HARMONIC SPACES

A Thesis presented

by

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INTRODUCTION

In 1940 a paper was published by E.T.Copson and H.S.Ruse [10] in which centrally and completely harmonic spaces were defined. This paper corrected an earlier one by Ruse [19] which claimed to give the 'elementary solution', in the sense of J.Hadamard [13], of Laplace's equation $\Delta_2 V = 0$ in a general Riemannian space. The properties of the particular spaces for which Ruse's work was valid were also investigated. These spaces, called harmonic spaces, are characterised by admitting a solution of Laplace's equation which is a function only of the geodesic distance from a given origin. If such a solution exists for one choice of origin alone the space is called centrally harmonic, if it exists for all choices of origin the space is completely harmonic. It was found that a necessary and sufficient condition for a space to be harmonic about a point is that $\Delta_2 S$ be a function of S alone, where S is the geodesic distance from the base point.

Much of this and subsequent work on harmonic spaces is contained in Chapter I. Although most of the results of this chapter are already well known we give several alternative proofs which unify the development of the subject. In particular we give an alternative derivation of the 'equations of Copson and Ruse'. Throughout this and most of the following chapters we are concerned only with positive definite Riemannian metrics. In some papers on harmonic spaces this restriction is not made and these we include in the bibliography.

The completely harmonic spaces (with positive definite metrics) already known are symmetric in the sense of E. Cartan [6,p.44] and although it is not known whether this is always true, the evidence does suggest that new examples of completely harmonic spaces may be found by assuming them to be symmetric. More generally one may look for the class of homogeneous spaces with Riemannian metrics which are completely harmonic. This is in fact our object in Chapters II and III.

Perhaps the simplest example of a homogeneous space is that of a Lie group. In Chapter II, after recalling some of the classical theory of Lie groups we derive properties of metrics associated with semi simple groups. With the additional assumption of compactness on the groups these metrics are positive definite and for the group spaces to be completely harmonic we then show that their Lie algebras must be simple and of rank one or zero. It follows immediately from Cartan's classification of the simple Lie algebras and their ranks [4 ch.VIII] that the only completely harmonic compact Lie group is the three dimensional rotation group.

In Chapter III we first consider some general properties of homogeneous spaces and symmetric spaces mainly from a global point of view. We then consider two point homogeneous Riemannian spaces or (*) spaces and show that these are completely harmonic. By inspecting H. C. Wang's classification of compact and connected (*) spaces we thus obtain a class of completely harmonic spaces. Since these are also symmetric it follows that the corresponding non-compact symmetric spaces are also (*) spaces and hence completely harmonic.

CHAPTER I

THE GENERAL THEORY OF HARMONIC SPACES

Summary. This first chapter is devoted to the definition of centrally harmonic and completely harmonic spaces with positive definite Riemannian metrics and to results which have been obtained on such spaces.

In §1 we define a centrally harmonic V_n and prove a necessary and sufficient condition due to Copson and Ruse, in order that a V_n should possess this property.

In §2 we obtain several identities, some of which hold only with respect to a system of normal coordinates. By making use of these identities we find explicitly in §3 the solution of Laplace's equation in a centrally harmonic V_n which depends only on the geodesic distance S from the base point. We then define the concept of a completely harmonic V_n and prove that in such a space the harmonic function $\chi(s)$ is independent of the coordinates of the base point - a result which greatly simplifies the study of these spaces. We then state without proof two theorems due to T.J.Willmore [30] by means of which one obtains an alternative characterization of completely harmonic V_n's.

In §4 we describe a method by which any required number of the 'equations of Copson and Ruse' may be obtained in a centrally harmonic V_n. The method used is different from that of A.G.Walker [23], following more naturally from our initial definitions. Since $\chi(s)$ has a singularity at the origin it is found more convenient in this section to replace it by $f(\Omega)$ where $\Omega = \frac{1}{2}s^2$.

In the case of a V_n which is symmetric in the sense of Cartan, Walker has obtained a necessary and sufficient set of at most n-1 conditions in order that it be completely harmonic [26]. We deduce these conditions in § 5 from the equations of Copson and Ruse and hence show that spaces of constant curvature are completely harmonic. Conversely all completely harmonic V_n's for n = 2 or 3 are of constant curvature. When n = 4, however, there exist completely harmonic spaces other than those of constant curvature; we give their most general metric as obtained by Walker.

Finally, we prove that decomposable completely harmonic spaces and simply harmonic spaces are necessarily flat and in addition show that in a completely harmonic space $f(\Omega)$

satisfies a set of inequalities. The simplest of these was obtained by Lichnerowicz and appears to be the only one of any practical significance, the others being extremely complicated.

1. Centrally harmonic spaces.

Let V_n denote an n-dimensional local space (n-cell) having a positive definite Riemannian metric

$$ds^2 = g_{ij}\,dx^i dx^j \dagger \qquad (i, j = 1, \dots n)$$

defined over it. Let O be a fixed point and P a variable point in V_n and write S for the geodesic distance OP. Then S is a function of position P and $\Delta_2 S$ can be calculated, $\Delta_2 S$ being the generalised Laplacian operator defined by the usual formulae

$$(1) \quad \Delta_2 V \equiv g^{ij} V_{,ij}$$

$$(2) \quad \equiv g^{ij}\left(\frac{\partial^2 V}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial V}{\partial x^k}\right)$$

where Γ^k_{ij} denotes the Christoffel symbols calculated from the metric tensor g_{ij} and comma denotes covariant differentiation. We say V_n is harmonic with respect to the base point O or centrally harmonic if

$$(3) \quad \Delta_2 S \equiv g^{ij} S_{,ij} \equiv \chi(S),$$

$\dagger$ Unless otherwise stated, a repeated index indicates summation over its range of values.

where the harmonic function $\chi(s)$ is a function of s alone; it does not involve the coordinates x^i of the variable point P explicitly, but the coordinates $\bar{x}^i$ of the base point O may appear as parameters. This definition of a centrally harmonic space is equivalent to that given initially by Copson and Ruse[10].

Theorem I. In order that a V_n be harmonic with respect to a point O it is necessary and sufficient that Laplace's equation has a solution depending only on the geodesic distance measured from O.

Denoting differentiation with respect to s by a dash, we have from (1) that

$$(4) \quad \Delta_2 \phi(s) = \phi'(s)\Delta_2 s + \phi''(s)\Delta_1 s = \phi'(s)\Delta_2 s + \phi''(s)$$

since as is well known, $\Delta_1 s$ has the value unity. Hence Laplace's equation

$$(5) \qquad \Delta_2 V = 0$$

has a solution of the form $\phi(s)$ provided $\phi(s)$ satisfies

$$(6) \qquad \phi'(s)\Delta_2 s + \phi''(s) = 0.$$

If the space is centrally harmonic and s the geodesic distance measured from the base point O then using (3) the solution of (6) can be written as

$$\phi(s) = A\int \exp\left(-\int \chi\, ds\right) ds \quad + B,$$

A and B being suitable constants; and it follows from (4) that Laplace's equation has the solution $\phi(S)$.

Conversely if, in a V_n, (5) has a solution depending on S alone, then from (4) we have

$$\Delta_2 S = -\frac{V''(S)}{V'(S)}$$

and so V_n is centrally harmonic. This completes the proof of theorem I.

§2. Identities relative to normal and general coordinate systems.

Before solving Laplace's equation is a centrally harmonic space we require several identities, some of which are valid only with respect to a system of normal coordinates.

With a normal coordinate system about a point $O \in V_n$ the coordinates of a general point P are given by

$$(1) \qquad y^{\alpha} = \lambda^{\alpha} S^{\dagger}$$

where λ^{α} are the components of the unit tangent vector to the geodesic of length S joining O to P. V_n is assumed small enough to ensure that geodesics from O to points of V_n are unique. Since (1) is the equation of the geodesic

† Greek and Roman suffixes will be used to denote the coordinates of a point with respect to systems of normal and general coordinates respectively.

through O in the direction λ^α it follows that at a point $P(y^\alpha)$ of V_n we have

$$\Gamma^\alpha_{\beta\gamma} y^\beta y^\gamma = 0$$

while at O

$$\Gamma^\alpha_{\beta\gamma} = 0$$

Let $\Psi_{\alpha\beta}$ denote the components of the fundamental tensor in the normal coordinate system and write $\overline{\Psi}_{\alpha\beta}$, $\overline{\Psi}^{\alpha\beta}$ for the values at O of $\Psi_{\alpha\beta}$, $\Psi^{\alpha\beta}$. Writing

$$\Omega = \tfrac{1}{2} S^2$$

it follows from (1) that

$$2\Omega = \overline{\Psi}_{\alpha\beta} y^\alpha y^\beta$$

Differentiating this equation we have

$$\frac{\partial \Omega}{\partial y^\alpha} = \overline{\Psi}_{\alpha\beta} y^\beta \tag{2}$$

$$\frac{\partial^2 \Omega}{\partial y^\alpha \partial y^\beta} = \overline{\Psi}_{\alpha\beta}$$

while the partial derivatives of higher order are all null.

In a general V_n with Riemannian metric

$$ds^2 = g_{ij}\, dx^i dx^j$$

Let S denote the geodesic distance between the points $O(\overline{x}^i)$ and $P(x^i)$. Then S is a function of the $\overline{x}$'s and of the x's. Denote covariant differentiation with respect to the coordinates of O by a barred suffix (i.e. $S_{,\bar{\iota}}$) and write $\bar{g}_{ij}$, $\bar{g}^{ij}$ for the values at O of g_{ij}, g^{ij} respectively. Now from the equations

$$g^{ij} S_{,i} S_{,j} = 1$$

$$g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 1$$

$$S_{,i} \frac{dx^i}{ds} = 1$$

we have that the unit contravariant vectors $\frac{dx^i}{ds}$, $g^{ij}S_{,j}$ are in the same direction and are therefore equal. Hence

$$(3) \qquad \frac{dx^i}{ds} = g^{ij} S_{,j} = \frac{1}{s} g^{ij} \Omega_{,j}$$

and

$$(4) \qquad \frac{d\bar{x}^i}{ds} = -\left(\frac{dx^i}{ds}\right)_{x^i=\bar{x}^i} = \bar{g}^{ij} S_{,\bar{j}} = \frac{1}{s} \bar{g}^{ij} \Omega_{,\bar{j}}.$$

In normal coordinates, we have as a consequence of (3) and (4)

$$(5) \qquad y^\alpha = \psi^{\alpha\beta} \Omega_{,\beta}$$

$$(6) \qquad y^\alpha = -\bar{\psi}^{\alpha\beta} \Omega_{,\bar{\beta}}$$

and from (2) and (5)

$$(7) \qquad \psi_{\alpha\beta} y^{\beta} = \overline{\psi}_{\alpha\beta} y^{\beta}.$$

It then follows by differentiating (7) with respect to y^{β} that

$$y^{\gamma} \frac{\partial \psi_{\alpha\gamma}}{\partial y^{\beta}} = \overline{\psi}_{\alpha\beta} - \psi_{\alpha\beta}$$

whence $y^{\gamma} \frac{\partial \psi_{\alpha\gamma}}{\partial y^{\beta}}$ is symmetrical in α and β .

§3. Laplace's equation in a centrally harmonic space. Completely harmonic spaces.

With the notation of §2 take the base point of the centrally harmonic V_n as origin of the normal coordinate system and as the point with coordinates $\overline{x}^i$ in the general coordinate system, and let J denote the determinant $|\Omega_{,i\bar{j}}|$. Thus we prove the following

Theorem II. In a centrally harmonic V_n Laplace's equation (5) of §1 has a solution which depends on S alone and is given by

$$V = A \int_a^s \frac{J}{g^{1/2} \bar{g}^{1/2}} \frac{ds}{s^{n-1}}$$

where A and a are constant and the integral is taken along the geodesic arc joining $O(\overline{x}^i)$ to $P(x^i)$.

The symbols $g^{\frac{1}{2}}$, $\bar{g}^{\frac{1}{2}}$ in the above equation denote of course the square roots of determinants $|g_{ij}|$, $|\bar{g}_{ij}|$. It should also be remarked that O must not itself be a point of the arc of integration for if it were the integral would in general be divergent. From (1) of § 1 and (4) of § 2 it follows that

$$\Delta_2 s = \left(\frac{dx^i}{ds}\right)_{,i}$$

$$= \frac{\partial\left(\frac{dx^i}{ds}\right)}{\partial x^i} + \Gamma^i_{ij}\frac{dx^j}{ds}$$

$$(1) \qquad = \frac{\partial\left(\frac{dx^i}{ds}\right)}{\partial x^i} + \frac{d\log g^{1/2}}{ds}.$$

Now in a system of normal coordinates

$$\frac{\partial\left(\frac{dy^\alpha}{ds}\right)}{\partial y^\alpha} = \frac{\partial\left(\frac{y^\alpha}{s}\right)}{\partial y^\alpha} = \frac{n-1}{s}$$

whence from (1)

$$(2) \qquad \Delta_2 s = \frac{n-1}{s} + \frac{d\log \psi^{1/2}}{ds} = \frac{d\log s^{n-1}\psi^{1/2}}{ds},$$

where Ψ denotes the determinant $|\Psi_{\alpha\beta}|$.

Since

$$\Psi^{1/2} = \frac{g^{1/2}}{\left|\frac{\partial y^\alpha}{\partial x^i}\right|}$$

(2) can be written as

$$\Delta_2 s = \frac{d}{ds} \log \frac{s^{n-1} g^{1/2}}{\left|\frac{\partial y^\alpha}{\partial x^i}\right|} \tag{3}$$

Differentiating (6) of §2 with respect to x^i we have

$$\frac{\partial y^\alpha}{\partial x^i} = -\overline{\Psi}^{\alpha\beta} \Omega_{,\bar{\beta} i}$$

and therefore

$$\left|\frac{\partial y^\alpha}{\partial x^i}\right| = (-)^n \overline{\Psi} \, |\Omega_{,\bar{\beta} i}|. \tag{4}$$

But, as is well known,

$$\frac{\partial y^\alpha}{\partial x^i} = \delta^\alpha_i$$

when evaluated at the origin of normal coordinates; hence the components of a tensor at the point are the same in either of the two coordinate systems and in particular

(5) $$\bar{\Psi}^{\alpha\beta} = \bar{g}^{\alpha\beta}$$

In virtue of (4) and (5) equation (3) becomes

$$\Delta_2 S = \frac{d}{dS} \log \frac{S^{n-1} g^{1/2}}{(-)^n \bar{g} J}$$

where J is as previously defined. Since the space is assumed to be centrally harmonic about O we know by theorem I that Laplace's equation has a solution depending on S alone. It then results from (6) of § 1 that

$$V = \phi(S) = A \int_a^S \frac{J}{g^{1/2} \bar{g}^{1/2}} \frac{dS}{S^{n-1}}$$

which completes the proof.

