Multiparticle veneziano models

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MULTIPARTICLE VENEZIANO MODELS

by

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PREFACE

The work presented in this thesis was carried out in the Department of Mathematics at the University of Durham during the period October 1967 to July 1970, under the supervision of Dr. D.B.Fairlie. The author wishes to express his sincere thanks to Dr. Fairlie for skilful guidance and expert advice at all stages of this research.

The work in this thesis has not been submitted for any other degree in this or any other University. It is based essentially on one paper by the author and two papers by the author in collaboration with Dr. D.B.Fairlie. No claim of originality is made on chapters one and two. Chapter three contains work done by the author in collaboration with Dr. Fairlie. Most of chapter four and all of chapter five, including the appendix, are claimed to be original.

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ABSTRACT

We describe the construction of the Veneziano model for strong interaction scattering amplitudes and its generalisation to multiparticle production processes, and examine some of its properties.

In chapter one we introduce the ideas which form the input to the Veneziano model. In chapter two we study how they may all be simultaneously realised in a compact form and how we may extend the model to incorporate multiparticle production processes. We also indicate in this chapter how the model may be used in experimental applications, and what difficulties the simple model faces.

In chapter three we examine how we may incorporate all the terms of the complete Veneziano expression into one term, provided certain trajectory constraints are satisfied.

In chapter four we study an alternative five-point amplitude having a different structure to the conventional tree graph. In chapter five we extend this amplitude to the case with an arbitrary number of external lines.

CHAPTER I

INTRODUCTION

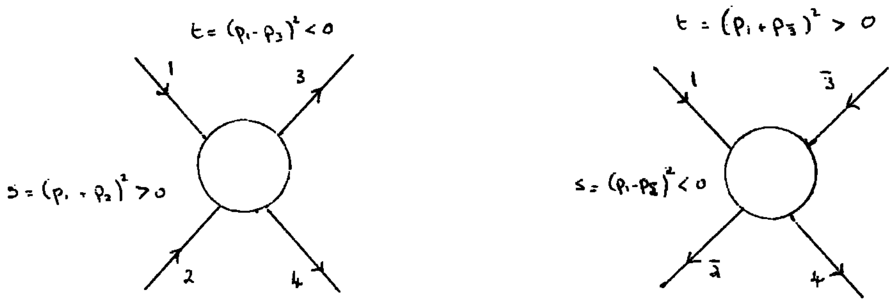
The S-matrix approach to the study of elementary particle theory was first proposed by Heisenberg (1). The guiding idea underlying it is that one should deal only with quantities that, as well as describing our microscopic system, are also directly measurable. Scattering probability amplitudes are such quantities. More recently this approach has been reconsidered and emphasized by several people, notably Chew (2), in the 'bootstrap' programme.

We review briefly the fundamental general assumptions underlying the present S-matrix theory:

(i) Unitarity: this is the statement of the conservation of probability. As probability is the square modulus of the amplitude, unitarity turns out to be a non-linear condition.

(ii) Analyticity: this is the mathematical expression for causality. It is mainly used in the form of dispersion relations, which hold in the classical theory of light dispersion as a consequence of a precise time-relation between cause and effect.

(iii) Crossing: this is a purely relativistic property that connects two different processes obtained one from the other, by interchange of one initial and one final particle of the process. Figure 1 shows two processes connected through crossing.



(a) (b)

Figure 1. Crossing. The two scattering processes (a) and (b) are described by the same scattering amplitude $A(s,t)$ in different regions of the (s,t) plane.

In addition to these fundamental assumptions, there are further constraints which we may impose on our scattering amplitudes. We recall some of the main features of the applications of Regge-pole theory.

The Regge-pole approach to strong interactions has been developed from analogy with potential scattering and has led to the well-known asymptotic formula for the scattering amplitude $A(s,t)$ at large energy s and fixed momentum transfer t.

$$A(s,t) \cong \beta(t) \frac{s^{\alpha(t)}[e^{-i\pi\alpha(t)} \pm 1]}{\sin \pi\alpha(t)} \tag{1.1}$$

where $\alpha(t)$ and $\beta(t)$ are the Regge trajectory and residue function respectively, and the intercepts

$$\alpha(t_n) = n \tag{1.2}$$

give the values of the masses $m_n = \sqrt{t_n}$ of the particles exchanged in the t-channel. In this way the Regge model gives rise to a very important connection between the s-channel asymptotic behaviour and the t-channel resonances.

In the early stages of Regge theory, in analogy with potential scattering, the following features were apparent:

(i) The Regge formula (1.1) had no direct relation with resonances in the s- and u-channels.

(ii) It was impossible to have reasonable crossing properties except in a very artificial way.

(iii) Trajectories reached a certain value of J and then started to decrease. This was due to the fact that in a two-body problem with short-range potential, the centrifugal barrier does not allow large J resonances to exist.

In recent years Regge theory has grown further and further away from its potential scattering origin. The changes in approach have been so great that it is now conceivable to think about representations in which resonances and asymptotic behaviour are taken into account simultaneously in a crossing symmetric way.

First of all it was realised, on the basis of both theoretical and experimental indications, that in elementary particle physics the trajectories could have a completely different shape from that suggested by potential scattering, and be regularly increasing with energy in a large energy range. The extreme idealisation of this point of view is provided by straight-line trajectories, $\alpha(t) = a + bt$, which interpolate an infinite number of equally spaced resonances of zero width. We may illustrate this last point by writing a dispersion relation for the trajectory function α. If α increases linearly, as above, then the dispersion relation will require two subtractions and may be written as, (3),

$$\alpha(t) = at + b + \frac{1}{\pi} \int dt' \, \frac{\operatorname{Im} \alpha (t')}{t' - t} \tag{1.3}$$

From equation 1.3 it can be seen that the condition for linearity is that $\operatorname{Im} \alpha$ be small. It is a well known result in Regge theory that the width of the resonance is proportional to $\operatorname{Im} \alpha$, so that a necessary and sufficient condition for the resonances to be narrow is that the trajectory be approximately linear. The present experimental evidence indicates that this is so.

We may note here that in the zero width approximation, the unitarity condition will be violated, since the scattering amplitude becomes arbitrarily large near a pole and thus gives eventually a probability greater than one.

There is also considerable theoretical evidence for indefinitely rising trajectories; as pointed out by Mandelstam (4), this could be connected with the essential many body nature of elementary particle physics. With increasing t, more and more channels open and the appearance of new thresholds compensates the decrease (due to centrifugal barrier) of the two-body part of $\alpha(t)$.

A second important new feature has come from the study of analyticity at $t = 0$ together with the Lorentz transformation properties of Regge poles. These new requirements have led to the prediction that Regge trajectories are members of large families (5). For example, each trajectory $\alpha(t)$ must be accompanied by an infinite set of daughters whose values at $t = 0$ are $\alpha(0) - n$. The possible intercepts of these daughters with integer values of J lead to extra particles, so that the elementary particle spectrum in this enlarged Regge-Lorentz scheme becomes more rich and complicated.

The analytic properties of scattering amplitudes coupled with their asymptotic behaviour have been further exploited in the form of finite energy sum rules (6). They were originally proposed by Igi (7) in 1962, but their general power has only been realised more recently (8, 9, 10, 11, 12).

For concreteness, we consider πN scattering, where the s- and u-channel processes are the same. Keeping t fixed, it is convenient to replace s by a variable with simple symmetry properties under $s \leftrightarrow u$. The usual choice is the variable

$$\nu = \frac{s-u}{4m}\,, \tag{1.4}$$

with m the nucleon mass. We consider an amplitude that is antisymmetric under the crossing $s \leftrightarrow u$..

$$F(\nu) = -F^*(-\nu) \tag{1.5}$$

and that satisfies the fixed t dispersion relation

$$F(\nu) = \frac{2\nu}{\pi}\int_0^\infty \frac{\mathrm{Im}\,F(\nu')d\nu'}{\nu'^2-\nu^2} \tag{1.6}$$

The integration is defined over the right-hand cut in s and includes the pole term, even if it occurs at negative ν. As t is fixed, we show only the variable ν of F.

The following πN amplitudes are antisymmetric under $s \leftrightarrow u$:

$$A'^{(-)},\quad \nu B^{(-)},\quad \nu A'^{(+)},\quad B^{(+)} \tag{1.7}$$

where the superscripts $(\pm)$ denote the linear combinations of the $I=\frac{1}{2}$ and $I=\frac{3}{2}$ amplitudes for which the isospin crossing matrix is diagonal,

$$F^{(+)} = \frac{1}{3}F^{(1/2)} + \frac{2}{3}F^{(3/2)}$$

$$F^{(-)} = \frac{1}{3}F^{(1/2)} - \frac{1}{3}F^{(3/2)} \qquad (1.8)$$

A' is defined by

$$A' = A + \frac{s - u}{4m^2 - t} \cdot mB \qquad (1.9)$$

where A and B are the invariant amplitudes of πN scattering, free of kinematic singularities and constraints. A' and B are usually referred to as the helicity-non-flip and helicity-flip amplitudes respectively.

At high energies we assume that the amplitudes $F(\upsilon)$ can be represented by a sum of Regge-pole contributions, each of the form

$$\frac{\beta(t)(\pm 1 - e^{-i\pi\alpha(t)})\upsilon^{\alpha(t)}}{\Gamma(1 + \alpha(t)) \sin \pi\alpha(t)} \qquad (1.10)$$

If the leading Regge term has $\alpha < -1$, then it will obey the superconvergence relation

$$\int_0^\infty \mathrm{Im}\, F(\upsilon)d\upsilon = 0 \qquad (1.11)$$

However, suppose that there are contributing Regge poles with α greater than -1. If we subtract these contributions from F then the resulting amplitude will satisfy the superconvergence relation

$$\int_0^\infty \left[\operatorname{Im} F(\upsilon) - \sum_{\alpha_i > -1} \frac{\beta_i \, \upsilon_i^{\alpha_i}}{\Gamma(1 + \alpha_i)} \right] d\upsilon = 0 \tag{1.12}$$

Both terms on the left hand side diverge if evaluated separately. To write the relation in a convergent form suitable for practical applications, we therefore cut the integration off at some $\upsilon = N$, assumed large enough so that the Regge representation (1.10) is valid for $\upsilon > N$, and express the high energy behaviour by the Regge terms whose α is below -1.

$$\int_0^N \left[\operatorname{Im} F(\upsilon) - \sum_{\alpha_i > -1} \frac{\beta_i}{\Gamma(1 + \alpha_i)} \upsilon_i^{\alpha_i} \right] d\upsilon + \int_N^\infty \sum_{\alpha_j < -1} \frac{\beta_j}{\Gamma(1 + \alpha_j)} \upsilon^{\alpha_j} \, d\upsilon = 0 \tag{1.13}$$

Performing the integration, we obtain the following finite energy sum rule,

$$S_o = \frac{1}{N} \int_0^N \operatorname{Im} F(\upsilon) d\upsilon = \sum_{\text{all } \alpha} \frac{\beta N^\alpha}{(1+\alpha)\Gamma(1 + \alpha)} \tag{1.14}$$

We observe that all the Regge terms enter the sum rule in the same form, regardless of whether α is above or below -1 .

The generalisation to sum rules for higher moments is straightforward. For even integer n, we obtain

$$S_n = \frac{1}{N^{n+1}} \int_0^N \upsilon^n \, \mathrm{Im}\, F(\upsilon) d\upsilon = \sum \frac{\beta N^{\alpha}}{(1+\alpha+n)\Gamma(1+\alpha)} \quad (1.15)$$

Finite energy sum rules can be similarly derived for symmetric functions; however, the above derivation only gives finite energy sum rules for even (odd) integer n for the antisymmetric (symmetric) part of the physical amplitude. Sum rules may also be derived for moments $\upsilon^{\gamma}\, F(\upsilon)$ where γ is not integral. These so-called continuous moment sum rules involve in general, both the real and imaginary parts of the amplitude (13).

The physical content of equation (1.14) is nothing more than just the analytic properties $F(\upsilon)$ and its assumed asymptotic behaviour. The important feature is that these properties have been expressed in a manner particularly convenient for phenomenological exploitation. The left-hand side is an integral over the 'low-energy' region from 0 up to the energy N, while the right-hand side is expressed in terms of those parameters that characterise the amplitude for high-energies, i.e. $\upsilon > N$. Equation 1.14 can therefore be regarded as a consistency relation connecting scattering at high- and low-energies imposed by analyticity, thus providing a new form of bootstrap.

The applications of finite energy sum rules to phenomenological analyses can be divided into three types, differing only by emphasis:

(a) using low energy data to predict high energy parameters, such as Regge intercepts and residue functions;

(b) resolving ambiguities in the low energy region by means of high energy data;

(c) making simultaneous fits of high and low energy data in a manner consistent with analyticity.

Obviously, since high energy data are in general less accurate than those at lower energies, (a) and (c) above far outweight (b) in importance.

We consider in particular πN charge exchange, $\pi^- p \to \pi^0 n$ (11). This process is described solely by the πN amplitudes $A'^{(-)}$ and $B^{(-)}$. At high energies we assume that the $\pi^- p \to \pi^0 n$ amplitudes are dominated by ρ Regge pole exchange. We may now write our finite energy sum rules for $A'^{(-)}$ and $\upsilon B^{(-)}$, S_0 and S_1 as:

$$\int_0^N \mathrm{Im}\, A'^{(-)}(\upsilon)d\upsilon = \beta_\rho \frac{N^{\alpha_\rho+1}}{(1+\alpha_\rho)\Gamma(1+\alpha_\rho)} \tag{1.16}$$

$$\int_0^N \mathrm{Im}\, \upsilon B^{(-)}(\upsilon)d\upsilon = \beta_\rho \frac{N^{\alpha_\rho+2}}{(2+\alpha_\rho)\ (1+\alpha_\rho)} \tag{1.17}$$

The usefulness and accuracy of the finite energy sum rules depend mainly on the available data. A particularly favourable case is the sum rule at $t = 0$, where by means of the optical theorem, the imaginary part of the amplitude is simply related to the total cross-section. With the accurate measurements of total cross-sections already available, one can make very accurate predictions of Regge parameters at $t = 0$.

To evaluate the left-hand side of these sum rules for $t \neq 0$, phase shift analyses are used, which for πN scattering are reliable up to $\nu_L \sim 1.13$ BeV. This is usually taken as the upper limit of integration N. Concerning the choice of N, we note that in the case $t = 0$, where one uses the known total cross-sections, one should choose N at that energy where the 'wiggles' become smaller than the systematic error. This occurs at around $\nu_L = 4.0$ BeV.

Using the results of phase shift analysis, Dolen, Horn and Schmid (11) were able to predict many of the high energy features of the $A'^{(-)}$ and $B^{(-)}$ amplitudes of πN charge exchange, for example;

(i) the spin-flip amplitude $\nu B^{(-)}$ is larger than the non-flip amplitude $A'^{(-)}$ by an order of magnitude at $t = 0$. This explains the near forward peak in πN charge exchange.

(ii) $B^{(-)}$ has a zero near $t = -0.5$ BeV^2. This explains the so-called 'wrong-signature nonsense' dip in $\frac{d\sigma}{dt}$ for πN charge exchange.

(iii) In an effective one-pole model, the ρ mass is predicted, and a trajectory α_{eff} which is 0.1 to 0.2 lower than the one measured at high energies.

Obviously sum rules are not limited to πN scattering or the ρ-pole alone. Such sum rules can be derived for any assumed asymptotic behaviour, such as sums of Regge poles, or even Regge cuts. The sum rules derived in each case will afford a means of checking phenomenologically whether the assumed high-energy behaviour is consistent with the low-energy scattering data.

However, based on only two theoretical concepts, namely analyticity and asymptotic behaviour, finite energy sum rules, being so general, have low predictive power, their use being restricted to that of a phenomenological tool for data analysis. Their use is greatly strengthened when supplemented by the following assumptions.

1. Down to fairly low energies (~ 2 GeV/c), scattering amplitudes are already well approximated by the exchange of a few Regge poles.

2. The imaginary part of a scattering amplitude is dominated entirely by direct channel resonances.

If we now consider again equations (1.16) and (1.17), taking into account our new assumptions, we see that on the right we have the ρ-exchange amplitude and on the left we assume that $\mathrm{Im}\, A'^{(-)}$ and

Im $B^{(-)}$ are dominated by the direct channel resonances which occur in πN scattering. Equations (1.16) and (1.17) then become a relation between the masses and widths of nucleonic resonances and the Regge parameters α and β of ρ-exchange.

The relation between resonances in the direct channel and the exchanged Regge pole is of great theoretical significance since the Regge poles themselves are supposed to be connected to resonances in the exchange channels. This significance is best appreciated in reactions such as $\pi\pi \rightarrow \pi\pi$ when the direct and exchange channels are identical. Equation (1.14) then becomes a consistency requirement involving the ρ-trajectory on both sides, which can be used to restrict the trajectory parameters. This is then the so-called finite energy sum rule bootstrap.

Further consequences of finite energy sum rules plus the assumptions (1) and (2) may be deduced. Analyticity implies in addition further sum rules for various moments of the amplitude, as we have seen. In each case, the contribution of resonances on the left must add in such a way as to build up the Regge exchange on the right. Now the higher moment sum rules will emphasise the higher mass resonances. The only way for all the sum rules to be satisfied will be to have the integrand itself approximately equal to the Regge amplitude.

Clearly, this 'duality' (14), or equivalence between the direct channel resonances and the Regge pole exchange should not be taken too literally, at least in the low energy region where the resonance amplitude shows large fluctuations as a function of the energy. It is supposed to hold only in the average sense when the resonance amplitude is integrated over a small (~ 1 Gev) interval. It is this 'semi-local average' over the resonance contribution which is supposed to be approximately equal to the Regge amplitude.

A demonstration of this surprising fact has been given by Schmid (12). He took the Regge parameters as determined from fits at high energy to extrapolate the ρ-exchange amplitude in πN scattering down to energies ~ 2 GeV. Then, performing a partial wave analysis in this, he obtained for each partial wave a loop on the Argand diagram very similar to those obtained by phase-shift analysis as evidence for nucleon resonances. Moreover, these 'pseudo-resonances' were shown to lie approximately on a linear rising trajectory. Indeed, on closer examination it was found that such a behaviour of partial wave phases is an almost automatic consequence of the Regge form of the amplitude, for an exchanged trajectory with finite slope (15).

At this point, we note that one notable exception to the assumptions (1) and (2) above is the Pomeranchuk trajectory which cannot be considered dual to resonances in the sense described above. It is at present unclear how the Pomeron will fit into any duality scheme. However, as a first approximation, we may assume that the Pomeron contribution is additive to the dual part of the amplitude. We write for the full amplitude $A = A_{\text{Dual}} + A_{\text{Pomeron}}$, where A_{Dual} has all the prescribed duality properties. In all that follows we assume that we are dealing with the dual part of the amplitude.

We may thus see that as a result of both theoretical and phenomenological studies, especially those based on finite energy sum rules and duality described above, we now have quite a good knowledge of the properties possessed by hadronic two-body scattering amplitudes. A simple closed expression, incorporating all these constraints, and providing a framework in which such loosely defined terms as duality can be given a clearer meaning, has been given by Veneziano (16). We now proceed to describe this representation.

CHAPTER II

THE VENEZIANO MODEL

2.1 The four-point amplitude.

We have indicated in our introductory chapter the properties which we expect our scattering amplitudes to satisfy, and have summarised the reasons for their assumption. These properties may be conveniently summarised as follows:

(1) analyticity,

(2) crossing symmetry,

(3) Regge asymptotic behaviour,

(4) the presence of daughters,

(5) resonances on linearly rising trajectories,

(6) 'duality' .

The Veneziano amplitude (16) is a particular example that satisfies all these properties. It is thus a good theoretical laboratory in which to study hadron collision amplitudes.

For simplicity we consider a system with only one family of neutral bosons, where all particles lie on the same Regge trajectory (or on its daughters), the lowest member of which has spin-parity $J^P = 0^+$. Our trajectory thus has a negative intercept (assuming a positive slope). We consider first the four-point amplitude describing the process

$$0^+ + 0^+ \rightarrow 0^+ + 0^+ \tag{2.1}$$

The amplitude is given as a sum of three terms, each corresponding to a particular permutation of the four external lines, i.e.

$$T_4 = V(1,2,3,4) + V(1,2,4,3) + V(1,3,2,4) \tag{2.2}$$

(see figure 2.1). Veneziano (16) gave for $V(1,2,3,4)$ the form:

$$V(1,2,3,4) = B_4(-1 - \alpha_{12}, -1 - \alpha_{23}) , \tag{2.3}$$

$$\left.\begin{aligned} &\text{where} \quad \alpha_{12} = \alpha_{34} = \alpha_0 + \alpha' s_{12} \\ &\text{and} \quad s_{12} = (p_1 + p_2)^2 \end{aligned}\right\} \tag{2.4}$$

$$\text{and} \quad B_4(x,y) = \int_0^1 du\, u^x v^y \tag{2.5a}$$

$$\text{or equivalently} \quad B_4(x,y) = \frac{\Gamma(1+x)\,\Gamma(1+y)}{\Gamma(2+x+y)} \tag{2.5b}$$

the variables u and v in (2.5a) being subject to the constraint

$$v = 1 - u \tag{2.6}$$

The other terms in (2.2) are readily obtained by a permutation of indices. We refer to the variables u and v in (2.5a) as being conjugate to the variables x and y. The properties of the amplitude (2.3) are well known. We review them briefly.

(i) The beta function defined in (2.5) is symmetric in x and y, which by (2.3) and (2.4) implies that $V(1,2,3,4)$ is invariant under a cyclic permutation or an inversion in ordering of the external lines. Thus,

$$V(1,2,3,4) = V(2,3,4,1) = V(4,3,2,1) \tag{2.7}$$

In what follows we shall regard all orderings related either by cyclic permutations or inversions to be equivalent. The three terms in (2.2) correspond to the three non-equivalent orderings of the external lines. The invariant properties (2.7) of B_4 thus imply that the amplitude (2.2) is completely crossing symmetric.

(ii) The function B_4 is analytic apart from a sequence of poles at negative intergers of x (hence also y by symmetry). The poles in x occur in the integration region near $u = 0$, and can best be studied by expanding the integrand in (2.5a) in a Taylor series about $u = 0$. Integrating the series term by term one sees that the residue at $x = -\ell - 1$ is given by

$$\operatorname*{Res}_{x=-\ell-1} \left\{ B_4(x,y) \right\} = \frac{1}{\ell!} \left[\frac{d^\ell (1-u)^y}{du^\ell} \right]_{u=0} \tag{2.8}$$

This is clearly a polynomial of degree ℓ in y. Hence the pole must represent some particle with maximum spin ℓ . However, the polynomial being in general not identical to the Legendre polynomials $P_\ell(y)$, there will also be components of lower spin. These are the daughter states, which are degenerate with the parent states.

(iii) There are no coincident poles in x and y. These double poles would only occur when both u and v are zero in the integrand (2.5a). Such unwanted features are prevented by the constraints of duality imposed on u and v, i.e. $v = 1-u$ (2.6).

(iv) The asymptotic behaviour of our amplitude (2.2) may be readily established. Because of symmetry, we need only show this for one channel, say $s_{12} \to \infty$ for fixed s_{23} . We must be careful in taking this limit as the amplitude has a sequence of poles along the real s_{12} axis, so that the limit $s_{12} \to \infty$ along the real axis cannot strictly exist. We therefore take the limit of s_{12} approaching ∞ along a ray at an infinitesimal angle to the real axis. We need only consider the first and third terms in (2.2), as the second term $V(1,2,4,3)$ vanishes exponentially as $s_{12} \to \infty$. Using the gamma function representation (2.5b), and the standard properties of the gamma function and taking the limit $s_{12} \to \infty$ as defined above, we obtain the result:

$$\underset{\begin{pmatrix} s_{12} \to \infty \\ s_{23} \text{ fixed} \end{pmatrix}}{T_4} \simeq \frac{\pi}{\Gamma(\alpha_{23})} \frac{[1 + e^{-i\pi\alpha_{23}}]}{\sin \pi\alpha_{23}} (\alpha' \; s_{12})^{\alpha_{23}} \qquad (2.9)$$

This is the usual Regge expression, with the appropriate signature factor.

Having now proved the Regge behaviour and also the existence of resonance poles in the amplitude, duality becomes an automatic consequence, since the same function has been shown explicitly to contain both the direct channel resonance poles and the crossed channel Regge exchange amplitude.

Thus all the properties (1,2,...6) listed at the beginning of this chapter are satisfied by the Veneziano amplitude given by equations (2.2) - (2.6).

We note here that each of the terms in the complete Veneziano expression (2.2) may be expressed as a summation over the poles in either of the two dual variables concerned, thus

$$V(1,2,3,4) = \sum_n \frac{c_n\ (s_{23})}{s_{12} - (s_{12})_n} = \sum_n \frac{c_n\ (s_{12})}{s_{23} - (s_{23})_n} \qquad (2.10)$$

If we consider an amplitude with only s- and t-channel resonances, for example $\pi^+\pi^- \rightarrow \pi^+\pi^-$, then the Veneziano expression will reduce to a single term $V(s,t)$, ($\equiv V(1,2,3,4)$ in figure 1), and allows the simplest interpretation of duality in terms of resonances only. It leads to a complete expansion of the amplitude either in terms of the resonances exchanged in the s-channel, or equivalently in terms of t-channel resonances.

