Characterisations of euler n-spheres

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Euler n-Spheres

by

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CONTENTS

SUMMARY

The purpose of the present work is to give a multidimensional generalisation of the Liebmann-Süss theorem by means of integral formulas. In order to achieve this it was first necessary to extend the classical local curvature theory. This was done by using the mean normal curvature vector (as a canonical cross-section of the normal bundle) to define generalised second and third fundamental forms from which curvature invariants could be obtained. Secondly, using these invariants, we derive a multidimensional generalisation of the classical integral formula of Minkowski.

As a final application of the integral formula technique we obtain an integral formula for a volume-preserving diffeomorphism between two compact immersed submanifolds. Using this we find conditions under which a diffeomorphism is an isometry. This generalises similar work of Chern and Hsiung.

INTRODUCTION

One of the most interesting classical results in global differential geometry is the

Liebmann H-theorem [1]: The only ovaloids of constant mean curvature H in Euclidean space E^3 are the spheres.

In 1901 Hilbert [2] gave an ingenious proof of this result and further proved the

Hilbert K-theorem: A closed surface of constant Gaussian curvature K in E^3 is a sphere.

These theorems characterise the sphere in E^3. In 1929 the H-theorem was generalised by W. Süss [3] for the case of convex n-dimensional hypersurfaces (assuming no self-intersections) in E^{n+1}.

In global differential geometry there are available three 'classical' methods of proving general uniqueness theorems: 1. the 'Index method', 2. the Maximum method, 3. the Integral formula method. The most effective tool used so far for closed strictly convex two-dimensional surfaces has been the Index method; but this does not readily generalise to higher dimensions.

The Maximum method is primarily the work of E. Hopf [4]; however in a long series of papers in 1958 A.D.Alexandrov [5] extended the method to show that convexity is unnecessary for the validity of the Liebmann-Süss theorem. Proofs up to 1958 have primarily used the elementary symmetric functions of the principal curvatures: however Alexandrov weakened the convexity condition by using more general functions.

The principal difficulty in using the third method is the derivation of the integral formulas themselves, - the actual proof which follows from them is mostly routine. Recently Hsiung [6] and Feeman and Hsiung [8] have used this method to further extend the Liebmann-Süss theorem for hypersurfaces embedded in a riemannian manifold of constant curvature. Hsiung was also able to weaken the convexity condition to a 'star-shape' condition.

The Liebmann-Süss theorem is closely related to the classical integral formulas of Minkowski. It is thus one of our main aims to develop multidimensional analogues of the Minkowski formulas for arbitrary co-dimension, and hence investigate whether the above mentioned theorems are special cases of more general theorems.

The essential difficulty in the translation of the known methods of global surface theory to multidimensional differential geometry is the formal generalisation of the second and third fundamental forms for codimension greater than one; for each normal vector to an embedded submanifold gives rise to a corresponding second and third fundamental form as well as to corresponding curvatures - in particular the r^{th}- mean curvature, corresponding to these normals.

The work of Chern [9], Hsiung[7], Schneider [12], Gulbinat [13] and Stong [14] clearly indicates the necessity of extending the known local theory of submanifolds and, in particular, to define curvature invariants which depend only on the Normal bundle of the embedding and not on the selection of special cross-sections of the normal bundle. The canonical second and third fundamental quadratic forms defined, from which we obtain such curvature invariants, are meaningful analogues of the second and third fundamental forms of classical surface theory. The third fundamental form can be interpreted simply as the metric of the 'spherical image'.

By analogy to the classical relation between the second and third fundamental forms it is possible to derive simple relations between the canonical fundamental forms and the global metric of the submanifold. The aim of this local part of the work is to help clarify the relationship of these forms to the intrinsic geometry of the submanifold.

It turns out that with suitable specialisations and restrictions many of the recent generalisations of the Minkowski formulas can be derived as special cases of our generalised integral formulas. Further, by restricting ourselves to co-dimension one we obtain the results of Süss [3], Hsiung [7] and Yano [15] directly. .

In 1963 Chern and Hsiung [17], using integral formulas, obtained conditions that a diffeomorphism between two compact submanifolds in Euclidean space should be an isometry. We extend their result to the case where the ambient space is an arbitrary riemannian manifold.

In Chapter I we develop the necessary analytic basis for our work, and for the first part of this chapter we depend mainly on the paper of H. Flanders [18]. In Chapter II we develop a local curvature theory of the mean normal curvature vector which overcomes the disadvantage of the earlier theory due to K. Voss [19], K. Leichtweiss [20] and W. Gulbinat [13] which depended on the choice of special cross-sections of the normal bundle. In Chapter III we derive the generalised Minkowski integral formulas; and in Chapter IV we apply them to give a global characterisation of riemannian n-spheres and their higher co-dimensional analogue Euler n-spheres. In Chapter V we generalise the isometry theorem of Chern and Hsiung [17].

We conclude by indicating possible lines of future research in this area.

CHAPTER I

In this chapter we develop the basic differential-geometric theory of isometrically immersed submanifolds of a connected riemannian manifold.

§ 1. Smooth Manifolds

Let M^n be an n-dimensional differentiable manifold of class C^∞, i.e. on M we assume an infinitely differentiable structure. Let $F(M)$ denote the space of all C^∞ real-valued functions on M. A tangent vector to M at a point $p \in M$ is a real-valued function $v : F(M) \to \mathbb{R}$ satisfying:

a) $v(f + g) = v(f) + v(g)$ $\qquad f,g \in F(M)$

b) $v(af) = av(f)$ $\qquad a \in \mathbb{R}$

c) $v(fg) = v(f)g(p) + f(p)v(g)$.

The set M_p of all tangent vectors at $p \in M$ forms an n-dimensional vector space called the tangent space at p. If $U,(x^1, \ldots, x^n)$ is a local coordinate neighbourhood of p, then the vectors e_i, $1 \leq i \leq n$, at p defined by: $e_i(f) := (\partial f/\partial x^i)_p$ form a basis for M_p. Any tangent vector v at p can thus be represented uniquely by $v = v^i e_i$ where v^i are unique constants.

The dual space M_p^* of M_p is called the space of one-forms at p, and it is easily seen that the basis of M_p^* dual to the basis $\{e_i\}$ of M_p is $dx^1, \ldots, dx^n$.

A mapping $X : M \to M_p$ such that $p \to X(p) \in M_p$ is called a vector field if in each coordinate neighbourhood U the expression $X(p) = v^i(x^1, \ldots, x^n)e_i$ defines C^∞ functions v^i on a neighbourhood of Euclidean n-space E^n. We denote by $\mathfrak{X}(M)$ the space of all vector fields on M over the ring $F(M)$. Similarly we define a differential 1-form as a function: $\omega : M \to M_p^*$.

DEF. 1: Tangent Bundle (TM, π, M): The tangent bundle of M is the 2n-dimensional fibre bundle $TM = \bigcup_{p \in M} M_p$; it has in a natural way a C^∞ structure in the following way: if $v = v^i e_i$ at $p \in M$ with local coordinates $x^1, \ldots, x^n$ then we take the coordinate system $\{x^1, \ldots, x^n, v^1, \ldots, v^n\}$ on the neighbourhood $\bigcup_{p \in M} M_p$. Clearly a vector field on M is simply a cross-section of this bundle.

DEF. 2: Frame Bundle (FM, π, M): A frame at $p \in M$ is any basis of the vector space M_p. The set of all frames at all points of M is called the frame bundle of M. Let $e_1, \ldots, e_n$ be a moving frame on U, then $(e_1)_p, \ldots, (e_n)_p$ is one basis of M_p. The most general basis of M_p stems from this one by applying an arbitrary non-singular transformation $B = (b_i^j)$ to it; we get

$$f = (f_1, \ldots, f_n) \quad \text{where} \quad f_i = b_i^j (e_j)_p .$$

Clearly the independent variables $\{x^1, \ldots, x^n, \ b_i^j\}$ coordinatise the neighbourhood of FM, and dim FM = $n + n^2$.

§ 2. Multivectors and Forms.

From the space M_p we can form the space $\Lambda^q M_p$ of q-vectors at $p \in M$. Similarly we can form the space $\Lambda^q M_p^*$ of q-forms at $p \in M$. By considering the union of these respective spaces over all of M, each of these spaces gives rise to a corresponding bundle on which the General Linear group acts. The C^∞ cross-sections of these bundles are called fields of q-vectors and q-forms respectively.

Note that F(M) acts on both of the vector spaces $\mathfrak{X}^q(M)$ and $\mathfrak{X}_q^*(M)$ of all q-vector fields and q-form fields respectively. Whence with F(M) as coefficients we have $\mathfrak{X}^q = \Lambda^q M$ over F(M) and $\mathfrak{X}_q^* = \Lambda^q M^*$.
Note also that $\mathfrak{X}_o^*(M)$ is just F(M).
We now form the tensor product:

$$\mathfrak{X}_q^p = \mathfrak{X}_q^* \otimes \mathfrak{X}^p$$

thus forming the space of all p-vectors with q-form coefficients. We can consider this space as the space of cross-sections of the bundle of all elements of

$$(\Lambda^q M_m^*) \otimes (\Lambda^p M_m) \qquad\qquad m \in M$$

this tensor product being taken over $\mathbb{R}$. Again the ring F(M) acts as a coefficient ring for $\mathfrak{X}_q^p$.

If we form the algebras $\mathfrak{X}^* = \sum_q \oplus \mathfrak{X}^*_q$, $\mathfrak{X} = \sum_p \oplus \mathfrak{X}^p$ over

$F(M)$ with the usual Grassmann products, we have defined, by linearity, an operation:

$$\mathfrak{X}^p_q \times \mathfrak{X}^r_s \to \mathfrak{X}^{p+r}_{q+s}$$

and

$$(\omega \otimes v) _\wedge (\theta \otimes \phi) = (\omega _\wedge \theta) \otimes (v _\wedge \phi) \qquad \omega \in \mathfrak{X}^*_q \quad \theta \in \mathfrak{X}^*_s$$
$$\phi \in \mathfrak{X}^r \quad v \in \mathfrak{X}^p$$

The operation is associative and distributive and satisfies:

$$v \otimes \omega = (-1)^{pr+qs} \omega \otimes v \qquad v \in \mathfrak{X}^p_q \quad \omega \in \mathfrak{X}^r_s .$$

§ 3. The Cartan Calculus

Let $\{e_1, \ldots, e_n\}$ be a moving frame (i.e. a cross-section of FM) on a coordinate neighbourhood U of M, and let $\{\omega^1, \ldots, \omega^n\}$ be the dual co-frame. We now form the identity transformation on M_p and denote it as usual by dP: $dP = \omega^i \otimes e_i \quad 1 \leqslant i \leqslant n$.

Clearly $dP \in \mathfrak{X}^1_1$, is intrinsic, and we shall in future omit the symbol $\otimes$. We can further facilitate calculation if we agree to use matrix notation. Let ω be the $1 \times n$ matrix $(\omega^1, \ldots, \omega^n)$ and e the $n \times 1$ matrix $(e_1, \ldots, e_n)^T$ and further agree to identify a 1×1 matrix with its single element (where superscript ^{T} denotes transpose). Then the above equation becomes:

$$dP = \omega e \tag{1.1}$$

We define the exterior derivative operation d to be a mapping: $\mathfrak{X}^*_q \to \mathfrak{X}^*_{q+1}$ characterised by:

a) Linearity.

b) $dd = 0$.

c) On a local coordinate neighbourhood $(U, x^1, \dots, x^n)$ we have:

$$df = \frac{\partial f}{\partial x^i} dx^i \qquad f \text{ a function on } U.$$

d) $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^q \omega \wedge d\theta \qquad \omega \in \mathfrak{X}^*_q$.

DEF. 3: <u>An affine connexion</u> on M is simply an operator $d : \mathfrak{X}^1_o \to \mathfrak{X}^1_1$ satisfying:

i) $d(v + \omega) = dv + d\omega$

ii) $d(fv) = df\, v + f\, dv$.

It can be proved from this that if d is an affine connexion on M^n then d induces a unique collection of operators, which we shall also simply denote by d, one per space $\mathfrak{X}^p_q \to \mathfrak{X}^p_{q+1}$ satisfying:

a) $d(v + \omega) = dv + d\omega \qquad v, \omega \in \mathfrak{X}^p_q$

b) $d(v \wedge \omega) = dv \wedge \omega + (-1)^q v \wedge d\omega \qquad v \in \mathfrak{X}^p_q \quad \omega \in \mathfrak{X}^r_s$

c) d coincides with the affine connexion on $\mathfrak{X}^1_o$ and with exterior derivative on $\mathfrak{X}^o_q = \mathfrak{X}^*_q$.

For a proof of this result see Flanders [18] .

Now each $e_i \in \mathfrak{X}^1_o$ hence $de_i \in \mathfrak{X}^1_1$, we can thus write $de_i = \omega_i^j e_j$ or in matrix form:

$$de = \Omega e \qquad \text{where } \Omega = (\omega_i^j) \tag{1.2}$$

Ω is called the connexion matrix and its elements (1-forms) the connexion forms. In a sense equations (1.1) and (1.2) and the operator d contain the entire calculus of an affinely connected n-manifold. For we now simply apply d to these equations and to the resulting equations.

We define the <u>torsion matrix</u> τ of M^n to be the $1 \times n$ matrix of 2-forms:

$$d(dP) = \tau e \qquad \text{where } \tau = d\omega - \omega\Omega \tag{1.3}$$

Its elements are called torsion forms and clearly $\tau e \in \mathfrak{X}^1_2$. Equation (1.3) is the first Cartan structural equation.

We define the curvature matrix Θ of M^n to be the $n \times n$ matrix of 2-forms:

$$d(de) = \Theta e$$

$$\text{where} \qquad \Theta = d\Omega - \Omega^2 \tag{1.4}$$

Clearly $\Theta e \in \mathfrak{X}^1_2$ and we call the elements of Θ curvature forms. Equation (1.4) is the second Cartan structural equation.

Applying d to (1.3) gives:

$$d\tau + \tau\Omega = \omega\Theta$$

and from (1.4) we get the Bianchi identity:

$$d\Theta = \Omega\Theta - \Theta\Omega$$

Clearly $\tau = 0 \Rightarrow \begin{cases} d\omega = \omega\Omega \\ \omega\Theta = 0 \end{cases}$.

Flanders has shown that the following identities hold:

1) $d\Theta^r = \Omega\Theta^r - \Theta^r\Omega$

2) $d(\text{Trace } \Theta^r) = 0$

3) $d(\omega\Theta^r) = \tau\Theta^r + \omega\Theta^r\Omega$

4) $d(\tau\Theta^r) = \omega\Theta^{r+1} - \tau\Theta^r\Omega$

5) $d^{2r-1}P = \omega\Theta^{r-1}e$
$d^{2r}P = \tau\Theta^{r-1}e \quad r = 1,2\ldots$

6) $d^{2r-1}e = \Omega\Theta^{r-1}e$
$d^{2r}e = \Theta^r e$.

The above system of equations is closed under d and expresses every possible result of iterated application of d to any of the basic quantities.

Let $v = \lambda e$ be a vector field (i.e. $v \in \mathfrak{X}^1_0$), where λ is the $1 \times n$ matrix of functions $(\lambda^1, \ldots, \lambda^n)$. Then

$$\left.\begin{aligned} d^{2r-1}v &= (D\lambda)\Theta^{r-1}e \\ d^{2r}v &= \lambda\Theta^r e \quad r = 1,2\ldots. \end{aligned}\right\} \text{ where } D\lambda = (D\lambda^1, \ldots, D\lambda^n) = d\lambda + \lambda\Omega$$

and we have $d(D\lambda) = D\lambda\Omega + \lambda\Theta$.

§ 4. Change of frame and global forms

In order to derive the local and global transformation equations of the above matrices let $\bar{e}$, e be two moving frames defined on the same local coordinate neighbourhood U.

Then $\bar{e} = Ae$ A a non-singular matrix defined on U.

From $dP = \omega e = \bar{\omega}\bar{e}$ we have $\bar{\omega} = \omega A^{-1}$

From $\tau e = \bar{\tau}\bar{e}$ follows $\bar{\tau} = \tau A^{-1}$.

Differentiation of the first equation yields

$$\bar{\Omega} = A\Omega A^{-1} + dA A^{-1} \tag{1.5}$$

Similarly

$$\bar{\Theta} = A\Theta A^{-1}$$

and from $\bar{A} = \lambda A^{-1}$ we have

$$D\bar{\lambda} = D\lambda A^{-1} \quad .$$

Equation (1.5) is basic to the theory; it tells us how the various matrices Ω associated with the various moving frames e must be related in the intersection of two neighbourhoods if we are to define an affine connexion on M^n globally.

Consider FM(DEF 2) the frame bundle, and define the forms $\tilde{\omega}^1, \ldots, \tilde{\omega}^n$.

$$\tilde{\omega} = \omega B^{-1} \qquad \text{where } \tilde{\omega} = (\tilde{\omega}^1, \ldots, \tilde{\omega}^n)$$

These are 1-forms on the part of FM over U; the values of the $\tilde{\omega}^i$ at a point $f \in$ FM given by $f_i = b_i^j (e_j)$ are

$$\tilde{\omega}|_f = (\omega|_p)B^{-1} \quad .$$

Note the $\tilde{\omega}^i$ are independent of any connexion.

If now $\bar{U}$ is a second coordinate neighbourhood, $\bar{e}$ a moving frame on it and $p \in U \cap \bar{U}$ then:

$$\bar{e} = Ae \quad \text{and} \quad \bar{\omega} = \omega A^{-1} \; .$$

If $f \in$ FM has coordinates B w.r.t. e and $\bar{B}$ w.r.t. $\bar{e}$ then

$$Be = \bar{B}\bar{e} = \bar{B}Ae$$

Thus $$\tilde{\omega} = \omega B^{-1} = (\bar{\omega}A)(\bar{B}A)^{-1} = \bar{\omega}\bar{B}^{-1} \; .$$

This implies that the (linearly independent) 1-forms $\tilde{\omega}^i$ are globally defined on FM and independent of the particular local coordinate neighbourhoods and moving frames used in their definition. In a similar way we can construct global connexion forms $\tilde{\omega}_i^j$:-

$$\tilde{\Omega} = (\tilde{\omega}_i^j) = B\Omega B^{-1} + (dB)B^{-1}$$

and hence finally we define the general torsion and general curvature forms respectively:

$$\tilde{\tau} = d\tilde{\omega} - \tilde{\omega}\tilde{\Omega}$$

$$\tilde{\Theta} = (\tilde{\theta}_i^j) = d\tilde{\Omega} - \tilde{\Omega}^2 \quad .$$

Note: The classical formulation of the above can be obtained by expressing all forms in terms of the basic forms ω^i. Thus

1) $\omega_i^{\ j} = \Gamma^{\ j}_{ik}\,\omega^k$ 2) $2\theta_i^{\ j} = R_{i\ k\ell}^{\ j}\,\omega^k\,\omega^\ell$

3) $2\tau^i = T^i_{\ jk}\,\omega^j\,\omega^k$ 4) $D\lambda^i = \lambda^i_{\ ,j}\,\omega^j$

define the connexion coefficients, riemann tensor, torsion tensor and covariant derivative respectively.