It clearly follows as a consequence of this theorem that the function ρ defined by

$$\rho^2 = \frac{\bar{g} g}{J^2}$$

is a function of S alone in a centrally harmonic V_n. Conversely if it is a function of S alone then V_n is centrally harmonic; ρ is usually called <u>Ruse's invariant</u>.

A V_n is said to be <u>completely harmonic</u> if it is harmonic about each of its points. We now prove the following result which is due to A. Lichnerowicz [16] :

<u>Theorem III In a completely harmonic V_n the function</u>

$\chi(s)$ is independent of the coordinates $\bar{x}^i$ of the base point chosen.

Since the solution of Laplace's equation (c.f. theorem II) may depend on $\bar{x}^i$ denote it provisionally by $\phi(s, \bar{x}^i)$. Differentiating ϕ with respect to s we have

$$\phi'(s, \bar{x}^i) = \frac{A\,J}{g^{1/2}\,\bar{g}^{1/2}}\,\frac{1}{s^{n-1}}.$$

This equation shows that $\phi'(s, \bar{x}^i)$ is a symmetric function of the two points with coordinates $\bar{x}^i$, x^i, and since s is also symmetric it follows immediately that ϕ' does not depend explicitly on the coordinates $\bar{x}^i$. Hence since

$$\chi(s) = \frac{-\phi''(s)}{\phi'(s)},$$

it is independent of the coordinates of the base point and the theorem is proved.

By extending some well known results of potential theory to completely harmonic V_n's, T. J. Willmore obtained an alternative characterisation of such spaces [30]. We state without proof the two main theorems from which this characterisation follows. Write $S(P,\rho)$ for the geodesic sphere of centre P and radius ρ. If u is a function of position the mean value $M(u, P, \rho)$ of u over the surface of the sphere $S(P,\rho)$ is defined by

$$M(u, P, \rho) \int_{n-1} d\sigma = \int_{n-1} u \, d\sigma$$

where both (n - 1) fold integrals are taken over the closed boundary of $S(P, \rho)$, $d\sigma$ being the element of area. Then we have the following

Theorem IV If u is a function of position with continuous second derivatives in a completely harmonic V_n such that the mean value of u over every geodesic sphere is the value at the centre, then u satisfies Laplace's equation in V_n.

Theorem V Let u be a function of position having continuous second derivatives and satisfying Laplace's equation in a V_n. If the mean value of every such function over the surface of every geodesic sphere centre P is equal to the value at P, then V_n is centrally harmonic with respect to P.

Thus theorem IV is true only if V_n is completely harmonic

§4. The equations of Copson and Ruse.

It was shown by Copson and Ruse[10] that at the base point of a centrally harmonic V_n, the values taken by the curvature tensor R_{hijk} and its covariant derivatives are not independent but satisfy an infinite sequence of relations; it being assumed that the fundamental tensor possessed an infinite number of derivatives at the base point. In a completely harmonic V_n these relations are satisfied everywhere and so form a system of differential equations for the metric tensor g_{ij}. It will be shown later that in a V_n which is symmetric as well as completely harmonic only the first $n - 1$ equations of the set are independent, but it is possible that this result is true without the assumption of symmetry.

A method for obtaining these equations has been derived by Walker [23] . By means of a recurrence relation, the coefficients of S in the expansion of ρ [c.f. §3] as a power series in S may be calculated up to any number of terms and expressed in invariant form. If the space is centrally harmonic these coefficients must in fact be constants and the set of conditions thus obtained leads immediately to the equations of Copson and Ruse.

The method which we shall use is to evaluate the successive derivatives of $\Delta_2 \Omega$ with respect to S, at the base point O. We do this by considering not $\Delta_2 \Omega$ itself but a matrix B whose trace is equal to $\Delta_2 \Omega$. B satisfies a recurrence relation, the initial conditions being the value of B at O. By means of this recurrence relation the derivatives of the trace of B and hence of $\Delta_2 \Omega$ can be evaluated at the base point and the equations of Copson and Ruse thereby obtained.

With the notation of §2 we consider the vector field

(1) $$\lambda^i = \frac{dx^i}{ds}$$

of unit tangent vectors to geodesics through O. With the exception of O itself there is a unique vector at each point of V_n. By means of the Ricci identity we have

(2) $$(s\lambda^i)_{,lm} = (s\lambda^i)_{,ml} - s R^i_{\ jlm}\lambda^j.$$

Multiplying both sides of (2) by λ^m and summing for m it results that

(3) $$D_s(s\lambda^i)_{,l} = (s\lambda^i)_{,ml}\lambda^m + s R^i_{\ jml}\lambda^j\lambda^m$$

where D_s denotes absolute differentiation along a geodesic. Now since λ^i are the components of a unit tangent vector to a geodesic,

$$\lambda^i_{,m}\lambda^m = D_s(\lambda^i) = 0$$

whence

$$(s\lambda^i)_{,m\ell}\lambda^m = [(s\lambda^i)_{,m}\lambda^m]_{,\ell} - (s\lambda^i)_{,m}\lambda^m_{,\ell}$$

$$= \lambda^i_{,\ell} - (s\lambda^i)_{,m}\lambda^m_{,\ell}$$

$$= \frac{1}{s}(s\lambda^i)_{,\ell} - \frac{1}{s}s_{,\ell}\lambda^i - (s\lambda^i)_{,m}\lambda^m_{,\ell}$$

(4) $$= \frac{1}{s}(s\lambda^i)_{,\ell} - \frac{1}{s}(s\lambda^i)_{,m}(s\lambda^m)_{,\ell}$$

Using matrix notation,

$$B = (B^i_j) = (s\lambda^i)_{,j}$$

and

$$\Gamma = (\Gamma^i_j) = (R^i{}_{hkj}\lambda^h\lambda^k)$$

are square matrices of order n. It follows using (4) that (3) may be written in matrix form as

(5) $$s D_s B = s^2\Gamma + B - B^2$$

If now we find the value of B at O then by means of (5) we can evaluate the successive covariant derivatives of B with respect to S at O. To evaluate B at O write (3) of § 2 as

(6) $$s\lambda^i = g^{ij}\Omega_{,j}$$

Then differentiating (6) covariantly we have

(7) $$B^i_k = (s\lambda^i)_{,k} = g^{ij}\Omega_{,jk},$$

whence, multiplying (7) by λ^k and summing for K, we obtain

(8) $$\lambda^i = g^{ij}\Omega_{,jk}\lambda^k.$$

Since λ^i, when evaluated at 0, is an arbitrary unit vector we deduce from (8) that at 0

$$g^{ij}\Omega_{,jk} = \delta^i_k$$

and hence from (7)

(9) $$B^i_k = \delta^i_k,$$

evaluation being again at 0. Operating on (5) with D_s we have

(10) $$sD_s(D_sB) = s^2 D_s\Gamma + 2s\Gamma - D_sB^2$$

and evaluating (10) at 0 it results using (9) that

(11) $$D_sB = 0$$

Denote by A_r the r^{th} absolute derivative of the elements of a matrix A. Then the equations obtained from (5) by successive operations of D_s may be written as

$$\begin{aligned}
SB_1 &= S^2\Gamma + B - B^2 \\
SB_2 &= S^2\Gamma_1 + 2S\Gamma - B_1^2 \\
SB_3 &= S^2\Gamma_2 + 4S\Gamma_1 + 2\Gamma - B_2^2 - B_2 \\
SB_4 &= S^2\Gamma_3 + 6S\Gamma_2 + 6\Gamma_1 - B_3^2 - 2B_3
\end{aligned}$$

....

It is easily seen that the general equation is

$$SB_r = S^2\Gamma_{r-1} + 2(r-1)S\Gamma_{r-2} + (r-1)(r-2)\Gamma_{r-3} - B^2_{r-1} - (r-2)B_{r-1} \tag{12}$$

where $r > 0$, a matrix with a negative suffix being zero† .
Evaluating (12) at (0) we have

$$(r-1)B_r = r(r-1)\Gamma_{r-2} - B^2_r \tag{13}$$

† B^2_r denotes the r^{th} absolute derivative of B^2 and must not be confused with the square of the r^{th} absolute derivative of B i.e. $(B_r)^2$.

where $r > 0$ †. Since at 0

$$B_r^2 = \sum_{s=0}^{r} \binom{r}{s} B_s B_{r-s}$$

$$= 2B_r + \sum_{s=1}^{r-1} \binom{r}{s} B_s B_{r-s}$$

(13) may be written in the more convenient form for purposes of calculation as

$$(14) \quad (r+1) B_r = r(r-1)\Gamma_{r-2} - \sum_{s=1}^{r-1} \binom{r}{s} B_s B_{r-s} .$$

Now by summing i and k in (7) we have

$$\operatorname{tr} B = \Delta_2 \Omega ,$$

where tr B denotes the trace of the matrix B; and since the two operators 'D_s' and 'tr' commute it follows that

$$(15) \quad \frac{d^r \Delta_2 \Omega}{ds^r} = \operatorname{tr} B_r$$

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† Equation (13) holds for the case r = 1 as a consequence of (11).

By means of (14) and (15) any number of derivatives of $\Delta_2 \Omega$ can be calculated at O fairly quickly. If the space is harmonic about O then it follows from (3) and (4) of §1 that $\Delta_2 \Omega$ is a function of S alone and hence its derivatives with respect to S are constants when evaluated at O; that is to say they are independent of the unit vector λ^i at O. It follows that the derivatives of tr B at O must then satisfy

$$(16) \qquad \frac{d^{2r-1}\, \mathrm{tr}\, B}{ds^{2r-1}} = 0 \qquad r = 1, 2, \cdots$$

$$(17) \qquad \frac{d^{2r}\, \mathrm{tr}\, B}{ds^{2r}} = K_r \qquad r = 1, 2 \cdots$$

where K's are scalars.

We shall illustrate the method of calculation by evaluating the first few derivatives. Unless otherwise stated it is to be understood that the matrices considered are evaluated at O. From (14) we have

$$3B_2 = 2\Gamma - 2B_1 B_1$$

$$4B_3 = 6\Gamma_1 - (3B_1 B_2 + 3B_2 B_1)$$

$$5B_4 = 12\Gamma_2 - (4B_1B_3 + 6B_2B_2 + 4B_3B_1)$$

$$6B_5 = 20\Gamma_3 - (5B_1B_4 + 10B_2B_3 + 10B_3B_2 + 5B_4B_1)$$

$$7B_6 = 30\Gamma_4 - (6B_1B_5 + 15B_2B_4 + 20B_3B_3 + 15B_4B_2 + 6B_5B_1).$$

Using (11) the above set of equations reduce to

(18) $$B_2 = \tfrac{2}{3}\Gamma$$

(19) $$B_3 = \tfrac{3}{2}\Gamma_1$$

(20) $$B_4 = \tfrac{4}{15}(9\Gamma_2 - 2\Gamma^2)$$

(21) $$B_5 = \tfrac{5}{3}(2\Gamma_3 - \Gamma\Gamma_1 - \Gamma_1\Gamma)$$

(22) $$B_6 = \tfrac{1}{7}\left(30\Gamma_4 - 24\Gamma_2\Gamma - 24\Gamma\Gamma_2 - 45\Gamma_1\Gamma_1 + \tfrac{32}{3}\Gamma^3\right)$$

and by substituting these values into equations (16) and (17) we obtain the first few equations of Copson and Ruse.

If the space is completely harmonic these equations hold at every point, the K's then being constants since it follows from theorem III and (4) of §1 that $\Delta_2\Omega$ is independent of the coordinates of the base point. Whence for r = 1,2 in (17) we have

$$\frac{d^t \operatorname{tr} B_2}{ds^t} = \frac{d^t \operatorname{tr} B_4}{ds^t} = 0 \qquad t \geq 1$$

and it follows using (18), (19), (20) and (21) that

$$\operatorname{tr} B_3 = \operatorname{tr} B_5 = 0 .$$

In addition by taking traces of matrices in (18), (20) and (22) we have, using (17), that

(23) $R_{ij}\lambda^i\lambda^j = \frac{3}{2}K_1$

(24) $R^h{}_{ijk}R^k{}_{lmh}\lambda^i\lambda^j\lambda^l\lambda^m = -\frac{15}{8}K_2$

(25) $R^h{}_{ijk}R^k{}_{lmr}R^r{}_{pqh}\lambda^i\lambda^j\lambda^l\lambda^m\lambda^p\lambda^q = \frac{21}{32}K_3 - \frac{9}{32}R^h{}_{ijk,l}R^k{}_{mph,q}\lambda^i\lambda^j\lambda^l\lambda^m\lambda^p\lambda^q.$

In deriving these equations we have used the additional equations obtained by equating derivatives of tr B_2 and tr B_4 to zero to give more concise results.

§5 Some consequences of the equations of Copson and Ruse.

Symmetric harmonic spaces. A V_n is called symmetric [6,p.44] if the symmetry with respect to any point $A \in V_n$ preserves distances. The symmetry is defined as follows : each point M (sufficiently near to A) is tranformed to the point M' obtained by extending the geodesic MA to a point M' such that the distance MA and M'A are equal.

An equivalent definition is that the covariant derivative of the curvature tensor vanishes at every point. We shall prove this in chapter III where symmetric spaces will be considered in more detail.

