Bäoklund haps and related connections

Kalim, Sheikh Muhammad

TO MY MOTHER

Bäcklund Maps and Related Connections

by

Sheikh Muhammad Kalim, M.Sc(Punjab)

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Abstract

The aim of this thesis is to show a link between solutions of Differential equations, and the integral submanifolds of sets of forms defined on jet bundles. The original idea behind Bäcklund maps was discovered by A. V. Bäcklund around 1875 during research into pseudospherical surfaces i.e. surfaces of constant negative curvature.

The central idea of this thesis is the Bäcklund map, which is a smooth map of jet bundles parameterised by the target manifold of its co-domain. The original system of differential equations appears as a system of integrability conditions for the Bäcklund map.

The map induces an horizontal distribution on its domain from the natural contact structure of its co-domain, which makes possible a geometrical description in terms of a connection, called here the Bäcklund connection; the system of integrability conditions reappears as the vanishing of the curvature of this connection.

The paper by Bäcklund was very obscurely written and perhaps for that reason was ignored for nearly a hundred years. Development of applied mathematics, hydrodynamics, mechanics and fluid mechanics published work raise the interest of Bäcklund maps and related topics.

Chapter I gives a brief account of Jet-Bundles (Pirani) and contact module on Jet-Bundles.

Chapter II summarises different ways of describing integrability conditions associated with Bäcklund maps. It also explains hash-operator, use of contact module and some examples.

Chapter III gives the idea of prolongation and explains with some examples.

Chapter IV discusses the idea of connections associated with Bäcklund maps, given by pull back of contact module of forms on $J^1(M,N_2)$ determine a carton connection. Then shows that the solution of differential equation corresponds to the zeros of curvature tensor of their connection.

I conclude the introduction with a summary of my notation and conventions. All objects and maps are assumed to be in C^{∞}; in the application they are generally real-analytic. If $f: M \to N$ is a map, then the domain of f is an open set in M, not necessarily the whole of M. If M is any set then Δ_M denotes the original map $M \to M \times M$ by $m \mapsto (m,m)$ for every $m \in M$. If ϕ is a map of manifold then ϕ^* is the induced map of forms and functions. If Θ is any collection of exterior forms then $\phi^* \Theta$ means $\{\phi^* \theta / \theta \in \Theta\}$, $d\Theta$ means $\{d\theta / \theta \in \Theta\}$, $I(\Theta)$ means the ideal $\{\eta \wedge \theta \mid \theta \in \Theta, \eta \text{ any form}\}$ generated by Θ, where denotes $\wedge$ the exterior product and $X \rfloor \Theta$ means $\{X \rfloor \theta / \theta \in \Theta\}$ where X is any vector field and $\rfloor$ denotes the interior product of a vector field and a form. Projection of a cartesian product on the i-th factor is nenoted by pr_i. The end of an example is denoted by $\square$.

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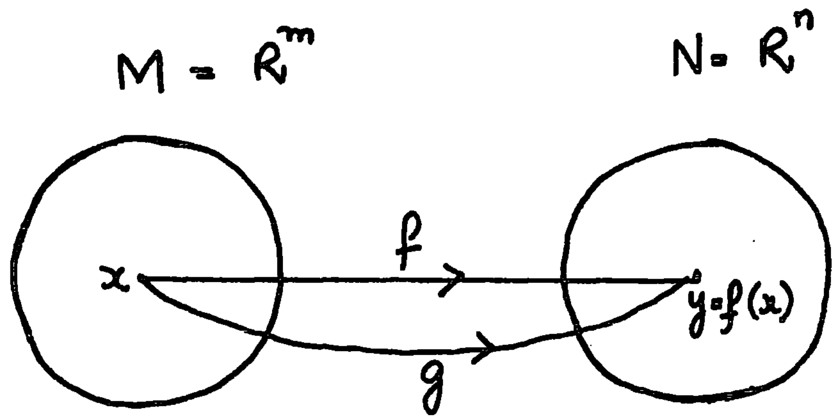
CHAPTER 1

Let M and N be C^∞ manifolds and let $C^\infty(M,N)$ denote the collection of C^∞ maps $f : U \longrightarrow N$ where U is an open set in M. The manifold M will be considered as a space of independent variables and N as a space of dependant variables.

Two functions f, g $\in C^\infty(M,N)$ are said to agree to order K at X$\in$M if they have the same value and if their derivatives up to the order K have the same value. It can be shown that this condition is independent of the particular co-ordinates chosen.

Let



write

$$\bar{y}^u = \bar{f}^u(x^1, x^2, \ldots\ldots\ldots, x^m)$$
$$\bar{y}^u = \bar{g}^u(x^1, x^2, \ldots\ldots\ldots, x^m)$$

$u = 1, 2, \ldots\ldots, n.$

f and g are equivalent at x if

$$f(x) = g(x), \quad \frac{\partial \bar{f}^u}{\partial x^a} = \frac{\partial \bar{g}^u}{\partial x^a}, \quad \frac{\partial^2 \bar{f}^u}{\partial x^a \partial x^b} = \frac{\partial^2 \bar{g}^u}{\partial x^a \partial x^b},$$

$$\ldots\ldots\ldots\ldots \quad \frac{\partial^K \bar{f}^u}{\partial x^a \partial x^b \ldots\ldots \partial x^{a_K}} = \frac{\partial^K \bar{g}^u}{\partial x^a \partial x^b \ldots\ldots \partial x^{a_K}}$$

This is a proper equivalence relation where a, b, a_1, - - a_k range over 1-- dim M. The maps f split into equivalence classes. The equivalence class of maps which agree with f to order K at x is called the <u>K-jet</u> of f at x and denoted by $j_x^K f$.

For to determine the elements of $j_x^K f$, let x^a are local co-ordinates around $X \in M$ and Z^u are local co-ordinates around $f(x) \in N$, then the elements of $j_x^K f$ are

$$x^a, \ \overset{u}{Z} = \overset{u}{f}(x), \ \overset{u}{Z}_a = \frac{\partial \overset{u}{f}(x)}{\partial x^a}, \ \overset{u}{Z}_{ab} = \frac{\partial^2 \overset{u}{f}(x)}{\partial x^a \partial x^b},$$

$$\ldots\ldots\ldots\ldots, \ \overset{u}{Z}_{a_1 \ldots\ldots a_K} = \frac{\partial^K \overset{u}{f}(x)}{\partial x^{a_1} \ldots\ldots \partial x^{a_K}}$$

where a, b, a_1, a_2, --- range over 1---- dim M and u range over 1---- dim N.

In particular case let K=3 then the elements of $j_x^3 f$ will be

$$x^a, \ \overset{u}{Z} = \overset{u}{f}(x), \ \overset{u}{Z}_a = \frac{\partial \overset{u}{f}}{\partial x^a}, \ \overset{u}{Z}_{ab} = \frac{\partial^2 \overset{u}{f}}{\partial x^a \partial x^b},$$

$$\overset{u}{Z}_{abc} = \frac{\partial^3 \overset{u}{f}}{\partial x^a \partial x^b \partial x^c}$$

Where a, b, c range over dim M, and u range over the dim of N.

<u>Jet Bundle</u>

The K-jet bundle of M and N denoted by $J^K(M,N)$ is the set of all K-jets $j_x^K f$, with K fixed, $X \in M$ and $f \in \overset{\infty}{C}(M,N)$. Jet bundle is a differentiable manifold having the differentiable structure pulled back from differentiable manifolds M and N.

Source Map

A point in $J^K(M,N)$ is represented by $j_x^K f$ for some f. Then the map

$$\alpha : J^K(M,N) \longrightarrow M$$
$$j_x^K f \longmapsto x$$

is called the source map and x is called the source of $j_x^K f$.

Target Map

A point in $J^K(M,N)$ is represented by $j_x^K f$ for some f. Then the map

$$\beta : J^K(M,N) \longrightarrow N$$
$$j_x^K f \longmapsto f(x)$$

is called the target map, and f(x) is called the target of $j_x^K f$.

Standard Co-ordinates

Now the question arises what will be the standard co-ordinates of $J^K(M,N)$ i.e. elements of $J^K(M,N)$ when the dim M = m and dim N = n. They will be

$$\left(x^a, \overset{u}{\widetilde{Z}}, \overset{u}{\widetilde{Z}}_a, \overset{u}{\widetilde{Z}}_{ab}, \ldots\ldots\ldots, \overset{u}{\widetilde{Z}}_{a_1 \ldots\ldots a_K} \right)$$

where u ranges and sum over n and a, b, a_1, a_2, ---- ranges over m.

e.g.

Let

$$M = R^2 \quad , \quad N = R^1$$
$$= (x^1, x^2) \qquad = Z$$

Then for given f the elements of $J^1(M,N)$ will be

$$\left(x^1, x^2, Z, \frac{\partial Z}{\partial x_1}, \frac{\partial Z}{\partial x_2}\right) \text{ where } Z = f(x_1, x_2)$$

The elements of $J^2(M,N)$ will be

$$\left(x_1, x_2, z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \frac{\partial^2 z}{\partial x_1^2}, \frac{\partial^2 z}{\partial x_1 \partial x_2}, \frac{\partial^2 z}{\partial x_2^2}\right)$$

where $z = f(x_1, x_2)$

Contact Form

To give the idea of contact form, let us consider $\dim M = R^2 = (U_1, U_2)$ $\dim N = R^1 = Z$ and $f : M \to N$ then 1-jet extension of f will be $j^1 f : M \to J^1(M,N)$ with the co-ordinate presentation

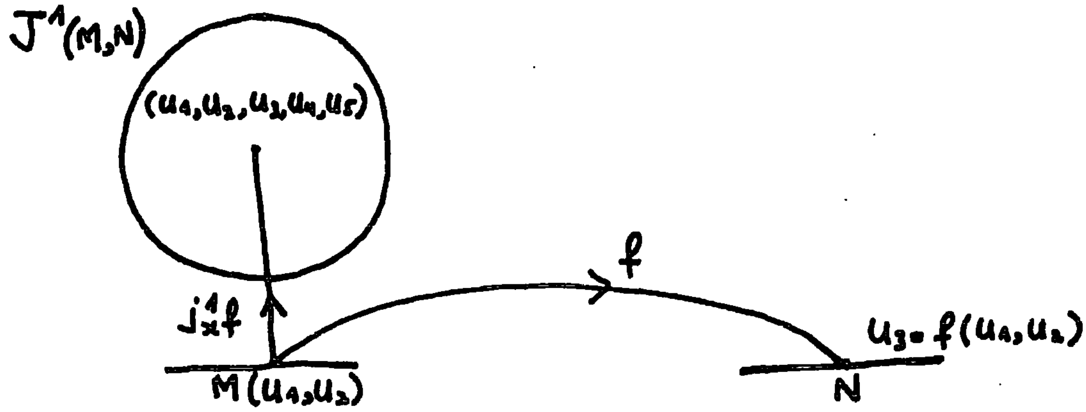
$$(u_1, u_2) \longmapsto \left(u_1, u_2, u_3 = f(u_1,u_2), u^4 = \frac{\partial f}{\partial u_1}, u^5 = \frac{\partial f}{\partial u_2}\right)$$

then the 1-form on $J^1(M,N)$ will be

$$a_1 du_1 + a_2 du_2 + a_3 du_3 + a_4 du_4 + a_5 du_5$$

where $a,^s$ are functions of U's.

Now the position is the following



$$(j^1_x f)^* : a_1 du_1 + a_2 du_2 + a_3 du_3 + a_4 du_4 + a_5 du_5$$

$$\longrightarrow a_1 du_1 + a_2 du_2 + a_3\left(\frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2\right)$$
$$+ a_4\left((j^1_x f)^* du_4\right) + a_5\left((j^1_x f)^* du_5\right)$$
$$\longrightarrow (A)$$

but the contact form is

$$(j^1_x f)^* \theta = 0 \qquad \text{for all } f.$$

this means $a_4 = 0$ and $a_5 = 0$

Hence

$$a_1 + a_3\left(\frac{\partial f}{\partial u_1}\right) = 0$$
$$a_2 + a_3\left(\frac{\partial f}{\partial u_2}\right) = 0$$

most general form on $J^1(M\ N)$ whose pull back is zero is

$$-a_3\left(\frac{\partial f}{\partial u_1}\right)du_1 - a_3\left(\frac{\partial f}{\partial u_2}\right)du_2 + a_3\,du_3 + 0\,du_4 + 0\,du_5$$

i.e.

$$a_3\left(du_3 - \left(\frac{\partial f}{\partial u_1}\right)du_1 - \left(\frac{\partial f}{\partial u_2}\right)du_2\right)$$

$$\theta = dz - p\,dx - q\,dy$$

If θ_1 and θ_2 are contact forms so is $U_1\,\theta_1 + U_2\theta_2$, where U_1 and U_2 are arbitrary functions on $J^K(M,N)$; thus contact forms comprise a module (Nelson 1967. p.1) over $C^\infty(J^K(M,N),R)$. This module will be denoted by $\Omega^K(M,N)$ or if no ambiguity arises by Ω^K and called the contact module of $J^K(M,N)$.

Each contact module is finitely generated. In fact Ω^K has basis (Johnson 1962) comprising the forms given in standard co-ordinates by

$$\overset{u}{\theta} = d\overset{u}{Z} - \overset{u}{Z}_c\,dx^c$$

$$\overset{u}{\theta}_b = d\overset{u}{Z}_b - \overset{u}{Z}_{bc}\,dx^c$$

$$\vdots$$

$$\overset{u}{\theta}_{b_1 \ldots\ldots b_{K-1}} = d\overset{u}{Z}_{b_1 \ldots\ldots b_{K-1}} - \overset{u}{Z}_{b_1 \ldots\ldots b_{K-1} c}\,dx^c$$

These basis will be called the standard basis for Ω^K.

To see whether an ideal of differential forms is closed under exterior differentiation. If Θ is any collection of forms then the exterior ideal $I(\Theta)$ generated by Θ is the collection of finite linear combinations $\sum \eta \wedge \theta$ where are arbitrary forms, θ are forms in Θ, and $\wedge$ denotes the exterior product. If Θ is any collection of forms, then $d\Theta$ means the collection of exterior derivatives $\{d\theta \mid \theta \in \Theta\}$

An ideal $I(\theta)$ is called closed or a differential ideal if $I(d\theta) \subset I(\theta)$.

<u>Theorem 1:1</u> Prove that $d\theta^u = dx^b \wedge \theta^u_b$

<u>Proof</u> Now we know

$$\theta^u = dZ^u - Z^u_c dx^c$$

$$d\theta^u = dd(Z^u) - dZ^u_c \wedge dx^c$$

But $dd(Z^u) = 0$

therefore $d\theta^u = -dZ^u_c \wedge dx^c \longrightarrow$ (A)

we have to prove that

$$d\theta^u = dx^b \wedge \theta^u_b \longrightarrow (i)$$

Now putting the value of θ^u_b in R.H.S. of (i) we get

R.H.S.

$$= dx^b \wedge (dZ^u_b - Z^u_{bc} dx^c)$$

$$= dx^b \wedge dZ^u_b - Z^u_{bc} dx^b \wedge dx^c$$

But $dx^b \wedge dx^c Z^u_{bc} = 0$

$$= dx^b \wedge dZ^u_b$$

changing "b" to "c" we get

$$= dx^c \wedge dZ^u_c$$

$$= -dZ^u_c \wedge dx^c \longrightarrow (B)$$

comparing A and B we get

L.H.S. = R.H.S.

$\square$

As we have already proved that $d\theta^u = dx^b \wedge \theta^u_b$
Similarly we can prove

$$d\theta^u_{b_1} = dx^b \wedge \theta^u_{bb_1}$$

$$\vdots$$

$$d\theta^u_{b_1 \ldots b_{K-2}} = dx^b \wedge \theta^u_{bb_1 \ldots \ldots b_{K-2}}$$

but that $d\theta^u_{bb_1 \ldots b_{K-2}}$ is not a linear combination of the forms in Ω^K. Thus $I(\Omega^K)$ is not closed.

