - \alpha_2^2)} \int_{C_5} \xi^{-\frac{1}{2}} \cdot e^{-\xi h} \cdot e^{-i\xi r} \cdot d\xi$$

after putting $\xi = Re^{i\theta}$,

$$I_2^{③} \simeq \frac{e^{\frac{i\pi}{4}} \cdot \sigma^2 \alpha_2^2}{(2\pi r)^{\frac{1}{2}} \beta_2^2 (\beta_2^2 - \alpha_2^2)} \int_{\frac{3\pi}{2}}^{2\pi} \Big[R^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}i\theta} \cdot e^{-Rr(\cos\theta + i\sin\theta)}$$

$$\times e^{-Rh(\cos\theta + i\sin\theta)} \cdot iRe^{i\theta} \Big] d\theta$$

$$\therefore \left| I_2^{③} \right| < \frac{\sigma^2 \alpha_2^2}{(2\pi r)^{\frac{1}{2}} \beta_2^2 (\beta_2^2 - \alpha_2^2)} \int_{\frac{3\pi}{2}}^{2\pi} e^{rR\sin\theta - Rh\cos\theta} \cdot d\theta$$

$$= \frac{\sigma^2 \alpha_2^2}{(2\pi r)^{\frac{1}{2}} \beta_2^2 (\beta_2^2 - \alpha_2^2)} \int_{\frac{3\pi}{2}}^{2\pi} e^{R\sin(\theta - \phi)} d\theta$$

$$\left[\tan\phi = \frac{h}{r} \right]$$

$$< \frac{\sigma^2 \alpha_2^2}{(2\pi r)^{\frac{1}{2}} \beta_2^2 (\beta_2^2 - \alpha_2^2)} \cdot \frac{\pi}{2R} \left[e^{-\frac{2R}{\pi}(\phi - \theta)} \right]_{\frac{3\pi}{2}}^{2\pi}$$

$$= \frac{\sigma^2 \alpha_2^2 \pi}{2(2\pi r)^{\frac{1}{2}} \cdot \beta_2^2 (\beta_2^2 - \alpha_2^2)} \cdot \frac{1}{R^{\frac{1}{2}}} \left[e^{\left(-\frac{2R\phi}{\pi} + 4R\right)} - e^{\left(-\frac{2R\phi}{\pi} + 3R\right)} \right]$$

$\to 0$ as $R \to \infty$

Hence $I_2^{③} \to 0$ as $R \to \infty$. (8·87)

We may deform the contour C into C ; see figure 11.

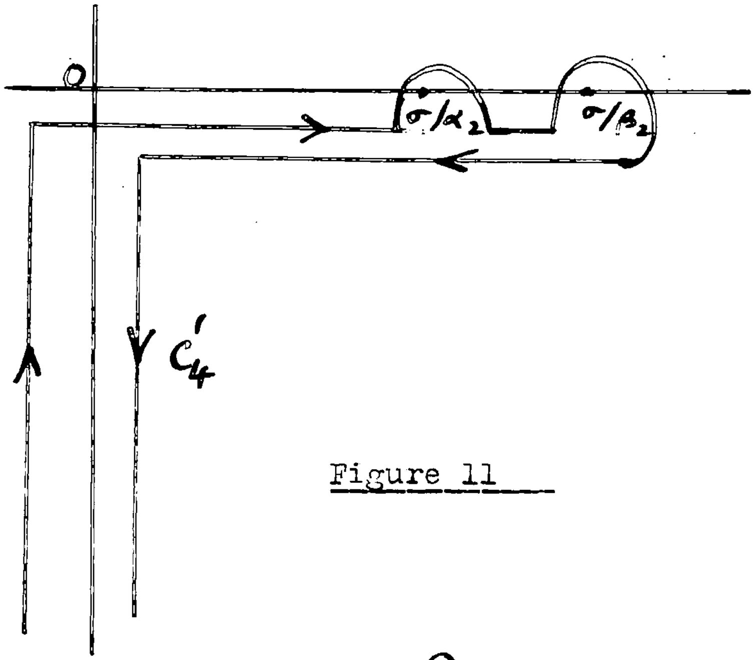


Figure 11

The contributions to $I_2^{(2)}$ from the straight portions of C_4' cancel each other and we are left with the contributions from approximately circular paths of small radius about

$\xi = \sigma/\alpha_2$ and $\xi = \sigma/\beta_2$.

With the values (10·3)

$$\frac{\sigma r}{\alpha_2} \simeq 2 \times 10^2 \quad ; \quad \frac{\sigma r}{\beta_2} \simeq 3.5 \times 10^2 .$$

Hence it is justifiable to use the asymptotic value of $H_0^{(2)}$ in the evaluation of $I_2^{(2)}$.

Hence $$I_2^{(2)} \simeq \frac{1}{2}\int_{\gamma_1} \frac{H_0^{(2)}(\xi r)\xi d\xi}{G(\xi)} + \frac{1}{2}\int_{\gamma_2} \frac{H_0^{(2)}(\xi r)\xi d\xi}{G(\xi)},$$

where γ_1 and γ_2 are circles of infinitesimal radius about the branch points σ/α_2 and σ/β_2 respectively. That is

$$I_2^{(2)} \simeq \frac{e^{i\frac{\pi}{4}}}{(2\pi r)^{\frac{1}{2}}}\int_{\gamma_1} \frac{\xi^{\frac{1}{2}} e^{-i\xi r} d\xi}{G(\xi)} + \frac{e^{\frac{i\pi}{4}}}{(2\pi r)^{\frac{1}{2}}}\int_{\gamma_2} \frac{\xi^{\frac{1}{2}} e^{-i\xi r} d\xi}{G(\xi)} \qquad (8.88)$$

Near the branch point $\frac{\sigma}{\alpha_2}$ we have $\xi = \frac{\sigma}{\alpha_2} + \rho e^{i\theta}$

and $G(\xi) \simeq \left(\frac{\beta_2}{\sigma}\right)^4 \cdot \left(\frac{2\rho\sigma e^{i\theta}}{\alpha_2}\right)^{-\frac{1}{2}} \cdot \frac{\sigma^4}{\alpha^4} \cdot \left(1 - \frac{\alpha_2^2}{2\beta_2^2}\right)^2$

$$= \frac{(2\beta_2^2 - \alpha_2^2)^2}{4(2\rho\sigma\alpha_2^7 e^{i\theta})^{1/2}} \cdot$$

