Stress systems in perforated aeolotropic and isotropic plates

Holgate, S.

STRESS SYSTEMS IN PERFORATED AEOLOTROPIC AND ISOTROPIC PLATES.

by S. HOLGATE.

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In all the problems a number of numerical results are given for specimens of spruce and oak and for an isotropic material, and certai of these results are compared with those obtained experimentally by different writers.

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CONTENTS

CONTENTS (cont.)

Stress Systems in Perforated Aeolotropic and Isotropic Plates.

A. Introduction.

1. The equilibrium of an elastic solid which is acted on by given forces is essentially a three-dimensional problem but, owing to analytical difficulties, very few complete solutions have so far been obtained. Considerable progress has, however, been made in solving two-dimensional problems and, although the conditions of these problems do not always accurately represent the physical facts, the solutions do throw some light on the corresponding three-dimensional problems.

It was shewn by Airy (1862, 1863) that in a two-dimensional problem the stress-components can be expressed as partial derivatives of the second order of a single function, and later Ibbetson (1886) shewed that this stress-function must satisfy the partial differential equation of the fourth order $\nabla_1^4 \chi = 0$. Commenting on these results, Jeffery (1920) says that, although it may have been expected that a fairly comprehensive general theory would be developed from them, no such development has in fact proved possible. He observes that elasticity appears to be a branch of mathematical physics in which knowledge comes by the patient accumulation of special solutions rather than by the establishment of great general propositions. Continuing, he suggests that the greater analytical difficulties involved in the solution of $\nabla_1^4 \chi = 0$ as compared with $\nabla_1^2 \chi = 0$ have probably been the greatest obstacles to progress but draws attention at the same time to a physical difficulty which has also restricted development to some extent. This difficulty is the one associated with the realization of a truly two-dimensional stress system. If the stresses are parallel to the xy plane and independent of z, there will in general be a varying displacement parallel to z, whereas a system in which the displacements are parallel to the xy plane and independent of z can only be maintained by a variable stress $\widehat{zz}$ perpendicular to the xy plane. Filon (1903) countered this difficulty to a great extent when he established the theorem of generalized plane stress which states that if, in a fairly thin plate parallel to the xy plane, the average value of $\widehat{zz}$ through the thickness of the plate be taken as zero, the two-dimensional theory will give accurately the average stresses through the thickness of the plate provided that the elastic constants are modified. Michell (1900a) had already shewn that, if the applied forces over any boundary in an isotropic plate are in equilibrium, the stresses are everywhere independent of the elastic constants, and, having due regard to these two results, the importance of a thorough investigation of the various two-dimensional plane problems becomes apparent.

When a plate is bent by transverse couples a two-dimensional theory for thin plates can also be developed by considering stress resultants and couples instead of actual stresses and by making the usual assumptions for thin plates. Although the conditions are not accurately realized in practice this two-dimensional theory gives some idea of the stress-distributions that arise in a number of problems.

2. The elementary theory of a thin isotropic plate is developed in most standard textbooks on elasticity in terms of real variables and until quite recently refinements and extensions of the theory have all been worked out in similar terms. These refinements have included a number of investigations, both theoretical and practical, into the stress-distributions that arise in thin isotropic plates containing holes. Kirsch (1898) solved the problem of a circular hole theoretically when the stress-distribution is produced by a uniform tension at infinity in a given direction. Inglis (1922) presented a solution of a similar type of problem for an elliptical hole, and also found the stress-distribution when the plate is subject to the action of uniform shears at infinity. Some attention to these problems for the elliptical hole had already been paid by Kolosoff (1910), Inglis (1913) and Pöschl (1921).

Tuzi (1928, 1930) examined optically the influence of a circular hole on the stress in a beam subject to pure bending in the plane of the hole and compared his results with the theoretical values for an infinite plate, while Jeffery (1920) used bipolar coordinates to discuss the effect of two circular boundaries and included the case of a semi-infinite plate containing a circular hole and subject to the action of uniform tensions at infinity parallel to the straight edge. In a subsequent investigation Bickley (1928) presented a more general solution of the problem of the circular hole when the hole itself is acted upon by any known tractions, and he used his method to discuss a number of special cases, including the problem of an isolated unbalanced force at the extremity of a diameter of the hole. Extensions of the theory for a circular hole have been published by Howland (1930) and Howland and Stevenson (1933), and problems involving groups of circular holes have been discussed by Howland (1935), Howland and Knight (1939) and Green (1940). In addition to these theoretical investigations, a number of experimental results for similar problems have been obtained by photoelastic methods by various writers. These results are quoted by Coker and Filon (1931).

The plane group of problems has, more recently, received attention from a number of writers employing complex variable methods and so avoiding some of the heavy analysis involved in solving the problem by real variable methods. The difficulties involved in the real variable method are particularly apparent when the perforation is not circular. Muschelišvili (1933) gave a complex variable method of solution for the circular hole and a number of other writers also used the complex variable. (See Sokolnikoff 1942) More recently Stevenson and Green, independently, have applied different complex variable methods to plane problems and obtained neater solutions for problems that have been discussed previously as well as a number of new solutions. Stevenson (1942, 1943) did not use Airy's stress function at all, but employed complex potentials to satisfy the boundary conditions, and also included some discussion of body forces and boundaries of a fairly general character, although some of his solutions are not finite. Green's method for the circular hole, which is outlined at the beginning of section B of this account, depends on the choice of two appropriate functions of the complex variable in a manner similar to that adopted by Bickley (1928), and has the advantage that it can readily be extended to aeolotropic materials. This method, published by Green in 1942, is employed in section B to investigate the stress-distributions that arise in an isotropic plate containing a circular hole when the plate is acted upon by isolated forces at points on, or near to, the boundary. A number of Bickley's (1928) results emerge as special cases in the course of this investigation and all the results are obtained in finite form. Green subsequently (1943a) extended his method to deal with the problem of a hole of fairly general shape in an isotropic plate under the action of tensions or shearing forces at infinity in its own plane and obtained, as a special case, the result for an elliptical hole much more neatly than the real variable method previously used had made possible.

In 1924, Bickley conducted a theoretical investigation into the stress-distribution in a moderately thick plate containing a circular hole and bent anticlastically or cylindrically by the action of transverse couples at infinity. His work in this direction was supplemented by Goodier (1936) who discussed the problem of a bent plate containing a circular or elliptical hole in five fundamental cases. Goodier concentrated his attention on the distribution of stress-couple and shearing force, whereas Bickley had dealt mainly with the stress-resultants. There has been little further discussion of this bending problem theoretically, although some photoelastic results have recently been published by Drucker (1942). The problem of a plate containing a hole of fairly general shape and bent transversely by couples at infinity is analytically similar to the plane problem, though the analysis is a little heavier; and a method of dealing with this type of problem is developed in section C along lines similar to those adopted by Green for the plane problem. The complex variable method is employed throughout and Goodier's results for circular and elliptical boundaries are obtained as special cases with less labour than is required if the real variable is used. Although some of the results are a little complicated they are all obtained in finite form, and, at the same time, the way is prepared in a number of cases for an extension to aeolotropic plates which are discussed in the final section.

3. The discussion of a thin aeolotropic plate in equilibrium under the action of various forces presents a problem of greater analytical difficulty than is encountered in the isotropic case, and the number of solutions that have so far been published is considerably less. Michell (1900b) gave the solution for an isolated force at the vertex of a wedge cut from a thin plate and in a subsequent paper (1900c) derived partial differential equations for a three-dimensional body possessing symmetry equivalent to that of a hexagonal crystal. A fundamental difference between the two types of material is that, in the plane problem, Airy's stress function for aeolotropic materials must satisfy the equation $\left(\frac{\partial^2}{\partial x^2} + \alpha_1 \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \alpha_2 \frac{\partial^2}{\partial y^2}\right)\chi = 0$ instead of $\nabla_1^4 \chi = 0$, where α_1, α_2 are constants depending on the nature of the material. Wolf (1935), Okubo (1937, 1939) and

Sen (1939) have all used an equation of this type but limited their analysis by assuming that either α_1 or α_2 was equal to unity, an assumption which is very inaccurate for highly aeolotropic materials such as spruce. Westergaard (1938) made an even more drastic assumption when he discussed a three-dimensional solid in which horizontal planes were inextensible.

It is only very recently that any progress has been made with the problem of an aeolotropic plate containing a hole when the plate is subject to various distributions of force in its own plane, and there has been no discussion of such a perforated plate bent by transverse couples. Huber (1938) obtained two fourth order differential equations which may be used for a system of plane stress in a plate which has any kind of aeolotropy, and Green has used one of these equations for the plane problem. Green and Taylor (1941) solved the problem of an aeolotropic plate containing a circular hole when the material of the plate has two perpendicular directions of symmetry and the stress-distribution is produced by uniform tensions at infinity parallel to these directions. They used real variables to obtain their results and encountered certain difficulties of convergence when the material was highly aeolotropic. It was possible, however, to modify the solution and express the result in a finite form which was completely satisfactory. Green subsequently (1942) developed a more satisfactory direct method of solving the tension problem by means of the complex variable and generalized it in order to deal with a much wider range of problems including those in which the force-resultant over the edge of the hole was not necessarily zero. This method, which gives a large number of results in finite form, is outlined at the beginning of section B of this account and used there to solve a group of problems involving isolated forces at points on or near to the circular boundary. Green later (1943a) extended his work to include certain problems of stress-distribution in a plate containing an elliptical hole, and has also indicated (1943b, 1943c) how his solution for the circular hole group of problems may be extended to plates having any kind of aeolotropy.

Huber (1938) also derived the equation which must be satisfied by the transverse displacement of a bent plate, but no subsequent writer appears to have used this equation to deal with a perforated aeolotropic plate which is bent transversely in any manner. A method of solving this equation when the material of the plate has any kind of aeolotropy is explained in the final section of this account, and the method is used to solve certain fundamental problems connected with a bent plate having two perpendicular directions of symmetry and

and containing a circular or elliptical hole. The results obtained are fairly complicated but they are all obtained in finite form.

4. The next section of the account has been published in the Proceedings of the Cambridge Philosophical Society (Vol. 40, p 172, 1944) and the final two sections, as separate papers, have been accepted for publication by the Royal Society of London.

B. The effect of a circular hole on certain plane stress-distributions in aeolotropic and isotropic plates.

INTRODUCTION.

1. This section contains a discussion of various problems associated with isolated forces in the neighbourhood of a circular hole. A solution is first obtained for two symmetrical problems consisting of equal and opposite forces at the extremities of any diameter of the hole, or at the extremities of any chord parallel to an axis of symmetry of the material. A special case of these two, when the forces act at the ends of a diameter perpendicular to the grain, is a first approximation to an experimental result obtained by Coker and Coleman (1930).

The group of problems arising from unbalanced isolated forces at points on the boundary are discussed next and, finally, the problem of an isolated force acting at a point near to but not on the edge of the circular hole, is solved for both isotropic and aeolotropic materials when the force acts along or perpendicular to the diameter through the point, this diameter being parallel to an axis of symmetry of the material. This solution may be regarded as a first approximation to the problem that arises when a force acts in the plane of the plate on a rivet near to a circular hole. As far as the writer is aware, this has not previously been discussed for any type of material. The solution may also be used, in the isotropic case, to determine the stresses round a circular hole, vertically below the axle, in a railway wheel, and, although the effect of the outer boundary is neglected, the general character of the stresses agrees with that determined experimentally by Coker and Filon (1931).

The hoop stress round the edge of the circular hole is evaluated numerically in various special cases of all the problems for specimens of spruce and oak, using for the elastic constants values obtained experimentally by various writers and quoted by Green and Taylor (1939).

GENERAL SOLUTION FOR ISOTROPIC MATERIALS.

2. It is known that the mean stresses can be expressed in terms of a stress-function χ by the equations

$$\widehat{xx} = \frac{\partial^2 \chi}{\partial y^2}, \quad \widehat{yy} = \frac{\partial^2 \chi}{\partial x^2}, \quad \widehat{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}, \qquad (2 \cdot 1)$$

or

$$\widehat{rr} = \frac{1}{r^2} \cdot \frac{\partial^2 \chi}{\partial \theta^2} + \frac{1}{r} \cdot \frac{\partial \chi}{\partial r}, \quad \widehat{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r} \cdot \frac{\partial \chi}{\partial \theta}\right),$$

$$\widehat{\theta\theta} = \frac{\partial^2 \chi}{\partial r^2}, \qquad (2 \cdot 2)$$

provided that $\nabla_1^4 \chi = 0.$ (2·3)

If E is Young's modulus and σ is Poisson's ratio, the formulae for the mean displacements are

$$u = \frac{1}{E} \cdot \frac{\partial \psi}{\partial y} - \frac{1+\sigma}{E} \cdot \frac{\partial \chi}{\partial x}, \quad v = \frac{1}{E} \cdot \frac{\partial \psi}{\partial x} - \frac{1+\sigma}{E} \cdot \frac{\partial \chi}{\partial y}, \qquad (2 \cdot 4)$$

where ψ is to be found from the equations

$$\nabla_1^2 \psi = 0, \quad \frac{\partial^2 \psi}{\partial x \partial y} = \nabla_1^2 \chi. \qquad (2 \cdot 5)$$

It has been shewn (Green 1942) that the stresses and displacements are singlevalued and the conditions at infinity are satisfied by taking the stress function to be the <u>real</u> part of

$$\chi = f(z) + \bar{z} g(z) - \frac{(3-\sigma)(A - iB) z \log z}{1+\sigma} + (A + iB) \bar{z} \log z, \qquad (2 \cdot 6)$$

where $z = x + iy, \quad \bar{z} = x - iy,$ (2·7) and $f'(z)$ and $g(z)$ are regular functions of z outside the circle $|z| = a$ which tend to zero at infinity. A and B are real constants such that the force resultant at the origin is given by

$$X = -8\pi A/(1+\sigma), \quad Y = -8\pi B/(1+\sigma), \tag{2.8}$$

and the mean stresses are now given by the formulae

$$\widehat{xx} = -f''(z) + 2g'(z) - \bar{z}g''(z) + \frac{(5+\sigma)A}{(1+\sigma)z} - \frac{(1-3\sigma)iB}{(1+\sigma)z} + \frac{(A+iB)\bar{z}}{z^2}, \tag{2.9}$$

$$\widehat{yy} = f''(z) + 2g'(z) + \bar{z}g''(z) - \frac{(1-3\sigma)A}{(1+\sigma)z} + \frac{(5+\sigma)iB}{(1+\sigma)z} - \frac{(A+iB)\bar{z}}{z^2}, \tag{2.10}$$

$$\widehat{xy} = -if''(z) - i\bar{z}g''(z) + \frac{i(3-\sigma)(A-iB)}{(1+\sigma)z} + \frac{i(A+iB)\bar{z}}{z^2}, \tag{2.11}$$

$$\widehat{rr} = -\frac{zf''(z)}{\bar{z}} + 2g'(z) - zg''(z) + \frac{3(A+iB)}{z} + \frac{(3-\sigma)(A-iB)}{(1+\sigma)\bar{z}}, \tag{2.12}$$

$$\widehat{\theta\theta} = \frac{z f''(z)}{\bar{z}} + 2g'(z) + z g''(z) + \frac{A+iB}{z} - \frac{(3-\sigma)(A-iB)}{(1+\sigma)\bar{z}}, \qquad (2\cdot13)$$

$$\widehat{r\theta} = -\frac{iz f''(z)}{\bar{z}} - iz g''(z) + \frac{i(A+iB)}{z} + \frac{i(3-\sigma)(A-iB)}{(1+\sigma)\bar{z}}, \qquad (2\cdot14)$$

where dashes denote differentiations with respect to z.

In any particular problem $\widehat{rr}$ and $\widehat{r\theta}$ must take prescribed values on the edge of the circular hole $|z| = a$, and, in order to satisfy these conditions, two functions of z, $V(z)$ and $W(z)$, are introduced. These functions are finite at infinity and are such that the real part of $V(z) = -\widehat{rr}_e$ and the imaginary part of $W(z) = -\widehat{r\theta}_e$ on the edge of the hole. Using equations (2·12) and (2·14) it is seen that $V(z)$ and $W(z)$ may also be written

$$V(z) = \frac{z^2 f''(z)}{a^2} + z g''(z) - 2g'(z) - \frac{3(A+iB)}{z} - \frac{(3-\sigma)(A+iB)}{(1+\sigma)z}, \qquad (2\cdot15)$$

$$W(z) = -\frac{z^2 f''(z)}{a^2} - z g''(z) + \frac{A+iB}{z} - \frac{(3-\sigma)(A+iB)}{(1+\sigma)z}, \qquad (2\cdot16)$$

and that, on the edge of the circular hole

$$\widehat{\theta\theta}_e = -V(z) - 2W(z) - 4(3-\sigma)(A+iB)/(1+\sigma)z. \qquad (2\cdot17)$$

In any particular problem, V and W are determined by considering the prescribed stresses on the circular boundary. The functions $f''(z)$, $g''(z)$ and $g'(z)$ may then be obtained from equations (2·15) and (2·16) and the complete stress system is then given by equations (2·9) to (2·14).

GENERAL SOLUTION FOR AEOLOTROPIC MATERIALS.

3. Rectangular coordinates x, y are taken in the plane of the plate parallel to the directions of symmetry of the material and with the origin at the centre of the circular hole. The mean stresses are still given by equations (2·1) and (2·2) but χ must now satisfy the equation

$$\left(\frac{\partial^2}{\partial x^2} + \alpha_1 \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \alpha_2 \frac{\partial^2}{\partial y^2}\right)\chi = 0, \tag{3·1}$$

where

$$\alpha_1 \alpha_2 = s_{11}/s_{22}, \quad \alpha_1 + \alpha_2 = (s_{66} + 2s_{12})/s_{22}, \tag{3·2}$$

s_{11}, s_{12}, s_{22} and s_{66} being the elastic constants.

The mean displacements are given by

$$u = (s_{12} - s_{11})\frac{\partial \chi}{\partial x} + s_{11}\frac{\partial \psi}{\partial y}, \quad v = (s_{12} - s_{22})\frac{\partial \chi}{\partial y} + s_{22}\frac{\partial \psi}{\partial x}, \tag{3·3}$$

where ψ is to be found from the equations

$$\frac{\partial^2\psi}{\partial x\,\partial y} = \nabla_1^2\chi\,, \quad \frac{\partial^2\psi}{\partial x^2} + \alpha_1\alpha_2\frac{\partial^2\psi}{\partial y^2} = (1-\alpha_1)(1-\alpha_2)\frac{\partial^2\chi}{\partial x\,\partial y}\,. \qquad (3\cdot4)$$

It is known (Green 1943b) that α_1, α_2, if real, are essentially positive, and this restriction is imposed on the subsequent analysis. (α_1, α_2 may be complex conjugates). Adopting, for convenience, the notation

$$\gamma_1 = (\alpha_1^{1/2}-1)/(\alpha_1^{1/2}+1)\,, \quad \gamma_2 = (\alpha_2^{1/2}-1)/(\alpha_2^{1/2}+1)\,, \qquad (3\cdot5)$$

it is possible to obtain a solution to equation (3·1) by taking the stress function to be the <u>real</u> part of

$$\chi = f(z+\gamma_1\bar{z}) + [A(1-\gamma_2)(s_{12}-\alpha_2 s_{22}) + iB(1+\gamma_2)(s_{12}-\alpha_1 s_{22})]$$
$$\times (z+\gamma_1\bar{z})\log(z+\gamma_1\bar{z})$$
$$+ g(z+\gamma_2\bar{z}) + [-A(1-\gamma_1)(s_{12}-\alpha_1 s_{22}) - iB(1+\gamma_1)(s_{12}-\alpha_2 s_{22})]$$
$$\times (z+\gamma_2\bar{z})\log(z+\gamma_2\bar{z})\,, \qquad (3\cdot6)$$

and Green (1942) has shewn that this stress function leads to singlevalued stresses and displacements if $f'(z+\gamma_1\bar{z})$ and $g'(z+\gamma_2\bar{z})$ are regular functions outside the circle which tend to zero at infinity. The quantities A and B are real constants such that the force resultant over the edge of the hole is

$$X = 2\pi A(1-\gamma_1)(1-\gamma_2)(\alpha_1-\alpha_2)s_{22},$$

$$Y = 2\pi B(1+\gamma_1)(1+\gamma_2)(\alpha_1-\alpha_2)s_{22}. \qquad (3.7)$$

The mean stresses are now given by the <u>real</u> parts of the expressions

$$\widehat{xx} = -(1-\gamma_1)^2 F(z+\gamma_1\bar{z}) - (1-\gamma_2)^2 G(z+\gamma_2\bar{z}), \qquad (3.8)$$

$$\widehat{yy} = (1+\gamma_1)^2 F(z+\gamma_1\bar{z}) + (1+\gamma_2)^2 G(z+\gamma_2\bar{z}), \qquad (3.9)$$

$$\widehat{xy} = -i(1-\gamma_1^2) F(z+\gamma_1\bar{z}) - i(1-\gamma_2^2) G(z+\gamma_2\bar{z}), \qquad (3.10)$$

$$\widehat{rr} = -\left[(z-\gamma_1\bar{z})^2 F(z+\gamma_1\bar{z}) + (z-\gamma_2\bar{z})^2 G(z+\gamma_2\bar{z})\right]/z\bar{z}, \qquad (3.11)$$

$$\widehat{\theta\theta} = \left[(z+\gamma_1\bar{z})^2 F(z+\gamma_1\bar{z}) + (z+\gamma_2\bar{z})^2 G(z+\gamma_2\bar{z})\right]/z\bar{z}, \qquad (3.12)$$

$$\widehat{r\theta} = -i\left[(z^2-\gamma_1^2\bar{z}^2) F(z+\gamma_1\bar{z}) + (z^2-\gamma_2^2\bar{z}^2) G(z+\gamma_2\bar{z})\right]/z\bar{z}, \qquad (3.13)$$

where

$$F(z+\gamma_1\bar{z}) = f''(z+\gamma_1\bar{z}) + \frac{A(1-\gamma_2)(s_{12}-\alpha_2 s_{22}) + iB(1+\gamma_2)(s_{12}-\alpha_1 s_{22})}{z+\gamma_1\bar{z}}, \qquad (3.14)$$

$$G(z+\gamma_2\bar{z}) = g''(z+\gamma_2\bar{z}) - \frac{A(1-\gamma_1)(s_{12}-\alpha_1 s_{22}) + iB(1+\gamma_1)(s_{12}-\alpha_2 s_{22})}{z+\gamma_2\bar{z}}. \qquad (3.15)$$

It is known (Green 1942) that $F(u)$ and $G(v)$ are given by

$$(\gamma_1-\gamma_2)(1-4\gamma_1 a^2/u^2)^{1/2}\, u F(u) =$$

$$= A(1-\gamma_2)\left[(1-\gamma_2)(s_{12}-\alpha_2 s_{22}) - (1-\gamma_1)(s_{12}-\alpha_1 s_{22})\right]$$

$$- iB(1+\gamma_2)\left[(1+\gamma_2)(s_{12}-\alpha_1 s_{22}) - (1+\gamma_1)(s_{12}-\alpha_2 s_{22})\right]$$

$$- \tfrac{1}{4}u\left[V(u)+W(u)\right]\left[1+(1-4\gamma_1 a^2/u^2)^{1/2}\right]$$

$$- \tfrac{1}{4}u\gamma_1^{-1}\gamma_2\left[V(u)-W(u)\right]\left[1-(1-4\gamma_1 a^2/u^2)^{1/2}\right], \qquad (3.16)$$

$$
\begin{aligned}
(\gamma_1-\gamma_2)(1-4\gamma_2 a^2/v^2)^{1/2}\, v\, G(v) \\
&= A(1-\gamma_1)\left[(1-\gamma_1)(s_{12}-\alpha_1 s_{22})-(1-\gamma_2)(s_{12}-\alpha_2 s_{22})\right] \\
&\quad - iB(1+\gamma_1)\left[(1+\gamma_1)(s_{12}-\alpha_2 s_{22})-(1+\gamma_2)(s_{12}-\alpha_1 s_{22})\right] \\
&\quad + \tfrac{1}{4} v\left[V(v)+W(v)\right]\left[1+(1-4\gamma_2 a^2/v^2)^{1/2}\right] \\
&\quad + \tfrac{1}{4} v\gamma_1\gamma_2^{-1}\left[V(v)-W(v)\right]\left[1-(1-4\gamma_2 a^2/v^2)^{1/2}\right],
\end{aligned}
\qquad (3\cdot17)
$$

where $V(u)$, etc., are obtained from $V(z)$, etc., by the substitutions

$$z=\tfrac{1}{2}u\left[1+(1-4\gamma_1 a^2/u^2)^{1/2}\right], \quad z=\tfrac{1}{2}v\left[1+(1-4\gamma_2 a^2/v^2)^{1/2}\right]. \qquad (3\cdot18)$$

The functions V and W may be determined in any particular problem, and the functions $F(u)$ and $G(v)$ then obtained from equations (3·7), (3·16) and (3·17). The required stresses are then given by equations (3·8) – (3·13) when u and v take the values

$$u=z+\gamma_1\bar{z}, \quad v=z+\gamma_2\bar{z}. \qquad (3\cdot19)$$

NUMERICAL WORK.

4. Calculations have been carried out for specimens of spruce and oak, with the elastic constants shewn in Table 1. The constants are chosen so that the grain is parallel to the y-axis and are measured in sq. mm./10^3 kg. The wood is cut from the tree so that the annular layers are parallel to the plane of the plate.

The notation

$$X_1 = \{1-4\gamma_1 a^2/(z+\gamma_1\bar{z})^2\}^{1/2}, \quad X_2 = \{1-4\gamma_2 a^2/(z+\gamma_2\bar{z})^2\}^{1/2}, \qquad (4.1)$$

will also be adopted, with the restriction that

$$-\tfrac{1}{2}\pi < \arg X_1 \leqslant \tfrac{1}{2}\pi, \quad -\tfrac{1}{2}\pi < \arg X_2 \leqslant \tfrac{1}{2}\pi. \qquad (4.2)$$

Table 1

	s_{11}	s_{22}	s_{12}	s_{66}	α_1	α_2	γ_1	γ_2
Oak	10·15	1·72	− 0·87	12·8	5·321	1·109	0·395	0·026
Spruce	15·5	0·587	− 0·33	11·5	16·91	1·56	0·608	0·111

ISOLATED RADIAL FORCES AT THE EXTREMITIES OF A DIAMETER

5. Consider two equal forces P acting radially at the points $(a, \alpha)(a, \alpha+\pi)$. These conditions require that V and W should be defined by

$$V(z) = \frac{P}{\pi a}\left\{1 + \frac{2a^2}{z^2 e^{-2i\alpha} - a^2}\right\}, \quad W(z) = -\frac{P}{\pi a}, \qquad (5.1)$$

$$V(u)+W(u) = \frac{2P}{\pi a}\left\{\frac{1-(1-4\gamma_1 a^2/u^2)^{1/2}}{\gamma_1 e^{-2i\alpha}[1+(1-4\gamma_1 a^2/u^2)^{1/2}] - [1-(1-4\gamma_1 a^2/u^2)^{1/2}]}\right\}, \qquad (5.2)$$

$$V(u)-W(u) = \frac{2P}{\pi a}\left\{\frac{\gamma_1 e^{-2i\alpha}[1+(1-4\gamma_1 a^2/u^2)^{1/2}]}{\gamma_1 e^{-2i\alpha}[1+(1-4\gamma_1 a^2/u^2)^{1/2}] - [1-(1-4\gamma_1 a^2/u^2)^{1/2}]}\right\}, \qquad (5.3)$$

with corresponding expressions for $V(v)+W(v)$, $V(v)-W(v)$. Equations (3.16)–(3.19), (5.2) and (5.3) give

$$F(z+\gamma_1\bar{z}) = \frac{2Pa}{\pi(\gamma_1-\gamma_2)r^2}\left\{\frac{\gamma_1(1+\gamma_2 e^{-2i\alpha})(e^{i\theta}+\gamma_1 e^{-i\theta})^{-2}}{X_1[1-\gamma_1 e^{-2i\alpha}-(1+\gamma_1 e^{-2i\alpha})X_1]}\right\}, \qquad (5.4)$$

$$G(z+\gamma_2\bar{z}) = \frac{-2Pa}{\pi(\gamma_1-\gamma_2)r^2}\left\{\frac{\gamma_2(1+\gamma_1 e^{-2i\alpha})(e^{i\theta}+\gamma_2 e^{-i\theta})^{-2}}{X_2[1-\gamma_2 e^{-2i\alpha}-(1+\gamma_2 e^{-2i\alpha})X_2]}\right\}, \qquad (5.5)$$

and the stresses are then obtained from equations (3.8)-(3.13).

The stresses on the edge of the hole take the forms

$$\widehat{\theta\theta}_e = \frac{2P(1-\gamma_1\gamma_2)}{\pi a}\left\{\frac{(\gamma_1+\gamma_2)\sin(\theta+\alpha)+(1+\gamma_1\gamma_2)\sin(\theta-\alpha)}{(1+\gamma_1^2-2\gamma_1\cos 2\theta)(1+\gamma_2^2-2\gamma_2\cos 2\theta)\sin(\theta-\alpha)}\right\}, \qquad (5.6)$$

$$\widehat{xx} = \widehat{\theta\theta}\sin^2\theta, \quad \widehat{yy} = \widehat{\theta\theta}\cos^2\theta, \quad \widehat{xy} = -\tfrac{1}{2}\widehat{\theta\theta}\sin 2\theta. \qquad (5.7)$$

Since equation (5.7) is true for all subsequent solutions, it will not be repeated. Formulae for $\widehat{\theta\theta}_e$ only are reproduced.

Calculations have been carried out in the three special cases when the angle between the direction of the forces and the grain is $0^\circ, 45^\circ, 90^\circ$. The results are shown in Table 2, which is compiled in such a way that the grain is parallel to the y-axis and perpendicular to $\theta=0$. The values of $\widehat{\theta\theta}$ are plotted in Fig. 1 for spruce, where, in the figure, the angles are measured in an anticlockwise direction from the direction of the forces.

