K-harmonic manifolds

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k-Harmonic Manifolds

by

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CONTENTS

Abstract

The present work is aimed toward the study of manifolds which admit k-harmonic metrics. These generalize the "classical" harmonic manifolds and in their definition, the k-th elementary symmetric polynomials of a certain endomorphism Φ of the fibres in the tangent bundle play a role similar to that of Ruse's invariant in classical harmonic spaces. We investigate some properties of k-harmonic manifolds analogous to those enjoyed by harmonic manifolds and obtain some results relating k-harmonic manifolds to harmonic ones. For instance we prove:

(a) a k-harmonic manifold is necessarily Einstein,

(b) a manifold is simply 1-harmonic iff it is simply n-harmonic.

We also work out a general formulation of k-harmonic manifolds in terms of the Jacobi fields on the manifold. This enables us, in particular, to generalize the equations of Walker, and obtain in the case of symmetric spaces, a finite set of necessary conditions for k-harmonicity. As an application of this we are able to show that if a locally symmetric space is n-harmonic then it is k-harmonic for all k. Under the further assumption of compactness we prove that an irreducible k-harmonic manifold is necessarily a symmetric space of rank one. Consequently:

(1) a compact simply connected riemannian symmetric manifold, k-harmonic for one k is k-harmonic for all k; and

(2) by a theorem of Avez we can drop the assumption of symmetry in (1) but assume instead that the manifold is n-harmonic.

INTRODUCTION

Harmonic riemannian spaces, as their name implies, are intimately related to the study in riemannian spaces of the generalized harmonic functions. A real-valued function Φ on an open set U in a riemannian manifold M is harmonic in U if

$$\Delta_2\Phi = g^{ij}\,\Phi_{,ij} = 0$$

where Δ_2 is the Laplace-Beltrami operator on U. It is natural to ask whether these functions possess certain properties enjoyed by harmonic functions in euclidean space. This question was attempted in 1930 by H.S.Ruse [1]who obtained an explicit formula for the elementary solution of Laplace's equation and claimed that it holds in a general riemannian space. His claim however was later shown to be false, Copson and Ruse [1], as it rests on the implicit assumption that a solution to Laplace's equation exists which depends only on the geodesic distance from the pole. The claim, nevertheless, holds for a substantial class of riemannian spaces, the harmonic riemannian spaces. The subsequent years witnessed the development of the subject in the work of Ruse, Walker, Willmore, Lichnerowicz and Ledger.

In Chapter I we give a resumé of the theory of locally harmonic riemannian manifolds as developed by its authors in the 40's and 50's. The chapter culminates in the recurrence formulae of Ledger which enable us to derive various properties of harmonic metrics. Here the main source of material is the book on "Harmonic Spaces" by H.S.Ruse, A.G.Walker and T.J.Willmore, hereafter referred to as RWW. We also develop in this chapter some basic differential geometric theory of affine connections - riemannian connections in particular - which serves as a logical background to the rest of the chapter and to the subsequent ones. Here we adopt the definition of connection due to J. Koszul.

We find that in a general riemannian manifold the conditions of harmonicity as derived from Ledger's or equivalent formulae are extremely complicated and only the first few of them are of practical use. However, in certain classes of manifolds, e.g. symmetric spaces, these conditions become more manageable as a consequence of the richness of the structure with which they are endowed.

In Chapter II we give a condensed account of the theory of symmetric spaces and the cannonical connection on them and discuss some of their properties. Here we make use of Helgason [1]and Kobayashi and Nomizu vol. II.

In Chapter III we outline some theory and properties of Jacobi fields which we use in the global description of harmonic manifold theory. We obtain Allamigeon's result that such manifolds are either diffeomorphic to euclidean space or cohomologous to a symmetric space of rank one. The chapter concludes with Avez's theorem that a compact simply connected globally harmonic manifold with definite metric is symmetric.

In Chapter IV k-harmonic manifolds are defined as a generalization of harmonic manifolds. We investigate some of their properties and obtain some results analogous to the harmonic ones. We then use the general formulation of k-harmonic manifolds in terms of Jacobi fields to generalize the equations of Walker [3], [4] which provide in the case of symmetric spaces a set of conditions for harmonicity. We also include the result, and certain corollaries thereof, that a compact simply connected riemannian symmetric space which is k-harmonic for one k is k-harmonic for all k. This latter result is contained in a joint paper by Willmore and El Hadi [1].

The present work, however, leaves open many questions. A paramount one is the fundamental conjecture of harmonic spaces, namely that a locally harmonic manifold with positive definite metric is locally symmetric. This remains unsettled. Secondly I believe that theorem (9) of Chapter IV is of an essentially local nature and it would be interesting to furnish a proof that dispenses with the assumption of compactness.

As indicated above, all this work arises from considerations of solutions of Laplace's equation. There are many other equations of interest in physics, e.g. the heat equation, the wave equation etc. Each of these will give rise in a similar manner to special classes of riemannian spaces. An interesting problem which I propose to investigate in the future is to classify such riemannian manifolds.

CHAPTER I

HARMONIC MANIFOLDS

1.1. Origin of harmonic manifolds

The study of harmonic spaces was initiated by H.S.Ruse [1] in an attempt to obtain a simple formula for the elementary solution of Laplace's equation

$$\Delta_2 V \equiv \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{g}\,\frac{\partial V}{\partial x^j}\right) = 0$$

in an n-dimensional analytic riemannian manifold with any given metric $ds^2 = |g_{ij}\, dx^i\, dx^j|$. He obtained a formula of the form

$$V(s) = A\int_a^s J/\sqrt{gg_o}\ \frac{ds}{s^{n-1}} ,$$

where A and a are constants, s the geodesic distance from the origin O (x_o^i) to the variable point $P(x^i)$, $g = \det(g_{ij})$, g_o = g evaluated at O and $J = \det\left(\frac{\partial^2\Omega}{\partial x^i\,\partial x^j}\right)$ where $\Omega = \frac{1}{2}\,es^2$. It was, however, shown in a subsequent paper, Copson and Ruse [1] that the above formula is not valid in general, as it was implicitly assumed that the manifold was homeomorphic to euclidean n-space and that a solution exists which depends only on Ω and not otherwise on the path from the base point. The particular class of manifolds for which Ruse's formula is valid for one choice of base point were then called centrally harmonic and those for which it holds for any choice of base point were called completely harmonic. Manifolds of constant curvature were shown to belong to this class.

An infinite set of necessary and sufficient conditions for a space to be harmonic were given; these imposed certain restrictions on the curvature of the metric to be admitted by a harmonic space. An immediate consequence is that a harmonic metric is necessarily Einstein. In view of these restrictions it was then conjectured that all harmonic metrics are of constant curvatures and this was shown to be the case in dimensions $\leqslant 3$. However Walker [2], [3], [4] soon gave examples of harmonic manifolds of dimensions $\geqslant 4$ which were not of constant curvature, thereby establishing the falsehood of the conjecture. Thus harmonic manifolds were of some interest in that they do not coincide with the "trivial" class of manifolds of constant curvature and are included in the class of Einstein manifolds.

1.2 Basic Connection Theory

a) Vector fields

Let M be a C^{∞} manifold, $m \in M$. We denote by $C^{\infty}(M,m)$ the set of all C^∞ real-valued functions with domain a neighbourhood of M.

<u>Definition (1)</u>. A <u>tangent</u> X_m to M at m is a map: $C^\infty(M,m) \to \mathbb{R}$ such that for all $a,b \in \mathbb{R}$ and $f,g \in C^\infty(M,m)$

(i) $X_m(af + bg) = a\,X_m(f) + b\,X_m(g)$

(ii) $X_m(fg) = X_m(f)g(m) + f(m)\,X_m(g)$.

The tangents to M at m form an n-dimensional linear space denoted by $T_m(M)$.

<u>Definition (2)</u>. A vector field X on $U \subset M$ is a mapping which assigns to each $m \in U$ a vector X_m in $T_m(M)$.

If $f \in C^\infty(M,m)$ then Xf is the function defined for $m \in U \cap \{\text{domain } f\}$ by $Xf(m) = X_m f$. The vector field X is C^∞ if U is open and for every $f \in C^\infty(M,m)$, $(Xf)(m) = X_m f$ is C^∞ on $U \cap \{\text{domain } f\}$.

We denote by $\mathfrak{X}(M)$ the set of all C^∞ vector fields on M. This forms a Lie algebra over $\mathbb{R}$ with bracket operation
$[X,Y]f = X(Yf) - Y(Xf), \quad f \in C^\infty(M)$.

Let M and N be C^∞ manifolds and $\Phi : M \to N$ be C^∞. For $m \in M$ we define the differential of Φ at m, $(\Phi_*)_m : T_m(M) \to T_{\Phi(m)}(N)$ by:
if $X \in T_m(M)$ and $f \in C^\infty(N,\Phi(m))$ then $(\Phi_*)_m(X)f = X(f \circ \Phi)$.
Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$; we say that X and Y are Φ-related if $(\Phi_*)_m(X_m) = Y_{\Phi(m)}$ for every m in the domain of X.

The C^∞-mapping $\phi : M \to N$ is a diffeomorphism if it is one-to-one, onto and ϕ^{-1} is C^∞.

A 1-parameter group of transformations of M is a mapping of $\mathbb{R} \times M$ into M, $(t,p) \in \mathbb{R} \times M \to \phi_t(p) \in M$ which satisfies:

(1) for each $t \in \mathbb{R}$, $\phi_t : p \to \phi_t(p)$ is a transformation of M,

(2) for all $t,s \in \mathbb{R}$ and $p \in M$, $\phi_{t+s}(p) = \phi_t(\phi_s(p))$.

Each 1-parameter group of transformations ϕ_t induces a vector field X as follows. For $p \in M$ X_p is the vector tangent to the curve $\phi_t(p)$, called the orbit of p, at $p = \phi_o(p)$. The orbit $\phi_t(p)$ is an integral curve starting at p. Conversely for every vector field X on M and each $p_o \in M$ there exists a neighbourhood of p_o, a positive number ϵ and a local 1-parameter group of transformations $\phi_t : U \to M$, $|t| < \epsilon$ which induces X (Cf Kobayashi and Nomizu I p. 13).

b) Affine Connections

Definition (3) An affine connection on a C^∞ manifold M is a function ∇ which assigns to each $X \in \mathfrak{X}(M)$ a linear mapping $\nabla_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying

(1) $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$

(2) $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$,

where $X,Y \in \mathfrak{X}(M)$ and $f,g \in C^\infty(M)$.

The operator ∇_X is called covariant differentiation w.r.t. X.