Hence $\displaystyle\int_{\gamma_1} \frac{\xi^{\frac{1}{2}} e^{-i\xi r} d\xi}{G(\xi)}$

$$\simeq \frac{(2\beta_2^2 - \alpha_2^2)^2}{4(2\rho\sigma\alpha_2^7)^{1/2}} \int_0^{2\pi} e^{-\frac{i\theta}{2}} \cdot \left(\frac{\sigma}{\alpha_2} + \rho e^{i\theta}\right)^{\frac{1}{2}} \cdot e^{-ir\left(\frac{\sigma}{\alpha_2} + \rho e^{i\theta}\right)} \cdot i\rho e^{i\theta} d\theta$$

$\longrightarrow 0$ as $\rho \to 0$ and r is large. (8.89)

Near the branch point σ/β_2 we have $\xi = \frac{\sigma}{\beta_2} + \rho e^{i\theta}$

and $G(\xi) \simeq \left(\frac{\beta_2}{\sigma}\right)^4 \cdot \left[\frac{\sigma^4}{4\beta_2^4} \cdot \frac{\alpha^2\beta_2}{\sigma(\alpha_2^2 - \beta_2^2)^{\frac{1}{2}}} - \frac{\sigma^4}{\beta_2^4} \cdot \left(\frac{2\sigma\rho e^{i\theta}}{\beta_2}\right)^{\frac{1}{2}}\right]$

$$= \frac{\alpha_2\beta_2}{4\sigma(\alpha_2^2 - \beta_2^2)^{\frac{1}{2}}} \left[1 - \frac{4\sigma(\alpha_2^2 - \beta_2^2)^{\frac{1}{2}}}{\alpha_2\beta_2} \cdot \left(\frac{2\sigma}{\beta_2}\right)^{\frac{1}{2}} \rho^{\frac{1}{2}} e^{i\theta/2}\right]$$

$$\therefore \frac{1}{G(\xi)} \simeq \frac{4\sigma(\alpha_2^2 - \beta_2^2)^{\frac{1}{2}}}{\alpha_2\beta_2} \left[1 + \frac{4\sigma(\alpha_2^2 - \beta_2^2)^{\frac{1}{2}}}{\alpha_2\beta_2} \cdot \left(\frac{2\sigma}{\beta_2}\right)^{\frac{1}{2}} \rho^{\frac{1}{2}} \cdot e^{i\theta/2}\right]$$

Hence $$\int_{\gamma_2} \frac{\xi^{\frac{1}{2}} . e^{-i\xi r} \, d\xi}{G(\xi)}$$

$$\simeq \frac{4\sigma(\alpha_2^2-\beta_2^2)^{\frac{1}{2}}}{\alpha_2\beta_2} \int_0^{2\pi} \left[1 + \frac{4\sigma(\alpha_2^2-\beta_2^2)^{\frac{1}{2}}}{\alpha_2\beta_2} . \left(\frac{2\sigma}{\beta_2}\right)^{\frac{1}{2}} \rho^{\frac{1}{2}} e^{i\theta/2} \right]$$

$$\times \left(\frac{\sigma}{\beta_2} + \rho e^{i\theta} \right)^{\frac{1}{2}} . e^{-ir\left(\frac{\sigma}{\beta_2} + \rho e^{i\theta}\right)} . i\rho e^{i\theta} . d\theta$$

$\rightarrow 0$ as $\rho \rightarrow 0$ and r is large. (8•90)

Hence by equations (8•88), (8•89) and (8•90) we see that the contribution of $I_2^{(2)}$ is very small. After reference to (8•80), (8•81), (8•82), (8•83), (8•85), (8•86), (8•87), (8•88), (8•89) and (8•90) we see that the only appreciable part of the integral I comes from the contribution of the integrand at the poles, that is (8•85). So that

$$\int_0^{\infty} \frac{J_0(\xi r) \xi \, d\xi}{G(\xi)} = -\frac{2\pi i \, e^{i\frac{\pi}{4}}}{(2\pi r)^{\frac{1}{2}}} \sum_{m=1}^{N} \frac{\xi_m^{\frac{1}{2}} . e^{-i\xi_m r}}{\left(\frac{dG}{d\xi}\right)_{\xi_m}} \qquad (8•91).$$

Hence e uation (8•56) becomes :

$$W(\sigma, r) e^{i\sigma t} = \frac{i \, e^{i\left(\frac{\pi}{4} + \sigma t\right)}}{\rho\sigma^2 (2\pi r)^{\frac{1}{2}}} \sum_{m=1}^{N} \frac{\xi_m^{\frac{1}{2}} . e^{-i\xi_m r}}{\left(\frac{dG}{d\xi}\right)_{\xi_m}}$$

$\therefore\ W(\sigma, r)\, e^{i\sigma t}$

$$= \frac{\sigma^{\frac{1}{2}}}{\rho_2\, \beta_2^{5/2}\, (2\pi r)^{\frac{1}{2}}} \sum_{m=1}^{N} i.\left(\frac{\beta_2}{\sigma}\right)^{5/2} \cdot \frac{\xi_m^{\frac{1}{2}} \cdot e^{i(\sigma t - \xi r + \frac{\pi}{4})}}{\left(\frac{dG}{d\xi}\right)_{\xi_m}}$$

$$= \frac{\sigma^{\frac{1}{2}}}{\rho_2\, \beta_2^{5/2}\, (2\pi r)^{\frac{1}{2}}} \sum_{m=1}^{N} C_m \cdot e^{i\left\{\sigma t - \xi_m r + (m + \frac{1}{4})\pi\right\}} \qquad (8\cdot 92)$$

where $$C_m = (-)^m \cdot \left(\frac{\beta_2}{\sigma}\right)^{5/2} \cdot \frac{\xi_m^{\frac{1}{2}}}{\left(\frac{dG}{d\xi}\right)_{\xi_m}} \qquad (8\cdot 93).$$

Each term in equation (8•92) represents a diverging wave of length $2\pi/\xi_m$ amd amplitude proportional to C_m .

CHAPTER 9.

The displacement of the ground due to Ocean Waves.

In chapter 5 we have shewn that a general wave motion, on the surface of an incompressible liquid, in a square S ($-R < x < R$, $-R < y < R$) where $2R \gg \lambda_g$, produces a total force with a frequency twice that of the mean frequency of the surface motion at all depths. This force sets up seismic waves in the sea-bed.

Inchapter 7 we have shewn that if the compressibility of the water is taken into account, the pressure variation can be regarded as due to gravity waves in the surface layer which is regarded as compressible. We can consider then a variable force applied to the surface of an incompressible sea instead of a motion at the surface of a compressible sea and producing the same variable force at the bed.