From the above discussion it has proved that the contact forms are on the jet bundles e.g. a jet bundle of order 1 i.e. $J^1(M,N)$ will have a contact form θ^u where u is the dim N and $\theta^u = dZ^u - Z^u_c dx^c$ similarly $J^2(M,N)$ will have contact forms θ^u, θ^u_a written by

$$\theta^u = dZ^u - Z^u_c\, dx^c$$

$$\& \quad \theta^u_a = dZ^u_a - Z^u_{ac}\, dx^c$$

Hence $\theta^u, \theta^u_a, \theta^u_{ab}, \ldots\ldots \theta^u_{a_1 \ldots a_{K-1}}$ will be the contact forms on $J^K(M,N)$.

Example 1:1 The Sine- Gordon equation

$$\frac{\partial^2 Z}{\partial x \partial y} = \sin Z$$

The equation is the zero set of the single function

$$F = Z_{12} - \sin Z$$

on $J^2(M,N)$ where dim N = 2 = (x,y) dim N = 1 = Z

And the contact ideal is generated by

$$\theta = dZ - Z_1\, dx^1 - Z_2\, dx^2$$

$$\theta_1 = dZ_1 - Z_{11}\, dx^1 - Z_{12}\, dx^2$$

$$\theta_2 = dZ_2 - Z_{21}\, dx^1 - Z_{22}\, dx^2$$

In the classical notation the last two would be written as

$$\theta_1 = dp - r\,dx - s\,dy$$

$$\theta_2 = dq - s\,dx - t\,dy$$

$\square$

Example 1:2 The Kortweg-devries equation

The equation may be written as

$$\frac{\partial Z}{\partial y} + 12 Z \frac{\partial Z}{\partial x} + \frac{\partial^3 Z}{\partial x^3} = 0$$

with dim M = 2 dim N = 1

The equation is the zero set of the single function

$$F = Z_2 + 12 Z Z_1 + Z_{111}$$

on $J^3(M,N)$ and the contact ideal is generated by

$$\theta = dz - z_a\, dx^a$$

$$\theta_a = dz_a - z_{ab}\, dx^b$$

$$\theta_{11} = dz_{11} - z_{111}\, dx^1 - z_{112}\, dx^2$$

$$\theta_{12} = dz_{12} - z_{112}\, dx^1 - z_{122}\, dx^2$$

$$\theta_{22} = dz_{22} - z_{122}\, dx^1 - z_{222}\, dx^2$$

$\square$

CHAPTER II

Prolongation

A map from a jet bundle to another manifold induces maps of higher jet bundles, called prolongations, which in co-ordinates amount merely to the taking of total derivatives. The construction is as follows:
Let M, N and P be manifolds and let $\phi : J^h(M,N) \longrightarrow P$ be a smooth map. The s-th prolongation of ϕ is the unique map

$$p^s\phi : J^{h+s}(M,N) \longrightarrow J^s(M,P)$$

with the property that for every $f \in C^\infty(M,N)$ the diagram

$$\begin{tikzcd} J^{h+s}(M,N) \arrow[rr] & & J^s(M,P) \\ & M \arrow[ul, "j^{h+s}f"] \arrow[ur, "j^s(\phi \circ j^h f)"'] & \end{tikzcd}$$

figure 1

Commutes. If $x^a, \tilde{z}^u, y^A$ are local co-ordinates on M, N and P respectively with the above conventions on a and u (and A ranging over 1, --- dim P) and if in the standard co-ordinates has the presentation

$$y^A = \phi^A(x^a, \tilde{z}^u, \tilde{z}^u_a, \dots\dots\dots, \tilde{z}^u_{a_1 \dots\dots a_h})$$

and f has the presentation

$$\tilde{z}^u = \tilde{f}^u(x^a)$$

then $\phi \circ j^h f$ has the presentation

$$y^A = \phi^A\left(x^a, \tilde{f}^u(x), \frac{\partial \tilde{f}^u(x)}{\partial x^a}, \dots\dots\dots, \frac{\partial^h \tilde{f}^u(x)}{\partial x^{a_1} \dots\dots \partial x^{a_h}}\right)$$

so that $j^s(\phi \circ j^h f)$ has the presentation

$$\left.\begin{aligned}
x^a &= x^a \\
y^A &= \phi^A\left(x^a, \tilde{f}^u(x), \frac{\partial \tilde{f}^u(x)}{\partial x^a}, \dots\dots\dots, \frac{\partial^h \tilde{f}^u(x)}{\partial x^{a_1} \dots\dots \partial x^{a_h}}\right) \\
y^A_b &= \left(\partial^{h+1\#}_b \pi^{h+1*}_h \phi^A\right)\left(x^a, \tilde{f}^u(x), \frac{\partial \tilde{f}^u(x)}{\partial x^a}, \dots\dots, \frac{\partial^{h+1} \tilde{f}^u(x)}{\partial x^{a_1} \dots\dots \partial x^{a_{h+1}}}\right) \\
&\vdots \\
y^A_{b_1 \dots\dots b_s} &= \left(\partial^{h+s\#}_{b_s} \pi^{h+s*}_{h+s-1} \dots\dots \partial^{h+1\#}_{b_1} \pi^{h+1*}_h \phi^A\right)\left(x^a, \tilde{f}^u(x), \frac{\partial \tilde{f}^u(x)}{\partial x^a}, \dots\dots\dots \frac{\partial^{h+s} \tilde{f}^u(x)}{\partial x^{a_1} \dots \partial x^{a_{h+s}}}\right)
\end{aligned}\right\} (2.1)$$

where

$$\partial^{K\#}_b = \frac{\partial}{\partial x^b} + \tilde{z}^u_b \frac{\partial}{\partial \tilde{z}^u} + \tilde{z}^u_{bb_1} \frac{\partial}{\partial \tilde{z}^u_{b_1}} + \dots + \tilde{z}^u_{bb_1 \dots b_{K-1}} \frac{\partial}{\partial \tilde{z}^u_{b_1 \dots b_{K-1}}} \quad (2.2)$$

is the total derivative or "hash aperator". It comprises a co-ordinate dependent collection of (dimM) vector fields defined on $J^1(M,H)$ for $1 \geqslant K$.

Solution of differential equation

In jet bundles, a differential equation defines a subset of the jet bundle. It is convenient to define a system of partial differential equations of order K to be a submanifold Z of $J^K(M,N)$ which is the zero set of a finitely generated ideal $\sum_Z$ of functions on $J^K(M,N)$. If $F_1, F_2 \text{ -- } F_q$ is a set of generators, then the system of equations may be written in terms of the co-ordinate presentations of these functions - say in standard co-ordinates

$$F_1(x^a, \bar{Z}^u, \bar{Z}^u_b, \ldots\ldots\ldots \bar{Z}^u_{b_1 \ldots\ldots b_K}) = 0$$
$$\vdots$$
$$F_q(x^a, \bar{Z}^u, \bar{Z}^u_b, \ldots\ldots\ldots, \bar{Z}^u_{b_1 \ldots\ldots b_K}) = 0$$

Thus the generators define a map

$$F : J^K(M,N) \longrightarrow R^q$$

and a solution of the system if a map

$$f \in C^\infty(M,N) \quad \text{s.t}$$
$$F \circ j^K f = 0$$

Example 2.1 : The Sine-Gordon equation

As an example of the application, consider the Sine-Gordon equation

$$Z_{12} = \sin Z$$
$$F \equiv Z_{12} - \sin Z$$

The function $F: M \to N$

by $(x_1, x_2) \longmapsto Z = f(x_1, x_2)$

is a solution of the Sine-Gordon equation if

$$F \circ j^2 f = 0$$

which means that

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \, \partial x_2} - \sin\left(f(x_1, x_2)\right) = 0$$

$\square$

Bäcklund Map

A Bäcklund map is a transformation of the dependent variables in a system of differential equations, whereby the first derivatives of the new variables are given in terms of the new variables themselves as well as of the old variables and their derivatives

Let M, N_1 and N_2 be C^∞ manifold, and let

$$\Psi : J^1(M, N_1) \times N_2 \longrightarrow J^1(M, N_2)$$

be a C^∞ map. Here M is the space of the independent variables N_1 is the space of the old dependent variables. N_2 is the space of the new dependent variables. We will impose conditions on Ψ which are appropriate for it to be called a "Backlund map".

The first condition on Ψ is that the new dependent variables - the co-ordinates on N_2 - should be unaltered by the map. This will be the case if Ψ acts trivially on N_2 i.e. if

figure 2

$$\begin{tikzcd} J^1(M, N_1) \times N_2 \arrow[r, "\Psi"] \arrow[dr, "Pr_2"'] & J^1(M, N_2) \arrow[d, "\beta"] \\ & N_2 \end{tikzcd}$$

commutes. Here β denotes the target map.

The second condition on Ψ that the independent variables i.e. the co-ordinates on M are also left unaltered by the map Ψ, which will be the case if Ψ acts trivially on M i.e. if

$$\begin{tikzcd}
J^1(M,N_1)\times N_2 \arrow[r, "\psi"] \arrow[d, "pr_1"'] & J^1(M,N_2) \arrow[d, "\alpha"] \\
J^1(M,N_1) \arrow[r, "\alpha"] & M
\end{tikzcd}$$

commutes. Here pr_1 means projection on the first factor and α denotes the source map.

Now since

$$\psi: J^1(M,N_1) \times N_2 \longrightarrow J^1(M,N_2)$$

On the domain of ψ, choose local co-ordinates

x^a on M a, b, c ----- = 1 ---- dim M

z^μ on N_1 μ, p = 1 ---- dim N_1

y^A on N_2 A.B.C. = 1 ---- dim N_2

on the co-domain of ψ, choose local co-ordinates

x'^a on M

y'^A on N_2

And standard associated co-ordinates

y'^A_b on $J^1(M,N_2)$

Thus the commuting of Fig. 2

$y^A = y^A$

And due to the commuting of Fig. 3

$x'^a = x^a$

The map ψ is fixed completely. If the co-ordinates y'^A_b are given as functions, say ψ^A_b on $J^1(M,N_1) \times N_2$. Thus under ψ

$$y'^A_b = \psi^A_b(x^a, \bar{z}^\mu, \bar{z}^\mu_a, y^B) \longrightarrow (2.3)$$

Now let f: $M \rightarrow N$, and g: $M \rightarrow N_2$ are given with local presentations $\bar{z}^\mu = \bar{f}^\mu(x)$ $y^A = g^A(x)$

Now $\quad y^A_b = \dfrac{\partial g^A(x)}{\partial x^b}$

Now since $y'^A_b = y^A_b$

Therefore by putting the value of y_b^A in (2.3)

We get

$$\frac{\partial g^A(x)}{\partial x^b} = \psi_b^A \left(x^a, f^u(x), \frac{\partial f^u(x)}{\partial x^a}, g^B(x) \right) \longrightarrow (2.4)$$

which, in co-ordinate-free terms would be

$$j^1 g = \psi \circ (j^1 f \times g) \circ \Delta_M \longrightarrow (2.5)$$

i.e. where Δ_M denotes diagonal map

$M \rightarrow M \times M \rightarrow J^1(M,N_1) \times N_2 \rightarrow J^1(M,N_2)$

In fact given ψ, a map g satisfying (2.5) can exist only if the integrability conditions for it are satisfied. These integrability conditions, which follow from

$$\frac{\partial}{\partial x^c}\frac{\partial g^A}{\partial x^b} - \frac{\partial}{\partial x^b}\frac{\partial g^A}{\partial x^c} = 0$$

are

$$g^A_{bc} = 0$$

where

$$g^A_{bc} = \left[\frac{\partial \psi_b^A}{\partial x^c} + \frac{\partial \psi_b^A}{\partial z^u}\frac{\partial f^u}{\partial x^c} + \frac{\partial \psi_B^A}{\partial z^u_\alpha}\frac{\partial^2 f^u}{\partial x^\alpha \partial x^c} + \frac{\partial \psi_b^A}{\partial y^B}\psi_c^B \right] - \left[\frac{\partial \psi_c^A}{\partial x^b} + \frac{\partial \psi_c^A}{\partial z^u}\frac{\partial f^u}{\partial x^b} + \frac{\partial \psi_c^A}{\partial z^u_\alpha}\frac{\partial^2 f^u}{\partial x^\alpha \partial x^b} + \frac{\partial \psi_c^A}{\partial y^B}\psi_b^B \right] \longrightarrow (2.5)$$

Here ψ_b^A and its derivatives are evaluated on the arguments exhibited in (2.4)

Integrability Conditions

There are three different co-ordinate-free formulations of the integrability conditions

(i) In terms of exterior ideals of differential forms

(ii) In terms of modules of 1-forms ∴ (pfaffians)

(iii) In terms of vector fields

First we develop the necessary machinery, for finding the integrability conditions. First of all, the projection

$Pr_1 : J^2(M,N_1) \times N_2 \longrightarrow J^2(M,N_1)$

induces on $J^2(M,N_1) \times N_2$ a naturally defined module $p\ell_1^* \Omega^1(M,N_1)$.

Next the projection

$$\tilde{\pi}_1^2 : J^2(M,N_1) \to J^1(M,N_1)$$

may be extended to

$$\tilde{\pi}_1^2 = \pi_1^2 \times \mathrm{id}_{N_2} : J^2(M,N_1) \times N_2 \to J^1(M,N_1) \times N_2$$

and then

$$\psi \circ \tilde{\pi}_1^2 : J^2(M,N_1) \times N_2 \to J^1(M,N_2)$$

induces on $J^2(M,N_1) \times N_2$ the module $\tilde{\pi}_1^{2*} \psi^* \Omega^1(M,N_2)$. The sum of these two induced modules is denoted $\tilde{\Omega}^{2,\psi}$

$$\tilde{\Omega}^{2,\psi} = p\ell_1^* \Omega^1(M,N_1) + \tilde{\pi}_1^{2*} \psi^* \Omega^1(M,N_2) \longrightarrow (2.6)$$

Let θ^u and θ^u_a denote standard contact forms on $J^2(M,N_1)$ and let θ^A denote standard contact forms on $J^1(M,N_2)$. Where no ambiguity arises, we shall denote $p\ell_1^* \hat{\theta}^u$, $p\ell_1^* \hat{\theta}^u_a$ and $\tilde{\pi}_1^{2*} \psi^* \hat{\theta}^A$ by θ^u, θ^u_a and θ^A respectively, Then θ^u and θ^u_a comprise a basis for $p\ell_1^* \Omega^1(M,N_1)$ and θ^A comprise a basis for $\tilde{\pi}_1^{2*} \psi^* \Omega^1(M,N_2)$. More over writing ψ^A_b for $\tilde{\pi}_1^2 \psi^A_b$ here and below the θ^A have the form

$$\theta^A = dy^A - \psi^A_b \, dx^b \longrightarrow (2.7)$$

Imitating the construction of $\partial^{\#}_a$, det

$$\tilde{\partial}^2_a = \partial^{\#2}_a + \psi^B_a \frac{\partial}{\partial y^B} = \frac{\partial}{\partial x^a} + z^u_a \frac{\partial}{\partial z^u} + z^u_{ad} \frac{\partial}{\partial z^u_d} + \psi^B_a \frac{\partial}{\partial y^B}$$
$$\longrightarrow (2.8)$$

we will prove the properties of this aperator in the form of the theoreums.

<u>Theorem</u> If u is any function on $J^1(M,N_1) \times N_2$ then

$$d(\pi_1^{2*} u) = \tilde{\partial}^2_a (\tilde{\pi}_1^{2*} u) \, dx^a \mod \tilde{\Omega}^{2,\psi}$$

<u>Proof</u> If u is a function on $J^1(M,N_1) \times N_2$ then π_1^{2*} is a function on $J^2(M,N_1) \times N_2$. Since

$$\pi_1^2 : J^2(M,N_1) \times N_2 \longrightarrow J^1(M,N_1) \times N_2$$

Now if we take the differential of any function which is on

$J^2(M,N_1) \times N_2$ which is pulled back from $J^1(M,N_1) \times N_2$ will be equal to

$$d(\overset{\#}{\pi}{}_1^2 u) = \tilde{\partial}_a^2(\tilde{\pi}_1^{2*} u)\,dx^a \mod \tilde{\Omega}^{2,\psi}$$

Because when we differentiate u with respect to co-ordinates on N_2 we get $\tilde{\partial}_a^2(\tilde{\pi}_1^2)\,dx^a \mod \tilde{\Omega}^{2,\psi}$ which is the required result.

