Some problems on random walks

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SOME PROBLEMS ON RANDOM WALKS

by

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CONTENTS

INTRODUCTION

Given a random variable X taking values on a space E, suppose that X_1, X_2, ... is a sequence of independent random variables each of which has the same distribution as X. If S_n denotes the nth partial sum $\sum_{j=1}^{n} X_j$ of this sequence, we can think of S_n as the position in E at time n of a particle which starts at the origin at time zero and receives the random displacement X_j at time j, and we refer to the sequence $\{S_n\}$ as the random walk (R.W.) generated by the random variable (R.V) X. Throughout this thesis the state-space E is either k-dimensional Euclidean space E_k or the lattice L_k of points in E_k with integral coordinates, and $k \geqslant 2$. This is the only feature that the three topics discussed have in common.

If $P^*(B) = P\{S_n \varepsilon B$ for an infinite number of integers $n\}$ it is known that for any R.W $P^*(B) = 1$ or 0 for every subset B of E, and the problem of deciding whether a given set B is a 1-set (i.e. $P^*(B) = 1$) or a 0-set (i.e. $P^*(B) = 0$) has received some attention. In the special case of the simple R.W on L_k for $k \geqslant 3$ this question was answered completely by Îto and McKean [19], and their solution was extended by Lamperti [21] to a large class of R.W's. It is obvious from the form of Îto and McKean's criterion that earlier attempts to find a condition of the form $\sum_{\underline{b} \varepsilon B} f(|\underline{b}|) = +\infty$ necessary and sufficient for B to be a 1-set were doomed to failure, yet conditions of this form which are either necessary or sufficient are not without interest. In chapter I such conditions are derived for the simple 3-dimensional R.W from Îto and McKean's criterion and are shown to be the best possible of this form. It is also proved that no condition of the more general type $\sum_{\underline{b} \varepsilon B} f(\underline{b}) = +\infty$ can be necessary and sufficient for B to a 1-set.

When we know that the particle is almost certain to visit a set we can define a R.V whose value is its position when it first does so. This R.V will be a function of the starting point of the R.W and if in the lattice case the absorbing set is taken to be a coordinate axis its characteristic function is easy to calculate in terms of the characteristic function of X. Thus in Chapter II we derive from the simple 3-dimensional R.W a doubly infinite sequence of R.V's F_{ab} which we investigate firstly for fixed a and b and then as $r = (a^2 + b^2)^{\frac{1}{2}} \rightarrow +\infty$. In the latter case one might expect there to exist a norming function d(r) such that $F_{ab}/d(r)$ has a non-degenerate limiting distribution as $r \rightarrow +\infty$. This fails to happen, and it fails to happen in such a way that we are lead first to postulate and then to prove that a non-degenerate limiting distribution exists for $\log|F_{ab}|/d(r)$ for suitable choice of $d(r)$. Though no attempt is made to extend this analysis to the general RW on L_3 it is not difficult to see that the argument leading to the non-existence of a limiting distribution for $F_{ab}/d(r)$ can be generalised. However the method used to investigate $\log|F_{ab}|$ depends upon specific properties of the simple R.W and does not seem to apply in general. It is interesting that, according to Ridler-Rowe [28], in the similar problem concerning the time at which a R.W first hits an axis the logarithmic change of scale leads to a limiting distribution in the general case.

When the state-space E is the positive half-line the renewal function $H(x) = \sum_{n=0}^{\infty} P\{S_n \leqslant x\}$ has an obvious physical interpretation and has been much studied. The central result is the Renewal Theorem, which, if X is a non-lattice R.V , says

$$\lim_{x \to +\infty} \{H(x + a) - H(x)\} = a\, m^{-1}, \text{ where } 0 < m = \mathcal{E}(X) \leqslant \infty.$$

From our point of view $H(x + a) - H(x)$ is the expected number of visits of the particle to the interval $(x, x + a)$ and as such it makes sense when X can take positive and negative values:

several authors have shown that the Renewal Theorem still holds in this case if $m > 0$. When $\underline{X}$ takes values in E_k for $k \geq 2$ the analogue of $H(x)$ may fail to exist but $G(A) = \sum_{n=0}^{\infty} P\{\underline{S_n} \in A\}$ exists for any bounded Borel-measurable set A, unless k = 2 and $\underline{X}$ has zero mean. This function was studied in 1952 by Chung [7] who proved that if $\underline{X}$ has non-zero mean $\lim_{|\underline{x}| \to \infty} G\{A + \underline{x}\} = 0$. Except for a slight weakening of the conditions under which this holds, due to Feller [16], the literature contains no improvements upon this 'Renewal Theorem in higher dimensions'. The bulk of the present work, however, is devoted to an investigation of the rate at which $G(A + \underline{x}) \to 0$ as $|x| \to +\infty$. Since when $\underline{X}$ has non-zero mean $\underline{m}$ one might expect the behaviour of $G(A + \underline{x})$ for large values of $|\underline{x}|$ to depend upon the angle between $\underline{x}$ and $\underline{m}$, the problem considered in Chapters III and IV is that of finding the asymptotic behaviour as $x \to +\infty$ of $G(A + x\underline{j})$, where $\underline{j}$ is any fixed vector. The solution presented uses a straight-forward Fourier analytic argument and applies under conditions which are not particularly restrictive unless $k > 4$, when the existence of moments of X of order higher than the second is required. In the zero mean case we are able, in Chapter V, to find an asymptotic estimate for $G(A + \underline{x})$ which holds as $|\underline{x}| \to +\infty$

in any manner, but again the result for $k > 4$ is marred by a probably superfluous condition on the higher moments of $\underline{X}$.

<u>Note</u> Since the completion of this thesis F. Spitzer has published a theorem [p. 307, Principles of Random Walk, Van Nostrand, 1964] which is stronger than theorem 1.1 of chapter V. He shows that our assumption (5.1.3) is superfluous.

CHAPTER I

§ 1. If the distribution of $\underline{X}$ on L_k is given by

$$P\{\underline{X} = (\pm 1, 0, \ldots, 0)\} = P\{\underline{X} = (0, \pm 1, 0, \ldots, 0)\} = \ldots$$
$$= P\{\underline{X} = (0, 0, \ldots, 0, \pm 1)\} = \frac{1}{2k},$$

the particle has at time n probability $\frac{1}{2k}$ of moving to each each of its 2k neighbours, and $\underline{X}$ generates the simple k-dimensional R.W. Blackwell [1] proved that any R.W on $L_k (K \geqslant 1)$ has the property that $P^*(B) = P\{S_n \varepsilon\ B$ for an infinite number of integers $n\} = 1$ or 0 for any subset B of L_k [this also follows from the 0 or 1 law of Hewitt and Savage [17]] and Îto and McKean [19] characterised the 0 and 1 sets for the simple k-dimensional R.W. $(k \geqslant 3)$ by what they called Wiener's test. This, for k = 3, is

$$(1.1) \quad \sum_{n=1}^{\infty} 2^{-n} \hat{C}(B_n) \begin{smallmatrix} = \\ < \end{smallmatrix} + \infty \iff P^*(B) = \begin{smallmatrix} 1 \\ 0 \end{smallmatrix},$$

where $\hat{C}(B_n)$ is the 'discrete capacity' of the set B_n, the intersection of B and the spherical shell $2^n \leqslant |\underline{a}| < 2^{n+1}$. Since Îto and McKean also showed that

$$\dagger(1.2) \quad k_1\ C(\hat{A}) \leqslant \hat{C}(A) \leqslant k_2 C(\hat{A}), \text{ where } C(\hat{A})$$

is the Newtonian capacity of the set $\hat{A}$ derived from the set of lattice points A by centreing at each point of A a unit cube with edges parallell to the coordinate axes, we may replace $\hat{C}(B_n)$ by $C(\hat{B}_n)$ in (1.1). In this chapter we use this

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† . k_1, k_2, ... denote positive constants.

'classical' form of (1.1) and some results of classical potential theory to investigate criteria of the form

$\sum_{\underline{b} \in B} f(|\underline{b}|) = +\infty$ for B to be a 1-set.

Plainly, since the capacity of a set depends intrinsically upon its geometrical configuration, we cannot hope to get a condition of this type which is both necessary and sufficient for B to be a 1-set. This is made explicit in §.3 where we prove;

(1.3) there is no positive f such that

$$\sum_{\underline{b} \in B} f(\underline{b}) \begin{matrix} = \\ < \end{matrix} +\infty \iff P^*(B) = \begin{matrix} 1 \\ 0 \end{matrix} .$$

[(1.3) is well-known: according to Breiman [4] it has been proved by P. Erdős and B. J. Murdoch (unpublished)]

In §.4 we prove;

(1.4) $\sum_{\underline{b} \in B} \frac{1}{|\underline{b}|^3} = +\infty \implies P^*(B) = 1,$

(1.5) $\sum_{\underline{b} \in B} \frac{1}{|\underline{b}|^2} = +\infty$ and B a set of coplanar points $\implies P^*(B) = 1,$

(1.6) $\sum_{\underline{b} \in B} \frac{1}{|\underline{b}| \log |\underline{b}|} = +\infty$ and B a set of collinear points $\implies P^*(B) = 1.$

[Here, and throughout this chapter we adopt the convention that, in sums of this form $\frac{1}{|b|} = 1$ when $\underline{b} = \underline{0}$ and $\log|\underline{b}| = 1$ when $|\underline{b}| \leqslant 1$.]

On account of (1.3) it is no surprise that there is quite a gap between (1.4)-(1.6) and our only necessary condition:

(1.7) $P^*(B) = 1 \Rightarrow \sum_{b \in B} \frac{1}{|\underline{b}|} = +\infty$. This gap cannot be narrowed: for we show in §.6 that, given an arbitrary positive-valued function f with $\lim_{x \to +\infty} f(x) + \infty$,

(1.8) there is a 0-set with $\sum_{\underline{b} \in B} \frac{f(|\underline{b}|)}{|\underline{b}|^3} = +\infty$,

(1.9) there is a 0-set in a plane with $\sum_{\underline{b} \in B} \frac{f(|\underline{b}|)}{|\underline{b}|^2} = +\infty$,

(1.10) there is a 0-set in a line with $\sum_{\underline{b} \in B} \frac{f(|\underline{b}|)}{|\underline{b}| \log |\underline{b}|} = +\infty$,

and given an arbitrary non-negative function f with

$\lim_{x \to +\infty} f(x) = 0$,

(1.11) there is a 1-set with $\sum_{\underline{b} \in B} \frac{f(|\underline{b}|)}{|\underline{b}|} < +\infty$.

§.2 In this section we gather together what we require from classical potential theory. If G is a compact region in E_3 and $v(\underline{x})$ is a function which is harmonic throughout E_3, vanishes as $|\underline{x}| \to +\infty$ and is equal to one on G, then G has Newtonian capacity C(G) given by

(2.1) $C(G) = \frac{1}{4\pi} \iint_S \frac{\partial v}{\partial n} \, ds$, where S is any smooth surface containing G. C(G) may also be characterised by (see, e.g. Brelot [5, p.50])

(2.2) $C(G) = \max \int_G e(d\underline{x})$: $e \geqslant 0$, $e = 0$ off G and $P(\underline{y}) = \int_G \frac{e(d\underline{x})}{|\underline{y}-\underline{x}|} \leqslant 1$ for all $\underline{y}$.

The geometric operation of symmetrization with respect to a plane P changes a solid B into another solid B' characterised by:

(2.3) B' is symmetric with respect to P.

(2.4) Any straight line perpendicular to P that intersects one of B and B' intersects the other. Both intersections have the same length, and the intersection with B' consists of just one line segment, which is bisected by P.

We will use the fact [Polya and Szegö, [25]) that symmetrization of a solid does not increase its capacity.

Lemma 2.5 The capacity of any solid consisting of n non-intersecting unit cubes is greater than or equal to $k_3 n^{\frac{1}{3}}$.

Proof This is a particular case of the Poincaré-Faber-Szegö inequality, $C(G) \geqslant (\frac{3V(G)}{4\pi})^{\frac{1}{3}}$, where V(G) is the volume of the region G. See, e.g. Polya and Szegö [26, p.63].

Lemma 2.6 The capacity of any solid consisting of n unit cubes whose centres are coplanar lattice points and whose faces are parallel to the coordinate axes is not less than $k_4 n^{\frac{1}{2}}$.

Proof First, note that we can reduce the general situation to the case when the centres lie in a coordinate plane by symmetrization with respect to a suitably chosen coordinate plane. Now the solid consisting of such an arrangement of cubes is cylindrical, and it is a special case of a theorem of Polya and Szegö on symmetrization with respect to a line {for a definition of this operation and a proof of the theorem see Polya and Szegö [25, pp.8-11]} that the capacity of any cylindrical solid is never less than the capacity of a right circular cylinder that has the same volume and the same altitude as the given solid. Thus (2.6) is a consequence of the right hand side of:

(2.7) $k_5 a \geqslant C\left(\begin{matrix}\text{right circular cylinder of} \\ \text{radius } a(>1) \text{ and altitude } 1\end{matrix}\right) \geqslant k_6 a.$

To establish (2.7) note that it is an obvious consequence of (2.2) that $G_1 \subseteq G_2 \rightarrow C(G_1) \leqslant C(G_2)$. Now the cylinder contains an oblate spheroid S_1 of semi-major axes $(\frac{1}{2}, \frac{a}{2}, \frac{a}{2})$ and is contained in an oblate spheroid S_2 of semi-major axes $(1, a, a)$.

Since $C(S_2) = 2C(S_1) = a \frac{\sin^{-1} e}{e}$, where $e^2 = 1 - \frac{1}{a^2}$ so that $0 < e < 1$, the fact that $\frac{\sin^{-1} e}{e}$ is bounded for $0 \leqslant e \leqslant 1$ proves (2.7) and hence the lemma.

<u>Lemma 2.8</u> The capacity of any solid consisting of n unit cubes whose faces are parallel to the co-ordinate axes and centres are collinear is not less than $\frac{k_7 n}{\log n}$. {Here again $\log n = 1$ when $n = 1$}.

<u>Proof</u> Symmetrization with respect to a suitably chosen co-ordinate plane reduces the general case to the case in

which the line of centres lies in a co-ordinate plane and then symmetrization with respect to a plane perpendicular to the line of centres allows us to consider rhe cubes to be adjacent, when they form a rectangular block. Thus (2.8) is a consequence of the right hand side of:

(2.9) $$\frac{k_8 n}{\log n} \geqslant C\{\text{rectangular } n \times 1 \times 1 \text{ block}\} \geqslant \frac{k_9 n}{\log n}.$$

Now the block contains a prolate spheroid S_1 of semi-major axes $(\frac{n}{2}, \frac{1}{2}, \frac{1}{2})$ and is contained in a prolate spheroid S_2 of semi-major axes $(n, 1, 1)$ and, if $n \geqslant 2$ $C(S_2) = 2C(S_1) =$ $ne/\log \frac{1+e}{1-e}$, where $e^2 = 1 - \frac{1}{n^2}$. Since $\frac{1}{2} < e < 1$ and $n^2 \leqslant \frac{1+e}{1-e} = \frac{(1+e)^2}{1-e^2} \leqslant n^4$, this establishes (2.9) for $n \geqslant 2$ and it can obviously be extended to cover the case $n = 1$ by suitable adjustment of k_8 and k_9.

Lemma 2.10 The capacity of a solid consisting of n unit cubes whose centres are collinear and equally spaced at a distance $2d + 2$ apart and whose faces are parallel is not less than $k_{10}n$ provided $d > \log n$.

Proof Calling the cubes $A_1, A_2, \ldots, A_n$ with centres $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$ write $A = \bigcup_{r=1}^{n} A_r$ and in the characterisation (2.2) of capacity set $e(d\underline{x}) = k_{10}d\underline{x}$ when $\underline{x} \in A$. Then given any $\underline{y}$ we may renumber $A_1, \ldots, A_n$ in such a way that

$$\min_{1 \leqslant r \leqslant n} |\underline{y} - \underline{a}_r| = |\underline{y} - \underline{a}_1| \text{ and } |\underline{y} - \underline{a}_r| \geqslant (r-1)(d+1)$$

for r = 2, 3, ..., n. If $|\underline{y} - \underline{a}_r| > 1$, then $P_r(\underline{y}) = \int_{A_r} \frac{e(d\underline{x})}{|\underline{y}-\underline{x}|}$

$\leqslant k_{10} \int_{A_r} \frac{d\underline{x}}{(|\underline{y}-\underline{a}_r| - |\underline{a}_r - \underline{x}|)} \leqslant \frac{k_{10}}{(|\underline{y}-\underline{a}_r| - \sqrt{3}/2)}$, and if $|\underline{y} - \underline{a}_1| \leqslant 1$

then $P_1(\underline{y}) \leqslant k_{10} \int_{(|\underline{x}-\underline{a}_1| \leqslant 1)} \frac{d\underline{x}}{|\underline{y}-\underline{x}|} = k_{10} \int_{(|\underline{z}+\underline{y}-\underline{a}_1| \leqslant 1)} \frac{d\underline{z}}{|\underline{z}|}$

$$\leqslant k_{10} \int_{(|\underline{z}| \leqslant 2)} \frac{d\underline{z}}{|\underline{z}|} = 4\pi\, k_{10}.$$

Thus $P_1(\underline{y}) \leqslant k_{10}.k_{11}$ and $P_r(\underline{y}) \leqslant \frac{k_{10}}{(r-1)d + (r-1-\sqrt{3}/2)} \leqslant \frac{k_{10}}{(r-1)d}$ for $r \geqslant 2$,

and so $P(\underline{y}) = \sum_{r=1}^{n} P_r(\underline{y}) \leqslant k_{10}(k_{11} + \sum_{r=1}^{n-1} \frac{1}{rd})$. Since $\sum_{r=1}^{n-1} \frac{1}{r}$

$\leqslant 1 + \log n$, $k_{11} + \frac{1}{d} \sum_{r=1}^{n-1} \frac{1}{r}$ is bounded for all n by the assumption that $d > \log n$, and so $P(\underline{y}) \leqslant 1$ for all $\underline{y}$ for some choice of $k_{10} > 0$. Then $C(A) \geqslant \int_A k_{10}\, d\underline{x} = nk_{10}$.

§.3 In this section we prove (1.3), which says that there is no positive-valued function f such that:

$$(3.1) \quad \sum_{\underline{b} \in B} f(\underline{b}) \mathrel{\substack{=\\<}} +\infty \iff P^*(B) = \substack{1\\0} .$$

We use a 3-dimensional version of the argument by which Breiman [4] proved (1.3) for a 1-dimensional R.W.

Denoting by A_n the spherical shell of lattice points $\{\underline{a} : 2^n \leqslant |\underline{a}| < 2^{n+1}\}$ and by $I(n, m, \alpha)$ the rectangular block of lattice points

$\{\underline{a} : 2^n + m < a_1 \leqslant 2^n(1 + \alpha) + m,\ 0 < a_2 \leqslant 2^n\alpha,\ 0 < a_3 \leqslant 2^n\}$, we have

<u>Lemma 3.2</u> If $I = \bigcup_{r=1}^{\infty} I_r$, where $I_r = I(n_r, m_r, \alpha_r)$ and $\{n_r\}$ is an increasing sequence of positive integers $\{m_r\}$ a sequence of non-negative integers and $\{\alpha_r\}$ a sequence of positive real numbers satisfying, for each $r \geqslant 1$,

(3.3): $I_r \subseteq A_{n_r}$ and $2^{n_r}\alpha_r > 1$,

then

(3.4): $\sum_{r=1}^{\infty} \frac{1}{\log\frac{1}{\alpha_r}} \begin{smallmatrix} = \\ < \end{smallmatrix} + \infty \iff P*(I) = \begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$.

<u>Proof</u>

Note that α_r is necessarily less than one (otherwise $I_r \nsubseteq A_{n_r}$) and that $\hat{I}_r$ is a solid rectangular block $[2^{n_r}\alpha_r] \times [2^{n_r}\alpha_r] \times 2^{n_r}$ (where $[x]$ denotes the largest positive integer $\leqslant x$). Thus the capacity $C(\hat{I}_r)$ of $\hat{I}_r$ is $[2^{n_r}\alpha_r] \times$ capacity of a rectangular block $1 \times 1 \times \frac{2^{n_r}}{[2^{n_r}\alpha_r]}$, and (2.9) implies that $2^{-n_r}C(\hat{I}_r)$ converges together with
$2^{-n_r} . [2^{n_r}\alpha_r] . \frac{2^{n_r}}{[2^{n_r}\alpha_r]} . \frac{1}{\log\frac{2^{n_r}}{[2^{n_r}\alpha_r]}} = \frac{1}{\log\frac{2^{n_r}}{[2^{n_r}\alpha_r]}}$; this plainly converges together with $\frac{1}{\log\frac{1}{\alpha_r}}$, and so (3.4) is a consequence of Wiener's test (1.1) and (1.2).

We now argue by contradiction, and assume the existence of a function f satisfying (3.1). Writing $F(n, m, \alpha) = \sum_{\underline{a} \in I(n,m,\alpha)} f(\underline{a})$, we define $g(\alpha)$ for $0 \leqslant \alpha \leqslant \alpha_0$ by

$$(3.5)\quad g(\alpha) = \lim_{n \longrightarrow +\infty} \inf \left\{ \inf_{m: I(n,m,\alpha) \subseteq A_n} F(n,m,\alpha) \right\},$$

where $1 > \alpha_0 = \sup\{\alpha : \text{for every } n \geq 1 \ \exists\ m \text{ with } I(n,m,\alpha) \subseteq A_n\} > 0$.

Lemma 3.6 Within its range of definition $g(\alpha)$ is non-decreasing and $g(\alpha + \beta) \geq g(\alpha) + g(\beta)$. $g(\alpha) < \infty$ for $0 \leq \alpha \leq \gamma$ for some $\gamma > 0$.

Proof The first assertion is immediate. As for the second, take $\alpha > 0$, $\beta > 0$ such that $\alpha + \beta < \alpha_0$ and write $m' = m + [2^n\alpha]$. Then $I(n, m', \beta) = \{\underline{a} : 2^n + m + [2^n\alpha] < a_1 \leq 2^n(1 + \beta) + m + [2^n\alpha],\ 0 < a_2 \leq 2^n\beta,\ 0 < a_3 \leq 2^n\} \subseteq \{\underline{a} : 2^n(1+\alpha) + m < a_1 \leq 2^n(1+\alpha+\beta)+n,\ 0 < a_2 \leq 2^n(\alpha+\beta),\ 0 < a_3 \leq 2^n\}$, so that $I(n, m, \alpha)$ and $I(n, m', \beta)$ are disjoint subsets of $I(n, m, \alpha+\beta)$. We therefore have,

$$(3.7)\quad F(n, m, \alpha+\beta) = \sum_{\underline{a} \in I(n,m,\alpha+\beta)} f(\underline{a}) \geq \sum_{I(n,m,\alpha)} f(\underline{a}) + \sum_{I(n,m',\beta)} f(\underline{a}) = F(n,m,\alpha) + F(n,m',\beta)$$

for all n and m. Now $\{m : I(n, m, \alpha+\beta) \subseteq A_n\} \subseteq \{m : I(n, m, \alpha) \subseteq A_n\}$ and $\{m' : I(n, m, \alpha+\beta) \subseteq A_n\} \subseteq \{m' : I(n, m', \beta) \subseteq A_n\}$, whence, for every n,

$$(3.8)\quad \inf_{m: I(n,m,\alpha+\beta) \subseteq A_n} F(n,m,\alpha) \geq \inf_{m: I(n,m,\alpha) \subseteq A_n} F(n,m,\alpha),$$

$$(3.9)\quad \inf_{m: I(n,m,\alpha+\beta) \subseteq A_n} F(n, m', \beta) \geq \inf_{m: I(n,m,\beta) \subseteq A_n} F(n,m,\beta).$$

(3.7), (3.8) and (3.9) in (3.5) yield

$$g(\alpha+\beta) = \liminf_{n \to +\infty} \left\{ \inf_{m:I(n,m,\alpha+\beta) \leq A_n} F(n,m,\alpha+\beta) \right\}$$

$$\geqslant \liminf_{n \to +\infty} \left\{ \inf_{m:I(n,m,\alpha) \leq A_n} F(n,m,\alpha) + \inf_{m:I(n,m,\beta) \leq A_n} F(n,m,\beta) \right\}$$

$$\geqslant \liminf_{n \to +\infty} \left\{ \inf_{m:I(n,m,\alpha) \leq A_n} F(n,m,\alpha) \right\} + \liminf_{n \to +\infty} \left\{ \inf_{m:I(n,m,\beta) \leq A_n} F(n,m,\beta) \right\}$$

$$= g(a) + g(\beta).$$

Suppose finally that $g(a) = +\infty$ for all $0 < \alpha \leqslant \alpha_0$. Then given any $0 < \alpha_r \leqslant \alpha_0$ with $\sum_{r=1}^{\infty} \frac{1}{\log \frac{1}{\alpha_r}} < +\infty$ we see that $F(n,0,\alpha_r) \to +\infty$ as $n \to +\infty$ for each α_r. Thus there is a sequence of positive integers N_r such that $F(n, 0, \alpha_r) \geqslant k_{12} > 0$ for all $n \geqslant N_r$ and hence an increasing sequence of integers n_r with $2^{n_r}\alpha_r > 1$ and $n_r \geqslant N_r$ for every r. Then by Lemma 3.2 $I = \bigcup_{r=1}^{\infty} I(n_r, 0, \alpha_r)$ is a 0-set yet $\sum_{\underline{a} \in I} f(\underline{a}) = \sum_{r=1}^{\infty} F(n_r, 0, \alpha_r) = +\infty$: this contradiction implies that for some γ in $[0, \alpha_0]$ $g(\gamma) < +\infty$, and hence by the first part of the lemma $g(a) < +\infty$ for $0 \leqslant a \leqslant \gamma$.

It follows from Lemma 3.6 that $g(\alpha) \leqslant k_{13}\alpha$ for $0 \leqslant \alpha \leqslant \gamma$ and some positive $k_{13} < +\infty$. For if $\alpha \in (0, \gamma)$ write $\gamma = n\alpha + \beta$ where $0 \leqslant \beta < \alpha$, and note that $g(\gamma) = g(n\alpha + \beta) \geqslant g(n\alpha)$ $\geqslant ng(\alpha)$ by repeated applications of the lemma. Thus $\frac{g(\alpha)}{\alpha}$

$$\leqslant \frac{g(\gamma)}{na} = \frac{n+1}{n}\,\frac{g(\gamma)}{(n+1)a} < \frac{2g(\gamma)}{\gamma} = k_{13} < \infty .$$

Now taking $\{a_r\}$ with $\sum_{r=1}^{\infty} \frac{1}{\log \frac{1}{a_r}} = +\infty$, $\sum_{r=1}^{\infty} a_r < +\infty$ and $0 < a_r \leqslant \gamma$ for each r we can find a sequence of increasing positive integers $\{n_r\}$ with $2^{n_r} a_r > 1$ and

$$\inf_{m: I(n_r, m, a_r) \subseteq A_{n_r}} F(n_r, m, a_r) \leqslant 2k_{13}\, a_r .$$

Thus there exists a sequence of positive integers $\{m_r\}$ such that $I(n_r, m_r, a_r) \subseteq A_{n_r}$ and $F(n_r, m_r, a_r) \leqslant 2k_{13} a_r$ for every r. Then $I = \bigcup_1^{\infty} I(n_r, m_r, a_r)$ is, by Lemma 3.2, a 1-set and yet

$$\sum_{\underline{a} \in I} f(\underline{a}) = \sum_{r=1}^{\infty} F(n_r, m_r, a_r) \leqslant 2k_{13} \sum_{r=1}^{\infty} a_r < +\infty .$$

This is the required contradiction and it establishes (1.3).

§.4 (1.4), (1.5) and (1.6) are easily deduced from the estimates of §.2.

Proof of (1.4)

Given a set B of lattice points with $\sum_{\underline{b} \in B} \frac{1}{|\underline{b}|} = +\infty$, let N_n be the number of points in B_n, the intersection of B and the spherical shell A_n. Then, since each $\underline{b} \in B_n$ has $|\underline{b}| \geqslant 2^n$,

$$(4.1) \qquad \sum_{n=1}^{\infty} \frac{N_n}{2^{3n}} = +\infty ,$$

and therefore, since $\sum_{n=1}^{\infty} \frac{N_n^{\frac{1}{3}}}{2^n} < \infty$ would contradict (4.1),

$$(4.2) \qquad \sum_{n=1}^{\infty} \frac{N_n^{\frac{1}{3}}}{2^n} = +\infty .$$

But, by Lemma 2.5, the capacity of $\hat{B}_n$ is not less than $k_3 N_n^{\frac{1}{3}}$, and so $\sum_{n=1}^{\infty} 2^{-n} C(\hat{B}_n) = +\infty$, making B a 1-set.

Proof of (1.5)

From the assumption that $\sum_{\underline{b} \in B} \frac{1}{|\underline{b}|2} = +\infty$ we have

(4.3) $\sum_{n=1}^{\infty} \frac{N_n}{2^{2n}} = +\infty,$

and hence

(4.4) $\sum_{n=1}^{\infty} \frac{N_n^{\frac{1}{2}}}{2^n} = +\infty.$

Since the points of B are coplanar, Lemma 2.6 applies and gives $C(\hat{B}_n) \geqslant k_4 N_n^{\frac{1}{2}}$: this in (4.4) means that B is a 1-set.

Proof of 1.6

This time we have, by assumption,

(4.5) $\sum_{n=1}^{\infty} \frac{N_n}{2^n n} = \log 2 \sum_{n=1}^{\infty} \frac{N_n}{2^n \log 2^n} = +\infty,$

and, by Lemma 2.8,

(4.6) $C(B_n) \geqslant k_7 N_n / \log N_n.$

Since N_n is necessarily less than $\frac{4\pi}{3}(2^{n+1}+1)^3$, which is less that e^{3n} for all but a finite number of values of n, (4.5) implies that $\sum_{n=1}^{\infty} \frac{N_n}{2^n \log N_n} = +\infty$, which with (4.6) means that B is a 1-set.

For (1.7) we do not need anything as complicated as Wiener's test.

Proof of (1.7)

It is well-known (and proved in Chapter V for a general class of R.W's) that $|\underline{b}|$. E(number of visits to $\underline{b}$) is bounded for all large enough $|\underline{b}|$. Thus the convergence of $\sum_{\underline{b} \varepsilon B} \frac{1}{|\underline{b}|}$ means that E(number of visits to B) is finite and hence $P^*(B) = 0$, and since B is either a 0-set or a 1-set this is equivalent to:

$$P^*(B) = 1 \implies \sum_{\underline{b} \varepsilon B} \frac{1}{|b|} = +\infty ,$$

which is (1.7).

§.5 In order to show that the conditions (1.3)-(1.7) are the best possible of their type, we need some information about series of positive terms.

Lemma 5.1 Given an arbitrary monotone sequence (λ_n) of positive terms with $\lim_{n \to +\infty} \lambda_n = +\infty$ there exists for each $\alpha > 1$ (b_n) with;

(5.2) $\quad 0 \leqslant b_n < 1,$

(5.3) $\quad \sum_{n=1}^{\infty} b_n < +\infty ,$

(5.4) $\quad \sum_{n=1}^{\infty} (b_n)^{\alpha} \lambda_n = +\infty .$

Proof

Let $m_o = 0$ and m_j for $j \geqslant 1$ be the first n for which $\lambda_n > j^{\alpha}$ and $m_j > m_{j-1}$. Let $b_n = \frac{1}{2_j}$ ~~$1+\delta$~~ when $n = m_j$ and $b_n = 0$ when $n \neq m_j$ for any j. Then (5.2) is clearly satisfied and since

$$\sum_{n=1}^{\infty} b_n = \sum_{j=1}^{\infty} b_{m_j} = \sum_{j=1}^{\infty} \frac{1}{j^{1+\delta}} ,$$

$$\sum_{n=1}^{\infty} (b_n)^{\alpha} \lambda_n = \sum_{j=1}^{\infty} (b_{m_j})^{\alpha} \lambda_{m_j} > \sum_{j=1}^{\infty} \frac{j^{\alpha}}{j^{(1+\delta)\alpha}} = \sum_{j=1}^{\infty} \frac{1}{j^{\alpha\delta}} .$$

(5.3) and (5.4) are also satisfied for each choice of δ in $0 < \delta \leqslant \frac{1}{\alpha}$

<u>Lemma 5.5</u> Given an arbitrary monotone sequence (λ_n) of positive terms with $\lim_{n \to +\infty} \lambda_n = +\infty$ there exists an increasing sequence of positive integers n_j with;

(5.6) $\sum_{j=1}^{\infty} \frac{1}{n_j} < +\infty$,

(5.7) $\sum_{j=1}^{\infty} \frac{\lambda_{n_j}}{n_j} = +\infty$.