Then in chapter 8 we have found an expression for the displacement of the sea bed at distance r due to a variable force applied to the surface of an incompressible sea.

Now since the wave lengths of the compression and seismic waves are comparable and $\lambda_g / \lambda_c \simeq 10^{-2}$, the square S can have a side very much greater than λ_g and yet be only a fraction of the length of a seismic wave.

We therefore divide a storm area into squares, such as S, and by considering the surface motion in each square as being equivalent to a suitable variable force at the centre of that square, make an estimate of the vertical displacement of the ground at a distant point due to each square. Summing the results for the whole storm area, we derive an estimate of the vertical displacement of a point distant from the storm area.

If microseisms recorded in Europe are generated by storms in the Atlantic, the estimate should accord with recorded measurements.

From chapter 8 we have that a variable force $e^{i\sigma t}$ applied to the surface of the sea above the origin produces a vertical displacement $W(\sigma, r)\, e^{i\sigma t}$ of the sea bed at a distance r from the origin. Hence a force

$$-R.4\rho\left(\frac{\pi}{k}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A'(u,v).\,A'(-u,-v)\,\sigma^2 e^{2i\sigma t}\,du\,dv \qquad (9\cdot 1)$$

applied to the surface will produse a vertical displacement, at distance r from the origin, of δ , where

$$\delta = -R.\,4\rho\left(\frac{\pi}{k}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A'(u,v).\,A'(-u,-v)\,\sigma^2\, W(2\sigma, r)\,e^{2i\sigma t}\,du\,dv \qquad (9\cdot 2)$$

But (9•1) is the total force at the bed due to the motion in a square S given by $-R < x < R$, $-R < y < R$; see equation (5•49).
Thus δ represents the vertical displacement of the ground at all points at a distance r from the centre of the square S.

The area of the square S is $4R^2 = \left(\frac{2\pi}{k}\right)^2$ after using equation (5•46). If E is the mean energy per unit area of the square S, then the total energy of S is $\left(\frac{2\pi}{k}\right)^2 E$.
Hence after reference to equation (5• 50) :

$$E = \frac{1}{2}\rho g\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A'(u,v).\,A'^{*}(u,v)\,du\,dv. \qquad (9\cdot 3)$$

We define the mean amplitude a of the motion within S as half the height, from trough to crest of the simple progressive wave train having the same mean energy per unit area.

By Lamb (§ 230) the mean energy of a progressive wave of amplitude a is $\frac{1}{2} g\rho a^2$,
hence $E = \frac{1}{2} g\rho a^2$.
That is, by equation (9.3)

$$a^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A'(u,v).\,A'^{*}(u,v)\,du\,dv. \qquad (9\cdot 4)$$

We have already postulated, in chapter 5, that for the motion to be wave-like, the energy of the motion must be confined to a narrow band of frequencies and directions characteristic of the group of waves. This range will be very nearly the same for the hypothetical spectrum A' as for the original spectrum A. Let Q denote the region and its area in which the point (-uk, -vk), defining the length and direction of the wave components of the group (see chapter 5) must lie. But this area is also $(2R)^2$,

hence $\frac{Q}{k^2} = (2R)^2 = \left(\frac{2\pi}{k}\right)^2$ by equation (5•46).

Let $\bar{A}$ denote the root mean square value of the modulus of the amplitude $A'(u,v)$, so that

$$\bar{A} = \left[\frac{1}{4R^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \left| A'(u,v) \right| \right\}^2 du\,dv \right]^{\frac{1}{2}}$$

or

$$\bar{A}^2 = \frac{k^2}{Q} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \left| A'(u,V) \right| \right\}^2 du\,dv$$

since $\mathcal{S}' = 0$, outside S,

$$= \frac{k^2}{Q} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A'(u,V)\, A'^{*}(u,v)\, du\,dv$$

$$= \frac{k^2}{Q} \cdot a^2 \qquad \text{by equation (9•4)}$$

so that $$\bar{A} = \frac{ka}{Q^{1/2}} . \qquad (9•5)$$

As we have seen in chapter 5 at least two separate wave groups are required to provide the opposite wave components needed to give rise to a pressure variation at the sea bed. We therefore suppose that the motion defined by $\mathfrak{J}'$ and A' comprises two distinct wave groups with spectra $A_1'(u,v)$ and $A_2'(u,v)$ Then $A'(u,v) = A_1'(u,v) + A_2'(u,v)$.

Let us suppose that the A_1' wave group has its energy in the region Q_1 ,which has an area Q_1 , and that the mean amplitude of this wave group in Q_1 is a_1 .
Let Q_2 and a_2 be the corresponding features of the A_2' wave group.

By analogy with equation (9•5) the root mean square values of the moduli of these two wave groups are $\overline{A_1}$ and $\overline{A_2}$, where $\overline{A_1} = \dfrac{ka_1}{Q_1^{1/2}}$, $\overline{A_2} = \dfrac{ka_2}{Q_2^{1/2}}$ (9•7).
Equation (9•2) may now be written

$$\delta = -R.\,4\rho\left(\frac{\pi}{k}\right)^2 \iint\limits_{Q_1+Q_2} \Big[\left\{A_1'(u,v) + A_2'(u,v)\right\} \times \left\{A_1'(-u,-v) + A_2'(-u,-v)\right\} \sigma^2 W(\varpi,r)\, e^{2i\sigma t}\Big]\, du\,dv \tag{9•8}$$

Since neither $A_1'(u,v)$ nor $A_2'(u,v)$ exist outside the region $Q_1 + Q_2$.