Coker and Coleman (1930) carried out an experiment, by photoelastic methods, on a tension member containing an incomplete circular hole acted upon by two forces at the extremities of a diameter perpendicular to the grain. They found that the stress normal to the grain reached its maximum value at points not on the

Table 2. *Values of $\pi a\widehat{\theta\theta}/P$ on the edge of the hole. Grain parallel to $\theta = \frac{1}{2}\pi$*

Direction of forces ...	Parallel to $\theta = 0$		Parallel to $\theta = \frac{1}{4}\pi$		Parallel to $\theta = \frac{1}{2}\pi$	
$\theta°$	Spruce	Oak	Spruce	Oak	Spruce	Oak
0	—	—	+ 5·37	+ 3·36	5·37	3·36
10	18·3	7·19	+ 0·413	+ 2·05	3·58	2·96
20	9·04	5·35	− 2·40	+ 0·402	1·76	2·20
30	4·85	3·82	− 4·39	− 1·50	0·946	1·57
40	2·95	2·80	− 11·82	− 7·44	0·576	1·15
50	2·01	2·17	+ 10·43	+ 8·825	0·391	0·893
60	1·50	1·78	3·15	3·21	0·2925	0·732
70	1·225	1·55	1·79	2·07	0·239	0·636
80	1·09	1·42	1·27	1·60	0·212	0·586
90	1·045	1·38	1·045	1·38	—	—
100	1·09	1·42	0·956	1·30	0·212	0·586
110	1·225	1·55	0·962	1·30	0·239	0·636
120	1·50	1·78	1·06	1·40	0·2925	0·732
130	2·01	2·17	1·27	1·59	0·391	0·893
140	2·95	2·80	1·66	1·91	0·576	1·15
150	4·85	3·82	2·38	2·39	0·946	1·57
160	9·04	5·35	3·70	3·04	1·76	2·20
170	18·3	7·19	5·76	3·59	3·58	2·96
180	—	—	5·37	3·36	5·37	3·36

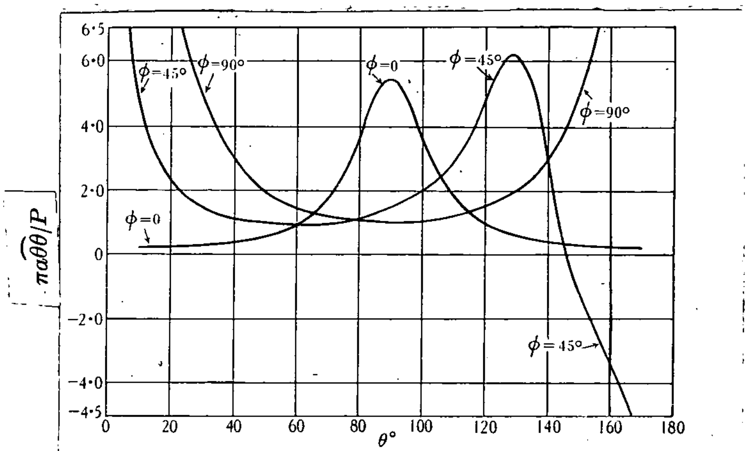

Fig. 1. Values of $\pi a\widehat{\theta\theta}/P$ for spruce. Angle between forces and grain $= \phi$.

central axis. The normal stress in the problem discussed above attains its maximum value, in the case of spruce, at points whose angular distance from the grain is about 48° when the forces are perpendicular to the grain. The physical conditions in the two problems are not identical, but both results indicate that the probable point of cleavage is not on the axis of symmetry, and Coker and Coleman state that this type of fracture is not uncommon in practice.

PARALLEL FORCES AT OPPOSITE ENDS OF A CHORD

6. If the forces are taken to be equal and opposite and to act at the points (a, β), $(a, \pi-\beta)$ parallel to the x-axis, the boundary conditions are satisfied by writing

$$V(z) = \frac{P\cos\beta}{\pi a}\left\{\frac{z^2+a^2}{z^2-a^2-2iaz\sin\beta}\right\}, \quad W(z) = -\frac{P\cos\beta}{\pi a}\left\{\frac{z^2-a^2}{z^2-a^2-2iaz\sin\beta}\right\}, \tag{6.1}$$

$$V(u)+W(u) = \frac{2P\cos\beta}{\pi a}\left\{\frac{1-(1-4\gamma_1 a^2/u^2)^{1/2}}{\gamma_1-1+(1+\gamma_1)(1-4\gamma_1 a^2/u^2)^{1/2}-4\gamma_1 a u^{-1}\sin\beta}\right\}, \tag{6.2}$$

$$V(u)-W(u) = \frac{2P\cos\beta}{\pi a}\left\{\frac{\gamma_1[1+(1-4\gamma_1 a^2/u^2)^{1/2}]}{\gamma_1-1+(1+\gamma_1)(1-4\gamma_1 a^2/u^2)^{1/2}-4\gamma_1 a u^{-1}\sin\beta}\right\}, \tag{6.3}$$

with equivalent expressions for $V(v)+W(v)$, $V(v)-W(v)$. Equations (3.16) – (3.19) give

$F(z+\gamma_1\bar{z})$

$$= \frac{2Pa\gamma_1(1+\gamma_2)\cos\beta}{\pi r(\gamma_1-\gamma_2)(e^{i\theta}+\gamma_1 e^{-i\theta})X_1\{r(e^{i\theta}+\gamma_1 e^{-i\theta})[1-\gamma_1-(1+\gamma_1)X_1]+4ai\gamma_1\sin\beta\}}, \tag{6.4}$$

$$G(z+\gamma_2\bar{z})$$

$$= \frac{-2Pa\gamma_2(1+\gamma_1)\cos\beta}{\pi r(\gamma_1-\gamma_2)(e^{i\theta}+\gamma_2 e^{-i\theta})X_2\left\{r(e^{i\theta}+\gamma_2 e^{-i\theta})[1-\gamma_2-(1+\gamma_2)X_2]+4ai\gamma_2\sin\beta\right\}}, \qquad (6.5)$$

and the stresses are obtained from equations (3.8)–(3.13).

Round the edge of the circular hole the stresses take the form

$$\widehat{\theta\theta}_e = \frac{2P\cos\beta}{\pi a}\left\{\frac{(1-\gamma_1\gamma_2)(1+\gamma_1)(1+\gamma_2)\sin\theta}{(1+\gamma_1^2-2\gamma_1\cos 2\theta)(1+\gamma_2^2-2\gamma_2\cos 2\theta)(\sin\theta-\sin\beta)}\right\}. \qquad (6.6)$$

For an isotropic material this reduces to

$$\widehat{\theta\theta}_e = \frac{2P\cos\beta\sin\theta}{\pi a(\sin\theta-\sin\beta)}, \qquad (6.7)$$

which has the constant value $2P/\pi a$ when the forces are at the ends of a diameter.

ISOLATED FORCE AT ANY POINT ON THE BOUNDARY.

7. Suppose the force is P and acts at the point (a,β). If P acts in a direction parallel to the x-axis, it is found that

$$V(z) = \frac{P(z+ae^{-i\beta})}{2\pi a(ze^{-i\beta}-a)}, \quad W(z) = \frac{-P(z-ae^{-i\beta})}{2\pi a(ze^{-i\beta}-a)}, \qquad (7.1)$$

$$V(u)+W(u) = \frac{2Pe^{-i\beta}}{\pi\{ue^{-i\beta}[1+(1-4\gamma_1a^2/u^2)^{1/2}]-2a\}}, \qquad (7.2)$$

$$V(u)-W(u) = \frac{Pu[1+(1-4\gamma_1a^2/u^2)^{1/2}]}{\pi a\{ue^{-i\beta}[1+(1-4\gamma_1a^2/u^2)^{1/2}]-2a\}}, \qquad (7.3)$$

with similar expressions for $V(r)+W(r)$, $V(r)-W(r)$.

The force resultant over the edge of the hole requires that A and B should be defined by

$$P = 2\pi A(1-\gamma_1)(1-\gamma_2)(\alpha_1-\alpha_2)s_{22}\,, \quad B = 0. \tag{7.4}$$

Using equations (3.16) – (3.19) and (7.2) and (7.3) it is found that

$$F(z+\gamma_1\bar{z}) = \frac{P}{2\pi(\alpha_1-\alpha_2)r s_{22}}\left\{\frac{s_{12}-\alpha_2 s_{22}}{(1-\gamma_1)X_1(e^{i\theta}+\gamma_1 e^{-i\theta})}\right\}$$
$$- \frac{Pa(1+\gamma_2)}{\pi r(\gamma_1-\gamma_2)X_1}\left\{\frac{(e^{i\theta}+\gamma_1 e^{-i\theta})^{-1}}{re^{-i\beta}(e^{i\theta}+\gamma_1 e^{-i\theta})(1+X_1)-2a}\right\}, \tag{7.5}$$

$$G(z+\gamma_2\bar{z}) = \frac{-P}{2\pi(\alpha_1-\alpha_2)r s_{22}}\left\{\frac{s_{12}-\alpha_1 s_{22}}{(1-\gamma_2)X_2(e^{i\theta}+\gamma_1 e^{-i\theta})}\right\}$$
$$+ \frac{Pa(1+\gamma_1)}{\pi r(\gamma_1-\gamma_2)X_2}\left\{\frac{(e^{i\theta}+\gamma_2 e^{-i\theta})^{-1}}{re^{-i\beta}(e^{i\theta}+\gamma_2 e^{-i\theta})(1+X_2)-2a}\right\}, \tag{7.6}$$

and the complete stress system is given by equations (3.8) – (3.13).

On the edge of the circular hole

$$\widehat{\theta\theta}_e = \frac{2P(1-\gamma_1\gamma_2)\{(1-\gamma_2)(s_{12}-\alpha_2 s_{22})-(1-\gamma_1)(s_{12}-\alpha_1 s_{22})\}\cos\theta}{\pi a(\alpha_1-\alpha_2)s_{22}(1+\gamma_1^2-2\gamma_1\cos 2\theta)(1+\gamma_2^2-2\gamma_2\cos 2\theta)}$$
$$+ \frac{P(1-\gamma_1\gamma_2)\{(1+\gamma_1\gamma_2)\sin\frac{1}{2}(\theta+\beta)+(\gamma_1+\gamma_2)\sin\frac{1}{2}(3\theta-\beta)\}}{\pi a(1+\gamma_1^2-2\gamma_1\cos 2\theta)(1+\gamma_2^2-2\gamma_2\cos 2\theta)\sin\frac{1}{2}(\theta-\beta)}. \tag{7.7}$$

If the force acts at the same point in a direction parallel to the y-axis, equations (7.5) and (7.6) must be replaced by

$$F(z+\gamma_1\bar{z}) = \frac{iP}{2\pi(\alpha_1-\alpha_2)r s_{22}}\left\{\frac{s_{12}-\alpha_1 s_{22}}{(1+\gamma_1)X_1(e^{i\theta}+\gamma_1 e^{-i\theta})}\right\}$$

$$-\frac{iPa(1-\gamma_2)}{\pi r(\gamma_1-\gamma_2)X_1}\left\{\frac{(e^{i\theta}+\gamma_1 e^{-i\theta})^{-1}}{re^{-i\beta}(e^{i\theta}+\gamma_1 e^{-i\theta})(1+X_1)-2a}\right\}, \quad (7\cdot8)$$

$$G(z+\gamma_2\bar{z}) = \frac{-iP}{2\pi(\alpha_1-\alpha_2)r s_{22}}\left\{\frac{s_{12}-\alpha_2 s_{22}}{(1+\gamma_2)X_2(e^{i\theta}+\gamma_2 e^{-i\theta})}\right\}$$

$$+\frac{iPa(1-\gamma_1)}{\pi r(\gamma_1-\gamma_2)X_2}\left\{\frac{(e^{i\theta}+\gamma_2 e^{-i\theta})^{-1}}{re^{-i\beta}(e^{i\theta}+\gamma_2 e^{-i\theta})(1+X_2)-2a}\right\}, \quad (7\cdot9)$$

and the stress round the edge of the circular hole is given by

$$\widehat{\theta\theta}_e = \frac{-2P(1-\gamma_1\gamma_2)\{(1+\gamma_2)(s_{12}-\alpha_1 s_{22})-(1+\gamma_1)(s_{12}-\alpha_2 s_{22})\}\sin\theta}{\pi a(\alpha_1-\alpha_2)s_{22}(1+\gamma_1^2-2\gamma_1\cos 2\theta)(1+\gamma_2^2-2\gamma_2\cos 2\theta)}$$

$$+\frac{P(1-\gamma_1\gamma_2)\{(\gamma_1+\gamma_2)\cos\frac{1}{2}(3\theta-\beta)-(1+\gamma_1\gamma_2)\cos\frac{1}{2}(\theta+\beta)\}}{\pi a(1+\gamma_1^2-2\gamma_1\cos 2\theta)(1+\gamma_2^2-2\gamma_2\cos 2\theta)\sin\frac{1}{2}(\theta-\beta)}. \quad (7\cdot10)$$

The stress system due to an isolated force in any direction at any point of the boundary may be obtained by combining these two results.

Taking the limit of the expressions as $\alpha_1, \alpha_2 \to 1$ and $\gamma_1, \gamma_2 \to 0$ gives the stresses in an isotropic material.

Equation (7·7) gives

$$\widehat{\theta\theta}_e = \frac{P}{\pi a}\left\{\frac{\sin\frac{1}{2}(\theta+\beta)}{\sin\frac{1}{2}(\theta-\beta)} + \frac{1}{2}(3-\sigma)\cos\theta\right\}, \quad (7\cdot11)$$

and (7·10) gives

$$\widehat{\theta\theta}_e = \frac{P}{\pi a}\left\{\frac{1}{2}(3-\sigma)\sin\theta - \frac{\cos\frac{1}{2}(\theta+\beta)}{\sin\frac{1}{2}(\theta-\beta)}\right\}. \quad (7\cdot12)$$

(7·11) is the formula apparently used by Bickley (1928) in the special case when $\beta = 0$ and the force is radial.

Calculations have been carried out for the cases of radial and tangential forces at the point (a, π). The values are tabulated in Tables 3a and 3b and the values for spruce are plotted in Figs. 2a and 2b.

Table 3*a*. *Isolated radial force at* (a, π). *Values of* $\pi a \widehat{\theta\theta}_e / P$

	Grain perpendicular to force			Grain parallel to force	
$\theta°$	Spruce	Oak	Isotropic	Oak	Spruce
0	−2·29	−0·962	−0·375	−0·163	−0·0872
10	−1·36	−0·780	−0·354	−0·161	−0·0877
20	−0·435	−0·432	−0·292	−0·152	−0·0888
30	−0·0249	−0·134	−0·191	−0·132	−0·0888
40	0·157	+0·0748	−0·053	−0·0914	−0·0825
50	0·251	0·223	0·116	−0·00624	−0·0556
60	0·313	0·340	0·3125	0·168	0·0340
70	0·368	0·446	0·530	0·509	0·322
80	0·433	0·559	0·761	1·08	1·21
90	0·522	0·691	1	1·68	2·68
100	0·653	0·864	1·24	1·88	2·35
110	0·857	1·10	1·47	1·69	1·44
120	1·19	1·44	1·69	1·40	0·912
130	1·76	1·95	1·88	1·16	0·631
140	2·80	2·73	2·05	0·984	0·474
150	4·88	3·95	2·19	0·865	0·381
160	9·47	5·78	2·29	0·789	0·328
170	19·6	7·97	2·35	0·746	0·300

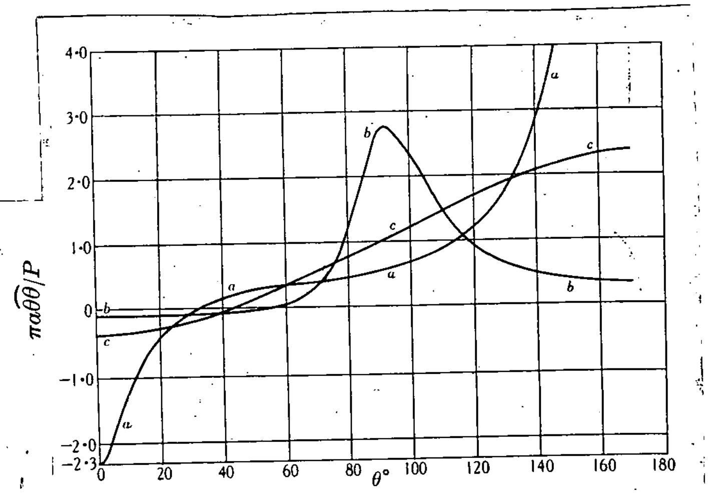

Fig. 2(*a*). Isolated radial force. Values of $\pi a \widehat{\theta\theta}_e / P$. (*a*) Spruce—grain perpendicular to force, (*b*) spruce—grain parallel to force, (*c*) isotropic material.

Table 3b. *Isolated tangential force at* (a, π). *Values of* $\pi a \widehat{\theta\theta}_e / P$

	Grain parallel to force			Grain perpendicular to force	
$\theta°$	Spruce	Oak	Isotropic	Oak	Spruce
0	0	0	0	0	0
10	0·417	0·274	0·151	0·0904	0·0625
20	0·404	0·398	0·294	0·190	0·136
30	0·312	0·407	0·420	0·311	0·236
40	0·239	0·373	0·520	0·467	0·387
50	0·187	0·330	0·587	0·673	0·631
60	0·151	0·287	0·613	0·941	1·05
70	0·125	0·248	0·592	1·23	1·79
80	0·105	0·208	0·515	1·36	2·83
90	0·0872	0·163	0·375	0·962	2·29
100	0·0674	0·105	0·162	0·0911	−0·391
110	0·0377	0·0158	−0·136	−0·714	−1·50
120	−0·0182	−0·136	−0·541	−1·26	−1·75
130	−0·141	−0·420	−1·09	−1·68	−1·85
140	−0·448	−1·00	−1·86	−2·12	−2·00
150	−1·33	−2·32	−3·05	−2·77	−2·36
160	−4·44	−5·65	−5·20	−4·06	−3·23
170	−19·8	−16·5	−11·2	−7·98	−6·10

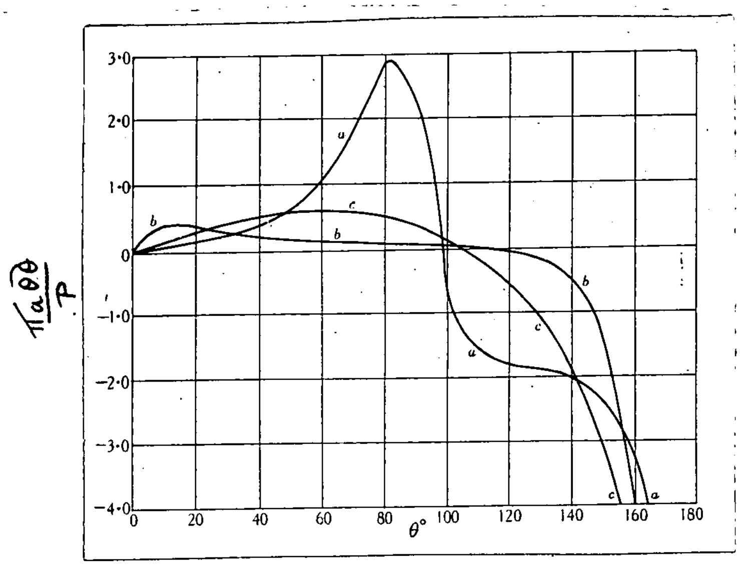

Fig. 2(b). Isolated tangential force. (a) Spruce — grain perpendicular to force. (b) Spruce — grain parallel to force. (c) Isotropic material.

8. The problem of an unstressed circular boundary presents some difficulty when the stress distribution is produced by pulling sideways on a rivet in a neighbouring hole, since this requires stresses which take prescribed values on each of two circular

boundaries. A first approximation to the solution, however, may be obtained by assuming that the force is concentrated at a point, and the remainder of the paper is devoted to the derivation of expressions giving the stress-systems that arise when isolated forces act at points in the neighbourhood of the circular hole instead of at points on the boundary. As far as the writer is aware, this problem has not so far been discussed for any type of material.

A stress function giving the required force when the hole is absent will be formed first, and the problem then becomes one of cancelling the values of $\widehat{rr}_e$ and $\widehat{r\theta}_e$ on the edge of the hole without breaking the various physical conditions. The forces are taken to act at the point $z=-b$, that is at a distance b from the centre of the circular hole in the negative direction of the x-axis. This involves imposing some restriction in the aeolotropic case. If the force is taken to act at any point, however, the method of solution is unchanged, though the analysis is increased in complexity. Solutions are therefore confined to the cases in which the forces act at points on a diameter parallel to an axis of symmetry.

FORCE PARALLEL TO THE x-AXIS. ISOTROPIC MATERIAL.

9. When the hole is absent, the stresses produced by the isolated force are

$$\widehat{xx} = -\frac{P}{8\pi}\left\{\frac{5+\sigma}{z+b} + \frac{(1+\sigma)(\bar{z}+b)}{(z+b)^2}\right\}, \tag{9·1}$$

$$\widehat{yy} = \frac{P}{8\pi}\left\{\frac{1-3\sigma}{z+b} + \frac{(1+\sigma)(\bar{z}+b)}{(z+b)^2}\right\}, \tag{9·2}$$

$$\widehat{xy} = -\frac{iP}{8\pi}\left\{\frac{3-\sigma}{z+b} + \frac{(1+\sigma)(\bar{z}+b)}{(z+b)^2}\right\}, \tag{9·3}$$

$$\widehat{rr} = -\frac{P}{8\pi}\left\{\frac{(3-\sigma)z}{\bar{z}(z+b)} + \frac{(1+\sigma)[3z\bar{z}+b(z+2\bar{z})]}{\bar{z}(z+b)^2}\right\}, \tag{9·4}$$

$$\widehat{\theta\theta} = \frac{P}{8\pi}\left\{\frac{(3-\sigma)z}{\bar{z}(z+b)} - \frac{(1+\sigma)[z\bar{z}+b(2\bar{z}-z)]}{\bar{z}(z+b)^2}\right\}, \tag{9·5}$$

$$\widehat{r\theta} = -\frac{iPz}{8\pi\bar{z}}\left\{\frac{3-\sigma}{z+b} + \frac{(1+\sigma)(\bar{z}+b)}{(z+b)^2}\right\}. \tag{9·6}$$

Remembering that only the real parts of these functions are to be taken, it is seen that the necessary conditions may be satisfied by writing

$$V(z) = -\frac{P}{8\pi}\left\{\frac{(3-\sigma)a^2}{z(a^2+bz)} + \frac{(1+\sigma)[3a^2z+b(a^2+2z^2)]}{(a^2+bz)^2}\right\}, \tag{9·7}$$

$$W(z) = -\frac{P}{8\pi}\left\{\frac{(3-\sigma)a^2}{z(a^2+bz)} + \frac{(1+\sigma)[b^2(a^2-2z^2)-a^2(2a^2+3bz)]}{b(a^2+bz)^2}\right\}. \tag{9·8}$$

The complete stress-system is then obtained from § 2, together with equations (9·1) – (9·6).

The hoop stress on the edge of the hole is given by

$$\begin{aligned} 2\pi b(a^2+b^2+2ab\cos\theta)^2\,\widehat{\theta\theta}_e/P &= (3-\sigma)ab^3\cos 3\theta \\ &+ 4b^2\{b^2+(1-\sigma)a^2\}\cos 2\theta + ab\{b^2(5-3\sigma)-4\sigma a^2\}\cos\theta \\ &- (1+\sigma)(a^4+b^4) + 2(1-\sigma)a^2b^2. \end{aligned} \tag{9·9}$$

The hoop stress has been evaluated numerically for the cases $b = 2a$, and $b = 3a$ and the values are shewn in Table 4 and plotted in Fig. 3.

By combining two of these solutions it is possible to obtain the hoop stress round the edge of a circular hole in a railway wheel when the hole is directly below the axle. The solution is approximate, since the effect of the outer boundary of the wheel is neglected, but the general character of the stresses is found to be the same as that discovered by Coker and Filon (1931) who investigated this problem experimentally.

FORCE PARALLEL TO THE y-AXIS. ISOTROPIC MATERIAL.

10. The stress system, when the hole is absent, is defined by

$$\widehat{xx} = \frac{iP}{8\pi}\left\{\frac{1-3\sigma}{z+b} - \frac{(1+\sigma)(\bar{z}+b)}{(z+b)^2}\right\}, \tag{10.1}$$

$$\widehat{yy} = -\frac{iP}{8\pi}\left\{\frac{5+\sigma}{z+b} - \frac{(1+\sigma)(\bar{z}+b)}{(z+b)^2}\right\}, \tag{10.2}$$

$$\widehat{xy} = -\frac{P}{8\pi}\left\{\frac{3-\sigma}{z+b} - \frac{(1+\sigma)(\bar{z}+b)}{(z+b)^2}\right\}, \tag{10.3}$$

$$\widehat{rr} = \frac{iP}{8\pi}\left\{\frac{(3-\sigma)z}{\bar{z}(z+b)} - \frac{(1+\sigma)[3z\bar{z}+b(z+2\bar{z})]}{\bar{z}(z+b)^2}\right\}, \tag{10.4}$$

$$\widehat{\theta\theta} = -\frac{iP}{8\pi}\left\{\frac{(3-\sigma)z}{\bar{z}(z+b)} + \frac{(1+\sigma)[z\bar{z}+b(2\bar{z}-z)]}{\bar{z}(z+b)^2}\right\}, \tag{10.5}$$

$$\widehat{r\theta} = -\frac{Pz}{8\pi\bar{z}}\left\{\frac{3-\sigma}{z+b} - \frac{(1+\sigma)(\bar{z}+b)}{(z+b)^2}\right\}. \tag{10.6}$$

It is now necessary to write

$$V(z) = -\frac{iP}{8\pi}\left\{\frac{(3-\sigma)a^2}{z(a^2+bz)} + \frac{(1+\sigma)a^2[2a^2-b^2+bz]}{b(a^2+bz)^2}\right\}, \tag{10.7}$$

$$W(z) = -\frac{iP}{8\pi}\left\{\frac{(3-\sigma)a^2}{z(a^2+bz)} - \frac{(1+\sigma)(z+b)a^2}{(a^2+bz)^2}\right\}. \tag{10.8}$$

Equations (2·9) – (2·14) give the compensating stresses and the complete system is found by adding in stresses (10·1) – (10·6). On the edge of the circular hole

$$2\pi(a^2+b^2+2ab\cos\theta)^2\,\widehat{\theta\theta}_e/P = (3-\sigma)ab^2\sin 3\theta$$
$$+2b[(3-\sigma)a^2+(1+\sigma)b^2]\sin 2\theta + a[4a^2+(3-5\sigma)b^2]\sin\theta. \tag{10.9}$$

The numerical values of these stresses are shewn in Table 5 and plotted in Fig. 4.

FORCE PARALLEL TO THE x-AXIS. AEOLOTROPIC MATERIAL.

11. Writing $P = -2\pi s_{22}(\alpha_1-\alpha_2)K$, (11·1)

and

$$\left.\begin{aligned} s_{12}-\alpha_2 s_{22} &= \phi_1(1-\gamma_1)\{z+\gamma_1\bar{z}+b(1+\gamma_1)\},\\ s_{12}-\alpha_1 s_{22} &= \phi_2(1-\gamma_2)\{z+\gamma_2\bar{z}+b(1+\gamma_2)\};\end{aligned}\right\} \tag{11.2}$$

the stresses when the hole is absent are given by

$$\widehat{xx} = K\{(1-\gamma_1)^2\phi_1 - (1-\gamma_2)^2\phi_2\}, \tag{11.3}$$

$$\widehat{yy} = -K\{(1+\gamma_1)^2\phi_1 - (1+\gamma_2)^2\phi_2\}, \tag{11.4}$$

$$\widehat{xy} = iK\{(1-\gamma_1^2)\phi_1 - (1-\gamma_2^2)\phi_2\}, \tag{11.5}$$

$$\widehat{rr} = K\left\{(z-\gamma_1\bar{z})^2\phi_1 - (z-\gamma_2\bar{z})^2\phi_2\right\}/z\bar{z}, \tag{11.6}$$

$$\widehat{\theta\theta} = -K\left\{(z+\gamma_1\bar{z})^2\phi_1 - (z+\gamma_2\bar{z})^2\phi_2\right\}/z\bar{z}, \tag{11.7}$$

$$\widehat{r\theta} = iK\left\{(z^2-\gamma_1^2\bar{z}^2)\phi_1 - (z^2-\gamma_2^2\bar{z}^2)\phi_2\right\}/z\bar{z}. \tag{11.8}$$

In order to cancel $\widehat{rr}_e$ and $\widehat{r\theta}_e$, $V(z)$ and $W(z)$ must now be chosen. If these functions are defined directly from equations (11.6) and (11.8) the solution obtained is impracticable since it cancels the stress system completely. If, on the other hand, $V(z)$ and $W(z)$ are defined to be the complex conjugates of (11.6) and (11.8) it is found from equations (3.16) – (3.19) that $F(z+\gamma_1\bar{z})$ and $G(z+\gamma_2\bar{z})$ are not regular functions outside the circle $|z| = a$, and the resulting stresses would not, consequently, be physically possible ones.

To overcome these difficulties, the following quantities are now defined:

$$\begin{aligned} \rho_1 &= \tfrac{1}{2}\left\{b(1+\gamma_1) - [b^2(1+\gamma_1)^2 - 4\gamma_1 a^2]^{1/2}\right\}, \\ \rho_2 &= \tfrac{1}{2}\left\{b(1+\gamma_1) + [b^2(1+\gamma_1)^2 - 4\gamma_1 a^2]^{1/2}\right\}; \end{aligned} \tag{11.9}$$

$$\begin{aligned} \sigma_1 &= \tfrac{1}{2}\left\{b(1+\gamma_2) - [b^2(1+\gamma_2)^2 - 4\gamma_2 a^2]^{1/2}\right\}, \\ \sigma_2 &= \tfrac{1}{2}\left\{b(1+\gamma_2) + [b^2(1+\gamma_2)^2 - 4\gamma_2 a^2]^{1/2}\right\}. \end{aligned} \tag{11.10}$$

It is easily verified that

$$\begin{aligned} \rho_1\rho_2 &= \gamma_1 a^2, \qquad \rho_1 + \rho_2 = b(1+\gamma_1), \\ \sigma_1\sigma_2 &= \gamma_2 a^2, \qquad \sigma_1 + \sigma_2 = b(1+\gamma_2). \end{aligned} \tag{11.11}$$

The expressions for $\widehat{rr}_e$ and $\widehat{r\theta}_e$ may now be expressed as partial fractions with denominators $z+\rho_1$, $z+\rho_2$, $z+\sigma_1$, $z+\sigma_2$. The functions $1/(z+\rho_1)$, $1/(z+\sigma_1)$ are regular outside the circle, since it may be established with some little difficulty that $|\rho_1| < a$, $|\rho_2| > a$, $|\sigma_1| < a$, $|\sigma_2| > a$. The functions $1/(z+\rho_2)$, $1/(z+\sigma_2)$ are not regular, however, and it is these terms which produce poles in $F(z+\gamma_1\bar{z})$, $G(z+\gamma_2\bar{z})$ after the necessary substitutions have been made. To obviate this, the functions must be replaced by regular functions which take the same real or imaginary values on the circular boundary according as the term occurs in $V(z)$ or $W(z)$.