DEF. 4. A Riemannian manifold is the structure consisting of a C^∞ manifold M together with an inner product on each tangent space such that whenever v, ω are vector fields then their inner product is a C^∞ function on M^n. One of the salient features of a riemannian manifold is the existence of a unique symmetric affine connexion d such that:

$$d(v.\omega) = dv.\omega + v.d\omega \quad .$$

§ 5. Submanifolds

Let M^n be an immersed submanifold of a riemannian manifold $\bar{M}^{n+N}$, with the induced metric. We shall work with the bundle of adapted frames: let $\{\bar{e}_A\}$ be an orthonormal frame at $p \in M^n$ such that $\{\bar{e}_1, \ldots, \bar{e}_n\}$ are tangent to M^n and $\{\bar{e}_{n+1}, \ldots, \bar{e}_{n+N}\}$ normal to M^n. We shall agree from now on to the following index convention:

$$\begin{aligned} A,B,C &= 1,2,\ \ldots,\ n,\ n+1,\ \ldots,\ n+N \\ i,j,k &= 1,2,\ \ldots,\ n. \\ \alpha,\beta,\gamma &= \qquad\qquad\quad n+1,\ \ldots,\ n+N \end{aligned} \tag{1.6}$$

From § 3

$$dP = \omega e + \bar{\omega}\bar{e} \quad \text{where} \quad \bar{e} = (e_{n+1}, \ldots, e_{n+N})^T \tag{1.7}$$

$$\left.\begin{aligned} \bar{d}e &= \Omega e + \bar{\Omega}\bar{e} \\ \bar{d}\bar{e} &= -\bar{\Omega}^T e + \bar{\bar{\Omega}}\bar{e} \end{aligned}\right\} \tag{1.8}$$

where $\bar{\omega} = (\omega^{n+1}, \ldots, \omega^{n+N})$, $\Omega = (\omega_i^{\ j})$, $\bar{\Omega} = (\omega_i^{\ \sigma})$, $\bar{\bar{\Omega}} = (\omega_\sigma^{\ \tau})$.

Also

$$\left.\begin{aligned} \bar{d}(dP) &= \tau e + \bar{\tau}\bar{e} \\ \bar{d}(\bar{d}e_A) &= \bar{\Theta}_A^{\ B}\,\bar{e}_B \end{aligned}\right\} \tag{1.9}$$

For a riemannian manifold torsion matrix = 0. Hence

$$d\bar{\omega}^A = \bar{\omega}^B\,\bar{\omega}_B^{\ A}$$

and

$$d\bar{\omega}_A^{\ B} = \bar{\omega}_A^{\ C}\,\bar{\omega}_C^{\ B} + \bar{\Theta}_A^{\ B} \tag{1.10}$$

The equations (1.7) through (1.10) contain the geometry of $\bar{M}^{n+N}$.

Now

$$\bar{g}_{AB} = \bar{e}_A \cdot \bar{e}_B$$

$$\Longrightarrow \qquad \bar{\omega}_A^{\ C}\,\bar{g}_{CB} + \bar{\omega}_B^{\ C}\,\bar{g}_{CA} = d\bar{g}_{AB} = 0$$

$$\bar{\omega}_A^{\ B} + \bar{\omega}_B^{\ A} = 0 \quad \text{if} \quad \bar{g}_{AB} = \delta_{AB}$$

Now on M^n $$\bar{\omega} = 0 \tag{1.11}$$

$$\Longrightarrow \quad d\omega = \omega\Omega \quad \text{and} \quad \omega\bar{\Omega} = 0 \,. \tag{1.12}$$

The equations (1.7) through (1.10) together with (1.11) contain the geometry of M^n .

DEF. 5. <u>Normal Bundle</u> $(TM^{\perp}, \pi, M)$: Let $(e_1,\ldots,e_n, e_{n+1},\ldots,e_{n+N})$ be an adapted frame then $(e_{n+1}, \ldots\ldots, e_{n+N})$ is a normal frame (i.e. an orthonormal basis for the normal space $M_p^{\perp}$). The set $\bigcup_{p\in M} M_p^{\perp}$ is called the normal bundle of M with structural group O(N) acting on the standard fibre $\mathbb{R}^N$.

DEF. 6. <u>Adapted Frame Bundle</u>: (AFM, π, M): Is the union of all adapted frames at all points of M, and is a principal bundle with group $O(n) \times O(N)$. It is clearly a sub bundle of the bundle of orthonormal frames over $\bar{M}^{n+N}$ restricted to M^n .

If we write $\bar{d}_j\, e_i$ for $\bar{d}_{e_j}\, e_i$, then we have the equation of Gauss:

$$\bar{d}_j e_i = \omega_i^{\,k}(e_j)e_k + \omega_i^{\,\sigma}(e_j)e_\sigma \tag{1.13}$$

The (induced) connexion on M is by definition the tangential component of this:

$$d_j e_i = \text{tang}(\bar{d}_j e_i) = \omega_i^{\,k}(e_j)e_k$$

i.e.

$$de = \Omega e \,.$$

Weingartens equation is just:

$$\bar{d}_j e_\sigma = \omega_\sigma^i (e_j) e_i + \omega_\sigma^\beta (e_j) e_\beta \tag{1.14}$$

Whence the connexion on the normal bundle $\overset{\perp}{d}$ – the so-called 'normal connexion' is just the normal component of (1.14):

$$\overset{\perp}{d}_j e_\sigma = \mathrm{Nor}(\bar{d}_j e_\sigma) = \omega_\sigma^\beta (e_j) e_\beta$$

i.e. $$\overset{\perp}{d}\bar{e} = \bar{\bar{\Omega}}\bar{e} \ .$$

Note

$$\text{torsion zero} \Longrightarrow \begin{cases} \omega_i^j (e_k) = \omega_k^j (e_i) \\ \omega_i^\sigma (e_j) = \omega_j^\sigma (e_i) \ . \end{cases}$$

It follows from (1.12) and a lemma of É. Cartan that

$$\omega_i^\sigma = A^\sigma_{ij} \omega^j \tag{1.15}$$

where $A^\sigma = (A^\sigma_{ij})$ is an $n \times n$ matrix. The quadratic differential forms:

$$\sum\nolimits_\sigma = -dP \cdot de_\sigma = A^\sigma_{ij} \omega^i \otimes \omega^j$$

are the so-called second fundamental forms of M.

The curvature matrix $\Theta = (\Theta_i^j)$ of the induced connexion on M is given by:

$$\Theta = d\Omega - \Omega^2$$

and this is related to the curvature matrix of $\bar{M}^{n+N}$ by the Gauss curvature equation:

$$\Theta = \Phi - \bar{\Omega}\bar{\Omega}^T \quad \text{where} \quad \Phi = (\bar{\Theta}_i^j) \,.$$

The Codazzi-Mainardi equation is:

$$d(\bar{\Omega}^T) = \bar{\Omega}^T\Omega + \bar{\bar{\Omega}}\bar{\Omega}^T + \bar{\Phi}^T \quad \text{where} \quad \bar{\Phi} = (\bar{\Theta}_i^\sigma)$$

which can also be written:

$$d\bar{\Omega} = \Omega\bar{\Omega} + \bar{\Omega}\bar{\bar{\Omega}} + \bar{\Phi} \,.$$

The connexion (matrix) of the normal bundle has curvature matrix $\overset{\perp}{\Theta} = (\overset{\perp}{\Theta}{}_\alpha^\beta)$

$$\overset{\perp}{\Theta} = d\bar{\bar{\Omega}} - \bar{\bar{\Omega}}^2$$

and this is related to the curvature matrix of $\bar{M}^{n+N}$ by

$$\overset{\perp}{\Theta}{}_\alpha^\beta = \bar{\Theta}_\alpha^\beta + \omega_\alpha^i\,\omega_i^\beta \,.$$

The Bianchi identity in § 3 becomes:

$$\begin{cases} d\Phi = \Omega\Phi - \Phi\Omega + \bar{\Phi}\bar{\Omega}^T - \bar{\Omega}\bar{\Phi}^T \\ d\bar{\Phi} = \Omega\bar{\Phi} - \Phi\bar{\Omega} + \bar{\Omega}\bar{\bar{\Phi}} - \bar{\Phi}\bar{\bar{\Omega}} \\ d\bar{\bar{\Phi}} = \bar{\Phi}^T\bar{\Omega} - \bar{\Omega}^T\bar{\Phi} + \bar{\bar{\Omega}}\bar{\bar{\Phi}} - \bar{\bar{\Phi}}\bar{\bar{\Omega}} \end{cases} \quad \text{where} \quad \bar{\bar{\Phi}} = (\bar{\Theta}_\alpha^\beta)$$

Note for an orthonormal basis we can write $\omega_A^{\ B} = \omega_{AB}$, and for a space of constant riemannian (sectional) curvature K:

$$\bar{\Theta}_{AB} = K\omega^A \wedge \omega^B .$$

§ 6. Some Definitions.

We end this chapter with some basic definitions that we will have recourse to later on.

(a) One-parameter transformation group: on a C^∞ n-manifold M^n is a set $\{\phi_t\}$ $t \in \mathbb{R}$ of diffeomorphisms of M onto itself such that the mapping

$$\Phi : \mathbb{R} \times M \to M \quad \text{defined by} \quad \Phi(t,p) = \phi_t(p) \quad p \in M,$$

satisfies:

(i) Φ is differentiable

(ii) $\Phi(s, \Phi(t,p)) = \Phi(s+t, p)$, i.e. $\phi_{s+t} = \phi_s \ \phi_t$ $\forall s,t$

(iii) $\Phi(0, p) = p$ i.e. ϕ_0 = identity.

Such a group defines a tangent vector field X on M in the following way. Let $f \in F(M)$ and set:

$$X_p(f) = \underset{t\to 0}{\mathrm{Lt}} \left\{ \frac{f(\phi_t(p)) - f(p)}{t} \right\} = \frac{d}{dt} f(\phi_t(p)) \Big|_{t=0} .$$

We can think of this field as follows: at every $p \in M$, the mapping $t \to \phi_t(p)$, is a curve through p; we define X_p to be the tangent vector to this curve at p. The fact that X_p varies smoothly with p (so that X is in fact a vector field) follows from (i).

The following theorem is a partial converse of the above:

Theorem: Let M^n be a smooth manifold, X a vector field on M, U an open set in M with compact closure K. Then we can find $\epsilon > 0$, and for each t with $|t| < \epsilon$, a map ϕ_t of $U \subset M$ such that

(i) Map $\phi : U \times E \to M \times \mathbb{R}$ (E is the set $|t| < \epsilon$) is a diffeomorphism onto an open submanifold.

(ii) If $|s|, |t|$ and $|s+t|$ are $< \epsilon$; p and $\phi_t(p)$ are in U, then: $$\phi_s \circ \phi_t(p) = \phi_{s+t}(p).$$

(iii) For each $p \in U$, $f \in F(M)$,
$$X_p(f) = \frac{d}{dt} f(\phi_t(p)) \Big|_{t=0}.$$

The map ϕ is completely determined by these conditions.

Corollary: If M^n is compact, each vector field generates a 1-parameter group of diffeomorphisms of M.

DEF. 7. A vector field on M is called complete if it generates a 1-parameter group of diffeomorphisms of M. Note that in the product

$M \times \mathbb{R}$ the field $\partial/\partial t$, which maps to zero on the first factor and to the standard field on the second, is complete, here

$$\phi_t(X,s) = (X, s+t)$$

This serves as partial justification of the term 'tangent' vector since clearly such vectors correspond to displacement along M.

(b) Lie derivative £

DEF. 8. The Lie derivative of a tensor T of type (r,s) with respect to a vector field X is defined:

$$\begin{aligned}(\underset{X}{\pounds}T)(\theta_1, \dots, \theta_r, X_1, \dots, X_s) = {} & X(T(\theta_1, \dots, \theta_r, X_1, \dots, X_s)) \\ & - T(\underset{X}{\pounds}\theta_1, \theta_2, \dots, \theta_r, X_1, \dots, X_s) \dots\dots \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & - T(\theta_1, \dots, \theta_r, X_1, \dots, X_{s-1}, [X, X_s])\end{aligned}$$

where $\theta_1, \dots, \theta_r$ and $X_1, \dots, X_s$ are respectively arbitrary 1-forms and vector fields on M; and where:

(i) $(\underset{X}{\pounds}\theta)(Y) = X(\theta(Y)) - \theta([X,Y])$

(ii) $\underset{X}{\pounds}f = Xf \qquad f \in F(M)$

(iii) $\underset{X}{\pounds}Y = [X,Y]$. In local coordinates $X = X^i\,\partial/\partial x^i$ and $Y = Y^i\,\partial/\partial x^i$ then

$$\underset{X}{\pounds}Y = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} .$$

Clearly for a symmetric connexion $d : \underset{X}{\pounds} Y = d_X Y - d_Y X$.

<u>Theorem</u>: If X is a vector field induced by a 1-parameter group of transformations $\{\phi_t\}$ then a tensor field T is invariant under X (or equivalently under ϕ_t for each t) iff $\underset{X}{\pounds} T = 0$.

Note that the Lie derivative is linear over sums and commutes with contraction; it is also a derivation in that for a product or a transvection of two quantities Φ, Ψ we have the Leibnitz rule:

$$\underset{X}{\pounds}(\Phi \Psi) = (\underset{X}{\pounds} \Phi)\Psi + \Phi(\underset{X}{\pounds} \Psi) .$$

DEF. 9: <u>Affine Mapping</u>: Let M, $\bar{M}$ be manifolds with linear connexions. We call the C^∞ map $\phi : M \to \bar{M}$ an affine mapping if the induced map $\phi_* : TM \to T\bar{M}$ maps every horizontal curve into a horizontal curve, i.e. if ϕ_* maps a parallel vector field along each curve τ of M into a parallel vector field along curve $\phi(\tau)$. An affine transformation is a diffeomorphism of M onto itself which is an affine mapping. Geodesics are preserved and the arc length s receives an affine transformation: $s \to as + b$ $a \neq 0$, b constants.

DEF. 10. <u>Conformal transformation</u>: Is a diffeomorphism ϕ from M onto itself such that the induced tensor $g^* = \phi^* g$ is another riemannian metric. Note that ϕ_* preserves orthogonality and dilates uniformly.

<u>Theorem</u>: A transformation on a C^∞ riemannian manifold (M^n,g) is conformal iff $\exists$ a positive real-valued C^∞ function f on M such that: $g^* = fg$ or $g^* = \exp(2\theta)g$ where $\theta \in C^\infty : M \to \mathbb{R}$.

A conformal vector field is one that generates a conformal 1-parameter transformation group.

DEF. 11. <u>Projective Transformation</u>: Is a C^∞ homeomorphism of M onto itself leaving geodesics invariant, the affine character of parameter s not necessarily being preserved.

A transformation ϕ on M is projective iff $\exists$ a covector $p(\phi)$ depending on ϕ such that:

$$\overset{*}{\Gamma}{}_j{}^i{}_k = \Gamma_j{}^i{}_k + p_j(\phi)\delta^i_k + p_k(\phi)\delta^i_j$$

If X is a vector field on (M^n,g) which induces a 1-parameter group of transformations on M. Then in order that X define:

(i) an isometry (motion) is:

$$\underset{X}{\pounds} g_{ij} \equiv X_{i,j} + X_{j,i} = 0$$

the comma denoting covariant derivative.

(ii) an affine collination is:

$$\underset{X}{\pounds}\Gamma_j{}^i{}_k = X^i{}_{,j,k} + R_j{}^i{}_{k\ell}\, X^\ell = 0$$

(iii) a projective motion is: $\underset{X}{\pounds}\bar{\Gamma}_{j\ k}^{\ i} = 0$

where

$$\bar{\Gamma}_{j\ k}^{\ i} = \Gamma_{j\ k}^{\ i} - \frac{1}{n+1}(\delta_j^i\,\Gamma_{k\ \ell}^{\ \ell} + \delta_k^i\,\Gamma_{j\ \ell}^{\ \ell})$$

(iv) a homothetic transformation is:

$$\underset{X}{\pounds} g_{ij} = 2c\, g_{ij} \qquad c = \text{constant}$$

(v) a conformal transformation is:

$$\underset{X}{\pounds} g_{ij} = 2\phi\, g_{ij} \quad .$$

Clearly for $c = 0$ in (iv) we have (i); we thus call (iv), such that $c \neq 0$, a proper homothety. Note also that from the definition

$$\underset{X}{\pounds}\Gamma_{j\ k}^{\ i} = \tfrac{1}{2} g^{i\ell}\{(\underset{X}{\pounds} g_{i\ell})_{;k} + (\underset{X}{\pounds} g_{\ell k})_{;j} - (\underset{X}{\pounds} g_{jk})_{;\ell}\}$$

thus (i) and (iv) are examples of (ii).

Proofs of these results are to be found in Yano's book 'The Theory of Lie Derivatives and its Applications'.

