## *Aspects of laminar boundary layers*

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# Aspects of Laminar Boundary Layers

by

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## Abstract

This thesis consists of three parts :-

**Part I** attempts to justify analytically some of Hartree's numerical findings for the Falkner-Skan equation and extends some of the results previously obtained by Stewartson to cover the cases of suction and injection.

**Part II** treats the problem of the flat plate in a uniform incompressible stream when there is homogeneous normal injection through the plate. A solution of the boundary layer equations valid in the neighbourhood of the separation point is found.

**Part III** is concerned with the numerical solution of the boundary layer equations for the problem in Part II. Results in the form of tables and graphs are given.


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# CONTENTS

## Part I

### The Falkner-Skan Equation

### I. Introduction

Falkner and Skan[1] found that the incompressible two-dimensional boundary-layer equations have a particularly simple solution when the free stream velocity is of a certain form. This result has been extended by Mangler[2] to cover the cases of normal suction or injection of fluid through the surface and a brief account is given here as the basis of this introduction.

After boundary layer approximations have been made the momentum and continuity equations for incompressible two-dimensional flow are :-

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \quad ; \tag{1.1}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad ; \tag{1.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \tag{1.3}$$

where $x$ and $y$ denote distances, and $u$ and $v$ velocities measured respectively along and perpendicular to the boundary, $p$ is the static pressure, $\rho$ the density and $\nu$ the kinematic viscosity. For a stationary flat plate in a stream moving with velocity $U(x)$ the boundary conditions are :-

$$\left.\begin{array}{ll} u(x,0) = 0 & , \\ v(x,0) = v_0(x) & , \end{array}\right\} (x>0)$$
$$u(0,y) = U(x) \quad , \quad (y>0) \tag{1.4}$$
$$u(x,y) \to U(x) \text{ as } y/\nu^{\frac{1}{2}} \to \infty .$$

Since, from (1.2), $p$ is a function of $x$ only we may write

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\frac{dp}{dx} = U\frac{dU}{dx} . \tag{1.5}$$

By introducing a stream function $\psi$, where

$$u = \frac{\partial \psi}{\partial y} \; ; \; v = -\frac{\partial \psi}{\partial x} , \tag{1.6}$$

equation (1.3) is satisfied.

We look for a solution of the form

$$\psi = U \sqrt{\nu}\, h\, f(\eta) \tag{1.7}$$

where $$\eta = y/h\sqrt{\nu} \tag{1.8}$$

and $h$ is a function of $x$ only.

From (1.6) we have

$$u = U f'(\eta) \; ; \quad \frac{v}{\sqrt{\nu}} = -\left( h \frac{dU}{dx} + U \frac{dh}{dx} \right) f + U \eta \frac{dh}{dx} f' , \tag{1.9}$$

the primes denoting differentiation with respect to $\eta$.

From (1.4) and (1.9) we have

$$f'(0) = 0 \; ; \; f'(\infty) = 1 . \tag{1.10}$$

In order to have boundary conditions which are independent of $x$ we must have (from (1.9))

$$\frac{v_0(x)}{\sqrt{\nu}} = -a \left( h \frac{dU}{dx} + U \frac{dh}{dx} \right) , \tag{1.11}$$

where $a$ is a constant, so that

$$f(0) = a . \tag{1.12}$$

Substitution from (1.9) into (1.1) gives a differential equation for $f$:

$$f''' + h \left[ h \frac{dU}{dx} + U \frac{dh}{dx} \right] f f'' + h^2 \frac{dU}{dx} (1 - f'^2) = 0 . \tag{1.13}$$

Since $f$ is a function of $\eta$ only we must have

$$\left. \begin{aligned} & h \left[ h \frac{dU}{dx} + U \frac{dh}{dx} \right] = \text{constant}, \\ \text{and} \quad & h^2 \frac{dU}{dx} = \text{constant}. \end{aligned} \right\} \tag{1.14}$$

The three equations (1.11) and (1.14) have two independent solutions :

$$U \propto (x-x_0)^m \; ; \; v_0 \propto (x-x_0)^{\frac{m-1}{2}} \tag{1.15}$$

and

$$U \propto e^{cx} \; ; \; v_0 \propto e^{\frac{1}{2}cx} \tag{1.16}$$

where $x_0$ and $c$ are constants. There is no loss in generality if we take $x_0 = 0$ since $x_0$ only implies a translation of the origin. $h$ is now obtained from (1.11) or (1.14). We consider only the first of these solutions and from (1.7) and (1.8) we may write

$$\psi = \left[ \frac{2\nu U x}{1+m} \right]^{\frac{1}{2}} f(\eta), \tag{1.17}$$

and

$$\eta = y \left[ \frac{U(1+m)}{2\nu x} \right]^{\frac{1}{2}} \tag{1.18}$$

the extra factor $(2/(1+m))^{\frac{1}{2}}$ being inserted to clear the equation ensuing from (1.13), viz.

$$f''' + ff'' + \beta(1-f'^2) = 0 \tag{1.19}$$

of superfluous constants. This is the Falkner-Skan equation, where $\beta = 2m/(1+m)$, with boundary conditions

$$f(0) = a, \; f'(0) = 0, \; f'(\eta) \to 1 \text{ as } \eta \to \infty \tag{1.20}$$

where $a > 0$ for suction and $a < 0$ for injection. Equation (1.19) is often termed the equation of similar profiles since the profiles at different stations along the plate are identical except for a multiplying factor.

The Falkner-Skan equation has been extensively investigated, particularly by Hartree[3] and Stewartson[4]. Hartree attempted to find numerical solutions for various values of $\beta$ with $a$ zero, his method being to begin the integration from $\eta = 0$ with initial conditions

$$f(0) = 0 = f'(0) \; ; \; f''(0) = c > 0 \tag{1.21}$$

and to vary $c$ until the solution for $f'$ is asymptotic to unity for large $\eta$.

With $2 \geq \beta \geq 0$ he found that his procedure gave a unique solution. The solutions with $\beta \geq 2$ do not correspond to real flow but Hartree tabulated them up to $\beta = 2.4$ to facilitate interpolation. If $\beta_0 = -0.1988 < \beta < 0$ however, he found that $f'$ approaches unity as $\eta \to \infty$ for all $c$ within a certain range. From considerations of continuity he chose the solution in which $f'$ approaches unity faster than in all the others. He then found that if $\beta = \beta_0 = -0.1988$ there is only one solution satisfying $|f'| < 1$ for all $\eta$, and in that $f''(0) = 0$, while if $\beta = -0.2$ there are no solutions with $f''(0)$ small having $|f'| < 1$ for all $\eta$.

It may be noted at this stage that normally we must have $|f'| < 1$ for all finite $\eta$ since otherwise we would have velocities inside the boundary layer greater than those outside (since $u = Uf'(\eta)$) which solutions we would reject on physical grounds. Cohen and Reshotko[5] show that it is possible to have velocities inside the boundary layer which are greater than the velocity in the main stream for the compressible case with heat transfer but we are not concerned here with compressible boundary layers. From now on, then, even if we find a solution of (1.19) which satisfies the boundary conditions (1.20) we will reject it if $f'$ exceeds unity for any value of $\eta$.

Stewartson[4], by replacing the outer boundary condition by $f' = 1$ when $\eta = \alpha$ and then letting $\alpha \to \infty$, found that Hartree's solutions in the range $\beta_0 < \beta < 0$ arose naturally without recourse to continuity arguments. In these solutions, in fact, $f'$ approaches unity exponentially from below whereas in the others the approach is algebraic. He then used Hartree's numerical results to show that if $\alpha = \infty$ and $\beta < \beta_0$ there are no solutions with $|f'| < 1$ for all $\eta$, and went on to show that in $\beta_0 < \beta < 0$ there is another family of solutions satisfying the condition that $f'$ approaches unity exponentially from below and in this family $f''(0) < 0$ corresponding to a region of reversed flow. The value $f''(0) = 0$ is particularly significant since it corresponds to separation ($\partial u/\partial y = 0$ at $y = 0$).

In Part I we firstly examine some of Hartree's findings for $\beta < 0$ and advance further arguments for their justification, and then present some new results dealing with suction. In § II we give analytical support for his numerical finding that for $\beta_0 < \beta < 0$ there is no unique solution of (1.19) with boundary conditions (1.20) and in § III show that his $\beta_0$ lies somewhere in the range $-1 < \beta < 0$ when $\alpha = 0$. In § IV we repeat Stewartson's discussion on the behaviour of the solution in the neighbourhood of $\beta_0$, extending the theory to cover the case of suction, with $\alpha > 0$, and show that to a given value of $\alpha$ there corresponds a $\beta_0(\alpha) < 0$ below which no solution exists and that when $\beta = \beta_0(\alpha)$ the solution has $f''(0) = 0$.

In § V we examine the solutions for $\beta > 0$ and $\alpha$ large and negative and show that the boundary layer is divided into a thin shear layer where viscosity is important in the outer region of the boundary layer and a comparatively thick layer adjacent to the surface where the effect of viscosity is small.

## §II The non-uniqueness of the solution when $\beta_0 < \beta < 0$

We take as boundary conditions for (1.19)

$$f(0) = a, \quad f'(0) = 0, \quad f''(0) = c > 0.$$

Firstly we see that $f'$ is initially positive and cannot have a maximum in $|f'| < 1$. For, from (1.19), when $f'' = 0$

$$f''' = (-\beta)(1 - f'^2) > 0 \quad \text{in } |f'| < 1,$$

whereas for a maximum we would need $f''' < 0$ when $f'' = 0$. Thus $f'$ either increases to a value greater than unity and thereafter tends to unity from above (or again decreases to a value less than unity), or it has a horizontal asymptote $f' = \alpha$, where $0 < \alpha \leq 1$. In this latter case $f'' > 0$ for all finite $\eta$ and, since $f' \leq \alpha \leq 1$ always, then

$$f \leq a + \eta$$

and, from (1.19)

$$f''' + (a+\eta)f'' \geq f''' + ff'' = (-\beta)(1 - f'^2) \geq (-\beta)(1 - \alpha^2) = b, \quad \text{say.}$$

We now show that $b = 0$ so that $\alpha = 1$.

Integrating the inequality

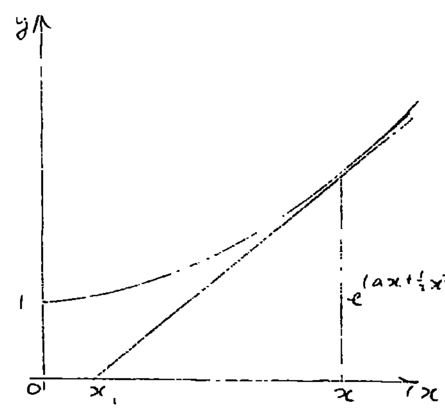
$$f''' + (a+\eta)f'' \geq b$$

we get

$$f'' \geq c\exp(-a\eta - \tfrac{1}{2}\eta^2) + b\exp(-a\eta - \tfrac{1}{2}\eta^2)\int_0^\eta \exp(ax + \tfrac{1}{2}x^2)\,dx.$$

If we consider the graph of

$$y = \exp(ax + \tfrac{1}{2}x^2)$$




(see diagram) firstly with $a > 0$, we see that if the tangent at the point $x$ meets the $x$ axis at $x_1$, then

$$\frac{\exp(ax + \tfrac{1}{2}x^2)}{x - x_1} = \frac{dy}{dx} = (a + x)\exp(ax + \tfrac{1}{2}x^2).$$

Therefore

$$x - x_1 = 1/(a + x)$$

Now $x_1 > 0$ if $x > \dfrac{-a + \sqrt{a^2 + 4}}{2}$.

Thus if $\eta > \eta_0 = \dfrac{-a + \sqrt{a^2+4}}{2}$ then

$$\int_0^\eta \exp(ax + \tfrac{1}{2}x^2)\,dx > \frac{\frac{1}{2}\exp(a\eta + \frac{1}{2}\eta^2)}{a+\eta}$$

by considering the integral as the area beneath the graph. If $a < 0$ then this inequality holds for $\eta > \eta_1$ where $\eta_1$ is less than $\eta_0$

Thus we have that if $\eta > \eta_0$

$$f'' > c\exp(-a\eta - \tfrac{1}{2}\eta^2) + \frac{\frac{1}{2}b}{a+\eta} > \frac{\frac{1}{2}b}{a+\eta}$$

since $c > 0$.

Therefore

$$f'(\eta) - f'(\eta_0) \geqslant \tfrac{1}{2}b\log\left(\frac{a+\eta}{a+\eta_0}\right).$$

Since the right hand side of this inequality tends to infinity as $\eta \to \infty$ if $b \neq 0$ and we have already stipulated that $f' \leq 1$ for all $\eta$, then $b$ must be zero and $\alpha = 1$.

Thus we see that with $c \geqslant 0$ $f'$ either assumes values greater than unity or tends to unity as $\eta$ tends to infinity.

It will be seen in § IV that if a solution exists with $f''(0) = c > 0$ with $|f'| < 1$ for all $\eta$, then the solutions with $0 \leq f''(0) \leq c$ all satisfy $|f'| < 1$ for all $\eta$ and from the above must also satisfy $f' \to 1$ as $\eta \to \infty$. Thus we have that if $\beta < 0$, then if a solution exists it will either have $f''(0) = 0$ (corresponding to Hartree's $\beta_0$) or there will be an infinity of solutions with $f''(0)$ taking values in the range $0 \leq f''(0) \leq c$ so that Hartree's numerical observation that there is no unique solution for $\beta_0 < \beta < 0$ is borne out.

## § III. The existence of $\beta_0$

We define $\beta_0$ as the lower bound of the values of $\beta$ such that the Falkner-Skan equation has a solution in which $|f'| \leq 1$ for all $\eta$ and show that if $\beta = 0$ equation (1.19) has a solution satisfying $|f'| \leq 1$ for all $\eta$ whereas if $\beta = -1$ there are no such solutions. We conclude from this that our $\beta_0$ as defined lies somewhere in the range $-1 < \beta < 0$. Stewartson[4] shows that if there are no solutions for $\beta = \beta_1 < 0$ with $|f'| \leq 1$ for all $\eta$, then there are no solutions for $\beta < \beta_1$ with $|f'| \leq 1$ for all $\eta$. Hence there are no solutions for $\beta < -1$.

### A. $\beta = 0$

When $\beta = 0$ equation (1.19) becomes

$$f''' + ff'' = 0, \tag{3.1}$$

the Blasius equation, and we take as boundary conditions

$$f(0) = f'(0) = 0, \; f''(0) = c > 0. \tag{3.2}$$

Integrating (3.1) we find

$$f'' = c\exp\left(-\int_0^\eta f\,d\eta\right),$$

so that $f'' > 0$ always and so $f' \geq 0$ and $f \geq 0$ always.

Thus $f'$ either increases indefinitely or has a horizontal asymptote.

From (3.1) $f''' \leq 0$ always, therefore

$$f'' \leq c;$$

$$f \leq \tfrac{1}{2}c\eta^2,$$

and from (3.1) again

$$f''' \geq -\tfrac{1}{2}c^2\eta^2;$$

$$f \geq \tfrac{1}{2}c\eta^2 - \tfrac{1}{5!}c^2\eta^5$$

by integration. Therefore

$$f(c^{-\frac{1}{3}}) \geq \tfrac{1}{2}c^{\frac{1}{3}} - \tfrac{1}{5!}c^{\frac{1}{3}} > \tfrac{1}{3}c^{\frac{1}{3}},$$

and so

$$\int_0^\eta f\,d\eta > \int_{c^{-\frac{1}{3}}}^\eta f\,d\eta > \int_{c^{-\frac{1}{3}}}^\eta \tfrac{1}{3}c^{\frac{1}{3}}\,d\eta = \tfrac{1}{3}\left[\eta c^{\frac{1}{3}} - 1\right] \quad \text{for } \eta > c^{-\frac{1}{3}},$$

since $f$ is an increasing function of $\eta$ and $f \geq 0$. Therefore

$$f'' < c\,e^{\frac{1}{3}}\exp\left[-\tfrac{1}{3}\eta c^{\frac{1}{3}}\right] \quad \text{for } \eta > c^{-\frac{1}{3}},$$

and by integration

$$f'(\eta) - f'(c^{-\frac{1}{3}}) \leq 3c^{\frac{2}{3}}\left[1 - \exp\left[-\tfrac{1}{3}(\eta c^{\frac{1}{3}} - 1)\right]\right] \leq 3c^{\frac{2}{3}}.$$

Also, since $f' \le c\eta$, then $f'(c^{-\frac{1}{3}}) \le c^{\frac{2}{3}}$. Therefore

$$f'(\eta) \le 4c^{\frac{2}{3}} \quad \text{for } \eta > c^{-\frac{1}{3}},$$

and so if $c < \frac{1}{8}$ then $f'(\eta) < 1$.

Thus when $\beta = 0$ solutions exist in which $f' \le 1$ for all $\eta$.

B $\underline{\beta = -1}$

When $\beta = -1$ equation (1.19) becomes

$$f''' + ff'' - 1 + f'^2 = 0. \tag{3.3}$$

We are intent on showing that (3.3) has no solutions consistent with $|f'| < 1$ for all $\eta$ and will take as boundary conditions

$$f(0) = f'(0) = f''(0) = 0. \tag{3.4}$$

Integrating twice we get

$$f^2 = \eta^2 - 2f'.$$

$f'''$ is initially positive and since, as in the last paragraph, $f'$ cannot have a maximum in $|f'| < 1$, $f'$ must be an increasing function of $\eta$ in $f' < 1$.

When $f' = \frac{1}{2}$, $f^2 = \eta^2 - 1$ and therefore at this point $\eta \geqslant 1$.

Therefore in $0 \le \eta \le 1$, $f' \le \frac{1}{2}$, and so

$$f \le \tfrac{1}{2}\eta \le \tfrac{1}{2} \quad \text{in } 0 \le \eta \le 1.$$

and

$$f(1) \le \tfrac{1}{2}.$$

In $\eta \geqslant 1$, if $f' \leqslant 1$ then

$$f^2 \geqslant \eta^2 - 2,$$

and also

$$f(\eta) - f(1) \le \eta - 1,$$

by integration of $f' \le 1$. Therefore

$$f \le \eta - \tfrac{1}{2} \quad \text{in } \eta \geqslant 1,$$

so that if $\eta \geqslant 1$ and $f' < 1$ we have

$$\eta - \tfrac{1}{2} \geqslant f \geqslant \sqrt{\eta^2 - 2};$$

but if $\eta > \frac{9}{4}$

$$\eta - \tfrac{1}{2} < \sqrt{\eta^2 - 2}$$

and therefore this inequality does not hold in $\eta > \frac{9}{4}$.

Therefore $f'$ must assume values greater than unity for some $\eta$ in $0 < \eta \leq \frac{9}{4}$. It was shown by Stewartson[4] that if in the solution for $f''(0) = 0$, $f'$ assumes values greater than unity, then there are in fact no solutions with $f''(0) \neq 0$ having $|f'| \leq 1$ for all $\eta$.

Thus there are no solutions when $\beta = -1$ and we conclude that $\beta_0$ lies somewhere in the range $-1 < \beta < 0$.

## IV. Behaviour near $\beta = \beta_0$

In this section we obtain Stewartson's results again, but this time with $f(0)$ not necessarily zero.

Following Stewartson[4] we make the transformation $f'(\eta) = z$ and $f(\eta) = F(z)$. Then

$$F'(z) = \frac{dF}{dz} = \frac{f'(\eta)}{f''(\eta)}, \quad F''(z) = \frac{f''(\eta) - f'''(\eta)\,F'(z)}{[f''(\eta)]^2},$$

so that (1.19) and (1.20) become

$$z^2 F'' - zFF'^2 - zF' - \beta F'^3 (1 - z^2) = 0 \; ; \qquad (4.1)$$

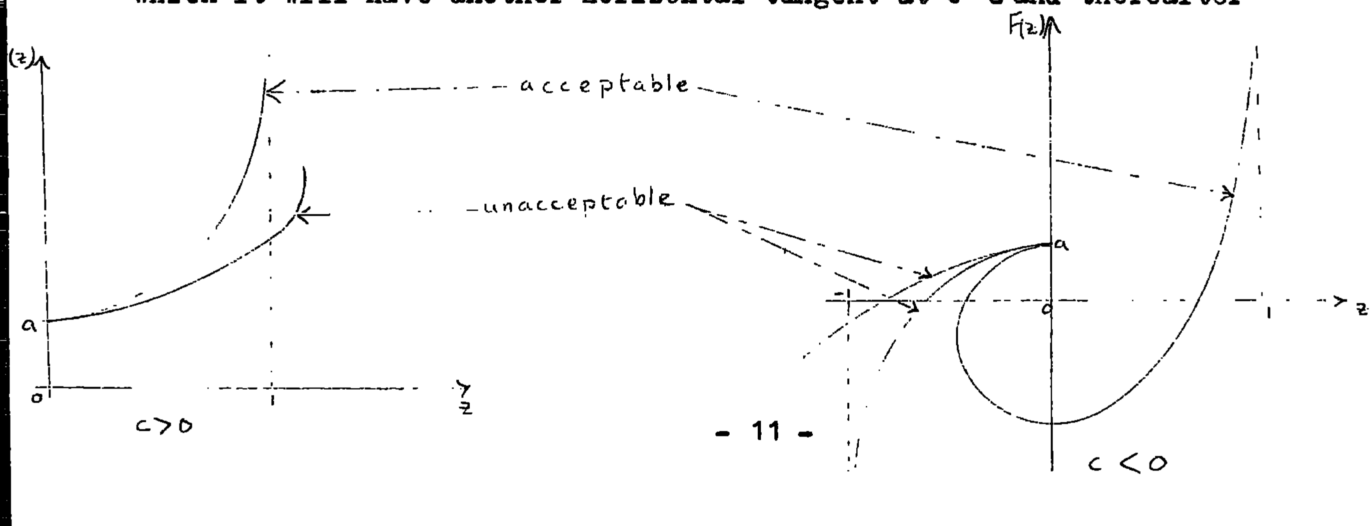
$$\left.\begin{aligned} F(0) &= a \, , \\ F'(0) &= 0 \, , \\ F''(0) &= 1/c \, , \\ F'''(0) &= 2(\beta + ac)/c^3 \, , \end{aligned}\right\} \qquad (4.2)$$

where $c = f''(0)$ and the primes now denote differentiation with respect to $z$. The third boundary condition becomes

$$F \to \infty \text{ as } z \to 1.$$

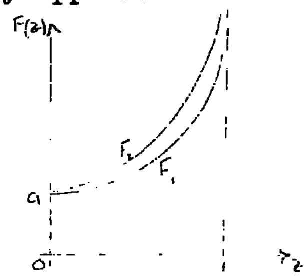
Considering the graph of $F$ as a function of $z$ we see first that if $c > 0$, $f'$ cannot have a maximum in $|f'| < 1$ (From § II), and so $F$ will have a horizontal tangent at $z = 0$ and then increase steadily, and either have a vertical asymptote at $z = 1$, or will cross the line $z = 1$, in which case it will correspond to an unacceptable solution (see sketches below).

If $c < 0$ the graph will again have a horizontal tangent at $z = 0$ but then both $z$ and $F$ decrease and $F$ will either continue to decrease, in which case it is unacceptable, or will have a vertical tangent in $z > -1$, after which it will have another horizontal tangent at $z = 0$ and thereafter



behave in a similar way to the solutions with $c>0$. Since $f'$ cannot have a minimum in $|f'|>1$ then if $F$ decreases below $z=-1$ it cannot again cross the line $z=-1$. $c<0$ corresponds to reversed flow near the plate.

With the aid of a few preliminary lemmas we will repeat Stewartson's proof that, subject to Hartree's numerical results being accepted, there are no solutions of the Falkner-Skan equation with $f(0)=f'(0)=0$ and $|f'|\leq 1$ for all $\eta$ if $\beta<\beta_0=-0.1988$ and then show that for any $f(0)=a>0$ there corresponds a similar $\beta_0(a)$ such that there are no solutions with $f(0)=a$, $f'(0)=0$ and $|f'|\leq 1$ for all $\eta$ if $\beta<\beta_0(a)$. The various sketches given refer to the Lemma or Corollary opposite which they appear.




__Lemma 1__ If $0\leq F_1''(0)<F_2''(0)$ and $F_1(0)=F_2(0)$, then the graphs of $F_1$ and $F_2$ do not cross in $0<z<1$. For, when $z=0$, we have $F_1=F_2$, $F_1'=F_2'$, $0\leq F_1''<F_2''$, so that there is a range of values of $z$, $0<z<z_0<1$, say, for which $F_1'<F_2'$. So long as $z<1$, $F'$ cannot become singular, as $f''(\eta)>0$. If at $z=z_0$ $F_1'=F_2'$, then also at $z=z_0$ we will have $F_1<F_2$ and $F_1''>F_2''$, because $F_1'$ must be increasing more rapidly than $F_2'$. But from (4.1), when $F_1'=F_2'$

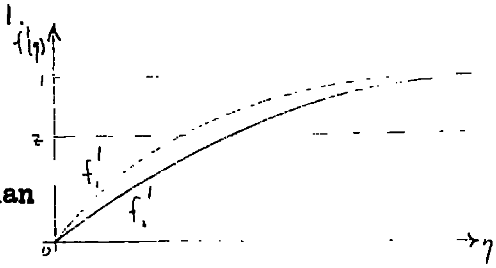
$$z_0^2(F_1''-F_2'')-z_0F'^2(F_1-F_2)=0,$$

so that if $F_1<F_2$ then $F_1''<F_2''$ giving a contradiction.

Thus the graphs cannot cross in $0<z<1$.

__Corollary 1__ If $f_1(\eta)=F_1(z)$ and $f_2(\eta)=F_2(z)$, then in the $f'$, $\eta$ plane $f_1'(\eta)$ and $f_2'(\eta)$ do not cross in $0<f'(\eta)<1$.

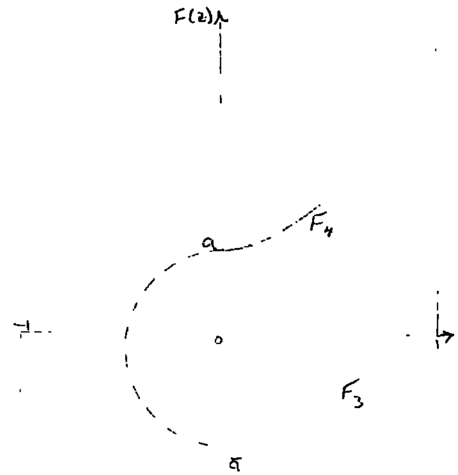
For $$\eta=\int_0^z\frac{F'(z)}{z}\,dz$$




and $F_1'(z)<F_2'(z)$ in $0<z<1$. Hence the value of $\eta$ for which $f_1'=z$ is always less than the value for which $f_2'=z$.

__Corollary 2__ If $f_1(0)=f_2(0)=a$, $f_1'(0)=f_2'(0)=0$ and $0\leq f_2''(0)<f_1''(0)$ then if $f_2'$ exceeds unity, so does $f_1'$, otherwise the graphs would cross.

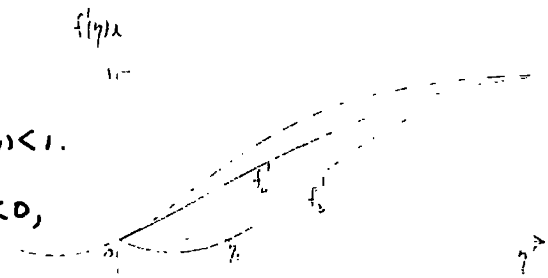
If now $c<0$ so that initially $F<a, z<0$ but $-c$ is sufficiently small for $z$ to have a minimum value greater than $-1$, then the graph will eventually recross the line $z=0$, when $F'=0$, and at this point let $F=\bar{a}(<a)$, $F''=\bar{c}>0$ and $\eta=\eta_0$. We will characterise this solution in the range $0<z<1$ by $F_3$ and move the origin of $\eta$ so that $F_3=\bar{a}$ at $\eta=0$. Let us designate any solution with $F=a, F''>\bar{c}$ at $z=0$, $\eta=0$ by $F_4$.




Then Lemma 2. $F_3'<F_4'$ in $0<z<1$.

For, at $z=0$, $F_3<F_4$, $F_3'=F_4'$, $F_3''<F_4''$ and hence in the range $0<z<z_1$, say, $F_3'<F_4'$. As in Lemma 1. we get a contradiction if $z_1<1$ and so $F_3'<F_4'$ in $0<z<1$.

Also if $f_3(\eta)=F_3(z)$ and $f_4(\eta)=F_4(z)$ then $f_3'(\eta)$ and $f_4'(\eta)$ do not cross in $0<f'(\eta)<1$.




Corollary 3 If $f_3(0)=a$, $f_3'(0)=0$, $f_3''(0)=c<0$,

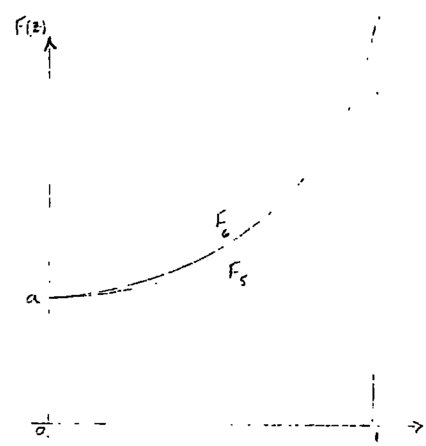
$$f_3'(\eta_0)=0,\ f_3''(\eta_0)=\bar{\bar{c}}>0$$

and
$$f_4(0)=a,\ f_4'(0)=0,\ f_4''(0)<\bar{\bar{c}}$$

then if $f_4'$ exceeds unity, so does $f_3'$.

Lemma 3 If $F_5$ is the solution of (4.1) with $F_5(0)=a$, $F_5'(0)=0$, $F_5''(0)=1/c>0$ and $\beta=\beta_5$, and $F_6$ is the solution with $F_6(0)=a$, $F_6'(0)=0$, $F_6''(0)=1/c>0$ and $\beta=\beta_6$, and $\beta_5<\beta_6<0$ then

$$F_5'<F_6' \text{ in } 0<z<1.$$




For when $z=0$, $F_5=F_6$, $F_5'=F_6'$, $F_5''=F_6''$, and

$$F_5'''-F_6'''=z(\beta_5-\beta_6)/c^3<0,$$

so that $F_5'<F_6'$ in the range $0<z<z_2$, say.

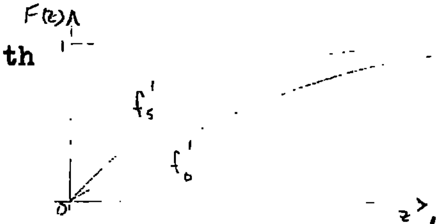
At $z=z_2$ $F_5<F_6$, $F_5'=F_6'$, $F_5''>F_6''$.

But from (4.1)

$$z_2^2(F_5''-F_6'')-z_2(F_5-F_6)-F'^3(\beta_5-\beta_6)(1+z_2^2)=0,$$

so that $F_5''<F_6''$ if $0<z_2<1$.

Therefore the graphs do not cross in $0<z<1$.




Corollary 4  If $f_5$ is the solution of (1.19) with
$f_5(0) = a$, $f_5'(0) = 0$, $f_5''(0) = c > 0$ and $\beta = \beta_5$
and $f_6$ the solution with
$f_6(0) = a$, $f_6'(0) = 0$, $f_6''(0) = c > 0$ and $\beta = \beta_6$
and $\beta_5 < \beta_6 < 0$ then if $f_6'(\eta)$ assumes values greater than unity, so does $f_5'(\eta)$.

Lemma 4  If $F_7(0) < F_8(0)$, $F_7'(0) = F_8'(0) = 0$ and $F_7''(0) = F_8''(0) = 1/c > 0$
then $F_7$ and $F_8$ do not cross in $0 < z < 1$.

For when $z = 0$ $F_7 < F_8$, $F_7' = F_8'$, $F_7'' = F_8''$ and $F_7''' - F_8''' = 2(F_7 - F_8)/c^2 < 0$,
so that $F_7' < F_8'$ in $0 < z < z_3$, say. As in Lemma 1 we get a contradiction if $z_3 < 1$ and so $F_7' < F_8'$ in $0 < z < 1$.

Corollary 5  If $f_7(0) < f_8(0)$, $f_7'(0) = f_8'(0) = 0$ and $f_7''(0) = f_8''(0) = c > 0$
then if $f_8'$ exceeds unity, so does $f_7'$.

This result is not needed for the theorem but will be used later.

We now consider the case of $f(0) = 0$.

Hartree found, by numerical computation, that when $\beta = \beta_0 = -0.1988$ and $f''(0) = 0$ then $|f'(\eta)| \leq 1$ for all $\eta$, but that if $f''(0) = \epsilon > 0$, where $\epsilon$ is small, then no matter how small $\epsilon$ is chosen, $f'(\eta) > 1$ for some $\eta$. He also found that if $\beta = -0.2$ and $f''(0) = 0$ then $f'$ assumes values greater than unity. If we accept these results then the following theorem, given by Stewartson,[4] follows :-

Theorem  If $\beta < \beta_0 = -0.1988$, then in all the solutions of the Falkner-Skan equation with $f(0) = f'(0) = 0$ there is a range of values of $\eta$ for which $f'(\eta) > 1$.

From the above conjecture and Corollary 2 we deduce that for all $f''(0) \geqslant \epsilon$ (and so for all $f''(0) > 0$ if $\epsilon$ is vanishingly small) there exists an $\eta$ at which $f' > 1$. From this result and from Corollary 4 we deduce that for any $\beta < \beta_0$ and $f''(0) > 0$ there exists an $\eta$ for which $f' > 1$. When $f''(0) < 0$, since $f'' > 0$ when $f'$ becomes zero for the second time, then from the above result and Corollary 3 there are no solutions when $\beta < \beta_0$ and $f''(0) < 0$ in which $|f'| \leq 1$ for all $\eta$. There remain only the solutions when $f''(0) = 0$. In § III we showed that when $\beta=-1$ and $f''(0)=0$, $f'$ exceeds unity for some $\eta$ and so we can consider the theorem is proven for $\beta\leq-1$, also, if we accept Hartree's second conjecture, the theorem is proven for $\beta\leq-0.2$.

<u>Lemma 5</u> If we take (1.19) with boundary conditions $f(0)=a>0$, $f'(0)=f''(0)=0$ then given any value of $a$ it is possible to choose a $\beta$ (with $-\beta$ large enough) such that $f'>1$ for some $\eta$.

We note that $f'''(0)=-\beta\,(>0)$ and $f''>0$ in $0<f'<1$, as before.

Hence in $|f'|<1$

$$f'''=(-\beta)(1-f'^2)-ff''\leq-\beta.$$

Therefore

$$f''\leq-\beta\eta \quad ,$$
$$f\leq a-\tfrac{1}{3!}\beta\eta^3$$

by integration, and so

$$f'''\geqslant(-\beta)(1-\tfrac{1}{4}\beta^2\eta^4)+(a-\tfrac{1}{6}\beta\eta^3)\beta\eta \quad ,$$
$$f'\geqslant(-\beta)(\tfrac{1}{2}\eta^2-\tfrac{1}{120}\beta^2\eta^6)+\beta(\tfrac{1}{6}a\eta^3-\tfrac{1}{180}\eta^6\beta).$$

Therefore

$$f'\left(\frac{2}{\sqrt{-\beta}}\right)\geqslant 2-\frac{8}{15}-\frac{4}{3}\left(\frac{a}{\sqrt{-\beta}}+\frac{4}{15(-\beta)}\right)$$

$$>\frac{22}{15}-\frac{4}{3}\left(\frac{a}{\sqrt{-\beta}}+\frac{1}{(-\beta)}\right).$$

The second term may be made as small as we please by making $(-\beta)$ large enough. For example if $-\beta>2(2a+1)$

$$f'\left(\frac{2}{\sqrt{-\beta}}\right)>\frac{17}{15}$$

and so we see that given any $a>0$, we can choose $-\beta$ large enough to make $f'>1$ for some $\eta$.

We see from the Theorem and Corollary 5 that in $-0.1988\leq\beta<0$ all the solutions with $f''(0)=0$ and $a>0$ satisfy $|f'|<1$ for all $\eta$ (from § II, of course, in all these solutions $f'\to1$ as $\eta\to\infty$).

We can now deduce from Lemma 5 that for any given $a>0$, there exists a $\beta_a$ in $-2(2a+1)<\beta_a<0$ such that in all the solutions with $f''(0)=0$ and $\beta<\beta_a$, $f'$ exceeds unity for some $\eta$, whereas if $\beta\geqslant\beta_a$ $|f'|\leq1$ for all $\eta$.

From Corollarys 2 and 3 we see from this that if $\beta < \beta_a$ then no solutions with $f(0)=a$ satisfy $|f'| < 1$ for all $\eta$.

Physically this means that when the suction velocity $-v_0(x) = A\sqrt{\frac{U}{x}}$ when $U \propto x^m (m<0)$ then for any value of $A$ we choose we get attached flow only if $m$ is greater than some value $m_A$. As $A$ is increased the permissible range of $m$ for attached flows is extended (i.e. $m_A$ decreases).

We can conjecture the converse from this, that for any value of $m$ there exists an $A_m$ such that we get attached flows only if $A > A_m$, although it has not been possible to prove this explicitly.

We note that the same should be true if $0 > m > m_0$ where $m_0 = \beta_0/(2-\beta_0) = -0.0904$ when the injection velocity must not exceed a certain value to retain attached flow.

## §V The solutions for positive $\beta$ and large rates of injection

Iglisch[6] proves that equation (1.19) with boundary conditions (1.20) with $\beta > 0$ possesses at least one solution $f(\eta)$ although he does not prove the uniqueness of this solution. We will examine the nature of this solution when $\alpha$ is large and negative.

Near the plate (i.e. $1-f'$ not small) we obtain a trial approximate solution by neglecting $f'''$ in the equation (1.19). Reference to the manner in which (1.19) was derived reveals that viscosity asserts itself only through the term $f'''$. This term being neglected we can integrate the equation to give

$$f = -(-a)(1-f'^2)^{1/2\beta} \;;$$

$$f'' = \frac{\beta}{(-a)}(1-f'^2)^{(1-1/2\beta)} \;,$$

and so

$$f''' = -\frac{\beta^2}{(-a)^2}\left(1-\frac{1}{2\beta}\right)f'(1-f'^2)^{(1-1/\beta)}$$

so that when the injection rate is large $(-a \gg 0)$ and $1-f'$ is not small

$$f''' = O\left(\frac{1}{(-a)^2}\right) \;,$$

and so is in fact small compared with the other terms in equation (1.19) which are of order (1) in $\alpha$. Thus when $1-f'$ is not small we are justified in assuming that the viscous term is negligible. As $f' \to 1$, $f \to 0$ in this solution and so the thickness of this region will be roughly of order $(-a)$.