After some reduction it is found that all the physical conditions are satisfied by writing

$$a^2K^{-1}V(z) = \frac{s_{12}-\alpha_2 s_{22}}{1-\gamma_1}\left\{\frac{a^2(1+\gamma_1)}{z} - b(1+\gamma_1) + (\rho_1-\rho_2)\left[\frac{\rho_1}{z+\rho_1} - \frac{z\rho_2}{a^2+z\rho_2}\right]\right\}$$

$$- \frac{s_{12}-\alpha_1 s_{22}}{1-\gamma_2}\left\{\frac{a^2(1+\gamma_2)}{z} - b(1+\gamma_2) + (\sigma_1-\sigma_2)\left[\frac{\sigma_1}{z+\sigma_1} - \frac{z\sigma_2}{a^2+z\sigma_2}\right]\right\}, \quad (11{\cdot}12)$$

$$a^2K^{-1}W(z) = \frac{s_{12}-\alpha_2 s_{22}}{1-\gamma_1}\left\{\frac{a^2(1+\gamma_1)}{z} + (\rho_1-\rho_2) - b(1+\gamma_1)\left[\frac{\rho_1}{z+\rho_1} - \frac{z\rho_2}{a^2+z\rho_2}\right]\right\}$$

$$- \frac{s_{12}-\alpha_1 s_{22}}{1-\gamma_2}\left\{\frac{a^2(1+\gamma_2)}{z} + (\sigma_1-\sigma_2) - b(1+\gamma_2)\left[\frac{\sigma_1}{z+\sigma_1} - \frac{z\sigma_2}{a^2+z\sigma_2}\right]\right\}. \quad (11{\cdot}13)$$

The functions $F(z+\gamma_1\bar{z})$, $G(z+\gamma_2\bar{z})$ have been obtained from the above equations, and the complete stress system is thus determined as before, although the expressions are extremely complicated. The boundary stresses, however, admit of a certain amount of reduction, and, as the main interest of the problem lies in them, they alone are reproduced.

Round the edge of the circular hole the stresses are given by

$$\tfrac{1}{2}\left(1+\gamma_1^2-2\gamma_1\cos 2\theta\right)\left(1+\gamma_2^2-2\gamma_2\cos 2\theta\right)\pi s_{22}\left(\alpha_1-\alpha_2\right)\widehat{\theta\theta}_e / Pa\left(1-\gamma_1\gamma_2\right)$$

$$\begin{aligned}
= {} & \left(s_{12}-\alpha_2 s_{22}\right)\left(1+\gamma_1\right)\left(a^2+\rho_1^2+2a\rho_1\cos\theta\right)^{-1}\left(a^2+\rho_2^2+2a\rho_2\cos\theta\right)^{-1} \\
& \times \left\{ \begin{array}{l} \gamma_1 a^2\cos 3\theta + \left[b(1+\gamma_1)+\rho_1(\gamma_1-\gamma_2)\right] a\cos 2\theta \\ +\left[a^2(1+\gamma_1-\gamma_2-2\gamma_1\gamma_2)+\rho_1^2(1-\gamma_2)\right]\cos\theta + a\left[\rho_1(1-\gamma_1\gamma_2)-\gamma_2 b(1+\gamma_1)\right] \end{array} \right\} \\
& -\left(s_{12}-\alpha_1 s_{22}\right)\left(1+\gamma_2\right)\left(a^2+\varpi_1^2+2a\varpi_1\cos\theta\right)^{-1}\left(a^2+\varpi_2^2+2a\varpi_2\cos\theta\right)^{-1} \\
& \times \left\{ \begin{array}{l} \gamma_2 a^2\cos 3\theta + \left[b(1+\gamma_2)+\varpi_1(\gamma_2-\gamma_1)\right] a\cos 2\theta \\ +\left[a^2(1+\gamma_2-\gamma_1-2\gamma_1\gamma_2)+\varpi_1^2(1-\gamma_1)\right]\cos\theta + a\left[\varpi_1(1-\gamma_1\gamma_2)-\gamma_1 b(1+\gamma_2)\right] \end{array} \right\}.
\end{aligned} \qquad (11.14)$$

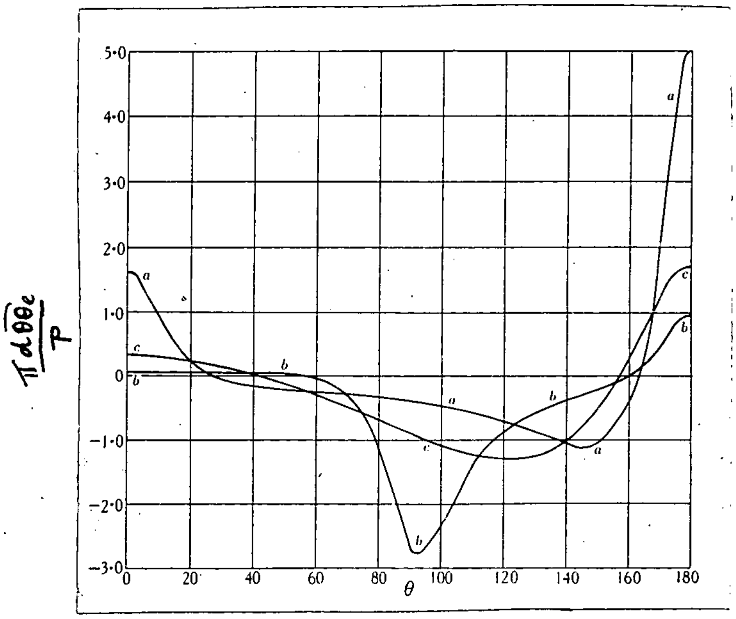

Fig. 3. Values of $\pi d\,\widehat{\theta\theta}_e/P$. (a) Spruce - grain perpendicular to force, (b) spruce - grain parallel to force, (c) isotropic material.

The hoop stresses have been evaluated numerically for the two cases $b=2a$, $b=3a$ for specimens of spruce and oak. These values

Table 4. *Force parallel to x-axis. Values of* $\pi d\widehat{\theta\theta}_e/P$

	Grain parallel to force						Grain perpendicular to force			
	Oak		Spruce		Isotropic		Oak		Spruce	
$\theta°$	$d=a$	$d=2a$	$d=a$	$d=2a$	$d=a$	$d=2a$	$d=a$	$d=2a$	$d=a$	$d=2a$
0	0·165	0·295	0·0872	0·164	0·354	0·583	0·794	1·24	1·65	2·48
10	0·163	0·290	0·0877	0·165	0·334	0·547	0·638	0·985	0·950	1·41
20	0·154	0·272	0·0888	0·166	0·274	0·439	0·338	0·499	0·263	0·353
30	0·135	0·232	0·0888	0·164	0·176	0·265	0·0830	0·0871	−0·0389	−0·108
40	0·0943	0·151	0·0824	0·148	0·0440	0·0319	−0·0952	−0·197	−0·163	−0·304
50	0·00844	−0·0135	0·0554	0·0902	−0·117	−0·247	−0·220	−0·393	−0·236	−0·397
60	−0·162	−0·341	−0·0341	−0·0925	−0·302	−0·558	−0·317	−0·538	−0·278	−0·452
70	−0·497	−0·961	−0·321	−0·660	−0·503	−0·881	−0·402	−0·658	−0·314	−0·495
80	−1·06	−1·95	−1·20	−2·35	−0·710	−1·19	−0·487	−0·770	−0·355	−0·543
90	−1·64	−2·92	−2·66	−5·07	−0·913	−1·47	−0·581	−0·882	−0·409	−0·602
100	−1·82	−3·09	−2·32	−4·28	−1·10	−1·67	−0·691	−0·998	−0·483	−0·678
110	−1·60	−2·56	−1·40	−2·48	−1·24	−1·75	−0·822	−1·11	−0·584	−0·773
120	−1·28	−1·86	−0·871	−1·44	−1·31	−1·67	−0·970	−1·21	−0·721	−0·886
130	−0·975	−1·21	−0·577	−0·846	−1·27	−1·39	−1·11	−1·23	−0·892	−0·997
140	−0·692	−0·629	−0·390	−0·446	−1·04	−0·874	−1·16	−1·08	−1·06	−1·04
150	−0·361	−0·0655	−0·230	−0·0984	−0·526	−0·148	−0·895	−0·593	−1·06	−0·863
160	0·123	0·485	−0·00367	0·279	0·308	0·661	0·0594	0·459	−0·390	−0·0139
170	0·800	0·936	0·461	0·683	1·24	1·32	1·80	1·96	2·12	2·40
180	1·21	1·12	0·955	0·890	1·69	1·58	2·90	2·80	5·03	4·99

are tabulated in Table 4, and the values for spruce ($b=2a$) are plotted in Fig. 3. The actual values shewn are those of $\pi d\,\widehat{\theta\theta}_e/P$ where $d = b-a$.

FORCE PARALLEL TO THE y-AXIS. AEOLOTROPIC MATERIAL.

12. When the hole is absent the stresses are given by equations (11·3) — (11·8) where it is now necessary to write

$$i(s_{12}-\alpha_1 s_{22}) = \phi_1(1+\gamma_1)\{z+\gamma_1\bar{z}+b(1+\gamma_1)\},$$
$$i(s_{12}-\alpha_2 s_{22}) = \phi_2(1+\gamma_2)\{z+\gamma_2\bar{z}+b(1+\gamma_2)\}. \qquad (12\cdot1)$$

Equations (11·12) and (11·13) must be replaced by

$$a^2K^{-1}V(z) = \frac{i(s_{12}-\alpha_1 s_{22})}{1+\gamma_1}\left\{-\frac{a^2(1-\gamma_1)}{z}+(\rho_1-\rho_2)\left(\frac{\rho_1}{z+\rho_1}+\frac{z\rho_2}{a^2+z\rho_2}-1\right)\right\}$$
$$-\frac{i(s_{12}-\alpha_2 s_{22})}{1+\gamma_2}\left\{-\frac{a^2(1-\gamma_2)}{z}+(\sigma_1-\sigma_2)\left(\frac{\sigma_1}{z+\sigma_1}+\frac{z\sigma_2}{a^2+z\sigma_2}-1\right)\right\}, \qquad (12\cdot2)$$

$$a^2K^{-1}W(z) = \frac{-i(s_{12}-\alpha_1 s_{22})}{1+\gamma_1}\left\{\frac{a^2(1-\gamma_1)}{z}+b(1+\gamma_1)\left(\frac{\rho_1}{z+\rho_1}+\frac{z\rho_2}{a^2+z\rho_2}-1\right)\right\}$$
$$+\frac{i(s_{12}-\alpha_2 s_{22})}{1+\gamma_2}\left\{\frac{a^2(1-\gamma_2)}{z}+b(1+\gamma_2)\left(\frac{\sigma_1}{z+\sigma_1}+\frac{z\sigma_2}{a^2+z\sigma_2}-1\right)\right\}, \qquad (12\cdot3)$$

and, after further reduction it is found that, on the edge of the circular hole,

$$\tfrac{1}{2}\left(1+\gamma_1^2-2\gamma_1\cos 2\theta\right)\left(1+\gamma_2^2-2\gamma_2\cos 2\theta\right)\pi s_{22}\left(\alpha_1-\alpha_2\right)\widehat{\theta\theta}_e \big/ Pa\left(1-\gamma_1\gamma_2\right)$$

$$= -\left(s_{12}-\alpha_1 s_{22}\right)\left(1-\gamma_1\right)\left(a^2+\rho_1^2+2a\rho_1\cos\theta\right)^{-1}\left(a^2+\rho_2^2+2a\rho_2\cos\theta\right)^{-1}$$
$$\times\left\{\begin{array}{l}\gamma_1 a^2 \sin 3\theta + a\left[b(1+\gamma_1)+\rho_1(\gamma_1+\gamma_2)\right]\sin 2\theta \\ \quad + \left[a^2(1+\gamma_1+\gamma_2)+\rho_1^2(1+\gamma_2)\right]\sin\theta\end{array}\right\}$$

$$+\left(s_{12}-\alpha_2 s_{22}\right)\left(1-\gamma_2\right)\left(a^2+\sigma_1^2+2a\sigma_1\cos\theta\right)^{-1}\left(a^2+\sigma_2^2+2a\sigma_2\cos\theta\right)^{-1}$$
$$\times\left\{\begin{array}{l}\gamma_2 a^2 \sin 3\theta + a\left[b(1+\gamma_2)+\sigma_1(\gamma_1+\gamma_2)\right]\sin 2\theta \\ \quad + \left[a^2(1+\gamma_1+\gamma_2)+\sigma_1^2(1+\gamma_1)\right]\sin\theta\end{array}\right\}. \qquad (12.4)$$

The hoop stresses have been evaluated numerically for the two cases $b=2a$, $b=3a$ and the values are shewn in Table 5. The values for spruce ($b=2a$) and the isotropic values are plotted in Fig. 4.

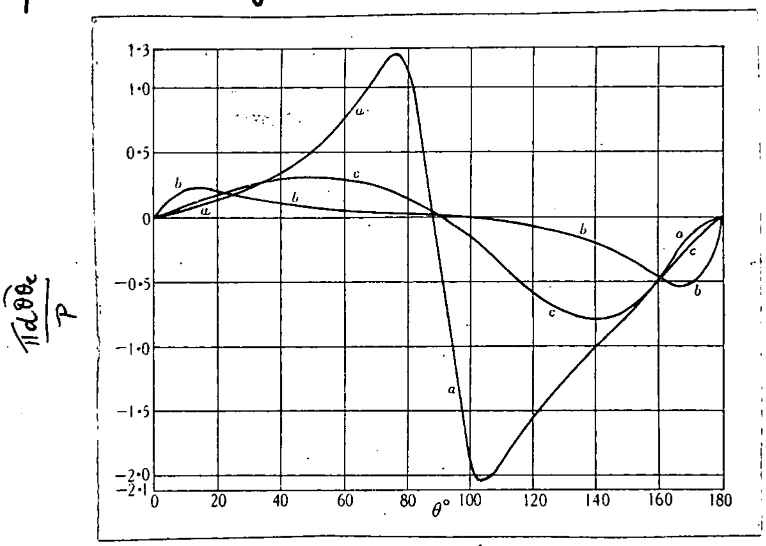

Fig. 4. Values of $\pi d\widehat{\theta\theta}_e/P$. (a) Spruce — grain perpendicular to force, (b) spruce — grain parallel to force (c) isotropic material.

Table 5. *Force parallel to y-axis. Values of* $\pi d\widehat{\theta\theta}_e/P$

	Grain parallel to force						Grain perpendicular to force			
	Oak		Spruce		Isotropic		Oak		Spruce	
$\theta°$	$d=a$	$d=2a$	$d=a$	$d=2a$	$d=a$	$d=2a$	$d=a$	$d=2a$	$d=a$	$d=2a$
0	0	0	0	0	0	0	0	0	0	0
10	0·153	0·220	0·220	0·315	0·0955	0·139	0·0668	0·102	0·0634	0·0833
20	0·218	0·311	0·208	0·297	0·183	0·265	0·139	0·212	0·135	0·180
30	0·216	0·305	0·156	0·220	0·254	0·364	0·223	0·336	0·226	0·305
40	0·188	0·261	0·113	0·157	0·302	0·424	0·326	0·481	0·349	0·484
50	0·154	0·209	0·0823	0·112	0·319	0·436	0·450	0·645	0·524	0·751
60	0·119	0·157	0·0592	0·0779	0·300	0·393	0·586	0·806	0·775	1·16
70	0·0847	0·108	0·0409	0·0512	0·242	0·295	0·679	0·869	1·09	1·73
80	0·0480	0·0528	0·0248	0·0282	0·141	0·144	0·569	0·608	1·13	2·05
90	0·00652	−0·00291	0·00843	0·00566	0	−0·0500	0·0770	−0·156	−0·412	0·0618
100	−0·0437	−0·0653	−0·0108	−0·0193	−0·177	−0·268	−0·585	−1·01	−1·95	−2·42
110	−0·108	−0·138	−0·0364	−0·0502	−0·377	−0·482	−1·02	−1·43	−1·90	−2·55
120	−0·190	−0·222	−0·0730	−0·0915	−0·577	−0·658	−1·18	−1·45	−1·55	−2·07
130	−0·293	−0·318	−0·128	−0·149	−0·739	−0·759	−1·18	−1·26	−1·25	−1·60
140	−0·411	−0·420	−0·209	−0·232	−0·808	−0·754	−1·07	−0·989	−1·00	−1·18
150	−0·511	−0·507	−0·326	−0·348	−0·724	−0·639	−0·854	−0·680	−0·766	−0·792
160	−0·523	−0·524	−0·470	−0·493	−0·486	−0·444	−0·515	−0·384	−0·473	−0·437
170	−0·353	−0·370	−0·512	−0·537	−0·203	−0·220	−0·156	−0·155	−0·121	−0·159
180	0	0	0	0	0	0	0	0	0	0

STRESSES ALONG STRAIGHT BOUNDARIES.

13. By considering the limits of the various expressions already obtained, the stresses along straight edges parallel to the y-axis are obtained. It is necessary to allow both b and a to tend to infinity in such a manner that $b-a$ remains constant and equal to d.

From equation (7·7) it is found that $\widehat{yy}=0$ when the force is perpendicular to the edge and acts at a point on it. When the force is parallel to the edge, equation (7·10) gives

$$\widehat{yy} = -P\left(\alpha_1^{1/2}+\alpha_2^{1/2}\right)/\pi y. \tag{13·1}$$

If the force acts perpendicular to the straight boundary and at a distance d from it, equation (9·9) gives for an isotropic material

$$\widehat{yy} = 2Pd\left(d^2-\sigma y^2\right)/\pi\left(y^2+d^2\right)^2; \tag{13·2}$$

and (11·14) gives for an aeolotropic material

$$\widehat{yy} = \frac{Pd}{\pi s_{22}\left(\alpha_1^{1/2}-\alpha_2^{1/2}\right)}\left\{\frac{s_{12}-\alpha_2 s_{22}}{d^2+y^2/\alpha_1} - \frac{s_{12}-\alpha_1 s_{22}}{d^2+y^2/\alpha_2}\right\}. \tag{13·3}$$

When the force is parallel to the edge, the corresponding results are, from equations (10·9) and (12·4),

$$\widehat{yy} = 2Py\left(\sigma d^2-y^2\right)/\pi\left(y^2+d^2\right)^2, \tag{13·4}$$

$$\widehat{yy} = \frac{Py}{\pi s_{22}\left(\alpha_1^{1/2}-\alpha_2^{1/2}\right)}\left\{\frac{s_{12}-\alpha_1 s_{22}}{y^2+\alpha_1 d^2} - \frac{s_{12}-\alpha_2 s_{22}}{y^2+\alpha_2 d^2}\right\}. \tag{13·5}$$

Apart from a slight change of notation, the above results agree with those obtained by Green (1939) using real variable methods.

C. The Transverse Flexure of Perforated Isotropic Plates

INTRODUCTION

1. This section contains the solutions of three fundamental problems connected with the transverse flexure of a large thin isotropic plate containing a hole of fairly general shape. The complex variable method which is employed throughout is similar to that used by Green (1943a) for the stretching of such a plate, and its use greatly simplifies the analysis and gives Goodier's (1936) results as special cases with much less labour.

The method consists of finding a conformal transformation of the type

$$z = F(\zeta), \tag{1.1}$$

or, for the special problems of this paper

$$F'(\zeta) = \frac{dz}{d\zeta} = Je^{i\phi} = a_0 e^{-i\zeta} + b_n e^{in\zeta}, \tag{1.2}$$

where z $(= x + iy)$ is the plane of the plate, and ζ $(= \xi + i\eta)$ is real on the edge of the hole. a_0 and b_n are complex constants, and ϕ is the angle between the tangent to the curve $\eta =$ constant and the x-axis.

For a circular hole, $n=0$, $a_0 = -ia$ and the radius is $\underline{a}$.

For an elliptical hole with the major axis parallel to the x-axis, $a_0 = -\frac{1}{2}ice^{\alpha}$, $b_n = \frac{1}{2}ice^{-\alpha}$, and the semi-axes are $c\cosh\alpha$ and $c\sinh\alpha$.

A more general type of curve with continuous curvature is obtained by writing $a_0 = -ina\,e^{i\beta}$, $b_n = inb\,e^{i\beta}$ provided that b is less than a, where the factor $e^{i\beta}$ affects only the orientation of the curve.

When $n=2$, $a=2b$ the curve is a very close approximation to an equilateral triangle with rounded corners.

When $n=3$, $a=3b$ the boundary is a square with rounded corners which has its diagonals parallel to the axes of coordinates if $\beta=0$, and its sides parallel to the axes of coordinates if $\beta = \frac{1}{4}\pi$.

The main practical value of the problems lies in the concentration of tangential stress-couple on the periphery of the hole and the factor of stress-concentration has been evaluated for square and triangular boundaries when the plate is subject to the action of bending or torsional couples about the axes of coordinates.

THE GENERAL METHOD OF SOLUTION

2. The middle surface of the plate is taken as the xy plane and the plate is bent by the action of couples at its edges only. As the plate is infinite this means that the stress-resultant perpendicular to the plate is everywhere zero, and the two faces of the plate are free from traction (Love 1927)

The displacement of the middle surface, $\underline{w}$, must satisfy the biharmonic equation

$$\nabla_1^4 w = 0, \tag{2.1}$$

and a solution of this equation is obtained by writing

$$w = f(z) + \bar{z}\, g(z), \tag{2.2}$$

where $\bar{z} = x - iy$ and only the <u>real</u> part of the expression (2.2) is required.

Stress-couples and shearing forces are defined in the usual way and may be obtained from well-known equations (See Love). From equation (2.2) it is found that the couples and shearing forces are given by the equations

$$G_1 = -D\left\{(1-\sigma)\left[f''(z) + \bar{z}g''(z)\right] + 2(1+\sigma)\, g'(z)\right\}, \tag{2.3}$$

$$G_2 = D\left\{(1-\sigma)\left[f''(z) + \bar{z}g''(z)\right] - 2(1+\sigma)\, g'(z)\right\}, \tag{2.4}$$

$$H_1 = -H_2 = i(1-\sigma)D\left\{f''(z) + \bar{z}g''(z)\right\}, \tag{2.5}$$

$$N_1 = -4Dg''(z), \quad N_2 = -4iDg''(z), \tag{2.6}$$

where D is a constant known as the flexural rigidity of the plate and dashes denote differentiations with respect to z_f. The functions $f'(z)$ and $g(z)$ must be regular functions of z outside the hole which are finite at infinity, and which do not produce many-valued displacements.

The stress-couples in any direction may be obtained by combining the above equations in the usual way, and it is easily verified that

$$\frac{G_\xi}{D} = -2(1+\sigma)g'(z) - (1-\sigma)\left\{f''(z) + \bar{F}(\bar{\zeta})g''(z)\right\}\frac{F'(\zeta)}{\bar{F}'(\bar{\zeta})}, \qquad (2.7)$$

$$\frac{G_\eta}{D} = -2(1+\sigma)g'(z) + (1-\sigma)\left\{f''(z) + \bar{F}(\bar{\zeta})g''(z)\right\}\frac{F'(\zeta)}{\bar{F}'(\bar{\zeta})}, \qquad (2.8)$$

$$\frac{H_\xi}{D} = -\frac{H_\eta}{D} = i(1-\sigma)\left\{f''(z) + \bar{F}(\bar{\zeta})g''(z)\right\}\frac{F'(\zeta)}{\bar{F}'(\bar{\zeta})}, \qquad (2.9)$$

where a bar placed over a function denotes the complex conjugate of the function, and σ is the ordinary Poisson's Ratio.

To satisfy the physical conditions the boundary must be free from normal couple and shearing force, and allowance must be made also for the effect of the torsional couple H_η. Love (1927) has pointed out that this distribution of couple is statically equivalent to a distribution of shearing force $-\partial H_\eta/\partial s$ in a direction perpendicular to the plane of the plate, and so the complete boundary conditions are

$$G_\eta = 0, \quad N_\eta - \frac{\partial H_\eta}{\partial s} = 0. \qquad (2.10)$$

Goodier (1936), however, has shewn that the second condition may be expressed in the form

$$H_\eta + \chi = \text{constant}, \qquad (2.11)$$

where χ is a harmonic function such that $N_\eta = -\partial\chi/\partial s$.

It is easily established from equation (2.6) that when the solution is of the form (2.2),

$$\chi = -4iDg'(z), \qquad (2.12)$$

and so, from equations (2.9) and (2.12),

$$\frac{H_\eta + \chi}{D} = -4ig'(z) - i(1-\sigma)\left\{f''(z) + \bar{F}(\bar{\zeta})g''(z)\right\}\frac{F'(\zeta)}{\bar{F}'(\bar{\zeta})}. \qquad (2.13)$$

In order to satisfy the boundary conditions two functions of the complex variable, $V(\zeta)$ and $W(\zeta)$, are now defined. The real part of $V(\zeta)$ is equal to $-G_\eta/D$ on the edge of the hole, and the imaginary part of $W(\zeta)$ is equal to the variable part of $(H_\eta + \chi)/D$ on the edge of the hole.

From equations (2.8) and (2.13) it is seen that $V(\zeta)$ and $W(\zeta)$ may also be expressed in the forms

$$V(\zeta) = 2(1+\sigma)g'(z) - (1-\sigma)\left\{f''(z) + \bar{F}(\zeta)g''(z)\right\}\frac{F'(\zeta)}{\bar{F}'(\zeta)}, \qquad (2.14)$$

$$W(\zeta) = 4g'(z) + (1-\sigma)\left\{f''(z) + \bar{F}(\zeta)g''(z)\right\}\frac{F'(\zeta)}{\bar{F}'(\zeta)}, \qquad (2.15)$$

as ζ is real on the boundary. The discussion is confined to cases in which $F'(\zeta)$ has no zeros outside the boundary. $\bar{F}'(\zeta)$ however may have zeros at points in the plate, and so $V(\zeta)$ and $W(\zeta)$ will have poles at these zeros, but $V(\zeta)+W(\zeta)$ must have no poles as

$$V(\zeta) + W(\zeta) = 2(3+\sigma)\, g'(z), \tag{2·16}$$

and is therefore regular outside the boundary.

After some reduction it is found that the stress-couples and shearing forces at any point in the plate are given by the real parts of the following expressions in which dashes denote differentiations with respect to ζ.

$$(3+\sigma)\bar{F}'(\bar{\zeta})\, G_\xi / D = V(\zeta)\left\{2\bar{F}'(\zeta) - (1+\sigma)\bar{F}'(\bar{\zeta})\right\}$$

$$- (1+\sigma) W(\zeta)\left\{\bar{F}'(\bar{\zeta}) + \bar{F}'(\zeta)\right\} + \tfrac{1}{2}(1-\sigma)\left\{\bar{F}(\zeta) - \bar{F}(\bar{\zeta})\right\}\left\{V'(\zeta) + W'(\zeta)\right\}, \tag{2·17}$$

$$(3+\sigma)\bar{F}'(\bar{\zeta})\, G_\eta / D = -V(\zeta)\left\{2\bar{F}'(\zeta) + (1+\sigma)\bar{F}'(\bar{\zeta})\right\}$$

$$- (1+\sigma) W(\zeta)\left\{\bar{F}'(\bar{\zeta}) - \bar{F}'(\zeta)\right\} - \tfrac{1}{2}(1-\sigma)\left\{\bar{F}(\zeta) - \bar{F}(\bar{\zeta})\right\}\left\{V'(\zeta) + W'(\zeta)\right\}, \tag{2·18}$$

$$(3+\sigma)\bar{F}'(\bar{\zeta})\, H_\xi / D = -i\bar{F}'(\zeta)\left\{2V(\zeta) - (1+\sigma)W(\zeta)\right\}$$

$$- \tfrac{1}{2} i (1-\sigma)\left\{\bar{F}(\zeta) - \bar{F}(\bar{\zeta})\right\}\left\{V'(\zeta) + W'(\zeta)\right\}, \tag{2·19}$$

$$(3+\sigma)\, J N_\xi / D = -2\left\{V'(\zeta) + W'(\zeta)\right\}, \tag{2·20}$$

$$(3+\sigma)\, J N_\eta / D = 2i\left\{V'(\zeta) + W'(\zeta)\right\}. \tag{2·21}$$

On the boundary of the hole

$$(3+\sigma)\,G_\xi/D = (1-\sigma)\,V(\zeta) - 2(1+\sigma)\,W(\zeta). \tag{2.22}$$

Equations (2.17) to (2.21) represent a formal solution of any problem when $F(\zeta)$ is known and $V(\zeta)$ and $W(\zeta)$ are also known. When the plate is subject to the action of certain couples at infinity it is possible to choose $V(\zeta)$ and $W(\zeta)$ in order to cancel the transmitted stresses along the boundary which then becomes free from all stress except the tangential couple G_ξ. The sum of the two stress-distributions then satisfies all the physical conditions for the perforated plate.

ALL-ROUND BENDING.

3. When the plate is bent by a couple G at infinity the displacement of the middle surface in the absence of the hole is given by

$$w = \frac{-Gr^2}{2D(1+\sigma)}, \tag{3.1}$$

from which it is easily seen that

$$G_1 = G_2 = G_\xi = G_\eta = G, \qquad H_\xi = H_\eta = \gamma = 0. \tag{3.2}$$

To cancel the stress-couples on the boundary it is required to superimpose a system such that $G_\eta = -G$ on the edge of the hole, and recalling the restrictions which have been imposed on $V(\zeta)$ and $W(\zeta)$ it is seen that all the conditions are satisfied by writing

$$\frac{DV(\zeta)}{G} = \frac{\bar{a}_0 e^{i\zeta} F'(\zeta) + \bar{F}'(\zeta)\{F'(\zeta) - a_0 e^{-i\zeta}\}}{F'(\zeta)\bar{F}'(\zeta)}, \qquad (3.3)$$

$$\frac{DW(\zeta)}{G} = -\frac{\bar{a}_0 e^{i\zeta} F'(\zeta) - \bar{F}'(\zeta)\{F'(\zeta) - a_0 e^{-i\zeta}\}}{F'(\zeta)\bar{F}'(\zeta)}. \qquad (3.4)$$

The complete distribution in the perforated plate is now obtained directly from equations (2.17) – (2.21) together with (3.2).