<u>Proof</u> Let ℓ_r be the first value of n for which $\lambda_n > r$, let $m_o = 0$ and $k_o = 2$ and define m_r and k_r inductively for $r \geqslant 1$ by:

(5.8) $m_r = \max\{2r^2, \ell_r, m_{r-1} + k_{r-1} + 1\}$,

(5.9) $\sum_{n=m_r}^{m_r+k_r} \frac{1}{n} < \frac{1}{r^2} \leqslant \sum_{n=m_r}^{m_r+k_r+1} \frac{1}{n}$.

Now let (n_j) consist of all numbers of the form $m_r + s$, where $0 \leqslant s \leqslant k_r$, arranged in increasing order. Then (5.6) holds, for, by (5.9)

$$\sum_{j=1}^{\infty} \frac{1}{n_j} = \sum_{r=1}^{\infty} \sum_{s=o}^{k_r} \frac{1}{m_r+s} < \sum_{r=1}^{\infty} \frac{1}{r^2} < \infty .$$

Since (5.8) and (5.9) together give

$$\sum_{s=o}^{k_r} \frac{1}{m_r+s} = \sum_{s=o}^{k_r+1} \frac{1}{m_r+s} - \frac{1}{m_r+k_r+1}$$

$$\geqslant \sum_{s=o}^{k_r+1} \frac{1}{m_r+s} - \frac{1}{2r^2}$$

$$\geqslant \frac{1}{2r^2} ,$$

(5.7) holds, for

$$\sum_{j=1}^{\infty} \frac{\lambda_{n_j}}{n_j} = \sum_{r=1}^{\infty} \sum_{s=m_r}^{m_r+k_r} \frac{\lambda_s}{s} \geqslant \sum_{r=1}^{\infty} \lambda_{m_r} \sum_{s=o}^{k_r} \frac{1}{m_r+s}$$

$$\geqslant \sum_{r=1}^{\infty} r . \frac{1}{2r^2} = + \infty .$$

<u>Lemma 5.10</u> Given an arbitrary monotone sequence (λ_n) of positive terms with $\lim_{n \to +\infty} \lambda_n = 0$ there exists an increasing sequence of positive integers n_j with;

(5.11) $\displaystyle\sum_{j=1}^{\infty} \frac{1}{n_j} = + \infty ,$

(5.12) $\displaystyle\sum_{j=1}^{\infty} \frac{\lambda_{n_j}}{n_j} < + \infty .$

<u>Proof</u> Let ℓ_r be the first value of n for which $\lambda_n < \frac{1}{r}$, let $m_o = k_o = 0$ and define k_r and m_r inductively for $r \geqslant 1$ by:

(5.13) $m_r = \max \{2r , \ell_r , m_{r-1} + k_{r-1} + 1\} ,$

(5.14) $\displaystyle\sum_{s=o}^{k_r} \frac{1}{m_r+s} < \frac{1}{r} \leqslant \sum_{s=o}^{k_r+1} \frac{1}{m_r+s} .$

Now let (n_j) consist of all numbers of the form $m_r + s$, with $0 \leqslant s \leqslant k_r$, arranged in increasing order. Then (5.12) holds, for by (5.13) and (5.14)

$$\sum_{j=1}^{\infty} \frac{\lambda_{n_j}}{n_j} = \sum_{r=1}^{\infty} \sum_{s=m_r}^{m_r+k_r} \frac{\lambda_s}{s} \leqslant \sum_{r=1}^{\infty} \lambda_{m_r} \sum_{s=m_r}^{m_r+k_r} \frac{1}{s} \leqslant \sum_{r=1}^{\infty} \frac{1}{r}.\frac{1}{r} < +\infty .$$

Also we have, from (5.13), $m_r \geqslant 2r$, and therefore

$$\sum_{s=o}^{k_r} \frac{1}{m_r+s} = \sum_{s=o}^{k_r+1} \frac{1}{m_r+s} - \frac{1}{m_r+k_r+1} > \frac{1}{r} - \frac{1}{2r} = \frac{1}{2r} ,$$

whence

$$\sum_{j=1}^{\infty} \frac{1}{n_j} = \sum_{r=1}^{\infty} \sum_{s=o}^{k_r} \frac{1}{m_r+s} > \tfrac{1}{2} \sum_{r=1}^{\infty} \frac{1}{r} = +\infty ,$$

so that (5.11) holds.

§.6 The assertions (1.8)-(1.11) are now proved using the results of §.5.

Proof of (1.8) Given an arbitrary positive function f with $\lim_{x \to +\infty} f(x) = +\infty$, we are required to exhibit a O-set B with $\sum_{\underline{b}\varepsilon B} \frac{f(|\underline{b}|)}{|\underline{b}|^3} = +\infty$. It is sufficient to do this in the case that f(x) is monotone: for if $g(x) = \inf_{y \geqslant x} f(y)$, g(x) increases monotonically to $+\infty$ and is less than or equal to f(x), so that any O-set with $\sum_{\underline{b}\varepsilon B} \frac{g(|\underline{b}|)}{|\underline{b}|^3} = +\infty$ necessarily has $\sum_{\underline{b}\varepsilon B} \frac{f(|\underline{b}|)}{|\underline{b}|^3} = +\infty$.

Now if in Lemma 5.1 we put $\lambda_n = f(2^n)$ and $\alpha = 3$ we get a sequence (b_n) with;

(6.1) $0 \leqslant b_n < 1$,

(6.2) $\sum_{n=1}^{\infty} b_n < +\infty$, (6.3) $\sum_{n=1}^{\infty} b_n^3 f(2^n) = +\infty$.

Letting B_n be the set of all lattice points lying within a sphere of radius $b_n 2^{n-1}$ centred at the point $(3.2^{n-1}, 0, 0)$, it is easy to see that $B = \bigcup_{n=1}^{\infty} B_n$ is a 0-set, for $\hat{B}_n$ is certainly contained in a sphere of radius $2^{n-1} b_n + 1$, and so has capacity less than or equal to $2^{n-1}b_n+1$, whence, by (6.2),

$$\sum_{n=1}^{\infty} 2^{-n} C(\hat{B}_n) \leqslant \sum_{n=1}^{\infty} \frac{b_n}{2} + 2^{-n} < +\infty .$$

Moreover $N_n \geqslant k_{14}(2^{n-1}b_n)^3$ for some $k_{14} > 0$, thus

$$\sum_{\underline{b}\in B} \frac{f(|\underline{b}|)}{|\underline{b}|^3} = \sum_{n=1}^{\infty} \sum_{\underline{b}\in B_n} \frac{f(|\underline{b}|)}{|\underline{b}|^3} \geqslant \sum_{n=1}^{\infty} N_n \frac{f(2^n)}{(2^{n+1})^3} \geqslant \frac{k_{14}}{64} \sum_{n=1}^{\infty} b_n^3 f(2^n),$$

and by (6.3) this last series diverges.

<u>Proof of (1.9)</u> Again we may take $f(x)$ to be monotone, but this time we put $\lambda_n = f(2^n)$ and $\alpha = 2$ in Lemma 5.1 to get a sequence (b_n) with:

(6.4) $0 \leqslant b_n < 1$,

(6.5) $\sum_{n=1}^{\infty} b_n < +\infty$,

(6.6) $\sum_{n=1}^{\infty} b_n^2 f(2^n) = +\infty$.

Let B_n consist of all lattice points of the form $(a_1, a_2, 0)$ lying within a sphere of radius $2^{n-1}b_n$ centred at the point $(3.2^{n-1}, 0, 0)$. Then $N_n \geqslant k_{15}(2^{n-1}b_n)^2$ and since $\hat{B}_n$ is certainly contained in a right circular cylinder of height 1 and radius $2^{n-1}b_n+1$ its capacity is not more than $k_5(2^{n-1}b_n+1)$, by (2.7). Therefore, by (6.5) and (6.6) respectively,

$$\sum_{n=1}^{\infty} 2^{-n} C(\hat{B}_n) \leqslant k_5 \sum_{n=1}^{\infty} \frac{b_n}{2} + 2^{-n} < +\infty$$

and

$$\sum_{\underline{b}\in B} \frac{f(|\underline{b}|)}{|\underline{b}|^2} = \sum_{n=1}^{\infty} \sum_{\underline{b}\in B} \frac{f(|\underline{b}|)}{|\underline{b}|^2} \geqslant \sum_{n=1}^{\infty} N(n) \frac{f(2^n)}{(2^{n+1})^2}$$

$$\geqslant \frac{k_{15}}{16} \sum_{n=1}^{\infty} f(2^n)\, b_n^2 = +\infty,$$

so that B is the required O-set.

Proof of (1.10) Once more we may assume that $f(x)$ is monotone and so can use Lemma 5.5 with $\lambda_n = f(2^n)$ to find an increasing sequence of positive integers n_j with;

(6.7) $$\sum_{j=1}^{\infty} \frac{1}{n_j} = +\infty,$$

(6.8) $$\sum_{j=1}^{\infty} \frac{f(2^{n_j})}{n_j} < +\infty.$$

Let B_n consist of the 2^n lattice points of the form $(a_1, 0, 0)$ with $2^n \leqslant a_1 < 2^{n+1}$ if $n = n_j$ for some j: let B_n be the empty set if $n \notin (n_j)$. Then by (2.8) the capacity of $\hat{B}_{n_j}$ is not more than $\frac{k_8 2^{n_j}}{n_j \log 2}$, so that

$$\sum_{n=1}^{\infty} 2^{-n} C(\hat{B}_n) = \sum_{j=1}^{\infty} 2^{-n_j} C(\hat{B}_{n_j}) \leqslant \frac{k_8}{\log 2} \sum_{j=1}^{\infty} \frac{1}{n_j} < +\infty$$

and B is, by (6.7), a 0-set. However, by (6.8),

$$\sum_{\underline{b}\,\varepsilon\, B} \frac{f(|\underline{b}|)}{|\underline{b}|\log|\underline{b}|} = \sum_{j=1}^{\infty} \sum_{\underline{b}\varepsilon B_{n_j}} \frac{f(|\underline{b}|)}{|\underline{b}|\log|\underline{b}|} \geqslant \sum_{j=1}^{\infty} \frac{f(2^{n_j})\, 2^{n_j}}{2^{n_j+1}\log 2^{n_j+1}}$$

$$\geqslant \sum_{j=1}^{\infty} \frac{f(2^{n_j})}{n_j \log 2} = +\infty .$$

<u>Proof of (1.11)</u> Given an arbitrary non-negative function $f(x)$ with $\lim_{x\to+\infty} f(x) = 0$, consider $g(x) = \sup_{y\geqslant x} f(y)$. Then $g(x)$ decreases monotonically to zero and is always greater than or equal to $f(x)$, so that any 1-set with $\sum_{\underline{b}\varepsilon B} \frac{g(|\underline{b}|)}{|\underline{b}|} < +\infty$ necessarily has $\sum_{\underline{b}\,\varepsilon\, B} \frac{f(|\underline{b}|)}{|\underline{b}|} < +\infty$. Thus we need only establish (1.11) for an arbitrary $f(x)$ decreasing monotonically to zero. Writing in Lemma 5.10 $\lambda_n = f(2^n)$ we find an increasing sequence of positive integers n_j with;

(6.9) $$\sum_{j=1}^{\infty} \frac{1}{n_j} = +\infty ,$$

(6.10) $$\sum_{j=1}^{\infty} \frac{f(2^{n_j})}{n_j} < +\infty .$$

Let B_n be the empty set if $n \not\in (n_j)$: let B_n for $n = n_j$ consist of all points of the form $(2^n+2rn, 0, 0)$ for $0 < r < [\frac{2^n}{2n}] - 1$. Then $\hat{B}_{n_j}$ consists of $[\frac{2^{n_j}}{2n_j}]$ unit cubes whose centres are collinear and equally spaced at a distance $2n_j$ apart and whose faces are parallel. Since $\log[\frac{2^{n_j}}{2n_j}] \leqslant n_j\log 2 - \log 2n_j < n_j - \log 2n_j$, Lemma 2.10 applies if $n_j \geqslant 2$ and gives $C(\hat{B}_{n_j}) \geqslant k_{10}[\frac{2^{n_j}}{2n_j}]$ $\geqslant k_{16}\frac{2^{n_j}}{n_j}$. Thus

$$\sum_{n=1}^{\infty} 2^{-n}\, C(\hat{B}_n) \geqslant \sum_{j=2}^{\infty} 2^{-n_j} C(\hat{B}_{n_j}) \geqslant k_{16} \sum_{j=1}^{\infty} \frac{1}{n_j}\ ,$$

$$\sum_{\underline{b}\in B} \frac{f(|\underline{b}|)}{|\underline{b}|} = \sum_{j=1}^{\infty} \sum_{\underline{b}\in B_{n_j}} \frac{f(|\underline{b}|)}{|\underline{b}|} \leqslant \sum_{j=1}^{\infty} \left[\frac{2^{n_j}}{2n_j}\right] \frac{f(2^{n_j})}{2^{n_j}} \leqslant \sum_{j=1}^{\infty} \frac{f(2^{n_j})}{2n_j}\ ,$$

so that, by (6.9) and (6.10), B is the required 1-set with

$$\sum_{\underline{b}\in B} \frac{f(|\underline{b}|)}{|\underline{b}|} < +\infty\ .$$

CHAPTER II

§.1 If $(\underline{S}_n)$ is the simple 3-dimensional R.W and D is any line of lattice points parallel to an axis, the fact that the 2-dimensional R.W $(\underline{S}'_n)$ derived by projecting $\underline{S}_n$ onto a plane perpendicular to D is recurrent (that is, $P\{\underline{S}'_n = \underline{a}$ for an infinite number of values of $n\} = 1$ for every $\underline{a} \in L_2$) is sufficient to show that $P^*(D) = 1$. Thus a particle which starts at any point $\underline{a}$ of L_3 and performs the simple R.W is almost certain to visit D, and in this chapter we are interested in its position when it first does so. Plainly there is no loss of generality in taking D to be the x_3 axis and the particle to start at a point (a, b, 0) in the x_1, x_2 plane. Then we can define a R.V. F_{ab} as the x_3 coordinate of the point at which $(\underline{S}_n)$ first visits D by

$$(1.1) \quad P\{F_{ab} = k\} = f^k_{ab} = P\{\underline{S}_1 \notin D, \ldots, \underline{S}_{n-1} \notin D, \underline{S}_n = (0,0,k) \text{ for some } n \geq 1 \mid \underline{S}_0 = (a, b, 0)\}.$$

In §.2 the characteristic function $\phi_{ab}(\theta)$ of F_{ab} is found, and its behaviour as $\theta \longrightarrow 0$ and $r = (a^2 + b^2)^{\frac{1}{2}} \longrightarrow +\infty$ is studied. In §.3 we deduce results about f^k_{ab} as $k \longrightarrow \pm \infty$ with a and b fixed and in §.4 and §.5 investigate the R.V's F_{ab} as $r \longrightarrow +\infty$.

§.2 Let P^n_{abc} be the n-step transition probabilities, defined by,

$$P^n_{abc} = 1 \text{ if } n = 0 \text{ and } (a, b, c) = (0, 0, 0)$$
$$0 \text{ if } n = 0 \text{ and } (a, b, c) \neq (0, 0, 0) \quad .$$
(2.1)
$$= P\{\underline{S}_n = (a, b, c) | \underline{S}_0 = (0, 0, 0)\} \quad \text{for } n \geq 1.$$

Define q^n_{abc} by,

$$q^n_{abc} = 0 \text{ if } n = 0$$
(2.2)
$$= P\{\underline{S}_1 \notin D, \underline{S}_2 \notin D, .. \underline{S}_{n-1} \notin D, \underline{S}_n = (0, 0, c) | \underline{S}_0 = $$
$$= (a, b, 0)\} \quad \text{for } n \geq 1,$$

so that, if $(a, b) \neq (0, 0)$, $\begin{Bmatrix} q^n_{abc} \\ q^n_{00c} \end{Bmatrix}$ is the probability that a particle starting at $\begin{Bmatrix} (a, b, 0) \\ (0, 0, 0) \end{Bmatrix}$ first $\begin{Bmatrix} \text{hits} \\ \text{returns to} \end{Bmatrix}$ the x_3 axis at time n and position (0, 0, c).

Note that $P\{\underline{S}_n = (0, 0, c) | \underline{S}_0 = (a, b, 0)\} = P\{\underline{S}_n = (0, 0, c) | \underline{S}_0 = (-a, -b, 0)\} = P^n_{abc}$, and let A_n be the event $\{ \underline{S}_0 = (a, b, 0), \underline{S}_n = (0, 0, c)\}$. Then for $n \geq 1$ we can decompose A_n by time and place of first hitting D into the mutually exclusive events $B^k_r = \{ \underline{S}_0 = (a, b, 0), \underline{S}_1 \notin D, .., \underline{S}_{r-1} \notin D, \underline{S}_r = (0, 0, k), \underline{S}_n = (0, 0, c)\}$. By the Markov property $P\{B^k_r | \underline{S}_0 = (a, b, 0)\} = P\{\underline{S}_n = (0, 0, c) | \underline{S}_r = (0, 0, k)\}$ $.P\{\underline{S}_1 \notin D, .. \underline{S}_{r-1} \notin D, \underline{S}_r = (0, 0, k) | \underline{S}_0 = (a, b, 0)\} =$

$$= P^{n-r}_{00\overline{c-k}} \, Q^r_{abk} \quad ,$$

so that, for $n \geq 1$,

(2.3) $P^n_{abc} = P\{A_n | \underline{S}_o = (a, b, 0)\} = \sum_{r=1}^{n} \sum_{k=-\infty}^{\infty} P^{n-r}_{00\overline{c-k}}\ q^r_{abk}$.

If (a, b) $\neq$ (0, 0) we also have, by definition,

(2.4) $P^o_{abc} = q^o_{abc}$ (= 0).

Writing $P^n_{ab}(\theta) = \sum_{c=-\infty}^{\infty} e^{ic\theta}\ P^n_{abc}$,

$$Q^n_{ab}(\theta) = \sum_{c=-\infty}^{\infty} e^{ic\theta}\ q^n_{abc},$$

we notice that $|P_{ab}(\theta)| \leq 1$, $|Q_{ab}(\theta)| \leq 1$ for all real θ so we may multiply (2.3) and (2.4) by $e^{ic\theta}$ and sum over all integers c to get

(2.5) $P^n_{ab}(\theta) = \sum_{r=1}^{n} P^{n-r}_{oo}(\theta)\ Q^r_{ab}(\theta)$ for $n \geq 1$,

(2.6) $P^o_{ab}(\theta) = Q^o_{ab}(\theta)$ (= 0).

Taking real s with $|s| < 1$ we can introduce the double generating functions

$P_{ab}(s,\theta) = \sum_{n=o}^{\infty} P^n_{ab}(\theta) s^n$, $Q_{ab}(s,\theta) = \sum_{n=o}^{\infty} Q^n_{ab}(\theta) s^n$,

and in terms of these (2.5) and (2.6) become

(2.7) $P_{ab}(s,\theta) = Q_{ab}(s,\theta)\ P_{oo}(s,\theta)$.

Now $P_{ab}(s,\theta) = \sum_{n=o}^{\infty} \left(\sum_{c=-\infty}^{\infty} P^n_{abc}\ e^{ic\theta} \right) s^n$

$= \sum_{c=-\infty}^{\infty} e^{ic\theta} \sum_{n=o}^{\infty} P^n_{abc} s^n$, the interchange of order of summation being justified for $|s|<1$ by the absolute convergence of the sum, since $\sum_{c=-\infty}^{\infty} P^n_{abc} \leq 1$. If $\phi(\underline{\theta})$ is the characteristic function of the R.V. $\underline{X}$ which generates $(\underline{S}_n)$, $\phi(\underline{\theta}) = E(e^{i\underline{X}.\underline{\theta}}) = \frac{1}{3}(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)$, and, since $\underline{S}_n$ has characteristic function $\phi^n(\underline{\theta})$ for $n \geqslant 1$ if $\underline{S}_0 = (0, 0, 0)$,

$$(2.8)\quad P^n_{abc} = \frac{1}{(2\pi)^3}\iiint_{-\pi}^{\pi} \phi^n(\underline{\theta})e^{-i(a\theta_1+b\theta_2+c\theta_3)}d\theta_1 d\theta_2 d\theta_3 \text{ for } n \geqslant 0.$$

Thus
$$\sum_{n=0}^{\infty} P^n_{abc}s^n = \frac{1}{(2\pi)^3}\lim_{N\to+\infty}\iiint_{-\pi}^{\pi} e^{-i(a\theta_1+b\theta_2+c\theta_3)} \frac{1-(s\phi(\underline{\theta}))^{N+1}}{1-s\phi(\underline{\theta})}d\theta_1 d\theta_2 d\theta_3$$

$$= \frac{1}{(2\pi)^3}\iiint_{-\pi}^{\pi}\frac{e^{-i(a\theta_1+b\theta_2+c\theta_3)}}{1-s\phi(\underline{\theta})}d\theta_1 d\theta_2 d\theta_3 \text{ for } |s|<1,$$

since $|s\phi(\underline{\theta})|^{N+1} \leqslant |s|^{N+1} \longrightarrow 0$, so that if we write

$$\psi_{ab}(s, \theta_3) = \frac{1}{(2\pi)^2}\iint_{-\pi}^{\pi}\frac{e^{-i(a\theta_1+b\theta_2)}}{1-s\phi(\underline{\theta})}d\theta_1 d\theta_2$$

we have

$$(2.9)\quad P_{ab}(s,\theta) = \sum_{c=-\infty}^{\infty} e^{ic\theta}\frac{1}{2\pi}\int_{-\pi}^{\pi}\psi_{ab}(s, \theta_3)e^{-ic\theta_3}d\theta_3.$$

For each s with $|s|< 1$,

$$\frac{\partial}{\partial\theta_3}\{\psi_{ab}(s,\theta_3)\} = \frac{-\sin\theta_3}{3}\frac{s}{(2\pi)^2}\iint_{-\pi}^{\pi}\frac{e^{-i(a\theta_1+b\theta_2)}}{\{1-s\,\phi\,(\underline{\theta})\}^2}d\theta_1 d\theta_2$$

exists and is finite for all θ_3, so that Dini's convergence theorem applies to (2.9) to yield

$$(2.10)\quad P_{ab}(s, \theta) = \psi_{ab}(s, \theta).$$

From (2.7) and (2.10) we have an explicit expression for $Q_{ab}(s,\theta)$. Moreover, since $\sum_{c=-\infty}^{\infty} q^n_{abc} \leqslant 1$, (2.11) $Q_{ab}(s,\theta) = \sum_{n=0}^{\infty} s^n \sum_{c=-\infty}^{\infty} q^n_{abc} e^{ic\theta} = \sum_{c=-\infty}^{\infty} e^{ic\theta} \sum_{n=0}^{\infty} q^n_{abc} s^n,$

and as $s \uparrow 1$ $\sum_{n=0}^{\infty} q^n_{abc} s^n \uparrow$ to $\sum_{n=0}^{\infty} q^n_{abc} = f^c_{ab} \leqslant 1$.

Since $\sum_{c=-\infty}^{\infty} f^c_{ab} = P\{\text{particle starting at } (a, b, 0) \text{ hits } x_3 \text{ axis}\}$ $= 1$, we can let s increase to one in (2.11) and apply the theorem of dominated convergence to get

(2.12) $$\lim_{s\uparrow 1} Q_{ab}(s,\theta) = \sum_{c=-\infty}^{\infty} e^{ic\theta} f^c_{ab} = \phi_{ab}(\theta),$$

where $\phi_{ab}(\theta)$ is the characteristic function of the R.V F_{ab} defined on p.26. But if $\theta_3 \neq 0 \pmod{2\pi}$

$$\lim_{s\uparrow 1} \psi_{ab}(s,\theta_3) = \frac{1}{(2\pi)^2} \iint_{-\pi}^{\pi} \frac{e^{-i(a\theta_1+b\theta_2)}}{1 - \frac{1}{3}(\cos\theta_1+\cos\theta_2+\cos\theta_3)}\, d\theta_1 d\theta_2$$

$$= \frac{1}{(2\pi)^2} \iint_{-\pi}^{\pi} \frac{\cos a\theta_1 \cos b\theta_2}{1 - \frac{1}{3}(\cos\theta_1+\cos\theta_2+\cos\theta_3)}\, d\theta_1 d\theta_2$$

$$= g_{ab}(\theta_3),$$

so that, if $(a, b) \neq (0, 0)$, by virtue of (2.7), (2.10) and (2.12)

(2.13) $$\phi_{ab}(\theta) = \frac{g_{ab}(\theta)}{g_{oo}(\theta)} \quad \text{if } \theta \neq 0 \pmod{2\pi}$$
$$= 1 \quad \text{if } \theta = 0 \pmod{2\pi}.$$

In the case that the particle starts from the origin $(a, b) = (0, 0)$ and although (2.3) and (2.5) still hold for $n \geqslant 1$ we have $P^o_{ook} = \delta_{ko}$ (this, the Kronecker delta, is one if $k = 0$ and zero otherwise) while $q^o_{ook} = 0$. Therefore the analogue of (2.7) is

$$(2.14)\quad P_{oo}(s,\theta) - P^{o}_{oo}(\theta) = P_{oo}(s,\theta)Q_{oo}(s,\theta) - P^{o}_{oo}(\theta)Q^{o}_{oo}(\theta)$$

and since $P^{o}_{oo}(\theta) = 1$, $Q^{o}_{oo}(\theta) = 0$, this reduces to

$$(2.15)\quad Q_{oo}(s,\theta) = 1 - \frac{1}{P_{oo}(s,\theta)}\ .$$

Just as (2.13) follows from (2.7), (2.15) leads to

$$(2.16)\quad \phi_{oo}(\theta) = 1 - \frac{1}{g_{oo}(\theta)} \quad \text{if } \theta \neq 0 \pmod{2\pi}$$
$$= 1 \qquad\qquad \text{if } \theta = 0 \pmod{2\pi}.$$

Thus the behaviour of the characteristic functions $\phi_{ab}(\theta)$ is completely determined by the behaviour of the functions $g_{ab}(\theta)$, some of whose properties are the content of:

<u>Lemma 2.17</u> For all (a,b) and $\theta \neq 0 \pmod{2\pi}$

$$(2.18)\quad 0 < g_{ab}(\theta) < +\infty\ .$$

There exists constants k_1 and k_2 independent of θ, a, b such that:

$$(2.19)\ :\ |g_{ab}(\theta) - \frac{3}{\pi} K_o(r|\theta|)| < k_1 \quad \text{for } 0 < |\theta| \leqslant \pi \text{ and } (a,b) \neq (0,\ 0),$$

$$(2.20)\ :\ |g_{oo}(\theta) - \frac{3}{\pi} \log \frac{1}{|\theta|}\ | < k_2 \quad \text{for } 0 < |\theta| \leqslant \pi,$$

where $r = (a^2 + b^2)^{\frac{1}{2}}$ and K_o is the Bessel coefficient of zero order and imaginary argument.

Finally we have the asymptotic estimates:

$$(2.21)\quad \frac{d}{d\theta} g_{oo}(\theta) \sim \frac{-3}{\pi\theta} \quad \text{as } \theta \downarrow 0$$

(2.22) $\frac{d^2}{d\theta^2} \; g_{oo}(\theta) \sim \frac{3}{\pi\theta^2}$ as $\theta \downarrow 0$.

<u>Corollary to Lemma 2.17</u> For all (a, b) and all $\theta \neq 0 \pmod{2\pi}$

(2.23) $0 < \phi_{ab}(\theta) \leqslant 1$.

<u>Proof of Lemma 2.17</u>

If we write, for $\theta \neq 0 \pmod{2\pi}$

$$g_{ab}(\theta) = \frac{3}{(2\pi)^2} \int_{-\pi}^{\pi}\int \frac{\cos a\alpha \cos b\beta}{3-\cos\theta-\cos\alpha-\cos\beta}$$

$$= \frac{3}{(2\pi)^2} \int_{-\pi}^{\pi}\int \frac{e^{-i(a\alpha+b\beta)}d\alpha d\beta}{3-\cos\theta-\cos\alpha-\cos\beta}$$

$$= \frac{3}{(2\pi)^2} \int_{-\pi}^{\pi}\int e^{-i(a\alpha+b\beta)} \int_0^\infty e^{-t(3-\cos\theta-\cos\alpha-\cos\beta)}dt\, d\alpha\, d\beta,$$

the fact that $\int_{-\pi}^{\pi}\int \int_0^\infty e^{-t(3-\cos\theta-\cos\alpha-\cos\beta)}dt\, d\alpha\, d\beta < +\infty$

allows us to interchange the order of integration to get,

(2.24) $$g_{ab}(\theta) = 3\int_0^\infty e^{-t(3-\cos\theta)} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ia\alpha}e^{t\cos\alpha}d\alpha \,.$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ib\beta}e^{t\cos\beta}d\beta\, dt,$$

$$= 3\int_0^\infty e^{-t(3-\cos\theta)}I_a(t)I_b(t)dt,$$

where $I_a(t)$ is the modified Bessel coefficient of order a. Since $I_a(t)$ and $I_b(t)$ are positive when $t > 0$, the first assertion follows from (2.24).

Noting that $g_{ab}(\theta)$ is an even function of θ, take $0 < \theta \leqslant \pi$ and write

$$g_{ab} = \frac{3}{2\pi^2} \int_0^\pi\int_0^\pi \frac{\cos a\alpha \cos b\beta \, d\alpha \, d\beta}{\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\theta}{2}}$$

$$= \frac{6}{\pi^2} \int_0^\pi\int_0^\pi \frac{\cos a\alpha \cos b\beta}{\alpha^2 + \beta^2 + \theta^2} \, d\alpha d\beta + h_{ab}(\theta).$$

The inequalities

$$\frac{\theta}{\pi} \leqslant \sin\frac{\theta}{2} \leqslant \frac{\theta}{2},$$

$$0 \leqslant \left(\frac{\theta}{2}\right)^2 - \sin^2\frac{\theta}{2} = \tfrac{1}{2}\left(\cos\theta - 1 + \frac{\theta^2}{2}\right) \leqslant \tfrac{1}{2}\cdot\frac{\theta^4}{4!},$$

both hold in the range $0 \leqslant \theta \leqslant \pi$ and lead to

(2.26) $$|h_{ab}(\theta)| \leqslant \frac{3}{2\pi^2} \int_0^\pi\int_0^\pi \left\{\frac{1}{\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\theta}{2}} - \frac{4}{\alpha^2+\beta^2+\theta^2}\right\} d\alpha d\beta$$

$$\leqslant \frac{3}{4!} \int_0^\pi\int_0^\pi \frac{\alpha^4 + \beta^4 + \theta^4}{(\alpha^2+\beta^2+\theta^2)^2} \, d\alpha d\beta \leqslant \frac{3\pi^2}{4!}.$$

Since $\frac{2}{\pi} \int_0^\infty\int_0^\infty \frac{\cos a\alpha \cos b\beta}{\alpha^2 + \beta^2 + \theta^2} \, d\alpha d\beta = K_0(r|\theta|)$ if $r > 0$, (2.19) will follow from (2.26) if we can show that the error involved in replacing the region of integration $(0 \leqslant \alpha \leqslant \pi,\ 0 \leqslant \beta \leqslant \pi)$ in (2.25) by the region $(0 \leqslant \alpha,\ 0 \leqslant \beta)$ is, for $(a,b) \neq (0,0)$, bounded for all $0 < \theta \leqslant \pi$ uniformly in a and b. To do this we use the following version of the second mean value theorem for 2-dimensional integrals, which has been proved by Hobson [18, p. 572].