When $f=0$, $\eta = \eta_0$, say, and since $f < a+\eta$ in $f' < 1$, then $\eta_0 > (-a)$. Considering the region now where $1-f'$ is small we replace $f'$ by $1-g$, and $f$ by $(\eta-\eta_0)+h$ where $g$ and $h$ are so small that products of $h$ and $g$ may be neglected to the first order, and so obtain the equation

$$g'' + (\eta-\eta_0)g' - 2\beta g = 0 \qquad (5.1)$$

with $g(\infty) = 0$.

Integrating (5.1) it may be shown that a solution always exists, so that $g$ is never singular. When $(\eta-\eta_0)$ is large

$$g \sim e^{-\frac{1}{2}(\eta-\eta_0)^2}(\eta-\eta_0)^{-(2\beta+1)}$$

and approaches zero very rapidly as $\eta$ increases.

Since there is no singularity in $f'$ and $f'$ approaches unity as $\eta \to \infty$ it may easily be seen from (1.19) that $|f'| < 1$ for all $\eta$, since if $f'$ exceeds unity it must have a maximum in $f' > 1$ and at this maximum $f'' = 0$ and $f''' > 0$. But from (1.19), when $f'' = 0$

$$f''' = \beta(f'^2 - 1) > 0 \quad \text{if} \quad f' > 1$$

giving a contradiction. Thus $f' < 1$ for all $\eta$ as shown by Iglisch.[6]

The velocity profile is thus split up into two parts. In the outer part the viscosity term is important, and since the solution of (5.1) approaches zero rapidly as $\eta$ increases this outer region will be fairly narrow. In the inner region, where the longitudinal velocity changes from just below its free-stream value to zero at the plate viscosity plays a very small part if $(-\alpha)$ is large enough. The thickness of this region will be of order $(-\alpha)$.

## Concluding Remarks on Part I

An attempt has been made in this section to justify analytically some of Hartree's numerical conjectures when $f(0)=0$. The existence was shown of a $\beta_0$, defined as the lower bound of the values of $\beta$ such that the Falkner-Skan equation has a solution in which $|f'| \leq 1$ for all $\eta$, somewhere in the range $-1<\beta<0$. It was also shown that for $\beta_0<\beta<0$ there is no unique solution of the Falkner-Skan equation. The argument for the existence of $\beta_0$ would be more conclusive if it had been shown that a solution exists with $|f'| \leq 1$ for all $\eta$ for some small negative value of $\beta$ rather that for $\beta=0$, since the solutions for $\beta \geq 0$ are of one type, the solution in each case being unique (this has not been shown, but is generally accepted as being true), whereas the solutions for $\beta<0$ are of another type and are not unique. It was not, however, found possible to prove the existence of a solution for $\beta$ small and negative and until this can be done we can not really consider the argument to be a proof.

Stewartson's proof that there are no solutions of the Falkner-Skan equation with $|f'| \leq 1$ for all $\eta$ if $\beta<\beta_0=-0.1988$ has been extended to cover the case of suction with $f(0)>0$ and it has been shown that to a given $f(0)=a>0$ there corresponds a $\beta_a$, defined in a similar way as $\beta_0$, as the lower bound of the values of $\beta$ such that the Falkner-Skan equation has a solution in which $|f'| \leq 1$ for all $\eta$ when $f(0)=a$. It was shown that $\beta_a$ lies somewhere in the range $-2(2a+1)<\beta_a<0$, but it was not found possible to derive a much closer relationship between $a$ and $\beta_a$.

Finally it was shown that for $\beta>0$ and large rates of injection the profile is divided into a thin outer shear layer where viscosity is important and a comparatively thick layer adjacent to the surface where the effect of viscosity is small.

Part II

## The Flat Plate in a Uniform Incompressible Stream with Constant Injection through the Plate

### § VI. Introduction

The boundary layer equations relating to this problem are integrated numerically in Part III. §§VII to X of this section deal with the solution in the neighbourhood of the separation point obtained ~~by~~ in collaboration with Stewartson. (~~to be published soon~~).

The two dimensional incompressible laminar boundary layer equations, in the absence of a pressure gradient, reduce to

$$u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} = \nu \frac{\partial^2 u_1}{\partial y_1^2} \quad ; \tag{6.1}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} = 0 \quad , \tag{6.2}$$

where $x_1$ and $y_1$, and $u_1$ and $v_1$ denote distances and velocities measured along and perpendicular to the plate and $\nu$ is the kinematic viscosity. For a semi infinite flat plate placed edge on at zero incidence to the stream, if we imagine it to be possible to inject fluid at a constant rate at all points normal to the plate, the boundary conditions are

$$\left.\begin{array}{ll} \left.\begin{array}{l} u_1(x_1, 0) = 0 \\ v_1(x_1, 0) = v_0 = \text{constant} > 0 \end{array}\right\} (x_1 > 0) & , \\ u_1(x_1, \infty) = U & , \\ u_1(0, y_1) = U \quad (y_1 > 0) & , \end{array}\right\} \tag{6.3}$$

where $U$ is the (constant) velocity of the main stream.

If we convert to non-dimensional co-ordinates, putting

$$u_1 = Uu \;;\; v_1 = v_0 v \;;\; x_1 = \frac{U\nu}{v_0^2} x \;;\; y_1 = \frac{\nu}{v_0} y \;,$$

the equations become

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} \quad ; \tag{6.4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \tag{6.5}$$

with boundary conditions

$$\left.\begin{aligned} &\left.\begin{aligned} u(x,0)&=0 \\ v(x,0)&=1 \end{aligned}\right\} (x>0), \\ &u(x,\infty)=1 , \\ &u(0,y)=1 \quad (y>0) , \end{aligned}\right\} \quad (6.6)$$

The related problem with no suction or injection has been studied by Blasius[7]. The transformation to non-dimensional co-ordinates is slightly different but the equations (6.4) and (6.5) are the same and also the boundary conditions, except that the second one is replaced by $v(x,0)=0$. He puts $\eta = y/\sqrt{2x}$ and $u = \frac{\partial}{\partial\eta}(f(\eta))$ giving the equation

$$f''' + ff'' = 0$$

where the prime denotes differentiation with respect to $\eta$. The boundary conditions are

$$f(0) = f'(0) = 0 \ ; \ f'(\infty) = 1 .$$

He found that the third boundary condition could be satisfied by making $f''(0) = 0.46960$. The non-dimensional skin friction

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \tau_0 = \frac{1}{\sqrt{2x}} f''(0). \qquad \text{(Fig.I graph 1)}$$

It will be seen that $\tau_0$ tends asymptotically to zero as $x$ tends to infinity. Figures I - III are only schematic. The tranverse velocity $v_\infty$ at the edge of the boundary layer is given by

$$v_\infty = \frac{1}{\sqrt{2x}} \lim_{\eta\to\infty} (\eta f' - f).$$

Since, for large $\eta$,

$$f \sim \eta - b + a e^{-\frac{1}{2}\eta^2}/\eta^2 \ ; \ f' \sim 1 - a e^{-\frac{1}{2}\eta^2}/\eta ,$$

where $a$ and $b$ are positive constants, then

$$v_\infty = \frac{b}{\sqrt{2x}} . \qquad \text{(Fig.II graph 1)}$$

$b$ is related to the displacement thickness and equals 1.22 approx.

With suction ($v(x,0) = -1$), the behaviour for large $x$ may be obtained from (6.4) by assuming changes in the $x$ direction to die out so that $\frac{\partial u}{\partial x}$ tends to zero and from (6.5) $v \sim -1$.

This gives $u \sim 1 - e^{-y}$ and so $\tau_0 \sim 1$. Near the leading edge Blasius' solution still holds and the two solutions may be combined to give a picture of the complete behaviour (Figs. I and II graph 2). The 'asymptotic suction profile' $u \sim 1 - e^{-y}$ was first given by Thwaites.[8]

With injection a similar procedure cannot be followed. Near the leading edge the Blasius solution will still apply, but if we again assume that changes in the $x$ direction die out for large $x$ we arrive at the solution $u \sim e^{y} - 1$. Obviously we cannot accept this, or any other solution which gives values greater than unity to $u$, and so we must assume either that the flow for large $x$ does not become independent of $x$ or that separation occurs ( $\tau_0$ becomes zero). The latter situation would invalidate the boundary layer approximations after the separation point, since the boundary layer equations are obtained on the assumption that the boundary layer thickness remains small and it is well known that the boundary layer thickens considerably after the separation point.

Lew and Fanucci[9], using a method of Iglisch[10], integrated the equations (6.4), (6.5) with boundary conditions (6.6) numerically by a step-by-step method from $x = 0$ to $x = 0.25$, and from these results it can be seen that $v_\infty$ begins to increase again after about $x = 0.15$ (Fig.II graph 3).

An independent numerical integration was performed (see Part III) using a method due to Leigh[11] and the results agree with those of Lew and Fanucci as far as they went. It was found that $v_\infty$ continues to increase. Although it could not be concluded from values of $\tau_0$ obtained (Fig.I graph 3) that $\tau_0$ becomes zero for some finite $x$, since it appears to be approaching zero tangentially, it can be seen, by observing the graph of $1/v_\infty$ against $x$ (Fig.III), that both $\frac{d}{dx}\left(\frac{1}{v_\infty}\right)$ and $\frac{d^2}{dx^2}\left(\frac{1}{v_\infty}\right)$ are negative for $x > 0.15$ for as far as it was possible to continue the integration and so it may fairly safely be assumed that $1/v_\infty$ becomes zero for some finite $x$. Since

$$v_\infty = 1 + \frac{d}{dx}\int_0^1 (1-u)\frac{du}{\tau},$$

by integrating the continuity equation (6.5), where $\tau = \partial u/\partial y$, therefore if $v_\infty$ becomes infinite for some finite $x$, $\tau$ must become zero for some finite $x$, and since it is normally assumed that $\tau$ first vanishes at the wall then $\tau_0$ must become zero for some finite $x$. It was possible to integrate up to $x = 0{\cdot}7$ before the method broke down and by extrapolation separation can be assumed to occur at about $x = 0{\cdot}74$.

In order to investigate the nature of the mathematical solution of (6.4) - (6.6) in the neighbourhood of the separation point we note first that at $y = 0$, if $\partial u/\partial y = 0$, then since $u = 0$, $\partial^2 u/\partial y^2$ is also zero from (6.4), and, by continued differentiation of (6.4) $\partial^r u/\partial y^r = 0 \, (r = 1, 2, 3 \cdots)$.
Thus the Taylor series for $u$ at $y = 0$ does not exist and so the equations must have a singularity at this point.

In §VII we firstly describe the work of Goldstein,[12] Stewartson[13] and Terrill[14] on related problems when the external flow is decelerating and show that the method used by them does not work in our case. We then transform the boundary layer equations into Crocco's form which uses $u$ and $x$ as independent variables and $\tau$ ($= \partial u/\partial y$) as dependent variable. We make one further transformation of the independent variables to $x_s - x$ and $\eta$ ($= u/(x_s - x)$) where $x_s$ is the distance of the separation point from the leading edge and look for a series expansion solution in the neighbourhood of the separation point. We find that the leading term is $\eta$ but that it is impossible to find an expansion valid throughout this neighbourhood and so we divide the boundary layer into three regions. We are not concerned with the outer region, but in order that the solution in this region be well behaved we eliminate all exponentially large terms from the second region. The solution in the second region is not valid near the wall but is 'matched' with a solution which is valid only in an inner region near the wall.

## §VII The Solution near the Separation Point

### 1. Introduction

Goldstein[12] investigated the problem of the flat plate in a stream with adverse pressure gradient and zero suction at the wall. In this case it is not at first evident that there is a singularity in the equations at the separation point. However Hartree's numerical work indicated the presence of a singularity, the skin friction behaving like $(x_s - x)^{\frac{1}{2}}$ near the separation point; $x_s$ is the distance of the separation point from the leading edge. Goldstein transformed to variables $\xi$ and $\eta$ where $\xi = (x_s - x)^{\frac{1}{n}}$ and $\eta = y/2^{\frac{1}{2}}(x_s - x)^{\frac{1}{n}}$ and assumed that the stream function $\psi$ could be expanded in the form $\psi = 2^{\frac{1}{2}} \xi^m \sum_{r=0}^{\infty} \xi^r f_r(\eta)$ near the separation point. When this expansion is substituted in the first boundary layer equation, which in this case is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \tag{7.1}$$

where $\frac{\partial p}{\partial x}$ is assumed to be expandible in the form

$$\frac{\partial p}{\partial x} = p_0 + p_1(x_s - x) + p_2(x_s - x)^2 + \cdots$$

then $n$ and $m$ are found to have values of 4 and 3 respectively, if we demand that the inertia, viscous and pressure gradient terms in (7.1) have equal "weight", that is each contributes a term to the equation for the leading term $f_0$ in the expansion for $\psi$. A set of equations for the $f_r$ is obtained. The boundary conditions are that $f_r(0) = f_r'(0) = 0$ for all $r$, and that $f_r$ must not be exponentially large for large $\eta$. This solution is valid only in the neighbourhood of the separation point and the latter boundary condition is included to ensure that this solution may be 'matched' with the solution outside this region. Goldstein included one further boundary condition which ensures that the skin friction behaves like $(x_s - x)^{\frac{1}{2}}$ near the separation point, namely

$$f_0''(0) = 0. \tag{7.2}$$

He found, however, that if $f_6(\eta)$ is not to be exponentially large a certain integral must be zero. It was later shown by Terrill[14] that in fact this integral is not zero, and Stewartson[13] showed that the addition of logarithmic terms in the expansion for $\psi$ is necessary to continue the expansion past the sixth term, $\psi$ now being expanded in the form

$$\psi = 2^{\frac{3}{2}}\xi^{3}\sum_{r=0}^{6}\xi^{r}f_r(\eta) + 2^{\frac{3}{2}}\xi^{8}\log\xi\left[F_5(\eta)+\xi F_6(\eta)\right] + O(\xi^{10}\log\xi). \tag{7.3}$$

By this means an expansion for the skin friction near the separation point was found. Terrill[14] extended the work to cover the case of an arbitrary suction distribution $v_0(x)$ of the form

$$v_0(x) = V_0 + V_1(x_s - x) + V_2(x_s - x)^2 + \cdots$$

He uses the same variables and expansion for $\psi$ but the boundary conditions at the wall now become

$$\frac{\partial f}{\partial\eta} = 0\,;\quad \frac{1}{2^{\frac{1}{2}}\xi}\left(3f + \xi\frac{\partial f}{\partial\xi}\right) = -(V_0 + V_1\xi^4 + \cdots),$$

where $f = \psi / 2^{\frac{3}{2}}\xi^3$, so that

$$f_r'(0) = 0 \quad \text{and} \quad f_r(0) = 0\ (r \neq 4n+1);\quad f_r(0) = -V_n\ (r = 4n+1). \tag{7.4}$$

The equation for $f_0$ is

$$f_0''' - 3f_0f_0'' + 2f_0'^2 = p_0, \tag{7.5}$$

with

$$f_0(0) = f_0'(0) = 0 = f_0''(0)$$

from (7.2) and (7.4).

The solution to (7.5) with these boundary conditions is

$$f_0 = \tfrac{1}{6}p_0\eta^3. \tag{7.6}$$

2. <u>The solution when $\partial p/\partial x = 0$ and $v(x,0) = 1$.</u>

For the problem considered here we have $p_0 = 0$ and (7.6) becomes

$$f_0 \equiv 0$$

If the condition $f_0''(0) = 0$ is waived, it can be shown numerically that any solution of (7.5) with $p_0 = 0$ and $f_0''(0) = A > 0$ ($A$ must be positive for the skin friction to be positive before separation) has $f_0'$ reaching a maximum and then decreasing. Since

$$f_0''' = -2f_0'^2$$

when $f_0'' = 0$, $f_0'$ cannot have a minimum and it can be shown that $f_0' \to -\infty$ as $\eta \to \infty$. This is obviously unacceptable and so this method of tackling the solution in the neighbourhood of the separation point was abandoned. Instead attention was switched to Crocco's transformation of the boundary layer equations. In this $\tau = \partial u/\partial y$ is used as dependent variable and for independent variables we use $u$ and $\bar{x} = x_s - x$. Dropping the bar on $x$ the equations become

$$\tau^2 \frac{\partial^2 \tau}{\partial u^2} + u \frac{\partial \tau}{\partial x} = 0 , \tag{7.7}$$

with boundary conditions

$$\left.\begin{aligned} u = 1 \ , \ \tau = 0 \quad & ; \\ u = 0 \ , \ \frac{\partial \tau}{\partial u} = 1 \quad & . \end{aligned}\right\} \tag{7.8}$$

For the derivation of this equation see Howarth.[15]

A series expansion for $\tau$ is assumed of which the leading term is $\tau = x^a f(\eta)$ where $\eta = u/x^b$.
Substituting this into (7.7) we obtain

$$x^{3a-2b} f^2 f'' + \eta x^{a+b-1} (af - b\eta f') = 0 ,$$

the primes denoting differentiation with respect to $\eta$. If each term in (7.7) is to have equal 'weight', then

$$3a - 2b = a + b - 1.$$

The second boundary condition of (7.8) gives

$$x^{a-b} f'(0) = 1,$$

and so $a = b$.

Thus $a = b = 1$ and we will use as independent variables in the neighbourhood of the separation point $x$ and $\eta$ $(= u/x)$. The equation for $f$ is now

$$f^2 f'' + \eta (f - \eta f') = 0 \quad , \tag{7.9}$$

with

$$f'(0) = 1. \tag{7.10}$$

Since $\tau = x f(\eta)$ the condition that $\tau_0 = 0$ when $x = 0$ is satisfied, provided $f(0)$ is finite. In addition to (7.10) we therefore take

$$f(0) = A. \tag{7.11}$$

We now show that $A$ must be zero. Put

$$\eta f' - f = G$$

then

$$\eta f'' = G'$$

and

$$G(0) = -A.$$

Substituting in (7.9) and integrating,

$$G = -A \exp\left[\int_0^{\eta} \frac{\eta_1^2}{f^2}\, d\eta_1\right]$$

Therefore

$$f'' = -\frac{A\eta}{f^2} \exp\left[\int_0^{\eta} \frac{\eta_1^2}{f^2}\, d\eta_1\right]. \tag{7.12}$$

(i) If $A > 0$ then $f'' < 0$ from (7.12), and so

$$f' \leq 1\ ,$$

$$0 \leq f \leq A + \eta.$$

Therefore

$$f'' \leq -\frac{A\eta}{(A+\eta)^2}.$$

Integrating twice we have

$$f \leq A - 2A^2 \log(1+\eta/A) + \eta\left[1 + 2A - A \log(1+\eta/A)\right].$$

The right hand side of this expression becomes zero for some finite $\eta$. Therefore $f$ becomes zero for some finite $\eta$.

(ii) If $A < 0$ then $f'' \geqslant 0$ , from (7.12). Therefore

$$f' \geqslant 1\ ;$$

$$0 \geqslant f \geqslant A + \eta$$

so that again $f$ becomes zero for some finite $\eta\ (\leq -A)$.

From (7.12) we see that if $f$ becomes zero for some finite $\eta$ then $f''$ becomes infinite for some finite $\eta$ . Therefore, in order to restrict singularities to the separation point only we must take $A = 0$.

The solution of (7.9) with $f(0)=0$ is now

$$f = \eta \cdot \tag{7.13}$$

Writing $\tau$ now as

$$\tau = x(\eta + F(x,\eta)) \ , \tag{7.14}$$

then (7.7) becomes

$$(\eta + F)^2 \frac{\partial^2 F}{\partial \eta^2} + \eta \left( F - \eta \frac{\partial F}{\partial \eta} + x \frac{\partial F}{\partial x} \right) = 0 \tag{7.15}$$

with

$$\frac{\partial F(x,0)}{\partial \eta} = 0 \cdot \tag{7.16}$$

If we follow Goldstein and look for a solution of the form

$$F = x \sum_0^\infty x^r f(\eta) \quad + \quad \text{possibly logarithmic terms,}$$

we arrive at the following equations for the $f_r$'s

$$\eta f_0'' - \eta f_0' + 2 f_0 = 0$$

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$$\eta f_r'' - \eta f_r' + (r+2) f_r = \bar{f}_r$$

where $\bar{f}_r$ is some function of $(f_0 \cdots \cdots f_{r-1})$. If this solution is to be valid throughout the neighbourhood surrounding the separation point then we must have (from (7.16))

$$f_r'(0) = 0$$

and all exponentially large terms must be absent from $f_r$.
The left hand side of the equation for $f_r$ has two complementary functions, one of which is exponentially large and the other is

$$\alpha_r \sum_1^{r+2} \frac{(-)^{s-1}(r+1)!}{s!(s-1)!(r-s+2)!} \eta^s \cdot \tag{7.17}$$

For $r=0$ the exponential term must vanish, and, since $f_0'(0)=0$ then $\alpha_0 = 0$ and $f_0 \equiv 0$. If $f_0 \equiv 0$ we find that the equation for $f_1$ has $\bar{f}_1 = 0$ and in a similar way we can see that $f_1 \equiv 0$ and so on.

We note here that the expansion for $F$ can contain only integral powers of $x$, for if it contained a term $x^k f_k(\eta)$ where $k$ is not an integer then the equation for $f_k$ would be

$$\eta f_k'' - \eta f_k' + (k+2) f_k = 0$$

The complementary functions of this equation are both exponentially large, the one corresponding to (7.17) now being

$$\alpha_k \sum_{1}^{\infty} \frac{(s-k-1)!}{s!\,(s-1)!\,(-k)!} \eta^s .$$

It might be supposed that it would be possible to combine the two complementary functions in such a way that the exponential parts would cancel for large $\eta$. However the leading terms in the other complementary function for small $\eta$ are a constant plus $\eta \log \eta$. Since $\frac{\partial}{\partial \eta}(\eta \log \eta)$ is equal to a constant plus $\log \eta$ we see that this function must disappear if $\frac{\partial f}{\partial \eta} = 0$ at $\eta = 0$, and so again $f_k \equiv 0$.

It is thus not possible to find a series solution which is valid throughout an inner region of the boundary layer surrounding the separation point and can be matched with the outer part by eliminating all exponentially large terms. It may, however, be possible to find a solution which has a series expansion valid throughout all but the most inner part of this region, while in this most inner part the solution has the wall condition imposed on it. From now on we shall speak of three regions of the boundary layer. In <u>Region I</u> we impose the condition that $\partial \tau / \partial u = 1$ at $u = 0$ and by comparing the leading terms we match this with the solution in <u>Region 2</u>. The solution in this region is assumed to have a series expansion. The leading terms for small $\eta$ are matched with the solution in Region I, and by eliminating exponentially large terms we make it possible to match this solution with a solution (which however we do not find) in <u>Region 3</u>, the outer part of the boundary layer.

In §VIII we will consider the solution in Region I. We take as leading terms

$$\tau = u + \tau_0(x), \tag{7.18}$$

where $\tau_0(x)$ is of order $x^n$ and find a perturbation solution. Retaining the lowest order terms the solution in this Region is

$$\frac{\tau}{x} = \eta + \frac{\tau_0}{x} + \tau_0' \eta \left[2 - \log\eta + \log\left(\frac{\tau_0}{x}\right)\right] + \cdots \tag{7.19}$$

when expressed in terms of $x$ and $\eta$. This will be matched with the series solution in Region 2, and the presence of the term $\log(\tau_0/x)$ in (7.19) suggests that the expansion in Region 2 will contain terms in $\log x$. In §IX we take an expansion in Region 2 of the form

$$\tau/x = \eta + x^{n-1}\left[(\log x)^p F_0 + (\log x)^{p-1} F_1 + \cdots\right]. \tag{7.20}$$

We find that $n$ must be an integer greater than unity and we take initially $n=2$. After solving the equations for $F_0$ and $F_1$ and eliminating all exponentially large terms we match this solution with that in Region 1 and find that $p=-1$. The addition of terms containing $\log\log x$ to (7.20) is found to be necessary to effect a match. Several more terms of (7.20) are found in §IX and by matching with (7.19) we obtain an expansion for $\tau_0(x)$, the leading term in the non-dimensional skin-friction. In §X we find the first few terms in the expansion for $\tau_1(x)$, the second term in the expansion of the non-dimensional skin friction.

## §VIII The Solution in Region 1

It will be remembered (7.14) that we took

$$\tau = u + F(x,u) \tag{8.1}$$

when written in the original independent variables. In Region 2 $u$ is the dominating term, but near $u=0$ it obviously cannot dominate and so in Region 1 the dominating term is taken to be $u+\tau_0(x)$ and we write

$$\tau = u + \tau_0(x) + f(x,u) \tag{8.2}$$

and assume that $f$ is small compared with $u+\tau_0$ when $u$ and $x$ are small. Substituting (8.2) into (7.7) and retaining only lowest order terms, the linearised equation for $f$ becomes

$$(u+\tau_0)^2 \frac{\partial^2 f}{\partial u^2} + u\,\tau_0' = 0, \tag{8.3}$$

where the prime denotes differentiation with respect to $x$. The boundary condition at the wall (7.8) means that

$$\frac{\partial f}{\partial u} = 0 \quad \text{at } u=0. \tag{8.4}$$

The solution of (8.3) subject to (8.4) is

$$f = \tau_0'\left[2u - (u+2\tau_0)\log(1+u/\tau_0)\right] + \tau_1(x). \tag{8.5}$$

If now $\tau_0$ is of order $x^n$ then when $u$ is of the same order of magnitude we will make the assumption, since we have taken $u+\tau_0(x)$ to be the leading term, that $f$ is of order $x^m$ where $m>n$. From (8.5) we see that

$$m = 2n-1,$$

and so if $m>n$, then $n>1$. We will see when we study the solution in Region 2 and attempt a match that $n$ must be an integer. The lowest value $n$ can take, therefore, is 2. If we put $u=\eta x$ and take leading terms in $x$ and $\eta$, then

$$\frac{\tau}{x} = \eta + \frac{\tau_0}{x} + \tau_0'\eta\left[2 - \log\eta + \log\left(\frac{\tau_0}{x}\right)\right] + \cdots, \tag{8.6}$$

where we have firstly neglected terms of order $x^n$ in (8.6) and then terms of order $\eta^2$ and higher, since

$$\log\left(1+\frac{\eta x}{\tau_0}\right) = \log\eta - \log\left(\frac{\tau_0}{x}\right) + \log\left(1+\frac{\tau_0}{x\eta}\right),$$

and $\log(1+\tau_0/x\eta)$ is of order $x^{n-1}/\eta$ which is of higher order in $x$ than the first two terms if $n>1$.

The solution in Region 1 is thus given by (8.2) where $f$ is given by (8.5) plus higher order terms. When referred to $(x, \eta)$ co-ordinates the leading terms of the solution which are to be matched with the solution in Region 2 are given by (8.6).

## §IX. The Solution in Region 2

Since we will have to match (8.6) with the solution in this region we can expect powers of $\log x$ to be included in the series for $\tau$. We thus assume a series of the form

$$\tau/x = \eta + x^{n-1}[(\log x)^p F_0(\eta) + (\log x)^{p-1} F_1(\eta) + \cdots] + O(x^n). \quad (9.1)$$

We shall show first that $n$ must be an integer.

Substitution in (7.15) and comparison of coefficients of $x(\log x)^p$ etc. results in the following equations for $F_0$ and $F_1$ :-

$$\eta F_0'' - \eta F_0' + n F_0 = 0 \quad ; \quad (9.2)$$

$$\eta F_1'' - \eta F_1' + n F_1 = -p F_0 \quad . \quad (9.3)$$

As mentioned in §VII when $n$ is not an integer both complementary functions of (9.2) are exponentially large and these must be combined to give a solution which is not exponentially large, thus the leading terms in $F_0$ are

$$F_0 = a - na\eta\log\eta + b\eta + \cdots \quad (9.4)$$

where $a$ and $b$ are two constants which are related in such a way that the exponentially large parts of the two complementary functions cancel. Thus neither $a$ nor $b$ must be zero, otherwise $F_0$ is zero. On comparing the coefficients of $\eta^0$, $\eta\log\eta$ and $\eta$ in equations (9.4) and (8.6) we obtain

$$\tau_0 = a x^n (\log x)^p + \cdots \quad ; \quad (9.5)$$

$$\tau_0' = a n x^{n-1} (\log x)^p + \cdots \quad ; \quad (9.6)$$

$$\tau_0'(\lambda + \log(\tau_0/x)) = b x^{n-1} (\log x)^p + \cdots \quad . \quad (9.7)$$

(9.5) and (9.6) are mutually compatible, but from (9.5) the leading term in $\tau_0'(\lambda + \log(\tau_0/x))$ is $an(n-1)x^{n-1}(\log x)^{p+1}$. Thus from (9.7) $a$ is zero, and, as mentioned above, this implies that $F_0$ is zero. The same applies for $F_1$ since equation (9.3) now has zero on the right hand side, and $F_r$ for all $r$ will be zero in a similar fashion. Hence $n$ must be an integer.

There seems to be no way of discovering the value of $n$ other than by comparison with the numerical solution. However, it was not found possible to compute near enough to the separation point for the numerical solution to overlap with the derived expansion. Since we decided in § VIII that $n$ is greater than unity, the lowest value it can take is 2. We will calculate the first five terms in the expansion if $n=2$, and then mention later (§ X) the form of the expansion for $n = 3$.

If $n=2$ equations (9.2) and (9.3) become

$$\eta F_0'' - \eta F_0' + 2F_0 = 0 \quad ; \tag{9.8}$$

$$\eta F_1'' - \eta F_1' + 2F_1 = -pF_0 \quad . \tag{9.9}$$

The solutions of (9.8) and (9.9), eliminating exponentially large terms are

$$F_0 = -\alpha(2\eta - \eta^2) \quad ; \tag{9.10}$$

$$F_1 = \alpha_1(2\eta - \eta^2) - p\alpha[(2\eta - \eta^2)\log\eta - 1 + 3\eta] \quad ; \tag{9.11}$$

where $\alpha$ and $\alpha_1$ are arbitrary constants. Taking leading terms for small $\eta$ we have

$$\tau/x = \eta + p\alpha x(\log x)^{p-1} - 2\alpha x\eta(\log x)^p - 2p\alpha x\eta\log\eta(\log x)^{p-1} + \ldots \quad . \tag{9.12}$$

Comparison with (8.6) gives

$$\tau_0 = p\alpha x^2(\log x)^{p-1} + \ldots \quad ; \tag{9.13}$$

$$\tau_0' = 2p\alpha x(\log x)^{p-1} + \ldots \quad ; \tag{9.14}$$

$$2\tau_0' + \tau_0'\log(\tau_0/x) = -2\alpha x(\log x)^p + \ldots \quad . \tag{9.15}$$

The first two are compatible, but substitution of (9.13) into (9.15) gives

$$2p\alpha x(\log x)^p = -2\alpha x(\log x)^p + \ldots \quad ,$$

and so $p = -1$.

We are now almost in a position to write down the form of the expansion for $\tau$ in Region 2. We note that since $\tau_0$ contains a power of $\log x$, then from (8.6) we can expect $\tau_0$ to contain terms involving powers of $\log\log x$ . By trial we find that the most general expansion for $\tau$ in Region 2 is

$$\frac{\tau}{x} = \eta + \frac{x}{\theta}\left[F_{00} + \frac{F_{10} + F_{11}\log\theta}{\theta} + \frac{F_{20} + F_{21}\log\theta + F_{22}(\log\theta)^2}{\theta^2} + \cdots\right]$$

where $\theta = \log(1/\alpha x)$ ; or

$$\frac{\tau}{x} = \eta + \frac{x}{\theta}\sum_{r=0}^{\infty}\frac{1}{\theta^r}\sum_{s=0}^{r}F_{rs}(\eta)(\log\theta)^s + O(x^2). \tag{9.16}$$

The only boundary condition we impose on the $F_{rs}$'s is that $F_{rs}$ does not contain any exponentially large terms. The wall condition has already been applied in Region 1, and the leading terms for small $\eta$ in (9.16) are matched with the leading terms in the solution in Region 1 which are

$$\frac{\tau}{x} = \eta + \frac{\tau_0}{x} + \tau_0'\eta\left[2 - \log\eta + \log\left(\frac{\tau_0}{x}\right)\right] + \cdots\;, \tag{8.6}$$

where

$$\tau_0 = \frac{x^2}{\theta^2}\left\{\alpha + \frac{a_{10} + a_{11}\log\theta}{\theta} + \cdots\right\}$$

or

$$\tau_0 = \frac{x^2}{\theta^2}\left\{\alpha + \sum_{r=1}^{\infty}\frac{1}{\theta^r}\sum_{s=0}^{r}a_{rs}(\log\theta)^s\right\} + O(x^2). \tag{9.17}$$

Our object will be to determine the $a_{rs}$'s in terms of $\alpha$.

Substituting (9.16) into (7.7) and comparing coefficients of $\frac{x}{\theta^r}(\log\theta)^s$ the equation for $F_{rs}$ is

$$\eta F_{rs}'' - \eta F_{rs}' + 2F_{rs} = \bar{F}_{rs} \tag{9.18}$$

where the $\bar{F}_{rs}$'s are given by the table below.

| r = | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\bar{F}_{r0}$ | 0 | $-F_{00}$ | $-2F_{10} + F_{11}$ | $-3F_{20} + F_{21}$ | $-4F_{30} + F_{31}$ |
| $\bar{F}_{r1}$ | - | 0 | $-2F_{11}$ | $-3F_{21} + 2F_{22}$ | $-4F_{31} + F_{32}$ |
| $\bar{F}_{r2}$ | - | - | 0 | $-3F_{22}$ | $-4F_{32} + F_{33}$ |
| $\bar{F}_{r3}$ | - | - | - | 0 | $-4F_{33}$ |
| $\bar{F}_{r4}$ | - | - | - | - | 0 |

so that

$$\bar{F}_{rs} = -rF_{r-1,s} + (s+1)F_{r-1,s+1} \quad ; \quad (s < r-1)$$

$$\bar{F}_{r,r-1} = -rF_{r-1,r-1} \;;$$

$$\bar{F}_{rr} = 0.$$

The Equation $\eta F'' - \eta F' + 2F = \bar{F}$ (9.19)

The equation has two complementary functions :-

$$g = 2\eta - \eta^2$$

and

$$h = (2\eta - \eta^2)\log\eta - 1 + 5\eta - 2\sum_{3}^{\infty}\frac{\eta^r}{r!(r-1)(r-2)} .$$

$h$ is exponentially large for large $\eta$ and when $\bar{F}=0$ we of course omit it. However, in general, a particular integral of (9.19) will also be exponentially large and we must include $h$ with a constant multiplier which is chosen so that the exponential parts will cancel.

A particular integral in general consists of two power series viz.

$$p = \sum_{r=0}^{\infty}\bar{a}_r\eta^r + \log\eta\sum_{r=0}^{\infty}\bar{b}_r\eta^r .$$

For simplicity we shall put $\bar{a}_2 = 0$, which just means altering the constant multiplier of $g$.

Since $\bar{F}$ has $\bar{F}(0) \neq 0$ in general the particular integral will have $\bar{a}_0 \neq 0$. If the solution with the exponentially large terms missing is

$$F = ag + p + bh , \tag{9.20}$$

where $a$ is an arbitrary constant, then $b$ is given by

$$F(0) = \bar{a}_0 - b .$$

To find $F(0)$ such that (9.20) should not contain any exponentially large terms we write $F = (2\eta - \eta^2)G$. Then

$$G'' + \left(-1 + \frac{2}{\eta} - \frac{2}{2-\eta}\right)G' = \frac{\bar{F}}{\eta(2\eta - \eta^2)}$$

and

$$G'\eta^2(2-\eta)^2 e^{-\eta} = -\int_{\eta}^{\infty}(2-\eta_1)e^{-\eta_1}\bar{F}(\eta_1)\,d\eta_1 .$$

By taking the integral from $\eta$ to $\infty$ we assure that $G'$ and therefore $G$ and $F$ are not exponentially large as $\eta \to \infty$.

Now since $G = F/(2\eta - \eta^2)$ then

$$\eta^2(2-\eta)^2 G' = \eta(2-\eta)F' - (2-2\eta)F . \tag{9.21}$$

Since, from (9.20) the leading terms for small $\eta$ for $F$ are

$$F = \bar{a}_0 - b + (\bar{b}_1 + 2b)\eta\log\eta + \cdots\cdot$$

then $\eta F' \to 0$ as $\eta \to 0$. Therefore, from (9.21)

$$F(0) = -\tfrac{1}{2}\lim_{\eta\to 0}\eta^2(2-\eta)^2 G'.$$

Therefore

$$F(0) = \tfrac{1}{2}\int_0^\infty (2-\eta_1)\bar{F}(\eta_1)e^{-\eta_1}d\eta_1\,.$$

If $\bar{F}$ is written in the form

$$\bar{F} = \sum_0^\infty \frac{a_r\eta^r}{r!} + \log\eta\sum_0^\infty \frac{b_r\eta^r}{r!}$$

where $b_0 = 0$, and if we assume in all that follows that we can exchange freely the order of integration and summation, then

$$\bar{F} = \frac{1}{2\pi i}\int_C\left(\sum_0^\infty \frac{a_r}{z^r} + \log\eta\sum_0^\infty \frac{b_r}{z^r}\right)\frac{e^{\eta z}}{z}dz$$

where $C$ is a contour embracing the origin. Thus

$$F(0) = \tfrac{1}{2}\cdot\frac{1}{2\pi i}\int_C \lim_{N\to\infty}\frac{1}{z}\left\{\sum_0^N \frac{a_r}{z^r}\int_0^\infty (2-\eta_1)e^{-\eta_1(1-z)}d\eta_1 + \sum_0^N \frac{b_r}{z^r}\int_0^\infty (2-\eta_1)\log\eta_1 e^{-\eta_1(1-z)}d\eta_1\right\}dz.$$

The $\lim\limits_{N\to\infty}$ has been introduced since some terms are divergent when taken singly. Thus

$$F(0) = \frac{1}{4\pi i}\lim_{N\to\infty}\int_C \frac{1}{z}\left\{\sum_0^N \frac{a_r}{z^r}\left(\frac{2}{1-z} - \frac{1}{(1-z)^2}\right) + \sum_0^N \frac{b_r}{z^r}\left(\frac{-2\gamma}{1-z} - \frac{2\log(1-z)}{1-z} + \frac{\log(1-z)}{(1-z)^2} - \frac{(1-\gamma)}{(1-z)^2}\right)\right\}dz$$

where now $|z| < 1$ on $C$ and $\gamma$ is Euler's Constant

$$\gamma = -\int_0^\infty e^{-t}\log t\,dt \doteq 0.5772.$$

Hence

$$F(0) = \tfrac{1}{2}\lim_{N\to\infty}\sum_{r=0}^N\left[a_r(1-r) + b_r\left\{\gamma(r-1) - 1 + (1-r)\sum_1^r \frac{1}{s}\right\}\right].$$

Although $\sum_1^r \frac{1}{s}$ is divergent we find in practice that

$$\lim_{N\to\infty}\sum_{r=0}^N b_r\left\{\gamma(r-1) - 1 + (1-r)\sum_1^r \frac{1}{s}\right\}$$ is convergent.