(c) <u>Generalised vector product</u>

Considerable use will be made of the vector product due to Hsiung [6]. Namely, let (M^n, g) be a riemannian manifold $(n \geq 3)$, and consider a fixed orthonormal frame $\{p, e_1, \dots, e_n\}$ at $p \in M$. Let a_i $(1 \leqslant i \leqslant n-1)$ be n-1 vectors at a point in M^n whose contravariant components w.r.t. frame $p, e_1, \dots, e_n$ are a_i^σ $1 \leqslant \sigma \leqslant n$. We define the vector product of the n-1 vectors a_i to be a vector in M^n denoted $a_1 \times \dots \times a_{n-1}$, whose contravariant components are given by:

$$a_1 \times \dots \times a_{n-1} = (-1)^{n-1} \begin{vmatrix} e_1 & e_2 & , \dots & e_n \\ g_{\sigma 1} a_1^\sigma & , g_{\sigma 2} a_1^\sigma & , \dots & g_{\sigma n} a_1^n \\ g_{\sigma 1} a_2^\sigma & , g_{\sigma 2} a_2^\sigma & , \dots & g_{\sigma n} a_2^n \\ \vdots & \vdots & & \vdots \\ g_{\sigma 1} a_{n-1}^\sigma , & g_{\sigma 2} a_{n-1}^\sigma , & \dots & g_{\sigma n} a_{n-1}^\sigma \end{vmatrix}$$

From the definition of scalar product of any two vectors a_i and a_j viz. $a_i \cdot a_j = g_{\alpha\beta}\, a_i^\alpha\, a_j^\beta$ it follows that $a_1 \times \dots \times a_{n-1}$ is orthogonal to a_i $1 \leqslant i \leqslant n-1$.

Clearly if π is a permutation of the set $\{1, 2, \dots, n\}$ then:

$$a_{\pi(1)} \times \dots \times a_{\pi(n-1)} = (\text{sign } \pi) a_1 \times \dots \times a_{n-1}$$

where sign π is +1 or -1 as π is even or odd.

If (a_i^σ) are differentiable functions of the n-1 variables then from the definition:

$$\frac{\partial}{\partial x^i}(a_1 \times \dots \times a_{n-1}) = \sum_{j=1}^{n-1} (a_1 \times \dots \times a_{j-1} \times \frac{\partial a_j}{\partial x^i} \times a_{j+1} \times \dots \times a_{n-1})$$

In a determinant whose columns are the components of vectors or vector-valued differential forms, we make the convention that in the expansion of the determinant the multiplication of differential forms is in the sense of exterior multiplication. A similar convention will be observed for vector products involving differential vectors.

For the differential vectors $dP_i = \omega_i^{\ j} e_j \quad 1 \leq i \leq n$ we define:

$$\| dP_1, \dots, dP_n \| := \delta_{k_1 \dots k_n} \omega_1^{\ k_1} \wedge \dots \wedge \omega_n^{\ k_n} | e_1, \dots, e_n |$$

where $| \quad |$ means a determinant of order n whose columns are components of the respective vectors $e_1, \dots, e_n$.

(d) Newton's Formula

Let A^σ be matrix of second fundamental form in normal direction e_σ (see (1.15)). Then we define the principal curvatures $\lambda_i(\sigma)$ $1 \leq i \leq n$ at a point $p \in M$ to be the roots of the equation:

$$|A^\sigma - \lambda G| = 0 \qquad G = (g_{ij})$$

and consider the k^{th} elementary symmetric function of the $\lambda_i(\sigma)$:

$$S_k := \sum_{i_1 < \dots < i_k} \lambda_{i_1}(\sigma) \dots \lambda_{i_k}(\sigma) \qquad S_o := 1$$

Set $\quad P_k = \sum_{i=1}^{n} (\lambda_i(\sigma))^k$.

Then the relations between S_k and P_k are given by the formulas of Newton:

$$P_1 - S_1 = 0$$

$$P_2 - S_1P_1 + 2S_2 = 0$$

$$P_3 - S_1P_2 + S_2P_1 - 3S_2 = 0$$

$$\cdot \; \cdot \; \cdot \; \cdot \; \cdot$$

$$P_n - S_1P_{n-1} + S_2P_{n-2} - \ldots + (-1)^{n-1}S_{n-1}P_1 + n(-1)^n S_n = 0 .$$

We can thus express S_k in terms of $P_1, \ldots, P_k$ as:

$$S_1 = P_1$$

$$2!\, S_2 = -P_2 + P_1^2$$

$$3!\, S_3 = 2P_3 - 3P_1P_2 + P_1^3$$

$$4!\, S_4 = -6P_4 + 8P_3P_1 - 6P_1^2 P_2 + 3P_2^2 + P_1^4$$

$$\cdot \; \cdot \; \cdot \; \cdot \; \cdot$$

$$S_k = \sum_{\substack{t_1+2t_2+\ldots+nt_n=k \\ t_i \geq 0}} \frac{(-1)^{t_1+\ldots+t_n+k}}{(t_1!)(t_2!)\ldots(t_n!)2^{t_2}\ldots n^{t_n}} P_1^{t_1} \ldots P_n^{t_n}$$

$$\cdot \; \cdot \; \cdot \; \cdot \; \cdot$$

$$S_n = \sum_{\substack{t_1+2t_2+\ldots+nt_n=n \\ t_i \geq 0}} \frac{(-1)^{t_1+\ldots+t_n+n}}{(t_1!)(t_2!)\ldots(t_n!)2^{t_2}\ldots n^{t_n}} P_1^{t_1} \ldots P_n^{t_n}$$

On the other hand we have the identity:

$$\frac{n-k}{n} S_1 S_k - (k+1)S_{k+1} = \frac{1}{n} \sum_{\substack{i_1 < i_2 \\ i_3 < \dots < i_{k+1}}} (\lambda_{i_1}(\sigma) - \lambda_{i_2}(\sigma))^2 \lambda_{i_3}(\sigma) \dots \lambda_{i_{k+1}}(\sigma)$$

DEF. 12: For the normal direction e_σ we define the k^{th} mean curvature $M_k(\sigma)$ to be:

$$\binom{n}{k} M_k = S_k$$ where $\binom{n}{k}$ is the binomial coefficient.

Whence:

$$M_1 M_k - M_{k+1} = k! \frac{(n-k-1)!}{nn!} \sum_{\substack{i_1 < i_2 \\ i_3 < \dots < i_{k+1}}} (\lambda_{i_1}(\sigma) - \lambda_{i_2}(\sigma))^2 \lambda_{i_3} \dots \lambda_{i_{k+1}} .$$

CHAPTER II

We now develop the local curvature theory of a riemannian n-manifold immersed in Euclidean (n+N)-space; the results however are easily translated without any essential change to submanifolds of a space of constant curvature. We first develop the more classical theory with more recent extensions due to Chern [11], Hsiung [7], Gulbinat [13] and Leichtweiss [20].

§ 1. Standard Equations

Consider an immersion $x : M^n \to E^{n+N}$, we denote the position vector of x(M) relative to a fixed origin O by x. If $(u^1, \ldots, u^n)$ are local coordinates on M then the (induced) metric on x(M) is given by:

$$g_{ij} = \langle x_i, x_j \rangle \qquad x_i = \partial x/\partial u^i .$$

As before we define the outer product $[a_2, \ldots, a_{n+N}]$ of the vectors $a_2, \ldots, a_{n+N}$ in the (oriented) Euclidean space E^{n+N} by defining

$$\langle a, [a_2, \ldots, a_{n+N}] \rangle = |a, a_2, \ldots, a_{n+N}| \qquad \forall a .$$

We choose the orientation of the orthonormal basis $\{e_\sigma\}$ of the normal bundle so that:

$$|x_1, \ldots, x_n, e_{n+1}, \ldots, e_{n+N}| > 0$$

Clearly

$$|x_{i_1}, \ldots, x_{i_n}, e_{n+1}, \ldots, e_{n+N}| = \epsilon_{i_1 \ldots i_n}$$

and

$$[x_{i_1}, \ldots, x_{i_n}, e_{n+1}, \ldots, \hat{e}_\alpha, \ldots, e_{n+N}] = (-1)^{n+\alpha-1} \epsilon_{i_1 \ldots i_n} e_\alpha$$

where as usual the carat $\wedge$ denotes omission of that element, and where the Ricci ϵ-tensor on M^n is:

$$\epsilon_{i_1 \ldots i_n} = g^{\frac{1}{2}} \mathrm{sign}(i_1, \ldots, i_n) \qquad \text{where } g = \det(g_{ij})$$

and hence

$$\epsilon_{i_1 \ldots i_n} \epsilon^{i_1 \ldots i_k}{}_{j_{k+1} \ldots j_n} = k! \sum_\pi g_{j_{k+1} \pi(i_{k+1})} \cdots g_{j_n \pi(i_n)} (\mathrm{sign}\ \pi) \tag{2.1}$$

summed over all permutations π leaving $(i_1, \ldots, i_k)$ fixed.

We also have:

$$[x_{i_1}, \ldots, \hat{x}_{i_k}, \ldots, x_{i_n}, e_{n+1}, \ldots, e_{n+N}] = \epsilon^{i}{}_{i_1 \ldots \hat{i}_k \ldots i_n} x_i$$

corresponding to each normal vector e_σ we have defined the symmetric second fundamental form tensor:

$$A^\sigma_{ij} = - < x_i, d_j e_\sigma >$$

$$d_i e_\sigma = -g^{jk} A^\sigma_{ik} x_j + \omega_\sigma^{\ \beta}(x_i) e_\beta \tag{2.2}$$

where

$$\omega_\sigma^{\ \beta}(x_i) \ = \ < d_i\, e_\sigma,\ e_\beta > \ \text{ and } \ \omega_\sigma^{\ \beta}(x_i) + \omega_\beta^{\ \sigma}(x_i) \ = \ 0\ .$$

The conditions of integrability are:

$$\left\{\begin{aligned}
&R_{i\ell kj} \ = \ A^\sigma_{\ ik} A^\sigma_{\ j\ell} - A^\sigma_{\ ij} A^\sigma_{\ k\ell} \\
&A^\sigma_{\ ij,k} - A^\sigma_{\ ik,j} + \omega_\beta^{\ \sigma}(x_k) A^\beta_{\ ij} - \omega_\beta^{\ \sigma}(x_j) A^\beta_{\ ik} \ = \ 0 \\
&\omega_\sigma^{\ \gamma}(x_i)_{,k} - \omega_\sigma^{\ \gamma}(x_k)_{,i} + g^{\ell j} A^\sigma_{\ k\ell} A^\gamma_{\ ji} - g^{\ell j} A^\sigma_{\ i\ell} A^\gamma_{\ jk} \ + \\
&\qquad + \omega_\sigma^{\ \beta}(x_i)\,\omega_\beta^{\ \gamma}(x_k) - \omega_\sigma^{\ \beta}(x_k)\,\omega_\beta^{\ \gamma}(x_i) \ = \ 0
\end{aligned}\right. \tag{2.3}$$

where $R_{i\ell kj}$ is the curvature tensor.

DEF. 13. <u>Mean normal curvature vector η</u> : is defined by:

$$\left(\frac{1}{n}\right) \text{Trace } A^\sigma \ = \ g(e_\sigma,\ \eta) \qquad\qquad \forall\, e_\sigma$$

whence:

$$\eta \ = \ \left(\frac{1}{n}\right) (\text{Trace } A^\sigma) e_\sigma \ .$$

§ 2. <u>Generalised classical curvature theory</u>

We now define generalised second fundamental forms which enable us to define generalised mean curvatures. Let e_σ be normalised cross-sections of $TM^\perp$, and define:

$$A^{\sigma}{}_{I(r)J(r)} = A^{\sigma}{}_{i_1 j_1} A^{\sigma}_{i_2 j_2} \ldots A^{\sigma}{}_{i_r j_r}$$

and

$$A^{\sigma}{}_{I(r)}{}^{J(r)} = A^{\sigma}{}_{i_1}{}^{j_1} A^{\sigma}{}_{i_2}{}^{j_2} \ldots A^{\sigma}{}_{i_r}{}^{j_r}$$

where $I(r) := \{i_1, \ldots, i_r\}$, $J(r) := \{j_1, \ldots, j_r\}$ are ordered sets of numbers.

We can now define a tensor symmetric in (ij) as follows:

$$n!C\left(\begin{array}{c|c|c} \sigma_1 & \ldots & \sigma_k \\ r_1 & & r_k \end{array}\right)^{ij} = \epsilon^{i_1 \ldots i_{s_k} i\, i_{s_k+2} \ldots i_n} \epsilon^{j_1 \ldots j_{s_k} j}{}_{i_{s_k+2} \ldots i_n}$$

$$\times\ A^{\sigma_1}{}_{I(r_1)J(r_1)} \ldots A^{\sigma_k}{}_{I(r_k)J(r_k)} \qquad (2.4)$$

and where $\bigcup_{t=1}^{k} I(r_t) \cup \{i_{s_k+1}, \ldots, i_n\} = \{1, \ldots, n\}$

and $$I(r_s) \cap I(r_t) = \Phi \qquad s \neq t .$$

And clearly there exists an analogous relation to (2.4) for the sets of indices $J(r_1), \ldots, J(r_k)$.

From (2.4) we define the generalised mean curvature of the submanifold to be:

$$H\left(\begin{array}{c|c|c} \sigma_1 & \ldots & \sigma_k \\ r_1 & & r_k \end{array}\right) = C\left(\begin{array}{c|c|c} \sigma_1 & \ldots & \sigma_k \\ r_1 & & r_k \end{array}\right)^{ij} g_{ij} \quad .$$

Clearly the curvatures depend on the choice of the normal vectors e_σ. In case $k = 1$ the curvature functions $H\binom{\sigma}{r}$ are simply the usual curvature functions with respect to the normals e_σ, namely the r^{th} (normalised) elementary symmetric function of the principal curvatures $\lambda_i(\sigma)$. We note that

$$nC\begin{pmatrix}\sigma\\0\end{pmatrix} = G^{-1} \quad \text{and} \quad H\begin{pmatrix}\sigma\\0\end{pmatrix} = 1 \quad \text{where } G = (g_{ij}) .$$

If e_α and e_β are two orthonormal vectors then e

$$\text{where} \qquad \sqrt{2}\, e = e_\alpha + e_\beta$$

is a unit normal vector, and we have:

<u>Lemma 1</u> (Gulbinat): The r^{th} curvatures $H\binom{\alpha}{r}$, $H\binom{\beta}{r}$ and $H\binom{e}{r}$ are related to the unit normals e_α, e_β and e by the relation:

$$(\sqrt{2})^r H\begin{pmatrix}e\\r\end{pmatrix} = H\begin{pmatrix}\alpha\\r\end{pmatrix} + H\begin{pmatrix}\beta\\r\end{pmatrix} + \sum_{t=1}^{r-1}\begin{pmatrix}r\\t\end{pmatrix} H\left(\begin{array}{c|c}\alpha & \beta\\ t & s-t\end{array}\right).$$

From (2.3) we have the well-known

$$R = A^\sigma G^{-1} A^\sigma - nH\begin{pmatrix}\sigma\\1\end{pmatrix} A^\sigma \qquad (\text{where } R = (R_{ij})) \tag{2.5}$$

Whence for the scalar curvature K we have:

$$n(n-1)K = -\text{Trace } R = R^{ij}{}_{ij} .$$

Hence from the first equation of (2.3) we have the well known generalisation of Gauss' theorema egregium:

<u>Theorem 1:</u> On $x(M)$, $K = \sum_{\alpha} H\binom{\alpha}{2}$;

the right hand side depends only on the metric and not on the normal space.

<u>Proof</u> Transvect the first equation of (2.3) with

$$\epsilon^{iki_3\ldots i_n}\,\epsilon^{rj}{}_{i_3\ldots i_n}$$

and use the properties of the outer product defined earlier, together with the fact that:

$$\epsilon^{iki_3\ldots i_n} A^{\sigma}{}_{ik} = 0$$

recalling that ϵ is skew symmetric.

<u>Lemma 2:</u> Let e_α be an arbitrary normal to $x(M)$, then it follows:

(a) $n(n-1)C\binom{\alpha}{1}^{ij} = nH\binom{\alpha}{1}g^{ij} - g^{ik}\,g^{j\ell}\,A^{\alpha}{}_{k\ell}$.

(b) $n(n-1)(n-2)C\binom{\alpha}{2}^{ij} = n(n-1)H\binom{\alpha}{2}g^{ij} + 2g^{i\ell}A^{\alpha}{}_{\ell k}g^{km}g^{jn}A^{\alpha}{}_{mn}$

$- 2nH\binom{\alpha}{1}g^{\ell i}g^{kj}A^{\alpha}{}_{\ell k}$.

Proof: From definition of tensor C

$$n!C\binom{\alpha}{1}^{ij} = \epsilon^{iki_3\cdots i_n}\,\epsilon^{jr}{}_{i_3\cdots i_n}\,A^{\alpha}{}_{kr}$$

$$= (n-2)!(g^{ij}g^{kr} - g^{ir}g^{kj})A^{\alpha}{}_{kr}$$

$$= (n-2)!(nH\binom{\alpha}{1}g^{ij} - g^{ki}g^{\ell j}A^{\alpha}{}_{k\ell}) .$$

(b) is proved analogously using the fact that from (2.5) we get

$$n(n-1)H\binom{\alpha}{2} = n^2H^2\binom{\alpha}{1} - A^{\alpha}{}_{ij}\,g^{ki}\,g^{\ell j}\,A^{\sigma}{}_{k\ell} .$$

Note finally that we have:

$$2C\binom{\alpha}{1}^{ij}g^{\ell k}A^{\alpha}{}_{j\ell} = H\binom{\alpha}{2}g^{ik} - (n-2)C\binom{\alpha}{2}^{ik}$$

whence by (2.4) and (2.5)

$$n(n-1)\,H\left(\begin{smallmatrix}\alpha\\1\end{smallmatrix}\middle|\begin{smallmatrix}\beta\\1\end{smallmatrix}\right) = n^2H\binom{\alpha}{1}H\binom{\beta}{1} - A^{\alpha}{}_{ij}\,g^{ki}\,g^{\ell j}\,A^{\beta}{}_{k\ell} .$$

These two equations together with (2.4) have been used extensively in investigations in minimal surfaces and umbilic submanifolds.

§ 3. Canonical curvature theory

We now develop a curvature theory of the mean normal curvature vector. In order to do this we generalise the classical second and third fundamental forms of surface theory.