<u>Theorem 2.27</u> If $\psi(\alpha,\beta)$ is integrable and $\phi(\alpha,\beta)$ is non-negative, monotone decreasing in α and β and integrable in

$(A_1 \leq \alpha \leq A_2,\ B_1 \leq \beta \leq B_2)$, there exists A_3, B_3 with $A_1 \leq A_3 \leq A_2$ and $B_1 \leq B_3 \leq B_2$ such that

$$\int_{A_1}^{A_2}\int_{B_1}^{B_2} \phi(\alpha,\beta)\psi(\alpha,\beta)\,d\alpha d\beta = \phi(A_1, B_1)\int_{A_1}^{A_3}\int_{B_1}^{B_3}\psi(\alpha,\beta)\,d\alpha d\beta.$$

For if $a^2 + b^2 > 0$ there is no loss of generality in taking $|a| \geq 1$, since $g_{ab}(\theta) = g_{ba}(\theta)$, and taking $R > \pi$ we can apply Theorem 2.27 with $\phi(\alpha,\beta) = \dfrac{1}{\theta^2 + \alpha^2 + \beta^2}$ and $\psi(\alpha,\beta) = \cos a\alpha \cos b\beta$ in the region $(\pi \leq \alpha \leq R,\ 0 \leq \beta \leq \pi)$ to get

$$(2.28)\quad \int_{\pi}^{R}\int_{0}^{\pi} \frac{\cos a\alpha \cos b\beta}{\theta^2 + \alpha^2 + \beta^2}\,d\alpha d\beta = \frac{1}{\theta^2+\pi^2}\int_{\pi}^{A}\cos a\alpha\,d\alpha \int_{0}^{B}\cos b\beta\,d\beta.$$

Now $B \leq \pi$ so that $\left|\int_0^B \cos b\beta\,d\beta\right| \leq \pi$ and $a \neq 0$ so that $\left|\int_\pi^A \cos a\alpha\,d\alpha\right| \leq \dfrac{2}{|a|} \leq 2$. Moreover, as $\dfrac{1}{\theta^2 + \alpha^2 + \beta^2}$ is integrable in $(\pi \leq \alpha,\ 0 \leq \beta \leq \pi)$, $\lim\limits_{R\to+\infty}\int_\pi^R\int_0^\pi \dfrac{\cos a\alpha \cos b\beta}{\theta^2 + \alpha^2 + \beta^2}\,d\alpha d\beta$ exists and equals $\int_0^\pi\int_\pi^\infty \dfrac{\cos a\alpha \cos b\beta}{\theta^2 + \alpha^2 + \beta^2}\,d\beta d\alpha$. Using the fact (Erdélyi [12, p.7]) that $\int_0^\infty \dfrac{\cos a\alpha\,d\alpha}{\theta^2+\alpha^2+\beta^2} = \dfrac{\pi}{2}\,\dfrac{e^{-a(\theta^2+\beta^2)^{\frac{1}{2}}}}{(\theta^2 + \beta^2)^{\frac{1}{2}}}$,

we can let $R \to +\infty$ in (2.28) to get

$$(2.29)\quad \left|\frac{\pi}{2}\int_0^\pi \frac{\cos b\beta\, e^{-a(\theta^2+\beta^2)^{\frac{1}{2}}}}{(\theta^2 + \beta^2)^{\frac{1}{2}}}\,d\beta - \int_0^\pi\int_0^\pi \frac{\cos a\alpha \cos b\beta}{\theta^2 + \alpha^2 + \beta^2}\,d\alpha d\beta\right| \leq k_3,$$

where k_3 is a finite constant independent of θ, a, and b.

Since $\int_0^\infty \frac{\cos b\beta e^{-a(\theta^2+\beta^2)}}{(\theta^2+\beta^2)^{\frac{1}{2}}} d\beta = K_0(|\theta|r)$ (Erdélyi [12, p.17])

and $\left|\int_\pi^\infty \frac{\cos b\beta e^{-a(\theta^2+\beta^2)}}{(\theta^2+\beta^2)^{\frac{1}{2}}} d\beta\right| \leqslant \int_\pi^\infty \frac{e^{-a\beta}}{\beta} d\beta \leqslant \int_\pi^\infty \frac{e^{-\beta}}{\beta} d\beta < +\infty$,

(2.25), (2.26) and (2.29) prove (2.19).

When a = b = 0 the argument is more direct, for

$$g_{oo}(\theta) = \frac{3}{(2\pi)^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{d\alpha\, d\beta}{3-\cos\theta-\cos\alpha-\cos\beta}$$

$$= \frac{3}{2\pi} \int_{-\pi}^{\pi} \frac{d\alpha}{\{(3-\cos\theta-\cos\alpha)^2-1\}^{\frac{1}{2}}}$$

$$= \frac{3}{2\pi} \int_0^{\pi} \frac{d\alpha}{(\sin^2\frac{\theta}{2}+\sin^2\frac{\alpha}{2})^{\frac{1}{2}}(A^2\sin^2\frac{\theta}{2}+\sin^2\frac{\alpha}{2})^{\frac{1}{2}}}$$

where $A^2 = \dfrac{1+\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}$,

$$= \frac{3}{2\pi} \int_0^\infty \frac{2A\,dt}{\{t^2+(t^2+A^2)\sin^2\frac{\theta}{2}\}^{\frac{1}{2}}\{t^2+A^2\sin^2\frac{\theta}{2}(t^2+A^2)\}^{\frac{1}{2}}},$$

where $t = A\tan\frac{\alpha}{2}$,

$$= \frac{3k}{\pi} \int_0^\infty \frac{dt}{(1+t^2)^{\frac{1}{2}}(1+k'^2t^2)^{\frac{1}{2}}}, \text{ where } k = \frac{1}{1+\sin^2\frac{\theta}{2}}$$

and $k^2 + k'^2 = 1$. Thus we have

(2.30) $g_{oo}(\theta) = \frac{3k}{\pi} K(k)$,

K denoting the complete elliptic integral of the first kind.

Now
$$k' = \frac{\sin\frac{\theta}{2}\,(2+\sin^2\frac{\theta}{2})^{\frac{1}{2}}}{1+\sin^2\frac{\theta}{2}} \sim \frac{\theta}{\sqrt{2}} \quad \text{as } \theta \downarrow 0,$$

and it is not difficult to see that $|\log\frac{1}{k'} - \log\frac{1}{\theta}|$ is bounded for $0 \leqslant \theta \leqslant \pi$. Since it is known (Erdélyi [14, p.318]) that $|K(k) - \log\frac{1}{k'}|$ is bounded for all k, this is sufficient to establish (2.20).

The representation (2.30) allows us to write down expressions for the derivatives of $g_{oo}(\theta)$ in terms of K(k) and E(k), the complete elliptic integral of the second kind, for the derivatives of E and K may always be expressed in terms of E, K, and elementary functions. We can then use the known asymptotic estimates for E and K, the results being

(2.31) $$g'_{oo}(\theta) = -3\,E.\left\{\pi\tan\frac{\theta}{2}\,(2+\sin^2\frac{\theta}{2})\right\}^{-1} \sim -\frac{3}{\pi\theta} \quad \text{as } \theta \downarrow 0,$$

(2.32) $$g''_{oo}(\theta) = \frac{3}{2\pi}\left\{\frac{\sin^2\theta(1+\sin^2\theta)\,K}{\sin^2\frac{\theta}{2}(2+\sin^2\frac{\theta}{2})} + \frac{E}{\sin^2\frac{\theta}{2}(2+\sin^2\frac{\theta}{2})}\left\{\cos\theta - \frac{\sin^2\theta(\sin^4\frac{\theta}{2}+2\sin^2\frac{\theta}{2}-2)}{1+\sin^2\frac{\theta}{2}}\right\}\right\}$$

$$\sim \frac{3}{\pi\theta^2} \quad \text{as } \theta \downarrow 0,$$

which are (2.21) and (2.22) respectively.

§.3 By assertion (2.20) of Lemma 2.17, $\phi_{oo}(\theta) \sim 1 - \frac{\pi}{3\log\frac{1}{\theta}}$

as $\theta \downarrow 0$. Moreover $\lim_{\theta \downarrow 0} \{g_{ab}(\theta) - g_{oo}(\theta)\}$ exists and equals $-3\, c_{ab}$, where for $(a, b) \neq (0, 0)$

$$(3.1) \quad 0 < c_{ab} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi}\!\!\int \frac{1-\cos a\alpha \cos b\beta}{2-\cos\alpha - \cos\beta}\, d\alpha\, d\beta < \infty ,$$

so that $\phi_{ab}(\theta) \sim 1 - \frac{\pi c_{ab}}{\log\frac{1}{\theta}}$ as $\theta \downarrow 0$. It follows immediately from a theorem of Pitman [24] that if $F_{ab}(x)$ is the distribution function of the R.V F_{ab}

$$(3.2) \quad F_{ab}(x) \sim 1 - \frac{\pi c_{ab}}{2\log x} \quad \text{as } x \longrightarrow +\infty$$

$$\sim \frac{\pi c_{ab}}{2\log|x|} \quad \text{as } x \longrightarrow -\infty ,$$

where $c_{oo} = \frac{1}{3}$. However the behaviour of the individual probabilities f^k_{ab} for large k depends on deeper properties of $\phi_{ab}(\theta)$. Since $\phi_{ab}(\theta)$ is an even function of θ

$$f^k_{ab} = \frac{1}{\pi}\int_0^{\pi} \phi_{ab}(\theta)\cos k\theta d\theta,$$

$$= \frac{-1}{k\pi} \int_0^{\pi} \phi'_{ab}(\theta)\, \sin k\,\theta\, d\,\theta, \quad \text{if } k \neq 0,$$

so we are interested in the asymptotic behaviour of the Fourier coefficients of a function which has a singularity like $\frac{1}{\theta(\log\frac{1}{\theta})^2}$ at $\theta = 0$. As the standard theorems do not seem to apply in this case, we prove a lemma that is more general than we need and extends a theorem of Zygmund [30, p.190, theorem 2.24] to the case $\beta = 1$ (his notation).

The lemma does not seem to hold without a condition like (3.6).

Lemma 3.3 Let $b(x)$ be a non-negative function satisfying;

(3.4) $b(x)$ is of bounded variation in every interval (ε, π) with $\varepsilon > 0$;

(3.5) $b(x)$ is slowly varying as $x \downarrow 0$ (that is for every $\delta > 0$, $x^{\delta} b(x)$ is an increasing, and $x^{-\delta} b(x)$ is a decreasing, function of x for all small enough positive x);

(3.6) $\lim_{x \downarrow 0} b(x) = 0$, and the convergence is ultimately monotone.

Then $a_n = \int_0^{\pi} \frac{b(x)}{x} \sin nx \, dx \sim \frac{\pi}{2} b(\frac{1}{n})$ as $n \longrightarrow +\infty$.

Proof of Lemma 3.3

$$\text{Write } a_n = \left\{ \int_0^{\delta_1/n} + \int_{\delta_1/n}^{\delta_2/n} + \int_{\delta_2/n}^{\delta_3} + \int_{\delta_3}^{\pi} \right\} \frac{b(x)}{x} \sin nx \, dx;$$

and call the integrals I_1, I_2, I_3, and I_4 respectively.

Since, by assumption (3.6), $b(x)$ is an increasing function of x throughout some neighbourhood of zero, we have for all large enough n

$$|I_1| \leqslant n \int_0^{\delta_1/n} b(x) \left|\frac{\sin nx}{nx}\right| dx \leqslant \delta_1 b(\frac{\delta_1}{n}),$$

so that if $\delta_1 \leqslant 1$

(3.7) $\displaystyle |I_1| / b(\frac{1}{n}) \leqslant \delta_1$ for $n \geqslant n_1(\delta_1)$.

Again $b(x)$ is monotone in $(\frac{\delta_1}{n}, \frac{\delta_2}{n})$ for all large enough n, and the second mean value theorem {Hobson [18, p.565]} shows that for some $\delta_1 \leqslant \xi_n \leqslant \delta_2$

$$I_2 = \int_{\delta_1}^{\delta_2} \frac{\sin x}{x} b(\frac{x}{n})\, dx = b(\frac{\delta_1}{n}) \int_{\delta_1}^{\xi_n} \frac{\sin x}{x}\, dx + b(\frac{\delta_2}{n}) \int_{\xi_n}^{\delta_2} \frac{\sin x}{x}\, dx.$$

Now $\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$, and therefore

$$(3.8) \quad I_2 - \frac{\pi}{2} b(\frac{1}{n}) = \{b(\frac{\delta_1}{n}) - b(\frac{1}{n})\} \int_{\delta_1}^{\xi_n} \frac{\sin x}{x}\, dx + \{b(\frac{\delta_2}{n}) - b(\frac{1}{n})\} \int_{\xi_n}^{\delta_2} \frac{\sin x}{x}\, dx$$

$$+ b(\frac{1}{n}) \int_0^{\delta_1} \frac{\sin x}{x}\, dx + b(\frac{1}{n}) \int_{\delta_2}^{\infty} \frac{\sin x}{x}\, dx.$$

Using the facts that $\left| \int_A^B \frac{\sin x}{x} dx \right|$ is bounded for all A and B, $\left| \int_0^A \frac{\sin x}{x}\, dx \right| \leqslant A$, $\left| \int_B^{\infty} \frac{\sin x}{x}\, dx \right| \leqslant \frac{2}{B}$, in (3.8), we have

$$(3.9) \quad \left| I_2 / b(\frac{1}{n}) - \frac{\pi}{2} \right| \leqslant k_4 \left\{ \left| \frac{b(\frac{\delta_1}{n})}{b(\frac{1}{n})} - 1 \right| + \left| \frac{b(\frac{\delta_2}{n})}{b(\frac{1}{n})} - 1 \right| \right\} + \delta_1 + \frac{2}{\delta_2}$$

for $n \geqslant n_2(\delta_1, \delta_2)$.

By assumption (3.5) we can choose $\delta_3 > 0$ such that $\frac{b(x)}{x}$ is a decreasing function of x in $(0, \delta_3)$, and then we can apply Bonnet's mean value theorem {Hobson [18, p.565]} to I_3 to get for some $\lambda_n \leqslant \delta_3$,

$$I_3 = \int_{\delta_2/n}^{\delta_3} \frac{b(x)}{x} \sin nx \, dx = \frac{b(\frac{\delta_2}{n})}{\delta_2} . \; n \int_{\delta_2/n}^{\lambda_n} \sin nx \, dx,$$

and since $\left| n \int_{\delta_2/n}^{\lambda_n} \sin nx \, dx \right| = \left| \int_{\delta_2}^{n\lambda_n} \sin x \, dx \right| \leqslant 2,$

(3.10) $\left| I_3 / b(\frac{1}{n}) \right| \leqslant \frac{1}{\delta_2} \cdot \frac{b(\frac{\delta_2}{n})}{b(\frac{1}{n})}$ for $n \geqslant n_3(\delta_3, \delta_2)$.

Remembering that a well-known property of slowly varying functions is that $\lim_{x \downarrow 0} \frac{b(\delta x)}{b(x)} = 1$ for every fixed $\delta > 0$, {Zygmund.[30, p.186]} it is plain that given arbitrary $\varepsilon > 0$ we can make each of the right hand sides of (3.7) (3.9) and (3.10) less than $\frac{\varepsilon}{3}$ for all $n \geqslant n_0(\varepsilon)$ by choosing δ_1 small enough and δ_2 large enough. The proof is then concluded by noticing that I_4, being the Fourier coefficient of a function of bounded variation, is $O(\frac{1}{n})$ as $n \longrightarrow +\infty$, and therefore is $O(b(\frac{1}{n}))$ as $n \longrightarrow +\infty$.

<u>Theorem 3.11</u> For each fixed (a, b)

$$\lim_{k \to \mp\infty} \left\{ |k| (\log|k|)^2 \, f^k_{ab} \right\} = \frac{\pi}{2} C_{ab},$$

where the constant $C_{ab} = \frac{1}{3}$ if $a = b = 0$

$$= \frac{1}{(2\pi)^2} \iint_{-\pi}^{\pi} \frac{1-\cos a\alpha \; \cos b\beta}{2-\cos\alpha-\cos\beta} \, d\alpha \, d\beta$$

if $(a, b) \neq (0, 0)$.

<u>Proof</u> When $a = b = 0$, $\phi'_{ab}(\theta) = \dfrac{g'_{oo}(\theta)}{g^2_{oo}(\theta)}$, so that

$f^{-k}_{oo} = f^{k}_{oo} = \frac{1}{k\pi} \int_0^{\pi} \frac{b(\theta)}{\theta} \sin k\theta \, d\theta$, for $k > 0$, where

$b(\theta) = -\theta \dfrac{g'_{oo}(\theta)}{g^2_{oo}(\theta)} \sim \dfrac{\pi}{3} \dfrac{1}{(\log\frac{1}{\theta})^2}$ as $\theta \downarrow 0$ by

(2.20) and (2.21) of Lemma 2.17. Thus $b(\theta)$ is slowly varying as $\theta \downarrow 0$, and since

$$g'_{oo}(\theta) = -\frac{3\sin\theta}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{d\alpha \, d\beta}{(3-\cos\theta-\cos\alpha-\cos\beta)^2}$$

it is plain that $b(\theta)$ is non-negative in $(0,\pi)$ and of bounded variation in (ε, π) for every $\varepsilon > 0$. Moreover the asymptotic estimates of Lemma 2.17 show that

$$b'(\theta) = \frac{2\theta g'^2_{oo}(\theta)}{g^3_{oo}(\theta)} - \frac{g'_{oo}(\theta) + \theta g''_{oo}(\theta)}{g^2_{oo}(\theta)} \sim \frac{2\pi}{3\theta(\log\frac{1}{\theta})^3} \text{ as } \theta \downarrow 0,$$

so that $b(\theta)$ is monotone in some neighbourhood of zero. Thus Lemma 3.3 applies and establishes the theorem for $a = b = 0$.

For $(a, b) \neq (0, 0)$, write $\phi_{ab}(\theta) = \dfrac{g_{ab}(\theta)}{g_{oo}(\theta)} = \dfrac{g_{oo}(\theta)-3C_{ab}+h_{ab}(\theta)}{g_{oo}(\theta)}$,

where $h_{ab}(\theta) = \dfrac{3}{(2\pi)^2} \iint_{-\pi}^{\pi} (1-\cos a\alpha \cos b\beta) \Big\{\dfrac{1}{2-\cos\alpha-\cos\beta}$

$$- \frac{1}{3-\cos\theta-\cos\alpha-\cos\beta}\Big\} \, d\alpha d\beta \ .$$

Since, for $k \neq 0$,

$$f^k_{ab} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_{ab}(\theta)\cos k\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{1 - \frac{3Cab}{g_{oo}(\theta)}\right\}\cos k\theta d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_{ab}(\theta)}{g_{oo}(\theta)}\cos k\theta d\theta$$

$$= \frac{-3C_{ab}}{2\pi}\int_{-\pi}^{\pi}\frac{\cos k\theta}{g_{oo}(\theta)}d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_{ab}(\theta)}{g_{oo}(\theta)}\cos k\theta d\theta$$

$$= 3C_{ab}\, f^k_{oo} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h_{ab}(\theta)}{g_{oo}(\theta)}\cos k\theta d\theta,$$

it suffices to show that

$$(3.12)\quad k(\log k)^2\int_0^{\pi}\frac{h_{ab}(\theta)}{g_{oo}(\theta)}\cos k\theta\, d\theta \longrightarrow 0 \text{ as } k \longrightarrow +\infty .$$

Now,

$$0 \leqslant \frac{1}{2-\cos\alpha-\cos\beta} - \frac{1}{3-\cos\theta-\cos\alpha-\cos\beta} = \frac{1-\cos\theta}{(2-\cos\alpha-\cos\beta)(3-\cos\alpha-\cos\beta-\cos\theta)}$$

$$\leqslant \frac{k_5\theta^2}{(\alpha^2+\beta^2)(\alpha^2+\beta^2+\theta^2)}$$

for all θ, α, β in $(-\pi, \pi)$. Since $\frac{1-\cos a\alpha\cos b\beta}{\alpha^2+\beta^2}$ is bounded for $(a, b) \neq (0, 0)$ and $(\alpha^2+\beta^2)^{-\frac{1}{2}}$ is integrable in $(|\alpha| \leqslant \pi, |\beta| \leqslant \pi)$, it follows that $\frac{h_{ab}(\theta)}{\theta}$ is bounded in $(0, \pi)$.

Again, for $0 \leqslant \theta \leqslant \pi$,

$$0 \leqslant h'_{ab}(\theta) = \frac{3\sin\theta}{(2\pi)^2}\iint_{-\pi}^{\pi}\frac{1-\cos a\alpha\cos b\beta}{(3-\cos\theta-\cos\alpha-\cos\beta)^2}d\alpha d\beta$$

$$\leqslant \frac{3\theta}{(2\pi)^2}\iint_{-\pi}^{\pi}\frac{1-\cos a\alpha\cos b\beta}{\frac{4}{\pi}4(\theta^2+\alpha^2+\beta^2)^2}d\alpha\, d\beta$$

$$\leqslant \frac{3\pi^2}{16}\iint_{-\pi}^{\pi}\frac{1-\cos a\alpha\cos b\beta}{(\alpha^2+\beta^2)^{\frac{3}{2}}}d\alpha\, d\beta < +\infty,$$

and a similar argument shows that $-\theta h''_{ab}(\theta)$ is bounded in this range.

Then, if $V_{ab}(\theta) = \frac{d}{d\theta}\left\{\frac{h_{ab}(\theta)}{g_{oo}(\theta)}\right\} = \frac{g_{oo}(\theta)h'_{ab}(\theta)-h_{ab}(\theta)g'_{oo}(\theta)}{g^2_{oo}(\theta)}$

the above estimates, together with those of Lemma 2.17, show that $V'_{ab}(\theta).\theta(\log\frac{1}{\theta})^2$ is bounded in some neighbourhood of zero. Thus, for $k \neq 0$,

$$k\int_0^{\pi}\frac{h_{ab}(\theta)}{g_{oo}(\theta)}\cos k\theta\, d\theta = -\int_0^{\pi} V_{ab}(\theta)\sin k\theta\, d\theta,$$

and this latter, being the Fourier coefficient of a function of bounded variation, is $O(\frac{1}{k})$ as $k \longrightarrow +\infty$. This proves (3.12), and hence the theorem.

Theorem 3.11 naturally implies (3.2), which for (a, b) = (0, 0) is:

(3.13) $$F_{oo}(x) \sim 1 - \frac{\pi}{6\log x} \quad \text{as } x \longrightarrow +\infty,$$

$$\sim \frac{\pi}{6\log|x|} \quad \text{as } x \longrightarrow -\infty.$$

Now if S^n_{oo} denotes the x_3 co-ordinate of the lattice point at which the R.W. starting at (0, 0, 0) returns to the x_3 axis for the nth time, it is plain that

$$S^n_{oo} = F'_{oo} + F^2_{oo} + \dots + F^n_{oo},$$

where the F^j_{oo} are independent R.V's having common distribution function $F_{oo}(x)$. $F_{oo}(x)$ has infinite moments of all orders (that is, for every $\rho > 0$ $\int_{-\infty}^{\infty}|x|^{\rho}dF_{oo}(x) = +\infty$) and, as noted by Levy [22], this means that for no (a_n), (b_n) does $a_n S^n_{oo} + b_n$ have a non-degenerate limiting distribution function as $n \longrightarrow +\infty$. However Darling [10] showed that the fact ((3.13)) that $F_{oo}(x)$ and $1-F_{oo}(x)$ are slowly varying as $x \longrightarrow +\infty$ and $x \longrightarrow -\infty$, respectively, leads to a limit theorem of a different kind, and we can easily deduce from his Theorem 4.2 that

$$(3.14) \quad \lim_{n \to +\infty} \{P\{|S^n_{oo}|^{\frac{1}{n}} < x\} = 0 \text{ if } x \leqslant 1$$
$$= e^{\frac{-\pi}{3\log x}} \text{ if } x > 1.$$

Also, if S^n_{ab} denotes the x_3 coordinate of the lattice point at which the R.W starting at (a, b, 0) hits the x_3-axis for the n^{th} time,

$$S^n_{ab} = F_{ab} + F^1_{oo} + \ldots + F^{n-1}_{oo},$$

and (3.14) holds with S^n_{ab} in place of S^n_{oo}.

§.4 An obvious question to ask about the R.V's F_{ab} is whether or not there exists a norming function d(r) such that $F_{ab}/d(r)$ has a non-degenerate limiting distribution as $r \longrightarrow +\infty$. The analogous question for the simple 2-dimensional R.W is easily answered in the affirmative, the norming factor being merely the distance between the starting point and the absorbing axis, and the limit a Cauchy distribution. However, in our case it turns out that the only possible limiting distribution is degenerate, with distribution function

$G_o(x)$ given by

(4.1) $G_o(x) = 1$ if $x \geqslant 1$

$= 0$ if $x < 0$.

We employ the following standard theorem (see, for example, Lukacs [23, p.54]):

<u>Theorem 4.2</u> Let $(F_n(x))$ be a sequence of distribution functions and denote by $(\phi_n(\theta))$ the sequence of the corresponding characteristic functions. The sequence $(F_n(x))$ converges weakly to a distribution function $F(x)$ if, and only if, the sequence $(\phi_n(\theta))$ converges for every θ to a function $\phi(\theta)$ which is continuous at $\theta = 0$. $\phi(\theta)$ is then the characteristic function of $F(x)$.

Plainly, if the distribution of $\frac{F_{ab}}{d(r)}$ is to tend to a limit, $d(r)$ will be of constant sign for all large enough r, so without loss of generality we can take $d(r)$ to be positive. $F_{ab/d(r)}$ then has distribution function $F_{ab}(xd(r))$, and therefore characteristic function $\phi_{ab}(\frac{\theta}{d(r)})$: suppose $\rho(\theta) = \lim_{r \to +\infty} \phi_{ab}(\frac{\theta}{d(r)})$ exists for all θ. Note that $\rho(0) = 1$, and assume first of all that $\liminf_{r \to +\infty} d(r) = 2D < +\infty$. Then we can find a sequence (r_n) with $r_n \uparrow +\infty$ and $D \leqslant d(r_n) \leqslant 3D$ for all n. Since the representation (2.24) shows that $g_{ab}(\theta)$ is a positive monotone decreasing function for $0 < \theta \leqslant \pi$, for any

$0 < \theta \leqslant \pi D$ we have

(4.3) $\quad \emptyset_{ab}\left(\frac{\theta}{d(r_n)}\right) \leqslant \frac{g_{ab}(\frac{\theta}{3D})}{g_{oo}(\frac{\theta}{D})}$ for $(a, b) \neq (0, 0)$ and all n.

Now $0 < \frac{\theta}{3D} \leqslant \frac{\pi}{3}$, so $1 - \cos\frac{\theta}{3D}$ is bounded away from zero, and $g_{ab}(\frac{\theta}{3D})$ is the 2-dimensional Fourier coefficient of an integrable function. Thus, by the Riemann-Lebesgue Lemma, $\lim_{r \to \infty} g_{ab}(\frac{\theta}{3D}) = 0$, and therefore $\rho(\theta) = 0$ for $0 < \theta \leqslant \pi D$, and is plainly discontinuous at $\theta = 0$. Hence, by Theorem 4.2, if $\liminf_{r \to +\infty} d(r) < +\infty$ there is no limiting distribution.

If $\lim_{r \to +\infty} d(r) = +\infty$, it follows from Lemma 2.17 that for all $\theta > 0$,

(4,4) $\quad \rho(\theta) = \lim_{r \to +\infty} \frac{K_o(\frac{r\theta}{d(r)})}{\log d(r)}$.

Assume that $\liminf_{r \to +\infty} \frac{d(r)}{r} < +\infty$. We can then find a sequence (r_n) with $r_n \uparrow +\infty$ and $\frac{r_n}{d(r_n)}$ bounded away from 0 uniformly in n: since $0 < K_o(z) < +\infty$ for $z > 0$, $K_o(\frac{r_n\theta}{d(r_n)})$ is bounded uniformly in n for each $\theta > 0$, and therefore $\rho(\theta) = 0$ for all $\theta > 0$. Thus to get a non-zero limit we must have $\frac{d(r)}{r} \to +\infty$ as $r \to +\infty$. But $K_o(z) \sim \log\frac{1}{z}$ as $z \downarrow 0$, so that by (4.4)

$$\rho(\theta) = \lim_{r \to +\infty} \left\{\frac{\log\frac{d_r}{r\theta}}{\log d_r}\right\} = 1 - \frac{\log r}{\log d(r)} = c,$$

for every $\theta > 0$, where c is independent of θ and lies between zero and one. If c is less than one, $\rho(\theta)$ is again discontinuous at $\theta = 0$, and if c equals one $g(\theta)$ is one for all θ, and is therefore the characteristic function of the degenerate distribution function $G_o(x)$. This establishes our assertion that $F_{ab/d(r)}$ has no non-degenerate limiting distribution.

As a particular case of the above argument, consider what happens when $d(r) = r^\beta$. We have,

$$\phi_{ab}\left(\frac{\theta}{r^\beta}\right) = E\left(e^{i\frac{\theta F_{ab}}{r^\beta}}\right) \longrightarrow 0 \text{ for } \theta > 0 \text{ and } 0 < \beta \leqslant 1,$$

(4.5)

$$\longrightarrow 1 - \frac{1}{\beta} \text{ for } \theta > 0 \text{ and } \beta > 1.$$

This suggests that the distribution of F_{ab} is too spread out to lie completely within the interval $(-r^\beta, r^\beta)$ for large values of r however large β is. Moreover, if $L^\beta_{ab} = P\{|F_{ab}| < r^\beta\}$ and we write N_r for $[r^\beta] + \frac{1}{2}$,

$$L^\beta_{ab} = \sum_{|k|<r^\beta} f^k_{ab} = \sum_{|k|<r^\beta} \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_{ab}(\theta)\cos k\theta \, d\theta$$

$$= \frac{1}{\pi} \int_0^{\pi} \phi_{ab}(\theta) \frac{\sin N_r\theta}{\sin\frac{\theta}{2}} d\theta$$

(4.6)
$$= \frac{2}{\pi} \int_0^{\pi r^\beta} \phi_{ab}\left(\frac{\theta}{r^\beta}\right) \frac{\sin\left(\frac{N_r}{r^\beta}\theta\right)}{2r^\beta \sin\frac{\theta}{2r^\beta}} d\theta \quad .$$

According to (4.5) the integrand in (4.6) tends, as

$r \to +\infty$, for each $\theta > 0$ to 0 or $\frac{1-\beta}{\beta}\frac{\sin\theta}{\theta}$ according as $\beta \leqslant 1$ or $\beta > 1$.

Since $\int_0^\infty \frac{\sin\theta}{\theta}\,d\theta = \frac{\pi}{2}$, the obvious conjecture is

$$(4.7)\quad \lim_{r\to+\infty} P\{|F_{ab}| < r^\beta\} = 1 - \frac{1}{\beta} \text{ if } \beta > 1$$
$$= 0 \text{ if } \beta \leqslant 1.$$

(4.7) is proved in §.5, where we use additional properties of $\phi_{ab}(\theta)$ to show that the error involved in replacing the integrand in (4.6) by its limit is $O(1)$ as $r \to +\infty$.