Let $\gamma : I \to M$ be an arbitrary C^∞ curve in M and let $\dot\gamma(t)$ be the vector field along γ defined by $\dot\gamma(t) = (\gamma_*)_t\ (d/dt)$. Let $X(t)$ be an arbitrary vector field along γ. We say that $X(t)$ is parallel w.r.t. γ if $\nabla_{\dot\gamma(t)} X(t) = 0$.

<u>Definition (4)</u> A C^∞ curve $\gamma : t \to \gamma(t)$, $t \in I \subset \mathbb{R}$ is a geodesic if the family of tangent vectors $\dot\gamma(t)$ is parallel w.r.t. γ. A geodesic is maximal if it is not the proper restriction of any geodesic.

Let $\gamma(t)$ be a C^∞ curve in M with tangent field $\dot\gamma(t)$. For each vector $Y \in T_{\gamma(0)}(M)$ there is a unique C^∞ field $Y(t)$ along γ such that $Y(0) = Y$ and the field $Y(t)$ is parallel along γ (cf Hicks [1], p. 58). The mapping $\tau_{0,t} : T_{\gamma(0)}(M) \to T_{\gamma(t)}(M)$ defined by $Y \to Y(t)$ is thus a linear isomorphism called parallel transport along γ from $\gamma(0)$ to $\gamma(t)$.

As a consequence of the existence and local uniqueness for solutions of ordinary differential equations with prescribed initial values, we have the following:

<u>Proposition</u> Let $m \in M$, $X \in T_m(M)$. Then for any real number t_0 there exists an interval I containing t_0 and a unique maximal geodesic $\gamma(t)$ defined on I such that $\gamma(t_0) = m$ and $\dot\gamma(t_0) = X$. Such a maximal geodesic we denote by $\gamma_{m,X}(t)$.

Definition (5) Let M be a C^∞ manifold with an affine connection ∇ and let $X,Y \in \mathfrak{X}(M)$. We define the torsion and curvature tensor fields of ∇ as follows:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Thus $T(X,Y)$ is a vector field on M and $R(X,Y)$ a linear transformation on $\mathfrak{X}(M)$.

For $m \in M$ denote by D_m the set of those vectors $A \in T_m(M)$ for which the geodesic $\gamma_{m,A}(t)$ is defined on an interval containing the point $t = 1$.

Definition (6) For an arbitrary $X \in D_m$ we define the exponential mapping $\exp_m : D_m \to M$ by $\exp_m X = \gamma_{m,X}(1)$.

We say that an affine connection is complete if every geodesic can be infinitely extended or, equivalently if each exponential mapping is defined on the whole tangent space.

Definition (7) Let (M, ∇) and $(\bar{M}, \bar{\nabla})$ be two affinely connected C^∞ manifolds. A diffeomorphism

$$\phi : M \to \bar{M}$$

is called an affine mapping if

$$\phi_*(\nabla_X Y) = \bar{\nabla}_{\phi_* X}(\phi_* Y) \qquad \text{for all } X,Y \in \mathfrak{X}(M).$$

Let $\phi : (M, \nabla) \to (\bar{M}, \bar{\nabla})$ be an affine mapping. Let $T(\bar{T})$ and $R(\bar{R})$ be the torsion and curvature tensors of $\nabla(\bar{\nabla})$ and let $X, Y, Z \in \mathfrak{X}(M)$ be ϕ-related to $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\bar{M})$, then we have (Cf Kobayashi and Nomizu I p. 225):

(a) ϕ maps every geodesic of M into a geodesic of $\bar{M}$, and consequently ϕ commutes with the exponential mappings.

(b) The fields $\nabla_X Y$, $T(X,Y)$ and $R(X,Y)Z$ are ϕ-related to the fields $\bar{\nabla}_{\bar{X}}\bar{Y}$, $\bar{T}(\bar{X},\bar{Y})$ and $\bar{R}(\bar{X},\bar{Y})\bar{Z}$.

A diffeomorphism of M onto itself is called an affine transformation if it is an affine mapping.

c) Holonomy

Let M be a C^∞ manifold with an affine connection ∇ and let $m \in M$. Given any piecewise C^∞ closed curve τ at m then parallel transport along τ, is a linear transformation of $T_m(M)$. The totality of these linear transformations for all piecewise C^∞ closed curves at m forma group. It is called the holonomy group at m of the connection ∇ and denoted by $\Phi(m)$. It was shown by Borel and Lichnerowicz that $\Phi(m)$ is a Lie subgroup of the general linear group. If we restrict our attention only to those piecewise C^∞ closed curves at m which are homotopic to zero we get the restricted holonomy group Φ_o (m) which turns out to be the identity component of $\Phi(m)$.

Ambrose and Singer [1] proved the following theorem about holonomy:

<u>Theorem</u>: The Lie algebra of the holonomy group $\Phi(m)$ is equal to the subspace of linear endomorphisms of $T_m(M)$ spanned by all elements of the form:

$$(\tau R)(X,Y) = \tau^{-1} \circ R(\tau X, \tau Y) \circ \tau$$

where $X, Y \in T_m(M)$ and τ is parallel transport along an arbitrary piecewise C^∞ curve τ starting at m.

d) <u>Riemannian Connections</u>:

<u>Definition (8)</u> A Riemannian (pseudo-riemannian) structure on a C^∞ manifold M is a tensor field g of type (0,2) satisfying:

(a) $g(X,Y) = g(Y,X)$ for all $X,Y \in \mathfrak{X}(M)$

(b) for each $m \in M$, g_m is a positive definite (non-degenerate) bilinear form on $T_m(M) \times T_m(M)$.

A riemannian (pseudo-riemannian) manifold is a C^∞ manifold equipped with a riemannian (pseudo-riemannian) structure.

The fundamental theorem of riemannian geometry asserts that:

On a (pseudo-) riemannian manifold there exists one and only one affine connection - the (pseudo-) riemannian connection - ∇ satisfying:

(a) the torsion tensor is zero, i.e. $\nabla_X Y - \nabla_Y X = [X,Y]$ for all $X,Y \in \mathfrak{X}(M)$, and

(b) parallel transport preserves the inner product on the tangent spaces, i.e.

$$\nabla_Z\, g = 0 \qquad\qquad \text{for all } Z \in \mathfrak{X}(M).$$

Definition (9) Let M and $\bar{M}$ be two riemannian manifolds with riemannian structures g and $\bar{g}$ and ϕ a diffeomorphism of M onto $\bar{M}$. ϕ is called an isometry if $\phi^*\bar{g} = g$, i.e. if for all $X,Y \in \mathfrak{X}(M)$

$$(\phi^*\bar{g})(X,Y) \equiv \bar{g}(\phi_* X,\ \phi_* Y) = g(X,Y).$$

On a riemannian manifold the tensor field of type $(0,4)$ defined by

$$R(X,Y,Z,W) = g(R(X,Y)Z,W)$$

is called the riemannian curvature tensor field. It satisfies the following symmetry relations:

$$\begin{aligned} R(X,Y,Z,W) &= -R(Y,X,Z,W) \\ &= -R(X,Y,W,Z) \\ &= R(Z,W,X,Y)\ . \end{aligned}$$

A riemannian manifold is said to be flat if its curvature tensor vanishes identically.

Definition (10) The Ricci tensor S of a riemannian manifold with riemannian curvature tensor R is defined as follows:

$$S(X,Y) = \text{trace } U \mapsto R(X,U)Y, \qquad X,Y,U \in \mathfrak{X}(M).$$

A riemannian manifold with metric g is said to be Einstein if $S(X,Y) = \lambda g(X,Y)$ for some scalar λ and all $X,Y \in \mathfrak{X}(M)$.

Definition (11) Let $p \in M$ and Σ a 2-dimensional subspace of $T_p(M)$. The sectional curvature $K(\Sigma)$ at p is defined by

$$K(\Sigma) = \frac{g_p(R_p(X,Y)Y,X)}{(A(X,Y))^{\frac{1}{2}}}$$

where X,Y are two linearly independent vectors in Σ and A the area of the parallelogram spanned by X and Y.

1.3 Normal Neighbourhoods

Definition (12) A neighbourhood V_o of the zero vector in $T_m(M)$ is normal if it is starlike, i.e. $\forall$ $A \in V_o$ and $0 < s \leq 1$, $sA \in V_o$ and if the mapping $\exp_m$ is defined on V_o (i.e. $V_o \subset D_m$) and is a diffeomorphism onto some neighbourhood $V(m)$ of m.

The neighbourhood $V(m) = \exp_m V_o$ is called a normal neighbourhood of $m \in M$.

The significance of normal neighbourhoods stems out of the fact that an arbitrary point m' in a normal neighbourhood $V(m)$ can be joined to m in $V(m)$ by a unique geodesic. Specifically m and m' can be joined by the geodesic $\gamma_{m,A}(t)$ where A is a vector for which $\exp_m A = m'$.

It has been shown by Whitehead [1] that:

Every point p_0 in a space M with an affine connection has a neighbourhood V which is a normal neighbourhood of each of its points.

Since any two points p and $q \in V$ can be connected in V by a unique geodesic it then follows from Whitehead's theorem that:

Every point p_0 in a space with an affine connection has a neighbourhood V any two points of which can be joined in V by a unique geodesic. Such a neighbourhood V is called simple convex.

For $p \in M$ let $\{A_i\}$ be a basis for $T_p(M)$, then there is a linear isomorphism $\alpha : \mathbb{R}^n \to T_p(M)$ defined by

$$a = (a^1, \ldots, a^n) \in \mathbb{R}^n \mapsto a^i A_i \ . \quad \text{Suppose that}$$

$$E = \exp_p \circ \alpha : \mathbb{R}^n \to M \ .$$

Then the mapping E^{-1} is a diffeomorphism of the normal neighbourhood $V(p)$ onto the open set $\alpha^{-1}(V_0) \subseteq \mathbb{R}^n$. We shall call the corresponding local coordinates $(y^1, \ldots, y^n)$ at the point P (with coordinate neighbourhood $V(p)$) the <u>normal coordinates</u> (defined by the basis $\{A_1, \ldots, A_n\}$) at the point p. Thus

$$E^{-1}(q) = (y'(q), \ldots, y^n(q)), \qquad q \in V(p) .$$

From this it follows that, for an arbitrary vector $A = a^i A_i \in T_p(M)$,

$$y^i(\exp_p A) = a^i .$$

Using the property of geodesics that $\gamma_A(s) = \gamma_{sA}(1)$, it then follows that every geodesic $\gamma_A(s)$ is expressed in terms of normal coordinates $(y^1, \ldots, y^n)$ in the neighbourhood $V(p)$ by the linear function

$$y^i(s) = a^i s \tag{1}$$

Let $*\Gamma^i_{jk}(y)$ denote the components of affine connection relative to the system y^i of normal coordinates of origin p_o. Then since the differential equations

$$\frac{d^2y^i}{dt^2} + *\Gamma^i_{jk}(y)\frac{dy^j}{dt}\frac{dy^k}{dt} = 0 \tag{2}$$

are satisfied by $y^i = a^i t$, we have

$$*\Gamma^i_{jk}(at)a^j a^k = 0$$

for all values of a^i. In particular at $t = 0$ $(*\Gamma^i_{jk})_o a^i a^k = 0$ for all a^i and hence

$$(*\Gamma^i_{jk})_o = 0 \tag{3}$$

If now the connection is a metric connection and if $*g_{ij}$ are the components of the metric w.r.t. the normal system (y^i), then a necessary and sufficient condition for (y^i) to be a normal coordinate system is (Cf R.W.W. [1], p. 12)

$$(*g_{ij}\, y^j) = (*g_{ij})_o\, y^j \quad . \tag{4}$$

1.4 Two-point invariant functions

a) The distance function Ω

Let W be a simple convex neighbourhood on a riemannian manifold M and let $p_o \in W$. Then any point $p \in W$ determines a unique vector $u \in T_{p_o}(M)$ which is tangent to the geodesic $\gamma_u(r) = \exp_{p_o} ru$ and $p = \exp_{p_o} u$.