DEF. 14. The second fundamental form of $x(M)$ is defined to be the quadratic form:

$$A_{ij}\, du^i\, du^j = \langle x_{i,j}, \eta \rangle\, du^i\, du^j .$$

DEF. 15. The third fundamental form of $x(M)$ is defined to be the quadratic form:

$$E_{ij}\, du^i\, du^j = \langle x_{i,k}, x_{r,j} \rangle\, g^{rk}\, du^i\, du^j .$$

These forms are clearly independent of the choice of special sections of the tangent respectively normal bundle. In the case of co-dimension 1 (i.e. hypersurfaces) this third fundamental form coincides with the classical one but this second fundamental form is distinguished from the classical one $\bar{A}_{ij}\, du^i\, du^j$ by a factor:

$$A = nH(1)\bar{A} \qquad A = (A_{ij}) \quad \bar{A} = (\bar{A}_{ij})$$

where $H(1)$ is the mean curvature of the hypersurface.

Note: The above generalisation of the second fundamental form is meaningful in the sense that it is in general impossible to normalise η globally since there may be points where $\eta = 0$. In case $\eta \neq 0$ on $x(M)$ then we can normalise it and hence:

$$A_{ij} = - < d_i \eta, x_j >$$

thus

$$A = nH\binom{\alpha}{1}A^{\alpha}$$

$$E_{ij} = A^{\alpha}{}_{ik}\, g^{\ell k}\, A^{\alpha}{}_{\ell j} \quad .$$

DEF. 16. We define the principal curvatures of $x(M)$ to be the n roots $\lambda_i \quad 1 \leqslant i \leqslant n$ of:

$$|A - \lambda G| = 0$$

and we further define the (normalised) elementary symmetric functions of the principal curvatures; in particular we call:

$$H = H_1 = \frac{1}{n}\,\text{Trace } A = \frac{1}{n}\, g^{ij}\, A_{ij}$$

the (first) mean curvature of $x(M)$.

From this:

$$nH = < \eta, \eta > \geqslant 0 \quad .$$

A submanifold for which $H = 0$ identically globally is called a minimal submanifold.

We can now derive the derivative equations and the conditions of integrability (i.e. (2.5)) for η .

Let x_o be any point not in $x(M)$. We agree to identify points in E^{n+N} with their position vectors in the usual manner.

Then

$$x - x_o = (\tfrac{1}{2})g^{ij}\, d_i(< x-x_o, x-x_o >)x_j + < x-x_o, e_\alpha > e_\alpha .$$

Now from:

$$\eta = nH(\,{}^{\alpha}_{1}\,)e_\alpha$$

and $$< x-x_o, \eta > = nH(\,{}^{\alpha}_{1}\,) < x-x_o, e_\alpha >$$

we get the derivative equations:

$$d_i\eta = -g^{kj} A_{ik}\, x_j + n\{d_i H(\,{}^{\beta}_{1}\,) + \omega_\alpha{}^{\beta}(x_i)H(\,{}^{\alpha}_{1}\,)\}\, e_\beta \quad .$$

If we denote the term: $n\{\quad\}$ by $d\beta_i$ $(= < d_i\eta, e_\beta >)$ we get for the conditions of integrability:

$$A_{ij,k} - A_{ik,j} = A^{\beta}{}_{ij}\, d\beta_k - A^{\beta}{}_{ik}\, d\beta_j$$

$$d\beta_{i,k} - d\beta_{k,i} + g^{\ell j} A_{k\ell} A^{\beta}{}_{ji} - g^{\ell j} A_{i\ell} A^{\beta}{}_{jk} =$$

$$= d\alpha_k\, \omega_\alpha{}^{\beta}(x_i) - d\alpha_i\, \omega_\alpha{}^{\beta}(x_k) .$$

In what follows we use the:

<u>Theorem</u>: The mean curvature of a submanifold for any normal orthogonal to the mean normal curvature vector is zero.

In case $H \neq 0$ on a neighbourhood U of a point then η is normalisable in U. Let $\mu = + < \eta, \eta >^{-\frac{1}{2}} > 0$

$$\text{then} \qquad n\bar{H}\binom{1}{1} = \frac{1}{\mu} > 0 \ .$$

Further let e_σ, $n+1 \leqslant \sigma \leqslant n+N$, be an orthonormal basis for the normal space such that e_{n+1} is in the direction of η. Then

1) $\bar{A}(1) = \mu A$ where $\bar{A}(1)$ is the second fundamental form in direction e_{n+1}

2) $\bar{H}\binom{1}{1} = \mu H$

3) $< x-x_0, e_{n+1} > = \mu < x-x_0, \eta >$.

We now discuss the relationship of these generalised fundamental forms and curvatures derived from them to the intrinsic geometry of x(M). We first generalise the classical relation between the three fundamental forms of a surface in E^3.

<u>Theorem</u>: On x(M) $R = E - A$ $\qquad E = (E_{ij})$

<u>Proof</u>: From $R_{ij} = A^\alpha{}_{ik}\, g^{\ell k}\, A^\alpha{}_{j\ell} - nH\binom{\alpha}{1} A^\alpha{}_{ij}$

and DEF. 14 and 15.

using $\qquad n(n-1)K = -\text{Trace } R$

and theorem 1 gives

$$R + (n-1)KG = E - A + (n-1)\sum_{\alpha} H\binom{\alpha}{2}G \qquad \text{Q.E.D.}$$

Now every riemannian 2-manifold is an Einstein space, thus the left hand side vanishes in this case. Whence using

$$A = nH(1)\bar{A}$$

we get the classical relation:

$$E - 2H(1)\bar{A} + H(2)G = 0 .$$

We have thus proved:

<u>Theorem</u>: On any 2-manifold in E^{2+N} we have

$$E - A + KG = 0.$$

<u>Theorem</u>: The following three tensors are intrinsic:

(a) $E - A + (n-1)KG$

$\qquad n \geqslant 2$

(b) $\sum_{\alpha} C\binom{\alpha}{1}^{ij} g^{\ell k} A^{\alpha}{}_{j\ell}$

(c) $\sum_{\alpha} C\binom{\alpha}{2}^{ij}$. $\qquad n > 2$

Proof From Lemma 2. $n(n-1)C\binom{\alpha}{1}^{ij} = nH\binom{\alpha}{1}g^{ij} - g^{i\ell}\, g^{jk}\, A^{\alpha}_{\ell k}$

hence

$$n(n-1)\sum C\binom{\alpha}{1}^{ij}\, g^{\ell k}\, A^{\alpha}_{j\ell} = g^{i\ell}\, g^{jk}(A_{\ell j} - E_{\ell j})$$

$$= -g^{i\ell}\, g^{kj}\, R_{\ell j}$$

and

$$n(n-1)\sum_{\alpha} nH\binom{\alpha}{1}C\binom{\alpha}{1}^{ij} = nHg^{ij} - g^{i\ell}\, g^{jk}\, A_{\ell k}$$

and for $n > 2$, again from Lemma 2

$$n(n-1)(n-2)C\binom{\alpha}{2}^{ij} = n(n-1)H\binom{\alpha}{2}g^{ij} + 2g^{\ell i}\, A^{\alpha}_{k\ell}\, g^{km}\, g^{jn}\, A^{\alpha}_{mn}$$

$$- 2nH\binom{\alpha}{1}\, g^{i\ell}\, g^{jk}\, A^{\alpha}_{\ell k}$$

that

$$n(n-1)(n-2)\sum_{\alpha} C\binom{\alpha}{2}^{ij} = n(n-1)\sum H\binom{\alpha}{2}\, g^{ij} + 2g^{ik}\, g^{j\ell}(E_{k\ell} - A_{k\ell})$$

$$= n(n-1)K\, g^{ij} + 2R^{ij}$$

whence

$$(n-1)(n-2)(n\sum_{\alpha} C\binom{\alpha}{2}^{ij} - Kg^{ij}) = 2(R^{ij} + (n-1)Kg^{ij})$$

$$= 2g^{ik}\, g^{j\ell}\, (E_{k\ell} - A_{k\ell} + (n-1)Kg_{k\ell})$$

Q.E.D.

We note finally that it is possible to generalise the tensors (b) and (c) of this theorem as follows:

$$C(k)^{ij} = \sum_{\alpha} C\binom{\alpha}{k}^{ij} \qquad\qquad 0 \leqslant k \leqslant n$$

$$A(r)^{ij} = \sum_{\alpha} C\binom{\alpha}{r-1}^{ik} g^{\ell j} A^{\alpha}_{k\ell} \quad .$$

However in this generality little if anything appears to be known.

We end this chapter with a theorem which we shall use implicitly throughout the remainder of our work.

<u>Theorem</u>: Let M^n be a riemannian n-manifold and $e_1, \ldots, e_n$ be an orthonormal basis for the tangent space on a neighbourhood U of M, then

$$d(e_1 \wedge \cdots \wedge e_n) = 0 \ .$$

<u>Proof</u>: From (1.2) $\quad de_i = \omega_i^{\ j} e_j$

and since $< e_i, e_j > = \delta_{ij}$:

$$d < e_i, e_j > = < de_i, e_j > + < e_i, de_j > = 0$$

thus

$$< \omega_i^{\ k} e_k, e_j > + < e_i, \omega_j^{\ k} e_k > = \omega_i^{\ j} + \omega_j^{\ i} = 0 \ .$$

Hence

$$d(e_1 \wedge \cdots \wedge e_n) = \sum_{i=1}^{n} e_1 \wedge \cdots \wedge de_i \wedge \cdots \wedge e_n$$

$$= \sum_{i=1}^{n} e_1 \wedge \cdots \wedge e_{i-1} \wedge \omega_i^{\,k} e_k \wedge e_{i+1} \wedge \cdots \wedge e_n$$

$$= \left(\sum_i \omega_i^{\,i} \right) e_1 \wedge \cdots \wedge e_n \;=\; 0$$

Q.E.D.

CHAPTER III

In this chapter we derive our main integral formulae for a submanifold of a riemannian manifold admitting a complete vector field. We shall then derive the formulae in case of some special vector fields.

§ 1. The generalised Minkowski formula

Lemma: Let (M,g,d) be an n-dimensional riemannian manifold with a connexion d, and let $(U, u^1, \ldots, u^n)$ be an allowable coordinate chart on M with local coordinate field $e_i = \partial/\partial u^i$. Then the Lie derivative of g w.r.t. a vector field $X = X^i e_i$ on U is given by:

$$\text{(i)} \quad (\underset{X}{£} g)_{ij} = (\underset{X}{£} g)(e_i, e_j) = < d_i X,\ e_j > + < e_i,\ d_j X >$$

or

$$\text{(ii)} \quad (\underset{X}{£} g)_{ij} = d_i X_j + d_j X_i \ .$$

Proof: (i) From the definition of £ and the fact that M is riemannian

$$d_X Y - d_Y X = [X,Y] = £_X Y$$

and

$$d_X g = 0 \quad \text{equivalently} \quad d_X < Y,Z > = < d_X Y,\ Z > + < Y,\ d_X Z >$$

we have:

$$(\underset{X}{\pounds} g)(e_i\ e_j) = X < e_i, e_j > - <[X,e_i], e_j > - < e_i, [X,e_j]> \tag{3.1}$$

$$= < d_X e_i - [X,e_i], e_j > + < e_i, d_X e_j - [X,e_j]>$$

$$= < d_i X, e_j > + < e_i, d_j X > .$$

(ii) $d_j e_i = \omega_i^k (e_j) e_k \Rightarrow d_X e_i = X^\ell \omega_\ell^k (e_i) e_k$.

Also

$$[X,e_i] = - e_i(X^j) e_j \quad .$$

So in local coordinates (3.1) can be written

$$(\underset{X}{\pounds} g)_{ij} = (\underset{X}{\pounds} g)(e_i,e_j) = (e_i(X^k) + X^\ell \omega_\ell^k(e_i))g_{kj} + (e_j(X^k) + X^\ell \omega_\ell^k(e_j))g_{ik}$$

$$= g_{kj}\, d_i X^k + g_{ik}\, d_j\, X^k$$

$$= d_i X_j + d_j X_i$$

Q.E.D.

Let $f : M \to \bar{M}$ be an isometric embedding of the manifold (M^n,g) into the C^∞ manifold $(\bar{M}^{n+N}, \bar{g})$. In terms of local coordinates $(u^1, \ldots, u^n)$ on a neighbourhood of a point $p \in M$ and local coordinates $(x^1, \ldots, x^{n+N})$ on a neighbourhood of $f(p) \in \bar{M}$ we have

$$x^A = f^A(u^1, \ldots, u^n) \tag{3.1A}$$

where at each point

$$\text{rank } (x^A_i) = n \qquad \text{where } x^A_i = \frac{\partial x^A}{\partial u^i}$$

and

$$g_{ij} = \left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right\rangle_{\bar{g}} \qquad \bar{g} = (\bar{g}_{AB})$$

Let $\bar{M}$ admit a one-parameter group T of transformations generated by a vector field ξ, the transformation being given by:

$$\bar{x} = x + \xi \, dt \, .$$

We assume that the submanifold M^n lies in a domain of $\bar{M}$ which is simply covered by the orbits of the transformations generated by the vector field ξ. The field ξ is assumed of class* C^∞ and non-vanishing on M^n, and at no point of M^n does it lie entirely in the tangent space to M^n.

In this section we work with the adapted frame bundle AFM (DEF.6) $\{e_i, e_\sigma\}$, e_i being tangent to M^n and e_σ normal to M^n. Let $e(\xi)$ be

---

* Actually C^r $(r \geq 3)$ is sufficient.

a unit vector normal to M^n that lies in the $(n+1)$-dimensional vector space spanned by the linearly independent vectors:

$$\{e_1, \ldots, e_n, \xi\} .$$

We can choose a basis of the normal space to M^n so that the vector is included in it, and we will assume this basis ordered so that: $e_{n+1} = e(\xi)$, i.e. $e(\xi)$ is the first normal vector.

At each point of M^n we consider the following differential $(n-1)$-form A_{n-1} :

$$A_{n-1} := \| e(\xi), e_{n+2}, \ldots, e_{n+N}, \xi, \underbrace{dP, \ldots, dP}_{n-1} \| \qquad (3.2)$$

which is by definition:

$$= | e(\xi), e_{n+2}, \ldots, e_{n+N}, \xi, e_{i_1}, \ldots, e_{i_{n-1}} | \omega^{i_1} \wedge \cdots \wedge \omega^{i_{n-1}}$$

where dP is the identity transformation on the tangent space at each point.

Note that this form is independent of any origin and depends only on ξ and M^n .

Whence:

$$dA_{n-1} = d\| e_{n+1}, e_{n+2}, \ldots, e_{n+N}, \xi, dP, \ldots, dP \|$$

$$= \| de_{n+1}, e_{n+2}, \ldots, e_{n+N}, \xi, dP, \ldots, dP \|$$

$$+ \| e_{n+1}, de_{n+2}, \ldots, e_{n+N}, \xi, dP, \ldots, dP \|$$

$$+ \; \ldots\ldots \qquad\qquad \ldots\ldots \; +$$

$$+ \| e_{n+1}, \ldots, e_{n+N}, d\xi, dP, \ldots, dP \| \tag{3.3}$$

We note that since M^n is riemannian: $d(dP) = 0$

and $\quad dv = v_{,i}\,\omega^i \quad$ or $\quad Dv^i = v^i_{\ ,j}\,\omega^j$

Now

$$de_\sigma = \omega_\sigma^{\ i}\, e_i$$

$$= -A^\sigma_{\ ij}\, \omega^j\, e_i \tag{3.3A}$$

i.e. $\quad e_i \cdot d_j e_\sigma = -A^\sigma_{\ ij} \quad .$

Hence

$$\| de_{n+1}, e_{n+2}, \ldots, e_{n+N}, \xi, dP, \ldots, dP \| = n!\,(-1)^{N(N+n-1)} H\left(\begin{smallmatrix}\xi\\1\end{smallmatrix}\right) p(\xi) * 1$$

where

$H\left(\begin{smallmatrix}\xi\\1\end{smallmatrix}\right)$ is the (first) mean curvature in direction $e(\xi) = e_{n+1}$

$p(\xi) = \langle e_{n+1}, \xi \rangle$

$*1 =$ volume element on $M^n = \omega^1 \wedge \cdots \wedge \omega^n$.

Similarly for each normal vector e_σ , $n+2 \leqslant \sigma \leqslant n+N$ we have:

$$\|e_{n+1}, \ldots, de_\sigma, \ldots, e_{n+N}, \xi, dP, \ldots, dP\|$$

$$= n!(-1)^{N(N+n-1)}H(\begin{smallmatrix}\sigma\\1\end{smallmatrix}) < e_\sigma, \xi > * 1 \qquad (3.4)$$

Now by definition the vector ξ lies in the $(n+1)$-dimensional vector space spanned by:

$$\{e_1, \ldots, e_n, e(\xi)\}$$

Hence the right hand side of (3.4) is zero. Thus

$$\|e(\xi), e_{n+2}, \ldots, de_\sigma, \ldots, e_{n+N}, \xi, dP, \ldots, dP\| = 0$$

$$n+1 < \sigma \leqslant n+N .$$

On the other hand, from the lemma, the last term on the right hand side of (3.3) becomes:

$$\|e(\xi), e_{n+2}, \ldots, e_{n+N}, d\xi, dP, \ldots, dP\|$$

$$= (n-1)!(\tfrac{1}{2})(-1)^{N(N+n-1)} \operatorname{Trace}_g f^*(\mathop{\pounds}_{\xi} \bar{g})$$

where

$$\operatorname{Trace}_g f^*(\mathop{\pounds}_{\xi} \bar{g}) = g^{ij}(f^*(\mathop{\pounds}_{\xi} \bar{g}))_{ij} \quad .$$

Collecting together we have:

$$\frac{1}{n!}\, dA_{n-1} = \frac{1}{n!}\, d\| e(\xi), e_{n+2}, \ldots, e_{n+N}, \xi, dP, \ldots, dP \|$$

$$= (-1)^{N(N+n-1)} \left\{ H(\overset{\xi}{1}) p(\xi) *1 + \frac{1}{2n} \operatorname{Trace}_g f^*(\mathop{\pounds}_{\xi} \bar{g}) \right\}$$

Integration over M^n now gives:

$$\frac{1}{n!} \int_{\partial M} A_{n-1} = (-1)^{N(N+n-1)} \left\{ \int_{M^n} H(\overset{\xi}{1}) p(\xi) *1 + \left(\frac{1}{2n}\right) \int_{M^n} \operatorname{Trace}_g f^*(\mathop{\pounds}_{\xi} \bar{g}) \right\} *1$$

where ∂M is the boundary of M^n.