2.2 Interpretation via quark duality diagrams.

One of the most important steps in the study of the Veneziano model was its interpretation via legal quark duality diagrams (17, 18). These diagrams exhibit in a simple visual form the duality between the s- and t-channel descriptions of strong interaction scattering amplitudes and the assumption that in every channel the scattering proceeds via non-exotic resonances. By non-exotic resonances, we mean here mesons which do not belong to SU(3) singlets or octets and baryons which do not belong to SU(3) singlets, octets, or decuplets, i.e. those resonances outside the naive quark model, for example $(q\bar{q})^N$ with $N \neq 1$.

We shall assume that all the incoming and outgoing particles as well as the poles in all channels are not exotic and may therefore be mathematically represented by three-quark or quark-antiquark combinations. The rules for drawing a legal diagram are very simple:

1. There are three types of lines, corresponding to the p, n, and λ quarks. Lines do not change their identity.

2. Every external baryon is represented by three lines running in the same direction.

3. Every external meson is represented by two lines running in opposite directions.

4. The two ends of a single line cannot belong to the same external particle.

5. In any B = 1 channel (s, t, or u) it is possible to "cut" the diagram into two by cutting only three quark lines (and not $4q + \bar{q}$, etc.). Similarly in any mesonic channel we should be able to split the diagram by cutting only two lines (and not $2q + 2\bar{q}$, etc.).

The physical assumptions involved are the following:

(a) All baryons are in the $\underline{1}$, $\underline{8}$, or $\underline{10}$ SU(3) multiplets and can be mathematically described as three-quark structures.

(b) All mesons are quark-antiquark structures in the $\underline{1}$ or $\underline{8}$ SU(3) multiplets.

(c) The scattering amplitude in any channel is given by a sum of single particle states. It is of course the alternative expansion of the functions B_4 given by equation (2.5), in terms of an infinite number of poles in one or other of the dual variables which characterises duality.

The diagrams describing forward meson-meson scattering (or at least all the independent ones) are shown in figure 2.2a. Diagrams describing forward and backward meson-baryon scattering are shown in figures 2.2b and c, respectively. The duality property is clearly demonstrated since the diagrams can be viewed either as a sum of single particle states in one channel or as a sum of such terms in the other channel. It is clear from the diagrams that in meson-meson and meson-baryon scattering there is at least one self-consistent set of amplitudes which satisfy duality and resonance dominance in the s- and t-channels. On the other hand, figure 2.2d demonstrates that in baryon - antibaryon scattering there is no way of describing the amplitude by a sum of non-exotic mesons in both the s- and t-channels. At least one of these channels must have $(2q + 2\bar{q})$ intermediate states contrary to the basic rules. The inconsistency between the duality requirement and the resonance dominance assumption in baryon-antibaryon scattering is not a new result (19), but here it is deduced in a trivial way from the simple diagrams.

These diagrams enable us to make a number of experimental predictions. If a scattering amplitude with a two-particle final state is completely explained by a sum of direct channel resonances, its imaginary part at a given energy may be approximated by the contributions of the resonances in the neighbourhood of that energy. The real part of the amplitude at the same energy will not be described in terms of nearby resonances. The reason for this distinction is, of course, the fact that the imaginary part of a resonant amplitude is large around the resonance energy, while the main influence of a resonance on the real part of the amplitude is spread over a wider energy range and actually vanishes at the resonance energy. If a process such as elastic K^+p scattering does not exhibit any s-channel resonances, only the imaginary part of its forward amplitude will vanish (except for the Pomeranchuk term). The real part of the K^+p amplitude will not vanish and will have contributions arising from resonances occurring in the u-channel - those that appear in K^-p scattering. Only if both the s- and u-channels of such a process do not show any resonances will the real part of the amplitude also vanish in the resonance-dominance approximation. We may thus see that if a certain process cannot be described by the duality diagrams, only the imaginary part of its amplitude is predicted to vanish. The real part may be fed by the u-channel process, and only when the latter also corresponds to an illegal diagram, both the real and imaginary part of the amplitude will vanish (again, except for the Pomeranchuk term).

All predictions that can be made, are based on searching for diagrams which cannot be described by a legal diagram, for example:

1. All processes of the form

$$K^-B \to \pi^-B'; \quad \pi^+B \to \bar{K}^oB'; \quad K^+B \to K^oB' ,$$

where B, B' are any one-exotic baryons, cannot be represented by legal diagrams and hence are predicted to have purely real amplitudes at small t values. Examples of the above types are:

$$K^-p \to \pi^-\Sigma^+; \quad \pi^+\Sigma^o \to \bar{K}^op; \quad K^+n \to K^op .$$

One way of testing the above predictions is to measure the polarisation of the final particles. A purely real amplitude leads to zero polarisation. However, such tests are especially inadequate since a negligible imaginary part ($\sim 5\%$ of the total cross-section) can lead to an appreciable polarisation ($\sim 20\%$).

2. The transitions $\pi^{\pm} \to \phi$ are not allowed by the diagrams. Hence $\sigma(\pi N \to \phi N) = 0$, $\sigma(\pi N \to \phi\Delta) = 0$, etc. This is in good agreement with experiment, as the production of ϕ mesons in πN scattering is known to be rare (20).

On the basis of these diagrams, many other predictions can be made, simply by searching for processes which cannot be described by a legal diagram. However, due to the difficulty of separating the real and imaginary parts of most scattering amplitudes, it is extremely hard to test many of the predictions. The ones that can easily be tested, for example (2) above, agree with experiment, but can also be derived using various combinations of assumptions, such as SU(3) invariance, universality, additivity of single-quark amplitudes, and cannot be considered as crucial tests of the basic diagram rules.

The assumptions involved in our basic diagram rules are stronger than the requirement that the exotic SU(3) amplitudes vanish in all channels. What we assume in addition is that $(4q + \bar{q})$ or

$(2q + 2\bar{q})$ intermediate states are illegal even if they happen to belong to a singlet or an octet.

We now examine the idea of exchange degeneracy which essentially stems from the idea of duality introduced in Chapter I. Consider, for example, the charge-exchange process $K^{o}p \rightarrow K^{+}n$. Now it is impossible to draw any legal diagram for this process, the channel being exotic, and indeed no resonances are known to exist in this channel, implying that the imaginary part of the amplitude vanishes. Duality suggests that the resonance contributions and the Regge exchanges are essentially the same. From the discussion of finite energy sum rules given earlier, we see that

$$\int_{0}^{N} \mathrm{Im}\, F_{res}\, d\upsilon = \int_{0}^{N} \mathrm{Im}\, F_{Regge}\, d\upsilon \qquad (2.11)$$

if $\mathrm{Im}\, F = \mathrm{Im}\, F_{Regge}$ for $\upsilon > N$ and $\mathrm{Im}\, F = \mathrm{Im}\, F_{Res}$ for $\upsilon < N$. Thus we require that $\mathrm{Im}\, F_{Regge} = 0$. This is readily achieved if the two t-channel trajectories ρ and A_2, of opposite signature, satisfy

$$\alpha_{\rho}(t) = \alpha_{A_2}(t), \qquad \beta_{\rho}(t) = -\beta_{A_2}(t) \qquad (2.12)$$

and if there is a similar relation between the u-channel Y_o^* and Y_1^* trajectories. The contributions of the exchange-degenerate partners to Im F then cancel. The mesons and also the Y^* resonances appear consistent with these requirements (see ref. 14).

Difficulties arise when we consider processes for which Pomeranchon exchange is allowed, as we indicated in Chapter I. It is clear from the positivity of cross-sections, that there is no trajectory which is exchange degenerate with the Pomeranchon. Thus, for example, the exotic $K^+p \to K^+p$ channel will have a regular Pomeranchon contribution, whereas the absence of resonances implies that the K^+p amplitude should be purely real. To overcome this problem, Harari (21) suggested that the Pomeranchon should not contribute to the direct channel resonance, but instead its contribution should be associated with the non-resonating background. The Pomeranchon is therefore excluded from the finite energy sum rule bootstrap and its contribution to an elastic amplitude may be regarded as the cumulative effect of all the inelastic channels. Such a separation between the Pomeranchon and the other Regge trajectories is possible because the finite energy sum rule bootstrap involves only a linear constraint. The situation obviously becomes much more complicated if one considers the non-linear relationships required by unitarity.

2.3 Incorporation of isospin.

We may use this interpretation of the Veneziano model via quark duality diagrams to incorporate isospin. We follow the treatment of Chan and Paton (22). As we have introduced it in section 2.1, the Veneziano amplitude is limited to only neutral mesons. Chan and Paton have proposed a very simple general method for incorporating isospin into the model, preserving all the desirable properties, and giving no unwanted states of high isospin, avoiding the presence of exotic resonances.

The Veneziano amplitude for the four-point function for spinless neutral particles takes the form (2.2),

$$T_4 = V(1,2,3,4) + V(1,2,4,3) + V(1,3,2,4) \qquad (2.2)$$

each term corresponding to a distinct non-equivalent ordering of the particles, and itself invariant under any cyclic permutation of the external lines.

We wish to introduce isospin into this four-point amplitude in a way which will ensure (i) cyclic symmetry for identical external particles; this guarantees crossing symmetry, and (ii) absence of poles with isospin larger than one. These poles would correspond to exotic resonances. This is dictated by experimental evidence. The solution takes the form of simple isospin factors multiplying the terms in the sum, (2.2), with each term and hence each cyclic ordering of the external lines, corresponding to a different isospin factor. In terms of the duality diagrams, we associate each incoming particle with the corresponding isospin (SU(2)) matrix, and the quark lines linking these particles with SU(2) contractions, each diagram representing an SU(2) trace. Each trace is then multiplied by the appropriate Veneziano term exhibiting dual poles in the Mandelstam variables concerned. We distinguish the following cases:

(a) Pions (I = 1) only as external lines. Give each external particle i the isospin label a_i ($a_i = 1,2,3$) and let τ_a ($a = 1,2,3$) denote the 2×2 Pauli matrices, where as usual $\pi_1 = \frac{1}{2}(\pi^+ + \pi^-)$, $\pi_2 = \frac{1}{2i}(\pi^+ - \pi^-)$, $\pi_3 = \pi^0$. Then the desired isospin factor for the term corresponding to the ordering (1 2 3 4) is simply the trace

$$\tfrac{1}{2}\,\mathrm{Tr}(\tau_{a_1}\ \tau_{a_2}\ \tau_{a_3}\ \tau_{a_4}) \tag{2.13}$$

The property (i) follows immediately from the properties of traces. The property (ii) follows since the product of any number of 2×2 matrices is again a 2×2 matrix and hence can represent only a combination of isospin 0 and isospin 1.

This four-point amplitude incorporating isospin has also been given by Lovelace (23).

As an immediate consequence of the isospin factor (2.13), we obtain the generalised exchange degeneracy between the isospin zero and one trajectories coupled to two pions (the ρ-f degeneracy). This follows simply from the identity

$$\tau_a\ \tau_b\ =\ \delta_{ab} + i\,\epsilon_{abx}\ \tau_x \tag{2.14}$$

which implies that any two-body pole occurs in both isospin states. However, when the summation in (2.2) is performed, the I = 0 and I = 1 poles will occur on opposite signature trajectories because of the opposite symmetry in the two terms of (2.14) under the interchange of a and b.

(b) Pions (I = 1) and kaons (I = $\frac{1}{2}$) as external lines. The rules in this case are most conveniently stated by introducing the 3×3 λ matrices of SU(3). We use these matrices, however, only as a device to ensure the correct isospin structure without making any SU(3) assumptions of mass degeneracy or coupling constant relations. Two each external line we associate a λ-matrix as follows

$$\pi_a \sim \begin{pmatrix} \tau_a & 0 \\ 0 & 0 \end{pmatrix} ; \quad K \sim \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} ; \quad \bar{K} \sim \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix} \tag{2.15}$$

Then the isospin factor corresponding to the ordering (1 2 3 4) is simply

$$\mathrm{Tr}(\lambda_{a_1} \lambda_{a_2} \lambda_{a_3} \lambda_{a_4}) \tag{2.16}$$

Equation (2.14) obviously satisfies conditions (i) and (ii) above, and in addition forbids poles with strangeness greater than one.

Taking particular combinations of external particles will imply further exchange degeneracies. For example, in the case where

K_1, $\bar{K}_2$, π_3, π_4 are the external lines. The isospin factors for the orderings (1 2 3 4) and (1 2 4 3) are given by (2.16) as $K_1 \bar{K}_2 \tau_{a_3} \tau_{a_4}$ and $K_1 \bar{K}_2 \tau_{a_4} \tau_{a_3}$, but zero for the ordering (1 3 2 4). The absence of a Veneziano term corresponding to the ordering (1 3 2 4) implies exchange degeneracy of the $I = \frac{1}{2}$ trajectories (K*(890) and K*(1400)).

2.4 Satellite terms.

We observe here that a term like

$$V(1\ 2\ 3\ 4) = \frac{\Gamma(m-\alpha_{12})\Gamma(n-\alpha_{23})}{\Gamma(m+n+p-\alpha_{12}-\alpha_{23})} \tag{2.17}$$

has the same basic properties as the expression given in equation (2.5). The first poles appear at $\alpha_{12} = m$ and $\alpha_{23} = n$ instead of zero (we will take m, n $\geqslant$ 1) and the asymptotic behaviours correspond respectively to

$$s_{12}^{\alpha_{23}-n-p} \quad \text{and} \quad s_{23}^{\alpha_{12}-m-p} \tag{2.18}$$

with $n+p$, $(m+p) > 0$; this corresponds to daughter behaviour. Poles in s_{12} (s_{23}) will have polynomial residuesin s_{23} (s_{12}) provided $p \leqslant 0$. We are therefore at liberty to add such terms, called "satellite", to the initially considered solution (2.5) without modifying any of the desired properties we require. Satellites can be added to eliminate unwanted daughter contributions and in particular odd daughters (24), if we want to show trajectories spaced by two units of spin as corresponding to the minimum number of trajectories. Satellites will not affect the leading high energy behaviour directly obtained from equation (2.5).

2.5 The Lovelace amplitude for $\pi\pi$ scattering.

We now consider the four-point amplitude of Lovelace (23) for pions, incorporating isospin. We consider $\pi^+\pi^-$ elastic scattering. The amplitude should show a string of poles in both s and t channels located on the linearly rising ρ and f trajectories. The $\pi^+\pi^+$ u-channel is exotic, implying exchange degeneracy of these two trajectories and resonances in both s- and t- channels should then be spaced by one unit of spin instead of two. We take for the $\pi^+\pi^-$ amplitude

$$V_{xy}(s,t) \;=\; -\beta\,\frac{\Gamma(1-\alpha_x(s))\;\Gamma(1-\alpha_y(t))}{\Gamma(1-\alpha_x(s)-\alpha_y(t)} \qquad (2.19)$$

where β is a constant and x and y label the trajectories in the s- and t-channel. Here both correspond to the ρ-f degenerate trajectories. We have taken (2.19) as the simplest tentative amplitude for $\pi^+\pi^-$ scattering, as opposed to more complicated terms involving satellites. Using the isospin formulation of section (2.3), we may now write the following amplitudes,

$$A(\pi^+\pi^- \to \pi^+\pi^-) = -\beta V_{\rho\rho}(s,t)$$

$$A(\pi^+\pi^0 \to \pi^+\pi^0) = -\frac{\beta}{2}(V_{\rho\rho}(s,t) + V_{\rho\rho}(t,u) - V_{\rho\rho}(u,s))$$

$$A(\pi^0\pi^0 \to \pi^0\pi^0) = -\frac{\beta}{2}(V_{\rho\rho}(s,t) + V_{\rho\rho}(t,u) + V_{\rho\rho}(u,s))$$

$$(2.20)$$

An important point concerning equation (2.19) is that the amplitude vanishes when

$$\alpha_\rho(s) + \alpha_\rho(t) = 1 \qquad (2.21)$$

Now we recall that Adler's self-consistency condition (25) derived from current algebra, states that the amplitude for $\pi\pi$ scattering vanishes in the limit of zero four-momentum of one of the pions, the remaining pions being kept on the mass-shell. In this limit $s = t = u = m_\pi^2$. Assuming linear trajectories, we see that (2.21) can be identified with the Adler zero if $\alpha_\rho(\mu^2) = \frac{1}{2}$, provided $\mu^2 = m_\pi^2$, where m_π is the physical pion mass. Taking the ρ-mass as 764 MeV, this gives $\alpha_\rho(0) = 0.48$ which is in good agreement with the intercept of the ρ-trajectory, determined empirically from Regge fits.

The Veneziano model was greatly strengthened when it was generalised, in certain idealised cases, to multiparticle production processes involving an arbitrary number of external lines, where all the properties (1, ..., 6) listed at the beginning of this chapter are consistently maintained. With these generalisations, the real power of the Veneziano amplitude become evident, as they allow to treat on a similar footing two-body collisions and multiparticle reactions, for which we need a common framework before unitarity can be fully used. We now describe this extension.

2.6. Generalisation to multiparticle production processes

This extension has been given by a number of authors independently (26, 27, 28, 29, 30, 31, 32). The treatment in this section follows that of Chan (28, 29).

We again consider an idealised system with only bosons and without internal symmetry. All bosons in the model belong to the same family, i.e. they lie either on the parent trajectory which contains a spinless particle, or on its daughters. The extension of the Veneziano model to the N-point function with spinless external particles is a straightforward generalisation of the equations (2.2) - (2.6). In analogy to (2.2) one first writes

$$T_N = \sum V(1, 2, \ldots, N) \tag{2.22}$$

where the sum runs over all the non-equivalent orderings of the external lines, there being altogether $(N-1)!/2$ terms. We need consider for the moment only one such term, say that corresponding to the ordering $(1, 2, \ldots, N)$. This is expected to be invariant under any cyclic permutations of the external lines so that complete crossing symmetry of (2.22) is guaranteed.

We require that V should contain poles corresponding to the trajectory α in all possible Mandelstam channels which can be formed in the diagram of figure 2.3 without changing the order of the external lines. We may enumerate these channels by means of dual diagrams (33). To the diagram of figure 2.3 we associate an N-sided polygon, as shown in figure 2.4. Possible Mandelstam channels for figure 2.3 are then in one-one correspondence with diagonal lines joining any two vertices of the polygon. To each diagonal $P = (i,j)$, we associate a dynamical variable

$$x_P = x_{i,j} = -1 - \alpha_{i,j} \tag{2.23}$$

where
$$\left.\begin{aligned} \alpha_{i,j} &= \alpha_0 + \alpha' s_{i,j} \\ s_{i,j} &= (P_i + P_{i+1} + \dots + P_j)^2 \end{aligned}\right\} \tag{2.24}$$
and

For each $P = (i,j)$, we introduce a variable u_P conjugate to x_P .

Two Mandelstam channels are said to be dual to each other if they correspond to diagonals which intersect on the dual diagram. This is just a generalisation of the usual concept of duality in the four-point function (14) between, say, the s and t channels which are clearly dual in the sense defined above.

Clearly dual channels cannot have coincident poles, since no Feynman tree diagram exists with both trajectories as internal lines, whereas between non-dual channels, coincident poles are possible. To ensure this, the equation (2.6) generalises to the set of equations

$$u_P = 1 - \prod_{\bar{P}} u_{\bar{P}} \qquad (2.25)$$

where $\bar{P}$ runs over all the channels dual to P. It is clear then that two variables corresponding to dual channels cannot vanish simultaneously. Although (2.24) represents as many equations as there are variables, (i.e. $N(N-3)/2$), they are not all independent. The whole set can be solved in terms of (N-3) independent variables. Now, by definition, independent variables can vanish simultaneously. They must therefore correspond to allowed coincident poles, or equivalently to non-intersecting lines on the dual diagram. Although in principle any set of N-3 non-dual variables can be chosen as independent, it happens that the most convenient are those sets corresponding to poles in a multiperipheral diagram, for example $u_{1,j}$ $(j = 2,3,\ldots,N-2)$ corresponding to the diagram figure 2.5, or the associated dual diagram figure 2.6. In terms of $u_{1,j}$ as independent variables, the solution of (2.25) yields:

$$u_{p,q} = \frac{(1 - w_{p,q-1})(1 - w_{p-1,q})}{(1 - w_{p-1,q-1})(1 - w_{p,q})} \qquad (2.26)$$

where $w_{r,s} = u_{1,r}\, u_{1,r+1} \cdots u_{1,s}$ (2.27)

and $u_{1,1} = u_{1,N-1} = 0$, by definition.

We may illustrate this for the case N = 5. Equations (2.25) then become

$$u_{1,2} = 1 - u_{5,1}\, u_{2,3}\ ; \qquad u_{2,3} = 1 - u_{1,2}\, u_{3,4}$$

$$u_{3,4} = 1 - u_{2,3}\, u_{4,5}\ ; \qquad u_{4,5} = 1 - u_{3,4}\, u_{5,1}$$

$$u_{5,1} = 1 - u_{4,5}\, u_{1,2} \tag{2.28}$$

The solutions for $u_{2,3}$, $u_{3,4}$ and $u_{5,1}$ in terms of $u_{1,2}$ and $u_{4,5}$ ($\equiv u_{1,3}$) are from equation (2.26),

$$u_{2,3} = \frac{1 - u_{1,2}}{1 - u_{1,2}\, u_{4,5}}\ ; \qquad u_{3,4} = \frac{1 - u_{4,5}}{1 - u_{1,2}\, u_{4,5}}$$

$$u_{5,1} = 1 - u_{1,2}\, u_{4,5} \tag{2.29}$$

Under a cyclic change of independent variables from $u_{1,j}$ $(j = 2,\ldots,N-2)$, (figure 2.5), to $u_{2,j}$ $(j = 3,\ldots,N-1)$, (figure 2.7), the Jacobian can be written as

$$\Delta_N = (-1)^N \left(\frac{J_2}{J_1} \right) \tag{2.30}$$

where $$J_1 = \prod_{i<j} (u_{i,j})^{j-i-1}, \qquad \begin{array}{l} i = 2,3,\ldots,N-2 \\ j = 3,4,\ldots,N-1 \end{array}$$

and $$J_2 = \prod_{i<j} (u_{i,j})^{j-i-1}, \qquad \begin{array}{l} i = 3,4,\ldots,N-1 \\ j = 4,5,\ldots,N \end{array}$$

$$(2.31)$$

Moreover, in terms of $u_{2,j}$ $(j = 3, \ldots, N-1)$ we may express the dependent variables as

$$u_{p,q} = \frac{(1 - v_{p,q-1})(1 - v_{p-1,q})}{(1 - v_{p-1,q-1})(1 - v_{p,q})} \tag{2.32}$$

where $$v_{r,s} = u_{2,r}\, u_{2,r+1} \cdots u_{2,s} \tag{2.33}$$

and $$u_{2,2} = u_{2,N} = 0 \quad \text{by definition,}$$

which is the same as (2.26) after a cyclic permutation of the external lines.

We may now write the generalisation of (2.5a) as

$$B_N(1,2,\ldots,N) = \int_0^1 \cdots \int_0^1 \prod_{j=2}^{N-2} du_{1,j} \left(\frac{1}{J_1} \right) \prod_P (u_P)^{x_P} \tag{2.34}$$

where the last product is taken over all the $N(N-3)/2$ channels. Under a change in variables in (2.34) from $u_{1,j}$ to $u_{2,j}$, we obtain from (2.30)

$$B_N(1,2,\ldots,N) = \int_0^1 \cdots \int_0^1 \prod_{j=3}^{N-1} du_{2,j} \left(\frac{1}{J_2} \right) \prod_P (u_P)^{x_P} \tag{2.35}$$

with u_P given by (2.32). This means that B_N is invariant under a cyclic permuation of the N external lines, thus guaranteeing complete crossing symmetry of our full amplitude (2.22). Using the same arguments as references (26), (28) and (29), which are similar to those for the four-point function we have already given, we may show that B_N as given by (2.34) is analytic in $x_{1,j}$ except for simple poles at $x_{1,j} = -1, -2, \ldots$. By cyclic invariance just proved above this is extended to x_P for all channels P, (analyticity and duality). Moreover, because of the conditions (2.25) there are no unwanted coincident poles.

The residue at the $(\ell+1)^{th}$ pole in $x_{1,j}$ is given, in analogy to (2.8) as,

$$\operatorname*{Res}_{\alpha_{1,j}=\ell} \left\{ B_N(1,2,\ldots,N) \right\} = \left[\frac{1}{\ell!} \frac{d^\ell}{du_{1,j}^{\ \ell}} (Q) \right]_{u_{1,j}=0} \tag{2.36}$$

where $$Q = \int_0^1 \prod_{\substack{k=2\\k\neq j}}^{N-2} du_{1,k} \left(\frac{1}{J_1} \right) \prod_{P' \neq (1,j)} (u_{P'})^{x_{P'}} \tag{2.37}$$

Since when $u_{1,j} = 0$, $u_{\bar{P}} = 1$ for all $\bar{P}$ dual to $(1,j)$, the differentiation in (2.36) can yield at most a polynomial of total degree ℓ in the variables $x_{\bar{P}}$, which are momentum transfers in the channel $(1,j)$. This means of course that the pole at $\alpha_{1,j} = \ell$ has maximum spin ℓ, giving rise to a linear trajectory. The cyclic symmetry of B_N then implies such a trajectory for all Mandelstam channels.