If we write $F'_{ab} = \log|F_{ab}|$ when $|F_{ab}| > 0$

$$= 0 \text{ when } F_{ab} = 0,$$

the events $\{|F_{ab}| < r^\beta\}$ and $\{\frac{F'_{ab}}{\log r} < \beta\}$ coincide, so that (4.7) is essentially a limit theorem of the standard kind for the R.V's F'_{ab}. However it seems to be impossible to get any information about the characteristic function of F'_{ab} from our knowledge of ϕ_{ab}, so the usual methods of proving such a theorem do not apply.

§.5 For the proof of (4.7), two lemmas are required, in each of which $r > 0$ and $\beta > 1$. Also, without loss of generality we can and do take $a \geqslant b \geqslant 0$, since $\phi_{ab}(\theta) = \phi_{AB}(\theta)$, where $A = \max(|a|, |b|)$, $B = \min(|a|,|b|)$.

<u>Lemma 5.1</u> There exists $\delta(r)$ such that $0 \leqslant \phi_{ab}(\theta) \leqslant \delta(r)$ for $\frac{\log r}{r} \leqslant \theta \leqslant \pi$, and $\delta(r)\log r \longrightarrow 0$ as $r \longrightarrow +\infty$.

<u>Proof</u>

The relation $\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos n\alpha \; d\alpha}{z - \cos\alpha} = (z^2-1)^{-\frac{1}{2}}(z+\sqrt{z^2-1})^{-n}$

(Edwards [11, p.207]) gives

$$g_{ab}(\theta) = \frac{3}{\pi}\int_0^{\pi}\{(3-\cos\theta-\cos\beta)^2-1\}^{-\frac{1}{2}}\{3-\cos\theta-\cos\beta+\sqrt{(3-\cos\theta-\cos\beta)^2-1}\}^{-a} \cos b\beta d\beta$$

so that

$$g_{ab}(\theta) \leqslant \max_{0\leqslant\beta\leqslant\pi} \{3-\cos\theta-\cos\beta + \sqrt{(3-\cos\theta-\cos\beta)^2-1}\}^{-a}.$$

$$\frac{3}{\pi}\int_0^{\pi}\frac{d\beta}{\{(3-\cos\beta-\cos\theta)^2-1\}^{\frac{1}{2}}}$$

$$= \{2-\cos\theta+\sqrt{(2-\cos\theta)^2-1}\}^{-a} g_{oo}(\theta).$$

Thus $\phi_{ab}(\theta) \leqslant \{2-\cos\theta+\sqrt{(2-\cos\theta)^2-1}\}^{-a}$; moreover $2-\cos\theta+\sqrt{(2-\cos\theta)^2-1}$ is an increasing function of θ in $(0, \pi)$ and $a \geqslant \frac{r}{2} > 0$, so

$$\max_{\frac{\log r}{r}\leqslant\theta\leqslant\pi} \phi_{ab}(\theta) \leqslant \{2-\cos\frac{\log r}{r}+\sqrt{(2-\cos\frac{\log r}{r})^2-1}\}^{-\frac{r}{2}} = \delta(r).$$

Since $2-\cos\theta+\sqrt{(2-\cos\theta)^2-1} \sim 1+\theta$ as $\theta \downarrow 0$,

$$\delta(r) \sim (1+\frac{\log r}{r})^{-\frac{r}{2}} = e^{-\frac{r}{2}\log(1+\frac{\log r}{r})} \sim e^{-\frac{r}{2}(\frac{\log r}{r})} = r^{-\frac{1}{2}},$$

and this proves the lemma.

<u>Lemma 5.2</u> If the total variation of $\phi_{ab}(\theta)$ in the interval $(\frac{1}{r\beta}, \frac{\log r}{r})$ is V_{ab}^{β}, then $V_{ab}^{\beta} \leqslant 3\log\beta$ for all $r \geqslant r_o$, where r_o depends only on β.

Proof For $\theta \neq 0 (\bmod 2\pi)$

$$g'_{ab}(\theta) = \frac{-3\sin\theta}{(2\pi)^2} \int_{-\pi}^{\pi}\!\!\int \frac{\cos a\alpha \cos b\beta}{3-\cos\alpha-\cos\beta-\cos\theta} \, d\alpha \, d\beta,$$

so for $0 < \theta \leqslant \pi$ $|g'_{ab}(\theta)| \leqslant -g'_{oo}(\theta)$. In this range we also have, from (2.24), $0 < g_{ab}(\theta) \leqslant g_{oo}(\theta)$, and therefore

$$|\phi'_{ab}(\theta)| = \frac{1}{g^2_{oo}(\theta)} \, |g_{oo}(\theta)g'_{ab}(\theta) - g'_{oo}(\theta)g_{ab}(\theta)| \leqslant \frac{-2g'_{oo}(\theta)}{g_{oo}(\theta)} \, .$$

Thus

$$V^{\beta}_{ab} = \int_{r^{-\beta}}^{\frac{\log r}{r}} |\phi'_{ab}(\theta)| d\theta \leqslant -2 \int_{r^{-\beta}}^{\frac{\log r}{r}} \frac{g'_{oo}(\theta)}{g_{oo}(\theta)} \, d\theta = 2\log\left\{\frac{g_{oo}(r^{-\beta})}{g_{oo}(\frac{\log r}{r})}\right\}.$$

Since this last function is independent of a and b and, by assertion (2.20) of Lemma 2.17, tends to $2\log\beta$ as $r \rightarrow +\infty$, the lemma is established.

Lemma 5.2 will be used in conjunction with the following version of the second mean value theorem for functions of bounded variation (Hobson [18, p.570]).

Theorem 5.3 Let $v(x)$ be a function of bounded variation in the interval (s, t) and $u(x)$ be any function integrable in (s, t). Then if $V(s, t)$ is the total variation of $v(x)$ in (s, t) and $M = \max_{s \leqslant \sigma \leqslant \tau \leqslant t} \left|\int_{\sigma}^{\tau} u(x)dx\right|$,

$$\left|\int_{s}^{t} v(x) \, u(x) \, dx\right| \leqslant M\{|v(s)| + V(x, t)\}.$$

Proof of (4.7)

Take $\beta > 1$ and consider, in the notation of p.47 ,

$$L^{\beta}_{ab} = \frac{1}{\pi} \int_{0}^{\pi} \phi_{ab}(\theta) \, \frac{\sin N_r\theta}{\sin\frac{\theta}{2}} \, d\theta.$$

$\sin N_r\theta / \sin\frac{\theta}{2}$ is less in absolute value than $\left|\frac{1}{\sin\frac{\theta}{2}}\right|$, which in $(0, \pi)$ is less than $\frac{\pi}{\theta}$. Therefore, by Lemma 5.1,

$$(5.4)\quad \frac{1}{\pi}\left|\int_{\frac{\log r}{r}}^{\pi} \phi_{ab}(\theta)\,\frac{\sin N_r\theta}{\sin\frac{\theta}{2}}\,d\theta\right| \leqslant \delta(r)\,\log\frac{\pi r}{\log r} \to 0 \text{ as } r \to +\infty.$$

Since $|N_r - r^\beta| \leqslant 1$ for all r,

$$\left|\sin\frac{N_r}{r^\beta}\theta - \sin\theta\right| = 2\left|\cos\left(\frac{N_r}{r^\beta}+1\right)\frac{\theta}{2}\,\sin\left(\frac{N_r - r^\beta}{r^\beta}\right)\frac{\theta}{2}\right|$$

$$\leqslant 2\sin\frac{\theta}{2r^\beta} \quad \text{for } 0 \leqslant \theta \leqslant \pi,$$

and in this range we also have

$$0 \leqslant \frac{1}{2r^\beta \sin\frac{\theta}{2r^\beta}} - \frac{1}{\theta} = \frac{\left\{\frac{\theta}{2r^\beta} - \sin\frac{\theta}{2r^\beta}\right\}}{\theta\,\sin\frac{\theta}{2r^\beta}}$$

$$\leqslant \frac{\frac{1}{3!}\left(\frac{\theta}{2r^\beta}\right)^3}{\theta\cdot\frac{\theta}{\pi r^\beta}} = \frac{\pi\theta}{48 r^{2\beta}}.$$

If $\beta = 1 + \alpha$

$$\int_0^{\frac{\log r}{r}} \phi_{ab}(\theta)\frac{\sin N_r\theta}{\sin\frac{\theta}{2}}\,d\theta = \frac{1}{r^\beta}\int_0^{r^\alpha \log r} \phi_{ab}\left(\frac{\theta}{r^\beta}\right)\frac{\sin\frac{N_r}{r^\beta}\theta}{\sin\frac{\theta}{2r^\beta}}\,d\theta,$$

and if we write

$$\frac{\sin\frac{N_r}{r^\beta}\theta}{2r^\beta \sin\frac{\theta}{2r^\beta}} = \frac{\sin\theta}{\theta} + \left\{\frac{\sin\frac{N_r}{r^\beta}\theta - \sin\theta}{2r^\beta\,\sin\frac{\theta}{2r^\beta}}\right\} + \sin\theta\left\{\frac{1}{2r^\beta \sin\frac{\theta}{2r^\beta}} - \frac{1}{\theta}\right\}$$

the above estimates show that for all large enough r

$$\left|\int_0^{r^\alpha\log r} \phi_{ab}\left(\frac{\theta}{r^\beta}\right)\left\{\frac{\sin\frac{N_r}{r^\beta}\theta - \sin\theta}{2r^\beta\,\sin\frac{\theta}{2r^\beta}}\right\}d\theta\right| \leqslant r^{-\beta}\int_0^{r^\alpha\log r} d\theta = \frac{\log r}{r},$$

$$\left|\int_0^{r^\alpha \log r} \phi_{ab}\left(\frac{\theta}{r\beta}\right) \sin\theta\left\{\frac{1}{2r^\beta \sin\frac{\theta}{2r\beta}} - \frac{1}{\theta}\right\}d\theta\right| \leqslant \frac{\pi}{48r^2\beta}\int_0^{r^\alpha \log r} \theta d\theta \leqslant \frac{\pi}{96}\left(\frac{\log r}{r}\right)^2,$$

so that

$$(5.5)\quad \lim_{r\to+\infty}\left\{\int_0^{\frac{\log r}{r}} \phi_{ab}(\theta)\,\frac{\sin N_r\theta}{\sin\frac{\theta}{2}}d\theta - 2\int_0^{r^\alpha \log r} \phi_{ab}\left(\frac{\theta}{r\beta}\right)\frac{\sin\theta}{\theta}d\theta\right\}=0.$$

Now take any $R > 1$ and recall that $0 \leqslant \phi_{ab}(\theta) \leqslant 1$ for all θ (Corallary to Lemma 2.17) and that $\lim_{r\to+\infty} \phi_{ab}(\frac{\theta}{r\beta})$ is $1 - \frac{1}{\beta}$ for $\beta > 1$ and all $\theta > 0$ ((4.5)). Then, by the theorem of dominated convergence, we have

$$\lim_{r\to+\infty}\int_0^R \phi_{ab}\left(\frac{\theta}{r\beta}\right)\,\frac{\sin\theta}{\theta}\,d\theta = \left\{1 - \frac{1}{\beta}\right\}\int_0^R \frac{\sin\theta}{\theta}\,d\theta,$$

and since $\lim_{R\to+\infty}\int_0^R \frac{\sin\theta}{\theta}\,d\theta = \frac{\pi}{2}$,

$$(5.6)\quad \lim_{r\to+\infty}\left\{\int_0^R \phi_{ab}\left(\frac{\theta}{r\beta}\right)\frac{\sin\theta}{\theta}\,d\theta - \frac{\pi}{2}\,\frac{\beta-1}{\beta}\right\} = f_1(R),$$

where $\lim_{R\to+\infty} f_1(R) = 0.$

Since $\left|\int_\alpha^\beta \frac{\sin\theta}{\theta}\,d\theta\right| \leqslant \frac{2}{\alpha}$ for all $\beta > \alpha > 0$, Lemma 5.2 and Theorem 5.3 together show that for all large enough r

$$\left|\int_R^{r^\alpha \log r} \phi_{ab}\left(\frac{\theta}{r\beta}\right)\,\frac{\sin\theta}{\theta}\,d\theta\right| \leqslant \frac{2}{R}\left\{\phi_{ab}\left(\frac{R}{r\beta}\right) + 3\log\beta\right\}$$

so that

$$(5.7)\quad \limsup_{r\to+\infty}\left|\int_R^{r^\alpha \log r} \phi_{ab}\left(\frac{\theta}{r\beta}\right)\frac{\sin\theta}{\theta}\,d\theta\right| \leqslant f_2(R),$$

where $\lim_{R \to +\infty} f_2(R) = 0.$

It follows from (5.6) and (5.7) that

$$(5.8)\quad \limsup_{r \to +\infty} \left| \int_0^{r^\alpha \log r} \phi_{ab}\left(\frac{\theta}{r\beta}\right) \frac{\sin\theta}{\theta}\, d\theta - \frac{\pi}{2}\, \frac{\beta-1}{\beta} \right| \leqslant f_3(R),$$

where $\lim_{R \to +\infty} f_3(R) = 0$. Since the left hand side of (5.8) is independent of R, (5.8) implies that

$$(5.9)\quad \lim_{r \to +\infty} \int_0^{r^\alpha \log r} \phi_{ab}\left(\frac{\theta}{r\beta}\right) \frac{\sin\theta}{\theta}\, d\theta = \frac{\pi}{2}\, \frac{\beta-1}{B} .$$

(5.9), together with (5.5) and (5.4), is

$$\lim_{r \to +\infty} L^{\beta}_{ab} = 1 - \frac{1}{\beta} ,$$

so we have proved (4.7) for $\beta > 1$.

However, for every $\varepsilon > 0$, $0 \leqslant L^{1}_{ab} \leqslant L^{1+\varepsilon}_{ab}$, whence

$$0 \leqslant \liminf_{r \to +\infty} L^{1}_{ab} \leqslant \limsup_{r \to +\infty} L^{1}_{ab} \leqslant \frac{\varepsilon}{1+\varepsilon} \quad \text{for every } \varepsilon > 0,$$

and (4.7) for $\beta = 1$ follows. Since $0 \leqslant L^{\beta}_{ab} \leqslant L^{1}_{ab}$ for $\beta < 1$, this means that $\lim_{r \to +\infty} L^{\beta}_{ab} = 0$ for $\beta < 1$, so that (4.7) is established for all β.

CHAPTER III

§.1 If (S_n) is a R.W on E_1 generated by a RV X and I is the half-open interval (a, b], define G(I) by

$$G(I) = E(n : S_n \in I) = \sum_{n=0}^{\infty} P\{a < S_n \leqslant b\}.$$

Suppose that F(x), the distribution function of X, increases only at multiples of a fixed number (the "lattice" case) and that d is the largest positive number with this property. Then the Renewal Theorem asserts that if $0 < m = E(X) \leqslant \infty$ and I + x is the shifted interval (a + x, b+ x]

$$(1.1) \quad \lim_{x \to +\infty} G(I \pm [x]d) = \begin{cases} m^{-1}N(I) \\ 0 \end{cases},$$

where N(I) is the number of points of I of the form rd for some integer r and m^{-1} is to be interpreted as zero if $m = +\infty$. If there is no d with the above property we have the "non-lattice" ease, and, with a similar proviso about m, the Renewal theorem takes the form

$$(1.2) \quad \lim_{x \to +\infty} G(I \pm x) = \begin{cases} m^{-1}|I| \\ 0 \end{cases},$$

where $|I|$ is the length of I(= b-a).

{In the case that X can take only positive values (1.1) was proved by Erdös, Feller and Pollard [15], but can be derived from Kolmogorov's earlier work [20] on Markov chains, and (1.2) is due to Blackwell [2]. Chung and Wolfowitz [8] extended (1.1) to the general case, and Balckwell [3] did the same for (1.2)}.

It is natural to look for a similar result for a R.W $(\underline{S}_n)$ on E_k when $k \geq 2$. This was first done by Chung [7] in 1952. He showed that if $(\underline{S}_n)$ is generated by a R.W $\underline{X}$ whose distribution function $F(\underline{x})$ does not degenerate into a one-dimensional distribution function, and if at least one component of $\underline{m} = E(\underline{X})$ is finite and non-zero, then for any compact set A in E_k

$$(1.3) \quad \lim_{|\underline{x}| \to +\infty} G(A + \underline{x}) = 0,$$

where $G(A) = \sum_{n=0}^{\infty} P\{\underline{S}_n \in A\}$, $\underline{x}$ is a k-dimensional vector of length $|\underline{x}|$, and $A + \underline{x}$ denotes the translated set $\{\underline{a} + \underline{x} : \underline{a} \in A\}$. Feller [16] showed that (1.3) also holds when each component of $\underline{m}$ is infinite, but apparently no other investigations of $G(A + \underline{x})$ have been made.

In this chapter and the following one we prove that if $0 < |\underline{m}| < \infty$ and certain other conditions are satisfied $x^{\frac{k-1}{2}} G(A + x\underline{j})$ has, for each fixed vector $\underline{j}$, a limit as $x \longrightarrow +\infty$ which is non zero if and only if $\underline{j}$ is parallel to the mean vector $\underline{m}$. This theorem is in some ways analogous to (1.1) and (1.2), since, like them, it makes explicit the intuitively obvious fact that G(A) behaves differently when

A is moved off in the direction of the mean and when it is moved off in any other direction.

Our proof involves Fourier inversion and the use of Green's Theorem, and unfortunately it only applies in the following two situations:

(A): $(\underline{S}_n)$ is an aperiodic R.W on the lattice L_k,

(B): $(\underline{S}_n)$ is a R.W on E_k generated by a R.V $\underline{X}$ satisfying

(1.4) $\limsup_{|\underline{u}| \to +\infty} |\phi(\underline{u})| < 1$, where $\phi(\underline{u}) = E(e^{i\underline{X}.\underline{u}})$.

However there is a natural mapping from L_k onto any k-dimensional lattice L which would allow us to extend our results for case (A) to the more general situation in which $(\underline{S}_n)$ is an aperiodic R.W. on L. (1.4) is the k-dimensional version of a condition under which the Renewal Theorem (1.2) was first proved: the k-dimensional Riemann-Lebesgue Lemma shows that it is satisfied in the important case when the distribution function $F(\underline{x})$ of $\underline{X}$ has a non-vanishing absolutely continuous component.

In §.2 we present some definitions and preliminary results for a k-dimensional R.W, but the rest of the chapter is devoted to the planar case, k = 2.

§.2 A R.V $\underline{X}$ on E_k is said to be strictly k-dimensional if there is no k - 1 dimensional hyperplane D of E_k such that $P\{\underline{X} \in D\} = 1$.

<u>Lemma 2.1</u> If $\underline{X}$ is strictly k-dimensional and $\sigma^2 = E(|\underline{X}|^2) < +\infty$, $Q(u) \equiv \frac{1}{2}E((\underline{X}.\underline{u})^2)$ is a positive definite quadratic form.

<u>Proof</u> Suppose the lemma is false: then for some $\underline{u}^o \neq \underline{O}$ $E((\underline{X}.\underline{u}^o)^2) = 0$. Let D^o be the k-1 dimensional hyperplane which contains $\underline{O}$ and is perpendicular to $\underline{u}^o$. Then $\underline{X}.\underline{u}^o = |\underline{u}^o|.$(perpendicular distance between $\underline{X}$ and D^o), and $E((\underline{X}.\underline{u}^o)^2) = 0 \implies P\{$perpendicular distance between $\underline{X}$ and $D^o > 0\} = 0$, so that $\underline{X}$ lies on D^o with probability one, which contradicts the above definition.

(2.2) <u>Corollary to Lemma 2.1</u> For some constants $0 < k_1 \leqslant k_2 < +\infty$,

$$k_1|\underline{u}|^2 \leqslant Q(\underline{u}) \leqslant k_2|\underline{u}|^2 \text{ for all } \underline{u}.$$

<u>Lemma 2.3</u> If $\sigma^2 = E(|\underline{X}|^2) < +\infty$ and $\rho(\underline{u})$

$$= E\{e^{i\underline{u}.\underline{X}} - (1 + i\underline{u}.\underline{X} - \tfrac{1}{2}(\underline{u}.\underline{X})^2)\} = \phi(\underline{u}) - (1 + i\,\underline{m}.\underline{u} - Q(\underline{u})),$$

then $|\rho(\underline{u})| = O(|\underline{u}|^2)$ as $|\underline{u}| \to 0$.

<u>Proof</u> Rao and Kendall [27] prove the 1-dimensional analogue of Lemma 2.3, and we borrow from the following remark: for each integer $n \geqslant 1$ and for all real y

$$(2.4) \quad e^{iy} = \sum_{r=0}^{n} \frac{(iy)^r}{r!} + \theta_n(y)\frac{(iy)^{n+1}}{(n+1)!},$$

where $|\theta_n(y)| \leqslant 1$ and $|\theta_n(y)-1| \leqslant \frac{|y|}{n+2}$. Putting n = 1 and $y = \underline{u}.\underline{X}$ in (2.4), the existence of σ^2 means that the expectation of both sides exist, whence

$$\phi(\underline{u}) = E(e^{i\underline{X}.\underline{u}}) = 1 + i\,\underline{m}.\underline{u} - E\{\theta_1(\underline{X}.\underline{u})\frac{(\underline{X}.\underline{u})^2}{2}\},$$

so that

$$\rho(\underline{u}) = \tfrac{1}{2}E\{(\theta_1(\underline{x}.\underline{u}) - 1)(\underline{X}.\underline{u})^2\},$$

and therefore

$$(2.5) \quad \frac{|\rho(\underline{u})|}{|\underline{u}|^2} \leqslant \tfrac{1}{2} \int\int_{-\infty}^{\infty}\cdots\int |\underline{x}|^2 |\theta_1(\underline{u}.\underline{x}) - 1| dF(\underline{x}).$$

Given arbitrary $\varepsilon > 0$, we can find $X(\varepsilon)$ such that $\int\cdots\int_{|\underline{x}|> X} |\underline{x}|^2 dF(\underline{x}) < \frac{\varepsilon}{2}$, and since $|\theta_1(y) - 1| \leqslant 2$ for all real y this means that $\int\int\cdots\int_{|\underline{x}|>X} |\underline{x}|^2|\theta_1(\underline{u}.\underline{x})-1|dF(\underline{x})< \varepsilon$. Now $|\theta_1(\underline{u}.\underline{x}) - 1| \leqslant \frac{|\underline{u}||\underline{x}|}{3}$, so there exists $\delta(\varepsilon) > 0$ such that $|\theta_1(\underline{u}.\underline{x})-1| \leqslant \frac{\varepsilon}{\sigma^2}$ for $|\underline{x}|\leqslant X$ and $|\underline{u}| \leqslant \delta$. In (2.5) this is, for $|\underline{u}| \leqslant \delta$,

$$\frac{|\rho(\underline{u})|}{|\underline{u}|^2} < \frac{\varepsilon}{2} + \tfrac{1}{2} \int\cdots\int_{|\underline{x}|\leqslant X} \frac{\varepsilon}{\sigma^2} |\underline{x}|^2 \, dF(\underline{x}) \leqslant \varepsilon ,$$

and this proves the lemma.

<u>Lemma 2.6</u> If $\underline{X}$ is a R.V taking values on the lattice L_k and $E(|\underline{x}|^n) < +\infty$, then for r = 0, 1, ...n, each derivative of $\phi(\underline{u})$ of the rth order exists and has period 2π in each of the coordinate variables $u_1, \ldots u_k$.

<u>Proof</u> Since $\underline{X}$ takes values on L_k there are non-negative numbers $p_{\underline{a}} = P\{\underline{X} = \underline{a}\}$ such that

(2.7) $\phi(\underline{u}) = \sum_{\underline{a} \varepsilon L_k} P_{\underline{a}} e^{i\underline{a}.\underline{u}}$,

and as each $\underline{a} \varepsilon L_k$ has integer coordinates, the conclusion for r = 0 follows. Since $E(|\underline{X}|^n) = \sum_{\underline{a} \varepsilon L_k} P_{\underline{a}} |\underline{a}|^n < +\infty$, we may differentiate under the summation sign in (2.7) r times (r = 1, 2, ..., n) to conclude the proof.

A random walk $(\underline{S}_n)$ on a lattice L is said to be aperiodic if the set $H = \{\underline{a} : P\{\underline{S}_n = \underline{a}\} > 0$ for some $n\}$ is not contained in any proper sub-lattice of L.

Lemma 2.8 If $(\underline{S}_n)$ is an aperiodic R.W on L_k generated by a R.V $\underline{X}$ with characteristic function $\phi(\underline{u})$, then:

(2.9) $\phi(\underline{u}) = 1 \implies u_s = 0 \pmod{2\pi}$ for s = 1, ..., k;

(2.10) if S is any closed subset of $E_k^\pi = \bigcap_{s=1}^{k} \{|u_s| \leqslant \pi\}$ which does not contain $\underline{O}$, $\inf_{\underline{u} \varepsilon S} |1-\phi(\underline{u})| > 0$.

Proof The smallest lattice containing H is L(H) = $\{\underline{b} : \underline{b} = \sum_{s=1}^{r} \lambda_s \underline{a}_s$ for some integers λ_s and vectors $\underline{a}_s \varepsilon H\}$. If H_o is the subset of H consisting of all $\underline{a}$ with $P_{\underline{a}} > 0$, then plainly $L(H) = L(H_o)$, and the aperiodicity assumption means that $L(H) = L_k$. Thus every member of L_k is expressible as $\sum_{s=1}^{r} \lambda_s \underline{a}_s$ with $\underline{a}_s \varepsilon H_o$. Now take any $\underline{u}$ with $\phi(\underline{u}) = 1$, and notice that

$$R\{1 - \phi(\underline{u})\} = \sum_{H_o} (1 - \cos \underline{u}.\underline{a}) P_{\underline{a}},$$

so that $\underline{u}.\underline{a} = 0 \pmod{2\pi}$ for every $\underline{a} \in H_0$, and hence, by the previous remark, for every $\underline{a} \in L_k$. In particular, taking $\underline{a}$ to have s^{th} coordinate one and all other coordinates zero, we have

$$u_s = 0 \pmod{2\pi}, \text{ for } s = 1, 2, \dots k.$$

This proves (2.9) and implies that if $\phi(\underline{u}) = 1$ and $\underline{u} \in E_k^\pi$, then $\underline{u} = \underline{0}$. Since $|1-\phi(\underline{u})| \geqslant 0$ for all $\underline{u}$ and $\phi(\underline{u})$ is continuous in E_k^π, this establishes (2.10).

<u>Lemma 2.11</u> If $(\underline{S}_n)$ is an aperiodic R.W on L_k generated by a strictly k-dimensional R.V $\underline{X}$ with $E(|\underline{X}|^2) = \sigma^2 < +\infty$, then for any $\underline{a} \in L_k$

$$G(\{\underline{a}\}) = \lim_{\rho \uparrow 1} \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{-i(\underline{a}.\underline{u})} \frac{d\underline{u}}{1-\rho\phi(\underline{u})} = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{e^{-i\underline{a}.\underline{u}} d\underline{u}}{1-\phi(\underline{u})}$$

provided that if $k = 2$ $\underline{m} = E(\underline{X}) \neq \underline{0}$

<u>Proof</u> Since $\underline{S}_n$ has characteristic function $\phi^n(\underline{u})$, $P\{\underline{S}_n = \underline{a}\} = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \phi^n(\underline{u})\, e^{-i\underline{u}.\underline{a}} d\underline{u}$. Take real ρ with $0 < \rho < 1$ and look at

$$\sum_{n=0}^{\infty} \rho^n P\{\underline{S}_n = \underline{a}\} = \frac{1}{(2\pi)^k} \sum_{n=0}^{\infty} \rho^n \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \phi^n(\underline{u}) e^{-i\underline{u}.\underline{a}} d\underline{u}$$

$$= \frac{1}{(2\pi)^k} \lim_{N \to +\infty} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{1-\{\rho\phi(\underline{u})\}^{N+1}}{1-\rho\phi(\underline{u})} e^{-i\underline{u}.\underline{a}} d\underline{u}$$

$$= \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{e^{-i\underline{u}.\underline{a}}}{1-\rho\phi(\underline{u})} d\underline{u},$$

for $(\rho\phi(\underline{u}))^{N+1} \longrightarrow 0$ for each $\underline{u}$, and the interchange of limiting processes is legalised by the Theorem of Dominated Convergence, the dominating function being $2(1-\rho)^{-1}$. To complete the proof we have to show that

$$\lim_{\rho\uparrow 1}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{e^{-i\underline{u}.\underline{a}}}{1-\rho\phi(\underline{u})}\,d\underline{u}=\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{e^{-i\underline{u}.\underline{a}}}{1-\phi(\underline{u})}\,d\underline{u}.$$

Since $R\{1-\phi(\underline{u})\} = \sum_{\underline{a}\,\varepsilon\, L_k}(1-\cos\underline{u}\,\underline{a})\,P_{\underline{a}} \geqslant 0$, $|1-\rho\phi(\underline{u})| = |1-\rho+\rho(1-\phi(\underline{u})| \geqslant$

$$\rho|1-\phi(\underline{u})|,$$

and it therefore suffices to show that $(1-\phi(\underline{u}))^{-1}$ is absolutely integrable in E_k^{π}. Now $|1-\phi(\underline{u})| = |Q(\underline{u})-i\underline{m}.\underline{u}-\rho(\underline{u})|$ $\geqslant ||Q(\underline{u})-i\underline{m}.\underline{u}|-|\rho(\underline{u})||$, and by (2.2) and Lemma 2.3 $|Q(\underline{u})-i\underline{m}.\underline{u}| \geqslant k_1|\underline{u}|^2$ and $|\rho(\underline{u})| = O(|\underline{u}|^2)$ as $|\underline{u}|\longrightarrow 0$. Therefore for some $\delta>0$ $|1-\phi(\underline{u})| > \frac{1}{2}|Q(\underline{u})-i\underline{m}.\underline{u}|$ for all $|\underline{u}|\leqslant\delta$. Since, by (2.10) $\sup_{\underline{u}\,\varepsilon\,[E_k^{\pi}\setminus\{|\underline{u}|>\delta\}]}|1-\phi(\underline{u})|^{-1} < +\infty$, it merely remains to check that $(Q(\underline{u})-i\underline{m}.\underline{u})^{-1}$ is integrable in $|\underline{u}|\leqslant\delta$. If $k\geqslant 3$ this follows from the fact that $|Q(\underline{u})|\geqslant k_1|\underline{u}|^2$, and if $k=2$ we have to apply a change of variable and note that for any $0<\delta'<+\infty$.

$$\iint_{-\delta'}^{\delta'}\frac{dv_1\,dv_2}{\{k_1^2(V_1^2+V_2^2)^2+m_2v_1^2\}^{\frac{1}{2}}} < \frac{4}{mk_1}\iint_{0}^{\delta'}\frac{dv_1\,dv_2}{\{2v_1(v_1^2+v_2^2)\}^{\frac{1}{2}}} < +\infty .$$

Turning now to the 'non-lattice' case (B), we need the following consequence of (1.4).

<u>Lemma 2.12</u> If $\limsup_{|\underline{u}|\to+\infty} |\phi(\underline{u})| < 1$, then $\sup_{|\underline{u}|>\delta} |\phi(\underline{u})| < 1$ for every $\delta > 0$.