<u>Theorem 2.2</u> Prove that

$$[\tilde{\partial}_a^2, \tilde{\partial}_b^2] = (\tilde{\partial}_a^2 \psi_b^B - \tilde{\partial}_b^2 \psi_a^B)\frac{\partial}{\partial y^B}$$

When the differentiation is with respect to the co-ordinates on N_2.

<u>Proof</u>

$$[\tilde{\partial}_a^2, \tilde{\partial}_b^2] = \left[\overset{\#}{\partial}{}_a^2 + \psi_a^A \frac{\partial}{\partial y^A}, \overset{\#}{\partial}{}_b^2 + \psi_b^B \frac{\partial}{\partial y^B}\right]$$

$$= \overset{\#}{\partial}{}_a^2 \overset{\#}{\partial}{}_b^2 + \overset{\#}{\partial}{}_a^2\left(\psi_b^B \frac{\partial}{\partial y^B}\right) + \psi_a^A \frac{\partial}{\partial y^A}\left(\overset{\#}{\partial}{}_b^2\right) + \psi_a^A \frac{\partial}{\partial y^A}\left(\psi_b^B \frac{\partial}{\partial y^B}\right)$$
$$- \overset{\#}{\partial}{}_b^2 \overset{\#}{\partial}{}_a^2 - \overset{\#}{\partial}{}_b^2\left(\psi_a^A \frac{\partial}{\partial y^A}\right) - \psi_b^B \frac{\partial}{\partial y^B}\left(\overset{\#}{\partial}{}_a^2\right) - \psi_b^B \frac{\partial}{\partial y^B}\left(\psi_a^A \frac{\partial}{\partial y^A}\right)$$

$$= \overset{\#}{\partial}{}_a^2 \overset{\#}{\partial}{}_b^2 + \overset{\#}{\partial}{}_a^2\left(\psi_b^B \frac{\partial}{\partial y^B}\right) + \psi_a^A \frac{\partial}{\partial y^A}\left(\overset{\#}{\partial}{}_b^2\right)$$
$$+ \psi_a^A \frac{\partial \psi_b^B}{\partial y^A} \frac{\partial}{\partial y^B} + \psi_a^A \psi_b^B \frac{\partial}{\partial y^A}\frac{\partial}{\partial y^B} - \overset{\#}{\partial}{}_b^2 \overset{\#}{\partial}{}_a^2$$
$$- \overset{\#}{\partial}{}_b^2\left(\psi_a^A \frac{\partial}{\partial y^A}\right) - \psi_b^B \frac{\partial}{\partial y^B}\left(\overset{\#}{\partial}{}_a^2\right) - \psi_b^B \frac{\partial \psi_a^A}{\partial y^B}\frac{\partial}{\partial y^A}$$
$$- \psi_b^B \psi_a^A \frac{\partial}{\partial y^B}\frac{\partial}{\partial y^A}$$

$$= \overset{\#}{\partial}{}_a^2 \psi_b^B \frac{\partial}{\partial y^B} + \psi_a^A \psi_b^B \frac{\partial}{\partial y^A}\frac{\partial}{\partial y^B} - \overset{\#}{\partial}{}_b^2 \psi_a^A \frac{\partial}{\partial y^A}$$
$$- \psi_b^B \psi_a^A \frac{\partial}{\partial y^B}\frac{\partial}{\partial y^A}$$

$$= \left(\overset{\#}{\partial}{}_a^2 + \psi_a^A \frac{\partial}{\partial y^A}\right)\psi_b^B \frac{\partial}{\partial y^B} - \left(\overset{\#}{\partial}{}_b^2 + \psi_b^B \frac{\partial}{\partial y^B}\right)\psi_a^A \frac{\partial}{\partial y^A} \longrightarrow (A)$$

But

$$\overset{*}{\partial}{}^{2}_{a} + \overset{A}{f}_{a} \frac{\partial}{\partial y^{A}} = \tilde{\partial}^{2}_{a} \quad \text{and}$$

$$\overset{*}{\partial}{}^{2}_{b} + \overset{B}{f}_{b} \frac{\partial}{\partial y^{B}} = \tilde{\partial}^{2}_{b}$$

therefore (A) becomes

$$\left(\tilde{\partial}^{2}_{a} \overset{B}{f}_{b} - \tilde{\partial}^{2}_{b} \overset{B}{f}_{a} \right) \frac{\partial}{\partial y^{B}}$$

which is the required result.

<u>Theorem 2.3</u> If X is a vector field of the form

$$X = \overset{a}{X} \tilde{\partial}^{2}_{a}$$

then

$$X \rfloor \tilde{\Omega}^{2,4} = 0$$

<u>Proof</u> $\tilde{\Omega}^{2,4}$ contains the basis of type

$$\overset{u}{\theta} = d\overset{u}{z} - \overset{u}{z}_{c}\, dx^{c}$$

$$\overset{u}{\theta}_{b} = d\overset{u}{z}_{b} - \overset{u}{z}_{bc}\, dx^{c}$$

and $\overset{A}{\theta} = d\overset{A}{y} - \overset{A}{f}_{b}\, dx^{b}$

and $\tilde{\partial}^{2}_{a} = \frac{\partial}{\partial x^{a}} + \overset{u}{z}_{a} \frac{\partial}{\partial z^{u}} + \overset{u}{z}_{ad} \frac{\partial}{\partial \overset{u}{z}_{d}} + \overset{B}{f}_{a} \frac{\partial}{\partial y^{B}}$

Now we will show that the contraction of $\overset{a}{X} \tilde{\partial}^{2}_{a}$ with each of the basis of $\tilde{\Omega}^{2,4}$ is zero

$$\overset{a}{X} \tilde{\partial}^{2}_{a} \rfloor \tilde{\Omega}^{2,4} \qquad \text{i.e}$$

(i) $X^a \tilde{\partial}^2_a \lrcorner \widehat{\theta}^u$

$$= X^a\left(\frac{\partial}{\partial x^a} + \widehat{Z}^u_a \frac{\partial}{\partial Z^u} + \widehat{Z}^u_{ad}\frac{\partial}{\partial Z^u_d} + y^B_a \frac{\partial}{\partial y^B}\right) \lrcorner \left(d\widehat{Z}^u - \widehat{Z}^u_c dx^c\right)$$

$$= X^a\left(-\widehat{Z}^u_a + \widehat{Z}^u_a\right) = 0$$

(ii) $X^a \tilde{\partial}^2_a \lrcorner \widehat{\theta}^u_b$

$$= X^a\left(\frac{\partial}{\partial x^a} + \widehat{Z}^u_a \frac{\partial}{\partial Z^u} + \widehat{Z}^u_{ad}\frac{\partial}{\partial Z^u_d} + y^B_a \frac{\partial}{\partial y^B}\right) \lrcorner \left(d\widehat{Z}^u_b - \widehat{Z}^u_{bc} dx^c\right)$$

$$= X^a\left(\widehat{Z}^u_{ab} - \widehat{Z}^u_{ab}\right) = 0$$

Similarly $X^a \tilde{\partial}^2_a \lrcorner \theta^A$

$$= X^a\left(\frac{\partial}{\partial x^a} + \widehat{Z}^u_a \frac{\partial}{\partial Z^u} + \widehat{Z}^u_{ad}\frac{\partial}{\partial Z^u_d} + y^B_a \frac{\partial}{\partial y^B}\right) \lrcorner \left(dy^A - y^A_b dx^b\right)$$

$$= X^a\left(y^A_a - y^A_a\right) = 0$$

Theorem 2.4 To prove that

$$\tilde{\partial}^2_a \lrcorner d\widehat{\theta}^u = \widehat{\theta}_a \longrightarrow \text{i}$$

$$\tilde{\partial}^2_a \lrcorner d\widehat{\theta}^u_b = d\widehat{Z}^u_{ab} \longrightarrow \text{ii}$$

$$\tilde{\partial}^2_a \lrcorner d\theta^A = \left(\tilde{\partial}^2_b y^A_a - \tilde{\partial}^2_a y^A_b\right) dx^b \bmod \tilde{\Omega}^{2,+} \longrightarrow \text{iii}$$

<u>Proof</u> Let us first take L.H.S. of (i)

$$\tilde{\partial}^{2}_{a} \lrcorner\, d\overset{u}{\theta}$$

$$= \left(\frac{\partial}{\partial x^a} + \overset{u}{Z}_a \frac{\partial}{\partial Z^u} + \overset{u}{Z}_{ad} \frac{\partial}{\partial Z^u_d} + \overset{B}{\psi}_a \frac{\partial}{\partial y^B} \right) \lrcorner\, (dx^c \wedge d\overset{u}{Z}_c)$$

$$= d\overset{u}{Z}_a - \overset{u}{Z}_{ac}\, dx^c$$

$= \theta^u_a$ which is the R.H.S. of (i)

Now for (ii)

taking the L.H.S. of (ii)

$$\tilde{\partial}^{2}_{a} \lrcorner\, d\overset{u}{\theta}_b$$

$$= \left(\frac{\partial}{\partial x^a} + \overset{u}{Z}_a \frac{\partial}{\partial Z^u} + \overset{u}{Z}_{ad} \frac{\partial}{\partial Z^u_d} + \overset{B}{\psi}_a \frac{\partial}{\partial y^B} \right) \lrcorner\, \left(dx^c \wedge d\overset{u}{Z}_{bc} \right)$$

$= dZ^u_{ab}$ which is R.H.S. of (ii)

Similarly for (iii)

taking the L.H.S. of (iii)

$$\tilde{\partial}^{2}_{a} \lrcorner\, d\overset{A}{\theta}$$

$$= \left(\frac{\partial}{\partial x^a} + \overset{u}{Z}_a \frac{\partial}{\partial Z^u} + \overset{u}{Z}_{ad} \frac{\partial}{\partial Z^u_d} + \overset{B}{\psi}_a \frac{\partial}{\partial y^B} \right) \lrcorner$$

$$\left(dx^b \wedge \left(\frac{\partial \overset{A}{\psi}_b}{\partial x^a} dx^a + \frac{\partial \overset{A}{\psi}_b}{\partial Z^u} d\overset{u}{Z} + \frac{\partial \overset{A}{\psi}_b}{\partial y^B} \right) \right)$$

$$= \left(\frac{\partial}{\partial x^a} + \overset{u}{Z}_a \frac{\partial}{\partial Z^u} + \overset{u}{Z}_{ad} \frac{\partial}{\partial Z^u_d} + \overset{B}{\psi}_a \frac{\partial}{\partial y^B} \right) \lrcorner$$

$$\left(\frac{\partial \overset{A}{\psi}_b}{\partial x^a} dx^b \wedge dx^a + \frac{\partial \overset{A}{\psi}_b}{\partial Z^u} dx^b \wedge d\overset{u}{Z} + \frac{\partial \overset{A}{\psi}_b}{\partial \overset{u}{Z}_a} dx^b \wedge d\overset{u}{Z}_a + \frac{\partial \overset{A}{\psi}_b}{\partial y^B} dx^b \wedge dy^B \right)$$

$$= \left(\frac{\partial \psi^A_a}{\partial x^a}dx^a - \frac{\partial \psi^A_b}{\partial x^a}dx^b\right) + \left(\widehat{Z}^u_a \frac{\partial \psi^A_a}{\partial \widehat{Z}^u} d\widehat{Z}^u - \widehat{Z}^u_a \frac{\partial \psi^A_b}{\partial \widehat{Z}^u} dx^b\right)$$

$$+ \frac{\partial \psi^A_a}{\partial \widehat{Z}^u_a} d\widehat{Z}^u_a - \widehat{Z}^u_{cd}\frac{\partial \psi^A_b}{\partial \widehat{Z}^u_c}dx^b + \frac{\partial \psi^A_a}{\partial y^B}dy^B$$

$$- \psi^B_a \frac{\partial \psi^A_b}{\partial y^B} dx^b$$

$$= -\left(\frac{\partial \psi^A_b}{\partial x^a} + \widehat{Z}^u_a \frac{\partial \psi^A_b}{\partial \widehat{Z}^u} + \widehat{Z}^u_{cd}\frac{\partial \psi^A_b}{\partial \widehat{Z}^u_c} + \psi^B_a \frac{\partial \psi^A_b}{\partial y^B}\right)dx^b$$

$$+ \left(\frac{\partial \psi^A_a}{\partial x^a}dx^a + \frac{\partial \psi^A_a}{\partial \widehat{Z}^u}d\widehat{Z}^u + \frac{\partial \psi^A_a}{\partial \widehat{Z}^u_a}d\widehat{Z}^u_a + \frac{\partial \psi^A_a}{\partial y^B}dy^B\right)$$

$$= -\left(\frac{\partial \psi^A_b}{\partial x^a} + \widehat{Z}^u_a \frac{\partial \psi^A_b}{\partial \widehat{Z}^u} + \widehat{Z}^u_{cd}\frac{\partial \psi^A_b}{\partial \widehat{Z}^u_c} + \psi^B_a \frac{\partial \psi^A_b}{\partial y^B}\right)dx^b$$

$$+ \left(\frac{\partial \psi^A_a}{\partial x^a}dx^a + \widehat{Z}^u_a \frac{\partial \psi^A_a}{\partial \widehat{Z}^u}dx^a + \widehat{Z}^u_{ab}\frac{\partial \psi^A_a}{\partial \widehat{Z}^u_a}dx^a + \psi^A_a \frac{\partial \psi^A_a}{\partial y^B}dx^a\right)$$

$$= -\left(\frac{\partial \psi^A_b}{\partial x^a} + \widehat{Z}^u_a \frac{\partial \psi^A_b}{\partial \widehat{Z}^u} + \widehat{Z}^u_{cd}\frac{\partial \psi^A_b}{\partial \widehat{Z}^u_c} + \psi^B_a \frac{\partial \psi^A_b}{\partial y^B}\right)dx^b$$

$$+ \left(\frac{\partial \psi^A_a}{\partial x^b} + \widehat{Z}^u_a \frac{\partial \psi^A_a}{\partial \widehat{Z}^u} + \widehat{Z}^u_{ab}\frac{\partial \psi^A_a}{\partial \widehat{Z}^u_b} + \psi^A_a \frac{\partial \psi^A_a}{\partial y^B}\right)dx^b$$

$$= \left(\tilde{\partial}^2_b \psi^A_a - \tilde{\partial}^2_a \psi^A_b\right)dx^b \bmod \tilde{\Omega}^{2,\psi}.$$

After having this additional material which we have proved in the form of theorems, we come to our original problem.

A differential equation of older K is a submanifold $\widetilde{Z}$ of $J^K(M, N_1) \times N_2$ which is the zero set of finitely generated ideal $\widetilde{\Sigma}_Z$ of functions on $J^K(M, N_1) \times N_2$. If F_1, F_2 --- F_q is a set of generators, then the system of equations may be written in terms of the co-ordinate presentations of these functions - say in standard co-ordinates

$$F_\alpha(x^a, Z^u, Z^u_b, \ldots\ldots\ldots, Z^u_{b_1 \ldots\ldots b_K}, y^A) = 0$$

$$\alpha = 1, \ldots\ldots, q$$

Then the generators define a map

$$F : J^K(M, N_1) \times N_2 \longrightarrow R^q$$

and a solution of the system is a pair of maps $f \in C^\infty(M, N_1)$, $g \in C^\infty(M, N_2)$ s. t

$$F \circ (j^k_f \times g) \circ \Delta_M = 0$$

Now to find integrability conditions compare (2.5) with (2.8), one sees that (2.5) may be written as

$$g^A_{bc} = (\widetilde{\partial}^2_c \psi^A_b - \widetilde{\partial}^2_b \psi^A_c) \circ (j^2 f \times g) \circ \Delta_M$$

Consequently the pair (f,g) must be the solution of the system

$$\widetilde{\partial}^2_c \psi^A_b - \widetilde{\partial}^2_b \psi^A_c = 0 \longrightarrow (2.9)$$

on $J^2(M, N_1) \times N_2$.