Let T(W) be the restriction to W of the tangent bundle of M. We define a map

$$\Omega : T(W) \to \mathbb{R} \qquad \text{by} \qquad ru \to \tfrac{1}{2} r^2 \; .$$

In terms of an allowable coordinate system (x^i) covering W, Ω is a symmetric function of the coordinates (x_o^i), (x^i) of p_o and p. If in particular we have a normal coordinate system y^i of origin p_o and let y^i be the coordinates of p then

$$y^i = a^i s$$

where $$*g_{ij}\, a^i a^j = e,$$

the indicator of the geodesic arc (p_o, p), and $\Omega(p_o, p) = \frac{1}{2} er^2$.

If the geodesic arc (p_o, p) is not null then $e \neq 0$ and the parameter s above is numerically equal to r, the arc length of (p_o, p). Hence

$$\Omega = \frac{1}{2} es^2 = \frac{1}{2} (*g_{ij})_o y^j y^j \quad . \tag{5}$$

Now

$$\frac{\partial \Omega}{\partial y^i} = (*g_{ij})_o y^j = *g_{ij} y^j \tag{6}$$

hence

$$*g^{ij} \frac{\partial \Omega}{\partial y^j} = y^i = a^i s = s \frac{dy^i}{ds} \tag{7}$$

where $\frac{d}{ds}$ is differentiation along (p_o, p).

This is a tensor equation and therefore relative to any coordinate system (x^i),

$$g^{ij} \frac{\partial \Omega}{\partial x^j} = s \frac{dx^i}{ds} \quad . \tag{8}$$

From (6)

$$*g^{ij} \frac{\partial \Omega}{\partial y^i} \frac{\partial \Omega}{\partial y^j} = (*g_{ij})_o y^i y^j = 2\,\Omega$$

and hence in any coordinate system,

$$\Delta_1 \Omega \equiv g^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} = 2\,\Omega \tag{9}$$

where Δ_1 is the first differential parameter of Beltrami. In (8) p_o is regarded as fixed and p variable. If instead p is kept fixed and p_o varied, we get by the symmetry of Ω,

$$(g^{ij})_o \frac{\partial \Omega}{\partial y_o^j} = s\left(-\frac{dy^i}{ds}\right)_o$$

and by virtue of (7)

$$y^i = -(g^{ij})_o \frac{\partial \Omega}{\partial x_o^j} \tag{10}$$

where (y^i) are the normal coordinates of p derived from (x^i) and having p_o as origin.

b) <u>Ruse's Invariant ρ</u>

Let (x^i) be an allowable coordinate system covering a simple convex neighbourhood W on a riemannian manifold M. Let Ω be the distance function. For $P_o\ (x_o{}^i)$ and $P(x^i)$ denote by J the determinant

$$J = \left| \left(\frac{\partial^2 \Omega}{\partial x_o{}^i\, \partial x^j} \right) \right|$$

and let $|J|$ be its modulus.

Definition (13) Ruse's invariant is the scalar function

$$\rho : W \times W \to \mathbb{R}$$

given by

$$(p_o, p) \in W \times W \mapsto +\sqrt{\det (g_p) \det(g_{p_o})} \; / \; |J| \qquad (11)$$

where we take the positive square root.

It is clear that ρ is everywhere positive. That it is in fact a 2-point invariant may be shown (Cf RWW pp.18-19) by considering its behaviour under a coordinate transformation.

In terms of normal coordinates (11) reduces to

$$\rho = \sqrt{\det(*g_p) \; / \; \det(*g_{p_o})} = \sqrt{\det(*g_{p_o}^{-1} \; *g_p)} \qquad (12)$$

from which it follows that $\rho \to 1$ as $p \to p_o$.

The relation between ρ and Ω is moreover expressed by (Cf RWW. p21)

$$\Delta_2 \Omega = n + \Omega^k \frac{\partial}{\partial x^k} \log \rho \qquad (13)$$

where $\Delta_2\Omega$ is the laplacian of Ω given by

$$\Delta_2\Omega = g^{ij} \, \Omega_{,ij} = \frac{1}{\sqrt{|\det g|}} \, \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{|\det g|} \, \frac{\partial \Omega}{\partial x^j} \right) .$$

1.5 Locally harmonic riemannian spaces

Let U and V be any two C^∞ manifolds and let Φ and ψ be two C^∞

mappings of U into V. We say that Φ is a function of ψ or Φ factors through V by ψ if there exists a C^{∞} mapping f of V into itself such that the following diagram commutes,

$$\begin{tikzcd} U \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & V \\ & V \arrow[ur, "f"'] & \end{tikzcd} \qquad \text{i.e.} \quad \Phi = f \circ \psi$$

<u>Definition (14)</u> A riemannian manifold M is said to be <u>locally harmonic</u> (denoted by H_n) if one of the following three conditions is satisfied for every point $P_o \in M$ in some normal neighbourhood W of origin P_o :

(A) Laplace's equation $\Delta_2 u = 0$ possesses a non-constant solution which is a function of Ω alone. Such a function is called a harmonic function on W and $u = \psi(\Omega)$ is called the elementary function.

(B) $\Delta_2\Omega$ is a function of Ω only, $\Delta_2\Omega = X(\Omega)$ called the characteristic function.

(C) Ruse's invariant is a function of Ω only.

Proofs that these three conditions are in fact equivalent are given in (RWW pp. 36-40). It turns out that in a harmonic space H_n the characteristic and the elementary functions satisfy:

$$X(\Omega) = \Delta_2\Omega = n + 2\Omega \frac{d}{d\Omega} \log \rho(\Omega) \tag{14}$$

and

$$\psi(\Omega) = A\int_a^{\Omega} \frac{d\omega}{|\omega|^{n/2}\,\rho(\omega)} + B \tag{15}$$

where a, A and B are arbitrary constants.

From (14) it follows that

$$\rho(\Omega) = \exp\int_0^{\Omega} \frac{\chi(\omega) - n}{2\omega}\, d\omega \tag{16}$$

the constant of integration being determined by the fact that $\rho(\Omega) \to 1$ as $\Omega \to 0$. Thus if any of the functions $\rho(\Omega)$, $\chi(\Omega)$ or $\psi(\Omega)$ is known the other two may be determined by the above formulae.

<u>Definition (15)</u> An $\underline{H}_n$ is simply harmonic if one of the following conditions is satisfied:

(A') $$\psi(\Omega) = \begin{cases} A/|\Omega|^{(n-2)/2} + B & (n > 2) \\ A\log|\Omega| + B & n = 2 \end{cases}$$

A,B being arbitrary constants.

(B') $\chi(\Omega)$ is a constant.

(C') $\rho(\Omega)$ is a constant.

The equivalence of (A'), (B'), (C') follows from the equivalence of (A), (B) and (C) in definition (14). The constants in (B') and (C') are respectively n and 1.

Mean Value Theorem Willmore [1]

Let M be a riemannian manifold with positive definite metric , $p_o \in M$ and N any normal neighbourhood of origin p_o. Let c be a positive number small enough that on any geodesic ray emanating from p_o there is a point of N whose geodesic distance in N from p_o is equal to c. The geodesic sphere $S^{n-1}(p_o, c)$ of centre p_o and radius c is the (n-1)-hypersurface consisting of all points in N at geodesic distance c from p_o.

Now let $\mu(u; p_o; r)$ denote the mean value of a harmonic function u over $S^{n-1}(p_o; r)$ then

$$\mu(u; p_o; r) = \frac{1}{C_{n-1}(r)} \int_{S^{n-1}(p_o,r)} u \, dV_{n-1}$$

where $C_{n-1}(r)$ and dV_{n-1} denote respectively the volume and volume element of $S^{n-1}(p_o,r)$.

Theorem (Willmore [1]):

M is harmonic at $p_o \in M$ iff the mean value over every geodesic sphere centre p_o of every function harmonic in any neighbourhood containing $S^{n-1}(p_o,r)$ is equal to the value of the function at p_o .

1.6 Conditions for harmonic manifolds

Let (y^i) be a system of normal coordinates of origin p_o covering a simple convex neighbourhood W of p_o in an analytic riemannian manifold M. M is not assumed harmonic for the moment. W.r.t. this system:

$$\Delta_2 \Omega = *\Omega^i_{\ ,i} \tag{17}$$

and by (7) we have $*\Omega^i = y^i$. Therefore

$$\Delta_2 \Omega = \text{trace}(*\Omega^i_{\ ,j}) = \text{trace}(\delta^i_j + *\Gamma^i_{jk}\, y^k) \tag{18}$$

Now $*\Gamma^i_{jk}$ are analytic functions of the (y^i) and admit therefore a convergent Maclaurin's series representation in a sufficiently small neighbourhood of p_o:

$$*\Gamma^i_{jk}(y) = \sum_{p=1}^{\infty} \frac{1}{p!}\, (A^i_{jk\ell_1\ell_2\ldots\ell_p})_o\, y^{\ell_1} \ldots y^{\ell_p} \tag{19}$$

where $*\Gamma^i_{jk}(0) = 0$ and

$$(A^i_{jk\ell_1\ldots\ell_p})_o = \left\{ \frac{\partial}{\partial y^{\ell_p}} \cdots \frac{\partial}{\partial y^{\ell_1}} (*\Gamma^i_{jk}) \right\}_o \tag{20}$$

are the affine normal tensors at p_o (Veblen [1], p. 89-90).
If (a^i) is the tangent vector at p_o to the geodesic $y^i = a^i s$, where $s = 0$ at p_o and $(g_{ij})_o\, a^i a^j = e$, then

$$\Delta_2\Omega = n + \sum_{p=1}^{\infty} \frac{1}{p!} (A^i_{ik\ell_1\ldots\ell_p})_o \, a^k a^{\ell_1} \ldots a^{\ell_p} s^{p+1} \tag{21}$$

If we now assume M to be harmonic we must have

$$\Delta_2\Omega = \chi(\Omega) = \sum_{m=0}^{\infty} \frac{1}{m!} \chi^{(m)}(0)\Omega^m \tag{22}$$

$$= \sum_{m=0}^{\infty} \frac{1}{2^m . m!} \chi^{(m)}(0) e^m S^{2m}$$

as $\Omega = \frac{1}{2} es^2$.

