Operational methods of solving ordinary and partial differential equations

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Operational Methods of Solving Ordinary and Partial Differential Equations.

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CONTENTS.

Introduction.

Newton, in 1676, solved a differential equation by the use of an infinite series; and in the year 1693 when he finally published his results, a differential equation occurred for the first time in the work of Leibnitz.

In the years following progress was very rapid and various methods of solution became known. The main features of this advancement being:

(i) 1694-7 John Bernoulli introduced the method of "Seperation of Variables" and showed how to reduce a homogeneous differential equation of the first order.

(ii) 1734 Euler and Clairut discovered independently the use of "Integrating Factors", and at about the same time Clairut first made a study of "Singular Solutions". The latter theory in its present form, however, is due mainly to the work of Cayley (1872) and M.J.M.Hill (1888).

(iii) Differential equations of second and higher orders with constant coefficients were first discussed by Euler, followed by Lobatto (1837) and Boole (1854).

The first partial differential equation to be noticed was that giving the form of a vibrating string. It was initially studied by Euler and D'Alembert in 1747, the work being finally completed by Lagrange.

From 1800 onwards the study of differential equations was transformed and became closely allied to the "Theory of Functions". Cauchy (1823) proved that the infinite series obtained from a differential equation was convergent and thus really did define a function satisfying the equation. Cauchy's investigations were extended and new methods introduced including the operational method first expounded by Heaviside (1850-1925).

Heaviside's method was mainly intuitive and it was much later before it was finally put on a rigorous mathematical basis and became known as the Laplace Transformation. Since then various operational methods have been used to solve many different types of differential equations. We intend , in the following pages, to outline and compare the different methods of solving both ordinary and partial differential equations.

In the first part of the dissertation we propose to study the solution of ordinary differential equations. It is clear that we cannot possibly enumerate all the known methods of solution, but the few given will be used as a comparison with the more "modern" methods.

The second part of the work will be a discussion of partial differential equations and the operational methods used, namely; Laplace, Hankel, Fourier and Legendre transforms. In this section we intend to study the Laplace Transformation in some detail.

The third and final section will be devoted to a comparison of the above methods and the different types of equations for which they are most suitable.

1.00. Ordinary Differential Equations.

An ordinary differential equation is an equation involving only one independent variable. The solution of an equation of this type containing the full number of arbitrary constants is called the Complete Primitive. Any solution derived from this with particular values to the arbitrary constants is known as a Particular Integral.

e.g. $y = x$ is an obvious Particular Integral (P.I.) of

$$2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 5 + 2x$$

Suppose we now consider the general case and let $y = w$ be a P.I. of

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = f(x) \qquad (1)$$

hence

$$a_0\frac{d^nw}{dx^n} + a_1\frac{d^{n-1}w}{dx^{n-1}} + \cdots + a_{n-1}\frac{dw}{dx} + a_n w = f(x) \qquad (2)$$

If we put $y = w + z$ in equation (1) and subtract (2) from it we obtain

$$a_0\frac{d^nz}{dx^n} + a_1\frac{d^{n-1}z}{dx^{n-1}} + \cdots + a_{n-1}\frac{dz}{dx} + a_n z = 0 \qquad (3)$$

Let the solution of (3) be $z = \phi(x)$ containing n arbitrary constants. The general solution of (1) is then

$y = w + \phi(x)$ where $\phi(x)$ is known as the Complementary Function (C.F.).

Thus the general solution of a linear differential equation with constant coefficients is the sum of the P.I. and the C.F.; the latter being the solution of the equation with the right hand side equated to zero.

The Operator 'D'.

This operator stands for $\frac{d}{dx}$, and is a useful symbol in writing down differential equations.

D^2 is used for $\frac{d^2}{dx^2}$, D^3 for $\frac{d^3}{dx^3}$, etc.

Thus the equation

$$2\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 3y = 0 \qquad (4)$$

may be written $2D^2y + 7Dy + 3y = 0$

or $(2D^2 + 7D + 3)y = 0.$ (5)

It can be quite easily shown that the operator obeys the fundamental laws of algebra except that it is not commutative with variables.

i.e. $D(vu) = v(Du)$ if v' is a constant but not if a variable.

It can also be shown that if

$$F(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$$

then (i) $F(D)e^{ax} = e^{ax}F(a)$.

(ii) $F(D)\ e^{ax}.V = e^{ax}F(D+a)V$ where V is a function of x.

(iii) $F(D^2)\cos ax = F(-a^2)\cos ax$, similarly for sin ax.

In the following work a knowledge of the usual operator theory is assumed.

1.10. First Method of Solution.

Given any linear differential equation with constant coefficients, we can find the general solution by obtaining a P.I. and the C.F.

We must now consider special cases of the general equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = f(x)$$

1.11. f(x) an exponential.

e.g. $$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = e^{3x} \qquad (1)$$

or $$(D-2)(D-4)y = e^{3x} \qquad (2)$$

Assuming that the R.H.S. of the equation is zero, the C.F. is obviously $Ae^{2x} + Be^{4x}$.

Eliminating the R.H.S. we obtain

$$(D-2)(D-4)(D-3)y = 0$$

i.e. $\therefore$ $y = Ae^{2x} + Be^{4x} + Ce^{3x}$ where the first two terms are the C.F. If we neglect these and put $y = Ce^{3x}$ in the original equation, we obtain a value for C.

Thus $C = -1$.

Hence a P.I. of the equation is $-e^{3x}$, and therefore

$y = Ae^{2x} + Be^{4x} - e^{3x}$ is the general solution.

n.b. If the R.H.S. is an exponential e^{mx} we assume that $y = Ce^{mx}$ immediately. If m happened to be a root of F(D) then we assume that $y = Cxe^{mx}$ to find the P.I.

e.g. if $(D-3)(D+1)y = e^{3x}$

then $y = Ae^{3x} + Be^{-x} + Cxe^{3x}$ where C is found as above.

1.12. <u>f(x) a mixture of exponentials and polynomials.</u>

e.g. $$(D^2 - 6D + 8)y = (x^2+1)e^{3x} \qquad (3)$$

As before, the C.F. is $Ae^{2x} + Be^{4x}$; put $y = ve^{3x}$,

$$\therefore \quad (D^2 - 6D + 8)ve^{3x} = (x^2+1)e^{3x}$$

giving $$(\overline{D+3}^2 - 6\,\overline{D+3} + 8)v = (x^2+1) \qquad (4)$$

i.e. $D^3(D^2 - 1)v = 0$ after eliminating the R.H.S.

$$\therefore \quad v = Ae^{x} + Be^{-x} + a + bx + cx^2 \qquad (5)$$

The first two terms are the C.F. and hence we assume that

$$v = a + bx + cx^2$$

Substitute for <u>v</u> in (4) obtaining

$$2c - a - bx - cx^2 = x^2 + 1$$

Hence $c = -1$, $b = 0$, and $a = -3$

$\therefore \quad v = -(x^2+3)$ and the P.I. is

$y = -e^{3x}(x^2+3)$ giving us the general solution

$$y = Ae^{2x} + Be^{4x} - e^{3x}(x^2+3)$$

1.13. <u>f(x) contains sine and cosine functions.</u>

e.g. $$(D^2 - 6D + 8)y = 3\sin 5x \qquad (6)$$

The C.F. is obviously $Ae^{2x} + Be^{4x}$

To eliminate the R.H.S. of the equation we differentiate twice with respect to <u>x</u>, thus

$$D^2(D-4)(D-2)y = -25.3\sin 5x$$

i.e. $$(D^2+25)(D-2)(D-4)y = 0 \qquad (7)$$

Hence $y = Ae^{2x} + Be^{4x} + L\sin 5x + M\cos 5x$ where the first two terms comprise the C.F. We therefore assume that

$y = L\sin 5x + M\cos 5x$ and substitute in (6)

Omitting the algebra, this gives

$$M = \frac{90}{1189} \quad \text{and} \quad L = \frac{51}{1189}$$

Hence the P.I. is $\frac{3}{1189}(17\sin 5x + 30\cos 5x)$

and the general solution

$$y = Ae^{2x} + Be^{4x} + \frac{3}{1189}(30\cos 5x + 17\sin 5x)$$

1.14. f(x) a mixture of exponentials, polynomials, and sine or cosine functions.

$$\text{e.g.} \quad (D^2 - 6D + 25)y = 2e^{3x}\cos 4x + 8e^{3x}(1-2x)\sin 4x \qquad (8)$$

$$\text{i.e.} \quad (\overline{D-3}^2 + 16)y = 2e^{3x}\cos 4x + 8e^{3x}(1-2x)\sin 4x$$

The C.F. is $e^{3x}(A\cos 4x + B\sin 4x)$ and if we put $y = ve^{3x}$, we can eliminate e^{3x} and obtain

$$(D^2 + 16)v = 2\cos 4x + 8(1-2x)\sin 4x \qquad (9)$$

To eliminate the R.H.S. we note that the polynomial is of the first degree, and hence one application of the method of 1.13 will reduce (9) to

$$(D^2 + 16)^2 v = 2.4.8.-2.\cos 4x$$

and hence to

$$(D^2 + 16)^3 v = 0$$

$$\therefore \quad v = (a + a_1x + a_2x^2)\cos 4x + (b + b_1x + b_2x^2)\sin 4x$$

where $a\cos 4x + b\sin 4x$ are the terms of the C.F.

Substitute the rest of the expression for y in (9) to give us the values $a_1 = -1$, $b_1 = 0$, $b_2 = 0$, $a_2 = 1$.

Therefore $v = (a - x + x^2)\cos 4x + b\sin 4x$ and the general solution is

$$y = e^{3x}\left[A\cos 4x + B\sin 4x + (x^2 - x)\cos 4x\right]$$

The few examples given in the above section show us the main steps in the method. They are:

(i) Assume that the right-hand-side of the equation is zero and thus write down the Complementary Function.

(ii) Reduce the R.H.S. of the equation to zero by either differentiation or by substitution or by a combination of both.

(iii) Write down the solution of the equation so obtained.

(iv) From the latter solution neglect the terms which correspond to the C.F. and substitute the remainder in the original equation thus obtaining values for the constants, that is, a particular integral

(v) Having thus obtained the C.F. and a P.I. the general solution is written down as a sum of the above.

1.20. Second method of Solution.

Im this section we agaim use the operator 'D' and we note that:

(i) $F(D)e^{ax} = e^{ax}F(a)$.

(ii) $F(D)e^{ax}.V = e^{ax}.F(D+a).V$

(iii) $F(D^2)\cos ax = F(-a^2)\cos ax$. (similarly for sin ax)

We now develop the idea of treating the operator 'D' as if it were an ordinary algebraic quantity, and we make the further suggestion that if 'D' is equivalent to the process of differentiation with respect to a variable then '1/D' is the equivalent of integration. No attempt is made to justify the above assumptionthough the solutions found by its use are easily verified. The following method is covered very adequately by Piaggio in his treatise on differential equations[1].

Suppose we consider the general equation

$$F(D)y = f(x)$$

and use the motation $\frac{1}{F(D)}f(x)$ to denote a P.I. of that equation.

1.21. <u>f(x) an exponential</u> i.e. e^{ax} say.

There are two cases to consider:

(i) <u>$F(a) \neq 0$.</u>

Since $F(D)e^{ax} = e^{ax}F(a)$ it suggests that $e^{ax}\cdot\frac{1}{F(a)}$ may be a value of $\frac{1}{F(D)}\cdot e^{ax}$ i.e. a P.I.

To verify this, we see that

$$F(D)\left\{e^{ax}\cdot\frac{1}{F(a)}\right\} = e^{ax}\cdot\frac{F(a)}{F(a)} = e^{ax}$$

[illegible] Equations - H.T.H. Piaggio (1920)

(ii) F(a) = 0. i.e. (D-a) is a factor of F(D).

We assume that

$$F(D) = (D-a)^r \phi(D) \quad \text{where } \phi(a) \neq 0$$

then $$\frac{1}{F(D)} \cdot e^{ax} = \frac{1}{(D-a)^r \phi(D)} \cdot e^{ax} = \frac{1}{(D-a)^r} \cdot \frac{e^{ax}}{\phi(a)}$$

$$= \frac{e^{ax}}{\phi(a)} \cdot \frac{1}{D^r} \cdot 1$$

$$= \frac{e^{ax}}{\phi(a)} \cdot \frac{x^r}{\lfloor r}$$

assuming that $\frac{1}{D^r}$ is equivalent to integrating r times.

The verification is once again quite straightforward.

e.g. $$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = e^{3x} \qquad \text{(see P.3.)}$$

or $$(D-2)(D-4)y = e^{3x}.$$

The P.I. is $$\frac{1}{(D-2)(D-4)} \cdot e^{3x} = e^{3x} \cdot \frac{1}{1.-1.} = -e^{-3x}$$

Hence the complete solution $y = Ae^{2x} + Be^{4x} - e^{-3x}$

e.g. $$(D-2)^2 y = 50e^{2x}$$

The P.I. is $$\frac{1}{(D-2)^2} \cdot 50e^{2x} = 50 . e^{2x} \cdot \frac{1}{D^2} \cdot 1.$$

$$= 25x^2 e^{2x}$$

and the general solution $y = (A + Bx)e^{2x} + 25x^2 e^{2x}$

1.22. <u>$f(x) = \cos ax$.</u>

Since $F(D)\cos ax = F(-a^2)\cos ax$ we are led to suggest that the P.I. may be obtained by writing $-a^2$ for D^2 wherever it occurs. If an odd power of 'D' occurs in $F(D)$ we can eliminate it quite easily as follows:

$$\frac{1}{D^3+3} = \frac{D^3-3}{D^6-9}$$ giving us an even power in the denominator.

e.g. $(D^2-6D+8)y = 3\sin 5x$ (see P.4.)

The P.I. is $$\frac{1}{(D^2-6D+8)}\,.3\sin 5x = 3\cdot\frac{1}{-6D-17}\cdot\sin 5x$$

$$= -3.\frac{(6D+17)}{36D^2-289}\cdot\sin 5x$$

$$= +3.\frac{(6D+17)}{1189}\cdot\sin 5x = \frac{3}{1189}\left(30\cos 5x + 17\sin 5x\right)$$

Hence the general solution is

$$y = Ae^{2x} + Be^{4x} + \frac{3}{1189}(30\cos 5x + 17\sin 5x)$$

1.23. <u>$f(x)$ a polynomial.</u>

The suggested method[1] is to expand $\frac{1}{F(D)}$ as a power series in the operator 'D'.

e.g. $(D^2-3D+2)y = x^2$

The P.I. is $$\frac{1}{(D-2)(D-1)}\cdot x^2 = \left[\frac{1}{1-D} - \frac{1}{2-D}\right]\cdot x^2$$

1. The verification is rather long. — see Piaggio pp. 37-38

$$= \left(\tfrac{1}{2} + \tfrac{3}{4}D + \tfrac{7}{8}D^2 + \tfrac{15}{16}D^3 + \cdots\right)x^2$$

$$= \tfrac{1}{2}x^2 + \tfrac{3}{2}x + \tfrac{7}{4}$$ and the general

solution is $$y = Ae^x + Be^{2x} + \tfrac{1}{2}x^2 + \tfrac{3}{2}x + \tfrac{7}{4}$$

1.24. f(x) = sin ax where $F(D^2) = 0$ when $D^2 = -a^2$

e.g. $$(D^2 + 9)y = \sin 3x$$

The method used in 1.22. is obviously useless here. We note however that sin 3x is the complex part of e^{3ix}.

Hence $$\frac{1}{D^2+9} \cdot e^{3ix} = \frac{1}{D-3i}\left\{\frac{1}{D+3i} \cdot e^{3ix}\right\}$$

$$= \frac{1}{6i} \cdot \frac{1}{D-3i} \cdot e^{3ix}$$

$$= \frac{1}{6i} \cdot e^{3ix} \cdot \frac{1}{D} \cdot 1. = \frac{x}{6i} \cdot e^{3ix}$$

$$= -\frac{x}{6} i(\cos 3x + i.\sin 3x)$$

The P.I. is therefore $-\frac{x}{6}\cos 3x$

and the general solution $$y = A\sin 3x + B\cos 3x - \frac{x}{6}\cos 3x.$$

1.25. f(x) a mixture of above. (see P.5.)

$$(D^2 - 6D + 25)y = 2e^{3x}\cos 4x + 8(1 - 2x)e^{3x}\sin 4x$$

The P.I. is $$\frac{1}{\{\overline{D-3}^2 + 16\}} \cdot \left[2e^{3x}\cos 4x + 8(1 - 2x)e^{3x}\sin 4x\right]$$

The first part immediately gives us $\tfrac{1}{4}x \sin 4x.e^{3x}$

The second part reduces to $$e^{3x} \cdot \frac{1}{D^2+16}(1-2x)\sin 4x$$

$$= I\left[e^{3x} \cdot \frac{1}{(D+4i)(D-4i)} \cdot e^{4ix} \cdot (1-2x)\right]$$

$$= I\left[e^{3x} \cdot e^{4ix} \cdot \frac{1}{D}(8i + D)^{-1} \cdot (1 - 2x)\right]$$

$$= I\left[e^{3x} \cdot e^{4ix} \cdot \frac{1}{8i}\left(\frac{1}{D} - \frac{1}{8i} + \frac{D}{16i^2}\right)(1 - 2x)\right]$$

$$= I\left[e^{3x} \cdot e^{4ix} \cdot \frac{1}{8i}\left(x - x^2 + \frac{1}{8}i - \frac{x}{4}i + \frac{1}{8}\right)\right]$$

giving us $\left[-(x - x^2)\cos 4x - \frac{x}{4}\sin 4x + \frac{1}{8}\cos 4x - \frac{1}{8}\sin 4x\right] e^{3x}$

Hence the complete solution is

$$y = \left[A\cos 4x + B\sin 4x - (x - x^2)\cos 4x\right] e^{3x}$$

The above procedure can be divided into two main steps:

(i) Write down the C.F. by equating the R.H.S. of the equation to zero.

(ii) Using the rules suggested to obtain a P.I. of the equation.

The complete solution can then be written down immediately.

Note that no attempt has been made to justify the rules suggested in the above section. Their justification lies in the fact that the solutions so obtained do indeed satisfy the original equation. In the examples chosen the verification of the solution is a simple step.

1.30. The 'δ' Method.

This procedure is an extension of the method of Frobenius[1] although the exact details are rather different. The great advantage of the 'δ' method is that it enables us to solve ordinary differential equations having variable coefficients. In the following section we intend to give an outline of the method and its applications to various well-known equations.

1.31. The operator δ.

We define the operator δ to be $x\frac{d}{dx}$; it follows immediately that $\delta(\delta-1) = x^2 \frac{d^2}{dx^2}$

since $\delta . \delta = x\frac{d}{dx}\left(x\frac{d}{dx}\right) = x\frac{d}{dx} + x^2\frac{d^2}{dx^2} = \delta + x^2\frac{d^2}{dx^2}$ etc.

Using the above definition of δ the following are at once obvious:

(a) $\delta^r(u \pm v) = \delta^r u \pm \delta^r v$

(b) $\delta^r(\delta^s u) = \delta^s(\delta^r u) = \delta^{r+s} u$

(c) $\delta(uv) = v\,\delta u + u\,\delta v$

(d) $\delta^n(uv) = \sum_0^n {}_nC_r\, \delta^r u . \delta^{n-r} v.$

(e) $f(\delta)x^n = f(n)x^n$

(f) $\delta^r(x^n y) = x^n(\delta + n)^r y$

(g) $f(\delta)x^n y = x^n f(\delta + n) y.$

} f a polynomial.

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We will also need

(h) $$f(\delta)u.v. = u.\,f(\delta)v + \delta u.\,f'(\delta).v. + \frac{\delta^2 u}{\lfloor 2}\cdot f''(\delta)v + ---$$

which is an extension of (d).

It can also be seen that

$\delta(\log x) = 1$ and therefore $\delta^2(\log x) = \delta.1 = 0$

and hence $\delta^r(\log x) = 0$ if $r \geq 2$.

From (h) we see that

$$f(\delta)y.\log x = \log x.f(\delta)y + 1.f'(\delta)y$$

Hence
$$f(\delta)x^r.\log x = \log x.f(\delta)x^r + 1.f'(\delta)x^r$$
$$= \log x.f(r).x^r + 1.f'(r)x^r$$
$$= 0 \qquad \text{if } f(r) = f'(r) = 0$$

i.e. if r is a repeated root of $f(\delta)$ then $f(\delta)x^r.\log x = 0$

We note also that $\delta(\log x)^n = n(\log x)^{n-1}$

$$\delta^r(\log x)^n = n(n-1)---(n-r+1)(\log x)^{n-r} \quad r \leq$$

$$\delta^{n+1}(\log x)^n = 0$$

1.32. Solution of Equations.

Consider the equation $F(\delta)y = x.g(\delta)y$ and suppose that we wish to find the solution as a series in ascending powers of x.

Assume that

$$y = a_0 x^r + a_1 x^{r+1} + --- \qquad \text{where } a_0 \neq 0$$

then
$$f(\delta)y = a_0 f(r)x^r + a_1 f(r+1)x^{r+1} + --- + a_{n+1}f(r+n+1)x^{r+n+1} + ---$$

and $$x.g(\delta)y = x\left[a_0 g(r)x^r + a_1 g(r+1)x^{r+1} + --- + a_n g(r+n)x^n + ---\right]$$

Thus, equating coefficients we have

$a_0 f(r) = 0$ i.e. $f(r) = 0$ assuming that $a_0 \neq 0$

$a_1 f(r+1) = a_0 g(r)$ etc.