Since a wave does not possess opposite pairs of wave components, then either $A_1'(u,v)$ or $A_1'(-u,-v)$ is zero and either $A_2'(u,v)$ or $A_2'(-u,-v)$ is zero.
We suppose the two wave groups to be motion in opposite senses and set

$$A_1'(-u,-v) = 0 \quad \text{and} \quad A_2'(u,v) = 0 \; .$$

Then δ

$$= -R\, 4\rho\left(\frac{\pi}{k}\right)^2 \iint_{Q_{12}} A_1'(u,v)\, A_2'(-u,-v)\, \sigma^2\, W(2\sigma, r)\, e^{2i\sigma t}\, du\, dv$$

where Q_{12} denotes theregion common to Q_1 and $-Q_2$ since this common region is the only region in which opposite pairs of wave components can exist, and so the contribution to δ from other parts of $(Q_1 + Q_2)$ must be zero.
If we let σ_{12} denote the mean value of σ over Q_{12}, then since u,v, r and σ are independent of t we can write

$$\delta e^{-2i\sigma_{12}t} = -R \cdot 4\rho\left(\frac{\pi}{k}\right)^2 \iint_{Q_{12}} A_1'(u,v)\, A_2'(-u,-v)\, \sigma^2\, W(2\sigma, r)\, e^{2i(\sigma-\sigma_{12})t}\, du\, dv \qquad (9\cdot9)$$

It may be assumed that there is no correlation between the phases of the wave components of the true spectrum A(u,v) at different points of the (u,v) plane. Owing to the fact that neighbouring wave components of the original spectrum A(u,v) contribute to wave components of the new spectrum $A'(u,v)$ there may be some correlation for points which are close together in the (u,v) plane for the phases of the wave components of $A'(u,v)$. For points more than unit distance apart, equation (5·48) indicates that the correlation will be very slight, whereas for those closer it will be appreciable. So we may divide the region Q_{12} into Q_{12}/k^2 unit squares and carry out the integration of equation (9·9) over each square Separately. The result will be the sum of Q_{12}/k^2 vectors of random phase and each of order of magnitude of

$$8\rho\left(\frac{\pi}{k}\right)^2 \bar{A}_1(u,v)\ \bar{A}_2(u,v)\,\sigma_{12}^2\ W(2\sigma_{12},r)$$

So that the order of magnitude of the right hand member of equation (9.9) is

$$\left[\frac{Q_{12}}{k^2}\right]^{\frac{1}{2}} \cdot 8\rho\left(\frac{\pi}{k}\right)^2 \bar{A}_1(u,v)\ \bar{A}_2(u,v)\,\sigma_{12}^2\ W(2\sigma_{12},r)$$

since the sum of n vectors in random phase relation increases as $n^{\frac{1}{2}}$. Hence

$$\delta \doteq 8\rho\left(\frac{\pi}{k}\right)^2 \bar{A}_1(u,v)\ \bar{A}_2(u,v).\ \frac{Q_{12}^{\frac{1}{2}}}{k}.\sigma_{12}^2\ W(2\sigma_{12},r)e^{2i\sigma_{12}t} \qquad (9.11)$$

If the total storm area is Λ, then this may be divided into $\frac{\Lambda}{4R^2} = \frac{\Lambda k^2}{4\pi^2}$ squares like S.

Then the total displacements of the ground, Δ, from the whole storm area is of the order

$$\Delta \doteq \left(\frac{\Lambda k^2}{4\pi^2}\right)^{\frac{1}{2}} \cdot \delta$$

That is

$$\Delta \doteq 8\rho\left(\frac{\pi}{k}\right)^2 \bar{A}_1(u,v)\ \bar{A}_2(u,v)\ \frac{\Lambda^{\frac{1}{2}} Q_{12}^{\frac{1}{2}}}{2\pi}.W(2\sigma_{12},r)e^{2i\sigma_{12}t} \qquad (9.12)$$

Let $\overline{W}^2(\sigma,r)$ denote the sum of the squared moduli of the terms in the asymptotic expansion of $W(\sigma,r)\,e^{i\sigma t}$. Then

$$\overline{W}(\sigma,r) = \frac{\sigma^{\frac{1}{2}}}{\rho_2\,\beta_2^{5/2}\,(2\pi r)^{\frac{1}{2}}}\left[\sum_{m=1}^{N} c_m^2\right]^{\frac{1}{2}} \qquad (9.13)$$

Then to the same order of approximation

$$\dot{W}(2\sigma_{12}, r) \doteq \overline{W}(2\sigma_{12}, r).$$

Hence

$$\Delta \doteq \frac{4\pi\rho}{k^2} \cdot \bar{A}_1(u,v) \cdot \bar{A}_2(u,v) \cdot \sigma_{12}^2 \cdot (\Lambda\, Q_{12})^{\frac{1}{2}} \cdot \overline{W}(2\sigma_{12}, r) \cdot e^{2i\sigma_{12}t}.$$

But from equation (9·7)

$$\bar{A}_1(u,v) \cdot \bar{A}_2(u,v) = \frac{k^2 a_1 a_2}{(Q_1 Q_2)^{1/2}},$$ hence

$$\Delta \doteq 4\pi\rho\, a_1 a_2\, \sigma_{12}^2 \left(\frac{\Lambda\, Q_{12}}{Q_1 Q_2}\right)^{\frac{1}{2}} \cdot \overline{W}(2\sigma_{12}, r)\, e^{2i\sigma_{12}t} \quad (9·14)$$

We notice that the displacement Δ is periodic with a frequency twice the mean frequency of the generating wave groups and with an amplitude which depends on the product of mean amplitudes of the two generating wave groups and the square of the mean frequency. In this it bears a marked similarity to the mean pressure variation produced by the ~~mean pressure variation produced by the~~ interferen-ce of two wave trains travelling in opposite directions (see equation 4·5) as is to be expected. Further Δ is independent of the sizes of squares used for subdivision of the area Λ , but depends only on the area Λ of the generating region. It will be noticed that Δ increases with Q_{12} and decreases with $Q_1 Q_2$; so that the greater the area of interference the greater is the displacement; but the greater Q_1 and Q_2, that is, the more widely distributed is the energy of each spectrum the smaller is the resulting disturbance.

CHAPTER 10.

Practical Application of the results of chapter 9.

Chapter 9 indicates that a periodic vertical displacement of the ground will occur if two groups of waves of the same wave-lengths but travelling in opposite directions interfere, So that in order to explain the generation of microseisms by this theory it is necessary to look for conditions which will give rise to opposing wave groups of surface waves.

When Bernard (1941) suggested that microseisms were the consequence of standing waves he considered that suitable standing waves would be generated at the centre of a cyclonic depression or off a steep coast where there was interference between the incident and reflected waves.

Microseisms from a circular depression:

Since the lowest pressure in a cyclone is near the middle the winds necessarily blow inwards from all sides, but because they are deflected to the right (in the northern hemisphere) they do not blow directly towards the centre (Lake). At each point of an isobar there is a considerable component towards the centre of the depression. Observation suggests that when a wind blows steadily in a particular direction there is eventually generated a swell travelling more or less in the direction of the wind. So that the centre of a depression should be receiving swells from several radial directions. This would be the necessary condition for the generation of large standing waves, and may be the reason for the large pyramidal waves reported from centres of low pressure and low wind velocity.