By this means all the $F_{rs}$'s may be obtained.

The following functions occur in the solutions :-

$$g_0 = 2\eta - \eta^2 ;$$

$$g_1 = (2\eta - \eta^2)\log\eta - 1 + 3\eta ;$$

$$g_2 = \left\{(2\gamma - 3)\eta + \left(\frac{5}{2} - \gamma\right)\eta^2 - 2\sum_3^\infty \frac{\eta^r}{r!(r-1)(r-2)}\right\}\log\eta$$
$$+ 1 - \gamma + \left(5\gamma - \frac{15}{2}\right)\eta + 2\sum_3^\infty \frac{\eta^r}{r!(r-1)(r-2)}\left\{-\gamma + \frac{1}{r-1} + \frac{1}{r-2} + \sum_1^r \frac{1}{s}\right\} ;$$

$$g_3 = \left[\left\{\frac{19}{2} - 8\gamma + 2\left(\gamma^2 + \frac{\pi^2}{6}\right)\right\}\eta + \left\{-\frac{25}{4} + 5\gamma - \left(\gamma^2 + \frac{\pi^2}{6}\right)\right\}\eta^2 + \sum_3^\infty \frac{\eta^r}{r!(r-1)(r-2)}\left\{5 - 2\gamma - 2\sum_1^{r-3}\frac{1}{s}\right\}\right]\log\eta$$
$$-\frac{1}{4}\left[17 - 14\gamma + 4\left(\gamma^2 + \frac{\pi^2}{6}\right)\right] + \left[\frac{83}{4} - 18\gamma + 5\left(\gamma^2 + \frac{\pi^2}{6}\right)\right]\eta$$
$$+ \sum_3^\infty \frac{\eta^r}{r!(r-1)(r-2)}\left\{-2\left(\gamma^2 + \frac{\pi^2}{6}\right) + 5\gamma + \frac{2\gamma + 3}{r} + \frac{4\gamma - 3}{r-1} + \frac{4\gamma - 10}{r-2}\right.$$
$$\left. - \frac{2}{r^2} - \frac{4}{(r-1)^2} - \frac{4}{(r-2)^2} - \left(5 + \frac{2}{r}\right)\sum_1^r \frac{1}{s} + 4\sum_1^r \frac{1}{s}\sum_1^s \frac{1}{t}\right\} ;$$

$$g_4 = \frac{35}{4} - 13\gamma + 6\gamma^2 + \left(\frac{7}{2} - 2\gamma\right)\frac{\pi^2}{6} - \gamma^3 + \zeta(3) + O(\eta\log\eta) ;$$

where $\zeta(s) = \sum_{n=1}^\infty n^{-s}$.

In terms of these functions the $F_{rs}$'s up to $r = 4$ are

$F_{00} = -\alpha g_0$ .

$F_{10} = \alpha_{10} g_0 - \alpha g_1$ .

$F_{11} = \alpha_{11} g_0$ .

$F_{20} = \alpha_{20} g_0 + (2\alpha_{10} - \alpha_{11}) g_1 + 2\alpha g_2$ .

$F_{21} = \alpha_{21} g_0 + 2\alpha_{11} g_1$ .

$F_{22} = \alpha_{22} g_0$ .

$F_{30} = \alpha_{30} g_0 + (3\alpha_{20} - \alpha_{21}) g_1 + (5\alpha_{11} - 6\alpha_{10}) g_2 - 6\alpha g_3$ .

$F_{31} = \alpha_{31} g_0 + (3\alpha_{21} - 2\alpha_{22}) g_1 - 6\alpha_{11} g_2$ .

$F_{32} = \alpha_{32} g_0 + 3\alpha_{22} g_1$ .

$F_{33} = \alpha_{33} g_0$ .

$F_{40} = \alpha_{40} g_0 + (4\alpha_{30} - \alpha_{31}) g_1 + (-12\alpha_{20} + 7\alpha_{21} - 2\alpha_{22}) g_2 + (-26\alpha_{11} + 24\alpha_{10}) g_3 + 24\alpha g_4$ .

$F_{41} = \alpha_{41} g_0 + (4\alpha_{31} - 2\alpha_{32}) g_1 + (-12\alpha_{21} + 14\alpha_{22}) g_2 + 24\alpha_{11} g_3$ .

$F_{42} = \alpha_{42} g_0 + (4\alpha_{32} - 3\alpha_{33}) g_1 - 12\alpha_{22} g_2$ .

$F_{43} = \alpha_{43} g_0 + 4\alpha_{33} g_1$ .

$F_{44} = \alpha_{44} g_0$ .

The $\alpha_{rs}$'s are abitrary constants.

Substituting these relationships into the expansion (9.16) we can find the coefficient of $\eta^0$ and $\eta$ in $\tau/x$ up to order $\frac{x}{\theta^3}$. These are equated to the corresponding coefficients from Region 1, which, from (8.6) are $\frac{\tau_0}{x}$ and $1+\tau_0'(2+\log\tau_0/x)$ respectively, where $\tau_0$ has the expansion (9.17). To demonstrate this we will find the constants $a_{10}$ and $a_{11}$ in (9.17).

The coefficient of $\eta^0$ is $\tau_0/x$ and

$$\frac{\tau_0}{x} = \frac{x}{\theta^2}\left(\alpha + \frac{1}{\theta}(a_{10}+a_{11}\log\theta) + \cdots\right). \tag{9.22}$$

The coefficient of $\eta$ is

$$\begin{aligned} 1 + \tau_0'(2+\log\tau_0/x) &= 1 + \frac{2x}{\theta^2}\left[\alpha + \frac{1}{\theta}(\alpha + a_{10}+a_{11}\log\theta) + \cdots\right]\left[-\theta + 2 - 2\log\theta + \cdots\right] \\ &= 1 + \frac{2x}{\theta^2}\left[-\alpha\theta + \alpha - a_{10} - (2\alpha + a_{11})\log\theta + \cdots\right]. \end{aligned} \tag{9.23}$$

In Region 2 we have from (9.16)

$$\frac{\tau}{x} = \eta + \frac{x}{\theta}\left[F_{00} + \frac{F_{10}+F_{11}\log\theta}{\theta} + \frac{F_{20}+F_{21}\log\theta + F_{22}(\log\theta)^2}{\theta^2}\cdots\right]. \tag{9.16}$$

The coefficient of $\eta^0$, using the relationships for the $F_{rs}$'s given above, is

$$\frac{x}{\theta^2}\left[\alpha + \frac{1}{\theta}(2\alpha(1-\gamma) - (2\alpha_{10} - \alpha_{11}) - 2\alpha_{11}\log\theta) + \cdots\right] \quad , \tag{9.24}$$

and the coefficient of $\eta$ is

$$1 + \frac{x}{\theta}\left[-2\alpha + \frac{1}{\theta}(2\alpha_{10} - 3\alpha + 2\alpha_{11}\log\theta) + \cdots\right]. \tag{9.25}$$

Comparing (9.22) and (9.24) we have

$$a_{10} = 2\alpha(1-\gamma) - (2\alpha_{10} - \alpha_{11}) \quad ;$$

$$a_{11} = -2\alpha_{11} .$$

Also, from (9.23) and (9.25)

$$2\alpha - 2a_{10} = 2\alpha_{10} - 3\alpha \quad ;$$

$$-2\alpha - a_{11} = \alpha_{11} .$$

From these four equations we find

$$\alpha_{11} = 2\alpha \quad ; \quad \alpha_{10} = \alpha\left(\tfrac{3}{2} - 2\gamma\right) \,,$$

and

$$a_{11} = -4\alpha \; ; \quad a_{10} = \alpha\,(2\gamma + 1)\,.$$

By this means we may find all the $a_{rs}$'s up to $a_{3s}$ and all the $\alpha_{rs}$'s up to $\alpha_{3s}$ from (9.16), (9.17) and the expressions for the $F_{rs}$'s given above. The expression we find for $\tau_0$ is

$$\tau_0 = \alpha\frac{x^2}{\theta^2}\Big[1 + \frac{1}{\theta}\left(1+2\gamma-4\log\theta\right) + \frac{1}{\theta^2}\left\{-1+7\gamma+3\gamma^2-\tfrac{1}{2}\pi^2-(14+12\gamma)\log\theta+12(\log\theta)^2\right\}$$
$$+\frac{1}{\theta^3}\Big\{-\frac{14}{3}+10\gamma+20\gamma^2-\left(\frac{26}{9}+2\gamma\right)\pi^2+4\gamma^3-8\zeta(3)$$
$$+(-20-80\gamma-24\gamma^2+4\pi^2)\log\theta+(80+48\gamma)(\log\theta)^2-32(\log\theta)^3\Big\}$$
$$+O\left(\frac{1}{\theta^4}\right)\Big] + O(x^3)\,, \qquad (9.26)$$

or

$$\tau_0 = \alpha\frac{x^2}{\theta^2}\Big[1+\frac{1}{\theta}\left(2.1544-4\log\theta\right)+\frac{1}{\theta^2}\left\{-0.8948-20.9266\log\theta+12(\log\theta)^2\right\}$$
$$+\frac{1}{\theta^3}\left\{-40.9541-34.6951\log\theta+107.7064(\log\theta)^2-32(\log\theta)^3\right\}$$
$$+O\left(\frac{1}{\theta^4}\right)\Big] + O(x^3)\,. \qquad (9.27)$$

## §X Extension to higher powers of $x$.

The complete expansion for $\tau$ near the separation point will be of the form

$$\frac{\tau}{x} = \eta + x f_1(\eta,\theta) + x^2 f_2(\eta,\theta) + O(x^3). \tag{10.1}$$

The leading terms for $f_1$ have been found above.

Substituting (10.1) into (7.7) the equation for $f_2$ is found to be

$$\eta \frac{\partial^2 f_2}{\partial \eta^2} - \eta \frac{\partial f_2}{\partial \eta} + 3 f_2 = \frac{\partial f_2}{\partial \theta} - 2 f_1 \frac{\partial^2 f_1}{\partial \eta^2}. \tag{10.2}$$

In Region 1, if we add another term of order $(x^4)$ to the series (8.2), solve, and take out the leading terms we find that

$$\frac{\tau}{x} = \eta + x\left[\frac{\tau_0}{x^2} + \frac{\tau_0'}{x}\eta\left(2 - \log\eta + \log\left(\frac{\tau_0}{x}\right)\right) + O(\eta^2)\right]$$

$$+ x^2\left[\frac{\tau_1}{x^3} + \frac{\tau_0\tau_0'}{x^3}\left(2\log\left(\frac{\tau_0}{x}\right) \pm 1\right) - 2\frac{\tau_0\tau_0'}{x^3}\log\eta\right.$$

$$+\eta\left\{\frac{\tau_1'}{x^2}\left(\log\left(\frac{\tau_0}{x}\right)+2\right) + \frac{\tau_0'\tau_1}{x^2\tau_0} - 5\left(\frac{\tau_0\tau_0''}{x^2} + \frac{\tau_0'^2}{x^2}\right)\log\left(\frac{\tau_0}{x}\right) - 12\frac{\tau_0'^2}{x^2} - \frac{15}{2}\frac{\tau_0\tau_0''}{x^2}\right\}$$

$$\left. + \eta\log\eta\left\{-\frac{\tau_1'}{x^2} + 5\frac{\tau_0\tau_0''}{x^2} + 5\frac{\tau_0'^2}{x^2}\right\} + O(\eta^2)\right]$$

$$+ O(x^3) \quad , \tag{10.3}$$

where $\tau_1 = \tau_1(x) = O(x^3)$ so that when $\alpha = 0$

$$\tau = \tau_0 + \tau_1 + O(x^4).$$

In Region 2, if we assume a series for $f_2$ of the form

$$f_2 = \frac{1}{\theta^m}\left[G_{00}(\eta) + \frac{G_{10} + G_{11}\log\theta}{\theta} + \cdots\right] \tag{10.4}$$

we note first that, since $f_1 \dfrac{\partial^2 f_1}{\partial \eta^2} = \dfrac{F_{00}F_{00}''}{\theta^2} + O\left(\dfrac{1}{\theta^3}\right)$, then from (10.2) we must have $m \leq 2$ if $\alpha \neq 0$.

Adopting the same procedure as in §IX we find that

$$m = 1.$$

The equation for $G_{rs}$ is

$$\eta G_{rs}'' - \eta G_{rs}' + 3 G_{rs} = \bar{G}_{rs} \tag{10.5}$$

where $\bar{G}_{rs}$ is some function of the previous $G$'s and some of the $F$'s.

Again solving for the $G_{rs}$'s and comparing with (10.3) we find

$$G_{00} = -\beta(6\eta - 6\eta^2 + \eta^3) \ ,$$

$$G_{10} = \beta_{10}(6\eta - 6\eta^2 + \eta^3) + 4\alpha^2(2\eta - \eta^2) - \beta\left[(6\eta - 6\eta^2 + \eta^3)\log\eta - 2 + 14\eta - 5\eta^2\right] ,$$

and so on, where $\beta$ and $\beta_{10}$ are constants, and

$$\tau_1 = 2\beta\frac{x^3}{\theta^2}\left[1 + \frac{2\gamma + \frac{4}{3} \mp 4\log\theta}{\theta} + O\left(\frac{1}{\theta^2}\right)\right] . \qquad (10.6)$$

## Conclusion

The laminar boundary layer equations have been studied for the case of incompressible two dimensional flow over a flat plate when there is normal injection of fluid through the plate at a constant rate. It was found numerically (Part III) that the skin friction vanishes when $x$ equals about ~~1.4~~ 0.74. The solution in the neighbourhood of the separation point was considered and it was found that the usual technique as used by Goldstein,[12] Stewartson[13] and Terrill[14] for cases where there is a pressure gradient in the main stream broke down in this example. It was found necessary to divide the boundary layer into three regions. The solution valid in the first region was made to satisfy the wall condition and then matched with a series solution valid in a second region, which in turn was matched with the external solution (which was not found) by eliminating all exponentially large terms from each term in the series. By this means the first few terms in the expansion for the skin friction in the neighbourhood of the separation point were found.

## Part III

## The Numerical Solution

### §XI Introduction

The numerical integration was performed in three parts. Firstly a method due to Iglisch[10] was used (§XII). This method permits the integration to be started at the leading edge, but is not very suitable for use with an automatic computor, as it is difficult to make it fully automatic. After a few steps instability set in near the upper boundary (main stream) and so after $x = 0.005$ Leigh's[11] method was used (§XIII). This involves an implicit difference scheme which is known to be stable for any choice of step length in either the $x$ or $y$ directions, but was found to be very time-consuming as $x$ increased, so that after $x = 0.5$ the boundary layer equations were transformed by Crocco's method (see Ref. 15) and a similar method to Leigh's used to integrate this new equation (§XIV). This was much faster and no difficulties were encountered up to $x = 0.65$ after which the method appeared to be unstable for small step lengths in the $x$ direction. Up to $x = 0.65$ the results are accurate to four decimal places, but cannot be guaranteed to more than three decimal places (one significant figure in the case of the skin friction) past this. The separation point was estimated to be at about $x = 0.74$ but it was not possible to get near enough to it to make comparisons with the series for the skin friction given in Part II.

## §XII The Numerical Integration from $x=0$ to $x=0.005$.

The method of Iglisch[10] transforms the boundary layer equations into one equation, first using the stream function $\psi$ as independent variable in addition to $x$, where

$$u = \frac{\partial \psi}{\partial y} \; ; \; v = -\frac{\partial \psi}{\partial x} \; . \tag{12.1}$$

This satisfies the continuity equation (6.5), and the boundary conditions become

$$\left.\begin{aligned} u(x,-x) &= 0 \; , \\ u(x,\infty) &= u(0,\psi) = 1 \; , \end{aligned}\right\} \tag{12.2}$$

since

$$\begin{aligned} \psi &= -x + \int_0^y u(x,y_1)\,dy_1 \; , \\ &= -x \text{ when } y=0 \text{ , and} \\ \psi &= y - \int_0^x v(x_1,y)\,dx_1 \; , \end{aligned}$$

which tends to infinity as $y$ tends to infinity. We now put

$$V = 4u^2 \; , \tag{12.3}$$

and

$$\phi = \psi + x \tag{12.4}$$

in order to obtain boundaries of integration which are rectangular. The momentum equation (6.4) now becomes

$$\frac{\sqrt{V}}{2}\frac{\partial^2 V}{\partial \phi^2} = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial \phi} \; , \tag{12.5}$$

with boundary conditions

$$\left.\begin{aligned} V(x,0) &= 0 \; , \\ V(x,\infty) &= V(0,\phi) = 4 \; . \end{aligned}\right\} \tag{12.6}$$

To obtain an equation suitable for numerical integration from the leading edge we make one final substitution :

$$\tau = 2^{\frac{1}{4}}\sqrt{\frac{\phi}{\sqrt{x}}} \; , \quad \sigma = -\sqrt{2x} \; . \tag{12.7}$$

This $\tau$ should not be confused with the $\tau$ used in Part II; it is used here only to be consistent with Iglisch's original notation. (12.5) and (12.6) now become

$$\sqrt{V}\frac{\partial^2 V}{\partial \tau^2} + \frac{\partial V}{\partial \tau}\left(2\sigma\tau + \tau^3 - \frac{\sqrt{V}}{\tau}\right) - 2\tau^2\sigma\frac{\partial V}{\partial \sigma} = 0 \; , \tag{12.8}$$

$$
\left.\begin{aligned}
V(\sigma, 0) &= 0 \quad , \\
V(\sigma, \infty) &= 4 \quad .
\end{aligned}\right\} \tag{12.9}
$$

To reduce this to an ordinary differential equation $\frac{\partial V}{\partial \sigma}$ is replaced by the difference quotient $\frac{V-\bar{V}}{k}$ where $\bar{V}$ is the known solution at $\sigma - k$ and $k$ is small. The resulting non-linear ordinary differential equation is, for $\sigma = \sigma_n$

$$
\sqrt{V}\,\frac{d^2V}{d\tau^2} + \frac{dV}{d\tau}\left(2\sigma\tau - \tau^3 - \frac{\sqrt{V}}{\tau}\right) = \frac{2\tau^2\sigma_n}{k}(V - \bar{V}). \tag{12.10}
$$

This was solved by a step-by-step method of profile continuation using Adam's method. The integration formulae, neglecting fifth and higher differences are

$$
\left.\begin{aligned}
V_{i+1} &= V_i + h\left[2.6402778\,V_i' - 3.8527778\,V_{i-1}' + 3.6333\,V_{i-2}' - 1.769444\,V_{i-3}' \right. \\
&\qquad \left. + 0.3486111\,V_{i-4}'\right] \quad ; \\
V_{i+1}' &= V_i' + h\left[2.6402778\,V_i'' - 3.8527778\,V_{i-1}'' + 3.6333\,V_{i-2}'' - 1.769444\,V_{i-3}'' \right. \\
&\qquad \left. + 0.3486111\,V_{i-4}''\right] \quad ; \\
V_{i+1}'' &= \frac{1}{\sqrt{V_{i+1}}}\left[\frac{2\tau_{i+1}^2\sigma_n}{k}\left(V_{i+1} - \bar{V}_{i+1}\right) + V_{i+1}'\left(-2\sigma_n\tau_{i+1} - \tau_{i+1}^3 + \frac{\sqrt{V_{i+1}}}{\tau_{i+1}}\right)\right] \quad ,
\end{aligned}\right\} \tag{12.11}
$$

where $i$ refers to steps in the $\tau$ direction, $k$ is the interval in this direction and the primes denote differentiation with respect to $\tau$.

One further boundary condition is necessary. Now

$$
\left(\frac{\partial V}{\partial \tau}\right)_\sigma = -\tau\sigma\left(\frac{\partial V}{\partial \phi}\right)_x = -\tau\sigma\left(\frac{\partial V}{\partial \psi}\right)_x = -\gamma\tau\sigma\left(\frac{\partial u}{\partial y}\right)_x
$$

where $\left(\frac{\partial V}{\partial \phi}\right)_x$ indicates differentiation keeping $x$ constant. Since $\left(\frac{\partial u}{\partial y}\right)_x$ is finite when $y=0$ and $\tau=0$ when $y=0$, therefore

$$
\frac{\partial V}{\partial \tau} = 0 \quad \text{at } \tau = 0. \tag{12.12}
$$

This should be added to the boundary conditions (12.9).

In order to set the integration going we expand $V$ etc. in a power series in $\tau$ near $\tau = 0$:

$$
\left.\begin{aligned}
V &= d_2\tau^2 + d_3\tau^3 + d_4\tau^4 + \cdots \quad , \\
\sqrt{V} &= c_1\tau + c_2\tau^2 + c_3\tau^3 + \cdots \quad , \\
\bar{V} &= b_2\tau^2 + b_3\tau^3 + b_4\tau^4 + \cdots \quad .
\end{aligned}\right\} \tag{12.13}
$$

The $b_r$'s are assumed to be known, and by substituting in (12.10) and comparing coefficients of like powers of $\tau$ we find that

$$d_3 = -\frac{4\sigma_n}{3c_1} d_2 \;; \; d_4 = -\frac{3(2\sigma_n + c_2)d_3}{8c_1} \;;$$

etc., also

$$c_1 = \sqrt{d_2} \;; \; c_2 = d_3/2c_1 \;;$$

etc., so that all the $c_r$'s and $d_r$'s may be found in terms of $d_2$ and known constants. After selecting $d_2$ the first five values of $V$, $V'$ and $V''$ may be calculated from (12.13) and then formulae (12.11) used.

The procedure for each step is to guess a value of $d_2$, calculate the first five values of $V$, $V'$ and $V''$ from (12.13) and use these as initial values in the formulae (12.11). $d_2$ is varied until $V \to H$ as $\tau \to \infty$.

At $\sigma = 0$ (12.10) reduces to

$$\sqrt{V}\frac{d^2V}{d\tau^2} + \frac{dV}{d\tau}\left(\tau^3 - \frac{\sqrt{V}}{\tau}\right) = 0 \tag{12.14}$$

with

$$\left.\begin{aligned} &V = dV/d\tau = 0 \text{ at } \tau = 0 \;, \\ &V \to H \text{ as } \tau \to \infty \;. \end{aligned}\right\} \tag{12.15}$$

Equation (12.14) may be solved in a similar way to that outlined above, or, by suitable substitution, may be transformed into Blasius' equation, the solution of which is known.

Equation (12.8) was solved by this method at $\sigma = 0, -0.05, -0.1$ using the Durham University Ferranti Pegasus Computer. At this point instability set in for large values of $\tau$ and it was decided to proceed no further with this method. The solution at $\sigma = -0.1\,(x = 0.005)$ was converted to the original variables $u$, $v$, $x$ and $y$ so as to act as a starting profile for an integration using Leigh's method. The conversion formulae are

$$\left.\begin{aligned} u &= \frac{\sqrt{V}}{2} \;, \\ v &= \frac{1}{\sigma}\left[-\frac{\tau^2}{2} - \sigma - \frac{\sqrt{V}}{2}\left(\frac{\partial y}{\partial \sigma}\right)_\tau\right] \;, \\ x &= \frac{\sigma^2}{2} \;, \\ y &= -2\sigma\int_0^\tau \frac{\tau_1}{\sqrt{V}}\, d\tau_1 \;. \end{aligned}\right\} \tag{12.16}$$

The above method was used by Lew and Fanucci[9] for the same problem and they obtained profiles up to $\sigma = -0.7 (x = 0.245)$. These results agree with the ones we find in §XIII.

## §XIII The Numerical Integration from $x = 0.005$ to $x = 0.5$

The simplest method of integrating equations of parabolic type is to replace derivatives in the $x$ direction by forward differences and derivatives in the $y$ direction by central differences so that if we know the solution $u_1$, $v_1$ at $x = x_1$ we may find the values of $u$ and $v$ at $x = x_2$ for each value of $y$ directly from known values. A stability condition connecting the step lengths in the $x$ and $y$ directions has to be observed and in our case this is

$$\delta x < \tfrac{1}{2} u (\delta y)^2, \tag{13.1}$$

where $\delta x$ and $\delta y$ are the step lengths in the $x$ and $y$ directions. Since $u = 0$ at the wall the method will be unstable near the wall for any choice of step lengths.

Richtmyer[16] considers a general implicit difference scheme for solving the simplest parabolic equation, the heat conduction equation, in which derivatives in the $x$ direction, $\partial f / \partial x$, say, are replaced by

$$\frac{\partial f}{\partial x} = \frac{f_2 - f_1}{\delta x}, \tag{13.2}$$

and other quantities by weighted means :

$$f = (1 - \theta) f_1 + \theta f_2 , \tag{13.3}$$

where $\theta$ is a constant between zero and unity. If $\theta = 0$ the method degenerates to the explicit method mentioned above, but for $\frac{1}{2} \leq \theta \leq 1$ the method can be shown to be unconditionally stable. Furthermore if $\theta = \frac{1}{2}$ the truncation errors are less than for any other value of $\theta$. While the same result has not yet been proved for a general parabolic equation it is generally believed that the same conclusions may be drawn. The main drawback with an implicit scheme is that all the values at $x = x_2$ have to be found simultaneously, which usually involves an iteration process since the equations are not linear, and so the method consumes much more time and machine space than the explicit method. However, for a stable process there is no alternative and so this method, with $\theta = \frac{1}{2}$ was used.

The complete method used was

first given by Leigh[11], who in addition eliminates $v$ from equations (6.4), (6.5), which means in our case replacing $v$ in (6.4) by

$$v = 1 - \int_0^y \frac{\partial u}{\partial x}\, dy \tag{13.4}$$

from (6.5). Equation (6.5) now becomes

$$\tfrac{1}{2}(u_1+u_2)\frac{(u_2-u_1)}{x_2-x_1} + \left(1 - \int_0^y \frac{u_2-u_1}{x_2-x_1}\, dy\right)\tfrac{1}{2}(u_2'+u_1') = \tfrac{1}{2}(u_2''+u_1''), \tag{13.5}$$

where the primes denote differentiation with respect to $y$. Putting

$$w = u_2 + u_1 \tag{13.6}$$

we get

$$w'' - w'\left\{1 - \frac{1}{x_2-x_1}\int_0^y (w-2u_1)\, dy\right\} - \frac{1}{x_2-x_1}(w-2u_1)w = 0, \tag{13.7}$$

with boundary conditions

$$w(0) = 0,\; w(\infty) = 2. \tag{13.8}$$

An iteration process is now introduced which is found to converge. If $w_m$ is an approximation to the solution of (13.7) then let a function $w_{m+1}$ be determined by the linear third order differential equation

$$w''_{m+1} - w'_m\left\{1 - \frac{1}{x_2-x_1}\int_0^y (w_{m+1}-2u_1)\, dy\right\} - \frac{1}{x_2-x_1}(w_{m+1}-2u_1)w_m = 0. \tag{13.9}$$

Now make the following substitutions :

$$\left.\begin{aligned}
&w''_{m+1,j} = \frac{1}{h^2}\left(w_{m+1,j+1} - 2w_{m+1,j} + w_{m+1,j-1}\right) \;; \\
&\int_0^{jh} w_{m+1}\, dy = h\left(w_{m+1,1} + w_{m+1,2} + \cdots + w_{m+1,j-1} + \tfrac{1}{2}w_{m+1,j}\right); \\
&\quad \text{since } w_{m+1,0} = 0 \text{ from (13.8)}, \\
&w'_{m,j} = \frac{1}{2h}\left(w_{m,j+1} - w_{m,j-1}\right) = \frac{\alpha_j}{h} \;; \\
&\int_0^{jh} 2u_1\, dy = h\left(\delta_1 + \delta_2 + \cdots + \delta_{j-1} + \tfrac{1}{2}\delta_j\right) = h\gamma_j \;; \\
&\text{where } 2u_{1,j} = \delta_j, \text{ and} \\
&\beta = (x_2-x_1)/h^2,
\end{aligned}\right\} \tag{13.10}$$

where $h$ is the step length in the $y$ direction and $j$ refers to the $j^{th}$ mesh point in the $y$ direction. Substituting the relations (13.10) into (13.9) and re-arranging the order of the terms we have for the $j^{th}$ mesh point

$$\alpha_j w_{m+1,1} + \alpha_j w_{m+1,2} + \cdots + \alpha_j w_{m+1,j-2} + A_j w_{m+1,j-1} + B_j w_{m+1,j} + \beta w_{m+1,j+1} = C_j \,, \tag{13.11}$$

where

$$\left.\begin{aligned} A_j &= \alpha_j + \beta \,, \\ B_j &= \tfrac{1}{2}\alpha_j - 2\beta - w_{m,j} \,, \\ C_j &= \alpha_j \gamma_j + \beta h \alpha_j - \delta_j w_{m,j} \,. \end{aligned}\right\} \tag{13.12}$$

At the lower end of the range, i.e. at $j=1$ we have

$$B_1 w_{m+1,1} + \beta w_{m+1,2} = C_1 \,, \tag{13.13}$$

since $w_{m+1,0} = 0$ from (13.8).

We cannot, of course, integrate out to $y=\infty$ and so we assume the second boundary condition of (13.8) is satisfied, within the limits of the accuracy of the integration, at $y=(N+1)h$ so that

$$w_{m+1,N+1} = 2 \,. \tag{13.14}$$

Hence the equation for the $N^{th}$ mesh point is

$$\alpha_N w_{m+1,1} + \cdots + \alpha_N w_{m+1,N-2} + A_N w_{m+1,N-1} + B_N w_{m+1,N} = C_N - 2\beta \,. \tag{13.15}$$

Writing the set of $N$ simultaneous equations (13.11), (13.13) and (13.15) in matrix form we have

$$\underset{\sim}{A}\, \underset{\sim}{W}_{m+1} = \underset{\sim}{C} \tag{13.16}$$

where

$$\underset{\sim}{A} = \begin{pmatrix} B_1 & \beta & 0 & 0 & 0 & - & - & - \\ A_2 & B_2 & \beta & 0 & 0 & - & - & - \\ \alpha_3 & A_3 & B_3 & \beta & 0 & - & - & - \\ \alpha_4 & \alpha_4 & A_4 & B_4 & \beta & - & - & - \\ \vdots & \vdots & \vdots & \vdots & \vdots & - & - & - \\ \alpha_{N-1} & \alpha_{N-1} & - & - & - & A_{N-1} & B_{N-1} & \beta \\ \alpha_N & \alpha_N & - & - & - & \alpha_N & A_N & B_N \end{pmatrix} \tag{13.17}$$

and

$$\underset{\sim}{W}_{m+1} = \begin{bmatrix} w_{m+1,1} \\ w_{m+1,2} \\ \vdots \\ w_{m+1,N-1} \\ w_{m+1,N} \end{bmatrix} \;; \qquad \underset{\sim}{C} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{N-1} \\ C_N - 2\beta \end{bmatrix} \,. \tag{13.18}$$

The equation (13.16) may be solved more easily if we first of all factorise $\underset{\sim}{A}$ by Choleski's method (see Ref. 17). We write $\underset{\sim}{A} = \underset{\sim}{L}\,\underset{\sim}{U}$ where $\underset{\sim}{L}$ is lower triangular and $\underset{\sim}{U}$ is upper triangular, and first solve

$$\underset{\sim}{L}\,\underset{\sim}{Y} = \underset{\sim}{C} \tag{13.19}$$

and then

$$\underset{\sim}{U}\,\underset{\sim}{W}_{m+1} = \underset{\sim}{Y}\,. \tag{13.20}$$

It will be seen later that by this method far fewer numbers need to be stored by the machine at any one time. In our case $\underset{\sim}{L}$ and $\underset{\sim}{U}$ may be expressed in the form

$$\underset{\sim}{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & - \\ l_{21} & 1 & 0 & 0 & - \\ l_{31} & l_{32} & 1 & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \quad \underset{\sim}{U} = \begin{bmatrix} u_1 & \beta & 0 & 0 & - \\ 0 & u_2 & \beta & 0 & - \\ 0 & 0 & u_3 & \beta & - \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{13.21}$$

where

$$\left.\begin{aligned} & u_j = B_j - \beta l_{j,j-1} \ (j>1)\ ; \ u_1 = B_1 \ , \\ & l_{j,j-1} = \frac{A_j - \beta l_{j,j-2}}{u_{j-1}} \quad (j>2)\ ; \quad l_{21} = \frac{A_2}{u_1}\ , \\ & l_{j,s} = \frac{\alpha_j - \beta l_{j,s-1}}{u_s} \quad \begin{pmatrix} 1<s<j-1 \\ j>2 \end{pmatrix}\ ; \ l_{j,1} = \frac{\alpha_j}{u_1} \quad (j>2). \end{aligned}\right\} \tag{13.22}$$

$\underset{\sim}{Y}$ is the column vector

$$\underset{\sim}{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \tag{13.23}$$

and

$$y_j = C_j - \sum_{s=1}^{j-1} l_{j,s}\, y_s \quad (j<N)\ ; \quad y_N = C_N - 2\beta - \sum_{s=1}^{N-1} l_{N,s}\, y_s\,. \tag{13.24}$$

The equation (13.20) now has the solution

$$w_{m+1,N} = \frac{y_N}{u_N}\ ; \qquad w_{m+1,j} = \frac{y_j - \beta w_{m+1,j+1}}{u_j} \quad (j<N). \tag{13.25}$$

It can now be seen that very little storage space is needed during the solving of (13.16), since $y_j$ and $u_j$ can be calculated from the same set of $l_{js}$'s at each step and only $\underset{\sim}{U}$ and $\underset{\sim}{Y}$ need be stored (i.e. $2N$ numbers as compared with the $N^2$ members of $\underset{\sim}{A}$). $\underset{\sim}{W}_{m+1}$ may then be calculated very simply from (13.25).

The above method was programmed for the Ferranti Pegasus computer, the set of operations being, once the step lengths have been decided :

(i) Feed in $u_1$ from the last step (initially the profile at $x=0.005$).

(ii) Set $w_m = w_1 = 2u_1$ .

(iii) Calculate $w_{m+1}$ as above.

(iv) If $\max|w_{m+1,j} - w_{m,j}| > 10^{-6}$ say, return to (iii) with the ~~near~~ new values of $w_m$ ; if not, calculate $u_2$ and return to (ii) for the next step.

Richardson[18] showed that the truncation error arising from using central differences may often be reduced by using a process known as $h^2$-extrapolation. This is the case when the approximation to a function $f(\eta)$ due to using a step length can be expressed in the form of a series

$$\phi(\eta,h) = f(\eta) + h^2 f_2(\eta) + h^4 f_4(\eta) + \cdots \quad , \tag{13.26}$$

odd powers of $h$ being absent. $\phi(\eta, h)$ is the solution of the finite difference equation when the step length is $h$ and $f_2(\eta)$ , $f_4(\eta)$ etc. are generally un-known. If this is the case then we can repeat the integration using a step length $2h$ obtaining

$$\phi(\eta,2h) = f(\eta) + 4h^2 f_2(\eta) + \cdots \tag{13.27}$$

so that $f_2(\eta)$ may be eliminated and a better approximation to $f(\eta)$ will be

$$f(\eta) = \phi(\eta,h) + \tfrac{1}{3}[\phi(\eta,h) - \phi(\eta,2h)] \quad , \tag{13.28}$$

the error now being of order $h^4$ instead of order $h^2$.

This technique was used in the $y$ direction, and was found indeed to eliminate the leading term in the truncation error. The procedure which was adopted at each step was as follows :

(i) Solve with $y_j - y_{j-1} = 2h$ taking $w_m = 2u_1$ as a first approximation and obtain the solution $u_2(y, 2h)$.

(ii) Interpolate in $u_2(y, 2h)$ obtaining additional values for $u_2$ at the points $y_{j+\frac{1}{2}}$.

(iii) Solve with $y_j - y_{j-1} = h$ using the results of (ii) to give a starting set of values for $\omega_m$ and obtain $u_2(y,h)$.

(iv) Compute $u_2 = u_2(y,h) + \frac{1}{3}[u_2(y,h) - u_2(y,2h)]$.

The value of $h$ to be used must be determined by trial using the criterion that we will use the largest value of $h$ consistent with $\max.|u_2(y,h) - u_2(y,\frac{1}{2}h)| < 10^{-5}$, say. As the integration proceeded downstream it was found that larger values of $h$ could be used. The integration must be performed between $y=0$ and $y=Nh$ with the stipulation that $1-u$ should be less than some small number, say $10^{-7}$ for $y=(N-4)h$, say. Since the boundary layer thickness as we go downstream the value of $Nh$ must be increased every few steps, but the value of $h$ may be increased as mentioned above, so that it was found sufficient to use between 80 and 160 steps in the $y$ direction for the whole range, even though the value of $Nh$ increased ten fold between $x=0.005$ and $x=0.5$.

In the $x$ direction a similar criterion may be adopted for the step interval, although no $h^2$-extrapolation process was used in this direction. It was again found that this step length could be increased as $x$ increased.

The integrations were performed on the Ferranti Pegasus Computer for $x = 0.005 - (\times 0.0003125) - 0.01 - (\times 0.000625) - 0.02 - (\times 0.00125) - 0.04 - (\times 0.0025) - (\times 0.005) - 0.5$.
For the same accuracy (four to five decimal places) the distance $0.005 - 0.01$ had to be traversed in sixteen steps, while only twenty steps were needed for $x=0.4$ to $x=0.5$.

Near $x=0.005$ only about four iterations were needed to obtain $u_2(y,2h)$ to the desired accuracy and a further five to obtain $u_2(y,h)$, each step taking about ten minutes to perform. However near $x=0.5$ many more iterations were needed and each step took thirty minutes or more, with the prospect of this time growing even larger as $x$ increased more. It was therefore decided to integrate equation (7.7) instead and a similar technique was employed, which is briefly described below.

The advantage of integrating Crocco's form of the boundary layer equations is that no quadrature formulae need be used and all the $\ell_{js}$'s in (13.21) with $s < j-1$ vanish. This reduces the time of integration considerably.