The first equation $f(r) = 0$ is known as the "indicial equation" and it tells us the index at which to start the series.

Thus, supposing that $f(r) = 0$ we can then take $a_0 = 1$ and therefore

$$a_n = \frac{g(r).\, g(r+1).\, ---.\, g(r+n-1)}{f(r+1).\, f(r+2).\, ---.\, f(r+n).}$$

hence giving us a formula for the coefficients of the terms of the series.

We must also note that if

(a) one of the numbers $g(r)$, $g(r+1)$ etc. vanish, then the series terminates and is known as the polynomial solution.

(b) one of the numbers $f(r+1)$, $f(r+2)$ etc. is zero, then the formula fails. Hence if f has a group of roots differing by an integer, then, in general, the method only gives a solution led by x^s, where s is the algebraically greatest of the group. ?

(c) the above (a) and (b) occur together.

i.e. $f(r+n) = g(r+n-1) = 0$ and thus a_n is indeterminate. In this case we must consider each example on its merits.

(d) If f has a repeated root r, there is naturally only one series led by x^r.

1.33. Convergence of the Series.

The series will converge for all values of x, if the degree of f > the degree of g.

for a limited range if the degree of f = degree of g.

for no value of x (other than zero) if the degree of f is less than the degree of g.

Now, we know that $a_{n+1} f(r+n+1) = a_n g(r+n)$

hence $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x| = \left|\frac{g(r+n)}{f(r+n+1)}\right| |x|$$

and if $f(\delta) = c_0\delta^s + \cdots$ lower powers

$g(\delta) = d_0\delta^t + \cdots$ lower powers

then $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{d_0\delta^t + \cdots}{c_0\delta^s + \cdots}\right| |x| \to \infty$$ if t>s for all x.

$\to 0$ if t<s for all x.

$\to \left|\frac{d_0}{c_0}\right| |x|$ if t = s and the series is convergent if $|x| < \left|\frac{c_0}{d_0}\right|$.

e.g.1. $$4x \cdot \frac{d^2y}{dx^2} + 2 \cdot \frac{dy}{dx} + y = 0$$

i.e. $$4x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + xy = 0$$

and rewriting in terms of the operator δ we have

$$2\delta(2\delta - 1)y = -xy$$

Therefore $f(r) = 2r(2r - 1) = 0$

giving us two solutions commencing with 1, and $x^{\frac{1}{2}}$ where both are convergent for all values of x.

For the first solution put

$$y = 1 + a_1 x + a_2 x^2 + \cdots$$

hence $a_1 . 2 . 1 . = -1$ i.e. $a_1 = -\frac{1}{\underline{|2}}$

$a_2 . 4 . 3 . = -a_1$ i.e. $a_2 = \frac{1}{\underline{|4}}$ etc

and the series is $y = 1 - \frac{1}{\underline{|2}} x + \frac{1}{\underline{|4}} x^2 - \frac{1}{\underline{|6}} x^3 + \cdots$

For the second solution put

$$y = x^{\frac{1}{2}}z \qquad \text{thus giving}$$

$$2\delta(2\delta + 1)z = -x^{\frac{1}{2}}z.$$

hence $a_1 = -\frac{1}{2.3}$ $\qquad a_2 = +\frac{1}{2.3.4.5.}$ etc.

and therefore $y = x^{\frac{1}{2}}(1 - \frac{1}{\lfloor 3}x + \frac{1}{\lfloor 5}x^2 - \text{----})$.

e.g.2. The above method applied to the hyper-geometric ~~series~~ equation

$$x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{dy}{dx} - \alpha\beta y = 0$$

If we multiply the equation by x and rewrite in terms of the operator we have

$$\delta(\delta-1)y - x\delta(\delta-1)y + \gamma\delta y - x(\alpha+\beta+1)\delta y - \alpha\beta xy = 0$$

giving $\qquad \delta(\delta+\gamma-1)y = x(\delta+\alpha)(\delta+\beta)y$

and thus the indicial equation is $\qquad r(r+\gamma-1) = 0$

i.e. 2 ascending solutions led by 1, $x^{1-\gamma}$ and both convergent if $|x| < 1$.

To find the first solution put

$$y = a_0 + a_1x + a_2x^2 + \text{---}$$

hence $\qquad a_{n+1}(n+1)(n+\gamma) = (n+\alpha)(n+\beta).a_n$

and if $a_0 = 0$ then

$$a_n = \frac{\alpha(\alpha+1)(\alpha+2)\text{---}(\alpha+n-1)\beta(\beta+1)\text{--}(\beta+n-1)}{\lfloor n.\ \gamma(\gamma+1)\text{---}(\gamma+n-1)}$$

therefore $\quad y = 1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1).\beta(\beta+1)}{\lfloor 2.\gamma(\gamma+1)}x^2 + \text{---}$ $\qquad$ i.e. $F(\alpha\beta\gamma x)$

To find the second solution put $y = x^{1-\gamma}z$ and z must satisfy

$$\delta(\delta+1-\gamma)z = x(\delta+\alpha+1-\gamma)(\delta+\beta+1-\gamma)z$$

therefore $z = F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x)$ and the second solution is

$$y = x^{1-\gamma} F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, x).$$

1.34. Descending Series.

In a similar manner to the above we can find series solutions in descending powers of x. To find these we proceed as follows.

If $x = \frac{1}{x'}$, $\frac{dy}{dx} = -\frac{1}{x^2}\frac{dy}{dx'}$, $x\frac{dy}{dx} = -x'\frac{dy}{dx'}$, i.e. $\delta = -\delta'$

Thus our initial simple equation becomes

$g(-\delta')y = x f(-\delta')y$ and we then apply the foregoing theory to obtain solutions in ascending powers of x'. Therefore each root of $g(\delta)$ leads to a descending series unless

(i) $g(\delta)$ has equal roots; only one series.

(ii) $g(\delta)$ has a group of roots differing by an integer; in which only the smallest root will, in general, lead to a solution.

It is worth noting that the descending series will terminate if there exists a root of f differing from a root of g by an integer and the root of f is the smaller.

The convergence of the series is found in the same manner, giving

(i) convergent for all x, if the degree of g > degree of f.

(ii) convergent $|x| > R$, if the degree of g = degree of f.

(iii) convergent nowhere, if the degree of g < degree of f.

e.g. The hyper-geometric equation.

$\delta(\delta+\gamma-1)y = x(\delta+\alpha)(\delta+\beta)y$ has 2 descending solutions led by $x^{-\alpha}$, $x^{-\beta}$ if $\alpha-\beta$ is not an integer or zero, and convergent when $|x| > 1$.

Put $x = \frac{1}{x'}$, and $\delta = -\delta'$

i.e. $(\delta'-\alpha)(\delta'-\beta)y = x'\delta'(\delta'-\gamma+1)y$

Now, if we put $y = x'^{\alpha} z$ we obtain

$$\delta'(\delta'+\alpha-\beta)z = x'(\delta'+\alpha)(\delta'+\alpha-\gamma+1)z$$

or $$\delta'(\delta'+\overline{\alpha-\beta+1}-1)z = x'(\delta'+\alpha)(\delta'+\alpha-\gamma+1)z$$

which is the same form as the original equation and has a solution

$$z = F(\alpha, \alpha-\gamma+1, \alpha-\beta+1, x')$$

or $$y = x^{-\alpha}F(\alpha, \alpha-\gamma+1, \alpha-\beta+1, \tfrac{1}{x})$$

and obviously the second solution is

$$y = x^{-\beta}F(\beta, \beta-\gamma+1, \beta-\alpha+1, \tfrac{1}{x})$$

There are, in fact, 24 solutions of the hypergeometric-equation but the above give a clear indication of our method.

1.35. The equation $f(\delta)y = x^r g(\delta)y$

The difference between this and the former equation considered is that we have a power of x greater than unity on the right-hand-side.

Our procedure with this equation is to let

$$x^r = x' \quad \text{then} \quad \frac{dy}{dx} = \frac{dy}{dx'}\cdot rx^{r-1} \quad \text{or} \quad x\frac{dy}{dx} = rx'\frac{dy}{dx'} \quad \text{i.e.} \quad \delta = r\delta'$$

and the equation becomes

$f(r\delta')y = x'g(r\delta')y$ allowing us to obtain series solutions as before.

Note however that the roots of the indicial equation are reduced in the ratio r:1, and the causes of failure will now be (i) when $\underline{f}$ has repeated roots.

(ii) when $\underline{f}$ has a group of roots differing by a multiple of $\underline{r}$.

The polynomial solution will occur when $\underline{f}$ and $\underline{g}$ have roots differing by a multiple of $\underline{r}$.

e.g.1. $\delta(\delta-1)y = x^2(\delta-n)(\delta+n+1)y$ i.e. Legendre's equation with $\underline{n}$ a positive integer.

There are 2 solutions led by 1 and x. Following the above method with the exception that we now have

$$a_{n+1}f\left[s+r(n+1)\right] = a_n g(s+r.n)$$

the first solution is

$$y = 1 + \frac{-n(n+1)}{2.1.}x^2 + \frac{-n(n+1)(2-n)(3+n)}{2.1.4.3}x^4 + ---$$

i.e. $$y = 1 - \frac{n(n+1)}{\underline{|2}}x^2 + \frac{n(n-2)(n+1)(n+3)}{\underline{|4}}x^4 - --- \text{ etc.}$$

The second solution is got by putting $y = xz$, giving

$$y = x\left[1 - \frac{(n-1)(n+2)}{\underline{|3}}x^2 + \frac{(n-1)(n-3)(n+2)(n+5)}{\underline{|5}}x^4 - ---\right]$$

One of these solutions will obviously terminate since either $\underline{n}$ or $\underline{n-1}$ must be a multiple of 2; therefore one of them is a polynomial solution.

There are 2 descending solutions led by x^n and x^{-n-1} one of which is the polynomial solution started from the other end.

The polynomial solution is known as Legendre's function of order n i.e. $P_n(x) = \frac{1}{2^n \underline{|n}} \cdot D^n(x^2-1)^n$

e.g.2. Bessel's equation of order n.

$$(\delta^2 - n^2)y + x^2 y = 0$$

The roots of $f(\delta)$ are $-n$ and $+n$; and if n is an integer the roots must differ by a multiple of 2and the lower root must therefore fail to give a solution. Hence we have a solutiom in ascending powers led by x^n. If, 2n is non-integral then there are 2 solutions.

The above method gives us for the first solution

$$y = x^n - \frac{1}{2(2n+2)} \cdot x^{n+2} + \frac{1}{2(2n+2)4(2n+4)} \cdot x^{n+4} - \cdots$$

i.e. $$y = x^n \left[1 - \frac{1}{4(1+n)} x^2 + \frac{1}{4.8.(n+1)(n+2)} x^4 - \cdots \right]$$

We may conclude then that the δ method is quite powerful if the original equation can be expressed in δ form. With practice, the first solution can be written down almost at sight and the other solutions follow automatically.

Its great disadvantage is that its use is limited to the range of equations which can be expressed in δ form.

1.40. The Laplace Transformation.

Over 50 years ago Oliver Heaviside (1850-1925) devised his operational calculus for the solution of differential equations. Since that time it has held a prominent place in the treatment of problems in applied mathematics. It was not ,however, until Jeffrey's published his work on "Operational Methods in Mathematical Physics" that the calculus became more widely known and used. We shall see that although Heaviside's methods gave him the results he required, his rules of procedure were not very rigorous. For all practical purposes this was not essential.

To follow Heaviside's method we must consider the

equation $$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + a_2 \frac{d^{n-2} x}{dt^{n-2}} + \cdots + a_n x = 1 \qquad t > 0 \qquad (1)$$

where $a_0, a_1, a_2, \cdots a_n$ are constants and

$$x = \frac{dx}{dt} = \frac{d^2 x}{dt^2} = \cdots = \frac{d^n x}{dt^n} = 0 \quad \text{when } t = 0 \qquad (2)$$

If we replace $\frac{dx}{dt}$ by 'p', $\frac{d^2x}{dt^2}$ by p^2 etc. then we obtain

$$f(p)x = 1 \qquad (3)$$

where $f(p) = a_0 p^n + a_1 p^{n-1} + \cdots + a_n$.

Heaviside regarded 'p' as an operator and equation (3) as his operational equation. Thus the operational solution would be

$$x = 1/f(p) \qquad (4)$$

This solution he interpreted by various rules, the most important being the "Expansion Theorem". The fact that he attached great importance to this theorem is seen when he stated that "its use, even in comparatively elementary problems, leads to a considerable saving of labour, while in cases involving partial differential equations it is invaluable".

The Expansion Theorem states that:

If $p_1, p_2, p_3, --- p_n$, be the roots of $f(p) = 0$ assumed all different and none zero, then

$$x = \frac{1}{f(0)} + \sum_{s=1}^{n} \frac{e^{p_s t}}{p_s f'(p_s)}$$

A more precise statement of equation(4) is

$$x = \frac{1}{f(p)} \cdot H(t) \qquad (5)$$

where H(t) is Heaviside's unit function, defined by

H(t) = 1 for t > 0; H(t) = 0 for t < 0

Now, since f(p) is a polynomial in p we can expand $\frac{1}{f(p)}$ in ascending powers of 1/p.

$$\text{i.e.} \quad \frac{1}{f(p)} = \frac{b_n}{p^n} + \frac{b_{n+1}}{p^{n+1}} + --- + \frac{b_{n+r}}{p^{n+r}} + ---$$

thus, if we regard $\frac{1}{p^n} H(t) = \frac{t^n}{\lfloor n}$ (c.f. 1.21.(ii)

then $x = b_n \cdot \frac{t^n}{\lfloor n} + b_{n+1} \cdot \frac{t^{n+1}}{\lfloor n+1} + ---$ giving us the actual solution.

As we stated before, Heaviside made no effort to prove the above rules; that they gave him the result seemed to be sufficient proof.

When he turned to partial differential equations Heaviside became even more obscure and it was probably because of this inadequate mathematical treatment that the importance of his theoretical work was not recognised in his lifetime. He solved certain equations and compared his results with the known solutions thus arriving at his rules of procedure.

Bromwich was the first to attempt to explain Heaviside's methods. He noted that the solution of the above problem is

$$x = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \cdot \frac{d\lambda}{\lambda f(\lambda)}$$

where the integral is taken along the line $R(\lambda) = \gamma$ in the λ-plane, γ being real and positive and the roots of $f(\lambda)$ all lying to the left of $\lambda = \gamma$

Bromwich extended Heaviside'S theory to include the cases in which the right-hand-side of the initial equation is a function of the independent variable and the initial values of $x, \frac{dx}{dt}, \frac{d^2x}{dt^2}$, etc. to have arbitrary values when $t = 0$.

Doetsch later showed that if

$$f(p) = \int_0^\infty e^{-pt} F(t)\, dt \qquad , \quad p > 0$$

then

$$F(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \cdot f(\lambda) \cdot d\lambda$$

where $F(t)$ and $f(p)$ fulfil certain conditions.

Carson and Van der Pol both worked along similar lines except that they used pe^{-pt} as the multiplier

i.e. $$f(p) = \int_0^\infty pe^{-pt} F(t)\, dt.$$

This latter operation tends to preserve the correspondence with Heaviside's ideas and has the advantage that it transforms a constant into itself. In general, however, the extra term p is just an added complication and has no mathematical significance. We intend, therefore, to follow the method of Doetsch and keep the term e^{-pt} i.e. in modern parlance, apply the Laplace Transformation.

1.41. The Laplace Transform and Inversion Integral.

Let us first consider a function f(t) defined for all values of t > 0. If we now multiply this function by the term e^{-pt} and integrate the expression thus obtained from zero to infinity we arrive at a new function of the variable p which we denote by $\bar{f}(p)$. i.e.

$$\int_0^\infty e^{-pt} f(t).dt = \bar{f}(p).$$

This new function $\bar{f}(p)$ is known as the Laplace Transform of f(t) and the whole operation as the Laplace Transformation. We will always denote the Laplace Transform (L.T.) of f(t) as $\bar{f}(p)$, or of y(t) as $\bar{y}(p)$ etc., and for present purposes we shall assume that the variable p is real and positive.

The conditions for the L.T. of f(t) to exist are that

(i) f(t) is sectionally continuous in every finite interval in the range $t \geq 0$.

(ii) f(t) is of exponential order as $t \to \infty$.

Thus, if

$$f(t) = t^2 \quad \text{then} \quad \bar{f}(p) = \frac{1}{p^3}\cdot\underline{|2}.$$

$$f(t) = \cos kt \quad \text{then} \quad \bar{f}(p) = \frac{p}{p^2+k^2}, p > 0.$$

Before we can begin using the Laplace Transformation we need a certain number of simple but important theorems. These follow automatically from the definition of the transform.

Theorem 1.

If $f_1(t)$ and $f_2(t)$ have transforms $\bar{f}_1(p)$ and $\bar{f}_2(p)$, then the transform of

$$f_1(t) \pm f_2(t) \quad \text{is} \quad \bar{f}_1(p) \pm \bar{f}_2(p).$$

The proof is obvious from the definition.

Theorem 2.

If the function $f(t)$ is continuous and has a sectionally continuous derivative $f'(t)$ in every finite interval $0 \leq t \leq T$; and also if $f(t) = O(e^{\alpha t})$ as $t \to \infty$; then, when $p > \alpha$ the transform of $f'(t)$ exists and is $p\bar{f}(p) - f(0)$.

The proof is again straightforward. By a continuous application of theorem (2) we obtain the following:

Theorem 3.

If the function $f(t)$ has continuous derivatives $f'(t)$, $f^2(t)$, $f^3(t)$, - - - $f^{n-1}(t)$ and a sectionally continuous derivative $f^n(t)$ in every finite interval $0 \leq t \leq T$; and if $f(t)$, $f'(t)$, - - - $f^{n-1}(t)$ are of order $(e^{\alpha t})$ as $t \to \infty$, then, when $p > \alpha$, the transform of $f^n(t)$ exists and is

$$p^n\bar{f}(p) - p^{n-1}f(+0) - p^{n-2}f'(+0) - - - f^{n-1}(+0).$$

Theorem 4.

If $\bar{f}(p)$ is the transform of $f(t)$ and <u>k</u> is any constant then $\bar{f}(p+k)$ is the transform of $e^{-kt}f(t)$.

e.g. $f(t) = \cos at$ then $\bar{f}(p) = p/(p^2 + a^2)$

$f(t) = e^{-kt}\cos at$ then $\bar{f}(p) = \frac{p+k}{(p+k)^2 + a^2}$

Theorem 5. (Translation theorem)

If $\bar{f}(p)$ is the transform of $f(t)$ then for any positive constant $\underline{k}$, $e^{-kp}\bar{f}(p)$ is the transform of $f_k(t)$ where

$$f_k(t) = 0 \quad \text{for} \quad 0 < t < k$$
$$= f(t - k) \quad \text{for} \quad t > k.$$

Since $$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)\,dt.$$

then $$e^{-kp}\,\bar{f}(p) = \int_0^\infty e^{-(t+k)p} f(t)\,dt \qquad \text{where } k > 0$$

Let $T = t + k$, therefore
$$e^{-kp}\,\bar{f}(p) = \int_0^\infty e^{-pt} f_k(t)\,dt$$ where $f_k(t)$ is defined as above. $= \text{L.T. of } f_k(t).$

Theorem 6.

If $\bar{f}_1(p)$ and $\bar{f}_2(p)$ are the transforms of $f_1(t)$ and $f_2(t)$ then $\bar{f}_1(p).\bar{f}_2(p)$ is the L.T. of

$$\int_0^t f_1(T).f_2(t - T)\,dT = \int_0^t f_1(t - T).f_2(T)\,dT.$$

The proof of this theorem is rather difficult, but it holds good if both $f_1(t)$ and $f_2(t)$ satisfy the conditions for the transforms to exist. This theorem is often referred to as the Convolution Theorem and the combination of the two functions under the integral sign as the convolution of those functions.

With this short list of theorems it is possible for us to study the Laplace Transformation in some detail.

Consider the equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_n y = f(t) \qquad (1)$$

where $y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \cdots, \frac{d^{n-1}y}{dt^{n-1}}$ have the values $y_0, y_1, y_2, \cdots y_{n-1}$, at $t = 0$; and $a_1, a_2, a_3, \cdots a_n$ are arbitrary constants.

If we now apply the transformation and assume that y(t) and its derivatives satisfy the conditions of Theorem 3., then we see that

$$\int_0^\infty e^{-pt} \cdot \frac{d^n y}{dt^n} \cdot dt = p^n \bar{y}(p) - p^{n-1} y_0 - p^{n-2} y_1 - \cdots - y_{n-1}$$

$$a_1 \int_0^\infty e^{-pt} \cdot \frac{d^{n-1} y}{dt^{n-1}} \cdot dt = a_1 (p^{n-1} \bar{y}(p) - p^{n-2} y_0 - p^{n-3} y_1 - \cdots - y_{n-2})$$

$$\vdots \qquad\qquad \vdots$$

$$a_{n-1} \int_0^\infty e^{-pt} \cdot \frac{dy}{dt} \cdot dt = a_{n-1} (p\bar{y}(p) - y_0)$$

$$a_n \int_0^\infty e^{-pt} \cdot y \cdot dt = a_n \bar{y}(p)$$

and $\int_0^\infty e^{-pt} f(t)\,dt = \bar{f}(p)$ assuming that ~~the~~ the transform exists.