Adding in the couple transmitted from infinity, the complete tangential stress-couple on the edge of the hole is given by the real part of the expression

$$\frac{G_\xi}{G} = \frac{2}{3+\sigma}\left\{1 - \sigma + 2(1+\sigma)\frac{a_0 e^{-i\zeta}}{F'(\zeta)}\right\}. \qquad (3.5)$$

In all the bending problems which are discussed the additional displacement due to the hole is found to contain a logarithmic term which is infinite at infinity. Such a term, however, is quite admissible in spite of its behaviour at infinity because the additional deflection is insignificant in comparison with the undisturbed deflection. Goodier (1936) has justified the inclusion of a logarithmic term in the case of the circular hole, and its presence in the case of a more general type of boundary admits of a similar justification.

4. Using equations (1·2) and (3·5) the results for a circular or elliptical hole follow at once. For a circular hole

$$G_\xi = 2G, \tag{4·1}$$

and for the ellipse

$$\frac{G_\xi}{G} = \frac{2}{3+\sigma}\left\{2 + \frac{(1+\sigma)\sinh 2\alpha}{\cosh 2\alpha - \cos 2\xi}\right\}, \tag{4·2}$$

both of which agree with Goodier's results.

When the transformation is of the more general type described in §1, the tangential stress-couple takes the form

$$\frac{G_\xi}{G} = \frac{2}{3+\sigma}\left\{2 + \frac{(1+\sigma)(a^2-b^2)}{a^2+b^2-2ab\cos(n+1)\xi}\right\}, \tag{4·3}$$

whatever the value of β.

Setting $n=2$, the result for a triangular hole is

$$\frac{G_\xi}{G} = \frac{2}{3+\sigma}\left\{2 + \frac{3(1+\sigma)}{5-4\cos 3\xi}\right\}, \tag{4·4}$$

and when $n=3$, for a square hole, it is found that

$$\frac{G_\xi}{G} = \frac{2}{3+\sigma}\left\{2 + \frac{4(1+\sigma)}{5-3\cos 4\xi}\right\}. \tag{4·5}$$

The last two expressions have been evaluated numerically using $\sigma = 0{\cdot}25$, a common value for steel. The results are given in Table 6, and plotted in Figs. 5 and 6.

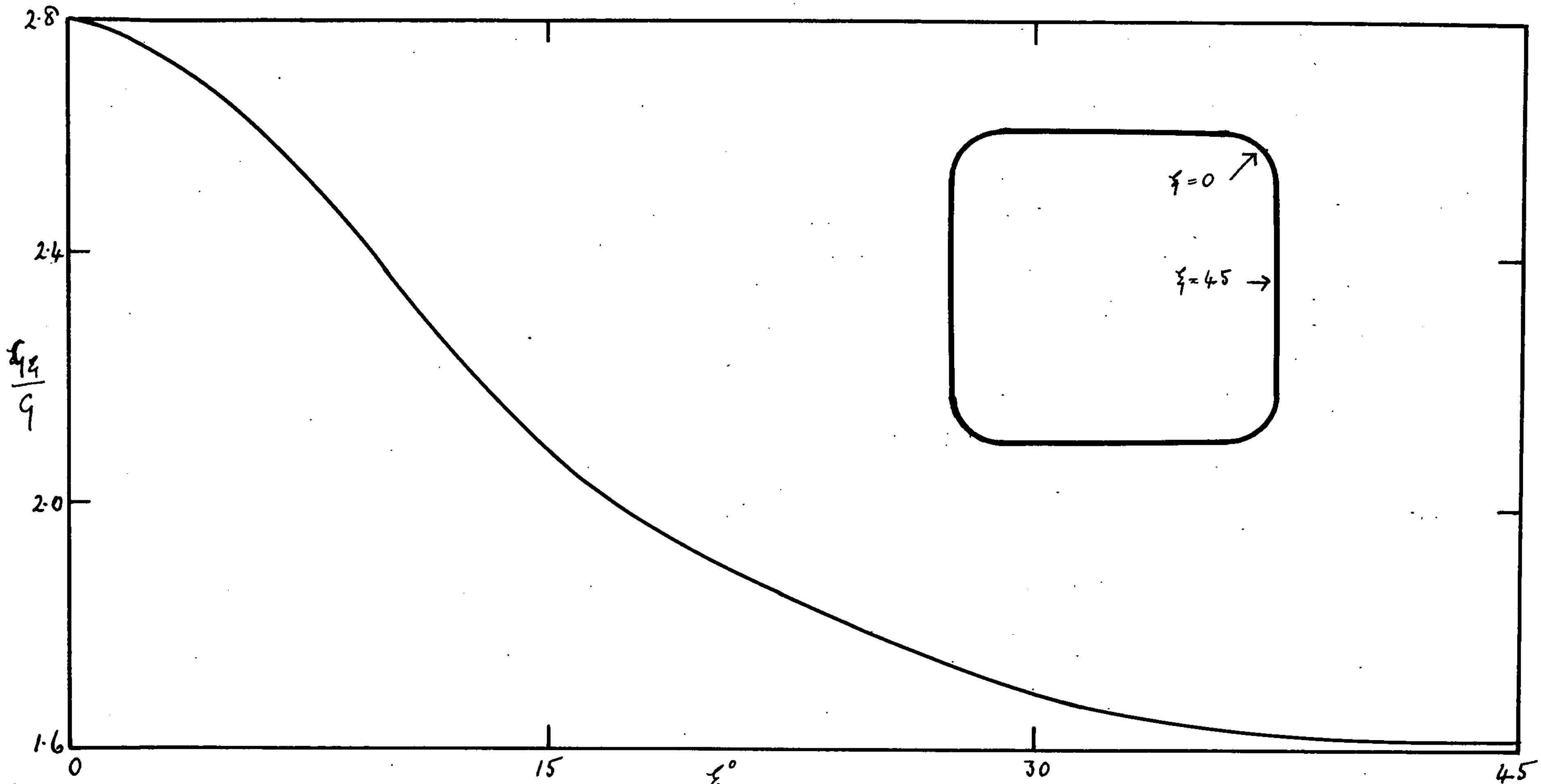

Fig 5. Values of G_ξ on the edge of a square hole in a plate bent by all-round couples.

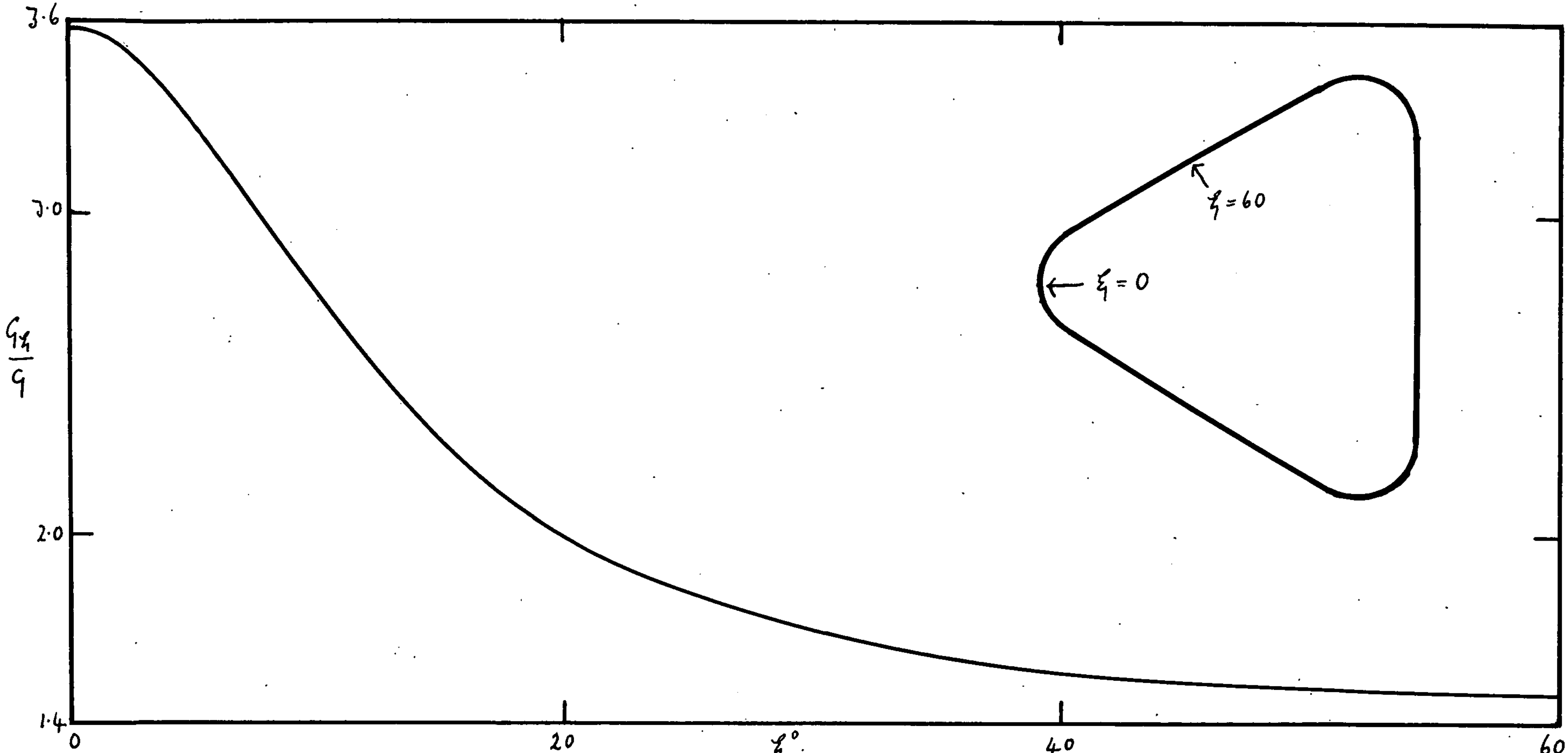

Fig. 6. Values of G_ξ on the edge of a triangular hole in a plate bent by all-round couples.

Table 6.

Values of G_ξ on the edge of the hole in a plate bent by all-round couples

(a) Square hole

ξ	0°	5°	10°	20°	30°
G_ξ/G	2·77	2·64	2·37	1·92	1·70
ξ	40°	45°			
G_ξ/G	1·62	1·615			

(b) Triangular hole

ξ	0°	10°	20°	30°	40°
G_ξ/G	3·54	2·73	2	1·69	1·56
ξ	50°	60°			
G_ξ/G	1·50	1·49			

CYLINDRICAL BENDING ABOUT THE x-AXIS

5. When the plate is bent into the form of a cylinder whose generators are parallel to the x-axis, the displacement of the middle surface, in the absence of the hole, is

$$w = -Gy^2/2D, \tag{5·1}$$

which can be maintained by the following stress-couples

$$G_1 = \sigma G, \quad G_2 = G, \quad H_1 = H_2 = N_1 = N_2 = \chi = 0, \tag{5.2}$$

$$2G_\xi / G = 1 + \sigma - (1-\sigma)\cos 2\phi, \tag{5.3}$$
$$2G_\eta / G = 1 + \sigma + (1-\sigma)\cos 2\phi, \tag{5.4}$$
$$2H_\eta / G = (1-\sigma)\sin 2\phi. \tag{5.5}$$

It is therefore required to introduce a further stress-distribution such that on the edge of the hole

$$G_\eta = -\tfrac{1}{2} G\{1 + \sigma + (1-\sigma)\cos 2\phi\}, \tag{5.6}$$

$$H_\eta = -\tfrac{1}{2} G\{1-\sigma\}\sin 2\phi, \tag{5.7}$$

and in order to achieve this $V(\zeta)$ and $W(\zeta)$ are defined by the equations

$$\frac{2DV(\zeta)}{G} = (1+\sigma)\left\{\frac{\bar{a}_0 e^{i\zeta} F'(\zeta) + \bar{F}'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}\right\}$$
$$+ (1-\sigma)\left\{\frac{\bar{a}_0 e^{i\zeta} \bar{F}'(\zeta) + F'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}\right\}, \tag{5.8}$$

$$\frac{2DW(\zeta)}{G} = -(1+\sigma)\left\{\frac{\bar{a}_0 e^{i\zeta} F'(\zeta) - \bar{F}'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}\right\}$$
$$+ (1-\sigma)\left\{\frac{\bar{a}_0 e^{i\zeta} \bar{F}'(\zeta) - F'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}\right\}. \tag{5.9}$$

The complete stress-system is now obtained from equations (2.17) — (2.21) together with (5.3) — (5.5) provided that $n \leq 2$ when all the conditions are satisfied. When $n \geqslant 3$, however, these expressions for $V(\zeta)$ and $W(\zeta)$ are found to lead to infinite stress-couples at infinity, a difficulty which arises because the term of highest degree in $f''(z)$ is of $O(z^{n-3})$ and so is inadmissible if $n \geqslant 2$. The coefficient of this term vanishes identically, however, when $n = 2$, and the difficulty is overcome for values of $n \geqslant 3$ if the ratio of the coefficients of $e^{2i\zeta}$ in $V(\zeta)$ and $W(\zeta)$ is $\{n(1+\sigma)+1-\sigma\}/(2n-1+\sigma)$. This requires that $V(\zeta)$ and $W(\zeta)$ shall be modified by the addition of extra terms given by

$$\frac{DV(\zeta)}{GP} = (1-\sigma)\left\{\frac{\bar{a}_0 e^{2i\zeta}}{a_0} - \frac{a_0 e^{-2i\zeta}}{\bar{a}_0}\right\}$$
$$-(1-\sigma)\left\{\frac{\bar{a}_0 e^{2i\zeta}}{a_0} + \frac{a_0 e^{-2i\zeta}}{\bar{a}_0}\right\}\left\{\frac{\bar{a}_0 e^{i\zeta}}{\bar{F}'(\zeta)} - \frac{a_0 e^{-i\zeta}}{F'(\zeta)}\right\}$$
$$+(1+\sigma)Q\left\{\frac{\bar{a}_0 e^{i\zeta} F'(\zeta) + \bar{F}'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}\right\}, \qquad (5.10)$$

$$\frac{DW(\zeta)}{GP} = (1-\sigma)\left\{\frac{\bar{a}_0 e^{2i\zeta}}{a_0} + \frac{a_0 e^{-2i\zeta}}{\bar{a}_0}\right\}\left\{\frac{\bar{a}_0 e^{i\zeta}}{\bar{F}'(\zeta)} + \frac{a_0 e^{-i\zeta}}{F'(\zeta)} - 1\right\}$$
$$-(1+\sigma)Q\left\{\frac{\bar{a}_0 e^{i\zeta} F'(\zeta) - \bar{F}'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}\right\}, \qquad (5.11)$$

where
$$4P(1-\sigma-2n) = (n-2)(1-\sigma), \qquad (5.12)$$
$$Q = 1 + \frac{1-\sigma}{3+\sigma}\cdot\frac{b_n \bar{b}_n}{n a_0 \bar{a}_0}, \qquad (5.13)$$

and the total couple about the x-axis is now $G[1+2PQ]$.

Then the tangential couple on the boundary is the real part of

$$\frac{G_\xi}{G} = \frac{1+\sigma}{3+\sigma}\left\{1-\sigma+2\,\frac{(1+\sigma)a_0 e^{-i\zeta}-(1-\sigma)\bar{a}_0 e^{i\zeta}}{F'(\zeta)}\right\}, \qquad (5.14)$$

when $n \leq 2$, and when $n \geqslant 3$ it is necessary to add the real part of

$$\frac{4(1+\sigma)PQ}{3+\sigma} - \frac{2P(1-\sigma^2)}{3+\sigma}\left\{\frac{\bar{a}_0 e^{i\zeta}}{\bar{F}'(\zeta)}+\frac{a_0 e^{-i\zeta}}{F'(\zeta)}-1\right\}\left\{\frac{\bar{a}_0 e^{2i\zeta}}{a_0}+\frac{a_0 e^{-2i\zeta}}{\bar{a}_0}-\frac{1+\sigma}{1-\sigma}Q\right\},$$

(5.15)

corresponding to a couple $G[1+2PQ]$.

6. Equation (5.14) gives Goodier's results for a circular or elliptical boundary at once in the forms

$$\frac{G_\xi}{G} = (1+\sigma)\left\{1+\frac{2(1-\sigma)}{3+\sigma}\cos 2\eta\right\}, \qquad (6.1)$$

for the circle and

$$\frac{G_\xi}{G} = (1+\sigma)\left\{1-\frac{1-\sigma+(1+\sigma)e^{-2\alpha}}{3+\sigma}\cdot\frac{1-e^{2\alpha}\cos 2\eta}{\cosh 2\alpha-\cos 2\eta}\right\}, \qquad (6.2)$$

for the ellipse.

When the boundary is of the more general type, the tangential stress-couple for all values of $n \geqslant 2$ is given by the formula

$$\left\{n^2a^2-(n-2)m^2b^2\right\}\left\{a^2+b^2-2ab\cos(n+1)\xi\right\}\left\{\frac{(3+\sigma)G_\xi}{(1+\sigma)G}-2\right\}$$

$$\begin{aligned}= (1+\sigma)(a^2-b^2)\left\{n^2a^2-(n-2)m^2b^2\right\} \\ + 2mna^2\left\{na^2+(n-2)mb^2\right\}\cos 2(\xi-\beta) \\ - 4mna^3b\,(2n-1+\sigma)\cos\left[(n-1)\xi+2\beta\right],\end{aligned} \qquad (6\cdot3)$$

where G is the total couple at infinity and

$$(3+\sigma)\,m = 1-\sigma. \qquad (6\cdot4)$$

For a triangular hole with one altitude parallel to the x-axis, equation (6·3) leads to the expression

$$\frac{G_\xi}{G} = \frac{1+\sigma}{3+\sigma}\left\{2+\frac{3(1+\sigma)+4(1-\sigma)(2\cos 2\xi-\cos\xi)}{5-4\cos 3\xi}\right\}, \qquad (6\cdot5)$$

for the couple on the boundary, while equations (6·6) and (6·7) give the corresponding results for two different orientations of the square hole.

When the diagonals of the square are parallel to the axes it is found that

$$\frac{G_\xi}{G} = \frac{2(1+\sigma)}{3+\sigma}+\frac{1+\sigma}{2(3+\sigma)(7+2\sigma)}\left\{\frac{8(1+\sigma)(7+2\sigma)+27(1-\sigma)(3+\sigma)\cos 2\xi}{5-3\cos 4\xi}\right\}, \qquad (6\cdot6)$$

and when the sides of the square are parallel to the axes

$$\frac{G_\xi}{G} = \frac{2(1+\sigma)}{3+\sigma}+\frac{2(1+\sigma)}{(3+\sigma)(13+5\sigma)}\left\{\frac{2(1+\sigma)(13+5\sigma)+27(1-\sigma)(3+\sigma)\sin 2\xi}{5-3\cos 4\xi}\right\}. \qquad (6\cdot7)$$

The numerical values of these expressions are considered in § 8 after the corresponding equations for bending about the y-axis have been deduced.

CYLINDRICAL BENDING ABOUT THE y-AXIS.

7. When the plate is bent into the form of a cylinder whose generators are parallel to the y-axis, the displacement and couples, in the absence of the hole, are given by

$$w = -Gx^2/2D, \qquad G_1 = G, \quad G_2 = \sigma G. \tag{7.1}$$

and, when $n \leq 2$, it is necessary to write

$$\frac{2DV(\zeta)}{G} = (1+\sigma)\left\{\frac{\bar{a}_0 e^{i\zeta} F'(\zeta) + \bar{F}'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}\right\}$$

$$-(1-\sigma)\left\{\frac{\bar{a}_0 e^{i\zeta} \bar{F}'(\zeta) + F'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}\right\}, \tag{7.2}$$

$$\frac{2DW(\zeta)}{G} = -(1+\sigma)\left\{\frac{\bar{a}_0 e^{i\zeta} F'(\zeta) - \bar{F}'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}\right\}$$

$$-(1-\sigma)\left\{\frac{\bar{a}_0 e^{i\zeta} \bar{F}'(\zeta) - F'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}\right\}. \tag{7.3}$$

Extra terms will again be required when $n \geq 3$, and these are given by equations (7.4) and (7.5).

$$\frac{DV(\zeta)}{GP} = -(1-\sigma)\left\{\frac{\bar{a}_0 e^{2i\zeta}}{a_0} - \frac{a_0 e^{-2i\zeta}}{\bar{a}_0}\right\}$$

$$+(1-\sigma)\left\{\frac{\bar{a}_0 e^{2i\zeta}}{a_0} + \frac{a_0 e^{-2i\zeta}}{\bar{a}_0}\right\}\left\{\frac{\bar{a}_0 e^{i\zeta}}{\bar{F}'(\zeta)} - \frac{a_0 e^{-i\zeta}}{F'(\zeta)}\right\}$$

$$+(1+\sigma)Q\left\{\frac{\bar{a}_0 e^{i\zeta}F'(\zeta) + \bar{F}'(\zeta)\left[F'(\zeta) - a_0 e^{-i\zeta}\right]}{F'(\zeta)\bar{F}'(\zeta)}\right\}, \quad (7.4)$$

$$\frac{DW(\zeta)}{GP} = -(1-\sigma)\left\{\frac{\bar{a}_0 e^{2i\zeta}}{a_0} + \frac{a_0 e^{-2i\zeta}}{\bar{a}_0}\right\}\left\{\frac{\bar{a}_0 e^{i\zeta}}{\bar{F}'(\zeta)} + \frac{a_0 e^{-i\zeta}}{F'(\zeta)} - 1\right\}$$

$$-(1+\sigma)Q\left\{\frac{\bar{a}_0 e^{i\zeta}F'(\zeta) - \bar{F}'(\zeta)\left[F'(\zeta) - a_0 e^{-i\zeta}\right]}{F'(\zeta)\bar{F}'(\zeta)}\right\}, \quad (7.5)$$

where P, Q have the same values as before and the total couple about the y-axis is now $G[1+2PQ]$.

Equations (2.17) to (2.21) in conjunction with (7.1) determine the modified stress-distribution. On the edge of the hole the stress-couple is the real part of

$$\frac{G_\eta}{G} = \frac{1+\sigma}{3+\sigma}\left\{1-\sigma+2\,\frac{(1+\sigma)a_0 e^{-i\zeta} + (1-\sigma)\bar{a}_0 e^{i\zeta}}{F'(\zeta)}\right\}, \quad (7.6)$$

when $n \leq 2$, with extra terms arising from the real part of

$$\frac{4PQ(1+\sigma)}{3+\sigma} + \frac{2P(1-\sigma^2)}{3+\sigma}\left\{\frac{\bar{a}_0 e^{i\zeta}}{\bar{F}'(\zeta)} + \frac{a_0 e^{-i\zeta}}{F'(\zeta)} - 1\right\}\left\{\frac{\bar{a}_0 e^{2i\zeta}}{a_0} + \frac{a_0 e^{-2i\zeta}}{\bar{a}_0} + \frac{1+\sigma}{1-\sigma}\cdot Q\right\}, \quad (7.7)$$

when $n \geq 3$ and the total couple is $G[1+2PQ]$.

8. The result for the ellipse is

$$\frac{G_\xi}{G} = (1+\sigma)\left\{1 + \frac{1-\sigma-(1+\sigma)e^{-2\alpha}}{3+\sigma}\cdot\frac{1-e^{2\alpha}\cos 2\xi}{\cosh 2\alpha - \cos 2\xi}\right\}, \quad (8.1)$$

which agrees with the expression obtained by Goodier.

When the hole is of the more general shape and $n \geqslant 2$, the couple G_ξ along the boundary is given by

$$\{n^2a^2-(n-2)m^2b^2\}\{a^2+b^2-2ab\cos(n+1)\xi\}\left\{\frac{(3+\sigma)G_\xi}{(1+\sigma)G}-2\right\}$$

$$= (1+\sigma)(a^2-b^2)\{n^2a^2-(n-2)m^2b^2\}$$
$$-2mna^2\{na^2+(n-2)mb^2\}\cos 2(\xi-\beta)$$
$$+4mna^3b(2n-1+\sigma)\cos[(n-1)\xi+2\beta], \quad (8.2)$$

which reduces in the special case of the triangular hole to

$$\frac{G_\xi}{G} = \frac{1+\sigma}{3+\sigma}\left\{2 + \frac{3(1+\sigma)-4(1-\sigma)(2\cos 2\xi - \cos\xi)}{5-4\cos 3\xi}\right\}. \quad (8.3)$$

The results for the square and the circle are as given in §6.

The numerical values of these expressions are given in tables 7, 8 and 9, and are plotted in figs. 7, 8 and 9.

Table 7.

Concentration of stress-couple round a square hole bent about a diagonal.

ξ	$0°$	$10°$	$20°$	$30°$	$40°$
G_ξ/G	2.57	2.07	1.49	1.19	1.05
ξ	$50°$	$60°$	$70°$	$80°$	$90°$
G_ξ/G	0.978	0.935	0.910	0.894	0.887

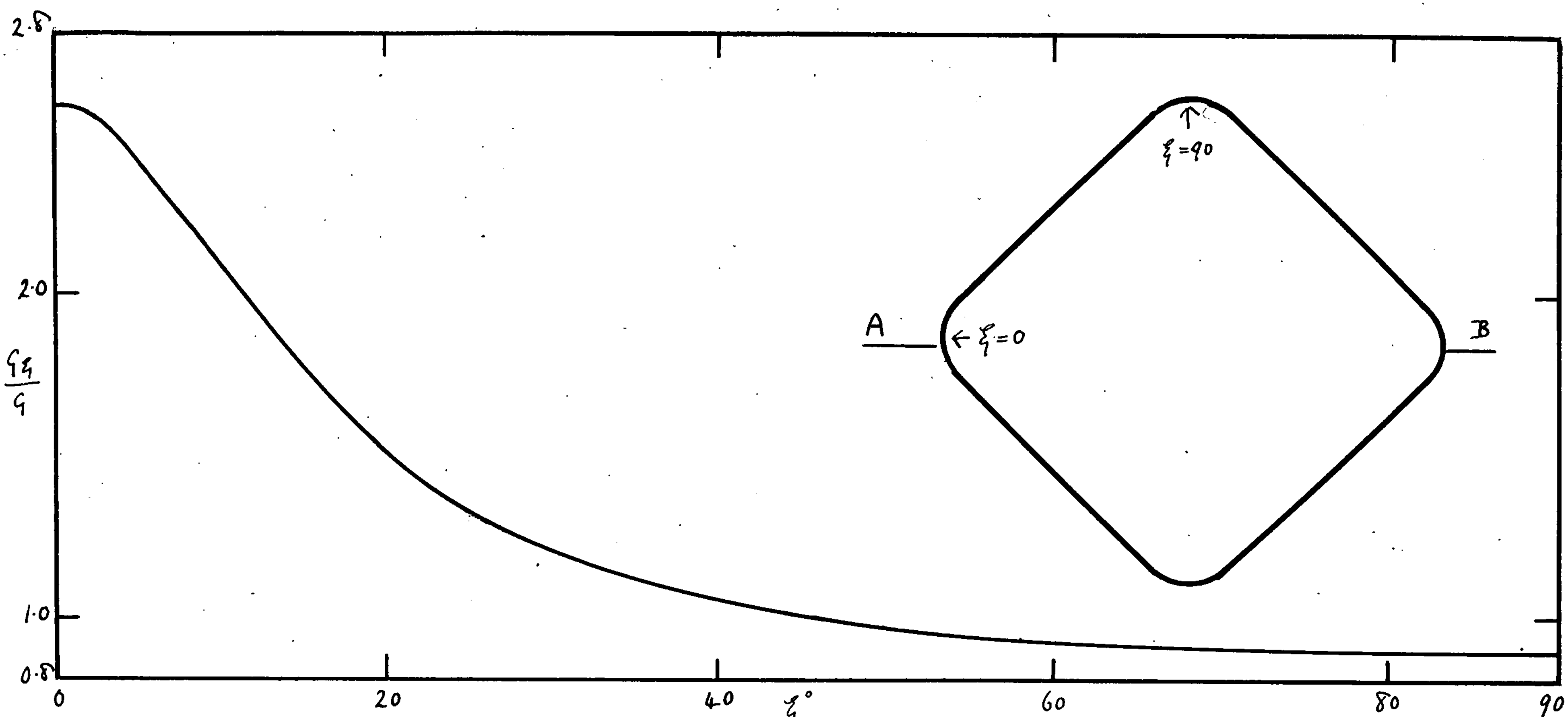

Fig. 7. Values of G_ξ on the edge of a square hole in a plate bent into the form of a cylinder about the diagonal AB.

Table 8.