## §XIV The Numerical Integration from $x=0.5$ to near separation

Crocco's form of the boundary layer equations, from (7.7), with the original meaning attached to $x$ is

$$\tau^2 \frac{\partial^2 \tau}{\partial u^2} = u \frac{\partial \tau}{\partial x} \quad , \tag{14.1}$$

and the boundary conditions are

$$\left.\begin{aligned} u=1\ ,\ \tau=0 \quad ; \\ u=0\ ,\ \partial\tau/\partial u=1 \quad . \end{aligned}\right\} \tag{14.2}$$

Replacing derivatives in the $x$ direction by central differences and others by averages (14.1) reduces to

$$\tfrac{1}{8} w^2 w'' = u\left(\frac{w-2\tau_1}{x_2-x_1}\right) \tag{14.3}$$

where $w=\tau_2+\tau_1$ , $\tau_1$ being the known solution at $x=x_1$ and $\tau_2$ the solution at $x=x_2$ . The boundary conditions are

$$w'(0)=2,\ w(1)=0 \quad , \tag{14.4}$$

the prime denoting differentiation with respect to $u$.

The iteration process

$$\tfrac{1}{8} w_m^2 w''_{m+1} = \frac{u}{x_2-x_1}\left(w_{m+1}-2\tau_1\right) \tag{14.5}$$

was found to converge.

The only finite difference substitution we have to make is

$$w''_{m+1,j} = \frac{1}{h^2}\left(w_{m+1,j+1}-2w_{m+1,j}+w_{m+1,j-1}\right) \tag{14.6}$$

and at the $j^{\text{th}}$ mesh point we have

$$\alpha_j w_{m+1,j+1} + A_j w_{m+1,j} + \alpha_j w_{m+1,j-1} = C_j \tag{14.7}$$

where

$$\left.\begin{aligned} \alpha_j &= (w_{m,j})^2 \quad ; \\ A_j &= -(2\alpha_j + j.\beta) \quad ; \\ C_j &= -2j.\beta\tau_{1,j} \quad ; \\ \beta &= 8h^3/(x_2-x_1) \quad , \end{aligned}\right\} \tag{14.8}$$

and $h$ is the step length in the $u$ direction.

At the ends of the range the equations are

$$2\alpha_0 w_{m+1,1} + A_0 w_{m+1,0} = 4h\alpha_0 \tag{14.9}$$

since

$$\frac{1}{2h}\left(w_{m+1,1} - w_{m+1,-1}\right) = 2 \;;$$

and

$$\alpha_{N-1} w_{m+1,N-2} + A_{N-1} w_{m+1,N-1} = C_{N-1} \tag{14.10}$$

since $w_{m+1,N} = 0$ , where $Nh = 1$.

In matrix form equations (14.8) to (14.10) are

$$\underset{\sim}{A}\,\underset{\sim}{W}_{m+1} = \underset{\sim}{C} \;, \tag{14.11}$$

where

$$\underset{\sim}{A} = \begin{bmatrix} A_0 & 2\alpha_0 & 0 & 0 & - & - & - & - \\ \alpha_1 & A_1 & \alpha_1 & 0 & - & - & - & - \\ 0 & \alpha_2 & A_2 & \alpha_2 & - & - & - & - \\ \vdots & \vdots & \vdots & \vdots & - & - & - & - \\ 0 & 0 & 0 & 0 & - & \alpha_{N-2} & A_{N-2} & \alpha_{N-2} \\ 0 & 0 & 0 & 0 & - & 0 & \alpha_{N-1} & A_{N-1} \end{bmatrix} ; \tag{14.12}$$

$$\underset{\sim}{W}_{m+1} = \begin{bmatrix} w_{m+1,0} \\ w_{m+1,1} \\ \vdots \\ w_{m+1,N-1} \end{bmatrix} \;; \qquad \underset{\sim}{C} = \begin{bmatrix} 4h\alpha_0 \\ C_1 \\ \vdots \\ C_{N-1} \end{bmatrix} . \tag{14.13}$$

$\underset{\sim}{A}$ is factorised by Choleski's method into

$$\underset{\sim}{A} = \underset{\sim}{L}\,\underset{\sim}{U} \tag{14.14}$$

and

$$\underset{\sim}{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & - & - & - \\ l_1 & 1 & 0 & 0 & - & - & - \\ 0 & l_2 & 1 & 0 & - & - & - \\ \vdots & \vdots & \vdots & \vdots & - & - & - \\ 0 & 0 & - & - & 0 & l_{N-1} & 1 \end{bmatrix}, \quad \underset{\sim}{U} = \begin{bmatrix} u_0 & 2\alpha_0 & 0 & 0 & - & - \\ 0 & u_1 & \alpha_1 & 0 & - & - \\ \vdots & \vdots & \vdots & \vdots & - & - \\ 0 & 0 & - & 0 & u_{N-2} & \alpha_{N-2} \\ 0 & 0 & - & 0 & 0 & u_{N-1} \end{bmatrix} , \tag{14.15}$$

where

$$\left.\begin{aligned} & u_p = A_p - l_p\alpha_{p-1} \;(p>1);\quad u_0 = A_0;\quad u_1 = A_1 - l_1 . 2\alpha_0 \;; \\ & l_p = \frac{\alpha_p}{u_{p-1}} \quad (p>0). \end{aligned}\right\} \tag{14.16}$$

We first solve

$$\underset{\sim}{L}\,\underset{\sim}{Y} = \underset{\sim}{C} \tag{14.17}$$

and then

$$\underset{\sim}{U}\,\underset{\sim}{W}_{m+1} = \underset{\sim}{Y} \tag{14.18}$$

where

$$\underset{\sim}{Y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix}. \tag{14.19}$$

It is found that

$$y_p = c_p - l_p y_{p-1} \quad (p>0) \quad ; \quad y_0 = k h d_0 \quad ; \tag{14.20}$$

and

$$w_{m+1,N-1} = \frac{y_{N-1}}{u_{N-1}} \quad ; \quad w_{m+1,p} = \frac{y_p - d_p w_{m+1,p+1}}{u_p} \quad (p < N-1)\,. \tag{14.21}$$

The same techniques as regards $h^2$- extrapolation and the size of the step lengths as before were used.

A little difficulty arises near $u \approx 1$. Near here we may find the leading term of the solution by setting

$$\tau = \frac{f(u)}{\sqrt{\lambda x}}$$

where $f$ is given by the equation

$$f f'' = -u \tag{14.22}$$

From this it may be seen that when $u=1$ and $f=0$, $f''$ becomes singular, which makes numerical integration round that point very difficult. However it was found that slight changes in the value of $\tau$ at $u=1$ made a negligible difference to the values of $\tau$ obtained four or five mesh points back, and after that no difference at all, so that, since the region near $u=0$ is of prime interest, nothing was done to improve the accuracy near $u=1$, even though the last few values computed would be in error.

The integration was carried out on the Pegasus computer from $x = 0.5 - (\times 0.0025) - 0.65$, each step taking only about five minutes.

After this point the method became unstable. It may be remembered that earlier we used the largest step length in $x$ consistent with the solution agreeing to within 4 or 5 decimal places with the solution obtained when the step length was halved. This was done in the interests of saving time. After about 6.5, however, the smaller the step length was made, the more the solution diverged from the solution for double this step length (near $u=0$, that is - elsewhere good agreement was noticed). This is thought to be due to the rounding errors becoming more important than the truncation errors. Values of $\tau$ at $u=0$ using different step lengths are shown in Fig. IV. Even when the integration was performed independently from $x=0.5$ the values of $\tau$ at $u=0$ when $x_2-x_1=0.005$ and when $x_2-x_1=0.0025$ agree to within $4\frac{1}{2}$ decimal places up to about $x=0.65$. When rounding errors become important it is advantageous to use a large step length, and so the results obtained for $x_2-x_1=0.005$ were accepted as being the most accurate after $x=0.65$.

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## § XV The Results

The results from §s XII and XIII are given in Table 1 where $u$, $v$ and $\psi$ are tabulated against $x$ from $x = 0.01$ to $x = 0.5$, $v$ and $\psi$ being calculated from

$$v = 1 - \frac{\partial}{\partial x}\int_0^y u\,dy \tag{15.1}$$

and

$$\psi = -x + \int_0^y u\,dy. \tag{15.2}$$

Table 2 gives the results obtained by the method of § XIV from $x = 0.5$ to $x = 0.7$, $\tau$ and $y$ being tabulated against $u$. $y$ is calculated from

$$y = \int_0^u \frac{du}{\tau}. \tag{15.3}$$

Table 3 shows values of the skin friction $\left.\frac{\partial u}{\partial y}\right|_{y=0}$, the displacement thickness $\delta^*$, momentum thickness $\theta^*$, the transverse velocity at the outer edge of the boundary layer, $v_\infty$ and the reciprocal of $v_\infty$ tabulated against $x$. $\delta^*$, $\theta^*$ and $v_\infty$ are calculated from

$$\delta^* = \int_0^\infty (1-u)\,dy = \int_0^1 (1-u)\frac{du}{\tau}\;; \tag{15.4}$$

$$\theta^* = \int_0^\infty u(1-u)\,dy = \int_0^1 u(1-u)\frac{du}{\tau}\;; \tag{15.5}$$

$$v_\infty = 1 + \frac{d}{dx}\int_0^\infty (1-u)\,dy = 1 + \frac{d}{dx}\int_0^1 (1-u)\frac{du}{\tau}. \tag{15.6}$$

As a check the continuity and momentum equations ( (6.5) and (6.4) ) may be integrated between $y=0$ and $y=\infty$ to give the following two relationships :

$$v_\infty = 1 + \frac{d\delta^*}{dx}. \tag{15.7}$$

$$\frac{d\theta^*}{dx} = 1 + \left.\frac{\partial u}{\partial y}\right|_{y=0}. \tag{15.8}$$

It is estimated that the results given will be accurate to within the number of significant figures given in Tables 1, 2 and 3 up to about $x = 0.65$, but the accuracy of the values given after $x = 0.65$ cannot be guaranteed, for the reasons given at the end of § XIV.

Figures V to IX depict graphically some of the results contained in Tables 1, 2 and 3. Figure V shows velocity profiles plotted against the 'scaled' distance from the wall, $y/\delta^*$ at the stations $x = 0.1, 0.3, 0.5$ and $0.7$. The skin friction curve plotted in Figure VI demonstrates the difficulty in determining the position of the separation point from the calculated values of $\left.\frac{\partial u}{\partial y}\right|_{y=0}$ since the graph shows every appearance of approaching the axis in an asymptotic fashion. In fact, as we know from the treatment in Part II, the graph is tangential to the axis at the separation point. The rapid increase in the displacement thickness and $v_\infty$ as we pass through $x = 0.7$, as demonstrated in Figures VII and VIII, displays the nearness of the separation point and Figure IX, showing $1/v_\infty$ against $x$, clearly shows that $1/v_\infty$ and thus $\left.\frac{\partial u}{\partial y}\right|_{y=0}$ becomes zero soon after $x = 0.7$. By extrapolation the separation point is found to lie somewhere between $x = 0.73$ and $x = 0.74$.

One would like to check the series for the skin friction in the neighbourhood of the separation point obtained in Part II, and in particular to obtain the value of $\alpha$ in (9.27), from the numerical integration. However this was not found to be possible. (9.27) may be written in the form

$$\frac{\tau_0}{\bar{x}} = \frac{e^{-\theta}}{\theta^2}\left[1 + \frac{A(\theta)}{\theta} + \frac{B(\theta)}{\theta^2} + \frac{C(\theta)}{\theta^3} + O\left(\frac{1}{\theta^4}\right)\right], \tag{15.9}$$

where $A$, $B$ and $C$ are given in (9.27). Large values of $\theta$ correspond to small values of $\bar{x}$ ($\bar{x}$ being measured from the separation point in the upstream direction). If, for a certain value of $\theta$, the first terms in this series are each less than the preceding term, then there is some hope that the series (15.9) converges for this value of $\theta$, and that a value for $\frac{\tau_0}{\bar{x}}$ may be calculated from the first four terms. However for $\theta = 10$ it was found that the third and fourth terms in (15.9) are of comparable magnitude, and it is no longer justifiable to assume that the value of the term $O\left(\frac{1}{\theta^4}\right)$ is negligible compared to the values of the first four terms. Hence we cannot use (15.9) with the values of $A$, $B$ and $C$ from (9.27) for $\theta < 10$. When $\theta = 10$ we find from (15.9) that

$$\frac{\tau_0}{\bar{x}} \simeq 5 \times 10^{-7}. \tag{15.10}$$

There is thus no possibility of matching the solution (15.9) to the numerical solution without performing the numerical integration to a very much greater accuracy, since if we take the separation point to lie at $x = 0{\cdot}74$, then at $x = 0{\cdot}7$ we have $\frac{\tau(x,0)}{\bar{x}} \simeq 0{\cdot}04$, a long way from the value given in (15.10).

## Acknowledgements

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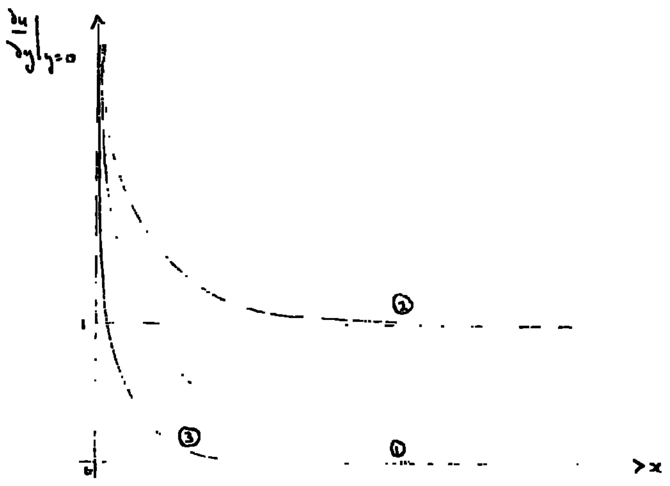



Fig I  Skin friction against $x$

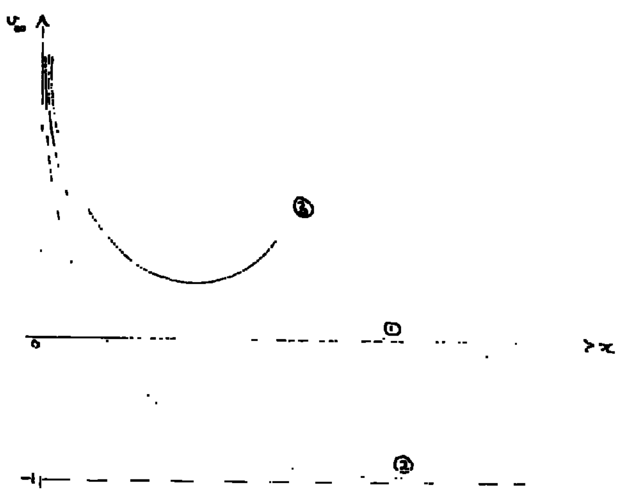



Fig II  $v_\infty$ against $x$

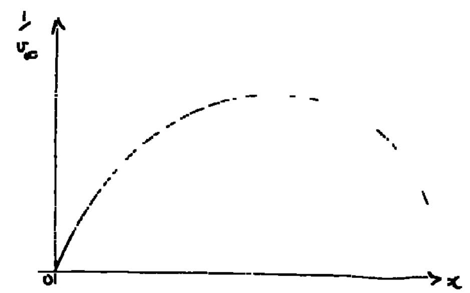
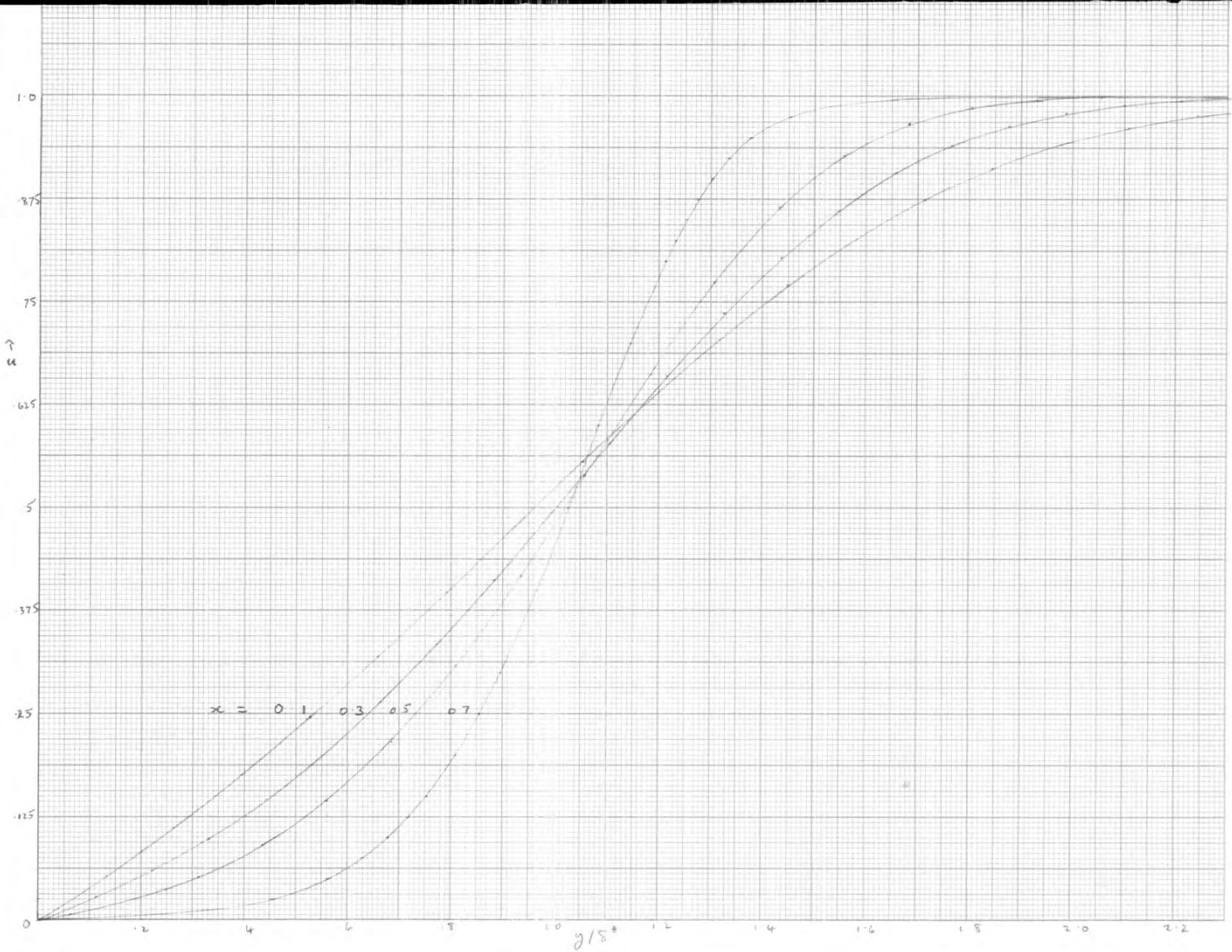
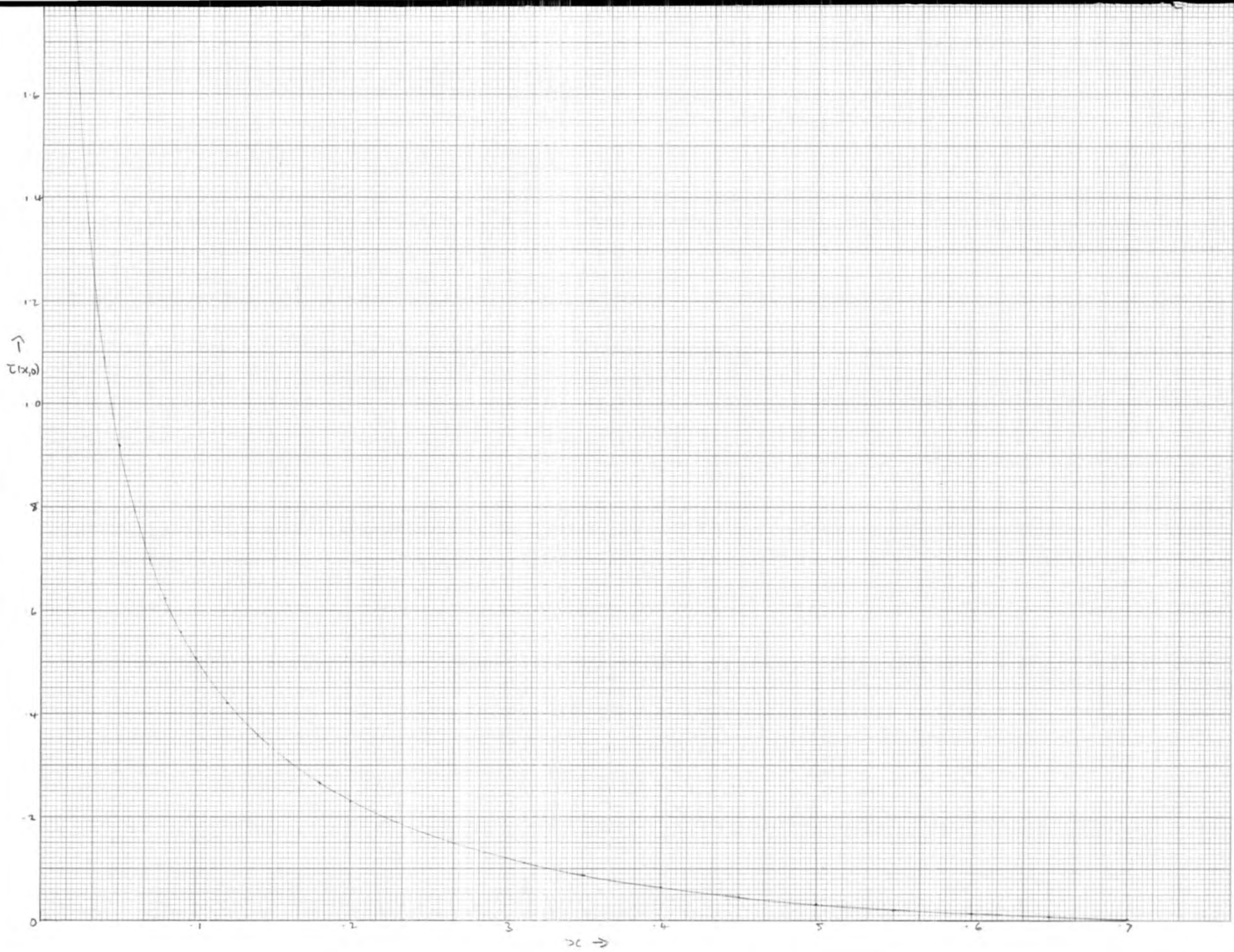


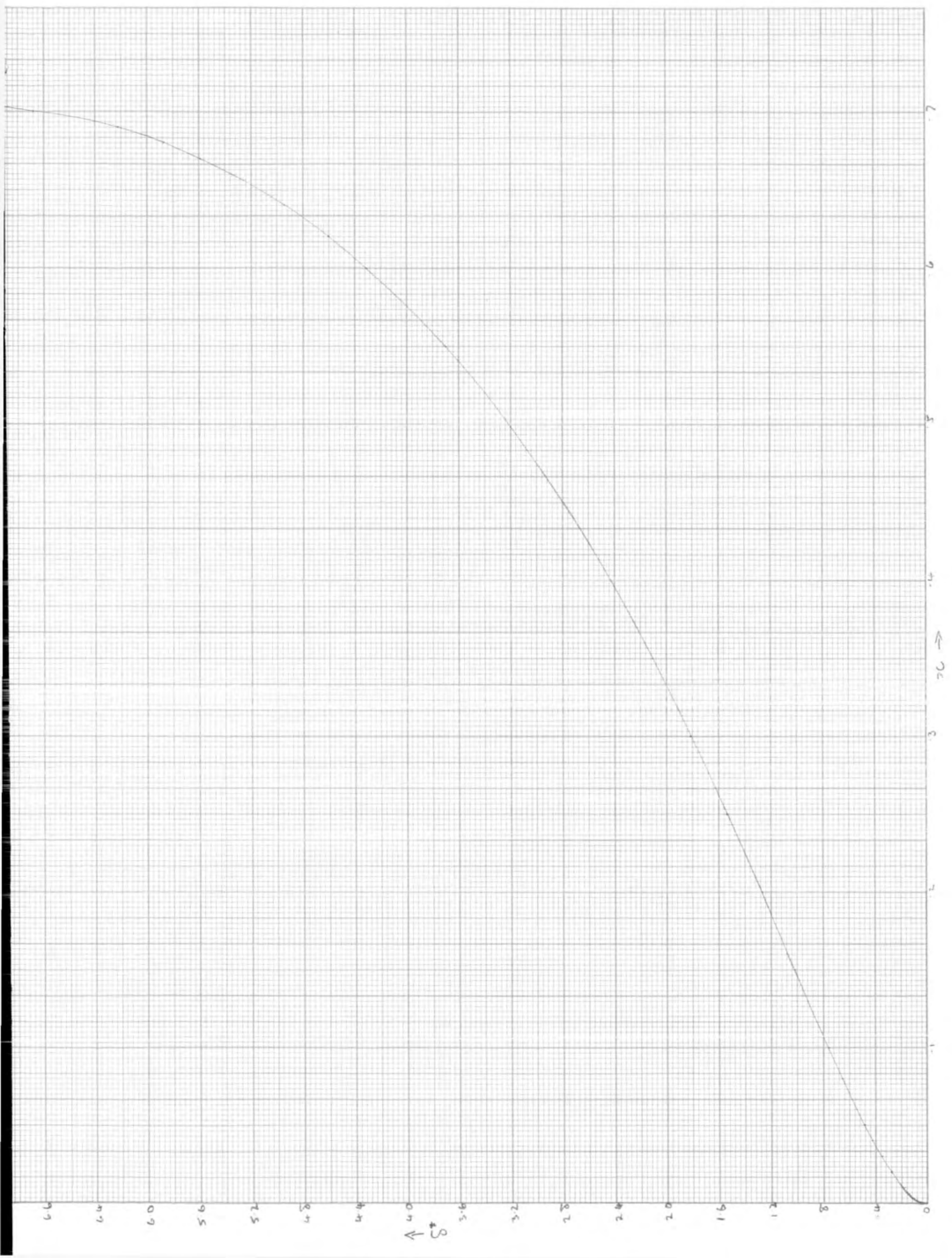

Fig III  $1/v_\infty$ against $x$

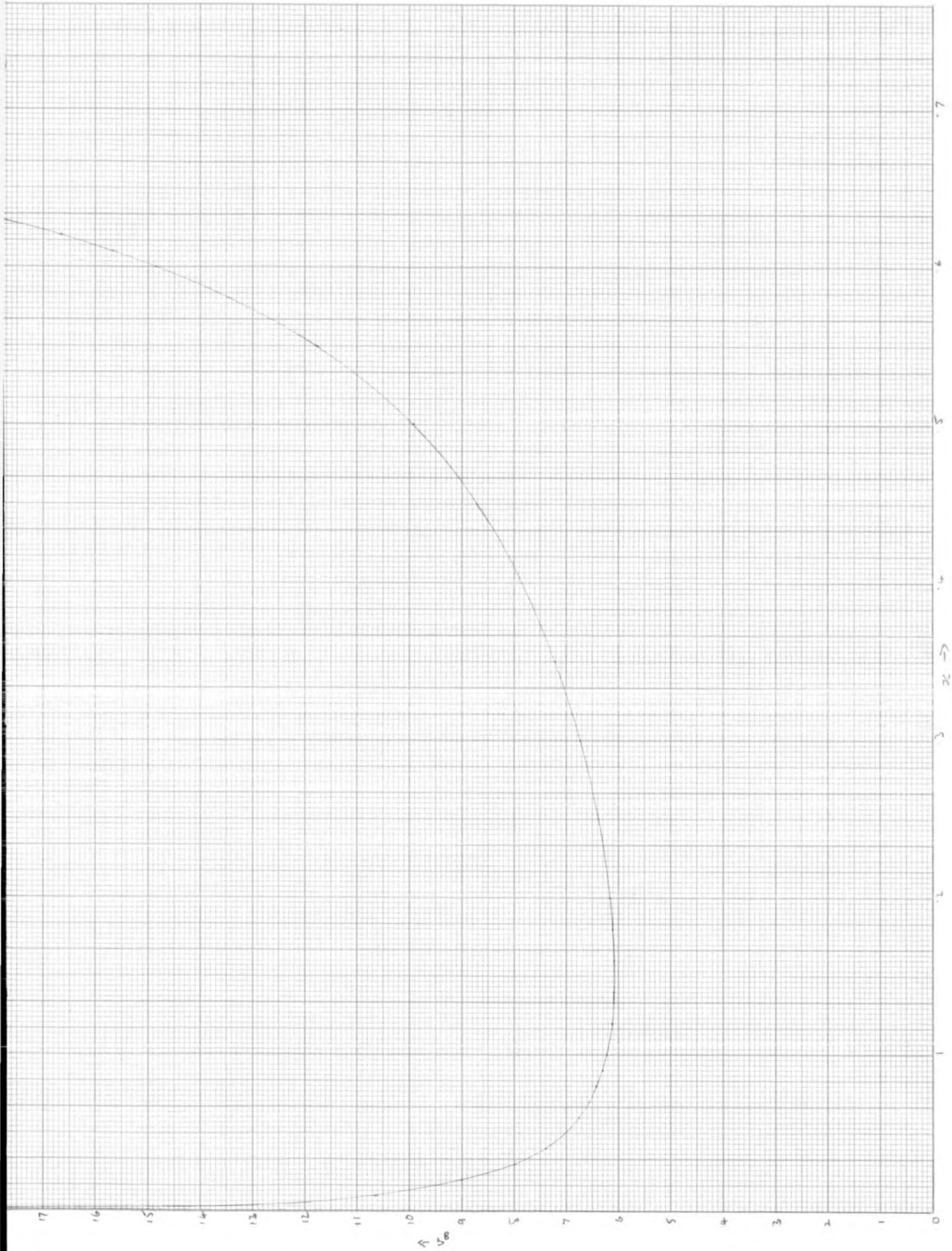
FIGS. I, II & III (Schematic)

h = .0025
h = .005
h = .000625
h = .00125
$x \rightarrow$
$z(x,0) \uparrow$
.6
.61
.62
.63
.64
.65
.66
.67
.68
.69
.7
0
.001
.002
.003
.004
.005
.006
.007
.008
.009
.010
.011
.012


1·0
·875
·75
u
·625
·5
·375
·25
·125
0
·2
·4
·6
·8
1·0
1·2
1·4
1·6
1·8
2·0
2·2
y/δ*
x = 0·1 0·3 0·5 0·7


τ(x,0)
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
.1
.2
.3
.4
.5
.6
.7
x





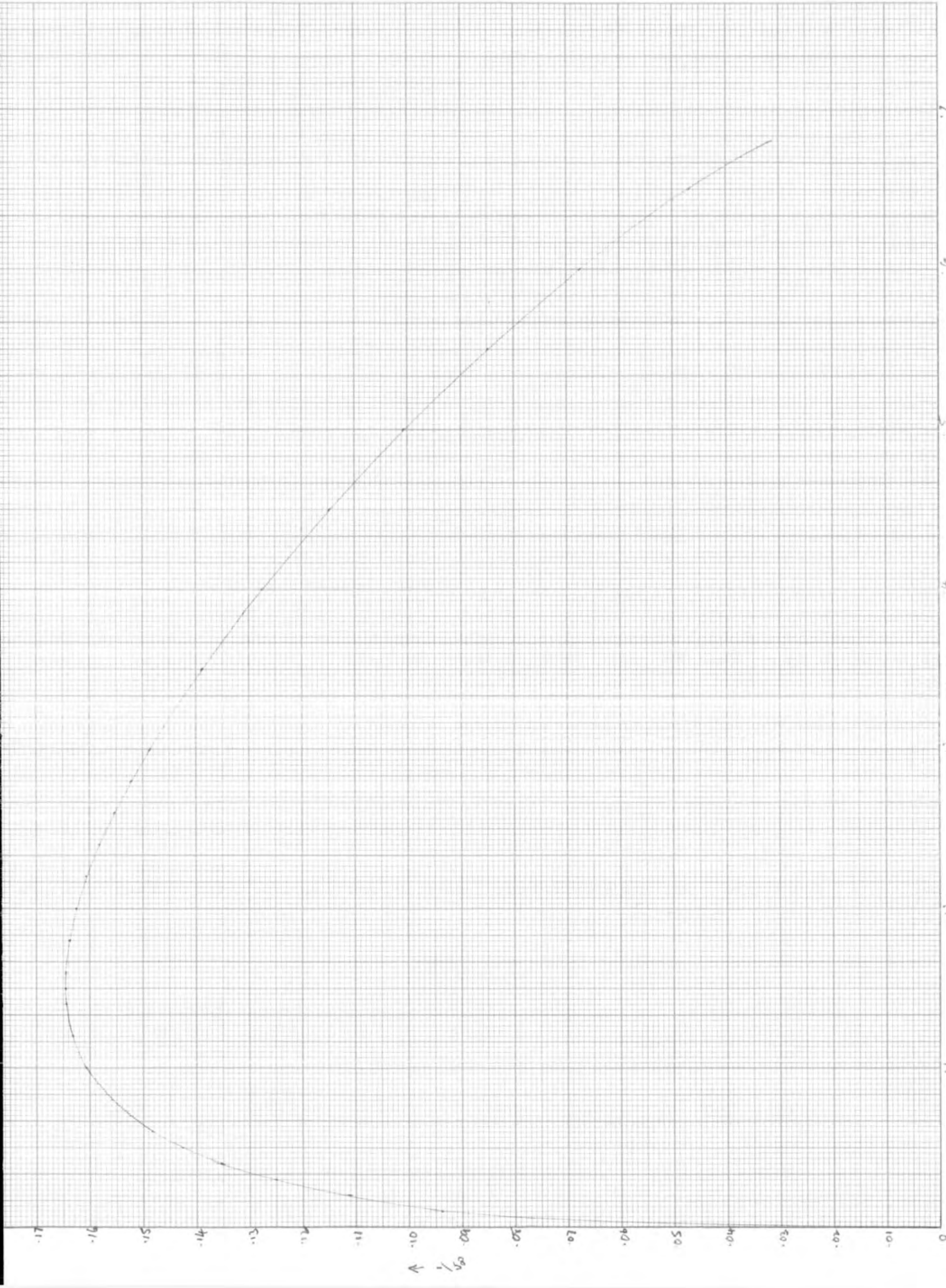


TABLE 1.– U, V AND $\psi$

| | $x=+0.01$ | | | $x=+0.02$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | $\psi$ | U | V | $\psi$ |
| +0.00 | +0.0000 | +1.0000 | −0.0100 | +0.0000 | +1.0000 | −0.0200 |
| +0.05 | +0.1397 | +1.1089 | −0.0065 | +0.0904 | +1.0728 | −0.0178 |
| +0.10 | +0.2849 | +1.3765 | +0.0041 | +0.1851 | +1.2956 | −0.0109 |
| +0.15 | +0.4313 | +1.8124 | +0.0220 | +0.2832 | +1.6716 | +0.0008 |
| +0.20 | +0.5720 | +3.3192 | +0.0471 | +0.3831 | +2.1975 | +0.0175 |
| +0.25 | +0.6987 | +5.0614 | +0.0790 | +0.4825 | +2.8597 | +0.0391 |
| +0.30 | +0.8036 | +6.7341 | +0.1166 | +0.5785 | +3.6312 | +0.0657 |
| +0.35 | +0.8828 | +8.1211 | +0.1589 | +0.6680 | +4.4718 | +0.0968 |
| +0.40 | +0.9364 | +9.1407 | +0.2045 | +0.7481 | +5.3310 | +0.1323 |
| +0.45 | +0.9689 | +9.8174 | +0.2522 | +0.8166 | +6.1547 | +0.1715 |
| +0.50 | +0.9863 | +10.2274 | +0.3011 | +0.8722 | +6.8948 | +0.2137 |
| +0.55 | +0.9946 | +10.4546 | +0.3506 | +0.9152 | +7.5177 | +0.2585 |
| +0.60 | +0.9981 | +10.5691 | +0.4005 | +0.9464 | +8.0086 | +0.3051 |
| +0.65 | +0.9994 | +10.6207 | +0.4504 | +0.9678 | +8.3713 | +0.3529 |
| +0.70 | +0.9998 | +10.6410 | +0.5004 | +0.9817 | +8.6227 | +0.4017 |
| +0.75 | +1.0000 | +10.6475 | +0.5504 | +0.9902 | +8.7864 | +0.4510 |
| +0.80 | +1.0000 | +10.6475 | +0.6004 | +0.9950 | +8.8867 | +0.5007 |
| +0.85 | +1.0000 | +10.6475 | +0.6504 | +0.9976 | +8.9445 | +0.5505 |
| +0.90 | +1.0000 | +10.6475 | +0.7004 | +0.9989 | +8.9760 | +0.6004 |
| +0.95 | +1.0000 | +10.6475 | +0.7504 | +0.9995 | +8.9917 | +0.6504 |
| +1.00 | +1.0000 | +10.6475 | +0.8004 | +0.9998 | +8.9993 | +0.7004 |
| +1.05 | +1.0000 | +10.6475 | +0.8504 | +0.9999 | +9.0027 | +0.7503 |
| +1.10 | +1.0000 | +10.6475 | +0.9004 | +1.0000 | +9.0040 | +0.8003 |
| +1.15 | +1.0000 | +10.6475 | +0.9504 | +1.0000 | +9.0045 | +0.8503 |