Thus, collecting all the terms we have

$$\left.\begin{aligned} \phi(p) \cdot \bar{y}(p) = \bar{f}(p) &+ (p^{n-1} y_0 + p^{n-2} y_1 \cdots y_{n-1}) \\ &+ a_1 (p^{n-2} y_0 + p^{n-3} y_1 \cdots y_{n-2}) \\ &+ \cdots + a_{n-2} (p y_0 + y_1) + a_{n-1} y_0 \end{aligned}\right\} \qquad (2)$$

where $\phi(p) = p^n + a_1 p^{n-1} + a_2 p^{n-2} + \cdots + a_n$.

The above equation (2) is known as the subsidiary equation corresponding to the given differential equation and initial conditions. The problem is thus reduced to finding the expression $\bar{y}(p)$ and hence obtaining the required solution y(t).

A few simple examples will help to show the above procedure more clearly. A short table of transforms is contained in Appendix 1.

e.g.1. $$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 2 \qquad \text{where } y_0 = 1,\ y_1 = 0.$$

Assuming that the transform exists etc., we can apply the transformation to the equation, thus

$$(p^2 - p - 6)\bar{y}(p) = \frac{2}{p} + (py_0 + y_1) - y_0$$

$$= \frac{2}{p} + p - 1$$

$$= \frac{p^2 - p + 2}{p}$$

i.e. $$\bar{y}(p) = \frac{p^2 - p + 2}{p(p-3)(p+2)}$$

Expressing this in partial fractions, we obtain

$$\bar{y}(p) = -\frac{1}{3p} + \frac{8}{15(p-3)} + \frac{4}{5(p+2)}$$

and hence $y(t) = -\frac{1}{3} + \frac{8}{15}e^{3t} + \frac{4}{5}e^{-2t}$ from our table of transforms. The solution is easily verified by substituting in the original equation.

e.g.2. $$\frac{d^2y}{dt^2} + n^2y = a.\sin nt \qquad t \geq 0$$

where $y = y_0$ and $dy/dt = y_1$ when $t = 0$.

Applying the transform, we have

$$(p^2 + n^2)\bar{y}(p) = a\int_0^\infty e^{-pt}\sin nt.dt + (py_0 + y_1)$$

$$= \frac{an}{p^2 + n^2} + py_0 + y_1$$

i.e. $\bar{y}(p) = \frac{an}{(p^2+n^2)^2} + \frac{py_0}{(p^2+n^2)} + \frac{y_1}{(p^2+n^2)}$

Hence $y(t) = \frac{a}{2n}\left(\frac{1}{n}\sin nt - t\cos nt\right) + y_0\cos nt + \frac{y_1}{n}\sin nt.$

e.g. 3. $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \frac{1}{2}t^2e^t \qquad t \geq 0$

where $y_0' = 0$, and $y_1 = 0$.

Therefore $(p^3 + 3p^2 + 3p + 1)\bar{y}(p) = \frac{1}{2}\int_0^\infty t^2 e^{-(p-1)t}\,dt.$

$$= \frac{1}{(p-1)^3} \qquad p>1, \quad \text{(Th.4.)}$$

Hence $\bar{y}(p) = \frac{1}{(p+1)^3(p-1)^3}$

$$= \frac{1}{2(p-1)^3} - \frac{3}{4(p-1)^2} + \frac{3}{8(p-1)} - \frac{1}{24(p+1)} - \frac{p-2}{3\left[(p-\frac{1}{2})^2+\frac{3}{4}\right]}$$

and $y(t) = \frac{1}{4}(t^2 - 3t + \frac{3}{2})e^t - \frac{1}{24}e^{-t} - \frac{1}{3}(\cos\frac{1}{2}\sqrt{3}t - \sqrt{3}\sin\frac{1}{2}\sqrt{3}t)e^{\frac{1}{2}t}$

From the examples shown we note that the method of procedure falls into 4 distinct steps:

(i) Transformation of the given differential equation assuming that the conditions for existence are satisfied.

(ii) Insertion of initial conditions to obtain the expression for $\bar{y}(p)$.

(iii) Expressing this in partial fractions.

(iv) Using the table of transforms to obtain the expression for $y(t)$.

The procedure outlined can easily be extended to cover the case of simultaneous differential equations. The method is so straightforward that one example should make it quite clear.

e.g.4. $$\frac{dy}{dt} - \frac{dz}{dt} - 2y + 2z = 1 - 2t$$

$$\frac{d^2y}{dt^2} + 2\frac{dz}{dt} + y = 0$$

where $y(t) = z(t) = \frac{dy}{dt} = 0$ when $t = 0$.

Transforming each equation we obtain

$$(p - 2)\bar{y} - (p - 2)\bar{z} = \frac{1}{p} - \frac{2}{p^2}$$

$$2p\bar{z} + (p^2 + 1)\bar{y} = 0$$

<u>or</u> $$\bar{y} - \bar{z} = 1/p^2$$

$$2p\bar{z} + (p^2 + 1)\bar{y} = 0$$

Eliminating $\bar{z}$ we have

$$(p^2 + 2p + 1)\bar{y} = 2/p$$

and hence $$\bar{y} = \frac{2}{p} - \frac{2}{p+1} - \frac{2}{(p+1)^2}$$

and $\therefore$ $y(t) = 2(1 - e^{-t} - te^{-t})$ with the expression for $z(t)$ being obtained in a similar manner.

The foregoing theory can be extended slightly to include a very useful theorem which enables us to solve equations with variable coefficients.

Theorem 7.

If $f(t)$ is sectionally continuous and of exponential order as $t \to \infty$, then

$$\frac{d}{dp}\left[\int_0^\infty e^{-pt} f(t).\,dt\right] = \int_0^\infty f(t).\,\frac{\partial}{\partial p}\left(e^{-pt}\right)dt$$ the latter integral being uniformly convergent and the former convergent.

Therefore $\frac{d}{dp}\left[\bar{f}(p)\right] = \int_0^\infty f(t).-t.e^{-pt}.dt = \bar{\phi}(p)$ where $\phi(t) = -t.f(t)$

If we repeat this <u>n</u> times we obtain

$$\int_0^\infty e^{-pt}.t^n f(t).dt = (-1)^n \frac{d^n}{dp^n}\bar{f}(p)$$

e.g.5. $\quad t.\frac{d^2y}{dt^2} - (2t+1)\frac{dy}{dt} + (t+1)y = 0$ where $y_0 = 0$.

Transforming the equation

$$(-1).\frac{d}{dp}(p^2\bar{y} - y_1) - (-1)2.\frac{d}{dp}(p\bar{y}) - (p\bar{y}) + (-1)\frac{d}{dp}(\bar{y}) + \bar{y} = 0$$

reducing to $\quad \frac{d\bar{y}}{dp}(p-1)^2 + 3\bar{y}(p-1) = 0$

i.e. $\quad \frac{d\bar{y}}{dp} = -\frac{3\bar{y}}{(p-1)}$

and $\quad \bar{y} = C(p-1)^{-3}$

$\therefore \quad y(t) = Ct^2e^t.$

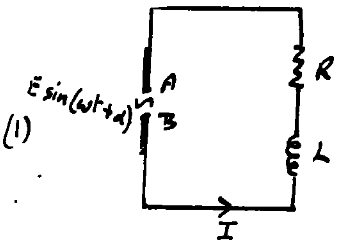
These examples are quite simple and are designed solely to show the method of approach. However, without extending our very limited knowledge it is possible to solve a number of problems of varying types in applied mathematics.

1.43. <u>Elementary Applications.</u>

Heaviside originally devised his operational calculus for use in problems arising in the theory of electric circuits. So it would seem appropriate to begin with a problem of that type.

e.g.1. An e.m.f. $E\sin(\omega t + \alpha)$ is applied at $t = 0$ to an induct-ive resistance L,R. The initial current being zero. Find the subsequent current at time t. (Interpret V, I, L, R, C and Q as the e.m.f., current, inductance, resistance, capacitance and charge. The same letters with the suffix o, i.e. I_o, representing the values at $t = 0$.)

The potential drop round the circuit must be the same as that across AB.



i.e. $$L\frac{dI}{dt} + R.I. = E\sin(\omega t + \alpha) \qquad (1)$$

where $I_o = 0$.

Applying the L.T. we immediately obtain

$$(Lp + R)\bar{I} = E\left\{\frac{p\sin\alpha + \omega\cos\alpha}{p^2+\omega^2}\right\}$$ using the initial condition.

Therefore $$\bar{I} = \frac{E(p\sin\alpha + \omega\cos\alpha)}{(Lp+R)(p^2+\omega^2)}.$$

This may be expressed in partial fractions, thus

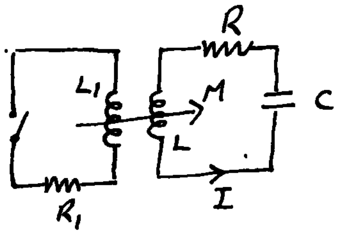
$$\bar{I} = E\left[\frac{L\sin(\gamma-\alpha)}{(L^2\omega^2+R^2)^{\frac{1}{2}}}\cdot\frac{1}{Lp+R} - \frac{\sin(\gamma-\alpha)}{(L^2\omega^2+R^2)^{\frac{1}{2}}}\cdot\frac{p}{p^2+\omega^2} + \frac{\omega\cos(\gamma-\alpha)}{(L^2\omega^2+R^2)^{\frac{1}{2}}}\cdot\frac{1}{p^2+\omega^2}\right]$$

where $\tan\gamma = L\omega/R.$, and hence

$$I = \frac{E}{(L^2\omega^2+R^2)^{\frac{1}{2}}}\left[\sin(\gamma-\alpha)\,e^{-\frac{R}{L}t} + \sin(\omega t + \alpha - \gamma)\right]$$

giving us the value of the current in the circuit at time t. The expression shows immediately that when <u>t</u> is very small the first term of the expression is the larger and actually represents the initial surge of current. As <u>t</u> increases this becomes negligible and the current alternates with period $\frac{2\pi}{\omega}$.

e.g2. A circuit L_1, R_1 is coupled by mutual inductance M to a secondary circuit consisting of R, L, and C in series. At $t = 0$, when steady current E_0/R_1 is flowing in it, the primary circuit is opened. Find the subsequent secondary current.



For the potential in the secondary circuit we have

$$L\frac{dI}{dt} + RI + \frac{Q}{C} + M.\frac{dI_1}{dt} = 0 \qquad (2)$$

where $I_1 = 0$ for $t > 0$; and therefore $\bar{I}_1 = 0$. (3)

and $I_1 = E_0/R_1$, $I_2 = 0$ when $t = 0$. (4)

also $dQ/dt = I$. (5)

Applying the transform to both (2) and (5) and using (3) and (4).

$$(Lp + R)\bar{I} + \frac{\bar{Q}}{C} = M.\frac{E_0}{R_1}$$

and $$p\bar{Q} = \bar{I}$$

Therefore $$(Lp + R + \frac{1}{pC})\bar{I} = ME_0/R_1$$

i.e. $$\bar{I} = \frac{ME_0}{LR_1} \cdot \frac{p}{p^2 + p.\frac{R}{L} + \frac{1}{LC}}$$

Hence, if $\mu = R/2L$ and $n^2 = 1/LC - \mu^2$, then

$$\bar{I} = \frac{ME_0}{LR_1} \cdot \frac{p}{(p+\mu)^2 + n^2} = \frac{ME_0}{LR_1}\left[\frac{(p+\mu) - \mu}{(p+\mu)^2 + n^2}\right]$$

$$I = \frac{ME_0}{LR_1} e^{-\mu t}(n \cos nt - \mu \sin nt).$$

giving us the subsequent current in the secondary circuit. We can see at once that as the time increases the current dies away.

The use of the L.T. is not, however, confined to this type of problem. Consider the following example.

e.g.3. A particle of mass m and charge e is projected from the origin with velocity (u,o,o) and is subject to a magnetic field H along the z-axis; the resistance to motion being km times the velocity. Find the coordinates at time t.

With the usual notation, the equations of motion are

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= \frac{eH}{c}\cdot\frac{dy}{dt} - km.\frac{dx}{dt} \\ m.\frac{d^2y}{dt^2} &= -\frac{eH}{c}\cdot\frac{dx}{dt} - km.\frac{dy}{dt} \\ m.\frac{d^2z}{dt^2} &= 0 \end{aligned}\right\} \qquad (6)$$

where $x = y = z = 0$ at $t = 0$ and $\frac{dx}{dt} = u$, $\frac{dy}{dt} = \frac{dz}{dt} = 0$ at $t = 0$. (7)

Applying the transform to (6) and using the initial conditions we obtain

$$\left.\begin{aligned} p^2\bar{x} &= \lambda p\bar{y} - kp\bar{x} + u \\ p^2\bar{y} &= -\lambda p\bar{x} - kp\bar{y} \\ p^2\bar{z} &= 0 \end{aligned}\right\} \text{ where } \lambda = eH/cm.$$

Obviously $z = 0$ and therefore solving the equations for $\bar{x}$ and $\bar{y}$ we have

$$\bar{x} = \frac{(p^2+kp)}{(p^2+kp)^2+\lambda^2p^2} \quad \text{and} \quad \bar{y} = -\frac{\lambda pu}{(p^2+kp)^2+\lambda^2p^2}.$$

hence

$$\bar{x} = \frac{u}{(\lambda^2+k^2)}\left[\frac{k}{p} - \frac{k(p+k)}{(p+k)^2+\lambda^2} + \frac{\lambda^2}{(p+k)^2+\lambda^2}\right]$$

and

$$x = \frac{u}{(\lambda^2+k^2)}(k - ke^{-kt}\cos\lambda t + \lambda e^{-kt}\sin\lambda t).$$

Similarly

$$y = \frac{u}{(\lambda^2+k^2)}\left[-\lambda + e^{-kt}(\lambda\cos\lambda t + k\sin\lambda t)\right].$$

We have made no attempt to justify the solutions obtained in the previous examples but they may easily be proved true by substitution in the original equation.

With the limited theory at our disposal we have been able to solve a few types of differential equations. These show us that one of the great advantages of the L.T. is that it brings the initial conditions into play almost immediately; and thus in a number of cases shortens the algebra. The method,,so far, is almost stereotyped and in the problems considered has reduced the equation to a known transform. We must now consider the cases in which the inverse transformation is not so obvious.

This inverse transformation is effected by the use of the Inversion Theorem. It is an integral formula from which the solution may be obtained by insertion of the final transform. The use of the Inversion Theorem immediately enlarges the scope of the L.T. and eliminates much of the tedious algebra necessary for resolution into partial fractions.

In using the Inversion Theorem we must assume a slight knowledge of the theory of the Complex Variable -- contour integration etc.

1.44. The Inversion Theorem.

This theorem states that:

If $\bar{y}(p)$ is any analytic function of the variable p and of the order $p^{-\ell}$ in that part of the plane $R(p) \geqslant x_\circ$, and where $x_\circ$ and k are real constants, $k > 1$; then

$$\lim_{\beta \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{\lambda t} \, \bar{y}(\lambda) \, d\lambda$$ converges to a function

$y(t)$ that is independent of γ, where $\gamma \geq x_0$.

i.e. if $$\bar{y}(p) = \int_0^\infty e^{-pt} y(t) \, dt \qquad R(p) > 0$$

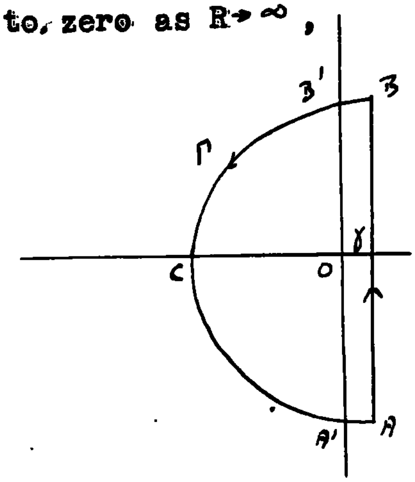
then $$y(t) = \lim_{\beta \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{\lambda t} \, \bar{y}(\lambda) \, d\lambda$$

The value of the theorem is at once self-evident. In solving problems the procedure is as usual until we arrive at the final form for $\bar{y}(p)$. If this satisfies the conditions of the above theorem then we can immediately find the solution by contour integration.

The line-integral in the Inversion Theorem is usually evaluated by transforming it into a closed contour and applying the calculus of residues. The following lemma is very useful in many cases:-

Lemma.[1]

If $|y(\lambda)| < CR^{-k}$, when $\lambda = Re^{i\theta}$, $-\pi \leq \theta \leq \pi$, $R > R_0$, and R_0, C, k are constants with $k > 0$, then $\int e^{\lambda t} y(\lambda) d\lambda$ taken over the arcs $BB'C$ and $AA'C$ of the circle Γ of radius R tends to zero as $R \to \infty$, provided $t > 0$.



Consider the integrals over BB' and $B'C$ seperately and let $\cos\alpha = \gamma/R$.

Then $$|I_{BB'}| = \left| \int_\alpha^{\frac{\pi}{2}} e^{Re^{i\theta}t} \cdot y(Re^{i\theta}) \cdot d(Re^{i\theta}) \right|$$

$$< CR^{-k+1} e^{\gamma t} \int_{\alpha}^{\frac{\pi}{2}} d\theta .$$

$$= CR^{-k+1} e^{\gamma t} \sin^{-1} \left(\frac{\gamma}{R} \right) \to 0 \quad \text{as } R \to \infty$$

$$\left| I_{B'C} \right| < CR^{-k+1} \int_{\frac{\pi}{2}}^{\pi} e^{Rt\cos\theta} d\theta . = CR^{-k+1} \int_{0}^{\frac{\pi}{2}} e^{-Rt\sin\phi} d\phi \text{ . where } \theta - \frac{\pi}{2} = \phi$$

$$< CR^{-k+1} \int_{0}^{\frac{\pi}{2}} e^{-\frac{2Rt\theta}{\pi}} d\theta \quad \text{using } 1 > \frac{\sin\theta}{\theta} \geq \frac{2}{\pi} \text{ if } 0 < \theta \leq \frac{\pi}{2}$$

$$< \frac{\pi C R^{-k}}{2t} \to 0 \quad \text{as } R \to \infty .$$

The integrals over the arcs CA' and $A'A$ follow in the same manner. Thus, if $\bar{y}(\lambda)$ is a function of λ satisfying the above lemma and is analytic except at a finite number of poles to the left of $R(\lambda) = \gamma$ then we may replace the line-integral by the semi-circle and hence by a circle, centre the ogigin, including all the poles of $\bar{y}(\lambda)$.

e.g.1. Suppose we wish to solve

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 2 \quad \text{where } y_0 = 1, \ y_1 = 0. \quad \text{(See P. 28)}$$

Applying the transform, we obtain as before

$$\bar{y}(p) = \frac{p^2 - p + 2}{p(p-3)(p+2)}$$

and hence
$$y(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t}(\lambda^2 - \lambda + 2)}{\lambda(\lambda-3)(\lambda+2)} d\lambda$$

Since $\bar{y}(\lambda)$ is of order (λ^{-1}) we may replace the line-integral by a circle C containing the poles $\lambda = 0, 3, -2,$ and hence

$y(t) = 2\pi i$ (sum of residues at these poles)

Pole $\lambda = 3$.

$$\text{Residue} = \lim_{\lambda \to 3} \frac{e^{\lambda t}(\lambda^2-\lambda+2)(\lambda-3)}{\lambda(\lambda-3)(\lambda+2)} = \frac{8}{15}e^{3t}$$

Pole $\lambda = -2$.

$$\text{Residue} = \frac{4}{5}e^{-2t}$$

Pole $\lambda = 0$.

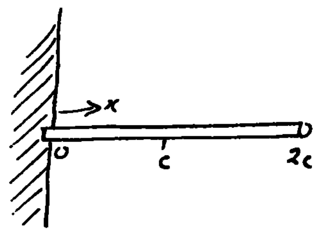
The expansion of the integrand gives us

$$\left[-\frac{1}{6}\cdot\frac{1}{\lambda}\cdot(\lambda^2-\lambda+2)\left(1+\frac{\lambda t}{\underline{1}}+\cdots\right)\left(1+\frac{\lambda}{3}+\cdots\right)\left(1-\frac{\lambda}{2}+\cdots\right)\right]$$

and hence residue $= -\frac{2}{6} = -\frac{1}{3}$

and $\quad y(t) = -\frac{1}{3} + \frac{4}{5}e^{-2t} + \frac{8}{15}e^{3t}$

e.g.2. To find the formula for the deflection in a uniform beam with a load bx per unit length in the interval $0 < x < c$, and load $b(2c - x)$ in the interval $c < x < 2c$; if the end $x = 0$ is built in and the end $x = 2c$ is hinged.



We know that

$$\frac{d^4y}{dx^4} = \frac{1}{EI}\,w(x) \qquad (1)$$

where E is Young's modulus, I the moment of inertia of cross-section of the beam, w(x) the load per unit length, and y(x) the deflection.

At any point where there is no support the function y(x) and its first 3 derivatives must be continuous.