<u>Proof</u> Since $R\{1 - \phi(\underline{u})\} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (1 - \cos \underline{u}.\underline{x})dF(\underline{x})$

and $1-\cos\alpha = 2\sin^2\frac{\alpha}{2} \geqslant 2\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2} = \frac{1}{2}\sin^2\alpha = \frac{1}{4}(1-\cos 2\alpha)$ for all α, $R\{1 - \phi(\underline{u})\} \geqslant \frac{1}{4} R\{1 - \phi(2\underline{u})\}$ and hence

(2.13) $\quad R\{1 - \phi(\underline{u})\} \geqslant \frac{1}{4^n} R\{1 - \phi(2^n\underline{u})\}$

for any $n \geqslant 1$. Let $\limsup_{|\underline{u}|\to+\infty} |\phi(\underline{u})| = 1 - \varepsilon$. Then for some $u < +\infty$ $\sup_{|\underline{u}|>u} |\phi(\underline{u})| \leqslant 1 - \frac{\varepsilon}{2}$. Now $\overline{\phi(\underline{u})} = E(e^{i(-\underline{X}.\underline{u})})$ is a characteristic function, and so also is $|\phi(\underline{u})|^2 = \phi(\underline{u}).\overline{\phi(\underline{u})}$. Given $\delta > 0$ we can pick N such that $2^N > \frac{u}{\delta}$ and apply (2.13) to $|\phi(\underline{u})|^2$ with $n = N$ to get for every $|\underline{u}| > \delta$,

$$1 - |\phi(\underline{u})|^2 \geqslant \frac{1}{4^N}\{1 - |\phi(2^N\underline{u})|^2\} > \frac{1}{4^N}\frac{\varepsilon}{2}.$$

In case (A) Lemma 2.11 provides a representation of G(A) as a Fourier integral whenever A is a bounded subset of L_k, and we would like a similar formula for case (B). It transpires that we need only consider G(A) when A is an interval of E_k (that is, for some $\underline{a}$, $\underline{t}$ $A = I(\underline{a}, \underline{t}) = \{\underline{x}: |x_s - a_s| \leqslant t_s \text{ for } s = 1, 2, \ldots, k\}$) and we also employ the device (used by Chung and Pollard [7] for 1-dimensional Renewal Theory) of considering the integrated version of $G(I(\underline{a}, \underline{t}))$;

$$(2.14)\quad L(\underline{a}, \underline{h}) = \int_0^{h_1}\cdots\int_0^{h_k} G(I(\underline{a}, \underline{t}))d\underline{t} = \int_0^{h_1}\cdots\int_0^{h_k} \sum_{n=0}^{\infty} P\{\underline{S}_n \varepsilon\, I(\underline{a},t)\}d\underline{t}.$$

<u>Lemma 2.15</u> If $\underline{S}_n$ is a R.W on $\underline{E}_k$ generated by a strictly k-dimensional R.V. $\underline{X}$ with $E(|\underline{X}|^2) = \sigma^2 < +\infty$ and $\limsup_{|\underline{u}|\to+\infty} |\phi(\underline{u})| < 1$, then for any $\underline{a} \varepsilon E_k$ and $\underline{h}$ with $0 < h_s < +\infty\ (s = 1, 2, \ldots k)$

$$L(\underline{a}, \underline{h}) = \frac{1}{\pi^k}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{D(\underline{h}, \underline{u})}{1-\phi(\underline{u})}\, e^{-i\underline{a}\cdot\underline{u}}\, d\underline{u},$$

where $D(\underline{h}, \underline{u}) = \prod_{s=1}^{k} \frac{1-\cosh_s u_s}{u_s^2}$, provided that if $k = 2$ $\underline{m} = E(\underline{X}) \neq \underline{0}$.

<u>Proof</u> Starting from the standard inversion formula for k-dimensional characteristic functions we may derive, just as Lukacs [23, p.51] does for $k = 1$, the integrated version:

$$(2.16)\quad \int_0^{h_1}\cdots\int_0^{h_k} P\{\underline{X}\ \varepsilon\, I(\underline{a}, \underline{t})\}d\underline{t} = \frac{1}{\pi^k}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} D(\underline{h}, \underline{u})\phi(\underline{u})e^{-i\underline{a}\cdot\underline{u}}\, d\underline{u}.$$

Applying (2.16) to each of the characteristic functions $\phi^n(\underline{u})$, we have for $0 < \rho < 1$,

$$\sum_{n=0}^{\infty} \rho^n \int_0^{h_1}\cdots\int_0^{h_k} P\{\underline{S}_n\ \varepsilon\ \ I(\underline{a}, \underline{t})\}d\underline{t} = \frac{1}{\pi^k}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} D(\underline{h},\underline{u})\{\rho\phi(\underline{u})\}^n e^{-i\underline{a}\cdot\underline{u}} d\underline{u}$$

$$= \frac{1}{\pi^k}\lim_{N\to+\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} D(\underline{h},\underline{u})\,\frac{1-\{\rho\phi(\underline{u})\}^{N+1}}{1-\rho\phi(\underline{u})}\, e^{-i\underline{a}\cdot\underline{u}}\, d\underline{u}$$

$$= \frac{1}{\pi^k}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} D(\underline{h}, \underline{u})\,\frac{e^{-i\underline{a}\cdot\underline{u}}}{1-\rho\phi(\underline{u})}\, d\underline{u},$$

since $\left|\dfrac{1-\{\rho\phi(\underline{u})\}^{N+1}}{1-\rho\phi(\underline{u})}\right| \leqslant \dfrac{2}{1-\rho}$ for all $\underline{u}$ and N and $D(\underline{h}, \underline{u})$ is a non-negative function which is integrable over E_k. This last remark, together with Lemma 2.12, shows that for every $\delta > 0$

$$\lim_{\rho \uparrow 1} \int_{|\underline{u}| \geqslant \delta} \cdots \int \frac{D(\underline{h}, \underline{u})e^{-i\underline{\alpha}.\underline{u}}}{1-\rho\phi(\underline{u})} d\underline{u} = \int_{|\underline{u}| \geqslant \delta} \cdots \int \frac{D(\underline{h},\underline{u})}{1-\phi(\underline{u})} e^{-i\underline{\alpha}.\underline{u}} d\underline{u}$$

and since $D(\underline{h}, \underline{u}) = \prod_{s=1}^{k} \dfrac{2\sin^2 \frac{h_s u_s}{2}}{u_s^2} \leqslant \prod_{s=1}^{k} \dfrac{h_s^2}{2}$, the estimates used in the proof of Lemma 2.11 show that for some $\delta > 0$

$$\lim_{\rho \uparrow 1} \int_{|\underline{u}|<\delta} \cdots \int \frac{D(\underline{h},\underline{u})}{1-\rho\phi(\underline{u})} e^{-i\underline{\alpha}\underline{u}} d\underline{u} = \int_{|\underline{u}|<\delta} \cdots \int \frac{D(\underline{h},\underline{u})}{1-\phi(\underline{u})} e^{-i\underline{\alpha}.\underline{u}} d\underline{u}.$$

Thus $\lim_{\rho \uparrow 1} \sum_{n=0}^{\infty} \rho^n \int_0^{h_1} \cdots \int_0^{h_k} P\{\underline{S}_n \varepsilon\, I(\underline{\alpha},\underline{t})\} \underline{dt}$ exists, and since the integrand is positive it equals $\int_0^{h_1} \cdots \int_0^{h_k} \{\lim_{\rho \uparrow 1} \sum_{n=0}^{\infty} \rho^n P\{\underline{S}_n \varepsilon I(\underline{\alpha},\underline{t})\}\} \underline{dt}$ $= L(\underline{\alpha}, \underline{h})$.

In investigating the Fourier integrals which occur in Lemmas 2.11 and 2.15, repeated use will be made of the following version of Green's Theorem {Courant and Hilbert [9, p.257]}.

Theorem 2.17 If S is a bounded subset of E_k with piecewise smooth boundary σ and $\psi_1(\underline{u})$ and its first and second derivatives and $\psi_2(\underline{u})$ and its first derivatives are all continuous and integrable in $S + \sigma$, then

(2.18) $\int_S \cdots \int \{ \psi_2 \nabla^2 \psi_1 + \underline{\nabla} \psi_1 \cdot \underline{\nabla} \psi_2 \} d\underline{u} = \int_\sigma \cdots \int \psi_2 \frac{\partial \psi_1}{\partial \underline{n}} d\underline{\sigma}$,

where $\frac{\partial}{\partial \underline{n}}$ denotes differentiation along the outward drawn normal to σ. If, in addition, the second derivatives of $\psi_2(\underline{u})$ exist and are continuous and integrable in $S + \sigma$, then

(2.19) $\int_S \cdots \int \{ \psi_2 \nabla^2 \psi_1 - \psi_1 \nabla^2 \psi_2 \} d\underline{u} = \int_\sigma \cdots \int \{ \psi_2 \frac{\partial \psi_1}{\partial \underline{n}} - \psi_1 \frac{\partial \psi_2}{\partial \underline{n}} \} d\underline{\sigma}$

§.3 When k = 2 it is obvious from Lemmas 2.3 and 2.11 that for case (A) the following question is crucial to our investigation:

(3.1): if $f(\underline{u}) \sim \frac{1}{Q(\underline{u}) - i\underline{m} \cdot \underline{u}}$ as $|\underline{u}| \to 0$ and $F(\underline{R}) = \int_{-\pi}^{\pi}\int e^{i\underline{u}\cdot\underline{R}} f(\underline{u})\, d\underline{u}$, how does $F(\underline{R})$ behave as $|\underline{R}| \to +\infty$?

In this section we give an answer to (3.1) which also applies to case (B).

<u>Theorem 3.2</u> Suppose $f(\underline{u}, \underline{R}) = \frac{k_2}{P(\underline{u}) - i\underline{e}_1 \cdot \underline{u}} \{1 + g(\underline{u}, \underline{R})\}$, where P is a positive definite quadratic form, $\underline{e}_1$ is a unit vector and g satisfies:

(3.3) for each $\underline{R}$ $\frac{\partial g}{\partial u_1}$ and $\frac{\partial g}{\partial u_2}$ exists and g, $\frac{\partial g}{\partial u_1}$ and $\frac{\partial g}{\partial u_2}$ are continuous in $S(d, d) = \{ |u_1| \leqslant d, |u_2| < d \}$;

(3.4) $g \to 0$ uniformly in $\underline{R}$ as $|\underline{u}| \to 0$;

(3.5) $|g|$ and $|P(\underline{u}) - i\underline{e}_1 \cdot \underline{u}| \, |\underline{\nabla} g|$ are bounded in $S(d, d)$ uniformly in $\underline{R}$.

If $\underline{e}_2$ is a unit vector orthogonal to $\underline{e}_1$, denote by $T(\delta_1, \delta_2)$ the region $\{|v_1| \leqslant \delta_1, |v_2| \leqslant \delta_2\}$, where $v_1 = \underline{e}_1 \cdot \underline{u}$ and $v_2 = \underline{e}_2 \cdot \underline{u}$. Then for any $\delta > 0$

$$(3.6) \quad |\underline{R}|^{\frac{1}{2}} \left\{ \iint_{S(d,d)} e^{i\underline{u}\cdot\underline{R}} f(\underline{u},\underline{R})\,d\underline{u} - k_3 \iint_{T(\delta,\delta)} \frac{e^{i\underline{u}\cdot\underline{R}}}{P(\underline{u}) - i\underline{e}_1\cdot\underline{u}}\,d\underline{u} \right\} \to 0$$

as $|\underline{R}| \to +\infty$.

Proof Writing R for $|\underline{R}|$, let ε be a function of R to be chosen later with $\varepsilon(R) \downarrow 0$ as $R \uparrow +\infty$. Then if $S_1 = T(\varepsilon, \varepsilon^{\frac{1}{2}})$, $S_2 = S(d, d) \setminus T(\varepsilon, \varepsilon^{\frac{1}{2}})$ and $S_3 = T(\delta,\delta) \setminus T(\varepsilon, \varepsilon^{\frac{1}{2}})$ (3.6) takes the form

$$(3.7) \quad \lim_{R \to +\infty} R^{\frac{1}{2}} \{ k_3 I_1 + I_2 - k_3 I_3 \} = 0,$$

where $I_1 = \iint_{S_1} \frac{g(\underline{u},\underline{R})e^{i\underline{u}\cdot\underline{R}}}{P(\underline{u}) - i\underline{e}_1\cdot\underline{u}}\,d\underline{u}$, $I_2 = \iint_{S_2} f(\underline{u}, \underline{R})\, e^{i\underline{u}\cdot\underline{R}}\, d\underline{u}$

and $I_3 = \iint_{S_3} \frac{e^{i\underline{u}\cdot\underline{R}}}{P(\underline{u}) - i\underline{e}_1\cdot\underline{u}}\,d\underline{u}$.

Now let $M(\varepsilon) = \sup_{\underline{u} \in S_1} \{ \sup_{\underline{R} \in E_2} |g(\underline{u}, \underline{R})| \}$: then assumption (3.4) implies that $M(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.

If $P_1(\underline{v}) = P(\underline{u})$, then P_1 is also positive definite, so that $P_1(\underline{v})(v_1^2 + v_2^2)^{-1}$ is bounded away from zero, whence

$$|I_1| \leqslant M(\varepsilon) \iint_{S_1} \frac{d\underline{u}}{|P(\underline{u}) - i\underline{e}_1\cdot\underline{u}|} = M(\varepsilon) \int_{v_1=-\varepsilon}^{\varepsilon} \int_{v_2=-\varepsilon^{\frac{1}{2}}}^{\varepsilon^{\frac{1}{2}}} \frac{dv_1\,dv_2}{|P_1(\underline{v}) - iv_1|}$$

$$\leq k_4 M(\varepsilon) \int_{v_1=0}^{\varepsilon} \int_{v_2=0}^{\varepsilon^{\frac{1}{2}}} \frac{dv_1\, dv_2}{(v_2^4 + v_1^2)^{\frac{1}{2}}}$$

$$= k_4 M(\varepsilon) \int_0^{\varepsilon} \int_0^{\varepsilon} \frac{dw_1\, dw_2}{2(w_2(w_1^2+w_2^2))^{\frac{1}{2}}}$$

$$(3.8) \quad = k_5\, \varepsilon^{\frac{1}{2}} M(\varepsilon).$$

Applying Green's Theorem (2.18) to I_2 with $\psi_1 = \frac{-1}{R^2} e^{i\underline{u}.\underline{R}}$ and $\psi_2 = f(\underline{u}, \underline{R})$, we have for every $\underline{R}$,

$$I_2 = \frac{i}{R^2} \iint_{S_2} e^{i\underline{u}.\underline{R}}\, \underline{R}.\, \underline{\nabla f}.\, d\underline{u} - \frac{1}{R^2} \int_{\sigma(d,d)\cup\tau(\varepsilon,\varepsilon^{\frac{1}{2}})} \frac{\partial}{\partial \underline{n}} (e^{i\underline{u}.\underline{R}}) f\, d\underline{\sigma} ,$$

where σ and τ are the boundaries of S and T, respectively. Plainly $|\frac{\partial}{\partial \underline{n}} e^{i\underline{u}.\underline{R}}| \leq R$ for all $\underline{u}$ and $\underline{n}$, and since $P(\underline{u})$ is bounded away from zero on $\sigma(d,d)$, assumption (3.5) means that

$$R^{-1} \left|\int_{\sigma(d,d)} \frac{\partial}{\partial \underline{n}}(e^{i\underline{u}.\underline{R}}) f(\underline{u}, \underline{R})\, d\underline{\sigma}\right|$$ is bounded for all $\underline{R}$. Also from (3.5) we have

$$R^{-1}\left|\int_{\tau(\varepsilon,\varepsilon^{\frac{1}{2}})} \frac{\partial}{\partial \underline{n}}(e^{i\underline{u}.\underline{R}})\, f(\underline{u}, \underline{R})\, d\underline{\sigma}\right| \leq k_6 \left|\int_{\tau(\varepsilon,\varepsilon^{\frac{1}{2}})} \frac{d\underline{\sigma}}{|P(\underline{u})-i\underline{e}_1.\underline{u}|}\right|$$

$$= k_6 \int_{v_1=-\varepsilon}^{\varepsilon} \left\{\frac{1}{|P_1(v_1,\varepsilon^{\frac{1}{2}})-iv_1|} + \frac{1}{|P_1(v_1,-\varepsilon^{\frac{1}{2}})-iv_1|}\right\} dv_1$$

$$+ k_6 \int_{v_2=-\varepsilon^{\frac{1}{2}}}^{\varepsilon^{\frac{1}{2}}} \left\{\frac{1}{|P_1(\varepsilon, v_2)-i\varepsilon|} + \frac{1}{|P_1(-\varepsilon, v_2)+i\varepsilon|}\right\} dv_2$$

$$\leq k_7 \int_0^{\varepsilon} \frac{dv_1}{\varepsilon} + 4k_6 \int_0^{\varepsilon^{\frac{1}{2}}} \frac{dv_2}{\varepsilon} = k_7 + 4k_6\, \varepsilon^{-\frac{1}{2}}.$$

A third consequence of (3.5) is that

$$R^{-1}\left|\iint_{S_2} e^{i\underline{u}.\underline{R}}\ \underline{R}.\ \underline{\nabla f}\ d\underline{u}\right|$$

$$\leqslant k_3\left|\iint_{S_2}\left\{\frac{(1+|g|)}{P^2(\underline{u})+(\underline{e}_1.\underline{u})^2}\ |\underline{\nabla}P(\underline{u}) - i\underline{e}_1| + \frac{|\nabla g|}{|P(\underline{u})-i\underline{e}_1.\underline{u}|}\right\} d\underline{u}\right|$$

$$\leqslant k_8 \iint_{S_2} \frac{d\underline{u}}{P^2(\underline{u})+(\underline{e}_1.\underline{u})^2}\ ,$$

for plainly $|\underline{\nabla}\ P(\underline{u})|$ is bounded in S_2. Now for some $D < +\infty$, $S_2 = S(d,\ d)\setminus T(\varepsilon,\varepsilon^{\frac{1}{2}}) \subseteq T(D,\ D^{\frac{1}{2}})\setminus T(\varepsilon,\ \varepsilon^{\frac{1}{2}})$, and

$$\iint_{\{\varepsilon<|v_1|\leqslant D,\ \varepsilon^{\frac{1}{2}}<|v_2|\leqslant D^{\frac{1}{2}}\}} \frac{dv_1 dv_2}{P_1^2(\underline{v})+v_1^2} \leqslant k_9\iint_{v_1=\varepsilon\, v_2=\varepsilon^{\frac{1}{2}}}^{D\ D} \frac{dv_1\ dv_2}{v_1^4+v_2^2}$$

$$= \frac{k_9}{2}\iint_{\varepsilon\,\varepsilon}^{D\,D} \frac{dw_1\ dw_2}{w_1^{\frac{1}{2}}(w_1^2+w_2^2)}$$

$$\leqslant k_{10}\ \varepsilon^{-\frac{1}{2}}\ ,$$

so that we finally have, for all $\underline{R}$,

(3.9) $|I_2| \leqslant (k_{12} + k_{11}\ \varepsilon^{-\frac{1}{2}})\ R^{-1}$

An exactly similar calculation on I_3 leads to

(3.10) $|I_3| \leqslant \{k_{14} + k_{13}\ \varepsilon^{-\frac{1}{2}}\}R^{-1}$

and if we now put $\varepsilon = \frac{\lambda}{R}$, (3.8), (3.9) and (3.10) yield

$$\limsup_{R\to+\infty} R^{\frac{1}{2}}\left|k_3 I_1 + I_2 - k_3 I_3\right| \leqslant \frac{k_{11}+k_3 k_{13}}{\lambda^{\frac{1}{2}}}\ ,$$

and since we can take λ arbitrarily large, (3.7) is established and with it the theorem.

The second part of the answer to (3.1) is contained in:

<u>Theorem 3.11</u> If $P(\underline{u}) = P_1u_1^2+2P_{12}u_1u_2 + P_2u_2^2$ is a positive definite quadratic form and $\underline{e}$ is a unit vector, then for any $\delta > 0$

$$\lim_{R\to+\infty} R^{\frac{1}{2}} \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \frac{e^{-iR(\underline{u}.\underline{e})}}{P(\underline{u})-iu_1}\, du_1du_2 = 0 \text{ if } \underline{e} \neq (1, 0)$$

$$= \frac{2\pi^{\frac{3}{2}}}{\sqrt{P_2}} \text{ if } \underline{e} = (1, 0).$$

<u>Proof</u> In Theorem 3.2 write $f(\underline{u}, \underline{R}) = \dfrac{1}{P_2u_2^2-iu_1}$, $\underline{e}_1 = (1, 0)$, $\underline{e}_2 = (0, 1)$, $k_3 = 1$ and $d = \delta$. Then $g(\underline{u}, \underline{R}) = \dfrac{P_1u_1^2 + 2P_{12}u_1u_2}{P_2u_2^2-iu_1}$, which is less absolute value than $P_1u_1 + P_{12}u_2$, so that (3.4) is satisfied. Since g has continuous first derivatives and is bounded in $S(\delta, \delta)$, (3.3) and (3.5) will be satisfied if $|P(\underline{u}) - iu_1||\underline{\nabla g}|$ is bounded in $S(\delta,\delta)$. Since $\dfrac{|P(\underline{u})-iu_1|}{|P_2u_2^2-iu_1|}$ is bounded in this region, it plainly is. The conclusion of Theorem 3.2 when $\underline{R} = R\underline{e}$ is

$$(3.12)\quad \lim_{R\to+\infty} R^{\frac{1}{2}} \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \left\{\frac{1}{P(\underline{u})-iu_1} - \frac{1}{P_2u_2^2-iu_1}\right\} e^{-iR(\underline{u}.\underline{e})}d\underline{u} = 0.$$

Consider the case $e_1 = 0$. Since $\displaystyle\int_{-\delta}^{\delta}\frac{du}{A-iu} = 2A\int_0^{\delta}\frac{du}{A^2+u^2} = 2\tan^{-1}\frac{\delta}{A}$,

$$\int_{-\delta}^{\delta}\int \frac{e^{-iRu_2e_2}}{P_2u_2^2-iu_1}\,du_1\,du_2 = 2\int_{-\delta}^{\delta} e^{-iRu_2e_2}\tan^{-1}\frac{\delta}{P_2u_2^2}\,du_2$$

$$= 4\int_0^{\delta}\cos(Ru_2e_2)\tan^{-1}\frac{\delta}{P_2u_2^2}\,du_2$$

$$= 4\cdot\frac{\pi}{2}\cdot\int_0^{\delta'(R)}\cos(Ru_2e_2)du_2,$$

where we have used Bonnet's mean value theorem (Hobson [18, p.565]), $\tan^{-1}\frac{\delta}{P_2u_2^2}$ being monotone decreasing in $(0,\delta)$. Since $e_2 = \pm 1$, this last integral is $O(\frac{1}{R})$ as $R \to +\infty$, and this, together with (3.12), proves the theorem when $e_1 = 0$.

If $e_1 \neq 0$, consider for $x > \delta$,

$$J(x) = \int_{u_2=-\delta}^{\delta} e^{-iRu_2e_2}\,du_2\int_{\delta}^{x} e^{-iRu_1e_1}\frac{du_1}{P_2u_2^2-iu_1}.$$

Since $\int_{\delta}^{x}\frac{e^{-iRu_1e_1}}{P_2u_2^2-iu_1}du_1 = \frac{i}{Re_1}\left[\frac{e^{-iRe_1x}}{P_2u_2^2-ix} - \frac{e^{-iRe_1\delta}}{P_2u_2^2-i\delta}\right]$

$$+\frac{1}{Re_1}\int_{\delta}^{x}\frac{e^{-iRu_1e_1}}{(P_2u_2^2-iu_1)^2}\,du_1,$$

and $\int_{u_2=-\delta}^{\delta}\int_{\delta}^{\infty}\frac{du_1\,du_2}{P_2^2u_2^4+u_1^2} = \int_{-\delta}^{\delta}\frac{2}{P_2u_2^2}\tan^{-1}\frac{2P_2u_2^2}{\delta}\,du_2 < +\infty,$

we have, by dominated convergence and the Riemann-Lebesgue Lemma,

$$\lim_{x\to+\infty} J(x) = \frac{1}{Re_1}\int_{u_2=-\delta}^{\delta}\int_{\delta}^{\infty}\frac{e^{-iRue}}{P_2u_2^2-iu_1}\,du_1\,du_2$$

$$-\frac{ie^{-iRe_1\delta}}{Re_1}\int_{-\delta}^{\delta}\frac{e^{-iRe_2u_2}}{P_2u_2^2-i\delta}\,du_2 = O(\tfrac{1}{R})\text{ as } R\to+\infty.$$

Similarly $\int_{u_2=-\delta}^{\delta} \int_{-\infty}^{-\delta} e^{-iR(\underline{u}\cdot\underline{e})} \frac{du_1\, du_2}{P_2 u_2^2 - iu_1} = O(\frac{1}{R})$ as $R \to +\infty$,

and therefore, if $e_1 \neq 0$,

$$(3.13) \quad \lim_{R \to +\infty} R^{\frac{1}{2}} \left\{ \int_{-\delta}^{\delta}\int \frac{e^{-iR\underline{u}\cdot\underline{e}}}{P_2 u_2^2 - iu_1} du_1 du_2 - \int_{u_2=-\delta}^{\delta} \int_{-\infty}^{\infty} \frac{e^{-iR\underline{u}\cdot\underline{e}}}{P_2 u_2^2 - iu_1} du_1\, du_2 \right\}$$

$$= 0.$$

Now if $A > 0$,

$$\int_{-\infty}^{\infty} \frac{e^{-iBu} du}{A - iu} = 2\pi e^{-AB} \quad \text{if } B > 0$$

$$= 0 \qquad \text{if } B < 0.$$

Thus if $e_1 < 0$ the second integral in (3.13) vanishes for $R > 0$ and then (3.13) and (3.12) prove the theorem. If $e_1 > 0$ we have, for $R > 0$,

$$\int_{u_2=-\delta}^{\delta} \int_{-\infty}^{\infty} \frac{e^{-iR\underline{u}\cdot\underline{e}}}{P_2 u_2^2 - iu_1} du_1\, du_2 = 2\pi \int_{-\delta}^{\delta} e^{-iRu_2 e_2}\, e^{-RP_2 u_2^2 e_1} du_2$$

$$= 2\pi \left\{ \int_{-\infty}^{\infty} e^{-iRu_2 e_2}\, e^{-Re_1 P_2 u_2^2} du_2 - I(R) \right\},$$

where

$$R^{\frac{1}{2}} |I(R)| = R^{\frac{1}{2}} \left| \int_{|u|>\delta} e^{-iRe_2 u}\, e^{-Re_1 P_2 u^2} du \right|$$

$$\leq 2R^{\frac{1}{2}} \int_{\delta}^{\infty} e^{-Re_1 P_2 u^2} du \leq 2 \int_{\delta R^{\frac{1}{2}}}^{\infty} e^{-e_1 P_2 u^2} du.$$

Since $P_2 e_1 > 0$, this last integral vanishes as $R \to +\infty$, and the theorem is established if we notice that [Erdélyi [12, p.121]]

$$R^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-iRu_2e_2} e^{-Re_1P_2u_2^2} du_2 = \sqrt{\frac{\pi}{e_1P_2}} \; e^{\frac{-Re_2^2}{4e_1P_2}},$$

for this last only has a non-zero limit as $R \to +\infty$ when $\underline{e} = (1, 0)$, the limit then being $\sqrt{\frac{\pi}{P_2}}$.

§.4 We are now in a position to state and prove our results for $k = 2$. In case (A) we have:

Theorem 4.1 Let $(\underline{S}_n)$ be an aperiodic R.W on L_2 generated by a strictly 2-dimensional R.V $\underline{X}$. Assume

(4.2) $\underline{m} = E(\underline{X}) = m\,\ell_1$, where $|\ell_1| = 1$ and $0 < m < +\infty$;

(4.3) $\sigma^2 = E(|\underline{X}|^2) < +\infty$.

Suppose that ℓ_2 is a unit vector orthogonal to ℓ_1, and if $Q(\underline{u}) = E(\frac{1}{2}(\underline{X}.\underline{u})^2)$ let $Q(\underline{u}) = Q_1(\underline{v})$ when $v_1 = \underline{u}.\ell_1$, $v_2 = \underline{u}.\ell_2$. Then, if $Q_1(0, v_2) = \Delta^2 v_2^2$ and $[x\underline{j}]$ denotes the vector with components $[xj_1]$, $[xj_2]$, for each unit vector $\underline{j}$,

$$(4.4) \quad \lim_{x \to +\infty} \{x^{\frac{1}{2}} G(A + [x\underline{j}])\} = \frac{N(A)}{2\Delta\sqrt{m\pi}} \text{ if } \underline{j} = \ell_1$$

$$= 0 \text{ if } \underline{j} \neq \ell_1,$$

where A is any bounded subset of L_2 having $N(A)$ members.

Proof Since $G(A) = \sum_{\underline{a} \in A} G(\{\underline{a}\})$, it is sufficient to prove the theorem when A has a single member $\underline{a}$. Therefore, by Lemma 2.11, we have to evaluate $\lim_{x \to +\infty} \frac{x^{\frac{1}{2}} J(x)}{(2\pi)^2}$, where

$$J(x) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{e^{-i\underline{u}(\underline{a}+[x\underline{j}])}}{1-\phi(\underline{u})} \, d\underline{u}$$

$$= \frac{1}{m}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-ix\underline{u}.\underline{j}} \frac{1+g(\underline{u},\, x)}{m^{-1}Q(\underline{u})-i\underline{\ell}_1.\underline{u}} \, d\underline{u}$$

and $g(\underline{u},\, x) = \frac{Q(\underline{u}) - i\underline{m}.\underline{u}}{1-\phi(\underline{u})} e^{-i\underline{u}.(\underline{a}+[x\underline{j}]-x\underline{j})} - 1.$

Now $\phi(\underline{u})$ and its first derivatives are continuous in $S(\pi, \pi)$, and therefore for each x so are $g(\underline{u}, x)$ and its first derivatives. Since $|\underline{a} + [x\underline{j}] - x\underline{j}| \leq |\underline{a}| + \{(xj_1-[xj_1])^2 + (xj_2-[xj_2])^2\}^{\frac{1}{2}} \leq |\underline{a}| + \sqrt{2}$, and, by Lemma 2.3, $\lim_{|\underline{u}|\downarrow 0} \frac{Q(\underline{u})-i\underline{m}.\underline{u}}{1-\phi(\underline{u})} = 1$, $g \to 0$ uniformly in x as $|\underline{u}| \to 0$. Now Lemmas 2.3 and 2.8 imply that $\frac{Q(\underline{u})-i\underline{m}.\underline{u}}{1-\phi(\underline{u})}$ is bounded in $S(\pi, \pi)$, and since $|\nabla\phi(\underline{u})| \leq m < +\infty$, it follows that $|g(\underline{u}, x)|$ and $|Q(\underline{u})-i\underline{m}.\underline{u}||\nabla g(\underline{u},x)|$ are bounded in $S(\pi, \pi)$ uniformly in x. Thus the conditions of Theorem 3.2 are satisfied with $\underline{R} = -x\underline{j}$, and its conclusion is that for any $\delta > 0$

$$(4.5) \quad \lim_{x\to+\infty} x^{\frac{1}{2}}\left\{J(x) - \frac{1}{m}\iint_{T(\delta,\delta)} e^{-ix\underline{j}.\underline{u}} \frac{d\underline{u}}{m^{-1}Q(\underline{u})-i\underline{\ell}_1.\underline{u}}\right\} = 0.$$

Now

$$\iint_{T(\delta,\delta)} \frac{e^{-ix\underline{j}.\underline{u}}\, d\underline{u}}{m^{-1}Q(\underline{u})-i\underline{\ell}_1.\underline{u}} = \int\int_{-\delta}^{\delta} \frac{e^{-ix\underline{j}'.\underline{v}} d\underline{v}}{m^{-1}Q_1(\underline{v})-iv_1}$$

where $\underline{j}'$ is a unit vector which equals (1, 0) if and only if

$\underline{j} = \ell_1$, and Theorem 3.11 shows that

$$\lim_{x\to+\infty} x^{\frac{1}{2}} \iint_{-\delta}^{\delta} \frac{e^{-ixj'\cdot v}}{m^{-1}Q_1(\underline{v})-iv_1}\, d\underline{v} = 0 \text{ if } \underline{j}' \neq (1,0)$$

$$= \frac{2\pi^{3/2} m^{\frac{1}{2}}}{\Delta} \text{ if } \underline{j}' = (1,0)$$

This, in (4.5), proves the theorem.