| | $x = +0.03$ | | | $x = +0.04$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | $\psi$ | U | V | $\psi$ |
| +0.00 | +0.0000 | +1.0000 | −0.0300 | +0.0000 | +1.0000 | −0.0400 |
| +0.05 | +0.0687 | +1.0397 | −0.0283 | +0.0558 | +1.0257 | −0.0386 |
| +0.10 | +0.1408 | +1.1614 | −0.0231 | +0.1145 | +1.1043 | −0.0344 |
| +0.15 | +0.2159 | +1.3678 | −0.0142 | +0.1758 | +1.2382 | −0.0271 |
| +0.20 | +0.2936 | +1.6601 | −0.0014 | +0.2395 | +1.4286 | −0.0168 |
| +0.25 | +0.3729 | +2.0359 | +0.0152 | +0.3053 | +1.6759 | −0.0031 |
| +0.30 | +0.4525 | +2.4888 | +0.0359 | +0.3724 | +1.9784 | +0.0138 |
| +0.35 | +0.5309 | +3.0075 | +0.0604 | +0.4400 | +2.3323 | +0.0341 |
| +0.40 | +0.6064 | +3.5757 | +0.0889 | +0.5072 | +2.7313 | +0.0578 |
| +0.45 | +0.6773 | +4.1725 | +0.1210 | +0.5729 | +3.1667 | +0.0848 |
| +0.50 | +0.7420 | +4.7742 | +0.1565 | +0.6360 | +3.6272 | +0.1150 |
| +0.55 | +0.7993 | +5.3566 | +0.1951 | +0.6952 | +4.0998 | +0.1483 |
| +0.60 | +0.8484 | +5.8972 | +0.2363 | +0.7497 | +4.5703 | +0.1845 |
| +0.65 | +0.8889 | +6.3785 | +0.2798 | +0.7987 | +5.0249 | +0.2232 |
| +0.70 | +0.9212 | +6.7890 | +0.3251 | +0.8416 | +5.4508 | +0.2643 |
| +0.75 | +0.9460 | +7.1242 | +0.3718 | +0.8781 | +5.8375 | +0.3073 |
| +0.80 | +0.9642 | +7.3865 | +0.4196 | +0.9085 | +6.1777 | +0.3520 |
| +0.85 | +0.9771 | +7.5828 | +0.4681 | +0.9330 | +6.4676 | +0.3980 |
| +0.90 | +0.9859 | +7.7237 | +0.5172 | +0.9521 | +6.7068 | +0.4452 |
| +0.95 | +0.9916 | +7.8204 | +0.5666 | +0.9667 | +6.8978 | +0.4932 |
| +1.00 | +0.9952 | +7.8841 | +0.6163 | +0.9774 | +7.0454 | +0.5418 |
| +1.05 | +0.9973 | +7.9242 | +0.6661 | +0.9851 | +7.1558 | +0.5909 |
| +1.10 | +0.9986 | +7.9485 | +0.7160 | +0.9905 | +7.2358 | +0.6403 |
| +1.15 | +0.9993 | +7.9626 | +0.7660 | +0.9941 | +7.2919 | +0.6899 |
| +1.20 | +0.9996 | +7.9704 | +0.8160 | +0.9964 | +7.3300 | +0.7396 |
| +1.25 | +0.9998 | +7.9746 | +0.8659 | +0.9979 | +7.3550 | +0.7895 |
| +1.30 | +0.9999 | +7.9767 | +0.9159 | +0.9988 | +7.3710 | +0.8394 |
| +1.35 | +1.0000 | +7.9777 | +0.9659 | +0.9993 | +7.3808 | +0.8894 |
| +1.40 | +1.0000 | +7.9782 | +1.0159 | +0.9996 | +7.3867 | +0.9393 |
| +1.45 | +1.0000 | +7.9784 | +1.0659 | +0.9998 | +7.3901 | +0.9893 |
| +1.50 | +1.0000 | +7.9785 | +1.1159 | +0.9999 | +7.3921 | +1.0393 |
| +1.55 | +1.0000 | +7.9785 | +1.1659 | +1.0000 | +7.3931 | +1.0893 |
| +1.60 | +1.0000 | +7.9785 | +1.2159 | +1.0000 | +7.3937 | +1.1393 |
| +1.65 | +1.0000 | +7.9785 | +1.2659 | +1.0000 | +7.3939 | +1.1893 |

| | $x = +0.05$ | | | $x = +0.06$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | $\psi$ | U | V | $\psi$ |
| +0.00 | +0.0000 | +1.0000 | −0.0500 | +0.0000 | +1.0000 | −0.0600 |
| +0.05 | +0.0471 | +1.0182 | −0.0488 | +0.0407 | +1.0138 | −0.0590 |
| +0.10 | +0.0966 | +1.0741 | −0.0452 | +0.0835 | +1.0560 | −0.0559 |
| +0.15 | +0.1484 | +1.1694 | −0.0391 | +0.1284 | +1.1279 | −0.0506 |
| +0.20 | +0.2025 | +1.3053 | −0.0304 | +0.1753 | +1.2308 | −0.0430 |
| +0.25 | +0.2586 | +1.4826 | −0.0188 | +0.2241 | +1.3654 | −0.0330 |
| +0.30 | +0.3163 | +1.7012 | −0.0045 | +0.2746 | +1.5321 | −0.0206 |
| +0.35 | +0.3752 | +1.9598 | +0.0128 | +0.3264 | +1.7307 | −0.0056 |
| +0.40 | +0.4347 | +2.2559 | +0.0331 | +0.3794 | +1.9600 | +0.0121 |
| +0.45 | +0.4941 | +2.5855 | +0.0563 | +0.4329 | +2.2184 | +0.0324 |
| +0.50 | +0.5526 | +2.9430 | +0.0825 | +0.4865 | +2.5028 | +0.0554 |
| +0.55 | +0.6095 | +3.3214 | +0.1115 | +0.5395 | +2.8095 | +0.0810 |
| +0.60 | +0.6638 | +3.7124 | +0.1434 | +0.5915 | +3.1336 | +0.1093 |
| +0.65 | +0.7149 | +4.1071 | +0.1778 | +0.6416 | +3.4694 | +0.1401 |
| +0.70 | +0.7620 | +4.4922 | +0.2148 | +0.6894 | +3.8107 | +0.1734 |
| +0.75 | +0.8046 | +4.8705 | +0.2540 | +0.7342 | +4.1510 | +0.2090 |
| +0.80 | +0.8424 | +5.2219 | +0.2952 | +0.7756 | +4.4835 | +0.2468 |
| +0.85 | +0.8752 | +5.5439 | +0.3381 | +0.8132 | +4.8021 | +0.2865 |
| +0.90 | +0.9030 | +5.8315 | +0.3826 | +0.8468 | +5.1012 | +0.3280 |
| +0.95 | +0.9261 | +6.0819 | +0.4283 | +0.8763 | +5.3762 | +0.3711 |
| +1.00 | +0.9449 | +6.2944 | +0.4751 | +0.9017 | +5.6239 | +0.4156 |
| +1.05 | +0.9597 | +6.4701 | +0.5228 | +0.9232 | +5.8423 | +0.4612 |
| +1.10 | +0.9711 | +6.6117 | +0.5710 | +0.9409 | +6.0308 | +0.5079 |
| +1.15 | +0.9798 | +6.7229 | +0.6198 | +0.9554 | +6.1900 | +0.5553 |
| +1.20 | +0.9861 | +6.8079 | +0.6690 | +0.9669 | +6.3217 | +0.6034 |
| +1.25 | +0.9907 | +6.8712 | +0.7184 | +0.9758 | +6.4282 | +0.6519 |
| +1.30 | +0.9939 | +6.9172 | +0.7680 | +0.9827 | +6.5126 | +0.7009 |
| +1.35 | +0.9961 | +6.9498 | +0.8178 | +0.9878 | +6.5779 | +0.7502 |
| +1.40 | +0.9975 | +6.9723 | +0.8676 | +0.9916 | +6.6274 | +0.7997 |
| +1.45 | +0.9985 | +6.9874 | +0.9175 | +0.9943 | +6.6641 | +0.8493 |
| +1.50 | +0.9991 | +6.9973 | +0.9675 | +0.9962 | +6.6908 | +0.8991 |
| +1.55 | +0.9995 | +7.0036 | +1.0174 | +0.9975 | +6.7098 | +0.9489 |
| +1.60 | +0.9997 | +7.0076 | +1.0674 | +0.9984 | +6.7230 | +0.9988 |
| +1.65 | +0.9998 | +7.0100 | +1.1174 | +0.9990 | +6.7320 | +1.0488 |
| +1.70 | +0.9999 | +7.0114 | +1.1674 | +0.9994 | +6.7380 | +1.0987 |
| +1.75 | +0.9999 | +7.0123 | +1.2174 | +0.9996 | +6.7419 | +1.1487 |
| +1.80 | +1.0000 | +7.0127 | +1.2674 | +0.9998 | +6.7444 | +1.1987 |
| +1.85 | +1.0000 | +7.0130 | +1.3174 | +0.9999 | +6.7460 | +1.2487 |
| +1.90 | +1.0000 | +7.0131 | +1.3674 | +0.9999 | +6.7470 | +1.2987 |
| +1.95 | +1.0000 | +7.0132 | +1.4174 | +1.0000 | +6.7476 | +1.3487 |
| +2.00 | +1.0000 | +7.0132 | +1.4674 | +1.0000 | +6.7479 | +1.3987 |
| +2.05 | +1.0000 | +7.0133 | +1.5174 | +1.0000 | +6.7481 | +1.4487 |

| | $x = +0.07$ | | | $x = +0.08$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | ψ | U | V | ψ |
| +0.00 | +0.0000 | +1.0000 | −0.0700 | +0.0000 | +1.0000 | −0.0800 |
| +0.05 | +0.0358 | +1.0108 | −0.0691 | +0.0319 | +1.0088 | −0.0792 |
| +0.10 | +0.0734 | +1.0441 | −0.0664 | +0.0654 | +1.0358 | −0.0768 |
| +0.15 | +0.1129 | +1.1008 | −0.0617 | +0.1005 | +1.0819 | −0.0726 |
| +0.20 | +0.1542 | +1.1819 | −0.0551 | +0.1373 | +1.1479 | −0.0667 |
| +0.25 | +0.1973 | +1.2883 | −0.0463 | +0.1758 | +1.2345 | −0.0589 |
| +0.30 | +0.2420 | +1.4205 | −0.0353 | +0.2158 | +1.3423 | −0.0491 |
| +0.35 | +0.2882 | +1.5785 | −0.0221 | +0.2572 | +1.4716 | −0.0373 |
| +0.40 | +0.3356 | +1.7622 | −0.0065 | +0.3000 | +1.6224 | −0.0234 |
| +0.45 | +0.3839 | +1.9706 | +0.0115 | +0.3438 | +1.7945 | −0.0073 |
| +0.50 | +0.4329 | +2.2023 | +0.0319 | +0.3885 | +1.9871 | +0.0110 |
| +0.55 | +0.4819 | +2.4551 | +0.0548 | +0.4337 | +2.1990 | +0.0316 |
| +0.60 | +0.5307 | +2.7262 | +0.0801 | +0.4792 | +2.4284 | +0.0544 |
| +0.65 | +0.5786 | +3.0119 | +0.1079 | +0.5244 | +2.6732 | +0.0795 |
| +0.70 | +0.6252 | +3.3083 | +0.1380 | +0.5690 | +2.9305 | +0.1068 |
| +0.75 | +0.6701 | +3.6106 | +0.1703 | +0.6127 | +3.1973 | +0.1364 |
| +0.80 | +0.7126 | +3.9140 | +0.2049 | +0.6550 | +3.4700 | +0.1681 |
| +0.85 | +0.7525 | +4.2134 | +0.2416 | +0.6954 | +3.7448 | +0.2019 |
| +0.90 | +0.7894 | +4.5039 | +0.2801 | +0.7337 | +4.0178 | +0.2376 |
| +0.95 | +0.8229 | +4.7811 | +0.3204 | +0.7696 | +4.2850 | +0.2752 |
| +1.00 | +0.8531 | +5.0410 | +0.3624 | +0.8027 | +4.5428 | +0.3145 |
| +1.05 | +0.8797 | +5.2804 | +0.4057 | +0.8329 | +4.7879 | +0.3554 |
| +1.10 | +0.9028 | +5.4971 | +0.4503 | +0.8600 | +5.0172 | +0.3977 |
| +1.15 | +0.9226 | +5.6896 | +0.4959 | +0.8842 | +5.2286 | +0.4414 |
| +1.20 | +0.9392 | +5.8576 | +0.5425 | +0.9053 | +5.4205 | +0.4861 |
| +1.25 | +0.9530 | +6.0016 | +0.5898 | +0.9235 | +5.5918 | +0.5318 |
| +1.30 | +0.9642 | +6.1226 | +0.6377 | +0.9390 | +5.7425 | +0.5784 |
| +1.35 | +0.9731 | +6.2226 | +0.6862 | +0.9520 | +5.8728 | +0.6257 |
| +1.40 | +0.9801 | +6.3036 | +0.7350 | +0.9627 | +5.9838 | +0.6736 |
| +1.45 | +0.9855 | +6.3680 | +0.7842 | +0.9714 | +6.0767 | +0.7219 |
| +1.50 | +0.9896 | +6.4184 | +0.8335 | +0.9783 | +6.1533 | +0.7707 |
| +1.55 | +0.9927 | +6.4569 | +0.8831 | +0.9838 | +6.2154 | +0.8197 |
| +1.60 | +0.9949 | +6.4860 | +0.9328 | +0.9881 | +6.2650 | +0.8690 |
| +1.65 | +0.9965 | +6.5074 | +0.9826 | +0.9914 | +6.3039 | +0.9185 |
| +1.70 | +0.9977 | +6.5230 | +1.0324 | +0.9938 | +6.3340 | +0.9682 |
| +1.75 | +0.9985 | +6.5341 | +1.0823 | +0.9956 | +6.3568 | +1.0179 |
| +1.80 | +0.9990 | +6.5418 | +1.1323 | +0.9970 | +6.3739 | +1.0677 |
| +1.85 | +0.9994 | +6.5472 | +1.1822 | +0.9979 | +6.3865 | +1.1176 |
| +1.90 | +0.9996 | +6.5508 | +1.2322 | +0.9986 | +6.3956 | +1.1675 |
| +1.95 | +0.9997 | +6.5532 | +1.2822 | +0.9991 | +6.4021 | +1.2174 |
| +2.00 | +0.9998 | +6.5547 | +1.3322 | +0.9994 | +6.4067 | +1.2674 |
| +2.05 | +0.9999 | +6.5557 | +1.3822 | +0.9996 | +6.4098 | +1.3174 |
| +2.10 | +0.9999 | +6.5563 | +1.4322 | +0.9997 | +6.4120 | +1.3674 |
| +2.15 | +1.0000 | +6.5567 | +1.4822 | +0.9998 | +6.4134 | +1.4174 |
| +2.20 | +1.0000 | +6.5570 | +1.5322 | +0.9999 | +6.4144 | +1.4674 |

| | $x=+0.09$ | | | $x=+0.10$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | ψ | U | V | ψ |
| +0.0 | +0.0000 | +1.0000 | −0.0900 | +0.0000 | +1.0000 | −0.1000 |
| +0.1 | +0.0587 | +1.0298 | −0.0871 | +0.0532 | +1.0252 | −0.0974 |
| +0.2 | +0.1234 | +1.1230 | −0.0781 | +0.1118 | +1.1042 | −0.0892 |
| +0.3 | +0.1941 | +1.2850 | −0.0622 | +0.1759 | +1.2417 | −0.0748 |
| +0.4 | +0.2704 | +1.5195 | −0.0390 | +0.2453 | +1.4413 | −0.0538 |
| +0.5 | +0.3512 | +1.8271 | −0.0080 | +0.3194 | +1.7046 | −0.0256 |
| +0.6 | +0.4351 | +2.2038 | +0.0313 | +0.3970 | +2.0298 | +0.0102 |
| +0.7 | +0.5198 | +2.6397 | +0.0791 | +0.4765 | +2.4111 | +0.0538 |
| +0.8 | +0.6028 | +3.1192 | +0.1352 | +0.5559 | +2.8379 | +0.1055 |
| +0.9 | +0.6814 | +3.6209 | +0.1995 | +0.6328 | +3.2947 | +0.1649 |
| +1.0 | +0.7529 | +4.1204 | +0.2713 | +0.7050 | +3.7627 | +0.2319 |
| +1.1 | +0.8153 | +4.5928 | +0.3497 | +0.7703 | +4.2210 | +0.3057 |
| +1.2 | +0.8673 | +5.0167 | +0.4340 | +0.8270 | +4.6496 | +0.3857 |
| +1.3 | +0.9085 | +5.3770 | +0.5228 | +0.8744 | +5.0319 | +0.4708 |
| +1.4 | +0.9397 | +5.6667 | +0.6153 | +0.9122 | +5.3565 | +0.5602 |
| +1.5 | +0.9620 | +5.8868 | +0.7105 | +0.9410 | +5.6188 | +0.6529 |
| +1.6 | +0.9772 | +6.0448 | +0.8075 | +0.9620 | +5.8202 | +0.7482 |
| +1.7 | +0.9869 | +6.1519 | +0.9057 | +0.9765 | +5.9671 | +0.8451 |
| +1.8 | +0.9929 | +6.2204 | +1.0048 | +0.9861 | +6.0689 | +0.9433 |
| +1.9 | +0.9963 | +6.2618 | +1.1042 | +0.9921 | +6.1359 | +1.0422 |
| +2.0 | +0.9982 | +6.2855 | +1.2040 | +0.9958 | +6.1778 | +1.1417 |
| +2.1 | +0.9991 | +6.2982 | +1.3038 | +0.9978 | +6.2026 | +1.2413 |
| +2.2 | +0.9996 | +6.3047 | +1.4038 | +0.9989 | +6.2167 | +1.3412 |
| +2.3 | +0.9998 | +6.3079 | +1.5038 | +0.9995 | +6.2242 | +1.4411 |
| +2.4 | +0.9999 | +6.3093 | +1.6037 | +0.9998 | +6.2280 | +1.5411 |
| +2.5 | +1.0000 | +6.3100 | +1.7037 | +0.9999 | +6.2299 | +1.6411 |
| +2.6 | +1.0000 | +6.3102 | +1.8037 | +1.0000 | +6.2307 | +1.7410 |
| +2.7 | +1.0000 | +6.3103 | +1.9037 | +1.0000 | +6.2311 | +1.8410 |

| | $x = +0.11$ | | | $x = +0.12$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | ψ | U | V | ψ |
| +0.0 | +0.0000 | +1.0000 | −0.1100 | +0.0000 | +1.0000 | −0.1200 |
| +0.1 | +0.0484 | +1.0217 | −0.1076 | +0.0443 | +1.0189 | −0.1178 |
| +0.2 | +0.1018 | +1.0896 | −0.1002 | +0.0932 | +1.0780 | −0.1110 |
| +0.3 | +0.1603 | +1.2079 | −0.0871 | +0.1468 | +1.1811 | −0.0990 |
| +0.4 | +0.2238 | +1.3802 | −0.0679 | +0.2051 | +1.3314 | −0.0815 |
| +0.5 | +0.2919 | +1.6083 | −0.0422 | +0.2679 | +1.5313 | −0.0579 |
| +0.6 | +0.3638 | +1.8921 | −0.0094 | +0.3346 | +1.7811 | −0.0278 |
| +0.7 | +0.4383 | +2.2281 | +0.0307 | +0.4043 | +2.0792 | +0.0092 |
| +0.8 | +0.5137 | +2.6091 | +0.0783 | +0.4757 | +2.4209 | +0.0532 |
| +0.9 | +0.5882 | +3.0243 | +0.1334 | +0.5472 | +2.7983 | +0.1043 |
| +1.0 | +0.6597 | +3.4591 | +0.1958 | +0.6172 | +3.2006 | +0.1626 |
| +1.1 | +0.7262 | +3.8966 | +0.2652 | +0.6838 | +3.6142 | +0.2276 |
| +1.2 | +0.7861 | +4.3193 | +0.3408 | +0.7454 | +4.0243 | +0.2991 |
| +1.3 | +0.8380 | +4.7108 | +0.4221 | +0.8005 | +4.4157 | +0.3765 |
| +1.4 | +0.8814 | +5.0581 | +0.5082 | +0.8482 | +4.7753 | +0.4590 |
| +1.5 | +0.9161 | +5.3526 | +0.5981 | +0.8880 | +5.0927 | +0.5459 |
| +1.6 | +0.9428 | +5.5915 | +0.6911 | +0.9200 | +5.3616 | +0.6363 |
| +1.7 | +0.9624 | +5.7764 | +0.7864 | +0.9447 | +5.5801 | +0.7296 |
| +1.8 | +0.9763 | +5.9130 | +0.8834 | +0.9631 | +5.7504 | +0.8251 |
| +1.9 | +0.9856 | +6.0093 | +0.9815 | +0.9762 | +5.8775 | +0.9221 |
| +2.0 | +0.9916 | +6.0741 | +1.0804 | +0.9852 | +5.9684 | +1.0202 |
| +2.1 | +0.9953 | +6.1157 | +1.1798 | +0.9912 | +6.0307 | +1.1190 |
| +2.2 | +0.9975 | +6.1412 | +1.2794 | +0.9949 | +6.0716 | +1.2183 |
| +2.3 | +0.9987 | +6.1561 | +1.3792 | +0.9972 | +6.0973 | +1.3180 |
| +2.4 | +0.9994 | +6.1644 | +1.4791 | +0.9985 | +6.1128 | +1.4177 |
| +2.5 | +0.9997 | +6.1689 | +1.5791 | +0.9992 | +6.1217 | +1.5176 |
| +2.6 | +0.9999 | +6.1711 | +1.6791 | +0.9996 | +6.1267 | +1.6176 |
| +2.7 | +0.9999 | +6.1722 | +1.7790 | +0.9998 | +6.1293 | +1.7175 |
| +2.8 | +1.0000 | +6.1727 | +1.8790 | +0.9999 | +6.1307 | +1.8175 |
| +2.9 | +1.0000 | +6.1730 | +1.9790 | +1.0000 | +6.1313 | +1.9175 |
| +3.0 | +1.0000 | +6.1731 | +2.0790 | +1.0000 | +6.1316 | +2.0175 |

| | $x = +0.13$ | | | $x = +0.14$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | $\psi$ | U | V | $\psi$ |
| +0.0 | +0.0000 | +1.0000 | −0.1300 | +0.0000 | +1.0000 | −0.1400 |
| +0.1 | +0.0407 | +1.0166 | −0.1280 | +0.0375 | +1.0147 | −0.1382 |
| +0.2 | +0.0856 | +1.0685 | −0.1217 | +0.0789 | +1.0608 | −0.1324 |
| +0.3 | +0.1349 | +1.1593 | −0.1107 | +0.1245 | +1.1413 | −0.1222 |
| +0.4 | +0.1887 | +1.2917 | −0.0946 | +0.1742 | +1.2590 | −0.1073 |
| +0.5 | +0.2468 | +1.4683 | −0.0728 | +0.2280 | +1.4162 | −0.0873 |
| +0.6 | +0.3087 | +1.6900 | −0.0451 | +0.2856 | +1.6143 | −0.0616 |
| +0.7 | +0.3739 | +1.9562 | −0.0110 | +0.3466 | +1.8533 | −0.0300 |
| +0.8 | +0.4413 | +2.2638 | +0.0298 | +0.4102 | +2.1315 | +0.0078 |
| +0.9 | +0.5097 | +2.6075 | +0.0773 | +0.4753 | +2.4450 | +0.0521 |
| +1.0 | +0.5776 | +2.9789 | +0.1317 | +0.5408 | +2.7878 | +0.1029 |
| +1.1 | +0.6434 | +3.3675 | +0.1928 | +0.6052 | +3.1516 | +0.1602 |
| +1.2 | +0.7056 | +3.7609 | +0.2602 | +0.6672 | +3.5263 | +0.2238 |
| +1.3 | +0.7628 | +4.1459 | +0.3337 | +0.7254 | +3.9005 | +0.2935 |
| +1.4 | +0.8137 | +4.5098 | +0.4126 | +0.7785 | +4.2626 | +0.3687 |
| +1.5 | +0.8576 | +4.8415 | +0.4962 | +0.8257 | +4.6018 | +0.4490 |
| +1.6 | +0.8943 | +5.1331 | +0.5839 | +0.8663 | +4.9092 | +0.5337 |
| +1.7 | +0.9238 | +5.3798 | +0.6748 | +0.9002 | +5.1782 | +0.6220 |
| +1.8 | +0.9468 | +5.5806 | +0.7684 | +0.9275 | +5.4055 | +0.7135 |
| +1.9 | +0.9640 | +5.7380 | +0.8640 | +0.9489 | +5.5909 | +0.8073 |
| +2.0 | +0.9764 | +5.8564 | +0.9610 | +0.9650 | +5.7367 | +0.9031 |
| +2.1 | +0.9851 | +5.9421 | +1.0592 | +0.9768 | +5.8472 | +1.0002 |
| +2.2 | +0.9909 | +6.0017 | +1.1580 | +0.9851 | +5.9280 | +1.0983 |
| +2.3 | +0.9946 | +6.0416 | +1.2573 | +0.9907 | +5.9850 | +1.1971 |
| +2.4 | +0.9969 | +6.0672 | +1.3568 | +0.9944 | +6.0236 | +1.2964 |
| +2.5 | +0.9983 | +6.0831 | +1.4566 | +0.9967 | +6.0489 | +1.3960 |
| +2.6 | +0.9991 | +6.0925 | +1.5565 | +0.9981 | +6.0649 | +1.4957 |
| +2.7 | +0.9995 | +6.0978 | +1.6564 | +0.9990 | +6.0746 | +1.5956 |
| +2.8 | +0.9998 | +6.1008 | +1.7564 | +0.9995 | +6.0803 | +1.6955 |
| +2.9 | +0.9999 | +6.1024 | +1.8564 | +0.9997 | +6.0836 | +1.7954 |
| +3.0 | +0.9999 | +6.1031 | +1.9564 | +0.9999 | +6.0853 | +1.8954 |
| +3.1 | +1.0000 | +6.1035 | +2.0564 | +0.9999 | +6.0863 | +1.9954 |
| +3.2 | +1.0000 | +6.1037 | +2.1564 | +1.0000 | +6.0867 | +2.0954 |
| +3.3 | +1.0000 | +6.1038 | +2.2564 | +1.0000 | +6.0870 | +2.1954 |

| | $x = +0.15$ | | | $x = +0.16$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | $\psi$ | U | V | $\psi$ |
| +0.0 | +0.0000 | +1.0000 | −0.1500 | +0.0000 | +1.0000 | −0.1600 |
| +0.1 | +0.0347 | +1.0131 | −0.1483 | +0.0322 | +1.0118 | −0.1584 |
| +0.2 | +0.0730 | +1.0543 | −0.1429 | +0.0677 | +1.0488 | −0.1535 |
| +0.3 | +0.1151 | +1.1263 | −0.1336 | +0.1068 | +1.1136 | −0.1448 |
| +0.4 | +0.1612 | +1.2316 | −0.1198 | +0.1495 | +1.2084 | −0.1320 |
| +0.5 | +0.2112 | +1.3725 | −0.1012 | +0.1960 | +1.3355 | −0.1147 |
| +0.6 | +0.2649 | +1.5506 | −0.0774 | +0.2461 | +1.4965 | −0.0927 |
| +0.7 | +0.3220 | +1.7664 | −0.0481 | +0.2996 | +1.6921 | −0.0654 |
| +0.8 | +0.3819 | +2.0189 | −0.0129 | +0.3560 | +1.9223 | −0.0326 |
| +0.9 | +0.4437 | +2.3057 | +0.0283 | +0.4147 | +2.1852 | +0.0059 |
| +1.0 | +0.5066 | +2.6222 | +0.0758 | +0.4748 | +2.4778 | +0.0504 |
| +1.1 | +0.5692 | +2.9621 | +0.1296 | +0.5354 | +2.7952 | +0.1009 |
| +1.2 | +0.6304 | +3.3172 | +0.1896 | +0.5953 | +3.1307 | +0.1574 |
| +1.3 | +0.6889 | +3.6780 | +0.2556 | +0.6534 | +3.4764 | +0.2199 |
| +1.4 | +0.7434 | +4.0340 | +0.3273 | +0.7086 | +3.8234 | +0.2880 |
| +1.5 | +0.7929 | +4.3752 | +0.4041 | +0.7597 | +4.1623 | +0.3615 |
| +1.6 | +0.8367 | +4.6924 | +0.4857 | +0.8060 | +4.4843 | +0.4398 |
| +1.7 | +0.8743 | +4.9780 | +0.5713 | +0.8467 | +4.7816 | +0.5225 |
| +1.8 | +0.9057 | +5.2272 | +0.6603 | +0.8817 | +5.0480 | +0.6089 |
| +1.9 | +0.9311 | +5.4376 | +0.7522 | +0.9108 | +5.2796 | +0.6986 |
| +2.0 | +0.9510 | +5.6093 | +0.8463 | +0.9345 | +5.4750 | +0.7909 |
| +2.1 | +0.9661 | +5.7448 | +0.9422 | +0.9531 | +5.6346 | +0.8853 |
| +2.2 | +0.9772 | +5.8482 | +1.0394 | +0.9672 | +5.7609 | +0.9814 |
| +2.3 | +0.9851 | +5.9244 | +1.1376 | +0.9778 | +5.8578 | +1.0787 |
| +2.4 | +0.9906 | +5.9787 | +1.2364 | +0.9853 | +5.9297 | +1.1768 |
| +2.5 | +0.9942 | +6.0160 | +1.3356 | +0.9905 | +5.9815 | +1.2756 |
| +2.6 | +0.9965 | +6.0409 | +1.4352 | +0.9941 | +6.0175 | +1.3749 |
| +2.7 | +0.9980 | +6.0569 | +1.5349 | +0.9964 | +6.0418 | +1.4744 |
| +2.8 | +0.9989 | +6.0668 | +1.6348 | +0.9979 | +6.0577 | +1.5741 |
| +2.9 | +0.9994 | +6.0728 | +1.7347 | +0.9988 | +6.0677 | +1.6740 |
| +3.0 | +0.9997 | +6.0762 | +1.8346 | +0.9993 | +6.0739 | +1.7739 |
| +3.1 | +0.9998 | +6.0782 | +1.9346 | +0.9996 | +6.0775 | +1.8738 |
| +3.2 | +0.9999 | +6.0792 | +2.0346 | +0.9998 | +6.0796 | +1.9738 |
| +3.3 | +1.0000 | +6.0798 | +2.1346 | +0.9999 | +6.0808 | +2.0738 |
| +3.4 | +1.0000 | +6.0801 | +2.2346 | +1.0000 | +6.0814 | +2.1738 |
| +3.5 | +1.0000 | +6.0802 | +2.3346 | +1.0000 | +6.0818 | +2.2738 |
| +3.6 | +1.0000 | +6.0803 | +2.4346 | +1.0000 | +6.0819 | +2.3738 |

| | $x = +0.17$ | | | $x = +0.18$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | ψ | U | V | ψ |
| +0.0 | +0.0000 | +1.0000 | −0.1700 | +0.0000 | +1.0000 | −0.1800 |
| +0.1 | +0.0299 | +1.0107 | −0.1685 | +0.0278 | +1.0097 | −0.1786 |
| +0.2 | +0.0629 | +1.0442 | −0.1639 | +0.0585 | +1.0401 | −0.1743 |
| +0.3 | +0.0992 | +1.1027 | −0.1558 | +0.0924 | +1.0934 | −0.1668 |
| +0.4 | +0.1390 | +1.1886 | −0.1440 | +0.1294 | +1.1715 | −0.1558 |
| +0.5 | +0.1823 | +1.3038 | −0.1279 | +0.1699 | +1.2764 | −0.1408 |
| +0.6 | +0.2291 | +1.4500 | −0.1074 | +0.2137 | +1.4098 | −0.1217 |
| +0.7 | +0.2793 | +1.6282 | −0.0820 | +0.2607 | +1.5728 | −0.0980 |
| +0.8 | +0.3324 | +1.8387 | −0.0514 | +0.3107 | +1.7659 | −0.0694 |
| +0.9 | +0.3879 | +2.0805 | −0.0154 | +0.3633 | +1.9887 | −0.0358 |
| +1.0 | +0.4453 | +2.3513 | +0.0262 | +0.4179 | +2.2398 | +0.0033 |
| +1.1 | +0.5037 | +2.6475 | +0.0737 | +0.4739 | +2.5163 | +0.0479 |
| +1.2 | +0.5620 | +2.9638 | +0.1270 | +0.5305 | +2.8142 | +0.0981 |
| +1.3 | +0.6193 | +3.2938 | +0.1860 | +0.5866 | +3.1282 | +0.1539 |
| +1.4 | +0.6745 | +3.6297 | +0.2508 | +0.6414 | +3.4518 | +0.2154 |
| +1.5 | +0.7266 | +3.9632 | +0.3208 | +0.6939 | +3.7777 | +0.2821 |
| +1.6 | +0.7746 | +4.2861 | +0.3959 | +0.7431 | +4.0983 | +0.3540 |
| +1.7 | +0.8179 | +4.5905 | +0.4756 | +0.7883 | +4.4062 | +0.4306 |
| +1.8 | +0.8559 | +4.8698 | +0.5593 | +0.8288 | +4.6944 | +0.5115 |
| +1.9 | +0.8885 | +5.1190 | +0.6466 | +0.8644 | +4.9573 | +0.5962 |
| +2.0 | +0.9156 | +5.3350 | +0.7369 | +0.8948 | +5.1910 | +0.6842 |
| +2.1 | +0.9377 | +5.5170 | +0.8296 | +0.9201 | +5.3931 | +0.7750 |
| +2.2 | +0.9551 | +5.6658 | +0.9242 | +0.9407 | +5.5632 | +0.8681 |
| +2.3 | +0.9684 | +5.7839 | +1.0204 | +0.9570 | +5.7023 | +0.9630 |
| +2.4 | +0.9783 | +5.8749 | +1.1178 | +0.9695 | +5.8131 | +1.0594 |
| +2.5 | +0.9855 | +5.9429 | +1.2160 | +0.9789 | +5.8987 | +1.1568 |
| +2.6 | +0.9906 | +5.9922 | +1.3148 | +0.9858 | +5.9631 | +1.2551 |
| +2.7 | +0.9940 | +6.0269 | +1.4141 | +0.9906 | +6.0101 | +1.3539 |
| +2.8 | +0.9963 | +6.0506 | +1.5136 | +0.9940 | +6.0435 | +1.4531 |
| +2.9 | +0.9978 | +6.0663 | +1.6133 | +0.9962 | +6.0665 | +1.5526 |
| +3.0 | +0.9987 | +6.0764 | +1.7131 | +0.9977 | +6.0820 | +1.6523 |
| +3.1 | +0.9993 | +6.0827 | +1.8130 | +0.9986 | +6.0920 | +1.7522 |
| +3.2 | +0.9996 | +6.0865 | +1.9130 | +0.9992 | +6.0984 | +1.8521 |
| +3.3 | +0.9998 | +6.0887 | +2.0129 | +0.9995 | +6.1024 | +1.9520 |
| +3.4 | +0.9999 | +6.0900 | +2.1129 | +0.9998 | +6.1047 | +2.0520 |
| +3.5 | +0.9999 | +6.0907 | +2.2129 | +0.9999 | +6.1061 | +2.1519 |
| +3.6 | +1.0000 | +6.0911 | +2.3129 | +0.9999 | +6.1069 | +2.2519 |
| +3.7 | +1.0000 | +6.0913 | +2.4129 | +1.0000 | +6.1073 | +2.3519 |
| +3.8 | +1.0000 | +6.0914 | +2.5129 | +1.0000 | +6.1075 | +2.4519 |

| | $x = +0.19$ | | | $x = +0.20$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | $\psi$ | U | V | $\psi$ |
| +0.0 | +0.0000 | +1.0000 | −0.1900 | +0.0000 | +1.0000 | −0.2000 |
| +0.1 | +0.0259 | +1.0089 | −0.1887 | +0.0242 | +1.0081 | −0.1988 |
| +0.2 | +0.0546 | +1.0366 | −0.1847 | +0.0509 | +1.0336 | −0.1951 |
| +0.3 | +0.0861 | +1.0853 | −0.1777 | +0.0804 | +1.0781 | −0.1885 |
| +0.4 | +0.1207 | +1.1566 | −0.1674 | +0.1127 | +1.1436 | −0.1789 |
| +0.5 | +0.1585 | +1.2525 | −0.1535 | +0.1481 | +1.2316 | −0.1659 |
| +0.6 | +0.1995 | +1.3747 | −0.1356 | +0.1865 | +1.3439 | −0.1492 |
| +0.7 | +0.2436 | +1.5243 | −0.1135 | +0.2279 | +1.4816 | −0.1285 |
| +0.8 | +0.2907 | +1.7020 | −0.0868 | +0.2722 | +1.6456 | −0.1035 |
| +0.9 | +0.3404 | +1.9079 | −0.0552 | +0.3193 | +1.8363 | −0.0740 |
| +1.0 | +0.3924 | +2.1410 | −0.0186 | +0.3686 | +2.0531 | −0.0396 |
| +1.1 | +0.4460 | +2.3993 | +0.0233 | +0.4199 | +2.2946 | −0.0002 |
| +1.2 | +0.5006 | +2.6798 | +0.0706 | +0.4724 | +2.5585 | +0.0444 |
| +1.3 | +0.5553 | +2.9780 | +0.1234 | +0.5255 | +2.8414 | +0.0943 |
| +1.4 | +0.6094 | +3.2885 | +0.1817 | +0.5785 | +3.1386 | +0.1495 |
| +1.5 | +0.6618 | +3.6052 | +0.2452 | +0.6304 | +3.4450 | +0.2100 |
| +1.6 | +0.7116 | +3.9211 | +0.3139 | +0.6805 | +3.7544 | +0.2756 |
| +1.7 | +0.7582 | +4.2293 | +0.3875 | +0.7280 | +4.0604 | +0.3460 |
| +1.8 | +0.8008 | +4.5230 | +0.4654 | +0.7721 | +4.3567 | +0.4211 |
| +1.9 | +0.8388 | +4.7963 | +0.5475 | +0.8122 | +4.6372 | +0.5003 |
| +2.0 | +0.8721 | +5.0444 | +0.6330 | +0.8480 | +4.8967 | +0.5833 |
| +2.1 | +0.9005 | +5.2641 | +0.7217 | +0.8792 | +5.1313 | +0.6697 |
| +2.2 | +0.9242 | +5.4537 | +0.8130 | +0.9059 | +5.3384 | +0.7590 |
| +2.3 | +0.9435 | +5.6131 | +0.9064 | +0.9281 | +5.5167 | +0.8508 |
| +2.4 | +0.9588 | +5.7436 | +1.0016 | +0.9461 | +5.6666 | +0.9445 |
| +2.5 | +0.9706 | +5.8477 | +1.0981 | +0.9605 | +5.7893 | +1.0399 |
| +2.6 | +0.9795 | +5.9285 | +1.1956 | +0.9717 | +5.8874 | +1.1365 |
| +2.7 | +0.9860 | +5.9895 | +1.2939 | +0.9801 | +5.9638 | +1.2341 |
| +2.8 | +0.9907 | +6.0344 | +1.3927 | +0.9863 | +6.0217 | +1.3324 |
| +2.9 | +0.9940 | +6.0665 | +1.4920 | +0.9908 | +6.0646 | +1.4313 |
| +3.0 | +0.9962 | +6.0889 | +1.5915 | +0.9940 | +6.0956 | +1.5306 |
| +3.1 | +0.9976 | +6.1041 | +1.6912 | +0.9961 | +6.1173 | +1.6301 |
| +3.2 | +0.9986 | +6.1141 | +1.7910 | +0.9976 | +6.1322 | +1.7298 |
| +3.3 | +0.9991 | +6.1206 | +1.8909 | +0.9985 | +6.1421 | +1.8296 |
| +3.4 | +0.9995 | +6.1246 | +1.9908 | +0.9991 | +6.1486 | +1.9295 |
| +3.5 | +0.9997 | +6.1270 | +2.0908 | +0.9995 | +6.1527 | +2.0294 |
| +3.6 | +0.9998 | +6.1285 | +2.1908 | +0.9997 | +6.1553 | +2.1293 |
| +3.7 | +0.9999 | +6.1293 | +2.2907 | +0.9998 | +6.1568 | +2.2293 |
| +3.8 | +1.0000 | +6.1298 | +2.3907 | +0.9999 | +6.1577 | +2.3293 |
| +3.9 | +1.0000 | +6.1301 | +2.4907 | +1.0000 | +6.1582 | +2.4293 |
| +4.0 | +1.0000 | +6.1302 | +2.5907 | +1.0000 | +6.1585 | +2.5293 |
| +4.1 | +1.0000 | +6.1303 | +2.6907 | +1.0000 | +6.1587 | +2.6293 |