Also $\quad y(0) = y'(0) = y(2c) = y''(2c) = 0 \qquad (2)$

To find the transform of w(x) we integrate from 0 to ∞

putting $w(x) = 0$ for all $x > 2c$.

i.e. $$\bar{w}(p) = b\int_0^c e^{-px}.x\,dx + b\int_c^{2c} e^{-px}(2c - x)\,dx + 0$$

$$= \frac{b}{p^2}(e^{-2cp} - 2e^{-cp} + 1)$$

Transforming (1)

$$EIp^4\,\bar{y} = \frac{b}{p^2}(e^{-2cp} - 2e^{-cp} + 1) + pA + B$$ using the first two conditions of (2).

i.e. $$EI\,\bar{y} = \frac{b}{p^6}(e^{-2cp} - 2e^{-cp} + 1) + \frac{A}{p^3} + \frac{B}{p^4}$$

and hence $$EIy(x) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda x}\left\{\frac{b}{\lambda^6}\left(e^{-2c\lambda} - 2e^{-c\lambda} + 1\right) + \frac{A}{\lambda^3} + \frac{B}{\lambda^4}\right\}d\lambda$$

Since $\bar{y}(\lambda)$ is obviously of order λ^{-k}, where $k > 0$, we can replace the integral by a circle C etc.

Pole $\lambda = 0$.

$$\text{Residue} = \frac{Bx^3}{\lfloor 3} + \frac{Ax^2}{\lfloor 2} + \frac{b}{120}x^5 - \frac{b}{60}(x-c)^5 + \frac{b}{120}(x-2c)^5$$

and thus $$EIy(x) = \frac{b}{120}\left[x - 2c\right]^5 - \frac{b}{60}(x-c)^5 + \frac{b}{120}x^5 + \frac{Ax^2}{2} + \frac{Bx^3}{6}$$

where the term $[x - 2c]$ must be interpreted as $[x - 2c] = 0 \quad x<2c$
and $= (x-2c) \quad x>2c$

and may be omitted here.

The correct solution is therefore

$$EIy(x) = -\frac{b}{60}(x-c)^5 + \frac{b}{120}x^5 + \frac{Ax^2}{2} + \frac{Bx^3}{6}$$

where $y(x) = y''(x) = 0$ when $x = 2c$.

Hence $$\frac{EI}{b}\cdot y(x) = \frac{5}{32}c^3x^2 - \frac{7}{64}c^2x^3 + \frac{1}{120}x^5 - \frac{1}{60}\{x-c\}^5$$ where $\{x-c\} = 0,\ x<c$

Equation (4) could have been written down immediately from (3) using the table of transforms. This problem is often solved by successive integrations of equation (1), but the continuity conditions which must be applied at $x = c$ make the method less direct than the above.

1.50. Conclusions.

We have so far outlined methods for solving ordinary differential equations. Let us now compare the methods for equations with constant coefficients.

Methods 1. and 2. are roughly equivalent with the latter slightly the quicker. The first method, however, has the advantage that it makes none of the assumptions that are made in 2. and thus is easier to follow. They both have the disadvantage of a number of rules which must be memorised before the methods can be applied. In solving equations rather than problems the L.T. has very little advantage except that the procedure has few variations. If, however, we wish to solve a problem in which the solution must satisfy certain conditions, then the Laplace Transformation is the quickest and the most direct procedure. This is because the L.T. brings the initial conditions into play immediately and in many cases they help to shorten the algebra. Since the majority of ordinary differential equations met with in applied mathematics arise naturally with given initial conditions we are led to the conclusion that for these equations the Laplace

Transformation is usually the quickest and most direct method. From the few examples given it can be seen that the transformation may be applied to a large range of problems although our theoretical knowledge is slight.

The last example given shows that we may apply the transformation to equations with variables which have a finite range instead of the more usual infinite range. In these cases the L.T. has not got the obvious advantages over the classical methods, since a number of equations derived from the conditions at one boundary still remain to be solved at the end.

When we come to compare the methods of solving equations with variable coefficients, we find that no one method has any great virtue. Although, in the example chosen (1.42. e.g.5.) to show the L.T. the solution followed immediately; this is by no means the general rule. In quite a number of cases, replacing an equation by the appropriate transforms leads to an equation which is as complex as the original. The δ method has the advantage that the solution can be written down at sight from the equation in δ form <u>if</u> it reduces to one of the types considered. We are led to the conclusion, however, that there is no hard and fast method for these equations and we must resort to trial and error to find the appropriate procedure.

2.00. Partial Differential Equations.

Equations of this type i.e. involving two or more variables, occur very frequently in applied mathematics and are of far greater importance than the ordinary equations already discussed. Naturally, the methods of solution are correspondingly more important and are rather more varied than the above.

The equations we shall be concerned with will be of the type involving two variables (usually x and t), since these occur most frequently in practice. Almost invariably these equations arise naturally with initial and boundary conditions attached to them which the required solution must satisfy. In fact, finding a solution which will satisfy the given conditions is one of the most important aspects of the work, and it is in this that we see the full power of the operational methods.

For partial differential equations we are not confined to the operational method already discussed i.e. the Laplace Transformation, but have a choice of several. In particular we intend to describe the Fourier, Hankel and Legendre transforms. Even this list is not exhaustive but it will, at least, give a general idea of the procedure involved.

2.10. The Laplace Transformation.

It is possible with the procedure already discussed to solve a number of partial differential equations[1]. We propose, however, to omit these and proceed immediately to the use of the Inversion Integral.

The Inversion Theorem is sometimes referred to as the "Mellin Inversion Theorem" but although it bears Mellin's name it was originally given by Poisson in a memoir read before the Academy of Sciences, Paris, in 1815. Mellin gave a rigorous discussion of it in 1896.

We have already used the Inversion Theorem, but since it is of far greater importance in this section we append the proof.

The Inversion Theorem.[2]

If $y(t)$ has a continuous derivative, and if $|y(t)| < Ke^{ct}$ where K and c are positive constants, and if

$$\overline{y}(p) = \int_0^\infty e^{-pt} y(t)\,dt. \qquad R(p) > c$$

then

$$y(t) = \frac{1}{2\pi i} \lim_{\omega \to \infty} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda t}\, \overline{y}(\lambda)\, d\lambda \qquad \gamma > c$$

Proof.

$$\frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda t}\, \overline{y}(\lambda).\, d\lambda = \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda t}.\, d\lambda \int_0^\infty e^{-\lambda u} y(u)\, du.$$

$$= \frac{1}{2\pi i} \int_0^\infty y(u)\, du \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda(t-u)}\, d\lambda.$$

(where we have inverted the order of integration because of the uniform convergence)

1. See "Modern Operational Methods in Engineering" — R.V. Churchill. Chapt. IV

2. [illegible]

$$= \frac{1}{\pi} \int_0^{\infty} y(u) \cdot e^{\lambda(t-u)} \cdot \frac{\sin w(t-u)}{t-u} \cdot du = \frac{1}{\pi} \int_{-t}^{\infty} f(s) \cdot \frac{\sin ws}{s} \cdot ds$$

where we put $u = t + s$ and $f(s) = e^{-\gamma s} y(t+s)$.

We can break up the integral into $\int_0^{\infty}$ and $\int_{-t}^{0}$, say I_1 and I_2.

$$\text{i.e. } I_1 = \int_0^{\infty} f(s) \frac{\sin ws}{s} ds = f(0) \int_0^{\delta} \frac{\sin ws}{s} \cdot ds + \int_0^{\delta} \frac{f(s) - f(0)}{s} \cdot \sin ws \cdot ds + \int_{\delta}^{\Delta} f(s) \cdot \frac{\sin ws}{s} \cdot ds + \int_{\Delta}^{\infty} f(s) \cdot \frac{\sin ws}{s} \cdot ds$$

For the first integral

$$\int_0^{\delta} \frac{\sin ws}{s} \cdot ds = \int_0^{w\delta} \frac{\sin z}{z} \cdot dz = \frac{\pi}{2} + O\left(\frac{1}{w}\right)$$

For the second integral

$$\left| \int_0^{\delta} \frac{f(s) - f(0)}{s} \cdot \sin ws \cdot ds \right| \leq \int_0^{\delta} \left| \frac{f(s) - f(0)}{s} \right| \cdot ds$$ and we can choose δ

such that $\left| \frac{f(s) - f(0)}{s} \right| \leq \frac{\varepsilon}{\delta}$ and hence the integral $\leq \varepsilon$.

For the third integral

$$\int_{\delta}^{\Delta} f(s) \cdot \frac{\sin ws}{s} \cdot ds = \left[- \frac{\cos ws}{ws} \cdot f(s) \right]_{\delta}^{\Delta} + \frac{1}{w} \int_{\delta}^{\Delta} \cos ws \cdot \frac{d}{ds} \left\{ \frac{f(s)}{s} \right\} ds = O\left(\frac{1}{w}\right)$$

For the fourth integral $\left| \int_{\Delta}^{\infty} f(s) \cdot \frac{\sin ws}{s} \cdot ds \right| \leq \int_{\Delta}^{\infty} \left| \frac{f(s)}{s} \right| \cdot ds.$

and hence we obtain $\lim\limits_{w \to \infty} \int_0^{\infty} f(s) \cdot \frac{\sin ws}{s} \cdot ds = \frac{1}{2} \pi f(0) = \frac{1}{2} \pi y(t)$

Similarly $\lim\limits_{w \to \infty} \int_{-t}^{0} f(s) \cdot \frac{\sin ws}{s} \cdot ds = \frac{1}{2} \pi y(t)$ and therefore

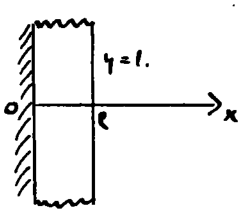
$$y(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}\bar{y}(\lambda)\,d\lambda .$$

As we stated before, the line integral in the theorem is usually evaluated by transforming it to a closed contour integral and using the calculus of residues. To make the solution completely rigorous it is necessary to verify the result obtained in the original equation and conditions. Thus, we actually assume that a solution exists with the necessary properties and verify it later; this seems to be adequate for most purposes in applied mathematics. We may note here that R.V.Churchill uses an entirely different method of approach and proves the Inversion Theorem under conditions on $\bar{y}(p)$ instead of, as above, conditions on $y(t)$.[1]

The Laplace Transformation can be applied to a number of different types of equations, and we propose now to discuss a few of these applications.

2.11. Heat Conduction.

e.g.1. Derive a formula for the temperature in a wall with its face $x = 0$ insulated and its face $x = \ell$ kept at a constant temperature $y(x,t) = 1$, the initial temperature being zero.



If $y(x,t)$ is the temperature function ~~function~~, then the problem is equivalent to solving

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \qquad (1)$$

with the initial condition $y(x,0) = 0$ (2)

1. R. V. Churchill P. 159 →.

and the boundary conditions $\left.\begin{array}{ll} y(\ell,t) = 1 & t > 0 \\ \frac{\partial y}{\partial x} = 0 \quad x = 0, & t > 0 \end{array}\right\}$ (3)

Assuming that y(x,t) satisfies the necessary conditions for the transform to exist, we follow the usual procedure obtaining

$$p\bar{y}(x,p) = \frac{\partial^2 \bar{y}(x,p)}{\partial x^2} \quad \text{using (2)} \qquad (4)$$

$$\text{and} \quad \bar{y}(\ell,p) = 1/p \; ; \; \left\{ \frac{\partial \bar{y}(x,p)}{\partial x} \right\}_{x=0} = 0 \qquad (5)$$

The solution of (4) is obviously

$\bar{y} = Ae^{-\sqrt{p}.x} + Be^{+\sqrt{p}.x}$ and hence from (5) we have

$$\bar{y} = \frac{ch.\sqrt{p}.x}{p.ch.\sqrt{p}.\ell} \qquad (6)$$

and from the Inversion Theorem

$$y(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{ch\sqrt{\lambda}.x}{\lambda\, ch.\sqrt{\lambda}.\ell}.d\lambda \qquad (7)$$

where the integrand has simple poles at $\lambda = 0, -\frac{(2n+1)^2\pi^2}{4\ell^2}$, n = 0,1,2,3,---

We can replace the line integral by a semi-circle as before if the circle has a radius $R = \frac{n^2\pi^2}{\ell^2}$ and therefore cannot pass through any one of the poles.

The integral (7) must therefore equal $2\pi i$ (sum of residues)

Pole $\lambda = 0$.

$$\text{Residue} = \lim_{\lambda\to 0} \frac{ch.\sqrt{\lambda}.x}{ch.\sqrt{\lambda}.\ell} e^{\lambda t} = 1.$$

Pole $\lambda = -\frac{(2n+1)^2\pi^2}{4\ell^2}$

$$\text{Residue} = \left[\frac{ch.\sqrt{\lambda}.x.\,e^{\lambda t}}{\frac{d}{d\lambda}\{\lambda\, ch.\sqrt{\lambda}.\ell\}}\right]_{\lambda=-\frac{(2n+1)^2\pi^2}{4\ell^2}} = -\frac{4\cos\frac{(2n+1)\pi x}{2\ell}\, e^{-\frac{(2n+1)^2\pi^2}{4\ell^2}.t}}{(-1)^n(2n+1).\pi.}$$

and thus $$y(x,t) = 1 - 4\sum_{n=0}^{\infty} (-1)^n e^{-\frac{(2n+1)^2\pi^2 t}{4\ell^2}} \cdot \frac{\cos\frac{(2n+1)\pi x}{2\ell}}{(2n+1)\pi}$$;

To complete the solution we must prove that when $0 < x < 1$ and $t > 0$

then $\int e^{\lambda t} \frac{ch.\, x\sqrt{\lambda}}{\lambda\, ch.\, \ell\sqrt{\lambda}}\, d\lambda$ vanishes over the arc BCA, i.e. it satisfies the conditions of the lemma (P.36) on the circle Γ of radius $R = \frac{n^2\pi^2}{\ell^2}$.

Since $2\cosh(a+ib)\cosh(a-ib) = \cosh 2a + \cos 2b$ and $\lambda = Re^{i\theta}$

then $$2\left|\cosh \ell\sqrt{\lambda}\right|^2 = \cosh(2n\pi\cos\tfrac{1}{2}\theta) + \cos(2n\pi\sin\tfrac{1}{2}\theta)$$
$$= \cosh(2n\pi\cos\tfrac{1}{2}\theta)\left[1 + \operatorname{sech}(2n\pi\cos\tfrac{1}{2}\theta)\cos(2n\pi\sin\tfrac{1}{2}\theta)\right]. \quad (8)$$

Let $\sin\frac{1}{2}\beta = \frac{2n-\frac{1}{2}}{2n}$ so that $\cos\frac{1}{2}\beta = \frac{\sqrt{8n-1}}{4n}$ and when $\pi \geq \theta \geq \beta$

then $$2n\pi \geq 2n\pi\sin\tfrac{1}{2}\theta \geq 2n\pi\sin\tfrac{1}{2}\beta = (2n - \tfrac{1}{2})\pi,$$

and therefore $$\cos(2n\pi\sin\tfrac{1}{2}\theta) \geq 0.$$

Hence when $\pi \geq \theta \geq \beta$
$$2\left|\cosh \ell\sqrt{\lambda}\right|^2 \geq \cosh(2n\pi\cos\tfrac{1}{2}\theta). \quad (9)$$

Also, when $\beta \geq \theta \geq 0$,
$$\left|\operatorname{sech}(2n\pi\cos\tfrac{1}{2}\theta)\cos(2n\pi\sin\tfrac{1}{2}\theta)\right| \leq \operatorname{sech}(2n\pi\cos\tfrac{1}{2}\theta)$$
$$\leq \operatorname{sech}(2n\pi\cos\tfrac{1}{2}\beta)$$
$$= \operatorname{sech}\tfrac{1}{2}\pi\sqrt{(8n-1)}$$
$$< \operatorname{sech}\tfrac{1}{2}\pi\sqrt{7}, \text{ when } n > 1. \quad (10)$$

Using (9) and (10) in (8), we have
$$2\left|\cosh \ell\sqrt{\lambda}\right|^2 > (1 - \operatorname{sech}\tfrac{1}{2}\pi\sqrt{7})\cosh(2n\pi\cos\tfrac{1}{2}\theta) \quad \text{when } \pi \geq \theta \geq 0.$$

Therefore $\left|\cosh \ell\sqrt{\lambda}\right| > C\cosh^{\frac{1}{2}}(2n\pi\cos\tfrac{1}{2}\theta)$, $\pi \geq \theta \geq 0$, where C is a constant independent of n. Also
$$\left|\cosh x\sqrt{\lambda}\right| = \left|\cosh\{(n\pi/\ell)x(\cos\tfrac{1}{2}\theta + i\sin\tfrac{1}{2}\theta)\}\right| \leq \cosh\{(n\pi/\ell)x\cos\tfrac{1}{2}\theta\}$$

Hence $$\left|\frac{\cosh x\sqrt{\lambda}}{\cosh \ell\sqrt{\lambda}}\right| < \frac{\cosh\{(n\pi/\ell)\, x\cos\frac{1}{2}\theta\}}{C\sqrt{[\cosh 2n\pi(\cos\frac{1}{2}\theta)]}}$$

$$< C' \frac{e^{(n\pi/\ell)\,x\cos\frac{1}{2}\theta}}{e^{n\pi\cos\frac{1}{2}\theta}}$$

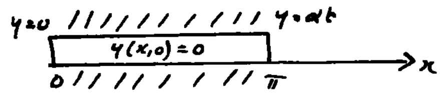
$$< C' e^{-\{(\ell-x)/\ell\}\,n\pi\cos\frac{1}{2}\theta}$$

$$< \epsilon', \text{ when } \pi \geq \theta \geq 0.$$

Since this also holds for $\pi \leq \theta \leq 2\pi$, the conditions are satisfied and the result follows.

e.g.2. An example involving arbitrary end temperatures.

Let the temperature of the end $x = \pi$ be a function of <u>t</u> i.e. αt.



Thus, $y(x,t)$ must satisfy

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \qquad t > 0, \text{ and } 0 < x < \pi \tag{1}$$

where $y(x,0) = 0$ and $y(0,t) = 0$, $y(\pi,t) = \alpha t$ $\quad t > 0$ (2) & (3)

Applying the transform and using the initial condition, we obtain

$$p\bar{y} = \frac{\partial^2 \bar{y}}{\partial x^2} \tag{4}$$

$$\text{and } \bar{y}(0,p) = 0, \qquad \bar{y}(\pi,p) = \alpha/p^2 \tag{5}$$

The solution of (4) is obviously

$$\bar{y} = Ae^{-x\sqrt{p}} + Be^{+x\sqrt{p}}$$ and incorporating (5) we have

$$\bar{y} = \frac{\alpha \sinh x\sqrt{p}}{p^2 \sinh \pi\sqrt{p}}.$$

$$\text{hence } y(x,t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}\cdot\frac{\alpha\,\text{sh}\,x\sqrt{\lambda}}{\lambda^2\,\text{sh}\,\pi\sqrt{\lambda}}\cdot d\lambda.$$

Replacing the line integral by the usual contour we see that the integrand has poles at $\lambda = 0$ (second order), $\lambda = -n^2$ where $n = 1,2,3,\ldots$

Pole $\lambda = 0$.

$$\text{Residue} = \frac{\alpha}{\pi}\left(\frac{1}{6}x^3 - \frac{1}{6}x\pi^2 + tx\right)$$

Pole $\lambda = -n^2$

$$\text{Residue} = -\frac{2\alpha \sin nx}{n^2\pi(-1)^n}\cdot e^{-n^2t}$$

and hence the solution

$$y(x,t) = \frac{\alpha}{\pi}\left[\frac{1}{6}x(x^2-\pi^2) + tx\right] - \frac{2\alpha}{\pi}\sum_1^\infty (-1)^n \frac{\sin nx}{n^3}\cdot e^{-n^2t}$$

n.b. In this and succeeding problems we have taken the limits of $\underline{x}$ to be $0, \pi$ to facilitate comparison with problems involving Fourier Transforms. It is obvious that the procedure is exactly the same if the limits were $0, \ell$.

To complete the proof we ought to show that it is permissable to change the contour of integration and, of course, verify the solution. In the last example we proved that the former was permissable; here we intend to show the verification.

When $x = \pi$, $y(x,t) = \frac{\alpha}{\pi}(\pi t) = \alpha t$.

$x = 0$, $y(x,t) = 0$.

For $t = 0$, $y(x,t) = \frac{\alpha}{\pi}\left[\frac{1}{6}x(x^2-\pi^2)\right] - \frac{2\alpha}{\pi}\sum_1^\infty (-1)^n \frac{\sin nx}{n^3}\cdot e^{-n^2t}$

and if we expand $\frac{1}{6}(x^3 - x\pi^2)$ in a sine series between 0 and π, we obtain $2\sum(-1)^r\frac{\sin rx}{r^3}$ and thus $y(x,t) = 0$ for $t = 0$.

$$\frac{\partial y}{\partial t} = \frac{\alpha x}{\pi} + \frac{2\alpha}{\pi}\sum_1^\infty (-1)^n \frac{\sin nx}{n}\cdot e^{-n^2t}$$

and $$\frac{\partial^2 y}{\partial x^2} = \frac{\alpha}{\pi}(x) + \frac{2\alpha}{\pi}\sum_1^\infty (-1)^n \frac{\sin nx}{n}\cdot e^{-n^2t}$$

The solution therefore satisfies the original equation and conditions e.g.3. The flux of heat into an infinite cylinder through its surface $r=1$ is a constant i.e. $\left(\frac{\partial y(r,t)}{\partial r}\right)_{r=1} = A$. If the initial temperature is zero, find a formula for the heat function.

If the heat function is $y(x,t)$ it must satisfy the following.