If the metric of a V_n is analytic then it is easily seen that $B^i_j = g^{il}\Omega_{lj}$ is also analytic. Its trace may then be expressed as a series in S, the coefficients being given by means of (14) of §4. Hence the necessary and sufficient conditions for V_n to be completely harmonic are that (16) and (17) of §4 should be satisfied. We now prove the following [26]

<u>Theorem VI. The necessary and sufficient conditions for a symmetric V_n having an analytical metric to be completely harmonic are that the latent roots of Γ should be constants.</u>

Let V_n be a symmetric space having an analytic metric. Then (14) of §4 becomes

$$(1) \qquad 3B_2 = 2\Gamma$$

$$(r+1)B_r = -\sum_{s=1}^{r-1}\binom{r}{s}B_s B_{r-s} \qquad r > 2.$$

Since B_1 is zero (c.f. (11) of §4) it can easily be proved by induction that

$$B_{2r+1} = 0 \qquad r \geqslant 0$$

and

$$(2) \qquad B_{2r} = b_r (B_2)^r \qquad r \geqslant 1$$

for some constants b_r. Hence using (1) and (2) we have that V_n is completely harmonic if and only if at a general point O of V_n,

$$\operatorname{tr} \Gamma^r = c_r$$

for all r, where c_r is independent of direction. By a well known theroem on square matrices, if $\omega_1, \ldots \omega_n$ are the latent roots of Γ,

$$(3) \qquad \operatorname{tr} \Gamma^r = \sum_{i=1}^{n} (\omega_i)^r.$$

The above infinite set of conditions are then easily seen to reduce to the finite set that ω_i (i = 1,...n) are constants. This completes the proof of the theorem. We note that at least one of the latent roots of Γ is zero since

$$\Gamma^i_j \lambda^j = 0.$$

It follows as a consequence of this theorem that spaces of constant curvature are completely harmonic, for at every point of such a space

$$R_{hijk} = K(g_{hk}g_{ij} - g_{hj}g_{ik})$$

and when Γ is formed and its latent roots calculated we find

$$\omega_1 = \omega_2 = \cdots\cdots = \omega_{n-1} = K.$$

The problem of characterizing all completely harmonic V_n's has been solved for $n = 2, 3$ or 4. In fact when $n = 2$ or 3 they are of constant curvature. To prove this we take first the case $n = 2$. Choose a system of coordinates for V_n such that at a point P we have $g_{ij} = \delta_{ij}$. Then with $\lambda^i = \delta^i_1$, $\mu^i = \delta^i_2$, it easily follows from (24) of §4 that

$$R_{hijk}\lambda^h\mu^i\lambda^j\mu^k = \pm\sqrt{-\tfrac{15}{8}K_2}$$

and the space being two dimensional the curvature is therefore a constant at P. By varying λ^i and μ^i continuously over V_n we see that the curvature is everywhere a constant equal to that at P. For the case $n = 3$ we note from (23) of §4 that a completely harmonic V_n is an Einstein space and it is well known that an Einstein space of three dimensions is necessarily of constant curvature.

The case $n = 4$ is more difficult. It has however been proved by Walker in [24] and [25] that every completely harmonic V_4 is symmetric and with the exception of spaces of constant curvature, has a metric of the form

$$ds^2 = f^{-2}(dx^2+dy^2+dz^2+dt^2) + \tfrac{1}{2}Kf^{-2}(z\,dx+t\,dy-x\,dz-y\,dt)^2 + \tfrac{1}{2}Kf^{-2}(t\,dx-z\,dy+y\,dz-x\,dt)^2$$

where

$$f = 1+\tfrac{1}{2}K(x^2+y^2+z^2+t^2).$$

K is any real constant but for $K<0$ the coordinates are restricted so that $f>0$.

Several results have been obtained by Lichnerowicz [16], using normal coordinates. We shall now deduce some of these from the first few equations of Copson and Ruse.

Decomposable spaces. A V_n is said to be decomposable if a decomposable coordinate system can be defined over it. That is to say a coordinate system in which the

Riemannian metric takes the form

$$ds^2 = g_{ij}dx^i dx^j = g_{i'j'}dx^{i'}dx^{j'} + g_{i''j''}dx^{i''}dx^{j''}$$

where $i', j' = 1, \ldots r$, $i'', j'' = r+1, \ldots n$ and $g_{i'j'}, g_{i''j''}$ are functions only of the first r and last n - r coordinates respectively. In such a coordinate system a component of a tensor is said to be mixed if its indices include numbers from both the ranges $1, \ldots r$ and $r+1, \ldots n$. If the mixed components of a tensor are all zero then it is said to be separable and if in additon its components whose indices range over $1, \ldots r$ and $r+1, \ldots n$ are functions of the first r and last n - r coordinates respectively then it is said to be decomposable. It is easily seen that a tensor which is decomposable remains so under a transformation of coordinates from one decomposable system to another. Also the sum, contraction, contracted product and covariant derivatives of decomposable tensors are decomposable tensors. In particular it can be verified that if g_{ij} is decomposable then so are R_{ij}, R_{hijk} and their covariant derivaties.

Theorem VII. A decomposable completely harmonic Vn is flat.

With the above metric for the decomposable V_n

let λ', λ'' be unit vectors whose last n-r and first r components respectively are zero. Then if the non-zero components of λ' and λ'' are $\lambda^{i'}$ and $\lambda^{j''}$ it follows using (24) of § 4 that

$$R^{h'}{}_{i'j'k'} R^{k'}{}_{l'm'h'} \lambda^{i'}\lambda^{j'}\lambda^{l'}\lambda^{m'} = R^{h''}{}_{i''j''k''} R^{k''}{}_{l''m''h''} \lambda^{i''}\lambda^{j''}\lambda^{l''}\lambda^{m''} = -\frac{15}{8}K_2.$$

But $\frac{1}{2}(\lambda' + \lambda'')$, being a unit vector, also satisfies (24) of § 4 and it results that $K_2 = 0$.

Since the V_n is completely harmonic K_2 is a constant defined over it and is independent of direction at a point, whence

$$(4) \qquad \Gamma^h_k \Gamma^k_h = -\frac{15}{8}K_2 = 0.$$

Let P be a general point of V_n. Then by a suitable transformation of coordinates the metric tensor can be chosen such that at P

$$g_{ij} = \delta_{ij}$$

In this case (4) becomes at P

$$\sum_{h,k} (\Gamma_{hk})^2 = 0$$

and since Γ_{hk} is real it follows that

$$\Gamma_{hk} = R_{hijk}\lambda^i\lambda^j = 0.$$

This equation holds for all unit vectors λ^i at P and therefore

$$(5) \qquad R_{hijk} + R_{hjik} = 0$$

and it is easily verified that (5) implies that

$$R_{hijk} = 0.$$

The curvature tensor thus vanishes at P and hence at every point of V_n. The space is therefore flat which proves the theorem.

Simply harmonic spaces

Let V_n be a completely harmonic space. Then

$$\Delta_2\Omega = s\Delta_2 s + 1 = s\chi(s) + 1,$$

and by writing $-s$ for s in this equation it follows that χ is an odd function of s. Writing

$$\Delta_2\Omega = f(\Omega)$$

we have from (15) and (17) of §4 that

$$K_r = \frac{d^{2r} f(0)}{d s^{2r}} = \frac{d^r f(0)}{d \Omega^r}.$$

A completely harmonic V_n is said to be <u>simply harmonic</u> if

$$f(\Omega) = n.$$

In the case of indefinite metrics such spaces are still of some interest but it can easily be shown that with a positive definite metric a simply harmonic space is flat. For if $f(\Omega)$ is constant it follows that K_r is zero for all $r > 0$, and in particular K_2 is zero. Using the same argument as in the proof of theorem VII the result then follows.

<u>Restrictions on the function $f(\Omega)$.</u> It will be noticed that the left hand sides of equations (23), (24) and (25) of §4 are respectively $\operatorname{tr}\Gamma$, $\operatorname{tr}\Gamma^2$, $\operatorname{tr}\Gamma^3$ and it can easily be seen that in the general case when $r = 2m$ the matrix $(B_2)^m$ will occur in (14) of §4 when the matrices within the summation sign are successively evaluated by means of this recurrence relation. Hence equations (17) of §4 may be written only with $\operatorname{tr}(B_2)^m$ i.e. ~~tr~~ $(\Gamma)^m$

on the left hand side. Now let P be a general point of a completely harmonic V_n and choose a system of coordinates such that at P

$$g_{ij} = \delta_{ij}$$

Then it follows that at this point

$$\Gamma^i_j = \Gamma_{ij} = \Gamma_{ji}.$$

Thus Γ is a real, symmetric matrix and its latent roots are therefore real. By an elementary theorem of algebra a necessary and sufficient set of conditions that the roots of a polynomial of degree n should be real are that a set of n - 1 inequalities between sums of powers of the roots be satisfied. Hence using (3) we can obtain (n - 1) inequalities between traces of powers of Γ. Evaluating these traces by means of the equations of Copson and Ruse we obtain inequalities which must be satisfied by f and the curvature tensor at P.

The simplest inequality for a polynomial

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

is that

$$2 n a_2 \leq (n-1) a_1^2$$

or in terms of powers of the roots $w_1, \ldots w_n$, that

$$n \sum w^2 \geq \left(\sum w\right)^2.$$

Applying this inequality to the latent roots of Γ it follows that

$$(5) \qquad n \sum_{i,j} \Gamma^i_j \Gamma^j_i \geqslant \left(\sum_i \Gamma^i_i\right)^2.$$

Hence since

$$K_1 = f'(0) \quad , \quad K_2 = f''(0),$$

where dash denotes differentiation with respect to Ω, we have from (23) and (24) of § 4.

$$f'^2(0) \leqslant -\frac{5}{6} n f''(0).$$

If the equality sign holds in (5) then

$$\left(\delta^i_j \Gamma^l_m - \delta^l_m \Gamma^i_j\right)\left(\delta^j_i \Gamma^m_l - \delta^m_l \Gamma^j_i\right) = 0$$

and therefore since the metric is positive definite

$$\delta^i_j \Gamma^l_m - \delta^l_m \Gamma^i_j = 0.$$

Summing i with j in this equation it follows from (21) of §4 that

$$R_{hijk} \lambda^i \lambda^j = \frac{3}{2n} K_1 g_{hk}$$

for all unit vectors λ^i at P. Consequently, if μ^i is a general vector at P then

$$\left(R_{hijk} - \frac{3}{2n} K_1 g_{hk} g_{ij}\right) \mu^i \mu^j = 0$$

and therefore

$$(6) \quad R_{hijk} + R_{hjik} - \frac{3K_1}{n} g_{hk} g_{ij} = 0.$$

Subtracting from (6) a similar equation with j and k interchanged we find

$$2 R_{hijk} + R_{hjik} + R_{hkji} = \frac{3K_1}{n}(g_{hk} g_{ij} - g_{hj} g_{ik}),$$

whence it is easily verified that

$$R_{hijk} = \frac{K_1}{n}(g_{hk} g_{ij} - g_{hj} g_{ik})$$

and the space is of constant curvature.

CHAPTER II

LIE GROUPS

Summary. In this chapter we consider the general theory of Lie groups before going on to discuss simple and semi-simple Lie groups and Riemannian metrics associated with them, the object being to enquire into the possible existence of semi-simple Lie groups which are completely harmonic.

We define in § 1 the concepts of a differentiable manifold and a Lie group. Initially no restriction is made that such a group be connected, although several results which come later are true only for connected Lie groups. We next introduce the important auxiliary functions and the constants of structure. In addition we prove in § 2 the existence and uniqueness of a local one parameter subgroup for any given direction through the identity e. We then define, for later use, two special coordinate systems which can be shown to exist in sufficiently small neighbourhoods of e.

In § 3, the concept of the Lie algebra is introduced, and several important consequences stated. § 4 is concerned mostly with the adjoint and linear adjoint groups. We show in particular that the question of the existence

of normal subgroups of a Lie group G can be reduced to that of subalgebras of its Lie algebra invariant under the linear adjoint group. Many of these subalgebras can be obtained from a study of systems of homogeneous linear equations whose coefficients are 'comitants'. We define a comitant in §5, and also the characteristic equation relative to a given Lie algebra. The coefficients in this equation are again comitants. In this way several restrictions can be found on the constants of structure in order that a Lie algebra be simple or semi-simple. We finally introduce the concept of the rank of a Lie algebra, due initially to Killing [14]

For semi-simple Lie groups, Cartan and Schouten have shown the existence of a Riemannian metric invariant under both right and left translations. We consider this metric in §6, calling it the (o) metric, since its associated affine connection is the (o) connection of Cartan and Schouten [8]. For a semi-simple Lie group which is not simple, the (o) metric is decomposable. In addition a group space over which it is defined is symmetric.

In §7, we consider the (o) metrics of compact simple groups, the reason being that only for these groups is it positive definite and indecomposable. We then define a completely harmonic manifold, and show that the only compact simple group which is completely harmonic with respect to the (o) metric is the three dimensional rotation group.

§1. DIFFERENTIABLE MANIFOLD, LIE GROUP, AUXILIARY FUNCTIONS.

An n-dimensional manifold M is a Hausdorff space covered by a system of neighbourhoods S, each neighbourhood of the system being homeomorphic to some open set in Euclidean n-space E_n. Such a neighbourhood will be called a coordinate neighbourhood. The correspondence between the points of a coordinate neighbourhood in M and the points of the corresponding open set in E_n defines a system of coordinates in the neighbourhood.

Let $(x^1, \dots x^n)$, $(y^1, \dots y^n)$ be the coordinate systems of two intersecting coordinate neighbourhoods in M. At points common to these neighbourhoods there is a (1-1) correspondence between the two systems of coordinates which can be expressed by the equations

$$x^i = x^i(y^1, \dots y^n) \qquad i = 1, \dots n$$

$$y^i = y^i(x^1, \dots x^n) \qquad i = 1, \dots n.$$

If all such functions $x^i(y^1, \dots y^n)$, $y^i(x^1, \dots x^n)$ have continuous derivatives of order r, then S is said to be of class r. Similarly, S is said to be of class ∞ or analytic if the functions possess an infinite number of derivatives or are analytic.

Two such systems S and S' of class r are said to be r-equivalent if the composite system of neighbourhoods is of class r. An n-dimensional differentiable manifold of class r is an n-dimensional manifold together with an r-equivalent class of systems S.

A Lie group is in the first place an analytic manifold of some dimension r which we assume satisfies the second axiom of countability. Secondly, the manifold carries a group structure. That is to say multiplication is defined between the points of the manifold, satisfying the usual group axioms. We denote the product of a and b by a.b or ab. Lastly, the analytic structure and group structure are related through the requirements that the product a.b and the inverse a^{-1} depend analytically on a and b, i.e. the coordinates of a.b and a^{-1} are analytic functions of the coordinates of a and b.