On putting

$$h_m = \chi^{(m)}(0) \,/\, (2^{m+1}.m.m!), \qquad m = 1,2,3, \ldots \tag{23}$$

we get

$$\Delta_2\Omega = \chi(0) + \sum_{m=1}^{\infty} (2m) h_m \, e^m S^{2m} \tag{24}$$

For a harmonic space (21) and (24) must be equal and hence by equating coefficients of S^k we get:

$$(A^i_{ik\ell_1\ell_2\ldots\ell_{2m-1}})_o = h_m \, S(g_{k\ell_1} \, g_{\ell_2\ell_3} \ldots g_{\ell_{2m-2}\,\ell_{2m-1}})$$

$$(A^i_{ik\ell_1\ell_2\ldots\ell_{2m}})_o = 0 \qquad m = 1, 2, 3, \ldots \tag{25}$$

where S denotes summation over all permutations of the free indices.

Ledger's Formulae, Ledger [1]

Consider a coordinate system (x^i) about $p_o(x_o) \in M$ and let $x^i = \Phi(x_o,y)$, $y^i = a^i s$, be the equation of a geodesic a through p_o, where

$$a^i = \left(\frac{dx^i}{ds}\right)_{s=0} \quad \text{and} \quad (g_{ij})_o\, a^i a^j = e .$$

Regarding (x^i_o) as fixed, $\Delta_2\Omega$ becomes a function of the x^i, so if we keep the a^i constant, we obtain a function of s having for sufficiently small values of $|s|$ the Maclaurin expansion:

$$\Delta_2\Omega = \sum_{r=0}^{\infty} \frac{1}{r!}\lambda_r s^r , \tag{26}$$

where $\lambda_r = \left(\dfrac{d^r}{ds^r}\Delta_2\Omega\right)_{s=0}$.

This is the same as (21) if we put:

$$\lambda_r = r(A^i_{ik\ell_1\ldots\ell_{r-1}})a^k a^{\ell_1} \ldots a^{\ell_{r-1}} . \tag{27}$$

Hence according to (25) the required conditions for a harmonic manifold are:

$$\begin{cases} \partial^{2r}/\partial a^k\, \partial a^{\ell_1} \ldots \partial a^{\ell_{2r-1}})\lambda_{2r} = (2r).(2r)!h_r S(g_{k\ell_1} \ldots g_{\ell_{2r-2}\,\ell_{2r-1}}) \\ \text{and} \quad (\partial^{2r+1}/\, \partial a^k\, \partial a^{\ell_1} \ldots \partial a^{2r})\lambda_{2r+1} = 0 \quad . \end{cases} \tag{28}$$

Now let D denote covariant differentiation along a. For any tensor of type (1,1)

$$D^r\, T^i_j = T^i_{j,\ell_1 \ldots \ell_r}\, a^{\ell_1} \ldots a^{\ell_r} \quad .$$

Using matrix notation $T = (T^i_j)$ let us denote $(D^r\, T^i_j)$ by T_r; and let us introduce the following matrices:

$$\Lambda = (\Omega^i_{,j}) \tag{29}$$

and

$$\Pi = (\Pi^i_j) = R^i_{k\ell j}\, a^k\, a^\ell \, . \tag{30}$$

Since the operations of covariant differentiation and taking the trace commute we get:

$$\text{trace}\, [\Lambda_r]_{s=0} = [D^r \text{ trace } \Omega^i_{,j}]_{s=0}$$

$$= [D^r \Delta_2 \Omega]_{s=0} \qquad = \lambda_r \quad .$$

Also

$$\text{trace}\ \Pi = R^i_{k\ell i}\, a^k\, a^\ell = R_{k\ell}\, a^k\, a^\ell \quad ,$$

where $R_{k\ell}$ is the Ricci tensor.

By twice differentiating the identity

$$\Omega^i \, \Omega_{,i} = 2\,\Omega$$

and raising suffixes when appropriate, Ledger obtains, by a judicious use of the Ricci identity and Leibniz theorem, the following recurrence formulae:

$$(r+1)\Lambda_r = r(r-1)\,\Pi_{r-2} - \sum_{q=2}^{r-2} \binom{r}{q} \Lambda_q \, \Lambda_{r-q} \qquad r \geqslant 2, \tag{31}$$

where the matrices Λ_r and Π_r are evaluated at p_o .

These impose certain conditions on the curvature of a harmonic space which we now proceed to investigate.

1.7 Some consequences of the curvature conditions

On putting $r = 2,3,\ldots$ we get from (31) at p_o ,

$$\lambda_2 = \text{trace } \Lambda_2 = \frac{2}{3} \text{ trace } \Pi = \frac{2}{3} R_{k\ell} \, a^k a^\ell .$$

Hence if the manifold is H_n it then follows from (28) that

$$\frac{2}{3} R_{k\ell} = 4h_1 \, g_{k\ell} \tag{32}$$

i.e. $R_{k\ell} = k_1 \, g_{k\ell}$ for some constant k_1 and this demonstrates that any H_n is Einstein.

Putting r = 3 in (31) we see

$$\lambda_3 = \text{trace } \Lambda_3 = \frac{3}{2} \text{ trace } \Pi_1 = \frac{3}{2} R_{k\ell,m} a^k a^\ell a^m ,$$

and therefore by (28) for an H_n ,

$$R_{k\ell,m} + R_{\ell m,k} + R_{mk,\ell} = 0 \qquad (33)$$

which is a consequence of (32).

Putting r = 4 in (31) we get

$$\lambda_4 = \frac{4}{15} \text{ trace } (9\Pi_2 - 2\Pi^2).$$

Since trace $\Pi_2 = 0$ by (32) we get, using (28)

$$S(R^p_{ijq} R^q_{k\ell p}) = k_2 S(g_{ij} g_{k\ell}) \qquad (34)$$

for some constant k_2 .

For r = 5 we get

$$S(R_{ij,k\ell m} - R^p_{ijq} R^q_{k\ell p,m}) = 0 \qquad (35)$$

which follows because of (32) and (34) and thus yields nothing new.

By the same process we get for r = 6 in (31),

$$S(32R^p_{ijq} R^q_{k\ell r} R^r_{mnp} + 9R^p_{ijq,m} R^q_{k\ell p,n}) = k_3 S(g_{ij} g_{k\ell} g_{mn}) \qquad (36)$$

where k_3 is constant.

The calculation of more such conditions is straightforward but becomes, with larger r, exceedingly laborious. Unless some additional properties are enjoyed by the manifold - e.g. local symmetry - the above conditions, essentially (32), (34) and (36) are almost all the tools at our disposal in dealing with locally harmonic spaces. As a consequence of them we have:

Theorem 1 An H_n is of constant curvature if

(a) $n = 2,3$

(b) H_n is conformally flat, i.e. locally conformal to a flat manifold.

(c) H_n is of normal hyperbolic metric.

Proof (a) For any M^2 the curvature K at any point is given by

$$K = R_{1212}/g = -\tfrac{1}{2}R = -\tfrac{1}{2} g^{ij} g^{k\ell} R_{ijk\ell}$$

Hence, if M^2 is an H_2 then from (32)

$$R = 2k_1 \quad \text{and thus} \quad K = -k_1 .$$

By a result of Schouten and Struick [1] every Einstein 3-manifold and also every conformally flat n-manifold are of constant curvature. This proves (a) for $n = 3$ and also (b).

(c) has been proved by Lichnerowicz and Walker [1].

<u>Corollary</u>: A simply harmonic SH_n is locally flat if

(a) $n = 2,3$.

(b) SH_n is conformally flat.

(C) SH_n is of normal hyperbolic metric.

CHAPTER II

SYMMETRIC SPACES

2.1 Basic facts about Lie algebras

Definition (1) A Lie group is a group G which is also an analytic manifold such that the mapping $(g_1, g_2) \mapsto g_1 g_2^{-1}$ of $G \times G \to G$ is analytic.

Let G be a Lie group and $\rho \in G$. The left translation $L_\rho : g \to \rho g$ of G onto itself is an analytic diffeomorphism. A vector field Z on G is called left invariant if $(L_\rho)_* Z_\sigma = Z_{\rho\sigma}$ for all $\sigma, \rho \in G$. Given a tangent vector X at the identity $X \in T_e(G)$, there exists exactly one left invariant vector field $\tilde{X}$ on G such that $\tilde{X}_e = X$. $\tilde{X}$ is necessarily analytic and is given by (Cf Chevalley [1], p. 102)

$$\tilde{X}_\rho = (L_\rho)_* X \quad .$$

It is easy to see that the sum of two left invariant vector fields, their bracket and the product of a left invariant vector field by a scalar are again left invariant vector fields.

Definition (2) The Lie algebra $\underline{g}$ of a Lie group G is the set of all left invariant vector fields on G. As a vector space $\underline{g}$ is isomorphic to $T_e(G)$.

More generally let $\underline{g}$ be a vector space over a field K of characteristic zero. Then $\underline{g}$ is a Lie algebra over K if there is given a rule of composition $(X,Y) \to [X,Y]$ in $\underline{g}$ satisfying:

(a) $[X,X] = 0$ for all $X \in \underline{g}$ and

(b) The Jacobi identity $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ for $X,Y,Z \in \underline{g}$.

The Lie algebra of definition (2) is clearly a Lie algebra over $\mathbb{R}$.