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial y}{\partial r} \qquad 0 \leq r < 1, \quad t > 0. \qquad (1)$$

$$y(r,t) = 0 \qquad t = 0. \qquad (2)$$

$$\frac{\partial y}{\partial r} = A \qquad t > 0, \quad r = 1. \qquad (3)$$

Applying the transform

$$p\,\bar{y}(r,p) = \frac{\partial^2 \bar{y}}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial \bar{y}}{\partial r} \qquad \text{using (2)} \qquad (4)$$

and

$$\frac{\partial \bar{y}}{\partial r} = A/p \qquad \text{when } r = 1. \qquad (5)$$

The solution of (4) must be finite and a solution of Bessel's equation. The solution that is finite at $r=0$ is

$$\bar{y}(r,p) = CI_0(r\sqrt{p}) \qquad \text{where } I_0(r\sqrt{p}) = J_0(ir\sqrt{p})$$

and thus employing (4) we see that

$$\bar{y}(r,p) = \frac{A\,I_0(r\sqrt{p})}{p^{3/2}\,I_1(\sqrt{p})}$$

Let the positive solutions of $J_1(z)$ be $\beta_1 \beta_2 \beta_3$ etc., then the solutions of $I_1(\sqrt{p})$ are $p = -\beta_n^2$ where $n = 0, 1, 2, 3, \ldots$

$$\text{Therefore } y(r,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\cdot\frac{A\,I_0(r\sqrt{\lambda})}{\lambda^{3/2}\,I_1(\sqrt{\lambda})}\,d\lambda.$$

We replace by the usual contour and we have poles at $\lambda = 0, -\beta_n^2, n = 0, 1, 2, \ldots$

Pole $\lambda = 0$.

Expansion of integrand gives $\frac{2A}{\lambda^{3/2}\lambda^{1/2}}\left[1+\frac{\lambda t}{1!}+\cdots\right]\left[1+\frac{r^2\lambda}{2^2}+\cdots\right]\left[1-\frac{\lambda}{1.4}+\cdots\right.$

and residue $= 2A(t - \frac{1}{8} + \frac{r^2}{4})$

Pole $\lambda = -\beta_n^2$

$$\text{Residue} = \left[\frac{A I_0(r\sqrt{\lambda})}{\frac{d}{d\lambda}\left\{\lambda^{3/2} . I_1(\sqrt{\lambda})\right\}}\right]_{\lambda = -\beta_n^2} . e^{-\beta_n^2 t}$$

$$= \frac{A I_0(r i \beta_n)}{\frac{1}{2} \cdot -\beta_n^2 \cdot I_1'(i\beta_n)} \cdot e^{-\beta_n^2 t} = \frac{2A. J_0(r\beta_n)\, e^{-\beta_n^2 t}}{-\beta_n^2 . J_1'(\beta_n)}$$

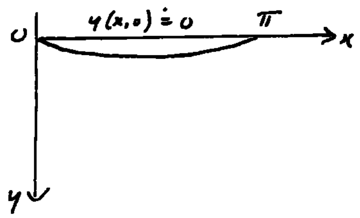
Hence $$y(r,t) = 2A\left[\frac{r^2}{4} - \frac{1}{8} + t - \sum_0^\infty \frac{J_0(r\beta_n)}{\beta_n^2 . J_1'(\beta_n)} \cdot e^{-\beta_n^2 . t}\right]$$

where $\beta_1 \beta_2 \beta_3 \cdots$ are the positive roots of $J_1(z) = 0$.

2.12. Vibrating Strings.

The Laplace Transformation may also be used to solve differential equations arising in the study of vibrating strings. (c.f. Fourier Transforms)

e.g.1. Vibrations of a string with fixed ends.



Let the string be stretched between the origin and the point $(\pi, 0)$ and assume it is released from rest in the position $y = 0$ where $y(x,t)$ is the displacement function.

Therefore $$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} + g \qquad 0 < x < \pi, \quad t > 0. \tag{1}$$

$$y(x,0) = \left\{\frac{\partial y(x,t)}{\partial t}\right\}_{t=0} = 0. \tag{2}$$

and $$y(0,t) = y(\pi,t) = 0. \tag{3}$$

Apply the transform to (1) and thus, using (2), we have

$$p^2 \cdot \bar{y}(x,p) = a^2 \cdot \frac{\partial^2 \bar{y}(x,p)}{\partial x^2} + \frac{g}{p} \qquad (4)$$

$$\text{and} \quad \bar{y}(0,p) = \bar{y}(\pi,p) = 0. \qquad (5)$$

The solution of (4) is

$$\bar{y}(x,p) = \frac{g}{p^3} + Be^{-\frac{px}{a}} + Ce^{+\frac{px}{a}}$$ and hence from (5),

$$\bar{y}(x,p) = \frac{g}{p^3} + \frac{g(1-e^{-\frac{p\pi}{a}})}{2p^3 \cdot sh\frac{p\pi}{a}} \cdot e^{-\frac{px}{a}} + \frac{g(e^{-\frac{p\pi}{a}}-1)}{2p^3 \cdot sh\frac{p\pi}{a}} \cdot e^{+\frac{px}{a}}.$$

reducing to
$$\bar{y}(x,p) = \frac{g}{p^3} - \frac{g \cdot \cosh\frac{p}{a}(x-\frac{\pi}{2})}{p^3 \cosh\frac{p\pi}{2a}}$$

hence
$$y(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \cdot \frac{g}{\lambda^3}\left[1 - \frac{ch\frac{\lambda}{a}(x-\frac{\pi}{2})}{ch\frac{\lambda\pi}{2a}}\right]d\lambda.$$

with poles at $\lambda = 0,\ (2n-1)ai,\ -(2n-1)ai$ for $n = 1, 2, 3, \ldots$

Pole $\lambda = 0$.

$$\text{Residue} = \tfrac{1}{2}t^2 + \frac{1}{2a^2}(x^2 - \pi x)$$

Pole $\lambda = (2n-1)ai$.

$$\text{Residue} = \left.\frac{ch\frac{\lambda}{a}(x-\frac{\pi}{2})}{\frac{d}{d\lambda}\left\{\lambda^3 \cdot ch\frac{\lambda\pi}{2a}\right\}}\right]_{\lambda=(2n-1)ai} \cdot e^{(2n-1)ait}$$

$$= \frac{2\sin(2n-1)x}{\pi a^2 (2n-1)^3} \cdot e^{(2n-1)ait}.$$

Pole $\lambda = -(2n-1)ai$.

$$\text{Residue} = \frac{2\sin(2n-1)x}{\pi a^2 (2n-1)^3} \cdot e^{-(2n-1)ait}.$$

Hence $y(x,t) = \frac{S}{2a^2}\left\{\pi x - x^2\right\} - \frac{4S}{\pi a^2}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x.\cos(2n-1)at}{(2n-1)^3}$

The change of contour and the solution must be verified as usual.

We have touched upon a few of the different problems which can be solved using the Laplace Transformation though it is by no means an exhaustive list. The transformation may also be applied to problems in Hydrodynamics, electric transmission lines (Heaviside's original use of the transform) etc.

From a study of the procedure used in the above examples we can resolve it into three distinct steps:-

(i) Application of the transform to reduce the given equation by one variable; in the examples taken, to an equation of one variable.

(ii) Solution of this equation to obtain a formula for the transform $\bar{y}(p)$.

(iii) Use of the table of transforms or the Inversion Theorem to give the required solution.

To make the solution rigorous we must verify it and the change of contour if the Inversion Theorem is used.

There are other transforms which are of importance in applied mathematics and mathematical physics, and although their range of usefulness is more limited than the above we must include them for completeness.

2.20. Fourier Transforms.

The theory of Fourier Integrals originated in Fourier's "Analytical Theory of Heat". It was pointed out later by Cauchy that the formulae used by Fourier lead to reciprocal relations between pairs of functions.

e.g. if $F_c(u) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos ut.dt.$

then $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(u) \cos xu.du.$

The relation between $f(x)$ and $F_c(x)$ is reciprocal and they are usually known as Fourier cosine transforms of each other. In a similar manner we have the Fourier sine transforms.

if $F_s(u) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin ut.dt.$

then $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(u) \sin ux.du$

The more common infinite Fourier transform is unsymmetrical:-

i.e. if $F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{iut}.dt$

then $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(u).\, e^{-iux}.du.$

and we note that if $f(x)$ is even then $F(x) = F_c(x)$

and if $f(x)$ is odd then $F(x) = iF_s(x)$.

However the condition that $f(t)$ should be integrable over an infinite range is rather a severe one and this feature combined with several others rather limits the types of boundary value problems to which these transforms are applicable. A more useful Fourier transform is one which is integrable over a finite range (usually taken to be $0, \pi$), and this we propose to discuss.

2.21. Fourier Finite Sine Transforms.

If f(x) denotes a function that is sectionally continuous over some finite interval of x, we can, by an appropriate choice of origin and unit of length, arrange the interval so that the end points become $x = 0$ and $x = \pi$.

i.e. if x is sectionally continuous between $0, \ell$ then by putting $x' = \frac{\pi x}{\ell}$ we see that x' is sectionally continuous in the interval $0, \pi$.

We define the Fourier Sine Transformation of f(x) in the interval $0, \pi$ to be the operation

$$\int_0^{\pi} f(x) \sin nx . dx \qquad n = 1, 2, 3, ---$$

and this function of n we call $f_s(n)$

$$\text{Thus} \quad f_s(n) = \int_0^{\pi} f(x) \sin nx . dx \qquad n = 1, 2, 3, --- \qquad (1)$$

e.g. if $f(x) = x \qquad (0 < x < \pi)$

$$\text{then} \quad f_s(n) = \frac{\pi (-1)^{n+1}}{n} \qquad n = 1, 2, 3, ---$$

If the first derivative of f(x) is also sectionally continuous in the interval $0, \pi$ and if $f'(x)$ is defined at each point of discontinuity so that

$$f(x_0) = \tfrac{1}{2}\left[f(x_0+0) + f(x_0-0)\right] \qquad 0 < x_0 < \pi$$

then the Fourier sine series corresponding to f(x) converges to

$$f(x) = \frac{2}{\pi}\sum_{1}^{\infty} \sin nx \int_0^{\pi} f(u) \sin nu.du. \qquad 0 < x < \pi$$

and hence using (1)

$$f(x) = \frac{2}{\pi}\sum_{1}^{\infty} f_s(n) \sin nx \qquad 0 < x < \pi, \; n = 1,2,3,\ldots \quad (2$$

giving us the inversion formula for the Fourier transformation.

e.g. if $f_s(n) = \frac{\pi.(-1)^{n+1}}{n}$ $\qquad 0 < x < \pi,$

$$\text{then } f(x) = \frac{2}{\pi}\sum_{1}^{\infty} \frac{\pi.(-1)^{n+1}}{n}.\sin nx = 2\sum_{1}^{\infty} (-1)^{n+1}.\frac{1}{n} \sin nx \qquad 0<x<\pi$$

$$= 2.\frac{x}{2} = \underline{x}.$$

Before we can attempt to apply the Fourier transformation to partial differential equations we need the following property.

<u>Transforms of derivatives of even order</u>

Let $f'(x)$ be continuous and $f''(x)$ sectionally continuous inthe interval $0, \pi$. Then

$$\int_0^{\pi} f''(x) \sin nx.dx = \Big[f'(x) \sin nx\Big]_0^{\pi} - n\int_0^{\pi} f'(x) \cos nx.dx$$

$$= \Big[-n f(x) \cos nx\Big]_0^{\pi} - n^2\int_0^{\pi} f(x) \sin nx.dx$$

$$= \Big[-n^2 f_s(n)\Big] + n\Big[f(0) - (-1)^n f(\pi)\Big]$$

thus giving us the transform of $f''(x)$.

By repeated application of the above we arrive at the following theorem:-

If $f(x)$ has a sectionally continuous derivative of order $2m$ and continuous derivative of order $2m - 1$, $(m = 1,2,3,\ldots)$, then

$$\int_0^{\pi} f^{2m}(x) \sin nx \, dx = (-n^2)^m f_s(n) - (-1)^m n^{2m-1}\left[f(0) - (-1)^n f(\pi)\right]$$

$$- (-1)^{m-1} n^{2m-3}\left[f''(0) - (-1)^n f''(\pi)\right] - \cdots$$

$$+ n\left[f^{2m-2}(0) - (-1)^n f^{2m-2}(\pi)\right]. \qquad (3)$$

and we note that the values of even-ordered derivatives appear at the end points.

2.22. Finite Fourier Cosine Transforms.

If $f(x)$ is defined in the interval $0, \pi$ in the same manner as for the sine transformation then we say that

$$f_c(n) = \int_0^{\pi} f(x) \cos nx . dx. \qquad n = 0,1,2,3,\ldots \qquad (4)$$

where $f_c(n)$ is known as the finite cosine transform of $f(x)$.

e.g. if $f(x) = x$ then $f_c(n) = \frac{\pi^2}{2}$, $n = 0$,

$$= \frac{-\{1-(-n)^n\}}{n^2}, \quad n = 1,2,3,\ldots$$

or, if $f(x) = \frac{1}{c} e^{cx}$ then $f_c(n) = \frac{(-1)^n . e^{c\pi} - 1}{n^2 + c^2}$

In a similar way to that of the sine transformation we have the inversion formula

$$f(x) = \frac{1}{\pi} f_c(0) + \frac{2}{\pi} \sum_{1}^{\infty} f_c(n) \cos nx \qquad 0 < x < \pi \qquad (5)$$

where f(x) and f'(x) are sectionally continuous functions in the interval.

To find the cosine transform of a derivative of f(x) we integrate by parts, thus

$$\int_0^{\pi} f'(x) \cos nx \, dx = \Big[f(x) \cos nx\Big]_0^{\pi} + n \int_0^{\pi} f(x) \sin nx . dx$$

$$= n \int_0^{\pi} f(x) \sin nx . dx + (-1)^n f(\pi) - f(0) \, , \; n = 0,1,2,---$$

where f(x) is continuous and f'(x) sectionally continuous in $0, \pi$. Continuing the above process we obtain the following:-

If f(x) has a sectionally continuous derivative of order 2m and continuous derivative of order 2m - 1, (m = 1,2,3,--), then

$$\int_0^{\pi} f^{2m}(x) \cos nx \, dx = (-n^2)^m f_c(n) - (-1)^{m-1} n^{2m-2} \left[f'(0) - (-1)^n f'(\pi) \right]$$

$$- (-1)^{m-2} n^{2m-4} \left[f'''(0) - (-1)^n f'''(\pi) \right] - \cdots$$

$$- \left[f^{2m-1}(0) - (-1)^n f^{2m-1}(\pi) \right] \qquad (6)$$

and we note that the values of the odd ordered derivatives appear at the end points.

Although the Fourier transform is not applicable to such a wide range of problems as the Laplace transform, there are problems to which it is more suited.

2.23. Applications of the Fourier transform.

e.g.1. Vibrations of a horizontal string with fixed ends (c.f. P.51)

Assume that the string is released from rest at t = 0, and that y(x,t) is the displacement function.

$$\text{Then} \quad \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} + g \qquad 0 < x < \pi, \quad t > 0. \tag{7}$$

$$y(x,0) = \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \tag{8}$$

$$y(0,t) = y(\pi,t) = 0 \qquad t \geq 0. \tag{9}$$

The point to note is that the values of y(x,t) at the end-points are given and we must therefore use the sine transformation.

Applying the transform we have, using the conditions (9)

$$\frac{d^2 y_s(n,t)}{dt^2} = -a^2 n^2 y_s(n,t) + g\int_0^\pi \sin nx . dx$$

$$\text{i.e.} \quad \frac{d^2 y_s(n,t)}{dt^2} + a^2 n^2 y_s(n,t) = \frac{g[1-(-1)^n]}{n} \tag{10}$$

$$\text{and} \quad y_s(n,0) = 0 : \qquad \left\{\frac{d}{dt} y_s(n,t)\right\}_{t=0} = 0 \tag{11}$$

Equation (10) is an ordinary differential equation and its solution is obviously

$$y_s(n,t) = \frac{g}{a^2 n^3}\left[1-(-1)^n\right] + B \cos ant + C \sin ant$$

where the constants A and B are evaluated from (11).

$$\text{i.e.} \quad y_s(n,t) = \frac{g}{a^2 n^3}\left[1-(-1)^n\right](1-\cos ant) \tag{12}$$

To find the value of y(x,t) we make the inverse transformation

$$y(x,t) = \frac{2}{\pi}\cdot\frac{g}{a^2}\sum_1^\infty \frac{1-(-1)^n}{n^3}\cdot(1-\cos ant)\sin nx.$$

The above solution is purely formal since we have assumed throughout that $y(x,t)$ and its derivatives satisfy the conditions in the preceding theoretical work. The solution, however, is quite easily verified thus justifying our assumptions. It is interesting to compare this solution with that found for the same problem using the Laplace transformation.

The L.T. gives us

$$y(x,t) = \frac{g}{2a^3}(\pi x - x^2) - \frac{4g}{\pi a^2}\sum_{1}^{\infty}\frac{\sin(2n-1)x \,.\, \cos(2n-1)at}{(2n-1)^3}$$

and we see that the solutions are identical if

$$\frac{\pi}{8}(\pi x - x^2) = \sin x + \frac{\sin 3x}{3^3} + \cdots \qquad (13)$$

this is so, since the expansion of $\pi x - x^2$ as a Fourier sine series in the interval $0, \pi$ is

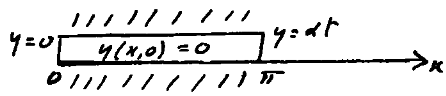
$$(\pi x - x^2) = \frac{4}{\pi}\sum_{1}^{\infty}\frac{1-(-1)^n}{n^3}\sin nx \qquad (14)$$

which is identical with (13).

Hence both solutions are the same though it can be seen that for this particular problem the Fourier transformation leads us to the solution in a more simple way.

e.g.2. The temperature in a solid bar. (c.f. P, 48)

Consider a bar initially at zero temperature. The end $x = 0$ is



maintained at zero temperature and the end $x = \pi$ has a temperature which varies with <u>t</u>, If $y(x,t)$ is the temperatuse function

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \tag{15}$$

and $\quad y(x,0) = 0 \qquad 0 \le x \le \pi$ (16)

$y(0,t) = 0, \qquad y(\pi,t) = \alpha t \qquad t > 0$ (17)

Once again we are given the values of y(x,t) at the end points and we need the sine transform.

i.e. $\frac{dy_s}{dt} = -n^2 y_s - n\left[\alpha t(-1)^n\right]$ using (17) (18)

and $\quad y_s(n,0) = 0$ (19)

The solution of (18) is straightforward and since it must satisfy (19) we obtain

$$y_s(n,t) = \frac{\alpha}{n^3}(-1)^n - \frac{\alpha}{n}\cdot t\cdot(-1)^n - \frac{\alpha}{n^3}\cdot(-1)^n\cdot e^{-n^2 t}$$

and hence $\quad y(x,t) = \frac{2\alpha}{\pi}\sum_1^\infty \frac{(-1)^n}{n^3}\cdot\sin nx - \frac{2\alpha t}{\pi}\sum_1^\infty \frac{(-1)^n}{n}\cdot\sin nx - \frac{2\alpha}{\pi}\sum_1^\infty \frac{(-1)^n}{n^3}\cdot\sin nx\cdot e^{-n^2 t}$

<u>Verification.</u>

$$\frac{\partial y}{\partial t} = -\frac{2\alpha}{\pi}\sum_1^\infty \frac{(-1)^n}{n}\cdot\sin nx + \frac{2\alpha}{\pi}\sum_1^\infty \frac{(-1)^n}{n}\sin nx\cdot e^{-n^2 t}$$

and $\quad \frac{\partial^2 y}{\partial x^2} = -\frac{2\alpha}{\pi}\sum_1^\infty \frac{(-1)^n}{n}\cdot\sin nx + \frac{2\alpha}{\pi}\sum_1^\infty \frac{(-1)^n}{n}\sin nx\cdot e^{-n^2 t}$

since the second term (which = x) disappears. Thus the solution satisfies the original equation.

Also at t = 0,

$$y(x,t) = \frac{2\alpha}{\pi}\sum_1^\infty \frac{(-1)^n}{n^3}\cdot\sin nx - \frac{2\alpha}{\pi}\sum_1^\infty \frac{(-1)^n}{n^3}\cdot\sin nx = 0$$

at $\quad x = 0, \qquad y(x,t) = 0.$

at $\quad x = \pi, \qquad y(x,t) = \frac{2\alpha t}{\pi}\cdot\frac{\pi}{2} = \alpha t$ thus satisfying the initial and boundary conditions. The solution is therefore verified.

To compare it with the Laplace transformation we note that its solution is

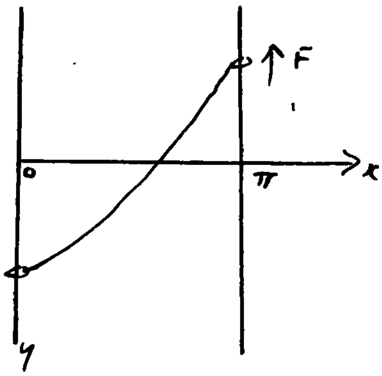
$$y(x,t) = \frac{\alpha}{\pi}\left[\frac{1}{6}x(x^2-\pi^2) + tx\right] - \frac{2\alpha}{\pi}\sum_{1}^{\infty}(-1)^n \frac{\sin nx}{n^3} \cdot e^{-n^2 t}$$

and this is identical with that obtained by the Fourier transform since it can be shown that

$$\frac{1}{6}x(x^2 - \pi^2) = 2\sum_{1}^{\infty}\frac{(-1)^n}{n^3}\sin nx \qquad 0 < x < \pi$$

and $x = 2\sum_{1}^{\infty}\frac{(-1)^n}{n}\sin nx$ when expanded as a Fourier sine series in the interval $0, \pi$.

e.g.3. Horizontal string with sliding ends. The ends of the string are looped about vertical supports at $x = 0$, $x = \pi$. A constant upward force acts on the right hand loop. Therefore if $y(x,t)$ is the displacement as the string falls from rest, it must satisfy



$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} + g \qquad 0 < x < \pi, \quad t > 0 \qquad (20)$$

$$y(x,0) = 0, \quad \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad (21)$$

$$\left(\frac{\partial y}{\partial x}\right)_{x=0} = 0, \quad \left(\frac{\partial y}{\partial x}\right)_{x=\pi} = -b \qquad (22)$$

where b is the magnitude of the force divided by the tension. Since we now have the values of an odd derivative at the end points we must use the cosine transform.