Now taking the exterior derivative of (2.7) we get

$$d\theta^A = dx^b \wedge d\psi^A_b \longrightarrow (2.10)$$

which by theorem (2.1) can be written as

$$d\theta^A = \widetilde{\partial}^2_c \psi^A_b \, dx^b \wedge dx^c \bmod I(\widetilde{\Omega}^2, \psi)$$

$$\longrightarrow (2.11)$$

Also

$$d\theta^A = \frac{1}{2}\left[\tilde{\partial}^2_c \overset{A}{\Psi}_b dx^b \wedge dx^c - \tilde{\partial}^2_c \overset{A}{\Psi}_b dx^c \wedge dx^b\right]$$
$$= \frac{1}{2}\left[\tilde{\partial}^2_c \overset{A}{\Psi}_b dx^b \wedge dx^c - \tilde{\partial}^2_b \overset{A}{\Psi}_c dx^b \wedge dx^c\right]$$
$$= \frac{1}{2}\left[\tilde{\partial}^2_c \overset{A}{\Psi}_b - \tilde{\partial}^2_b \overset{A}{\Psi}_c\right] dx^b \wedge dx^c$$

Therefore when comparing (2.11) with (2.9) one may conclude that the system (2.9) is equivalent to

$$\tilde{\pi}^{2*}_1 \Psi^* d\Omega^1(M, N_2) \subset I(\tilde{\Omega}^{2,\Psi}) \longrightarrow (2.12)$$

This is the co-ordinate free formulation of the integrability conditions in terms of ideals of differential forms.

(ii) <u>Integrability conditions in terms of the properties of pfaffian modules</u>

$$\tilde{\Omega}^{2,\Psi} \doteq \tilde{\Omega}^{2,\Psi} + \left\{ X \lrcorner \tilde{\pi}^{2*}_1 \Psi^* d\Omega^1(M,N_2) \mid X \lrcorner \tilde{\Omega}^{2,\Psi} = 0 \right\} \longrightarrow (2.13)$$

Thus $\tilde{\Omega}^{2,\Psi}$ is constructed by adding to $\tilde{\Omega}^{2,\Psi}$ the contractions of forms on the left hand side of (2.12) with vectors which annihilate $\tilde{\Omega}^{2,\Psi}$ then by theorem (2.4), (2.12) is equivalent to $\tilde{\Omega}^{2,\Psi} \subset \tilde{\Omega}^{2,\Psi}$

(iii) From equation (2.8) theorem (2.2) and equation (2.9) it is seen at once the integrability conditions $\tilde{Z}$ may be characterised as the submanifold on which the distribution $\tilde{\Delta}^{2,\Psi}$ is completely integrable.

<u>Example 2.2</u> <u>The Sine-Gordon Equation: Backlund's original map</u>

Let dim M = 2 dim N_1 = dim N_2 = 1

The co-ordinates are x, y on M i.e. independent variables. z on N_1 and $\acute{z}$ on N_2 where $\acute{z}$ is new dependent variable.

Also

$$p = \frac{\partial z}{\partial x} \qquad q = \frac{\partial z}{\partial y} \qquad r = \frac{\partial^2 z}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} \qquad t = \frac{\partial^2 z}{\partial y^2}$$

$$p' = \frac{\partial z'}{\partial x} \qquad q' = \frac{\partial z'}{\partial y}$$

The map originally constructed by Backlund (1875, 1883) is

$$p' = p + 2a \sin\left(\frac{z'+z}{2}\right) \longrightarrow (i)$$

$$q' = -q + \frac{2}{a} \sin\left(\frac{z'-z}{2}\right) \longrightarrow (ii)$$

Now the integrability condition is

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z'}{\partial x \partial y}$$

Differentiating (i) w. r. t "y"

$$\frac{\partial^2 z'}{\partial y \partial x} = \frac{\partial p}{\partial y} + 2a \cos\left(\frac{z'+z}{2}\right) \cdot \frac{1}{2}(q'+q)$$

putting the value of $(q'+q)$ from (ii)

$$s' = s + a \cos\left(\frac{z'+z}{2}\right) \frac{2}{a} \sin\left(\frac{z'-z}{2}\right)$$

$$s' = s + \sin z' - \sin z$$

$$\text{i.e } (s' - \sin z') = (s - \sin z) \longrightarrow (A)$$

Differentiating (ii) by x we get

$$s' = -s + \frac{2}{a} \cos\left(\frac{z'-z}{2}\right) \cdot \frac{1}{2}\, 2a \sin\left(\frac{z'+z}{2}\right)$$

$$s' = -s + \sin z' + \sin z$$

$$(s' - \sin z') = -(s - \sin z) \longrightarrow (B)$$

From (A) and (B) we conclude that

$$s' = \sin z' \qquad s = \sin z$$

This means that if Z satisfies $s = \sin z$ and z' is related to Z by Bäcklund map i.e. by equations (i) and (ii) then z' is a solution.

CHAPTER III

In this chapter we will discuss the idea of prolongation in detail. We have already seen in the II chapter that if we consider M, N_1 and N_2 be C^∞ manifolds and $\psi: J^1(M; N_1) \times N_2 \longrightarrow J^1(M; N_2)$ be a C^∞ map. If the co-ordinates on M and N_2 are unaltered i.e. to fix the map ψ. If the co-ordinates y'^A_b are given as functions say ψ^A_b on $J^1(M; N_1) \times N_2$.

Thus under ψ

$$y'^A_b = \psi^A_b \left(x^a, \bar{z}^\mu, \bar{z}^\mu_a, y^B \right)$$

And if x, y, are co-ordinates on M and Z is co-ordinate on N_1 and Z' is co-ordinate on N_2 then we find the integrability condition

$$\tilde{\partial}^2_c \psi^A_b - \tilde{\partial}^2_b \psi^A_c = 0$$

and the Bäcklund map

$$p' = p + 2a \sin \tfrac{1}{2} (z' + z)$$
$$q' = -q + \tfrac{2}{a} \sin \tfrac{1}{2} (z' - z)$$

changes to Sine-Gordon equation. Now we shall see what happens when we get the first prolongation.

Let

$$\psi: J^1(M; N_1) \times N_2 \longrightarrow J^1(M; N_2)$$

be a map for which

$$\begin{tikzcd}
J^1(M,N_1)\times N_2 \arrow[r, "\Psi"] \arrow[dr, "ph_2"'] & J^1(M,N_2) \arrow[d, "\beta"] \\
& N_2
\end{tikzcd}$$

figure 3.1

$$\begin{tikzcd}
J^1(M,N_1)\times N_2 \arrow[r, "\Psi"] \arrow[d, "ph_1"'] & J^1(M,N_2) \arrow[d, "\alpha"] \\
J^1(M,N_1) \arrow[r, "\alpha"] & M
\end{tikzcd}$$

figure 3.2

commutes. A map

$\psi^1 : J^2(M; N_1) \times N_2 \longrightarrow J^2(M; N_2)$

is said to be compatible with if the diagram

$$\begin{tikzcd} J^2(M, N_1) \times N_2 \arrow[r, "\psi^1"] \arrow[d, "\tilde{\pi}^2_1"'] & J^2(M, N_2) \arrow[d, "\pi^2_1"] \\ J^1(M, N_1) \times N_2 \arrow[r, "\psi"] & J^1(M, N_2) \end{tikzcd}$$

commutes. In Fig (3.3) we have done the following process

$\pi^2_1 : J^2(M; N_1) \longrightarrow J^1(M; N_1)$

may be extended to

$\tilde{\pi}^2_1 = \pi^2_1 \times id_{N2} : J^2(M; N_1) \times N_2 \longrightarrow J^1(M; N_1) \times N_2$

A map ψ^1 compatible with ψ is completely determined by the specification of functions ψ^A_{bc} on $J^2(M; N_1) \times N_2$

Under ψ^1

$$y'^A_{bc} = \psi^A_{bc}(x^a, \tilde{z}^u, \tilde{z}^u_a, \tilde{z}^u_{ad}, y^B) \longrightarrow (3.1)$$

Hence the y'^A_{bc} are standard co-ordinates on $J^2(M; N_2)$ and the arguments of ψ^A_{bc} are standard co-ordinates on $J^2(M; N_1) \times N_2$. The appropriate choice of the functions ψ^A_{bc} is

$$\psi^A_{bc} = \tilde{\partial}^2(b\, \psi^A_c) \longrightarrow (3.2)$$

The map ψ^1 is called the first prolongation of ψ. This definition of prolongation is consistant with the definition of prolongation which is defined in chapter II i.e.

$$J^h(M, N) \longrightarrow P$$

Now the following example will clear the idea of prolongation.

Example 3.1: The Sine-Gordon equation: Bäcklund's original map

Let dim M = 2, dim N_1 = dim N_2 = 1.

The co-ordinates are x^1 and x^2 on M, Z on N_1 and $y = y'$ on N_2.

The map originally constructed by Bäcklund is

$$\left.\begin{aligned} y_1' &= \psi_1 = Z_1 + 2a \sin\tfrac{1}{2}(y+Z) \\ y_2' &= \psi_2 = -Z_2 + 2a^{-1} \sin\tfrac{1}{2}(y-Z) \end{aligned}\right\} \quad (3.3)$$

$$\psi : J^1(M,N_1) \times N_2 \longrightarrow J^1(M,N_2)$$

$$(x^1, x^2, Z, Z_1, Z_2, y) \longmapsto (x^1, x^2, y, y_1, y_2)$$

$$\psi^1 : J^2(M,N_1) \times N_2 \longrightarrow J^2(M,N_2)$$

$$(x^1, x^2, Z, Z_1, Z_2, Z_{11}, Z_{12}, Z_{22}, y) \longrightarrow (x^1, x^2, y, y_1, y_2, y_{11}, y_{12}, y_{22})$$

Now we have to find the value of y_{11}, y_{12}, y_{22} when the value of y_1 and y_2 are given by equation (3.3). We can easily find the value of y_{11}, y_{12}, y_{22} from the (3.2) relation

$$\psi^A_{bc} = \tilde{\partial}^2(b\,\psi^A_c)$$

here A = 1 Since dim y = 1

$$\psi_{bc} = \tilde{\partial}^2(b\,\psi_c)$$

in general

$$\tilde{\partial}^2_a = \frac{\partial}{\partial x^a} + Z^u_a \frac{\partial}{\partial Z^u} + Z^u_{ad} \frac{\partial}{\partial Z^u_d} + \psi^A_a \frac{\partial}{\partial y^A}$$

but u = 1 Since dim N_1 = 1

$$\tilde{\partial}^2_a = \frac{\partial}{\partial x^a} + Z_a \frac{\partial}{\partial Z} + Z_{ad} \frac{\partial}{\partial Z_d} + \psi_a \frac{\partial}{\partial y} \longrightarrow (A)$$

Now from equation (3.1)

$$y_{bc} = \psi_{bc} \quad \text{and}$$

$$y_{bc} = \psi_{bc} = \tilde{\partial}^2_b \psi_c \quad \text{so}$$

$$y_{11} = \psi_{11} = \tilde{\partial}^2_1 \psi_1$$

From (A)

Now

$$\tilde{\partial}_1^2 = \frac{\partial}{\partial x^1} + z_1 \frac{\partial}{\partial z} + z_{11} \frac{\partial}{\partial z_1} + z_{12} \frac{\partial}{\partial z_2} + \psi_1 \frac{\partial}{\partial y}$$

Therefore

$$y_{11} = \left(\frac{\partial}{\partial x^1} + z_1 \frac{\partial}{\partial z} + z_{11} \frac{\partial}{\partial z_1} + z_{12} \frac{\partial}{\partial z_2} + \psi_1 \frac{\partial}{\partial y}\right)\left(z_1 + 2a \sin \tfrac{1}{2}(y+z)\right)$$

$$y_{11} = \left(\frac{\partial}{\partial x^1} + z_1 \frac{\partial}{\partial z} + z_{11} \frac{\partial}{\partial z_1} + z_{12} \frac{\partial}{\partial z_2} + \left(z_1 + 2a \sin \tfrac{1}{2}(y+z)\right)\frac{\partial}{\partial y}\right)\left(z_1 + 2a \sin \tfrac{1}{2}(y+z)\right)$$

$$= z_{11} + 2a \cos \tfrac{1}{2}(y+z)\cdot \tfrac{1}{2}(y_1 + z_1) + z_1 \cdot 2a \cos \tfrac{1}{2}(y+z)\cdot \tfrac{1}{2} + z_{11} + \left(z_1 + 2a \sin \tfrac{1}{2}(y+z)\right)\left(2a \cos \tfrac{1}{2}(y+z)\cdot \tfrac{1}{2}\right)$$

Putting the value of y_1 we get

$$= z_{11} + a \cos \tfrac{1}{2}(y+z)\left[2z_1 + 2a \sin \tfrac{1}{2}(y+z)\right] + a z_1 \cos \tfrac{1}{2}(y+z) + z_{11}$$
$$+ \left(z_1 + 2a \sin \tfrac{1}{2}(y+z)\right)\left(a \cos \tfrac{1}{2}(y+z)\right)$$
$$= 2z_{11} + z_1 a \cos \tfrac{1}{2}(y+z)\left[2+1+1\right] + 2a^2 \sin (y+z)$$
$$= 2\left(z_{11} + 2a z_1 \cos \tfrac{1}{2}(y+z) + a^2 \sin (y+z)\right)$$

$$y_{22} = \tilde{\partial}_2^2 \psi_2 = \left(\frac{\partial}{\partial x^2} + z_2 \frac{\partial}{\partial z} + z_{21} \frac{\partial}{\partial z_1} + z_{22} \frac{\partial}{\partial z_2} + \psi_2 \frac{\partial}{\partial y}\right)(\psi_2)$$
$$= \left(\frac{\partial}{\partial x^2} + z_2 \frac{\partial}{\partial z} + z_{21} \frac{\partial}{\partial z_1} + z_{22} \frac{\partial}{\partial z_2} + \left(-z_2 + 2a^{-1} \sin \tfrac{1}{2}(y-z)\right)\frac{\partial}{\partial y}\right)$$
$$\left(-z_2 + 2a^{-1} \sin \tfrac{1}{2}(y-z)\right)$$
$$= -z_{22} + 2a^{-1} \cos \tfrac{1}{2}(y-z)\cdot \tfrac{1}{2}(y_2 - z_2) + z_2 2a^{-1} \cos \tfrac{1}{2}(y-z)(-\tfrac{1}{2})$$
$$- z_{22} + \left(-z_2 + 2a^{-1} \sin \tfrac{1}{2}(y-z)\right)\left(2a^{-1} \cos \tfrac{1}{2}(y-z)\cdot \tfrac{1}{2}\right)$$

Putting the value of y_2 we get

$$= -z_{22} + a^{-1} \cos \tfrac{1}{2}(y-z)\left(-2z_2 + 2a^{-1} \sin \tfrac{1}{2}(y-z)\right) - a^{-1} z_2 \cos \tfrac{1}{2}(y-z)$$
$$- z_{22} + \left(-z_2 + 2a^{-1} \sin \tfrac{1}{2}(y-z)\right)\left(a^{-1} \cos \tfrac{1}{2}(y-z)\right)$$
$$= -2z_{22} - a^{-1} z_2 \cos \tfrac{1}{2}(y-z)\left[2+1+1\right] + 2a^{-2} \sin (y-z)$$
$$= 2\left(-z_{22} - 2a^{-1} z_2 \cos \tfrac{1}{2}(y-z) + a^{-2} \sin (y-z)\right)$$

Similarly we can get

$$y_{12} = \psi_{12} = \tfrac{1}{2}\left(\tilde{\partial}_1^2 \psi_2 + \tilde{\partial}_2^2 \psi_1\right) = \sin y$$