In case (B) we have:

Theorem 4.6 Let $(\underline{S}_n)$ be a R.W on E_2 generated by a strictly 2-dimensional R.V $\underline{X}$. Assume

(4.7) $\underline{m} = E(\underline{X}) = m\underline{\ell}_1$, where $|\underline{\ell}_1| = 1$ and $0 < m < +\infty$;

(4.8) $\sigma^2 = E(|\underline{X}|^2) < +\infty$;

(4.9) $\limsup_{|\underline{u}|\to\infty} |\phi(\underline{u})| < 1$, where $\phi(\underline{u}) = E(e^{i\underline{X}\cdot\underline{u}})$.

Suppose that $\underline{\ell}_2$ is a unit vector orthogonal to $\underline{\ell}_1$, and if $Q(\underline{u}) = E(\frac{1}{2}(\underline{X}\cdot\underline{u})^2)$ let $Q(\underline{u}) = Q_1(\underline{v})$ when $v_1 = \underline{u}\cdot\underline{\ell}_1$, $v_2 = \underline{u}_2\cdot\underline{\ell}_2$. Then, if $Q_1(0, v_2) = \Delta^2 v_2^2$ and A is any Jordan measurable subset of E_2 with measure $|A|$, for any unit vector $\underline{j}$

(4.10) $$\lim_{x\to+\infty} \{x^{\frac{1}{2}} G(A + x\underline{j})\} = \frac{|A|}{2\Delta\sqrt{m\pi}} \text{ if } \underline{j} = \underline{\ell}_1$$

$$= 0 \text{ if } \underline{j} \neq \underline{\ell}_1.$$

Proof It is sufficient to establish (4.10) when A is a bounded interval of E_2. For if (4.10) holds for every such interval I, consider the case when $A = \bigcup_{s=1}^{r} I_s$. Then, since $\bigcup_{s=1}^{r} I_s = \bigcup_{s=1}^{n} I_s'$, where the I_s' are mutually disjoint intervals, and plainly $G(\bigcup_{s=1}^{n} I_s^{\prime}) = \sum_{s=1}^{n} G(I_s^{\prime})$, (4.10)

holds for every A of this form. But given an arbitrary Jordan-measurable A and an arbitrary $\varepsilon > 0$ there exists sets A_1 A_2 such that $A_1 \subseteq A \subseteq A_2$, A_1 and A_2 are unions of finite numbers of intervals and

$|A_1| + \varepsilon \geq |A| \geq |A_2| - \varepsilon$. Plainly

$P\{\underline{S}_n \in A_1 + x\underline{j}\} \leq P\{\underline{S}_n \in A + x\underline{j}\} \leq P\{\underline{S}_n \in A_2 + x\underline{j}\}$

for all x and $\underline{j}$, and so, for each $x > 0$

$$x^{\frac{1}{2}}G\{A_1 + x\underline{j}\} \leq x^{\frac{1}{2}}G\{A + x\underline{j}\} \leq x^{\frac{1}{2}}G\{A_2 + x\underline{j}\}.$$

Now let $x \to +\infty$, and denote the right hand side of (4.10) by $\lambda(\underline{j})$ to get for every $\underline{j}$,

$$(|A| - \varepsilon)\lambda(\underline{j}) \leq |A_1|\lambda(\underline{j}) \leq \liminf_{x \to +\infty} x^{\frac{1}{2}} G\{A + x\underline{j}\}$$

$$\leq \limsup_{x \to +\infty} x^{\frac{1}{2}} G\{A + x\underline{j}\} \leq |A_2|\lambda(\underline{j}) \leq (|A| + \varepsilon)\lambda(\underline{j})$$

Since ε is arbitrary, this implies that $\lim_{x \to +\infty} x^{\frac{1}{2}}G(A+x\underline{j}) = |A|\lambda(\underline{j})$, so that (4.10) holds for arbitrary Jordon-measurable A.

We are thus left to prove that for every $\underline{a}$ and $\underline{h}$ with $0 < h_1 < +\infty$, $0 < h_2 < +\infty$,

(4.11) $$\lim_{x \to +\infty} x^{\frac{1}{2}} G\{I(\underline{a}, \underline{h}) + x\underline{j}\} = 4h_1h_2\lambda(\underline{j}).$$

Instead of (4.11), we prove

(4.12) $$\lim_{x \to +\infty} x^{\frac{1}{2}}L(\underline{a} + x\underline{j}, \underline{h}) = h_1^2 h_2^2 \lambda(\underline{j}),$$

where $L(\underline{a}, \underline{h}) = \int_0^{h_1} \int_0^{h_2} G\{I(\underline{a}, \underline{t})\}dt_1\, dt_2$. For, if $0 < g_1 < h_1$, $0 < g_2 < h_2$,

$$\int_{g_1}^{h_1} \int_{g_2}^{h_2} G\{I(\underline{a}, \underline{t})\}\, dt_1\, dt_2 = L(\underline{a}, \underline{h}) + L(\underline{a}, \underline{g}) - L(\underline{a},(h_1, g_2)) - L(\underline{a},(g_1, h_2)),$$

so that (4.12) implies

$$(4.13) \quad \lim_{x \to +\infty} x^{\frac{1}{2}} \int_{g_1}^{h_1} \int_{g_2}^{h_2} G\{I(\underline{a}+x\underline{j}, \underline{t})\}dt_1\, dt_2 = \lambda(\underline{j})(h_1^2 - g_1^2)(h_2^2 - g_2^2).$$

Since $G\{I(\underline{a}, \underline{t})\}$ is a non-decreasing function of t_1 and t_2 we have, for every $x > 0$,

$$(4.14) \quad x^{\frac{1}{2}}G\{I(\underline{a}+x\underline{j},\underline{g})\}(h_1-g_1)(h_2-g_2) \leq x^{\frac{1}{2}} \int_{g_1}^{h_1} \int_{g_2}^{h_2} G\{I(\underline{a}+x\underline{j},\underline{t})\}dt_1dt_2$$

$$\leq x^{\frac{1}{2}}\, G\{I(\underline{a}+x\underline{j}, \underline{h})\}\, (h_1-g_1)(h_2-g_2),$$

and it therefore follows from (4.13) that

$$\limsup_{x \to +\infty} x^{\frac{1}{2}}\, G\{I(\underline{a}+x\underline{j}, \underline{g})\} \leq (h_1 + g_1)(h_2 + g_2)\, \lambda(\underline{j})$$

$$\leq \liminf_{x \to +\infty} x^{\frac{1}{2}}G\{I(\underline{a} + x\underline{j}, \underline{h})\}.$$

If we let $h_1 \downarrow g_1$ and $h_2 \downarrow g_2$ in the first inequality and $g_1 \uparrow h_1$ and $g_2 \uparrow h_2$ in the second, this becomes

$$\limsup_{x \to +\infty} x^{\frac{1}{2}}G\{I(\underline{a} + x\underline{j}, \underline{g})\} \leq 4g_1g_2\, \lambda(\underline{j})$$

$$\liminf_{x \to +\infty} x^{\frac{1}{2}}\, G\{I(\underline{a} + x\underline{j}, \underline{h})\} \geq 4h_1h_2\, \lambda(\underline{j}).$$

Thus (4.11) is a consequence of (4.12).

In order to prove (4.12) we have to evaluate, according to Lemma 2.15, $\lim_{x \to +\infty} \frac{x^{\frac{1}{2}} J(x)}{\pi^2}$, where

$$J(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(\underline{h}, \underline{u}) \frac{e^{-i\underline{u}.(x\underline{j}+\underline{a})}}{1 - \phi(\underline{u})} \, d\underline{u}.$$

Taking any $\delta > 0$, split the region of integration in the above formula into $S(\delta, \delta)$ and $E_2 \setminus S(\delta, \delta)$, calling the corresponding integrals $J_1(x)$ and $J_2(x)$. Then $J_2(x)$ $= \lim_{R \to +\infty} J_2(x, R)$, where

$$J_2(x, R) = \iint_{S(R,R) \setminus S'\delta,\delta)} e^{-ix\underline{j}.\underline{u}} . \frac{D(\underline{h},\underline{u}) e^{-i\underline{a}.\underline{u}}}{1 - \phi(\underline{u})} \, d\underline{u}.$$

Applying Green's Theorem (2.18) with $\psi_1(\underline{u}) = -\frac{1}{x^2} e^{-ix\underline{j}.\underline{u}}$,

$\psi_2(\underline{u}) = \frac{D(\underline{h}, \underline{u})}{1-\phi(\underline{u})} e^{-i\underline{a}.\underline{u}}$, we have for every $R > \delta$,

$$(4.15) \quad J_2(x, R) = \frac{-i}{x} \iint_{S(R,R) \setminus S(\delta,\delta)} e^{-ix\underline{j}.\underline{u}} \underline{j}.\underline{\nabla}\psi_2(\underline{u}) \, d\underline{u} +$$

$$\frac{i}{x} \int_{\sigma(R,R) \cup \sigma(\delta,\delta)} \underline{n}.\underline{j} e^{-ix\underline{j}.\underline{u}} \psi_2(\underline{u}) \, d\sigma .$$

If $w(\delta) = \sup_{\underline{u} \notin S(\delta,\delta)} |1-\phi(\underline{u})|^{-1}$, by Lemma 2.12 $w(\delta) < +\infty$ for every $\delta > 0$. Since the second integrand of (4.15) is dominated by $w(\delta) D(\underline{h}, \underline{u})$ and

$$\int_{u_1=-R}^{R} D(\underline{h}, \underline{u})\Big|_{|u_2|=R} du_1 \leq \frac{2}{R^2} \int_{-R}^{R} \frac{1-\cos h_1 u_1}{u_1^2} du_1 \leq \frac{2h_1\pi}{R^2} ,$$

the contribution from $\sigma(R,R)$ vanishes as $R \to +\infty$. This remark also shows that

$$\left| \int_{\sigma(\delta,\delta)} \underline{n}.\underline{j}\, e^{-ix\underline{j}.\underline{u}}\, \psi_2(\underline{u})\, d\underline{\sigma} \right| \leqslant \frac{8(h_1 + h_2)\pi w(\delta)}{\delta^2},$$

and since it is easy to check that $|\underline{\nabla}\, \psi_2(\underline{u})|$ is integrable in $E_2 \setminus S(\delta, \delta)$, $J_2(x) = \lim_{R \to +\infty} J_2(x, R)$ exists and is $O(\frac{1}{x})$ as $x \to +\infty$ for every $\delta > 0$.

We can now apply Theorem 3.2 to

$$J_1(x) = \iint_{S(\delta,\delta)} e^{-ix\underline{j}.\underline{u}} \frac{D(\underline{h}, \underline{u})}{1 - \phi(\underline{u})} e^{-i\underline{a}.\underline{u}}\, d\underline{u}.$$

For if we notice that without loss of generality we can take $\ell_1 = (1, 0)$, $\ell_2 = (0, 1)$, (the theorem in the general case following by a change of coordinates from this particular case), and write $\frac{D(\underline{h}, \underline{u})}{1-\phi(\underline{u})} e^{-i\underline{a}.\underline{u}} = \frac{h_1^2 h_2^2}{4m} \frac{(1 + g(\underline{u}))}{m^{-1}Q(\underline{u})-iu_1}$, it follows from Lemma 2.3 that $\lim_{|\underline{u}| \to 0} g(\underline{u}) = 0$. Plainly $g(\underline{u})$ and its first derivatives are continuous in $S(\delta, \delta)$, and since Lemmas 2.3 and 2.12 imply that $\frac{Q - imu_1}{1-\phi(\underline{u})}$ is bounded in $S(\delta, \delta)$, the fact that the first derivatives of $D(\underline{h}, \underline{u})$ are bounded in this region allows us to check that $g(\underline{u})$ and $|Q(\underline{u}) - imu_1||\underline{\nabla}\, g(\underline{u})|$ are bounded in $S(\delta, \delta)$. The conclusion of Theorem 3.2, with $\underline{R} = -x\underline{j}$, is that for any $\delta_1 > 0$

$$(4.16)\quad \lim_{x \to +\infty} x^{\frac{1}{2}} \left\{ J_1(x) - \int_{-\delta_1}^{\delta_1} \int_{-\delta_1}^{\delta_1} \frac{h_1^2 h_2^2 e^{-ix\underline{j}.\underline{u}}}{4(Q(\underline{u})-iu_1 m)}\, d\underline{u} \right\} = 0.$$

Now Theorem 3.11 tells us that

$$\lim_{x \to +\infty} x^{\frac{1}{2}} \int_{-\delta_1}^{\delta_1} \int_{-\delta_1}^{\delta_1} \frac{e^{-ix\underline{j}.\underline{u}}}{Q(\underline{u})-imu_1}\, d\underline{u} = 4\pi^2 \lambda(\underline{j}) ,$$

and this in (4.16) establishes (4.12), and hence Theorem 4.6.

CHAPTER IV

§.1 Theorems 4.1 and 4.6 of the previous chapter are the special cases k = 2 of the following theorems.

<u>Theorem 1.1</u> Let $(\underline{S}_n)$ be an aperiodic R.W on $L_k (k \geqslant 2)$ generated by a strictly k-dimensional R.V $\underline{X}$.

Assume

(1.2) $\underline{m} = E(\underline{X}) = m|\ell_1|$, where $|\ell_1| = 1$ and $0 < m < +\infty$;

(1.3) $\sigma^2 = E(|\underline{X}|^2) < +\infty$;

(1.4) $E\{|\underline{X}|[\frac{k+1}{2}]\} < +\infty$.

Suppose that $\ell_1, \ell_2, \ldots, \ell_k$ are orthonormal vectors and if $Q(\underline{u}) = E\{\frac{1}{2}(\underline{X}.\underline{u})^2\}$ let $Q(\underline{u}) = Q_1(\underline{v})$ when $v_s = \underline{u}.\ \ell_s$ for s = 1, 2, ..., k. Then $Q_0(v_2, v_3, \ldots, v_k) = Q_1(0, v_2, \ldots v_k)$ is a positive definite quadratic form with non-zero determinant Δ^2, and if $\underline{j}$ is any unit vector and $[x\underline{j}]$ has components $[xj_s]$ for s = 1, 2, ...k,

$$(1.5)\quad \lim_{x \to +\infty} \{x^{\frac{k-1}{2}} . G(A + [x\underline{j}])\} = \frac{N(A)}{2^{k-1}\pi\Delta} . \left(\frac{m}{\pi}\right)^{\frac{k-3}{2}} \text{ if } \underline{j} = \ell_1$$

$$= 0 \text{ if } \underline{j} \neq \ell_1,$$

where A is any bounded subset of L_k having N(A) members.

<u>Theorem 1.6</u> Let $(\underline{S}_n)$ be a R.W on $E_k (k \geqslant 2)$ generated by a strictly k-dimensional R.V $\underline{X}$. Assume that (1.2), (1.3), (1.4) and

(1.7): $\limsup_{|\underline{u}| \to \infty} |\phi(\underline{u})| < 1$, where $\phi(\underline{u}) = E(e^{i\underline{u}.\underline{X}})$;

hold. Then if A is any Jordan measurable subset of E_k having measure $|A|$ and $\underline{j}$ is any unit vector,

(1.8) $$\lim_{x \to +\infty} \{ x^{\frac{k-1}{2}} G(A+x\underline{j}) \} = \frac{|A|}{2^{k-1}\pi\Delta} \left(\frac{m}{\pi}\right)^{\frac{k-3}{2}} \quad \text{if } \underline{j} = \ell_1$$

$$= 0 \quad \text{if } \underline{j} \neq \ell_1 .$$

In §.2 and §.3 we give a detailed proof of these theorems when k = 3. Since the limits in (1.5) and (1.8) involve only the first and second moments of $\underline{X}'$, assumption (1.4) means that our conditions for $k \geq 4$ are unlikely to be best possible. We therefore restrict ourselves to giving, in §.4, only a sketch of the proof for this case.

At first sight the appearance of the factor $m^{\frac{k-3}{2}}$ in the above results is rather surprising, especially for the case k = 3. In one-dimensional renewal theory it is plain that by increasing the mean of the R.V X one increases the average size of the steps which the particle takes, and since with probability one the particle drifts off to $+\infty$, the effect of this will be to diminish the probability of the particle visiting any fixed set. Though this effect is still present in $k(\geq 2)$-space, there is another one as well. To see this, consider the lattice case when the conditions of Theorem 1.1 hold with $\ell_1 = (1, 0, 0, \ldots 0)$ so that

$\mathbb{E}(X_1) = m$ and $\mathbb{E}(X_s) = 0$ for $s = 2, 3, \ldots, k$.

If $S_n^{(1)}$, $\underline{S}_n^{(2)}$ denote the projections of $\underline{S}_n$ onto the x_1 axis and the $(x_2, x_3, \ldots x_k)$ hyperplane, respectively, it is reasonable to suppose that the long term behaviour of $S_n^{(1)}$ and $\underline{S}_n^{(2)}$ will be independent. Then

$$(1.9) \quad G(\{\underline{O}\} + [x\,\underline{\ell}_1]) = \sum_{n=0}^{\infty} P(S_n^{(1)} = [x], \underline{S}_n^{(2)} = \underline{O})$$

$$\sim \sum_{n=0}^{\infty} P(S_n^{(1)} = [x])\, P(\underline{S}_n^{(2)} = \underline{O}) \text{ as } x \to +\infty.$$

Just as in the 1-dimensional case, one effect of increasing m will be to diminish $\sum_{n=0}^{\infty} P(S_n^{(1)} = [x])$. However, the Central Limit Theorem indicates that only those terms in (1.9) with $nm - x = O(\sqrt{x})$ make a significant contribution for large x. Thus increasing m will pick out terms $P\{\underline{S}_n^{(2)} = O\}$ with smaller values of n, and these terms will therefore be larger. We can carry this crude argument a stage further by noting that the local version of the Multidimensional Central Limit Theorem gives

$$P\{\underline{S}_n^{(2)} = \underline{O}\} \sim \frac{1}{2^{k-1}\Delta} \frac{1}{(\pi n)^{\frac{k-1}{2}}} \text{ as } n \to +\infty.$$

It is therefore plausible that

$$G\{\{\underline{O}\} + [x\,\underline{\ell}_1]) \sim \sum_{|nm-x|=O(x^{\frac{1}{2}})} P(S_n^{(1)} = [x]).\, P(\underline{S}_n^{(2)} = \underline{O})$$

$$\sim \frac{1}{2^{k-1}\Delta} \sum_{|nm-x|=O(x^{\frac{1}{2}})} P(S_n^{(1)} = [x]) \frac{1}{(\pi n)^{\frac{k-1}{2}}}$$

$$\sim \left(\frac{m}{\pi x}\right)^{\frac{k-1}{2}} \frac{1}{\Delta 2^{k-1}} \sum_{|nm-x|=O(x^{\frac{1}{2}})} P\{S_n^{(1)} = \{x]\}$$

$$\sim \left(\frac{m}{\pi x}\right)^{\frac{k-1}{2}} \cdot \frac{1}{\Delta 2^{k-1}} \cdot \frac{1}{m} \quad \text{as } x \to +\infty,$$

the last step depending upon the one-dimensional Renewal Theorem.

§.2 When $k = 3$ Lemmas 3.2.11 and 3.2.15 again show that we must investigate the asymptotic behaviour of the Fourier transform of a function which has a singularity like $\{Q(\underline{u}) - i\underline{m}.\underline{u}\}^{-1}$ at the origin. However, if we try to prove a straightforward analogue of Theorem 3.3.2 it turns out that when we apply Green's Theorem we get a non-vanishing contribution from the surface integrals. To overcome this difficulty we introduce a further technical device in the following theorem.

Theorem 2.1 Suppose that the coordinates $\underline{u}$ and $\underline{v}$ are connected by $v_s = \underline{u}.\underline{e}_s$, where the $\underline{e}_s$ form an orthonormal triad, and denote by $S(\delta_1, \delta_2, \delta_3)$ and $T(\delta_1, \delta_2, \delta_3)$ the regions $\{|u_s| \leq \delta_s,\ s = 1, 2, 3\}$ and $\{|V_s| \leq \delta_s,\ s = 1, 2, 3\}$ respectively. Let $e^{i\underline{u}.\underline{R}} f(\underline{u}, \underline{R}) = \dfrac{k_1 e^{i\underline{v}.\underline{R}'}\{1 + g(\underline{v}, \underline{R})\}}{P(v_2, v_3) - iv_1}$, where P is a positive definite quadratic form, $|R'|-|R|$ is bounded for all $\underline{R}$ and g satisfies:

(2.2) for each $\underline{R}$, g, $\frac{\partial g}{\partial v_s}$, and $\frac{\partial^2 g}{\partial v_s \partial v_t}$ exist and are continuous in $S(\pi, \pi, \pi)$ for s = 1, 2, 3 and t = 1, 2, 3;

(2.3) $|g|$, $|P-iv_1||\frac{\partial g}{\partial v_s}|$ and $|P-iv_1|^2|\frac{\partial^2 g}{\partial v_s \partial v_t}|$ are bounded in $S(\pi, \pi, \pi)$ uniformly in $\underline{R}$ for s = 1, 2, 3 and t = 1, 2, 3;

(2.4) $|g| \to 0$ uniformly in $\underline{R}$ as $|\underline{v}| \to 0$.

Then, if $f(\underline{u}, \underline{R})$ and $\frac{\partial f}{\partial u_s}(\underline{u}, \underline{R})$ (s = 1, 2, 3) are periodic with period 2π in each u_s (s = 1, 2, 3)

$$(2.5)\quad \lim_{\delta \downarrow 0}\left\{\limsup_{|\underline{R}| \to +\infty, \underline{R}\in L_3} |\underline{R}|\left|\iiint_{S(\pi,\pi,\pi)} e^{+i\underline{u}.\underline{R}} f(\underline{u}.\underline{R})\, d\underline{u} - k_1 \iiint_{T(\delta^{1+\beta},\delta,\delta)} \frac{e^{+i\underline{v}.\underline{R}'} d\underline{v}}{P(v_2, v_3) - iv_1}\right|\right\} = 0,$$

for every $0 < \beta < 1$.

<u>Proof</u> Since $P(v_2, v_3)$ is positive definite, $k_2(v_2^2 + v_3^2) \leq P(v_2, v_3) \leq k_3(v_2^2 + v_3^2)$ for all v_2, v_3, where $0 < k_2 \leq k_3 < +\infty$. Writing R and R′ for $|\underline{R}|$ and $|\underline{R}'|$ respectively, let $\varepsilon_1(R)$ and $\varepsilon_2(R)$ be functions, to be chosen later, which decrease to zero as R increases to $+\infty$ and satisfy

$$(2.6)\quad \varepsilon_2^2 \leq \varepsilon_1 \min(1, k_2^{-1})$$

for all R. Then if $S_1 = T(\varepsilon_1, \varepsilon_2, \varepsilon_2)$, $S_2 = S(\pi, \pi, \pi) \setminus S_1$ and $S_3 = T(\delta^{1+\beta}, \delta, \delta) \setminus S_1$, (2.5) takes the form

$$(2.7)\quad \lim_{\delta \downarrow 0}\left\{\limsup_{R \to +\infty, \underline{R}\in L_3} R|k_1 I_1 + I_2 - k_1 I_3|\right\} = 0$$

where $I_1 = \iiint_{S_1} \frac{e^{+i\underline{v}.\underline{R}'} g(\underline{v},\underline{R})}{P(v_2,v_3)-iv_1} d\underline{v}$, $I_2 = \iiint_{S_2} e^{+i\underline{u}.\underline{R}} f(\underline{u},\underline{R}) d\underline{u}$

and $I_3 = \iiint_{S_3} \frac{e^{+i\underline{v}.\underline{R}'}}{P(v_2,v_3)-iv_1} d\underline{v}$.

Defining $M(\varepsilon) = \sup_{\underline{v}\varepsilon T(\varepsilon,\varepsilon^{\frac{1}{2}},\varepsilon^{\frac{1}{2}})} \{\sup_{\underline{R}\varepsilon L_3} |g(\underline{v},\underline{R})|\}$, we see that (2.4) implies that $M(\varepsilon)\downarrow 0$ as $\varepsilon\downarrow 0$. On account of (2.6), $S_1 \subseteq T(\varepsilon_1,\varepsilon_1^{\frac{1}{2}},\varepsilon_1^{\frac{1}{2}})$ and therefore

$$|I_1| \leq M(\varepsilon_1) \iiint_{S_1} \frac{d\underline{v}}{|P(v_2,v_3)-iv_1|}$$

$$\leq M(\varepsilon_1) \iiint_{S_1} \frac{d\underline{v}}{\{k_2^2(v_2^4+v_3^4)+v_1^2\}^{\frac{1}{2}}}$$

$$= M(\varepsilon_1) \int_{-k_2\varepsilon_1}^{k_2\varepsilon_1} \int_0^{\varepsilon_2^2} \int_0^{\varepsilon_2^2} \frac{dw_1 dw_2 dw_3}{\{(w_1^2+w_2^2+w_3^2)w_2 w_3\}^{\frac{1}{2}}}$$

(3.8) $\leq k_4\varepsilon_1 M(\varepsilon_1)$.

By an application of Green's Theorem (3.2.19),

$$I_2 = \iiint_{S_2} f(\underline{u}, \underline{R}) \nabla^2\{-\frac{1}{R^2} e^{i\underline{u}.\underline{R}}\} d\underline{u}$$

$$= -\frac{1}{R^2} \iiint_{S_2} e^{i\underline{u}.\underline{R}} \nabla^2 f(\underline{u}, \underline{R}) d\underline{u} + \frac{1}{R^2} \iint_{\sigma\cup\tau} e^{i\underline{u}.\underline{R}}\{\underline{n}.\underline{\nabla} \underline{f} - i\underline{R}.\underline{n}f\} d\sigma$$

where σ and τ are the boundaries of $S(\pi, \pi, \pi)$ and $T(\varepsilon_1, \varepsilon_2, \varepsilon_2)$ respecitively. σ consists of the six plane faces $\sigma^{\pm}(s)$ ($s = 1, 2, 3$), where $\sigma^{+}(s) = \{u_s = +\pi, |u_t| \leq \pi$ for $t \neq s\}$. By assumption $f(\underline{u},\underline{R})$ and $\underline{\nabla} f(\underline{u}, \underline{R})$ are periodic with period 2π in each variable, and if $\underline{R} \varepsilon L_3$ the same is true of $e^{i\underline{u}.\underline{R}}$. Therefore the contributions from the faces $\sigma^+(s)$ and $\sigma^-(s)$ cancel for each s, and the total contribution from the integral over σ is zero. Now assumption (2.3) means that the contribution from τ is less in absolute value than

$$\iint_{\tau}\left\{\frac{k_5 R}{|P(v_2,v_3)-iv_1|} + \frac{k_6}{|P(v_2,v_3)-iv_1|^2}\right\}d\underline{\sigma}\ .$$ Since

$$\tau_1 = \{\,|v_1| = \varepsilon_1, |v_2| \leq \varepsilon_2, |v_3| \leq \varepsilon_2\}$$

has area $8\varepsilon_2^2$ and on it $|P(v_2,v_3)-iv_1| \geq \varepsilon_1$,

$$\tau_2 = \{\,|v_1| \leq \varepsilon_1,\ |v_2| = \varepsilon_2,\ |v_3| \leq \varepsilon_2\}$$

has area $8\varepsilon_1\varepsilon_2$ and on it $|P(v_2,v_3)-iv_1| \geq k_2\varepsilon_2^2$, and $\tau_3 = \{|v_1| \leq \varepsilon_1, |v_2| \leq \varepsilon_2, |v_3| = \varepsilon_2\}$ has area $8\varepsilon_1\varepsilon_2$ and on it $|P(v_2,v_3)-iv_1| \geq k_2\varepsilon_2^2$, this last integral is dominated by

$$8k_5 R\left\{\frac{\varepsilon_2^2}{\varepsilon_1} + \frac{2\varepsilon_1}{k_2\varepsilon_2}\right\} + 8k_6\left\{\frac{\varepsilon_2^2}{\varepsilon_1^2} + \frac{2\varepsilon_1}{k_2^2\varepsilon_2^3}\right\}.$$ Again from (2.3) and the fact that $P(v_2,v_3)-iv_1$ and its first two derivatives are bounded in $S(\pi, \pi, \pi)$ we have

$$|\nabla^2 f(\underline{u}, \underline{R})| \leq \frac{k_7}{|P(v_2,v_3)-iv_1|^3}$$ for all $\underline{u} \in S_2$ and all $\underline{R}$.

Thus

$$\left|\iiint_{S_2} e^{i\underline{u}.\underline{R}}\ \nabla^2 f(\underline{u}, \underline{R})\ d\underline{u}\right| \leq k_7 \iiint_{\{\underline{v}\notin S_1\}} \frac{d\underline{v}}{|P(v_2,v_3)-iv_1|^3}$$

$$\leq k_7 \iiint_{\{|\underline{w}|\geq \varepsilon_2^2\}} \frac{dw_1\, dw_2\, dw_3}{(w_2 w_3)^{2}\{w_1^2+w_2^2+w_3^2\}^{3/2}}$$

$$\leq k_8/\varepsilon_2^2 \;,$$

and therefore

$$(2.9)\quad |I_2| \leq \frac{k_8}{R^2\varepsilon_2^2} + \frac{8k_6}{R^2}\left\{\frac{\varepsilon_2^2}{\varepsilon_1^2} + \frac{2\varepsilon_1}{k_3^3\varepsilon_2^3}\right\} + \frac{8k_5}{R}\left\{\frac{\varepsilon_2^2}{\varepsilon_1} + \frac{2\varepsilon_1}{k_3\varepsilon_2}\right\}$$

for all $\underline{R} \in L_3$.

Again by Green's Theorem,

$$I_3 = \iiint_{S_3} \nabla^2 \left\{-\frac{1}{R'^2}\, e^{+i\underline{R}'\cdot\underline{v}}\right\} \frac{d\underline{v}}{P(v_2,v_3)-iv_1}$$

$$= -\frac{1}{R'^2}\iiint_{S_3} e^{i\underline{R}'\cdot\underline{v}}\; \nabla^2\left\{\frac{1}{P(v_2,v_3)-iv_1}\right\} d\underline{v}$$

$$+\frac{1}{R'^2}\iint_{\Upsilon(\delta)\cup\Upsilon(\varepsilon)} e^{i\underline{R}'\cdot\underline{v}}\left\{\underline{n}\,\underline{\nabla}\left\{\frac{1}{P(v_2,v_3)-iv_1}\right\} - \frac{\underline{R}'\underline{n}}{P(v_2,v_3)-iv_1}\right\} d\underline{\sigma}$$

Since $|P(v_2,v_3)-iv_1|^3\,|\nabla^2\{\frac{1}{P(v_2,v_3)-iv_1}\}|$ and $|P(v_2,v_3)-iv_1|\,|\underline{\nabla}\{\frac{1}{P_2(v_2,v_3)-iv_1}\}|$ are bounded in $S(\pi,\pi,\pi)$ and $S_\delta \subseteq S(\pi,\pi,\pi)$ for all small enough δ, our previous estimates apply and show that

$$(2.10)\quad |I_3 - \frac{1}{R'^2} I_\delta| \leq k_9 \{\text{right hand side of } (2.9)\} \text{ for all}$$

$\delta \leq \delta_0$, where

$$I_\delta = \iint_{\Upsilon(\delta)} e^{i\underline{R}'\underline{v}}\left\{\underline{n}\cdot\underline{\nabla}\left\{\frac{1}{P(v_2,v_3)-iv_1}\right\} - \frac{\underline{R}'\underline{n}}{P(v_2,v_3)-iv_1}\right\} d\underline{\sigma}\;.$$

Plainly $|I_\delta| \le \iint_{\Upsilon(\delta)} \{\frac{k_{10}}{|P(v_2,v_3)-iv_1|^2} + \frac{R^\ell}{|P(v_2,v_3)-iv_1|}\} d\sigma$, and if we split $\Upsilon(\delta)$ up just as we split $\Upsilon(\varepsilon)$ up and apply the same estimates, we find that

(2.11) $|I_\delta| \le k_{10}\, f_1(\delta) + R^\ell f_2(\delta)$

where $f_1(\delta) = 8\{\delta^{-2\beta} + \frac{2\delta^{\beta-2}}{k_2^2}\}$, $f_2(\delta) = 8\{\delta^{1-\beta} + \frac{2\delta^{\beta}}{k_2}\}$,

so that for $0 < \beta < 1$ $\lim_{\delta \downarrow 0} f_2(\delta) = 0$.