Hence $$\frac{d^2}{dt^2} y_c + a^2 n^2 y_c = -a^2 b(-1)^n + g\int_0^{\pi} \cos nx \, dx \qquad (23)$$

where we have used (22).

Also $$y_c(n,t) = \frac{dy_c}{dt} = 0 \qquad t = 0 \qquad (24)$$

When $n = 0$, $$\int_0^{\pi} \cos nx . dx = \pi$$

and the solution of (23) and (24) is

$$y_c(0,t) = \tfrac{1}{2}(\pi g - a^2 b)t^2 \qquad (25)$$

When $n = 1, 2, 3, \ldots$ $$\int_0^{\pi} \cos nx . dx = 0$$

and the solution is

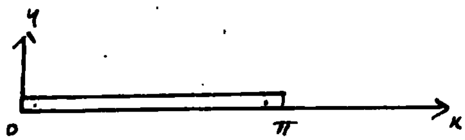
$$y_c(n,t) = b\left[\frac{(-1)^n}{n^2} \cos ant - \frac{(-1)^n}{n^2}\right] \qquad (26)$$

and thus, using the inversion formula

$$y(x,t) = \frac{1}{\pi} y_c(0,t) + \frac{2}{\pi}\sum_{1}^{\infty} y_c(n,t) \cos nx$$

where $y_c(0,t)$ and $y_c(n,t)$ are given by the equations (25) and (26). As usual, the solution must be verified before it can be said to be complete.

e.g.4. Transverse Vibrations of a Beam.



A constant transverse force F(x) acts at each point of a beam, with ends 0, π hinged. The transverse displacement must then satisfy

$$\frac{\partial^2 y}{\partial t^2} = -a^2 \frac{\partial^4 y}{\partial x^4} + F(x) \qquad (27)$$

$$y(x,t) = \frac{\partial^2 y}{\partial x^2} = 0 \quad \text{at} \quad x = 0 \text{ and } \pi. \qquad (28)$$

$$y(x,t) = \frac{\partial y}{\partial t} = 0 \qquad \text{at} \quad t = 0 \qquad (29)$$

Applying the sine transform we obtain

$$\frac{d^2 y_s}{\partial t^2} + a^2 n^4 y_s = f_s(n) \qquad \text{using (28), where}$$

$f_s(n)$ is the sine transform of $F(x)$.

Hence $y_s = \frac{f_s(n)}{a^2 n^4} + A \sin an^2 t + B \cos an^2 t.$

But $y(n,t) = \frac{\partial y(n,t)}{\partial t} = 0 \qquad \text{at} \quad t = 0$

Therefore $y_s = -\frac{f_s(n)}{a^2 n^4} + \frac{g_s(n)}{a^2}$ where $G^4(x) = F(x)$ and $G(x) = G''(x)$ at $x = 0, \pi$

and $y(x,t) = \frac{g(x)}{a^2} - \frac{2}{a^2 \pi} \sum_1^\infty \frac{f_s(n)}{n^4} \cos an^2 t . \sin nx$

The above examples are designed to show the procedure used in applying the Fourier transforms. They by no means constitute an exhaustive list of the different problems which can be solved by means of the F.T.s, but they do give us some idea of its usefulness.

The essentials of the above procedure are

1. Note the boundary conditions on $y(x,t)$.

(a) If the values of even derivatives are given at the end points use the sine transform.

(b) If the values of odd derivatives are given at the end points use the cosine transform.

2. Transformation of the given differential equation to reduce it to a simple ordinary equation which is easily solvable.

3. Use of the inverse transformation formulae to obtain the required solution.

The Fourier Transform is similar to the Laplace Transform in that they both depend on eliminating one variable, and in the cases considered this means reducing the equation to an equation of only one variable.

2.30. Hankel Transforms.

In the transforms already discussed, the procedure of transformation is to multiply the given equation by a function of one of the variables and then to integrate term by term over a given range of values of that variable.

Laplace Transform $\int_0^\infty f(x)\ e^{-px}\,dx.$

Fourier Transform $\int_0^\pi f(x)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} nx\,dx.$

Exactly the same procedure is followed for the Hankel Transform but in this case our multiplying factor is a solution of Bessel's equation. There are two different types of Hankel transforms.

(i) Finite transforms in which the interval is finite.

(ii) Infinite transforms inwhich the range of integration is from zero to infinity.

Firstly we intend to study the finite transform and its applications.

2.31. <u>Finite Hankel Transforms.</u>[1]

The finite transform is defined by

$$\bar{f}_J(\xi_i) = \int_0^a x.\,f(x).\,J_\mu(\xi_i x).\,dx = T_\mu\{f(x)\} \tag{1}$$

where T implies 'the transformation of' and $\bar{f}_J$ is the transform. The transform holds for all f(x) integrable in the interval $0 \leq x \leq a$.

1. Phil. Mag: Ser. 7. vol. XXXVII. p.17. Jan. 1946.

The paremeter ξ_i can be chosen in more than one way depending upon the particular problem to be solved. Naturally, the form of the inversion theorem used will depend upon the paremeter ξ_i .

If we choose ξ_i $(i = 1,2,3,- - -)$ to be the positive roots of the equation

$$J_\mu(a\,\xi_i) = 0 \tag{2}$$

regarded as an equation in ξ_i , then the appropriate inversion theorem is

$$f(x) = T^{-1}\{\bar{f}_J\} = \frac{2}{a^2}\sum_i \bar{f}_J(\xi_i)\cdot \frac{J_\mu(x\xi_i)}{\{J'_\mu(a\xi_i)\}^2} \tag{3}$$

the summation extending over all the positive roots of (2). Assuming then that ξ_i is a positive root of the equation (2) the following properties are immediately obtained:-

$$T_\mu(x^{\mu+1}) = a^{\mu+1}J_{\mu+1}(a\,\xi_i)/\xi_i \tag{4}$$

$$T_0(a^2 - x^2) = 4aJ_1(a\,\xi_i)/\xi_i^3 \tag{5}$$

$$T_0\left\{\frac{J_0(x\alpha)}{J_0(a\alpha)} - 1\right\} = -\frac{a\,J_1(a\xi_i)}{\xi_i(1-\xi_i^2/\alpha^2)} \tag{6}$$

$$T_\mu\left\{x^{-1}\frac{\partial f}{\partial x}\right\} = \frac{1}{2}\cdot\xi_i\left[T_{\mu+1}\{f(x)\} - T_{\mu-1}\{f(x)\}\right] \qquad \mu \neq 0 \tag{7}$$

$$T_0\left\{x^{-1}\frac{\partial f}{\partial x}\right\} = f(0) + \xi_i\cdot T_1\{f(x)\} \tag{8}$$

$$T_\mu\left\{\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\cdot\frac{\partial f}{\partial x}\right\} = -\xi_i^2\cdot T_\mu\{f(x)\} + \mu^2\cdot T_\mu\left\{\frac{1}{x^2}\cdot f(x)\right\} - a\,\xi_i\cdot f(a)\cdot J'_\mu(\xi_i a) \tag{9}$$

and if $\mu = 0$, $f(a) = 0$ the latter reduces to

$$T_0\left\{\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\cdot\frac{\partial f}{\partial x}\right\} = -\xi_i^2\cdot T_0\{f(x)\} = -\xi_i^2\,\bar{f}_J \tag{10}$$

The above properties are obtained from the definition of the transform (1) and the wellknown properties of the Bessel function $J_\mu(\xi_i x)$. Take, for example (9).

$$T_\mu\left\{\frac{\partial^2 f}{\partial x^2}+\frac{1}{x}\cdot\frac{\partial f}{\partial x}\right\} = \int_0^a x\left\{\frac{\partial^2 f}{\partial x^2}+\frac{1}{x}\cdot\frac{\partial f}{\partial x}\right\}\cdot J_\mu(\xi_i x)\,dx$$

$$= \int_0^a \frac{\partial}{\partial x}\left\{x\frac{\partial f}{\partial x}\right\}\cdot J_\mu(\xi_i x)\cdot dx$$

$$= \left[x\cdot\frac{\partial f}{\partial x}\cdot J_\mu(\xi_i x)\right]_0^a - \xi_i\int_0^a x\cdot\frac{\partial f}{\partial x}\cdot J'_\mu(\xi_i x)\,dx$$

$$= -\xi_i\left[x f(x)\cdot J'_\mu(\xi_i x)\right]_0^a + \xi_i\int_0^a f(x)\left[J'_\mu + x\,\xi_i\,J''_\mu\right]dx$$

$$= -\xi_i\,a f(a)\,J'_\mu(\xi_i a) + \int_0^a f(x)\cdot\frac{1}{x}\left[\mu^2-\xi_i^2x^2\right]J_\mu(\xi_i x)\,dx \quad \text{from Bessel's equation}$$

$$= -\xi_i\,a f(a)\,J'_\mu(\xi_i a) + \int_0^a \mu^2\cdot\frac{1}{x}\cdot J_\mu(\xi_i x)\cdot f(x)\cdot dx - \xi_i^2\int_0^a x\cdot f(x)\cdot J_\mu\cdot dx$$

$$= -\xi_i\,a f(a)\cdot J'_\mu(\xi_i a) + \mu^2 T_\mu\left[\frac{1}{x^2}\cdot f(x)\right] - \xi_i^2\,T_\mu\{f(x)\}$$

There is no necessity to outline the procedure as it is much the same as that of the preceding transforms. As an example of the above transform let us consider the vibrations of a circular lamina.

e.g.1. A thin, perfectly flexible, circular lamina of radius a is stretched by a tension T and has uniform surface density σ throughout. If the coordinates of a point are (x, y, z) then the motion must satisfy

$$\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial z}{\partial r} = \frac{1}{c^2}\cdot\frac{\partial^2 z}{\partial t^2} \qquad 0 \le r < a, \quad t > 0 \qquad (11)$$

where (x, y, 0) is the equilibrium position and where $c^2 = \frac{T}{\sigma}$, $r^2 = x^2 + y^2$.

Let us also suppose that

$$z = 0, \quad \frac{\partial z}{\partial t} = f(r) \quad \text{when} \quad t = 0, \quad 0 \leqslant r < a \tag{12}$$

and

$$z = 0 \quad \text{when} \quad r = a, \quad t > 0. \tag{13}$$

If we multiply equation (11) by $rJ_0(\xi_i r)$ and integrate with respect to r from 0 to a we have, using (10)

$$\frac{d^2}{dt^2}\cdot \bar{z}_J + c^2 \xi_i^2 \bar{z}_J = 0 \tag{14}$$

$$\text{and.} \qquad \bar{z}_J = 0, \qquad \frac{d\bar{z}}{dt} = \int_0^a \eta f(\eta) \cdot J_0(\xi_i \eta) \cdot d\eta \qquad t = 0 \tag{15}$$

The solution of (14) is $\bar{z}_J = A \sin(c\xi_i t) + B \cos(c\xi_i t)$

and from (15)

$$\bar{z}_J = \frac{\sin(c\xi_i t)}{c\xi_i}\int_0^a \eta f(\eta) J_0(\xi_i \eta) d\eta$$

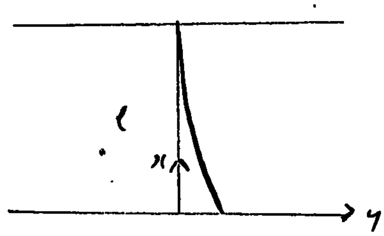
Hence, the inversion formula (3) gives us

$$z(r,t) = \frac{2}{ca^2}\sum_i \frac{\sin(c\xi_i t)}{\xi_i} \cdot \frac{J_0(\xi_i r)}{[J_1(\xi_i a)]^2}\int_0^a \eta f(\eta) \cdot J_0(\xi_i \eta) d\eta$$

where the summation extends over all the positive roots of $J_0(\xi_i a) = 0$

The same transform may be used to solve Bernoulli's equation for the motion of a heavy finite chain.[21]

e.g.2. Consider the small transverse oscillations of a heavy chain line density ρ, hung from one end.



Let the axes be as shown. If the vibrations are free, the displacement y(x,t) must satisfy

$$\frac{\partial}{\partial x}\left(gx\frac{\partial y}{\partial x}\right) - \frac{\partial^2 y}{\partial t^2} = 0 \tag{16}$$

and suppose that $y = f(x), \; \frac{\partial y}{\partial t} = 0$, when $t = 0$. (17)

If we change the independent variable from x to z, where

$z = \sqrt{\frac{4x}{g}}$ then (16) reduces to

$$\left(\frac{\partial^2}{\partial z^2} + \frac{1}{z}\cdot\frac{\partial}{\partial z} - \frac{\partial^2}{\partial t^2}\right)y = 0 \tag{18}$$

where $y = f\left(\frac{gz^2}{4}\right), \; \frac{\partial y}{\partial t} = 0, \quad t = 0 \qquad 0 < z < \sqrt{\frac{4\ell}{g}} = \beta$ say. (19)

Applying the transform as in e.g.1. with the range of integration from 0 to β, we have using (10)

$$\frac{d^2}{dt^2}\cdot\bar{y}_J + \xi_i^{\,2}\bar{y}_J = 0 \qquad \text{where } \xi_i \text{ is a solution of } J_0(\xi_i\beta) = 0 \tag{20}$$

and $\bar{y}_J = \int_0^\beta z.f\left(\frac{gz^2}{4}\right)J_0(\xi_i z)\,dz. \qquad \frac{\partial \bar{y}_J}{\partial t} = 0 \qquad t = 0$ (21)

Hence, from (20) and (21) we obtain

$$\bar{y}_J = \cos \xi_i t \int_0^\beta z.f\left(\frac{gz^2}{4}\right)J_0(\xi_i z)\,dz.$$

Using the inversion formula

$$y(z,t) = \frac{2}{\beta^2}\sum_i \cos \xi_i t \cdot \frac{J_0(\xi_i z)}{[J_1(\xi_i\beta)]^2}\cdot \int_0^\beta z.f\left(\frac{gz^2}{4}\right).\,J_0(\xi_i z).dz$$

where the sum extends over all the positive roots of $J_0(\xi_i\beta) = 0$.

Therefore, in the original coordinates

$$y(x,t) = \frac{1}{\ell}\sum_i \cos\left\{\tfrac{1}{2}\alpha_i t\sqrt{\tfrac{g}{\ell}}\right\}\cdot\frac{J_0\left(\alpha_i\sqrt{\frac{x}{\ell}}\right)}{[J_1(\alpha_i)]^2}\cdot\int_0^\ell f(\eta).\,J_0\left(\alpha_i\sqrt{\tfrac{\eta}{\ell}}\right)d\eta$$

where the summation is now over the positive roots of $J_0(\alpha_i) = 0$.

So far, we have taken the parameter ξ_i to be a root of the equation $J_\mu(a\xi_i) = 0$; we can, however, take ξ_i to be a root of the equation $\qquad \xi_i J'_\mu(\xi_i a) + h.J_\mu(\xi_i a) = 0. \qquad (22)$

In this case, our inversion formula would be

$$f(x) = T^{-1}(\bar{f}_J) = \frac{2}{a^3}\sum_i \frac{\xi_i^2 . \bar{f}_J(\xi_i)}{h^2 + (\xi_i^2 - \mu^2/a^2)} \cdot \frac{J_\mu(\xi_i x)}{\{J_\mu(\xi_i a)\}^2} \qquad (23)$$

From the equations (1) and (22) we obtain the following property:

$$T_0\left\{\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\cdot\frac{\partial f}{\partial x}\right\} = -\xi_i^2 \bar{f}_J + a J_0(\xi_i a)\left[\frac{\partial f}{\partial x} + hf\right]_{x=a} \qquad (24)$$

[Since $T_0\left\{\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\cdot\frac{\partial f}{\partial x}\right\} = \int_0^a \frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial x}\right) . J_0(\xi_i x)\,dx$

$$= \left[x\cdot\frac{\partial f}{\partial x}\cdot J_0(\xi_i x)\right]_{x=a} - \xi_i\int_0^a x\cdot\frac{\partial f}{\partial x} . J_0'(\xi_i x)\,dx$$

$$= \left[x\,\frac{\partial f}{\partial x} . J_0\right]_{x=a} - \xi_i\left[x . f . J_0'(\xi_i x)\right]_{x=a} + \xi_i\int_0^a f(x)\left[J_0' + x\,\xi_i\,J_0''\right]dx$$

$$= \left[x\cdot\frac{\partial f}{\partial x}\cdot J_0\right]_{x=a} + a . f(a) . h . J_0(\xi_i a) - \int_0^a f(x) . \frac{1}{x} . (\xi_i^2 x^2) . J_0(\xi_i x)\,dx \quad \text{using (22) and Bessel's eq.}$$

Hence $T_0\left\{\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\cdot\frac{\partial f}{\partial x}\right\} = a J_0(\xi_i a)\left\{\frac{\partial f}{\partial x} + hf\right\}_{x=a} - \xi_i^2 \bar{f}_0$]

A typical example of the above transform is given in the theory of the conduction of heat in a circular cylinder of solid section.

e.g.3. A cylinder of radius <u>a</u> has an initial temperature $f(r)$, and radiation takes place at the surface $r = a$ into a medium maintained at zero temperature. Thus, if $y(r,t)$ is the temperature function, it must satisfy

$$\frac{\partial y}{\partial t} = K\left(\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial y}{\partial r}\right) \qquad 0 \le r < a,\quad t > 0. \qquad (25)$$

$$y(r,t) = f(r) \qquad 0 \le r < a,\quad t = 0. \qquad (26)$$

$$\frac{\partial y}{\partial r} = -hy \qquad r = a,\quad t > 0. \qquad (27)$$

Applying the transform with $\mu = 0$ and choosing ξ_i to be a root of equation (22) we have

$$\frac{\partial \bar{y}_s}{\partial t} = K\left[a\, J_0(\xi_i a)\left\{\frac{\partial y}{\partial r} + h y\right\}_{r=a} - \xi_i^2\, \bar{y}_s\right] \qquad \text{using (24).}$$

Hence, from (27)

$$\frac{\partial \bar{y}_s}{\partial t} + k\,\xi_i^2\, \bar{y}_s = 0$$

$$\therefore \quad \bar{y}_s = A e^{-\xi_i^2 k t} = e^{-\xi_i^2 k t} \int_0^a \eta\, f(\eta)\, J_0(\xi_i \eta)\,d\eta \quad \text{from (26)}$$

and therefore from the inversion formula,

$$y = \frac{2}{a^2} \sum_i \frac{\xi_i^2\, e^{-k\xi_i^2 t}}{h^2 + \xi_i^2} \cdot \frac{J_0(\xi_i r)}{\{J_0(\xi_i a)\}^2} \cdot \int_0^a \eta\, f(\eta) \cdot J_0(\xi_i \eta)\,d\eta.$$

where the summation extends over all the positive zeros of the function $\xi_i J_0'(\xi_i a) + h.J_0(\xi_i a)$[1]

2.32. Infinite Hankel Transforms.

The method here corresponds more closely to that of the Laplace transform since the range of integration is from zero to infinity.

i.e. $$\bar{y}_\mu(x) = \int_0^\infty x . y(x) . J_\mu(\xi x)\, dx. = T_\mu[y(x)] \qquad (1)$$

and the inversion formula[2] is now

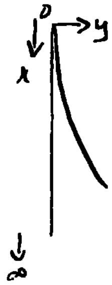
$$y(x) = \int_0^\infty \xi .\, \bar{y}(x)\, J_\mu(\xi x)\, d\xi \qquad (2)$$

The procedure is very similar to that for the finite transform and thus we can proceed immediately to the applications of the transform

1. Carslaw and Jaeger. op. cit. p.128-9

2. [illegible] "... to the Theory of Fourier Integrals" 1937 p.240.

As an example let us consider the vibrations of a chain (c.f. 2.31. e.g.2.) but let the chain be semi-infinite.[1]



e.g.1. Suppose that the chain is initially drawn aside to the position

$$y(x,t) = f(x) = \frac{\epsilon}{\sqrt{1+\frac{x}{a}}} \qquad (3)$$

and released from rest at $t = 0$.