Up till now we have got the idea of first prolongation, we can generalize this idea and can find the second, third, -- --

s th prolongation. For example in the above case the second prolongation will consist of the components y_{111}, y_{112}, y_{122}, y_{222} which we can write

$$y_{111} = \psi_{111} = \tilde{\partial}_1^2 \psi_{11}$$
$$y_{112} = \psi_{112} = \tilde{\partial}_1^2 \psi_{12}$$
$$y_{122} = \psi_{122} = \tilde{\partial}_1^2 \psi_{22}$$
$$y_{222} = \psi_{222} = \tilde{\partial}_2^2 \psi_{22}$$

$$y_{111} = \psi_{111} = \tilde{\partial}_1^2 \psi_{11} = \left(\frac{\partial}{\partial x^1} + z_1 \frac{\partial}{\partial z} + z_{11}\frac{\partial}{\partial z_1} + z_{12}\frac{\partial}{\partial z_2} + \psi_1 \frac{\partial}{\partial y}\right)\left(z_{11} + 2a z_1 \cos\tfrac{1}{2}(y+z) + a^2 \sin(y+z)\right)$$

$$\begin{aligned} y_{111} = {} & z_{111} - 2a z_1 \sin\tfrac{1}{2}(y+z)\cdot\tfrac{1}{2}(y_1+z_1) + a^2\cos(y+z)(y_1+z_1) \\ & + z_1\left(-2a z_1 \sin\tfrac{1}{2}(y+z)\cdot\tfrac{1}{2}\right) + z_1 a^2 \cos(y+z) \\ & + z_{11} 2a \cos\tfrac{1}{2}(y+z) + \left(z_1 + 2a\sin\tfrac{1}{2}(y+z)\right)\left(-2a z_1 \sin\tfrac{1}{2}(y+z)\cdot\tfrac{1}{2} + a^2\cos(y+z)\right) \end{aligned}$$

$$\begin{aligned} = {} & z_{111} + (y_1+z_1)\left[-a z_1 \sin\tfrac{1}{2}(y+z) + a^2\cos(y+z)\right] \\ & + z_1\left(a^2\cos(y+z) + a^2\cos(y+z) - 2a^2\sin^2\tfrac{1}{2}(y+z)\right) \\ & + z_1^2\left(-a\sin\tfrac{1}{2}(y+z) - a\sin\tfrac{1}{2}(y+z)\right) \\ & + z_{11}\left(2a\cos\tfrac{1}{2}(y+z)\right) + 2a^3\cos(y+z)\sin\tfrac{1}{2}(y+z) \end{aligned}$$

putting the value of "y_1", we get

$$\begin{aligned} = {} & z_{111} + \left(2z_1 + 2a\sin\tfrac{1}{2}(y+z)\right)\left(-a z_1 \sin\tfrac{1}{2}(y+z) + a^2\cos(y+z)\right) \\ & + z_1\left(a^2\cos(y+z) + a^2\cos(y+z) - 2a^2\sin^2\tfrac{1}{2}(y+z)\right) \\ & + z_1^2\left(-a\sin\tfrac{1}{2}(y+z) - a\sin\tfrac{1}{2}(y+z)\right) \\ & + z_{11}\left(2a\cos\tfrac{1}{2}(y+z)\right) + 2a^3\cos(y+z)\sin\tfrac{1}{2}(y+z) \end{aligned}$$

$$\begin{aligned} y_{111} = \psi_{111} = {} & z_{111} + z_1\left(4a^2\cos(y+z) - 4a^2\sin^2(y+z)\right) \\ & + z_1^2\left(-4a\sin\tfrac{1}{2}(y+z)\right) + z_{11}\left(2a\cos\tfrac{1}{2}(y+z)\right) \\ & + 4a^3\sin\tfrac{1}{2}(y+z)\cos(y+z) \end{aligned}$$

Similarly we can find the other components.

Example 3.2 The differential equation generated by the single function

$$F = Z_{111} + Z_2 - 6Z_1^2 \tag{i}$$

is called the potential equation for the Korteweg-deVries equation because its first prolongation comprises (i) and

$$\left.\begin{aligned} F_1 &= Z_{1111} + Z_{12} - 12 Z_1 Z_{11} \cdots\cdots (a) \\ F_2 &= Z_{112} + Z_{22} - 12 Z_1 Z_{12} \cdots\cdots (b) \end{aligned}\right] \tag{ii}$$

and (ii) (a) is just the Korteweg-deVries equation for $-Z_1$.

A Bäcklund map whose integgrability condition is the Korteweg-deVries potential equation (i) was discovered by Wahlquist and Estabrook (1973).

Let dim M = 2   dim N_1 = dim N_2 = 1

The co-ordinates are x^1 and x^2 on M, Z on N_1 and y = y′ on N_2. The map discovered by Wahlquist and Estabrook is

$$\Psi(t) : J^2(M, N_1) \times N_2 \longrightarrow J^1(M, N_2)$$

by

$$\left.\begin{aligned} y_1 &= \psi_1(t) = -Z_1 - 2t + (y-Z)^2 \\ y_2 &= \psi_2(t) = -Z_2 + 4\left\{4t^2 + 2tZ_1 - 2t(y-Z)^2 + Z_1^2 + Z_1(y-Z)^2 + Z_{11}(y-Z)\right\} \end{aligned}\right] \tag{iii}$$

The first prolongation is

$$y_{11} = \psi_{11} = \tilde{\partial}_1^2 \psi_1 = -Z_{11} + 2(y-Z)\left\{-2Z_1 - 2t + (y-Z)^2\right\}$$

$$y_{12} = \psi_{12} = \tilde{\partial}_1^2 \psi_2 = -Z_{12} + 2(y-Z)(16t^2 + 8t^2 Z_1 - 2Z_1^2 + Z_{11} - Z_2) + 8(y-Z)^2 Z_{11} + 8(y-Z)^3(Z_1 - 2t)$$

$$y_{22} = \psi_{22} = \tilde{\partial}_2^2 \psi_2 = -Z_{22} + 8tZ_{12} + 2Z_1 Z_{12} + Z_{112}(y-Z) + Z_{12}(y-Z)^2 + \left(Z_{11} + (2Z_1 - 4t)(y-Z)\right)$$

$$\left[-2z_1+4\left\{4t^2+2tz_1-2t(y-z)^2+z_1^2+z_1(y-z)^2 + z_{11}(y-z)\right\}\right]$$

Similarly we can find the second prolongation and can find the components $y_{111}, y_{112}, y_{122}, y_{222}$ as follows

$$y_{111} = t_{111} = \tilde{\partial}_1^2 t_{11}$$

$$y_{112} = t_{112} = \tilde{\partial}_1^2 t_{12}$$

$$y_{122} = t_{122} = \tilde{\partial}_1^2 t_{22}$$

$$y_{222} = t_{222} = \tilde{\partial}_2^2 t_{22}$$

CHAPTER IV

Connections

In this chapter there are two related formulations of Backlund maps in terms of connections. We begin by reproducing Ehresmann's definition of a connection, more or less as expounded by R. Hermann (1975 Ch. 4). This version is convenient because in the applications it is natural to introduce the connection first and the structure group afterwards, so that one can not begin, as is usual in most standard treatments by defining the connection on a principal bundle.

Let E and M be (smooth) manifolds and $\pi : E \to M$ a surjective map of maximal rank. We shall give conditions for E = (E, M, π) to be a local product fibre space, and define a connection to be an horizontal distribution on E. M is the space of independent variables (spare-time) and E is either M x N_2 or the product ($J(M; N_1)$) x N_2 where $J(M; N_1)$ is the jet bundle of infinite order defined below. For the present, however, E is assumed to be finite-dimensional.

For every $\eta \in E$, assume $\pi(\eta)$ has neighbourhood $\mathcal{U}$ s.t there exists a diffeomorphism.

$$P_{\mathcal{U}} : \mathcal{U} \times F \longrightarrow \pi^{-1}(\mathcal{U}) \longrightarrow \quad (4.1)$$

where F is a fixed manifold. Then E = (E, M, π) is called a local product fibre space with typical fibre F. For every $X \in M$ the fibre $\pi^{-1}(x)$ is diffeomorphic to F.

A vector X tangent to E at η is called vertical if it is tangent to the fibre through η i.e. if $\pi_* X = 0$. The subspace of vertical vectors at η is denoted by $V_\eta E$. A vector field is called vertical if it is vertical at each point.

If x^a are local co-ordinates on M and y^A are local co-ordinates on F, then $\overset{*}{\pi} x^a$ and $\overset{*}{p_u} y^A$ may be chosen as local co-ordinates on E, which will be abbreviated to x^a and y^A and called adapted co-ordinates. It follows that $\pi_* \left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial x^a}$ and $\pi_* \left(\frac{\partial}{\partial y^A}\right) = 0$, so that every vertical vector field has the co-ordinate presentation $\xi^A \frac{\partial}{\partial y^A}$, where ξ^A are functions on E.

A distribution D on E is an assignment of a subspace $D_\eta E \subset T_\eta E$ at each $\eta \in E$. We shall be concerned only with regular distributions, for which dim $D_\eta E$ has a fixed value, independent of η, denoted dim D. More over we shall suppose that all the distributions which arise are differentiable, which means that each $\eta \in E$ has a neighbourhood W in which there are dim D differentiable vector fields spanning $D_\xi E$ for which $\xi \in W$. If X is a vector field on E and $X(\eta) \in D_\eta E$ for each η then we can say that X belongs to D. A distribution D is called involutive if $[X, Y]$ belongs to D whenever X and Y belong to D. If $i : E' \rightarrow E$ is the natural injection of a submanifold E' of E and if $i_*(T_{\eta'} E') = D^E_{i(\eta')}$ for every $\eta' \in E'$, then E' is called an integral manifold of D. A distribution D on E is called integrable if an integral manifold of D passes through every point of E. Frobenius's theorem asserts that a distribution is integrable if and only if it is involutive (Bricknell and Clark 1970).

An horizontal distribution H on E is a distribution which is complementary to the vertical in the sense that for each $\eta \in E$

$$T_\eta E = V_\eta E \oplus H_\eta E$$

if follows that

$$\dim H_\eta E = \dim M$$

and that $H_\eta E \cap V_\eta E = 0$

If H is a given Horizontal distribution on E than a vector X tangent to E at any point η has a unique decomposition

$$X = hX + vX \longrightarrow (4.2)$$

where $hX \in H_\eta E$ and $vX \in V_\eta E$

A vector X at η is called horizontal if it lies in $H_\eta E$

In adapted local co-ordinates, horizontal vectors have a basis of the form

$$H_a = \frac{\partial}{\partial x^a} + \Gamma_a^A \frac{\partial}{\partial y^A} \longrightarrow (4.3)$$

where Γ_a^A are functions on E which determine and are determined by H. In these co-ordinates the decomposition (4.2) of $X = \xi^a \frac{\partial}{\partial x^a} + \xi^A \frac{\partial}{\partial y^A}$ is given by putting the value of $\frac{\partial}{\partial x^a}$ from (4.3)

$$X = \xi^a \left(H_a - \Gamma_a^A \frac{\partial}{\partial y^A} \right) + \xi^A \frac{\partial}{\partial y^A}$$

$$= \xi^a H_a + \left(\xi^A - \xi^a \Gamma_a^A \right) \frac{\partial}{\partial y^A} \longrightarrow (4.4)$$

A vector field X is called horizontal if it belongs to H. Thus an horizontal vector field is of form

$$X = \xi^a H_a \longrightarrow (4.5)$$

where ξ^a are functions on E. A curve is called horizontal if its tangent vector at each point is horizontal. The 1-forms which annihilate and are annihilated by, horizontal vectors are called vertical forms. The vertical forms comprise a $\overset{\infty}{C}(E)$ module

$$\overset{*}{H} = \left\{ \theta \in \Lambda^1(E) \setminus X \lrcorner \theta = 0 \, , \, \forall X \in H \right\} \rightarrow (4.6)$$

Theorem 4.1 To prove that vertical forms has a local basis in adapted co-ordinates of the form

$$\theta^A = dy^A - \Gamma_a^A \, dx^a \longrightarrow (4.7)$$

Proof Now the 1-form can be written as

$$\theta^A = A_B^A \, dy^B + B_b^A \, dx^b$$

But since A_B^A is non singular matrix therefore we can write

$$\theta^A = dy^A + C_b^A \, dx^b \longrightarrow (A)$$

But since there are vertical forms, and by definition of vertical forms

$$H \lrcorner \theta^A = 0 \longrightarrow i$$

where H is horizontal vectors, and horizontal vectors have the basis of the form

$$H_a = \frac{\partial}{\partial x^a} + \Gamma_a^A \frac{\partial}{\partial y^A}$$

So putting the value of H and θ^A in (i) we get

$$\left(\frac{\partial}{\partial x^a} + \Gamma_a^A \frac{\partial}{\partial y^A}\right) \lrcorner \left(dy^B + C_b^B \, dx^b\right) = 0$$

$$C_a^B + \Gamma_a^B = 0 \qquad \therefore \; C_a^B = -\Gamma_a^B$$

Now putting the value of C_a^B in 'A' we get

$$\theta^A = dy^A - \Gamma_a^A \, dx^a$$

where Γ_a^A are functions introduceced in (4.3).

The $\overset{\infty}{C}(E)$-bilinear map

$$\Omega : T_\eta E \times T_\eta E \longrightarrow V_\eta E$$

$$(X_1, X_2) \longmapsto v[hX_1, hX_2] \longrightarrow (4.8)$$

is called the curvature of H at η. The bilinearity may be verified by the computation

$$v\left[hX_1, h(fX_2)\right] = v\left(f\left[hX_1, hX_2\right] + (hX_1)f(hX_2)\right)$$
$$= f\,v\left[hX_1, hX_2\right]$$

where f is any function on E.

It follows from Frobenius's theorem that an horizontal distribution is integrable if and only if its curvature vanishes.

<u>Theorem 4.2</u> To prove that

$$\Omega(H_a, H_b) = \Omega^B_{ab}\,\frac{\partial}{\partial y^B} \longrightarrow (4.9)$$

where

$$\Omega^B_{ab} = \left(\frac{\partial}{\partial x^a} + \Gamma^A_a \frac{\partial}{\partial y^A}\right)\Gamma^B_b - \left(\frac{\partial}{\partial x^b} + \Gamma^A_b \frac{\partial}{\partial y^A}\right)\Gamma^B_a \rightarrow (4.10)$$

and which are called the components of curvature tensor.