Now let $\varepsilon_1 = \frac{c^3}{R}$, $\varepsilon_2 = \frac{c}{R^{\frac{1}{2}}}$ $\{$where $c \geq \frac{1}{\min\{1, k_2^{-1}\}}$ so that (2.6) is satisfied$\}$ and let $R \to +\infty$ with $\underline{R} \in L_3$ in (2.8), (2.9), (2.10) and (2.11) to get

$$\limsup_{R \to +\infty, \underline{R} \in L_3} \{R|k_1 I_1 + I_2 - k_1 I_3|\}$$
$$\le (1 + k_1 k_9)(\frac{k_8}{c^2} + \frac{8k_6}{c^4} + \frac{8k_5}{c}) + k_1 f_2(\delta).$$

Since we can choose c to be arbitrarily large, this reduces to

$$\limsup_{R \to +\infty, \underline{R} \in L_3} \{R|k_1 I_1 + I_2 - k_1 I_3|\} \le f_2(\delta),$$

and since $\lim_{\delta \downarrow 0} f_2(\delta) = 0$, this is (2.7), which proves the theorem.

Obviously Theorem 2.1 will not apply to the non-lattice case, so we need:

<u>Theorem 2.12</u> Suppose that $\underline{u}$ and $\underline{v}$ are connected as in Theorem 2.1 and that $S(\delta_1, \delta_2, \delta_3)$ and $T(\delta_1,\delta_2,\delta_3)$ denote the same regions as in Theorem 2.1. Let $e^{iR\underline{u}.\underline{j}} f(\underline{u}) = e^{iR\underline{v}.\underline{j}'} \dfrac{k_1(1 + g(\underline{v}))}{P(v_2,v_3)-iv_1}$, where P is a positive definite quadratic form, $|\underline{j}| = 1$ and g satisfies;

(2.13) g, $\dfrac{\partial g}{\partial v_s}$ and $\dfrac{\partial^2 g}{\partial v_s \partial v_t}$ exist and are continuous and $|g|$, $|P(v_2,v_3)-iv_1|\left|\dfrac{\partial g}{\partial v_s}\right|$ and $|P(v_2,v_3)-iv_1|^2\left|\dfrac{\partial^2 g}{\partial v_s \partial v_t}\right|$ are bounded in $S(d, d, d)$ for some $d > 0$ for $s = 1, 2, 3$, $t = 1, 2, 3$;

(2.14) $\lim\limits_{|\underline{v}|\to 0} |g(\underline{v})| = 0.$

Then, if $f(\underline{u})$, $\dfrac{\partial f(\underline{u})}{\partial u_s}$ and $\dfrac{\partial^2 f(\underline{u})}{\partial u_s \partial u_t}$ are continuous and absolutely integrable in $\overline{S}(d, d, d) = \{\underline{u} \notin S(d, d, d)\}$

$$(2.15)\quad \lim_{\delta \downarrow 0} \ \limsup_{R\to+\infty} \left\{ R \left| \iiint_{-\infty}^{\infty} e^{iR\underline{u}.\underline{j}} f(\underline{u})d\underline{u} - k_1 \iiint_{T(\delta^{1+\beta},\delta,\delta)} \frac{e^{iR\underline{v}.\underline{j}'}}{P(v_2,v_3)-iv_1}\, d\underline{v} \right| \right\} = 0$$

for every $0 < \beta < 1$.

<u>Proof</u> If we examine the proof of Theorem 2.1 we see that the periodicity of $f(\underline{u}, \underline{R})$ and its derivatives and the fact that $\underline{R} \in L_3$ are used only to show that the integral over the boundary $\sigma(\pi)$ of $S(\pi, \pi, \pi)$ vanishes. Since the $f(\underline{u})$ and $g(\underline{v})$ of Theorem 2.12 satisfy, in $S(d, d, d)$ instead

of $S(\pi, \pi, \pi)$, the other assumptions of Theorem 2.1, the argument used to prove Theorem 3.1 applies again and yields

$$(2.16)\quad \lim_{\delta\downarrow 0}\ \limsup_{R\to+\infty}\left(R\left|\iiint_{S(d,d,d)} e^{iR\underline{u}.\underline{j}} f(\underline{u})d\underline{u} -\right.\right.$$

$$\left.\left. k_1 \iiint_{T(\delta^{1+\beta},\delta,\delta)} \frac{e^{iR\underline{v}.\underline{j}'}}{P(v_2,v_3)-iv_1}\, d\underline{v} - \frac{I(d)}{R^2}\right|\right) = 0,$$

where $I(d) = \iint_{\sigma(d)} e^{iR(\underline{u}.\underline{j})} \{\underline{n}.\underline{\nabla f}(\underline{u}) - iR\underline{r}.\underline{j}\ f(\underline{u})\}d\underline{u}$.

If $J(Y) = \iiint_{S(Y,Y,Y)\setminus S(d,d,d)} e^{iR(\underline{u}.\underline{j})} f(\underline{u})d\underline{u}$, (2.15) would follow from (2.16) and

$$(2.17)\quad \limsup_{R\to+\infty}\{R|\lim_{Y\to+\infty} J(Y) + \frac{I(d)}{R^2}|\} = 0.$$

However Green's Theorem (3.2.19) shows that for eacy $y > d$ and each R,

$$J(Y) = -\frac{1}{R^2}\iiint_{S(Y,Y,Y)\setminus S(d,d,d)} e^{iR\underline{u}.\underline{j}}\ \nabla^2 f(\underline{u})d\underline{u} - \frac{I(d)}{R^2}$$

$$+ \frac{1}{R^2}\iint_{\sigma(Y)} e^{iR\underline{u}.\underline{j}}\{\underline{n}.\ \underline{\nabla f}(\underline{u}) - iR\underline{n}.\underline{j}f(\underline{u})\}d\underline{u}.$$

Now, by assumption, $|f(\underline{u})|$ and $|\underline{\nabla f}(\underline{u})|$ are integrable in $\overline{S}(d,d,d)$, and this implies that

$$\lim_{Y\to+\infty}\iint_{\sigma(Y)} e^{iR(\underline{u}.\underline{j})}\ \{\underline{n}.\underline{\nabla f}(\underline{u}) - iR\underline{n}.\underline{j}f(\underline{u})\}d\underline{\sigma} = 0.$$

Since $\nabla^2 f(\underline{u})$ is also integrable in $\bar{S}(d, d, d)$,

$\lim_{Y\to+\infty} J(Y) + \frac{I(d)}{R^2}$ exists and is $O(\frac{1}{R^2})$ as $R \to +\infty$ by the

Rieman-Lebesgue Lemma. This is more than enough to establish (2.17), and hence the Theorem.

Our analogue of Theorem 3.3.11 is:

<u>Theorem 2.18</u> If $\underline{e}$ is a unit vector and $0 < \beta < 1$

$$(2.19)\quad \lim_{\delta\downarrow 0}\left\{\limsup_{R\to+\infty}\left|R\iiint_{T(\delta^{1+\beta},\delta,\delta)}\frac{e^{-iR\underline{e}.\underline{v}}}{P(v_2,v_3)-iv_1}\,d\underline{v} - w(\underline{e})\right|\right\}=0$$

for any positive definite quadratic form P, where $w(\underline{e}) = 0$ when $\underline{e} \neq (1, 0, 0)$, $= \frac{2\pi^2}{D}$ when $\underline{e} = (1, 0, 0)$, and $\det(P) = D^2$.

<u>Proof</u> For the moment write $\delta^{1+\beta} = \delta'$, and consider first the case $e_1 = 0$. Since $\int_{-\delta'}^{\delta'}\frac{dv}{A-iv} = 2\tan^{-1}\frac{\delta'}{A}$, the integral in (2.19) reduces to

$$2\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} e^{-iR(e_2v_2+e_3v_3)}\tan^{-1}\delta'/_{P(v_2,v_3)}\,dv_2\,dv_3.$$

Calling this integral I_1, the 2-dimensional version of Green's Theorem (3.2.18) shows that $RI_1 = -iI_2+ie_3I_3+ie_2I_4$ where

$$I_2 = \iint_{-\delta}^{\delta} e^{-iR(e_2v_2+e_3v_3)}\left\{e_2\frac{\partial}{\partial v_2}\tan^{-1}\frac{\delta'}{P(v_2,v_3)} + e_3\frac{\partial}{\partial v_3}\tan^{-1}\frac{\delta'}{P(v_2,v_3)}\right\}dv_2dv_3,$$

$$I_3 = \int_{-\delta}^{\delta} e^{-iRe_2v_2}\left\{e^{-iRe_3\delta}\tan^{-1}\frac{\delta'}{P(v_2,\delta)} - e^{+iRe_3\delta}\tan^{-1}\frac{\delta'}{P(v_2,-\delta)}\right\}dv_2,$$

$$I_4 = \int_{-\delta}^{\delta} e^{-iRe_3v_3}\{e^{-iRe_2\delta}\tan^{-1}\frac{\delta'}{P(\delta,v_3)} - e^{iRe_2\delta}\tan^{-1}\frac{\delta'}{P(-\delta,v_3)}\}dv_3.$$

Since $\frac{\partial}{\partial v_2}\tan^{-1}\frac{\delta'}{P(v_2,v_3)} = \frac{-\delta'\frac{\partial P(v_2,v_3)}{\partial v_2}}{\delta'^2+P^2(v_2,v_3)}$, which is integrable near $\underline{O}$, $\lim_{R\to+\infty} I_2 = 0$ by the Riemann-Lebesgue Lemma. Plainly the bracketed term in I_3 is integrable in $(-\delta, \delta)$ so

$\lim_{R\to+\infty} I_3 = 0$ by the Riemann Lebesgue Lemma unless $e_2 = 0$.

In that case

$$|I_3| \leq 2|e_3|\int_{-\delta}^{\delta}\{\tan^{-1}\frac{\delta'}{P(v_2,\delta)} + \tan^{-1}\frac{\delta'}{P(v_2,-\delta)}\}dv_2 \leq 4\pi\delta,$$

so that $\lim_{\delta\downarrow 0}\limsup_{R\to+\infty}|I_3| = 0$. The same remarks apply to I_4 and therefore $\lim_{\delta\downarrow 0}\limsup_{R\to+\infty}\{R|I_1|\} = 0$; this proves the theorem when $e_1 = 0$.

If $e_1 \neq 0$, consider for $X > \delta'$

$$J_+(X) = \int_{v_1=\delta'}^{X}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\frac{e^{-iR\underline{v}.\underline{e}}}{P(v_2,v_3)-iv_1}\,d\underline{v}.$$

Since

$$\int_{\delta'}^{X}\frac{e^{-iRv_1e_1}}{P(v_2,v_3)-iv_1}dv_1 = \frac{i}{Re_1}\left[\frac{e^{-iRe_1X}}{P(v_2,v_3)-iX} - \frac{e^{-iRe_1\delta'}}{P(v_2,v_3)-i\delta'}\right]$$

$$+\frac{1}{Re_1}\int_{\delta'}^{X}\frac{e^{-iRv_1e_1}}{(P(v_2,v_3)-iv_1)^2}dv_1,$$

and

$$\int_{v_1=\delta'}^{\infty}dv_1\iint_{-\delta}^{\delta}\frac{dv_2\,dv_3}{P^2(v_2,v_3)+v_1^2} \doteq \iint_{-\delta}^{\delta}\frac{\tan^{-1}\frac{P(v_2,v_3)}{\delta'}}{P(v_2,v_3)}dv_2dv_3 < +\infty,$$

we have, for every $\delta > 0$

$$(2.20)\quad R \lim_{X\to+\infty} J_+(X) = \frac{e^{-iRe_1\delta'}}{ie_1} \iint_{-\delta}^{\delta} \frac{e^{-iR(e_2v_2+e_3v_3)}}{P(v_2,v_3)-i\delta'}\,dv_2dv_3$$

$$+\frac{1}{j_1}\int_{v_1=\delta'}^{0}\iint_{-\delta}^{\delta}\frac{e^{-iR\underline{e}.\underline{v}}\,d\underline{v}}{(P(v_2,v_3)-iv_1)^2}\,.$$

This last term vanishes when $R\to+\infty$, by the Riemann-Lebesgue Lemma, and since the other term on the right hand side of (2.20) is dominated by $\frac{1}{|e_1|}\iint_{-\delta}^{\delta}\frac{dv_2\,dv_3}{|P(v_2,v_3)-i\delta'|}\leq\frac{4\delta^2}{|e_1|\delta'}$,

$$(2.21+)\quad \lim_{\delta\downarrow 0}\ \limsup_{R\to+\infty}\{R|\lim_{X\to+\infty} J_+(X)|\} = 0.$$

Similarly, if $J\_(X) = \int_{v_1=-X}^{-\delta'}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\frac{e^{-iR\underline{e}.\underline{v}}}{P(v_2,v_3)-iv_1}\,d\underline{v}$, we have

$$(2.21-)\quad \lim_{\delta\downarrow 0}\ \limsup_{R\to+\infty}\{R|\lim_{X\to+\infty} J\_(X)|\} = 0.$$

The theorem will follow from (2.21+) and (2.21-) if we can show that whenever $e_1 > 0$,

$$(2.22)\quad \lim_{\delta\downarrow 0}\ \limsup_{R\to+\infty}\{|RJ(R) - w(\underline{e})|\} = 0,$$

where $J(R) = \int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\int_{v_1=-\infty}^{\infty}\frac{e^{-iR\underline{e}.\underline{v}}}{P(v_2,v_3)-iv_1}\,d\underline{v}$. Since, when $A > 0$,

$$\int_{-\infty}^{\infty}\frac{e^{-iBv}}{A-iv}\,dv = \begin{cases} 2\pi e^{-AB} & \text{if } B > 0 \\ 0 & \text{if } B < 0\,, \end{cases}$$

$J(R)$ vanishes when $e_1 < 0$, and so equals $w(\underline{e})$ identically, and when $e_1 > 0$

$$J(R) = 2\pi \iint_{-\delta}^{\delta} e^{-iR(e_2 v_2 + e_3 v_3)} e^{-Re_1 P(v_2, v_3)} dv_2 \; dv_3$$

$$= 2\pi\Big\{ \iint_{-\infty}^{\infty} e^{-iR(e_2 v_2 + e_3 v_3)} e^{-Re_1 P(v_2, v_3)} dv_2 \; dv_3 - J_1(R)\Big\};$$

Here $R|J_1(R)| \leq R \iint_{v_2^2 + v_3^2 \geq \delta^2} e^{-Re_1 k_2 (v_2^2 + v_3^2)} dv_2 \; dv_3$

$$= \iint_{w_2^2 + w_3^2 \geq R\delta^2} e^{-e_1 k_2 (w_2^2 + w_3^2)} dw_2 \; dw_3,$$

so plainly $\lim_{R \to +\infty} R|J_1(R)| = 0.$

Since $P(v_2, v_3)$ is positive definite there is a rotation of coordinate axes taking $P(v_2, v_3)$ into $p_2^1 w_2^2 + p_3^1 w_3^2$, where w_2, w_3 are new rectangular coordinates and $p_2^1 > 0$, $p_3^1 > 0$. Then, writing $z_2 = (p_2^1)^{\frac{1}{2}} w_2$, $z_3 = (p_3^1)^{\frac{1}{2}} w_3$ and noticing that $p_2^1 p_3^1 = \mathrm{Det}(P) = D^2$, we have

$$\iint_{-\infty}^{\infty} e^{-iR(e_2 v_2 + e_3 v_3)} e^{-Re_1 P(v_2, v_3)} dv_2 dv_3$$

$$= \frac{1}{D} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-iR(e_2^1 z_2 + e_3^1 z_3)} e^{-Re_1 (z_2^2 + z_3^2)} dz_2 dz_3$$

$$= \frac{4}{D} \int_0^{\infty} \cos(Re_2^1 z_2) e^{-Re_1 z_2^2} dz_2 \int_0^{\infty} \cos(Re_3^1 z_3) e^{-Re_1 z_3^2} dz_3$$

$$= \frac{\pi}{RDe_1} e^{-\frac{(e_2^{1\,2} + e_3^{1\,2})R}{4e_1}};$$

Thus $\lim_{R \to +\infty} RJ(R) = 0$ unless $e_2^{\frac{1}{2}a} + e_3^{\frac{1}{2}\lambda} = 0$, when

$\lim_{R \to +\infty} RJ(R) = \frac{2\pi^2}{De_1}$. However $e_2^{\frac{1}{2}2} + e_3^{\frac{1}{2}2} = 0$ if and only

$e_2 = e_3 = 0$, and in this case $e_1 = 1$ and $\frac{2\pi^2}{De_1} = w(e)$.

(2.22) is therefore established, and with it the theorem.

§.3 Proof of Theorem 1.1 when k = 3.

Since $G(A) = \sum_{\underline{a} \in A} G(\{\underline{a}\})$, it is sufficient to prove the theorem when A has a single member $\underline{a}$. Therefore, by Lemma 3.2.11, we have to evaluate $\lim_{x \to +\infty} \frac{xJ(x)}{(2\pi)^3}$, where

$$J(x) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{e^{-i\underline{u}.(\underline{a}+[x\underline{j}])}}{1 - \phi(\underline{u})} d\underline{u}.$$

In Theorem 2.1 put $\underline{R} = -(\underline{a} + [x\underline{j}])$ (so that $\underline{R} \in L_3$ for all x), $f(\underline{u}, \underline{R}) = (1 - \phi(\underline{u}))^{-1}$, $P(v_2, v_3) = m^{-1}Q_0(v_2, v_3)$, $k_1 = m$ and $\underline{R}' = -x\underline{j}'$, where $j_s' = \underline{\ell}_s . \underline{j}$ for $s = 1, 2, 3$ (and the $\underline{\ell}_s$ are those of Theorem 1.1). Then by Lemma 3.2.1 P is positive definite and by Lemma 3.2.6 the periodicity condition holds for $\frac{1}{1-\phi(\underline{u})}$ and its derivatives.

Also $|\underline{R}'| = x$ and $|x\underline{j} - [x\underline{j}]| = \{\sum_{s=1}^{3} (xj_s - [xj_s])^2\}^{\frac{1}{2}} \leq \sqrt{3}$,

so that $||\underline{R}| - |\underline{R}'|| \leq |\underline{a}| + \sqrt{3}$ for all x.

Now $g(\underline{v}, \underline{R}) = \frac{Q_0(v_2, v_3) - imv_1}{Q_1(\underline{v}) - imv_1 + \rho_1(\underline{v})} e^{i\underline{v}.\underline{b}(x)} - 1,$

where $\rho_1(\underline{v}) = \rho(\underline{u})$ (the function introduced in Lemma 3.2.3) and $\underline{b}(x)$ has components $b_s(x) = \underline{\ell}_s.(x\underline{j}-\underline{a}-[x\underline{j}])$, each of which is bounded for all x. Thus $e^{i\underline{v}.\underline{b}(x)} \rightarrow 1$ uniformly in x as $|\underline{v}| \downarrow 0$, and since, using Lemma 3.2.3,

$$\frac{Q_0(v_2,v_3)-imv_1}{Q_1(\underline{v})-imv_1+\rho(\underline{v})} = \frac{Q_0(v_2,v_3)-imv_1}{Q_1(\underline{v})-imv_1} \cdot \frac{Q_1(\underline{v})-imv_1}{Q_1(v)-imv_1+\rho_1(\underline{v})}$$

$$= \left\{1 - \frac{v_1\{q^1_{11}v_1+2q^1_{12}v_2+q^1_{13}v_3\}}{Q_1(\underline{v}) - imv_1}\right\}\left\{1 - \frac{\rho_1(\underline{v})}{Q_1(\underline{v})-imv_1+\rho_1(\underline{v})}\right\}$$

$$= \{1 + O\ (1)\}\ \left\{1 + \frac{o(|\underline{v}|^2)}{O(|\underline{v}|^2)}\right\} \text{ as } |\underline{v}| \rightarrow 0,$$

$$= 1 + O(1) \qquad \text{as } |\underline{v}| \rightarrow 0,$$

$g(\underline{v}, \underline{R}) \rightarrow 0$ as $|\underline{v}| \rightarrow 0$ uniformly in $\underline{R}$. This is condition (2.4), and clearly the existence of σ^2 ((1.3)) implies that (2.2) is satisfied. Since $\left|\frac{Q_0-imv_1}{Q_1(\underline{v})-imv_1+\rho_1(\underline{v})}\right| =$ $\left|\frac{Q_0(v_2,v_3)-imv_1}{1 - \phi(\underline{u})}\right|$ is bounded in any subset of $S(\pi, \pi, \pi)$ not containing the origin (Lemma 3.2.8), and it tends to one as $|\underline{v}| \rightarrow 0$, it is bounded throughout $S(\pi, \pi, \pi)$. This, together with the fact that $|\underline{b}(x)|$ is bounded for all x shows that (2.3) is satisfied. The conclusion of Theorem 2.1 is

$$(3.1) \qquad \lim_{\delta \downarrow 0}\ \limsup_{x \rightarrow +\infty} \{x|J(x) - I(x,\delta)|\} = 0,$$

where $I(x, \delta) = \int_{v_1=-\delta^{1+\beta}}^{\delta^{1+\beta}} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{e^{-ix(\underline{j}'.\underline{v})}\, d\underline{v}}{Q_0(v_2, v_3)-imv_1}$ and $0 < \beta < 1$.

Now Theorem 2.18 tells us that

$$(3.2) \quad \lim_{\delta \downarrow 0} \limsup_{x \to +\infty} |xI(x, \delta) - \frac{1}{m} w(\underline{j}')| = 0,$$

where $w(\underline{j}') = 0$ if $\underline{j}' \neq (1, 0, 0)$,

$$= \frac{2\pi^2}{D} \text{ if } \underline{j}' = (1, 0, 0).$$

Noting that $D^2 = \text{Det}(m^{-1}Q_0(v_2, v_3)) = \frac{1}{m^2} \Delta^2$ and that $\underline{j}' = (1, 0, 0)$ if and only if $\underline{j} = \underline{\ell}_1$, (3.1) and (3.2) together imply

$$(3.3) \quad \lim_{\delta \downarrow 0} \limsup_{x \to +\infty} |xJ(x) - (2\pi)^3 \lambda(\underline{j})| = 0,$$

where $\lambda(\underline{j})$ denotes the right hand side of (1.5) when $k = 3$ and $N(A) = 1$. Since both $J(x)$ and $\lambda(\underline{j})$ are independent of δ, (3.3) reduces to

$$\lim_{x \to +\infty} \frac{xJ(x)}{(2\pi)^3} = \lambda(\underline{j}),$$

and this is Theorem 1.1 for $k = 3$ and $N(A) = 1$.

Proof of Theorem 1.6 when $k = 3$

The argument used on pp.74-76 of Chapter III needs only trivial modifications to apply to the case $k = 3$, when it shows firstly that we need only establish (1.8) when A is a bounded interval $I(\underline{a}, \underline{h})$, secondly that (1.8) when $A = I(\underline{a}, \underline{h})$ is a consequence of

$$(3.4)\quad \lim_{x \to +\infty} x\, L\{\underline{a} + x\underline{j},\ \underline{h}\} = h_1^2 h_2^2 h_3^2\, \lambda(\underline{j}),$$

and finally that we can take $\underline{\ell}_1 = (1, 0, 0)$, $\underline{\ell}_2 = (0, 1, 0)$ and $\underline{\ell}_3 = (0, 0, 1)$ without loss of generality. From Lemma 3.2.15 we have the representation

$$(3.5)\quad L(\underline{a} + x\underline{j},\ \underline{h}) = \frac{1}{\pi^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{D(\underline{h},\underline{u})\, e^{-i\underline{u}(x\underline{j}+\underline{a})}}{1 - \phi(\underline{u})}\, d\underline{u} = \frac{J(x)}{\pi^3},$$

and if we write $f(\underline{u}) = \dfrac{D(\underline{h},\underline{u})e^{-i\underline{u}.\underline{a}}}{1 - \phi(\underline{u})} = \dfrac{h_1^2 h_2^2 h_3^2}{8m} \dfrac{(1 + g(\underline{u}))}{m^{-1}Q_0(u_2, u_3) - iu_1}$

in Theorem 2.12 we have,

$$(3.6)\quad \lim_{\delta \downarrow 0} \limsup_{x \to +\infty} \left\{x\left|J(x) - \frac{h_1^2 h_2^2 h_3^2}{8} I(x, \delta)\right|\right\} = 0,$$

where $I(x, \delta) = \displaystyle\int_{u_1=-\delta^{1+\beta}}^{\delta^{1+\beta}} \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \frac{e^{-ix(\underline{j}.\underline{u})}\, d\underline{u}}{Q_0(u_2, u_3) - imu_1}$, provided that the conditions of Theorem 2.12 are satisfied. Since $D(\underline{h}, \underline{u})$ and its derivatives of the first two orders are continuous, bounded and absolutely integrable in E_3 and, by Lemma 3.2.12, $\sup_{|\underline{u}| > d} |1-\phi(\underline{u})| < +\infty$ for every $d > 0$, the calculations of p.96 together with the fact that $\lim_{|\underline{u}| \to 0} D(\underline{h}, \underline{u}) = \dfrac{h_1^2 h_2^2 h_3^2}{8}$ show that these conditions are satisfied for any $d > 0$.

Just as before Theorem 2.18 yields

$$\lim_{\delta \downarrow 0} \limsup_{x \to +\infty} |xI(x,\delta) - (2\pi)^3 \lambda(\underline{j})| = 0,$$

and this in (3.6) is

$$\lim_{\delta \downarrow 0} \limsup_{x \to +\infty} |xJ(x) - \pi^3 h_1^2 h_2^2 h_3^2 \lambda(\underline{j})| = 0,$$

which establishes (3.4) and hence Theorem 1.6 when $k = 3$.

§.4 When k is even and greater than three the proofs of Theorems 1.1 and 1.6 follow the same lines as those given in chapter III for the case $k = 2$: the proofs when k is odd and greater than three are similar to those of §.3 of the present chapter.

Consider, for example, the lattice case 1.1 when $k = 2\ell$. Assumption (1.4) and Lemma 3.2.6 mean that all derivatives of $\phi(\underline{u})$ of the first ℓ orders exist, are bounded, have period 2π in each variable and are continuous in E_k^π. Since $\{1 - \phi(\underline{u})\}^{-s}$ is integrable in E_k^π for $s = 1, 2, \ldots \ell$, we can apply Green's Theorem (3.2.18) $\ell - 1$ times to the representation in Lemma 3.2.11 of $G(\{\underline{a}\})$ to get, the surface integrals cancelling out in the usual way,

$$(4.1) \quad G(\{\underline{a}\}) = \frac{1}{(2\pi)^k} \left\{\frac{-i}{|\underline{a}|^2}\right\}^{\ell-1} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{-i\underline{a}\cdot\underline{u}} \, \Box^{\ell-1}\{(1-\phi(\underline{u}))^{-1}\} d\underline{u},$$

where $\Box = \underline{a} \cdot \underline{\nabla}$ and $\underline{a} \in L_k$. Now

$f(\underline{u}, \underline{a}) = \{\Box^{\ell-1}\{\frac{1}{1-\phi(\underline{u})}\} - \frac{(\ell-1)!(\Box\phi)^{\ell-1}}{\{1-\phi(\underline{u})\}^{\ell}}\}$ is a polynomial of degree $\ell-1$ in $\{1 - \phi(\underline{u})\}^{-1}$ with coefficients which are functions of $\underline{a}$ and the derivatives of $\phi(\underline{u})$ of the first $\ell-1$ orders. Since the ℓ^{th} derivatives of $\phi(\underline{u})$ exist and are bounded, $\Box f(\underline{u},\underline{a})$ is integrable in E_k^{π} and therefore, by Green's Theorem and the Riemann-Lebesgue Lemma

$$(4.2) \quad G(\{a\}) = \frac{(\ell-1)!}{(2\pi)^k}\{\frac{-i}{|\underline{a}|^2}\}^{\ell-1} \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} e^{-i\underline{a}.\underline{u}} \frac{(\Box\phi(\underline{u}))^{\ell-1}}{\{1-\phi(\underline{u})\}^{\ell}} d\underline{u}$$

$$+ O\{|\underline{a}|^{-\ell}\} \text{ as } |a| \to +\infty .$$

If we write $\frac{\{\Box\phi(\underline{u})\}^{\ell-1}}{\{1-\phi(\underline{u})\}^{\ell}} = \frac{\{-i\underline{a}.\underline{m}\}^{\ell-1}}{\{Q(\underline{u})-i\underline{m}.\underline{u}\}^{\ell}}\{1 + g(\underline{u},\underline{a})\}$

the usual estimates for $\phi(\underline{u})$ (Lemma 3.2.3) and its derivatives show that $g(\underline{u}, \underline{a}) \to 0$ uniformly in $\underline{a}$ as $|\underline{u}| \to 0$. As the first derivatives of $g(\underline{u}, \underline{a})$ are well-behaved, we have a situation analogous to that treated in Theorem 3.3.2, and the conclusion of that Theorem is, essentially, that when $\ell = 1$

$$(4.3) \quad \lim_{|\underline{a}|\to\infty} \{|\underline{a}|^{\frac{1}{2}} \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} e^{-i\underline{u}.\underline{a}} \frac{g(\underline{u},\underline{a})}{(Q(\underline{u})-i\underline{m}.\underline{u})^{\ell}} d\underline{u}\} = 0.$$

However a similar calculation shows that (4.3) holds when $\ell > 1$ and that the error incurred in replacing the region of integration by $T(\delta, \delta, \ldots, \delta)$ is also $O(|\underline{a}|^{-\frac{1}{2}})$ as $|\underline{a}| \to +\infty$ for any $\delta > 0$. The same argument also shows that for each $\underline{b} \in L_k$ and each unit bector $\underline{j}$,

$$(4.4)\quad \lim_{x \to +\infty} \left\{ x^{\frac{1}{2}} \int_{T(\delta,..\delta)} \!\!\cdots\! \int \left\{ e^{-i\underline{u}.(\underline{b}+[x\underline{j}])} - e^{-ix\underline{u}.\underline{j}} \right\} \frac{d\underline{u}}{\{Q(\underline{u}) - i\underline{m}.\underline{u}\}^{\ell}} \right\} = 0,$$

so that, by (4.2), (4.3) and (4.4)

$$(4.5)\quad G(\{\underline{b} + [x\underline{j}]\}) \sim \frac{(\ell-1)!}{(2\pi)^k} \left\{\frac{\underline{m}.\underline{j}}{x}\right\}^{\ell-1} \int_{T(\delta,..\delta)} \!\!\cdots\! \int \frac{e^{-ix\underline{u}.\underline{j}}}{\{Q(\underline{u}) - i\underline{m}.\underline{u}\}^{\ell}} d\underline{u}$$

as $x \to +\infty$.

All that is required now is the following 2 ℓ-dimensional version of Theorem 3.3.11:

$$(4.6)\quad \lim_{x \to +\infty} x^{\frac{1}{2}} \int_{T(\delta,..\delta)} \!\!\cdots\! \int \frac{e^{-ix\underline{u}.\underline{j}}}{\{Q(\underline{u}) - i m\underline{u}.\underline{\ell}_1\}^{\ell}} d\underline{u} = \frac{2\pi^{\ell}}{(\ell-1)!\Delta} \sqrt{\frac{\pi}{m}} \text{ if } j = \underline{\ell}_1$$

$$= 0 \text{ if } \underline{j} \neq \underline{\ell}_1$$

and this is proved in the same way as Theorem 3.3.11.