Every $A \in \underline{g}$ generates a global 1-parameter group of transformations of G. Indeed if ϕ_t is a local 1-parameter group of transformations generated by A and $\phi_t e$ is defined for $|t| < \epsilon$, then $\phi_t a$ **can be** defined for $|t| < \epsilon$ for every $a \in G$ and is equal to $L_a(\phi_t e)$ as ϕ_t commutes with every L_a. Since $\phi_t a$ is defined for $|t| < \epsilon$ for every $a \in G$, $\phi_t a$ is defined for $|t| < \infty$ for every $a \in G$. We set $a_t = \phi_t e$; then $a_{t+s} = a_t\, a_s$ for all $t,s \in \mathbb{R}$, and a_t is the 1-parameter subgroup of G generated by A. It is the unique curve in G such that its tangent vector $\dot{a}_t$ at a_t is equal to $L_{a_t} A_e$ and that $a_o = e$. Denote $a_1 = \phi_1 e$ by exp A. It follows that $\exp tA = a_t$ for all t. The mapping $A \to \exp A$ of $\underline{g}$ into G is the exponential map.

A vector subspace $\underline{h} \subseteq \underline{g}$ is a subalgebra of $\underline{g}$ if $[\underline{h}, \underline{h}] \subseteq \underline{h}$ and is an ideal if $[\underline{h}, \underline{g}] \subseteq \underline{h}$. A linear mapping $\sigma : \underline{g}_1 \to \underline{g}_2$ is a Lie algebra homomorphism if $\sigma([X,Y]) = [\sigma X, \sigma Y]$ for all $X,Y \in \underline{g}_1$;

$\sigma(\underline{g}_1)$ is a subalgebra of $\underline{g}_2$ and the kernel $\sigma^{-1}\{0\}$ is an ideal in $\underline{g}_1$. If $\sigma^{-1}\{0\} = \{0\}$ then σ is an isomorphism into.

For $X \in \underline{g}$ the linear transformation $Y \mapsto [X,Y]$ of $\underline{g}$ denoted by ad X satisfies: $\mathrm{ad}([X,Y]) = \mathrm{ad}X\mathrm{ad}Y - \mathrm{ad}Y\mathrm{ad}X = [\mathrm{ad}X,\mathrm{ad}Y]$ i.e. ad is a Lie algebra homomorphism. Now let V be a vector space over a field K and let $\underline{g\ell}(V)$ denote the vector space of all endomorphisms of V with bracket operation $[A,B] = AB - BA$. Then $\underline{g\ell}(V)$ is a Lie algebra over K. Let $\underline{g}$ be a Lie algebra over K, then a homomorphism

$$\rho: \ \underline{g} \to g\ell(V)$$

is called a representation of $\underline{g}$ on V. In particular the mapping $X \mapsto \mathrm{ad}X$ $(X \in \underline{g})$ is a representation of $\underline{g}$ on $\underline{g}$ called the adjoint representation and is denoted by $\mathrm{ad}\underline{g}$ or simply ad. The kernel of $\mathrm{ad}\underline{g}$ is called the centre of $\underline{g}$. If the centre of $\underline{g}$ equals $\underline{g}$ then $\underline{g}$ is said to be abelian. Thus $\underline{g}$ is abelian iff $[\underline{g},\underline{g}] = \{0\}$.

The derived series of a Lie algebra $\underline{g}$ is the decreasing sequence of ideals $D^0\underline{g}$, $D^1\underline{g}$, ... of $\underline{g}$ defined inductively by:

$$D^0\underline{g} = \underline{g}, \quad D^{p+1}\underline{g} = [D^p\underline{g}, D^p\underline{g}] .$$

The descending central series of $\underline{g}$ is the decreasing sequence of ideals $C^0\underline{g}$, $C^1\underline{g}$, ... of $\underline{g}$ defined inductively by

$$C^0\underline{g} = \underline{g}, \quad C^{p+1}\underline{g} = [\underline{g}, C^p\underline{g}] .$$

Evidently $D^p\underline{g} \subset C^p\underline{g}$ and $D^1\underline{g} = 0$ iff $\underline{g}$ is abelian.

Definition (3) $\underline{g}$ is nilpotent if $C^p\underline{g} = 0$ for some p. It is solvable if $D^p\underline{g} = 0$ for some p.

Every Lie algebra $\underline{g}$ has a unique maximum nilpotent ideal $\underline{n}$, i.e. a nilpotent ideal which contains every nilpotent ideal of $\underline{g}$; and a unique maximum solvable ideal $\underline{r}$ called the radical of $\underline{g}$. $\underline{g}$ is called semi-simple if its radical is zero and simple if it is not abelian and has no non-zero ideal other than $\underline{g}$ itself. (Cf Jacobson [1], p. 24-26).

Definition (4) The Killing form B of a Lie algebra $\underline{g}$ is the symmetric bilinear form on $\underline{g}$ defined by

$$B(X,Y) = \operatorname{trace} \operatorname{ad}X \operatorname{ad}Y \qquad \text{for } X,Y \in \underline{g}.$$

If α is any automorphism of $\underline{g}$, we have $\operatorname{ad}\alpha X = \alpha \circ \operatorname{ad}X \circ \alpha^{-1}$ and so

$$\begin{aligned} B(\alpha X, \alpha Y) &= \operatorname{trace}(\operatorname{ad}\alpha X \operatorname{ad}\alpha Y) = \operatorname{trace}(\alpha \operatorname{ad}X \operatorname{ad}Y \alpha^{-1}) \\ &= \operatorname{trace}(\operatorname{ad}X \operatorname{ad}Y) = B(X,Y). \end{aligned}$$

Since $\operatorname{trace}(AB) = \operatorname{trace}(BA)$ for any endomorphisms A and B, we have,

$$\text{trace}(\text{ad}[Z,X]\text{ad}Y) = \text{trace}((\text{ad}Z\ \text{ad}X - \text{ad}X\ \text{ad}Z)\text{ad}Y)$$

$$= \text{trace}((\text{ad}Y\ \text{ad}Z - \text{ad}Z\ \text{ad}Y)\text{ad}X) = \text{trace}(\text{ad}[Y,Z]\text{ad}X)$$

and thus we have

$$B(X,[Y,Z]) = B(Y,[Z,X]) = B(Z,[X,Y]) .$$

If $\underline{a}$ is an ideal of $\underline{g}$, the Killing form of $\underline{a}$ is the restriction of B to $\underline{a}$. For a subalgebra $\underline{a}$ of $\underline{g}$, denote by $\underline{a}^{\perp}$ the subspace:

$$a^{\perp} = \{X \in \underline{g},\ B(X,Y) = 0 \quad \forall \quad Y \in \underline{a}\}$$

If $\underline{a}$ is an ideal of $\underline{g}$ then so is $\underline{a}^{\perp}$. Denoting by $\underline{n}$ and $\underline{r}$ the maximum nilpoint ideal and radical of $\underline{g}$ respectively, we have:

$$\underline{g} \supset \underline{r} \supset \underline{g}^{\perp} \supset \underline{n} .$$

Equivalent to the semi-simplicity of $\underline{g}$ are the following conditions (Kobayashi and Nomizu II)

(1) Its radical $\underline{r} = 0$ (definition).

(2) Its maximum nilpotent ideal $\underline{n} = 0$. This is so because any nilpoint ideal is solvable (Jacobson [1], p. 25).

(3) $\underline{g}^{\perp} = 0$, i.e. its Killing form is non-degenerate. This is Cartan's Criterion for semi-simplicity (Cf Jacobson [1], p.69).

(4) every abelian ideal of $\underline{g}$ is zero.

(5) $\underline{g}$ is isomorphic to the direct sum of semi-simple Lie albegras.

(6) Every finite-dimensional representation of $\underline{g}$ is semi-simple, i.e. completely reducible.

2.2 Symmetric Spaces

Let M be an affinely connected n-manifold. For $x \in M$ the symmetry S_x at x is an involutive diffeomorphism of a neighbourhood of x onto itself which sends $\exp X \to \exp(-X)$, $X \in T_x(M)$. If $(x^1, \ldots, x^n)$ is a normal coordinate system at x then S_x sends $(x^1,\ldots,x^n) \to (-x^1,\ldots,-x^n)$. The differential of S_x is equal to $-I_x$, the identity transformation on $T_x(M)$.

Definition (5) An affinely connected manifold M is said to be affinely locally symmetric if each $x \in M$ has an open neighbourhood on which the symmetry S_x is an affine transformation. If for each $x \in M$, S_x can be extended to a global affine transformation of M, then M is called affine symmetric.

The definition of affine local symmetry is equivalent to the vanishing of the torsion tensor and $\nabla_Z R = 0$ for all $Z \in \mathfrak{X}(M)$. (Helgason [1], p. 163).

It is known (Cf Kobayashi and Nomizu II p.224) that every affine symmetric space is complete and that a complete simply connected affine locally symmetric space is affine symmetric. Also if A(M) is the Lie group of affine transformations of an affine symmetric space M and G its identity component, then both $A(M)$ and G act transitively on M.

For $m \in M$ we define the isotropy subgroup H of G at m by

$$H = \{g \in G \mid g(m) = m\}.$$

H is a closed subgroup of G and the map $G/H \to M$ given by $gH \mapsto gm$ is C^{∞}, one to one and onto. Thus M may be regarded as a homogeneous space G/H.

Definition (6) The linear isotropy representation is the homomorphism $\lambda : h \mapsto (dh)_m$ of H into the group of linear transformations of $T_m(M)$. $\lambda(H) = H'$ is called the linear isotropy group at m.

Proposition (1) Let M be an affine symmetric space and G the largest connected group of affine transformations of M. Let $0 \in M$ and H the isotropy subgroup at 0, so that $M = G/H$. Let S_0 be the symmetry at 0 and σ the automorphism of G given by

$$g \mapsto S_0 \circ g \circ S_0^{-1} \qquad , \quad g \in G.$$

Let G_σ be the closed subgroup of G consisting of those elements fixed by σ. Then H lies between G_σ and its identity component.

Proof See (Kobayashi and Nomizu II, p. 224).

In view of the above proposition we give the following definition:

Definition (7) A symmetric space is a triple (G, H, σ) where σ is an involutive automorphism of a connected Lie group G and H a closed subgroup of G lying between G_σ and its identity component.

(G, H, σ) is effective if the largest normal subgroup of G contained in H is the identity element.

Given a symmetric space (G, H, σ) we construct for each point x in the quotient space $M = G/H$ an involutive diffeomorphism S_x, the symmetry at x for which x is an isolated fixed point. For the origin $0 \in G/H$, S_0 is defined to be the involutive diffeomorphism of G/H onto itself induced by the automorphism σ of G. To show that 0 is in fact an isolated fixed point of S_0; let $g(0)$ be a fixed point of S_0, $g \in G$. This means $\sigma(g) \in gH$. Set

$$h = g^{-1}\sigma(g) \in H. \quad \text{Since } \sigma h = h, \text{ we have}$$

$$h^2 = h\sigma(h) = g^{-1}\sigma(g)\sigma(g^{-1}\sigma(g)) = g^{-1}\sigma(g)\sigma(g^{-1})g = e.$$

If g is sufficiently close to the identity so that h is also near the identity element, then h itself must be the identity and hence $\sigma(g) = g$. Being invariant by σ and near the identity element, g lies in the identity component of G_σ and hence in H. This implies that $g(0) = 0$.