Suppose then that in the centre of a circular depression in the Atlantic, wave energy is being received equally from all directions with a range of periods between 10 and 16 seconds.

The speed of propagation of waves in deep water V is given approximately by $V^2 = \frac{g\lambda}{2\pi}$ (Milne-Thomson 14·17) whilst the period T of the wave of length λ is given by $V = \lambda / T$

Hence approximately for waves in deep water $\lambda = \frac{gT^2}{2\pi}$

If λ_1 and λ_2 are the lengths of waves of periods 10 seconds and 16 seconds respectively,

$$\lambda_1 = \frac{981 \times 10^2}{2\pi} = 1{\cdot}54 \times 10^4 \text{ cm.}$$

$$\lambda_2 = \frac{981 \times 16^2}{2\pi} = 4{\cdot}00 \times 10^4 \text{ cm.}$$

Referring to chapter 5 and taking the centre of the circular depression as the origin we see that the energy of the frequency spectrum entering the area of the depression is contained between the two circles with centres at the origin and radii $\frac{2\pi}{\lambda_1}$ and $\frac{2\pi}{\lambda_2}$.

In the half of this annular region formed by drawing any common diameter of the two circles, the two wave groups will be moving in opposite directions. Such a region is the Q_1, the Q_2 and the region Q_{12} of equation (9·14).

Hence

$$Q_1 = Q_2 = Q_{12} = \frac{1}{2}\pi\left[\left(\frac{2\pi}{\lambda_1}\right)^2 - \left(\frac{2\pi}{\lambda_2}\right)^2\right]$$

$$= \frac{2\pi^3}{10^8}\left[\frac{1}{(1{\cdot}54)^2} - \frac{1}{4^2}\right]$$

$$= \underline{2{\cdot}16 \times 10^{-7} \text{ sq. cm.}}$$

Taking a storm area of 1000 sq. Km., a mean period of 13 secs. and a mean amplitude of 3 metres;

$\Lambda = 1000 \times 10^{10}$ sq. cm.; $\sigma_{12} = \frac{2\pi}{13}$ sec^{-1}; $a_1 = a_2 = 300$ cm (10·1)

Then the coefficient of $\overline{W}(2\sigma_{12}, r)\, e^{2i\sigma_{12}t}$ in equation (9·14) is

$$4\pi \times 300^2 \times \frac{4\pi^2}{169} \times \left(\frac{1000 \times 10^{10}}{2\cdot 16 \times 10^{-7}}\right)^{\frac{1}{2}}$$

$$= \underline{1\cdot 826 \times 10^{15} \text{ dynes.}}$$

Hence $|\Delta| \simeq 1\cdot 8 \times 10^5 \times \overline{W}(2\sigma_{12}, r)$ (10·2)

To evaluate $\overline{W}(2\sigma_{12}, r)$ we assume the values

$\rho_1 = 1\cdot 0$ gm/cm^3, $\rho_2 = 2\cdot 5$ gm/cm^3,

$c = 1\cdot 4 \times 10^5$ cm/sec, $\beta_2 = 2\cdot 8 \times 10^5$ cm/sec

$\alpha_2 = \sqrt{3} \times 2\cdot 8 \times 10^5$ cm/sec

(According to Poisson's Hypothesis -- Bullen § 4·12)

$h = 3$Km $= 3 \times 10^5$ cm, r = 2000 Km. $= 2 \times 10^8$ cm.

With these values $G(\xi)$ vanishes only once, and

$$\left.\begin{aligned} &G(\xi_1) = 0, \\ \text{when } \quad &\xi_1 = 5\cdot 29125 \times 10^{-6} \end{aligned}\right\} \quad (10\cdot 4)$$

Equation (9·13) becomes

$$\overline{W}(\sigma, r) = -\frac{\sigma^{\frac{1}{2}}}{\rho_2\, \beta_2^{5/2}\, (2\pi r)^{\frac{1}{2}}} \cdot C_1$$

$$= -\frac{\sigma^{\frac{1}{2}}}{\rho_2\, \beta_2^{5/2}\, (2\pi r)^{\frac{1}{2}}} \quad \frac{(\beta_2/\sigma)^{5/2} \cdot \xi_1^{\frac{1}{2}}}{\left[dG(\xi)/d\xi\right]_{\xi=\xi_1}}$$

From equation (8 47) :

$$\frac{d\,G(\xi)}{d\xi} = \left(\frac{\beta_2}{\sigma}\right)^4 \left[\left(2\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^2 \left(\xi^2 - \frac{\sigma^2}{\alpha_2^2}\right)^{-\frac{1}{2}} - 4\xi^2 \left(\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}}\right]$$

$$\times \left(\frac{\sigma^2}{c^2} - \xi^2\right)^{-\frac{1}{2}} \xi h .\ \sin\left(\frac{\sigma^2}{\alpha_1^2} - \xi^2\right)^{\frac{1}{2}} h$$

$$+ \left(\frac{\beta_2}{\sigma}\right)^4 \Big[-\left(2\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^2 \left(\xi^2 - \frac{\sigma^2}{\alpha_2^2}\right)^{-3/2} \xi$$

$$+ 8\xi \left(2\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)\left(\xi^2 - \frac{\sigma^2}{\alpha_2^2}\right)^{-\frac{1}{2}} - 4\xi^3 \left(\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^{-\frac{1}{2}}$$

$$- 8\xi \left(\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}} \Big] \cos\left(\frac{\sigma^2}{c^2} - \xi^2\right)^{\frac{1}{2}} h$$

$$-\left(\frac{\rho_1}{\rho_2}\right) \cdot \left(\frac{\sigma^2}{c^2} - \xi^2\right)^{-\frac{1}{2}} \xi h \left(\frac{\sigma^2}{c^2} - \xi^2\right)^{-\frac{1}{2}} \cos\left(\frac{\sigma^2}{c^2} - \xi^2\right)^{\frac{1}{2}} h$$

$$+\left(\frac{\rho_1}{\rho_2}\right) \cdot \xi .\ \left(\frac{\sigma^2}{c^2} - \xi^2\right)^{-3/2} \sin\left(\frac{\sigma^2}{\alpha_1^2} - \xi^2\right)^{\frac{1}{2}} h$$

Hence

$$\left[\frac{dG(\xi)}{d\xi}\right]_{\xi=\xi_1} = -1527.19 \times 10^8 \qquad (10.6)$$

Hence using (10 3), (10 4), (10 5) and (10 6) :

$$\bar{W}(2\sigma_a, r) = \frac{13^2}{2.5 \times (2\pi)^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 10^4 \times (4\pi)^2} \times \frac{(5.291)^{\frac{1}{2}} \times 10^{-3}}{1527 \times 10^8}$$

$$= \underline{1.82 \times 10^{-19} \text{ cm/dynes.}}$$

Substituting in equation (10 2)

$$|\Delta| \simeq 1{\cdot}8 \times 10^{5} \times 1{\cdot}82 \times 10^{-19}$$

$$= 3{\cdot}276 \times 10^{-4} \text{ cm.} \qquad (10.7)$$

Hence the amplitude from peak to trough of the vertical displacement of the ground is

$$6{\cdot}5 \times 10^{-4} \text{ cm.} = 6{\cdot}5\mu$$

at a distance of 2000 Km. from a wave in water of depth 3Km., and the period of the displacement is approximately 6·5 seconds.