<u>Proof</u>

$$\Omega(H_a, H_b) = v\left[H_a, H_b\right]$$
$$= v\left[\frac{\partial}{\partial x^a} + \Gamma^A_a \frac{\partial}{\partial y^A}, \frac{\partial}{\partial x^b} + \Gamma^B_b \frac{\partial}{\partial y^B}\right]$$

Now

$$\left[\frac{\partial}{\partial x^a} + \Gamma^A_a \frac{\partial}{\partial y^A}, \frac{\partial}{\partial x^b} + \Gamma^B_b \frac{\partial}{\partial y^B}\right]$$
$$= \left(\frac{\partial}{\partial x^a} + \Gamma^A_a \frac{\partial}{\partial y^A}\right)\left(\frac{\partial}{\partial x^b} + \Gamma^B_b \frac{\partial}{\partial y^B}\right) - \left(\frac{\partial}{\partial x^b} + \Gamma^B_b \frac{\partial}{\partial y^B}\right)\left(\frac{\partial}{\partial x^a} + \Gamma^A_a \frac{\partial}{\partial y^A}\right)$$
$$= \frac{\partial^2}{\partial x^a \partial x^b} + \frac{\partial \Gamma^B_b}{\partial x^a}\frac{\partial}{\partial y^B} + \Gamma^B_b \frac{\partial^2}{\partial x^a \partial y^B} + \Gamma^A_a \frac{\partial^2}{\partial y^A \partial x^b} + \Gamma^A_a \Gamma^B_b \frac{\partial^2}{\partial y^A \partial y^B}$$
$$+ \Gamma^A_a \frac{\partial \Gamma^B_b}{\partial y^A}\frac{\partial}{\partial y^B} - \frac{\partial^2}{\partial x^a \partial x^b} - \frac{\partial \Gamma^A_a}{\partial x^b}\frac{\partial}{\partial y^A} - \Gamma^A_a \frac{\partial^2}{\partial y^A \partial x^b}$$
$$- \Gamma^B_b \frac{\partial^2}{\partial y^B \partial x^a} - \Gamma^B_b \Gamma^A_a \frac{\partial^2}{\partial y^A \partial y^B} - \Gamma^B_b \frac{\partial \Gamma^A_a}{\partial y^B}\frac{\partial}{\partial y^A}$$
$$= \left(\frac{\partial \Gamma^B_b}{\partial x^a} + \Gamma^A_a \frac{\partial \Gamma^B_b}{\partial y^A}\right)\frac{\partial}{\partial y^B} - \left(\frac{\partial \Gamma^A_a}{\partial x^b} + \Gamma^B_b \frac{\partial \Gamma^A_a}{\partial y^B}\right)\frac{\partial}{\partial y^A}$$

Now changing (A to B) and (B to A) in 2nd term

$$= \left(\frac{\partial \Gamma^B_b}{\partial x^a} + \Gamma^A_a \frac{\partial \Gamma^B_b}{\partial y^A}\right)\frac{\partial}{\partial y^B} - \left(\frac{\partial \Gamma^B_a}{\partial x^b} + \Gamma^A_b \frac{\partial \Gamma^B_a}{\partial y^A}\right)\frac{\partial}{\partial y^B}$$

$$= \left[\left(\frac{\partial}{\partial x^a} + \Gamma_a^A \frac{\partial}{\partial y^A} \right) \Gamma_b^B - \left(\frac{\partial}{\partial x^b} + \Gamma_b^A \frac{\partial}{\partial y^A} \right) \Gamma_a^B \right] \frac{\partial}{\partial y^B} \longrightarrow (B)$$

Hence

$$\mathcal{N}(H_a, H_b) = \mathcal{N}_{ab}^B \frac{\partial}{\partial y^B}$$

where $\mathcal{N}_{ab}^B$ is given by "B"

If X is tangent to M at x, the horizontal lift of X to any point $\eta \in \overset{-1}{\Pi}(x)$ is the unique horizontal vector $\overline{X} \in H_\eta E$, s.t

$$\Pi_* \overline{X} = X$$

If in local co-ordinates

$$X = \overset{a}{X}(x) \frac{\partial}{\partial x^a}$$

then from (4.3)

$$\overline{X} = \overset{a}{X}(x) \left(\frac{\partial}{\partial x^a} + \Gamma_a^A \frac{\partial}{\partial y^A} \right) \longrightarrow (4.11)$$

If γ is a curve through $x \in M$ the horizontal lift of γ through $\eta \in \overset{-1}{\Pi}(x)$ is the unique horizontal curve $\overline{\gamma}$ through η such that

$$\Pi \circ \overline{\gamma} = \gamma$$

If it exists, η is said to be parallel transported along γ to other points of $\overline{\gamma}$.

If every curve on M has an horizontal lift through each point above it, the horizontal distribution is called a connection on E. From now on we assume that all the horizontal distributions introduced are connections.

The functions Γ_a^A introduced in (4.3) are in this case called the connection co-efficients.

<u>Corollary 4.1</u> If the presentation of γ in local co-ordinates is $t \longmapsto \overset{a}{\gamma}(t)$ and that of $\overline{\gamma}$ is $t \longmapsto (\overset{a}{\overline{\gamma}}(t), \overset{A}{\gamma}(t))$ then

$$\bar{\gamma}^a(t) = \gamma^a(t) \longrightarrow (4.12)$$

$$\frac{d\gamma^A}{dt} = \Gamma_a^A\left(\gamma^b(t), \gamma^B(t)\right) \frac{d\gamma^a}{dt} \longrightarrow (4.13)$$

Proof

$$\bar{\gamma}^a(t) = \gamma^a(t) \quad \text{(By definition)}$$

Now if $\bar{X} \in H_v E$ then the components of

$$\bar{X} = \left(X^a \frac{\partial}{\partial x^a}, X^a \Gamma_a^A \frac{\partial}{\partial y^A}\right)$$

$$= X^a \frac{\partial}{\partial x^a} + \Gamma_a^A X^a \frac{\partial}{\partial y^A}$$

Now in this case

$$\gamma^A = \Gamma_a^A X^a$$

$$\frac{d\gamma^A}{dt} = \Gamma_a^A \frac{dx^a}{dt}$$

$$= \Gamma_a^A\left(\gamma^b(t), \gamma^B(t)\right) \frac{d\gamma^a}{dt}$$

The equations (4.12) and (4.13) are called the equations of parallel transport.

If $\sigma : U \to E$ is a local section of π, U an open set in M, then the co-variant derivative $\nabla_X \sigma$ of σ with respect to a vector X tangent to u at x is defined by

$$\nabla_X \sigma = v(\sigma_* X) \longrightarrow (4.14)$$

<u>Theorem 4.3</u>

If $X = \overset{a}{X} \frac{\partial}{\partial x^a}$ and $\sigma(x) = (\overset{a}{x}, \overset{A}{\sigma}(x)) \longrightarrow (4.15)$

then
$$\nabla_X \sigma = (\nabla_X \sigma)^A \frac{\partial}{\partial y^A}$$
where
$$(\nabla_X \sigma)^A = \overset{a}{X}\left(\frac{\partial \sigma^A}{\partial x^a} - \Gamma^A_a(x, \overset{B}{\sigma}(x))\right) \longrightarrow (4.16)$$

<u>Proof</u>

$$E = M \times F = (\overset{a}{x}, \overset{A}{y})$$
$$a = 1, \ldots\ldots\ldots\ldots, \dim M$$
$$A = 1, \ldots\ldots\ldots\ldots, \dim F$$

$$P_u : u \times F \longrightarrow \pi^{-1}(u)$$
$$H_a = \frac{\partial}{\partial x^a} + \Gamma^A_a \frac{\partial}{\partial y^A}$$

Now
$$X = vX + hX$$
$$vX = X - hX$$
$$= \left(\xi^a \frac{\partial}{\partial x^a} + \overset{A}{\xi} \frac{\partial}{\partial y^A}\right) - \overset{a}{\xi}\left(\frac{\partial}{\partial x^a} + \Gamma^A_a \frac{\partial}{\partial y^A}\right)$$
$$= (\xi^A - \Gamma^A_a \xi^a) \frac{\partial}{\partial y^A} \longrightarrow (A)$$

Now
$$\sigma : u \longrightarrow E$$
$$\overset{a}{x} \overset{\sigma}{\longmapsto} (\overset{a}{x}, \overset{A}{y})$$
$$\sigma_* X = \left(\overset{a}{X}, \frac{\partial \sigma^A}{\partial x^b} \overset{b}{X}\right)$$

Now putting the value of $\overset{a}{\xi}$ and $\overset{A}{\xi}$ in "A"

we get

$$v(\sigma_* X) = \left(\frac{\partial \sigma^A}{\partial x^b} \overset{b}{X} - \Gamma^A_a \overset{a}{X}\right) \frac{\partial}{\partial y^A}$$

The section σ is called integrable if $\nabla_X \sigma = 0$ for all vector fields X tangent to U. It is no more than a restatement of Ferobenius's theorem to remark that there is an integrable (local) section through each point if and only if the curvature vanishes.

In the cases of interest where the connection will be associated with a Bäcklund map, it may be possible to endow the local product fibre space with some additional structure.
(1) E may be a (differentiable) fibre bundle, with (Lie) structure group G acting on the Fibre F; in this case the Lie algebra of G will be denoted $\mathfrak{G}$ and the Lie algebra of vector fields on F generating the action of G will be denoted $\mathfrak{G}_F$.
(2) The fibre bundle E may be a vector bundle, with F a vector space, G a subgroup of the group of linear transformations $GL(F)$ and H a linear connection.

If E is a fibre bundle, the action of G must be compatible with the diffeomorphisms (4.1) which means that if

$$P_u : U \times F \longrightarrow \pi^{-1}(U)$$
$$P_{u'} : U' \times F \longrightarrow \pi^{-1}(U')$$

are two such diffeomorphisms, with $U \cap U'$ not empty, and if for each $x \in U \cap U'$

$$P_x : F \longrightarrow \pi^{-1}(x) \text{ by } \quad y \longmapsto P_u(x, y)$$
$$P_x' : F \longrightarrow \pi^{-1}(x) \text{ by } \quad y \longmapsto P_{u'}(x, y)$$

then $P_x^{-1} \circ P_x' : F \longrightarrow F$
is an action of some $g_x \in G$ on F, depending smoothly on X.

The possibility of endowing E with the structure of a fibre bundle is of interest here only if the action of G is compatible also with parallel transport by the connection

H. This means that parallel transport, suitably composed with the diffeomorphisms (4.1) should yield an action of G. Explicitly let x_1 and x_2 be points of M, γ a curve joining them with $\gamma(t_1)=x_1$ and $\gamma(t_2)=x_2$. For each $\eta \in \pi^{-1}(x_1)$, let $\bar{\gamma}_\eta$ be the horizontal lift of γ through η. Then the parallel transport along γ is given by

$$T_\gamma : \pi^{-1}(x_1) \longrightarrow \pi^{-1}(x_2) \longrightarrow (4.17)$$
$$\text{by } \eta \longmapsto \bar{\gamma}_\eta(t_2)$$

Let U_1 and U_2 be neighbourhoods of x_1 and x_2 respectively for which diffeomorphisms (4.1) are defined and let P_1 and P_2 be the maps $P_i : F \longrightarrow \pi^{-1}(x_i)$ by $y \longmapsto P_{U_i}(x_i, y)$ implies that

$$P_2^{-1} \circ T_\gamma \circ P_1 : F \longrightarrow F \longrightarrow (4.18)$$

is an action of $g \in G$ on F. If this is the case, H is sometimes called a G-connection.

If H is a G-connection then (4.18) imposes conditions on the connection co efficients, for it implies that to each curve γ through $x \in M$ there is a curve g_γ in G s.t

$$\bar{\gamma}(t) = P_U(\gamma(t), g_\gamma(t) y) \longrightarrow (4.19)$$

where $g_\gamma(0) = e$ the identity of G, $\gamma(0) = x$ and $\bar{\gamma}(0) = P_U(x, y)$ is any point in $\pi^{-1}(x)$.

<u>Theorem 4.4</u> To prove that if f is a function on E, then differentiating it along $\bar{\gamma}$ yields

$$\omega^\alpha_\gamma = \dot{\gamma}^a \, \omega^\alpha_a$$

and $\Gamma^A_{a}(x, y) = \omega^\alpha_a(x) \, X^A_\alpha(y)$

where ω^α_γ are functions depending on the choice of γ, and $X^A_\alpha \frac{\partial}{\partial y^A}$ are basis for $\underset{\sim}{G} F$.

<u>Proof</u> Now

$$\bar{\gamma}(t) = P_U(\gamma(t), g_\gamma(t) y)$$

But we know

$$\bar{\gamma}(t) = \left(\overset{a}{\gamma}(t), \overset{a}{\gamma}(t)\Gamma_a^A\right)$$

$$= \overset{a}{\gamma}(t)\frac{\partial f}{\partial x^a} + \overset{a}{\gamma}(t)\Gamma_a^A \frac{\partial f}{\partial y^A}$$

$$\frac{d\bar{\gamma}(t)}{dt} = \dot{\gamma}^a(t)\left(\frac{\partial f}{\partial x^a} + \Gamma_a^A \frac{\partial f}{\partial y^A}\right) \longrightarrow \text{i}$$

Now when

$$\bar{\gamma}(t) = P_u\left(\gamma(t), g_\gamma(t)\, y\right)$$

$$\frac{d\bar{\gamma}(t)}{dt} = \dot{\gamma}^a \frac{\partial f}{\partial x^a} + \omega_\gamma^\alpha \overset{A}{X}_\alpha(y)\frac{\partial f}{\partial y^A} \longrightarrow \text{ii}$$

Equating (1) and (2) we get

$$\dot{\gamma}^a \Gamma_a^A \frac{\partial f}{\partial y^A} = \omega_\gamma^\alpha \overset{A}{X}_\alpha(y) \frac{\partial f}{\partial y^A} \longrightarrow (A)$$

Since $\overset{A}{X}_\alpha \frac{\partial}{\partial y^A}$ are basis of $\underline{G_F}$.

It follows that $\overset{A}{X}_\alpha$ is non-singular materix. Then there exist inverse materix $\overset{\beta}{X}_A$.

Multiply both sides of (A) with $\overset{\beta}{X}_A$ we get

$$\overset{\beta}{X}_A \dot{\gamma}^a \Gamma_a^A = \omega_\gamma^\alpha \overset{A}{X}_\alpha \overset{\beta}{X}_A = \omega_\gamma^\alpha$$

$$\omega_\gamma^\alpha = \dot{\gamma}^a\left(\overset{\beta}{X}_A \Gamma_a^A\right)$$

$$\omega_\gamma^\alpha = \dot{\gamma}^a \overset{\alpha}{\omega}_a \longrightarrow (B)$$

where

$$\overset{\alpha}{\omega}_a = \left(\Gamma_a^A \overset{\beta}{X}_A\right)$$

Now putting the value of (B) in (A) we get

$$\dot{\gamma}^a \Gamma_a^A = \dot{\gamma}^a \overset{\alpha}{\omega}_a \overset{A}{X}_\alpha(y)$$

$$\Gamma_a^A = \overset{\alpha}{\omega}_a \overset{A}{X}_\alpha(y)$$

Now to sum up: if now E = (E, M, π, F, G, H) is a fibre bundle with typical fibre F structural group G, and G-connection H, then in adapted co-ordinates x^a (from the base) and y^A (from the fibre) a basis for horizontal vector fields has the form

$$H_a = \frac{\partial}{\partial x^a} + \omega_a^\alpha(x)\overset{A}{X}_\alpha(y)\frac{\partial}{\partial y^A} \longrightarrow (4\ 21)$$

where

$$X_\alpha = \overset{A}{X}_\alpha \frac{\partial}{\partial y^A} \longrightarrow \qquad (4.22)$$

is a basis for the lie algebra $\underset{\sim}{G}_F$ and the $\overset{\alpha}{\omega}_a$ are (lifted from) functions on the base M. The module $\overset{*}{H}$ of vertical forms is generated by

$$\theta^A = dy^A - \overset{A}{X}_\alpha(y)\,\overset{\alpha}{\omega}(x) \longrightarrow \qquad (4.23)$$

where

$$\overset{\alpha}{\omega} = \overset{\alpha}{\omega}_a(x)\, dx^a \longrightarrow \qquad (4.24)$$

are 1-forms which may also be considered to be lifted from M.

Theorem 4.5 To prove that if

$$H_a = \frac{\partial}{\partial x^a} + \omega_a^\alpha(x)\, X_\alpha^A(y) \frac{\partial}{\partial y^A}$$

where $X_\alpha = \overset{A}{X}_\alpha \frac{\partial}{\partial y^A}$

then the components of the curvature tensor are

$$\Omega_{ab}^{\beta} = \left(\partial\omega_b^\gamma|\partial x^a - \partial\omega_a^\gamma|\partial x^b + \omega_a^\alpha \omega_b^\beta C_{\alpha\beta}^\gamma\right) \overset{\beta}{X}_\gamma \longrightarrow (4.25)$$

where $C_{\alpha\beta}^\gamma$ are the structure constants of $\underline{G}$.