For $x = g(0)$ we define

$$S_x = g \circ S_0 \circ g^{-1} .$$

Thus S_x is independent of g such that $g(0) = x$. Indeed if
$x = g_1(0) = g_2(0) \Rightarrow 0 = g_1^{-1} \circ g_2(0)$ so that
$S_0 = g_1^{-1}g_2 \circ S_0 \circ g_2^{-1}g_1$ and thus

$$g_1 \circ S_0 \circ g_1^{-1} = g_2 \circ S_0 \circ g_2^{-1} .$$

2.3 Transvections

Let M be an affine symmetric space and τ a geodesic joining two points $x, y \in M$. Then the product of two symmetries S_x and S_y is called a transvection along τ. If G is the largest connected group of affine transformations of M and if we write

$$S_x = g \circ S_0 \circ g^{-1} \qquad \text{where} \quad x = g \circ 0, \quad g \in G$$

and $$S_y = g' \circ S_0 \circ g'^{-1} \qquad \text{for} \quad y = g' \circ 0, \quad g' \in G$$

then

$$S_x \circ S_y = g \circ (S_0 \circ g^{-1} \circ g' \circ S_0) \circ g'^{-1}.$$

On the other hand by definition of S_0 we have

$$S_0 \circ g^{-1} \circ g' \circ S_0 = \sigma(g^{-1} \circ g') \in G, \qquad \text{and thus}$$

$$S_x \circ S_y = g \circ \sigma(g^{-1} \circ g') \circ g'^{-1} \in G,$$

i.e. every transvection belongs to G whereas a symmetry may not. Let us write $\tau = x_t \quad 0 \leq t \leq 4a$ with $x = x_0, \; y = x_{4a}$ and consider for each t the symmetries S_{x_t}, $S_{x_{3t}}$ and set

$$f_t = S_{x_{3t}} \cdot S_{x_t} .$$ Then f_o is the identity,

and $f_a(x) = y$, which shows that the set of transvections generates a transitive subgroup of G.

Definition (8) A symmetric Lie algebra (or involutive Lie algebra) is a triple $(\underline{g}, \underline{h}, \sigma)$ where σ is an involutive automorphism of the Lie algebra $\underline{g}$ and $\underline{h}$ is the subalgebra of $\underline{g}$
$\underline{h} = \{x \in \underline{g} | \sigma(x) = x\}$.

$(\underline{g}, \underline{h}, \sigma)$ is called effective if $\underline{h}$ contains no non-zero ideal of $\underline{g}$.

We shall use the abbreviation SLA for a symmetric Lie algebra. To every symmetric space (G, H, σ) there corresponds a SLA $(\underline{g},\underline{h},\sigma)$ where $\underline{g}$ and $\underline{h}$ are the Lie algebras of H and G respectively and the automorphism σ of $\underline{g}$ is that induced by σ of G. Conversely if $(\underline{g}, \underline{h}, \sigma)$ is a SLA and G a connected, simply connected Lie group with Lie algebra $\underline{g}$, then the automorphism σ of $\underline{g}$ induces an automorphism σ of G (Cf Chevalley [1], p. 113) and for any subgroup H lying between G_σ and its identity component the triple (G, H, σ) is a symmetric space.

Let $(\underline{g}, \underline{h}, \sigma)$ be a SLA. Since σ is involutive its eigenvalues as a linear transformation of $\underline{g}$ are ± 1 , and $\underline{h}$ is the eigenspace for +1. Let $\underline{m}$ be the eigenspace for -1. Then the decomposition

$$\underline{g} = \underline{h} + \underline{m}$$

is called the cannonical decomposition of $(\underline{g}, \underline{h}, \sigma)$. Under the usual identification of $\underline{g}$ with $T_e(G)$, $\underline{m}$ is seen to be isomorphic to $T_{\pi(e)}(G/H)$ under the differential of the natural projection

$$\pi : G \to G/H , \quad \text{and} \quad \underline{h} = \ker d\pi .$$

Proposition (2)

(a) $[\underline{h},\underline{h}] \subseteq \underline{h}$

(b) $[\underline{h},\underline{m}] \subseteq \underline{m}$

(c) $[\underline{m},\underline{m}] \subseteq \underline{h}$.

Proof

(a) holds because $\underline{h}$ is a subalgebra.

(b) Let $X \in \underline{h}$, $Y \in \underline{m}$ then

$$\sigma([X,Y]) = [\sigma X, \sigma Y] = [X, -Y] = -[X,Y].$$

(c) Let $X,Y \in \underline{m}$ then

$$\sigma([X,Y]) = [\sigma X, \sigma Y] = [-X, -Y] = [X,Y].$$

The above inclusion relations characterize a SLA in the following sense. Given a Lie algebra $\underline{g}$ and a vector space direct sum

$\underline{g} = \underline{h} + \underline{m}$ satisfying (a), (b) and (c), let σ be the linear transformation of $\underline{g}$ defined by $\sigma(X) = X$, $X \in \underline{h}$ and $\sigma(Y) = -Y$, $Y \in \underline{m}$. Then σ is an involutive automorphism of $\underline{g}$ and $(\underline{g}, \underline{h}, \sigma)$ is a SLA. If G is moreover effective and semi-simple then we can strengthen (c) by $[\underline{m},\underline{m}] = \underline{h}$. (Cf Lichnerowicz [1], p.183).

Proposition (3) Let (G, H, σ) be a symmetric space and $(\underline{g}, \underline{h}, \sigma)$ its SLA. If $\underline{g} = \underline{h} + \underline{m}$ is the cannonical decomposition then $ad(H)\underline{m} \subseteq \underline{m}$.

Proof Let $X \in \underline{m}$, $h \in H$. Then $\sigma(adh.X) = ad\sigma(h).\sigma(X) = adh(-X) = -adh.X$.

2.4 The Cannonical connection on a symmetric space

Definition (9) Let $M = G/H$ be a homogeneous space on which a connected Lie group G acts transitively and effectively. We say that M is reductive if the Lie algebra $\underline{g}$ of G is decomposed into a vector space direct sum:

$$\underline{g} = \underline{h} + \underline{m}, \quad \underline{h} \cap \underline{m} = 0 \qquad \text{such that}$$

$ad(H)\underline{m} \subseteq \underline{m}$.

The existence and properties of invariant affine connections on homogeneous spaces were studied by Nomizu [1]. A generalisation of this study has been made by Wang [1]. Here we state a specialisation to the reductive case of a theorem by Wang [1]

(Cf Kobayashi and Nomizu II, p.191).

<u>Theorem (1)</u> Let G/H be a reductive homogeneous space with cannonical decomposition $\underline{g} = \underline{h} + \underline{m}$. Then there is a one to one correspondence between the set of G-invariant connections on G/H and the set of linear mappings:

$$\Lambda m : \underline{m} \to \underline{g\ell}(n, \mathbb{R}) \qquad \text{such that}$$

$$\Lambda m(\mathrm{ad}h(X)) = \mathrm{ad}(\lambda(h))(\Lambda m(X)) \qquad \text{for } X \in \underline{m},\ h \in H,$$

where λ denotes the linear isotropy representation $H \to GL(n,\mathbb{R})$.

<u>Definition (10)</u> The G-invariant connection on G/H defined by $\Lambda m = 0$ is called the cannonical connection (w.r.t. the decomposition $\underline{g} = \underline{h} + \underline{m}$).

The torsion and curvature tensors at $0 \in G/H$ of the cannonical connection satisfy (Cf Kobayashi and Nomizu, p. 193)

(1) $(T(X,Y))_o = -[X,Y]_{\underline{m}}$

(2) $(R(X,Y)Z)_o = -[[X,Y]_{\underline{h}}, Z]$

(3) $\nabla T = \nabla R = 0$

where $X,Y,Z \in \underline{m}$ and $[X,Y]_{\underline{m}}$ (resp $[X,Y]_{\underline{h}}$) denotes the $\underline{m}$ (resp. $\underline{h}$)-component of $[X,Y] \in \underline{g}$.

Since for a symmetric space $[\underline{m},\underline{m}] \subset \underline{h}$ it follows that the cannonical connection on a symmetric space is torsion free. Moreover it is the only affine connection on $M = G/H$ invariant by the symmetries of M.

Many of the nice properties of the cannonical connection are summarized in the following theorem (Cf Kobayashi and Nomizu II, p.231)

<u>Theorem (2)</u> With respect to the cannonical connection of a symmetric space (G, H, σ), the homogeneous space $M = G/H$ is a (complete) affine symmetric space with symmetries S_x and has the following properties:

(1) $T = 0$, $\nabla R = 0$ and $R(X,Y)Z = -[[X,Y], Z]$, $X,Y,Z \in \underline{m}$ where $\underline{m}$ is identified with $T_o(M)$, 0 being the origin of M.

(2) For each $X \in \underline{m}$, parallel displacement along $\pi(\exp tX)$ coincides with the differential of the transformation $\exp tX$ on M.

(3) For each $X \in \underline{m}$, $\pi(\exp tX) = (\exp tX) \circ 0$ is a geodesic starting from 0; conversely every geodesic from 0 is of this form.

(4) Every G-invariant tensor field on M is parallel.

(5) The Lie algebra of the linear holonomy group σ with reference point 0 is spanned by $\{R(X,Y)_o = \mathrm{ad}_{\underline{m}}([X,Y]), \quad X,Y \in \underline{m}\}$.

<u>Corollary</u> If G is assumed effective and semi-simple then the Lie algebra of the restricted linear holonomy group coincides with the

Lie algebra of the Linear isotropy group (Cf Lichnerowicz [1], p.183).

2.5 Riemannian symmetric spaces

Definition (11) A riemannian manifold M is called riemannian locally (globally) symmetric if it is affine locally (globally) symmetric w.r.t. the riemannian connection.

For each x in a riemannnina locally symmetric space the symmetry S_x is an isometry. From properties of affinely symmetric spaces it follows that a riemannian manifold is locally symmetric iff its curvature tensor field is parallel. The properties of affine symmetric spaces referred to at the beginning of this chapter could easily be formulated in the language of riemannian symmetric spaces; for instance every riemannian globally symmetric space is complete and a complete simply connected riemannian locally symmetric space is globally symmetric.

Theorem (3) Let (G, H, σ) be a symmetric space. A G-invariant pseudo-riemannian structure on $M = G/H$, if there exists any, induces the cannonical connection on M.