In the limit when $x_{i,i+1} \to \infty$ $(i = 2,\dots,N-2)$ at fixed values of $x_{1,i}$ $(i = 2,\dots,N-2)$ and

$$K_j = (x_{j-1,j} \cdot x_{j,j+1}) / x_{j-1,j+1}, \qquad j = 3,\dots,N-2 ,$$

namely the multi-Regge limit corresponding to figure 2.5 with 1 and N as incoming particles, it has been shown by a straightforward extension of the proof given by Bardakci and Ruegg (26) for the Reggeisation of the five-point function that $B_N(1,\dots,N)$ has the proper Regge limit

$$B_N \to \prod_{i=2}^{N-2} (x_{i,i+1})^{-x_{1,i-1}} G_N(x_{1,2},\dots,x_{1,N-2};\ K_3,\dots,K_{N-2}) \tag{2.38}$$

where the Residue G_N factorises (34). The asymptotic form of (2.38) is by itself insufficient to guarantee the correct multi-Regge behaviour of the full amplitude (2.22). For the special case of $N = 5$, it has been shown that of the twelve terms in (2.22), four have the double-Regge limit and give the signature factors as required (35, 36), while the rest vanish exponentially as the term $V(1, 2, 4, 3)$ does in the four-point function. For general N, a similar result is extremely likely.

Our general N-point amplitude thus has all the expected properties (1, ... 6) listed in section 2.1 .

The functions B_N also have the attractive property of being consistent with the bootstrap hypothesis. Consider, for example, the pole in $B_N\,(1,2,\ldots,N)$ at $\alpha_{1,j} = 0$. Since in our present idealised system, all trajectories are identical, this pole should represent the same spinless particle as the external particles we started with. Bootstrap consistency would require that the residue at the pole factorises into beta functions of lower orders, or explicitly,

$$\operatorname*{Res}_{\alpha_{1,j}=0} B_N\,(1,2,\ldots,N) = B_{j+1}\,(1,2,\ldots,j;\, I) \times B_{N-j+1}(I,\, j+1,\ldots,N) \tag{2.39}$$

That this is indeed so may readily be seen from equations (2.36) and (2.37) with $\ell = 0$. Since, on putting $u_{1,j} = 0$ according to (2.25) $u_{\bar{P}} = 1$ for all $\bar{P}$ dual to $P = (1,j)$, one essentially cuts the N-point tree diagram into halves corresponding exactly to the reduction in figure (2.8).

We may extend the bootstrap idea further. The same arguments as we have used above would lead us to interpret the residue (2.36) at $\alpha_{1,j} = \ell$ as the amplitude for a (j+1)-particle process in which a particle with a 'spin-like' quantum number ℓ is produced, the particle then decaying into (N-j) spinless ones. By a 'spin-like' quantum number, we mean here, as in the original Veneziano model, not a pure spin state of angular momentum ℓ, but a mixed state with maximum spin ℓ, plus a whole sequence of daughters with angular momentum less than ℓ. Moreover, each component with a definite spin may still contain several levels. The question then arises whether it is consistent at all to consider the pole at $\alpha_p = \ell$ as representing a finite number of single particle states, and if so, what is the spectrum or level structure.

If a pole represents a single level only, then unitarity requires that its residue should factorise. If it represents a finite number of single particle states, then the residue should be expressible as the sum of a finite number of factorised terms. A first requirement for consistency in our present problem, therefore, would be that the residue at $\alpha = \ell$ be expressible as a sum of factors, where the number of terms does not increase indefinitely with the number of external lines. That this requirement is satisfied has been shown by Fubini and Veneziano (37) and Bardakci and Mandelstam (38).

Because of the cyclic symmetry of the functions B_N, it is sufficient to establish this assertion for one Mandelstam channel, say, $P = (1,j)$. Starting from an expression for B_N first derived by Bardakci and Ruegg(34)

$$B_N(1,2,\ldots,N) = \int_0^1 du_{1,2}\, du_{1,3} \ldots du_{1,N-2} \cdot u_{1,2}^{-\alpha_{1,2}-1} \ldots u_{1,N-2}^{-1-\alpha_{1,N-2}}$$

$$\times\ (1-u_{1,2})^{-1-\alpha_{2,3}}(1-u_{1,3})^{-1-\alpha_{3,4}} \ldots (1-u_{1,N-2})^{-1-\alpha_{N-2,N-1}}$$

$$\times\ (1-w_{2,3})^{-2\alpha'(p_2\cdot p_4)+\alpha_0+\alpha'} \ldots (1-w_{N-3,N-2})^{-2\alpha'(p_{N-3}\cdot p_{N-1})+\alpha_0+\alpha'}$$

$$\times (1-w_{2,4})^{-2\alpha' p_2\cdot p_5} \ldots (1-w_{N-4,N-2})^{-2\alpha'(p_{N-4}\cdot p_{N-1})}$$

$$\times\ \ldots (1-w_{2,N-2})^{-2\alpha'(p_2\cdot p_{N-1})}\ , \qquad (2.40)$$

using the notation introduced earlier in this section, these authors show that B_N can be written in the form

$$B_N(1,2,\ldots,N) = \int_0^1 du' \int_0^1 du''\ \Phi'(u',p')\ \Phi''(u'',p'')$$

$$\times \int_0^1 du_{1,j}\ F(u_{1,j};\ u',\ p';\ u'',\ p'')\ (u_{1,j})^{-\alpha_{1,j}-1}$$

$$(2.41)$$

where u' and p' refer to variables to the left of the line (1,j) in figure 2.5, namely: $u_{1,k'}$ ($k' = 2,\ldots,j-1$) and $p_{i'}$ ($i' = 1,\ldots,j$), while u" and p" refer to the variables to the right of (1,j), namely: $u_{1,k''}$ ($k'' = j+1,\ldots,N-2$) and $p_{i''}$ ($i'' = j,\ldots,N$). Moreover, they were able to show that the function F in equation (2.41) can be expressed in the form:

$$F = \exp\left[\sum_{n=1}^{\infty} \frac{(u_{1,j})^n}{n} G_n \right] \tag{2.42}$$

with

$$G_n = P'^{(n)}_{\mu} P''^{(n)}_{\mu} + c \tag{2.43}$$

where c is a constant and $P'^{(n)}$ ($P''^{(n)}$) are four-vectors depending only on the variables u', p' (u", p"). Expanding F in (2.42) as a power series in $u_{1,j}$ thus gives us the residues automatically as a sum of factorised terms. In particular, the residue at $\alpha_{1,j} = \ell$ is given by the terms with $(u_{1,j})^{\ell}$, in the expansion of (2.42). Clearly, the number of factors of the form (2.43) contained in such a term depends only on the functional form of (2.42) and not on the number of external lines. This is what we wished to prove.

Although from the arguments above one is able to show that any pole in the model can be consistently interpreted as a superposition of a finite number of factorised levels, the number of levels required is extremely large. It can readily be seen that the number of factors for a given ℓ is the same as the number of ways of choosing non-negative integers ℓ_k which satisfy the equation

$$\ell_1 + 2\ell_2 + 3\ell_3 + \ldots + k\ell_k + \ldots = \ell \quad . \tag{2.44}$$

For large ℓ, this number increases as $\exp a\sqrt{\ell}$ with $a = 2\pi/\sqrt{6}$. Thus the number of levels increases exponentially with the centre of mass energy.

We may note further, from equations (2.41) - (2.43) that the residues at the poles are expressed as scalar products of four-dimensional tensors. This reflects the Lorentz invariance of the present approach and explains the lack of kinematical singularities. However, because of the minus sign of the Lorentz metric, this also means that the time components of these tensors will in general have negative residues, and correspond to ghost states. Fubini and Veneziano (37) were able to show, with the help of a Ward-like identity inherent in the beta function, that leading ghosts at least are compensated by similar poles with positive residues. However, ghosts on lower trajectories are not so compensated.

2.7 Alternative forms.

Several equivalent forms of the generalised beta function have been suggested which are found to be useful for different purposes

(a) Series form of Hopkinson and Plahte (30).

By expanding the integral of (2.34) in a power series in various ways, one can obtain B_N in terms of an infinite series of beta functions of lower order. Such series expansions, considered in some detail by Hopkinson and Plahte (30) yield a practical iterative method for the numerical evaluation of the beta functions. In particular, for the five-point function of Bardakci and Ruegg (26), (see also (39)),

$$B_5(1,..,5) = \int_0^1 \int_0^1 du_{12}\, du_{45}\, (u_{12})^{-\alpha_{12}-1} (u_{45})^{-\alpha_{45}-1}$$

$$\times \left(\frac{1 - u_{12}}{1 - u_{12}u_{45}} \right)^{-\alpha_{23}-1} \left(\frac{1 - u_{45}}{1 - u_{12}u_{45}} \right)^{-\alpha_{34}-1}$$

$$\times\ (1 - u_{12}u_{45})^{-\alpha_{51}-2} \tag{2.45}$$

by expanding the term in $(1 - u_{12}u_{45})$ in a binomial series, we may obtain

$$B_5(1,\ldots,5) = \sum_{k=0}^{\infty} (-1)^k \binom{z}{k} B_4(x_{34},\ x_{45}+k)\ B_4(x_{12}+k,\ x_{23}) \tag{2.46}$$

where $$z = x_{51} - x_{23} - x_{34} , \tag{2.47}$$

and where we have written x_{ij} for $x_{i,j}$.

Using the gamma function representation of B_4 , this may be rewritten as

$$B_5(1,\ldots,5) = B_4(x_{12}, x_{23})\, B_4(x_{34}, x_{45}) \times {}_3F_2(x_{12}, x_{45}, -z;\ x_{12}+x_{23}, x_{34}+x_{45};\ 1) \tag{2.48}$$

where ${}_3F_2(a_1, a_2, a_3;\ b_1, b_2;\ 1)$ is a standard hypergeometric series. This series converges when

$$\mathrm{Re}\,[\,1 + (x_{12} + x_{23}) + (x_{34} + x_{45}) - x_{23} + z - x_{45}\,] > 1 \tag{2.49}$$

i.e. when $\mathrm{Re}(x_{51})$ is positive. Thus we have found a representation for B_5 which has a much larger range of convergence than the integral. This series (2.48), is the starting point for any method of calculating B_5 numerically (40). This has been done, and the programme used successfully in the phenomenological study of five-line processes (41, 42) .

(b) Koba-Nielsen parametrisation.

An interesting representation considered by Koba and Nielsen (32) is based on their deduction that each variable u_P introduced in section 2.6 can be expressed as an anharmonic ratio of four out of N-points on the unit circle, where the N points correspond to the N external lines, such that the duality constraints (2.25) are automatically satisfied. We write

$$u_{ij} = \frac{(z_i - z_j)(z_{j+1} - z_{i-1})}{(z_i - z_{j+1})(z_j - z_{i-1})} \tag{2.50}$$

where the points z_i are in one-one correspondence with the external particles. This parametrisation enables us to write the term in the N-point amplitude corresponding to the ordering of the particles $(1,2,\ldots,N)$ in the form of a cyclic symmetric contour integral

$$B_N(1,2,\ldots,N) = \int \cdots \int dG_3^{-1} \prod_{k=1}^{N} dz_k \prod_{1 \leqslant i < j \leqslant N} (z_j - z_i)^{Y_{ij}} \tag{2.51}$$

where $z_j = \exp(i\,\theta_j)$ are N ordered points on the unit circle in the complex z-plane such that:

$$\theta_1 < \theta_2 \ldots < \theta_N < \theta_1 + 2\pi \tag{2.52}$$

The Y_{ij} are defined by

$$Y_{ij} = \begin{cases} -\alpha_{ij} - 1 & ; \quad i,j \text{ adjacent lines} \\ -\alpha_{ikj} + \alpha_{ik} + \alpha_{kj} & ; \quad j = k+1 = i+2 \\ -\alpha_{ik_1\ldots k_r j} + \alpha_{ik_1\ldots k_r} + \alpha_{k_1\ldots k_r j} - \alpha_{k_1\ldots k_r} & ; \end{cases}$$

$$j = k_r + 1 = \ldots = k_1 + r = i + r + 1 \tag{2.53}$$

Here, we have labelled a channel with the indices of all those particles appearing in that channel. Where a term is undefined, such as $\alpha_{ik_1\ldots k_r j}$ for $r+2 > N-2$, it is set equal to zero.

Taking the case $r \geqslant 2$ and $j - i = r+1 \leqslant N-3$, so that each of the α's in the last line of equation (2.53) is defined, and assuming all trajectories are identical, linear of the form

$$\alpha_{i_1 i_2 \ldots i_r} = \alpha_0 + \alpha'(p_{i_1} + p_{i_2} + \ldots + p_{i_r})^2 \qquad (2.54)$$

then the kinematic conditions arising from four-momentum conservation demand that Y_{ij} takes the simple form

$$Y_{ij} = -2\alpha' p_i \cdot p_j \qquad (2.55)$$

The ordering of the points z_k is the same as the ordering of the particles, and z_k may be thought of as representing particle k. The differential dG_3 is defined by

$$dG_3 = \frac{dz_r \, dz_s \, dz_t}{(z_s - z_r)(z_t - z_s)(z_t - z_r)} \qquad (2.56)$$

where z_r, z_s and z_t are three arbitrary but fixed points such that:

$$\theta_r < \theta_s < \theta_t < \theta_r + 2\pi \qquad (2.57)$$

The integration in the remaining (n-3) variables is over those parts of the circle consistent with the constraint (2.52). Our expression (2.51) is independent of the choice of r, s and t and of the values assigned to z_r, z_s and z_t. It may readily be shown to be exactly equivalent to the Chan form, equation (2.34).

The expression (2.51) has also been given by Plahte (43). Apart from a constant factor, it is equal to the formula given by Koba and Nielsen (32). An elegant interpretation of this representation has been given by these authors using the formalism of projective geometry.

The representation of the N-point amplitude given here is very compact and manifestly crossing symmetric, and has been particularly useful in certain aspects of the formal developments of dual theories, for example in the problem of factorisation (44) and in the functional approach to dual theories (45).

2.8 An application of the multi-Veneziano amplitude.

An interesting application of the five-point function has been given by Rubinstein, Squires and Chaichian (46) in the study of the process ($\bar{p}n$) threshold annihilation into three pions.

The process was first studied by Lovelace (23) who used the fact that at threshold the $\bar{p}n$ system has the same quantum numbers as the pion to relate this process to $\pi\pi$ scattering, one of the pions having $(\text{mass})^2$ of $4M^2$, M the nucleon mass. However, the method of extrapolation was arbitrary and no justification was given. A single Veneziano $\pi\pi$ term was used, but instead of a leading one, a satellite without the leading trajectory was taken , so the amplitude did not have the correct Regge behaviour. Although some of the features of the data were predicted, particularly the dip in the centre of the Dalitz plot of the two final $\pi^+\pi^-$ combinations, the fit was not very satisfactory.

Rubinstein, Squires and Chaichian start from the assumption that, when the external particles lie on leading trajectories, a good approximation to the amplitude is provided by the leading Veneziano terms. It is now necessary to construct physically acceptable five-point functions. They must satisfy the following conditions: all desired poles, leading Regge behaviour in all channels, no spin-zero ghosts when trajectories have positive intercepts.

These authors then demand that the relevant piece of the five-point function, i.e. the invariant non-flip amplitude, reduces to the leading term in each channel when we go to a pole on a leading trajectory. In particular, this gives the important restriction that the amplitude does not have the nucleon pole in both baryon channels simultaneously, since otherwise we would obtain an incorrect $\pi N \rightarrow \pi N$ non-flip amplitude.

They take for that part of the amplitude which has poles in the $N\bar{N}$-channel, corresponding to the configuration of figure 2.9

$$A = \alpha_{12}^{\rho}\, F(\alpha_{12}^{\rho},\ \alpha_{23}^{\rho} - 1,\ \alpha_{34}^{B} - \tfrac{1}{2},\ \alpha_{45}^{\pi}\ \ \alpha_{15}^{B} - 3/2)$$
$$+\ c(\alpha_{34}^{B} - \tfrac{1}{2})F(\alpha_{12}^{\rho} - 1,\ \alpha_{23}^{\rho} - 1,\ \alpha_{34}^{B} - \tfrac{1}{2},\ \alpha_{45}^{\pi} - 1,\ \alpha_{15}^{B} - \tfrac{1}{2})$$
$$+\ \ldots\ldots \qquad (2.58)$$

where c is a constant and the terms not written come from non-cyclic reordering of the external particles of figure 2.9, and

$$F(\alpha_{12},\ \alpha_{23},\ \alpha_{34},\ \alpha_{45},\ \alpha_{51}) \equiv B_5(1,\ 2,\ 3,\ 4,\ 5) \qquad (2.59)$$

with B_5 given by (2.45). α^{B} refers to either the N or the Δ trajectory.

It is now necessary to evaluate A at threshold, i.e. $s_{45} = 4M^2$. At this point, assuming linear trajectories of universal slope, it is a good approximation to take $\alpha_{45}^{\pi} = 3 (\approx \alpha' \,.\, 4M^2)$ so that the amplitude is given by the residue of the pole at $\alpha_{45}^{\pi} = 3$. This immediately eliminates all terms coming from different reordering in figure 2.9, as the only other reordering which contains the s_{45} poles are those having exotic mesons, which we assume do not exist.

Using this amplitude, a very good fit to the data is obtained, with our one free parameter c. This may be regarded as good evidence for the validity of using the leading five-point term.

2.9 Difficulties of the naive model.

In spite of incorporating many desirable properties in a simple compact form, the Veneziano model is not free from difficulties. By far the most serious of these are (i) the question of unitarity and (ii) the problem of ambiguities.

(i) The model amplitude (2.34) is not unitary. This can easily be seen since the function has only poles on the real axis, whereas a unitary amplitude should have cuts corresponding to thresholds of elastic and inelastic channels, while its poles, corresponding to resonances, should move off the real axis onto the unphysical Riemann sheet.

Attempts to 'unitarise' the Veneziano model have been made. On a phenomenological level one may simulate the effects of unitarity by introducing an imaginary part to the trajectory function α. The poles of the model amplitude will then move off the real axis, but their residues will now, in general, not be polynomials in the momentum transfers, and the foregoing arguments will only be approximately valid.

A further approach aims at what amounts to a complete theory of strong interactions. The Veneziano amplitude is considered as the Born term in a perturbation expansion in which duality is consistently maintained. General rules for constructing loop diagrams consistent with duality which correspond to the higher terms in the perturbation series, have been given by Kikkawa, Sakita and Virosoro (47). In order to guarantee unitarity in the sense of perturbation theory, the internal propagators of loop diagrams should represent a complete sum over all possible intermediate states, while the coupling constants of the internal lines should be identical to those occurring in the tree diagrams. A study of the level structure and factorisation properties of tree diagrams is therefore necessary. This was initiated by Fubini and Veneziano (37) and Bardakçi and Mandelstam (38), as we have discussed in section 2.6, and extended by Fubini, Gordon and Veneziano (48) using an elegant operator formalism. This allows one to construct all diagrams with any number of closed loops for this idealised system. However, such diagrams were found to be badly divergent (49). Possible cases have recently been proposed by Oleson (50) and by Neveu and Scherk (51).

(ii) A natural question to ask is whether our expression (2.34), satisfying the properties (1,...,6) listed at the beginning of this chapter is in any sense unique. That this is not so has been demonstrated for the four-point amplitude by the ambiguity due to satellite terms, discussed earlier in section 2.4. A similar ambiguity exists in the generalised Veneziano model. One may choose to modify the integrand in equation (2.34) by a function $f_N(u_P)$. So long as $f_N(u_P)$ is cyclic invariant and well behaved in the region of integration, the resultant function will have the same pole structure and asymptotic behaviour. The factorisation requirement will impose additional constraints on $f_N(u_P)$. However, considerable freedom still remains. The problem has been investigated by Gross (52) who found that the number of levels increases in general when satellites are introduced, but remains finite for a wide class of functions $f_N(u_P)$ which may be used to modify the integrand in (2.34). In other words, the consistency condition concerning the number of daughter levels removes but little of the large ambiguity due to satellites.

In addition to the above theoretical difficulties, the idealised system which provides the framework for the Veneziano model is over simplified and should be treated only as a theoretical laboratory. To construct a more realistic system we should have to take into account properties such as (a) internal symmetry, (b) positive intercept trajectories, (c) meson trajectories with abnormal parity, (d) baryon trajectories.

The solution to (a) to incorporate isospin has already been discussed for the four-point function in section 2.3. The extension to the n-point amplitude is straightforward and is given by Chan and Paton (22). The solution again takes the form of simple isospin factors multiplying the terms in the summation (equation (2.22)) with each term and hence each cyclic ordering of the external lines, corresponding to a different isospin factor. As well as maintaining the conditions (i) and (ii) given in section 2.3, we also wish to ensure factorisation so as to preserve bootstrap consistency.

For the case of pions (I = 1) only as external lines, the desired isospin factor for the term corresponding to the ordering (1, 2, ..., N) is simply the trace,

$$\tfrac{1}{2}\,\mathrm{Tr}(\tau_{a_1}\;\tau_{a_2}\;\ldots\;\tau_{a_N}) \tag{2.60}$$

where the τ's are as defined in section 2.3.

The properties (i) and (ii) follow as before. To see the factorisation property, we note the following identity:

$$\begin{aligned}\tfrac{1}{2}\,\mathrm{Tr}(\tau_{a_1}\;\ldots\;\tau_{a_N}) &= [\tfrac{1}{2}\,\mathrm{Tr}(\tau_{a_1}\;\ldots\;\tau_{a_M})]\;[\tfrac{1}{2}\,\mathrm{Tr}(\tau_{a_{M+1}}\;\ldots\;\tau_{a_N})]\\ &+\sum_{a_I}[\tfrac{1}{2}\,\mathrm{Tr}(\tau_{a_1}\;\ldots\;\tau_{a_M}\;\tau_{a_I})]\;[\tfrac{1}{2}\,\mathrm{Tr}(\tau_{a_I}\;\tau_{a_{M+1}}\;\ldots\;\tau_{a_N})]\end{aligned} \tag{2.61}$$

where τ_{a_I} are again the Pauli matrices. The two terms in equation (2.61) correspond respectively to intermediate states of isospin 0 and 1.

The extension to include kaons as external lines is straightforward using the Gell-Mann λ matrices. The isospin factor corresponding to the ordering $(1,\ldots,N)$ is then simply

$$\mathrm{Tr}(\lambda_{a_1}\ \lambda_{a_2}\ \ldots\ \lambda_{a_N}) \qquad (2.62)$$

One might attempt a similar solution for the problems (b) and (c), say, for example, by taking traces of γ-matrices instead of τ and λ. Unfortunately this leads to unphysical solutions with parity degenerate doublets and ghost states. However, if we regard our model as only the Born term to a future unitarised theory, such features might not be unacceptable and even necessary in a purely mesonic system with linear trajectories and without unitarity (53).

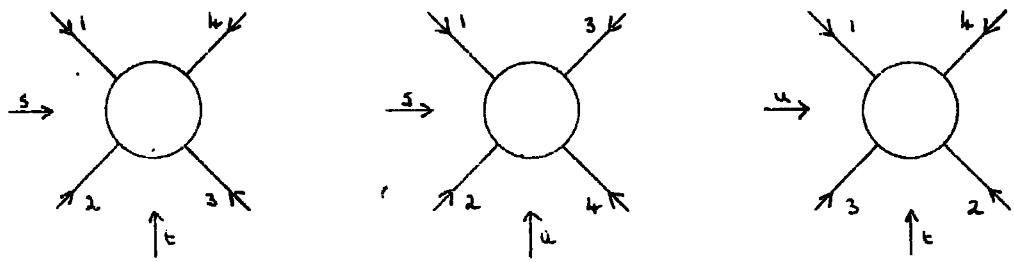


Figure 2.1 Diagrams corresponding to the three terms in the summation (2.2). We may equivalently write

$$T = V(s,t) + V(u,s) + V(t,u).$$

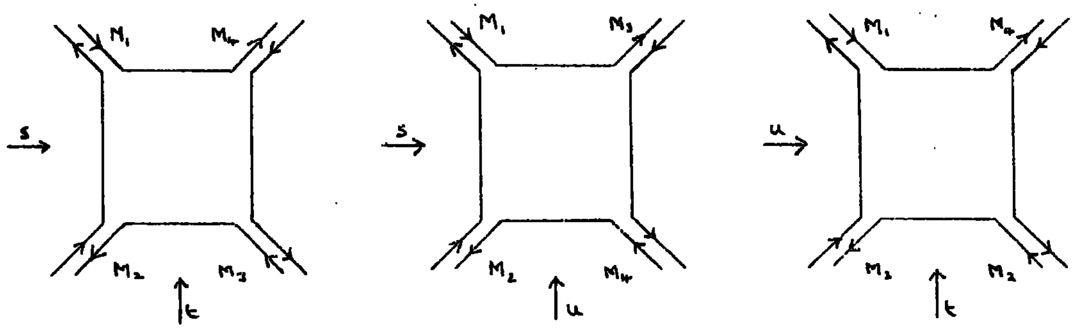


Figure 2.2a. Quark duality diagrams for forward meson-meson scattering.