Let $y(x,t)$ be the displacement at any time $\underline{t}$; it must then satisfy

$$\frac{\partial}{\partial x}\left(g x \frac{\partial y}{\partial x}\right) - \frac{\partial^2 y}{\partial t^2} = 0 \qquad (4)$$

If we replace x by z by means of the substitution

$$z = \sqrt{\frac{4x}{g}}$$

equation (4) becomes $\left[\frac{\partial^2}{\partial z^2} + \frac{1}{z}\cdot\frac{\partial}{\partial z} - \frac{\partial^2}{\partial t^2}\right] y = 0$ (5)

where $y(x,t) = f(x)$, $\frac{\partial y}{\partial t} = 0$ at $t = 0$. (6)

Now let
$$\bar{y}_\mu = \int_0^\infty z.y.J_\mu(\xi z)\,dz.$$

then
$$\int_0^\infty z\left(\frac{\partial^2 y}{\partial z^2} + \frac{1}{z}\cdot\frac{\partial y}{\partial z}\right).J_\mu(\xi z).dz = \int_0^\infty \frac{\partial}{\partial z}\left(z\cdot\frac{\partial y}{\partial z}\right).J_\mu(\xi z).dz$$

$$= \left[z\cdot\frac{\partial y}{\partial z}\cdot J_\mu(\xi z)\right]_0^\infty - \xi\left[y.z.J_\mu'(\xi z)\right]_0^\infty + \int_0^\infty y\cdot\frac{1}{z}(\mu^2 - \xi^2 z^2).J_\mu(\xi z)\,dz$$

$$= \int_0^\infty y\cdot\frac{1}{z}(\mu^2 - \xi^2 z^2).J_\mu(\xi z)\,dz$$

And thus, applying the transform to (5) we get

1. Phil. Mag. Ser. 7. Vol. XXXIX. P. 229. 1948.

$$\int_0^\infty y.\tfrac{1}{2}(\mu^2 - \xi^2 z^2).J_\mu(\xi z)\,dz - \frac{\partial^2}{\partial t^2}\bar{y}_\mu = 0 \qquad (7)$$

and if we take $\mu = 0$ this becomes

$$\xi^2 \bar{y}_\mu - \frac{\partial^2}{\partial t^2}\bar{y}_\mu = 0$$

$$\text{i.e. } \bar{y}_\mu = A(\xi)\cos\xi t + B(\xi)\sin\xi t \qquad (8)$$

where $\bar{y} = \bar{f}(\xi)$ and $\frac{\partial \bar{y}}{\partial t} = 0$ at $t = 0.$ (9)

Therefore $\bar{y}_\mu - \bar{f}(\xi)\cos\xi t.$ (10)

and $\bar{f}(\xi) = \int_0^\infty z.f(\tfrac{1}{4}gz^2).J_0(\xi z)\,dz = \alpha\epsilon \int_0^\infty \frac{z.J_0(\xi z)}{\sqrt{\alpha^2 + z^2}}dz$ where $\alpha^2 = 4a/g$

$$\text{i.e. } \bar{f}(\xi) = \frac{\epsilon.\alpha.e^{-\alpha\xi}}{\xi}$$

Inserting in equation (10) and using the inversion formula we obtain

$$y = \epsilon\alpha.\int_0^\infty e^{-\alpha\xi}\cos\xi t.J_0(\xi z)\,d\xi. = \epsilon.\mathcal{R}\left[\int_0^\infty e^{-(1+\frac{it}{\alpha})\xi}.\ J_0\left(\frac{\xi z}{\alpha}\right).d\xi\right]$$

Using the known result $\int_0^\infty e^{-\omega x} J_0(\rho x)\,dx = (\rho^2+\omega^2)^{-\frac{1}{2}}$

we get

$$y = \epsilon.\mathcal{R}\left[\frac{z^2}{\alpha^2} + \left(1 + \frac{it}{\alpha}\right)^2\right]^{-\frac{1}{2}}$$

and therefore $y = \epsilon M^{-\frac{1}{2}}\cos\tfrac{1}{2}\phi$

where $M^2 = \left(\frac{x}{\alpha} + 1 - \frac{gt^2}{4a}\right)^2 + \frac{gt^2}{a}$ and $\phi = \tan^{-1}\left\{\frac{\left(\frac{gt^2}{a}\right)^{\frac{1}{2}}}{\frac{x}{a} + 1 - \frac{gt^2}{4a}}\right\}$

These problems are but a few of the different types of equations which can be solved by means of the Hankel transform (see papers by I.N.Sneddon in the Bibliography).

The Hankel transformation unfortunately will not solve problems that cannot be solved by the more usual Laplace transformation. It does however eliminate the use of contour integration in the solution and this, in some cases, is a great advantage. Although the above problems are only a small selection it is still fairly obvious that the range of application of the H.T. is much smaller than that of the L.T.

2.40. Legendre Transforms.

We have discussed so far integral transforms with kernels as exponential, trigonometrical or Bessel functions. The Legendre transformation follows the same procedure in which the kernel is a Legendre polynomial. As in the previous cases the application of the transform reduces the partial differential equation in n independent variables to one in n - 1 variables. In the problems considered, from two variables we reduce to one.

We define the Legendre transform to be

$$\bar{V}_n = \int_0^1 V.P_n(\mu)\,d\mu. \tag{1}$$

where V. is a function of μ and $\bar{V}_n$ a function of n.

Now, if $P_n(\mu)$ is the Legendre polynomial of degree n, where n is a positive integer, then[1.]

$$\int_0^1 P_m(\mu).P_n(\mu)\,d\mu = \begin{cases} 0 & m - n \quad \text{even} \\ 1/2n+1 & m = n \end{cases} \tag{2}$$

From (1) we define the odd Legendre transform

$$\text{i.e.} \quad \bar{V}_{2n+1} = \int_0^1 V.P_{2n+1}(\mu)\,d\mu \tag{3}$$

and immediately from (2) we obtain the inversion theorem

$$V = \sum_{n=0}^{\infty} (4n+3).\bar{V}_{2n+1}.P_{2n+1}(\mu) \tag{4}$$

That this is the inversion theorem can be easily seen if we expand the summation; multiply by $P_{2n+1}(\mu)$ and then integrate from zero to infinity. Allexcept the $(n+1)^{th}$ term disappear leaving us with the required result.

Similarly we define the even transforms as

$$\overline{V}_{2n} = \int_0^1 V.P_{2n}(\mu)\,d\mu \tag{5}$$

and the corresponding inversion theorem

$$V = \sum_{n=0}^{\infty} (4n+1).\overline{V}_{2n}.P_{2n}(\mu) \tag{6}$$

In the following applications we will need the transform of $\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\}$. If we write Legendre's equation in the form

$$\frac{d}{d\mu}\left\{(1-\mu^2).P_m'(\mu)\right\} = -m(m+1).P_m(\mu) \tag{7}$$

we can, integrating by parts, obtain

$$\int_0^1 \frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\}P_m(\mu)\,d\mu = \left[(1-\mu^2).\frac{\partial V}{\partial\mu}.P_m(\mu)\right]_0^1 - \int_0^1 (1-\mu^2).\frac{\partial V}{\partial\mu}.P_m'(\mu)\,d\mu$$

$$= -P_m(0)\left[\frac{\partial V}{\partial\mu}\right]_{\mu=0} - \int_0^1 (1-\mu^2).\frac{\partial V}{\partial\mu}.P_m'(\mu).\,d\mu.$$

$$= -P_m(0)\left[\frac{\partial V}{\partial\mu}\right]_{\mu=0} - \left[(1-\mu^2).V.P_m'(\mu)\right]_0^1 - m(m+1)\int_0^1 V.P_m(\mu)\,d\mu.$$

from (7).

i.e $$\int_0^1 \frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\}.P_m(\mu)\,d\mu = -P_m(0)\left[\frac{\partial V}{\partial\mu}\right]_{\mu=0} + m.P_{m-1}(0)\left[V\right]_{\mu=0} - m(m+1)\overline{V}_m \tag{8}$$

where $P_m'(0) = mP_{m-1}(0)$, and $\overline{V}_m = \int_0^1 V.P_m(\mu)\,d\mu$.

2.41. Applications.

(a) Odd Legendre Transform.

We consider the problem of finding a function V which will satisfy Laplace's equation in the semi-infinite solid

$z > 0$ such that, on the surface $z = 0$, $\frac{\partial V}{\partial z} = f(\rho)$ inside the circle $\rho = a$, and $V = F(\rho)$ outside the circle. F and f are given functions of ρ and we assume that the various integrals used in the solution do, in fact, exist.

If we write $z = a\mu\zeta$ and $\rho = a(1-\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}}$ then Laplace's equation becomes

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\} + \frac{\partial}{\partial\zeta}\left\{(1+\zeta^2)\frac{\partial V}{\partial\zeta}\right\} = 0, \quad 0<\mu<1, \quad \zeta>0 \qquad (9)$$

with the boundary conditions

$$\frac{1}{a\mu}\cdot\frac{\partial V}{\partial\zeta} = f\left\{a\sqrt{1-\mu^2}\right\} \qquad \zeta = 0 \qquad (10)$$

$$V = F\left\{a\sqrt{1+\zeta^2}\right\} \qquad \mu = 0 \qquad (11)$$

Applying the transform to (9), i.e. multiply throughout by $P_m(\mu)$ and integrate with respect to μ from 0 to 1, we obtain

$$\int_0^1 \frac{\partial}{\partial\zeta}\left\{(1+\zeta^2)\frac{\partial V}{\partial\zeta}\right\}.P_m(\mu)d\mu + \int_0^1 \frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\}P_m(\mu)d\mu = 0$$

Hence, from (8)

$$\frac{d}{d\zeta}\left\{(1+\zeta^2)\frac{d\bar{V}_m}{d\zeta}\right\} - P_m(0)\left[\frac{\partial V}{\partial\mu}\right]_{\mu=0} + m.P_{m-1}(0).\left[V\right]_{\mu=0} - m(m+1)\bar{V}_m = 0$$

$$\text{i.e. } \frac{d}{d\zeta}\left\{(1+\zeta^2)\frac{d\bar{V}_m}{d\zeta}\right\} - m(m+1)\bar{V}_m = P_m(0)\left[\frac{\partial V}{\partial\mu}\right]_{\mu=0} - m.P_{m-1}(0).F\left\{a\sqrt{1+\zeta^2}\right\} \text{ from (11)}$$

In the initial conditions we are not given the value of $\frac{\partial V}{\partial\mu}$ at $\mu = 0$ and we therefore take $m = 2n + 1$ to eliminate this term (since when n is an integer, $P_{2n+1}(0) = 0$.)

$$\therefore \quad \frac{d}{d\zeta}\left\{(1+\zeta^2)\frac{d\bar{V}_m}{d\zeta}\right\} - (2n+1)(2n+2)\bar{V}_m = R_{2n+1}(\zeta) \qquad (12)$$

where
$$R_{2n+1}(\zeta) = -(2n+1)P_{2n}(0).\ F\left\{a\sqrt{1+\zeta^2}\right\} \qquad (13)$$

and we note that after the transformation (10) becomes

$$\frac{d\bar{V}_{2n+1}}{d\zeta} = a\int_0^1 \mu . f\left\{a\sqrt{1-\mu^2}\right\}.P_{2n+1}(\mu)\,d\mu \qquad \zeta = 0 \qquad (14)$$

Equation (12) is solved by the method of variation of parameters and the use of the oblate spheroidal harmonics introduced by J.W.Nicholson in 1924[1]. These harmonics are denoted by $p_n(x)$, $q_n(x)$ and are the solutions of

$$\frac{d}{dx}\left\{(1+x^2)\frac{dy}{dx}\right\} - n(n+1)y = 0 \qquad (15)$$

We require the following properties of these harmonics

$$p_n'(x).q_n(x) - q_n'(x).p_n(x) = \frac{1}{1+x^2} \qquad (16)$$

and
$$p_{2n+1}'(0) = -\frac{2}{\pi}q_{2n+1}'(0) = \frac{\lfloor 2n+1}{2^{2n}(\lfloor n)^2} = (-1)^n(2n+1).P_{2n}(0) \qquad (17)$$

The solution of (12) is now written

$$\bar{V}_{2n+1} = A_{2n+1}(\zeta).p_{2n+1}(\zeta) + B_{2n+1}(\zeta).q_{2n+1}(\zeta) \qquad (18)$$

and hence, differentiating

$$\frac{d\bar{V}_{2n+1}}{d\zeta} = A_{2n+1}(\zeta).p_{2n+1}'(\zeta) + B_{2n+1}(\zeta).q_{2n+1}'(\zeta) \qquad (19)$$

if
$$p_{2n+1}(\zeta).A_{2n+1}'(\zeta) + q_{2n+1}(\zeta).B_{2n+1}'(\zeta) = 0 \qquad (20)$$

We multiply (19) by $1+\zeta^2$, differentiate with respect to ζ and substitute in (12), thus obtaining

1. J.W.Nicholson ...

$$A_{2n+1}(\zeta)\frac{d}{d\zeta}\left\{(1+\zeta^2)p'_{2n+1}(\zeta)\right\}+A'_{2n+1}(\zeta).p'_{2n+1}(\zeta)(1+\zeta^2)+B_{2n+1}(\zeta)\frac{d}{d\zeta}\left\{(1+\zeta^2)q'_{2n+1}(\zeta)\right\}$$
$$+B'_{2n+1}(\zeta).q'_{2n+1}(\zeta)-(2n+1)(2n+2)(A_{2n+1}p_{2n+1}+B_{2n+1}q_{2n+1})=R_{2n+1}(\zeta)$$

Therefore, using (15) where $(2n+1)$ is <u>n</u>,

$$A'_{2n+1}\cdot p'_{2n+1}+B'_{2n+1}\cdot q'_{2n+1}=\frac{R_{2n+1}}{1+\zeta^2} \tag{21}$$

Solving (21) and (20)

$$A'(p'q-pq')=\frac{R}{1+\zeta^2}\cdot q$$
$$B'(pq'-p'q)=\frac{R}{1+\zeta^2}\cdot p$$

This gives, using (16)

$$A'_{2n+1}=R.q_{2n+1}(\zeta)\,;\quad B'_{2n+1}=-R.p_{2n+1}(\zeta) \tag{22}$$

When $\zeta\to\infty$, V must vanish and hence $\overline{V}_{2n+1}(\zeta)$ and therefore $A_{2n+1}(\zeta)$ must also vanish. From (22) we see that

$$A_{2n+1}(\zeta)-A_{2n+1}(0)=\int_0^\zeta R_{2n+1}(x)\cdot q_{2n+1}(x)\,dx$$

$$\text{i.e.}\quad -A_{2n+1}(0)=\int_0^\infty R_{2n+1}(x).q_{2n+1}(x)\,dx$$

$$\therefore\quad A_{2n+1}(\zeta)=\int_0^\zeta R_{2n+1}(x)\cdot q_{2n+1}(x)\,dx-\int_0^\infty R_{2n+1}(x).q_{2n+1}(x)\,dx=-\int_\zeta^\infty R_{2n+1}\cdot q_{2n+1}\cdot dx \tag{23}$$

From (19) and (16) we obtain

$$A_{2n+1}(0).p'_{2n+1}(0)+B_{2n+1}(0)q'_{2n+1}(0)=a\int_0^1\mu f\left\{a\sqrt{1-\mu^2}\right\}P_{2n+1}(\mu)\,d\mu \tag{24}$$

i.e. from (17)

$$A_{2n+1}(0).(-1)^n(2n+1)P_{2n}(0)-B_{2n+1}(0)\cdot\frac{\pi}{2}(-1)^n(2n+1)P_{2n}(0)=a\int_0^1\mu f\left\{a\sqrt{1-\mu^2}\right\}P_{2n+1}(\mu)d\mu.$$

$$\therefore\quad \frac{\pi}{2}\cdot B_{2n+1}(0)=A_{2n+1}(0)-\frac{(-1)^n.a}{(2n+1)P_{2n}(0)}\int_0^1\mu f\left\{a\sqrt{1-\mu^2}\right\}P_{2n+1}(\mu)\,d\mu \tag{25}$$

and therefore (22) gives

$$B_{2n+1}(\zeta) = B_{2n+1}(0) - \int_0^\zeta R_{2n+1}(x).\, p_{2n+1}(x).\, dx \tag{26}$$

and since $\quad \overline{V}_{2n+1} = A_{2n+1}(\zeta).\, p_{2n+1}(\zeta) + B_{2n+1}(\zeta).\, q_{2n+1}(\zeta)$

the inversion theorem gives us

$$V = \sum_{n=0}^{\infty} (4n+3)\left[A_{2n+1}(\zeta).\, p_{2n+1}(\zeta) + B_{2n+1}(\zeta).\, q_{2n+1}(\zeta)\right] P_{2n+1}(\mu) \tag{27}$$

where $A_{2n+1}(\zeta)$ is given by (23), $B_{2n+1}(0)$ by (25) and (23) with $\zeta = 0$ and $B_{2n+1}(\zeta)$ by (26).

(b) Even Legendre Transform.

A similar problem to the above where the function V must now satisfy Laplace's equation in $z > 0$ such that on $z = 0$, $V = F(\rho)$ inside the circle $\rho = a$ and $\frac{\partial V}{\partial z} = f(\rho)$ outside. Transforming the equation as above, we obtain

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\} + \frac{\partial}{\partial\zeta}\left\{(1+\zeta^2)\frac{\partial V}{\partial\zeta}\right\} = 0 \qquad 0<\mu<1,\ \zeta>0 \tag{28}$$

with $$V = F\left\{a\sqrt{1-\mu^2}\right\} \qquad \zeta = 0 \tag{29}$$

and $$\frac{1}{a\zeta} \cdot \frac{\partial V}{\partial\mu} = f\left\{a\sqrt{1+\mu^2}\right\} \qquad \mu = 0 \tag{30}$$

As before, we arrive at the equation

$$\frac{d}{d\zeta}\left\{(1+\zeta^2).\frac{d\overline{V}_m}{d\zeta}\right\} - P_m(0).\left\{\frac{\partial V}{\partial\mu}\right\}_{\mu=0} + m.P_{m-1}(0)\left[V\right]_{\mu=0} - m(m+1)\overline{V}_m = 0$$

and since V is not known at $\mu = 0$ we take $m = 2n$.

i.e., $$\frac{d}{d\zeta}\left\{(1+\zeta^2)\frac{dV_{2n}}{d\zeta}\right\} - 2n(2n+1)V_{2n} = P_{2n}(0).\,a\zeta.\,f\{a\sqrt{1+\mu^2}\} \tag{31}$$

$$= R_{2n}(\zeta)$$

The analysis is similar to the preceding section and we finally obtain the result

$$V = \sum_{n=0}^{\infty}(4n+1)\left[A_{2n}(\zeta).\,p_{2n}(\zeta) + B_{2n}(\zeta).\,q_{2n}(\zeta)\right]P_{2n}(\mu) \tag{32}$$

where $$A_{2n}(\zeta) = -\int_{\zeta}^{\infty} R_{2n}(x).\,q_{2n}(x).\,dx \tag{33}$$

$$B_{2n}(\zeta) = B_{2n}(0) - \int_0^{\zeta} R_{2n}(x).\,p_{2n}(x).\,dx \tag{34}$$

and $$\tfrac{1}{2}\pi\,.\,B_{2n}(0) = -A_{2n}(0) + \frac{(-1)^n}{P_{2n}(0)}\cdot\int_0^1 F\{a\sqrt{1-\mu^2}\}.\,P_{2n}(\mu)\,d\mu \tag{35}$$

The procedure given above is strictly formal and to complete the solution we should actually verify the final result. To verify the solution for the general case is rather a difficult task but it is possible to show that for certain values of $F(\rho)$ and $f(\rho)$ the solution does hold[1].

The above procedure is on similar lines to that of the preceding sections and it is fairly clear that no result can be obtained by the use of Legendre transforms that could not be obtained by the use of expansions in Legendre polynomials. The advantage here, as in the case of the other transforms, is that the analysis is reduced almost to a drill. The obvious disadvantage of the Legendre transform is that there is only a limited number of problems which can be put in the required form.

1. Quart. J. Maths. Oxford (2), 1 (1950) 1-8

3.00. Conclusion.

We have discussed four different types of transform applicable to the solution of partial differential equations; with the Laplace transformation studied in some detail. Before proceeding we must state that:

(i) the Fourier transformation does not solve problems that cannot be solved by the use of the ordinary theory of Fourier series

(ii) the Hankel transformation cannot solve problems which are intractable to the use of Fourier-Bessel expansions.

(iii) the Legendre transformation can only obtain results which could have been obtained by the direct use of Legendre polynomials.

Although these three points seem to diminish the importance of the transformations we can say that they do at least reduce the analysis involved almost to a "drill", and in most cases give a quicker and neater solution.

In general, all the methods are of a set pattern i.e. the use of an integral transform of the type

$$\int_0^\infty f(t) \cdot K\{\alpha, t\}\, dt$$

to reduce an equation of n independent variables to one of $n - 1$ independent variables. The methods differ in that they use different kernels $K\{\alpha, t\}$ and different ranges of integration.

Taking a very general view of the transforms it is clear that the Laplace transformation has the greatest range of application though, of course, the others may be more important in their ~~book~~ sphere. It will be interesting to compare each of the other transforms with the Laplace transform admitting at once the greater usefulness of, in general, of the latter.

1. Fourier.

It can be seen at once in the problems to which it can be applied that the Fourier transformation gives a neater solution. For example, the solution of the vibrating string problem is much quicker and easier than that of the corresponding solution using the Laplace transform.

The Fourier transformation has the serious disadvantage that it can only be applied to problems which involve even powers of derivatives. Although this does include a very large number of boundary value problems it must still be regarded as a serious disadvantage.

2. Hankel.

The method of finite Hankel transforms will not solve problems intractable to the Laplace transform method but it does reduce the amount of calculation involved and, as in the other cases eliminates the use of contour integration. In the problems considered the Hankel transform is much superior to the L.T. but we must once again note the limitation of its scope.

3. Legendre.

The above criticisms are true in this case; the Legendre transform is superior if the problem under consideration lies within its range.