Concentration of stress-couple round a triangular hole

ξ°	G_ξ. Bending about $\xi = 0$.	G_ξ. Bending about $\xi = 90^\circ$.
0	3.37	1.06
10	2.38	1.04
20	1.48	1.02
30	1.09	1.03
40	0.906	1.04
50	0.805	1.07
60	0.737	1.12
70	0.684	1.20
80	0.637	1.31
90	0.596	1.52
100	0.594	1.90
110	0.814	2.60
112	–	2.75
115	–	2.91
120	1.63	2.79
125	1.93	–
130	1.93	1.49
140	1.68	0.822
150	1.49	0.627
160	1.38	0.568
170	1.33	0.549
180	1.31	0.545

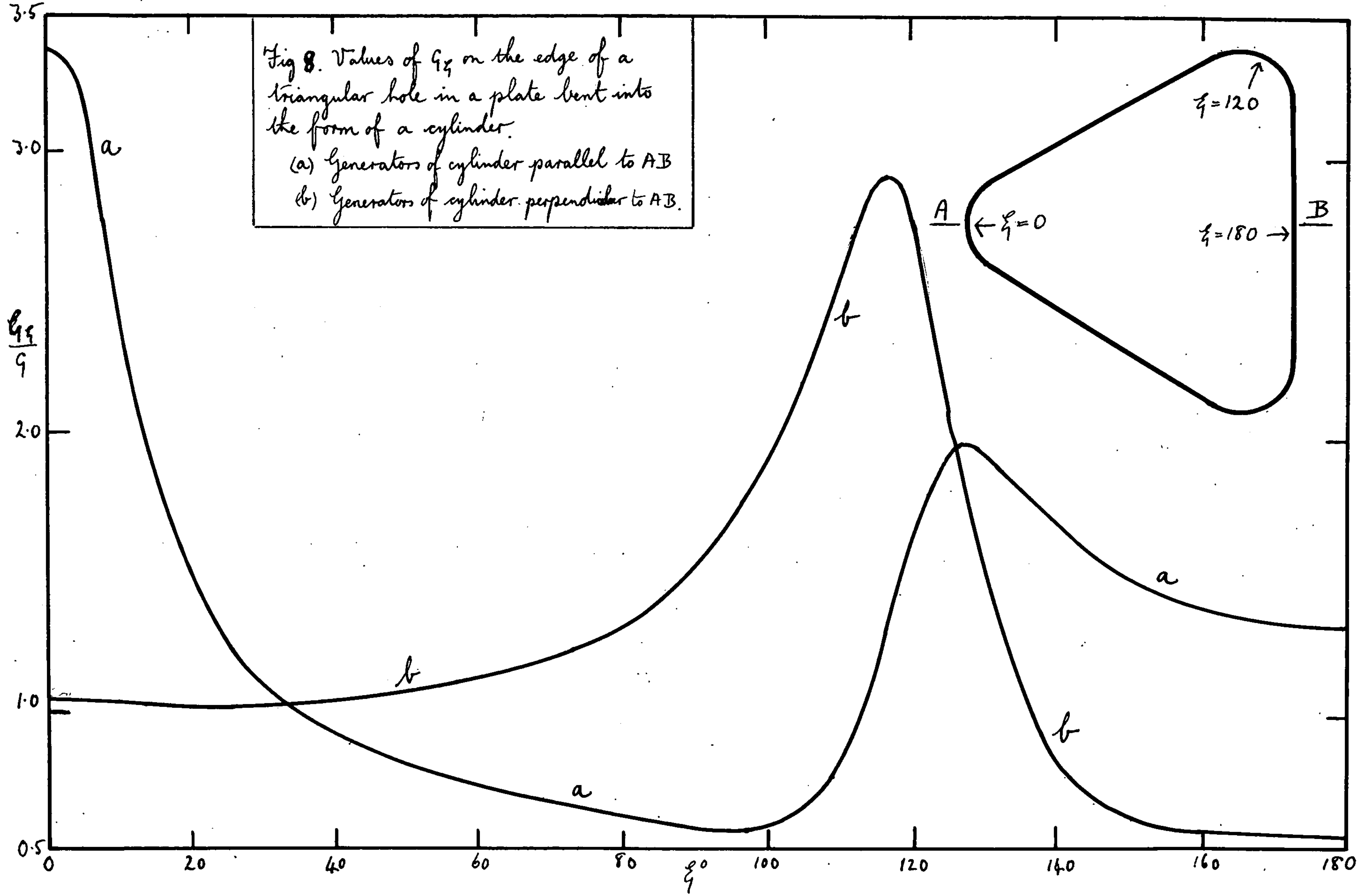

Fig 8. Values of G_ξ on the edge of a triangular hole in a plate bent into the form of a cylinder.
(a) Generators of cylinder parallel to AB
(b) Generators of cylinder perpendicular to AB.

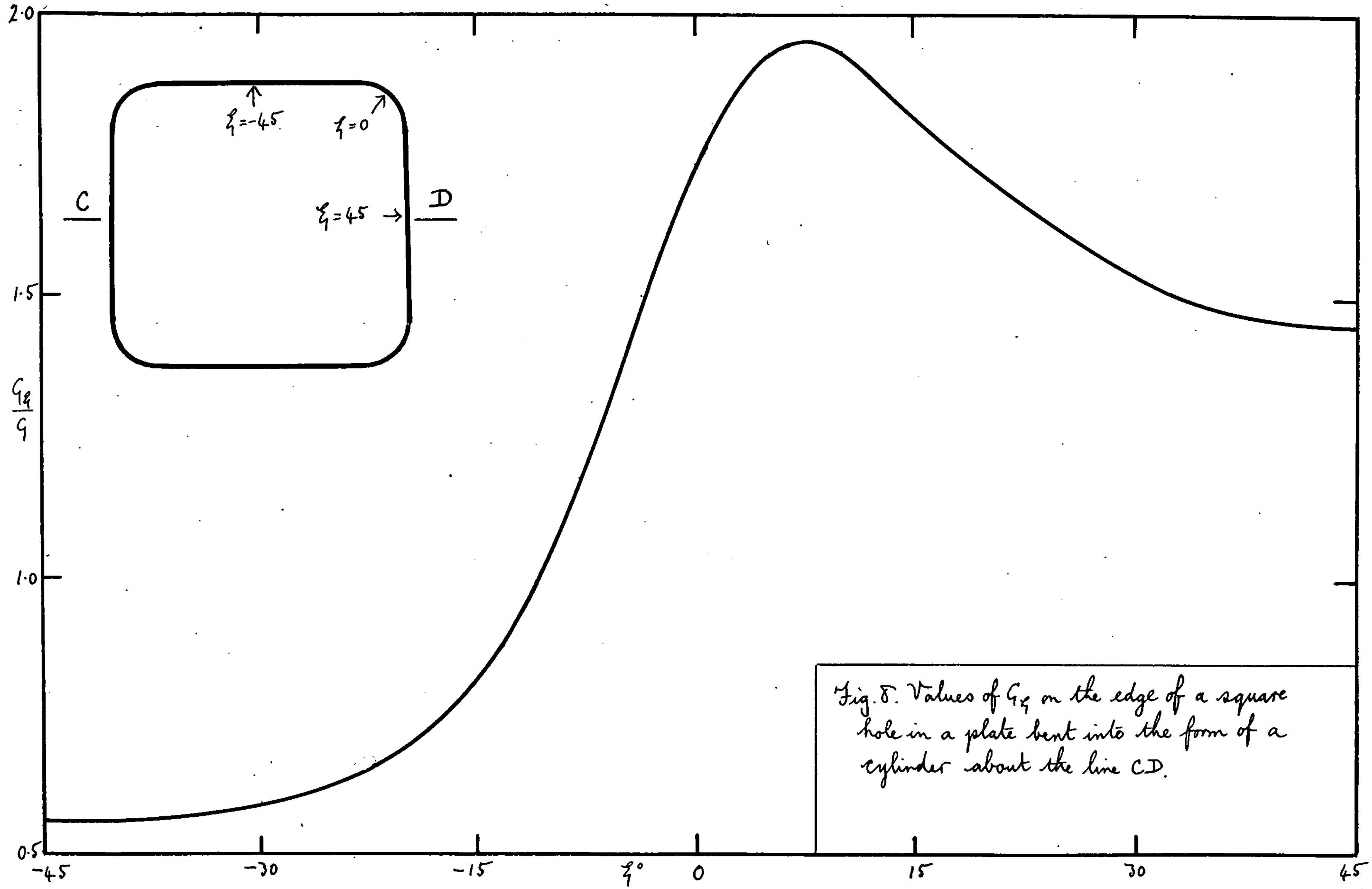

Fig. 8. Values of G_η on the edge of a square hole in a plate bent into the form of a cylinder about the line CD.

Table 9

Concentration of stress-couple round a square hole bent about the line $\xi = 45°$, which is the direction of two sides of the square.

ξ	$-45°$	$-40°$	$-30°$	$-20°$	$-10°$	$0°$
G_ξ/G	0·566	0·568	0·592	0·689	1·03	1·73
ξ	$10°$	$20°$	$30°$	$40°$	$45°$	
G_ξ/G	1·93	1·71	1·54	1·46	1·45	

THE TWISTED PLATE

9. When the unperforated plate is acted upon by torsional couples H, the displacement of the middle surface is

$$w = Hxy/D(1-\sigma), \tag{9·1}$$

and the stress-distribution is given by

$$H_1 = -H_2 = H, \quad G_1 = G_2 = N_1 = N_2 = \gamma = 0, \tag{9·2}$$

$$G_\xi = -H\sin 2\phi, \quad G_\eta = H\sin 2\phi, \quad H_\xi = -H_\eta = H\cos 2\phi. \tag{9·3}$$

In order to satisfy the boundary conditions for a number of transformations it is necessary to write

$$\frac{DV(\zeta)}{iH} = \frac{\bar{a}_0 e^{i\zeta}\bar{F}'(\zeta) - F'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}, \tag{9·4}$$

$$\frac{DW(\zeta)}{iH} = \frac{\bar{a}_0 e^{i\zeta}\bar{F}'(\zeta) + F'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}, \tag{9·5}$$

from which the complete stress-distribution may be found as before.

When $n \geq 3$, the necessary extra terms to avoid the introduction of infinite stress-couples at infinity are

$$\frac{DV(\zeta)}{iH} = 2P\left\{\frac{\bar{a}_0 e^{2i\zeta}}{a_0} + \frac{a_0 e^{-2i\zeta}}{\bar{a}_0}\right\}$$

$$-2P\left\{\frac{\bar{a}_0 e^{2i\zeta}}{a_0} - \frac{a_0 e^{-2i\zeta}}{\bar{a}_0}\right\}\left\{\frac{\bar{a}_0 e^{i\zeta}}{\bar{F}'(\zeta)} - \frac{a_0 e^{-i\zeta}}{F'(\zeta)}\right\}, \qquad (9\cdot6)$$

$$\frac{DW(\zeta)}{iH} = 2P\left\{\frac{\bar{a}_0 e^{2i\zeta}}{a_0} - \frac{a_0 e^{-2i\zeta}}{\bar{a}_0}\right\}\left\{\frac{\bar{a}_0 e^{i\zeta}}{\bar{F}'(\zeta)} + \frac{a_0 e^{-i\zeta}}{F'(\zeta)} - 1\right\}, \qquad (9\cdot7)$$

where P, Q have the same values as before and the total twisting couple is now $H[1+2PQ]$.

On the boundary of the hole, the stress-couple is the real part of

$$\frac{G_\xi}{H} = \frac{-4i(1+\sigma)\,\bar{a}_0 e^{i\zeta}}{(3+\sigma)\,F'(\zeta)}, \qquad (9\cdot8)$$

when $n = 0$ or 2, and the extra portion when $n \geq 3$ is the real part of the expression

$$-\frac{4i(1+\sigma)P}{3+\sigma}\left\{\frac{\bar{a}_0 e^{2i\zeta}}{a_0} - \frac{a_0 e^{-2i\zeta}}{\bar{a}_0}\right\}\left\{\frac{\bar{a}_0 e^{i\zeta}}{\bar{F}'(\zeta)} + \frac{a_0 e^{-i\zeta}}{F'(\zeta)} - 1\right\}, \qquad (9\cdot9)$$

corresponding to a couple $H[1+2PQ]$.

The expressions (9·4) and (9·5) are found to produce many-valued displacements when $n=1$, that is in the case of the ellipse, and so some modification is required for this particular boundary.

It is found necessary to write as <u>additional</u> terms to (9·4) and (9·5)

$$\frac{DV(\zeta)}{iH} = K\left\{\frac{\bar{a}_0 e^{i\zeta}F'(\zeta) - \bar{F}'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}\right\}, \qquad (9\cdot10)$$

$$\frac{DW(\zeta)}{iH} = -K\left\{\frac{\bar{a}_0 e^{i\zeta}F'(\zeta) + \bar{F}'(\zeta)[F'(\zeta) - a_0 e^{-i\zeta}]}{F'(\zeta)\bar{F}'(\zeta)}\right\}, \qquad (9\cdot11)$$

where

$$K\left\{(3+\sigma)e^{2\alpha} + (1-\sigma)e^{-2\alpha}\right\} = 4, \qquad (9\cdot12)$$

and $\operatorname{sech}\alpha$ is the eccentricity of the ellipse.

It will be noticed that this involves adding an imaginary constant to $W(\zeta)$ and therefore a real constant to the value of $H_\eta + \chi$ on the boundary. This is immediately seen to be physically admissible as $H_\eta + \chi$ on the boundary in general involves an indeterminate real constant.

The extra stress-couple on the boundary is given by the real part of

$$\frac{G_\xi}{H} = -\frac{4iK(1+\sigma)a_0 e^{-i\zeta}}{(3+\sigma)F'(\zeta)}. \qquad (9\cdot13)$$

10. The stresses round a circular or elliptical boundary are

$$\frac{G_\xi}{H} = -\frac{4(1+\sigma)\sin 2\xi}{3+\sigma}, \qquad (10\cdot1)$$

for the circle and

$$\frac{G_\xi}{H} = -\frac{4(1+\sigma)\sinh 2\alpha \sin 2\xi}{\{3+\sigma+(1-\sigma)e^{-4\alpha}\}\{\cosh 2\alpha - \cos 2\xi\}}, \qquad (10\cdot2)$$

for the ellipse, both results agreeing with those given by Goodier.

When the boundary is of the more general type and $n \geqslant 2$ the tangential stress-couple is given by

$$
\begin{aligned}
&\left\{n^2a^2(3+\sigma)^2-(n-2)(1-\sigma)^2b^2\right\}\left\{a^2+b^2-2ab\cos(n+1)\xi\right\}G_\xi/H \\
&= -4(1+\sigma)na^2\left\{(3+\sigma)na^2-(n-2)(1-\sigma)b^2\right\}\sin 2(\xi-\beta) \\
&\qquad +8(1+\sigma)(2n-1+\sigma)na^3b\sin\left\{(n-1)\xi+2\beta\right\},
\end{aligned}
\tag{10.3}
$$

which reduces in the special case of the equilateral triangle to

$$\frac{G_\xi}{H} = -\frac{8(1+\sigma)(\sin\xi+2\sin 2\xi)}{(3+\sigma)(5-4\cos 3\xi)}, \tag{10.4}$$

For a square with one diagonal parallel to the x-axis the result is

$$\frac{G_\xi}{H} = -\frac{108(1+\sigma)\sin 2\xi}{(13+5\sigma)(5-3\cos 4\xi)}, \tag{10.5}$$

and the result for a square with its sides parallel to the axes is

$$\frac{G_\xi}{H} = -\frac{27(1+\sigma)\cos 2\xi}{(7+2\sigma)(5-3\cos 4\xi)}. \tag{10.6}$$

The numerical values of the stress-concentration in the case of square or triangular boundaries are given in tables 10, 11 and 12, and are plotted in figs. 10, 11 and 12.

Table 10

Concentration of stress-couple round the edge of a square hole in a plate twisted about a diagonal of the square.

ξ	0°	5°	10°	15°	20°
G_ξ/H	0	0·754	1·20	1·35	1·36
ξ	25°	30°	35°	40°	45°
G_ξ/H	1·31	1·26	1·22	1·19	1·18

Table 11

Concentration of stress-couple round the edge of a square hole in a plate twisted about the sides of the square.

ξ	0°	10°	20°	30°	40°	45°
G_ξ/H	2·25	1·57	0·770	0·346	0·0999	0.

Table 12

Concentration of stress-couple round the edge of a triangular hole in a twisted plate.

ξ	0°	10°	13°	15°	20°	30°	40°	50°
G_ξ/H	0	1·72	1·79	1·78	1·67	1·37	1·15	0·995
ξ	60°	70°	80°	90°	100°	110°	120°	123°
G_ξ/H	0·888	0·809	0·734	0·615	0·309	−0·693	−2·66	−2·90
ξ	125°	130°	140°	150°	160°	170°	180°	
G_ξ/H	−2·87	−2·41	−1·36	−0·758	−0·414	−0·186	0	

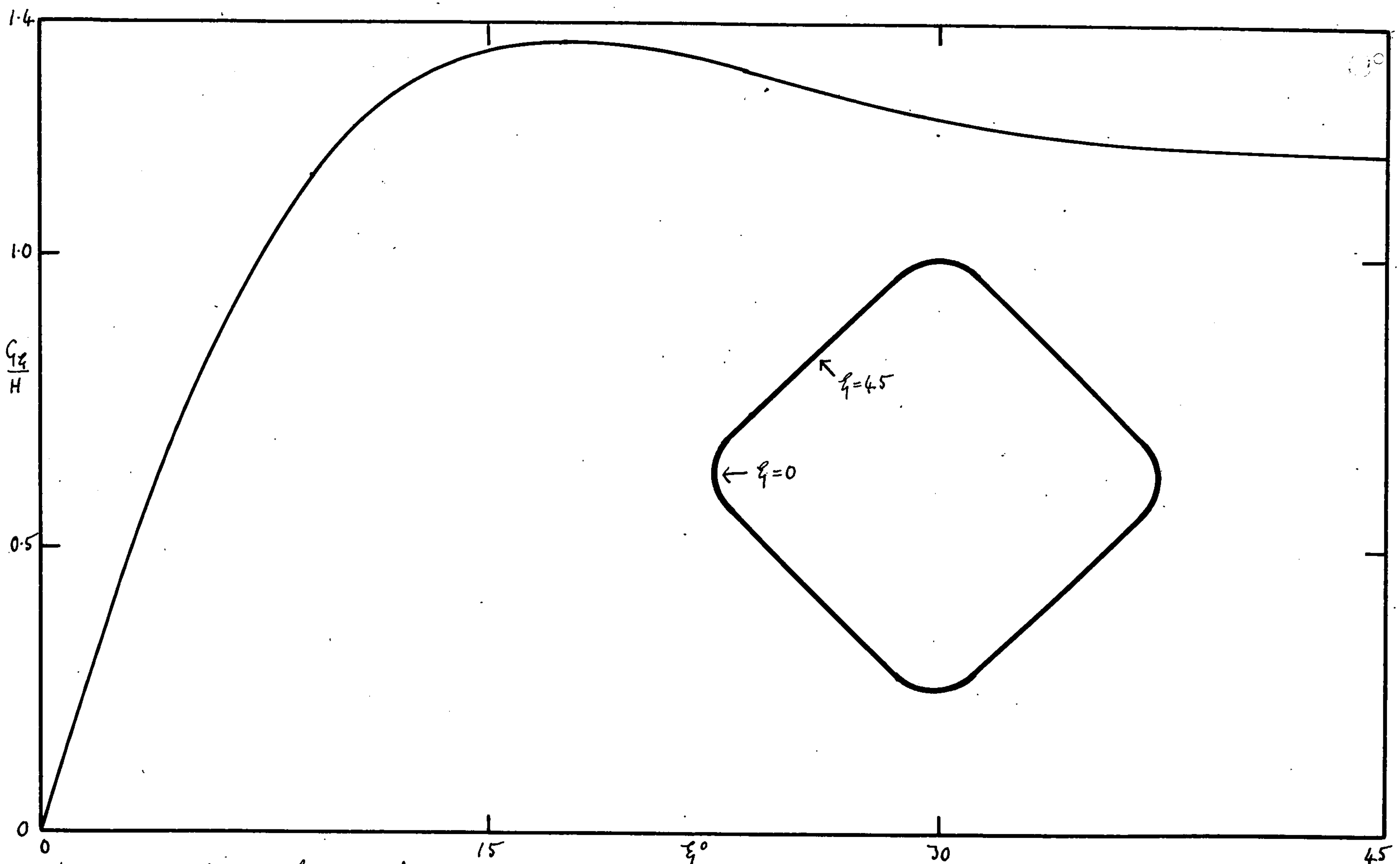

Fig. 10. Values of G_ξ on the edge of a square hole in a plate twisted about the diagonals of the square.

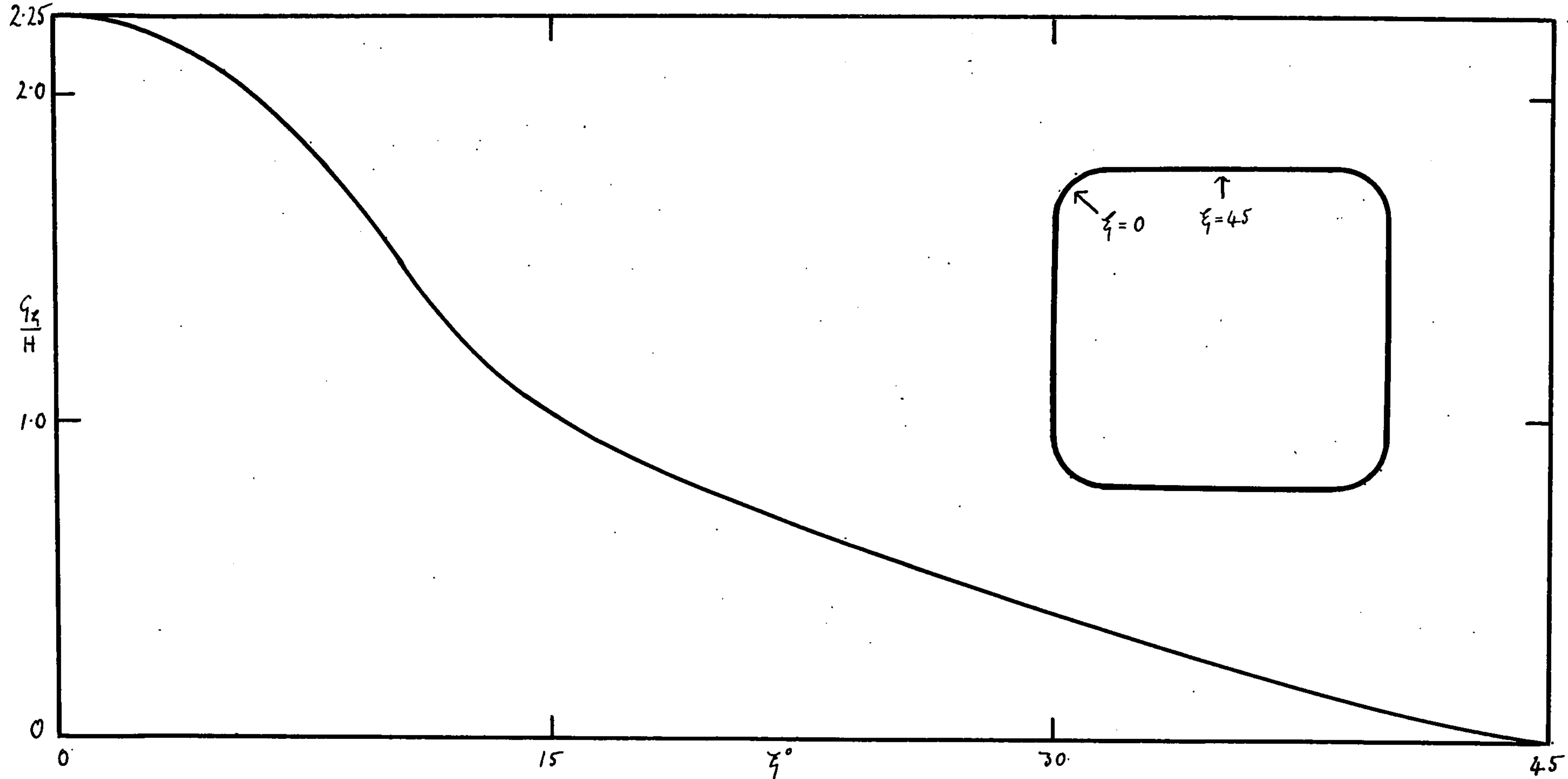

Fig. 11. Values of G_ξ on the edge of a square hole in a plate twisted about the sides of the square.

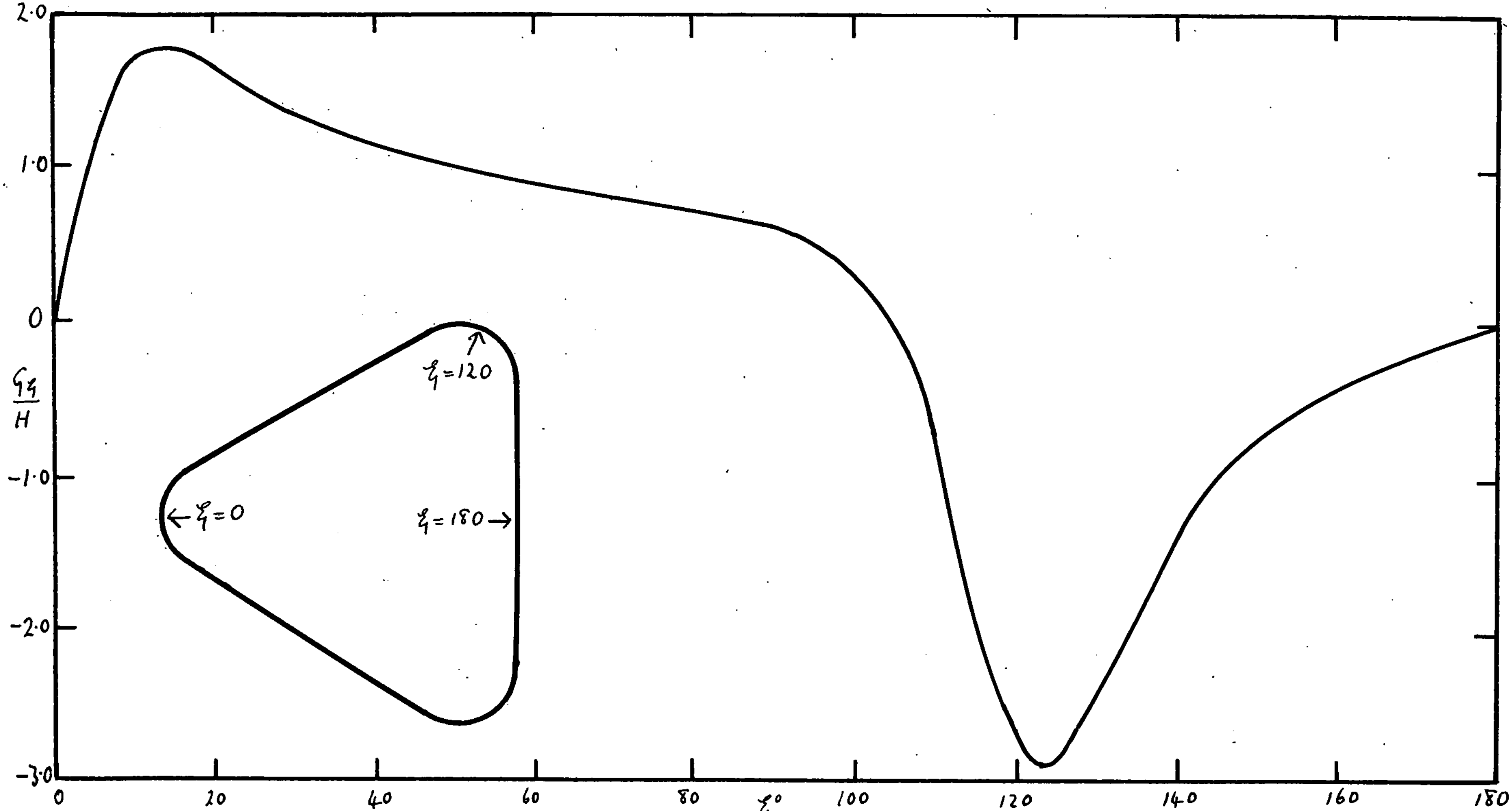

Fig. 12 Values of G_ξ on the edge of a triangular hole in a twisted plate.

D. The Transverse Flexure of Perforated Aeolotropic Plates

INTRODUCTION.

1. The method of section C is now extended to apply to aeolotropic plates containing circular or elliptical holes, and solutions are obtained in a number of fundamental cases for the bending and twisting of such a plate. Although some of these solutions are unwieldy they are all obtained in finite form.

No previous attempt appears to have been made to solve this group of problems either theoretically or experimentally, although Green (1942) has dealt with the problem of a perforated aeolotropic plate under the action of various forces in its own plane. The two problems are analytically similar in that each involves finding an appropriate solution of a fourth order partial differential equation, and the method used is based on Green's method for the plane problem. When the forces act in the plane of the plate, the solution of the equation is χ, Airy's stress-function, from which the displacements are derived by differentiation. It is possible therefore for the equation to have many-valued solutions provided that the corresponding displacements are single-valued and for a number of problems Green used many-valued stress-functions. The differential equation for a bent plate, on the other hand, has to be satisfied by the transverse displacement itself and so only single-valued solutions are admissible, a fact which makes the solution of certain bending problems a little more obscure.

As before, the complex variable is employed throughout but each boundary is dealt with separately, plane polar coordinates being used for the circular hole, and the transformation

$$z = c \cos(\zeta + i\alpha), \qquad (1 \cdot 1)$$

for the elliptical hole.

The general theory is first of all developed for an aeolotropic material of any nature, but in the subsequent work it is assumed that the material has two perpendicular directions of symmetry. Solutions are then obtained in the case of the circular hole when the plate is bent cylindrically about axes parallel to these directions or twisted by torsional couples about these axes. In the case of the elliptical hole, one example only is given. The principal axes of the ellipse are taken to be parallel to the directions of symmetry of the material and couples are applied in order to bend the plate into the form of a cylinder about the major-axis of the ellipse.

Some numerical work is carried out for a specimen of spruce wood, which is highly aeolotropic. It is found that the tangential stress-couple along the boundary is considerably modified by the aeolotropy of the material, although the maximum value of the stress concentration factor is in no case as great as the corresponding maximum in the plane tension problem.

THE GENERAL DISPLACEMENT EQUATION.

2. Co-ordinate axes are defined in the same manner as in the isotropic case, and, with a similar notation for the various stress-couples it is known that

$$G_1 = -\frac{2h^2}{3}\left\{c_{11}\frac{\partial^2 w}{\partial x^2} + c_{12}\frac{\partial^2 w}{\partial y^2} + 2c_{16}\frac{\partial^2 w}{\partial x \partial y}\right\}, \qquad (2\cdot1)$$

$$G_2 = -\frac{2h^2}{3}\left\{c_{12}\frac{\partial^2 w}{\partial x^2} + c_{22}\frac{\partial^2 w}{\partial y^2} + 2c_{26}\frac{\partial^2 w}{\partial x \partial y}\right\}, \qquad (2\cdot2)$$

$$H_1 = -H_2 = \frac{2h^2}{3}\left\{c_{16}\frac{\partial^2 w}{\partial x^2} + c_{26}\frac{\partial^2 w}{\partial y^2} + 2c_{66}\frac{\partial^2 w}{\partial x \partial y}\right\}, \qquad (2\cdot3)$$

where $2h$ is the thickness of the plate and the 'c''s are the elastic constants defined in the usual way.