| | $x = +0.21$ | | | $x = +0.22$ | | |
|---|---|---|---|---|---|---|
| $y$ | $U$ | $V$ | $\psi$ | $U$ | $V$ | $\psi$ |
| +0.0 | +0.0000 | +1.0000 | −0.2100 | +0.0000 | +1.0000 | −0.2200 |
| +0.1 | +0.0226 | +1.0075 | −0.2089 | +0.0212 | +1.0069 | −0.2190 |
| +0.2 | +0.0476 | +1.0309 | −0.2054 | +0.0445 | +1.0285 | −0.2157 |
| +0.3 | +0.0752 | +1.0719 | −0.1993 | +0.0703 | +1.0663 | −0.2100 |
| +0.4 | +0.1054 | +1.1321 | −0.1903 | +0.0986 | +1.1219 | −0.2015 |
| +0.5 | +0.1385 | +1.2132 | −0.1781 | +0.1296 | +1.1968 | −0.1902 |
| +0.6 | +0.1745 | +1.3166 | −0.1625 | +0.1634 | +1.2924 | −0.1755 |
| +0.7 | +0.2134 | +1.4438 | −0.1431 | +0.2000 | +1.4102 | −0.1574 |
| +0.8 | +0.2552 | +1.5956 | −0.1197 | +0.2393 | +1.5510 | −0.1354 |
| +0.9 | +0.2996 | +1.7726 | −0.0920 | +0.2813 | +1.7156 | −0.1094 |
| +1.0 | +0.3464 | +1.9745 | −0.0597 | +0.3257 | +1.9040 | −0.0791 |
| +1.1 | +0.3953 | +2.2006 | −0.0226 | +0.3723 | +2.1157 | −0.0442 |
| +1.2 | +0.4458 | +2.4489 | +0.0194 | +0.4206 | +2.3496 | −0.0046 |
| +1.3 | +0.4972 | +2.7169 | +0.0666 | +0.4702 | +2.6034 | +0.0400 |
| +1.4 | +0.5488 | +3.0009 | +0.1189 | +0.5204 | +2.8743 | +0.0895 |
| +1.5 | +0.6000 | +3.2963 | +0.1763 | +0.5706 | +3.1584 | +0.1440 |
| +1.6 | +0.6499 | +3.5979 | +0.2388 | +0.6200 | +3.4512 | +0.2036 |
| +1.7 | +0.6978 | +3.8998 | +0.3062 | +0.6680 | +3.7475 | +0.2680 |
| +1.8 | +0.7430 | +4.1962 | +0.3783 | +0.7138 | +4.0419 | +0.3371 |
| +1.9 | +0.7848 | +4.4811 | +0.4547 | +0.7568 | +4.3289 | +0.4107 |
| +2.0 | +0.8227 | +4.7492 | +0.5351 | +0.7964 | +4.6030 | +0.4884 |
| +2.1 | +0.8564 | +4.9961 | +0.6191 | +0.8323 | +4.8596 | +0.5698 |
| +2.2 | +0.8857 | +5.2184 | +0.7062 | +0.8641 | +5.0948 | +0.6547 |
| +2.3 | +0.9108 | +5.4140 | +0.7961 | +0.8917 | +5.3059 | +0.7425 |
| +2.4 | +0.9316 | +5.5822 | +0.8883 | +0.9153 | +5.4911 | +0.8329 |
| +2.5 | +0.9486 | +5.7234 | +0.9823 | +0.9349 | +5.6501 | +0.9254 |
| +2.6 | +0.9621 | +5.8392 | +1.0779 | +0.9509 | +5.7836 | +1.0197 |
| +2.7 | +0.9727 | +5.9318 | +1.1746 | +0.9637 | +5.8930 | +1.1155 |
| +2.8 | +0.9807 | +6.0042 | +1.2723 | +0.9736 | +5.9808 | +1.2124 |
| +2.9 | +0.9866 | +6.0594 | +1.3707 | +0.9812 | +6.0495 | +1.3101 |
| +3.0 | +0.9909 | +6.1004 | +1.4696 | +0.9869 | +6.1021 | +1.4086 |
| +3.1 | +0.9940 | +6.1302 | +1.5688 | +0.9911 | +6.1414 | +1.5075 |
| +3.2 | +0.9961 | +6.1513 | +1.6683 | +0.9940 | +6.1701 | +1.6067 |
| +3.3 | +0.9975 | +6.1659 | +1.7680 | +0.9961 | +6.1906 | +1.7063 |
| +3.4 | +0.9985 | +6.1757 | +1.8678 | +0.9975 | +6.2049 | +1.8059 |
| +3.5 | +0.9991 | +6.1822 | +1.9677 | +0.9984 | +6.2147 | +1.9057 |
| +3.6 | +0.9994 | +6.1864 | +2.0676 | +0.9990 | +6.2211 | +2.0056 |
| +3.7 | +0.9997 | +6.1890 | +2.1676 | +0.9994 | +6.2253 | +2.1055 |
| +3.8 | +0.9998 | +6.1906 | +2.2676 | +0.9997 | +6.2280 | +2.2055 |
| +3.9 | +0.9999 | +6.1916 | +2.3676 | +0.9998 | +6.2297 | +2.3055 |
| +4.0 | +0.9999 | +6.1921 | +2.4676 | +0.9999 | +6.2307 | +2.4054 |
| +4.1 | +1.0000 | +6.1924 | +2.5675 | +0.9999 | +6.2313 | +2.5054 |
| +4.2 | +1.0000 | +6.1926 | +2.6675 | +1.0000 | +6.2317 | +2.6054 |
| +4.3 | +1.0000 | +6.1927 | +2.7675 | +1.0000 | +6.2319 | +2.7054 |

| | $x = +0.23$ | | | $x = +0.24$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | ψ | U | V | ψ |
| +0.0 | +0.0000 | +1.0000 | −0.2300 | +0.0000 | +1.0000 | −0.2400 |
| +0.1 | +0.0198 | +1.0064 | −0.2290 | +0.0186 | +1.0059 | −0.2391 |
| +0.2 | +0.0417 | +1.0264 | −0.2260 | +0.0391 | +1.0245 | −0.2362 |
| +0.3 | +0.0658 | +1.0614 | −0.2206 | +0.0617 | +1.0569 | −0.2312 |
| +0.4 | +0.0924 | +1.1128 | −0.2127 | +0.0866 | +1.1047 | −0.2238 |
| +0.5 | +0.1215 | +1.1822 | −0.2020 | +0.1139 | +1.1691 | −0.2138 |
| +0.6 | +0.1532 | +1.2708 | −0.1883 | +0.1436 | +1.2514 | −0.2009 |
| +0.7 | +0.1875 | +1.3801 | −0.1713 | +0.1760 | +1.3531 | −0.1850 |
| +0.8 | +0.2246 | +1.5110 | −0.1507 | +0.2109 | +1.4750 | −0.1657 |
| +0.9 | +0.2643 | +1.6643 | −0.1263 | +0.2483 | +1.6181 | −0.1427 |
| +1.0 | +0.3064 | +1.8404 | −0.0978 | +0.2882 | +1.7829 | −0.1159 |
| +1.1 | +0.3507 | +2.0390 | −0.0650 | +0.3304 | +1.9693 | −0.0850 |
| +1.2 | +0.3969 | +2.2592 | −0.0276 | +0.3745 | +2.1769 | −0.0498 |
| +1.3 | +0.4446 | +2.4996 | +0.0145 | +0.4202 | +2.4045 | −0.0101 |
| +1.4 | +0.4932 | +2.7577 | +0.0613 | +0.4672 | +2.6502 | +0.0343 |
| +1.5 | +0.5422 | +3.0304 | +0.1131 | +0.5148 | +2.9115 | +0.0834 |
| +1.6 | +0.5909 | +3.3138 | +0.1698 | +0.5626 | +3.1851 | +0.1373 |
| +1.7 | +0.6386 | +3.6034 | +0.2312 | +0.6098 | +3.4671 | +0.1959 |
| +1.8 | +0.6847 | +3.8942 | +0.2974 | +0.6559 | +3.7531 | +0.2592 |
| +1.9 | +0.7286 | +4.1811 | +0.3681 | +0.7002 | +4.0383 | +0.3270 |
| +2.0 | +0.7695 | +4.4589 | +0.4430 | +0.7422 | +4.3178 | +0.3992 |
| +2.1 | +0.8072 | +4.7229 | +0.5219 | +0.7813 | +4.5869 | +0.4754 |
| +2.2 | +0.8411 | +4.9688 | +0.6044 | +0.8170 | +4.8412 | +0.5553 |
| +2.3 | +0.8712 | +5.1932 | +0.6900 | +0.8492 | +5.0771 | +0.6387 |
| +2.4 | +0.8972 | +5.3940 | +0.7785 | +0.8776 | +5.2917 | +0.7250 |
| +2.5 | +0.9194 | +5.5698 | +0.8693 | +0.9023 | +5.4830 | +0.8141 |
| +2.6 | +0.9379 | +5.7205 | +0.9622 | +0.9232 | +5.6503 | +0.9054 |
| +2.7 | +0.9530 | +5.8469 | +1.0568 | +0.9407 | +5.7935 | +0.9986 |
| +2.8 | +0.9651 | +5.9507 | +1.1527 | +0.9550 | +5.9135 | +1.0934 |
| +2.9 | +0.9745 | +6.0340 | +1.2497 | +0.9664 | +6.0122 | +1.1895 |
| +3.0 | +0.9818 | +6.0994 | +1.3476 | +0.9754 | +6.0915 | +1.2866 |
| +3.1 | +0.9872 | +6.1497 | +1.4460 | +0.9823 | +6.1539 | +1.3845 |
| +3.2 | +0.9912 | +6.1874 | +1.5450 | +0.9875 | +6.2020 | +1.4830 |
| +3.3 | +0.9941 | +6.2152 | +1.6442 | +0.9913 | +6.2383 | +1.5820 |
| +3.4 | +0.9961 | +6.2351 | +1.7437 | +0.9941 | +6.2651 | +1.6812 |
| +3.5 | +0.9974 | +6.2491 | +1.8434 | +0.9961 | +6.2845 | +1.7807 |
| +3.6 | +0.9984 | +6.2587 | +1.9432 | +0.9974 | +6.2982 | +1.8804 |
| +3.7 | +0.9990 | +6.2652 | +2.0431 | +0.9983 | +6.3077 | +1.9802 |
| +3.8 | +0.9994 | +6.2694 | +2.1430 | +0.9990 | +6.3141 | +2.0801 |
| +3.9 | +0.9996 | +6.2721 | +2.2429 | +0.9994 | +6.3184 | +2.1800 |
| +4.0 | +0.9998 | +6.2739 | +2.3429 | +0.9996 | +6.3212 | +2.2799 |
| +4.1 | +0.9999 | +6.2749 | +2.4429 | +0.9998 | +6.3230 | +2.3799 |
| +4.2 | +0.9999 | +6.2756 | +2.5429 | +0.9999 | +6.3241 | +2.4799 |
| +4.3 | +1.0000 | +6.2759 | +2.6429 | +0.9999 | +6.3248 | +2.5799 |
| +4.4 | +1.0000 | +6.2762 | +2.7429 | +1.0000 | +6.3252 | +2.6799 |
| +4.5 | +1.0000 | +6.2763 | +2.8429 | +1.0000 | +6.3254 | +2.7799 |
| +4.6 | +1.0000 | +6.2763 | +2.9429 | +1.0000 | +6.3255 | +2.8799 |

| $y$ | $x=$ +0.2[illegible] | | | $x=$ +0.[illegible] | | |
|---|---|---|---|---|---|---|
| +0.0 | +0.0000 | +1.0000 | −0.2500 | +0.0000 | +1.0000 | −0.2500 |
| +0.2 | +0.0396 | +1.0337 | −0.2465 | +0.0384 | +1.0312 | −0.2567 |
| +0.4 | +0.0[illegible] | +1.0779 | −0.2[illegible] | +0.07[illegible] | +1.0907 | −0.2459 |
| +0.6 | +0.13[illegible] | +1.[illegible] | −0.2134 | +0.1[illegible] | +1.2[illegible] | −0.2336 |
| +0.8 | +0.1[illegible] | +1.[illegible] | −0.1[illegible] | +0.1[illegible] | +1.[illegible] | −0.1945 |
| +1.0 | +0.27[illegible] | +1.7307 | −0.1555 | +0.2[illegible] | +1.[illegible] | −0.1506 |
| +1.2 | +0.3330 | +2.101[illegible] | −0.0712 | +0.3333 | +2.0339 | −0.0919 |
| +1.4 | +0.[illegible] | +2.[illegible] | +0.00[illegible] | +0.[illegible] | +2.4[illegible] | −0.0[illegible] |
| +1.6 | +0.555[illegible] | +3.0[illegible] | +0.1060 | +0.5097 | +2.[illegible] | +0.0760 |
| +1.8 | +0.[illegible] | +3.[illegible] | +0.[illegible] | +0.[illegible] | +3.[illegible] | +0.[illegible] |
| +2.0 | +0.714[illegible] | +4.1[illegible] | +0.35[illegible] | +0.[illegible] | +4.0470 | +0.3155 |
| +2.2 | +0.7931 | +4.71[illegible] | +0.5075 | +0.7[illegible] | +4.5[illegible] | +0.[illegible] |
| +2.4 | +0.[illegible] | +5.1[illegible] | +0.[illegible] | +0.[illegible]344 | +5.075[illegible] | +0.[illegible] |
| +2.6 | +0.[illegible] | +5.57[illegible] | +0.[illegible] | +0.[illegible] | +5.4907 | +0.79[illegible] |
| +2.8 | +0.[illegible] | +5.[illegible] | +1.[illegible] | +0.[illegible] | +5.[illegible] | +0.[illegible] |
| +3.0 | +0.[illegible]77 | +[illegible].0774 | +1.[illegible] | +0.[illegible] | +[illegible].057[illegible] | +1.1[illegible] |
| +3.2 | +0.[illegible] | +[illegible] | +1.[illegible] | +0.[illegible]770 | +[illegible].[illegible] | +1.35[illegible] |
| +3.4 | +0.[illegible]015 | +[illegible] | +1.[illegible] | +0.[illegible] | +[illegible] | +1.555[illegible] |
| +3.6 | +0.[illegible] | +[illegible] | +1.[illegible] | +0.[illegible] | +[illegible] | +1.75[illegible] |
| +3.8 | [illegible] | [illegible] | [illegible] | +0.[illegible] | [illegible] | +1.[illegible] |
| +4.0 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +4.2 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +4.4 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +4.6 | +1.0000 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +4.8 | +1.0000 | [illegible] | [illegible] | +1.0000 | [illegible] | [illegible] |

| $y$ | $x$ = +0.3[illegible] U | V | W | $x$ = +0.3[illegible] U | V | W |
|---|---|---|---|---|---|---|
| +0.0 | +0.0000 | +1.0000 | −0.3[illegible]00 | +0.0000 | +1.0000 | −0.3[illegible]00 |
| +0.2 | +0.033[illegible] | +1.0198 | −0.3[illegible] | +0.030[illegible] | +1.018[illegible] | −0.3771 |
| +0.4 | +0.0715 | +1.0847 | −0.3[illegible] | +0.0671 | +1.079[illegible] | −0.367[illegible] |
| +0.6 | +0.119[illegible] | +1.203[illegible] | −0.297[illegible] | +0.111[illegible] | +1.1707 | −0.3107 |
| +0.8 | +0.1749 | +1.3[illegible] | −0.20[illegible] | +0.1[illegible] | +1.[illegible] | −0.[illegible] |
| +1.0 | +0.240[illegible] | +1.6[illegible] | −0.1[illegible] | +0.2[illegible] | +1.[illegible]000 | −0.1[illegible] |
| +1.2 | +0.3144 | +1.7[illegible] | −0.111[illegible] | +0.2[illegible] | +1.[illegible]111 | −0.1[illegible] |
| +1.4 | +0.3961 | +2.[illegible] | −0.0[illegible] | +0.37[illegible] | +2.[illegible] | −0.0[illegible] |
| +1.6 | +0.4[illegible] | +2.[illegible]7 | +0.0[illegible] | +0.4[illegible] | +2.[illegible]78 | +0.0100 |
| +1.8 | +0.5[illegible] | +2.[illegible] | +0.15[illegible] | +0.5[illegible] | +2.[illegible] | +0.110[illegible] |
| +2.0 | +0.[illegible] | +3.0[illegible] | +0.2757 | +0.6[illegible] | +3.7[illegible] | +0.237[illegible] |
| +2.2 | +0.7[illegible] | +3.[illegible] | +0.[illegible] | +0.715[illegible] | +4.[illegible]57 | +0.371[illegible] |
| +2.4 | +0.[illegible]111 | [illegible] | +0.[illegible] | +0.7[illegible]70 | [illegible] | +0.5321 |
| +2.6 | +0.[illegible]7 | [illegible] | +0.7[illegible] | +0.[illegible] | [illegible] | +0.[illegible] |
| +2.8 | +0.[illegible] | +5.7[illegible] | +0.[illegible] | +0.[illegible] | +5.[illegible]1 | +0.[illegible] |
| [illegible] | +0.[illegible] | [illegible] | +1.[illegible] | +0.[illegible] | [illegible] | +1.[illegible] |
| [illegible] | +0.[illegible] | [illegible] | +1.[illegible] | +0.[illegible] | [illegible] | +1.[illegible] |
| [illegible] | +0.[illegible] | [illegible] | +1.[illegible] | +0.[illegible] | [illegible] | +1.[illegible] |
| [illegible] | +0.[illegible] | [illegible] | +1.[illegible] | +0.[illegible] | [illegible] | +1.[illegible] |
| [illegible] | +0.[illegible] | [illegible] | +1.[illegible] | +0.[illegible] | [illegible] | +1.[illegible] |
| [illegible] | +0.[illegible] | [illegible] | +[illegible] | +0.[illegible] | [illegible].5[illegible] | +2.[illegible] |
| [illegible] | +0.[illegible] | [illegible] | +[illegible] | +0.[illegible] | [illegible] | +[illegible] |
| [illegible] | +0.[illegible] | [illegible] | +[illegible] | +0.[illegible] | [illegible] | +[illegible] |
| [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | +[illegible] |
| [illegible] | +1.0000 | [illegible] | +[illegible] | +0.[illegible] | [illegible] | +[illegible] |
| [illegible] | +1.0000 | +[illegible].3098 | +[illegible] | +1.0000 | +[illegible] | +[illegible] |

| $y$ | $x =$ [illegible] | | | $x =$ [illegible] | | |
|---|---|---|---|---|---|---|
| +0.0 | +0.0000 | +1.0000 | [illegible] | +0.0000 | +1.0000 | [illegible] |
| +0.2 | [illegible] | +1.0173 | [illegible] | [illegible] | [illegible] | [illegible] |
| +0.4 | [illegible] | +1.0734 | [illegible] | [illegible] | +1.0608 | [illegible] |
| +0.6 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +0.8 | [illegible] | [illegible] | [illegible] | +0.1123 | [illegible] | −0.3401 |
| +1.0 | [illegible] | [illegible] | [illegible] | +0.2002 | [illegible] | [illegible] |
| +1.2 | [illegible] | [illegible] | −0.1501 | [illegible] | +1.8074 | [illegible] |
| +1.4 | +0.3531 | [illegible] | [illegible] | [illegible] | [illegible] | −0.1087 |
| +1.6 | +0.4350 | [illegible] | −0.0080 | [illegible] | [illegible] | −0.0341 |
| +1.8 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +2.0 | [illegible] | [illegible] | [illegible] | [illegible] | +3.5540 | [illegible] |
| +2.2 | [illegible] | [illegible] | [illegible] | +0.6601 | [illegible] | [illegible] |
| +2.4 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +2.6 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +2.8 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +3.0 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +3.2 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +3.4 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +3.6 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | +1.4054 |
| +3.8 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +4.0 | +0.9961 | [illegible] | +1.7571 | [illegible] | [illegible] | +1.8707 |
| +4.2 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| +4.4 | [illegible] | [illegible] | [illegible] | [illegible] | +6.7107 | [illegible] |
| +4.6 | [illegible] | [illegible] | [illegible] | [illegible] | +6.7191 | [illegible] |
| +4.8 | [illegible] | [illegible] | +2.7561 | [illegible] | +6.7319 | [illegible] |
| +5.0 | +1.0000 | [illegible] | +2.9561 | [illegible] | [illegible] | [illegible] |
| +5.2 | +1.0000 | [illegible] | +3.1561 | +1.0000 | [illegible] | +3.0893 |

| y | U ($x = +0.31$) | V ($x = +0.31$) | ψ ($x = +0.31$) | U ($x = +0.32$) | V ($x = +0.32$) | ψ ($x = +0.32$) |
|---|---|---|---|---|---|---|
| +0.0 | +0.0000 | +1.0000 | -0.3100 | +0.0000 | +1.0000 | -0.3200 |
| +0.2 | +0.0251 | +1.0153 | -0.3076 | +0.0235 | +1.0144 | -0.3177 |
| +0.4 | +0.0556 | +1.0654 | -0.2996 | +0.0522 | +1.0615 | -0.3102 |
| +0.6 | +0.0925 | +1.1576 | -0.2849 | +0.0869 | +1.1483 | -0.2964 |
| +0.8 | +0.1365 | +1.2996 | -0.2621 | +0.1283 | +1.2820 | -0.2750 |
| +1.0 | +0.1884 | +1.4987 | -0.2298 | +0.1772 | +1.4699 | -0.2446 |
| +1.2 | +0.2483 | +1.7612 | -0.1862 | +0.2339 | +1.7183 | -0.2036 |
| +1.4 | +0.3160 | +2.0906 | -0.1299 | +0.2983 | +2.0313 | -0.1505 |
| +1.6 | +0.3905 | +2.4862 | -0.0594 | +0.3695 | +2.4094 | -0.0839 |
| +1.8 | +0.4700 | +2.9415 | +0.0266 | +0.4463 | +2.8479 | -0.0024 |
| +2.0 | +0.5520 | +3.4432 | +0.1288 | +0.5263 | +3.3359 | +0.0949 |
| +2.2 | +0.6334 | +3.9713 | +0.2473 | +0.6068 | +3.8560 | +0.2082 |
| +2.4 | +0.7107 | +4.5008 | +0.3818 | +0.6843 | +4.3855 | +0.3374 |
| +2.6 | +0.7807 | +5.0051 | +0.5311 | +0.7562 | +4.8989 | +0.4816 |
| +2.8 | +0.8409 | +5.4595 | +0.6935 | +0.8192 | +5.3713 | +0.6393 |
| +3.0 | +0.8900 | +5.8459 | +0.8667 | +0.8718 | +5.7828 | +0.8086 |
| +3.2 | +0.9276 | +6.1552 | +1.0487 | +0.9133 | +6.1311 | +0.9873 |
| +3.4 | +0.9548 | +6.3878 | +1.2371 | +0.9443 | +6.3831 | +1.1732 |
| +3.6 | +0.9733 | +6.5519 | +1.4300 | +0.9660 | +6.5740 | +1.3644 |
| +3.8 | +0.9851 | +6.6604 | +1.6260 | +0.9803 | +6.7046 | +1.5591 |
| +4.0 | +0.9921 | +6.7276 | +1.8237 | +0.9892 | +6.7886 | +1.7562 |
| +4.2 | +0.9961 | +6.7667 | +2.0226 | +0.9944 | +6.8393 | +1.9546 |
| +4.4 | +0.9982 | +6.7879 | +2.2220 | +0.9973 | +6.8680 | +2.1538 |
| +4.6 | +0.9992 | +6.7987 | +2.4218 | +0.9987 | +6.8832 | +2.3534 |
| +4.8 | +0.9997 | +6.8039 | +2.6217 | +0.9994 | +6.8908 | +2.5532 |
| +5.0 | +0.9999 | +6.8062 | +2.8216 | +0.9998 | +6.8944 | +2.7531 |
| +5.2 | +1.0000 | +6.8072 | +3.0216 | +0.9999 | +6.8960 | +2.9531 |
| +5.4 | +1.0000 | +6.8075 | +3.2216 | +1.0000 | +6.8966 | +3.1531 |

| $y$ | $x = +0.33$ U | V | $\psi$ | $x = +0.34$ U | V | $\psi$ |
|---|---|---|---|---|---|---|
| +0.0 | +0.0000 | +1.0000 | -0.3300 | +0.0000 | +1.0000 | -0.3400 |
| +0.2 | +0.0221 | +1.0135 | -0.3279 | +0.0207 | +1.0127 | -0.3380 |
| +0.4 | +0.0490 | +1.0579 | -0.3208 | +0.0460 | +1.0546 | -0.3314 |
| +0.6 | +0.0816 | +1.1397 | -0.3079 | +0.0766 | +1.1317 | -0.3192 |
| +0.8 | +0.1206 | +1.2657 | -0.2878 | +0.1133 | +1.2507 | -0.3004 |
| +1.0 | +0.1667 | +1.4432 | -0.2592 | +0.1567 | +1.4184 | -0.2735 |
| +1.2 | +0.2203 | +1.6784 | -0.2206 | +0.2073 | +1.6412 | -0.2372 |
| +1.4 | +0.2814 | +1.9759 | -0.1706 | +0.2653 | +1.9241 | -0.1901 |
| +1.6 | +0.3495 | +2.3372 | -0.1076 | +0.3303 | +2.2692 | -0.1306 |
| +1.8 | +0.4234 | +2.7591 | -0.0304 | +0.4013 | +2.6748 | -0.0576 |
| +2.0 | +0.5013 | +3.2329 | +0.0620 | +0.4768 | +3.1342 | +0.0302 |
| +2.2 | +0.5805 | +3.7438 | +0.1702 | +0.5545 | +3.6347 | +0.1333 |
| +2.4 | +0.6581 | +4.2712 | +0.2941 | +0.6317 | +4.1582 | +0.2520 |
| +2.6 | +0.7311 | +4.7911 | +0.4331 | +0.7055 | +4.6822 | +0.3858 |
| +2.8 | +0.7965 | +5.2789 | +0.5861 | +0.7729 | +5.1829 | +0.5337 |
| +3.0 | +0.8524 | +5.7134 | +0.7511 | +0.8316 | +5.6381 | +0.6943 |
| +3.2 | +0.8976 | +6.0796 | +0.9263 | +0.8804 | +6.0310 | +0.8657 |
| +3.4 | +0.9323 | +6.3713 | +1.1094 | +0.9188 | +6.3520 | +1.0458 |
| +3.6 | +0.9574 | +6.5902 | +1.2986 | +0.9474 | +6.5999 | +1.2326 |
| +3.8 | +0.9745 | +6.7450 | +1.4919 | +0.9676 | +6.7806 | +1.4242 |
| +4.0 | +0.9856 | +6.8480 | +1.6880 | +0.9810 | +6.9049 | +1.6192 |
| +4.2 | +0.9922 | +6.9124 | +1.8858 | +0.9894 | +6.9854 | +1.8163 |
| +4.4 | +0.9960 | +6.9504 | +2.0847 | +0.9944 | +7.0346 | +2.0147 |
| +4.6 | +0.9981 | +6.9714 | +2.2841 | +0.9972 | +7.0629 | +2.2139 |
| +4.8 | +0.9991 | +6.9823 | +2.4838 | +0.9987 | +7.0782 | +2.4135 |
| +5.0 | +0.9996 | +6.9877 | +2.6837 | +0.9994 | +7.0861 | +2.6134 |
| +5.2 | +0.9998 | +6.9902 | +2.8837 | +0.9997 | +7.0898 | +2.8133 |
| +5.4 | +0.9999 | +6.9913 | +3.0837 | +0.9999 | +7.0916 | +3.0132 |
| +5.6 | +1.0000 | +6.9917 | +3.2836 | +1.0000 | +7.0923 | +3.2132 |
| +5.8 | +1.0000 | +6.9919 | +3.4836 | +1.0000 | +7.0926 | +3.4132 |

| | $x = +0.35$ | | | $x = +0.36$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | $\psi$ | U | V | $\psi$ |
| +0.0 | +0.0000 | +1.0000 | -0.3500 | +0.0000 | +1.0000 | -0.3600 |
| +0.2 | +0.0195 | +1.0120 | -0.3481 | +0.0182 | +1.0114 | -0.3582 |
| +0.4 | +0.0432 | +1.0515 | -0.3419 | +0.0405 | +1.0487 | -0.3524 |
| +0.6 | +0.0719 | +1.1243 | -0.3305 | +0.0674 | +1.1174 | -0.3417 |
| +0.8 | +0.1063 | +1.2367 | -0.3128 | +0.0998 | +1.2236 | -0.3251 |
| +1.0 | +0.1472 | +1.3953 | -0.2876 | +0.1383 | +1.3737 | -0.3014 |
| +1.2 | +0.1950 | +1.6065 | -0.2534 | +0.1834 | +1.5740 | -0.2693 |
| +1.4 | +0.2500 | +1.8754 | -0.2091 | +0.2354 | +1.8298 | -0.2276 |
| +1.6 | +0.3119 | +2.2051 | -0.1530 | +0.2943 | +2.1446 | -0.1747 |
| +1.8 | +0.3800 | +2.5948 | -0.0839 | +0.3594 | +2.5188 | -0.1095 |
| +2.0 | +0.4530 | +3.0396 | -0.0007 | +0.4299 | +2.9490 | -0.0306 |
| +2.2 | +0.5290 | +3.5290 | +0.0975 | +0.5039 | +3.4265 | +0.0627 |
| +2.4 | +0.6054 | +4.0468 | +0.2110 | +0.5792 | +3.9374 | +0.1710 |
| +2.6 | +0.6795 | +4.5728 | +0.3395 | +0.6533 | +4.4632 | +0.2943 |
| +2.8 | +0.7484 | +5.0838 | +0.4824 | +0.7233 | +4.9821 | +0.4321 |
| +3.0 | +0.8098 | +5.5575 | +0.6384 | +0.7868 | +5.4719 | +0.5832 |
| +3.2 | +0.8618 | +5.9753 | +0.8057 | +0.8419 | +5.9128 | +0.7462 |
| +3.4 | +0.9039 | +6.3251 | +0.9824 | +0.8875 | +6.2906 | +0.9193 |
| +3.6 | +0.9361 | +6.6026 | +1.1666 | +0.9233 | +6.5980 | +1.1006 |
| +3.8 | +0.9595 | +6.8109 | +1.3563 | +0.9500 | +6.8350 | +1.2880 |
| +4.0 | +0.9755 | +6.9586 | +1.5499 | +0.9689 | +7.0081 | +1.4800 |
| +4.2 | +0.9859 | +7.0575 | +1.7461 | +0.9815 | +7.1277 | +1.6752 |
| +4.4 | +0.9923 | +7.1200 | +1.9440 | +0.9896 | +7.2058 | +1.8723 |
| +4.6 | +0.9960 | +7.1573 | +2.1428 | +0.9944 | +7.2541 | +2.0708 |
| +4.8 | +0.9980 | +7.1783 | +2.3423 | +0.9971 | +7.2823 | +2.2700 |
| +5.0 | +0.9991 | +7.1895 | +2.5420 | +0.9986 | +7.2979 | +2.4696 |
| +5.2 | +0.9996 | +7.1951 | +2.7419 | +0.9994 | +7.3060 | +2.6694 |
| +5.4 | +0.9998 | +7.1978 | +2.9418 | +0.9997 | +7.3100 | +2.8693 |
| +5.6 | +0.9999 | +7.1989 | +3.1418 | +0.9999 | +7.3119 | +3.0692 |
| +5.8 | +1.0000 | +7.1995 | +3.3418 | +1.0000 | +7.3127 | +3.2692 |
| +6.0 | +1.0000 | +7.1997 | +3.5418 | +1.0000 | +7.3131 | +3.4692 |

| $y$ | $x = +0.37$ U | V | $\psi$ | $x = +0.38$ U | V | $\psi$ |
|---|---|---|---|---|---|---|
| +0.0 | +0.0000 | +1.0000 | -0.3700 | +0.0000 | +1.0000 | -0.3800 |
| +0.2 | +0.0171 | +1.0107 | -0.3683 | +0.0160 | +1.0101 | -0.3785 |
| +0.4 | +0.0380 | +1.0460 | -0.3629 | +0.0356 | +1.0435 | -0.3734 |
| +0.6 | +0.0632 | +1.1109 | -0.3529 | +0.0593 | +1.1049 | -0.3639 |
| +0.8 | +0.0936 | +1.2114 | -0.3373 | +0.0878 | +1.2000 | -0.3493 |
| +1.0 | +0.1298 | +1.3536 | -0.3150 | +0.1218 | +1.3347 | -0.3285 |
| +1.2 | +0.1723 | +1.5436 | -0.2849 | +0.1618 | +1.5150 | -0.3002 |
| +1.4 | +0.2215 | +1.7869 | -0.2457 | +0.2082 | +1.7465 | -0.2633 |
| +1.6 | +0.2774 | +2.0875 | -0.1959 | +0.2613 | +2.0335 | -0.2165 |
| +1.8 | +0.3397 | +2.4466 | -0.1343 | +0.3206 | +2.3780 | -0.1584 |
| +2.0 | +0.4074 | +2.8622 | -0.0597 | +0.3857 | +2.7790 | -0.0879 |
| +2.2 | +0.4793 | +3.3273 | +0.0290 | +0.4553 | +3.2313 | -0.0038 |
| +2.4 | +0.5533 | +3.8300 | +0.1322 | +0.5276 | +3.7248 | +0.0944 |
| +2.6 | +0.6269 | +4.3538 | +0.2502 | +0.6005 | +4.2450 | +0.2072 |
| +2.8 | +0.6977 | +4.8783 | +0.3828 | +0.6716 | +4.7729 | +0.3345 |
| +3.0 | +0.7630 | +5.3818 | +0.5289 | +0.7383 | +5.2878 | +0.4756 |
| +3.2 | +0.8208 | +5.8440 | +0.6875 | +0.7984 | +5.7690 | +0.6294 |
| +3.4 | +0.8696 | +6.2485 | +0.8566 | +0.8503 | +6.1989 | +0.7944 |
| +3.6 | +0.9089 | +6.5856 | +1.0347 | +0.8931 | +6.5652 | +0.9689 |
| +3.8 | +0.9391 | +6.8523 | +1.2196 | +0.9267 | +6.8623 | +1.1510 |
| +4.0 | +0.9610 | +7.0526 | +1.4097 | +0.9519 | +7.0913 | +1.3390 |
| +4.2 | +0.9762 | +7.1952 | +1.6036 | +0.9698 | +7.2591 | +1.5313 |
| +4.4 | +0.9861 | +7.2913 | +1.7999 | +0.9818 | +7.3756 | +1.7265 |
| +4.6 | +0.9923 | +7.3527 | +1.9978 | +0.9896 | +7.4524 | +1.9237 |
| +4.8 | +0.9959 | +7.3898 | +2.1966 | +0.9943 | +7.5004 | +2.1222 |
| +5.0 | +0.9979 | +7.4111 | +2.3960 | +0.9970 | +7.5289 | +2.3213 |
| +5.2 | +0.9990 | +7.4226 | +2.5957 | +0.9985 | +7.5449 | +2.5209 |
| +5.4 | +0.9995 | +7.4285 | +2.7956 | +0.9993 | +7.5534 | +2.7207 |
| +5.6 | +0.9998 | +7.4314 | +2.9955 | +0.9997 | +7.5578 | +2.9206 |
| +5.8 | +0.9999 | +7.4328 | +3.1955 | +0.9999 | +7.5598 | +3.1205 |
| +6.0 | +1.0000 | +7.4333 | +3.3955 | +0.9999 | +7.5608 | +3.3205 |
| +6.2 | +1.0000 | +7.4336 | +3.5955 | +1.0000 | +7.5614 | +3.5205 |