Proof Such a metric is parallel w.r.t. the cannonical connection by theorem (2). Since the cannonical connection is also torsion free it must be the riemannian connection. q.e.d.

There is a large class of symmetric spaces which admit invariant pseudo-riemannian structures as shown by:

Theorem (4) Let (G, H, σ) be a symmetric space with G semi-simple and let $\underline{g} = \underline{h} + \underline{m}$ be the cannonical decomposition. The restriction of the Killing form B of $\underline{g}$ to $\underline{m}$ defines a G-invariant pseudo-riemannian metric on G/H by:

$$B(X,Y) = g(X,Y)_0 \qquad \text{for} \quad X,Y \in \underline{m}.$$

Proof Since the Killing form of a semi-simple Lie algebra is non-degenerate and invariant by all automorphisms of $\underline{g}$, the theorem follows from the following

Lemma Let $(\underline{g}, \underline{h}, \sigma)$ be a SLA with cannonical decomposition $\underline{g} = \underline{h} + \underline{m}$. If B is a symmetric bilinear form on $\underline{g}$ invariant by σ then $B(\underline{h},\underline{m}) = 0$. If B is moreover non-degenerate, so are its restrictions $B_{\underline{h}}$ and $B_{\underline{m}}$ to $\underline{h}$ and $\underline{m}$.

Proof If $X \in \underline{h}$ and $Y \in \underline{m}$ then

$$B(X,Y) = B(\sigma X,\sigma Y) = B(X, -Y) = -B(X,Y) \Rightarrow B(\underline{h}, \underline{m}) = 0.$$

It is clear that B is non-degenerate iff both $B_{\underline{h}}$ and $B_{\underline{m}}$ are so. q.e.d.

Theorem (5) Let M be a riemannian symmetric space, G the largest connected group of isometries of M and H the isotropy group at a point $0 \in M$. Let S_0 be the symmetry of M at 0 and σ the involutive automorphism of G defined by $g \mapsto S_0 \circ g \circ S_0^{-1}$, $g \in G$. Let G_σ be the closed subgroup of G consisting of elements left fixed by σ. Then

(1) G is transitive on M so that $M = G/H$.

(2) H is compact and lies between G_σ and its identity component.

<u>Proof</u>

(1) Let x,y be any two points of M. Since M is complete these could be joined by a geodesic $\tau(t)$, $0 \leqslant t \leqslant 4a$, say. Then as before

$$f_t = S_{x_{3t}} \circ S_{x_t} \qquad \text{defines a 1-parameter}$$

group of transvections along τ which is transitive since $f_a(x) = y$. Now $S_{x_{3t}}$ and S_{x_t} being isometries shows that the group of transvections is a subgroup of G and thus G itself is transitive.

(2) Let I(M) be the group of isometries of M, then $I_0(M)$ the isotropy subgroup of I(M) at 0 is compact (Cf Kobayashi-Nomizu I p.49). Being the identity component of I(M), G is closed in I(M). Hence $H = G \cap I_0(M)$ is compact.

That H lies between G_σ and its component of the identity follows from proposition (1). q.e.d.

2.6 Decomposition of Riemannian Symmetric Spaces

Let M be a connected riemannian manifold with metric g and $\Phi(X)$ its linear holonomy group at $x \in M$. Then M is said to be reducible (irreducible) according as $\Phi(X)$ is reducible (irreducible) as a linear group acting on $T_x(M)$. From the deRham decomposition theorem (Cf Kobayashi and Nomizu I, p. 187) we get the following analogous theorem for symmetric spaces.

Theorem (6) Let M be a simply connected riemannian symmetric space and $M = M_0 \times M_1 \times \ldots \times M_k$ its de Rham decomposition, where M_0 is euclidean and each M_i is irreducible. Then each M_i is riemannian symmetric.

Another important decomposition of a simply connected riemannian symmetric space is the following

$$M = M_0 \times M_- \times M_+ \qquad \text{where } M_0 \text{ is euclidean}$$

and M_+ (resp. M_-) is a compact (resp. non-compact) riemannian symmetric space. (Cf Helgason [1], p. 208). To achieve this we introduce the following definition.

Definition (12) Let $(\underline{g}, \underline{h}, \sigma)$ be a SLA. If the connected Lie group of linear transformations of $\underline{g}$ generated by $ad_{\underline{g}}(\underline{h})$ is compact then $(\underline{g}, \underline{h}, \sigma)$ is called an orthogonal symmetric Lie algebra (OSLA).

Proposition (4) (Kobayashi and Nomizu II, p. 247-250).
Let ($\underline{g}$, $\underline{h}$, σ) be an OSLA with decomposition $\underline{g} = \underline{h} + \underline{m}$. Let B be the Killing form of $\underline{g}$. Then

(A) If $\underline{h} \cap$ centre of $\underline{g} = 0$ then B is negative definite on $\underline{h}$.

(B) If $\underline{g}$ is simple then

(1) ad($\underline{h}$) is irreducible on $\underline{m}$.

(2) B is (negative or positive) definite on $\underline{m}$.

Definition (13) An OSLA ($\underline{g}$, $\underline{h}$, σ) with $\underline{g}$ semi-simple is of the compact (resp. non compact) type according as the Killing form B of $\underline{g}$ is negative (resp. positive) definite on $\underline{m}$.

Let ($\underline{g}$, $\underline{h}$, σ) be an OSLA and suppose (G, H, σ) is the symmetric space associated with it. Then (G, H, σ) is of the compact (non compact) type according to ($\underline{g}$, $\underline{h}$, σ). A riemannian symmetric space M = G/H with a connected semi-simple group of isometries is of the same type as the symmetric space (G, H, σ).

Theorem (7) (Helgason p. 205) Let ($\underline{g}$, $\underline{h}$, σ) be an OSLA with an associated symmetric space (G, H, σ) with H connected and closed. Let g be an arbitrary G-invariant riemannian structure on G/H. If (G, H, σ) is of the compact (resp. non compact) type then G/H has sectional curvature everywhere ≥ 0 (resp. ≤ 0).

2.7 Rank of a symmetric space and two-point homogeneous spaces

Definition (14) Let M be a riemannian manifold and S a connected submanifold of M. S is said to be geodesic at $p \in M$, if each M-geodesic which is tangent to S at p is contained in S. S is totally geodesic if it is geodesic at each of its points.

Definition (15) The rank of a riemannian symmetric space M is the maximal dimension of a flat totally geodesic submanifold of M.

Theorem (8) (Helgason p.210) Let M be a riemannian symmetric space of the compact or non-compact type of rank ℓ. Let A, A' be two flat totally geodesic submanifolds of M of dimension ℓ.

(1) Let $q \in A$, $q' \in A'$. Then there exists an element $x \in G$ such that $x.A = A'$ and $x.q = q'$.

(2) Let $X \in T_q(M)$. Then there exists $k \in G$ such that $k.q = q$ and $dk(X) \in T_q(A)$.

(3) A and A' are closed topological subspaces of M.

There is another equivalent definition of rank which is algebraic. For this we need to define a Cartan subalgebra $\underline{h}$ of a semi-simple Lie algebra $\underline{g}$; $\underline{h}$ is a maximal abelian subalgebra of $\underline{g}$ such that for each $H \in \underline{h}$, ad(H) is semi-simple, i.e. each of its invariant subspaces has a complementary invariant subspace.

It is known that every semi-simple Lie algebra has a Cartan subalgebra and that if $\underline{h}_1$ and $\underline{h}_2$ are two Cartan subalgebras of $\underline{g}$; then there is an automorphism σ of $\underline{g}$ such that $\sigma\underline{h}_1 = \underline{h}_2$. In particular all Cartan subalgebras of a semi-simple Lie algebra have the same dimension.

Definition (16) Let $M = G/H$ be a riemannian symmetric space with cannonical decomposition $\underline{g} = \underline{h} + \underline{m}$ of its OSLA. Since $\underline{m}$ is semi-simple (the Killing form being non-degenerate on $\underline{m}$) it has a Cartan subalgebra of dimension ℓ, say. Then ℓ is the rank of M.

Definition (17) A riemannian manifold is two-point homogeneous if for any two point pairs (p_1, p_2), $(q_1, q_2) \in M$ satisfying $d(p_1, p_2) = d(q_1, q_2)$ there is an isometry g of M such that $g.p_i = q_i$. $i = 1,2$.

Theorem (9) Let M be a riemannian symmetric space of rank one. Then M is two-point homogeneous.

Proof Write $M = G/H$ where H is compact. The symmetric space (G, H, σ) is then either euclidean, in which case $M = \mathbb{R}$ or S^1 or else is of the compact or non-compact type. We thus invoke theorem (8), with $\ell = 1$ and the conclusion immediately follows. q.e.d.

Two-point homogeneous spaces have been classified completely by Wang [2] in the compact case and Tits in the non-compact case. The results show that the two-point homogeneous spaces are the euclidean spaces the circle S^1 and the symmetric spaces of rank one of the compact and non-compact type.

CHAPTER III

JACOBI VECTOR FIELDS

3.1 Variations

Let M be an n-dimensional riemannian manifold and $\gamma : I \to M$ an arbitrary segment of a regular curve in M with end points $p = \gamma(t_1)$ and $q = \gamma(t_2)$. Let G denote the subset of the plane $\mathbb{R}^2$ consisting of all points (t,ϵ) for which

$$t_1(\epsilon) \leqslant t \leqslant t_2(\epsilon)$$

$$-\epsilon_0 < \epsilon < \epsilon_0 \ , \qquad\qquad \epsilon_0 > 0,$$

where $t_1(\epsilon)$, $t_2(\epsilon)$ are C^∞ functions defined for $|\epsilon| < \epsilon_0$ such that:

$$t_1(\epsilon) < t_2(\epsilon), \quad t_1(0) = t_1, \quad t_2(0) = t_2 \ .$$

Definition (1) The surface

$$\Phi : G \to M$$

is called a variation of γ if the curve $\Phi_0(t) = \Phi(t,0) = \gamma(t)$.

Every variation Φ defines on M two (not necessarily regular) bounding curves:

$$\Phi_p(\epsilon) = \Phi(t_1(\epsilon), \epsilon) \quad \text{and} \quad \Phi_q(\epsilon) = \Phi(t_2(\epsilon), \epsilon),$$

passing through the points p and q respectively.

If these curves are degenerate, i.e. $\phi_p(\epsilon) = p$, $\phi_q(\epsilon) = q$ for all $|\epsilon| < \epsilon_o$, then the surface Φ defines a variation with fixed end points.