Using equations $(2\cdot1)-(2\cdot3)$ and the equations of equilibrium, the general equation for the transverse displacement is obtained in the form

$$\frac{\partial^4 w}{\partial x^4} + k_1\frac{\partial^4 w}{\partial x^3 \partial y} + k_2\frac{\partial^4 w}{\partial x^2 \partial y^2} + k_3\frac{\partial^4 w}{\partial x \partial y^3} + k_4\frac{\partial^4 w}{\partial y^4} = \frac{p}{D}, \qquad (2\cdot4)$$

where

$$k_1 = \frac{4c_{16}}{c_{11}}, \quad k_2 = \frac{2(c_{12}+2c_{66})}{c_{11}}, \qquad (2\cdot5)$$

$$k_3 = \frac{4c_{26}}{c_{11}}, \quad k_4 = \frac{c_{22}}{c_{11}}, \quad D = \frac{2h^3 c_{11}}{3}, \qquad (2\cdot6)$$

and p is the normal load per unit area on the xy plane.

This equation for the displacement has already been given by Huber (1938).

SOLUTION OF THE GENERAL EQUATION.

3. When the normal load is everywhere zero, it is possible to obtain physically admissible solutions of equation (2·4) provided that the roots of

$$c_{11}\rho^4 + 4c_{16}\rho^3 + 2(c_{12}+2c_{66})\rho^2 + 4c_{26}\rho + c_{22} = 0, \qquad (3\cdot1)$$

are all complex.

Equation (2·4) may then be written in the form

$$\left(\frac{\partial^2}{\partial x^2} + l_1\frac{\partial^2}{\partial x\partial y} + \eta_1\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + l_2\frac{\partial^2}{\partial x\partial y} + \eta_2\frac{\partial^2}{\partial y^2}\right)\omega = 0, \quad (3\cdot2)$$

where

$$\eta_1 + \eta_2 + l_1 l_2 = 2(c_{12}+2c_{66})/c_{11}, \qquad \eta_1\eta_2 = c_{22}/c_{11}, \qquad (3\cdot3)$$

$$l_1 + l_2 = \frac{4c_{16}}{c_{11}}, \qquad l_1\eta_2 + l_2\eta_1 = \frac{4c_{26}}{c_{11}}, \qquad (3\cdot4)$$

and equation (3·2) then has a general solution

$$\omega = f\{z + (\lambda_1 + i\mu_1)\bar{z}\} + g\{z + (\lambda_2 + i\mu_2)\bar{z}\}, \qquad (3\cdot5)$$

where $$z = x + iy, \quad \bar{z} = x - iy, \tag{3.6}$$
and $f\{z+(\lambda_1+i\mu_1)\bar{z}\}$ and $g\{z+(\lambda_2+i\mu_2)\bar{z}\}$ are regular functions outside the circle of the order of logarithmic terms at the most. The constants λ_1, etc. are related to l_1, etc. by the expressions

$$\lambda_1 = \frac{\eta_1 - 1}{\eta_1 + 1 + 2(\eta_1 - \frac{1}{4}l_1^2)^{1/2}}, \quad \mu_1 = \frac{-l_1}{\eta_1 + 1 + 2(\eta_1 - \frac{1}{4}l_1^2)^{1/2}}, \tag{3.7}$$

$$\lambda_2 = \frac{\eta_2 - 1}{\eta_2 + 1 + 2(\eta_2 - \frac{1}{4}l_2^2)^{1/2}}, \quad \mu_2 = \frac{-l_2}{\eta_2 + 1 + 2(\eta_2 - \frac{1}{4}l_2^2)^{1/2}}, \tag{3.8}$$

where it may be convenient to express the relations in the forms

$$\eta_1 = \frac{(1+\lambda_1)^2 + \mu_1^2}{(1-\lambda_1)^2 + \mu_1^2}, \quad \frac{l_1}{\eta_1 - 1} = -\frac{\mu_1}{\lambda_1}, \tag{3.9}$$

$$\eta_2 = \frac{(1+\lambda_2)^2 + \mu_2^2}{(1-\lambda_2)^2 + \mu_2^2}, \quad \frac{l_2}{\eta_2 - 1} = -\frac{\mu_2}{\lambda_2}. \tag{3.10}$$

Equations (2.1) etc. then lead directly to the following relations in which dashes denote differentiations.

$$\begin{aligned}\frac{G_1}{D} = &\left\{\frac{c_{12}}{c_{11}}(1-\lambda_1-i\mu_1)^2 - (1+\lambda_1+i\mu_1)^2 - 2i\frac{c_{16}}{c_{11}}\left[1-(\lambda_1+i\mu_1)^2\right]\right\} \\ &\qquad\qquad \times f''\{z+(\lambda_1+i\mu_1)\bar{z}\} \\ &+\left\{\frac{c_{12}}{c_{11}}(1-\lambda_2-i\mu_2)^2 - (1+\lambda_2+i\mu_2)^2 - \frac{2ic_{16}}{c_{11}}\left[1-(\lambda_2+i\mu_2)^2\right]\right\} \\ &\qquad\qquad \times g''\{z+(\lambda_2+i\mu_2)\bar{z}\},\end{aligned} \tag{3.11}$$

$$\frac{G_2}{D} = \left\{ \frac{c_{22}}{c_{11}}\left(1-\lambda_1-i\mu_1\right)^2 - \frac{c_{12}}{c_{11}}\left(1+\lambda_1+i\mu_1\right)^2 - 2i\,\frac{c_{26}}{c_{11}}\left[1-\left(\lambda_1+i\mu_1\right)^2\right] \right\}$$

$$\times\, f''\left\{z+\left(\lambda_1+i\mu_1\right)\bar{z}\right\}$$

$$+ \left\{ \frac{c_{22}}{c_{11}}\left(1-\lambda_2-i\mu_2\right)^2 - \frac{c_{12}}{c_{11}}\left(1+\lambda_2+i\mu_2\right)^2 - 2i\,\frac{c_{26}}{c_{11}}\left[1-\left(\lambda_2+i\mu_2\right)^2\right] \right\}$$

$$\times\, g''\left\{z+\left(\lambda_2+i\mu_2\right)\bar{z}\right\}, \qquad (3.12)$$

$$\frac{H_1}{D} = \left\{ 2i\,\frac{c_{66}}{c_{11}}\left[1-\left(\lambda_1+i\mu_1\right)^2\right] + \frac{c_{16}}{c_{11}}\left(1+\lambda_1+i\mu_1\right)^2 - \frac{c_{26}}{c_{11}}\left(1-\lambda_1-i\mu_1\right)^2 \right\}$$

$$\times\, f''\left\{z+\left(\lambda_1+i\mu_1\right)\bar{z}\right\}$$

$$+ \left\{ 2i\,\frac{c_{66}}{c_{11}}\left[1-\left(\lambda_2+i\mu_2\right)^2\right] + \frac{c_{16}}{c_{11}}\left(1+\lambda_2+i\mu_2\right)^2 - \frac{c_{26}}{c_{11}}\left(1-\lambda_2-i\mu_2\right)^2 \right\}$$

$$\times\, g''\left\{z+\left(\lambda_2+i\mu_2\right)\bar{z}\right\}, \qquad (3.13)$$

$$\frac{N_1}{D} = \left\{ i\left(\frac{\eta_1-\eta_2}{2\eta_1}\right)\left(\eta_1-\frac{1}{4}\ell_1^2\right) + \left(\eta_1-\frac{1}{4}\ell_1^2\right)^{1/2}\left[\ell_1\left(\eta_1+\eta_2\right)-2\ell_2\eta_1\right]\frac{1}{4\eta_1} \right\}$$

$$\times\, i\left(1-\lambda_1-i\mu_1\right)^2\left(1+\lambda_1+i\mu_1\right) f'''\left\{z+\left(\lambda_1+i\mu_1\right)\bar{z}\right\}$$

$$+ \left\{ i\left(\frac{\eta_2-\eta_1}{2\eta_2}\right)\left(\eta_2-\frac{1}{4}\ell_2^2\right) + \left(\eta_2-\frac{1}{4}\ell_2^2\right)^{1/2}\left[\ell_2\left(\eta_1+\eta_2\right)-2\ell_1\eta_2\right]\frac{1}{4\eta_2} \right\}$$

$$\times\, i\left(1-\lambda_2-i\mu_2\right)^2\left(1+\lambda_2+i\mu_2\right) g'''\left\{z+\left(\lambda_2+i\mu_2\right)\bar{z}\right\}, \qquad (3.14)$$

$$\frac{N_2}{D} = -\left\{i\left(\frac{\eta_1-\eta_2}{2\eta_1}\right)\left(\eta_1-\frac{1}{4}l_1^2\right)+\left(\eta_1-\frac{1}{4}l_1^2\right)^{\frac{1}{2}}\left[l_1(\eta_1+\eta_2)-2l_2\eta_1\right]\frac{1}{4\eta_1}\right\}$$

$$\times\left(1-\lambda_1-i\mu_1\right)\left(1+\lambda_1+i\mu_1\right)^2 f'''\{z+(\lambda_1+i\mu_1)\bar{z}\}$$

$$-\left\{i\left(\frac{\eta_2-\eta_1}{2\eta_2}\right)\left(\eta_2-\frac{1}{4}l_2^2\right)+\left(\eta_2-\frac{1}{4}l_2^2\right)^{\frac{1}{2}}\left[l_2(\eta_1+\eta_2)-2l_1\eta_2\right]\frac{1}{4\eta_2}\right\}$$

$$\times\left(1-\lambda_2-i\mu_2\right)\left(1+\lambda_1+i\mu_1\right)^2 g'''\{z+(\lambda_2+i\mu_2)\bar{z}\}. \qquad (3\cdot15)$$

When the normal load is everywhere zero one of the equations of equilibrium referred to in §2 is

$$\frac{\partial N_1}{\partial x}+\frac{\partial N_2}{\partial y}=0, \qquad (3\cdot16)$$

and therefore there exists a function ψ such that

$$N_1 = -\frac{\partial\psi}{\partial y}, \quad N_2 = \frac{\partial\psi}{\partial x}. \qquad (3\cdot17)$$

One of the boundary conditions is now easily seen to reduce, as in the isotropic case, to

$$H_\eta + \psi = \text{constant} \qquad (3\cdot18)$$

and it is found from equations (3·14) and (3·15) that

$$\frac{\psi}{D} = -\frac{1}{4\eta_1}\left(\eta_1-\frac{1}{4}l_1^2\right)^{\frac{1}{2}}\left\{l_1(\eta_1+\eta_2)-2l_2\eta_1+2i(\eta_1-\eta_2)\left(\eta_1-\frac{1}{4}l_1^2\right)^{\frac{1}{2}}\right\}$$

$$\times\left\{1-(\lambda_1+i\mu_1)^2\right\} f''\{z+(\lambda_1+i\mu_1)\bar{z}\}$$

$$-\frac{1}{4\eta_2}\left(\eta_2-\frac{1}{4}l_2^2\right)^{\frac{1}{2}}\left\{l_2(\eta_1+\eta_2)-2l_1\eta_2+2i(\eta_2-\eta_1)\left(\eta_2-\frac{1}{4}l_2^2\right)^{\frac{1}{2}}\right\}$$

$$\times\left\{1-(\lambda_2+i\mu_2)^2\right\} g''\{z+(\lambda_2+i\mu_2)\bar{z}\}. \qquad (3\cdot19)$$

From these fundamental equations the stress-couples and resultants in other directions may be obtained from the well-known formulae.

4. The relations of § 3 are valid whatever the nature of the aeolotropy of the plate, but the subsequent analysis is simplified by taking both c_{16} and c_{26} to be zero, which means that the material has two perpendicular directions of symmetry. From equation (3·4) it is seen that this restriction leads to two possible forms of equation (3·2). The possibilities are

(a) $l_1 = l_2 = 0$, and η_1, η_2 are both real and positive. From equations (3·9) and (3·10) it is found that

$$\mu_1 = \mu_2 = 0, \tag{4·1}$$

and the solution is

$$w = f(z+\lambda_1 \bar{z}) + g(z+\lambda_2 \bar{z}), \tag{4·2}$$

where λ_1, λ_2 are real.

(b) $l_1 = -l_2 = l, \quad \eta_1 = \eta_2 = \epsilon, \quad$ where $\epsilon > \frac{1}{4} l^2$ (4·3)

From equations (3·9) and (3·10)

$$\lambda_1 = \lambda_2 = \lambda, \quad \mu_1 = -\mu_2 = \mu, \tag{4·4}$$

and the general solution is

$$w = f\{z+(\lambda+i\mu)\bar{z}\} + g\{z+(\lambda-i\mu)\bar{z}\}. \tag{4·5}$$

In either case the equation may be written in the form

$$\left(\frac{\partial^2}{\partial x^2} + \epsilon_1 \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \epsilon_2 \frac{\partial^2}{\partial y^2}\right) w = 0, \tag{4·6}$$

where ϵ_1, ϵ_2 may be complex, and the solution is expressible in the form (4·2) where λ_1, λ_2 are either real or complex conjugate quantities.

The analysis is reproduced in terms of ϵ_1, ϵ_2 and λ_1, λ_2 and the relations may then be taken to apply to case (a) immediately. The corresponding equations for case (b) are obtainable directly in each instance by writing

$$\lambda_1 = \lambda + i\mu, \quad \lambda_2 = \lambda - i\mu, \tag{4·7}$$

$$\epsilon_1 = \epsilon - \tfrac{1}{2}l^2 - il\left(\epsilon - \tfrac{1}{4}l^2\right)^{\frac{1}{2}}; \quad \epsilon_2 = \epsilon - \tfrac{1}{2}l^2 + il\left(\epsilon - \tfrac{1}{4}l^2\right)^{\frac{1}{2}}. \tag{4·8}$$

THE PROBLEM OF THE CIRCULAR HOLE.

5. It will be assumed that the centre of the circular hole is the origin of the cartesian coordinates and that the axis of x is the direction $\theta = 0$. The circular boundary is then given by $|z| = a$.

With the conditions stated in §4, the various stress-couples and shearing forces are given by the following equations.

$$\frac{G_1}{D} = \left\{\frac{c_{12}}{c_{11}}(1-\lambda_1)^2 - (1+\lambda_1)^2\right\} f''(z+\lambda_1\bar{z})$$

$$+ \left\{\frac{c_{12}}{c_{11}}(1-\lambda_2)^2 - (1+\lambda_2)^2\right\} g''(z+\lambda_2\bar{z}), \tag{5·1}$$

$$\frac{G_2}{D} = \left\{k_4(1-\lambda_1)^2 - \frac{c_{12}}{c_{11}}(1+\lambda_1)^2\right\} f''(z+\lambda_1\bar{z})$$

$$+ \left\{k_4(1-\lambda_2)^2 - \frac{c_{12}}{c_{11}}(1+\lambda_2)^2\right\} g''(z+\lambda_2\bar{z}), \qquad (5.2)$$

$$\frac{H_1}{D} = -\frac{H_2}{D} = 2i\,\frac{c_{66}}{c_{11}}\left\{(1-\lambda_1^2) f''(z+\lambda_1\bar{z}) + (1-\lambda_2^2) g''(z+\lambda_2\bar{z})\right\}, \qquad (5.3)$$

$$\frac{K}{D} = -\frac{1}{2}i(\epsilon_1-\epsilon_2)\left\{(1-\lambda_1^2) f''(z+\lambda_1\bar{z}) - (1-\lambda_2^2) g''(z+\lambda_2\bar{z})\right\}, \qquad (5.4)$$

$$\frac{G_r}{D} = \left\{\frac{c_{12}}{c_{11}}\left(e^{i\theta}-\lambda_1 e^{-i\theta}\right) - (1-\lambda_1)\epsilon_1\cos\theta - i(1+\lambda_1)\epsilon_2\sin\theta\right\}$$

$$\times \left(e^{i\theta}-\lambda_1 e^{-i\theta}\right) f''(z+\lambda_1\bar{z})$$

$$+ \left\{\frac{c_{12}}{c_{11}}\left(e^{i\theta}-\lambda_2 e^{-i\theta}\right) - (1-\lambda_2)\epsilon_2\cos\theta - i(1+\lambda_2)\epsilon_1\sin\theta\right\}$$

$$\times \left(e^{i\theta}-\lambda_2 e^{-i\theta}\right) g''(z+\lambda_2\bar{z}), \qquad (5.5)$$

$$\frac{G_\theta}{D} = \left\{-\frac{c_{12}}{c_{11}}\left(e^{i\theta}+\lambda_1 e^{-i\theta}\right) + (1+\lambda_1)\epsilon_2\cos\theta + i(1-\lambda_1)\epsilon_1\sin\theta\right\}$$

$$\times \left(e^{i\theta}+\lambda_1 e^{-i\theta}\right) f''(z+\lambda_1\bar{z})$$

$$+ \left\{-\frac{c_{12}}{c_{11}}\left(e^{i\theta}+\lambda_2 e^{-i\theta}\right) + (1+\lambda_2)\epsilon_1\cos\theta + i(1-\lambda_2)\epsilon_2\sin\theta\right\}$$

$$\times \left(e^{i\theta}+\lambda_2 e^{-i\theta}\right) g''(z+\lambda_2\bar{z}), \qquad (5.6)$$

$$\frac{H_r+\chi}{D} = \left\{-\frac{c_{12}}{c_{11}}\left(e^{i\theta}+\lambda_1 e^{-i\theta}\right)+(1+\lambda_1)\epsilon_2\cos\theta+i(1-\lambda_1)\epsilon_1\sin\theta\right\}$$
$$\times\, i\left(e^{i\theta}-\lambda_1 e^{-i\theta}\right) f''(z+\lambda_1\bar{z})$$
$$+\left\{-\frac{c_{12}}{c_{11}}\left(e^{i\theta}+\lambda_2 e^{-i\theta}\right)+(1+\lambda_2)\epsilon_1\cos\theta+i(1-\lambda_2)\epsilon_2\sin\theta\right\}$$
$$\times\, i\left(e^{i\theta}-\lambda_2 e^{-i\theta}\right) g''(z+\lambda_2\bar{z}), \qquad (5{\cdot}7)$$

$$\frac{N_r}{D} = -\frac{1}{2}(\epsilon_1-\epsilon_2)\left\{(1-\lambda_1^2)\left(e^{i\theta}-\lambda_1 e^{-i\theta}\right)f'''(z+\lambda_1\bar{z})\right.$$
$$\left.-(1-\lambda_2^2)\left(e^{i\theta}-\lambda_2 e^{-i\theta}\right)g'''(z+\lambda_2\bar{z})\right\}, \qquad (5{\cdot}8)$$

$$\frac{N_\theta}{D} = \frac{1}{2}i(\epsilon_1-\epsilon_2)\left\{(1-\lambda_1^2)\left(e^{i\theta}+\lambda_1 e^{-i\theta}\right)f'''(z+\lambda_1\bar{z})\right.$$
$$\left.-(1-\lambda_2^2)\left(e^{i\theta}+\lambda_2 e^{-i\theta}\right)g'''(z+\lambda_2\bar{z})\right\}, \qquad (5{\cdot}9)$$

$$\frac{G_r+G_\theta}{D} = -\left\{4\lambda_1\frac{c_{12}}{c_{11}}-(\epsilon_2-1)(1+\lambda_1)^2\right\}f''(z+\lambda_1\bar{z})$$
$$-\left\{4\lambda_2\frac{c_{12}}{c_{11}}-(\epsilon_1-1)(1+\lambda_2)^2\right\}g''(z+\lambda_2\bar{z}). \qquad (5{\cdot}10)$$

As in the isotropic case the torsional couple may be regarded as statically equivalent to a distribution of shearing force and the boundary conditions are

$$G_r=0, \quad H_r+\chi = \text{constant}. \qquad (5{\cdot}11)$$

In order to satisfy these conditions it is convenient to introduce two functions of the complex variable z, $V(z)$ and $W(z)$, such that the real part of $V(z) = -G_r/D$ and the imaginary part of $W(z)$ = the variable part of $(H_r + \chi)/D$ on the boundary. In general the presence of an imaginary constant in $W(z)$ is to be avoided as it will lead to many-valued displacements, but in certain cases it is necessary to insert an imaginary constant in order to make single-valued the displacements that arise from the remainder of the two functions. Both functions are required to be finite at infinity and, in addition, $V(z) + W(z)$ must tend to zero at infinity and must contain no terms of $O\left(\frac{1}{z}\right)$.

From equations (5·5) and (5·7) it is readily seen that $V(z)$ and $W(z)$ may be expressed in the forms

$$\begin{aligned} 2a^2 V(z) = &-\left\{2\frac{c_{12}}{c_{11}}\left(z-\lambda_1\frac{a^2}{z}\right)-(1-\lambda_1)\epsilon_1\left(z+\frac{a^2}{z}\right)-(1+\lambda_1)\epsilon_2\left(z-\frac{a^2}{z}\right)\right\} \\ &\qquad\times\left(z-\lambda_1\frac{a^2}{z}\right)f''\left(z+\lambda_1\frac{a^2}{z}\right) \\ &-\left\{2\frac{c_{12}}{c_{11}}\left(z-\lambda_2\frac{a^2}{z}\right)-(1-\lambda_2)\epsilon_2\left(z+\frac{a^2}{z}\right)-(1+\lambda_2)\epsilon_1\left(z-\frac{a^2}{z}\right)\right\} \\ &\qquad\times\left(z-\lambda_2\frac{a^2}{z}\right)g''\left(z+\lambda_2\frac{a^2}{z}\right), \end{aligned} \tag{5·12}$$

$$2a^2 W(z) = \left\{2\frac{c_{12}}{c_{11}}\left(z+\lambda_1\frac{a^2}{z}\right) - (1+\lambda_1)\epsilon_2\left(z+\frac{a^2}{z}\right) - (1-\lambda_1)\epsilon_1\left(z-\frac{a^2}{z}\right)\right\}$$
$$\times\left(z-\lambda_1\frac{a^2}{z}\right) f''\left(z+\lambda_1\frac{a^2}{z}\right)$$
$$+\left\{2\frac{c_{12}}{c_{11}}\left(z+\lambda_2\frac{a^2}{z}\right) - (1+\lambda_2)\epsilon_1\left(z+\frac{a^2}{z}\right) - (1-\lambda_2)\epsilon_2\left(z-\frac{a^2}{z}\right)\right\}$$
$$\times\left(z-\lambda_2\frac{a^2}{z}\right) g''\left(z+\lambda_2\frac{a^2}{z}\right). \qquad (5\cdot13)$$

A little manipulation leads to the comparatively simple forms

$$\frac{a^2\{V(z)-W(z)\}}{z} = -B_2\left(z-\lambda_1\frac{a^2}{z}\right) f''\left(z+\lambda_1\frac{a^2}{z}\right)$$
$$-B_1\left(z-\lambda_2\frac{a^2}{z}\right) g''\left(z+\lambda_2\frac{a^2}{z}\right), \qquad (5\cdot14)$$

$$z\{V(z)+W(z)\} = C_2\left(z-\lambda_1\frac{a^2}{z}\right) f''\left(z+\lambda_1\frac{a^2}{z}\right)$$
$$+C_1\left(z-\lambda_2\frac{a^2}{z}\right) g''\left(z+\lambda_2\frac{a^2}{z}\right), \qquad (5\cdot15)$$

where B_1, B_2, C_1, C_2 are constants depending only on the nature of the material, and defined by

$$B_1 = 2\frac{c_{12}}{c_{11}} - \epsilon_1(1+\lambda_2) - \epsilon_2(1-\lambda_2), \qquad (5\cdot16)$$

$$B_2 = 2\frac{c_{12}}{c_{11}} - \epsilon_2(1+\lambda_1) - \epsilon_1(1-\lambda_1), \qquad (5\cdot17)$$

$$C_1 = 2\lambda_2\frac{c_{12}}{c_{11}} + \epsilon_2(1-\lambda_2) - \epsilon_1(1+\lambda_2), \qquad (5\cdot18)$$

$$C_2 = 2\lambda_1\frac{c_{12}}{c_{11}} + \epsilon_1(1-\lambda_1) - \epsilon_2(1+\lambda_1), \qquad (5\cdot19)$$

In order to find the stress-couples at any point in the plate two new variables $\underline{u}$ and $\underline{v}$ are defined by the substitutions

$$2z = u\left\{1+\left(1-\frac{4\lambda_1 a^2}{a^2}\right)^{1/2}\right\}, \qquad (5{\cdot}20)$$

u

$$2z = v\left\{1+\left(1-\frac{4\lambda_2 a^2}{v^2}\right)^{1/2}\right\}, \qquad (5{\cdot}21)$$

and, using equations (5·14) and (5·15), it is found, after a certain amount of reduction, that

$$\begin{aligned} &8\lambda_1(\lambda_1-\lambda_2)K\left(1-\frac{4\lambda_1 a^2}{u^2}\right)^{1/2} f''(u) \\ &= \lambda_1 B_1\left\{1+\left(1-\frac{4\lambda_1 a^2}{u^2}\right)^{1/2}\right\}\left\{V(u)+W(u)\right\} \\ &\quad + C_1\left\{1-\left(1-\frac{4\lambda_1 a^2}{u^2}\right)^{1/2}\right\}\left\{V(u)-W(u)\right\}, \end{aligned} \qquad (5{\cdot}22)$$

$$\begin{aligned} &8\lambda_2(\lambda_1-\lambda_2)K\left(1-\frac{4\lambda_2 a^2}{v^2}\right)^{1/2} g''(v) \\ &= -\lambda_2 B_2\left\{1+\left(1-\frac{4\lambda_2 a^2}{v^2}\right)^{1/2}\right\}\left\{V(v)+W(v)\right\} \\ &\quad - C_2\left\{1-\left(1-\frac{4\lambda_2 a^2}{v^2}\right)^{1/2}\right\}\left\{V(v)-W(v)\right\}, \end{aligned} \qquad (5{\cdot}23)$$

where

$$K = \left(\frac{c_{12}}{c_{11}}\right)^2 + \frac{2(1+\lambda_1)(1+\lambda_2)}{(1-\lambda_1)(1-\lambda_2)}\left(\frac{c_{12}}{c_{11}}\right) - \frac{(1+\lambda_1)(1+\lambda_2)}{(1-\lambda_1)^3(1-\lambda_2)^3}\left\{3(1-\lambda_1\lambda_2)^2+(\lambda_1-\lambda_2)^2\right\}, \qquad (5{\cdot}24)$$

and is completely real whichever of the cases outlined in §4 is applicable.

The tangential stress-couple on the edge of the circular hole may be obtained from the formula

$$\left(e^{i\theta}-\lambda_1 e^{-i\theta}\right)\left(e^{i\theta}-\lambda_2 e^{-i\theta}\right)\frac{KG_\theta}{D}$$

$$= -P\left(e^{i\theta}+\lambda_1 e^{-i\theta}\right)\left(e^{i\theta}+\lambda_2 e^{-i\theta}\right)V(z)-2Q\left(e^{2i\theta}-\lambda_1\lambda_2 e^{-2i\theta}\right)W(z)$$

$$+2R\left\{\left(\lambda_1 e^{i\theta}+e^{-i\theta}\right)\left(\lambda_2 e^{i\theta}+e^{-i\theta}\right)V(z)+\left(\lambda_1\lambda_2 e^{2i\theta}-e^{-2i\theta}\right)W(z)\right\}, \quad (5\cdot25)$$

where

$$P=\left(\frac{c_{12}}{c_{11}}\right)^2-\frac{2(1-\lambda_1\lambda_2)^2}{(1-\lambda_1)^2(1-\lambda_2)^2}\left(\frac{c_{12}}{c_{11}}\right)+\frac{(1+\lambda_1)(1+\lambda_2)}{(1-\lambda_1)^3(1-\lambda_2)^3}\left\{(1-\lambda_1\lambda_2)^2+(\lambda_1-\lambda_2)^2\right\}, \quad (5\cdot26)$$

$$Q=\left(\frac{c_{12}}{c_{11}}\right)^2-\frac{(\lambda_1-\lambda_2)^2}{(1-\lambda_1)^2(1-\lambda_2)^2}\left(\frac{c_{12}}{c_{11}}\right)-\frac{(1-\lambda_1\lambda_2)^2(1+\lambda_1)(1+\lambda_2)}{(1-\lambda_1)^3(1-\lambda_2)^3}, \quad (5\cdot27)$$

$$R=\frac{(\lambda_1-\lambda_2)^2}{(1-\lambda_1)^2(1-\lambda_2)^2}\left\{\frac{c_{12}}{c_{11}}+\frac{(1+\lambda_1)(1+\lambda_2)}{(1-\lambda_1)(1-\lambda_2)}\right\}. \quad (5\cdot28)$$

6. Any particular problem may be solved theoretically by choosing $V(z)$ and $W(z)$ so that they take prescribed values on the edge of the circular hole and also satisfy the necessary conditions at infinity. The values of $f''(u)$ and $g''(v)$ are then determined from equations (5·22) and (5·23) and the complete stress-distribution is given by equations (5·1) – (5·10) inclusive when $\underline{u}$ and $\underline{v}$ have the values

$$u=z+\lambda_1\bar{z}, \qquad v=z+\lambda_2\bar{z}. \qquad (6\cdot1)$$

In the particular problem of a perforated plate bent by couples at infinity, $V(z)$ and $W(z)$ must be chosen in order to cancel on the boundary the stress-couples transmitted from infinity, and it is to problems of this type that the remainder of this section is devoted.

7. Some numerical work has been carried out to evaluate the factor of stress-concentration round the edge of a hole in a wooden plank made of the highly aeolotropic material spruce using for the elastic constants values obtained by various writers and quoted in a number of recent papers. (See Green 1942). These constants are reproduced in table 13 where the grain is taken to be parallel to the y-axis.

Table 13

Values of the elastic constants for a specimen of spruce

c_{11}	c_{22}	c_{12}	c_{66}	ϵ	l	λ	μ
0.0653	1.72	0.0367	0.0870	5.14	1.96	0.405	0.191

These values have been calculated from the values of s_{11} etc previously used.