Microseisms from Coastal Reflection.

When a wave group is incident on a steep coast some measure of reflection occurs and the reflected wave group will contain the same frequencies as the incident wave group and the necessary conditions for the generation of microseisms are realised. It has been demonstrated experimentally by Cooper and Longuet-Higgins (1950) that there is a sharp decline in the value of the coefficient of reflexion against a plane surface, when the plane is inclined at less than 45 degrees to the horizontal. At 15 degrees the coefficient of reflexion is less than 10% and the foremost edge of the incident wave is becoming turbullent. The coasts of Europe are anything but plane suffaces and the beaches are frequently shelving, so that a high degree of reflexion is not to be expected. Exactly how much energy is reflected is difficult to assess and we shall assume that the mean amplitude of the reflected wave is 5% of that of the incoming wave.

Let us suppose that a swell of mean amplitude 2 metres and period 12 to 16 seconds whose direction of propagation lies within an angle of 30 degrees is approaching a coast, so that the shore-line makes 10 degrees with the mean direction of the incoming waves.

The direction of the reflected wave is also spread over an angle of 30 degrees, so that only one third of the angle of the reflected waves overlaps that of the incoming waves. We assume that the effective shore-line is 600 Km. and that the reflected wave extends outwards a distance of 10 Km., this gives us a value 6000 sq. Km. for Λ. Normally the depth up to 10 Km. from the shore is negligible compared with that at the storm centre in the Atlantic, so we may take $k=0$. For the quantities in equation (9 14) we have the values :

$$\left.\begin{array}{llll} a_1 = 200 \text{ cm.}, & a_2 = 10\text{cm.}, & k = 0, & r = 1000 \text{ Km. (say)} \\ \rho = 1 \text{ gm / cm}^3, & \sigma_{12} = 14 \text{ sec}^{-1}, & \Lambda = 6000 \text{ sq. Km.} & \end{array}\right\} \quad (10\cdot8)$$

$$\lambda_1 = \frac{g.12^2}{2\pi}, \qquad \lambda_2 = \frac{g.16^2}{2\pi}$$

Then
$$Q_1 = \frac{3}{18}\cdot\frac{\pi}{2}\left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}\right)\cdot 4\pi^2$$

$$= \frac{4\pi^5}{3g^2}\left(\frac{1}{12^4} - \frac{1}{16^4}\right)$$

$$= 1\cdot396 \times 10^{-8} \text{ sq. cm.}$$

hence
$$\left.\begin{array}{l} Q_1 = Q_2 = 1\cdot396 \times 10^{-8} \\ \text{and } \quad Q_{12} = 0\cdot465 \times 10^{-8} \end{array}\right\} \quad (10\cdot9)$$

With the values (10·8) and (10·9) the ceefficient of

$\overline{W}(2\sigma_{12}, r)\, e^{2i\sigma_{12}t}$ in equation (9·14) is

$$4\pi \times 200 \times 10 \times \left(\frac{2\pi}{14}\right)^2 \cdot \left[\frac{6\times 10^3 \times 0\cdot47 \times 10^{-8}}{1\cdot4 \times 1\cdot4 \times 10^{-16}}\right]^{\frac{1}{2}}$$

$$= 1\cdot945 \times 10^{14} \text{ dynes.} \quad (10\cdot10)$$

When $k = 0$

$$G(\xi) = \left(\frac{\beta_2}{\sigma}\right)^4 \left[\left(2\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^2 \left(\xi^2 - \frac{\sigma^2}{\alpha_2^2}\right)^{-\frac{1}{2}} - 4\xi^2 \left(\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}} \right]$$

$$= 0,$$

when $\xi_1 = 3{\cdot}755 \times 10^{-6}$ (10•11)

also
$$\frac{dG}{d\xi} = \left(\frac{\beta_2}{\sigma}\right)^4 \xi \left[-\left(2\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^2 \left(\xi^2 - \frac{\sigma^2}{\alpha_2^2}\right)^{-3/2} \right.$$

$$\left. + 8\left(2\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)\left(\xi^2 - \frac{\sigma^2}{\alpha_2^2}\right)^{-\frac{1}{2}} - 4\xi^2\left(\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^{-\frac{1}{2}} - 8\left(\xi^2 - \frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}} \right]$$

$= - 4574 \times 10^8$ when ξ has the value of (10•11)

Hence

$$\overline{W}(2\sigma_{12}, r) = \frac{1}{\rho_2 \cdot (2\pi r)^{\frac{1}{2}} \cdot \sigma_{12}^2} \cdot \frac{\xi_1^{\frac{1}{2}}}{\left(\frac{dG}{d\xi}\right)_{\xi=\xi_1}}$$

$$= \frac{13^2}{2{\cdot}5 \times (2\pi)^{\frac{1}{2}} \times 10^4 \times (4\pi)^2} \times \frac{(3{\cdot}755)^{\frac{1}{2}} \times 10^{-3}}{4574 \times 10^8}$$

$= 8{\cdot}393 \times 10^{-20}$ cm./ dynes (10•12)

From (10•10) and (10•12)

$$|\Delta| \simeq 1{\cdot}945 \times 8{\cdot}393 \times 10^{-6}$$

$= 16{\cdot}32 \times 10^{-6}$ cm.

$= 0{\cdot}1632\ \mu$

$\therefore\ 2|\Delta| \simeq 0{\cdot}33$ micron (10•13)

So that the reflexion of a wave group, of mean amplitude 2 metres, will produce microseisms of amplitude 0·33 micron.

The effect of Resonance.

It has been shewn in chapter 7 that we may expect resonance at certain depths.