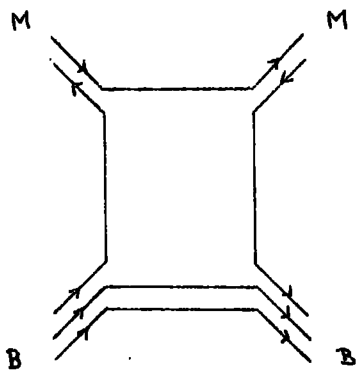


Figure 2.2b

Quark duality diagram for forward meson-baryon scattering

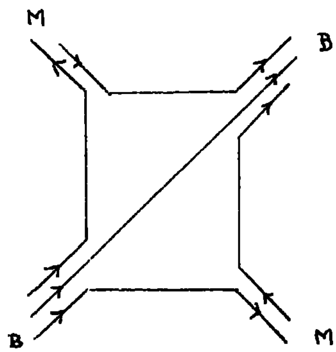


Figure 2.2c

Quark duality diagram for backward meson-baryon scattering

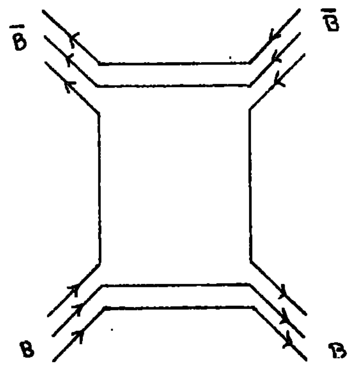


Figure 2.2d

Illegal quark duality diagram for baryon-antibaryon scattering

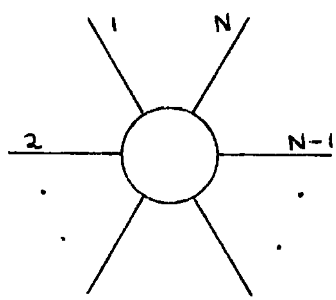


Figure 2.3 Diagram representing the term in the N-point amplitude which corresponds to the ordering (1, 2, ..., N) of the external lines

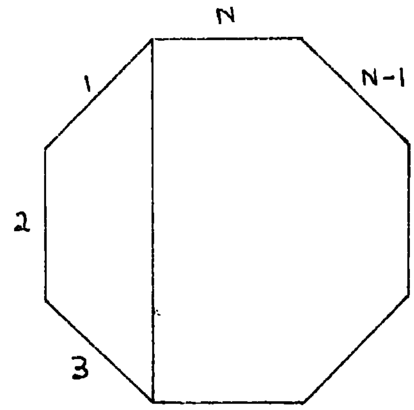


Figure 2.4 The dual diagram associated with figure 2.3.
The diagonal shown is denoted by the indices (1,3).

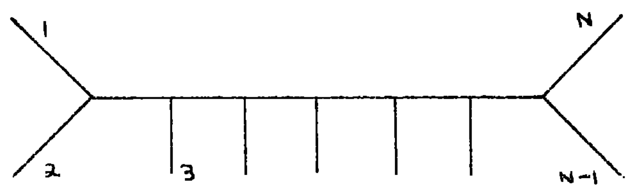


Figure 2.5 Diagram representing the set of independent variables $u_{1,j}$ $(j = 2,3,\dots,N-2)$

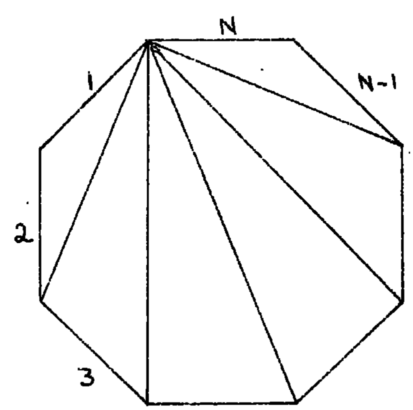


Figure 2.6 The set of variables $u_{1,j}$ $(j = 2,\dots,N-2)$ of
of figure 2.5, represented on the dual diagram.

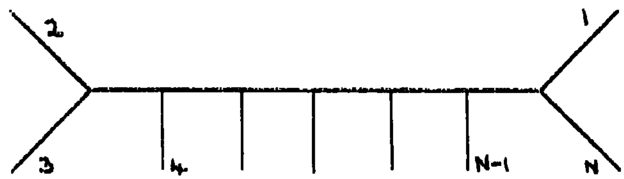


Figure 2.7 Diagram representing the set of independent variables $u_{2,j}$ $(j = 3,\ldots,N-1)$

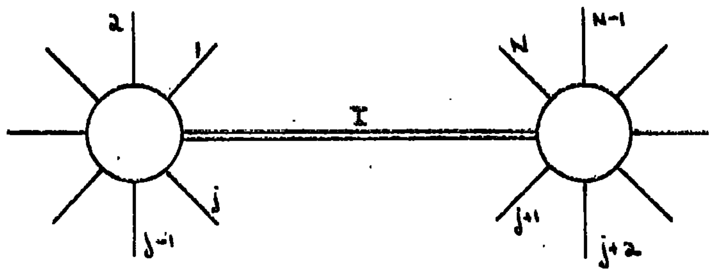


Figure 2.8 Diagram illustrating the bootstrap consistency of the N-point amplitude

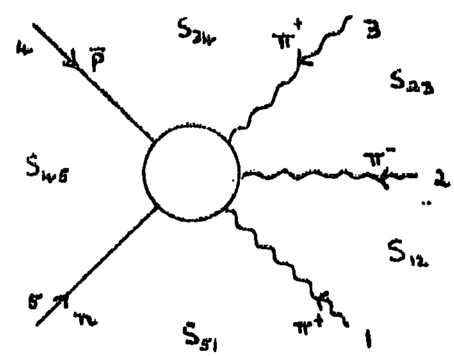


Figure 2.9 $\bar{p}n$ threshold annihilation into three pions

CHAPTER III

A CONTRACTED FORM OF THE GENERALISED VENEZIANO AMPLITUDE

In this chapter we show how we may incorporate all the $\frac{1}{2}(N-1)!$ terms of the multi-Veneziano formula, (2.34), in one term, provided certain trajectory constraints are satisfied, thus constructing an amplitude invariant under all $N!$ permutations of the external lines. This work is based on a paper by Fairlie and Jones (54).

For simplicity, we take first the four-point function. Consider the expression

$$T_4 = \int_{-\infty}^{\infty} |x|^{-1-\alpha_{12}} \, |1-x|^{-1-\alpha_{23}} \, dx \qquad (3.1)$$

We split the integration range into three parts and write

$$T_4 = \int_1^{\infty} (x)^{-1-\alpha_{12}} (x-1)^{-1-\alpha_{23}} dx + \int_0^1 (x)^{-1-\alpha_{12}} (1-x)^{-1-\alpha_{23}} dx$$

$$+ \int_{-\infty}^{0} (-x)^{-1-\alpha_{12}} (1-x)^{-1-\alpha_{23}} dx \qquad (3.2)$$

In the first term we change the integration variable from x to $1/x$, and in the third to $\frac{x}{x-1}$. We may now write

$$T_4 = \int_0^1 x^{\alpha_{12}+\alpha_{23}} (1-x)^{-\alpha_{23}-1} dx + \int_0^1 x^{-\alpha_{12}-1} (1-x)^{-\alpha_{23}-1} dx$$

$$+ \int_0^1 x^{-\alpha_{12}-1} (1-x)^{\alpha_{12}+\alpha_{23}} dx \qquad (3.3)$$

We now write the complete Veneziano four-point amplitude (2.2 - 2.6) as

$$T_4 = \int_0^1 x^{-\alpha_{12}-1} (1-x)^{-\alpha_{23}-1} dx + \int_0^1 x^{-\alpha_{12}-1} (1-x)^{-\alpha_{13}-1} dx$$

$$+ \int_0^1 x^{-\alpha_{13}-1} (1-x)^{-\alpha_{23}-1} dx \qquad (3.4)$$

We see that the two expressions (3.3) and (3.4) are identical, provided the trajectory constraint

$$\alpha_{12} + \alpha_{23} + \alpha_{31} + 1 = 0 \qquad (3.5)$$

is satisfied.

Thus the expression (3.1) represents the complete Veneziano four-point function, invariant under all 24 permutations of the external lines, providing the trajectory constraint (3.5) is satisfied. We may rewrite this as

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} + 1 = 0 \qquad (3.6)$$

for any three external lines.

This representation of the amplitude may not be too useful as the integrand is non-analytic. However, it does exhibit the correct signature factor when the asymptotic behaviour of equation (3.1) is examined for $s_{12} \to \infty$. Splitting it up into three terms as in (3.3), we may write

$$T_4 = \frac{\Gamma(-\alpha_{12})\,\Gamma(-\alpha_{23})}{\Gamma(-\alpha_{12} - \alpha_{23})} + \frac{\Gamma(-\alpha_{12})\,\Gamma(\alpha_{12} + \alpha_{23} + 1)}{\Gamma(\alpha_{23} + 1)}$$

$$+ \frac{\Gamma(-\alpha_{23})\,\Gamma(\alpha_{12} + \alpha_{23} + 1)}{\Gamma(\alpha_{12} + 1)} \tag{3.7}$$

Now, taking the limit of s_{12} approaching ∞ along a ray at an infinitesimal angle to the real axis, and using the standard properties of the gamma function, we see that the second term vanishes, and the remaining two terms combine to give

$$\underset{\left(\begin{smallmatrix} s_{12} \to \infty \\ s_{23}\ \text{fixed} \end{smallmatrix}\right)}{T_4} \simeq \frac{\pi}{\Gamma(\alpha_{23})} \frac{[1 + e^{-i\pi\alpha_{23}}]}{\sin \pi\alpha_{23}} (\alpha' s_{12})^{\alpha_{23}} \tag{3.8}$$,

exactly the expression we obtained previously, (equation (2.9)).

We may extend this contracted form of the amplitude to the five-point function by writing, in analogy with (2.45),

$$T_5 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dz |t|^{-\alpha_{12}-1} |z|^{-\alpha_{45}-1} |1-t|^{-\alpha_{23}-1} |1-z|^{-\alpha_{34}-1} \times |1-tz|^{\alpha_{23}+\alpha_{34}-\alpha_{51}} \tag{3.9}$$

We now split the integration range into twelve regions and transform the variables in the twelve regions as shown in table 3.1. Then just as equation (3.1) decomposes into three parts, equation (3.9) decomposes into twelve contributions of the form (2.45), each contribution being associated with a definite non-cyclic permutation of the external lines (1, 2, 3, 4, 5) corresponding to the twelve terms in the summation (2.34), provided the trajectory function satisfies the constraints

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} - \alpha_{ijk} + 1 = 0 \tag{3.10}$$

where i, j and k are any three external lines. The non-cyclic permutations associated with the various transformations are given in the last column of table 3.1. Thus, provided our constraints (3.10) hold, the expression (3.9) will represent the complete five-point Veneziano amplitude, invariant under all 120 permutations of the external lines.

On the basis of the two cases given, it would seem reasonable to suppose that in order to incorporate all $\frac{1}{2}(N-1)!$ terms of the N-point Veneziano amplitude, (2.34), into one term, all we have to do is extend the limits of integration (2.34) from (0, 1) to $(-\infty, \infty)$ for each of the $N-3$ integration variables u_{ij},

($j = 2, \ldots, N-2$), and introduce moduli into the integrand. We write

$$T_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=2}^{N-2} du_{ij} \frac{1}{|J_1|} \prod_P |u_P|^{x_P} \tag{3.11}$$

The integration range is now split into $\frac{1}{2}(N-1)!$ regions, and under a particular transformation of variables, each term in the summation (2.22) is recovered, provided certain trajectory constraints are satisfied, of which (3.6) and (3.10) are particular examples for the case of $N = 4$ and $N = 5$ respectively.

In order to demonstrate this contraction explicitly, it is convenient to use the Koba-Nielsen parametrisation (32). In this representation, we may write the four-point amplitude corresponding to the ordering of the external lines (1,2,3,4) according to (2.51) as:

$$B_4(1,2,3,4) = \int \cdots \int dG_3^{-1} \prod_{k=1}^{4} dz_k \prod_{1 \leqslant i < j \leqslant 4} (z_j - z_i)^{Y_{ij}} \tag{3.12}$$

where $z_j = \exp i\,\theta_j$ ($j = 1, \ldots, 4$) are four ordered points on the unit circle in the complex z-plane such that

$$\theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_1 + 2\pi \tag{3.13}$$

The Y_{ij} are given by equation (2.53).

Using the fact that

$$(z_j - z_i) = i \exp\left(\frac{i(\theta_i + \theta_j)}{2}\right) |z_j - z_i| \qquad (3.14)$$

we may rewrite (3.12) as

$$B_4(1,2,3,4) = \int \cdots \int dG_3^{-1} \prod_{k=1}^{4} d\theta_k \prod_{1 \leqslant i < j \leqslant 4} |z_j - z_i|^{Y_{ij}} \qquad (3.15)$$

The ordering of the points z_j is the same as the ordering of the external particles. The differential dG_3 is defined by (2.56) as

$$dG_3 = \frac{dz_r \, dz_s \, dz_t}{(z_s - z_r)(z_t - z_s)(z_t - z_r)} \qquad (2.56)$$

or equivalently by

$$dG_3 = \frac{d\theta_r \, d\theta_s \, d\theta_t}{|z_s - z_r||z_t - z_s||z_t - z_r|} \qquad (3.16)$$

where z_r, z_s and z_t are three arbitrary but fixed points such that

$$\theta_r < \theta_s < \theta_t < \theta_r + 2\pi .$$

The integration in the remaining variable is over that part of the circle consistent with (3.13).

Writing (3.15) out in full, we obtain

$$B_4(1,2,3,4) = \int \cdots \int dG_3^{-1} \prod_{k=1}^{4} d\theta_k \Big\{ |z_4 - z_3|^{-\alpha_{34}-1} |z_3 - z_2|^{-\alpha_{23}-1}$$

$$\times |z_2 - z_1|^{-\alpha_{12}-1} |z_4 - z_1|^{-\alpha_{14}-1} |z_4 - z_2|^{\alpha_{23}+\alpha_{34}} |z_3 - z_1|^{\alpha_{12}+\alpha_{23}} \Big\} \tag{3.17}$$

The complete four-point amplitude is written as a sum of three terms, (equation (2.2), as

$$T_4 = B_4(1,2,3,4) + B_4(1,2,4,3) + B_4(1,3,2,4) \tag{2.2}$$

Now, $B_4(1,2,4,3)$ and $B_4(1,3,2,4)$ may be written in the form (3.17) simply by interchanging the indices ($3 \longleftrightarrow 4$) and ($2 \longleftrightarrow 3$) in (3.17) respectively. The integrand in all three terms will now be identical provided our trajectory constraint (3.6) is satisfied,

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} + 1 = 0 \tag{3.6}$$

with i, j, k any three external lines. Now, as we have stated, the ordering of the points z_k is the same as the ordering of the external lines, therefore:

in $B(1,2,3,4)$: $\theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_1 + 2\pi$ (3.18a)

in $B(1,2,4,3)$: $\theta_1 < \theta_2 < \theta_4 < \theta_3 < \theta_1 + 2\pi$ (3.18b) in $B(1,3,2,4)$: $\theta_1 < \theta_3 < \theta_2 < \theta_4 < \theta_1 + 2\pi$ (3.18c)

Let us fix θ_1, θ_2, and θ_4 subject to

$$\theta_1 < \theta_2 < \theta_4 < \theta_1 + 2\pi \tag{3.19}$$

consistent with equations (3.18a,b, and c). We now integrate with respect to θ_3 in each of the three terms in the summation (2.2) over the range defined by the equations (3.18), i.e. for $B_4(1,3,2,4)$ over the range (θ_1,θ_2), for $B_4(1,2,3,4)$ over the range (θ_2, θ_4) and for $B_4(1,2,4,3)$ over the range $(\theta_4, \theta_1 + 2\pi)$.

Choosing θ_1 as zero, we may now write

$$T_4 = \int_0^{2\pi} dG_3^{-1} \prod_{k=1}^{4} d\theta_k \prod_{1 \leqslant i < j \leqslant 4} |z_j - z_i|^{-\alpha_{ij}-1} \tag{3.20}$$

We emphasize that this is conditional on our constraint (3.6) being satisfied. We state again that three of the θ_k are fixed, the integration in the remaining variable being taken over the whole of the unit circle $(0, 2\pi)$. Our result is, in fact, independent of the choice of the variables kept fixed, and of the values assigned to them.

We may easily demonstrate that equation (3.20) is exactly equivalent to the expression (3.1), by making the substitution, according to (2.50),

$$u_{12} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \tag{3.21}$$

Equation (3.20) then reduces to

$$T_4 = \int_{-\infty}^{\infty} |u_{12}|^{-\alpha_{12}-1} \; |1 - u_{12}|^{-\alpha_{23}-1} \; du_{12} \tag{3.22}$$

The extension to the N-point amplitude in this notation is clear: using the identity (3.14), we may express the N-point amplitude corresponding to the ordering of the external lines (1,2,...,N), equation (2.51) as:

$$B_N(1,2,\ldots,N) = \int \cdots \int dG_3^{-1} \prod_{k=1}^{N} d\theta_k \prod_{1 \leq i < j \leq N} |z_j - z_i|^{Y_{ij}} \tag{3.23}$$

Y_{ij} is defined by equation (2.53). The differential dG_3 is defined as before in terms of three arbitrary, but fixed points z_r, z_s and z_t. The integration in the remaining (N-3) variables is over those parts of the circle consistent with

$$\theta_1 < \theta_2 < \ldots < \theta_N < \theta_1 + 2\pi \tag{2.52}$$

where the ordering of the points z_k is the same as the ordering of the external lines.

Similarly each of the $\frac{1}{2}(N-1)!$ terms in the summation of the complete amplitude (2.22), corresponding to non-equivalent orderings of the external lines, may be written in this form, obtained from (3.23), simply by interchanging indices. The integrand in each term will be identical, provided a certain set of trajectory constraints are satisfied. These may be deduced from (2.53).

For $N = 2n$ and $2n + 1$ (n integral), there are $(n-1)$ separate constraints,

$$
\begin{aligned}
-\alpha_{ij} - 1 &= -\alpha_{ikj} + \alpha_{ik} + \alpha_{kj} \\
&= -\alpha_{ik_1k_2j} + \alpha_{ik_1k_2} + \alpha_{k_1k_2j} - \alpha_{k_1k_2} \\
&\;\;\vdots \\
&= -\alpha_{ik_1\ldots k_rj} + \alpha_{ik_1\ldots k_r} + \alpha_{k_1\ldots k_rj} - \alpha_{k_1\ldots k_r} \\
&\;\;\vdots \\
&= -\alpha_{ik_1\ldots k_{n-1}j} + \alpha_{ik_1\ldots k_{n-1}} + \alpha_{k_1\ldots k_{n-1}j} - \alpha_{k_1\ldots k_{n-1}}
\end{aligned}
\tag{3.24}
$$

For $n = 2$, equations (3.6) and (3.10) result for the four-and five-point amplitudes respectively.

For $n \geqslant 3$, using equation (2.55), we see that these constraints simply reduce to ,

$$-\alpha_{ij} - 1 = -\alpha_{ikj} + \alpha_{ik} + \alpha_{kj}$$

$$= -2\alpha' \, p_i \cdot p_j \tag{3.25}$$

for any three external lines i, j, k.

The region of integration in the (N-3) variables in each of the terms in the summation will be over a different part of the unit circle for each variable, defined by the ordering of the external lines, as in (2.52) for the ordering (1 2 ... N). When we sum over all the terms in (2.22) the integration regions will sum to the whole of the unit circle for each of the (N-3) variables independently.

The complete amplitude T_N, in (2.22), may now be written as

$$T_N = \int_0^{2\pi} \cdots \int_0^{2\pi} dG_3^{-1} \prod_{k=1}^{N} d\theta_k \prod_{1 \leq i < j \leq N} |z_j - z_i|^{-\alpha_{ij}-1} \tag{3.25}$$

This expression is exactly equivalent to that given earlier,(3.11).

As we have already mentioned, though elegant, this representation suffers from the considerable drawback of having a non-analytic integrand. However, as we demonstrated explicitly for the four-point amplitude, the correct signature factors are exhibited in the appropriate asymptotic limits. Although we do not demonstrate it explicitly here, it seems very likely that this will also be true for the higher-point functions.

We emphasize that the expression (3.25) is only valid provided our trajectory constraints (3.24) are satisfied. We now examine these constraints in greater detail.

The constraints resulting from (3.24) for the four- and five-point functions, as given by equations (3.6) and (3.10) are:

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} + 1 = 0 \tag{3.6}$$

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} - \alpha_{ijk} + 1 = 0 \tag{3.10}$$

These constraints have also been given by Koba and Nielsen (55), and are similar in structure to the constraint invoked by Veneziano (15) to incorporate wrong signature zeros in the $\pi\pi \rightarrow \pi\omega$ helicity-flip amplitude:

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 2 \tag{3.27}$$

If the scattering particles all have mass μ and the trajectory functions are given by

$$\left.\begin{aligned} \alpha_{ij} &= \alpha_o + \alpha'(p_i + p_j)^2 \\ \alpha_{ijk} &= \alpha_o + \alpha'(p_i + p_j + p_k)^2 \end{aligned}\right\} \tag{3.28}$$

then the kinematic conditions arising from four-momentum conservation demand that:

$$\left.\begin{array}{ll} & \alpha_o = 1 \\ \text{and} & \alpha'\mu^2 = -1 \end{array}\right\} \qquad (3.29)$$

The constraints on the trajectories given by the higher order functions ($N \geqslant 6$) in equation (3.25) are obviously consistent with this.

This enables us to rewrite the expression for the complete N-point function, invariant under all N! permutations of the external lines as

$$T_N = \int_0^{2\pi} \cdots \int_0^{2\pi} dG_3^{-1} \prod_{k=1}^{N} d\theta_k \prod_{1 \leqslant i < j \leqslant N} |z_j - z_i|^{-2\alpha' p_i \cdot p_j} \qquad (3.30)$$

In this form we see that T_N extrapolates off the mass shell in the simplest possible way; by setting $p_N = 0$, the integral over θ_N gives 2π multiplied by the (N-1) point function. This was first pointed out by Fairlie (56).

Also, using (3.14), we may rewrite (3.30) as a contour integral,

$$T_N = \int \cdots \int dG_3^{-1} \prod_{k=1}^{N} dz_k \prod_{1 \leqslant i < j \leqslant N} (z_j - z_i)^{-2\alpha' p_i \cdot p_j} \qquad (3.31)$$

However, the price to be paid for all this simplicity is too high. Either α' or μ^2 is negative, which is physically unacceptable.

This model with $\alpha_0 = 1$ has been used, however, quite frequently by a number of authors in the harmonic oscillator approach to dual resonance models (57), because of its simplicity.

TABLE 3.1

	New variables	Integration range original variables t	z	Permutations associated with transformations
1	(x, y)	$(0, 1)$	$(0, 1)$	1 2 3 4 5
2	$\left(\frac{1}{1-xy}, 1-y\right)$	$(1, \frac{1}{z})$	$(0, 1)$	1 3 2 4 5
3	$\left(\frac{1}{xy}, y\right)$	$(\frac{1}{z}, \infty)$	$(0, 1)$	1 3 4 2 5
4	$\left(xy, \frac{1}{y}\right)$	$(0, \frac{1}{z})$	$(1, \infty)$	1 2 4 3 5
5	$\left(1-xy, \frac{1}{1-y}\right)$	$(\frac{1}{z}, 1)$	$(1, \infty)$	1 4 2 3 5
6	$\left(\frac{1}{x}, \frac{1}{y}\right)$	$(1, \infty)$	$(1, \infty)$	1 4 3 2 5
7	$\left(1-\frac{1}{x}, \frac{1}{1-y}\right)$	$(-\infty, 0)$	$(1, \infty)$	1 2 5 3 4
8	$\left(\frac{x}{x-1}, 1-y\right)$	$(-\infty, 0)$	$(0, 1)$	1 2 5 4 3
9	$\left(\frac{1}{1-x}, 1-\frac{1}{y}\right)$	$(1, \infty)$	$(-\infty, 0)$	1 3 2 5 4
10	$\left(1-x, \frac{y}{y-1}\right)$	$(0, 1)$	$(-\infty, 0)$	1 2 3 5 4
11	$\left(\frac{xy}{xy-1}, 1-\frac{1}{y}\right)$	$(\frac{1}{z}, 0)$	$(-\infty, 0)$	1 2 4 5 3
12	$\left(1-\frac{y}{xy}, \frac{y}{y-1}\right)$	$(-\infty, \frac{1}{z})$	$(-\infty, 0)$	1 4 2 5 3

The first column gives the transformations of the variables t and z in equation (3.9) operative in the regions specified by the entries in the second and third columns.

CHAPTER IV

A FIVE-POINT AMPLITUDE INVOLVING THE INTERACTION OF TWO CLASSES OF HADRONS

4.1 Introduction

The interpretation of the Veneziano model via quark duality diagrams introduced in chapter two for the four-point function, extends in a straightforward manner to multiparticle production processes. For example, a five-point meson process can be represented by the quark duality diagram shown in figure 4.1. The entire amplitude can, in principle, be described in terms of any of the five ordinary diagrams in figures 4.2 (a) ... (e). Figure 4.1 includes all these possibilities, and indicates that every one of them may be a complete description. Quark duality diagrams for five-point meson-baryon processes, MB $\rightarrow$ MMB, are shown in figure 4.3. Figures 4.3(a) and (b) are characterised by the property that in figure 4.3(a) two of the quarks constituting the baryon have no interaction with the external mesons, while in figure 4.3(b), only one of the baryon quarks is such a 'spectator' quark.

The usual Veneziano five-point function, constructed in chapter two may be regarded as corresponding to the Feynman diagram of figure 4.4 in the sense that the Veneziano amplitude has resonances in those channels where the Feynman diagram has intermediate states. By interchanging the external lines, we may obtain eleven further Veneziano amplitudes which make up the complete Veneziano expression.