CHAPTER V

§.1 It follows from Lemmas 2.11 and 2.15 of chapter III that, when $\underline{X}$ has zero mean and finite second moments and $k \geqslant 3$, $G(A)$ exists for every bounded subset A of L_k in the lattice case and for every Jordan measurable subset A of E_k in the non-lattice case. We now investigate the behaviour of $G(A + \underline{x})$ as $|\underline{x}| \to +\infty$ in this case, and once again can consider only the situations (A) and (B) of chapter III. Our results are:

Theorem 1.1 Let $(\underline{S}_n)$ be an aperiodic R.W on $L_k (k \geqslant 3)$ generated by a strictly k-dimensional R.V $\underline{X}$. Assume

(1.2) $E(\underline{X}) = \underline{0}$;

(1.3) if $k \leq 4$ $E(|\underline{X}|^{2+\gamma}) < +\infty$ for some $\gamma > 0$;

(1.4) if $k > 4$ $E(|\underline{X}|^{k-2}) < +\infty$.

Then $Q(\underline{u}) = \frac{1}{2} E\{(\underline{X}.\underline{u})^2\}$ is a positive definite quadratic form so for some positive real numbers Q_s and orthonormal vectors $\underline{j}_s$,

(1.5) $$Q(\underline{u}) = \sum_{s=1}^{k} Q_s (\underline{u}.\underline{j}_s)^2.$$

If, for any $\underline{x}$, $\underline{x}^*$ denotes the vector having sth component $\frac{\underline{x}.\underline{j}_s}{Q_s^{\frac{1}{2}}}$, and A is any bounded subset of L_k having N(A) members, (1.6) $$\lim_{|\underline{x}| \to +\infty, \underline{x} \in L_k} \{ |\underline{x}^*|^{k-2} G(A + \underline{x}) \} = \frac{\Gamma(\frac{k-2}{2})}{(\sqrt{\pi})^{k-2}} \cdot \frac{N(A)}{4\pi\Delta}$$

where $\Delta^2 = \text{Det}(Q) = \prod_{s=1}^{k} Q_s$.

<u>Theorem 1.7</u> Let $(\underline{S}_n)$ be a R.W on $E_k (k \geq 3)$ generated by a strictly k-dimensional R.V $\underline{X}$. Assume that (1.2), (1.3), (1.4) and

(1.8): $\limsup_{|\underline{u}| \to +\infty} |\phi(\underline{u})| < 1$, where $\phi(\underline{u}) = E(e^{i\underline{u}\cdot\underline{X}})$;

hold. Then if A is any Jordan measurable subset of E_k having measure $|A|$,

(1.9) $$\lim_{|\underline{x}| \to +\infty} \{ |\underline{x}^*|^{k-2} G(A + \underline{x}) \} = \frac{\Gamma(\frac{k-2}{2})}{(\sqrt{\pi})^{k-2}} \frac{|A|}{4\pi\Delta} .$$

These two theorems are proved in §.2 and §.3 when k = 3, and in §.4 we indicate how the method of proof extends to the general case.

An interesting consequence of Theorems 1.1 and 1.7 is that, according to Lamperti [21], his generalization of Wiener's test holds for every R.W satisfying the conditions of one of these theorems. A note at the end of Lamperti's paper implies that Spitzer has proved a theorem equivalent to 1.1 when k = 3, but the other results seem to be new.

§.2 Assumption (1.3) appears to be essential: it allows us to make use of:

<u>Lemma 2.1</u> If $\rho(\underline{u}) = E\{e^{i\underline{u}.\underline{X}} - (1 + i\underline{u}.\underline{X} - \frac{1}{2}(\underline{u}.\underline{X})^2)\}$ $= \phi(\underline{u}) - \{1 + i\underline{u}.\underline{m} - Q(\underline{u})\}$ and $|\underline{X}|^{2+\gamma} < +\infty$ for some $1 \geq \gamma > 0$, $\rho(\underline{u}) = O(|\underline{u}|^{2+\gamma})$ as $|\underline{u}| \to 0$ and $\frac{\partial \rho(\underline{u})}{\partial u_s} = O(|\underline{u}|^{1+\gamma})$ as $|\underline{u}| \to 0$, for $s = 1, 2, .., k$.

<u>Proof</u> For $|\underline{u}| > 0$ and $0 < \gamma \leq 1$,

$$|\rho(\underline{u})| \leq \int_{|\underline{x}| \leq |\underline{u}|^{-1}} \!\!\cdots\int \left|e^{i\underline{u}.\underline{x}} - (1 + i\underline{u}.\underline{x} - \tfrac{1}{2}(\underline{u}.\underline{x})^2)\right| dF(\underline{x})$$

$$+ \int_{|\underline{x}| > |\underline{u}|^{-1}} \!\!\cdots\int \{2 + |\underline{u}||\underline{x}| + \tfrac{1}{2}|\underline{u}|^2|\underline{x}|^2\} dF(\underline{x})$$

$$\leq \int_{|\underline{x}||\underline{u}| \leq 1} \!\!\cdots\int \frac{|\underline{u}|^3|\underline{x}|^3}{3!} dF(x) + \int_{|\underline{u}||\underline{x}| > 1} \!\!\cdots\int \frac{7}{2}|\underline{u}|^2|\underline{x}|^2 \, dF(\underline{x})$$

$$\leq \frac{|\underline{u}|^{2+\gamma}}{3!} \int_{-\infty}^{\infty} \!\!\cdots \int_{-\infty}^{\infty} |\underline{x}|^{2+\gamma} dF(\underline{x}) + \frac{7}{2}|\underline{u}|^{2+\gamma} \int_{-\infty}^{\infty} \!\!\cdots\int |\underline{x}|^{2+\gamma} dF(\underline{x})$$

$$\leq k_1 |\underline{u}|^{2+\gamma} .$$

The existence of the second moments allows us to differentiate under the integral sign to get

$$\frac{\partial \rho(\underline{u})}{\partial u_s} = i \int_{-\infty}^{\infty} \!\!\cdots \int_{-\infty}^{\infty} x_s\{e^{i\underline{u}.\underline{x}} - 1 - i\underline{u}.\underline{x}\} dF(\underline{x}),$$

and a similar argument now establishes the second part of the lemma.

Proof of Theorem 1.1 when k = 3

Obviously we need only establish (1.6) when A has a single member, and we can take this member to be $\underline{0}$, without loss of generality. For if

$$(2.2) \quad \lim_{|\underline{x}| \to +\infty, \underline{x} \in L_3} \{ |\underline{x}^*| \; G_0(\underline{x}) \} = \frac{1}{4\pi\Delta} ,$$

where $G_0(\underline{x}) = G(\{\underline{0}\} + \underline{x})$, then $\lim_{|\underline{x}| \to +\infty, \underline{x} \in L_3} \{ |\overset{*}{\underline{x}}| G(\{\underline{a}\} + \underline{x}) \}$

$$= \lim_{|\underline{x}| \to +\infty, \; \underline{x} \in L_3} \{ \frac{|\underline{x}^*|}{|\underline{a}^* + \underline{x}^*|} \cdot |\underline{x}^* + \underline{a}^*| G_0(\underline{x} + \underline{a}) \} = \frac{1}{4\pi\Delta} .$$

Lemma 3.2.11 now tells us that $G_0(\underline{x}) = \lim_{\varepsilon \downarrow 0} \frac{I_\varepsilon(\underline{x})}{(2\pi)^3}$ for $\underline{x} \in L_3$, where

$$I_\varepsilon(\underline{x}) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{-i\underline{x} \cdot \underline{u}}}{\varepsilon + 1 - \phi(\underline{u})} \, d\underline{u}.$$

Since $\phi(\underline{u})$ and its first derivatives are continuous and, if $\varepsilon > 0$, $|\{\varepsilon + 1 - \phi(\underline{u})\}^{-1}|$ is bounded throughout $S(\pi, \pi, \pi)$, we can apply Green's Theorem (3.2.19) to get for every $\varepsilon > 0$,

$$I_\varepsilon(\underline{x}) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \nabla^2 \{ \frac{-e^{-i\underline{x} \cdot \underline{u}}}{|\underline{x}|^2} \} \frac{d\underline{u}}{\varepsilon + 1 - \phi(\underline{u})}$$

$$= -i \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{-i\underline{x} \cdot \underline{u}}}{|\underline{x}|^2} \frac{\underline{x} \cdot \nabla \phi(\underline{u})}{\{\varepsilon + 1 - \phi(\underline{u})\}^2} \, d\underline{u} + \iint_{\sigma(\pi)} \frac{i \underline{n} \cdot \underline{x}}{|\underline{x}|^2} \frac{e^{-i\underline{x} \cdot \underline{u}}}{\varepsilon + 1 - \phi(\underline{u})} d\underline{u}.$$

When $\underline{x} \in L_3$, $e^{-i\underline{u}\,\underline{x}}$ has period 2π in each variable, and as the same is true of $\phi(\underline{u})$ (Lemma 3.2 6), the second integral vanishes and for every $\varepsilon > 0$ and $\underline{x} \in L_3$

$$(2.3) \quad I_\varepsilon(\underline{x}) = \frac{-i}{|\underline{x}|^2} \iiint_{S(\pi)} \frac{\underline{x}\,.\,\nabla \phi}{\{\varepsilon+1-\phi(\underline{u})\}^2}\, e^{-i\underline{u}.\underline{x}}\, d\underline{u}.$$

Now let $V(\delta) = \{\underline{u}: Q(\underline{u}) \leqslant \delta^2\}$: then by Lemma 3.2.8 $\inf\limits_{\underline{u}\in S(\pi,\pi,\pi)\setminus V(\delta)} |1-\phi(\underline{u})| > 0$ for each $\delta > 0$. Since $\Re\{1-\phi(\underline{u})\} \geqslant 0$, $\left|\frac{1}{\varepsilon+1-\phi(\underline{u})}\right|^2 \leqslant \frac{1}{|1-\phi(\underline{u})|^2}$,

and this is integrable in $S(\pi, \pi, \pi) \setminus V(\delta)$. Moreover $|\underline{\nabla}\, \underline{\phi}(\underline{u})| \leqslant E(|\underline{X}|) < +\infty$, so that by dominated convergence, for every $\delta > 0$,

$$\lim_{\varepsilon \to 0} \iiint_{S(\pi,\pi,\pi)\setminus V(\delta)} \frac{e^{-i\underline{u}.\underline{x}}\,\underline{x}\,.\,\underline{\nabla}\,\underline{\phi}(\underline{u})}{\{\varepsilon+1-\phi(\underline{u})\}^2}\, d\underline{u} =$$

$$\iiint_{S(\pi,\pi,\pi)\setminus V(\delta)} e^{-i\underline{u}.\underline{x}}\, \frac{\underline{x}\cdot\underline{\nabla}\,\underline{\phi}(\underline{u})}{\{1-\phi(\underline{u})\}^2}\, d\underline{u}$$

and this last is $o\{|\underline{x}|\}$ as $|\underline{x}| \to +\infty$, by the Riemann-Lebesgue Lemma. Therefore

$$(2.4) \quad G_0(\underline{x}) = \frac{-i}{(2\pi)^3|\underline{x}|^2} \lim_{\varepsilon \downarrow 0} \iiint_{V(\delta)} \frac{\underline{x}\,.\,\nabla\phi(\underline{u})e^{-i\underline{u}.\underline{x}}}{\{\varepsilon+1-\phi(\underline{u})\}^2}\, d\underline{u}$$

$$+\ o\left(\frac{1}{|\underline{x}|}\right) \text{ as } |\underline{x}| \to +\infty .$$

The crux of the proof is that we can now replace $\phi(\underline{u})$ by its asymptotic estimate, $1 - Q(\underline{u})$. To see this, let

$$f_\varepsilon(\underline{u}) = \frac{\underline{x}\cdot\nabla Q(\underline{u})}{\{\varepsilon+Q(\underline{u})\}^2} + \frac{\underline{x}\cdot\nabla\phi(\underline{u})}{\{\varepsilon+1-\phi(\underline{u})\}^2}$$

$$= \frac{\underline{x}\cdot\nabla\rho(\underline{u})}{\{\varepsilon+1-\phi(\underline{u})\}^2} + \underline{x}\cdot\nabla Q(\underline{u})\left\{\frac{1}{(\varepsilon+Q(\underline{u}))^2} - \frac{1}{(\varepsilon+1-\phi(\underline{u}))^2}\right\}$$

$$= \frac{\underline{x}\cdot\nabla\rho(\underline{u})}{\{\varepsilon+1-\phi(\underline{u})\}^2} - \frac{\underline{x}\cdot\nabla Q(\underline{u})\cdot\rho(\underline{u})}{\{\varepsilon+Q(\underline{u})\}\{\varepsilon+1-\phi(\underline{u})\}}\left\{\frac{1}{\varepsilon+Q(\underline{u})} + \frac{1}{\varepsilon+1-\phi(\underline{u})}\right\}.$$

Then for all $\varepsilon \geqslant 0$,

$$|f_\varepsilon(\underline{u})| \leqslant \frac{|\underline{x}|\,|\nabla\rho(\underline{u})|}{|1-\phi(\underline{u})|^2} + \frac{|\underline{x}|\cdot|\nabla Q(\underline{u})|\,|\rho(\underline{u})|}{Q(\underline{u})\,|1-\phi(\underline{u})|}\left\{\frac{1}{Q(\underline{u})} + \frac{1}{|1-\phi(\underline{u})|}\right\},$$

and Lemma 2.1 shows that this last function is $\mathcal{O}(|\underline{u}|^{\gamma-3})$ as $|\underline{u}|\to 0$, and is therefore integrable over $V(\delta)$ for all small enough δ. Hence, by dominated convergence,

$$\lim_{\varepsilon\downarrow 0}\iiint_{V(\delta)} e^{-i\underline{u}\cdot\underline{x}} f_\varepsilon(\underline{u})d\underline{u} = \iiint_{V(\delta)} e^{-i\underline{u}\cdot\underline{x}} f_0(\underline{u})d\underline{u},$$

and Riemann-Lebesgue Lemma now shows that this last is $O(|\underline{x}|)$ as $|\underline{x}| \to +\infty$. Hence, when $\underline{x} \in L_3$,

(2.5) $$G_0(\underline{x}) = \frac{i}{(2\pi)^3|\underline{x}|^2}\lim_{\varepsilon\downarrow 0} J(\varepsilon, \underline{x}) + O\left(\frac{1}{|\underline{x}|}\right) \text{ as } |x|\to+\infty,$$

where $$J(\varepsilon, \underline{x}) = \iiint_{V(\delta)} \frac{\underline{x}\cdot\nabla Q(\underline{u})\,e^{-i\underline{u}\cdot\underline{x}}}{\{\varepsilon+Q(\underline{u})\}^2}\,d\underline{u}.$$

To evaluate $\lim_{\varepsilon\downarrow 0} J(\varepsilon, \underline{x})$, we introduce new variables of integration by writing $v_s = \underline{u}\cdot\underline{j}_s = \sum_{r=1}^{3} u_r j_{sr}$, or, in view of the orthogonality of the $\underline{j}_s$, $u_s = \sum_{r=1}^{3} v_r j_{rs}$.

Then

$$J(\varepsilon,\underline{x}) = \iiint_{\{\underline{v}:\ \sum_{s=1}^{3} Q_s v_s^2 \leq \delta^2\}} e^{-i\underline{Y}.\underline{v}} \cdot \frac{2\sum_{s=1}^{3} Y_s Q_s v_s}{(\varepsilon+\sum_{s=1}^{3} Q_s v_s^2)^2}\, d\underline{v},$$

where $\underline{y} = \underline{x}.\underline{j}_s$, so that $y_s = Q_s^{\frac{1}{2}} x_s^*$. Hence, writing $w_s = Q_s^{\frac{1}{2}} v_s$

$$J(\varepsilon,\underline{x}) = \frac{2}{\Delta} \iiint_{\{|\underline{w}| \leq \delta\}} \frac{\sum_{s=1}^{3} Q_s x_s^* w_s}{\{\varepsilon+|\underline{w}|^2\}^2} e^{-i(\underline{x}^*.\underline{w})}\, d\underline{w}.$$

Now there is another orthogonal change of variables which takes $\sum_{s=1}^{3} Q_s x_s^* w_s$ into $X z_3$ (where $X^2 = \sum_{s=1}^{3} Q_s^2 x_s^{*2}$) and $\underline{x}^*.\underline{w}$ into $\underline{y}^*.\underline{z}$. Then $|\underline{y}^*| = |\underline{x}^*|$,

$$(2.6)\quad J(\varepsilon,\underline{x}) = \frac{2X}{\Delta} \iiint_{\{|\underline{z}| \leq \delta\}} \frac{z_3 e^{-i\underline{y}^*.\underline{z}}}{\{\varepsilon+|\underline{z}|^2\}^2}\, dz,$$

and it is easy to check that

$$(2.7)\quad y_3^* = \frac{1}{X} \sum_{s=1}^{3} Q_s x_s^{*2} = \frac{|\underline{x}|^2}{X}.$$

Introducing spherical polar co-ordinates ρ, θ, ϕ, (2.6) becomes

$$J(\varepsilon,\ \underline{x}) = \frac{2X}{\Delta} \int_0^{\delta} \int_{-\frac{\pi}{2}}^{\pi/2} \int_0^{2\pi} e^{-i\rho(y_1^*\cos\theta\cos\phi + y_2^*\cos\theta\sin\phi + y_3^*\sin\theta)} \cdot \frac{\rho^3 \sin\theta\cos\theta\, d\rho\, d\theta\, d\phi}{(\varepsilon + \rho^2)^2}$$

$$= \frac{2\chi}{\Delta} \int_0^{\delta} \frac{\rho^3 d\rho}{(\varepsilon+\rho^2)^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\rho y_3^* \sin\theta} \sin\theta\cos\theta d\theta \int_0^{2\pi} e^{-i\rho y_4^* \cos(\phi-\alpha)\cos\theta} d\phi$$

where $y_4^{*2} = y_1^{*2} + y_2^{*2}$, $\tan\alpha = \frac{y_2^*}{y_1^*}$. If $J_\lambda(x)$ denotes the Bessel function of the first kind of order λ,

$$\int_0^{2\pi} e^{-i\rho y_4^* \cos(\phi-\alpha)\cos\theta} d\phi = \int_0^{2\pi} e^{-i\rho y_4^* \cos\theta\cos\phi} d\phi = 2\pi J_0(\rho y_4^* \cos\theta),$$

and by Sonine's second finite integral [29, p.376]

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\rho y_3^* \sin\theta} J_0(\rho y_4^* \cos\theta) \sin\theta\cos\theta d\theta$$

$$= -2i \int_0^{\frac{\pi}{2}} \sin(\rho y_3^* \sin\theta) J_0(\rho y_4^* \cos\theta) \sin\theta\cos\theta d\theta$$

$$= -2i \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{y_3^*}{|\underline{y}|^{3/2}} \frac{J_{3/2}(\rho|\underline{y}^*|)}{\rho^{\frac{1}{2}}}$$

Making use of (2.7), it follows from (2.6) that

$$(2.8) \quad J(\varepsilon, \underline{x}) = \frac{4|\underline{x}|^2 2^{\frac{1}{2}} \pi^{3/2}}{i\Delta |\underline{x}^*|^{3/2}} \int_0^{\delta} \frac{\rho^{\frac{5}{2}}}{(\varepsilon+\rho^2)^2} J_{3/2}(\rho|\underline{x}^*|) d\rho.$$

Now $J_{3/2}(\rho) \sim \rho^{3/2}$ as $\rho \downarrow 0$ and $|J_{3/2}(\rho)|$ is bounded for all real ρ, so $|\rho^{-3/2} J_{3/2}(\rho|\underline{x}^*|)|$ is integrable in $(0, \delta)$ for each $\underline{x}$ and δ. Therefore, by dominated convergence,

$$\lim_{\varepsilon \downarrow 0} J(\varepsilon, \underline{x}) = \frac{4|\underline{x}|^2 2^{\frac{1}{2}} \pi^{3/2}}{i\Delta |\underline{x}^*|^{3/2}} \int_0^{\delta} J_{3/2}(\rho|\underline{x}^*|) \rho^{-3/2} d\rho$$

$$(2.9) \qquad = \frac{4|\underline{x}|^2 2^{\frac{1}{2}} \pi^{3/2}}{i\Delta |\underline{x}^*|} \int_0^{\delta|\underline{x}^*|} J_{3/2}(\rho) \rho^{-3/2} d\rho.$$

Our previous remarks show that $J_{3/2}(\rho)\rho^{-3/2}$ is absolutely integrable in (0, ∞), and [13, p.22] $\int_0^\infty J_{3/2}(\rho)d\rho = \frac{1}{2}(\frac{\pi}{2})^{\frac{1}{2}}$,

so that (2.9) in (2.5) gives

$$\lim_{|\underline{x}|\to+\infty, \underline{x}\varepsilon L_3} \{|\underline{x}^*|G_0(\underline{x})\} = \frac{i}{(2\pi)^3|\underline{x}|^2} \cdot \frac{4|\underline{x}|^2 2^{\frac{1}{2}}\pi^{3/2}}{i\Delta} \cdot \frac{1}{2}(\frac{\pi}{2})^{\frac{1}{2}}$$

$$= \frac{1}{4\pi\Delta}.$$

This is (2.2), and it establishes the theorem when k = 3.

§.3 Proof of Theorem 1.7 when k = 3

With some slight modifications the argument of pp.74-76 of chapter III applies again and shows firstly that we need only establish (1.9) when A is a bounded interval $I(\underline{a}, \underline{h})$ and secondly that (1.9) when $A = I(\underline{a},\underline{h})$ follows from

$$(3.1) \quad \lim_{|\underline{x}|\to+\infty} \{|\underline{x}^*|L\{\underline{a} + \underline{x}, \underline{h}\}\} = \frac{h_1^2 h_2^2 h_3^2}{4\pi\Delta}.$$

Since $\lim_{|\underline{x}|\to+\infty} \{\frac{|\underline{x}^* + \underline{a}^*|}{|\underline{x}^*|}\} = 1$ for each $\underline{a}$, there is no loss of generality in taking $\underline{a} = \underline{0}$, and Lemma 3.2.15 shows that (3.1) then becomes

$$(3.2) \quad \lim_{|\underline{x}|\to+\infty} \{|\underline{x}^*| \lim_{\varepsilon\downarrow 0} I(\varepsilon, \underline{x})\} = \frac{\pi^2 h_1^2 h_2^2 h_3^2}{4\Delta},$$

where

$$I(\varepsilon, \underline{x}) = \iiint_{-\infty}^{\infty} \frac{D(\underline{h},\underline{u})}{\varepsilon+1-\phi(\underline{u})} e^{-i\underline{u}.\underline{x}} d\underline{u}.$$

Now we can apply Green's Theorem (3.2.19) to get for each $\varepsilon > 0$,

$$I(\varepsilon, \underline{x}) = \lim_{X \to +\infty} \iiint_{-X}^{X} \frac{D(\underline{h},\underline{u})e^{-i\underline{u}.\underline{x}}}{\varepsilon+1-\phi(\underline{u})} d\underline{u}$$

$$= \lim_{X\to+\infty} \left\{\frac{-i}{|\underline{x}|^2} \iiint_{-X}^{X} \underline{x}.\nabla \left\{\frac{D(\underline{h},\underline{u})}{\varepsilon+1-\phi(\underline{u})}\right\} e^{-i\underline{u}.\underline{x}}\right.$$

$$\left. + \frac{i}{|\underline{x}|^2} \iint_{\sigma(X)} \underline{n}.\underline{x}\; e^{-i\underline{x}.\underline{u}} \frac{D(\underline{h},\underline{u})}{\varepsilon+1-\phi(\underline{u})} d\underline{\sigma}\right\}$$

$$= \frac{-i}{|\underline{x}|^2} \iiint_{-\infty}^{\infty} \underline{x}.\nabla \left\{\frac{D(\underline{h},\underline{u})}{\varepsilon+1-\phi(\underline{u})}\right\} e^{-i\underline{u}.\underline{x}} d\underline{u},$$

since $\left|\frac{1}{\varepsilon+1-\phi(\underline{u})}\right| \leq \varepsilon^{-1}$ and $\lim_{X \to +\infty} \iint_{\sigma(X)} |D(\underline{h},\underline{u})| d\underline{\sigma} = 0.$

Some slight calculation shows that for each s $\left|\frac{\partial}{\partial u_s} D(\underline{h},\underline{u})\right|$ is bounded near $\underline{0}$ and integrable over E_3. Since, by Lemmas 2.7 and 3.2.12, $|1-\phi(\underline{u})|^{-1}$ is integrable near $\underline{0}$ and bounded in any closed region not containing $\underline{0}$, the Theorem of Dominated Convergence shows that

$$\lim_{\varepsilon\downarrow 0} \iiint_{-\infty}^{\infty} \frac{\underline{x}.\nabla D(\underline{h},\underline{u})}{\varepsilon+1-\phi(\underline{u})} e^{-i\underline{u}.\underline{x}} d\underline{u} = \iiint_{-\infty}^{\infty} \frac{\underline{x}.\nabla D(\underline{h},\underline{u})}{1-\phi(\underline{u})} e^{-i\underline{u}.\underline{x}} d\underline{u},$$

and the Riemann-Lebesgue Lemma shows that this last is $o(|\underline{x}|)$ as $|\underline{x}| \to +\infty$.

If $V(\delta)$ once again denotes the set $\{\underline{u}: Q(\underline{u}) \leq \delta^2\}$, $\sup_{\underline{u} \notin V(\delta)} |1-\phi(\underline{u})|^{-1} < +\infty$ for each $\delta > 0$, by Lemma 3.2.12.

Then, since $D(\underline{h}, \underline{u})$ is a positive integrable function and $|\underline{\nabla\emptyset(u)}|$ is bounded, the same argument applies again and shows that for any $\delta > 0$,

$$\lim_{|\underline{x}|\to+\infty}\left\{\frac{|\underline{x}^*|}{|\underline{x}|^2}\lim_{\varepsilon\downarrow 0}\iiint_{\{\underline{u}\notin V(\delta)\}} e^{-i\underline{u}.\underline{x}}\,\frac{D(\underline{h},\underline{u}).\underline{x}.\underline{\nabla\emptyset(u)}}{\{\varepsilon+1-\emptyset(\underline{u})\}^2}\,d\underline{u}\right\} = 0.$$

Thus (3.2) is equivalent to

$$(3.3)\quad \lim_{|\underline{x}|\to+\infty}\left\{\frac{|\underline{x}^*|}{|\underline{x}|^2}\lim_{\varepsilon\downarrow 0}\iiint_{V(\delta)} e^{-i\underline{u}.\underline{x}}\,\frac{D(\underline{h},\underline{u})\;\underline{x}.\underline{\nabla\emptyset(u)}}{\{\varepsilon+1-\emptyset(\underline{u})\}^2}\,d\underline{u}\right\} = \frac{i\pi^2 h_1^2h_2^2h_3^2}{4\Delta}.$$

Since $D(\underline{h}, \underline{u}) = \frac{h_1^2h_2^2h_3^2}{8} + O(|\underline{u}|^2)$, as $|\underline{u}|\to 0$, we can again use Lemma 2.7 to approximate to the integrand of (3.3) For if

$$g_\varepsilon(\underline{u}) = D(\underline{h}, \underline{u})\,\frac{\underline{x}.\underline{\nabla\emptyset}(\underline{u})}{\{\varepsilon+1-\emptyset(\underline{u})\}^2} + \frac{h_1^2h_2^2h_3^2}{8}\cdot\frac{\underline{x}.\underline{\nabla Q}(\underline{u})}{\{\varepsilon+Q(\underline{u})\}^2},$$

then

$$g_\varepsilon(\underline{u}) = D(\underline{h}, \underline{u})\, f_\varepsilon(\underline{u}) - \frac{\underline{x}.\underline{\nabla Q(u)}}{(\varepsilon+Q(\underline{u}))^2}\left\{D(\underline{h},\underline{u})-\frac{h_1^2h_2^2h_3^2}{8}\right\},$$

where $f_\varepsilon(\underline{u})$ is the function of p.107. On that page it is shown that $f_\varepsilon(\underline{u})$ is dominated by a function which is integrable in $V(\delta)$ for all small enough δ. Plainly

$$\left|\frac{\underline{x}.\underline{\nabla Q(u)}}{Q^2(\underline{u})}\right|\;\left|D(\underline{h}, \underline{u}) - \frac{h_1^2h_2^2h_3^2}{8}\right| = O\left(\frac{1}{|\underline{u}|}\right),\text{ as } |\underline{u}|\to 0,$$

and therefore, by dominated convergence and the Riemann-Lebesgue Lemma,

$$(3.4)\quad \lim_{|\underline{x}|\to+\infty}\left\{\frac{|x^*|}{|\underline{x}|^3}\lim_{\varepsilon\downarrow 0}\iiint_{V(\delta)} g_\varepsilon(\underline{u})e^{-i\underline{u}\cdot\underline{x}}\,d\underline{u}\right\}$$

$$= \lim_{|\underline{x}|\to+\infty}\left\{\frac{|x^*|}{|\underline{x}|^3}\lim_{\varepsilon\downarrow 0}\iiint_{V(\delta)} g_0(\underline{u})e^{-i\underline{u}\cdot\underline{x}}d\underline{u}\right\} = 0.$$

It was established in the previous section that

$$(3.5)\quad \lim_{|\underline{x}|\to+\infty}\left\{\frac{|x^*|}{|\underline{x}|^3}\lim_{\varepsilon\downarrow 0}\iiint_{V(\delta)}\frac{\underline{x}\cdot\underline{\nabla}\ Q(\underline{u})}{\{\varepsilon+Q(\underline{u})\}^3}\,d\underline{u}\right\} = \frac{2\pi^3}{i\Delta},$$

and (3.4) and (3.5) prove (3.3), and hence the theorem when k=3.

§.4 The proofs of Theorems 1.1 and 1.7 when $k > 3$ follow the same lines as the proofs given in §.2 and §.3 for the case $k = 3$: consider the (easier) lattice case 1.1. Assumption (1.4) means that all derivatives of $\phi(\underline{u})$ of the first k-2 orders exist, are bounded, have period 2π in each variable and are continuous in E_k^π. We can therefore apply Green's Theorem k-2 times to the Fourier integral for $G_0(\underline{x})$ (Lemma 3.2.11) to get

$$(4.1)\quad G_0(\underline{x}) = \frac{1}{(2\pi)^k}\left\{\frac{-i}{|\underline{x}|^2}\right\}^{k-2}\lim_{\varepsilon\downarrow 0}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} e^{-i\underline{u}\cdot\underline{x}}\ \square^{k-2}\left\{\frac{1}{\varepsilon+1-\phi(\underline{u})}\right\}d\underline{u},$$

where $\square = \underline{x}\cdot\underline{\nabla}$ and $\underline{x}\in L_k$. (4.1) is the analogue of (2.3), and once again the error incurred in replacing the region of integration by $V(\delta)$ and $1-\phi(\underline{u})$ by $Q(\underline{u})$ is

$O\{|\underline{x}|^{-(k-2)}\}$ as $|\underline{x}| \to +\infty$. In §2 we showed that

$$\lim_{\varepsilon \downarrow 0} \iiint_{V(\delta)} \Box\left\{\frac{1}{\varepsilon + Q(\underline{u})}\right\} e^{-i\underline{u}.\underline{x}}\, d\underline{u} \sim \frac{2\pi^2 i|\underline{x}|}{\Delta\, |\underline{x}^*|} \text{ as } |\underline{x}| \to +\infty,$$

and a more complicated argument of a similar type shows that in k-dimensions

$$\lim_{\varepsilon \downarrow 0} \int_{V(\delta)} \cdots \int \Box^{k-2}\left\{\frac{1}{\varepsilon + 1 - Q(\underline{u})}\right\} e^{-i\underline{u}.\underline{x}}\, d\underline{u} \sim \frac{\pi}{\Delta}\left\{\frac{2\pi^{\frac{1}{2}} i|\underline{x}|^2}{|\underline{x}^*|}\right\}^{k-2} \Gamma\left(\frac{k-2}{2}\right)$$

$$\text{as } |\underline{x}| \to +\infty,$$

which is all that is needed to conclude the proof.

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