Now in case M^n is closed (i.e. compact and without boundary) we use Stokes' theorem to get:

<u>Theorem A</u>: Let $\bar{M}$ be a C^∞ riemannian (n+N)-manifold admitting a complete vector field ξ, and M^n a closed n-submanifold lying in a domain which is simply covered by the orbits of ξ. If M^n does not contain any singular points of ξ and the set of points of M where ξ lies entirely tangent to M is null; then:

$$\int_{M^n} H(\overset{\xi}{1}) p(\xi) * 1 + \frac{1}{2n} \int_{M^n} \operatorname{Trace}_g f^*(\mathop{\pounds}_{\xi} \bar{g}) * 1 = 0 \qquad (3.5)$$

This integral formula generalises to arbitrary dimensional submanifolds in riemannian spaces of arbitrary co-dimension an integral formula obtained by Chern [11] in 1952.

We may interpret A_{n-1} in the following way: consider the tangential component of ξ

$$\operatorname{tang} \xi = \xi^i e_i .$$

Now take its dual 1-form and denote it by K

$$\therefore \qquad K = \xi^i \omega^i .$$

If we now apply the Hodge star operator to this we get:

$$*K = \sum_i \xi^i (-1)^{i-1} \omega^1 \wedge \cdots \wedge \hat{\omega}^i \wedge \cdots \wedge \omega^n$$

which is precisely A_{n-1}.

§ 2. The second generalised integral formula

We first specialise the ambient manifold $\bar{M}^{n+N}$ to be one of constant riemannian (sectional) curvature. On the submanifold M^n we consider the following differential (n-1)-form $B(r)_{n-1}$:

$$B(r)_{n-1} := \| e(\xi), e_{n+2}, \ldots, e_{n+N}, \xi, \underbrace{de(\xi), \ldots, de(\xi)}_{r}, \underbrace{dP, \ldots, dP}_{n-r-1} \|$$

where $1 \leqslant r \leqslant n-1$

Whence to each fixed r, there corresponds a differential (n-1)-form $B(r)_{n-1}$.

From the classical Codazzi-Mainardi equation it follows that on the submanifold M we have:

$$\left.\begin{aligned} d\omega_\sigma^{\ i} - \omega_\sigma^{\ k}\,\omega_k^{\ i} - \omega_\sigma^{\ \beta}\,\omega_\beta^{\ i} &= \bar{\Theta}_\sigma^{\ i} \\ \text{i.e.} \qquad d(\bar{\Omega}^T) &= \bar{\Omega}^T\Omega - \bar{\bar{\Omega}}\bar{\Omega}^T + \bar{\Phi}^T \end{aligned}\right\} \qquad (3.6)$$

where $\bar{\Theta}$ is the curvature tensor of $\bar{M}$. Now $\bar{M}$ has by assumption constant riemannian curvature, hence:

$$d(de(\xi)) = 0 \quad . \qquad (3.7)$$

Hence by applying the operator d to $B(r)_{n-1}$ we have:

$$\begin{aligned} dB(r)_{n-1} &= d\|e(\xi),e_{n+2},\ldots,e_{n+N},\xi,de(\xi),\ldots,de(\xi),dP,\ldots,dP\| \\ &= \|de(\xi),e_{n+2},\ldots,e_{n+N},\xi,de(\xi),\ldots,de(\xi),dP,\ldots,dP\| \\ &\quad + \|e(\xi),de_{n+2},\ldots,e_{n+N},\xi,de(\xi),\ldots,de(\xi),dP,\ldots,dP\| \\ &\quad + \ldots\ldots \qquad\qquad \ldots\ldots + \\ &\quad + \|e(\xi),e_{n+2},\ldots,de_{n+N},\xi,de(\xi),\ldots,de(\xi),dP,\ldots,dP\| \\ &\quad + \|e(\xi),e_{n+2},\ldots,e_{n+N},d\xi,de(\xi),\ldots,de(\xi),dP,\ldots,dP\| \\ &\quad + \|e(\xi),e_{n+2},\ldots,e_{n+N},\xi,d(de(\xi)),\ldots,de(\xi),dP,\ldots,dP\| \\ &\quad + \ldots\ldots \qquad\qquad \ldots + \end{aligned}$$

$$+ \| e(\xi), e_{n+2}, \ldots, e_{n+N}, \xi, de(\xi), \ldots, d(de(\xi)), dP, \ldots, dP \| .$$

Now from (3.6) we have:

$$\| e(\xi), e_{n+2}, \ldots, e_{n+N}, \xi, d(de(\xi)), \ldots, de(\xi), dP, \ldots, dP \| = 0 .$$

And from (3.3A) we have:

$$\| de(\xi), e_{n+2}, \ldots, e_{n+N}, \xi, de(\xi), \ldots, de(\xi), dP, \ldots, dP \| =$$

$$= n!(-1)^{N(N+n-1)-r} H\binom{\xi}{r+1} p(\xi) * 1$$

where $H\binom{\xi}{r+1}$ is the $(r+1)^{th}$ mean curvature of M^n in the direction $e(\xi)$, i.e. if $k_i(\xi)$ $1 \leq i \leq n$ are the solutions of

$$|A^{n+1} - \lambda G| = 0$$

then $H\binom{\xi}{r+1}$ is the $(r+1)^{th}$ elementary symmetric function of the $k_i(\xi)$ divided by the number of terms:

$$\binom{n}{r+1} H\binom{\xi}{r+1} = \sum_{i_1 < i_2 < \ldots < i_{r+1}} k_{i_1}(\xi)\, k_{i_2}(\xi) \ldots k_{i_{r+1}}(\xi) .$$

Note: From a theorem in linear algebra it is known that the r^{th} elementary symmetric function of the characteristic roots of matrix A^{n+1} is equal to the sum of all r-rowed principal minors of A^{n+1}.

Hence $\binom{n}{r+1} H \binom{\xi}{r+1}$ defined above is equal to the sum of all (r+1)--rowed principal minors of A^{n+1}. In particular

$$nH\left(\begin{smallmatrix}\xi\\1\end{smallmatrix}\right) = \text{Trace } A^{n+1}$$

$$H\left(\begin{smallmatrix}\xi\\n\end{smallmatrix}\right) = \det A^{n+1}$$

$$nH\begin{pmatrix}\xi\\n-1\end{pmatrix} = \sum_{i=1}^{n} A^{ii}$$

where A^{ii} is the cofactor of A_{ii}^{n+1} in $\det A^{n+1}$.

It follows further from (3.3A) and the definition of generalised vector product, given earlier, that the vectors:

$$e(\xi) \times de_{n+2} \times e_{n+3} \times \ldots \times e_{n+N} \times \underbrace{de(\xi) \times \ldots \times de(\xi)}_{r} \times \underbrace{dP \times \ldots \times dP}_{n-r-1}$$

$$e(\xi) \times e_{n+2} \times de_{n+3} \times \ldots \times e_{n+N} \times de(\xi) \times \ldots \times de(\xi) \times dP \times \ldots \times dP$$

. . . .

. . . .

$$e(\xi) \times e_{n+2} \times \ldots \times de_{n+N} \times de(\xi) \times \ldots \times de(\xi) \times dP \times \ldots \times dP$$

have the same direction as e_{n+2}, e_{n+3}, ..., e_{n+N} respectively.

Hence:

$$\|e(\xi), de_{n+2}, e_{n+3}, \ldots, e_{n+N}, \xi, de(\xi), \ldots, de(\xi), dP, \ldots, dP\| = 0$$

$$\|e(\xi), e_{n+2}, de_{n+3}, \ldots, e_{n+N}, \xi, de(\xi), \ldots, de(\xi), dP, \ldots, dP\| = 0$$

$$\cdots\cdots\cdots$$

$$\|e(\xi), \ldots, de_{n+N}, \xi, de(\xi), \ldots, de(\xi), dP, \ldots, dP\| = 0$$

since the vector ξ lies in the (n+1)-dimensional vector space spanned by the vectors:

$$\{e_1, \ldots, e_n, e(\xi)\} .$$

Now the vector:

$$e(\xi) \times e_{n+2} \times \ldots \times e_{n+N} \times de(\xi) \times \ldots \times de(\xi) \times dP \times \ldots \times dP$$

is orthogonal to the vectors $e(\xi)$, $e_{n+2}, \ldots, e_{n+N}$, and from (3.3A) $de(\xi) = \omega_{n+1}^{\ \ i} e_i$, we have:

$$\|e(\xi), e_{n+2}, \ldots, e_{n+N}, d\xi, de(\xi), \ldots, de(\xi), dP, \ldots, dP\|$$

$$= n!(-1)^{N(N+n-1)-r}\left(\frac{1}{2n}\right) \mathrm{Trace}[C(\begin{smallmatrix}\xi\\ r\end{smallmatrix})\ f^*(\underset{\xi}{\pounds}\,\bar{g})]*1$$

where $C(\begin{smallmatrix}\xi\\ r\end{smallmatrix}) = (C(\begin{smallmatrix}\xi\\ r\end{smallmatrix})^{ij})$

$$C(\begin{smallmatrix}\xi\\ r\end{smallmatrix})^{ij} = \frac{1}{(n-1)!}\epsilon^{ii_1\cdots i_{n-1}}\epsilon^{jj_1\cdots j_r} A^{n+1}_{i_1j_1}\cdots A^{n+1}_{i_rj_r} g_{i_{r+1}j_{r+1}} \cdots$$

$$\cdots g_{i_{n-1}j_{n-1}}$$

where as before $\epsilon^{ii_1\cdots i_{n-1}}$ is the Ricci ϵ-symbol of the submanifold M^n.

Hence we have finally:

$$\frac{1}{n!}\, dB(r)_{n-1} = \frac{1}{n!}\, d\|e(\xi),e_{n+2},\ldots,e_{n+N},\xi,de(\xi),\ldots,de(\xi),dP,\ldots,dP\|$$

$$= (-1)^{N(N+n-1)-r}\left\{H\binom{\xi}{r+1}p(\xi)*1 + \left(\frac{1}{2n}\right)\right.$$

$$\left.\times\ \mathrm{Trace}[C(\tfrac{\xi}{r})\ f^*(\mathop{\pounds}_{\xi}\bar{g})]*1\right\} \qquad .$$

Integrate both sides of this equation over M^n

$$\frac{1}{n!}\int_{\partial M^n}\|e(\xi),e_{n+1},\ldots,e_{n+N},\xi,de(\xi),\ldots,de(\xi),dP,\ldots,dP\|$$

$$= (-1)^{N(N+n-1)-r}\left\{\int_{M^n} H\binom{\xi}{r+1}p(\xi)*1 + \left(\frac{1}{2n}\right)\int_{M^n}\mathrm{Trace}[C(\tfrac{\xi}{r})\,f^*(\mathop{\pounds}_{\xi}\bar{g})]*1\right.$$

and in case M^n is closed we use Stokes' theorem to get:

Theorem B: Let $\bar{M}$ be a C^∞ $(n+N)$-dimensional riemannian manifold of constant curvature admitting a complete vector field ξ, and M^n a closed n-submanifold lying in a domain which is simply covered by the orbits of ξ. If M^n does not contain any singular points of ξ and the set of points on M^n where ξ lies entirely tangent to M^n is null; then:

$$\int_{M^n} H\binom{\xi}{r+1} p(\xi) * 1 + \left(\frac{1}{2n}\right) \int_{M^n} \mathrm{Trace}[C(\underset{r}{\xi}) f^*(\pounds \bar{g})] * 1 = 0 \tag{3.8}$$

This is the second of our two generalised Minkowski integral formulas. The remainder of our work will involve the various specialisations of (3.5) and (3.8) and the applications which follow therefrom.

§ 3. Specialisations of ξ

Theorem 1: If $\bar{M}$ is a C^∞ riemannian (n+N)-manifold admitting a one-parameter C^0 group T of conformal transformations generated by a vector field ξ. If further M^n is an embedded closed orientable C^∞ n-submanifold which is simply covered by the orbits of T, and does not contain any singular point of ξ. Then

$$\int_M H(\underset{1}{\xi})\, p(\xi) * 1 + \int \phi * 1 = 0 \tag{3.9}$$

where ϕ is the characteristic function of ξ.

and in case $\bar{M}$ is a riemannian manifold of constant curvature we have also:

$$\int_{M^n} H\binom{\xi}{r+1} p(\xi) * 1 + \int_{M^n} \phi H\binom{\xi}{r} * 1 = 0 \qquad (3.10)$$

$$1 \leqslant r \leqslant n-1$$

<u>Proof</u>: In case T is conformal ξ satisfies:

$$\underset{\xi}{\pounds}\bar{g} = \xi_{i,j} + \xi_{j,i} = 2\phi\,\bar{g}$$

$$\Longrightarrow \quad \underset{g}{\mathrm{Trace}}\, f^*(\underset{\xi}{\pounds}\bar{g}) = 2n\phi \qquad (3.11)$$

Also $$\mathrm{Trace}[C\binom{\xi}{r} f^*(\underset{\xi}{\pounds}\bar{g})] = 2n\phi H\binom{\xi}{r} \qquad (3.12)$$

Substitution of (3.11) and (3.12) in (3.5) and (3.8) respectively gives the result. Q.E.D.

<u>Theorem 2</u>: If $\bar{M}$ is a C^∞ riemannian (n+N)-manifold admitting a one-parameter C^0 group T of homothetic transformations generated by a vector field ξ, and M^n is an embedded C^∞ closed orientable n-manifold which is simply covered by the orbits of T and does not contain any singular points of ξ, then:

$$\int_{M^n} H\binom{\xi}{1} p(\xi) * 1 + c\, V(M) = 0 \qquad (3.13)$$

where c = constant and V(M) is the total volume of M^n.

In case $\bar{M}$ is of constant riemannian curvature

$$\int_{M^n} H\binom{\xi}{r+1} p(\xi) * 1 + c \int_{M^n} H\binom{\xi}{r} * 1 = 0 \qquad (3.14)$$

$$1 \leqslant r \leqslant n-1 .$$

<u>Proof</u>: In case T is homothetic then Φ = constant in (3.11) and (3.12) above. Hence:

$$\underset{\xi}{\pounds}\, \bar{g} = 2c\, \bar{g}$$

$$\Rightarrow \begin{cases} \operatorname{Trace}_g f^* (\underset{\xi}{\pounds}\, \bar{g}) = 2nc \\ \operatorname{Trace} [\, C\binom{\xi}{r} f^* (\underset{\xi}{\pounds}\, \bar{g})] = 2nc\, H\binom{\xi}{r} \end{cases}$$

And again the result follows by substitution in (3.5) and (3.8).

Q.E.D.

Note that in this case $\bar{M}$ becomes Euclidean (n+N)-space since:

<u>Theorem</u>: If $\bar{M}^{n+N}$ is a riemannian manifold of constant curvature admitting a one-parameter group T of homothetic transformations, then either $\bar{M}$ is E^{n+N} or T is isometric.

Theorem 2 has the following important corollary:

<u>Corollary</u>: With the conditions as in theorem 2, if $\bar{M}^{n+N}$ is taken to be E^{n+N} and assume M^n contains the origin O of E^{n+N}, then if we take for ξ the (homothetic) radial vector field, where the orbits are simply the lines through O then $\pounds_{\xi}\bar{g}=2\bar{g}$ giving for (3.5) and (3.8) respectively:

$$\int_{M^n} H\binom{\xi}{1} p(\xi) * 1 + V(M^n) = 0$$

and $$\int_{M^n} H\binom{\xi}{r+1} p(\xi) * 1 + \int_{M^n} H\binom{\xi}{r} * 1 = 0 \qquad 1 \leqslant r \leqslant n-1 .$$

These generalise the two corresponding formulas of Hsuing [7] for hypersurfaces to arbitrary co-dimension. The formulas of Hsuing themselves being generalisations of two integral formulas due to Chern [11] for surfaces in E^3.

<u>Theorem 3</u>: If $\bar{M}^{n+N}$ is a C^∞ riemannian (n+N)-manifold admitting a one-parameter C^0 group T of isometric transformations generated by a vector field ξ, and M^n is an embedded C^∞ closed orientable n-manifold which is simply covered by the orbits of T and does not contain any singular point of ξ, then:

$$\int_{M^n} H\binom{\xi}{1} p(\xi) * 1 = 0 \tag{3.15}$$

In case $\bar{M}$ has constant riemannian curvature we have:

$$\int_{M^n} H\binom{\xi}{r+1} p(\xi) * 1 = 0 \qquad 1 \leqslant r \leqslant n-1 \tag{3.16}$$

Proof: In case T is isometric then $\Phi = 0$ in (3.11) and (3.12) whence the result follows by substitution as before. Q.E.D.

As a final specialisation of the field ξ we can consider the so-called tension field of the embedding. This we explain briefly - for a more detailed exposition see Eells and Sampson [21] .