The asymptotic expansion of $W(\sigma, r)\, e^{i\sigma t}$

namely

$$(-)^m \cdot \frac{1}{\rho_2 \sigma^2 (2\pi r)^{\frac{1}{2}}} \sum_{m=1}^{N} \frac{\xi_m^{\frac{1}{2}}}{\left(\frac{dG}{d\xi}\right)_{\xi=\xi_m}} \cdot e^{i\left[\sigma t - \xi_m r + (m+\frac{1}{2})\pi\right]}$$

represents a set of waves of length $2\pi/\xi_m$ and amplitude proportional to $\xi_m^{1/2} \Big/ \left(\frac{dG}{d\xi}\right)_{\xi=\xi_m}$.

By giving ξ particular values and solving the equation $G(\xi) = 0$ the following table is obtained. There will be no real values of h for $\xi <$ 3·452.

It will be noticed that there are several values h_1, h_2, h_3, h_4, etc. at which particular values of ξ satisfy $G(\xi) = 0$; these correspond to the roots of the equivalent equation

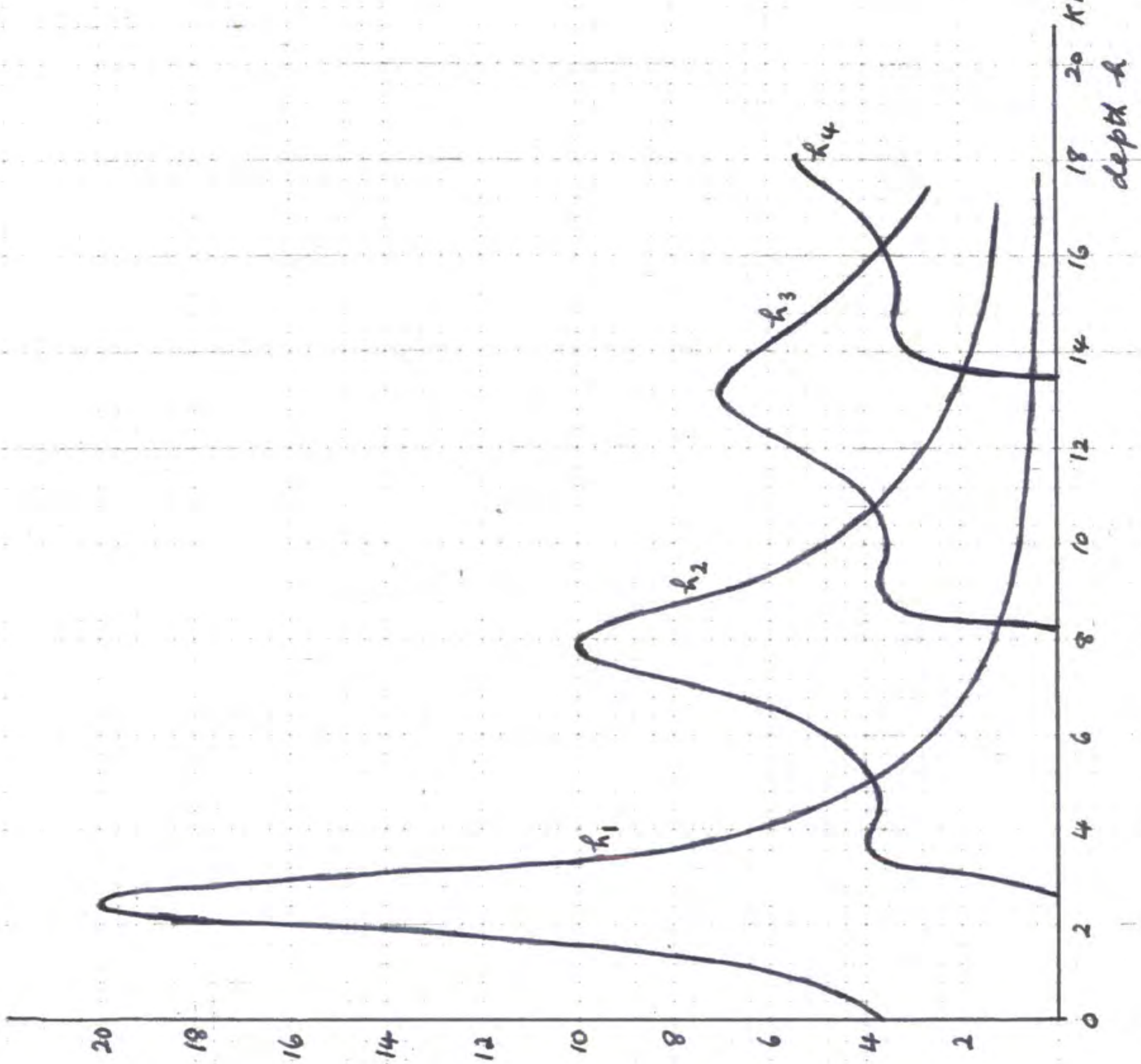
$$\tan\left(\frac{\sigma^2}{c^2}-\xi^2\right)^{\frac{1}{2}} h = \frac{\left(\frac{\beta_2}{\sigma}\right)^4 \left[4\xi^2\left(\xi^2-\frac{\sigma^2}{\beta_2^2}\right)^{\frac{1}{2}} - \left(2\xi^2-\frac{\sigma^2}{\beta_2^2}\right)^2\left(\xi^2-\frac{\sigma^2}{\alpha_2^2}\right)^{-\frac{1}{2}}\right]}{\rho_1/\rho_2 \cdot \left(\frac{\sigma^2}{c^2}-\xi^2\right)^{-1/2}}$$

The values of $\left(\xi^{\frac{1}{2}} \Big/ \frac{dG}{d\xi}\right)_{\xi_1}$ correspond to $h = h_1$, etc.

It is observed that $\left(\xi^{\frac{1}{2}} \Big/ \frac{dG}{d\xi}\right)_{\xi_1}$ rises to a maximum at $h_1 =$ 2·462 Km; $\left(\xi^{1/2} \Big/ \frac{dG}{d\xi}\right)_{\xi_2}$ has a maximum at $h_2 =$ 7·281 Km; these and the other maxima must correspond to resonance. So that we may expect [redacted] with a standing wave group of mean period $2\pi/13$ seconds to find resonance occurring when the depth is 2·462 Km, 7·821 Km., 13·12 Km., etc.