As is well known, the component of the identity element e of a topological group is generated by an arbitrary neighbourhood of e. In the case of a Lie group this affords a powerful means for obtaining properties of the component of e. We shall consider only coordinate neighbourhoods of e in which the coordinates of e are zero; this will be found to simplify many calculations

Let U_e and U_x be coordinate neighbourhoods of e and of a general point x respectively, in a Lie group G of dimension r. We denote symbolically by x+dx the point whose coordinates are $x^\alpha + dx^\alpha$, $(\alpha = 1, \cdots r)$, in U_x, where the dx^α are infinitesimally small. Then the point y given by

(1) $$y = (x + dx)x^{-1}$$

is infinitesimally near e, and can therefore be written in coordinate form as

(2) $$y^i = v^i_\alpha(x)\, dx^\alpha$$

where Roman suffixes indicate coordinates of points in the U_e coordinate system. The functions v^i_α are called the auxiliary functions; they are clearly defined at every point of G. Using (1), we see that they can be written as

(3) $$v^i_\alpha(x) = \left[\frac{\partial f^i(y, x^{-1})}{\partial y^\alpha}\right]_{y=x}$$

where $f^i(y, x^{-1})$ denotes the i^{th} coordinate of the point $yx^{-1} \in U_e$. Under a transformation of coordinates in U_x, $v^i_\alpha(x)$, for fixed i, transforms as a covariant vector, as follows immediately from the equation

$$v^i_\alpha(x)\, dx^\alpha = \bar{v}^i_\beta(\bar{x}) \frac{\partial \bar{x}^\beta}{\partial x^\alpha}\, dx^\alpha,$$

where the new coordinates of x are denoted by $\bar{x}^\alpha$. Similarly using (2) we see that under a transformation of coordinates in U_e, (or with a different choice of U_e) $v^i_\alpha(x)$, for fixed α, transforms as a contravariant vector at e. If U_x is itself a neighbourhood of e, then considering it also as U_e, we have, from (1) and (2),

$$v^i_\alpha(e)\,dx^\alpha = \delta^i_\alpha\,dx^\alpha$$

for all dx^α; hence

(4) $$v^i_\alpha(e) = \delta^i_\alpha.$$

If

$$z = xy$$

or, in coordinate form,

(5) $$z^\alpha = f^\alpha(x,y),$$

where x is a variable point and y a fixed point in G, then

$$\bigl(xy + d(xy)\bigr)(xy)^{-1} = (x+dx)\,y\,(xy)^{-1} = (x+dx)x^{-1}$$

or

(6) $$(z+dz)z^{-1} = (x+dx)\,x^{-1}.$$

Writing (6) in coordinate form, we have

(7) $$v^i_\alpha(z)\,dz^\alpha = v^i_\alpha(x)\,dx^\alpha$$

whence

(8) $$v^i_\beta(z)\,\frac{\partial z^\beta}{\partial x^\alpha} = v^i_\alpha(x).$$

In particular, if x = e, it follows from (4), (5) and (8) that

$$v^i_\beta(y)\,\frac{\partial f^\beta(0,y)}{\partial x^j} = \delta^i_j . \tag{9}$$

The matrix (v^i_α) is non-singular at every point of G, for if this were not so then for some y and dy we would have, from (1) and (2),

$$(y+dy)\,y^{-1} = e$$

which is clearly impossible. We see from (9) that its inverse matrix is

$$v^\alpha_i(y) = \frac{\partial f^\alpha(0,y)}{\partial x^i} ; \tag{10}$$

and multiplication in G being analytic, it follows that the elements of the matrix (v^α_i) are analytic, hence the elements of its inverse (v^i_α) are also analytic.

Since equations (8) admit a solution

$$z = xy$$

for arbitrary initial values of x and z, it follows that the integrability conditions of (8) are satisfied identically. These conditions can easily be seen to reduce to

$$v^i_{\alpha.\beta} - v^i_{\beta.\alpha} = C^i_{jk}\, v^j_\beta\, v^k_\alpha , \tag{11}$$

where a dot denotes partial differentiation and the C^i_{jk}

are constants called the <u>constants of structure</u> of the group G. These constants are not arbitrary, but satisfy the following equations;

(12) $\quad C^i_{jk} + C^i_{kj} = 0$

(13) $\quad C^i_{jk} C^k_{\ell m} + C^i_{mk} C^k_{j\ell} + C^i_{\ell k} C^k_{mj} = 0.$

Evaluating (11) at e, we have

(14) $\quad v^i_{\alpha.\beta} - v^i_{\beta.\alpha} = C^i_{jk}\, \delta^j_\beta\, \delta^k_\alpha$

and since the left hand side of (14) is skew symmetric in α and β , equations (12) must be satisfied. To prove (13), differentiate (11) with respect to x^γ and evaluate at e. We then have

(15) $\quad v^i_{\alpha.\beta\gamma} - v^i_{\beta.\alpha\gamma} \mp C^i_{jk}\left(v^j_{\beta.\gamma}\, \delta^k_\alpha + v^k_{\alpha.\gamma}\, \delta^j_\beta\right) = 0,$

and adding to (15) the two equations obtained by permuting the suffixes α, β, γ cyclicly, (13) follows by virtue of (14). The C^i_{jk} are said to define the <u>infinitesimal structure</u> of G.

2. One parameter subgroups.

In this and the following section we shall be concerned only with the coordinates of points in some neighbourhood of the identity. For this reason we use only one system of suffixes, writing $(v^i_j), (V^i_j)$ in place of (v^α_i), (v^i_α).

A <u>local one parameter subgroup</u> of a Lie group G is a differentiable curve

$$x^i = g^i(t), \qquad |t| \leqslant \alpha \qquad (i = 1, \cdots r),$$

which passes through e and whose points satisfy the equation

$$(1) \qquad g(s)\,g(t) = g(s+t) \qquad |s|, |t|, |s+t| \leqslant \alpha .$$

In (1), g(t) denotes the point whose coordinates are $g^i(t)$, the parameter t being chosen so that g(o) = e. By the <u>direction vector</u> of a local one parameter subgroup we mean its tangent vector at the identity, i.e. $\frac{d g^i(0)}{dt}$; we denote it by a^i or symbolically by a. We now prove the following:

<u>Theorem VIII. A local one parameter subgroup g(t), with direction vector a satisfies the system of equations</u>

$$(2) \qquad v^i_j(x)\,\frac{dx^j}{dt} = a^i$$

<u>having for initial conditions</u>

$$(3) \qquad g^i(0) = 0$$

<u>Conversely, the solution of (2) with initial conditions (3) defines a local one parameter subgroup g(t) having a as direction vector</u>.

The first part of the theorem follows immediately by noting that, along the curve, we have

$$(4) \qquad (x+dx)x^{-1} = g(t+dt)\,g(-t) = g(dt) = a\,dt.$$

Writing (4) in coordinate form by means of (1) and (2) of §1, we obtain (2). Since the solution of (2) and (3) is unique, there is a unique local one parameter subgroup having a as direction vector.

To prove the converse, let g(t) denote the solution of (2) with initial conditions (3). Then if z = xy, where x = g(t), y = g(s), we have, from (7) of §1 and (2),

$$(5) \qquad v^i_j(z)\,\frac{dz^j}{dt} = a^i$$

with the initial conditions

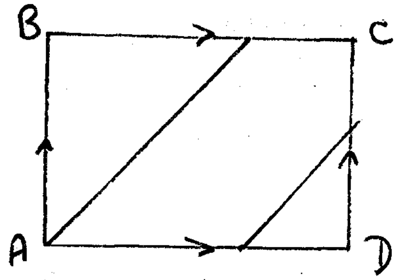
$$(6) \qquad z^i(0) = g^i(s)$$

Since $g^i(t+s)$ clearly satisfies (5) and (6), it follows from the uniqueness of the solution that

$$z = g(t)\,g(s) = g(t+s).$$

Finally, from (4) of §1 and (5), we see that the direction vector is a. This completes the proof of Theorem VIII

It is easily seen that a local one parameter subgroup of a Lie group G generates a connected subgroup of G, which we call a <u>one parameter subgroup.</u> It is a Lie group but is not necessarily closed in G. An



example of this is as follows. Let T^2 denote the two dimensional torus represented in the diagram by the rectangle ABCD with opposite sides identified. It is the manifold of a Lie group which is the product of two one dimensional abelian groups. If A represents the identity of T^2, a subgroup generated by a local one parameter subgroup is represented by a straight line through A, which on meeting a side of the rectangle is continued in the same direction from the corresponding point of the side opposite, as shown. The points of a line making an angle with AD whose tangent is an irrational multiple of $\frac{AB}{AD}$ can easily be seen to form an everywhere dense set, and hence the corresponding subgroup is not closed.

We now define, for later use, two types of coordinate systems which can be shown to exist in suitably small neighbourhoods of the identity of a Lie group G [18 Ch.VI].

In <u>canonical coordinates of the first kind</u>, the coordinates of a point x are given by

(7) $\quad x^i = a^i t$

where a is the direction vector and t a parameter for the local one parameter subgroup through x. The point t=o is the identity. It follows from (2) that the auxiliary functions satisfy

(8) $$v^i_j(at)a^j = a^i.$$

We note from (7) that under a linear transformation with constant coefficients canonical coordinates of the first kind are transformed into canonical coordinates of the first kind.

<u>Canonical coordinates of the second kind</u> are obtained by selecting a system of r one parameter subgroups

$$g_1(t^1),\ g_2(t^2), \cdots g_r(t^r), \qquad |t^k| \leqslant \alpha,$$

whose direction vectors are linearly independent. Then the points

$$x = g_1(t^1).\ g_2(t^2). \cdots . g_r(t^r), \quad |t^k| < \beta \leqslant \alpha$$

can be shown to form an allowable coordinate system, the coordinates of a point x being given by $x^i = t^i$. The coordinates of the identity are again zero.

§3. The Lie algebra.

Let L be the tangent space at the identity e, and let a and b be two vectors in L. We may suppose a and b to be tangent vectors at e to two curves x(t), y(t) passing through e. On evaluating

$$\text{(1)} \qquad \lim_{t \to 0} \frac{x(t)\, y(t)\, x^{-1}(t)\, y^{-1}(t)}{t^2}$$

we again get a vector at e depending only on a and b. This vector is called the <u>commutator</u> and written $[a, b]$.

In order to calculate (1), differentiate (9) of § 1 with respect to y^k and evaluate at e. Then we have

$$\text{(2)} \qquad v^{i}_{l.k}(0)\, \delta^{l}_{j} = - \frac{\partial^2 f^i(0,0)}{\partial x^j \partial y^k} .$$

Writing

$$p^i = f^i(x, y)$$

and expanding p^i as far as second order terms in x and y, we obtain

$$\text{(4)} \qquad p^i = x^i + y^i - x^l y^m v^{i}_{l.m}(0) .$$

Expanding $(z^{-1})^i$ as far as second order terms in z by means of the relation $zz^{-1} = e$, we have

$$\text{(5)} \qquad (z^{-1})^i = -z^i - z^l z^m v^{i}_{l.m}(0) .$$

Hence, using (4) and (5) it follows that

$$\text{(6)} \qquad q^i = f^i(x^{-1}, y^{-1}) = -x^i - y^i + y^l x^m v^{i}_{l.m}(0) - (x^l + y^l)(x^m + y^m) v^{i}_{l.m}(0),$$

where the expansion is up to second order terms in x and y. Finally we have, from (4) and (6)

$$f^i(p, q) = x^l y^m \left(v^{i}_{m.l}(0) - v^{i}_{l.m}(0) \right) + \cdots\cdots ,$$

and hence, from (14) of §1,

(7) $f^i(p,q) = c^i_{\ell m} x^\ell y^m + \cdots\cdots$

Writing $x^i = a^i t$, $y^i = b^i t$, where t is infinitesimally small, it follows from (1) and (7) that

(8) $[a,b]^i = c^i_{\ell m} a^\ell b^m.$

With this multiplication L is called the Lie algebra of G. From (12) and (13) of §1 and (8), we see that multiplication is bilinear, skew symmetric and satisfies the Jacobi identity

(9) $[a,[b,c]] + [c,[a,b]] + [b,[c,a]] = 0$

Under a homogeneous linear transformation p^i_j of the vector space at e, the constants of structure transform as tensors. For we have, from (8),

$$p^i_j c^j_{\ell m} a^\ell b^m = c'^i_{rs} p^r_\ell p^s_m a^\ell b^m,$$

where a and b are arbitrary vectors and c'^i_{rs} are the transformed constants of structure; consequently

(10) $c^i_{jk} = c'^\ell_{mn} p^m_j p^n_k \bar{p}^i_\ell$

where $(\bar{p}^i_j)$ is the inverse of (p^i_j).

Let L and L' be two Lie algebras. A map h of L on L' is homomorphic if for arbitrary real numbers α and β, and for arbitrary a and b in L

a) $h(\alpha a+\beta b) = \alpha h(a) + \beta h(b)$

b) $h[a,b] = [h(a), h(b)]$.

h is an isomorphism if, in addition, it is (1-1).

The importance of Lie algebras lies in the fact that if G and G' are two local Lie groups [18, Ch.VI] and R and R' their Lie algebras, then an isomorphism between R and R' implies a local isomorphism between G and G' [18 p.257]. It follows immediately that two Lie groups with isomorphic Lie algebras are locally isomorphic. The study of local properties of a Lie group is thereby reduced to a study of its Lie algebra.

Let L be a Lie algebra. A subset L' of L is called a subalgebra of L if the following conditions are satisfied:

1) L' is a vector subspace of L,

2) the conditions $a \in L'$, $b \in L'$ imply $[a,b] \in L'$.

If L' is distinct from L and the null space, then L' is said to be proper.

Let G be a Lie group. A Lie group H is called a Lie subgroup of G if the following conditions are satisfied:

1) The underlying group of H is a subgroup of the underlying group of G.

2) The identity mapping of H into G is regular, i.e.

is analytic and its Jacobian is of maximum rank. Condition 2) is in fact the condition that the underlying manifold of H is an analytic submanifold of the underlying manifold of G.

We state without proof the following result [9.p.109]:

Theorem IX. Let G be a Lie group. If H is a Lie subgroup of G, the Lie algebra of H is a subalgebra of the Lie algebra L of G. Every subalgebra of L is the Lie algebra of one and only one connected Lie subgroup of G.

By means of this theorem the question of the existence of connected Lie subgroups of a given Lie group G becomes a question of the existence of subalgebras of the Lie algebra of G. We shall see later that this method for determining Lie subgroups can be extended to the determination of normal Lie subgroups, provided the given Lie group is connected. The connected Lie subgroup corresponding to a subalgebra is not necessarily closed. On the other hand, a closed subgroup of a Lie group is automatically a Lie group which is not necessarily connected [9.p.135].

§4. The adjoint and linear adjoint groups.

Let G be a Lie group. To any element $x \in G$ there corresponds an inner automorphism T_x of the group G; i.e. if $z \in G$, then

$$T_x(z) = x z x^{-1}.$$

Since multiplication in G is analytic, it follows that the mapping T_x is analytic. The group A formed by the operations T_x is called the adjoint group of G.

The linear adjoint group is obtained by considering points infinitesimally near to the identity e of G. The mapping

$$T_x(dz) = x\, dz\, x^{-1}$$

then defines a mapping of the tangent space L at e onto itself. In coordinate form, we have

$$(x . dz . x^{-1})^i = \rho^i_j(x)\, dz^j.$$

If λ^i is a vector in L and x is a point of G, then we define the operator dT_x by

$$\left[dT_x(\lambda)\right]^i = \rho^i_j(x)\lambda^j.$$

The set of operators dT_x obtained as x varies over G clearly form a group which we call the linear adjoint group of G. We denote it by the symbol A_L.