| $y$ | $x = +0.39$ U | V | $\psi$ | $x = +0.40$ U | V | $\psi$ |
|---|---|---|---|---|---|---|
| +0.0 | +0.0000 | +1.0000 | −0.3900 | +0.0000 | +1.0000 | −0.4000 |
| +0.2 | +0.0150 | +1.0096 | −0.3885 | +0.0140 | +1.0091 | −0.3986 |
| +0.4 | +0.0333 | +1.0411 | −0.3838 | +0.0312 | +1.0389 | −0.3942 |
| +0.6 | +0.0555 | +1.0993 | −0.3750 | +0.0519 | +1.0940 | −0.3859 |
| +0.8 | +0.0822 | +1.1893 | −0.3613 | +0.0770 | +1.1793 | −0.3731 |
| +1.0 | +0.1141 | +1.3170 | −0.3417 | +0.1069 | +1.3003 | −0.3548 |
| +1.2 | +0.1518 | +1.4882 | −0.3152 | +0.1423 | +1.4629 | −0.3300 |
| +1.4 | +0.1956 | +1.7085 | −0.2806 | +0.1836 | +1.6725 | −0.2975 |
| +1.6 | +0.2458 | +1.9825 | −0.2366 | +0.2311 | +1.9340 | −0.2562 |
| +1.8 | +0.3023 | +2.3129 | −0.1819 | +0.2847 | +2.2506 | −0.2047 |
| +2.0 | +0.3647 | +2.6995 | −0.1153 | +0.3443 | +2.6231 | −0.1419 |
| +2.2 | +0.4318 | +3.1388 | −0.0357 | +0.4090 | +3.0490 | −0.0666 |
| +2.4 | +0.5023 | +3.6224 | +0.0577 | +0.4774 | +3.5218 | +0.0220 |
| +2.6 | +0.5742 | +4.1375 | +0.1653 | +0.5480 | +4.0304 | +0.1245 |
| +2.8 | +0.6452 | +4.6670 | +0.2873 | +0.6186 | +4.5594 | +0.2412 |
| +3.0 | +0.7129 | +5.1911 | +0.4232 | +0.6869 | +5.0904 | +0.3718 |
| +3.2 | +0.7750 | +5.6893 | +0.5721 | +0.7506 | +5.6032 | +0.5156 |
| +3.4 | +0.8296 | +6.1429 | +0.7327 | +0.8077 | +6.0786 | +0.6716 |
| +3.6 | +0.8757 | +6.5376 | +0.9034 | +0.8569 | +6.5008 | +0.8382 |
| +3.8 | +0.9128 | +6.8654 | +1.0824 | +0.8974 | +6.8591 | +1.0137 |
| +4.0 | +0.9413 | +7.1246 | +1.2679 | +0.9292 | +7.1493 | +1.1965 |
| +4.2 | +0.9621 | +7.1951 | +1.4584 | +0.9532 | +7.3733 | +1.3849 |
| +4.4 | +0.9766 | +7.4589 | +1.6523 | +0.9703 | +7.5380 | +1.5774 |
| +4.6 | +0.9862 | +7.5536 | +1.8487 | +0.9819 | +7.6531 | +1.7727 |
| +4.8 | +0.9922 | +7.6147 | +2.0466 | +0.9895 | +7.7296 | +1.9699 |
| +5.0 | +0.9958 | +7.6521 | +2.2454 | +0.9942 | +7.7780 | +2.1683 |
| +5.2 | +0.9978 | +7.6739 | +2.4448 | +0.9969 | +7.8071 | +2.3674 |
| +5.4 | +0.9989 | +7.6859 | +2.6445 | +0.9984 | +7.8238 | +2.5669 |
| +5.6 | +0.9995 | +7.6922 | +2.8443 | +0.9992 | +7.8328 | +2.7667 |
| +5.8 | +0.9998 | +7.6954 | +3.0443 | +0.9996 | +7.8375 | +2.9666 |
| +6.0 | +0.9999 | +7.6969 | +3.2442 | +0.9998 | +7.8398 | +3.1666 |
| +6.2 | +1.0000 | +7.6975 | +3.4442 | +0.9999 | +7.8409 | +3.3665 |
| +6.4 | +1.0000 | +7.6978 | +3.6442 | +1.0000 | +7.8414 | +3.5665 |
| +6.6 | +1.0000 | +7.6979 | +3.8442 | +1.0000 | +7.8416 | +3.7665 |

| $y$ | $x=+0.41$ U | V | ψ | $x=+0.42$ U | V | ψ |
|---|---|---|---|---|---|---|
| +0.0 | +0.0000 | +1.0000 | −0.4100 | +0.0000 | +1.0000 | −0.4200 |
| +0.2 | +0.0131 | +1.0086 | −0.4087 | +0.0123 | +1.0081 | −0.4188 |
| +0.4 | +0.0291 | +1.0369 | −0.4046 | +0.0272 | +1.0349 | −0.4149 |
| +0.6 | +0.0486 | +1.0890 | −0.3969 | +0.0454 | +1.0843 | −0.4077 |
| +0.8 | +0.0720 | +1.1698 | −0.3849 | +0.0673 | +1.1610 | −0.3965 |
| +1.0 | +0.1000 | +1.2847 | −0.3677 | +0.0935 | +1.2699 | −0.3805 |
| +1.2 | +0.1333 | +1.4391 | −0.3445 | +0.1247 | +1.4166 | −0.3588 |
| +1.4 | +0.1721 | +1.6386 | −0.3141 | +0.1612 | +1.6064 | −0.3303 |
| +1.6 | +0.2169 | +1.8881 | −0.2753 | +0.2035 | +1.8445 | −0.2939 |
| +1.8 | +0.2679 | +2.1913 | −0.2269 | +0.2517 | +2.1349 | −0.2485 |
| +2.0 | +0.3247 | +2.5499 | −0.1677 | +0.3057 | +2.4797 | −0.1929 |
| +2.2 | +0.3867 | +2.9624 | −0.0967 | +0.3651 | +2.8788 | −0.1259 |
| +2.4 | +0.4530 | +3.4238 | −0.0128 | +0.4291 | +3.3283 | −0.0465 |
| +2.6 | +0.5221 | +3.9248 | +0.0847 | +0.4964 | +3.8206 | +0.0460 |
| +2.8 | +0.5919 | +4.4517 | +0.1961 | +0.5652 | +4.3438 | +0.1521 |
| +3.0 | +0.6604 | +4.9874 | +0.3214 | +0.6335 | +4.8823 | +0.2720 |
| +3.2 | +0.7253 | +5.5126 | +0.4600 | +0.6993 | +5.4176 | +0.4054 |
| +3.4 | +0.7846 | +6.0078 | +0.6111 | +0.7603 | +5.9306 | +0.5514 |
| +3.6 | +0.8366 | +6.4560 | +0.7734 | +0.8149 | +6.4034 | +0.7091 |
| +3.8 | +0.8804 | +6.8446 | +0.9452 | +0.8618 | +6.8214 | +0.8769 |
| +4.0 | +0.9156 | +7.1665 | +1.1250 | +0.9004 | +7.1754 | +1.0532 |
| +4.2 | +0.9428 | +7.4212 | +1.3109 | +0.9309 | +7.4621 | +1.2365 |
| +4.4 | +0.9628 | +7.6134 | +1.5016 | +0.9539 | +7.6840 | +1.4251 |
| +4.6 | +0.9767 | +7.7515 | +1.6956 | +0.9704 | +7.8477 | +1.6176 |
| +4.8 | +0.9861 | +7.8461 | +1.8920 | +0.9813 | +7.9631 | +1.8129 |
| +5.0 | +0.9920 | +7.9078 | +2.0898 | +0.9893 | +8.0405 | +2.0101 |
| +5.2 | +0.9956 | +7.9460 | +2.2886 | +0.9939 | +8.0901 | +2.2085 |
| +5.4 | +0.9977 | +7.9687 | +2.4880 | +0.9967 | +8.1203 | +2.4075 |
| +5.6 | +0.9988 | +7.9814 | +2.6877 | +0.9983 | +8.1379 | +2.6071 |
| +5.8 | +0.9994 | +7.9882 | +2.8875 | +0.9992 | +8.1476 | +2.8068 |
| +6.0 | +0.9997 | +7.9917 | +3.0874 | +0.9996 | +8.1528 | +3.0067 |
| +6.2 | +0.9999 | +7.9934 | +3.2874 | +0.9998 | +8.1554 | +3.2066 |
| +6.4 | +1.0000 | +7.9942 | +3.4874 | +0.9999 | +8.1566 | +3.4066 |
| +6.6 | +1.0000 | +7.9945 | +3.6874 | +1.0000 | +8.1572 | +3.6066 |
| +6.8 | +1.0000 | +7.9947 | +3.8874 | +1.0000 | +8.1574 | +3.8066 |

| | $x = +0.43$ | | | $x = +0.44$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | Ψ | U | V | Ψ |
| +0.0 | +0.0000 | +1.0000 | −0.4300 | +0.0000 | +1.0000 | −0.4400 |
| +0.2 | +0.0114 | +1.0077 | −0.4289 | +0.0107 | +1.0073 | −0.4390 |
| +0.4 | +0.0254 | +1.0331 | −0.4253 | +0.0237 | +1.0314 | −0.4356 |
| +0.6 | +0.0423 | +1.0799 | −0.4185 | +0.0395 | +1.0758 | −0.4293 |
| +0.8 | +0.0628 | +1.1526 | −0.4081 | +0.0586 | +1.1447 | −0.4196 |
| +1.0 | +0.0873 | +1.2560 | −0.3931 | +0.0815 | +1.2428 | −0.4056 |
| +1.2 | +0.1165 | +1.3953 | −0.3728 | +0.1087 | +1.3751 | −0.3867 |
| +1.4 | +0.1508 | +1.5760 | −0.3462 | +0.1409 | +1.5471 | −0.3618 |
| +1.6 | +0.1906 | +1.8031 | −0.3122 | +0.1783 | +1.7637 | −0.3300 |
| +1.8 | +0.2361 | +2.0810 | −0.2696 | +0.2212 | +2.0296 | −0.2901 |
| +2.0 | +0.2874 | +2.4125 | −0.2173 | +0.2698 | +2.3479 | −0.2411 |
| +2.2 | +0.3442 | +2.7981 | −0.1543 | +0.3239 | +2.7201 | −0.1818 |
| +2.4 | +0.4058 | +3.2353 | −0.0793 | +0.3830 | +3.1448 | −0.1112 |
| +2.6 | +0.4711 | +3.7181 | +0.0083 | +0.4462 | +3.6172 | −0.0284 |
| +2.8 | +0.5386 | +4.2362 | +0.1092 | +0.5121 | +4.1288 | +0.0674 |
| +3.0 | +0.6064 | +4.7754 | +0.2238 | +0.5791 | +4.6672 | +0.1765 |
| +3.2 | +0.6726 | +5.3188 | +0.3517 | +0.6454 | +5.2163 | +0.2990 |
| +3.4 | +0.7350 | +5.8474 | +0.4925 | +0.7089 | +5.7583 | +0.4345 |
| +3.6 | +0.7919 | +6.3428 | +0.6453 | +0.7676 | +6.2745 | +0.5822 |
| +3.8 | +0.8417 | +6.7893 | +0.8088 | +0.8200 | +6.7482 | +0.7411 |
| +4.0 | +0.8836 | +7.1754 | +0.9815 | +0.8651 | +7.1660 | +0.9098 |
| +4.2 | +0.9174 | +7.4952 | +1.1617 | +0.9023 | +7.5197 | +1.0866 |
| +4.4 | +0.9436 | +7.7487 | +1.3479 | +0.9317 | +7.8065 | +1.2702 |
| +4.6 | +0.9629 | +7.9407 | +1.5387 | +0.9540 | +8.0291 | +1.4588 |
| +4.8 | +0.9766 | +8.0796 | +1.7327 | +0.9702 | +8.1945 | +1.6514 |
| +5.0 | +0.9858 | +8.1755 | +1.9290 | +0.9814 | +8.3118 | +1.8466 |
| +5.2 | +0.9917 | +8.2307 | +2.1268 | +0.9889 | +8.3914 | +2.0437 |
| +5.4 | +0.9954 | +8.2785 | +2.3256 | +0.9936 | +8.4429 | +2.2420 |
| +5.6 | +0.9975 | +8.3024 | +2.5249 | +0.9965 | +8.4749 | +2.4410 |
| +5.8 | +0.9987 | +8.3160 | +2.7245 | +0.9981 | +8.4937 | +2.6405 |
| +6.0 | +0.9994 | +8.3235 | +2.9243 | +0.9991 | +8.5044 | +2.8402 |
| +6.2 | +0.9997 | +8.3274 | +3.1242 | +0.9995 | +8.5101 | +3.0401 |
| +6.4 | +0.9999 | +8.3294 | +3.3242 | +0.9998 | +8.5131 | +3.2400 |
| +6.6 | +0.9999 | +8.3303 | +3.5242 | +0.9999 | +8.5146 | +3.4400 |
| +6.8 | +1.0000 | +8.3307 | +3.7242 | +1.0000 | +8.5153 | +3.6400 |
| +7.0 | +1.0000 | +8.3309 | +3.9242 | +1.0000 | +8.5156 | +3.8399 |

| | $x = +0.45$ | | | $x = +0.46$ | | |
|---|---|---|---|---|---|---|
| $y$ | U | V | $\psi$ | U | V | $\psi$ |
| +0.0 | +0.0000 | +1.0000 | -0.4500 | +0.0000 | +1.0000 | -0.4600 |
| +0.2 | +0.0099 | +1.0069 | -0.4490 | +0.0092 | +1.0066 | -0.4591 |
| +0.4 | +0.0220 | +1.0297 | -0.4459 | +0.0205 | +1.0282 | -0.4562 |
| +0.6 | +0.0367 | +1.0718 | -0.4401 | +0.0342 | +1.0681 | -0.4508 |
| +0.8 | +0.0545 | +1.1372 | -0.4310 | +0.0507 | +1.1301 | -0.4423 |
| +1.0 | +0.0759 | +1.2303 | -0.4180 | +0.0706 | +1.2185 | -0.4302 |
| +1.2 | +0.1014 | +1.3561 | -0.4003 | +0.0944 | +1.3380 | -0.4138 |
| +1.4 | +0.1314 | +1.5197 | -0.3771 | +0.1224 | +1.4936 | -0.3922 |
| +1.6 | +0.1665 | +1.7263 | -0.3474 | +0.1553 | +1.6905 | -0.3645 |
| +1.8 | +0.2069 | +1.9805 | -0.3102 | +0.1933 | +1.9335 | -0.3298 |
| +2.0 | +0.2529 | +2.2860 | -0.2643 | +0.2366 | +2.2265 | -0.2869 |
| +2.2 | +0.3043 | +2.6449 | -0.2087 | +0.2854 | +2.5722 | -0.2347 |
| +2.4 | +0.3608 | +3.0568 | -0.1422 | +0.3393 | +2.9711 | -0.1724 |
| +2.6 | +0.4217 | +3.5182 | -0.0640 | +0.3977 | +3.4211 | -0.0987 |
| +2.8 | +0.4858 | +4.0222 | +0.0267 | +0.4599 | +3.9164 | -0.0130 |
| +3.0 | +0.5518 | +4.5579 | +0.1304 | +0.5245 | +4.4479 | +0.0854 |
| +3.2 | +0.6178 | +5.1108 | +0.2474 | +0.5898 | +5.0026 | +0.1968 |
| +3.4 | +0.6819 | +5.6639 | +0.3774 | +0.6542 | +5.5646 | +0.3212 |
| +3.6 | +0.7421 | +6.1990 | +0.5199 | +0.7156 | +6.1162 | +0.4583 |
| +3.8 | +0.7969 | +6.6983 | +0.6739 | +0.7724 | +6.6395 | +0.6072 |
| +4.0 | +0.8449 | +7.1472 | +0.8382 | +0.8231 | +7.1184 | +0.7669 |
| +4.2 | +0.8854 | +7.5349 | +1.0114 | +0.8668 | +7.5404 | +0.9360 |
| +4.4 | +0.9182 | +7.8565 | +1.1918 | +0.9029 | +7.8979 | +1.1131 |
| +4.6 | +0.9436 | +8.1122 | +1.3781 | +0.9316 | +8.1886 | +1.2966 |
| +4.8 | +0.9625 | +8.3069 | +1.5688 | +0.9535 | +8.4154 | +1.4852 |
| +5.0 | +0.9761 | +8.4487 | +1.7628 | +0.9695 | +8.5850 | +1.6776 |
| +5.2 | +0.9853 | +8.5476 | +1.9590 | +0.9807 | +8.7064 | +1.8727 |
| +5.4 | +0.9913 | +8.6135 | +2.1567 | +0.9883 | +8.7896 | +2.0697 |
| +5.6 | +0.9950 | +8.6556 | +2.3553 | +0.9932 | +8.8442 | +2.2679 |
| +5.8 | +0.9973 | +8.6812 | +2.5546 | +0.9962 | +8.8786 | +2.4668 |
| +6.0 | +0.9986 | +8.6962 | +2.7542 | +0.9979 | +8.8992 | +2.6662 |
| +6.2 | +0.9993 | +8.7045 | +2.9540 | +0.9989 | +8.9111 | +2.8659 |
| +6.4 | +0.9997 | +8.7090 | +3.1539 | +0.9995 | +8.9176 | +3.0658 |
| +6.6 | +0.9998 | +8.7112 | +3.3538 | +0.9997 | +8.9211 | +3.2657 |
| +6.8 | +0.9999 | +8.7123 | +3.5538 | +0.9999 | +8.9228 | +3.4657 |
| +7.0 | +1.0000 | +8.7128 | +3.7538 | +0.9999 | +8.9237 | +3.6656 |
| +7.2 | +1.0000 | +8.7131 | +3.9538 | +1.0000 | +8.9241 | +3.8656 |
| +7.4 | +1.0000 | +8.7132 | +4.1538 | +1.0000 | +8.9242 | +4.0656 |

| y | U (x = +0.47) | V (x = +0.47) | ψ (x = +0.47) | U (x = +0.48) | V (x = +0.48) | ψ (x = +0.48) |
|---|---|---|---|---|---|---|
| +0.0 | +0.0000 | +1.0000 | −0.4700 | +0.0000 | +1.0000 | −0.4800 |
| +0.2 | +0.0086 | +1.0062 | −0.4692 | +0.0079 | +1.0059 | −0.4792 |
| +0.4 | +0.0190 | +1.0267 | −0.4664 | +0.0176 | +1.0253 | −0.4767 |
| +0.6 | +0.0317 | +1.0646 | −0.4614 | +0.0294 | +1.0612 | −0.4720 |
| +0.8 | +0.0471 | +1.1234 | −0.4536 | +0.0437 | +1.1170 | −0.4648 |
| +1.0 | +0.0656 | +1.2072 | −0.4424 | +0.0608 | +1.1966 | −0.4544 |
| +1.2 | +0.0877 | +1.3207 | −0.4271 | +0.0814 | +1.3043 | −0.4402 |
| +1.4 | +0.1139 | +1.4688 | −0.4070 | +0.1057 | +1.4451 | −0.4216 |
| +1.6 | +0.1446 | +1.6564 | −0.3813 | +0.1344 | +1.6239 | −0.3977 |
| +1.8 | +0.1802 | +1.8885 | −0.3489 | +0.1677 | +1.8455 | −0.3675 |
| +2.0 | +0.2210 | +2.1694 | −0.3088 | +0.2060 | +2.1144 | −0.3303 |
| +2.2 | +0.2671 | +2.5021 | −0.2601 | +0.2494 | +2.4343 | −0.2848 |
| +2.4 | +0.3183 | +2.8879 | −0.2017 | +0.2980 | +2.8070 | −0.2301 |
| +2.6 | +0.3743 | +3.3258 | −0.1325 | +0.3515 | +3.2325 | −0.1653 |
| +2.8 | +0.4343 | +3.8115 | −0.0517 | +0.4092 | +3.7078 | −0.0893 |
| +3.0 | +0.4973 | +4.3374 | +0.0415 | +0.4703 | +4.2268 | −0.0014 |
| +3.2 | +0.5617 | +4.8920 | +0.1473 | +0.5334 | +4.7795 | +0.0990 |
| +3.4 | +0.6259 | +5.4608 | +0.2661 | +0.5972 | +5.3528 | +0.2120 |
| +3.6 | +0.6881 | +6.0267 | +0.3976 | +0.6598 | +5.9307 | +0.3378 |
| +3.8 | +0.7466 | +6.5719 | +0.5411 | +0.7196 | +6.4958 | +0.4758 |
| +4.0 | +0.7997 | +7.0795 | +0.6959 | +0.7748 | +7.0306 | +0.6253 |
| +4.2 | +0.8463 | +7.5354 | +0.8606 | +0.8241 | +7.5196 | +0.7853 |
| +4.4 | +0.8857 | +7.9295 | +1.0339 | +0.8667 | +7.9509 | +0.9545 |
| +4.6 | +0.9178 | +8.2573 | +1.2144 | +0.9021 | +8.3171 | +1.1315 |
| +4.8 | +0.9428 | +8.5190 | +1.4006 | +0.9304 | +8.6162 | +1.3149 |
| +5.0 | +0.9616 | +8.7195 | +1.5911 | +0.9522 | +8.8510 | +1.5032 |
| +5.2 | +0.9751 | +8.8668 | +1.7848 | +0.9683 | +9.0279 | +1.6954 |
| +5.4 | +0.9845 | +8.9706 | +1.9809 | +0.9797 | +9.1558 | +1.8902 |
| +5.6 | +0.9907 | +9.0406 | +2.1784 | +0.9875 | +9.2445 | +2.0870 |
| +5.8 | +0.9946 | +9.0859 | +2.3770 | +0.9926 | +9.3035 | +2.2851 |
| +6.0 | +0.9970 | +9.1140 | +2.5762 | +0.9957 | +9.3412 | +2.4839 |
| +6.2 | +0.9984 | +9.1307 | +2.7757 | +0.9976 | +9.3642 | +2.6833 |
| +6.4 | +0.9992 | +9.1402 | +2.9755 | +0.9988 | +9.3777 | +2.8829 |
| +6.6 | +0.9996 | +9.1453 | +3.1754 | +0.9994 | +9.3853 | +3.0827 |
| +6.8 | +0.9998 | +9.1480 | +3.3753 | +0.9997 | +9.3895 | +3.2826 |
| +7.0 | +0.9999 | +9.1494 | +3.5753 | +0.9999 | +9.3916 | +3.4826 |
| +7.2 | +1.0000 | +9.1500 | +3.7753 | +0.9999 | +9.3926 | +3.6826 |
| +7.4 | +1.0000 | +9.1503 | +3.9753 | +1.0000 | +9.3931 | +3.8826 |
| +7.6 | +1.0000 | +9.1505 | +4.1753 | +1.0000 | +9.3934 | +4.0826 |

| y | U ($x=+0.49$) | V | ψ | U ($x=+0.50$) | V | ψ |
|---|---|---|---|---|---|---|
| +0.0 | +0.0000 | +1.0000 | -0.4900 | +0.0000 | +1.0000 | -0.5000 |
| +0.2 | +0.0073 | +1.0056 | -0.4893 | +0.0068 | +1.0053 | -0.4993 |
| +0.4 | +0.0163 | +1.0240 | -0.4870 | +0.0150 | +1.0228 | -0.4972 |
| +0.6 | +0.0272 | +1.0580 | -0.4826 | +0.0251 | +1.0550 | -0.4932 |
| +0.8 | +0.0404 | +1.1109 | -0.4759 | +0.0373 | +1.1051 | -0.4870 |
| +1.0 | +0.0563 | +1.1864 | -0.4663 | +0.0520 | +1.1767 | -0.4781 |
| +1.2 | +0.0753 | +1.2887 | -0.4532 | +0.0696 | +1.2739 | -0.4660 |
| +1.4 | +0.0980 | +1.4226 | -0.4359 | +0.0906 | +1.4010 | -0.4500 |
| +1.6 | +0.1246 | +1.5928 | -0.4137 | +0.1154 | +1.5630 | -0.4295 |
| +1.8 | +0.1557 | +1.8042 | -0.3858 | +0.1443 | +1.7646 | -0.4036 |
| +2.0 | +0.1916 | +2.0616 | -0.3511 | +0.1778 | +2.0107 | -0.3715 |
| +2.2 | +0.2325 | +2.3688 | -0.3088 | +0.2161 | +2.3054 | -0.3322 |
| +2.4 | +0.2784 | +2.7283 | -0.2578 | +0.2594 | +2.6519 | -0.2847 |
| +2.6 | +0.3292 | +3.1411 | -0.1971 | +0.3076 | +3.0516 | -0.2281 |
| +2.8 | +0.3845 | +3.6053 | -0.1258 | +0.3603 | +3.5040 | -0.1614 |
| +3.0 | +0.4435 | +4.1161 | -0.0431 | +0.4171 | +4.0056 | -0.0837 |
| +3.2 | +0.5052 | +4.6652 | +0.0518 | +0.4770 | +4.5495 | +0.0057 |
| +3.4 | +0.5681 | +5.2410 | +0.1591 | +0.5388 | +5.1257 | +0.1072 |
| +3.6 | +0.6307 | +5.8286 | +0.2790 | +0.6010 | +5.7207 | +0.2212 |
| +3.8 | +0.6914 | +6.4113 | +0.4112 | +0.6622 | +6.3186 | +0.3476 |
| +4.0 | +0.7483 | +6.9714 | +0.5553 | +0.7205 | +6.9020 | +0.4859 |
| +4.2 | +0.8002 | +7.4925 | +0.7102 | +0.7745 | +7.4539 | +0.6355 |
| +4.4 | +0.8458 | +7.9608 | +0.8749 | +0.8229 | +7.9589 | +0.7953 |
| +4.6 | +0.8845 | +8.3668 | +1.0481 | +0.8648 | +8.4053 | +0.9642 |
| +4.8 | +0.9162 | +8.7057 | +1.2283 | +0.8999 | +8.7860 | +1.1408 |
| +5.0 | +0.9411 | +8.9779 | +1.4141 | +0.9281 | +9.0987 | +1.3237 |
| +5.2 | +0.9600 | +9.1881 | +1.6043 | +0.9501 | +9.3460 | +1.5116 |
| +5.4 | +0.9738 | +9.3440 | +1.7977 | +0.9665 | +9.5340 | +1.7034 |
| +5.6 | +0.9834 | +9.4550 | +1.9935 | +0.9782 | +9.6715 | +1.8979 |
| +5.8 | +0.9899 | +9.5309 | +2.1909 | +0.9864 | +9.7681 | +2.0944 |
| +6.0 | +0.9940 | +9.5808 | +2.3893 | +0.9917 | +9.8332 | +2.2923 |
| +6.2 | +0.9966 | +9.6122 | +2.5884 | +0.9952 | +9.8755 | +2.4910 |
| +6.4 | +0.9981 | +9.6312 | +2.7879 | +0.9973 | +9.9018 | +2.6902 |
| +6.6 | +0.9990 | +9.6422 | +2.9876 | +0.9985 | +9.9176 | +2.8898 |
| +6.8 | +0.9995 | +9.6484 | +3.1875 | +0.9992 | +9.9266 | +3.0896 |
| +7.0 | +0.9998 | +9.6517 | +3.3874 | +0.9996 | +9.9316 | +3.2895 |
| +7.2 | +0.9999 | +9.6534 | +3.5874 | +0.9998 | +9.9343 | +3.4894 |
| +7.4 | +0.9999 | +9.6542 | +3.7873 | +0.9999 | +9.9356 | +3.6894 |
| +7.6 | +1.0000 | +9.6546 | +3.9873 | +1.0000 | +9.9363 | +3.8894 |
| +7.8 | +1.0000 | +9.6548 | +4.1873 | +1.0000 | +9.9366 | +4.0894 |

TABLE 2.- U, $\tau$ AND $y$

| $x = +0.51$ | | | $x = +0.52$ | | |
|---|---|---|---|---|---|
| U | $\tau$ | $y$ | U | $\tau$ | $y$ |
| +0.000 | +0.0282 | +0.0000 | +0.000 | +0.0259 | +0.0000 |
| +0.025 | +0.0528 | +0.6362 | +0.025 | +0.0505 | +0.6771 |
| +0.050 | +0.0761 | +1.0275 | +0.050 | +0.0738 | +1.0836 |
| +0.075 | +0.0982 | +1.3155 | +0.075 | +0.0957 | +1.3799 |
| +0.100 | +0.1190 | +1.5463 | +0.100 | +0.1164 | +1.6163 |
| +0.125 | +0.1386 | +1.7407 | +0.250 | +0.1359 | +1.8147 |
| +0.150 | +0.1571 | +1.9099 | +0.150 | +0.1543 | +1.9872 |
| +0.175 | +0.1745 | +2.0608 | +0.175 | +0.1716 | +2.1407 |
| +0.200 | +0.1908 | +2.1977 | +0.200 | +0.1878 | +2.2798 |
| +0.225 | +0.2061 | +2.3236 | +0.225 | +0.2030 | +2.4078 |
| +0.250 | +0.2204 | +2.4409 | +0.250 | +0.2172 | +2.5267 |
| +0.275 | +0.2336 | +2.5510 | +0.275 | +0.2304 | +2.6384 |
| +0.300 | +0.2458 | +2.6553 | +0.300 | +0.2425 | +2.7442 |
| +0.325 | +0.2569 | +2.7547 | +0.325 | +0.2536 | +2.8449 |
| +0.350 | +0.2670 | +2.8501 | +0.350 | +0.2636 | +2.9416 |
| +0.375 | +0.2761 | +2.9422 | +0.375 | +0.2727 | +3.0348 |
| +0.400 | +0.2841 | +3.0314 | +0.400 | +0.2806 | +3.1251 |
| +0.425 | +0.2909 | +3.1183 | +0.425 | +0.2875 | +3.2131 |
| +0.450 | +0.2967 | +3.2034 | +0.450 | +0.2932 | +3.2992 |
| +0.475 | +0.3014 | +3.2870 | +0.475 | +0.2979 | +3.3838 |
| +0.500 | +0.3048 | +3.3694 | +0.500 | +0.3014 | +3.4672 |
| +0.525 | +0.3071 | +3.4511 | +0.525 | +0.3037 | +3.5498 |
| +0.550 | +0.3082 | +3.5323 | +0.550 | +0.3047 | +3.6320 |
| +0.575 | +0.3080 | +3.6134 | +0.575 | +0.3046 | +3.7140 |
| +0.600 | +0.3065 | +3.6948 | +0.600 | +0.3031 | +3.7962 |
| +0.625 | +0.3036 | +3.7767 | +0.625 | +0.3003 | +3.8791 |
| +0.650 | +0.2993 | +3.8596 | +0.650 | +0.2961 | +3.9629 |
| +0.675 | +0.2935 | +3.9439 | +0.675 | +0.2904 | +4.0481 |
| +0.700 | +0.2862 | +4.0301 | +0.700 | +0.2831 | +4.1352 |
| +0.725 | +0.2772 | +4.1189 | +0.725 | +0.2743 | +4.2249 |
| +0.750 | +0.2665 | +4.2108 | +0.750 | +0.2637 | +4.3178 |
| +0.775 | +0.2539 | +4.3068 | +0.775 | +0.2512 | +4.4149 |
| +0.800 | +0.2393 | +4.4082 | +0.800 | +0.2368 | +4.5173 |
| +0.825 | +0.2225 | +4.5164 | +0.825 | +0.2202 | +4.6266 |
| +0.850 | +0.2033 | +4.6338 | +0.850 | +0.2012 | +4.7453 |
| +0.875 | +0.1814 | +4.7638 | +0.875 | +0.1795 | +4.8766 |
| +0.900 | +0.1563 | +4.9119 | +0.900 | +0.1548 | +5.0262 |
| +0.925 | +0.1276 | +5.0882 | +0.925 | +0.1263 | +5.2042 |
| +0.950 | +0.0942 | +5.3144 | +0.950 | +0.0933 | +5.4327 |
| +0.975 | +0.0543 | +5.6564 | +0.975 | +0.0537 | +5.7781 |

$x = +0.53$

| U | $\tau$ | $y$ |
|---|---|---|
| +0.000 | +0.0238 | +0.0000 |
| +0.025 | +0.0483 | +0.7209 |
| +0.050 | +0.0715 | +1.1428 |
| +0.075 | +0.0933 | +1.4475 |
| +0.100 | +0.1139 | +1.6894 |
| +0.125 | +0.1333 | +1.8920 |
| +0.150 | +0.1516 | +2.0676 |
| +0.175 | +0.1688 | +2.2238 |
| +0.200 | +0.1849 | +2.3651 |
| +0.225 | +0.2001 | +2.4950 |
| +0.250 | +0.2142 | +2.6157 |
| +0.275 | +0.2273 | +2.7290 |
| +0.300 | +0.2393 | +2.8361 |
| +0.325 | +0.2504 | +2.9382 |
| +0.350 | +0.2604 | +3.0361 |
| +0.375 | +0.2693 | +3.1305 |
| +0.400 | +0.2773 | +3.2219 |
| +0.425 | +0.2841 | +3.3110 |
| +0.450 | +0.2898 | +3.3980 |
| +0.475 | +0.2945 | +3.4836 |
| +0.500 | +0.2980 | +3.5680 |
| +0.525 | +0.3003 | +3.6515 |
| +0.550 | +0.3014 | +3.7346 |
| +0.575 | +0.3013 | +3.8175 |
| +0.600 | +0.2998 | +3.9007 |
| +0.625 | +0.2971 | +3.9844 |
| +0.650 | +0.2929 | +4.0691 |
| +0.675 | +0.2873 | +4.1552 |
| +0.700 | +0.2802 | +4.2433 |
| +0.725 | +0.2714 | +4.3339 |
| +0.750 | +0.2610 | +4.4278 |
| +0.775 | +0.2487 | +4.5258 |
| +0.800 | +0.2344 | +4.6293 |
| +0.825 | +0.2180 | +4.7398 |
| +0.850 | +0.1992 | +4.8596 |
| +0.875 | +0.1777 | +4.9922 |
| +0.900 | +0.1532 | +5.1433 |
| +0.925 | +0.1251 | +5.3231 |
| +0.950 | +0.0924 | +5.5539 |
| +0.975 | +0.0531 | +5.9027 |

$x = +0.54$

| U | $\tau$ | $y$ |
|---|---|---|
| +0.000 | +0.0217 | +0.0000 |
| +0.025 | +0.0462 | +0.7685 |
| +0.050 | +0.0693 | +1.2061 |
| +0.075 | +0.0910 | +1.5194 |
| +0.100 | +0.1115 | +1.7668 |
| +0.125 | +0.1308 | +1.9734 |
| +0.150 | +0.1490 | +2.1522 |
| +0.175 | +0.1661 | +2.3110 |
| +0.200 | +0.1822 | +2.4546 |
| +0.225 | +0.1972 | +2.5864 |
| +0.250 | +0.2112 | +2.7088 |
| +0.275 | +0.2242 | +2.8236 |
| +0.300 | +0.2362 | +2.9322 |
| +0.325 | +0.2472 | +3.0356 |
| +0.350 | +0.2572 | +3.1347 |
| +0.375 | +0.2661 | +3.2302 |
| +0.400 | +0.2740 | +3.3228 |
| +0.425 | +0.2808 | +3.4128 |
| +0.450 | +0.2866 | +3.5009 |
| +0.475 | +0.2912 | +3.5875 |
| +0.500 | +0.2947 | +3.6728 |
| +0.525 | +0.2970 | +3.7572 |
| +0.550 | +0.2982 | +3.8412 |
| +0.575 | +0.2981 | +3.9251 |
| +0.600 | +0.2967 | +4.0091 |
| +0.625 | +0.2940 | +4.0937 |
| +0.650 | +0.2899 | +4.1793 |
| +0.675 | +0.2844 | +4.2663 |
| +0.700 | +0.2773 | +4.3553 |
| +0.725 | +0.2687 | +4.4468 |
| +0.750 | +0.2584 | +4.5417 |
| +0.775 | 20.2462 | +4.6407 |
| +0.800 | +0.2321 | +4.7452 |
| +0.825 | +0.2158 | +4.8568 |
| +0.850 | +0.1972 | +4.9778 |
| +0.875 | +0.1760 | +5.1117 |
| +0.900 | +0.1517 | +5.2643 |
| +0.925 | +0.1239 | +5.4459 |
| +0.950 | +0.0914 | +5.6789 |
| +0.975 | +0.0521 | +6.0312 |

$x$ = +0.55

| U | τ | y |
|---|---|---|
| +0.000 | +0.0198 | +0.0000 |
| +0.025 | +0.0442 | +0.8201 |
| +0.050 | +0.0673 | +1.2740 |
| +0.075 | +0.0889 | +1.5958 |
| +0.100 | +0.1092 | +1.8488 |
| +0.125 | +0.1284 | +2.0595 |
| +0.150 | +0.1465 | +2.2415 |
| +0.175 | +0.1635 | +2.4029 |
| +0.200 | +0.1795 | +2.5487 |
| +0.225 | +0.1944 | +2.6824 |
| +0.250 | +0.2084 | +2.8065 |
| +0.275 | +0.2213 | +2.9228 |
| +0.300 | +0.2333 | +3.0328 |
| +0.325 | +0.2442 | +3.1375 |
| +0.350 | +0.2541 | +3.2378 |
| +0.375 | +0.2630 | +3.3345 |
| +0.400 | +0.2709 | +3.4281 |
| +0.425 | +0.2777 | +3.5193 |
| +0.450 | +0.2834 | +3.6083 |
| +0.475 | +0.2880 | +3.6958 |
| +0.500 | +0.2150 | +3.7821 |
| +0.525 | +0.2939 | +3.8675 |
| +0.550 | +0.2950 | +3.9524 |
| +0.575 | +0.2950 | +4.0371 |
| +0.600 | +0.2936 | +4.1220 |
| +0.625 | +0.2910 | +4.2075 |
| +0.650 | +0.2870 | +4.2940 |
| +0.675 | +0.2815 | +4.3819 |
| +0.700 | +0.2746 | +4.4718 |
| +0.725 | +0.2660 | +4.5642 |
| +0.750 | +0.2558 | +4.6600 |
| +0.775 | +0.2438 | +4.7600 |
| +0.800 | +0.2298 | +4.8656 |
| +0.825 | +0.2138 | +4.9782 |
| +0.850 | +0.1953 | +5.1004 |
| +0.875 | +0.1743 | +5.2356 |
| +0.900 | +0.1503 | +5.3897 |
| +0.925 | +0.1227 | +5.5730 |
| +0.950 | +0.0904 | +5.8083 |
| +0.975 | +0.0497 | +6.1642 |

$x$ = +0.56

| U | τ | y |
|---|---|---|
| +0.000 | +0.0180 | +0.0000 |
| +0.025 | +0.0424 | +0.8760 |
| +0.050 | +0.0653 | +1.3463 |
| +0.075 | +0.0868 | +1.6768 |
| +0.100 | +0.1070 | +1.9354 |
| +0.125 | +0.1261 | +2.1501 |
| +0.150 | +0.1441 | +2.3353 |
| +0.175 | +0.1610 | +2.4992 |
| +0.200 | +0.1769 | +2.6472 |
| +0.225 | +0.1918 | +2.7829 |
| +0.250 | +0.2057 | +2.9087 |
| +0.275 | +0.2185 | +3.0265 |
| +0.300 | +0.2304 | +3.1379 |
| +0.325 | +0.2413 | +3.2439 |
| +0.350 | +0.2511 | +3.3454 |
| +0.375 | +0.2600 | +3.4432 |
| +0.400 | +0.2678 | +3.5379 |
| +0.425 | +0.2746 | +3.6300 |
| +0.450 | +0.2803 | +3.7201 |
| +0.475 | +0.2849 | +3.8086 |
| +0.500 | +0.2884 | +3.8957 |
| +0.525 | +0.2908 | +3.9820 |
| +0.550 | +0.2920 | +4.0678 |
| +0.575 | +0.2919 | +4.1534 |
| +0.600 | +0.2906 | +4.2392 |
| +0.625 | +0.2880 | +4.3256 |
| +0.650 | +0.2841 | +4.4129 |
| +0.675 | +0.2787 | +4.5017 |
| +0.700 | +0.2719 | +4.5925 |
| +0.725 | +0.2634 | +4.6859 |
| +0.750 | +0.2533 | +4.7826 |
| +0.775 | +0.2414 | +4.8836 |
| +0.800 | +0.2276 | +4.9901 |
| +0.825 | +0.2117 | +5.1039 |
| +0.850 | +0.1935 | +5.2272 |
| +0.875 | +0.1727 | +5.3638 |
| +0.900 | +0.1489 | +5.5193 |
| +0.925 | +0.1215 | +5.7043 |
| +0.950 | +0.0891 | +5.9420 |
| +0.975 | +0.0401 | +6.3116 |