Let (M,g) and $(\bar{M},\bar{g})$ be complete C^∞ manifolds of dimension n and n+N respectively, and suppose M closed. For each point $p \in M^n$ let $<, >_p$ denote the inner product on the space of 2-covariant tensors of M_p defined:

$$< a,b >_p := a_{ij}\, b_{k\ell}\, g^{ik}\, g^{j\ell} \qquad \text{where } g^{ik} g_{kj} = \delta^i_j$$

and $a = (a_{ij})$, $b = (b_{ij})$. Let $f \in C^\infty : M \to \bar{M}$, then through f, $\bar{g}$ induces a metric $f^*\bar{g}$ on M. We can thus define the functional on M:

$$p \to < g(p), (f^*\bar{g})(p) >_p \quad .$$

We call

$$e(f)(p) = \tfrac{1}{2} < g(p),(f^{*}\bar{g})(p) >_{p}$$

the <u>energy density</u> of f at p. Its dual n-form $e(f) * 1$ can then be integrated over M, and we define <u>the energy of f, E(f)</u>:

$$E(f) = \int_{M^n} e(f) * 1 .$$

Then from (3.1A)

$$(f^{*}\bar{g})_{ij} = f_i^{\ A} f_j^{\ B} \bar{g}_{AB}$$

Hence

$$E(f) = \tfrac{1}{2} \int_{M^n} \bar{g}_{AB} f_i^{\ A} f_j^{\ B} g^{ij} * 1$$

i.e.

$$E(f) = \tfrac{1}{2} \int \operatorname{Trace}_g (f^{*}\bar{g}) * 1$$

where $\quad \text{integrand} = g^{ij} f_i^{\ A} f_j^{\ B} \bar{g}_{AB} \quad .$

Let $H(M,\bar{M})$ be the totality of smooth maps: $M \to \bar{M}$. Then for each $f \in H(M,\bar{M})$ the set of C^∞ maps $u : M \to T\bar{M}$ such that: $\pi \circ u = f$ is a vector space, which we denote $H(f)$, with algebraic operations defined point-wise. We call a typical element $u \in H(f)$ <u>a vector</u>

field along f. Space $H(f)$ is given the inner product (denoted $<, >_f$) defined:

$$< u, v >_f := \int_{M^n} < u(p), v(p) >_{f(p)} * 1 \qquad u, v \in H(f)$$

For any $u \in H(f)$ the directional derivative of E in direction u

i.e.

$$\nabla_u E(f) = \frac{d}{dt} (E(f_t))\Big|_{t=0} \qquad \text{where} \quad f_t(p) = \exp_{f(p)}(tv(p))$$

$$t \in \mathbb{R}$$

is the end point of the geodesic segment in $\bar{M}$ starting at $f(p)$ and determined in length and direction by vector $tv(p) \in \bar{M}_{f(p)}$. We can thus define a unique field $\tau(f)$ along f:

$$\nabla_u E(f) = - < \tau(f), u >_f \qquad u \in H(f)$$

This field is called the tension field of f. Whence:

$$\tau(f) = \Delta f + \operatorname{Trace}_g f^* \bar{\omega}$$

Where $\bar{\omega}$ is connexion matrix on $\bar{M}$ and where Δ is the Laplace-Beltrami operator on M.

Maps for which $\tau(f) = 0$ are called harmonic. In case $\bar{M}$ is flat then $\tau(f) = \Delta f$; in particular if $\bar{M}$ is E^{n+N} then a map $f : M^n \to E^{n+N}$ is harmonic iff f = constant, by Hopf's Maximum principal

Let $f : M \to \bar{M}$ be a riemannian immersion, whence for each $p \in M$, $f_*(p)$ maps M_p isometrically into $\bar{M}_{f(p)}$, i.e. $g = f^* \bar{g}$. We use the following theorem due to Eells and Sampson [21] .

<u>Theorem</u>: Let $f : M \to \bar{M}$ be a riemannian immersion. Then $e(f) = n/2$ and $\tau(f)$ coincides with the mean normal curvature vector.

Note that for any $N \in M_{f(p)}$ normal to M_p we have:

$$< \tau(f), N > \;=\; \text{Trace } A(N)$$

$A(N)$ being the second fundamental form for N. Hence in case ξ is the tension field of the (isometric) embedding:

$$< e(\xi), \xi > \;=\; H \;=\; \text{(first) mean curvature of } M^n$$

and where we assume implicitly that $\xi \neq 0$ on M^n.

Hence in this case (3.9) and (3.10) become respectively:

$$\int_{M^n} (H^2 + \phi) * 1 \;=\; 0$$

$$\int_{M^n} (H(r+1)H + \phi H(r)) * 1 \;=\; 0 \qquad 1 \leq r \leq n-1$$

(3.13), (3.14) become respectively:

$$\int_{M^n} H^2 * 1 + cV(M) = 0$$

$$\int_{M^n} (H(r+1)H + cH(r)) * 1 = 0 \qquad 1 \leqslant r \leqslant n-1$$

and finally (3.15), (3.16) become respectively:

$$\int_{M^n} H^2 * 1 = 0$$

$$\int_{M^n} H(r+1)H * 1 = 0 \qquad (1 \leqslant r \leqslant n-1).$$

CHAPTER IV

In this chapter we use the integral formulas (3.5) and (3.8) and their various specialisations to prove some characterisation theorems.

§ 1. Characterisation in codimension one

DEF. 17. Normal coordinates: Let O be a point in a riemannian (n+1)-manifold $\bar{M}^{n+1}$, and U a sufficiently small neighbourhood of O, such that in U there exists a unique geodesic γ joining O to every point $p \in U$. Let γ_i $1 \leq i \leq n+1$, be n+1 mutually orthogonal geodesics through O. Then the normal coordinates $\{y^i\}$ of p w.r.t. the geodesic from O, $\gamma_1, \ldots, \gamma_{n+1}$ are defined by: $y^i = s \cos(\gamma, \gamma_i)$ where (γ, γ_i) is the angle between γ and γ_i at O, and s is arc length from O to p along γ. Clearly $\sum_i \cos^2(\gamma, \gamma_i) = 1$. If on $\bar{M}^{n+1}$ there exists a unique geodesic arc with minimal length joining a fixed point O to every point p, then we have a global normal coordinate system at O.

In what follows we shall have need of the following two lemmas, the proof of which is to be found in Hardy, Littlewood and Pólya - Inequalities (pp. 52, 104, 105).

Lemma 1 (Newton): Let S_r be the r^{th} elementary symmetric function of n real non-zero numbers k_i, $1 \leq i \leq n$, and define $H_o = 1$ and $\binom{n}{r}H_r := S_r$ for $r = 1, \ldots, n$. Then :

$$H_{r-1}\, H_{r+1} - H_r^2 \leq 0 \qquad\qquad 1 \leq r \leq n-1$$

if in addition $H_r, H_{r-1}, \ldots, H_{r-i}$ are positive

$$H_{r-1}/H_r \geq H_{r-2}/H_{r-1} \geq \ldots \geq H_{r-i-1}/H_{r-i}$$

where equality for any $r \Rightarrow k_1 = \ldots = k_n$.

Lemma 2 (Maclaurin): Let k_i $1 \leq i \leq n$, H_o and H_r be as in the previous lemma. If $H_1, \ldots, H_s$, $1 \leq s \leq n$ are positive then:

$$H_1 \geq H_2^{\frac{1}{2}} \geq H_3^{\frac{1}{3}} \geq \ldots \geq H_s^{1/s} \qquad\qquad 1 \leq s \leq n$$

where equality at any stage $\Longrightarrow$ $k_1 = \ldots = k_n$.

DEF. 18. An umbilic point: Is a point p in a hypersurface M^n where $A_p = \lambda I$, where A_p is the second fundamental form at p, λ is a scalar and I is the identity transformation on M_p. At an umbilic point all principal curvatures are equal and all directions are directions of principal curvature. If all points of M^n are umbilic, M^n is called totally umbilic.

The following important lemma will be used:

Lemma 3: A point $p \in M^n$ embedded in a riemannian manifold M^{n+1} of dimension $n \geq 2$ is umbilic if $k_1 = \ldots = k_n$ at p.

Proof: The principal curvatures k_i $1 \leqslant i \leqslant n$ are extremal values of the quantity: $A_{ij}\, e^i e^j / g_{ij}\, e^i e^j$ at p for an arbitrary tangent vector $e = (e^i)$ of M^n. Hence if $k_1 = \ldots = k_n$ at p then the given ratio is independent of e so $A_{ij} = c\, g_{ij}$ $\forall\, i,j$ at p, where c is a scalar invariant. Hence p is umbilic.

DEF. 19. Riemann n-sphere: Is a closed hypersurface M^n in a riemannian manifold $\bar{M}^{n+1}$ $(n \geqslant 2)$ such that every point of M^n is umbilic.

Theorem 1: Let $\bar{M}^{n+1}$ be a riemannian (n+1)-manifold $(n \geqslant 2)$ with constant riemannian curvature such that there exists a global normal coordinate system at a fixed point O. Then for a closed orientable embedded C^3 hypersurface M^n

$$\int (H_{r-1} + H_r\, p) * 1 = 0 \qquad\qquad r = 1, \ldots, n$$

and where $p = Y \cdot e_{n+1}$ where e_{n+1} is the unit normal vector to M^n at a point q and Y is the position vector of q w.r.t. the global coordinate system.

Proof: The procedure is exactly as for theorem B where in this case we take $\xi = Y$ and $e_{n+1} = e(Y)$, and use the (n-1)-form:

$$B_{n-1} = \|\underbrace{dY, \ldots, dY}_{n-r}, Y, e_{n+1}, \underbrace{de_{n+1}, \ldots, de_{n+1}}_{r-1}\|$$

Applying the operator d to this and using the facts that

$$d(dY) = 0$$

and

$$dY \wedge \cdots \wedge dY \wedge de_{n+1} \wedge \cdots \wedge de_{n+1} = n!(-1)^r H_r \, e_{n+1} * 1$$

the result follows immediately by integration and application of Stokes' theorem. Q.E.D.

From this theorem several important characterization theorems may be deduced:

<u>Theorem 2</u> (Feeman - Hsiung): Let M^n $(n \geqslant 2)$ be a hypersurface of a riemannian manifold $\bar{M}^{n+1}$ satisfying the conditions of theorem 1. If there exists an integer s, $(1 \leqslant s \leqslant n)$, such that $H_s > 0$, and either $p \leqslant -H_{s-1}/H_s$ or $p \geqslant -H_{s-1}/H_s$ at all points of M^n then M^n is a riemann n-sphere.

<u>Proof</u> Case (i) $1 \leqslant s \leqslant n-1$

The inequalities in the theorem are respectively equivalent to:

$$H_s p + H_{s-1} \leqslant 0 \quad \text{and} \quad H_s p + H_{s-1} \geqslant 0 \qquad \text{since} \quad H_s > 0.$$

Now theorem 1 for $r = s$, together with either of these inequalities $\Longrightarrow p = -H_{s-1}/H_s$; and on substituting this value of p into theorem 1 for $r = s+1$ gives:

$$\int_{M^n} \frac{1}{H_s} (H_s^2 - H_{s-1} H_{s+1}) * 1 = 0 \tag{4.1}$$

From lemma 1 for r = s the above integrand is non-negative and hence (4.1) holds iff:

$$H_s^2 - H_{s-1} H_{s+1} = 0 .$$

Hence from lemma 1, $k_1 = \dots = k_n$ at all points of M^n, and it then follows from lemma 3 that M^n is a Riemann n-sphere.

Case (ii) s = n:

Apply theorem 1 with r = n

$$\int_{M^n} (H_{n-1} + H_n p) * 1 = 0$$

and since the integrand is of fixed sign then $p = -H_{n-1}/H_n$.

Now apply theorem 1 with r = n-1

$$\int_{M^n} H_{n-2} * 1 = -\int_{M^n} H_{n-1}\, p * 1$$

$$= \int_{M^n} (H_{n-1}^2 / H_n) * 1$$

or
$$\int \frac{1}{H_n} (H_{n-1}^2 - H_n H_{n-2}) * 1 = 0$$

and a similar argument to case (i) proves the result. Q.E.D.

Theorem 3 (Hsiung): Let M^n $(n \geq 2)$ be a hypersurface of a riemannian manifold $\bar{M}^{n+1}$ satisfying the conditions of theorem 1. Suppose there exists an integer s, $(1 \leq s \leq n)$ such that at all points of M^n, p is of the same sign, $H_i > 0$ for $1 \leq i \leq s$, and H_s is constant. Then M^n is a Riemann n-sphere.

Proof: Case (i) $s < n$.

From lemma 1 for $r = 1, \ldots, s$ and the assumption that $H_i > 0$ for $i = 1, \ldots, s$ we obtain:

$$\frac{H_1}{H_0} \geq \frac{H_2}{H_1} \geq \cdots \geq \frac{H_{s+1}}{H_s} .$$

In particular

$$H_1 H_s \geq H_{s+1} \tag{4.2}$$

where equality $\Longrightarrow k_1 = \cdots = k_n$. From theorem 1 for $r = 1$ and the assumption $H_1 > 0$ and p is of constant sign on all of M^n, it follows that p is negative. Multiply both sides of (4.2) by p and integrate over M^n and apply theorem 1 for case $r = 1$ and $r = s+1$; it thus follows from the assumption that H_s is constant:

$$-H_s \int_{M^n} *1 = \int_{M^n} H_1 H_s p*1 \leq \int_{M^n} H_{s+1} p*1 = -H_s \int_{M^n} *1$$

whence

$$\int (H_1 H_s - H_{s+1})\, p * 1 = 0 \tag{4.3}$$

Now (4.1) $\Longrightarrow$ this integrand is non-positive and hence

$$H_1 H_s - H_{s+1} = 0$$

whence by lemma 1 we have: $k_1 = \ldots = k_n$ on all of M^n and application of lemma 3 proves this part.

Case (ii) $s = n$:

Using the assumption $H_i > 0$ $(1 \leqslant i \leqslant n)$ and lemma 2 we have:

$$H_1 \geqslant H_2^{\frac{1}{2}} \geqslant \ldots \geqslant H_{n-1}^{1/(n-1)} \geqslant H_n^{1/n} = c \tag{4.4}$$

where c = constant > 0. From theorem 1 for $r = n$ and inequalities (4.4) we have:

$$\int_{M^n} H_n\, p * 1 = -\int_{M^n} H_{n-1} * 1 \leqslant -c^{n-1} \int_{M^n} * 1 \tag{4.5}$$

Also from theorem 1 for $r = 1$ and inequalities (4.4) and the fact that $p < 0$ we have:

$$\int_{M^n} H_n\, p * 1 = c^{n-1} \int_{M^n} H_n^{1/n}\, p * 1 \tag{4.6}$$

$$\geqslant c^{n-1} \int_{M^n} H_1\, p * 1 = -c^{n-1} \int_{M^n} * 1$$

From (4.5) and (4.6) we thus have:

$$\int_{M^n} (H_n^{1/n} - H_1)\, p * 1 = 0$$

Also (4.4) $\Longrightarrow$ this integrand is non-negative and hence

$$H_n^{1/n} = H_1$$

whence lemma 2 $\Longrightarrow$ $k_1 = \dots = k_n$ on all M^n. Hence by lemma 1 the result follows. Q.E.D.

In the case where $\bar{M}^{n+1}$ is Euclidean (n+1)-dimensional space E^{n+1} and M^n is a convex C^2 hypersurface the integral formula in theorem 1 was obtained by Minkowski [22] for the case n = 2 in the well-known form

$$\int_{M^n} (Kp + H) * 1 = 0$$

where H and K are the mean curvature and Gauss curvature respectively at a point $q \in M^2$, and p is the oriented distance from a fixed point $0 \in E^3$ to the tangent space M_q^2 of the ovaloid M^2. The theorem was extended for general n by Kubota [23].

Also in the case where $\bar{M}^{n+1}$ is E^{n+1} and M^n is a convex C^2 hypersurface theorem 2 with the restriction that s = 1 and the equality on p:

$$p + \frac{H_{s-1}}{H_s} = 0$$

being satisfied - (rather than the two corresponding inequalities in theorem 2) - was proved by Grotemeyer [24] for the case n = 2:

$$\int (H^2 - K)\, p *1 = \tfrac{1}{4} \int g^{ik} H_k (x^2)_i *1$$

where p is distance from tangent plane to the origin, x^2 is the square of the radius vector and $(x^2)_i$, H_k the derivatives of x^2 and H w.r.t. the surface parameters. Again this was extended to the case general n by Süss.

Finally theorem 3 is essentially the well-known Liebmann-Süss theorem, which was obtained by Liebmann [1] for the case n = 2 in the form:

Liebmann H-theorem: The only ovaloids with constant mean curvature H in E^3 are the spheres.

The extension of this theorem to a convex hypersurface in E^n was given by Süss [3].

Finally we note that by using different methods Hsiung [7] obtained theorem 1 for the cases r = 1 and r = n and some special cases of theorems 2 and 3; and also theorem 1 for $1 \leqslant r \leqslant n$, together with theorems 2 and 3 in the special case of a c^2 (instead of the c^3 condition of theorems 2 and 3) hypersurfaces in Euclidean space.

<u>Theorem 4</u> (Stong): Under the assumptions of theorem 1, if there are integers s and i $(1 \leqslant i < s \leqslant n)$, with $H_s, \ldots, H_i > 0$ and constants $c_j \geqslant 0$ for $i \leqslant j \leqslant s-1$ such that on all of $\bar{M}^n$ we have $H_s = \Sigma c_j H_j$ then M^n is a Riemann n-sphere.

<u>Proof</u>: From theorem 1

$$\frac{H_j}{H_s} - \frac{H_{j-1}}{H_{s-1}} = \left(\frac{H_j}{H_{s-1}}\right)\left(\frac{H_{s-1}}{H_s} - \frac{H_{j-1}}{H_j}\right) \geqslant 0 \qquad i \leqslant j \leqslant s-1$$

and equality holds everywhere only if M^n is a Riemann n-sphere.

Hence:

$$1 = \sum c_j \left(\frac{H_j}{H_s}\right) \geqslant \sum c_j \left(\frac{H_{j-1}}{H_{s-1}}\right)$$

or

$$H_{s-1} - \sum c_j H_{j-1} \geqslant 0$$

and again equality holds everywhere only when M^n is a Riemann n-sphere.

From theorem 1:

$$\int_{M^n} (H_{s-1} - \sum c_j H_{j-1}) * 1 = - \int_{M^n} p(H_s - \sum c_j H_j) * 1$$

$$= 0$$

so that $$H_{s-1} = \sum c_j H_{j-1} \qquad \text{on all of } M^n$$

Q.E.D.

Theorem 5 (Stong): Under the assumptions of theorem 1, if there exist integers s and i, $(0 \leqslant i < s < n)$, with $H_{s+1}, \ldots, H_{i+1} > 0$ and constants $c_j \geqslant 0$ for $i \leqslant j \leqslant s-1$ such that at all points of M^n, $H_s = \Sigma c_j H_j$ and if p is of fixed sign on all of M^n, then M^n is a Riemann-n-sphere.

Proof: Apply procedure of previous theorem in reverse

$$H_{s+1} - \sum c_j H_{j+1} \leqslant 0$$

and equality only if M^n is a Riemann n-sphere. By theorem 1

$$\int_{M^n} (H_{s+1} - \sum c_j H_{j+1}) p * 1 = - \int_{M^n} (H_s - \sum c_j H_j) * 1$$

$$= 0$$

and since the integrand on the left hand side is of fixed sign $H_{s+1} = \Sigma c_j H_{j+1}$ everywhere.

Q.E.D.