Let U be a system of canonical coordinates of the first kind [c.f. § 2] in G. Since A maps a one parameter subgroup on a one parameter subgroup it follows that, if $g^i(t) = c^i t$ is a local one parameter subgroup and if $x \in U$, then

(1) $$\left[T_x(g)\right]^i = \left[x g x^{-1}\right]^i = \rho^i_j(x)\, c^j t$$ †

† It is convenient to talk of points in U whereas we actually mean points in a sufficiently small neighbourhoos V of e such that $V^n \subset U$ for some required n.

where

$$[dT_x(c)]^i = p_j^i(x)\, c^j.$$

We now show that in U

(2) $$\frac{dp_j^i(at)}{dt} = C_{k\ell}^i a^k p_j^\ell .$$

It will then follow that, with the initial conditions $p_j^i(a0) = \delta_j^i$, we can determine the functions $p_j^i(x)$ for $x \in U$ by integrating the system (2). To prove (2), let z be a point in U. Then if

$$z = xyx^{-1} = (xyx^{-1}y^{-1})y,$$

it follows, using (4) and (7) of §3, that, with $x^i = a^i t$,

$$z^i = y^i + C_{jk}^i a^j y^k t + \epsilon^i,$$

where ϵ^i is of the third order in t and y. We see, however, from (1), that the coordinates of z are linear in the coordinates of y. Hence

(3) $$p_j^i(at) = \delta_j^i + C_{kj}^i a^k t + \epsilon_j^i(t),$$

where $\epsilon_j^i(t)$ is of the second order with respect to t. Since $a^i t$ is a local one parameter subgroup, we have

$$p_j^i(a(t+dt)) = p_k^i(a\,dt)\, p_j^k(at)$$

where dt is infinitesimally small. Whence, using (3),

$$p_j^i(a(t+dt)) = p_j^i(at) + C_{\ell k}^i a^\ell p_j^k(at)\, dt,$$

and (2) follows immediately.

With the matrix notation

$$P = (p^i_j), \quad \eta = (C^i_{kj}a^k), \quad I = (\delta^i_j)$$

(2) can be written

$$\frac{dP}{dt} = \eta P. \tag{4}$$

The solution of (4) with initial conditions as given is then

$$P = I + \eta t + \frac{\eta^2 t^2}{2!} + \cdots\cdots, \tag{5}$$

which we write symbolically as

$$P = \exp(\eta t).$$

It follows that the functions $p^i_j(x)$ are analytic for all points x in a sufficiently small neighbourhood of e. Since multiplication in G is analytic, the relation

$$p^i_j(x.y) = p^i_k(x)\, p^k_j(y) \tag{7}$$

shows that the p^i_j are analytic at every point of G. Equations (7) also show that, provided G is connected, the matrices $p^i_j(x)$ for all $x \in U$ generate A_L.

Let G be a connected Lie group and H a connected Lie subgroup of G. Then the Lie algebra L' of H is a subalgebra of the Lie algebra L of G (cf. Theorem IX). We now prove:

Theorem X. A necessary and sufficient condition for H to be a normal subgroup of G is that L' is invariant under A_L.

Let U be a system of canonical coordinates of the first kind. By allowing t to be infinitesimally small in (5), it follows that, if L' is invariant under A_L, then it is invariant under the linear transformation by η. Conversely, if L' is invariant under η, then it follows from the form of (5) that L' is invariant under P, and hence under A_L, since G is connected.

If H is normal in G and if $x \in H$ and $y \in G$, we have $yx^{-1}y^{-1} \in H$, and hence $xyx^{-1}y^{-1} \in H$. Whence, from the definition of the commutator [cf. p. 46] and (8) of § 3, if

$$x^i = a^i t, \qquad y^i = b^i t$$

where t is infinitesimally small, then $[a, b] \in L'$. Since $x \in H$, the vector a belongs to L', therefore L' is invariant under matrices $P(x)$ for all $x \in U$ and hence is invariant under A_L. This proves the necessity of the condition. Suppose now that L' is invariant under A_L and let H be the connected Lie subgroup of G determined by L'. If g(t) is a local one parameter subgroup in H whose direction vector is a, then for any $y \in G$ the set of points $T_y(g(t))$ lie on a one parameter subgroup whose direction vector is $dT_y(a)$. Since L' is invariant under

A_L, $dT_y(a) \in L'$ and therefore $T_y(g(t)) \subset H$. It follows that there is a neighbourhood V of e in H which is mapped into H by A. But H, being connected, is generated by V and therefore any point $x \in H$ can be written $x = x_1.x_2 \cdots x_t$, where $x_1, x_2, \cdots x_t$ are in V. Hence, for any $y \in G$,

$$T_y(x) = y.x_1.x_2 \cdots x_t.y^{-1} = yx_1y^{-1}yx_2y^{-1} \cdots yx_ty^{-1} \in H$$

H is then a normal subgroup of G. This completes the proof of the theorem.

An immediate consequence of this theorem is that a necessary and sufficient condition for a connected Lie subgroup H of G to be normal is that its Lie algebra L' should be an ideal in the Lie algebra L of G. For, if H is normal, then L' is invariant under A_L, which implies $\eta a \in L'$, i.e. $[a\ b] \in L'$ for any $b \in L$. Hence L' is an ideal. Conversely, if L' is an ideal, then clearly it is invariant under A_L, and H is normal, as proved above.

§5. The characteristic equation, simple and semi simple Lie algebras.

For the remainder of this chapter we consider only connected Lie groups.

By means of the matrix $\eta = (\eta^i_k(a)) = (C^i_{jk} a^j)$, we can define r linear operators

(1) $$E_K = C^i_{jK} a^j \frac{\partial}{\partial a^i} \qquad (K = 1, \ldots r)$$

on a given Lie algebra L. They are called the <u>infinitesimal operators</u> of the linear adjoint group A_L. Using (5) of § 4, it is easily seen that a necessary and sufficient condition for an analytic function f defined over L to be invariant under A_L is that

(2) $$E_K f = 0 \qquad (K = 1, \ldots r).$$

The determinant

$$\Delta(a, w) = |\eta^i_j(a) - w\delta^i_j|$$

is called the <u>characteristic determinant</u>, and

(3) $$\Delta(a, \omega) = 0$$

the <u>characteristic equation.</u> Since the constants of structure transform as tensors [cf. (10) of § 3], it follows that the form of (3) is invariant under a linear transformation of the coordinates of L. If the components a^i are given particular values a^i_o, $\Delta(a_o, w)$ is the characteristic determinant relative to the vector a_o. Similarly, by considering only vectors $a \in L'$, where L' is a subalgebra of L of some dimension n, we obtain the characteristic determinant relative to L'. This is not to be confused with the characteristic determinant of L' which is only of order n. Since C_{ij}^h is skew symmetric in i and j, (3) has at least one zero root. Writing (3) in full, we have

$$(4)\quad (-)^r\Delta(a,w) = w^r - \psi_1(a)w^{r-1} + \psi_2(a)w^{r-2} - \dots + (-)^{r-1}\psi_{r-1}(a)w,$$

where the coefficient $\psi_i(a)$ is a homogeneous polynomial in $a_1,\dots,a_r$ of degree i. It results from (4) that

$$\psi_1(a) = C_{ih}^h a^i$$

$$(5)\quad \psi_1^2(a) - 2\psi_2(a) = C_{ik}^h C_{jh}^k a^i a^j$$

....

We note that since A_L is a group of automorphisms of L, the constants of structure are invariants under a transformation of coordinates in L by a matrix of A_L. The ψ_i's, being scalars, are therefore invariants of A_L.

Let L_1 denote the minimal linear subspace of L which contains all elements of the form $[a, b]$. Clearly L_1 is an ideal in L, (which may of course be L itself); it is called the <u>derived algebra of L.</u> We now construct the sequence L, L_1, L_2,..., L_i,....

where L_{i+1} is the derived algebra of L_i. Clearly, if two consecutive members of the sequence are equal, then the remaining members coincide with them. Furthermore, if two members are not equal, then their dimensions differ by at least one. Since the dimensions of L are finite, the sequence must eventually become stationary. If it becomes stationary only with the null derived algebra, then L is said to be solvable. A Lie algebra is said to be semi-simple if it contains no solvable ideals other than the zero ideal and simple if it contains no ideals other than the zero ideal ~~,~~ and L itself. It follows that, if L is simple and if dim $L > 1$, then it is semi-simple. If dim $L=1$, the constants of structure are clearly zero, and hence L is solvable and cannot be semi-simple.

Any tensor formed by the usual methods from the tensor C^i_{jk} is called a comitant. For example, the coefficients of the a's in any of the functions Ψ_i in (4) are comitants. A comitant is clearly an invariant of A_L, since the C^i_{jk}'s are invariants of A_L. Let $C^{h \cdots \ell}_{k \cdots m}$ be a comitant; then by considering the equations

(6) $$C^{h \cdots \ell}_{j \cdots k} a^k = 0 ,$$

which are assumed to be consistent, we prove the following important result [11.p.161]:

Theorem XI. If equations (6) admit m independent sets of solutions, they determine an ideal L' in L of dimension m.

Suppose $p_i^k \in A_L$. Then if

$$p_i^k a^i = \bar{a}^k,$$

it follows from the invariance of the comitant in (6) that

$$C^{h\cdots l}_{j\cdots k}\,\bar{a}^k = 0.$$

Hence, the vector space generated by solutions of (6) is invariant under A_L and is therefore an ideal in L [cf. § 4]. Its dimension is clearly equal to the number of independent solutions of (6). This completes the proof.

By means of theorem XI, several important conditions can be obtained on the Lie algebra L in order that it be simple or semi-simple. For example, it follows from the equations

$$C^h_{ij}\,a^i = 0$$

that, if the matrix $\left(C^h_{ij}\right)$, where i indicates the column and h and j the row, is of rank $m < r$, then L admits an ideal of dimension r-m. This ideal is clearly solvable and hence L is not semi-simple.

Again, if $C^j_{ij} \neq 0$, the equation

$$C^j_{ij} a^i = 0$$

admits r-1 independent solutions, in which case L admits an ideal of dimension r-1. Hence, if L is simple, $C^j_{ij} = 0$.

An ideal is also defined by

$$C_{hij} a^j = 0$$

where

$$(7) \qquad C_{hij} = C^l_{jm} C^m_{kl} C^k_{hi}.$$

From the definition, it follows that C_{hij} is skew symmetric in h and i. It is also skew symmetric in i and j; for, by (12) and (13) of §1 and (7), we have

$$\begin{aligned} C_{hij} + C_{hji} &= C^l_{jm} C^m_{kl} C^k_{hi} + C^l_{im} C^m_{kl} C^k_{hj} \\ &= C^l_{mj} (C^m_{ki} C^k_{lh} - C^m_{kh} C^k_{li}) + C^l_{mi} (C^m_{kj} C^k_{lh} - C^m_{kh} C^k_{lj}) \\ &= 0 \end{aligned}$$

Another important comitant is

(8) $$g_{ij} = C_{ih}^{\ell} C_{j\ell}^{h},$$

which appears in the second equation of (5). We now state without proof the following two theorems of Cartan [4, p.52-3].

Theorem XII. A necessary and sufficient condition for L to be semi-simple is that the matrix (g_{ij}) should be non-singular.

Let L', L" be ideals in L. We say L decomposes into the direct sum of L' and L", and write L = L' + L" if L' $\cap$ L" contains only the zero element and if the sum of the vector spaces L' and L" is L.

Theorem XIII. A semi-simple Lie algebra decomposes into a direct sum of simple subalgebras.

Thus a study of semi-simple Lie algebras is reduced to a study of simple Lie algebras.

The rank of a Lie algebra is the number of functionally independent coefficients φ_i in the characteristic equation [cf. p.55]. If the rank of η is q, equations (2) admit at most r-q independent solutions, and since the coefficients φ_i are invariant under A_L, it follows that $\ell <$ r-q, where ℓ is the rank of L. If, however, the rank of η is q, equation (3) has a zero root of order r-q at least. Hence, the rank of a Lie algebra does not exceed the order of the zero root of its characteristic equation. If L is semi-simple, it can, in fact, be shown that equality holds, and we have the following theorem [4.p.55]:

<u>Theorem XIV. If a semi-simple Lie algebra is of rank ℓ, its characteristic equation has exactly ℓ zero roots. The non-zero roots are simple, and if w is root, then so is -w.</u>

§6. The (c) metric of a semi-simple Lie group.

In the theory of abstract groups, a simple group is one which has no normal subgroups and a semi-simple group is one which has no solvable normal subgroups. It will be more convenient for our purpose, however, to define a Lie group as being simple or semi-simple if its Lie algebra is respectively simple or semi-simple. The question of the relationship between the two concepts in the case of a Lie group will not be considered in the present work.

Let G be a semi-simple Lie group of dimensions r, and L its Lie algebra. By means of the non-singular matrix g_{ij} [c.f. theorem XII], we now define a Riemannian metric

$$ds^2 = \epsilon\, g_{\alpha\beta}\, dx^\alpha dx^\beta \quad ^{+}$$

over G. For a point dx infinitesimally near to the identity e, we define

(1) $$ds^2(e, dx) = \epsilon\, g_{ij}\, dx^i dx^j \, {}^{*},$$

where dx^i are coordinates of the point dx. Let x and x + dx be two infinitesimally near points in G. Right translation by the element x^{-1} takes x to e and x + dx to $(x+dx)x^{-1}$. We define $ds^2(x.\ x+dx)$ by

(2) $$ds^2(x, x+dx) = ds^2(e, (x+dx)x^{-1}).$$

By means of (1) and (2) of §2 and (1), this can be written in coordinate form as

(3) $$ds^2 = \epsilon\, g_{ij}\, v^i_\alpha(x)\, v^j_\beta(x)\, dx^\alpha dx^\beta = \epsilon\, g_{\alpha\beta}(x)\, dx^\alpha dx^\beta.$$

Since the $v^i_\alpha(x)$, for fixed i, are covariant vectors, $g_{\alpha\beta}(x)$ are the components of a second order covariant tensor. The metric defined by (3) will be called <u>the (0) metric of G</u>.

+ ϵ is chosen to be plus or minus one so as to make ds^2 positive.

* Greek indices indicate a general coordinate neighbourhood in G and Roman indices a particular coordinate neighbourhood of the identity.

A homeomorphism of a Riemannian space on itself which preserves distance is called an isometry or a motion of the space. It can easily be seen that a necessary and sufficient condition for a transformation to be an isometry is that it leaves ds^2 invariant.

We now prove:

Theorem XV. Left and right translations and transformations of the adjoint group of G are isometries.