A model has been proposed by Burnett and Schwarz (58) in which they attempt to write a Veneziano-type five-point amplitude describing the interaction between three 'elementary' mesons and a single baryon, where only meson-baryon-baryon couplings are allowed (see figure 4.5). Five other diagrams obtained by permuting the three meson lines amongst themselves are assumed dual with figure 4.5. This process may be described in terms of a quark duality diagram by figure 4.6, where mesons 1, 2 and 3 interact with quarks 1, 2 and 3 respectively. In this diagram, none of the quark constituents of the baryon are spectator quarks.

Such a process has resonances in six channels (in contrast to the usual five-point Veneziano amplitude which has resonances in only five channels), containing the particles (A,i) or (B,i), with i = 1, 2 or 3, and can be represented by the Feynman diagram of figure 4.7, where again resonances occur in those channels where the Feynman diagram has intermediate states. We require the amplitude associated with figure 4.5, or equivalently 4.6 or 4.7, to possess the property that we can exhibit it as a double sum of poles with polynomial residues in the channels $(p_A + p_1)^2$, $(p_B + p_3)^2$ say, or in any of the channels obtained by permuting the meson lines 1, 2 and 3.

We derive in this chapter a representation of the amplitude for such a process similar in structure to that of the usual Veneziano five-point amplitude, i.e. we write the amplitude in the form

$$F_5 = \int_0^1 \int_0^1 du_{A3}\, du_{B1}\, J^{-1} \prod_{i=1}^{3} \left\{ u_{Ai}^{-\alpha_{Ai}-1} \, u_{Bi}^{-\alpha_{Bi}-1} \right\} \tag{4.1}$$

Further motivation for the study of such an amplitude has been given by Susskind (59) in an harmonic oscillator theory of dual resonance models. Here the baryon quark constituents emit and absorb the external mesons one at a time, the process being governed by a quark-quark-meson coupling constant. A prescription is derived for obtaining meson-baryon scattering amplitudes, a separate contribution being given for each of the cases of zero, one and two spectator quarks. In the case of no spectator quarks, the expression derived is essentially the same as we derive here.

We will use techniques developed by Mandelstam (60) in the study of the amplitude associated with figure 4.7, and show how other representations such as that given by Fairlie and Jones (61) are related to Mandelstam's expression. We then examine in section three the decomposition of this amplitude in terms of a series representation similar to that of Hopkinson and Plahte (30) for the five-point amplitude. In section four we examine the asymptotic behaviour of our amplitude in the Regge limit. In section five we examine the parametrisation of this amplitude in terms of the Koba-Nielsen variables (32).

The generalisation of the problem proposed by Burnett and Schwarz is to an n-point process involving the interaction of (n-2) mesons with a single baryon, illustrated in figure 4.8, where all diagrams obtained by permuting the meson lines amongst themselves are assumed dual with figure 4.8. The process may equivalently be represented by the Feynman diagram of figure 4.9, where the amplitude has resonances in those channels where the Feynman diagram has intermediate states.

The generalisation derived by Mandelstam (60) to processes involving an arbitrary number of external lines ($\geqslant 6$) describes a different set of interactions to those discussed here. The generalisations coincide only for the case of five external particles. In chapter five, we clarify the distinction between the generalisations proposed by Mandelstam and those proposed here. We then use the techniques developed by Mandelstam to derive an expression for the amplitude illustrated by the Feynman diagram of figure 4.9. The treatment is as given by Jones (62).

Finally, we note here that within the unitarisation programme of Kikkawa, Sakita and Virosoro (47), we assume that the amplitudes we have constructed here should be added to the conventional tree graph amplitude, as part of the Born term.

4.2. The five-point process of Burnett and Schwarz.

We derive here a representation of the five-point amplitude associated with the Feynman diagram of figure 4.7. Associated with each of the six channels, (Ai), (Bi), (i = 1, 2, 3) will be a variable u_{Ai}, u_{Bi}. Only two of these will be independent. Each channel will have a trajectory function α_{Ai} or α_{Bi}. The five-point amplitude will then have the form of equation (4.1). We now derive formulas expressing the dependent u's in terms of the independent u's. The formulas must satisfy the following:

(i) It must be possible to set the u's for two non-overlapping channels simultaneously equal to zero, since the amplitude can have simultaneous resonances in two such channels.

(ii) If any u is set equal to zero, the u's for all overlapping channels must be equal to unity since the residue at a pole in any channel must be a polynomial in the overlapping variables.

(iii) If any u is set equal to zero, the remaining integration must reproduce the ordinary four-point Veneziano amplitude.

As a starting point to derive these constraints we take a formula proposed by Virosoro for a five-point amplitude with resonances in all ten channels (see figure 4.10), which has been generalised to processes with an arbitrary number of external lines by Collop (63). Five of his variables u_{ij} are independent; the other five are determined by the five equations,

$$u_{ij}\, u_{k\ell} + u_{ik}\, u_{j\ell} + u_{i\ell}\, u_{jk} = 2 \tag{4.2}$$

$$i,j,k,\ell = 1,\ldots,5; \qquad i < j < k < \ell .$$

Since our amplitude has resonances in only six channels, we must set the remaining u's equal to unity. We obtain in the notation of figure 4.7 the three equations

$$\begin{aligned} u_{A1}\, u_{B2} + u_{A2}\, u_{B1} &= 1 \\ u_{A2}\, u_{B3} + u_{A3}\, u_{B2} &= 1 \\ u_{A3}\, u_{B1} + u_{A1}\, u_{B3} &= 1 \end{aligned} \tag{4.3}$$

together with,

$$\sum_{i=1}^{3} u_{Ai} = 2; \qquad \sum_{i=1}^{3} u_{Bi} = 2 \tag{4.4}$$

The five equations have no non-trivial solution, and we cannot obtain our required amplitude as a special case.

An obvious modification we can make is to take equations (4.3) together with,

$$u_{A1} + u_{A2} + u_{A3} = u_{B1} + u_{B2} + u_{B3}. \tag{4.5}$$

Call this the set of equations (a).

It is a simple matter to verify that this set of equations satisfies the conditions (i) ... (iii) above. For example, if we take $u_{A3} = 0$, and solve equations (4.3) and (4.5), we obtain

$$\left.\begin{aligned} u_{A1} = u_{A2} = u_{B3} &= \pm 1 \\ u_{B1} + u_{B2} &= \pm 1 \end{aligned}\right\} \qquad (4.6)$$

The - sign is excluded by the range of integration in equation (4.1). It follows from (4.1) that the remaining integral is as follows:

$$\int_0^1 J^{-1}\, du_{B1}\, u_{B1}^{-\alpha_{B1}-1}\, (1 - u_{B1})^{-\alpha_{B2}-1} \qquad (4.7)$$

This is precisely the four-point Veneziano formula for the Feynman diagram obtained by contracting the external vertices A and 3 of figure 4.7. Provided that the Jacobian factor J behaves suitably, our conditions (i) ... (iii) are met .

However, this is not the only set of equations we can derive satisfying the conditions (i) ... (iii) above. There are various other possibilities:

(b) takes equations (4.3) together with

$$u_{A1} + u_{A2} + u_{A3} + u_{B1} + u_{B2} + u_{B3} = 4 \quad . \qquad (4.8)$$

However, the variables in this set of equations are related to the variables in the set of equations (a) by a simple transformation. Taking the set of equations (a) and making the transformation of variables

$$u_{Ai} = \lambda u_{Ai}' ; \qquad u_{Bi} = \frac{1}{\lambda} u_{Bi}' \tag{4.9}$$

with λ defined by

$$\lambda^2 = \left\{ \frac{1 \pm i\sqrt{\alpha\beta\gamma}}{1 \mp i\sqrt{\alpha\beta\gamma}} \right\} \tag{4.10}$$

where

$$\left.\begin{aligned} \alpha &= u_{A1}\, u_{B2} \\ \beta &= u_{A2}\, u_{B3} \\ \gamma &= u_{A3}\, u_{B1} \end{aligned}\right\} \tag{4.11}$$

the variables u_{Ai}', u_{Bi}' will satisfy the set of equations (b). As the variables in the two sets of equations differ only by a scaling factor, the representations we obtain for the amplitude in (4.1) will differ only in the Jacobian, which is in any case non-unique, and in a function

$$(\lambda)^{\sum_{i=1}^{3}(\alpha_{Bi} - \alpha_{Ai})} \tag{4.12}$$

multiplying the integrand. The exponent reduces simply to

$\alpha'(p_B^2 - p_A^2)$, so that in the case of equal mass external baryons the extra factor reduces to unity. The ambiguity may be regarded simply as a reflexion of the ambiguity due to satellites discussed in chapter two.

(c) A further possibility, with the conditions (i) ... (iii) above satisfied, is the set of equations given by Fairlie and Jones (61):

$$\left.\begin{aligned} u_{Ai} &= 1 - u_{Aj}(1 - u_{Bk}) - u_{Ak}(1 - u_{Bj}) \\ u_{Bi} &= 1 - u_{Bj}(1 - u_{Ak}) - u_{Bk}(1 - u_{Aj}) \end{aligned}\right\} \qquad (4.13)$$

with $i,j,k = 1, 2$ or 3

$i \neq j \neq k$.

The six equations (4.13) readily reduce to four independent equations

$$\begin{aligned} u_{A1} + u_{A2} + u_{A3} - 1 &= u_{B1} + u_{B2} + u_{B3} - 1 \\ &= u_{A1}\, u_{B2} + u_{A2}\, u_{B1} \\ &= u_{A2}\, u_{B3} + u_{A3}\, u_{B2} \\ &= u_{A3}\, u_{B1} + u_{A1}\, u_{B3} \end{aligned} \qquad (4.14)$$

so that four variables may be expressed in terms of two as desired. Again, by making a simple transformation of the variables in the set of equations (a), we may arrive at the set of equations given here. The transformation is:

$$u_{Ai} = u_{Ai}' \cdot \left\{ \sum_{i=1}^{3} u_{Ai}' - 1 \right\}^{-\frac{1}{2}}$$

$$u_{Bi} = u_{Bi}' \quad \left\{ \sum_{i=1}^{3} u_{Ai}' - 1 \right\}^{-\frac{1}{2}} \tag{4.15}$$

Again, we may interpret this ambiguity as a reflexion of the ambiguity regarding non-leading terms.

Yet further possibilities may be derived, such as that proposed originally by Burnett and Schwarz (58), but the equations here do not appear to generalise readily for arbitrarily large numbers of external particles.

We are now faced with a choice of which set of equations to take. Neither set of equations is distinguished by the simplicity of solution of the equations. All three sets of equations (a), (b) and (c) are readily generalised to the n-point amplitude. The ambiguity can probably be resolved by recourse to a consideration of the simplicity of the spectrum of the amplitude (59), and to the asymptotic behaviour. On these bases, we take the set of equations (a) to consider in detail.

The proof that this amplitude has single-particle poles at the correct positions and with the correct angular momenta goes through in identical manner to that for the conventional tree graph Veneziano amplitude.

For this set of equations (a), the Jacobian in equation (4.1) may be defined as:

$$J = \frac{u_{A1}\, u_{B3}\, (u_{A1} + u_{B3})}{u_{A1} + u_{A2} + u_{A3}} \tag{4.16}$$

This Jacobian transforms correctly when the pair of integration variables (u_{A3}, u_{B1}) is replaced by any other pair of non-overlapping variables, but it is not uniquely defined by this requirement since it is subject to the usual ambiguity associated with non-leading Veneziano terms. The Jacobian would have the correct transformation properties if the denominator were omitted, but such a choice is unsuitable when we generalise to the n-point function. If the variables u_{A3} or u_{B1} are set equal to zero, the Jacobian reduces to unity. We have thus completed our verification that (4.7) is identical to the four-point Veneziano formula. If we had omitted the denominator in the Jacobian, this would not hold, as with $u_{A3} = 0$, the Jacobian would then be equal to two.

We now give an interpretation of the set of equations (a) which is useful for generalisation to the n-point amplitude. The Feynman diagram of figure 4.7 contains three plane polygons AB12, AB23, AB31. Associate variables v_{ij}, w_{ij}, x_{ij} with each polygon, corresponding to the integration variables of the associated plane

Veneziano diagram. The v's, w's and x's are defined in terms of the u's as follows:

$$\left.\begin{array}{lll} v_{A1} = u_{A1}\, u_{B2} & & v_{B1} = u_{B1}\, u_{A2} \\ w_{A2} = u_{A2}\, u_{B3} & & w_{B2} = u_{B2}\, u_{A3} \\ x_{A3} = u_{A3}\, u_{B1} & & x_{B3} = u_{B3}\, u_{A1} \end{array}\right\} \qquad (4.17)$$

The rule for constructing the v's, w's and x's is to take the u for the channel in question, and to multiply it by the u's for all the channels which consist of the particles in the original channel together with particles which do not form part of the relevant polygon. We can now rewrite equations (4.3) as:

$$\left.\begin{array}{l} v_{A1} + v_{B1} = 1 \\ w_{A2} + w_{B2} = 1 \\ x_{A3} + x_{B3} = 1 \end{array}\right\} \qquad (4.18)$$

These equations are precisely the equations that the variables would satisfy if they were regarded as Veneziano integration variables for the plane polygons.

4.3 Series representation

We examine the solutions to the equations

$$\left.\begin{aligned} u_{A1}\,u_{B2} + u_{A2}\,u_{B1} &= 1 \\ u_{A2}\,u_{B3} + u_{A3}\,u_{B2} &= 1 \\ u_{A3}\,u_{B1} + u_{A1}\,u_{B3} &= 1 \end{aligned}\right\} \qquad (4.3)$$

$$u_{A1} + u_{A2} + u_{A3} = u_{B1} + u_{B2} + u_{B3} \qquad (4.5)$$

Introduce the notation

$$\left.\begin{aligned} u_{A1}\,u_{B2} &= \alpha\,, & u_{A2}\,u_{B1} &= 1-\alpha \\ u_{A2}\,u_{B3} &= \beta\,, & u_{A3}\,u_{B2} &= 1-\beta \\ u_{A3}\,u_{B1} &= \gamma\,, & u_{A1}\,u_{B3} &= 1-\gamma \end{aligned}\right\} \qquad (4.19)$$

Then $$\alpha\,\beta\,\gamma = (1-\alpha)(1-\beta)(1-\gamma) \qquad (4.20)$$

and $$\gamma^{-1} = 1 + \frac{\alpha\beta}{(1-\alpha)(1-\beta)} \qquad (4.21a)$$

$$(1-\gamma)^{-1} = 1 + \frac{(1-\alpha)(1-\beta)}{\alpha\beta} \qquad (4.21b)$$

We now solve for the u's in terms of α and β .

The solutions are

$$u_{A1}^{\;2} = \left(\frac{1+\alpha\beta-\alpha}{1+\alpha\beta-\beta}\right)\cdot\frac{\alpha(1-\gamma)}{\beta}\,; \quad u_{B1}^{\;2} = \left(\frac{1+\alpha\beta-\beta}{1+\alpha\beta-\alpha}\right)\cdot\frac{(1-\alpha)\gamma}{1-\beta}$$

$$u_{A2}^{2} = \left(\frac{1+\alpha\beta-\alpha}{1+\alpha\beta-\beta} \right) \cdot \frac{\alpha\beta}{1-\gamma} \; ; \qquad u_{B2}^{2} = \left(\frac{1+\alpha\beta-\beta}{1+\alpha\beta-\alpha} \right) \cdot \frac{(1-\alpha)(1-\beta)}{\gamma}$$

$$u_{A3}^{2} = \left(\frac{1+\alpha\beta-\alpha}{1+\alpha\beta-\beta} \right) \cdot \frac{(1-\beta)\gamma}{1-\alpha} \; ; \qquad u_{B3}^{2} = \left(\frac{1+\alpha\beta-\beta}{1+\alpha\beta-\alpha} \right) \cdot \frac{\beta(1-\gamma)}{\alpha}$$

$$(4.22)$$

with γ given in terms of α and β by (4.21)

The five-point amplitude under consideration is given by (4.1)

$$F_5 = \int_0^1 \int_0^1 du_{A3} \, du_{B1} \left\{ \frac{u_{A1} + u_{A2} + u_{A3}}{u_{A1} u_{B3} (u_{A1} + u_{B3})} \right\} \cdot (u_{A1})^{-\alpha_{A1}-1} (u_{A2})^{-\alpha_{A2}-1}$$

$$\times (u_{A3})^{-\alpha_{A3}-1} (u_{B1})^{-\alpha_{B1}-1} (u_{B2})^{-\alpha_{B2}-1} (u_{B3})^{-\alpha_{B3}-1} \tag{4.23}$$

where we have taken a specific form of Jacobian factor (4.16).

Now, using (4.22) above, after a little algebra, we can show that

$$\frac{u_{A1} + u_{A2} + u_{A3}}{u_{A1} \, u_{B3} (u_{A1} + u_{B3})} \, du_{A3} du_{B1} = \left(\frac{1-\gamma}{\alpha\beta} \right) d\alpha d\beta = \left(\frac{\gamma}{(1-\alpha)(1-\beta)} \right) d\alpha d\beta \tag{4.24}$$

Now, substitute (4.22) together with (4.24) into (4.23), and using (4.21), we obtain after some manipulation,

$$F_5 = \int_0^1 \int_0^1 d\alpha d\beta (\alpha)^{-\alpha_{A1}-1} (\beta)^{-\alpha_{B3}-1} (1-\alpha)^{\frac{1}{2}(\alpha_{A1}+\alpha_{A3}+\alpha_{B3}-\alpha_{A2}-\alpha_{B1}-\alpha_{B2})-1}$$

$$\times \quad (1-\beta)^{\frac{1}{2}(\alpha_{A1}+\alpha_{B1}+\alpha_{B3}-\alpha_{A2}-\alpha_{A3}-\alpha_{B2})-1}$$

$$\times \left\{ 1 + \frac{\alpha\beta}{(1-\alpha)(1-\beta)} \right\}^{\frac{1}{2}(\alpha_{A1}+\alpha_{A3}+\alpha_{B1}+\alpha_{B3}-\alpha_{A2}-\alpha_{B2})}$$

$$\times \left\{ \frac{1-\beta(1-\alpha)}{1-\alpha(1-\beta)} \right\}^{\frac{\alpha'}{2}(p_A^2 - p_B^2)} \tag{4.25}$$

where we have taken

$$\alpha_{Ai} = \alpha_o + \alpha'(p_A + p_i)^2 \tag{4.26}$$

and similarly for α_{Bi}, $(i = 1,2,3)$.

For the sake of simplicity we take

$$p_A^2 = p_B^2 \tag{4.27}$$

i.e. we take the external baryons identical, on mass shell. We now expand the term $\left\{ 1 + \frac{\alpha\beta}{(1-\alpha)(1-\beta)} \right\}$ in a series:

$$\left\{ 1 + \frac{\alpha\beta}{(1-\alpha)(1-\beta)} \right\}^z = \sum_{r=0}^{\infty} \binom{z}{r} \left(\frac{\alpha\beta}{(1-\alpha)(1-\beta)} \right)^r \tag{4.28}$$

where

$$z = \tfrac{1}{2}(\alpha_{A1} + \alpha_{A3} + \alpha_{B1} + \alpha_{B3} - \alpha_{A2} - \alpha_{B2}) \tag{4.29}$$

to obtain,

$$F_5 = \sum_{r=0}^{\infty} \int_0^1 \int_0^1 d\alpha d\beta \binom{z}{r} (\alpha)^{-\alpha_{A1}+r-1} (1-\alpha)^{z_1-r-1}$$

$$\times \ (\beta)^{-\alpha_{B3}+r-1} (1-\beta)^{z_2-r-1} \tag{4.30}$$

with

$$\left. \begin{aligned} z_1 &= \tfrac{1}{2}(\alpha_{A1} + \alpha_{A3} + \alpha_{B3} - \alpha_{A2} - \alpha_{B1} - \alpha_{B2}) \\ z_2 &= \tfrac{1}{2}(\alpha_{A1} + \alpha_{B1} + \alpha_{B3} - \alpha_{A2} - \alpha_{A3} - \alpha_{B2}) \end{aligned} \right\} \tag{4.31}$$

i.e.

$$F_5 = \sum_{r=0}^{\infty} \binom{z}{r} F_4(-\alpha_{A1} + r, \ z_1 - r) \, F_4(-\alpha_{B3} + r, \ z_2 - r) \tag{4.32}$$

where

$$F_4(x,y) = \int_0^1 u^{x-1} (1-u)^{y-1} du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{4.33}$$

i.e. the ordinary Veneziano four-point function.

So,

$$F_5 = \sum_{r=0}^{\infty} \binom{z}{r} \frac{\Gamma(-\alpha_{A1} + r)\Gamma(z_1 - r)}{\Gamma(z_1 - \alpha_{A1})} \cdot \frac{\Gamma(-\alpha_{B3} + r)\Gamma(z_2 - r)}{\Gamma(z_2 - \alpha_{B3})} \tag{4.34}$$

Now

$$\left.\begin{aligned} \Gamma(-\alpha_{A1} + r) &= \Gamma(-\alpha_{A1})(-\alpha_{A1})_r \\ \Gamma(-\alpha_{B3} + r) &= \Gamma(-\alpha_{B3})(-\alpha_{B3})_r \end{aligned}\right\} \qquad (4.35)$$

where

$$(-\alpha)_r \equiv (-\alpha)(-\alpha + 1) \ldots (-\alpha + r - 1) \qquad (4.36)$$

Also

$$\Gamma(z_1 - r) = \frac{\Gamma(z_1)}{(z_1 - 1)(z_1 - 2) \ldots (z_1 - r)}$$

$$= \frac{\Gamma(z_1)}{(-1)^r (-z_1 + 1)_r} \qquad (4.37)$$

and similarly for $\Gamma(z_2 - r)$.

Also we have

$$\binom{z}{r} \equiv \frac{(-z)_r (-1)^r}{r!} \qquad (4.38)$$

so that

$$F_5 = F_4(z_1, -\alpha_{A1}) F_4(z_2, -\alpha_{B3}) \sum_{r=0}^{\infty} \frac{(-\alpha_{A1})_r(-\alpha_{B3})_r(-z)_r(-1)^r}{(-z_1 + 1)_r(-z_2 + 1)_r \; r!}$$

$$(4.39)$$

The summation is a standard hypergeometric function, so that we may write,

$$F_5 = F_4(z_1, -\alpha_{A1})F_4(z_2, -\alpha_{B3})$$

$$\times \; {}_3F_2\{-\alpha_{A1}, -\alpha_{B3}, -z; \; -z_1+1, -z_2+1; -1\} \qquad (4.40)$$

where z, z_1, z_2 are given by (4.29) and (4.31) respectively.

Similar expressions for F_5 may be obtained by permuting the meson lines 1, 2 and 3 among themselves.

The hypergeometric function converges for

$$\mathrm{Re}\{-z_1 + 1 - z_2 + 1 + \alpha_{A1} + \alpha_{B3} + z\} > 0$$

$$\text{i.e.} \quad \mathrm{Re}\{\alpha_{A1} + \alpha_{A2} + \alpha_{A3} + \alpha_{B1} + \alpha_{B2} + \alpha_{B3}\} > -4 \qquad (4.41)$$

We have thus found a representation for our five-point amplitude which has a much larger range of convergence than the integral representation.

The series provides the simplest method of determining the pole structure of our amplitude. The beta function $F_4(x,y)$ has simple poles in each variable at the non-positive integers, so using the symmetry properties of our amplitude, it has simple poles at the non-positive integers in each argument separately. There are thus various advantages to be seen in the series representation: it exhibits clearly the physical interpretation of the amplitude and the dynamical assumption underlying it. The series may also be useful for numerical evaluation of the amplitude, to which the integral is not easily applicable.

4.4 Asymptotic behaviour.

We now examine the asymptotic behaviour of our amplitude in the double-Regge limit. Single Reggeisation can similarly be established.

From (4.3) and (4.5) the solutions for u_{A1}, u_{A2}, u_{B2}, u_{B3} in terms of u_{A3}, u_{B1} may be written:

$$u_{A1} = \frac{u_{A3}\, u_{B1}(u_{B1} - u_{A3}) \pm X^{\frac{1}{2}}}{2(u_{A3}\, u_{B1} - 1)} \tag{4.42a}$$

$$u_{B3} = \frac{u_{A3}\, u_{B1}(u_{A3} - u_{B1}) \pm X^{\frac{1}{2}}}{2(u_{A3}\, u_{B1} - 1)} \tag{4.42b}$$

$$u_{A2} = \frac{-u_{A3}(u_{B1}^{2} - 3u_{A3}\, u_{B1} + 2) \pm X^{\frac{1}{2}}}{2(1 - u_{A3}\, u_{B1})(1 - 2u_{A3}\, u_{B1})} \tag{4.42c}$$

$$u_{B2} = \frac{-u_{B1}(u_{A3}^{2} - 3u_{A3}\, u_{B1} + 2) \pm X^{\frac{1}{2}}}{2(1 - u_{A3}\, u_{B1})(1 - 2u_{A3}\, u_{B1})} \tag{4.42d}$$

where $$X = \{u_{A3}^{2}\, u_{B1}^{2}(u_{B1} - u_{A3})^{2} + 4(1 - u_{A3}\, u_{B1})^{3}\} \tag{4.43}$$

We take the positive roots in (4.42 a...d) to avoid results like $u_{A1} = -u_{B3}$ which we cannot allow, as we must have two independent variables. We take the integral representation for our amplitude as in (4.23). We consider first the double-Regge limit associated with the exchange of Regge poles with the quantum numbers of the meson-baryon system, choosing the particular graph of figure 4.11a. We take all α's real and negative, and consider the limit

$$s_{A3},\ s_{B3} \rightarrow -\infty\ ; \qquad \frac{s_{A3}\, s_{B3}}{s_{12}} = K, \text{ fixed}$$

$$s_{A1},\quad s_{B2},\quad \text{fixed.} \tag{4.44}$$

Now, there are various constraints amongst the s's from which we may deduce the behaviour of s_{B1} and s_{A2} in this limit. We have

$$s_{B1} + s_{B2} + s_{12} - s_{A3} = c, \quad \text{constant} \tag{4.45}$$

$$\text{so,} \qquad s_{B1} \sim -s_{12} \tag{4.46}$$

and

$$s_{A1} + s_{A2} + s_{12} - s_{B3} = c \tag{4.47}$$

$$\text{so,} \qquad s_{A2} \sim -s_{12} \tag{4.48}$$

We take the limit $s_{12} \rightarrow +\infty$ in order that we may have α_{B1} and α_{A2}

negative. Then K, defined above in (4.44) is positive.