Proof

$$\Omega_{ab}^{B} = v\left[H_a, H_b\right] f$$

$$\Omega_{ab}^{B} = v\left[\frac{\partial}{\partial x^a} + \omega_a^\alpha \overset{A}{X}_\alpha \frac{\partial}{\partial y^A}, \frac{\partial}{\partial x^b} + \omega_b^\beta \overset{B}{X}_\beta \frac{\partial}{\partial y^B}\right] f$$

$$= v\left\{\left(\frac{\partial}{\partial x^a} + \omega_a^\alpha \overset{A}{X}_\alpha \frac{\partial}{\partial y^A}\right)\left(\frac{\partial f}{\partial x^b} + \omega_b^\beta \overset{B}{X}_\beta \frac{\partial f}{\partial y^B}\right)\right\}$$

$$- v\left\{\left(\frac{\partial}{\partial x^b} + \omega_b^\beta \overset{B}{X}_\beta \frac{\partial}{\partial y^B}\right)\left(\frac{\partial f}{\partial x^a} + \omega_a^\alpha \overset{A}{X}_\alpha \frac{\partial f}{\partial y^A}\right)\right\}$$

$$= v\left\{\frac{\partial^2 f}{\partial x^a \partial x^b} + \frac{\partial \omega_b^\beta}{\partial x^a} \overset{B}{X}_\beta \frac{\partial f}{\partial y^B} + \frac{\partial}{\partial x^a}\overset{B}{X}_\beta\, \omega_b^\beta \frac{\partial f}{\partial y^B} + \omega_b^\beta \overset{B}{X}_\beta \frac{\partial^2 f}{\partial x^a \partial y^B} + \omega_a^\alpha \overset{A}{X}_\alpha \frac{\partial^2 f}{\partial y^A \partial x^b} + \omega_a^\alpha \overset{A}{X}_\alpha \left[\left(\frac{\partial \omega_b^\beta}{\partial y^A} \overset{B}{X}_\beta + \omega_b^\beta \frac{\partial \overset{B}{X}_\beta}{\partial y^A}\right)\frac{\partial f}{\partial y^B} + \omega_b^\beta \overset{B}{X}_\beta \frac{\partial^2 f}{\partial y^A \partial y^B}\right]\right\}$$

$$- v\left\{\frac{\partial^2 f}{\partial x^b \partial x^a} + \frac{\partial \omega_a^\alpha}{\partial x^b} \overset{A}{X}_\alpha \frac{\partial f}{\partial y^A} + \omega_a^\alpha \frac{\partial \overset{A}{X}_\alpha}{\partial x^b} \frac{\partial f}{\partial y^A} + \omega_a^\alpha \overset{A}{X}_\alpha \frac{\partial^2 f}{\partial x^b \partial y^A} + \omega_b^\beta \overset{B}{X}_\beta \frac{\partial^2 f}{\partial y^B \partial x^a} + \omega_b^\beta \overset{B}{X}_\beta \left[\left(\frac{\partial \omega_a^\alpha}{\partial y^B} \overset{A}{X}_\alpha + \omega_a^\alpha \frac{\partial \overset{A}{X}_\alpha}{\partial y^B}\right)\frac{\partial f}{\partial y^A} + \omega_a^\alpha \overset{A}{X}_\alpha \frac{\partial^2 f}{\partial y^A \partial y^B}\right]\right\}$$

$$= v\left\{\frac{\partial \omega_b^\gamma}{\partial x^a} \overset{B}{X}_\gamma \frac{\partial f}{\partial y^B} - \frac{\partial \omega_a^\gamma}{\partial x^b} \overset{B}{X}_\gamma \frac{\partial f}{\partial y^B} + \omega_a^\alpha \omega_b^\beta \overset{A}{X}_\alpha \frac{\partial \overset{B}{X}_\beta}{\partial y^A} \frac{\partial f}{\partial y^B} - \omega_a^\alpha \omega_b^\beta \overset{B}{X}_\beta \frac{\partial \overset{A}{X}_\alpha}{\partial y^B} \frac{\partial f}{\partial y^A}\right\}$$

$$= v\left\{\left(\frac{\partial \omega_b^\gamma}{\partial x^a} - \frac{\partial \omega_a^\gamma}{\partial x^b}\right)\overset{B}{X}_\gamma \frac{\partial f}{\partial y^B} + \omega_a^\alpha \omega_b^\beta \overset{B}{X}_\alpha \frac{\partial \overset{A}{X}_\beta}{\partial y^B} \frac{\partial f}{\partial y^A} - \omega_a^\alpha \omega_b^\beta \overset{B}{X}_\beta \frac{\partial \overset{A}{X}_\alpha}{\partial y^B} \frac{\partial f}{\partial y^A}\right\}$$

$$= \vartheta\left[\left(\frac{\partial \omega^\gamma_b}{\partial x^a} - \frac{\partial \omega^\gamma_a}{\partial x^b}\right)\overset{B}{X}_\gamma \frac{\partial f}{\partial y^B} + \overset{\alpha}{\omega}_a \omega^\beta_b \left\{\overset{B}{X}_\alpha \frac{\partial \overset{A}{X}_\beta}{\partial y^B} - \overset{B}{X}_\beta \frac{\partial \overset{A}{X}_\alpha}{\partial y^B}\right\}\frac{\partial f}{\partial y^A}\right] \longrightarrow (A)$$

Now

$$X_\alpha = X^B_\alpha \frac{\partial}{\partial y^B}, \qquad X_\beta = \overset{A}{X}_\beta \frac{\partial}{\partial y^A}$$

$$[X_\alpha, X_\beta] f = \overset{B}{X}_\alpha \frac{\partial}{\partial y^B}\left(\overset{A}{X}_\beta \frac{\partial}{\partial y^A}\right) - \overset{A}{X}_\beta \frac{\partial}{\partial y^A}\left(X^B_\alpha \frac{\partial}{\partial y^B}\right)$$

$$= X^B_\alpha \frac{\partial X^A_\beta}{\partial y^B}\frac{\partial}{\partial y^A} - \overset{A}{X}_\beta \frac{\partial X^B_\alpha}{\partial y^A}\frac{\partial}{\partial y^B}$$

$$= \left(X^B_\alpha \frac{\partial X^A_\beta}{\partial y^B} - X^B_\beta \frac{\partial X^A_\alpha}{\partial y^B}\right)\frac{\partial}{\partial y^A} f$$

Now putting this value in (A) we get

$$\Omega^B_{ab} = \vartheta\left[\left(\frac{\partial \omega^\gamma_b}{\partial x^a} - \frac{\partial \omega^\gamma_a}{\partial x^b}\right) X^B_\gamma \frac{\partial f}{\partial y^B} + \overset{\alpha}{\omega}_a \omega^\beta_b [X_\alpha, X_\beta] f\right]$$

Now

$$[X_\alpha, X_\beta] = C^\gamma_{\alpha\beta} X_\gamma$$

therefore

$$\Omega^B_{ab} = \left(\frac{\partial \omega^\gamma_b}{\partial x^a} - \frac{\partial \omega^\gamma_a}{\partial x^b} + \overset{\alpha}{\omega}_a \omega^\beta_b C^\gamma_{\alpha\beta}\right)\overset{B}{X}_\gamma$$

<u>Theorem 4.6</u> To prove that

$$d\theta^A = -X^A_\gamma \left[d\omega^\gamma + \frac{1}{2} C^\gamma_{\alpha\beta} \overset{\alpha}{\omega} \wedge \omega^\beta\right] \pmod{\theta^B}$$

and also

$$d\theta^A = -\frac{1}{2}\Omega^A_{ab}\, dx^a \wedge dx^b \longrightarrow (4.26)$$

<u>Proof</u>

$$\theta^A = dy^A - \overset{A}{X}_\alpha(y)\overset{\alpha}{\omega}(x)$$

$$d\theta^A = -\left[d\overset{A}{X}_\alpha(y) \wedge \overset{\alpha}{\omega}(x) + \overset{A}{X}_\alpha(y)\, d\overset{\alpha}{\omega}(x)\right]$$

$$= -\left[\overset{A}{X}_\alpha(y)\, d\overset{\alpha}{\omega}(x) - \overset{\alpha}{\omega}(x) \wedge \frac{\partial \overset{A}{X}_\alpha(y)}{\partial y^B} dy^B\right]$$

$$= -\left[\overset{A}{X}_\alpha(y)\, d\overset{\alpha}{\omega}(x) - \overset{\alpha}{\omega}(x) \wedge \frac{\partial \overset{A}{X}_\alpha(y)}{\partial y^B}\left(\theta^B + \overset{B}{X}_\beta(y)\, \omega^\beta(x)\right)\right]$$

$$= - \left[\overset{A}{X}_\alpha(y)\, d\overset{\alpha}{\omega}(x) - \overset{\alpha}{\omega}(x) \wedge \omega^\beta(x) \frac{\partial \overset{A}{X}_\alpha(y)}{\partial y^B} \overset{B}{X}_\beta(y) \right]$$

$$= - \left[\overset{A}{X}_\alpha\, d\overset{\alpha}{\omega}(x) - \frac{1}{2} \overset{\alpha}{\omega}(x) \wedge \omega^\beta(x) \left(X^B_\beta \frac{\partial \overset{A}{X}_\alpha}{\partial y^B} - \overset{B}{X}_\alpha \frac{\partial \overset{A}{X}_\beta}{\partial y^B} \right) \right]$$

$$= - \left[\overset{A}{X}_\alpha\, d\overset{\alpha}{\omega}(x) - \frac{1}{2} \overset{\alpha}{\omega}(x) \wedge \omega^\beta(x)\, C^\gamma_{\beta\alpha} \overset{A}{X}_\gamma \right]$$

$$= - \overset{A}{X}_\gamma \left[d\omega^\gamma + \frac{1}{2} \overset{\alpha}{\omega} \wedge \omega^\beta C^\gamma_{\alpha\beta} \right]$$

$$= - \overset{A}{X}_\gamma \left[d\omega^\gamma + \frac{1}{2} C^\gamma_{\alpha\beta}\, \overset{\alpha}{\omega} \wedge \omega^\beta \right]$$

For the (ii) part

$$d\theta^A = - \overset{A}{X}_\gamma \left[d\omega^\gamma + \frac{1}{2} C^\gamma_{\alpha\beta}\, \overset{\alpha}{\omega} \wedge \omega^\beta \right]$$

$$= - X^A_\gamma \left[\frac{\partial \omega^\gamma_b}{\partial x^c} dx^c \wedge dx^b + \frac{1}{2} C^\gamma_{\alpha\beta}\, \overset{\alpha}{\omega}_c\, dx^c \wedge \omega^\beta_b dx^b \right]$$

$$= - \overset{A}{X}_\gamma \left[\frac{1}{2} \left(\frac{\partial \omega^\gamma_b}{\partial x^c} - \frac{\partial \omega^\gamma_c}{\partial x^b} \right) + \frac{1}{2} C^\gamma_{\alpha\beta} \overset{\alpha}{\omega}_c \omega^\beta_b \right] dx^c \wedge dx^b$$

$$= - \frac{1}{2} \Omega^A_{ab}\, dx^a \wedge dx^b$$

After giving the idea of connection our problem is this, that how a connection may be associated to a Bäcklund map.

$\psi : J^h(M, N_1) \times N_2 \longrightarrow J^1(M, N_2)$

induces on $J^h(M, N_1) \times N_2$ a module of 1-forms $\overset{*}{\psi}\, \Omega^1(M, N_2)$ - The pull back of the contact module on $J^1(M, N_2)$. The forms of this module may be chosen as vertical forms defining a connection on the product space $(J^h(M, N_1) \times N_2,\ J^h(M, N_1)\ Pr_1)$ i. e. in this case

$E = (J^h(M, N_1) \times N_2,\ J^h(M, N_1),\ Pr_1)$

where $E = J^h(M, N_1) \times N_2$

$M = J^h(M, N_1)$ and $\pi = Pr_1$

$F = N_2$

we call this connection a Bäcklund connection. The integrability conditions for the Bäcklund map were the vanishing of the curvature of the Bäcklund connection. The vanishing of this curvature imposes additional conditions which one could not expect to satisfy. Alternatively one might attempt to construct a connection with module of vertical forms chosen to be $\tilde{\Omega}^{h,\Psi}$ on $(J^h(M; N_1) \times N_2, M, \alpha)$ or something of that kind, but $\tilde{\Omega}^{h,\Psi}$ is not quite large enough for this purpose, because of the intervention of the highest derivatives of the independent variable $\tilde{z}^u$.

To overcome the first difficulty one may eliminate the additional conditions by constructing sections of $(J^h(M; N_1) \times N_2, M \times N_2, dxid)$ namely jets of maps from M to N_1; thus connecting the $\tilde{z}^u$ and their derivatives into functions on the base M. To overcome the second difficulty, one may work with the projective limit $J(M; N_1) \times N_2$, where $J(M; N_1)$ is the jet bundle of infinite order; so that there is no longer any highest derivative. We will discuss these two procedures one by one.

First of all suppose that $\Psi: J^h(M; N_1) \times N_2 \to (J^1(M; N_2) \to$ (4.27) is an ordinary Bäcklund map with integrability condition

$$\tilde{\partial}^{h+1}_c \Psi^A_b - \tilde{\partial}^{h+1}_b \Psi^A_c = 0$$

If $f: M \to N$, then one can associate a map

$$\tilde{j}^h f = j^h f \times id_{N_2} : \overset{0}{J}(M \times N_2) \to \overset{h}{J}(M, N_1) \times N_2 \longrightarrow \quad (4.29)$$

Now

$$J^h(M, N_1) \times N_2 \xrightarrow{pr_1} J^h(M, N_1)$$

Any contact form in $pr_1^* \overset{h}{\Omega}(M, N_1)$ is annihilated by the map $\tilde{j}^h f$. Therefore the module $H_{f,\Psi} = \tilde{j}^h f^*(\tilde{\Omega}^{\Psi})$ on

$J^{0}(M, N_2)$ and hence $\tilde{j}^{h}f^{*}\psi^{*}\Omega^{1}(M,N_2)$ is also on $J^{0}(M, N_2)$

It has a basis

$$\theta^{A} = dy^{A} - \Gamma^{A}_{a}\, dx^{a}$$
$$\Gamma^{A}_{a} = \psi^{A}_{a} \circ \tilde{j}^{h}f \longrightarrow \qquad (4.30)$$

Because here the position is this

$$M \times N_2 = J^{0}(M, N_2) \longrightarrow J^{h}(M, N_1) \times N_2 \longrightarrow J^{1}(M, N_2)$$

This module may be chosen as a module of vertical forms defining an horizontal distribution $H_{f,\psi}$ on the trivial fibre bundle $E = (J^{0}(M, N_2), M, \alpha)$. This horizontal distribution depends on the choice of both the Backlund map ψ and the map f. If the Γ^{A}_{a} factorize as in (4.20) then the X^{A}_{α} may be identified with members of a subset of a basis for a lie algebra $\underset{\sim}{G}_{N_2}$ of vector fields acting on N_2, which may be supposed to be the lie algebra of a lie group G which is the structure group of the fibre bundle.

$$E = (J^{0}(M, N), M, \alpha, N_2, G, H_{f,\psi})$$

It is not justified to suppose that the X^{A}_{α} which actually occur in Γ^{A}_{a} are themselves a basis for G_{N_2}. Since some of the ω_{α} might vanish.

Since the Γ^{A}_{a} are constructed from the ψ^{A}_{a} of the Bäcklund map, according to (4.30) by substituting for the jet bundle variables $\bar{z}^{u}, \bar{z}^{u}_{a}$ -- - without altering the y^{A}, it follows that if the Γ^{A}_{a} factorize for a general f, then the ψ^{A}_{a} factorize, in the form

$$\psi^{A}_{a} = \tilde{\omega}^{\alpha}_{a}(x^{b}, \bar{z}^{u}, \bar{z}^{u}_{a}, \ldots\ldots)\, X^{A}_{\alpha}(y) \longrightarrow (4.31)$$

where $\tilde{\omega}^{\alpha}_{a}$ are some functions pulled back by P_{ℓ_1}, from $J^{h}(M, N_1)$. Conversely, if the ψ^{A}_{a} factorize as in (4.31) so trivially do the Γ^{A}_{a}.

Theorem 4.7 To prove that if

$$\overset{A}{\psi}_a = \tilde{\omega}_a^{\alpha}(x^b, \overset{u}{z}, \overset{u}{z}_a, \ldots\ldots) \overset{A}{X}_{\alpha}(y)$$

then the integrability conditions

$$\tilde{\partial}_c^{h+1} \overset{A}{\psi}_b - \tilde{\partial}_b^{h+1} \overset{A}{\psi}_c = 0$$

of the Bäcklund map, take the form

$$\tilde{\Omega}_{cb}^{\alpha} X_{\alpha}^{A} = 0 \longrightarrow \qquad (4.32)$$

where

$$\tilde{\Omega}_{cb}^{\alpha} = \overset{h+1}{\partial}{}^{\#}_c \tilde{\omega}_b^{\alpha} - \overset{h+1}{\partial}{}^{\#}_b \tilde{\omega}_c^{\alpha} + C_{\gamma\beta}^{\alpha} \tilde{\omega}_c^{\gamma} \tilde{\omega}_b^{\beta} \longrightarrow (4.33)$$

where $\overset{A}{X}_{\alpha}$ are among the basis vectors of $\underset{\sim}{G}_{N_2}$ while the $C_{\gamma\beta}^{\alpha}$ are structure constants of $\underset{\sim}{G}_{N_2}$.