The (o) metric is by definition invariant under right translations. To prove its invariance under A, suppose $T_y \in A$. Then, from (2), we have

$$ds^2(yxy^{-1}, y(x+dx)y^{-1}) = ds^2(e, y(x+dx)y^{-1}yx^{-1}y^{-1})$$

$$= ds^2(e, y(x+dx)x^{-1}y^{-1})$$

$$= dT_y(\epsilon\, g_{ij} w^i w^j),$$

where $dT_y \in A_L$ and

$$w^i = v^i_\alpha(x)\, dx^\alpha.$$

Since g_{ij} is a comitant, it is an invariant of A_L [c.f.p. 57] and therefore $g_{ij}\, w^i\, w^j$ is an invariant of $dT_y \in A_L$. It follows immediately that the (o) metric is an invariant of A. Invariance under left translations then holds as a result of the identity $yx = (yx\, y^{-1})y$. This completes the proof.

Theorem XVI. If L is semi-simple but not simple, then the (c) metric of G is decomposable.

A tensor defined over a Riemannian manifold M is said to be decomposable over M if it is decomposable at each point of M, that is to say, if each point has a coordinate neighbourhood in which the tensor is decomposable [c.f. § 5 of Ch.I.]. By a decomposable metric we mean, of course, that the fundamental tensor is decomposable. Since L is assumed to be semi-simple, we have, from theorem XIII, that $L = L' + L''$ where $\dim L' = n$, $\dim L'' = r-n$. Also, since L is not simple, $o < n < r$. Select in L a system of coordinates such that $a \in L'$ if and only if

$$a^{n+1} = a^{n+2} = \cdots = a^{r} = 0,$$

and $a \in L''$ if and only if

$$a^1 = a^2 = \cdots = a^n = 0.$$

Write a prime (') after indices which takes values 1,...n and a double prime (") after indices which take values n+1,...r. Then since L' and L" are normal subgroups of L, it is easily seen that

$$C^{i}_{j'k''} = C^{i''}_{j'k'} = C^{i'}_{j''k''} = 0,$$

whence

$$(4) \qquad (g_{ij}) = \left(\begin{array}{c|c} g_{i'j'} & 0 \\ \hline 0 & g_{i''j''} \end{array}\right).$$

If $g_1 \ldots g_n$ and $g_{n+1} \ldots g_r$ are local one parameter subgroups whose direction vectors span L' and L" respectively then the set of points

$$x = g_1(t^1) \cdots g_r(t^r), \quad |t^k| < \beta,$$

form a canonical coordinate neighbourhood U of the second kind in G [cf. p.45]. Similarly the sets $g_1(t^1) \ldots g_n(t^n)$ and $g_{n+1}(t^{n+1}) \ldots g_r(t^r)$ form canonical coordinate systems of the second kind for the local normal subgroups H and K corresponding to L' and L" [18. Ch.VI].

Let x + dx', x + dx" be points infinitesimally near to $x \in U$, where dx', dx" denote infinitesimal increments in the primed and doubly primed coordinates of x. Then

$$\begin{aligned}(x+dx')x^{-1} &= g_1(t^1+dt^1) \cdots g_n(t^n+dt^n)\, g_{n+1}(t^{n+1}) \cdots g_r(t^r)\, g_r(-t^r) \cdots g_1(-t^1) \\ &= g_1(t^1+dt^1) \cdots g_n(t^n+dt^n)\, g_n(-t^n) \cdots g_1(-t^1)\end{aligned}$$

whence, since H is a local subgroup,

$$(5) \qquad v_{\alpha'}^{i''}(x) = 0$$

and

(6) $$v^{i'}_{\alpha'}(x) = v^{i'}_{\alpha'}(x'),$$

where x' denotes the point whose last r-n coordinates are zero, and whose first n coordinates are those of x. We have also

(7) $$(x + dx'')x^{-1} = g_1(t^1).\cdots g_n(t^n).g_{n+1}(t^{n+1}+dt^{n+1}).\cdots g_r(t^r+dt^r)g_r(-t^r).\cdots g_1(-t^1)$$

and writing

$$y = g_1(t^1).\cdots g_n(t^n)$$

$$z = g_{n+1}(t^{n+1}+dt^{n+1})\cdots g_r(t^r+dt^r)g_r(-t^r).\cdots g_{n+1}(t^{n+1})$$

equation (7) becomes

(8) $$(x + dx'')x^{-1} = yzy^{-1}.$$

Clearly, $z \in K$ and hence $yzy^{-1} \in K$, from which it follows that

(9) $$v^{i}_{\alpha''}(x) = 0$$

(10) $$v^{i''}_{\alpha''}(x) = v^{i''}_{\alpha''}(x''),$$

where x" denotes the point whose first n coordinates are zero and whose last r-n coordinates are those of x. It results from (4), (5), (6), (9) and (10) that, at any point $x \in U$, we have

$$g_{\alpha\beta}(x) = \left(\begin{array}{c|c} g_{\alpha'\beta'}(x') & 0 \\ \hline 0 & g_{\alpha''\beta''}(x'') \end{array}\right),$$

and the metric tensor is therefore decomposable in U.

Let y be a general point of G. We obtain a coordinate system in the neighbourhood Uy of y as follows. If $z \in Uy$, we say the coordinates of z are those of the point zy^{-1} in U. Since multiplication is analytic, it is easily seen that the coordinate system thus obtained is allowable. Right translation by y is, however, an isometry [c.f. theorem XV], and it follows immediately that the metric tensor is decomposable in Uy. Hence it is decomposable over G, and the theorem is proved.

We now calculate the Christoffel symbols and the curvature tensor determined by the (o) metric (3). For the Christoffel symbols of the first kind [12. p.17], we have

$$[\alpha\, \beta, \gamma] = \frac{1}{2} g_{ij}\left[v^i_\alpha\left(v^j_{\gamma.\beta} - v^j_{\beta.\gamma}\right) + v^i_\gamma v^j_{\alpha.\beta} + v^i_\beta\left(v^j_{\gamma.\alpha} - v^j_{\alpha.\gamma}\right) + v^i_\gamma v^j_{\beta.\alpha}\right],$$

and hence, using (11) of § 1, it follows from the skew symmetry of C_{ijk} that

$$[\alpha\beta,\gamma] = \tfrac{1}{2} g_{ij} v^i_\gamma \left(v^j_{\alpha\cdot\beta} + v^j_{\beta\cdot\alpha}\right).$$

Therefore

$$\Gamma^\alpha_{\beta\gamma} = g^{ij} v^\alpha_i v^\epsilon_j [\beta\gamma,\epsilon]$$

(11) $$= \tfrac{1}{2} v^\alpha_i \left(v^i_{\beta\cdot\gamma} + v^i_{\gamma\cdot\beta}\right).$$

If we now consider a coordinate neighbourhood of e such that

$$\Gamma^\alpha_{\beta\gamma} = 0$$

at e, then it follows from (14) of § 1 and (11) that

$$v^i_{\alpha\cdot\beta}(e) = C^i_{jk} \delta^j_\beta \delta^k_\alpha ,$$

and a simple calculation gives

(12) $$R^\alpha_{\beta\gamma\epsilon}(e) = \tfrac{1}{4} C^i_{jk} C^k_{lm} \delta^\alpha_i \delta^j_\beta \delta^l_\gamma \delta^m_\epsilon .$$

Finally, since a right translation is an isometry, the curvature tensor at a general point $x \in G$ is

(13) $$R^\alpha_{\beta\gamma\epsilon}(x) = \tfrac{1}{4} C^i_{jk} C^k_{lm} v^\alpha_i(x) v^j_\beta(x) v^l_\gamma(x) v^m_\epsilon(x).$$

Summing α and ϵ in (13), we have

(14) $$R_{\beta\gamma} = \tfrac{1}{4} g_{\beta\gamma},$$

so that with respect to the (c) metric a semi-simple Lie group is an Einstein space with scalar curvature $\frac{n}{4}$.

A Riemannian manifold M is said to be symmetric if for each point $P \in M$ there is an isometry σ_P which is locally a symmetry about P [c.f. § 5 of Ch.I.]. Denote by PX the length of a minimal geodesic joining P to a general point $X \in M$. Continue this geodesic through X until a point Y is reached such that $PX = PY$; then clearly

(15) $$\sigma_P(x) = Y.$$

Conversely, let X and Y be general points in M and let P be the mid point of a minimal geodesic joining X to Y. Then (15) is satisfied, and it follows that M admits a transitive group of isometries. We now prove:

<u>Theorem XVII. The underlying manifold of a semi-simple Lie group is symmetric with respect to the (o) metric.</u>

It is clearly sufficient to prove symmetry about e, since left or right translations will then establish it about any other point. We first show that the geodesics through e are in fact the one parameter sub-groups. For if U is a system of canonical coordinates of the first kind, then writing $x^\alpha = a^\alpha t$, we have, from (8) of § 2 and (11),

$$\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} = v^\alpha_i \frac{d}{dt}(v^i_\beta a^\beta)$$

$$= 0.$$

This establishes that a one parameter subgroup is a geodesic near e, and since translations are isometries, it follows that the entire subgroup is a geodesic.

Let d(x,y) denote the distance between points x and y in G. From theorem XV we have

(16) $$d(x,y) = d(e, yx^{-1}) = d(y^{-1}, x^{-1}) = d(x^{-1}, y^{-1})$$

and

$$d(e,x) = d(x^{-1}, e) = d(e, x^{-1}).$$

Since the points x and x^{-1} lie in the same one parameter subgroup, it follows, using (16), that symmetry about e is an isometry. This completes the proof.

§7. Completely harmonic compact simple Lie groups.

If the form $\phi(a) = g_{ij}a^ia^j$ associated with a semi-simple Lie group is definite then it is necessarily negative definite; for, with a suitable choice of coordinates, we have

$$(g_{ij}) = \pm(\delta_{ij}),$$

and hence

$$\phi(a) = C^h_{ik} C^k_{jh} a^i a^j$$

$$= \sum_{h,k} C_{ikh} C_{jhk} a^i a^j$$

$$= - \sum_{h,k} C_{ihk} C_{jhk} a^i a^j$$

The terms in the last summation are positive, being the squares of real numbers and therefore ϕ (a) is negative definite. In this case it follows that the (o) metric is positive definite.

We now enquire into the existence of semi-simple Lie groups which are completely harmonic with respect to the (o) metric (assumed to be positive definite). A Riemannian manifold M is said to be completely harmonic if every point of M has a neighbourhood which is centrally harmonic about this point. By virtue of the following result we need only consider completely harmonic simple Lie groups.

Theorem XVIII. A completely harmonic semi-simple Lie group is simple.

Suppose G is a semi-simple Lie group which is not simple. Then its (o) metric is decomposable [cf. theorem XVI]. Assuming G to be completely harmonic, it follows from theorem VII of Ch.I that the manifold of G is locally flat, and hence, in particular, $R_{ij} = 0$. But the scalar curvature of the space of G is

$\frac{n}{4}$ [cf. p.68]. We therefore have a contradiction, whence G must be simple.

The simple Lie groups for which ϕ (a) is definite are compact. This result, which is of fundamental importance in the theory of linear representation of semi-simple Lie groups was first proved by H.Weyl [29] Moreover, E.Cartan has shown that conversely the quadratic form associated with a compact simple Lie group is definite [7.p.240].

Theorem XIX. The only connected compact simple Lie group which is completely harmonic is the three dimensional rotation group.

We first note that, since left or right translations are isometries, a group space which is harmonic about the identity is completely harmonic. Now the (o) metric is clearly analytic since the functions v^i_α are analytic [cf. p.40], and therefore we have, from theorem VI, theorem XVII and (12) of § 6, that the necessary and sufficient conditions for a compact simple Lie group G to be completely harmonic are that the latent roots of

(1) $$4\Gamma = C^i_{jk} C^k_{\ell m} \lambda^j \lambda^\ell,$$

where λ^i is a unit vector, are constants. An equivalent set of conditions, more convenient in this case, are that the latent roots of (1) for a general vector λ^i are of the form

(2) $\quad 0,\ a_1\Lambda,\ a_2\Lambda,\ \cdots\ a_{n-1}\Lambda,$

where the a's are constants and $\Lambda = g_{ij}\lambda^i\lambda^j$.

Now, if w is a latent root of a matrix A, then w^2 is a latent root of A^2, and conversely if w^2 is a latent root of A^2 then either w or -w is a latent root of A. Hence, if the latent roots of (1) are those given in (2), then the latent roots of the matrix

$$\gamma = (c_{ij}^{\ k}\lambda^i)$$

are functionally dependent, and consequently the coefficients ψ_i in the characteristic equation (3) of §5 are functionally dependent. Thus if G is completely harmonic, the rank ℓ of its Lie algebra is one or zero [cf. p.60]. The case $\ell = 0$ can be discarded immediately, since from (5) of §5 we would then have $g_{ij}a^ia^j = 0$, which is impossible. For $\ell = 1$, an inspection of Cartan's classification of the structures of simple Lie groups shows that the only connected group of rank one is the three dimensional rotation group. Being an Einstein space of three dimensions, it is of constant curvature and therefore completely harmonic [cf. p.25]. This completes the proof of theorem XIX.

CHAPTER III

COMPLETELY HARMONIC HOMOGENEOUS SPACES

Summary. We now use the theory of Chapter II to consider homogeneous spaces, and finally to obtain a class of these spaces which are completely harmonic.

In §1, a brief study is made of topological transformation groups and coset spaces with special reference to coset spaces of Lie groups. We show that such spaces can be given an analytic structure, from which it follows that any homogeneous space M (i.e. a manifold having a transitive Lie group of homeomorphisms) can also be given an analytic structure. The role of the auxiliary functions for a Lie group is now taken over, in the case of a homogeneous space, by the infinitesimal vectors. Several global properties of the homogeneous space then become properties of the infinitesimal vectors and conversely.

In §2, we have collected together several results of Cartan on symmetric spaces. Our proofs are in the main different from those of Cartan in that no use is made of the theory of forms. We first prove the equivalence of symmetry and the vanishing of the covariant derivative of the curvature tensor in a V_n having an analytic metric. It

is possible that this result is equally true for symmetric manifolds, but as far as we know the problem is as yet unsolved. If G/H is a coset space of a Lie group, with H compact, then, by theorem XXIV, G/H admits an analytic metric invariant under G. By further restricting G and H, we show in theorem XXV that G/H becomes a symmetric space with respect to this metric. If this is the case, then the constants of structure of G can be shown to satisfy certain important relations. Conversely, if these relations are satisfied for a coset space $\frac{G}{H}$ with G and H connected then G/H is symmetric.

Finally, we show in § 3 that those homogeneous spaces which are 'two point homogeneous' are completely harmonic. Using Wang's classification of compact two point homogeneous spaces, we thus obtain a class of completely harmonic spaces. It turns out that these spaces are also symmetric, so that we can derive the corresponding non-compact symmetric spaces from them and thus obtain another class of completely harmonic spaces which are again two point homogeneous.