We also define

$$\tilde{K} = \frac{\alpha_{A3}\,\alpha_{B3}}{\alpha_{12}} . \tag{4.49}$$

Now, substitute equations (4.42) into (4.23) and make the following change of variables:

$$u_{B1} = \exp(-x'y') \; ; \qquad u_{A3} = \exp(-y') \tag{4.50}$$

with $$x' = -\frac{x}{\alpha_{B3}} \quad ; \qquad y' = -\frac{y}{\alpha_{A3}} \tag{4.51}$$

The outline of the proof we give here is now almost identical to the treatment given in Bardakci and Ruegg (26), though the algebra is more complicated owing to the complex nature of the solutions (4.42). With these substitutions, we obtain

$$F_5 = (-\alpha_{A3})^{\alpha_{A1}} (-\alpha_{B3})^{\alpha_{B2}} \cdot I \tag{4.52}$$

$$I = \int_0^\infty \int_0^\infty dxdy \exp\left\{ -y + xy \frac{\alpha_{B1}}{\alpha_{A3}\,\alpha_{B3}} \right\} x^{-\alpha_{B2}-1} \, y^{-\alpha_{A1}-1}$$

$$\times \; [\lambda_1(x', y')]^{-\alpha_{A1}-2} \, [\lambda_2(x', y')]^{-\alpha_{B2}-1} \cdot \mu(x', y')$$

$$\times \; [\lambda_3(x', y', \alpha_{B3})] \cdot [\lambda_4(x', y', \alpha_{A2})]$$

(4.53)

where

$$\lambda_1(x',y') = \frac{1}{y'} u_{A1}(x',y') \; ; \qquad \lambda_2(x',y') = \frac{1}{x'} u_{B2}(x',y')$$

$$\lambda_3(x',y') = [u_{B3}(x',y')]^{-\alpha_{B3}-2} \; ; \qquad \lambda_4(x',y') = [u_{A2}(x',y')]^{-\alpha_{A2}-1}$$

$$\mu(x',y') = \sum_{i=1}^{3} \frac{u_{Ai}(x', y')}{u_{A1}(x',y') + u_{B3}(x',y')} \tag{4.54}$$

u_{A1}, u_{B2} etc. are obtained as functions of x' and y' from equations (4.42), using the substitutions (4.50) .

The range of both x and y and x' and y' is between 0 and ∞ . In this range, the following statements hold:

(i) $|\lambda_i| < M, \qquad i = 1,\ldots,4$

$|\mu| < M$, where M is a fixed constant, independent of all variables.

$$(0 \leqslant x' \leqslant \infty, \; 0 \leqslant y' \leqslant \infty, \; \alpha_{B3}, \alpha_{A2} < 0). \tag{4.55}$$

(ii) $\displaystyle \operatorname*{\ell t}_{x',y' \to 0} \lambda_i = 1 , \qquad i = 1,2.$

$$\operatorname*{\ell t}_{x',y' \to 0} \mu = 1 \tag{4.56}$$

$$\mathop{\ell t}_{\substack{x',y'\to 0 \\ x,y \text{ fixed} \\ \alpha_{B3}\to -\infty}} \lambda_3(x',y',\alpha_{B3}) = \exp(-x) \tag{4.57}$$

$$\mathop{\ell t}_{\substack{x',y'\to 0 \\ x,y \text{ fixed} \\ \alpha_{A2}\to -\infty}} \lambda_4(x',y',\alpha_{A2}) = \exp\left(\frac{-4xy}{K}\right) \tag{4.58}$$

The above limits are all uniform in both variables in the neighbourhood of x', $y' = 0$. We now write,

$$I = \left\{ \int_0^P \int_0^P + \int_P^\infty \int_0^P + \int_0^P \int_P^\infty + \int_P^\infty \int_P^\infty \right\} dxdy \times [\text{ Integrand}]$$

$$I = I_0(P) + I_1(P) + I_2(P) + I_3(P) \tag{4.59}$$

where P is a constant independent of all the s's. Now, given an ϵ, we choose P so large that

$$|I_1| < \epsilon, \quad |I_2| < \epsilon, \quad |I_3| < \epsilon \tag{4.60}$$

independent of s_{A3} and s_{B3}, which follows from the fact that the λ_i and μ are bounded.

At the same time, keeping P fixed and taking s_{A3} and s_{B3} sufficiently large, we can satisfy the following:

$$\left| I_0(P) - \int_0^P \int_0^P dxdy \exp\left\{ -x-y-\frac{5xy}{K} \right\} x^{-\alpha_{B2}-1} y^{-\alpha_{A1}-1} \right| < \epsilon \tag{4.61}$$

which follows from the existence of uniform limits for the λ_i and μ .
Finally, we can also have,

$$\left| \left\{ \int_0^\infty \int_0^\infty - \int_0^P \int_0^P \right\} dxdx \, \exp\left\{ -x-y-\frac{5xy}{\tilde{K}} \right\} \, x^{-\alpha_{B2}-1} \, y^{-\alpha_{A1}-1} \right| < \epsilon$$

(4.62)

Combining (4.59) ... (4.62), the following limit is derived,

$$\underset{\substack{s_{A3}, s_{B3} \to -\infty \\ \tilde{K}, \text{ fixed}}}{\ell t} I = \int_0^\infty \int_0^\infty dxdy \left\{ x^{-\alpha_{B2}-1} y^{-\alpha_{A1}-1} \exp\left(-x-y-\frac{5xy}{\tilde{K}} \right) \right\}$$

(4.63)

We may then write

$$\underset{\text{(double-Regge limit)}}{F_5} = (-s_{A3})^{\alpha_{A1}} (-s_{B3})^{\alpha_{B2}} \, g \qquad (4.64)$$

where

$$g = (a)^{\alpha_{B2}+\alpha_{A1}} \int_0^\infty \int_0^\infty x^{-\alpha_{B2}-1} \, y^{-\alpha_{A1}-1} \exp\left(-x-y-\frac{5xy}{aK} \right)$$

(4.65)

where K is given by (4.44) and we have taken

$$\alpha_{ij} = a \, s_{ij} + b \qquad (4.66)$$

This is the usual Regge behaviour.

We have here assumed that the procedure of changing the integration variables and taking the limit inside the integral is valid.

The above expressions are also valid for complex α_{A3} and α_{B3} as long as Re $\alpha_{A3} < 0$, Re $\alpha_{B3} < 0$.
To reach the right-half plane in these variables one has to rotate the line of integration of x and y in equation (4.53) from the positive axis to a complex direction. In this manner, one can establish (4.63) for any complex direction with the exception of the real axis.

Regge behaviour can similarly be proved for all other double-Regge limits associated with the exchange of Regge poles with the quantum numbers of the meson-baryon system. However, Regge behaviour does not hold for processes where the exchanged Regge pole does not occur in a meson-baryon channel. Consider as an example the diagram of figure 4.11b. We consider the Regge limit associated with this diagram,

$$s_{A2},\ s_{A3} \rightarrow -\infty \ ; \qquad \frac{s_{A2}\ s_{A3}}{s_{B1}} = K, \ \text{fixed}$$

$$s_{B2},\ s_{13} \ \text{fixed} \tag{4.67}$$

Constraints amongst the s's similar to (4.45) and (4.47) enable us to determine the behaviour of s_{A1} and s_{B3} in this limit. We obtain,

$$s_{A1} \sim -s_{A3} \,; \quad s_{B3} \sim -s_{B1} \tag{4.68}$$

Now, we wish to take all the α's (α_{Ai}, α_{Bi}, $i = 1,2,3$) real and negative in order that our expression (4.23) for the five-point amplitude converges. However, from the above equation (4.68), we see that if we take α_{A3} and α_{B1} real and negative, then α_{A1} and α_{B3} must be real and positive, and the integral diverges. An identical argument holds for the Regge limit associated with the graph of figure 4.11c.

4.5 A parametrisation using the Koba-Nielsen variables.

The usefulness of the Koba-Nielsen parametrisation of the variables u_{ij}, defined in equation (2.50), is that expressed in this way, the duality constraints such as (2.28) for the conventional five-point tree graph amplitude, are automatically satisfied.

For the amplitudes we are considering, the Koba-Nielsen variables do not form natural solutions of our constraint equations (4.3) and (4.5), and hence are not so immediately useful for the description of our amplitude. However, a modified form of the prescription has been suggested by Mandelstam (60). We associate the points z_A, z_B, z_1, z_2 and z_3 with each of our external lines; we define the harmonic conjugate of z_i (written $\tilde{z}_i$), with respect to z_A and z_B by

$$\frac{(z_i - z_A)(\tilde{z}_i - z_B)}{(z_i - z_B)(\tilde{z}_i - z_A)} = -1 \tag{4.69}$$

with $i = 1, 2, 3$.

We now express α, β, and γ of (4.19) as

$$\alpha = \frac{(\tilde{z}_1 - z_A)(z_2 - z_B)}{(\tilde{z}_1 - z_2)(z_A - z_B)} ; \quad \beta = \frac{(\tilde{z}_2 - z_A)(z_3 - z_B)}{(\tilde{z}_2 - z_3)(z_A - z_B)}$$

$$\gamma = \frac{(\tilde{z}_3 - z_A)(z_1 - z_B)}{(\tilde{z}_3 - z_1)(z_A - z_B)} \tag{4.70}$$

Defined in this way, the condition (4.20), $\alpha\beta\gamma = (1-\alpha)(1-\beta)(1-\gamma)$ is automatically satisfied, with the $\tilde{z}_i$'s defined in (4.69).
We now define $\tilde{\alpha}$ in the obvious way,

$$\tilde{\alpha} = \frac{(z_1 - z_A)(\tilde{z}_2 - z_B)}{(z_1 - \tilde{z}_2)(z_A - z_B)} \tag{4.71}$$

and similarly for $\tilde{\beta}$ and $\tilde{\gamma}$.

Using (4.22), we may now write the solutions for $(u_{Ai}\, \tilde{u}_{Ai})$, $(u_{Bi}\, \tilde{u}_{Bi})$ as:

$$(u_{A1}\tilde{u}_{A1}) = \frac{(z_A-z_1)(\tilde{z}_1-z_A)}{[(z_A-z_B)(z_B-z_A)]^{\frac{1}{2}}} \left\{ \frac{(z_3-\tilde{z}_2)(z_2-\tilde{z}_3)}{(z_3-\tilde{z}_1)(z_1-\tilde{z}_3)(z_2-\tilde{z}_1)(z_1-\tilde{z}_2)} \right\}^{\frac{1}{2}} \chi^{\frac{1}{2}} \tag{4.72}$$

and

$$(u_{B1}\tilde{u}_{B1}) = \frac{(z_B-z_1)(\tilde{z}_1-z_B)}{[(z_A-z_B)(z_B-z_A)]^{\frac{1}{2}}} \left\{ \frac{(z_3-\tilde{z}_2)(z_2-\tilde{z}_3)}{(z_3-\tilde{z}_1)(z_1-\tilde{z}_3)(z_2-\tilde{z}_1)(z_1-\tilde{z}_2)} \right\}^{\frac{1}{2}} \chi^{-\frac{1}{2}} \tag{4.73}$$

where

$$\chi = \frac{(z_B-z_2)(\tilde{z}_2-z_B)}{(z_A-z_2)(\tilde{z}_2-z_A)} \cdot \frac{(1+\alpha\beta-\alpha)(1+\tilde{\alpha}\tilde{\beta}-\tilde{\alpha})}{(1+\alpha\beta-\beta)(1+\tilde{\alpha}\tilde{\beta}-\tilde{\beta})} \tag{4.74}$$

and similarly for $(u_{A2}\,\tilde{u}_{A2})$, $(u_{A3}\,\tilde{u}_{A3})$, $(u_{B2}\,\tilde{u}_{B2})$ and $(u_{B3}\,\tilde{u}_{B3})$.

Our five-point amplitude may be written in the form

$$F_5 = \int_0^1\int_0^1 \left\{ (du_{A1}d\tilde{u}_{A1})(du_{B2}d\tilde{u}_{B2})(J^{-1}\,\tilde{J}^{-1}) \prod_{i=1}^{3} \left[(u_{Ai}\tilde{u}_{Ai})^{-\alpha_{Ai}-1} \right.\right.$$
$$\left.\left. \times\ (u_{Bi}\tilde{u}_{Bi})^{-\alpha_{Bi}-1} \right] \right\}^{\frac{1}{2}} \tag{4.75}$$

where $(du_{A1}d\tilde{u}_{A1})^{\frac{1}{2}}$ etc. are really only single differentials as $\tilde{u}_{A1}$ is a function of u_{A1}. This may be rewritten as,

$$F_5 = \int\cdots\int \left\{ (dz_A d\tilde{z}_A)(dz_B d\tilde{z}_B) \prod_{i=1}^{3} (dz_i d\tilde{z}_i) \right.$$

$$\times \prod_{i=1}^{3} \left[\left\{ (z_A-z_i)(\tilde{z}_i-z_A) \right\}^{-\alpha_{Ai}-1} \left\{ (z_B-z_i)(\tilde{z}_i-z_B) \right\}^{-\alpha_{Bi}-1} \right] \times$$

/ over

$$\times \prod_{\substack{i,j=1 \\ (i<j)}}^{3} \left[(z_i - \tilde{z}_j)(z_j - \tilde{z}_i) \right]^{\frac{1}{2}(\alpha_{Ai}+\alpha_{Aj}+\alpha_{Bi}+\alpha_{Bj}-\alpha_{Ak}-\alpha_{Bk})}$$

$$\times \left[(z_B - z_A)(z_A - z_B) \right]^{\frac{1}{2}\sum_p \alpha_p} [X]^{\frac{1}{2}\sum_{i=1}^{3}(\alpha_{Bi} - \alpha_{Ai})} \Big\}^{\frac{1}{2}}$$

(4.76)

The summation in the exponent of the term $[(z_B-z_A)(z_A-z_B)]$ is taken over all channels. Also, we take $d\tilde{z}_A \equiv dz_A$, and $d\tilde{z}_B \equiv dz_B$. The region of integration is as discussed earlier in section 2.7. We have omitted from the integrand a term analagous to (dG_3^{-1}) defined in (2.56) for the tree-graph amplitude. This essentially forms a constant multiplier of the amplitude. Up to a constant factor, our two expressions (4.75) and (4.76) are identical.

Assuming the trajectories are all identical, linear, of the form (4.26), then the exponent of X in the last term, under the kinematic conditions arising from four-momentum conservation, reduces simply to,

$$\frac{\alpha'}{2} (p_B^2 - p_A^2) \tag{4.77}$$

In keeping with section 4.3, we take $p_A^2 = p_B^2$ and ignore this term.

Assuming similar conditions the exponent of the term $[(z_i - \tilde{z}_j)(z_j - \tilde{z}_i)]$ reduces to

$$-2\alpha' \; p_i \cdot p_j \;\; + \;\; \alpha(M^2) \tag{4.78}$$

where we have put $p_A^2 = p_B^2 = M^2$, M the mass of the external baryons.

Similarly the exponent of the term $[(z_A - z_B)(z_B - z_A)]$ reduces to

$$-2\alpha' \; p_A \cdot p_B + \alpha' \; M^2 + 3\alpha(\mu^2) \tag{4.79}$$

where we have taken $p_1^2 = p_2^2 = p_3^2 = \mu^2$, μ the meson mass.

The important point about this representation is that the momentum dependence of each of the above two terms is simply $-2\alpha' \; p_i \cdot p_j$ and $-2\alpha' \; p_A \cdot p_B$, respectively. Such a representation enables us to write our amplitude in a very compact way, and will certainly be useful in any attempt to incorporate the process we have considered into the functional integration approach to dual theories that has been proposed recently (45).

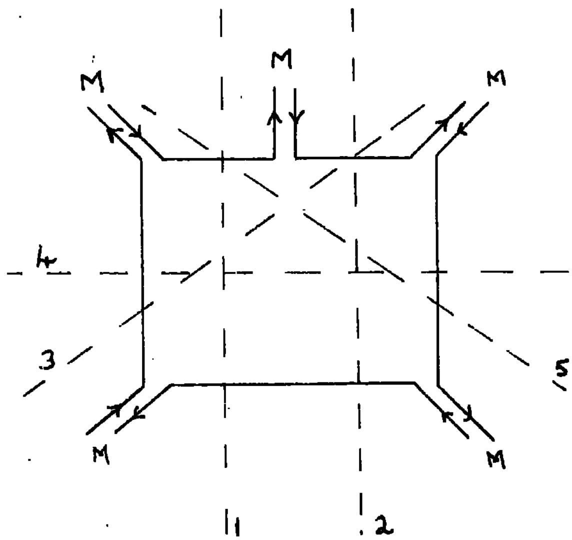


Figure 4.1 Quark duality diagram for a five-point meson process

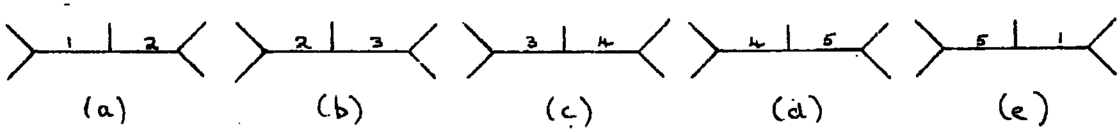


Figure 4.2 The various alternative descriptions of figure 4.1. Every one of the five descriptions (a)...(e) may in principle be a complete description of the amplitude. They should be summed over all possible intermediate states (1,...,5) which are marked by dashed lines in figure 4.1

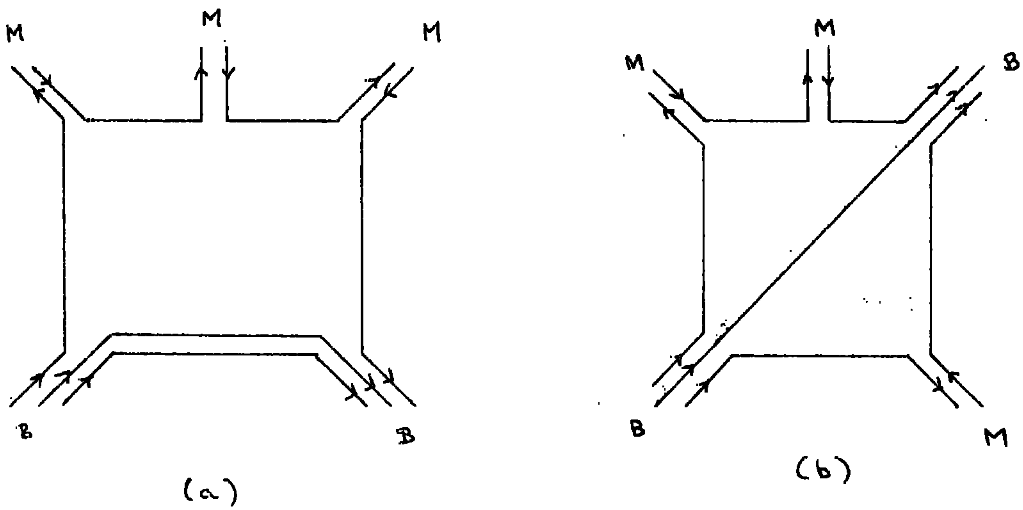


Figure 4.3 Quark duality diagrams for five-point meson-baryon processes

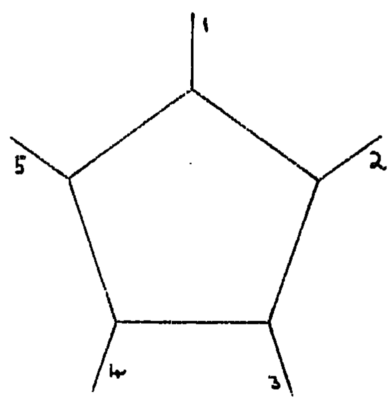


Figure 4.4 Feynman diagram representing the usual five-point tree graph amplitude

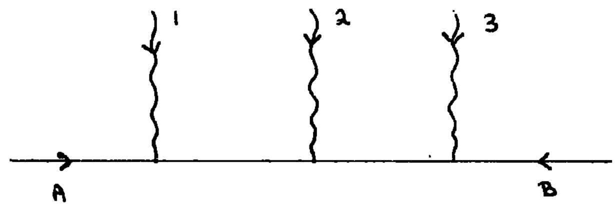


Figure 4.5 The Burnett and Schwarz five-point process. A and B are the baryon lines; 1, 2 and 3 the meson lines.

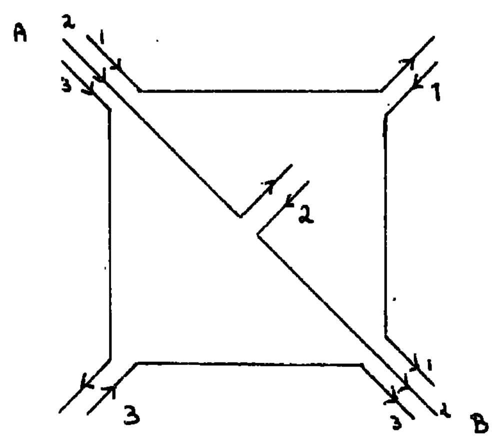


Figure 4.6 Quark duality diagram for the Burnett and Schwarz process

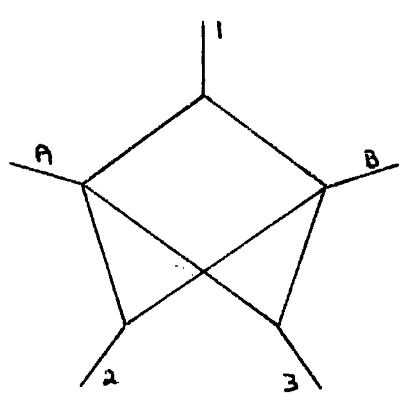


Figure 4.7 Feynman diagram for the Burnett and Schwarz five-point process

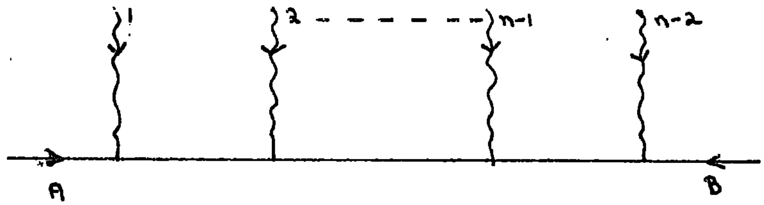


Figure 4.8 The n-point generalisation of the Burnett and Schwarz process

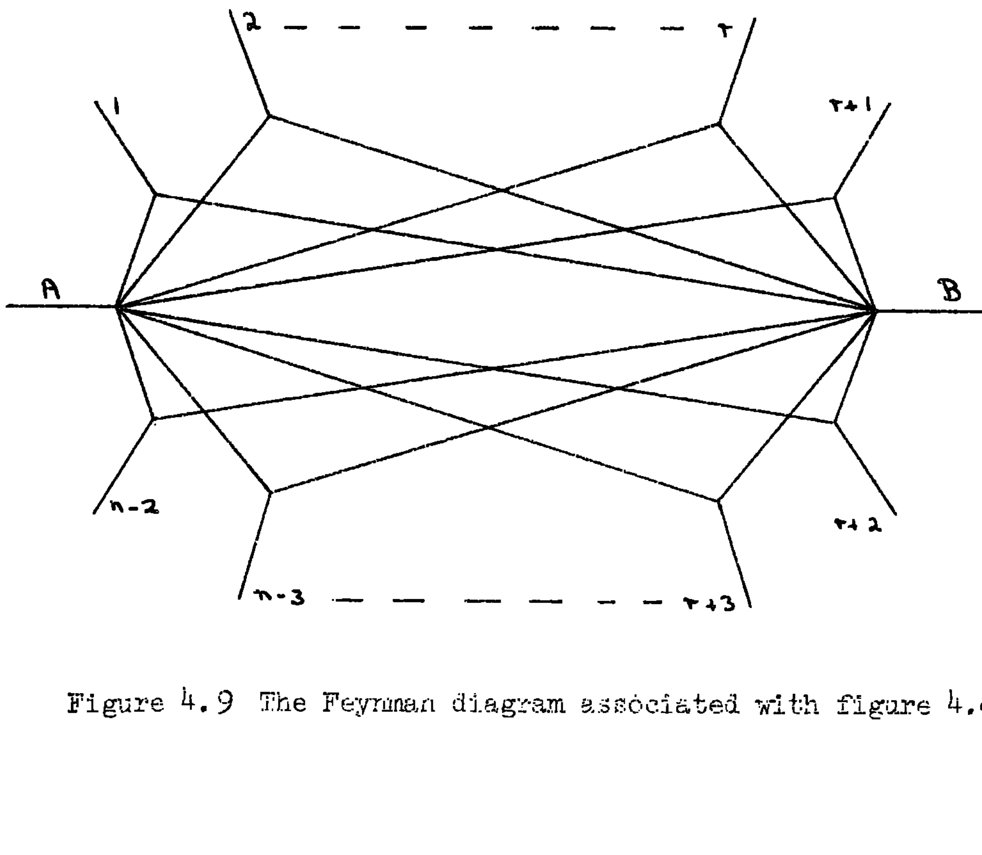


Figure 4.9 The Feynman diagram associated with figure 4.8

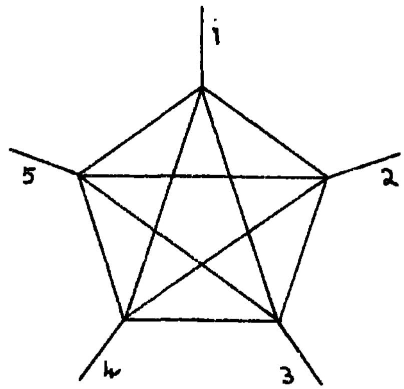


Figure 4.10 The Feynman diagram associated with the five-point function of Virozoro

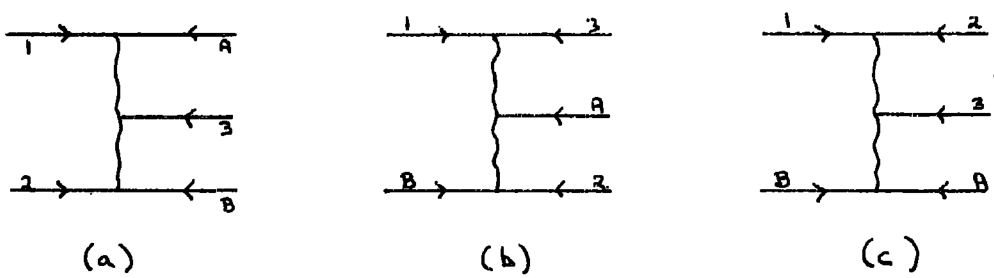


Figure 4.11 Relevant diagrams for examining double-Regge limits

CHAPTER V

THE GENERALISED AMPLITUDE INVOLVING THE INTERACTION OF TWO CLASSES OF HADRONS

5.1 Introduction

The natural extension of the Burnett and Schwarz process would describe the scattering of four mesons off a baryon target, represented equivalently by figure 5.1, or by the Feynman diagram of figure 5.2, resonances occurring in those channels where the Feynman diagram has intermediate states. The diagrams obtained by permuting the four mesons amongst themselves are assumed dual with figure 5.1. There are thus fourteen channels in which resonances can occur in this six-point amplitude:

u_{Ai}, u_{Bi} ; $i = 1,2,3$ - two particle channels

u_{Aij} ; $i,j = 1,2,3, \quad i < j$ - three particle channels.