Theorem 6: If there is an integer s, $(1 < s \leqslant n)$, with $H_s > 0$ and a constant c with $H_s = c\,H_{s-1}$ at all points of M^n, then M^n is a Riemann n-sphere.

Proof: $H_s > 0 \implies c \neq 0$ and H_{s-1} must be fixed in sign. From lemma 1

$$H_{s-1}(H_{s-1} - c\,H_{s-2}) = H^2_{s-1} - H_s\,H_{s-2} \geqslant 0$$

So $H_{s-1} - cH_{s-2}$ is of fixed sign and vanishes identically only if M^n is a Riemann n-sphere. Theorem 1 gives:

$$\int_{M^n} (H_{s-1} - c\,H_{s-2}) * 1 = \int_{M^n} (c\,H_{s-1} - H_s)p * 1$$

$$= 0$$

and this implies:

$$H_{s-1} = c\,H_{s-2} \qquad \text{on all } M^n$$

Q.E.D.

Theorem 7: If there is an integer s $(1 < s \leqslant n)$ with $H_i > 0$ for $(1 \leqslant i \leqslant s)$ and a constant c with:

$$H_{s-1}^{1/(s-1)} \geqslant c \geqslant H_s^{1/s}$$

on all of M^n and if p is of fixed sign on M^n then M^n is a Riemann n-sphere.

Proof: p of fixed sign and theorem 1 with $r = 1 \Longrightarrow p < 0$. By lemma 2, $H_1 \geqslant c$. Choose the orientation of M^n for which $g \geqslant 0$ throughout $M^n \Longrightarrow \int g * 1 \geqslant 0$. Then theorem 1 gives:

$$-\int_{M^n} c^{s-1} p H_1 * 1 \geqslant -\int c^s p * 1$$

$$\geqslant -\int H_s p * 1 = \int H_{s-1} * 1$$

$$\geqslant \int c^{s-1} * 1$$

$$= -\int c^{s-1} p H_1 * 1$$

$\Longrightarrow$ all terms are equal and: $\int p(H_1 - c) * 1 = 0.$

So $H_1 = c$. By theorem 5 with $s = 1$ and $i = 0$ we get the required result. Q.E.D.

Theorem 8: (Stong): If there is an integer s $(1 < s \leqslant n)$ with H_s, $H_{s-1} > 0$ and a constant c with:

$$\frac{H_{s-1}}{H_s} \geqslant c \geqslant \frac{H_{s-2}}{H_{s-1}}$$

on all M^n, and if p is of fixed sign throughout M^n then M^n is a Riemann n-sphere.

Proof: From theorem 1 with $r = s$ we have that $p < 0$.

Choose the orientation as in the previous proof, then from theorem 1 we get:

$$\int_{M^n} H_{s-2} * 1 = -\int_{M^n} p\,H_{s-1} * 1$$

$$\geqslant -\int_{M^n} p\,c\,H_s * 1 = \int_{M^n} c\,H_{s-1} * 1$$

$$\geqslant \int_{M^n} H_{s-2} * 1$$

Hence all these terms are equal, so

$$\int_{M^n} p(H_{s-1} - c\,H_s) * 1 = 0$$

and $p(H_{s-1} - cH_s) \leqslant 0 \implies H_{s-1} = c\,H_s$ on all of M^n. Hence by theorem 6, M^n is a Riemann n-sphere.

Q.E.D.

Corollary: If Σ is a closed orientable surface of class c^3 twice differentiably embedded in E^3 with $H_2 > 0$, then either

$$\inf_{q \in \Sigma} H_1^2 < \sup_{q \in \Sigma} H_2 \quad \text{or} \quad \Sigma \text{ is a sphere.}$$

Proof: Hadamard's theorem $\Longrightarrow$ Σ is the boundary of a convex body, thus, choosing the origin to be inside the body, p is of fixed sign. If $\inf H_1^2 \geqslant \operatorname{Sup} H_2$ $\exists$ a constant $c > 0$ such that

$$H_1^2 \geqslant c^2 \geqslant H_2$$

Since H_1 is continuous, either $H_1 \geqslant c$ or $H_1 \leqslant -c$. In the first case theorem 7 implies every point of Σ is umbilic $\Longrightarrow$ Σ is a sphere (see Willmore [25] , p. 128). In the second case p must be positive and choosing the orientation as before:

$$\begin{aligned} -\int_\Sigma cpH_1 * 1 &= \int_\Sigma c * 1 \\ &\leqslant -\int_\Sigma H_1 * 1 = \int_\Sigma pH_2 * 1 \\ &\leqslant \int_\Sigma p\,c^2 * 1 \\ &\leqslant -\int_\Sigma cp\, H_1 * 1 \end{aligned}$$

Thus all the terms are equal and we have:

$$H_2 = -c\,H_1 \quad .$$

By theorem 6 every point is umbilic and Σ is a sphere.

Q.E.D.

§ 2. Characterisation in arbitrary co-dimension

The characterisation in the previous section resulted from a specialisation of the integral formula (3.8) to the case of co-dimension one. In this section we shall study the question as to what similar characterisation is possible in arbitrary co-dimension.

In what follows we shall assume that the vector $e(\xi)$ coincides with the Euler-Schouten vector e (i.e. the unit vector in the direction of the mean normal curvature vector); we shall further assume that the vector field ξ is, at each point of M^n, contained in the $(n+1)$-dimensional space spanned by the tangent space to M^n and the vector e. In this case the integral formulas (3.5) and (3.8) become respectively:

$$\int_{M^n} H\binom{e}{1}\, p(\xi) * 1 + \left(\frac{1}{2n}\right) \int_{M^n} \operatorname{Trace}_g \left[f^*(\pounds_\xi\, \bar{g}) \right] *1 = 0 \qquad (4.7)$$

and

$$\int_{M^n} H\binom{e}{r+1}\, p(\xi) * 1 + \left(\frac{1}{2n}\right) \int_{M^n} \operatorname{Trace}\left[C\binom{e}{r} f^* (\pounds_\xi\, \bar{g}) \right] * 1 = 0$$

(4.8)

and in case the group T generated by ξ is conformal these become respectively:

$$\int_{M^n} H\binom{e}{1}\, p\,(\xi) * 1 + \int_{M^n} \phi * 1 \;=\; 0 \tag{4.9}$$

and

$$\int_{M^n} H\binom{e}{r+1} p(\xi) * 1 + \int_{M^n} \phi H\binom{e}{r} * 1 \;=\; 0 \tag{4.10}$$

By analogy to DEF. 19 we now define:

DEF. 20. An Euler n-sphere: Is a closed orientable embedded n-dimensional submanifold such that every point is umbilic with respect to the Euler-Schouten vector.

Theorem: Let $\bar{M}^{n+N}$ be an (n+N)-dimensional ($n+N \geqslant 3$) C^∞ riemannian space form*, admitting a C^0 one-parameter group of conformal transformations generated by a vector field ξ ; and M^n be a closed orientable embedded submanifold. If $p(\xi)$ is positive (or negative) at each point of M^n and M^n has constant (first) mean curvature H then M^n is an Euler n-sphere.

Proof: Multiply (4.9) by the constant H, giving:

---

* i.e. riemannian manifold of constant curvature.

$$\int_{M^n} H^2 \, p(\xi) * 1 + \int_{M^n} \phi H * 1 = 0$$

From (4.10) with $r = 1$ we have:

$$\int_{M^n} H\binom{e}{2} \, p(\xi) * 1 + \int_{M^n} \phi H * 1 = 0$$

Hence

$$\int_{M^n} \left(H^2 - H\binom{e}{2}\right) p(\xi) * 1 = 0$$

Whence from the assumption on $p(\xi)$ this is true iff

$$H^2 - H\binom{e}{2} = 0$$

since

$$H^2 - H\binom{e}{2} = \frac{1}{n^2(n-1)} \sum_{i<j} (\lambda_i - \lambda_j)^2 \geqslant 0$$

Hence at each point of M^n we have:

$$k_1 = k_2 = \dots = k_n$$

Hence each point of M^n is umbilic with respect to e.

Q.E.D.

<u>Theorem</u>: Let $\bar{M}^{n+N}$ be an (n+N)-dimensional ($n+N \geq 3$) C^∞ riemannian space form, admitting a C^0 one-parameter group of conformal transformations generated by a vector field ξ; and M^n be a closed orientable embedded submanifold. If $p(\xi)$ is positive (or negative) on M^n, and if the principal curvatures $k_1, k_2, \ldots, k_n$ at each point of M^n are positive and $H\binom{e}{r}$ is constant for any r ($1 < r \leq n-1$) then M^n is an Euler n-sphere.

<u>Proof</u>: Multiply (4.9) by the constant $H\binom{e}{r}$ and we get:

$$\int_{M^n} H\,H\binom{e}{r}\,p(\xi) * 1 + \int_{M^n} \phi\,H\binom{e}{r} * 1 = 0 \tag{4.11}$$

Hence from (4.10) and (4.11) we get:

$$\int \left(H\,H\binom{e}{r} - H\binom{e}{r+1} \right) p(\xi) * 1 = 0$$

From the assumption on $p(\xi)$, this holds iff

$$H\,H\binom{e}{r} - H\binom{e}{r+1} = 0$$

since

$$H\,H\binom{e}{r} - H\binom{e}{r+1} = \frac{r!\,(n-r-1)!}{n\,(n!)} \sum k_{i_1} \cdots k_{i_{r-1}} (k_r - k_{r+1})^2$$

$$\geq 0 .$$

Thus at each point of M^n we obtain:

$$k_1 = k_2 = \dots = k_n$$

Whence each point of M^n is umbilic with respect to the direction e. Hence M^n is an Euler n-sphere.

Q.E.D.

<u>Theorem</u>: Let $\bar{M}^{n+N}$ be an (n+N)-dimensional $(n+N \geqslant 3)$ C^∞ riemannian space form admitting C^o one-parameter group of conformal transformations generated by a vector field ξ, and M^n be a closed orientable embedded n-dimensional submanifold. If on all of M^n the following hold:

(i) $p(\xi)$ is positive (or negative)

(ii) $H\,p(\xi) + \phi \geqslant 0$ (or $\leqslant 0$)

then M^n is an Euler n-sphere.

<u>Proof</u>: We can rewrite (4.9) as:

$$\int (H\,p(\xi) + \phi) * 1 = 0$$

Then from the assumptions we have:

$$\phi + H\,p(\xi) = 0$$

Substitute this into (4.10) with $r = 1$. Whence we have:

$$\int (H^2 - H(\tfrac{e}{2})\ )\ p(\xi) * 1 = 0$$

and the required result follows.

Q.E.D.

<u>Theorem</u>: Let $\bar{M}^{n+N}$ be an (n+N)-dimensional $(n+N \geq 3)$ C^∞ riemannian space form admitting a C^0 one-parameter group T of conformal transformations generated by a vector field ξ, and M^n be a closed orientable embedded n-dimensional submanifold. If

(i) H is positive (or negative) on all of M^n

(ii) T is such that ϕ is positive (or negative), for which either $p + \frac{\phi}{H} \geq 0$ or $p + \frac{\phi}{H} \leq 0$

on all of M^n.

Then M^n is an Euler n-sphere.

<u>Proof</u>: Rewriting (4.9) as follows:

$$\int_{M^n} H(p(\xi) + \frac{\phi}{H}\ ) * 1 = 0$$

Hence from the assumptions $H > 0$ (or < 0) and $p(\xi) + \frac{\phi}{H} \geq 0$ (or ≤ 0) on all M^n we have:

$$p(\xi) + \frac{\phi}{H} = 0$$

Substituting this in (4.10) with $r = 1$, we have:

$$\int_{M^n} \frac{\phi}{H}\left(H^2 - H\binom{e}{2} \right) * 1 = 0$$

It now follows from our assumptions that this is true iff:

$$H^2 - H\binom{e}{2} = 0$$

Whence the required result follows. Q.E.D.

§ 3. Generalisations of § 1

In all our work so far we have been restricted to ambient manifolds of constant riemannian curvature. It is clearly of interest and importance to investigate to what degree this restriction can be removed and/or replaced by weaker assumptions on the ambient space. To this end we have the following theorem due to Feeman-Hsiung.

Theorem 9: Let $\bar{M}^{n+1}$ be a riemannian manifold of dimension $n \geq 2$ such that there is a normal coordinate system of Riemann at a fixed point 0 covering the whole of $\bar{M}^{n+1}$. Then for a closed orientable hypersurface M^n of class C^3 embedded in $\bar{M}^{n+1}$

$$\int_{M^n} H_{r-1} * 1 + \int_{M^n} H_r p * 1 = 0 \tag{4.12}$$

r is an odd integer less than or equal to n, and p is as defined in theorem 1.

Proof: The proof is exactly the same as that of theorem 1 except that this time r is an odd integer and the vanishing of the fourth member of the derivative of the right hand side of the form B_{n-1} in theorem 1 is due to the pairwise cancellation of its terms.

Q.E.D.

From this we have immediately:

Theorem (Feeman-Hsiung): Let M^n $(n \geq 3)$ be a hypersurface on a riemannian manifold $\bar{M}^{n+1}$ satisfying the conditions of theorem 9. Suppose that there exists an odd integer s, $(1 < s \leq n)$ such that $H_i > 0$ for $i = s, s-1, s-2$, and either $p \leq -H_{s-1}/H_s$ or $p \geq -H_{s-1}/H_s$ at all points of M^n. The M^n is a Riemann n-sphere.

Proof: By the same argument as in the proof of theorem 2, equation (4.12) for r = s, together with either of the two inequalities:

$H_s p + H_{s-1} \leqslant 0$ and $H_s p + H_{s-1} \geqslant 0 \Rightarrow p = -H_{s-1}/H_s$.

Substituting this value of p in equation (4.12) for $r = s-2$, which is an odd integer by assumption we obtain:

$$\int_{M^n} \frac{1}{H_s}\left(H_{s-3} H_s - H_{s-2} H_{s-1} \right) * 1 = 0 \tag{4.13}$$

Since H_{s-1} and H_{s-2} are positive, from lemma 1 for $r = s-1$ and $r = s-2$ we obtain:

$$\frac{H_{s-3}}{H_{s-2}} \leqslant \frac{H_{s-2}}{H_{s-1}} \leqslant \frac{H_{s-1}}{H_s}$$

from which it follows that

$$H_{s-3} H_s - H_{s-2} H_{s-1} \leqslant 0. \tag{4.14}$$

Thus the integrand on the left hand side of (4.13) is non-positive, and the equality in (4.14) holds. From lemma 1 it follows immediately that $k_1 = \dots = k_n$ at all points of M^n and thus by lemma 3 the theorem is proved.

Q.E.D.

<u>Theorem</u>: Let M^n $(n \geqslant 3)$ be a hypersurface on a riemannian manifold $\bar{M}^{n+1}$ satisfying the conditions of theorem 9. If there exists an even integer s, $(1 < s \leqslant n)$, with $H_s, H_{s-1} > 0$

and $p \leqslant -H_{s-1}/H_s$ at all points of M^n, then M^n is a Riemann n-sphere.

Proof: By lemma 1 $p \leqslant -H_{s-1}/H_s \leqslant -H_{s-2}/H_{s-1}$ and by theorem 9:

$$\int_{M^n} (H_{s-2} + p\,H_{s-1}) * 1 = 0$$

so $$p \leqslant -H_{s-1}/H_s \leqslant -H_{s-2}/H_{s-1} = p .$$

Thus $$\frac{H_{s-1}}{H_s} = \frac{H_{s-2}}{H_{s-1}} \quad \text{at all points of } M^n$$

and by the lemma M^n is a Riemann n-sphere. Q.E.D.

Theorem: with similar conditions to previous theorem. If there is an even integer s, $(1 < s < n)$ with H_s, $H_{s+1} > 0$ and $p \geqslant -H_{s-1}/H_s$ at all points of M^n, then M^n is a Riemann n-sphere.

Proof: By lemma 1, $p \geqslant -H_{s-1}/H_s \geqslant -H_s/H_{s+1}$ and by theorem 9:

$$\int_{M^n} (H_s + p\,H_{s+1}) * 1 = 0$$

so

$$p \geqslant -H_{s-1} / H_s \geqslant -H_s / H_{s+1} = p$$

Therefore

$$H_{s-1} / H_s = H_s / H_{s+1}$$

at all points of M^n, thus M^n is a Riemann n-sphere.

Q.E.D.

CHAPTER V

In this chapter we shall obtain, by the methods already developed, an integral formula for a pair of compact embedded submanifolds. We shall then use this integral formula to find conditions under which a volume-preserving diffeomorphism between the submanifolds is an isometry. This generalises somewhat a similar theorem by Chern and Hsiung [17].

§ 1. Algebraic Preliminaries

Let V be an n-dimensional vector space with a bilinear functional $G : V \times V \to \mathbb{R}$ on it. If $\{e_1, \ldots, e_n\}$ is any basis, then G is completely determined by the values:

$$g_{ij} = G(e_i, e_j) \qquad 1 \leq i, j \leq n.$$

If we change the basis with a matrix $T = (t_i^{\ j})$ such that

$$\bar{e} = Te$$

then the matrix $G = (g_{ij})$ is changed according to

$$T\,G\,{}^tT \qquad {}^tT \text{ denotes transpose.}$$

If H is a second bilinear functional and

$$h_{ij} = H(e_i, e_j)$$

we can consider the determinant:

$$\det(G + \lambda H) = \det G + n\lambda P(g_{ij}, h_{ij}) + \dots + \lambda^n \det H .$$

Note that under a change of basis the ratio of any two coefficients in this polynomial will be multiplied by $(\det T)^2$ and is thus independent of the choice of basis. In particular if G is non-singular:

$$H_G := P(g_{ij}, h_{ij}) / \det G \tag{5.1}$$

depends only on G and H. Clearly in case $G = I = \mathrm{id}$

$$H_G = \left(\frac{1}{n}\right) \text{Trace } H = \frac{1}{n} \sum h_{ii} .$$

Since the construction (5.1) is linear in H, a similar construction can be carried out in the case that H is a vector-valued bilinear function on V.

We shall have recourse to the following simplified form of a lemma due to Lars Gårding [26] .

<u>Lemma</u>: Let G and H be symmetric positive-definite bilinear real-valued functions over $V \times V$. Let $g = \det G$ and $h = \det H$. Then

$$H_G \geqslant \left(\frac{h}{g}\right)^{1/n}$$

where equality holds iff $H = \rho G$ for some ρ.