Proof

$$\overset{A}{\psi}_a = \tilde{\omega}_a^{\alpha}(x^b, \overset{u}{z}, \overset{u}{z}_a, \ldots\ldots) \overset{A}{X}_{\alpha}(y)$$

Similarly

$$\psi_b^B = \tilde{\omega}_b^{\beta} X_{\beta}^{B}(y)$$

$$\psi_c^C = \tilde{\omega}_c^{\gamma} X_{\gamma}^{C}(y)$$

we have already proved that

$$\tilde{\partial}_c^{h+1} = \overset{\#}{\partial}{}_c^{h+1} + \overset{C}{\psi}_c \frac{\partial}{\partial y^C}$$

$$\tilde{\partial}_b^{h+1} = \overset{\#}{\partial}{}_b^{h+1} + \psi_b^B \frac{\partial}{\partial y^B}$$

Now the integrability condition is

$$\tilde{\partial}_c^{h+1} \overset{A}{\psi}_b - \tilde{\partial}_b^{h+1} \overset{A}{\psi}_c = 0$$

putting the values

$$= \left(\overset{\#}{\partial}{}_c^{h+1} + \overset{C}{\psi}_c \frac{\partial}{\partial y^C}\right)\overset{A}{\psi}_b - \left(\overset{\#}{\partial}{}_b^{h+1} + \overset{B}{\psi}_b \frac{\partial}{\partial y^B}\right)\overset{A}{\psi}_c$$

$$= \overset{\#}{\partial}{}_c^{h+1} \overset{A}{\psi}_b + \overset{C}{\psi}_c \frac{\partial \overset{A}{\psi}_b}{\partial y^C} - \overset{\#}{\partial}{}_b^{h+1} \overset{A}{\psi}_c - \overset{B}{\psi}_b \frac{\partial \overset{A}{\psi}_c}{\partial y^B}$$

putting the values of ψ's

$$= \overset{\#}{\partial}{}_c^{h+1}(\tilde{\omega}_b^{\beta} \overset{A}{X}_{\beta}) + \tilde{\omega}_c^{\gamma} X_{\gamma}^{C} \frac{\partial}{\partial y^C}(\tilde{\omega}_b^{\beta} \overset{A}{X}_{\beta})$$

$$= \overset{\#}{\partial}{}_b^{h+1}(\tilde{\omega}_c^{\gamma} X_\gamma^A) - \tilde{\omega}_b^{\beta} X_\beta^B \frac{\partial}{\partial y^B}(\tilde{\omega}_c^{\gamma} X_\gamma^A)$$

$$= (\overset{\#}{\partial}{}_c^{h+1}\tilde{\omega}_b^{\beta} - \partial_b^{h+1}\tilde{\omega}_c^{\beta}) X_\beta^A + \tilde{\omega}_c^{\gamma}\tilde{\omega}_b^{\beta}\left[X_\gamma^C \frac{\partial}{\partial y^C}(X_\beta^A) - X_\beta^B \frac{\partial}{\partial y^B}(X_\gamma^A)\right]$$

$$= (\overset{\#}{\partial}{}_c^{h+1}\tilde{\omega}_b^{\beta} - \overset{\#}{\partial}{}_b^{h+1}\tilde{\omega}_c^{\beta}) X_\beta^A + \tilde{\omega}_c^{\gamma}\tilde{\omega}_b^{\beta}\left[X_\gamma^C \frac{\partial}{\partial y^C}(X_\beta^A) - X_\beta^C \frac{\partial}{\partial y^C} X_\gamma^A\right]$$

$$= (\overset{\#}{\partial}{}_c^{h+1}\tilde{\omega}_b^{\alpha} - \overset{\#}{\partial}{}_b^{h+1}\omega_c^{\alpha}) X_\alpha^A + \tilde{\omega}_c^{\gamma}\tilde{\omega}_b^{\beta} C_{\gamma\beta}^{\alpha} X_\alpha^A$$

$$= \overset{\#}{\partial}{}_c^{h+1}\tilde{\omega}_b^{\alpha} - \overset{\#}{\partial}{}_b^{h+1}\omega_c^{\alpha} + C_{\gamma\beta}^{\alpha}\tilde{\omega}_c^{\gamma}\tilde{\omega}_b^{\beta}$$

The integrability condition for the system of differential equation Z is the vanishing of the $\tilde{\Omega}_{cb}^{\alpha}$ which are functions on $J^h(M, N_1)$. Moreover the curvature of the connection $H_{f,\psi}$ is $\tilde{j}^h f^* \Omega_{cb}^{\alpha}$ and so the curvature vanishes if and only if f is a solution of the integrability condition for ψ. The vanishing of the curvature is necessary and sufficient for the integrability of the connection, in which case there exist maps $g: M \to N_2$ whose graphs are local sections $\sigma_g : M \longrightarrow \overset{\circ}{J}(M,N) \to$ (4.34) satisfying $\sigma_g^* \theta^A = 0$

If in local co-ordinates, g is given by

$$y^A = g^A(x^b) \longrightarrow \qquad (4.35)$$

then $\dfrac{\partial g^A}{\partial x^a} = \Gamma_a^A(x)$ where Γ_a^A is given by (4.30).

As an application of the ideas developed in this section we describe the construction of linear scattering equations with the help of a Bäcklund map.

Suppose that a Bäcklund map $\psi : J^h(M, N_1) \times N_2 \longrightarrow J^1(M, N_2)$

is a solution of the Bäcklund problem for a system Z of differential equations and that the functions ψ^A_a defining ψ factorize as in (4.31) so that ψ defines a G-connection for some lie group G, not necessarily unique. Then by Ado's theorem (Jacobsen 1962 p. 202), the Lie algebra $\mathcal{G}$ of G admits a faithful linear representation, so that if the representation defined by (4.31) is not already linear, it may be explained by a linear one. Thus there always exists a manifold N_2 and another Bäcklund map

$$\psi': J^h(M;N_1) \times N_2' \longrightarrow J^1(M;N_2') \longrightarrow \quad (4.36)$$

defined by functions

$$\psi'^A_a = \tilde{\omega}^\alpha_a X'^A_\alpha$$

Where X'^A_α satisfying the same commutation relation as the X^A_α in (4.31) but being of the form

$$X'^A_\alpha = \Gamma^A_{B\alpha}\, y'^B \longrightarrow \quad (4.37)$$

where y'^B are local co-ordinates on N_2' and $\Gamma^A_{B\alpha}$ are constants. To circumvent the second difficulty we work with the projective limit $J(M;N_1) \times N_2$ where $J(M;N_1)$ is the jet bundle of infinite order. The procedure is as follows.

The jet bundle of infinite order $J(M;N)$ is the projective limit (also called the inverse limit) (Lange 1965 p.55) of the bundles $J^k(M,N)$, defined as follows:
Consider the infinite product $\prod_{K=0}^{\infty} J^k(M;N)$, whose elements are sequences $\xi = (\xi_0, \xi_1, \ldots \xi_K, \ldots)$ with $\xi^K \in J^k(M;N)$. Then $J(M;N)$ is the subset consisting of sequences related by the natural projection:

$$J(M,N) = \left\{ (\xi_0, \xi_1, \ldots, \xi_K, \ldots) \in \prod_{K=0}^{\infty} J^K(M,N) \;\middle|\; \pi^K_\ell \xi_K = \xi_\ell \text{ for all } k \text{ and all } \ell \leq K \right\}$$

Two functions define the same point of $J(M;N)$ if they have the same derivative about a point of M, then there is a natural projection

for every K, by

$$(\xi_0, \xi_1, \ldots\ldots, \xi_K, \ldots) \longmapsto \xi_K$$

with the property that

$$K \geq l, \quad \pi_l^K \circ \pi_K = \pi_l$$

i.e. $J(M,N) \longrightarrow J^K(M,N) \longrightarrow J^l(M,N)$

The image of $j_x^0 f \times j_x^1 f \times \ldots \times j_x^K f \times \ldots$ in $\prod_{K=0}^{\infty} J^k(M,N)$ lies in $J(M,N)$ and defines the infinite jet $j f$ of any $f \in \overset{\infty}{C}(M,N)$. The ideas of differentiable function, vector field, form and so on generalize straight forwardly.

A function f on $J(M,N)$ is differentiable if for some K there is a differentiable function f_k on $J^k(M,N)$ such that

$$f = f_K \circ \pi_K$$

More generally a map $\phi: J(M,N) \longrightarrow P$ into a manifold P is differentiable if for some K there is a differentiable map $\phi_k = J^k(M,N) \longrightarrow P$ s.t. $\phi = \phi_K \circ \pi_K$

A curve γ at $\xi \in J(M,N)$ is a map

$$\gamma : I \longrightarrow J(M,N)$$

Where I is an open interval of R, $\gamma(0) = \xi$, and $f \circ \gamma$ is a differentiable (real) function for every differentiable function f on $J(M,N)$. Two curves γ_1 and γ_2 at ξ are equivalent if $\frac{d}{dt}(f \circ \gamma_1)_{t=0} = \frac{d}{dt}(f \circ \gamma_2)_{t=0}$ for every differentiable f. A tangent vector X at ξ is an equivalence class of curves at ξ.

If f is any (differentiable) function on $J(M,N)$ then Xf means $\frac{d}{dt}(f \circ \gamma)_{t=0}$ where γ is any curve in the equivalence class defining X. But there is a function f_k

on $J^k(M,N)$ for some k such that $f = f_k \circ \pi_k$ so that $Xf = \frac{d}{dt}(f_k \circ \pi_k \circ \gamma)\big|_{t=0}$, which is the derivative of f_k along the curve $\pi_k \circ \gamma$ on $J^k(M,N)$.

More over, as is easily seen, this derivative is independent of the choice of γ in the equivalence class, and so by the usual arguments may be written

$$Xf = \xi^a \frac{\partial f_k}{\partial x^a} + \overset{u}{\xi} \frac{\partial f_k}{\partial z^u} + \cdots\cdots + \overset{u}{\xi}_{a_1 \ldots a_k} \frac{\partial f^k}{\partial z^u_{a_1 \ldots a_k}} \qquad (4.38)$$

where all the partial derivatives of f_k are evaluated at $\pi_k(\xi)$.

If X is a vector field on $J(M,N)$ and f_k is any function on $J^k(M,N)$ then for some $\ell(k) \geq k$ there is a function g_ℓ on $J^\ell(M,N)$ such that $X(f_k \circ \pi_k) = g_\ell \circ \pi_\ell$, so that the action of x on the f_k is given by

$$X(f_k \circ \pi_k) = \left[\left(\xi^a \frac{\partial}{\partial x^a} + \overset{u}{\xi} \frac{\partial}{\partial z^u} + \overset{u}{\xi}_a \frac{\partial}{\partial z^u_a} + \cdots + \overset{u}{\xi}_{a_1 \ldots a_k} \frac{\partial}{\partial z^u_{a_1 \ldots a_k}}\right) f_k \circ \overset{\ell}{\pi}_k\right] \circ \pi_\ell$$

where $\xi^a, \overset{u}{\xi}, \overset{u}{\xi}_a, \ldots\ldots, \overset{u}{\xi}_{a_1 \ldots a_k}$ all functions on $J^\ell(M,N)$ independent of the choice of f_k. Thus the action of X may be specified by writing

$$X = \xi^a \frac{\partial}{\partial x^a} + \overset{u}{\xi} \frac{\partial}{\partial z^u} + \overset{u}{\xi}_a \frac{\partial}{\partial z^u_a} + \cdots\cdots + \overset{u}{\xi}_{a_1 \ldots a_k} \frac{\partial}{\partial z^u_{a_1 \ldots a_k}} + \cdots$$

where all the co-efficients $\xi^a, \overset{u}{\xi}, \ldots$ are differentiable functions on $J(M,N)$.

Let X denote the collection of vector fields on $J(M,N)$. Then a p-form on $J(M,N)$ is a skew-symmetric $C^\infty(J(M,N))$ - multilinear map

$$X \times X \times \cdots\cdots \times X \longrightarrow C^\infty(J(M,N))$$

Thus $X \rfloor \omega$ may be defined in the usual way.

If X is a vector at $\xi \in J(M,N)$ then the vector $\pi_{k*} X(\xi)$ at $\pi_k(\xi) \in J^k(M,N)$ is defined by $(\pi_{k*} X) f_k = X(f_k \circ \pi_k)$ for all $f_k \in C^\infty(J^k(M,N))$

if w_k is a p-form on $J^k(M,N)$ then $\overset{*}{\pi}_K\, w_k$ is a p-form on $J(M,N)$ defined pointwise by

$$(\overset{*}{\pi}_K \omega_K)(X_1,\ldots\ldots,X_p) = \omega_K(\pi_{K*}X_1,\ldots,\pi_{K*}X_p)$$

Thus the contact module $\Omega(M,N)$ is the module of 1-forms on $J(M,N)$ generated by $\displaystyle\sum_{K=0}^{\infty} \pi_{K^*}\, \Omega^K(M,N)$.

All these definitions generalize in a rather obvious way to $J(M,N_1) \times N_2$. For example if $\tilde{\pi}_K = \pi_K \times id$ where id is the identity map of N_2, then a differentiable function for $J(M,N_1)\times N_2$ is one of the form $f_K \circ \tilde{\pi}_K$ where f_K is a differentiable function on $J^k(M,N_1) \times N_2$.

A vector X at $\xi \in J(M,N_1) \times N_2$ is a derivation of differentiable functions, the components of a vector may be specified as in (4.38) above, with the addition of a term of the form $\eta^A \frac{\partial}{\partial y^A}$ (where y^A are local co-ordinates on N_2) and projections of vectors and pull backs of forms are defined by trifling modifications of the above definitions. Then if $\psi: J^h(M,N_1) \times N_2 \longrightarrow J^1(M,N_2)$ is a Bäcklund map, a module of 1-forms Ω^{ψ} on $J(M,N)$ is generated by $\displaystyle\sum_{K=h}^{\infty} \overset{*}{\tilde{\pi}}_K\, \tilde{\Omega}^{K,\psi}$. A straight forward computation shows that if $X \lrcorner \Omega^{\psi} = 0$ then X is a vector field of the form $X = \xi^a \tilde{\partial}_a$ where ξ^a are functions on $J(M,N)$ and

$$\tilde{\partial}_a = \overset{*}{\tilde{\pi}}_h\, \psi^A_a \frac{\partial}{\partial y^A} + \frac{\partial}{\partial x^a} + \bar{z}^u_a \frac{\partial}{\partial z^u} + \bar{z}^u_{ab} \frac{\partial}{\partial z^u_b} + \cdots\cdots$$

Then by extending the definition of a connection to infinite jet bundles, we can define a connection $H\psi$ on $(J(M,N_1) \times N_2,\ P r_1 \circ \pi_0, M)$ with $\tilde{\partial}_a$ as basis for horizontal vector fields and the forms in Ω^{ψ} as vertical forms. Since for any $f \in \overset{\infty}{C}(J(M,N))$

$$= [\tilde{\partial}_a, \tilde{\partial}_b] f = (\tilde{\partial}_a \tilde{\partial}_b - \tilde{\partial}_b \tilde{\partial}_a) f$$
$$= (\tilde{\partial}_a \tilde{\pi}_h^* f_b^A - \tilde{\partial}_b \tilde{\pi}_h^* f_a^A) \frac{\partial}{\partial y^A}$$

It is natural to identify (the vertical part of) $[\tilde{\partial}_a, \tilde{\partial}_b]$ as the curvature of this connection. The vanishing of the curvature is then exactly the system of integrability conditions for the Bäcklund map.

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