PURE BENDING ABOUT THE x-AXIS

8. In the absence of the hole the stress-couples required to bend the plate into the form of a cylinder about the x-axis are

$$G_1 = \frac{c_{12}}{c_{11}k_4} G, \quad G_2 = G, \quad H_1 = N_1 = N_2 = \gamma = 0, \tag{8.1}$$

corresponding to an initial deflection of

$$w = -\frac{G y^2}{2Dk_4}, \tag{8.2}$$

and it is easily verified that the corresponding stress-couples in polar coordinates are

$$\frac{2G_r}{G} = 1 + \frac{c_{12}}{c_{11}k_4} - \left(1 - \frac{c_{12}}{c_{11}k_4}\right)\cos 2\theta, \tag{8.3}$$

$$\frac{2G_\theta}{G} = 1 + \frac{c_{12}}{c_{11}k_4} + \left(1 - \frac{c_{12}}{c_{11}k_4}\right)\cos 2\theta, \tag{8.4}$$

$$2\frac{H_r + \gamma}{G} = \left(\frac{c_{12}}{c_{11}k_4} - 1\right)\sin 2\theta. \tag{8.5}$$

$V(z)$ and $W(z)$ must therefore be chosen so that, on the edge of the circular hole,

$$\frac{2G_r}{G} = -1 - \frac{c_{12}}{c_{11}k_4} + \left(1 - \frac{c_{12}}{c_{11}k_4}\right)\cos 2\theta, \tag{8.6}$$

$$2\frac{H_r + \gamma}{G} = \left(1 - \frac{c_{12}}{c_{11}k_4}\right)\sin 2\theta, \tag{8.7}$$

and it is readily seen that all the conditions are satisfied by writing

$$\frac{2DV(z)}{G} = 1 + \frac{c_{12}}{c_{11}k_4} - \left(1 - \frac{c_{12}}{c_{11}k_4}\right)\frac{a^2}{z^2}, \tag{8.8}$$

$$\frac{2DW(z)}{G} = -1 - \frac{c_{12}}{c_{11}k_4} - \left(1 - \frac{c_{12}}{c_{11}k_4}\right)\frac{a^2}{z^2}, \tag{8.9}$$

$$D\{V(u) - W(u)\} = D\{V(v) - W(v)\} = G\left(1 + \frac{c_{12}}{c_{11}k_4}\right), \tag{8.10}$$

$$D\{V(u) + W(u)\} = -G\left(1 - \frac{c_{12}}{c_{11}k_4}\right)\frac{\left\{1 - \left(1 - \frac{4\lambda_1 a^2}{u^2}\right)^{1/2}\right\}}{\lambda_1\left\{1 + \left(1 - \frac{4\lambda_1 a^2}{u^2}\right)^{1/2}\right\}}, \tag{8.11}$$

$$D\{V(v) + W(v)\} = -G\left(1 - \frac{c_{12}}{c_{11}k_4}\right)\frac{\left\{1 - \left(1 - \frac{4\lambda_2 a^2}{v^2}\right)^{1/2}\right\}}{\lambda_2\left\{1 + \left(1 - \frac{4\lambda_2 a^2}{v^2}\right)^{1/2}\right\}}. \tag{8.12}$$

From equations (5.22) and (5.23) it is found, after a little reduction, that

$$4\lambda_1(\lambda_1 - \lambda_2)K\left(1 - \frac{4\lambda_1 a^2}{u^2}\right)^{1/2} D f''(u) = GA_1\left\{1 - \left(1 - \frac{4\lambda_1 a^2}{u^2}\right)^{1/2}\right\}, \tag{8.13}$$

$$4\lambda_2(\lambda_1 - \lambda_2)K\left(1 - \frac{4\lambda_2 a^2}{v^2}\right)^{1/2} D g''(v) = -GA_2\left\{1 - \left(1 - \frac{4\lambda_2 a^2}{v^2}\right)^{1/2}\right\}, \tag{8.14}$$

where A_1, A_2 are constants defined by

$$A_1 = \frac{1+\lambda_2}{k_4}\left(\frac{c_{12}}{c_{11}}\right)^2 - 2\,\frac{1-\lambda_2}{1+\lambda_2}\left(\frac{c_{12}}{c_{11}}\right) + \frac{(1+\lambda_2)^2}{1-\lambda_2}, \tag{8.15}$$

$$A_2 = \frac{1+\lambda_1}{k_4}\left(\frac{c_{12}}{c_{11}}\right)^2 - 2\,\frac{1-\lambda_1}{1+\lambda_1}\left(\frac{c_{12}}{c_{11}}\right) + \frac{(1+\lambda_1)^2}{1-\lambda_1}. \tag{8.16}$$

The extra stress-couples and shearing forces due to the hole may now be obtained from equations (5.1)–(5.10) directly when

$$f''(z+\lambda_1\bar{z}) = \frac{\varsigma A_1(Y_1^{-1}-1)}{4\lambda_1(\lambda_1-\lambda_2)KD}, \tag{8.17}$$

$$g''(z+\lambda_2\bar{z}) = -\frac{\varsigma A_2(Y_2^{-1}-1)}{4\lambda_2(\lambda_1-\lambda_2)KD}, \tag{8.18}$$

where

$$Y_1 = \left\{1 - \frac{4\lambda_1 a^2}{(z+\lambda_1\bar{z})^2}\right\}^{1/2}, \quad Y_2 = \left\{1 - \frac{4\lambda_2 a^2}{(z+\lambda_2\bar{z})^2}\right\}^{1/2}, \tag{8.19}$$

with the added restriction that

$$-\tfrac{1}{2}\pi < \arg Y_1 \leqslant \tfrac{1}{2}\pi \quad , \quad -\tfrac{1}{2}\pi < \arg Y_2 \leqslant \tfrac{1}{2}\pi. \tag{8.20}$$

It is found that logarithmic terms with imaginary coefficients arise in both $f(z+\lambda_1\bar{z})$ and $g(z+\lambda_2\bar{z})$ when λ_1 and λ_2 are complex, and these terms would appear to lead to many-valued displacements. The total displacement, however, is $f(z+\lambda_1\bar{z}) + g(z+\lambda_2\bar{z})$ and it may be verified that the coefficient of the logarithmic term in the sum of the two functions is always real, and so the displacement is single-valued. The presence of such a logarithmic term in the additional displacement admits of a justification similar to that in the isotropic case.

The tangential stress-couple round the edge of the circular hole is of particular interest, and takes a form which lends itself readily to calculation when the coefficients have been evaluated. The formulae for G_θ henceforward are all reproduced in terms of λ and μ as, for a number of materials, including spruce, the quantities λ_1, λ_2 are complex conjugates $\lambda+i\mu$, $\lambda-i\mu$.

Writing in all cases $\lambda^2+\mu^2=d$, the tangential stress-couple is given by

$$\frac{KG_\theta}{G} = \frac{4(1-d)P_1\cos 4\theta + 2Q_1\cos 2\theta + R_1}{1+4\lambda^2+d^2-4\lambda(1+d)\cos 2\theta + 2d\cos 4\theta}, \qquad (8.21)$$

where

$$P_1 = \frac{\lambda-d}{1+2\lambda+d}\left(\frac{c_{12}}{c_{11}}\right)^2 - 2\,\frac{\lambda(1+d)+2d}{(1+d)^2-4\lambda^2}\left(\frac{c_{12}}{c_{11}}\right) + \epsilon\,\frac{(1-d)(\lambda-d)-2\mu^2}{(1-2\lambda+d)^2}, \qquad (8.22)$$

$$Q_1 = \left\{1-\left(\frac{c_{12}}{c_{11}\epsilon}\right)^2\right\}\left\{(1+2\lambda+d)\frac{c_{12}}{c_{11}} - \epsilon(1-3d-2\lambda d)\right\}, \qquad (8.23)$$

$$\begin{aligned} R_1 = {} & \frac{(1+2\lambda)^2-d^2}{\epsilon^2}\left(\frac{c_{12}}{c_{11}}\right)^3 + \frac{(c_{12}/c_{11})^2}{1+2\lambda+d}\left\{\begin{array}{l} -8\lambda^3-4\lambda^2(1-d)+2\lambda(8d-d^2-3) \\ \quad +(1-d)(3+6d-d^2) \end{array}\right\} \\ & + \frac{c_{12}/c_{11}}{(1+d)^2-4\lambda^2}\left\{\begin{array}{l} 16\lambda^4+16\lambda^3-8d(1+d)\lambda^2+4\lambda(1+d)(1-3d) \\ \quad -(1-d)(1-13d+3d^2+d^3) \end{array}\right\} \\ & - \frac{\epsilon}{(1-2\lambda+d)^2}\left\{\begin{array}{l} 16\lambda^4-16d\lambda^3+8\lambda^2(1-4d+d^2) \\ \quad -4\lambda(1-d-5d^2+d^3)+(3-d)(1-d)^3 \end{array}\right\}, \end{aligned} \qquad (8.24)$$

$$K = \left(\frac{c_{12}}{c_{11}}\right)^2 + 2\epsilon\left(\frac{c_{12}}{c_{11}}\right) - \frac{\epsilon}{(1-2\lambda+d)^2}\left\{\mathfrak{z}(1-d)^2 - 4\mu^2\right\}, \quad (8\cdot25)$$

which is the same value as was assigned to K in equation (5·24).

Equation (8·21) reduces to the known formula for an isotropic material when $\lambda, \mu \to 0$.

The numerical values of the above expression are tabulated at the end of § 9.

PURE BENDING ABOUT THE y-AXIS.

9. The displacement of the unperforated plate bent into the form of a cylinder about the y-axis is given by

$$w = -\frac{Gx^2}{2D}, \quad (9\cdot1)$$

and this displacement is maintained by the distribution of stress-couple

$$G_1 = G, \quad G_2 = \frac{c_{12}}{c_{11}} G, \quad H_1 = H_2 = N_1 = N_2 = \gamma = 0. \quad (9\cdot2)$$

The analysis is parallel with that of § 8 and the tangential stress-couple round the edge of the circular boundary is given by

$$\frac{KG_\theta}{G} = \frac{4(1-d)P_1'\cos 4\theta + 2Q_1'\cos 2\theta + R_1'}{1+4\lambda^2+d^2-4\lambda(1+d)\cos 2\theta + 2d\cos 4\theta}, \quad (9\cdot3)$$

where

$$P_1' = -\frac{\lambda+d}{1-2\lambda+d}\left(\frac{c_{12}}{c_{11}}\right)^2 + 2\epsilon\,\frac{\lambda(1+d)-2d}{(1-2\lambda+d)^2}\left(\frac{c_{12}}{c_{11}}\right)$$
$$\qquad - \frac{\epsilon}{(1-2\lambda+d)^2}\left\{(1-d)(\lambda+d)+2\mu^2\right\}, \qquad (9{\cdot}4)$$

$$Q_1' = \left\{\left(\frac{c_{12}}{c_{11}}\right)^2 - \epsilon^2\right\}\left\{(1-2\lambda+d)\frac{c_{12}}{c_{11}} - \epsilon(1-3d+2\lambda d)\right\}, \qquad (9{\cdot}5)$$

$$R_1' = \left\{(1-2\lambda)^2-d^2\right\}\left(\frac{c_{12}}{c_{11}}\right)^3 + \frac{(c_{12}/c_{11})^2}{1-2\lambda+d}\left\{\begin{array}{l}8\lambda^3-4\lambda^2(1-d)+2\lambda(3-8d+d^2)\\ \quad +(1-d)(3+6d-d^2)\end{array}\right\}$$
$$+ \frac{\epsilon\,(c_{12}/c_{11})}{(1-2\lambda+d)^2}\left\{\begin{array}{l}16\lambda^4-16\lambda^3-8\lambda^2 d(1+d)-4\lambda(1-3d)(1+d)\\ \quad -(1-d)(1-13d+3d^2+d^3)\end{array}\right\}$$
$$- \frac{\epsilon}{(1-2\lambda+d)^2}\left\{\begin{array}{l}16\lambda^4+16\lambda^3 d+8\lambda^2(1-4d+d^2)\\ \quad +4\lambda(1-d-5d^2+d^3)+(1-d)^3(3-d)\end{array}\right\}, \qquad (9{\cdot}6)$$

and K is given by equation (8·25)

The concentration of stress-couple round the edge of a circular hole in a bent plate is shown in table 14 and plotted in Fig. 13, the results for an isotropic material being included for comparison.

Table 14

Concentration of stress-couple round the edge of a circular hole in a plate bent into the form of a cylinder about the line $\theta = 0$.

Values of G_θ / G.

θ°	Spruce-grain parallel to $\theta = 0$	Isotropic	Spruce-grain parallel to $\theta = \frac{\pi}{2}$.
0	1·44	1·83	3·04
10	1·415	1·79	2·78
20	1·35	1·69	2·00
30	1·24	1·54	1·19
40	1·08	1·35	0·684
50	0·907	1·15	0·405
60	0·883	0·962	0·252
70	1·50	0·808	0·169
80	3·005	0·708	0·127
90	3·90	0·673	0·114
Couple about $\theta = 0$	G	G	G
Couple about $\theta = \frac{1}{2}\pi$	$0{\cdot}56G$	$0{\cdot}25G$	$0{\cdot}021G$.

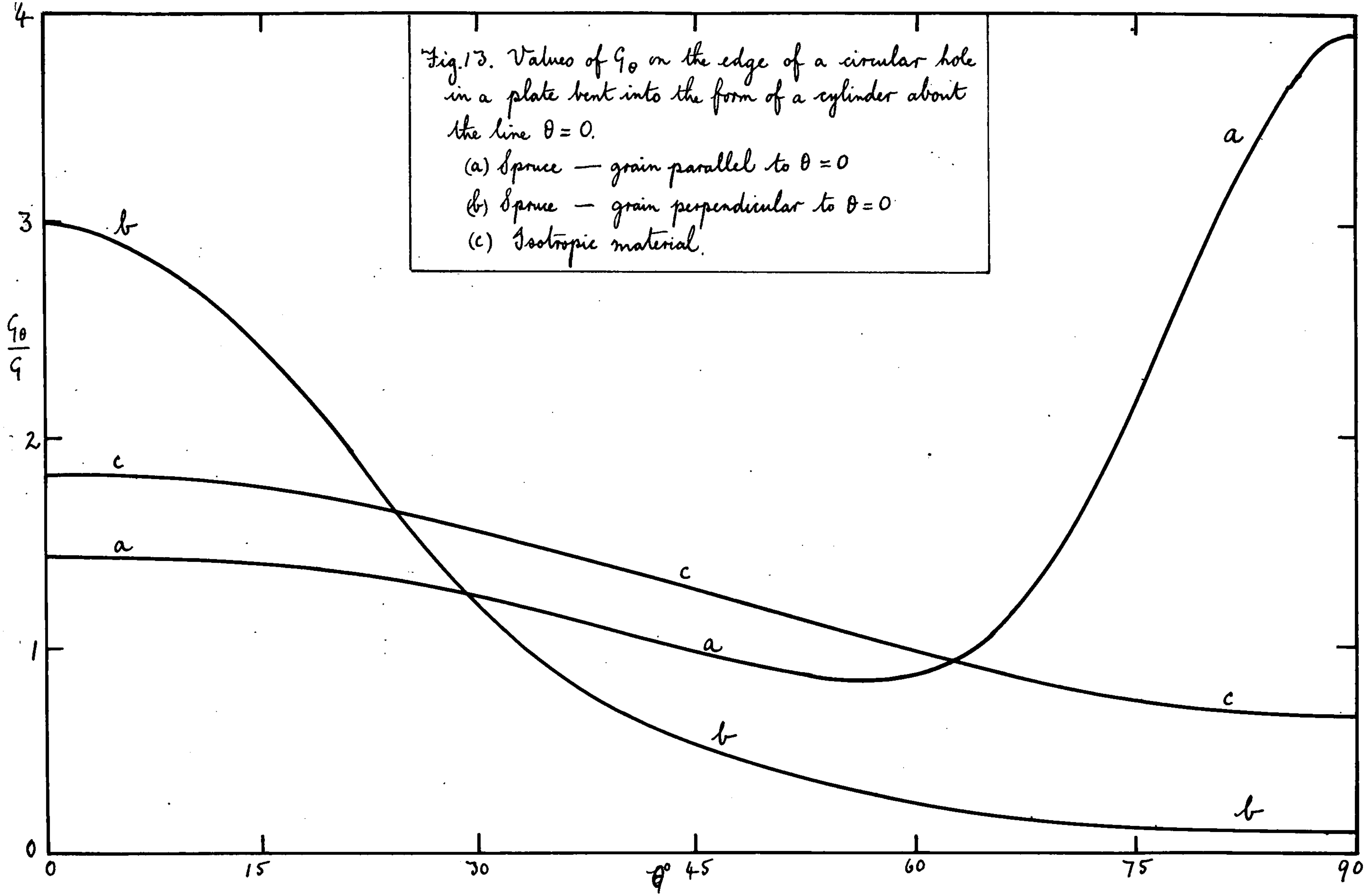

Fig. 13. Values of G_θ on the edge of a circular hole in a plate bent into the form of a cylinder about the line $\theta = 0$.

(a) Spruce — grain parallel to $\theta = 0$

(b) Spruce — grain perpendicular to $\theta = 0$

(c) Isotropic material.

ALL-ROUND BENDING

10. The deformation due to any distribution of bending couple may be obtained by combining the results of the two previous paragraphs, and the problem of a plate bent by couples G acting everywhere radially is a particular case. Before the hole is cut the displacement is given by the equation

$$w = -\frac{G}{2D}\left\{\left(k_4 - \frac{c_{12}}{c_{11}}\right)x^2 + \left(1 - \frac{c_{12}}{c_{11}}\right)y^2\right\} \Big/ \left\{k_4 - \left(\frac{c_{12}}{c_{11}}\right)^2\right\}, \qquad (10\cdot1)$$

and the stress-couples are

$$G_1 = G_2 = G_r = G_\theta = G, \qquad (10\cdot2)$$

$$H_1 = H_2 = N_1 = N_2 = \mathcal{H} = 0. \qquad (10\cdot3)$$

Round the edge of the circular hole the tangential stress-couple has a comparatively simple form. It is given by

$$\frac{KG_\theta}{G} = \frac{8(1-d)A\cos 4\theta + 8\lambda B\cos 2\theta + 2C}{1+4\lambda^2+d^2-4\lambda(1+d)\cos 2\theta + 2d\cos 4\theta}, \qquad (10\cdot4)$$

where

$$A = \frac{2\lambda^2 - d(1+d)}{(1-2\lambda+d)^2}\left(\frac{c_{12}}{c_{11}} + \epsilon\right) - \frac{2d(1-d)\epsilon}{(1-2\lambda+d)^2}, \qquad (10\cdot5)$$

$$B = -\left(\frac{c_{12}}{c_{11}}\right)^2 - \epsilon(1+d)\left\{\frac{c_{12}}{c_{11}} - \frac{2(1-3d-2\lambda d)}{(1-2\lambda+d)^2}\right\} + d\epsilon^3, \qquad (10\cdot6)$$

$$C = \left(1+4\lambda^2-d^2\right)\left(\frac{c_{12}}{c_{11}}\right)^2 - 2\,\frac{\left\{16\lambda^4+4\lambda^2(1-5d)-(1-d^2)(1+2d-d^2)\right\}}{(1-2\lambda+d)^2}$$
$$-\,\epsilon\,\frac{\left\{16\lambda^4+8\lambda^2(1-4d+d^2)+(1-d)^3(3-d)\right\}}{(1-2\lambda+d)^2}. \qquad (10.7)$$

The values of this expression are tabulated in table 15 and plotted in Fig 14.

Table 15

Values of the stress-couple round the edge of a circular hole in a plate bent by all-round couples. Grain perpendicular to $\theta = 0$

Spruce

θ°	0	10	20	30	40
G_θ/G	5·22	4·21	2·37	1·40	1·20
θ°	50	60	70	80	90
G_θ/G	1·25	1·34	1·41	1·46	1·47

For an isotropic material $G_\theta = 2G$ everywhere on the boundary.

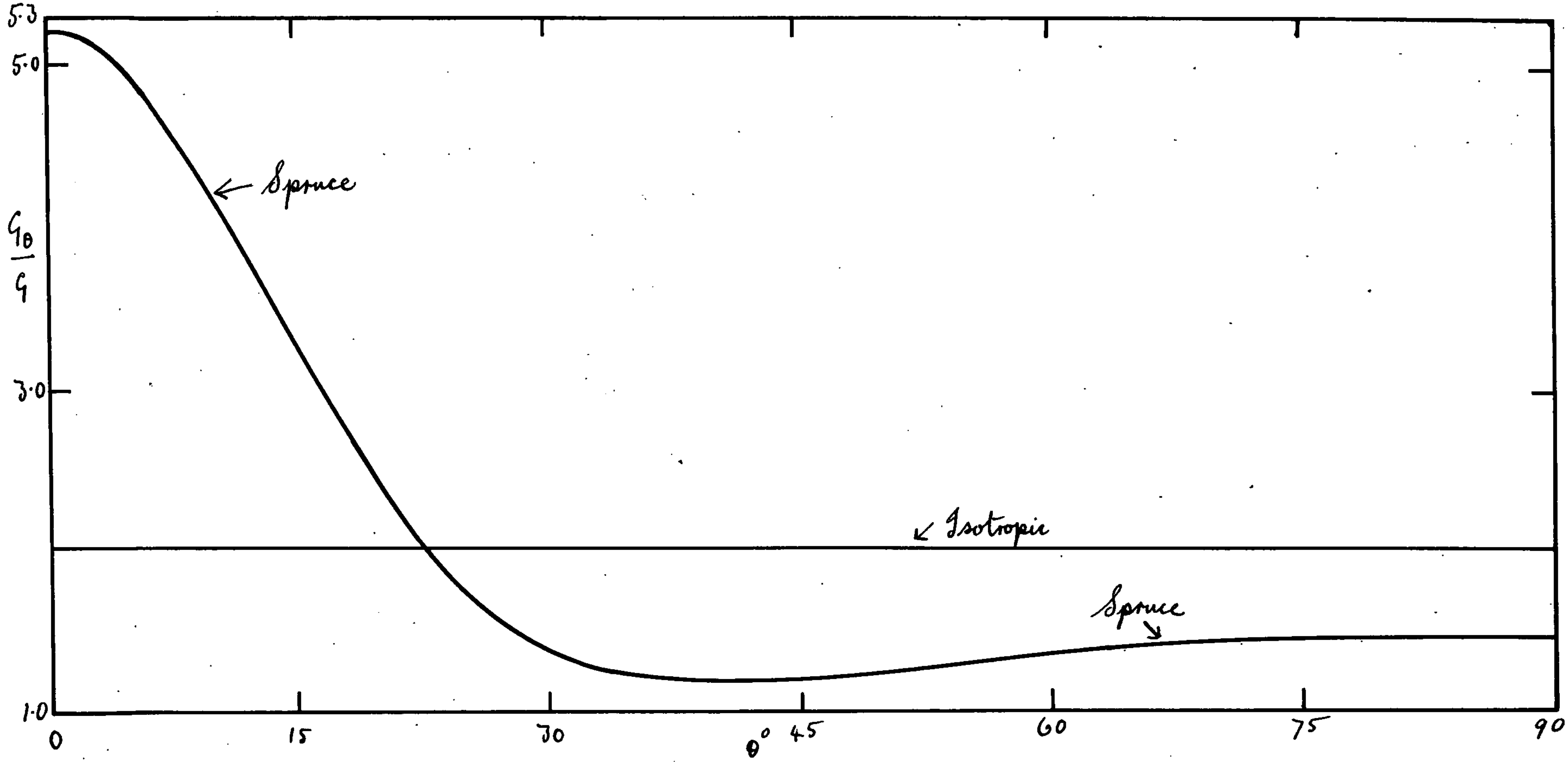

Fig. 14. Values of G_θ on the edge of a circular hole in a plate bent by all-round couples. Grain perpendicular to $\theta = 0$.

THE TWISTED PLATE

11. It will be assumed that the displacement of the plate in the absence of the hole is

$$w = \frac{H c_{11} xy}{2 D c_{66}}, \qquad (11{\cdot}1)$$

and it is easily verified that this displacement is maintained by torsional couples H acting at infinity in planes perpendicular to the directions of symmetry of the material, so that

$$H_1 = -H_2 = H, \quad G_1 = G_2 = N_1 = N_2 = \mathcal{T} = 0. \qquad (11{\cdot}2)$$

It is readily seen that the functions $V(z)$ and $W(z)$ must be chosen in order to produce on the boundary

$$G_r = H \sin 2\theta, \quad H_r + \mathcal{T} = -H \cos 2\theta, \qquad (11{\cdot}3)$$

and as they must also be finite at infinity, they will clearly involve terms in $\frac{ia^2}{z^2}$. This means that terms of order $(z+\lambda_1\bar{z})^{-2}$ will occur in $f''(z+\lambda_1\bar{z})$ and hence that the additional displacement will contain terms in $i \log(z+\lambda_1\bar{z})$ and $i\log(z+\lambda_2\bar{z})$ which are both many-valued and so are only physically admissible if their coefficients are equal and opposite. The simplest way of ensuring that the two coefficients are equal and opposite is to add an imaginary constant to $V(z)$ and $W(z)$, which means adding a real constant to the value of $H_r + \mathcal{T}$ on the boundary. The value of $H_r + \mathcal{T}$

on the boundary, however, is in general indeterminate to the extent of a real constant and so the modification of $V(z)$ and $W(z)$ in this manner is quite admissible. All the necessary conditions are found to be satisfied by defining a real constant M such that

$$(\lambda_1+\lambda_2)(1-\lambda_1\lambda_2)M^{-1} = (1-\lambda_1^2)(1-\lambda_2^2)\frac{c_{12}}{c_{11}} + 3-\lambda_1^2-\lambda_2^2-\lambda_1^2\lambda_2^2, \quad (11{\cdot}4)$$

and writing $V(z)$ and $W(z)$ in the forms

$$DV(z) = 2iHM - iH\frac{a^2}{z^2}, \quad (11{\cdot}5)$$

$$DW(z) = -2iHM - iH\frac{a^2}{z^2}, \quad (11{\cdot}6)$$

$$D\{V(u)+W(u)\} = -2iH\lambda_1^{-1}\left\{1-\left(1-\frac{4\lambda_1 a^2}{u^2}\right)^{\frac{1}{2}}\right\}\left\{1+\left(1-\frac{4\lambda_1 a^2}{u^2}\right)^{\frac{1}{2}}\right\}^{-1}, \quad (11{\cdot}7)$$

$$D\{V(v)+W(v)\} = -2iH\lambda_2^{-1}\left\{1-\left(1-\frac{4\lambda_2 a^2}{v^2}\right)^{\frac{1}{2}}\right\}\left\{1+\left(1-\frac{4\lambda_2 a^2}{v^2}\right)^{\frac{1}{2}}\right\}^{-1}, \quad (11{\cdot}8)$$

$$D\{V(u)-W(u)\} = D\{V(v)-W(v)\} = 4iHM, \quad (11{\cdot}9)$$

Using equations (5·22) and (5·23) it is found, after some reduction, that

$$f''(z+\lambda_1\bar{z}) = -\frac{iH(\gamma_1^{-1}-1)}{2\lambda_1(\lambda_1-\lambda_2)L}, \quad (11{\cdot}10)$$

$$g''(z+\lambda_2\bar{z}) = \frac{iH(\gamma_2^{-1}-1)}{2\lambda_2(\lambda_1-\lambda_2)L}, \quad (11{\cdot}11)$$

where L is a real constant given by the equation

$$(1-\lambda_1^2)(1-\lambda_2^2)L = (1-\lambda_1^2)(1-\lambda_2^2)\frac{c_{12}}{c_{11}} + 3-\lambda_1^2-\lambda_2^2-\lambda_1^2\lambda_2^2. \quad (11{\cdot}12)$$

The extra stress-couples may now be obtained directly from equations (5·1) – (5·10), but the main interest again lies in the tangential stress-couple on the boundary and it takes the simple form

$$\left\{(3-d)(1+d)-4\lambda^2+(1-2\lambda+d)^2\frac{c_{12}}{c_{11}}\right\}\frac{G_\theta}{H}$$

$$=\frac{8\lambda(1-d^2)\sin 4\theta-4(1-d)E\sin 2\theta}{1+4\lambda^2+d^2-4\lambda(1+d)\cos 2\theta+2d\cos 4\theta}, \qquad (11\cdot 13)$$

where $E=(1-2\lambda+d)^2\frac{c_{12}}{c_{11}}+(1+d)^2+4\lambda^2$, (11·14)

from which the known form for an isotropic material is obtained by allowing λ and μ to tend to zero.

The numerical values of this expression are shown in table 16 and plotted in fig. 15, the isotropic results being included for comparison.