In this chapter we give the extension of our prescription for this amplitude, both for the six-point and for the general n-point function associated with figure 4.8, or equivalently with the Feynman diagram of figure 4.9. The treatment is essentially that given by Jones (62), and uses the techniques developed by Mandelstam (60) in his generalisation to a different set of processes.

The five-point function given in chapter four of this thesis, and that considered by Mandelstam (60) coincide, i.e. the same amplitude is under consideration. For the six-point and higher amplitudes, the generalisations considered here, diverge from those derived by Mandelstam. The generalisation to the six-point amplitude suggested by Mandelstam describes an amplitude which may be associated with the Feynman diagram of figure 5.3. This amplitude has resonances in eleven channels. This is an example of a minimal non-planar diagram with only one pair of crossed internal lines in the nomenclature of Mandelstam. We use 'minimal' in the sense that the set of channels in which resonances occur do not contain as a subset any set of channels which form a complete set for the ordinary planar Veneziano amplitude with the same number of external lines. In this sense, the diagrams we consider are minimal non-planar, but they do contain more than one pair of crossed internal lines. The Feynman diagram associated with the general n-point amplitude of Mandelstam (60) is shown in figure 5.4, where again we understand that resonances occur in those channels where the Feynman diagram has intermediate states. The quark duality diagram for this extension describes a contribution to the n-point production amplitude of mesons off a baryon target in which n_1 mesons interact with quark number one, n_2 with quark number two, and n_3 with quark number three, where the three quarks are the baryon constituents and

$n_1 + n_2 + n_3 \;=\; n-2.$ Thus, if indeed a quark structure underlies hadronic processes, Mandelstam's extension should be necessary for the description of such production processes. It is clear that Mandelstam's extension and that given here are special cases of a more general mathematical problem to construct meson-baryon amplitudes with a single baryon line, where the baryon is constituted from N quarks.

5.2 Generalisation to the n-point amplitude.

We are concerned in this section with deriving expressions for the general n-point amplitude associated with figure 4.8 (or equivalently with the Feynman diagram of figure 4.9), but before we proceed to this general case, we shall consider the six-point diagram (figure 5.2). There are fourteen channels in which resonances can occur:

$$(A1),(A2),(A3),(A4),$$
$$(B1),(B2),(B3),(B4),$$

$$(A12),(A13),(A14),(A23),(A24),(A34),$$
$$\text{or equivalently}\quad (B34),(B24),(B23),(B14),(B13),(B12).$$

To each channel there will correspond an integration variable u, and a trajectory function α, and we subscript the variables with the indices of all the particles in the channel. The six-point amplitude will be given by the integral ,

$$\int_0^1\int_0^1\int_0^1 du_{A1}\, du_{B2}\, du_{A13}\, J^{-1} \left[\prod_p u_p^{-\alpha_p - 1} \right] \qquad (5.1)$$

The three integration variables may be replaced by any other triplet, and the Jacobian factor must be defined in such a way that it is independent of the choice. The product is taken over all fourteen channels. We wish to derive eleven constraint equations, so that we may express eleven of the u's in terms of three of them. It is here that the interpretation given at the end of section 4.2 becomes relevant. The Feynman diagram of figure 5.2 contains four 'non-planar' five-point diagrams: AB123, AB124, AB134, AB234. Associate variables v_{ij}, w_{ij}, x_{ij}, y_{ij} with each polygon, defined in terms of the u's according to the prescriptions given at the end of section 4.2. The required equations now result by demanding that the v's, w's, x's, y's are related by the rules for the five-point amplitude given by equations (4.3) and (4.5).

The equations that result are:

(1) $u_{A1}\, u_{B4}\, u_{A12}\, u_{A13} + u_{A4}\, u_{B1}\, u_{A34}\, u_{A24} = 1$

(2) $u_{A2}\, u_{B3}\, u_{A12}\, u_{A24} + u_{A3}\, u_{B2}\, u_{A13}\, u_{A34} = 1$

(3) $u_{A2}\, u_{B4}\, u_{A12}\, u_{A23} + u_{A4}\, u_{B2}\, u_{A14}\, u_{A34} = 1$

(4) $u_{A1}\, u_{B3}\, u_{A12}\, u_{A14} + u_{A3}\, u_{B1}\, u_{A23}\, u_{A34} = 1$

$$(5) \quad u_{A1}\, u_{B2}\, u_{A13}\, u_{A14} + u_{A2}\, u_{B1}\, u_{A23}\, u_{A24} = 1$$

$$(6) \quad u_{A3}\, u_{B4}\, u_{A13}\, u_{A23} + u_{A4}\, u_{B3}\, u_{A14}\, u_{A24} = 1$$

$$(7) \quad u_{A2}\, u_{A12} + u_{A3}\, u_{A13} + u_{A4}\, u_{A14} = u_{B2}\, u_{A34} + u_{B3}\, u_{A24} + u_{B4}\, u_{A23}$$

$$(8) \quad u_{A1}\, u_{A12} + u_{A3}\, u_{A23} + u_{A4}\, u_{A24} = u_{B1}\, u_{A34} + u_{B3}\, u_{A14} + u_{B4}\, u_{A13}$$

$$(9) \quad u_{A1}\, u_{A14} + u_{A2}\, u_{A24} + u_{A3}\, u_{A34} = u_{B1}\, u_{A23} + u_{B2}\, u_{A13} + u_{B3}\, u_{A12}$$

$$(10) \quad u_{A1}\, u_{A13} + u_{A2}\, u_{A23} + u_{A4}\, u_{A34} = u_{B1}\, u_{A24} + u_{B2}\, u_{A14} + u_{B4}\, u_{A12}$$

(5.2)

The final equation is determined by the condition that conditions (i) ... (iii) of section 4.2 with (iii) suitably modified, must be satisfied. The generalised condition (iii) now reads: if there is a pole in any of the two particle channels, i.e. if any of the u_{Ai} or u_{Bi} is equal to zero, the remaining integral must be the integral for the five-point amplitude derived in section 4.2, associated with the Feynman diagram of figure 4.7, since the diagram obtained by constracting any pair of vertices (Ai) or (Bi) $(i = 1 \ldots 4)$ of figure 5.2, is the diagram of figure 4.7. Also, if one of the u's associated with a three-particle channel is set equal to zero, the remaining integral must be the product of two ordinary four-point Veneziano integrals.

These conditions define our final equation as:

$$(11)\quad u_{A1}\,u_{B1} + u_{A2}\,u_{B2} + u_{A3}\,u_{B3} + u_{A4}\,u_{B4}$$

$$= u_{A12}\,u_{A34} + u_{A13}\,u_{A24} + u_{A14}\,u_{A23} \qquad (5.3)$$

We may thus express eleven of our u's in terms of three independent u's, and write our amplitude as a three dimensional integral as desired.

It is an elementary through rather tedious matter to verify that conditions (i) ... (iii) mentioned above, with (iii) in the modified form indicated, are satisfied by equations (5.2) and (5.3); (iii) is conditional of course on the Jacobian factor behaving suitably. We shall show this presently, though we shall first generalise our constraints to the n-point amplitude.

Associated with the n-point amplitude of figure 4.9, it is an elementary matter to show that there are $(2^{n-2} - 2)$ channels in which resonances occur. Again, to each channel there will correspond an integration variable u and a trajectory function α, and we subscript the variables with the indices of all the particles in the channel. We require to write the n-point amplitude in the form of an (n-3)-dimensional integral:

$$\int_0^1 \cdots \int_0^1 dV_{n-3} \left[\prod_p u_p^{-\alpha_p - 1} \right] \qquad (5.4)$$

where dV_{n-3} is a volume element, equivalent in the six-point case to

$$dV_3 \equiv du_{A1}\, du_{B2}\, du_{A13}\, J^{-1} \tag{5.5}$$

We must now write the constraint equations between the various u's to show that (n-3) of them are independent. The method should be obvious. For example the seven-point process associated with figure 5.5 contains five 'non-planar' six-point diagrams. There are thirty channels in which resonances occur in this amplitude. If we define the v_{ij}'s, w_{ij}'s etc. associated with each polygon in terms of the u's according to the prescription already given, and demand that they are related by the rules for our six-point amplitude, then we will obtain twenty-five equations relating the u's. The twenty-sixth equation is defined by demanding that the equations (i) ...(iii), with (iii) in the generalised form are satisfied. We thus have a self-generative process for determining the constraints for the n-point amplitude; the Feynman diagram associated with this process (figure 4.9) contains (n-2) non-planar (n-1)-point diagrams. Associated with this process there are $2^{n-2} - 2$ channels in which resonances can occur. Defining the v's, w's etc. as already indicated, and demanding that they are related by the rules for the (n-1) point function already derived, we obtain $(2^{n-2} - n)$ relations amongst the u's. One more relation is derived by demanding that the conditions (i) ...(iii) with the factorisation property (iii) in a suitable form are satisfied. We thus have

$2^{n-2} - n + 1$ relations among $2^{n-2} - 2$ u's, thus allowing (n-3) of them to be independent. This will be shown explicitly, presently.

We now examine the nature of the constraint equations amongst the u's for the general n-point function. In order to do this it is useful to look at the constraint equations in a suffix notation for the first few values of n.

<u>n = 4</u> - the ordinary four-point Veneziano amplitude.

There is only one $\binom{n-2}{2}$ constraint

$$\sum_{i=1}^{2} u_{Ai} = 1 \tag{5.6}$$

<u>n = 5</u> There are four constraints,

three $\binom{n-2}{2}$ of one type

$$\sum_{\substack{i=1 \\ i \neq j}}^{2} u_{Ai}\, u_{Aij} = 1 \tag{5.7}$$

Summations is over any two of the three meson lines, a different equation resulting from each of the three choices; j is the other meson line,

and one $\binom{n-2}{3}$ of a second type

$$\sum_{i=1}^{3} u_{Ai} = \sum_{i=1}^{3} u_{Bi} \tag{5.8}$$

<u>n = 6</u> There are eleven constraints,

six $\binom{n-2}{2}$ of one type

$$\sum_{i=1}^{2} u_{Ai}\, u_{Aik}\, u_{Ai\ell}\, u_{Aik\ell} = 1 \tag{5.9}$$

Summation is here over any two of the four meson lines, a different equation resulting from each of the six choices; k and ℓ are the other two meson lines;

four $\binom{n-2}{3}$ of a second type

$$\sum_{i=1}^{3} u_{Ai}\, u_{Aij} = \sum_{i=1}^{3} u_{Bi}\, u_{Bij} \tag{5.10}$$

and one $\binom{n-2}{4}$ of a third type

$$\sum_{i=1}^{4} u_{Ai}\, i_{Bi} = \sum_{\substack{i,j=1 \\ i \neq j}}^{4} u_{Aij}\, u_{Bij} \tag{5.11}$$

Summations are taken as already indicated. No terms are repeated on the right hand side of 5.11.

<u>n = 7</u> There are twenty-six constraints,

ten $\binom{n-2}{2}$ of one type

$$\sum_{i=1}^{2} (u_{Ai}\, u_{Aik}\, u_{Ai\ell}\, u_{Aim}\, u_{Aik\ell}\, u_{Ai\ell m}\, u_{Aimk}\, u_{Aik\ell m}) = 1 \tag{5.12}$$

ten $\binom{n-2}{3}$ of a second type

$$\sum_{i=1}^{3} u_{Ai}\, u_{Aij}\, u_{Aik}\, u_{Aijk} = \sum_{i=1}^{3} u_{Bi}\, u_{Bij}\, u_{Bik}\, u_{Bijk} \tag{5.13}$$

five $\binom{n-2}{4}$ of a third type

$$\sum_{i=1}^{4} u_{Ai}\, u_{Bi}\, u_{Aij}\, u_{Bij} = \sum_{\substack{i,k=1 \\ i\neq k}}^{4} u_{Aik}\, u_{Bik}\, u_{Aijk}\, u_{Bijk} \tag{5.14}$$

and one $\binom{n-2}{5}$

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{5} u_{Ai}\, u_{Bi}\, u_{Aij}\, u_{Bij} = \sum_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{5} u_{Aij}\, u_{Bij}\, u_{Aijk}\, u_{Bijk} \tag{5.15}$$

The summations are again taken as already indicated.

The pattern and inference is obvious; for the n-point function, the constraints on the u's can be separated into (n-3) distinct classes,

$\binom{n-2}{2}$ of type 1, $\binom{n-2}{3}$ of type 2, ..., $\binom{n-2}{n-2}$ (=1) of type n-3.

The total number of constraints is thus

$$\sum_{r=2}^{n-2} \binom{n-2}{r} = 2^{n-2} - n + 1,$$

as we stated earlier in this section without proof. We are thus able to show explicitly that we can express our n-point amplitude as an (n-3)-dimensional integral; this was not possible with the schemes proposed in (60) or (61), except in certain limiting cases.

We are now able to look at the constraint equations for the n-point amplitude. There are (n-3) distinct classes of equations. We label each class of equations by the number of equations in that class, i.e. the class r contains $\binom{n-2}{r}$ equations with $r = 2, \ldots, n-2$. We now look at each class of equations individually, for the n-point amplitude.

<u>Class of equation r = 2</u>

$$\sum_{i=1}^{2} \prod_{\substack{p \\ \{p\,:\,(Ai)(n-4)\}}} u_p = 1 \qquad (5.16)$$

The summation is taken over any two of the (n-2)-meson lines. For each term in the summation, the product is taken over all channels containing the two particles (A,i) together with any or none of the remaining (n-4) mesons. We see that for the cases

$n = 4, 5, 6, 7$ we reproduce equations (5.6, 7, 9, 12).

Class of equations $r = 3$

$$\sum_{i=1}^{3} \prod_{\substack{p \\ \{p\ :\ (Ai)(n-5)\}}} u_p = \sum_{i=1}^{3} \prod_{\substack{p \\ \{p\ :\ (Bi)(n-5)\}}} u_p \tag{5.17}$$

Summation is taken over any three of the (n-2) meson lines. For each term in the summation, the product is taken over all channels containing the two particles (A,i) or (B,i) together with any or none of the remaining (n-5) mesons. Equations (5.8, 10, 13) are reproduced for the cases $n = 5, 6, 7$.

Class of equations $r = 4$

$$\sum_{i=1}^{4} \prod_{\substack{p \\ \{p\ :\ (Ai)(n-6)\}}} u_p \prod_{\substack{p \\ \{p\ :\ (Bi)(n-6)\}}} u_p$$

$$= \sum_{\substack{j,k=1 \\ j\neq k}}^{4} \prod_{\substack{p \\ \{p\ :\ (Aij)(n-6)\}}} u_p \prod_{\substack{p \\ \{p\ :\ (Bij)(n-6)\}}} u_p \tag{5.18}$$

No terms on the right hand side are repeated. The interpretation in terms of sums and products is as already indicated. Equations (5.11) and (5.14) are reproduced for the cases $n = 6$ and $n = 7$.

Similar equations can be written down for higher values of r, $r \leq n-2$, but they suffer from considerable notational complexity, and we do not write them here. We content ourselves with the fact that we have developed a process whereby we can derive all the constraints on the u's for the general n-point process. This is essentially an adaptation of the programme given in (60).

There remain two questions: the first concerns the final equation in each case (the class $r = n-2$) which we must put in 'by hand' to ensure the conditions (i) ...(iii) of section 4.2, with (iii) in a suitably generalised form, are satisfied, e.g. equations (5.8) for $n = 5$, (5.11) for $n = 6$, (5.15) for $n = 7$. For the n-point function, this equation takes the form

$$\sum_{i_1,i_2,\ldots,i_{n-5}=1}^{n-2} (u_{Ai_1}u_{Bi_1})(u_{Ai_1i_2}u_{Bi_1i_2}) \ldots (u_{Ai_1i_2\,\ldots\,i_{n-5}}u_{Bi_1i_2\,\ldots\,i_{n-5}})$$

$$= \sum_{i_1\,i_2\,\ldots\,i_{n-4}}^{n-2} (u_{Ai_1i_2}u_{Bi_1i_2}) \ldots (u_{Ai_1i_2\,\ldots\,i_{n-4}}u_{Bi_1i_2\,\ldots\,i_{n-4}}) \tag{5.19}$$

This equation is only defined for $n \geq 6$. For $n-5 = 0, -1$, the equations are as listed (5.6) and (5.8). For the case of $n = 6, 7$ equation (5.19) reproduces (5.11) and (5.15) respectively. We state this last equation for completeness. This set of equations for the n-point function now has all the required properties.

The second question concerns the Jacobian for the generalised amplitude, which we now examine.

Jacobian factor

Consider first the five-point amplitude associated with figure 4.7. We may rewrite 4.1 as

$$\int_0^1 \int_0^1 dV_2 \prod_{i=1}^{3} \left\{ (u_{Ai})^{-\alpha_{Ai}-1} (u_{Bi})^{-\alpha_{Bi}-1} \right\} \tag{5.20}$$

where $dV_2 = du_{A1}\, du_{B2}\, J^{-1}$ (5.21)

with
$$J^{-1} = \frac{u_{A1} + u_{A2} + u_{A3}}{u_{A2}\, u_{B1}(u_{A2} + u_{B1})} \tag{5.22}$$

We now wish to generalise this expression first to the six-point amplitude of (5.1) associated with figure 5.2, then to the general n-point amplitude of (5.4) associated with figure 4.9.

Choose as integration variables for the six-point function those of (5.1), i.e. u_{A1}, u_{B2}, u_{A13}. We wish to find an expression for J^{-1} which reduces to an expression of the form (5.22) whenever u_{A1} or u_{B2} is equal to zero, and reduces to unity when $u_{A13} = 0$.

Consider the two five-point diagrams AB134 and AB234 contained in the Feynman diagram of figure 5.2. Associate variables x_{ij}, y_{ij} with these polygons, defined in terms of the u's according to the prescription given at the end of section 4.2. We may now write J^{-1} in (5.1) as

$$dV_3 = du_{A1}\, du_{B2}\, du_{A13}\, J^{-1}$$

$$dV_3 = du_{A1}\, du_{B2}\, du_{A13} \left(\frac{x_{A1} + x_{A3} + x_{A4}}{x_{B1}x_{A4}(x_{B1} + x_{A4})} \right) \left(\frac{y_{B2} + y_{B3} + y_{B4}}{y_{A2}y_{B3}(y_{A2} + y_{B3})} \right) \tag{5.23}$$

i.e. J^{-1} is the product of two five-point Jacobians associated with the two polygons. In terms of the u's

$$J^{-1} = \left(\frac{u_{A1}\, u_{A12} + u_{A3}\, u_{A23} + u_{A4}\, u_{A24}}{u_{A4}\, u_{B1}\, u_{A24}\, u_{A34}(u_{A4}\, u_{A24} + u_{B1}\, u_{A34})} \right)$$

$$\times \left(\frac{u_{B2}\, u_{A34} + u_{B3}\, u_{A24} + u_{B4}\, u_{A23}}{u_{B3}\, u_{A2}\, u_{A12}\, u_{A24}(u_{A2}\, u_{A12} + u_{B3}\, u_{A24})} \right) \tag{5.24}$$

This Jacobian transforms correctly when we replace the triplet of integration variables $(u_{A1}, u_{B2}, u_{A13})$ with any other triplet. Also, when we take $u_{A1} = 0$, J^{-1} becomes

$$\frac{u_{B2} + u_{B3} + u_{B4}}{u_{B3}\, u_{A12}(u_{B3} + u_{A12})} \tag{5.25}$$

- the usual five-point Jacobian, and similarly when $u_{B2} = 0$. When we take $u_{A13} = 0$, the Jacobian reduces to unity, as we would expect, (the ordinary four-point Veneziano formula has unit Jacobian factor).

This procedure can easily be extended to the general n-point function. The Jacobian can always be expressed as a product of (n-4) five-point like Jacobians, associated with (n-4) five-point diagrams contained in our n-point diagram of figure 4.9. We consider the case of odd and even n separately.

<u>2N-1 meson lines</u> i.e. n = 2N+1-point function.

We take as our variables of integration

$$u_{A1},\quad u_{A12}\quad \ldots\ldots\quad u_{A12\ldots N-1}$$

$$u_{N\,N+1},\ u_{B\,N+1\,N+2},\ \ldots\ldots,\ u_{B\,N+1\,N+2\,\ldots\,2N-1}$$

The five-point diagrams with which we associate our variables v_{ij}, w_{ij}, x_{ij}, y_{ij}, etc. are defined by the particles A, B together with one of the groups:

(1, 2, 3); (2, 3, 4); .... (N-2, N-1, N);

(N+1, N+2, N+3); (N+2, N+3, N+4); .... (2N-3, 2N-2, 2N-1);

(2N-2, 2N-1, N); (2N-1, N-1, N)

i.e. a total of 2N-3 (= n-4) five-point diagrams.

2N - meson lines i.e. n = 2N+2 - point function.

We take as our variables of integration,

$$u_{A1}, \quad u_{A12} \quad \ldots\ldots \quad u_{A12\ldots N}$$

$$u_{B\ N+1}, \ u_{B\ N+1\ N+2}, \ \ldots\ldots, \ u_{B\ N+1\ N+2\ \ldots\ 2N-1}$$

In this case the five-point diagrams with which we associate our variables v_{ij}, w_{ij}, x_{ij}, y_{ij}, etc. are defined by the particles A and B together with one of the groups

(1, 2, 3); (2, 3, 4); .... (N-2, N-1, N);

(N+1, N+2, N+3); (N+2, N+3, N+4); .... (2N-2, 2N-1, 2N)

(N-1, N, 2N); (2N-1, 2N, N)

i.e. a total of 2N-2 (= n-4) five-point diagrams.

In each case the Jacobian is written as a product of (n-4) five-point Jacobians associated with each of the (n-4) - five-point diagrams. Defined in this way the Jacobian factors have the right transformation properties when the integration variables are changed to an alternative set, and reduce to a suitable product of lower-point Jacobians when any one of the integration variables is set equal to zero.

This completes our prescription for generalising our amplitude to processes with an arbitrary number of meson lines.

5.3 Koba-Nielsen parametrisation

The representation we proposed in section 4.5 for our five-point amplitude may be extended in a straightforward way for the six-point amplitude. The details, which are rather complicated and tedious are given in the appendix. Here, we simply quote the result.