§ 1. Transformation groups. Homogeneous spaces. Infinitesimal transformations.

A topological transformation group of a topological space is an association of a topological group G and a topological space M, such that each $g \in G$ and $x \in M$ determines a unique point of M. If this point is x' we write: $g(x) = x'$ or $gx = x'$. This association is subject to the following conditions:

(a) if e denotes the identity of G, $ex = x$ for all $x \in M$,

(b) $$g(g'x) = (gg')x \text{ , where } g, g' \in G, \ x \in M,$$

(c) gx is continuous simultaneously in g and x.

G is said to be <u>transitive</u> on M if, for any x and y in M, there is a g in G with $g(x) = y$. It is said to be <u>effective</u> on M if $g(x) = x$ for all $x \in M$ implies g is the identity element of G. We note that, for any fixed g, the map $x \to gx$ is a homeomorphism of M onto itself, for it has the continuous inverse $x \longrightarrow g^{-1}x$.

Let H be a closed subgroup of a topological group G. A <u>left coset</u> of H in G is a set of the form $g.H$ where $g \in G$. Such a set is closed and any two either coincide or are disjoint. Let G/H denote the set whose elements are left cosets of H in G. Define the <u>natural map</u>

$$p : G \longrightarrow G/H$$

by

$$p(b) = b.H.$$

We say a set $U \subset G/H$ is open if $p^{-1}(U)$ is an open set of G. It is easily seen that these open sets define a topology on G/H. The topological space G/H is called the <u>coset space of G by H.</u> By definition, p is continuous. It is also an interior map, for if U is an open set in G, then UH is open in G, and hence p(U) is open in G/H.

The space G/H is Hausdorff. To see this, let $x_1 = p(g_1)$ and $x_2 = p(g_2)$ be distinct points in G/H; then $g_1^{-1}g_2$ is not in H. Let W be a neighbourhood of $g_1^{-1}g_2$ with $W \cap H = \phi$, and let U and V be neighbourhoods of g_1, g_2 such that $U^{-1}V \subset W$. Then UH and VH are open sets in G and have no common point. For if $uh_1 = vh_2$ where $u \in U$, $v \in V$, then $u^{-1}v \in H$, which is impossible. Hence p(U) and p(V) are disjoint neighbourhoods of x_1 and x_2 respectively, which completes the proof.

The transformation sending each element of G into its inverse defines a (1-1) correspondence $gH \longrightarrow Hg^{-1}$ between left and right cosets, which induces a homeomorphism between the spaces of right and left cosets. Unless otherwise stated, we shall consider the space of left cosets.

For $x \in G/H$ and $g \in G$, we define the left translation of x by g by

$$g(x) = p\left(g p^{-1}(x)\right)$$

Under this operation, G is easily seen to be a transitive topological transformation group of G/H. Conversely, if G is a transitive topological transformation group of a Hausdorff space M, choose a base point $x_0 \in M$ and define

$$q : G \longrightarrow M$$

by

$$q(g) = g(x_0)$$

Clearly q is continuous. The elements of G which leave x_0 fixed form a closed subgroup $H = G_{x_0}$ of G, called the <u>isotropy subgroup</u> of G at x_0. If $g(x_0) = x$, then $gH(x_0) = x$; and conversely, if $g_1(x_0) = x$, then $g^{-1}g_1(x_0) = x_0$, whence $g_1 \in gH$. Consequently, there is a (1-1) map

$$q' : G/H \to M$$

such that

$$q' p(g) = g(x_0).$$

In general q'^{-1} is not continuous; it is, however, for the case in which we shall be mainly interested, namely when G is a Lie group operating transitively on a manifold M* G/H is then homeomorphic to M, and it follows, using theorem XX, which we now prove, that M can be given an analytic structure. It is then called a <u>homogeneous space.</u>

<u>Theorem XX.</u> <u>If G is a Lie group and H a closed subgroup, then G/H is an analytic manifold.</u>

Since H is closed in G, it is a Lie group, and further, its topology as a manifold agrees with its induced topology as a subspace of G [9, p. 109]†. Hence, if

* This is well known and has recently been proved in a more general case [1, p. 607].

† Although the proof of this is given only for a connected subgroup, it is easily seen to extend to the more general case

dim $G = r$ and dim $H = \cancel{r-b}$ $r-n$, we can select a system of canonical coordinates of the second kind $g_1(t^1) \cdots g_r(t^r)$ and a neighbourhood U of e in G such that points $g_{n+1}(t^{n+1}) \cdots g_r(t^r)$ in U form a neighbourhood V of e in H, with $U \cap H = V$. We may further assume U small enough such that points of U^2 are defined by means of the above canonical coordinate system. Let W be the set $g_1(t^1) \cdots g_n(t^n)$. Writing f^{-1} for the restriction of p to W, we have

$$f^{-1}(W) = p(UH),$$

and consequently $f^{-1}(W)$ ($= X$, say) is an open set in G/H. In this case, f^{-1} is a (1-1) map of W onto X; for if f were not single valued, there would exist a relation of the form

$$aH = bH, \qquad a, b \in W$$

and therefore

$$a = bh, \qquad h \in V.$$

In coordinate form, this equation becomes

$$g_1(t^1) \cdots g_n(t^n) = g_1(s^1) \cdots g_n(s^n) g_{n+1}(s^{n+1}) \cdots g_r(s^r),$$

which is only possible if

$$s^{n+1} = s^{n+2} = \cdots = s^r = 0$$

in which case $a = b$. Hence f is single valued. Finally, since p is an interior map, it follows that f is a continuous map of X into G. We have therefore proved the existence of a <u>local cross section</u> of H in G, i.e. a function f mapping a neighbourhood X of $x_0 = p(H)$ continuously into G, and such that $pf(x) = x$ for each $x \in X$. It follows that f is a homeomorphism of X on W, and consequently that X is homeomorphic to an open set of Euclidean n-space. We take for the coordinates of each point in X the coordinates of its corresponding point in W. For a general point $x = p(b)$ in G/H, the set $X_b = bX$ is an open set containing x. Define $f_b : X_b \to G$ by

$$f_b(y) = b f(b^{-1} y) \qquad y \in X_b.$$

Then

$$p f_b(y) = y,$$

and X_b is homeomorphic to $f_b(X_b) = bW$, and hence to an open set of Euclidean n space. As before, we may then give to each point $y \in X_b$ the coordinates of the corresponding point in bW. We have therefore proved that G/H is a manifold.

Let X_b, X_c be two intersecting coordinate neighbourhoods. Then

$$X_b \cap X_c = p(bW) \cap p(cW)$$

and since multiplication in G is analytic, it follows immediately that the transformation of coordinates in $X_b \cap X_c$ is analytic. Thus G/H is an analytic manifold. This completes the proof of theorem XX.

In order that G should be an effective topological transformation group on the coset space G/H, it is necessary and sufficient that H should contain no non-trivial normal subgroup of G. For if H_0 is such a subgroup

$$H_0 g H = g H_0 H = g H,$$

and hence

$$(1) \qquad H_0 x = x$$

for all $x \in G/H$. Conversely the set H_0 of all $g \in G$ such that $g(x) = x$ for all x in G/H is clearly a subgroup of H and is normal in G; for if $h_0 \in H_0$ and $g \in G$ we have

$$g h_0 g^{-1} g_1 H = g g^{-1} g_1 H = g_1 H,$$

and therefore $g\, h_0\, g^{-1} \in H_0$.

Let G_x denote the maximal subgroup of G leaving the point $x \in G/H$ invariant. Then G_x is closed. The subgroup H_0 defined above is the intersection of all such subgroups, and is therefore closed. Let p_0 denote the natural map of G on G/H_0. If $x_0 = p_0(g)$ and if $x \in G/H$, define the transformation of x by x_0 by

$$x_0(x) = g(x).$$

Then the factor group G/H_0 is easily seen to act as an effective, transitive topological transformation group on G/H.

Let M be a homogeneous space of a Lie group G, and suppose $\dim M = n$ and $\dim G = r$. If $dg \in G$ is a point infinitesimally near to the identity e of G and if $x \in M$, then the transformation $dg(x)$ may be written in coordinate form as

(2) $$f^i(dg, x) = x^i + dg^\alpha \frac{\partial f^i(0,x)}{\partial g^\alpha} \qquad \alpha = 1, \dots r, \; i = 1, \dots n.$$

The functions

(3) $$\xi^i_\alpha(x) = \frac{\partial f^i(0,x)}{\partial g^\alpha}$$

are called the infinitesimal vectors of G. They serve to determine r linear operators

(4) $$X_\alpha = \xi^i_\alpha(x) \frac{\partial}{\partial x^i},$$

called the infinitesimal operators of G. The ξ^i_α are easily seen to satisfy the differential equations

(5) $$\frac{\partial y^i}{\partial g^\mu} = v^\alpha_\mu(g)\, \xi^i_\alpha(y)$$

where v^α_μ are the auxiliary functions of G†. For suppose $y = gx$; then

(6) $$(g+dg)x = (g+dg)g^{-1}y,$$

where $g + dg$ denotes a point infinitesimally near to g. Writing (6) in coordinate form, we have

† We use suffixes λ, μ, ν etc. to denote general coordinates in G and α, β, γ etc. to denote coordinates in some neighbourhood of e.

$$(7) \qquad dy^i = v^\alpha_\mu(g)\, \xi^i_\alpha(y)\, dg^\mu$$

whence (5) follows immediately with the boundary conditions y = x when g = e. Since G is transitive on M, equations (5) are completely integrable. Calculating the integrability conditions in the usual way and using (11) of § 1 of Ch.II, we get

$$\xi^i_{\alpha,j}(y)\, \xi^j_\beta(y) - \xi^i_{\beta,j}(y)\, \xi^j_\alpha(y) = C^\gamma_{\alpha\beta}\, \xi^i_\gamma(y),$$

or from (4)

$$(8) \qquad [X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha = C^\gamma_{\beta\alpha} X_\gamma .$$

If G is effective on M, then the infinitesimal vectors are independent to within constant coefficients. To prove this, we assume $c^\alpha \xi^i_\alpha(x) = 0$ for all $x \in M$ and for some constants c^α. Then if g(t) denotes the local one parameter subgroup of G having c^α as components of its direction vector, it follows from (2) of § 2 of Ch.II and from (5) that

$$(9) \qquad \frac{d}{dt} f(y) = c^\alpha X_\alpha(f)$$

where

$$(10) \qquad y = g(t).x .$$

We may then write (10) in coordinate form as

$$y^i = x^i + \sum_{r=1}^{\infty} \frac{t^r}{r!} c^{\alpha_1} \cdots c^{\alpha_r} X_{\alpha_1} \cdots X_{\alpha_r}(x^i),$$

the series being valid for small enough t. It follows from the initial assumption that, for these values of t, we have y = x and hence that G is not effective.

Conversely, if the infinitesimal vectors are independent, G contains no non-trivial subgroup H leaving M pointwise invariant; for in this case for all $x \in M$ we would have some relation $\frac{dg^\alpha(0)}{dt} \xi^i_\alpha(x) = 0$ where $g(t) \in H$. Unless otherwise stated, we assume transformation groups to be effective.

Theorem XXI. If G is a Lie group of transformations on a connected manifold M of dimension n, then for G to be transitive it is necessary and sufficient that the rank of the matrix $\left(\xi^i_\alpha(x)\right)$ should be n at every point of M.

Suppose $g \in G$ and $x \in M$. Then from the implicit function theorem [21. p.240-250] we have that, for fixed x, the necessary and sufficient condition for the equations $y^i = f^i(g, x)$ to have a solution for the g's in terms of the y's is that the matrix $\left(\frac{\partial f^i(0,x)}{\partial g^\alpha}\right) = \left(\xi^i_\alpha(x)\right)$ should be of rank n. This proves the necessity of the conditions and also shows that if $\left(\xi^i_\alpha(x)\right)$ is of rank n, then G is transitive in some neighbourhood of x. It follows that the orbit of a point in M is open and hence closed, being the complement of the union of the others. Since M is connected there is only one orbit. This completes the proof of the theorem.

§2. Symmetric spaces.

In this section, we give a brief account of some of the results obtained by E.Cartan on symmetric spaces. We restrict ourselves to positive definite analytic metrics.

Theorem XXII. *A necessary and sufficient condition for a V_n to be symmetric is that the covariant derivative of the curvature tensor vanishes* [cf. p. 23].

We first prove the necessity of the condition. With a system of normal coordinates, base point P, the symmetry about P is defined by

(1) $$y^i = -x^i.$$

The Jacobian of the transformation being

$$\frac{\partial y^i}{\partial x^j} = -\delta^i_j$$

it follows that

$$R^{(x)}_{hijk,l} = -R^{(-x)}_{hijk,l},$$

and so, from considerations of continuity,

$$R^{(P)}_{hijk,l} = 0$$

Since the space is symmetric about each of its points, our assertion is proved.

To prove the sufficiency of the condition, we again select a system of normal coordinates about a general point P. Since the partial derivatives $g_{ij.k_1 \cdots k_r}$ of the fundamental tensor at P are expressible as polynomials in g_{ij}, R_{hijk} and successive covariant derivatives of R_{hijk} at P [22. ch. VI] it follows immediately that the vanishing of $R_{hijk,\ell}$ implies

$$g_{ij.k_1 \cdots k_s} = 0$$

for s odd and ≥ 1. The functions g_{ij}, being analytic, are therefore even functions of the coordinates, and with the transformation (1), we have

$$g_{ij}(x)\,dx^i dx^j = g_{ij}(y)\,dy^i dy^j.$$

Hence symmetry about P is an isometry, and the theorem is proved.

Let M be a homogeneous space of a Lie group G, and suppose M has a Riemannian metric defined over it such that G is the largest continuous group of isometries. The subgroup of G leaving a point $P \in M$ fixed is called the isotropy subgroup at P, and we write it as H_p. Since H_p is a Lie group of isometries leaving P fixed, it is not only closed in G, but compact [6. p.43 [redacted]]

Suppose now that M is a symmetric space with respect to this metric, and denote by σ_p the symmetry about P. Then the transformation

$$a_p(g) = \sigma_p g \sigma_p$$

defines an automorphism of G, which is involutive, since

$$\sigma_p^2(x) = x$$

for any $x \in M$. At least locally, no element of G not in H_p is invariant under a_p; for in this case we would have

$$g(p) = \sigma_p g(p)$$

which is clearly impossible for $g(p)$ near enough to p. If $h \in H_p$, we have

$$h\sigma_p(x) = \sigma_p h(x)$$

for $x \in M$, and consequently

$$\sigma_p h \sigma_p(x) = \sigma_p^2 h(x) = h(x).$$

Thus a_p leaves H_p pointwise invariant. The converse of this result is also true, but before proving it, we must consider in more detail the linear adjoint group A_L of a Lie group G.