Table 16

Values of G_θ/H on the edge of a circular hole in a twisted plate

Grain parallel to $\theta=\frac{1}{2}\pi$

θ°	0	10	20	30	40
G_θ – spruce	0	0·791	1·85	2·29	2·12
G_θ – isotropic	0	0·526	0·989	1·33	1·51
θ°	50	60	70	80	90
G_θ – spruce	1·72	1·27	0·831	0·410	0
G_θ – isotropic	1·51	1·33	0·989	0·526	0

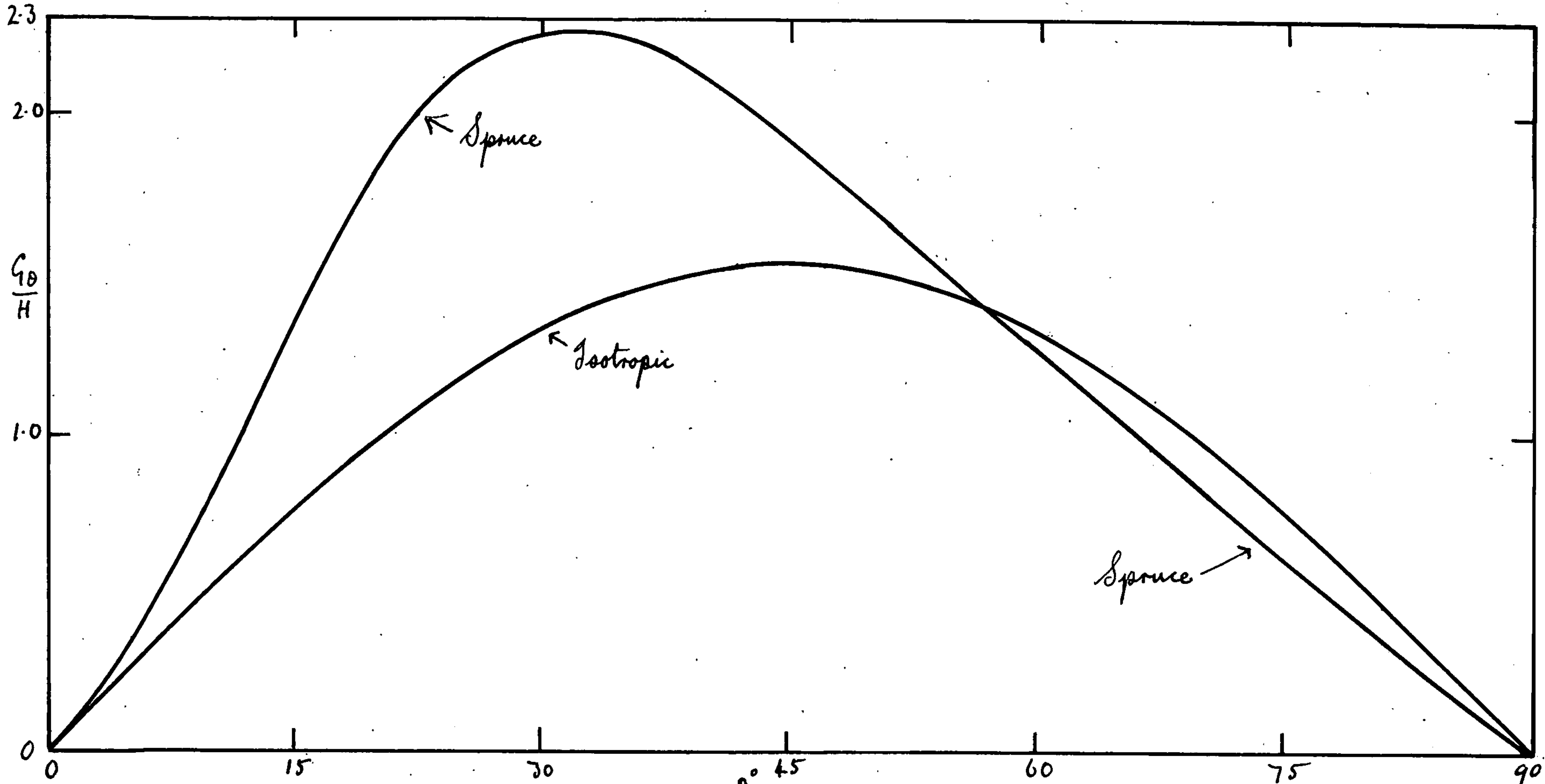

Fig. 15. Values of G_θ on the edge of a circular hole in a twisted plate. Grain perpendicular to $\theta = 0$.

THE PROBLEM OF THE ELLIPTICAL HOLE

12. When the hole is elliptical, it is convenient to express the stress-couples and shearing forces in terms of new coordinates ξ and η by means of the transformation

$$z = F(\zeta) = F(\xi + i\eta) = c\cos(\zeta + i\alpha), \qquad (12\cdot1)$$

$$\frac{dz}{d\zeta} = F'(\zeta) = Je^{i\phi} = a_0 e^{-i\zeta} + b_1 e^{i\zeta}, \qquad (12\cdot2)$$

where a_0 and b_1 are complex constants and ϕ is the angle between the tangent to the curve η = constant and the x-axis. The transformation represents a series of ellipses, the boundary being given by the special case $\eta = 0$, when the semi-axes of the hole are $c\cosh\alpha$, $c\sinh\alpha$, ζ is therefore real and equal to $\bar{\zeta}$ on the boundary, a bar over any function denoting the complex conjugate of the function.

The stress-couples in the new coordinates are obtained from the formulae

$$G_\xi = G_1\cos^2\phi + G_2\sin^2\phi - H_1\sin 2\phi, \qquad (12\cdot3)$$

$$G_\eta = G_1\sin^2\phi + G_2\cos^2\phi + H_1\sin 2\phi, \qquad (12\cdot4)$$

$$H_\xi = -H_\eta = \tfrac{1}{2}(G_1 - G_2)\sin 2\phi + H_1\cos 2\phi, \qquad (12\cdot5)$$

G_1, G_2, H_1 and χ are given by equations (5·1) to (5·4) when $z = F(\zeta)$, $\bar{z} = \bar{F}(\bar{\zeta})$ and a certain amount of reduction leads to the forms

$$
\begin{aligned}
\frac{2F'(\zeta)\bar{F}'(\bar{\zeta})G_\xi}{D} = {} & \left\{B_2 F'(\zeta) - C_2 \bar{F}'(\bar{\zeta})\right\}\left\{F'(\zeta) - \lambda_1 \bar{F}'(\bar{\zeta})\right\} \\
& \times f''\left\{F(\zeta) + \lambda_1 \bar{F}(\bar{\zeta})\right\} \\
& + \left\{B_1 F'(\zeta) - C_1 \bar{F}'(\bar{\zeta})\right\}\left\{F'(\zeta) - \lambda_2 \bar{F}'(\bar{\zeta})\right\} \\
& \times g''\left\{F(\zeta) + \lambda_2 \bar{F}(\bar{\zeta})\right\},
\end{aligned}
\qquad (12{\cdot}6)
$$

$$
\begin{aligned}
\frac{2F'(\zeta)\bar{F}'(\bar{\zeta})G_\eta}{D} = {} & -\left\{B_2 F'(\zeta) + C_2 \bar{F}'(\bar{\zeta})\right\}\left\{F'(\zeta) + \lambda_1 \bar{F}'(\bar{\zeta})\right\} \\
& \times f''\left\{F(\zeta) + \lambda_1 \bar{F}(\bar{\zeta})\right\} \\
& - \left\{B_1 F'(\zeta) + C_1 \bar{F}'(\bar{\zeta})\right\}\left\{F'(\zeta) + \lambda_2 \bar{F}'(\bar{\zeta})\right\} \\
& \times g''\left\{F(\zeta) + \lambda_2 \bar{F}(\bar{\zeta})\right\},
\end{aligned}
\qquad (12{\cdot}7)
$$

$$
\begin{aligned}
\frac{2F'(\zeta)\bar{F}'(\bar{\zeta})(H_\eta + \chi)}{D} = {} & i\left\{B_2 F'(\zeta) - C_2 \bar{F}'(\bar{\zeta})\right\}\left\{F'(\zeta) + \lambda_1 \bar{F}'(\bar{\zeta})\right\} \\
& \times f''\left\{F(\zeta) + \lambda_1 \bar{F}(\bar{\zeta})\right\} \\
& + i\left\{B_1 F'(\zeta) - C_1 \bar{F}'(\bar{\zeta})\right\}\left\{F'(\zeta) + \lambda_2 \bar{F}'(\bar{\zeta})\right\} \\
& \times g''\left\{F(\zeta) + \lambda_2 \bar{F}(\bar{\zeta})\right\},
\end{aligned}
\qquad (12{\cdot}8)
$$

$$\frac{G_\xi + G_\eta}{D} = -\left\{4\lambda_1 \frac{c_{12}}{c_{11}} - (\epsilon_2 - 1)(1+\lambda_1)^2\right\} f''\{F(\zeta) + \lambda_1 \bar{F}(\bar{\zeta})\}$$
$$- \left\{4\lambda_2 \frac{c_{12}}{c_{11}} - (\epsilon_1 - 1)(1+\lambda_2)^2\right\} g''\{F(\zeta) + \lambda_2 \bar{F}(\bar{\zeta})\}, \qquad (12{\cdot}9)$$

where B_1, B_2, C_1, C_2 are the constants defined in equations (5·16) to (5·19).

The boundary conditions are, as in the isotropic case,

$$G_\eta = 0, \quad H_\eta + \chi = \text{constant}, \qquad (12{\cdot}10)$$

and in order to satisfy them functions $V(\zeta)$ and $W(\zeta)$ are defined so that the real part of $V(\zeta) = -G_\eta/D$ and the imaginary part of $W(\zeta)$ = the variable part of $(H_\eta+\chi)/D$ on the boundary. From equations (12·7) and (12·8) it is seen that they may be written in the forms

$$2F'(\zeta)\bar{F}'(\zeta)V(\zeta) = \{B_2 F'(\zeta) + C_2 \bar{F}'(\zeta)\}\{F'(\zeta) + \lambda_1 \bar{F}'(\zeta)\}$$
$$\times f''\{F(\zeta) + \lambda_1 \bar{F}(\zeta)\}$$
$$+ \{B_1 F'(\zeta) + C_1 \bar{F}'(\zeta)\}\{F'(\zeta) + \lambda_2 \bar{F}'(\zeta)\}$$
$$\times g''\{F(\zeta) + \lambda_2 \bar{F}(\zeta)\}, \qquad (12{\cdot}11)$$

$$2F'(\zeta)\bar{F}'(\zeta)W(\zeta) = \{C_2 \bar{F}'(\zeta) - B_2 F'(\zeta)\}\{F'(\zeta) + \lambda_1 \bar{F}'(\zeta)\}$$
$$\times f''\{F(\zeta) + \lambda_1 \bar{F}(\zeta)\}$$
$$+ \{C_1 \bar{F}'(\zeta) - B_1 F'(\zeta)\}\{F'(\zeta) + \lambda_2 \bar{F}'(\zeta)\}$$
$$\times g''\{F(\zeta) + \lambda_2 \bar{F}(\zeta)\}, \qquad (12{\cdot}12)$$

$$V(\zeta)+W(\zeta) = C_2\left\{1+\lambda_1\frac{\bar{F}'(\zeta)}{F'(\zeta)}\right\} f''\{F(\zeta)+\lambda_1\bar{F}(\zeta)\}$$

$$+C_1\left\{1+\lambda_2\frac{\bar{F}'(\zeta)}{F'(\zeta)}\right\} g''\{F(\zeta)+\lambda_2\bar{F}(\zeta)\}, \qquad (12\cdot13)$$

$$V(\zeta)-W(\zeta) = B_2\left\{\lambda_1+\frac{F'(\zeta)}{\bar{F}'(\zeta)}\right\} f''\{F(\zeta)+\lambda_1\bar{F}(\zeta)\}$$

$$+B_1\left\{\lambda_2+\frac{F'(\zeta)}{\bar{F}'(\zeta)}\right\} g''\{F(\zeta)+\lambda_2\bar{F}(\zeta)\}. \qquad (12\cdot14)$$

The analysis is restricted by the fact that, although $F'(\zeta)$ has no zeros, $\bar{F}'(\zeta)$ has zeros but no poles outside the ellipse. $V(\zeta)$ and $W(\zeta)$ will therefore have poles at these zeros but $V(\zeta)+W(\zeta)$ must be free from poles as may be seen from equation (12·13).

Very complicated analysis would be involved if an attempt were made to find the distribution of stress-couple at any point of the plate, and the results would be of little practical value when obtained. Attention is therefore confined to the tangential stress-couple on the boundary for which it is only necessary to evaluate $f''\{F(\zeta)+\lambda_1\bar{F}(\zeta)\}$ and $g''\{F(\zeta)+\lambda_2\bar{F}(\zeta)\}$. From equations (12·13) and (12·14) these two functions are obtained in the forms

$$4(\lambda_1-\lambda_2)K\{F'(\zeta)+\lambda_1\bar{F}'(\zeta)\} f''\{F(\zeta)+\lambda_1\bar{F}(\zeta)\}$$

$$= B_1\{V(\zeta)+W(\zeta)\}F'(\zeta) - C_1\{V(\zeta)-W(\zeta)\}\bar{F}'(\zeta), \qquad (12\cdot15)$$

$$4(\lambda_1-\lambda_2)K\{F'(\zeta)+\lambda_2\bar{F}'(\zeta)\}\,g''\{F(\zeta)+\lambda_2\bar{F}(\zeta)\}$$
$$=-B_2\{V(\zeta)+W(\zeta)\}F'(\zeta)+C_2\{V(\zeta)-W(\zeta)\}\bar{F}'(\zeta), \qquad (12{\cdot}16)$$

where K has the value given in equation (5·24).

If the functions $V(\zeta)$ and $W(\zeta)$ are chosen in order to cancel on the boundary the values of G_η and $H_\eta+\chi$ produced by the stress distribution at infinity, the extra tangential stress-couple on the edge of the ellipse is obtained by writing $\zeta=\bar{\zeta}$ in equation (12·6) and using the values of $f''\{F(\zeta)+\lambda_1\bar{F}(\zeta)\}$ and $g''\{F(\zeta)+\lambda_2\bar{F}(\zeta)\}$ obtained from equations (12·15) and (12·16).

CYLINDRICAL BENDING ABOUT THE MAJOR AXIS.

13. The conditions in the absence of the hole are exactly as given in §8, the required displacement and stress-distribution being

$$w=-\frac{Gy^2}{2Dk_4}, \qquad (13{\cdot}1)$$

$$G_1=\frac{c_{12}}{c_{11}k_4}G,\quad G_2=G,\quad H_1=N_1=N_2=\chi=0. \qquad (13{\cdot}2)$$

Hence, from equations (12·3) – (12·5)

$$\frac{2G_\xi}{G}=1+\frac{c_{12}}{c_{11}k_4}-\left(1-\frac{c_{12}}{c_{11}k_4}\right)\cos 2\phi, \qquad (13{\cdot}3)$$

$$\frac{2G_\eta}{G} = 1 + \frac{c_{12}}{c_{11}k_4} + \left(1 - \frac{c_{12}}{c_{11}k_4}\right)\cos 2\phi, \tag{13·4}$$

$$\frac{2H_\eta}{G} = \left(1 - \frac{c_{12}}{c_{11}k_4}\right)\sin 2\phi, \tag{13·5}$$

and $V(\zeta)$ and $W(\zeta)$ must be defined in order to cancel the values of G_η and $H_\eta + \chi$ on the boundary $\eta = 0$.

All the necessary conditions are found to be satisfied by writing

$$\frac{2DV(\zeta)}{G} = \left(1 + \frac{c_{12}}{c_{11}k_4}\right)\left\{\frac{\bar{a}_0}{\bar{F}'(\zeta)} + \frac{b_1}{F'(\zeta)}\right\}e^{i\zeta} + \left(1 - \frac{c_{12}}{c_{11}k_4}\right)\left\{\frac{\bar{a}_0}{F'(\zeta)} + \frac{b_1}{\bar{F}'(\zeta)}\right\}e^{i\zeta}, \tag{13·6}$$

$$\frac{2DW(\zeta)}{G} = \left(1 + \frac{c_{12}}{c_{11}k_4}\right)\left\{-\frac{\bar{a}_0}{\bar{F}'(\zeta)} + \frac{b_1}{F'(\zeta)}\right\}e^{i\zeta} + \left(1 - \frac{c_{12}}{c_{11}k_4}\right)\left\{\frac{\bar{a}_0}{F'(\zeta)} - \frac{b_1}{\bar{F}'(\zeta)}\right\}e^{i\zeta}, \tag{13·7}$$

$$\frac{D\bar{F}'(\zeta)\{V(\zeta) - W(\zeta)\}}{G} = \left(1 + \frac{c_{12}}{c_{11}k_4}\right)\bar{a}_0 e^{i\zeta} + \left(1 - \frac{c_{12}}{c_{11}k_4}\right)b_1 e^{i\zeta}, \tag{13·8}$$

$$\frac{DF'(\zeta)\{V(\zeta) + W(\zeta)\}}{G} = \left(1 + \frac{c_{12}}{c_{11}k_4}\right)b_1 e^{i\zeta} + \left(1 - \frac{c_{12}}{c_{11}k_4}\right)\bar{a}_0 e^{i\zeta}. \tag{13·9}$$

Using equations (12·6), (12·15) and (12·16) and adding in the portion transmitted from infinity it is found after some very heavy reduction that the stress-couple on the periphery of the hole is given by equation (13·10).

$$4S(\xi)\,K\,G_\xi/G$$

$$= 2\{2(1-d)e^{4\alpha}P_1 + 2d(1-d)e^{-4\alpha}P_2 + P_3e^{2\alpha} + P_4e^{-2\alpha} + 2P_5\}\cos 4\xi$$
$$+2\{Q_1e^{4\alpha} + Q_2e^{-4\alpha} + Q_3e^{2\alpha} + Q_4e^{-2\alpha} + 3Q_5\}\cos 2\xi$$
$$+R_1e^{4\alpha} + R_2e^{-4\alpha} + 2R_3e^{2\alpha} + 2R_4e^{-2\alpha} + 4R_5, \qquad (13\cdot10)$$

where $\operatorname{sech}\alpha$ is the eccentricity of the ellipse and K, P_1, Q_1, R_1 are given by equations (8·22) – (8·25).

$$S(\xi) = (1+d^2+4\lambda^2)(\cosh 2\alpha - \cos 2\xi)^2$$
$$+d\,(\cosh 4\alpha\cos 4\xi - 4\cosh 2\alpha\cos 2\xi + 3)$$
$$+2\lambda(1+d)(\cosh 2\alpha\cos 4\xi - \cosh 4\alpha\cos 2\xi - 3\cos 2\xi + 3\cosh 2\alpha), \qquad (13\cdot11)$$

and the remaining coefficients may be obtained in turn from the equations

$$P_2 = \frac{1-\lambda}{1+2\lambda+d}\left(\frac{c_{12}}{c_{11}}\right)^2 + 2\,\frac{\lambda(1+d)+2d}{(1+d)^2-4\lambda^2}\left(\frac{c_{12}}{c_{11}}\right) + \epsilon\,\frac{\{\lambda(1-d)-2\lambda^2-1+3d\}}{(1-2\lambda+d)^2}, \qquad (13\cdot12)$$

$$Q_2 = \left\{1-\left(\frac{c_{12}}{c_{11}\epsilon}\right)^2\right\}\left\{d\,(1+2\lambda+d)\,\frac{c_{12}}{c_{11}} + \epsilon(2\lambda+3d-d^2)\right\}, \qquad (13\cdot13)$$

$$Q_3 = \left\{1-\left(\frac{c_{12}}{c_{11}\epsilon}\right)^2\right\}\left\{\left[8\lambda^2+2\lambda(3+d)+1-d^2\right]\frac{c_{12}}{c_{11}}\right.$$
$$\left. +\,\epsilon\left[8\lambda^2+2\lambda(3+d)+(5-3d)(1-d)\right]\right\}, \qquad (13\cdot14)$$

$$Q_4 = Q_3 + 2(1-d)(1+2\lambda+d)\left\{\left(\frac{c_{12}}{c_{11}\epsilon}\right)^2 - 1\right\}\left\{\frac{c_{12}}{c_{11}} + \epsilon\right\}, \qquad (13\cdot15)$$

$$Q_5 = Q_1 + Q_2, \tag{13·16}$$

$$(1+d)P_3 = 4\lambda(1-d)(dP_1 - P_2) - Q_5, \tag{13·17}$$

$$P_4 = -Q_5 - P_3, \tag{13·18}$$

$$P_5 = -(1-d)(P_1 + dP_2) - \tfrac{1}{8}(Q_3 + Q_4), \tag{13·19}$$

$$R_2 = 4P_5 - R_1, \tag{13·20}$$

$$R_3 = \tfrac{1}{2}(P_4 - P_3) - \tfrac{5}{2}Q_1 - \tfrac{1}{2}Q_2, \tag{13·21}$$

$$R_4 = -3Q_5 - R_3, \tag{13·22}$$

$$R_5 = -P_5 - \tfrac{3}{8}(Q_3 + Q_4). \tag{13·23}$$

Taking the limit of equation (13·10) when λ, μ tend to zero gives for the couple round an elliptical hole in an isotropic plate,

$$\frac{G_\xi}{G} = (1+\sigma)\left\{1 - \frac{(1-\sigma)e^{2\alpha} + 1 + \sigma}{3+\sigma} \cdot \frac{e^{-2\alpha} - \cos 2\xi}{\cosh 2\alpha - \cos 2\xi}\right\}, \tag{13·24}$$

which agrees with equation (6·2) of section C. The result for the circular hole is also obtained in the form (8·21) by allowing α to tend to infinity.

14. Calculations have been carried out to find the numerical values of the stress-concentration in a specimen of spruce. The results are tabulated in table 17 and plotted in fig. 16, the isotropic values being included for comparison.

Table 17

Concentration of stress-couple round the elliptical boundary. Plate bent about the major axis, which is parallel to $\xi = 0$.

ξ°	Spruce-grain parallel to the major axis.	Isotropic material.	Spruce-grain parallel to the minor axis.
0	1·63	2·21	4·04
10	1·57	2·11	3·28
20	1·38	1·84	1·73
30	1·10	1·52	0·863
40	0·784	1·23	0·478
50	0·633	0·991	0·294
60	0·982	0·816	0·198
70	1·96	0·698	0·145
80	3·06	0·631	0·119
90	3·53	0·609	0·111

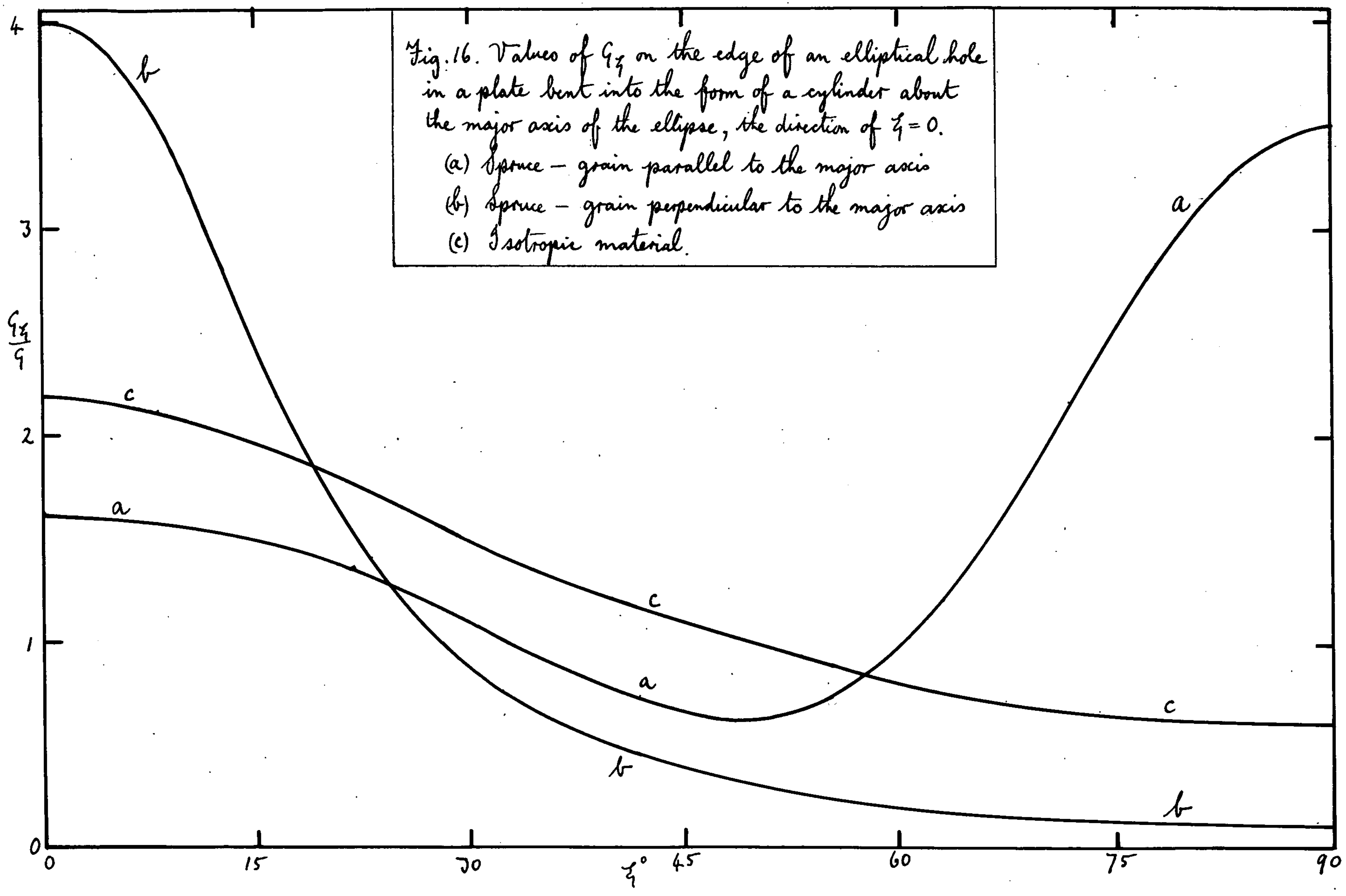

Fig. 16. Values of G_ξ on the edge of an elliptical hole in a plate bent into the form of a cylinder about the major axis of the ellipse, the direction of $\xi = 0$.
(a) Spruce – grain parallel to the major axis
(b) Spruce – grain perpendicular to the major axis
(c) Isotropic material.

Comparison with table 14 shews that there is a close resemblance between the stress-concentration in the cases of the ellipse and the circle, the difference being greatest at the ends of the major axis of the ellipse. The concentration at these two points increases without limit when the ellipse is very slender, but at other points of the boundary the factor of concentration tends to the finite constant value

$$G_{\xi} = -G\left\{\frac{c_{12}}{c_{11}\epsilon} + 1\right\}^2\left\{\epsilon - \frac{c_{12}}{c_{11}}\right\}\Big/ K, \qquad (14\cdot1)$$

as the length of the minor axis tends to zero, and this expression has the value $(1+\sigma)^2/(3+\sigma)$ in the isotropic case.

An interesting feature of the numerical results for both the circle and the ellipse is that the factor of stress-concentration depends more on the direction of the grain than on the direction of bending. For example, in the circular case, when the plate is bent about the line $\theta = 0$, the concentration is a minimum for an isotropic material at the point $\theta = \frac{1}{2}\pi$. For an aeolotropic material, however, this becomes a maximum value when the grain is parallel to $\theta = 0$, and is in fact greater than the highest value obtained when the grain is perpendicular to $\theta = 0$, a result which would not be anticipated from a cursory examination of the conditions.

It must be borne in mind, however, that the displacement is only maintained if there is an additional couple $\frac{c_{12} G}{c_{11} k_4}$ about the y-axis and the magnitude of this couple depends on the nature of the material. In the case of spruce it is $0{\cdot}56G$ when the grain is parallel to $\theta = 0$, and $0{\cdot}021G$ when the grain is perpendicular to $\theta = 0$, and this large variation in the magnitude of the compensating couple necessary to ensure that the plate is bent into the form of a cylinder will produce a correspondingly large variation in the value of the stress-concentration factor. In general the factor for a bent plate is lower than the corresponding value for a similar plate under tension, with the exception that the factor never becomes negative in the case of pure bending, whereas a portion of the boundary in a stretched plate is always under compression.

REFERENCES

Airy, G.B.	1862	Brit. Assoc. Rep. p. 82.
Airy, G.B.	1863	Phil. Trans. 153, 49.
Bickley, W.G.	1924	Phil. Mag. 6, 48, 1014.
Bickley, W.G.	1928	Phil. Trans. A, 227, 383.
Coker, E.G. and Coleman, G.P.	1930	Proc. Roy. Soc. A, 128, 418
Coker, E.G. and Filon, L.N.G.	1931	"Photoelasticity" (Cambridge).
Drucker, D.C.	1942	J. Appl. Mech. 9, A-161.
Filon, L.N.G.	1903	Phil. Trans. A, 201, 63.
Goodier, J.N.	1936	Phil. Mag. 7, 22, 69.
Green, A.E.	1939	Proc. Roy. Soc. A, 173, 173.
Green, A.E.	1940	Proc. Roy. Soc. A, 176, 121.
Green, A.E.	1942	Proc. Roy. Soc. A, 180, 173
Green, A.E.	1943a	Proc. Roy. Soc. Not yet published.
Green, A.E.	1943b	Phil. Mag. 7, 34, 416.
Green, A.E.	1943c	Phil. Mag. 7, 34, 420.
Green, A.E. and Taylor, G.I.	1939	Proc. Roy. Soc. A, 173, 162.
Green, A.E. and Taylor, G.I.	1940	Proc. Roy. Soc. Not yet published.
Howland, R.C.J.	1930	Phil. Trans. A, 229, 49.
Howland, R.C.J.	1935	Proc. Roy. Soc. A, 148, 471.

Howland, R.C.J. and Knight, R.C.	1939	Phil. Trans. A, 238, 357.
Howland, R.C.J. and Stevenson, A.C.	1933	Phil. Trans. A, 232, 155.
Huber, M.T.	1938	"Contributions to the mechanics of solids dedicated to S. Timoshenko" p.89.
Ibbetson, W.J.	1886	Proc. Lond. Math. Soc. 17, 296.
Inglis, C.E.	1913	Trans. Inst. Naval Arch. London. [Also "Engineering" v.95 p.415]
Inglis, C.E.	1922	Trans. Inst. Naval Arch.
Jeffery, G.B.	1920	Phil. Trans. A, 221, 265.
Kirsch, G.	1898	Z. des Ver. Deutsches. Ingen. v32. p797
Kolosoff, G.	1910	Dissertation, St. Petersburg.
Love, A.E.H.	1927	"The mathematical theory of elasticity." 4th ed. (Cambridge)
Michell, J.H.	1900a	Proc. Lond. Math. Soc. 31, 100.
Michell, J.H.	1900b	Proc. Lond. Math. Soc. 32, 35.
Michell, J.H.	1900c	Proc. Lond. Math. Soc. 32, 247.
Muschelisvili, N.	1933	Z. angew Math. Mech. 13, 264.
Okubo, H.	1937	Sci. Rep. Tôhoku Univ. (ser. 1.) 25, 1110.
Okubo, H.	1939	Phil. Mag. 7, 27, 508.
Pöschl, Th.	1921	Math. Z. 11, 95.
Sen, B.	1939	Phil. Mag. 7, 27, 596.
Sokolnikoff,	1942	Bull. Amer. Math. Soc. 48, 533.
Stevenson, A.C.	1942	Phil. Mag. 7, 33, 639.

Stevenson, A.C.	1943	Phil. Mag. 7, 34, 766.
Tuzi, Z.	1928	Sci. Pap. Inst. Phys. and Chem. Res. (Tokyo) No. 156
Tuzi, Z.	1930	Phil. Mag. 7, 9, 210.
Westergaard, H.M.	1938	"Contributions to the mechanics of solids dedicated to S. Timoshenko." p. 268.
Wolf, K.	1935	Z. angew Math. Mech. 15, 249.