§ 2. Geometric Preliminaries

Let $(M_1{}^n, g_1, \Omega_1)$, $(M_2{}^n, g_2, \Omega_2)$ be two C^∞ riemannian n-manifolds, and $f : M_1 \to M_2$ be a C^∞ immersion. On M_1 there are now two connexions - firstly the riemannian (metric) connexion obtained from g_1 and secondly the induced connexion derived from the induced metric $f^* g_2$. The second of the two connexions can also be obtained in the usual way, simply by using the pull back of the riemannian (metric) connexion on M_2, i.e. the second connexion is simply: $f^*\Omega_2$.

The difference between these two connexions on M_1 is also a connexion, which we denote simply by $D = (D_i{}^j)$. By means of the construction in § 1 we can, using D, construct a vector field $D(g_1)$ on M_1 as follows:

$$D(g_1) \;:=\; g_1^{ik}(\omega_i{}^j(e_k) - (f^*\Omega_2)_i{}^j(e_k))e_j \qquad (5.2)$$

Let $X = (X^i)$ be a tangent vector field on M_1. The divergence of X (denoted div X) can be defined in (at least) two entirely equivalent ways; on the one hand, as the <u>function</u> on M_1 defined by:

$$X(*1) \;=\; (\mathrm{div}\, X) * 1 \qquad (5.3)$$

where $*1$ is a fixed volume element on M_1. We are of course assuming implicitly that M_1 is orientable. It follows immediately from the definition of Lie derivative that (5.3) is equivalent to:

$$(\mathrm{div}\ X) * 1 \;=\; \underset{X}{\pounds}(*1)\ ;$$

on the other hand, on a local coordinate neighbourhood U, with local coordinates $(u^1, \ldots, u^n)$, the above definitions take the more classical form:

$$\mathrm{div}\ X \;=\; \frac{1}{\sqrt{g_1}} \sum \frac{\partial}{\partial u^i} \left(\sqrt{g_1}\ X_i \right) \tag{5.4}$$

where $X = X_i\ \ \partial/\partial u^i$ on U. In terms of covariant derivatives we can write (5.4) simply as:

$$\mathrm{div} X \;=\; \underset{1}{g}^{ij}\ X_{i,j}\ .$$

DEF. 21. By means of the induced metric on M_1 (i.e. $f^* g_2$) we can define an analogous function which we shall call the <u>induced divergence</u> with respect to mapping f of the field X, and we shall denote it simply by div(f)

$$\mathrm{div}(f)\ X \;:=\; (f^* g_2)^{ij}\ X_{i,j}\ .$$

Consider, now two embeddings x_1, x_2 of an abstract n-dimensional orientable differentiable manifold into a riemannian manifold $\bar{M}^{n+N}$

of dimension $n + N$. Further let $f : x_1(M) \to x_2(M)$ be a diffeomorphism. We thus have the commutative diagram as follows:

$$\begin{array}{ccc} & \overset{x_1}{\nearrow} & x_1(M) \subset \bar{M} \\ M & & \downarrow f \\ & \underset{x_2}{\searrow} & x_2(M) \subset \bar{M} \end{array}$$

We call the mapping f an isometry if at a point $p \in M$

$$< dx_1(p), dx_1(p) > = < dx_2(p), dx_2(p) > \qquad (5.5)$$

i.e. if it maps the induced riemannian metric of the embedded submanifold (M, x_1) into the induced riemannian metric of the embedded submanifold (M, x_2). In terms of the cotangent map f^* we can write (5.5) simply as:

$$g_1 = f^* g_2 \quad .$$

Over the abstract manifold M we now have two metrics namely

$$g_1 : < dx_1(p), dx_1(p) >$$

and

$$g_2 : < dx_2(p), dx_2(p) > = < d(f \circ x_1)(p), d(f \circ x_1)(p) >$$

In what follows, unless it is obvious or specifically stated to the contrary, we will work on the manifold M_1 with the (induced) metric g_1.

The notion of frames $e_1, \ldots, e_n$ having measure 1 and coherently orientated with that of M thus has a sense in both metrics. At a point $p \in M$ any such frame can be obtained from a fixed one by a linear transformation of determinant 1. The induced map dx_1 on the tangent spaces is univalent, so we can identify e_i and $dx_1(e_i)$. Let $\omega^1, \ldots, \omega^n$ be the dual coframe of $e_1, \ldots, e_n$, then the volume element on M is:

$$*1 = \omega^1 \wedge \ldots \wedge \omega^n .$$

We call the above mapping f volume-preserving if it maps the volume element of one manifold into that of the other, i.e.

$$f^* \omega_2 = \omega_1$$

where ω_1 and ω_2 are respectively the volume elements of M_1 and M_2.

It follows immediately from the definition that a volume-preserving diffeomorphism exists only if M is oriented and the diffeomorphism f is then oriention-preserving.

§ 3. Integral Formula

Let $(\tilde{M},\bar{g})$ be a C^∞ riemannian manifold of dimension $(n+N)$, which admits a C^∞ vector field ξ. Further let M^n be a C^∞ n-dimensional orientable manifold with two compact C^∞ immersions x_1

and x_2 into $\bar{M}$; and let $f : x_1(M) \to x_2(M)$ be a volume-preserving diffeomorphism. Assume ξ lies nowhere entirely in the tangent space of $x_1(M)$.

We shall work with the bundle AFM over $x_1(M)$. We shall write simply M_1, M_2 respectively for $x_1(M)$, $x_2(M)$.

As seen in § 1 the manifold M_1 inherits a metric g_1 from $\bar{M}$ as does M_2 inherit a metric g_2. On M_1 we thus have the two metrics:

$$g_1 = (\underset{1}{g}_{ij}), \qquad f^* g_2 = (f^* \underset{2}{g})_{ij} ,$$

At points of M_1 we can thus represent the vector field ξ :

$$\xi = \xi^i e_i + \xi^\sigma c_\sigma$$

Hence

$$p_i(\xi) = \xi^k \underset{1}{g}_{ki} = \xi_i \tag{5.6}$$

where

$$p_i(\xi) := \langle \xi, e_i \rangle_{\bar{g}} .$$

Thus

$$\langle d\xi, e_i \rangle_{\bar{g}} + \langle \xi, de_i \rangle_{\bar{g}} = d\xi_i$$

$$\therefore \quad \langle d_j\xi, e_i \rangle_{\bar{g}} + \langle \xi, d_je_i \rangle_{\bar{g}} = d_j\xi_i \tag{5.7}$$

Now from (1.13) we have:

$$de_i = \omega_i^{\ k} e_k + \omega_i^{\ \sigma} e_\sigma$$

$$\Longrightarrow \qquad d_j\, e_i = \omega_i^{\ k}(e_j)e_k + \omega_i^{\ \sigma}(e_j)e_\sigma$$

whence from (5.7)

$$< d_j\xi,\ e_i >_{\bar g} + < \xi,\ \omega_i^{\ k}(e_j)e_k >_{\bar g} + < \xi,\ \omega_i^{\ \sigma}(e_j)e_\sigma >_{\bar g}$$

$$= d_j\xi_i\ . \qquad (5.8)$$

Now use g_1 :

$$g_1^{\ ij} < d_j\xi,\ e_i >_{\bar g} + g_1^{\ ij} < \xi,\ \omega_i^{\ k}(e_j)e_k >_{\bar g}$$

$$+ g_1^{\ ij} < \xi,\ \omega_i^{\ \sigma}(e_j)e_\sigma >_{\bar g} = g_1^{\ ij}\, d_j\xi_i\ .$$

Hence

$$g_1^{\ ij} < d_j\xi,\ e_i >_{\bar g} + g_1^{\ ij} < \xi,\ \omega_i^{\ \sigma}(e_j)e_\sigma >_{\bar g} = \underset{1}{g}^{ij}\ \xi_{i,j}\ .$$

The right hand side of this equation is simply div ξ , and

$$\omega_i^{\ \sigma}(e_j) = A^\sigma_{\ ik}\, \omega^k(e_j) \qquad . \qquad \text{(from (1.15))}$$

Further, the mean curvature H^σ in the direction e_σ is by definition: $\underset{g_1}{\text{Trace}}\ A^\sigma$

i.e. $$\underset{1}{H}^{\sigma} := g_1^{ij} A^{\sigma}_{ij} = \operatorname{Trace}_{g_1} A^{\sigma} .$$

Thus we have:

$$g_1^{ij} < d_j \xi, e_i >_{\bar{g}} + \underset{1}{H}^{\sigma} p_{\sigma}(\xi) = \operatorname{div} \xi ,$$

where $$p_{\sigma}(\xi) := < \xi, e_{\sigma} >_{\bar{g}}$$

Note: We use $\underset{1}{H}$ to denote the (first) mean curvature with respect to the (induced) metric g_1 .

Finally the above equation becomes:

$$(\tfrac{1}{2}) \operatorname{Trace}_{g_1} x_1^*(\underset{\xi}{\pounds} \bar{g}) + \underset{1}{H}^{\sigma} p_{\sigma}(\xi) = \operatorname{div} \xi \tag{5.9}$$

Whence:

$$\int_{M_1^n} \left\{ (\tfrac{1}{2}) \operatorname{Trace}_{g_1} x_1^*(\underset{\xi}{\pounds} \bar{g}) + \underset{1}{H}^{\sigma} p_{\sigma}(\xi) \right\} * 1 = 0 .$$

In exactly the same manner, beginning with (5.8) and by using this time the second induced metric on M_1 viz. f^*g_2 we get

$$(f^* g_2)^{ij} < d_j \xi, e_i >_{\bar{g}} + (f^* g_2)^{ij} < \xi, \omega_i^{\ k}(e_j) e_k >_{\bar{g}}$$

$$+ (f^* g_2)^{ij} < \xi, \omega_i^{\ \sigma}(e_j) e_{\sigma} >_{\bar{g}} = (f^* g_2)^{ij} d_j \xi_i$$

If we agree to write simply $\underset{2}{g}^{ij}$ for $(f^* g_2)^{ij}$ we have

$$(\tfrac{1}{2}) \operatorname{Trace}_{g_2} x_1{}^*(\underset{\xi}{\pounds}\bar{g}) + g_2{}^{ij}\,\omega_i{}^k(e_j)p_k(\xi) + \underset{2}{H}^\sigma\, p_\sigma(\xi) = g_2{}^{ij}\, d_j \xi_i$$

where $\underset{2}{H}^\sigma$ is the (induced) mean curvature in the direction e_σ with respect to the second induced metric $g_2{}^{ij}$, i.e. $\underset{2}{H}^\sigma = \operatorname{Trace}_{g_2} A^\sigma$.

Finally we get:

$$(\tfrac{1}{2}) \operatorname{Trace}_{g_2} x_1{}^*(\underset{\xi}{\pounds}\bar{g}) + \underset{2}{H}^\sigma\, p_\sigma(\xi) = \operatorname{div}(f)\xi \tag{5.10}$$

whence:

$$\int_{M_1} \left\{ (\tfrac{1}{2}) \operatorname{Trace}_{g_2} x_1{}^*(\underset{\xi}{\pounds}\bar{g}) + \underset{2}{H}^\sigma\, p_\sigma(\xi) \right\} * 1 = \int_{M_1} \operatorname{div}(f)\xi * 1 \,.$$

This extends the corresponding integral formula of Chern and Hsiung [17].

If we now subtract (5.9) from (5.10) and use Stokes' theorem we arrive at our basic integral formula.

<u>Theorem 1</u>: If $(\bar{M},\bar{g})$ is a riemannian manifold admitting a C^∞ vector field ξ, and M_1, M_2 are a pair of compact C^∞ immersed equidimensional submanifolds which do not contain any singular points of ξ , and such that ξ is almost nowhere

(i.e. at most on a set of measure zero) tangent to M_1, then for a volume-preserving diffeomorphism $f : M_1 \to M_2$ we have:

$$(\tfrac{1}{2})\int_{M_1} (\operatorname{Trace}_{g_2} - \operatorname{Trace}_{g_1})\, x_1{*}(\pounds_{\xi}\bar{g}) * 1 + \int_{M_1} (H^{\sigma}_{2} - H^{\sigma}_{1})p_{\sigma}(\xi) * 1$$

$$= \int_{M_1} \operatorname{div}(f)\xi * 1$$

We note immediately that in case f is an isometry the right hand side of this equation is zero. By analogy to the work of Chern and Hsiung [17], we make the following:

DEF. 22. A mapping f for which $\operatorname{div}(f)\xi = 0$ is called an almost isometry.

§ 4. Specialisations

We now consider the integral formula in case f and ξ are specialised. In particular in all that follows we shall assume that f is an almost isometry.

In case ξ is a conformal field:

$$\int_{M_1} (\operatorname{Trace}_{g_2} g_1 - 1)\,\phi * 1 + \int_{M_1} (H^{\sigma}_{2} - H^{\sigma}_{1})\, p_{\sigma}(\xi) * 1 = 0 \tag{5.11}$$

Hence

$$\int_{M_1} \operatorname{Trace}_{g_2} g_1 \,\phi * 1 - \int_{M_1} \phi * 1 + \int_{M_1} (\underset{2}{H}^{\sigma} - \underset{1}{H}^{\sigma})\, p_\sigma(\xi) * 1 = 0 \tag{5.12}$$

In case ξ is proper homothetic we have:

$$\int_{M_1} \operatorname{Trace}_{g_2} g_1 * 1 + \frac{1}{c}\int_{M_1} (\underset{2}{H}^{\sigma} - \underset{1}{H}^{\sigma})\, p_\sigma(\xi) * 1 = V(M_1) \tag{5.13}$$

In case ξ is isometric:

$$\int_{M_1} (H_2{}^{\sigma} - H_1{}^{\sigma})\, p_\sigma(\xi) * 1 = 0 \tag{5.14}$$

We can now generalise a theorem of Chern and Hsiung [17] .

Theorem: Let x_1, x_2 : $M \to \bar{M}$ be two imbedded compact submanifolds of riemannian manifold $\bar{M}$ which admits a conformal vector field ξ with the properties of theorem 1. If f has the properties:

(i) ϕ and $p_\sigma(\xi)$ are positive (or negative),

(ii) $\underset{2}{H}^{\sigma} \geqslant \underset{1}{H}^{\sigma}$,

then f is an isometry.

Proof: From the conditions of the theorem and Gårding's lemma we get

$$\operatorname*{Trace}_{g_2} g_1 - \operatorname*{Trace}_{g_1} g_1 \geq 0$$

and conditions (i) and (ii) give us

$$g_1 g_2 = 1$$

whence Gårding's lemma gives us immediately that:

$$g_1 = g_2$$

Q.E.D.

Note that in case the ambient space is E^{n+N} and ξ is the homothetic Killing vector field on E^{n+N} with components $\xi^A = x^A$, x^A being rectangular coordinates with a point in the interior of M^n as origin, then the orbits of the transformations generated by ξ are simply the lines through the origin and we have

$$\mathop{\pounds}_{\xi} \bar{g} = 2\bar{g}$$

and our integral formula of theorem 1 reduces to that of Chern and Hsiung [17].

CONCLUSION

In the foregoing we have not concerned ourselves with the global existence of linearly independent sections of the normal bundle. Although in a neighbourhood of a point there are linearly independent sections, the existence of a single section globally is an open question. Work in this direction can be found in Stiefel [28], Hirsch [29] (who showed that the immersion problem is essentially a cross-section problem for the normal bundle), Kervaire [30] and Handel [31].

Problem 1: It is clear from the foregoing work that the mean normal curvature vector η contains much important geometric information about the immersion/embedding. It is clearly desirable in the future to study its properties in more detail. An obvious generalisation of η is: the r^{th} mean normal curvature vector $\eta(r)$, defined:

$$\eta(r) \;:=\; H\binom{\sigma}{r} e_\sigma \tag{C.1}$$

where $H\binom{\sigma}{r}$ is the r^{th} mean curvature in the normal direction e_σ to the submanifold. Clearly $\eta(1)$ is just the classical mean normal curvature vector. We can thus ask: what are the analogous properties of $\eta(r)$?

Problem 2: Do analogues of theorems A and B exist for $\eta(r)$?

Problem 3: In Chapter IV we have been restricted to a space of constant curvature, we can thus ask: to what extent can this condition be relaxed and/or replaced? Theorem 9 shows that in certain cases at least it is redundant.

Problem 4: What is the nature of the umbilic set of a submanifold? Can it only be either: (i) a discrete set

or (ii) the submanifold/nul set ?

Problem 5: What is the connection between curvature and the umbilic set? Concerning this we have the following example: consider a 3-dimensional space M^3 with fundamental form:

$$ds^2 = \left(\sum_{i=1}^{3} (y_i)^2 \right)^c \sum_{j=1}^{3} (dy_j)^2 \qquad c = \text{constant} \ (\geqslant 2)$$

Calculation shows that $M^3 - (0, 0, 0)$ is not of constant curvature and that every point of the surface in M^3 given by:

$$\left.\begin{aligned} y_1 &= \sin u \cos v \\ y_2 &= \sin u \sin v \\ y_3 &= \cos u \end{aligned}\right\} \quad \text{is umbilic.}$$

Problem 6: In Chapter V we have been concerned with a volume-preserving diffeomorphism. It is thus important to ask: what restrictions does the existence of such a diffeomorphism impose on the ambient manifold?

<u>Problem 7</u>: We say two riemannian manifolds (M,g), $(\bar{M},\bar{g})$ are 'isocurved' if $\exists$ a sectional-curvature-preserving diffeomorphism $f : M \to \bar{M}$ i.e. for every $p \in M$ and every 2-plane section π of M_p we have:

$$K(\pi) = \bar{K}(f_* \pi) .$$

Clearly two diffeomorphic manifolds of the same constant curvature are isocurved but need not be globally isometric. Kulkarni [27] has proved:

<u>Theorem</u>: If $\dim M \geqslant 4$ then isocurved manifolds with analytic metric are globally isometric except in the aforementioned example of diffeomorphic non-globally isometric manifolds of the same constant curvature.

We can thus ask what is the connexion between isocurvature and the volume-preserving property of a diffeomorphism?

The integral formula method is clearly a very powerful technique, particularly in the solution of uniqueness problems. Its further application in the future could lead to the solution of an even wider class of geometric problems.

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