$\xi \times 10^6$	h_1 Kms.	h_2 Kms.	h_3 Kms.	h_4 Kms.	$\left(\xi^{\frac{1}{2}} / \frac{dG}{d\xi}\right)_{\xi_1} \times 10^{15}$	$\left(\xi^{\frac{1}{2}} / \frac{dG}{d\xi}\right)_{\xi_2} \times 10^{15}$	$\left(\xi^{\frac{1}{2}} / \frac{dG}{d\xi}\right)_{\xi_3} \times 10^{15}$	$\left(\xi^{\frac{1}{2}} / \frac{dG}{d\xi}\right)_{\xi_4} \times 10^{15}$
3.452		2.925	8.198	13.47				0.686
3.46		3.012	8.285	13.53		1.683	1.55	1.439
3.469		3.070	8.343	13.58		2.391	2.17	1.993
3.486		3.157	8.429	13.71		3.122	2.79	2.546
3.493		3.302	8.604	13.90		3.582	3.344	3.056
3.530		3.476	8.808	14.10		3.941	3.609	3.321
3.586		3.708	9.01	14.34		3.986	3.676	3.432
3.621		3.940	9.298	14.63		3.917	3.654	3.453
3.653		4.230	9.616	14.97		3.830	3.609	3.432
3.685		4.577	9.965	15.36		3.763	3.587	3.410
3.718		4.982	10.37	15.76		3.806	3.631	3.476
3.750		5.388	10.8	16.22		3.982	3.763	3.631
3.781	0.2897	5.736	11.18	16.60	4.562	4.295	4.074	3.875

3.936	1.390	6.923	12.46	17.99	8.148	7.041	6.200	5.535
4.084	1.825	7.473	13.12		12.51	9.255	7.338	
4.228	2.085	7.821	13.58		16.12	10.05	7.307	
4.367	2.230	8.112	13.99		18.53	9.919	6.753	
4.501	2.375	8.371	14.17		19.79	9.322	6.087	
4.631	2.462	8.604	14.74		20.10	8.547	5.425	
4.757	2.548	8.863	15.12		19.71	7.771	4.824	
4.881	2.665	9.095	15.52		18.98	6.997	4.295	
5.121	2.867	9.645	16.43		16.81	5.667	3.410	
5.347	3.070	10.25	17.44		14.33	4.561	2.724	
5.565	3.266	10.97	18.47		12.00	3.654		
5.776	3.534	11.85	20.17		9.831	2.900		
5.978	3.853	12.95	22.05		7.859	2.237		
6.176	4.287	14.45	24.61		6.110	1.683		



20
18
16
14
12
10
8
6
4
2
0
2
4
6
8
10
12
14
16
18
20 Km.
depth h
h_1
h_2
h_3
h_4

Examination of the table shows that there is a very steep rise to and decline from the maximum values of the amplitude. That these do not become infinite as chapter 7 indicated must be due to the fact that energy is being continuously removed from the region of the disturbance. The graph shews how the amplitudes of the different wave components of the displacement $W(\sigma, r)\,e^{i\sigma t}$ come to a maximum at different depths. The amplitudes of successive wave components are of opposite sign. Remembering ~~Rembering~~ this fact a quantity proportional to the displacement may be obtained by adding algebraically the ordinates for any value of h. The effect of resonance is to increase the amplitude of the displacement at certain depths by some multiple of the average value. This factor may be as much as five for certain depths.

CHAPTER 11.

CONCLUSION.

Miche has indicated that a pressure fluctuation independent of depth is not produced by a swell (houle) but by a choppy sea (clapotis). This phenomenon is wide-spread and occurs whenever the motion of the sea comprises a frequency spectrum in which there are wave groups of similar characteristics moving in opposite directions; these opposite wave groups produce a generalised choppiness which is shewn by the tumultuous waves noted near the centre of a depression. Opposite wave groups produce a fluctuation of pressure with a mean frequency twice that of the mean frequency of the wave groups; this pressure variation is not attenuated with depth and will produce a periodic displacement of the ground with a frequency equal to that of the pressure variation.

The required opposite wave-groups occur in a region of depression and in coastal waters. The existence of more opposite wave groups in a circular depression than in coastal waters will give rise to microseisms of greater amplitude from interference in mid-ocean than near coasts. Owing to the damping of the higher frequencies by the viscosity a greater proportion of the energy will be carried by the lower frequency components near coasts than near storm centres, so the coastal microseisms are likely to be of smaller amplitude and lower mean frequency than the deep water microseisms. In both cases the mean frequency of the microseisms will be twice that of the mean frequency of the generating waves. Since resonance between the compression waves and the sea-bed will occur at certain depths, there will be microseisms of unusually large amplitudes from certain ocean depths.

This will account for the change in amplitudes noticed as a depression moves and also for the fact that depressions of equal intensity but of different location do not produce equal microseismic activity.

I should like to express my thanks to Mr. E. F. Baxter for suggesting the subject of this thesis and for his encouragement and supervision.

This thesis is not an account of an original investigation, but a synthesis of detailed treatments of several original papers.

M. Miche in a paper "Mouvements Ondulaires de la Mer en profondeur constante et décroissante" published in four parts in the Annales des Ponts et Chaussées (1944) discusses several problems. Chapter 2 in this thesis is a detailed presentation of the relevant parts of Miche's work on the existence of a second order pressure variation under a standing wave. Miche's work is very contracted and parts consist of statements of results. Miche's notation has been maintained and the missing steps in his mathematical treatment have been provided. Thus the values of the functions G_2 and $f_2(t)$ have been determined by considering the boundary conditions and the associated differential equations, whereas Miche is content to state a value for G_2 (his equation 66), to give no value for $f_2(t)$ and to state in his equation (67), our equations (2·38), (2·39) and (2·40), (values of the coordinates and pressure.)

Chapters 3-7, 9 and 10 are a detailed treatment of the paper in the Phil. Trans. Roy., Vol. 243, No. 857 September 1950 by M.S.Longuet-Higgins. This work is also very contracted and the details have been provided.

Longuet-Higgins in equation 178, (our equation (8·56)) states a result given by J.G.Scholte in a paper "Over het verband tussen zeegolven en microseismen" (Nederlandsche Akademie van Wetenschappen Vol. LII, 1943). Scholte starts his paper by assuming an equation given by K.Sezawa ("On the transmission of Seismic Waves on the Bottom Surface of an Ocean"-- Bulletin of the Earthquake Research Institute, Tokio Imperial University, Vol. IX,1931). Chapter 8 gives a full determination of Longuet-Higgins' equation 178 and the evaluation of the integral which he has stated in equations 183 and 184.

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