$x = +0.57$

| U | $\tau$ | $\vartheta$ |
|---|---|---|
| +0.000 | +0.0162 | +0.0000 |
| +0.025 | +0.0406 | +0.9369 |
| +0.050 | +0.0634 | +1.4240 |
| +0.075 | +0.0848 | +1.7631 |
| +0.100 | +0.1050 | +2.0273 |
| +0.125 | +0.1239 | +2.2461 |
| +0.150 | +0.1418 | +2.4344 |
| +0.175 | +0.1586 | +2.6009 |
| +0.200 | +0.1744 | +2.7510 |
| +0.225 | +0.1892 | +2.8885 |
| +0.250 | +0.2030 | +3.0160 |
| +0.275 | +0.2158 | +3.1354 |
| +0.300 | +0.2276 | +3.2481 |
| +0.325 | +0.2384 | +3.3554 |
| +0.350 | +0.2483 | +3.4581 |
| +0.375 | +0.2571 | +3.5570 |
| +0.400 | +0.2649 | +3.6528 |
| +0.425 | +0.2716 | +3.7460 |
| +0.450 | +0.2773 | +3.8370 |
| +0.475 | +0.2819 | +3.9264 |
| +0.500 | +0.2854 | +4.0145 |
| +0.525 | +0.2878 | +4.1017 |
| +0.550 | +0.2890 | +4.1884 |
| +0.575 | +0.2890 | +4.2749 |
| +0.600 | +0.2878 | +4.3615 |
| +0.625 | +0.2852 | +4.4488 |
| +0.650 | +0.2813 | +4.5370 |
| +0.675 | +0.2760 | +4.6267 |
| +0.700 | +0.2693 | +4.7183 |
| +0.725 | +0.2609 | +4.8126 |
| +0.750 | +0.2509 | +4.9102 |
| +0.775 | +0.2392 | +5.0122 |
| +0.800 | +0.2255 | +5.1198 |
| +0.825 | +0.2098 | +5.2346 |
| +0.850 | +0.1917 | +5.3591 |
| +0.875 | +0.1711 | +5.4969 |
| +0.900 | +0.1475 | +5.6539 |
| +0.925 | +0.1204 | +5.8407 |
| +0.950 | +0.0878 | +6.0809 |
| +0.975 | +0.0342 | +6.4670 |

$x = +0.58$

| U | $\tau$ | $\vartheta$ |
|---|---|---|
| +0.000 | +0.0146 | +0.0000 |
| +0.025 | +0.0389 | +1.0037 |
| +0.050 | +0.0616 | +1.5080 |
| +0.075 | +0.0829 | +1.8558 |
| +0.100 | +0.1030 | +2.1256 |
| +0.125 | +0.1218 | +2.3483 |
| +0.150 | +0.1396 | +2.5398 |
| +0.175 | +0.1563 | +2.7088 |
| +0.200 | +0.1720 | +2.8611 |
| +0.225 | +0.1867 | +3.0005 |
| +0.250 | +0.2004 | +3.1297 |
| +0.275 | +0.2132 | +3.2505 |
| +0.300 | +0.2249 | +3.3647 |
| +0.325 | +0.2357 | +3.4732 |
| +0.350 | +0.2454 | +3.5771 |
| +0.375 | +0.2542 | +3.6771 |
| +0.400 | +0.2620 | +3.7740 |
| +0.425 | +0.2687 | +3.8682 |
| +0.450 | +0.2744 | +3.9602 |
| +0.475 | +0.2790 | +4.0505 |
| +0.500 | +0.2825 | +4.1396 |
| +0.525 | +0.2849 | +4.2276 |
| +0.550 | +0.2861 | +4.3152 |
| +0.575 | +0.2861 | +4.4025 |
| +0.600 | +0.2849 | +4.4901 |
| +0.625 | +0.2824 | +4.5781 |
| +0.650 | +0.2786 | +4.6672 |
| +0.675 | +0.2734 | +4.7578 |
| +0.700 | +0.2667 | +4.8503 |
| +0.725 | +0.2585 | +4.9455 |
| +0.750 | +0.2486 | +5.0440 |
| +0.775 | +0.2369 | +5.1470 |
| +0.800 | +0.2234 | +5.2555 |
| +0.825 | +0.2078 | +5.3715 |
| +0.850 | +0.1899 | +5.4971 |
| +0.875 | +0.1695 | +5.6362 |
| +0.900 | +0.1462 | +5.7946 |
| +0.925 | +0.1192 | +5.9832 |
| +0.950 | +0.0860 | +6.2264 |
| +0.975 | +0.0319 | +6.6205 |

$x = +0.59$

| U | $\tau$ | y |
|---|---|---|
| +0.000 | +0.0131 | +0.0000 |
| +0.025 | +0.0374 | +1.0767 |
| +0.050 | +0.0600 | +1.5984 |
| +0.075 | +0.0811 | +1.9549 |
| +0.100 | +0.1010 | +2.2302 |
| +0.125 | +0.1198 | +2.4570 |
| +0.150 | +0.1375 | +2.6515 |
| +0.175 | +0.1541 | +2.8231 |
| +0.200 | +0.1697 | +2.9776 |
| +0.225 | +0.1843 | +3.1188 |
| +0.250 | +0.1980 | +3.2496 |
| +0.275 | +0.2106 | +3.3720 |
| +0.300 | +0.2223 | +3.4875 |
| +0.325 | +0.2330 | +3.5973 |
| +0.350 | +0.2427 | +3.7024 |
| +0.375 | +0.2515 | +3.8035 |
| +0.400 | +0.2592 | +3.9014 |
| +0.425 | +0.2659 | +3.9966 |
| +0.450 | +0.2716 | +4.0896 |
| +0.475 | +0.2762 | +4.1809 |
| +0.500 | +0.2797 | +4.2708 |
| +0.525 | +0.2821 | +4.3598 |
| +0.550 | +0.2833 | +4.4482 |
| +0.575 | +0.2834 | +4.5364 |
| +0.600 | +0.2822 | +4.6248 |
| +0.625 | +0.2797 | +4.7137 |
| +0.650 | +0.2760 | +4.8036 |
| +0.675 | +0.2708 | +4.8950 |
| +0.700 | +0.2642 | +4.9884 |
| +0.725 | +0.2561 | +5.0845 |
| +0.750 | +0.2463 | +5.1840 |
| +0.775 | +0.2348 | +5.2879 |
| +0.800 | +0.2214 | +5.3974 |
| +0.825 | +0.2059 | +5.5144 |
| +0.850 | +0.1882 | +5.6412 |
| +0.875 | +0.1680 | +5.7815 |
| +0.900 | +0.1448 | +5.9414 |
| +0.925 | +0.1179 | +6.1319 |
| +0.950 | +0.0840 | +6.3786 |
| +0.975 | +0.0311 | +6.7765 |

$x = +0.60$

| U | $\tau$ | y |
|---|---|---|
| +0.000 | +0.0116 | +0.0000 |
| +0.025 | +0.0359 | +1.1563 |
| +0.050 | +0.0583 | +1.6957 |
| +0.075 | +0.0794 | +2.0609 |
| +0.100 | +0.0992 | +2.3417 |
| +0.125 | +0.1178 | +2.5725 |
| +0.150 | +0.1354 | +2.7701 |
| +0.175 | +0.1519 | +2.9442 |
| +0.200 | +0.1674 | +3.1008 |
| +0.225 | +0.1820 | +3.2439 |
| +0.250 | +0.1955 | +3.3763 |
| +0.275 | +0.2081 | +3.5002 |
| +0.300 | +0.2198 | +3.6170 |
| +0.325 | +0.2304 | +3.7281 |
| +0.350 | +0.2401 | +3.8343 |
| +0.375 | +0.2488 | +3.9365 |
| +0.400 | +0.2565 | +4.0355 |
| +0.425 | +0.2632 | +4.1317 |
| +0.450 | +0.2688 | +4.2256 |
| +0.475 | +0.2734 | +4.3178 |
| +0.500 | +0.2769 | +4.4086 |
| +0.525 | +0.2793 | +4.4985 |
| +0.550 | +0.2860 | +4.5878 |
| +0.575 | +0.2807 | +4.6768 |
| +0.600 | +0.2795 | +4.7661 |
| +0.625 | +0.2771 | +4.8559 |
| +0.650 | +0.2734 | +4.9466 |
| +0.675 | +0.2683 | +5.0389 |
| +0.700 | +0.2618 | +5.1332 |
| +0.725 | +0.2538 | +5.2301 |
| +0.750 | +0.2441 | +5.3305 |
| +0.775 | +0.2327 | +5.4353 |
| +0.800 | +0.2194 | +5.5459 |
| +0.825 | +0.2041 | +5.6639 |
| +0.850 | +0.1866 | +5.7918 |
| +0.875 | +0.1665 | +5.9334 |
| +0.900 | +0.1434 | +6.0947 |
| +0.925 | +0.1165 | +6.2872 |
| +0.950 | +0.0824 | +6.5381 |
| +0.975 | +0.0307 | +6.9473 |

$x = +0.61$

| U | $\tau$ | y |
|---|---|---|
| +0.000 | +0.0103 | +0.0000 |
| +0.025 | +0.0344 | +1.2452 |
| +0.050 | +0.0568 | +1.8024 |
| +0.075 | +0.0778 | +2.1763 |
| +0.100 | +0.0974 | +2.4626 |
| +0.125 | +0.1160 | +2.6973 |
| +0.150 | +0.1334 | +2.8979 |
| +0.175 | +0.1498 | +3.0745 |
| +0.200 | +0.1653 | +3.2333 |
| +0.225 | +0.1797 | +3.3782 |
| +0.250 | +0.1932 | +3.5123 |
| +0.275 | +0.2057 | +3.6376 |
| +0.300 | +0.2173 | +3.7558 |
| +0.325 | +0.2279 | +3.8681 |
| +0.350 | +0.2375 | +3.9755 |
| +0.375 | +0.2462 | +4.0788 |
| +0.400 | +0.2539 | +4.1788 |
| +0.425 | +0.2605 | +4.2760 |
| +0.450 | +0.2661 | +4.3709 |
| +0.475 | +0.2707 | +4.4640 |
| +0.500 | +0.2742 | +4.5557 |
| +0.525 | +0.2766 | +4.6464 |
| +0.550 | +0.2779 | +4.7366 |
| +0.575 | +0.2780 | +4.8265 |
| +0.600 | +0.2769 | +4.9165 |
| +0.625 | +0.2746 | +5.0072 |
| +0.650 | +0.2709 | +5.0988 |
| +0.675 | +0.2659 | +5.1919 |
| +0.700 | +0.2595 | +5.2870 |
| +0.725 | +0.2515 | +5.3848 |
| +0.750 | +0.2419 | +5.4861 |
| +0.775 | +0.2306 | +5.5919 |
| +0.800 | +0.2175 | +5.7034 |
| +0.825 | +0.2023 | +5.8225 |
| +0.850 | +0.1849 | +5.9515 |
| +0.875 | +0.1650 | +6.0944 |
| +0.900 | +0.1421 | +6.2572 |
| +0.925 | +0.1151 | +6.4518 |
| +0.950 | +0.0820 | +6.7059 |
| +0.975 | +0.0292 | +7.1344 |

$x = +0.62$

| U | $\tau$ | y |
|---|---|---|
| +0.000 | +0.0090 | +0.0000 |
| +0.025 | +0.0331 | +1.3424 |
| +0.050 | +0.0554 | +1.9175 |
| +0.075 | +0.0762 | +2.3001 |
| +0.100 | +0.0957 | +2.5919 |
| +0.125 | +0.1142 | +2.8305 |
| +0.150 | +0.1315 | +3.0343 |
| +0.175 | +0.1478 | +3.2134 |
| +0.200 | +0.1632 | +3.3742 |
| +0.225 | +0.1775 | +3.5209 |
| +0.250 | +0.1910 | +3.6566 |
| +0.275 | +0.2034 | +3.7834 |
| +0.300 | +0.2149 | +3.9029 |
| +0.325 | +0.2255 | +4.0164 |
| +0.350 | +0.2351 | +4.1250 |
| +0.375 | +0.2437 | +4.2294 |
| +0.400 | +0.2513 | +4.3304 |
| +0.425 | +0.2579 | +4.4286 |
| +0.450 | +0.2635 | +4.5244 |
| +0.475 | +0.2681 | +4.6184 |
| +0.500 | +0.2716 | +4.7110 |
| +0.525 | +0.2740 | +4.8026 |
| +0.550 | +0.2753 | +4.8936 |
| +0.575 | +0.2755 | +4.9844 |
| +0.600 | +0.2744 | +5.0753 |
| +0.625 | +0.2721 | +5.1667 |
| +0.650 | +0.2685 | +5.2592 |
| +0.675 | +0.2635 | +5.3531 |
| +0.700 | +0.2572 | +5.4491 |
| +0.725 | +0.2493 | +5.5478 |
| +0.750 | +0.2398 | +5.6500 |
| +0.775 | +0.2286 | +5.7567 |
| +0.800 | +0.2156 | +5.8692 |
| +0.825 | +0.2006 | +5.9893 |
| +0.850 | +0.1833 | +6.1195 |
| +0.875 | +0.1635 | +6.2636 |
| +0.900 | +0.1406 | +6.4280 |
| +0.925 | +0.1138 | +6.6246 |
| +0.950 | +0.0811 | +6.8816 |
| +0.975 | +0.0277 | +7.3273 |

$x = +0.63$

| U | $\tau$ | $y$ |
|---|---|---|
| +0.000 | +0.0079 | +0.0000 |
| +0.025 | +0.0319 | +1.4516 |
| +0.050 | +0.0540 | +2.0448 |
| +0.075 | +0.0747 | +2.4361 |
| +0.100 | +0.0941 | +2.7333 |
| +0.125 | +0.1124 | +2.9758 |
| +0.150 | +0.1297 | +3.1826 |
| +0.175 | +0.1459 | +3.3642 |
| +0.200 | +0.1611 | +3.5271 |
| +0.225 | +0.1754 | +3.6757 |
| +0.250 | +0.1888 | +3.8130 |
| +0.275 | +0.2011 | +3.9412 |
| +0.300 | +0.2126 | +4.0620 |
| +0.325 | +0.2231 | +4.1768 |
| +0.350 | +0.2326 | +4.2865 |
| +0.375 | +0.2412 | +4.3920 |
| +0.400 | +0.2488 | +4.4940 |
| +0.425 | +0.2554 | +4.5931 |
| +0.450 | +0.2610 | +4.6899 |
| +0.475 | +0.2656 | +4.7848 |
| +0.500 | +0.2691 | +4.8783 |
| +0.525 | +0.2715 | +4.9708 |
| +0.550 | +0.2728 | +5.0626 |
| +0.575 | +0.2730 | +5.1542 |
| +0.600 | +0.2719 | +5.2459 |
| +0.625 | +0.2697 | +5.3382 |
| +0.650 | +0.2661 | +5.4315 |
| +0.675 | +0.2612 | +5.5262 |
| +0.700 | +0.2549 | +5.6231 |
| +0.725 | +0.2471 | +5.7226 |
| +0.750 | +0.2377 | +5.8257 |
| +0.775 | +0.2267 | +5.9333 |
| +0.800 | +0.2138 | +6.0468 |
| +0.825 | +0.1989 | +6.1680 |
| +0.850 | +0.1817 | +6.2993 |
| +0.875 | +0.1620 | +6.4447 |
| +0.900 | +0.1393 | +6.6106 |
| +0.925 | +0.1126 | +6.8093 |
| +0.950 | +0.0799 | +7.0695 |
| +0.975 | +0.0264 | +7.5297 |

$x = +0.64$

| U | $\tau$ | $y$ |
|---|---|---|
| +0.000 | +0.0068 | +0.0000 |
| +0.025 | +0.0307 | +1.5745 |
| +0.050 | +0.0527 | +2.1858 |
| +0.075 | +0.0732 | +2.5857 |
| +0.100 | +0.0925 | +2.8883 |
| +0.125 | +0.1107 | +3.1347 |
| +0.150 | +0.1279 | +3.3445 |
| +0.175 | +0.1440 | +3.5285 |
| +0.200 | +0.1592 | +3.6935 |
| +0.225 | +0.1734 | +3.8439 |
| +0.250 | +0.1866 | +3.9828 |
| +0.275 | +0.1989 | +4.1124 |
| +0.300 | +0.2103 | +4.2346 |
| +0.325 | +0.2208 | +4.3505 |
| +0.350 | +0.2303 | +4.4614 |
| +0.375 | +0.2388 | +4.5679 |
| +0.400 | +0.2464 | +4.6710 |
| +0.425 | +0.2530 | +4.7711 |
| +0.450 | +0.2585 | +4.8688 |
| +0.475 | +0.2631 | +4.9646 |
| +0.500 | +0.2666 | +5.0590 |
| +0.525 | +0.2690 | +5.1523 |
| +0.550 | +0.2704 | +5.2450 |
| +0.575 | +0.2705 | +5.3374 |
| +0.600 | +0.2695 | +5.4299 |
| +0.625 | +0.2673 | +5.5230 |
| +0.650 | +0.2638 | +5.6171 |
| +0.675 | +0.2590 | +5.7127 |
| +0.700 | +0.2527 | +5.8104 |
| +0.725 | +0.2450 | +5.9108 |
| +0.750 | +0.2357 | +6.0148 |
| +0.775 | +0.2247 | +6.1233 |
| +0.800 | +0.2119 | +6.2378 |
| +0.825 | +0.1972 | +6.3599 |
| +0.850 | +0.1802 | +6.4924 |
| +0.875 | +0.1606 | +6.6391 |
| +0.900 | +0.1380 | +6.8065 |
| +0.925 | +0.1114 | +7.0072 |
| +0.950 | +0.0786 | +7.2709 |
| +0.975 | +0.0254 | +7.7433 |

$x = +0.65$

| $U$ | $\tau$ | $y$ |
|---|---|---|
| +0.000 | +0.0057 | +0.0000 |
| +0.025 | +0.0296 | +1.7167 |
| +0.050 | +0.0514 | +2.3465 |
| +0.075 | +0.0719 | +2.7550 |
| +0.100 | +0.0910 | +3.0630 |
| +0.125 | +0.1091 | +3.3133 |
| +0.150 | +0.1262 | +3.5260 |
| +0.175 | +0.1422 | +3.7124 |
| +0.200 | +0.1573 | +3.8794 |
| +0.225 | +0.1714 | +4.0316 |
| +0.250 | +0.1846 | +4.1720 |
| +0.275 | +0.1968 | +4.3031 |
| +0.300 | +0.2081 | +4.4266 |
| +0.325 | +0.2185 | +4.5437 |
| +0.350 | +0.2280 | +4.6557 |
| +0.375 | +0.2365 | +4.7633 |
| +0.400 | +0.2440 | +4.8674 |
| +0.425 | +0.2506 | +4.9684 |
| +0.450 | +0.2561 | +5.0671 |
| +0.475 | +0.2607 | +5.1638 |
| +0.500 | +0.2642 | +5.2590 |
| +0.525 | +0.2666 | +5.3532 |
| +0.550 | +0.2680 | +5.4467 |
| +0.575 | +0.2682 | +5.5399 |
| +0.600 | +0.2672 | +5.6333 |
| +0.625 | +0.2650 | +5.7272 |
| +0.650 | +0.2615 | +5.8221 |
| +0.675 | +0.2568 | +5.9185 |
| +0.700 | +0.2506 | +6.0170 |
| +0.725 | +0.2430 | +6.1183 |
| +0.750 | +0.2338 | +6.2231 |
| +0.775 | +0.2229 | +6.3326 |
| +0.800 | +0.2102 | +6.4480 |
| +0.825 | +0.1955 | +6.5712 |
| +0.850 | +0.1786 | +6.7048 |
| +0.875 | +0.1592 | +6.8528 |
| +0.900 | +0.1367 | +7.0218 |
| +0.925 | +0.1102 | +7.2245 |
| +0.950 | +0.0773 | +7.4917 |
| +0.975 | +0.0246 | +7.9742 |

$x = +0.66$

| $U$ | $\tau$ | $y$ |
|---|---|---|
| +0.000 | +0.0048 | +0.0000 |
| +0.025 | +0.0285 | +1.8785 |
| +0.050 | +0.0503 | +2.5264 |
| +0.075 | +0.0705 | +2.9435 |
| +0.100 | +0.0896 | +3.2568 |
| +0.125 | +0.1076 | +3.5108 |
| +0.150 | +0.1245 | +3.7265 |
| +0.175 | +0.1404 | +3.9153 |
| +0.200 | +0.1554 | +4.0844 |
| +0.225 | +0.1695 | +4.2383 |
| +0.250 | +0.1826 | +4.3803 |
| +0.275 | +0.1948 | +4.5128 |
| +0.300 | +0.2060 | +4.6376 |
| +0.325 | +0.2164 | +4.7559 |
| +0.350 | +0.2258 | +4.8690 |
| +0.375 | +0.2342 | +4.9777 |
| +0.400 | +0.2417 | +5.0827 |
| +0.425 | +0.2483 | +5.1847 |
| +0.450 | +0.2538 | +5.2843 |
| +0.475 | +0.2583 | +5.3819 |
| +0.500 | +0.2618 | +5.4780 |
| +0.525 | +0.2643 | +5.5730 |
| +0.550 | +0.2656 | +5.6673 |
| +0.575 | +0.2658 | +5.7614 |
| +0.600 | +0.2649 | +5.8555 |
| +0.625 | +0.2627 | +5.9503 |
| +0.650 | +0.2593 | +6.0460 |
| +0.675 | +0.2546 | +6.1432 |
| +0.700 | +0.2485 | +6.2426 |
| +0.725 | +0.2409 | +6.3447 |
| +0.750 | +0.2318 | +6.4504 |
| +0.775 | +0.2210 | +6.5608 |
| +0.800 | +0.2084 | +6.6771 |
| +0.825 | +0.1939 | +6.8013 |
| +0.850 | +0.1771 | +6.9361 |
| +0.875 | +0.1578 | +7.0853 |
| +0.900 | +0.1354 | +7.2559 |
| +0.925 | +0.1089 | +7.4607 |
| +0.950 | +0.0761 | +7.7314 |
| +0.975 | +0.0240 | +8.2223 |

x = +0.67

| U | τ | y |
|---|---|---|
| +0.000 | +0.0040 | +0.0000 |
| +0.025 | +0.0276 | +2.0684 |
| +0.050 | +0.0491 | +2.7343 |
| +0.075 | +0.0693 | +3.1598 |
| +0.100 | +0.0882 | +3.4784 |
| +0.125 | +0.1061 | +3.7363 |
| +0.150 | +0.1229 | +3.9549 |
| +0.175 | +0.1387 | +4.1461 |
| +0.200 | +0.1536 | +4.3172 |
| +0.225 | +0.1676 | +4.4728 |
| +0.250 | +0.1806 | +4.6164 |
| +0.275 | +0.1927 | +4.7504 |
| +0.300 | +0.2039 | +4.8764 |
| +0.325 | +0.2142 | +4.9959 |
| +0.350 | +0.2236 | +5.1101 |
| +0.375 | +0.2320 | +5.2198 |
| +0.400 | +0.2395 | +5.3259 |
| +0.425 | +0.2460 | +5.4288 |
| +0.450 | +0.2515 | +5.5293 |
| +0.475 | +0.2560 | +5.6278 |
| +0.500 | +0.2595 | +5.7247 |
| +0.525 | +0.2620 | +5.8206 |
| +0.550 | +0.2630 | +5.9157 |
| +0.575 | +0.2636 | +6.0106 |
| +0.600 | +0.2627 | +6.1056 |
| +0.625 | +0.2605 | +6.2011 |
| +0.650 | +0.2572 | +6.2976 |
| +0.675 | +0.2525 | +6.3957 |
| +0.700 | +0.2465 | +6.4958 |
| +0.725 | +0.2390 | +6.5988 |
| +0.750 | +0.2299 | +6.7054 |
| +0.775 | +0.2192 | +6.8167 |
| +0.800 | +0.2068 | +6.9340 |
| +0.825 | +0.1923 | +7.0592 |
| +0.850 | +0.1756 | +7.1951 |
| +0.875 | +0.1564 | +7.3456 |
| +0.900 | +0.1341 | +7.5178 |
| +0.925 | +0.1077 | +7.7247 |
| +0.950 | +0.0749 | +7.9989 |
| +0.975 | +0.0235 | +8.4965 |

x = +0.68

| U | τ | y |
|---|---|---|
| +0.000 | +0.0032 | +0.0000 |
| +0.025 | +0.0267 | +2.3071 |
| +0.050 | +0.0481 | +2.9908 |
| +0.075 | +0.0681 | +3.4247 |
| +0.100 | +0.0869 | +3.7484 |
| +0.125 | +0.1046 | +4.0100 |
| +0.150 | +0.1213 | +4.2315 |
| +0.175 | +0.1371 | +4.4251 |
| +0.200 | +0.1519 | +4.5982 |
| +0.225 | +0.1658 | +4.7556 |
| +0.250 | +0.1787 | +4.9007 |
| +0.275 | +0.1908 | +5.0360 |
| +0.300 | +0.2019 | +5.1633 |
| +0.325 | +0.2122 | +5.2840 |
| +0.350 | +0.2215 | +5.3993 |
| +0.375 | +0.2298 | +5.5101 |
| +0.400 | +0.2373 | +5.6171 |
| +0.425 | +0.2438 | +5.7210 |
| +0.450 | +0.2493 | +5.8224 |
| +0.475 | +0.2538 | +5.9217 |
| +0.500 | +0.2573 | +6.0195 |
| +0.525 | +0.2597 | +6.1162 |
| +0.550 | +0.2611 | +6.2122 |
| +0.575 | +0.2614 | +6.3078 |
| +0.600 | +0.2605 | +6.4036 |
| +0.625 | +0.2584 | +6.4999 |
| +0.650 | +0.2551 | +6.5973 |
| +0.675 | +0.2505 | +6.6961 |
| +0.700 | +0.2445 | +6.7971 |
| +0.725 | +0.2370 | +6.9009 |
| +0.750 | +0.2281 | +7.0083 |
| +0.775 | +0.2175 | +7.1205 |
| +0.800 | +0.2051 | +7.2388 |
| +0.825 | +0.1907 | +7.3650 |
| +0.850 | +0.1742 | +7.5020 |
| +0.875 | +0.1550 | +7.6539 |
| +0.900 | +0.1328 | +7.8276 |
| +0.925 | +0.1065 | +8.0366 |
| +0.950 | +0.0739 | +8.3142 |
| +0.975 | +0.0232 | +8.8171 |

$x = +0.69$

| U | $\tau$ | $y$ |
|---|---|---|
| +0.000 | +0.0025 | +0.0000 |
| +0.025 | +0.0258 | +2.6108 |
| +0.050 | +0.0471 | +3.3119 |
| +0.075 | +0.0670 | +3.7540 |
| +0.100 | +0.0856 | +4.0829 |
| +0.125 | +0.1032 | +4.3482 |
| +0.150 | +0.1199 | +4.5726 |
| +0.175 | +0.1355 | +4.7685 |
| +0.200 | +0.1502 | +4.9435 |
| +0.225 | +0.1640 | +5.1027 |
| +0.250 | +0.1769 | +5.2493 |
| +0.275 | +0.1889 | +5.3860 |
| +0.300 | +0.2000 | +5.5146 |
| +0.325 | +0.2101 | +5.6365 |
| +0.350 | +0.2194 | +5.7529 |
| +0.375 | +0.2278 | +5.8646 |
| +0.400 | +0.2352 | +5.9726 |
| +0.425 | +0.2416 | +6.0775 |
| +0.450 | +0.2471 | +6.1797 |
| +0.475 | +0.2516 | +6.2800 |
| +0.500 | +0.2551 | +6.3786 |
| +0.525 | +0.2575 | +6.4761 |
| +0.550 | +0.2589 | +6.5729 |
| +0.575 | +0.2592 | +6.6694 |
| +0.600 | +0.2583 | +6.7659 |
| +0.625 | +0.2563 | +6.8631 |
| +0.650 | +0.2530 | +6.9612 |
| +0.675 | +0.2484 | +7.0608 |
| +0.700 | +0.2425 | +7.1626 |
| +0.725 | +0.2352 | +7.2673 |
| +0.750 | +0.2263 | +7.3756 |
| +0.775 | +0.2157 | +7.4886 |
| +0.800 | +0.2034 | +7.6079 |
| +0.825 | +0.1892 | +7.7352 |
| +0.850 | +0.1727 | +7.8733 |
| +0.875 | +0.1537 | +8.0264 |
| +0.900 | +0.1316 | +8.2017 |
| +0.925 | +0.1054 | +8.4128 |
| +0.950 | +0.0729 | +8.6939 |
| +0.975 | +0.0229 | +9.2012 |

$x = +0.70$

| U | $\tau$ | $y$ |
|---|---|---|
| +0.000 | +0.0018 | +0.0000 |
| +0.025 | +0.0251 | +3.0348 |
| +0.050 | +0.0461 | +3.7535 |
| +0.075 | +0.0659 | +4.2038 |
| +0.100 | +0.0844 | +4.5378 |
| +0.125 | +0.1019 | +4.8067 |
| +0.150 | +0.1184 | +5.0339 |
| +0.175 | +0.1340 | +5.2322 |
| +0.200 | +0.1486 | +5.4092 |
| +0.225 | +0.1623 | +5.5700 |
| +0.250 | +0.1751 | +5.7182 |
| +0.275 | +0.1870 | +5.8563 |
| +0.300 | +0.1981 | +5.9861 |
| +0.325 | +0.2082 | +6.1091 |
| +0.350 | +0.2174 | +6.2266 |
| +0.375 | +0.2257 | +6.3394 |
| +0.400 | +0.2331 | +6.4483 |
| +0.425 | +0.2395 | +6.5541 |
| +0.450 | +0.2450 | +6.6573 |
| +0.475 | +0.2495 | +6.7584 |
| +0.500 | +0.2530 | +6.8579 |
| +0.525 | +0.2554 | +6.9562 |
| +0.550 | +0.2568 | +7.0538 |
| +0.575 | +0.2571 | +7.1510 |
| +0.600 | +0.2563 | +7.2484 |
| +0.625 | +0.2543 | +7.3463 |
| +0.650 | +0.2510 | +7.4452 |
| +0.675 | +0.2465 | +7.5457 |
| +0.700 | +0.2406 | +7.6483 |
| +0.725 | +0.2333 | +7.7537 |
| +0.750 | +0.2245 | +7.8629 |
| +0.775 | +0.2141 | +7.9768 |
| +0.800 | +0.2018 | +8.0970 |
| +0.825 | +0.1877 | +8.2253 |
| +0.850 | +0.1713 | +8.3645 |
| +0.875 | +0.1524 | +8.5189 |
| +0.900 | +0.1304 | +8.6958 |
| +0.925 | +0.1043 | +8.9091 |
| +0.950 | +0.0720 | +9.1934 |
| +0.975 | +0.0227 | +9.7043 |

TABLE 3.

| $x$ | $(\partial u/\partial y)_{y=0}$ | $\delta^*$ | $\theta^*$ | $v_\infty$ | $1/v_\infty$ |
|---|---|---|---|---|---|
| .01 | 2.7281 | 0.1896 | 0.0704 | 10.647 | 0.0939 |
| .02 | 1.7644 | 0.2797 | 0.1021 | 9.005 | 0.1111 |
| .03 | 1.3404 | 0.3541 | 0.1274 | 7.979 | 0.1252 |
| .04 | 1.0893 | 0.4207 | 0.1494 | 7.394 | 0.1352 |
| .05 | 0.9192 | 0.4826 | 0.1694 | 7.013 | 0.1426 |
| .06 | 0.7946 | 0.5413 | 0.1880 | 6.748 | 0.1482 |
| .07 | 0.6986 | 0.5978 | 0.2054 | 6.557 | 0.1525 |
| .08 | 0.6217 | 0.6527 | 0.2220 | 6.416 | 0.1559 |
| .09 | 0.5586 | 0.7063 | 0.2379 | 6.310 | 0.1585 |
| .10 | 0.5056 | 0.7590 | 0.2532 | 6.231 | 0.1605 |
| .11 | 0.4604 | 0.8110 | 0.2680 | 6.173 | 0.1620 |
| .12 | 0.4213 | 0.8625 | 0.2824 | 6.132 | 0.1631 |
| .13 | 0.3871 | 0.9136 | 0.2965 | 6.104 | 0.1638 |
| .14 | 0.3569 | 0.9646 | 0.3102 | 6.087 | 0.1643 |
| .15 | 0.3300 | 1.0154 | 0.3236 | 6.080 | 0.1645 |
| .16 | 0.3060 | 1.0662 | 0.3368 | 6.082 | 0.1644 |
| .17 | 0.2842 | 1.1171 | 0.3498 | 6.091 | 0.1642 |
| .18 | 0.2646 | 1.1681 | 0.3625 | 6.108 | 0.1637 |
| .19 | 0.2466 | 1.2193 | 0.3751 | 6.130 | 0.1631 |
| .20 | 0.2302 | 1.2707 | 0.3874 | 6.159 | 0.1624 |
| .21 | 0.2152 | 1.3225 | 0.3997 | 6.193 | 0.1615 |
| .22 | 0.2013 | 1.3746 | 0.4117 | 6.232 | 0.1605 |
| .23 | 0.1885 | 1.4271 | 0.4237 | 6.276 | 0.1593 |
| .24 | 0.1766 | 1.4801 | 0.4355 | 6.326 | 0.1581 |
| .25 | 0.1656 | 1.5336 | 0.4472 | 6.380 | 0.1567 |
| .26 | 0.1553 | 1.5877 | 0.4588 | 6.440 | 0.1553 |
| .27 | 0.1458 | 1.6424 | 0.4703 | 6.503 | 0.1538 |
| .28 | 0.1368 | 1.6978 | 0.4818 | 6.572 | 0.1522 |
| .29 | 0.1284 | 1.7539 | 0.4931 | 6.645 | 0.1505 |
| .30 | 0.1206 | 1.8107 | 0.5043 | 6.724 | 0.1487 |
| .31 | 0.1132 | 1.8684 | 0.5155 | 6.808 | 0.1469 |
| .32 | 0.1063 | 1.9269 | 0.5266 | 6.897 | 0.1450 |
| .33 | 0.0998 | 1.9864 | 0.5376 | 6.992 | 0.1430 |
| .34 | 0.0937 | 2.0468 | 0.5486 | 7.093 | 0.1410 |
| .35 | 0.0879 | 2.1082 | 0.5595 | 7.200 | 0.1389 |

TABLE 3. (CONTINUED)

| $x$ | $(\partial u/\partial y)_{y=0}$ | $\delta^*$ | $\theta^*$ | $v_\infty$ | $1/v_\infty$ |
|---|---|---|---|---|---|
| .36 | 0.0824 | 2.1708 | 0.5703 | 7.313 | 0.1367 |
| .37 | 0.0773 | 2.2345 | 0.5811 | 7.434 | 0.1345 |
| .38 | 0.0724 | 2.2995 | 0.5919 | 7.561 | 0.1323 |
| .39 | 0.0678 | 2.3658 | 0.6026 | 7.698 | 0.1299 |
| .40 | 0.0634 | 2.4335 | 0.6133 | 7.842 | 0.1275 |
| .41 | 0.0593 | 2.5026 | 0.6239 | 7.995 | 0.1251 |
| .42 | 0.0554 | 2.5734 | 0.6344 | 8.158 | 0.1226 |
| .43 | 0.0517 | 2.6458 | 0.6450 | 8.331 | 0.1200 |
| .44 | 0.0482 | 2.7201 | 0.6555 | 8.516 | 0.1174 |
| .45 | 0.0448 | 2.7962 | 0.6659 | 8.713 | 0.1148 |
| .46 | 0.0417 | 2.8744 | 0.6763 | 8.924 | 0.1121 |
| .47 | 0.0387 | 2.9549 | 0.6867 | 9.151 | 0.1093 |
| .48 | 0.0358 | 3.0379 | 0.6971 | 9.394 | 0.1065 |
| .49 | 0.0332 | 3.1235 | 0.7075 | 9.654 | 0.1036 |
| .50 | 0.0306 | 3.2107 | 0.7179 | 9.926 | 0.1007 |
| .51 | 0.0282 | 3.3015 | 0.7282 | 10.23 | 0.0978 |
| .52 | 0.0259 | 3.3957 | 0.7385 | 10.61 | 0.0943 |
| .53 | 0.0238 | 3.4927 | 0.7488 | 10.91 | 0.0917 |
| .54 | 0.0217 | 3.5938 | 0.7590 | 11.33 | 0.0883 |
| .55 | 0.0198 | 3.6994 | 0.7692 | 11.77 | 0.0850 |
| .56 | 0.0180 | 3.8093 | 0.7796 | 12.23 | 0.0818 |
| .57 | 0.0162 | 3.9244 | 0.7899 | 12.81 | 0.0781 |
| .58 | 0.0146 | 4.0458 | 0.8004 | 13.44 | 0.0744 |
| .59 | 0.0131 | 4.1734 | 0.8110 | 14.13 | 0.0708 |
| .60 | 0.0116 | 4.3078 | 0.8217 | 14.84 | 0.0674 |
| .61 | 0.0103 | 4.4504 | 0.8317 | 15.66 | 0.0639 |
| .62 | 0.0090 | 4.6015 | 0.8419 | 16.66 | 0.0600 |
| .63 | 0.0079 | 4.7644 | 0.8523 | 17.79 | 0.0562 |
| .64 | 0.0068 | 4.9404 | 0.8628 | 19.50 | 0.0513 |
| .65 | 0.0057 | 5.1353 | 0.8732 | 21.34 | 0.0469 |
| .66 | 0.0048 | 5.3485 | 0.8836 | 23.42 | 0.0427 |
| .67 | 0.0040 | 5.5884 | 0.8940 | 27.01 | 0.0370 |
| .68 | 0.0032 | 5.8744 | 0.9043 | 32.10 | 0.0312 |
| .69 | 0.0025 | 6.2217 | 0.9146 | | |
| .70 | 0.0018 | 6.6833 | 0.9248 | | |