Generally then, we may say that for the boundary value problems considered, the Laplace transformation is the most widely applicable and may be expected to give the required result. It has the disadvantages in some problems of involved calculation and rather difficult contour integration which tend to make the procedure rather cumbersome. In their own spheres the other transformations are of great importance and generally lead to a neater solution.

Finally, the Laplace transformation provides a very simple method for the solution of ordinary differential equations. The actual procedure used in solving these equations is so straightforward that at some future date we may expect to see it used quite widely in the sixth-forms of our Grammar schools.

Appendix 1. Laplace Transforms. $\bar{x}(p) = \int_0^\infty e^{-pt} x(t)\, dt$

In this table a and b are real positive constants and ω is a complex constant.

$\bar{x}(p)$.	$x(t)$.
$\frac{1}{p}$	1.
$\frac{1}{p^{n+1}}$	$t^n/\lfloor n$, $n = 0,1,2,\ldots$
$\frac{1}{p^{\nu+1}}$	$\frac{t^\nu}{\Gamma(\nu+1)}$, $\nu > -1$
$\frac{1}{p+\omega}$	$e^{-\omega t}$
$\frac{p}{p^2+\omega^2}$	$\cos \omega t$
$\frac{\omega}{p^2+\omega^2}$	$\sin \omega t$
$\frac{p}{p^2-\omega^2}$	$\cosh \omega t$
$\frac{\omega}{p^2-\omega^2}$	$\sinh \omega t$
$\frac{p}{(p^2+\omega^2)^2}$	$\frac{t}{2\omega} \sin \omega t$
$\frac{\omega^2}{(p^2+\omega^2)^2}$	$\frac{1}{2\omega}(\sin \omega t - \omega t \cos \omega t)$.

Appendix 2. Fourier Transforms.

$f(x)$	$f_c(n)$	$f_s(n)$
1.	0 when $n = 1, 2, 3 \cdots$; $f_c(0) = \pi$	$\frac{1-(-1)^n}{n}$
x.	$-\frac{1-(-1)^n}{n^2}$; $f_c(0) = \frac{\pi^2}{2}$	$\frac{\pi(-1)^{n+1}}{n}$
x^2.	$\frac{2\pi(-1)^n}{n^2}$; $f_c(0) = \frac{\pi^3}{3}$	$\frac{\pi^2(-1)^{n-1}}{n} - \frac{2[1-(-1)^n]}{n^3}$
e^{cx}.	$c\left[\frac{(-1)^n e^{c\pi} - 1}{n^2+c^2}\right]$	$\frac{n}{n^2+c^2}\left[1-(-1)^n e^{c\pi}\right]$
sin kx.	$\frac{K}{n^2-K^2}\left[(-1)^n \cos \pi K - 1\right]$ $K \neq 0, 1, 2 \cdots$	$\begin{cases} \frac{\pi}{2} & \text{when } n = K \\ 0 & \text{when } n \neq K \end{cases}$ $(K = 1, 2, \cdots)$
cos kx.	0 when $n = 1, 2, 3 \cdots$ $f_c(K) = \frac{\pi}{2}$ $(K = 1, 2, 3 \cdots)$	$\frac{n}{n^2-K^2}\left[1-(-1)^n \cos K\pi\right]$, $K \neq 1, 2, \cdots$

$$f_c(n) = \int_0^{\pi} f(x) \cos nx \, dx \qquad n = 0, 1, 2, \cdots$$

$$f_s(n) = \int_0^{\pi} f(x) \sin nx \, dx \qquad n = 1, 2, 3, \cdots$$

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OPERATIONAL METHODS OF SOLVING ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS.

(Supplementary sheets submitted January 1952)

Operational Methods of Solving Ordinary and Partial Differential Equations. (Supplement).

1.0. Use of $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x).e^{ipx}dx = \bar{f}(p)$ the infinite Fourier Transform.

The examples to be considered require the use of the Inversion Formula and the Convolution Theorem.

1.1. The Inversion Formula.

If $\bar{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x).e^{ipx}dx$ then the required Inversion formula states that

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \bar{f}(p).\, e^{-ipx}dp \qquad (1)$$

1.2. The Convolution Theorem.

i.e.
$$\int_{-\infty}^{+\infty} \bar{f}(p).\,\bar{g}(p).\,e^{-ipx}.\,dp = \int_{-\infty}^{+\infty} g(u).\,f(x-u).\,du. \qquad (2)$$

If we suppose that f(x) and g(x) are functions that satisfy the requirements for the changes in the order of integration, then

$$\int_{-\infty}^{+\infty} g(u).\,f(x-u).\,du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(u)\,du \int_{-\infty}^{+\infty}\bar{f}(p).\,e^{-ip(x-u)}dp$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\bar{f}(p).\,e^{-ipx}dp\int_{-\infty}^{+\infty} g(u).\,e^{ipu}du.$$

$$= \int_{-\infty}^{+\infty}\bar{f}(p).\,\bar{g}(p).\,e^{-ipx}dp.$$

1.3. Example.

Suppose we wish to solve the heat equation

$$\frac{\partial\theta}{\partial t} = K\frac{\partial^2\theta}{\partial x^2} \qquad -\infty < x < \infty,\; t > 0 \qquad (3)$$

where $\theta = f(x)$ when $t = 0$, and $\theta \to 0$ when $x \to \pm\infty$ (4)

If we apply the transformation to equation (3), i.e. multiply the equation by e^{ipx} and integrate from $-\infty$ to $+\infty$ with respect to x, we obtain

$$\frac{d\bar{\theta}}{dt} = -Kp^2\bar{\theta} \qquad (5)$$

and for (4)

$$\bar{\theta} = \bar{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x).\,e^{ipx}\,dx \qquad t=0 \qquad (6)$$

Hence from (5) and (6)

$$\bar{\theta} = \bar{f}(p).\,e^{-Kp^2t}$$

Using the Inversion Formula we have

$$\theta(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \bar{f}(p).\,e^{-Kp^2t}.\,e^{-ipx}\,dp \qquad (7)$$

Now suppose that

$$g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-Kp^2t}.\,e^{-ipx}.\,dp$$

$$= \frac{1}{\sqrt{2Kt}}.\,e^{-\frac{x^2}{4Kt}}$$ (a well known integral)[1]

We see immediately that

$$\bar{g}(p) = e^{-Kp^2t}$$

and therefore

$$\theta(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \bar{f}(p).\,\bar{g}(p).\,e^{-ipx}.\,dp$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(u).\,g(x-u).\,du \qquad \text{from (2)}$$

$$= \frac{1}{\sqrt{4K\pi t}}\int_{-\infty}^{+\infty} f(u).\,e^{-\frac{(x-u)^2}{4Kt}}\,du.$$

Hence if $p = \frac{u-x}{2\sqrt{Kt}}$

then

$$\theta(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} f(x + 2\sqrt{Kt}.\,p)\,e^{-p^2}\,dp.$$

1. "Modern Operational Methods in Engineering" — R.V. Churchill P.59.

1.4. It is interesting to compare this solution with that obtained using only the Laplace Transform.

Given $$\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2} \qquad \begin{matrix} -\infty < x < \infty \\ t > 0 \end{matrix} \tag{8}$$

where $\theta = f(x)$ when $t = 0$, and $\theta \to 0$ when $x \to \pm\infty$ (9)

Apply the L.T. i.e. multiply by e^{-pt} and integrate as usual

obtaining $p\bar{\theta} = k \frac{d^2\bar{\theta}}{dx^2} + f(x)$ using the first part of (9)

Now if we put $\bar{\theta} = u$ we get $$k \frac{d^2 u}{dx^2} - pu = -f(x) \tag{10}$$

We must now apply the L.T. with respect to x, i.e. $\bar{u} = \int_0^\infty u . e^{-sx} dx$

Hence $$(ks^2 - p)\bar{u} - k(sA + B) = -\bar{f}(s) \tag{11}$$

where A and B are constants.

The complementary function of equation (10) is [1] (see foot note s.4).

$$u = De^{+x\sqrt{\frac{p}{k}}} + Ee^{-x\sqrt{\frac{p}{k}}} \tag{12}$$

And we obtain a Particular Integral from (14) with A and B zero.

Thus $$\bar{u} = -\frac{\bar{f}(s)}{(ks^2 - p)} = -\frac{\bar{f}(s)}{k(s^2 - \frac{p}{k})}$$

$$= -\frac{\bar{f}(s)}{2\sqrt{pk}} \left[\frac{1}{s - \sqrt{\frac{p}{k}}} - \frac{1}{s + \sqrt{\frac{p}{k}}} \right]$$

If we use the Convolution Theorem this gives us the P.I. to be

$$u = -\frac{1}{2\sqrt{pk}} \int_0^x \left\{ e^{\sqrt{\frac{p}{k}}(x-\tau)} - e^{-\sqrt{\frac{p}{k}}(x-\tau)} \right\} f(\tau)\, d\tau$$

and the complete solution of (10) is therefore

$$u = De^{x\sqrt{\frac{p}{k}}} + Ee^{-x\sqrt{\frac{p}{k}}} - \frac{1}{2\sqrt{pk}} \int_0^x \left\{ e^{+\sqrt{\frac{p}{k}}(x-\tau)} - e^{-\sqrt{\frac{p}{k}}(x-\tau)} \right\} f(\tau)\, d\tau \tag{13}$$

Now using the second part of (9), as $x \to +\infty$, $u \to 0$

therefore $$D - \frac{1}{2\sqrt{pK}} \int_0^{\infty} e^{-\sqrt{\frac{p}{K}}\cdot\tau} g(\tau)\, d\tau = 0$$

and as $x \to -\infty$, $u \to 0$

giving $$E + \frac{1}{2\sqrt{pK}} \int_0^{-\infty} e^{+\sqrt{\frac{p}{K}}\cdot\tau} g(\tau)\, d\tau = 0.$$

Thus our solution is now

$$u = \frac{1}{2\sqrt{pK}} \left[\int_0^x e^{-\sqrt{\frac{p}{K}}(x-\tau)} g(\tau)\, d\tau + \int_{-\infty}^0 e^{-\sqrt{\frac{p}{K}}(x-\tau)} g(\tau)\, d\tau - \int_0^x e^{\sqrt{\frac{p}{K}}(x-\tau)} g(\tau)\, d\tau + \int_0^{\infty} e^{\sqrt{\frac{p}{K}}(x-\tau)} g(\tau)\, d\tau \right]$$

i.e.

$$u = \frac{1}{2\sqrt{pK}} \left[\int_{-\infty}^x e^{-\sqrt{\frac{p}{K}}(x-\tau)} g(\tau)\, d\tau + \int_x^{\infty} e^{\sqrt{\frac{p}{K}}(x-\tau)} g(\tau)\, d\tau \right] = \bar{\theta} \qquad (14)$$

The table of transforms gives the transform of

$$\frac{1}{\sqrt{\pi t}} \cdot e^{-\frac{K^2}{4t}} \text{ to be } \frac{1}{\sqrt{p}} \cdot e^{-K\sqrt{p}}.$$

and if we apply this to our equation we obtain

$$\theta = \frac{1}{2\sqrt{\pi K t}} \left[\int_{-\infty}^x e^{-\frac{(x-\tau)^2}{4Kt}} g(\tau)\, d\tau + \int_x^{\infty} e^{-\frac{(\tau-x)^2}{4Kt}} g(\tau)\, d\tau \right]$$

i.e.

$$\theta = \frac{1}{2\sqrt{\pi K t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\tau)^2}{4Kt}} g(\tau)\, d\tau$$

which is the required solution.

n.b. this solution is that obtained in the second last line of 1.3.

1. The author gratefully acknowledges advice from Professor A.E.Green at this stage, which enabled him to complete the solution.

It is at once obvious which of the above two methods gives the neater solution.The advantages of the Fourier Transform method are:

(i) The general boundary condition f(x) is not introduced until the ordinary differential equation given by the transformation is solved. Note that the Laplace transformation brings this in immediately and thus introduces the awkward equation 1.4.(10).

(ii) As a result of the equation mentioned in (i) we introduce two constants A and B and use the conditions at infinity to evaluate them. This process is entirely eliminated in the Fourier transformation.

2.0. To compare the use of $\int_0^\infty \theta(x,t).e^{-pt}dt$ and $\int_0^\infty \theta(x,t).\sin \xi x \, dx$ for

solving $$\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2} \qquad \begin{matrix} 0 < x < \infty \\ t > 0 \end{matrix} \qquad (15)$$

where $$\theta = 0, \; t = 0, \; x > 0 \qquad (16)$$
$$\theta = \theta_0, \; t \geq 0, \; x = 0 \qquad (17)$$

2.1. Use of $\int_0^\infty \theta(x,t).e^{-pt}dt$

Apply the transform in the usual way to give us

$$p\bar{\theta} = k\frac{d^2\bar{\theta}}{dx^2} \quad \text{using (16)}$$

i.e. $$\frac{d^2\bar{\theta}}{dx^2} = \frac{p}{k}\bar{\theta}$$

and therefore $$\bar{\theta} = Ae^{-\sqrt{\frac{p}{k}}x} + Be^{+\sqrt{\frac{p}{k}}x}$$

But θ is finite when $x \to \infty$.

Hence $$\bar{\theta} = Ae^{-\sqrt{\frac{p}{k}}x} \qquad (18)$$

and $\bar{\theta} = \frac{\theta_0}{p}$ when $x = 0$ and $t \geq 0$ from (17)

Therefore $$\bar{\theta} = \frac{\theta_0}{p} e^{-\sqrt{\frac{p}{k}}x}$$

and thus $$\theta = \theta_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)$$

where erfc. x is the complementary error function defined by

$$\operatorname{erfc} x = 1 - \operatorname{erf} x$$
$$= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du.$$

2.2. Use of $\int_0^\infty \theta(x,t) . \sin \xi x \, dx$

Apply the transform and assume that $\theta \to 0$ when $x \to \infty$, arriving at

$$\frac{d\bar{\theta}}{dt} + k\xi^2 \bar{\theta} = k\xi\theta_0 \qquad \text{from (17)}$$

Since $\theta = 0$ when $t = 0$ then $\bar{\theta} = 0$ when $t = 0$

Therefore the solution is

$$\bar{\theta} = \frac{\theta_0}{\xi}\left(1 - e^{-k\xi^2 t}\right)$$

The Inversion theorem giving us

$$\theta(x,t) = \frac{2\theta_0}{\pi} \int_0^\infty \frac{\sin \xi x}{\xi}\left(1 - e^{-k\xi^2 t}\right) d\xi. \qquad (19)$$

Making use of the integral

$$\int_0^\infty e^{-p^2} . \frac{\sin 2py}{p} . dp = \frac{1}{2}\pi \operatorname{erf}(y)$$

where $$\operatorname{erf} y = \frac{2}{\sqrt{\pi}} \int_0^y e^{-u^2} du.$$

we have finally

$$\theta(x,t) = \frac{2}{\pi} . \theta_0 . \frac{\pi}{2} - \frac{2}{\sqrt{\pi}} \theta_0 \int_0^{\frac{x}{2\sqrt{kt}}} e^{-u^2} du.$$

$$= \theta_0 \left[1 - \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right)\right]$$

$$= \underline{\theta_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)}$$

We note here that the Laplace Transform gives us the neater solution. The fixed boundary conditions introduced in the transformation give us immediately a very simple equation. In each case the solution is quite straightforward and presents no real difficulties.

3.0. Mixed Value Boundary Problems.

These problems introduce different boundary conditions at each side of a contour, e.g. the flow of fluid through an aperture in a rigid screen giving different conditions inside and outside the aperture.

The example we shall consider will require the use of the Hankel Transform

$$\bar{f}(\xi) = \int_0^\infty r.\ f(r).\ J_0(\xi r)\,dr \qquad (20)$$

3.1. The Inversion Theorem (see p. 72 of previous work)

If

$$\bar{f}(\xi) = \int_0^\infty r.\ f(r).\ J_0(\xi r)\,dr$$

then the inversion formula is

$$f(r) = \int_0^\infty \xi.\ \bar{f}(\xi).\ J_0(\xi r)\,d\xi \qquad (21)$$

3.2. Dual Integral Equations

For the problem to be considered we need the solution of the equations

$$\left.\begin{aligned} \int_0^\infty y^\alpha.\ f(y).\ J_\nu(xy)\,dy &= g(x) \qquad 0<x<1 \\ \int_0^\infty f(y).\ J_\nu(xy)\,dy &= 0 \qquad x>1 \end{aligned}\right\} \qquad (22)$$

This solution is given as[1.]

1. See "Fourier Transforms" I.N.Sneddon pp. 65-70

$$f(x) = \frac{2^{-\frac{1}{2}\alpha}\cdot x^{-\alpha}}{\Gamma(1+\frac{1}{2}\alpha)}\left[x^{1+\frac{1}{2}\alpha}\cdot J_{\nu+\frac{1}{2}\alpha}(x)\int_0^1 y^{\nu+1}(1-y^2)^{\frac{1}{2}\alpha}\cdot g(y)\,dy\; + \right. \tag{23}$$

$$\left. + \int_0^1 u^{\nu+1}\cdot(1-u^2)^{\frac{1}{2}\alpha}\,du\int_0^1 g(yu)\cdot(xy)^{2+\frac{1}{2}\alpha}\,J_{\nu+1+\frac{1}{2}\alpha}(xy)\,dy\right]$$

which is valid for $\alpha > -2$.

3.3. Example. (Flow of liquid through an aperture in a rigid screen)

To solve the equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \begin{matrix} 0 < r < \infty \\ z > 0 \end{matrix} \tag{24}$$

subject to

$$\phi = g(r) \qquad r < a,\ z = 0 \tag{25}$$

$$\frac{\partial\phi}{\partial z} = 0 \qquad r > a,\ z = 0 \tag{26}$$

Multiply both sides of equation (24) by $rJ_0(\xi r)$ and integrate from 0 to ∞ with respect to r, obtaining (see previous pp 73, 74)

$$\frac{d^2\bar{\phi}}{dz^2} - \xi^2\bar{\phi} = 0 \tag{27}$$

where

$$\bar{\phi}(\xi z) = \int_0^\infty r.\,\phi(r z).\,J_0(\xi r).dr.$$

We are only interested in the flow in the half-plane $z \geq 0$ and thus $\phi \to 0$ as $z \to \infty$ and the solution of (27) must be

$$\bar{\phi} = A(\xi)\,e^{-\xi z} \tag{28}$$

and therefore

$$\frac{\partial\bar{\phi}}{\partial z} = -\xi.A(\xi).e^{-\xi z} = \int_0^\infty r.\frac{\partial\phi}{\partial z}\cdot J_0(\xi z)\,dr \tag{29}$$

If we now apply the Inversion theorem (21) we obtain

$$\phi = \int_0^\infty \xi.\,A(\xi).\,e^{-\xi z}\cdot J_0(\xi r)\,d\xi.$$

$$\frac{\partial\phi}{\partial z} = -\int_0^\infty \xi^2\cdot A(\xi).\,e^{-\xi z}\cdot J_0(\xi r)\,d\xi.$$

Hence from (25) and (26)

$$\int_0^\infty \xi . A(\xi). J_0(\xi r)\, d\xi = g(r)$$

$$\int_0^\infty \xi^2 . A(\xi). J_0(\xi r)\, d\xi = 0$$

Write $\rho = \frac{r}{a}$; $F(u) = u^2 A\left(\frac{u}{a}\right)$; $\varsigma(\rho) = a^2 g(r)$, $u = a\xi$ arriving at (30)

$$\left.\begin{aligned} \int_0^\infty u^{-1} . F(u). J_0(u\rho)\, du &= \varsigma(\rho) \qquad 0<\rho<1. \\ \int_0^\infty F(u). J_0(u\rho)\, du &= 0 \qquad \rho>1. \end{aligned}\right\} \tag{31}$$

Comparing this with (22) our solution must be

$$F(u) = \frac{\sqrt{2}.u}{\Gamma(\frac{1}{2})}\left[u^{\frac{1}{2}} . J_{-\frac{1}{2}}(u)\int_0^1 \frac{y}{(1-y^2)^{\frac{1}{2}}} . \varsigma(y)\, dy + \int_0^1 \frac{y}{(1-y^2)^{\frac{1}{2}}} . dy \int_0^1 \varsigma(yx). (yu)^{\frac{3}{2}}\, J_{\frac{1}{2}}(yu) dy\right]$$

and since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $J_{\frac{1}{2}}(u) = \left(\frac{2}{\pi u}\right)^{\frac{1}{2}} \cos u$, $J_{-\frac{1}{2}}(u) = \left(\frac{2}{\pi u}\right)^{\frac{1}{2}} \sin u.$

this reduces to

$$F(u) = \frac{2u}{\pi}\cos u \int_0^1 \frac{y.\varsigma(y)\, dy}{(1-y^2)^{\frac{1}{2}}} + \frac{2u}{\pi}\int_0^1 \frac{y\, dy}{(1-y^2)^{\frac{1}{2}}}\int_0^1 \varsigma(yx). yu. \sin(yu)\, dy \tag{32}$$

If we take the particular case in which $\varsigma(\rho)$ is a constant C we find that

$$F(u) = \frac{2C}{\pi} \sin u \tag{33}$$

From (30) we see that $g(r)$ is a constant γ where $C = a^2\gamma$

and thus $$A(\xi) = \frac{2}{\pi} . \frac{\gamma \sin(a\xi)}{\xi^2}$$

giving finally $\phi = \frac{2\gamma}{\pi}\int_0^\infty \frac{\sin(a\xi)}{\xi} . e^{-\xi z} . J_0(\xi r) d\xi$ a solution obtained

otherwise by Lamb.