<u>Theorem XXIII. A_L is a Lie group.</u>

Let dim G = r, and denote by GL(r) the general linear group of r^2 dimensions with real coefficients. We define the homomorphic map f of G into GL(r) by

$$f(g) = (\rho_j^i(g))$$

where (ρ_j^i) is an element of A_L [cf. p. 50]. Then f induces a homomorphism f' of the Lie algebra L of G into the Lie algebra R of GL(r) [18.p.243]. With a system of canonical coordinates of the first kind in G, this map is defined by

$$f'(a) = \left(\frac{d}{dt} p^i_j(a^i t)\right)_{t=0} = \left(C^i_{kj}\, a^k\right), \quad a \in L.$$

Since f' is a homomorphism, f'(L) is a subalgebra of R. Let K be the corresponding connected Lie subgroup of GL(r). Then points

$$x^i_j = C^i_{kj} a^k t,$$

for t small enough, form a coordinate neighbourhood of the identity of K. The transformation

$$y^i_j = \exp(x^i_j)$$

is easily seen to define another coordinate neighbourhood U of the identity in K; and since K is connected, it is generated by U. But A_L is generated by the set of matrices $\exp(C^i_{kj} a^k t)$ [cf. p.52]. Hence $A_L = K$, and the theorem is proved.

We note that, since the p^i_j's are analytic functions [cf.p.52], the homomorphism f is an analytic map of G onto A_L. Hence, in particular, if G is compact, then A_L is compact.

Let H be a compact subgroup of a Lie group G such that H contains no non-trivial normal subgroup of G. The subgroup of A_L formed by points having coordinates

$p^i_j(h)$ for $h \in H$ is called the <u>linear adjoint group relative to H.</u> We denote it by $A_{(L')}$, where L' denotes the Lie algebra of H. Then $A_{(L')}$ is easily seen to be a compact subgroup of A_L. We now prove the following:

<u>Theorem XXIV. The coset space G/H admits a Riemannian metric such that G is a group of isometries.</u>

A fundamental theorem, due to H.Weyl [29. p. 289] states: A compact linear group leaves invariant at least one positive definite Hermitian form. If the group has real coefficients, this form can, of course, be replaced by a real positive definite quadratic form. Thus the group $A_{(L')}$ defined above leaves invariant a real form, which we may take to be $\sum_{\alpha=1}^{r} a^\alpha a^\alpha$, where dim G = r. Suppose dim H = r-n, and choose a system of coordinates in L such that the first n coordinates of vectors in L' are zero. Since L' is invariant under $A_{(L')}$, it follows that the conjugate space of vectors whose last r-n coordinates are zero is invariant under $A_{(L')}$. Consequently the quadratic form $\sum_{i=1}^{n} a^i a^i$ is invariant under $A_{(L')}$.

Let W, X be respectively coordinate neighbourhoods in G, $\frac{G}{H}$ as defined in theorem XX. If O = p(H), where p is the natural map of G on G/H, and if A is ~~infinitely~~ infinitesimally near O, define the distance ds(O,A) by

$$ds(0, A) = \sqrt{\sum_{i=1}^{n} dx^i dx^i},$$

where dx^i are the coordinates of the point $dx = f(A)$ in W (dx^i are also the coordinates of A itself). For $h \in H$, we have

$$h\, dx\, H = h\, dx\, h^{-1} H$$

and consequently, the coordinates of $B = h(A)$ are $(h\, dx\, h^{-1})^i$. Since $\sum_{i=1}^{n} dx^i dx^i$ is an invariant of $A_{(n)}$, it follows that

$$ds(0, A) = ds(0, B).$$

This metric defined at 0 is therefore invariant under H. It is now an easy matter to define a positive definite metric over the manifold G/H. For let M and N be two infinitesimally near points in G/H such that

$$M = g_1(0), \quad N = g(0) \qquad g, g_1 \in G.$$

Define the distance $ds(M, N)$ by

$$\begin{aligned} ds(M, N) &= ds(g_1^{-1}(M), g_1^{-1}(N)) \\ &= ds(0, g_1^{-1} g(0)) \end{aligned} \tag{1}$$

Then if also $M = g_2(0)$, we have

$$g_2^{-1} g_1(0) = 0$$

and hence $g_2^{-1} . g_1 \in H$. Writing $g_2^{-1} . g_1 = h$, it follows using (1) that

$$ds(M, N) = ds(0, hg_1^{-1}g(0))$$
$$= ds(0, g_2^{-1}g(0)).$$

Thus the metric ds(M, N) is well defined over G/H and is invariant under G. Since transformations of G are analytic, and since the metric is positive definite at O, it is easily seen to be positive definite and analytic everywhere. This completes the proof.

We now prove the converse result to that on page 85

Theorem XXV. Let G be a Lie group, and α an involutive automorphism of G such that the vectors at the identity e invariant under α generate a compact subgroup H of G. Then the coset space G/H can be given a Riemannian metric such that it becomes a symmetric manifold.

By theorem XXIV, the manifold G/H admits an analytic Riemannian metric such that G is a group of isometries. We show that G/H is symmetric with respect to this metric.

With a suitable choice of coordinates in the Lie algebra L of G, the automorphism α effects a linear substitution on L of the form

$$(2) \quad \begin{cases} a'^i = -a^i & (i = 1, 2, \dots n) \\ a'^\alpha = a^\alpha & (\alpha = n+1, \dots r) \end{cases}$$

where a vector in the subspace L' of L corresponding to H has its first n components zero.

Since the form $\sum_{i=1}^{n} a^i a^i$ is invariant under the transformation (2), it follows, using the same notation as in the proof of theorem XXIV, that

$$ds(M,N) = ds(0, g_1^{-1} g(0))$$
$$= ds(0, a g_1^{-1} g(0)).$$

Thus if $M = g_1(0)$ is a general point in G/H, the transformation σ_0, defined by

$$\sigma_0(M) = a g_1(0),$$

is an isometry. It is locally a symmetry about 0, as follows using (2) and the special coordinate system in X [cf. p.80]. To see that the definition of σ_0 is unambiguous, suppose $M = g_2(0)$. Then we have

(3) $$g_2 = g_1 h$$

for some $h \in H$. Hence

$$a g_2(0) = a g_1 h(0)$$
$$= a g_1(0),$$

and $\sigma_0(M)$ is the same point for any choice of $g \in G$ sending 0 to M. We have therefore shown that G/H is symmetric about 0. For a general point $M = g_1(0)$, the isometry σ_M defined by

$$\sigma_M = g_1 a g_1^{-1}$$

is easily seen to be a symmetry about M. Again σ_M is independent of the choice of g_1 for suppose $M = g_2(0)$. Then since H is pointwise invariant under $\mathcal{a}$, we have, using (3), that

$$g_2 \mathcal{a} g_2^{-1} = g_1 h \mathcal{a} h^{-1} g_1^{-1}$$
$$= g_1 \mathcal{a} g_1^{-1}.$$

This completes the proof of theorem XXV.

With the same notation, it results, using (2), that the following relations hold in L:

$$a \in L', \; b \notin L' \text{ implies } [a, b] \notin L';$$
$$a \notin L', \; b \notin L' \text{ implies } [a, b] \in L',$$
$$a \in L', \; b \in L' \text{ implies } [a, b] \in L'.$$

Thus the constants of structure are such that

$$(3) \quad \begin{cases} C^{\beta}_{\alpha i} = 0 \\ C^{k}_{ij} = 0 \\ C^{i}_{\alpha\beta} = 0 \end{cases}$$

In addition, since the forms $\sum_{i=1}^{n} a^i a^i$, $\sum_{\alpha=n+1}^{r} a^\alpha a^\alpha$ are invariants of $A_{(L')}$, it follows that

$$(4) \quad \begin{cases} C^{j}_{\alpha i} + C^{i}_{\alpha j} = 0 \\ C^{\gamma}_{\alpha\beta} + C^{\beta}_{\alpha\gamma} = 0 \end{cases}$$

Conversely, if (3) and (4) are satisfied for a connected Lie group G and closed connected subgroup H then equations (2) define an involutive automorphism of the Lie algebra L of G from which follows the existence of an involutive automorphism of G leaving H pointwise invariant. Using (4), it results that $\sum_{i=1}^{n} a^i a^i$ and $\sum_{\alpha=n+1}^{r} a^\alpha a^\alpha$ are invariants of $A_{(L')}$. The coset space can then be given a Riemannian metric as in theorem XXIV, with respect to which it is a symmetric space.

Thus the determination of symmetric spaces is largely reduced to the problem of determining Lie algebras whose constants of structure satisfy (3) and (4). The complete classification of symmetric spaces (compact and non-compact) having indecomposable metrics is given in great detail in [5]. Cartan shows by a very elegant method that to every compact symmetric space there corresponds a non-compact symmetric space, and conversely. Under this correspondence, the constants of structure of an isotropy subgroup are unchanged, and hence the <u>linear isotropy subgroup</u> at a point P in G/H (i.e. the group of transformations on the tangent space at P induced by the isotropy subgroup at P) is the same for the corresponding compact and non-compact symmetric spaces.

§3. A class of completely harmonic spaces.

Theorem XXVI. Let M be a Riemannian manifold such that each point P in M has a neighbourhood U admitting a group of rotations about P which is transitive on any geodesic sphere centre P in U. Then M is completely harmonic.

Let $V \subset U$ be a spherical neighbourhood with centre P such that no geodesics in V through P intersect again in V. Let X be any point in V and S the geodesic distance from P to X. Now $\Delta_2 S$ [cf. p. 3] is a scalar, and its value is therefore unaltered by an isometry. But V admits a group of rotations sending X to any point on the geodesic sphere centre P through X. Hence $\Delta_2 S$ is constant on this sphere, and consequently on any sphere centre P in V. It is then a function of S alone. Since P is a general point in M, it follows that M is completely harmonic. Theorem XXVI is thereby proved.

We say that a Riemannian manifold M is two point homogeneous if, for any two pairs of points (x_1, x_2), (y_1, y_2) of the space such that $d(x_1, x_2) = d(y_1, y_2)$, where 'd' denotes distance, there is an isometry of the space carrying x_1, x_2 to y_1, y_2 respectively.† For simplicity

† This definition is the same as that initially given by G. Birkhoff [2], except that we restrict 'metric space' to 'Riemannian manifold'.

we shall call such a space a (*) space. By taking $x_1 = y_1$ we see that the isotropy subgroup at x_1 is transitive on any geodesic sphere centre x_1. Using theorem XXVI we then have the following:

Corollary. A (*) space is completely harmonic.

Compact, connected (*) spaces have been classified by Wang [28] . Although his definition of a two point homogeneous space merely requires it to be a metric space, he proves that if it is compact and connected, then the group G of all isometries is a compact Lie group whose isotropy subgroup at a point P acts effectively on spheres centre P. It follows using theorem XXIV that it admits a Riemannian metric invariant under G, and is therefore a (*) space as defined above.

Using Wang's classification and the corollary to theorem XXVI we have:

Theorem XXVII. The following spaces admit Riemannian metrics with respect to which they are completely harmonic: (i) spheres, (ii) real projective spaces, (iii) complex projective spaces, (iv) quaternionic projective spaces, and (v) the Cayley projective plane.

Theorem XXVIII. A Riemannian metric of a (*) space is uniquely determined to within a constant multiple.

Let $ds^2 = g_{ij}\, dx^i\, dx^j$ be a Riemannian metric for the (*) space invariant under the group G of all isometries, and suppose $d\bar{s}^2 = e\,\sigma_{ij}\, dx^i dx^j$ is another metric for the space (not necessarily positive definite) also invariant under G. Then, with suitable choice of coordinates in a neighbourhood of a point P in the (*) space, we have $ds^2 = \sum dx^i\, dx^i$. We then have two metrics defined on the tangent space at P, each being invariant under the linear isotropy subgroup at P. The intersection of the two hypersurfaces

$$\sum x^i x^i = K \quad , \quad e\,\sigma_{ij} x^i x^j = K$$

where K is any constant, is therefore an invariant subspace of the sphere $\sum x^i x^i = K$. But the linear isotropy subgroup is transitive on the sphere, and consequently this subspace is either null or the sphere itself. It follows that, for all x^i, we have

$$c \sum x^i x^i = e\,\sigma_{ij}\, x^i x^j$$

where c is a constant. Hence $c\, ds^2 = d\bar{s}^2$ at P, and G being transitive, this relation must hold at every point of the space. This completes the proof of theorem XXVIII.

The spaces mentioned in theorem XXVII are all symmetric. They are respectively of types BDII, BDII, AIV, CII(q=1) and FII in Cartan's classification [5].

Also, since they are compact, their curvature is everywhere positive or zero [5. p. 346]. They are not all of constant curvature. To see this, we may use the result that, for a compact n-dimensional space of constant curvature $K > 0$,

$$B_p = 0, \qquad 1 \leq p \leq n-1$$

where B_p is the pth Betti number [3]. Thus a consideration of the Betti numbers shows that the complex and quaternionic projective spaces (other than the projective lines) and the Cayley projective plane are completely harmonic without being of constant curvature.

The non-compact symmetric spaces corresponding to the above compact ones are again (*) spaces. For, under this correspondence, the linear isotropy subgroups are unchanged [cf. p 94], and hence there is a group of rotations defined about each point which is transitive on spheres centre ~~to~~ this point. Then for any two pairs of points (x_1, x_2), (y_1, y_2) with $d(x_1\ x_2) = d(y_1\ y_2)$, it is only necessary to take y_1 to x_1 by translation and y_2 to x_2 by a rotation. It follows as before that the non-compact (*) spaces thus obtained are completely harmonic. They are all homeomorphic with Euclidean space [6. p. 1221], although, of course, their metrics are not Euclidean.

One might conjecture that the class of completely harmonic spaces thus obtained is, in fact, the whole class of such spaces. We note in particular that there are (*) spaces of four dimensions, other than spaces of constant curvature, namely those of type C II in Cartan's classification. Being completely harmonic these metrics must be of the form given on page 26. There is, in fact, associated with this metric an eight parameter group of motions having a four parameter subgroup of rotations. The infinitesimal vectors are

$$\left.\begin{matrix} K(x^2-y^2)+2 & -2Kxy & K(xz+yt) & -K(xt-yz) & y & 0 & z & -t \\ 2Kxy & K(x^2-y^2)-2 & -K(xt-yz) & -K(xz+yt) & -x & 0 & -t & -z \\ K(xz+yt) & K(xt-yz) & K(z^2-t^2)+2 & -2Kzt & 0 & t & -x & y \\ K(xt-yz) & -K(xz+yt) & 2Kzt & K(z^2-t^2)-2 & 0 & z & y & x \end{matrix}\right).$$

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