Provided that we maintain our restriction $p_A^2 = p_B^2$, we may write for our six-point amplitude associated with the Feynman diagram of figure 5.2.

$$
\begin{aligned}
F_6 = \int \cdots \int \Bigg\{ & (dz_A d\tilde{z}_A)(dz_B d\tilde{z}_B) \prod_{i=1}^{4} (dz_i d\tilde{z}_i) \\
& \times \prod_{i=1}^{4} \left[\left\{ (z_A - z_i)(\tilde{z}_i - z_A) \right\}^{-\alpha_{Ai}-1} \left\{ (z_B - z_i)(\tilde{z}_i - z_B) \right\}^{-\alpha_{Bi}-1} \right] \\
& \times \prod_{\substack{i,j=1 \\ (i<j)}}^{4} \left[(z_i - \tilde{z}_j)(z_j - \tilde{z}_i) \right]^{K_{ij}} \\
& \times \left[(z_A - z_B)(z_B - z_A) \right]^{\frac{1}{2}(\alpha_{A12}+\alpha_{A13}+\alpha_{A14}+\alpha_{A23}+\alpha_{A24}+\alpha_{A34})} \Bigg\}^{\frac{1}{2}}
\end{aligned}
\tag{5.26}
$$

where
$$K_{ij} = \frac{1}{4}\Big\{ \alpha_{Ai} + \alpha_{Aj} + \alpha_{Bi} + \alpha_{Bj} - \alpha_{Ak} - \alpha_{A\ell} - \alpha_{Bk} - \alpha_{B\ell} - \alpha_{Aij} - \alpha_{Bij} + \alpha_{Aik} + \alpha_{Bik} + \alpha_{Ai\ell} + \alpha_{Bi\ell} \Big\} \qquad (5.27)$$

We have omitted from the integrand those functions corresponding to X in (4.66) for the five-point function, since their exponent is $\frac{\alpha'}{2}(p_A^2 - p_A^2)$.

Assuming the trajectories are all identical, linear of the form,

$$\alpha_{Ai} = \alpha_o + \alpha'(p_A + p_i)^2$$

$$\alpha_{Aij} = \alpha_o + \alpha'(p_A + p_i + p_j)^2 \qquad (5.28)$$

and similarly for α_{Bi}, α_{Bij}, then the kinematic conditions arising from four-momentum conservation demand that the exponent of the term $[(z_i - \tilde{z}_j)(z_j - \tilde{z}_i)]$ reduces to

$$K_{ij} = -2\alpha' p_i \cdot p_j + \tfrac{1}{2}\alpha(M^2) \qquad (5.29)$$

where we have taken $p_A^2 = p_B^2 = M^2$, M the baryon mass, and the exponent of the term $[(z_A - z_B)(z_B - z_A)]$ reduces to,

$$-2\alpha' p_i \cdot p_j + \alpha' M^2 + 4\alpha(\mu^2) \qquad (5.30)$$

where we have taken $p_1^2 = p_2^2 = p_3^2 = p_4^2 = \mu^2$, μ the meson mass. As for the five-point function, the momentum dependence of each of the above two terms is simply $-2\alpha' p_i \cdot p_j$ and $-2\alpha' p_A \cdot p_B$ respectively.

We conjecture that similar expressions can be generated for the higher order functions. The only modification that we make will be to extend the limits in the products from 1 to n-2 for the n-point process, in (5.26).

5.4 Concluding remarks.

We have shown in the last two chapters how we may construct Veneziano-like multiparticle amplitudes corresponding to processes involving the interaction of two classes of hadrons, where each channel in which resonances occur contains one particle from one class, and any number from the second class. The motivation for the study of such amplitudes was derived from the work of Burnett and Schwarz, and the consideration of a certain type of meson-baryon quark duality diagram. The method we use is essentially an adaptation of a prescription described by Mandelstam for a generalisation of the multi-Veneziano amplitude to a different set of processes. The two generalisations coincide for the case of $n=5$. For this case we have examined the asymptotic behaviour and found that a multi-Regge behaviour was obtained when the exchanged Regge poles occurred in meson-baryon channels. We were also able to derive a series representation for our five-point amplitude which would form the basis of any attempt to compute the function numerically. Although we do not examine these properties specifically for the n-point function, we know that any four- or five-point function that we factor out of the n-point amplitude will have Regge behaviour and will have a series representation.

A parametrisation of our amplitudes in terms of Koba-Nielsen variables is not immediately obvious. However, a modification proposed by Mandelstam (60) is used which enables us to express our amplitudes in a compact form, which is readily seen to generalise when the external baryons have the same mass.

Although we have examined the factorisation properties of our constraints in so far as when a resonance occurs in a particular channel, the appropriate lower point functions result, we have not examined the residues at any pole to determine the spectrum of resulting particles on the leading trajectory, (37, 38). Mandelstam (60) has examined this problem. For the five-point function his discussion is relevant here. In order to obtain the simplest possible spectrum various factors are introduced into the amplitude. For the five-point function they are of the form

$$[1 - \alpha(1-\alpha)]^{-p_1 \cdot p_2} \, [1 - \beta(1-\beta)]^{-p_2 \cdot p_3} \, [1 - \gamma(1-\gamma)]^{-p_3 \cdot p_1} \tag{5.31}$$

They do not alter the essential properties of the amplitude that we have examined, such as the pole structure, the symmetry properties and the asymptotic behaviour, but the series representation will be considerably complicated.

If we are to describe processes in which not all the particles are identical, we may well have to include amplitudes of the form we have constructed.

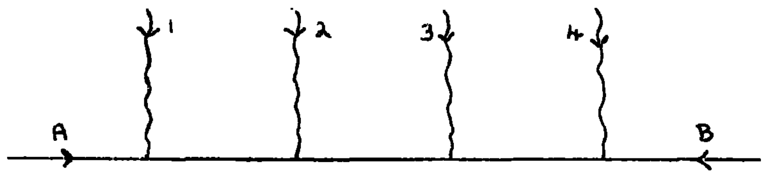


Figure 5.1 The six-point generalisation of the Burnett and Schwarz process

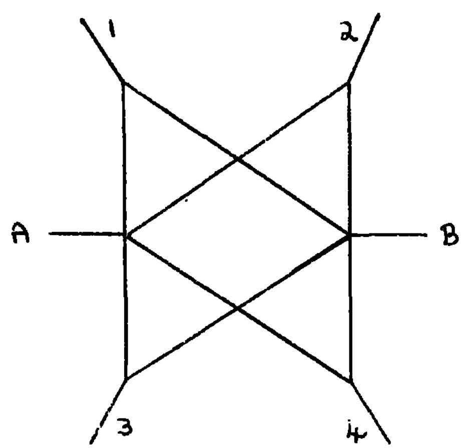


Figure 5.2 The Feynman diagram associated with figure 5.1

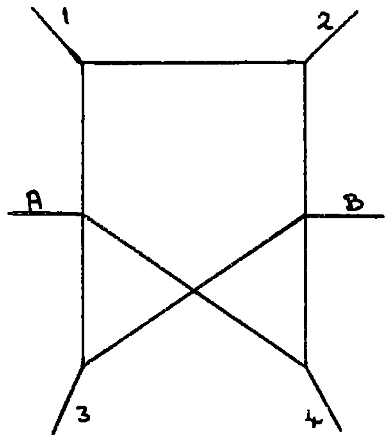


Figure 5.3 The Feynman diagram associated with the six-point generalisation of Mandelstam (60)

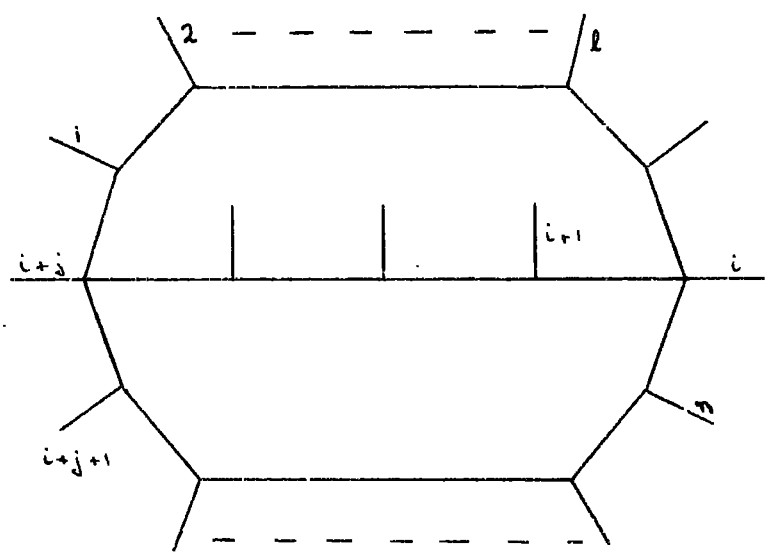


Figure 5.4 The Feynman diagram associated with the n-point generalisation of Mandelstam

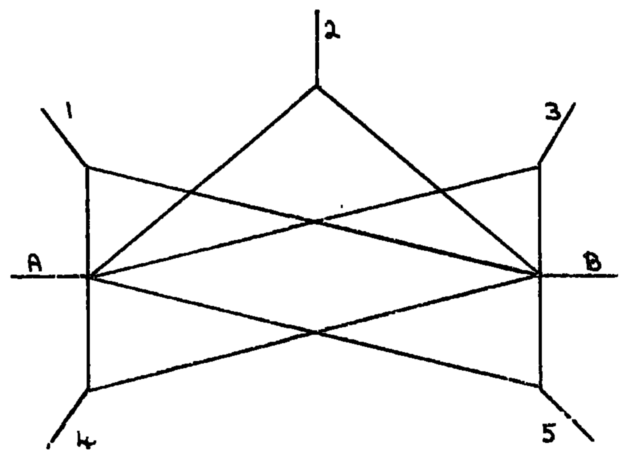


Figure 5.5 Feynman diagram associated with our seven-point amplitude

APPENDIX

The derivation of equation (5.26).

With reference to equation (5.2), we define

$$\alpha = u_{A1}u_{B2}u_{A13}u_{A14} \; ; \qquad 1-\alpha = u_{A2}u_{B1}u_{A23}u_{A24}$$

$$\beta = u_{A2}u_{B3}u_{A12}u_{A24} \; ; \qquad 1-\beta = u_{A3}u_{B2}u_{A13}u_{A34}$$

$$\gamma = u_{A3}u_{B4}u_{A13}u_{A23} \; ; \qquad 1-\gamma = u_{A4}u_{B3}u_{A14}u_{A24}$$

$$\delta = u_{A4}u_{B1}u_{A34}u_{A24} \; ; \qquad 1-\delta = u_{A1}u_{B4}u_{A12}u_{A13}$$

$$\lambda = u_{A2}u_{B4}u_{A12}u_{A23} \; ; \qquad 1-\lambda = u_{A4}u_{B2}u_{A14}u_{A34}$$

$$\mu = u_{A1}u_{B3}u_{A12}u_{A14} \; ; \qquad 1-\mu = u_{A3}u_{B1}u_{A23}u_{A34} \tag{A.1}$$

satisfying the following,

$$\alpha\beta\gamma\delta = (1-\alpha)(1-\beta)(1-\gamma)(1-\delta)$$

$$\alpha\lambda(1-\gamma)(1-\mu) = (1-\alpha)(1-\lambda)\gamma\mu$$

$$\mu(1-\beta)\lambda\delta = (1-\mu)\beta(1-\lambda)(1-\delta) \tag{A.2}$$

Effectively we may now solve for all u's in terms of any three of $\alpha, \beta, \gamma, \delta, \lambda, \mu$.

We now parametrise $\alpha, \beta,$ etc. in terms of the Koba-Nielsen variables

$$\alpha = \frac{(\tilde{z}_1 - z_A)(z_2 - z_B)}{(\tilde{z}_1 - z_2)(z_A - z_B)} \; ; \qquad \beta = \frac{(\tilde{z}_2 - z_A)(z_3 - z_B)}{(\tilde{z}_2 - z_3)(z_A - z_B)}$$

$$\gamma = \frac{(\tilde{z}_3 - z_A)(z_4 - z_B)}{(\tilde{z}_3 - z_4)(z_A - z_B)} \; ; \qquad \delta = \frac{(\tilde{z}_4 - z_A)(z_1 - z_B)}{(\tilde{z}_4 - z_1)(z_A - z_B)}$$

$$\lambda = \frac{(\tilde{z}_2 - z_A)(z_4 - z_B)}{(\tilde{z}_2 - z_4)(z_A - z_B)} \; ; \qquad \mu = \frac{(\tilde{z}_1 - z_A)(z_3 - z_B)}{(\tilde{z}_1 - z_3)(z_A - z_B)} \qquad \text{(A.3)}$$

such that the relations (A.2) are automatically satisfied, with $\tilde{z}_i$ defined as in (4.69).

$\tilde{\alpha}$ is defined in the obvious way;

$$\tilde{\alpha} = \frac{(z_1 - z_A)(\tilde{z}_2 - z_B)}{(z_1 - \tilde{z}_2)(z_A - z_B)} \qquad \text{(A.4)}$$

and similarly for $\tilde{\beta}$, $\tilde{\gamma}$, etc.

The solutions to the constraint equations (5.2) and (5.3) for the six-point amplitude may be written as:

$$(u_{A1})^4 = \frac{\alpha\mu(1 - \delta)}{\beta\,\gamma\,\lambda} \cdot \frac{\theta\;\phi\;\psi}{\xi} \cdot \Gamma \qquad \text{(a)}$$

$$(u_{A2})^4 = \frac{\alpha\,\beta\,\lambda}{\gamma\mu(1 - \delta)} \cdot \frac{\xi\;\phi\;\psi}{\theta} \cdot \Gamma \qquad \text{(b)}$$

$$(u_{A3})^4 = \frac{\beta\,\gamma\,\mu}{\alpha\lambda(1-\delta)} \cdot \frac{\xi\,\theta\,\psi}{\phi} \cdot \Gamma \qquad \text{(c)}$$

$$(u_{A4})^4 = \frac{\lambda(1-\delta)}{\alpha\,\beta\,\mu} \cdot \frac{(1-\gamma)^4}{\gamma^3} \cdot \frac{\xi\,\theta\,\phi}{\psi} \cdot \Gamma \qquad \text{(d)}$$

$$(u_{B1})^4 = \frac{\mu(1-\delta)}{\beta\,\gamma\,\lambda} \cdot \frac{(1-\alpha)^4}{\alpha^3} \cdot \frac{\xi}{\theta\,\phi\,\psi} \cdot \Gamma \qquad \text{(e)}$$

$$(u_{B2})^4 = \frac{\alpha\,\beta\,\lambda}{\gamma\mu(1-\delta)} \cdot \frac{\theta}{\xi\,\phi\,\psi} \cdot \Gamma \qquad \text{(f)}$$

$$(u_{B3})^4 = \frac{\beta\,\gamma\,\mu}{\alpha\lambda(1-\delta)} \cdot \frac{\phi}{\xi\,\theta\,\psi} \cdot \Gamma \qquad \text{(g)}$$

$$(u_{B4})^4 = \frac{\gamma\lambda(1-\delta)}{\alpha\,\beta\,\mu} \cdot \frac{\psi}{\xi\,\theta\,\phi} \cdot \Gamma \qquad \text{(h)}$$

$$(u_{A12})^4 = \frac{\mu\beta\lambda(1-\delta)}{\alpha\gamma} \cdot \frac{\theta\xi}{\phi\psi} \cdot \Gamma^{-1} \qquad \text{(i)}$$

$$(u_{A13})^4 = \frac{\alpha\beta\gamma(1-\delta)}{\lambda\mu} \cdot \frac{\phi\xi}{\theta\psi} \cdot \Gamma^{-1} \qquad \text{(j)}$$

$$(u_{A14})^4 = \frac{\alpha\,\gamma\,\mu\,\lambda}{\beta(1-\delta)} \cdot \frac{\psi\xi}{\theta\phi} \cdot \Gamma^{-1} \qquad \text{(k)}$$

$$(u_{A23})^4 = \frac{\alpha\,\gamma\,\mu\,\lambda}{\beta(1-\delta)} \cdot \frac{\theta\phi}{\psi\xi} \cdot \Gamma^{-1} \qquad (\ell)$$

$$(u_{A24})^4 = \frac{\alpha\beta\gamma(1-\delta)}{\lambda\mu} \cdot \frac{\theta\psi}{\phi\xi} \cdot \Gamma^{-1} \qquad (m)$$

$$(u_{A34})^4 = \frac{(1-\beta)^4}{\beta^3} \cdot \frac{\mu\lambda(1-\delta)}{\alpha\gamma} \cdot \frac{\phi\psi}{\theta\xi} \cdot \Gamma^{-1} \qquad (n)$$

(A.5)

where

$$\theta = \frac{1+\mu\gamma-\mu}{1+\mu\gamma-\gamma}\,; \qquad \phi = \frac{1+\alpha\lambda-\alpha}{1+\alpha\lambda-\lambda}\,;$$

$$\psi = \frac{1+\alpha\beta-\alpha}{1+\alpha\beta-\beta}\,; \qquad \xi = \frac{1+\beta\gamma-\beta}{1+\beta\gamma-\gamma}\,;$$

$$\Gamma = \frac{\lambda\mu(1-\delta)(1-\beta) + \alpha\gamma\lambda\mu + \alpha\beta\gamma(1-\delta)}{\mu(1-\alpha)(1-\delta) + \beta(\lambda\alpha+\gamma\mu) + \lambda(1-\gamma)(1-\delta)} \qquad (A.6)$$

Now, we wish to write our six-point function in the form

$$F_6 = \int\cdots\int\Big\{(du_{A1}d\tilde{u}_{A1})(du_{B2}d\tilde{u}_{B2})(du_{A13}d\tilde{u}_{A13})(J^{-1}\,\tilde{J}^{-1})$$

$$\times \prod_{i=1}^{4} (u_{Ai}\tilde{u}_{Ai})^{-\alpha_{Ai}-1} \cdot (u_{Bi}\tilde{u}_{Bi})^{-\alpha_{Bi}-1} \prod_{\substack{i,j=1\\ i<j}}^{4} (u_{Aij}\tilde{u}_{Aij})^{-\alpha_{Aij}-1}\Big\}^{\frac{1}{2}}$$

(A.7)

The exponent in the integrand of the term $(\theta\tilde{\theta})$ will be

$$\frac{1}{4}\Big\{-\alpha_{A1} + \alpha_{A2} - \alpha_{A3} - \alpha_{A4} + \alpha_{B1} - \alpha_{B2} + \alpha_{B3} + \alpha_{B4}$$

$$- \alpha_{A12} + \alpha_{A13} + \alpha_{A14} - \alpha_{A23} - \alpha_{A24} + \alpha_{A34}\Big\} \tag{A.8}$$

This reduces simply to

$$\tfrac{1}{4}\alpha'(p_B^2 - p_A^2) \tag{A.9}$$

using the kinematic constraint of four-momentum conservation and linear trajectories of the form (5.28). Exactly similar arguments apply to the exponents of Φ, ψ, ξ, and Γ in (A.7). Taking $p_A^2 = p_B^2$, we ignore these terms.

We may now write the remaining parts of the u_{Ai}'s and u_{Aij}'s as

$$(u_{Ai}\tilde{u}_{Ai})^4 = \Big[(\tilde{z}_i - z_A)(z_i - z_A)\Big]^4$$

$$\times \left\{ \frac{(z_j - \tilde{z}_k)(\tilde{z}_j - z_k)(z_j - \tilde{z}_\ell)(\tilde{z}_j - z_\ell)(z_k - \tilde{z}_\ell)(\tilde{z}_k - z_\ell)}{(z_i - \tilde{z}_j)(\tilde{z}_i - z_j)(z_i - \tilde{z}_k)(\tilde{z}_i - z_k)(z_i - \tilde{z}_\ell)(\tilde{z}_i - z_\ell)} \right\}$$

$$\times\ \chi_{Ai} \tag{A.10}$$

and similarly for $(u_{Bi}\,\tilde{u}_{Bi})^4$, and

$$(u_{Aij}\tilde{u}_{Aij})^4 = \frac{1}{[(z_A - z_B)(z_B - z_A)]^2}$$

$$\times \left\{ \frac{(\tilde{z}_i - z_j)(z_i - \tilde{z}_j)(\tilde{z}_k - z_\ell)(z_k - \tilde{z}_\ell)}{(\tilde{z}_i - z_k)(z_i - \tilde{z}_k)(\tilde{z}_i - z_\ell)(z_i - \tilde{z}_\ell)(\tilde{z}_j - z_k)(z_j - \tilde{z}_k)(\tilde{z}_j - z_\ell)(z_j - \tilde{z}_\ell)} \right\}$$

$$\times \; \chi_{Aij} \qquad \text{(A.11)}$$

where i, j, k, ℓ are the four meson lines,

and

$$\chi_{A1} = \chi_{A2} = \frac{1}{y} \left\{ \frac{(z_2 - z_B)(\tilde{z}_2 - z_B)}{(\tilde{z}_2 - z_A)(z_2 - z_A)} \right\}^2$$

$$\chi_{A3} = \chi_{A4} = \frac{1}{y} \left\{ \frac{(z_3 - z_B)(\tilde{z}_3 - z_B)}{(z_3 - z_A)(\tilde{z}_3 - z_A)} \right\}^2$$

$$\chi_{B1} = \chi_{B2} = \frac{1}{y} \left\{ \frac{(z_2 - z_A)(\tilde{z}_2 - z_A)}{(z_2 - z_B)(\tilde{z}_2 - z_B)} \right\}^2$$

$$\chi_{B3} = \chi_{B4} = \frac{1}{y} \left\{ \frac{(z_3 - z_A)(\tilde{z}_3 - z_A)}{(z_3 - z_B)(\tilde{z}_3 - z_B)} \right\}^2$$

$$\chi_{A12} = y \left\{ \frac{(z_2 - z_A)(\tilde{z}_2 - z_A)(z_3 - z_B)(\tilde{z}_3 - z_B)}{(z_2 - z_B)(\tilde{z}_2 - z_B)(z_3 - z_A)(\tilde{z}_3 - z_A)} \right\}^2$$

$$\chi_{A34} = y \left\{ \frac{(z_2 - z_B)(\tilde{z}_2 - z_B)(z_3 - z_A)(\tilde{z}_3 - z_A)}{(z_2 - z_A)(\tilde{z}_2 - z_A)(z_3 - z_B)(\tilde{z}_3 - z_B)} \right\}^2$$

$$\chi_{A13} = \chi_{A24} = \chi_{A14} = \chi_{A23} = y \qquad \text{(A.12)}$$

where

$$y = (\tilde{z}_1 - z_A)(z_1 - z_A)(z_2 - z_B)(\tilde{z}_2 - z_B)(z_3 - z_A)(\tilde{z}_3 - z_A)(z_4 - z_B)(\tilde{z}_4 - z_B) \qquad \text{(A.13)}$$

These terms appear very complicated. However, their exponents in the integral representation of F_6 are all of the form

$$\text{const.} \ (p_B^2 - p_A^2) \ ,$$

for example, the exponent of the term

$$\left(\frac{(z_2 - z_B)(\tilde{z}_2 - z_B)}{(z_2 - z_A)(\tilde{z}_2 - z_A)} \right) \qquad \text{is}$$

$$\tfrac{1}{2} (\alpha_{B1} + \alpha_{B2} - \alpha_{A1} - \alpha_{A2} + \alpha_{A12} - \alpha_{A34}) \qquad \text{(A.14)}$$

which reduces simply to $\frac{\alpha'}{2} (p_B^2 - p_A^2)$, and hence we may ignore these too.

So, finally, we may write for our amplitude F_6 ,

$$F_6 = \int \cdots \int \Big\{ (dz_A d\tilde{z}_A)(dz_B d\tilde{z}_B) \prod_{i=1}^{4} (dz_i d\tilde{z}_i)$$

$$\times \prod_{i=1}^{4} \left[\Big\{ (z_A - z_i)(\tilde{z}_i - z_A) \Big\}^{-\alpha_{Ai}-1} \Big\{ (z_B - z_i)(\tilde{z}_i - z_B) \Big\}^{-\alpha_{Bi}-1} \right]$$

$$\times \prod_{\substack{i,j=1 \\ (i<j)}}^{4} \left[(z_i - \tilde{z}_j)(z_j - \tilde{z}_i) \right]^{K_{ij}} \left[(z_A - z_B)(z_B - z_A) \right]^{L} \Big\}^{\frac{1}{2}}$$

(A.15)

where $K_{ij} = \frac{1}{4} \{ \alpha_{Ai} + \alpha_{Bi} + \alpha_{Aj} + \alpha_{Bj} - \alpha_{Ak} - \alpha_{Bk} - \alpha_{A\ell} - \alpha_{B\ell}$

$-\alpha_{Aij} - \alpha_{Bij} + \alpha_{Aik} + \alpha_{Bik} + \alpha_{Ai\ell} + \alpha_{Bi\ell} \}$

(A.16)

and $$L = \frac{1}{2} \sum_{\substack{i,j=1 \\ (i<j)}}^{4} \alpha_{Aij} \qquad \text{(A.17)}$$

which is just the expression (5.26).

As we stated in section 4.5, we have omitted from the integrand a term analagous to (dG_3^{-1}) defined in (2.56) for the tree graph amplitude, which forms a constant multiplier of the amplitude. Up to a constant factor, our two expressions (A.7) and (A.15) are identical.

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