There are two vector fields on M defined by the variation Φ:

$$\frac{\partial\phi}{\partial t}(t,\epsilon) \quad \text{and} \quad \frac{\partial\phi}{\partial\epsilon}(t,\epsilon) \qquad \text{for } (t,\epsilon) \in G.$$

In particular at each point of $\gamma(t)$ we have the vector fields:

$$\frac{\partial\phi}{\partial t}(t,0) = \dot{\gamma}(t), \quad \text{the tangent vector field to } \gamma(t),$$

and

$$\frac{\partial\phi}{\partial\epsilon}(t,0) = X(t), \quad \text{called the vector field associated with}$$

the variation Φ .

At the end points p and q we have:

$$\dot{\phi}_p(0) = \dot{t}_1(0)\,\dot{\gamma}(t_1) + X(t_1)$$

$$\dot{\phi}_q(0) = \dot{t}_2(0)\,\dot{\gamma}(t_2) + X(t_2) \,.$$

Thus if Φ is a variation with fixed end points,

$$X(t_1) = -\dot{t}_1(0)\,\dot{\gamma}(t_1)$$

$$X(t_2) = -\dot{t}_2(0)\,\dot{\gamma}(t_2).$$

In particular if $G \subset \mathbb{R}^2$ is a rectangle $t_1(\epsilon) = t_1$, $t_2(\epsilon) = t_2$ for all $|\epsilon| < \epsilon_o$ and Φ an arbitrary variation with fixed points:

$$X(t_1) = X(t_2) = 0.$$

3.2 Jacobi Variations

We recall that an affinely connected manifold is complete if geodesics are defined for the whole range of the affine parameter or equivalently if the exponential mappings $\exp_m$ are globally defined on each tangent space.

In what follows we assume that M is a complete riemannian manifold. Let $\gamma : \mathbb{R} \to M$ be an arbitrary non-degenerate maximal geodesic in M.

The variation

$$\phi : \mathbb{R} \times I \to M, \quad I = (-\epsilon_0, \epsilon_0) \quad \text{for some } \epsilon_0 > 0$$

of the geodesic γ is called a Jacobi variation if for each $\epsilon \in I$ the curve

$$\phi_\epsilon(t) = \phi(t, \epsilon); \quad t \in \mathbb{R} \quad \text{is a geodesic and}$$

$$\phi_0(t) = \gamma(t).$$

For each fixed $t_0 \in \mathbb{R}$ the Jacobi variation ϕ defines a smooth curve

$$\alpha(\epsilon) = \phi_{t_0}(\epsilon) = \phi(t_0, \epsilon), \qquad \epsilon \in I,$$

and a vector field

$$A(\epsilon) = \frac{\partial\phi}{\partial t}(t_0, \epsilon) \qquad \epsilon \in I$$

on the curve $\alpha(\epsilon)$. Since

$$A(\epsilon) = \dot{\phi}_\epsilon(t_0) \qquad \epsilon \in I$$

the vector $A(\epsilon)$ uniquely determines the geodesic $\phi_\epsilon(t)$ for arbitrary $\epsilon \in I$. Hence a Jacobi variation is uniquely determined by the curve $\alpha(\epsilon)$ and the field $A(\epsilon)$ along it. These are related to $\gamma(t)$ at t_o by:

$$\alpha(0) = \gamma(t_o), \qquad A(0) = \dot{\gamma}(t_o) . \tag{1}$$

These conditions are not only necessary but also sufficient for the curve $\alpha(\epsilon)$ and the field $A(\epsilon)$ to determine a Jacobi variation ϕ of the geodesic $\gamma(t)$. That is for an arbitrary smooth curve $\alpha(\epsilon)$ in M and an arbitrary vector field $A(\epsilon)$ on $\alpha(\epsilon)$ which are related to $\gamma(t)$ by (1), there exists a Jacobi variation ϕ of γ such that

$$\alpha(\epsilon) = \phi(t_o, \epsilon) \quad \text{and} \quad A(\epsilon) = \frac{\partial\phi}{\partial t}(t_o, \epsilon).$$

The variation is given by:

$$\phi(t, \epsilon) = \gamma_{\alpha(\epsilon),A(\epsilon)}(t), \qquad \text{the maximal geodesic through}$$

the point $\alpha(\epsilon)$ with initial tangent vector $A(\epsilon)$. Thus for each ϵ the variation may be regarded as the image under $\exp_{\alpha(\epsilon)}$ of the ray $t \to tA(\epsilon)$ in $T_{\alpha(\epsilon)}(M)$ into M, i.e.

$$\phi(t,\epsilon) = \exp_{\alpha(\epsilon)}(t\,A(\epsilon)) . \tag{2}$$

Definition (2) A vector field $X(t)$ along a geodesic $\gamma(t)$ is a Jacobi field if there exists a Jacobi variation ϕ of γ such that $X(t) = \frac{\partial\phi}{\partial\epsilon}(t,0)$ for every $t \in \mathbb{R}$, i.e. $X(t)$ is associated with a Jacobi variation ϕ .

It follows from the above discussion that:

<u>Proposition (1)</u> An arbitrary Jacobi field $X(t)$ along a geodesic $\gamma(t)$ is uniquely determined by giving a curve $\alpha(\epsilon)$ and a vector field $A(\epsilon)$ along it satisfying (1).

Consequently for any two vectors $A, B \in T_{\gamma(t_o)}(M)$ there exists on the geodesic $\gamma(t)$ a Jacobi field $X(t)$ such that

$$X(t_o) = A, \qquad \frac{\nabla X}{dt}(t_o) = B \tag{3}$$

where $\frac{\nabla}{dt}$ denotes covariant differentiation along γ.

To construct this field it is sufficient to construct a curve $\alpha(\epsilon)$ such that

$$\alpha(0) = \gamma(t_o), \qquad \dot{\alpha}(0) = A$$

and to construct on this curve a vector field $A(\epsilon)$ such that

$$A(0) = \dot{\gamma}(t_o), \qquad \frac{\nabla A}{dt}(0) = B,$$

and then obtain the Jacobi variation (2).

We then see that,

$$X(t_o) = \frac{\partial \phi}{\partial \epsilon}(t,\epsilon)\Big|_{\substack{\epsilon=0 \\ t=t_o}} = \dot{\alpha}(0) = A$$

and

$$\frac{\nabla X}{dt}(t_o) = \frac{\nabla}{dt}\left[\frac{\partial \phi}{\partial \epsilon}(t,\epsilon)\Big|_{\epsilon=0}\right]_{t=t_o} = \frac{\nabla A}{dt}(0) = B.$$

<u>Theorem (1)</u> A vector field X along $\gamma(t)$ is a Jacobi field iff

$$\frac{\nabla}{dt}\frac{\nabla}{dt} X(t) + R_{\gamma(t)}(X(t), \dot{\gamma}(t))\, \dot{\gamma}(t) = 0 \qquad (4)$$

<u>Proof</u> Let $\phi(t,\epsilon)$ be the Jacobi variation with which the field $X(t)$ is associated. On the surface $\phi(t,\epsilon)$ we have the operators $\frac{\nabla}{\partial t}$ and $\frac{\nabla}{\partial \epsilon}$ of covariant differentiation w.r.t. the parameters t and ϵ respectively. The vanishing of the torsion tensor implies that the vector fields $\frac{\partial \phi}{\partial t}$ and $\frac{\partial \phi}{\partial \epsilon}$ satisfy:

$$\frac{\nabla}{\partial t}\frac{\partial \phi}{\partial \epsilon}(t,\epsilon) = \frac{\nabla}{\partial \epsilon}\frac{\partial \phi}{\partial t}(t,\epsilon) \qquad (*)$$

since $\left[\frac{\partial}{\partial \epsilon}, \frac{\partial}{\partial t}\right]\phi = 0.$

Also the curvature tensor field satisfies

$$\frac{\nabla}{\partial \epsilon}\frac{\nabla}{\partial t} - \frac{\nabla}{\partial t}\frac{\nabla}{\partial \epsilon} = R_{\phi(t,\epsilon)}\left(\frac{\partial \phi}{\partial \epsilon}, \frac{\partial \phi}{\partial t}\right) \qquad (**)$$

which applied to the field $\frac{\partial \phi}{\partial t}(t,\epsilon)$ gives

$$\frac{\nabla}{\partial \epsilon}\frac{\nabla}{\partial t}\frac{\partial \phi}{\partial t}(t,\epsilon) - \frac{\nabla}{\partial t}\frac{\nabla}{\partial \epsilon}\frac{\partial \phi}{\partial t}(t,\epsilon) =$$

$$R_{\phi(t,\epsilon)}\left(\frac{\partial \phi}{\partial \epsilon}(t,\epsilon), \frac{\partial \phi}{\partial t}(t,\epsilon)\right)\frac{\partial \phi}{\partial t}(t,\epsilon).$$

Because of (*) this is equivalent to:

$$\frac{\nabla}{\partial\epsilon}\frac{\nabla}{\partial t}\frac{\partial\phi}{\partial t}(t,\epsilon) - \frac{\nabla}{\partial t}\frac{\nabla}{\partial t}\frac{\partial\phi}{\partial\epsilon}(t,\epsilon) =$$

$$R_{\phi(t,\epsilon)}\left(\frac{\partial\phi}{\partial\epsilon}(t,\epsilon), \frac{\partial\phi}{\partial t}(t,\epsilon)\right)\frac{\partial\phi}{\partial t}(t,\epsilon) .$$

Setting $\epsilon = 0$ and since $\frac{\partial\phi}{\partial t}(t,0) = \dot{\gamma}(t)$ and $\frac{\partial\phi}{\partial\epsilon}(t,0) = X(t)$, we get

$$-\frac{\nabla}{dt}\frac{\nabla}{dt}X(t) = R_{\gamma(t)}(X(t), \dot{\gamma}(t))\,\dot{\gamma}(t),$$

since $\frac{\nabla}{dt}\dot{\gamma}(t) = 0.$

On the other hand let $A_1(t), \ldots, A_n(t)$ be a linear frame along $\gamma(t)$ and let $X^i(t)$ denote the components of $X(t)$ w.r.t. this frame. Equation (4) would thus be equivalent to a system of n ordinary second order differential equations for the unknown functions $X^1(t), \ldots, X^n(t)$. By appealing to the theorem on the existence and uniqueness of solutions to such a system, every solution of (4) and in particular every Jacobi field is uniquely determined by the values at $t = t_0$ of $X^i(t)$ and $\frac{dX^i}{dt}$. But obviously these values are uniquely determined by conditions (3), which by themselves determine a unique Jacobi field. q.e.d.

<u>Corollary</u> The space $\epsilon(\gamma)$ of all Jacobi fields along γ is 2n-dimensional - this being the space of solution of (4) which is linear.

Of particular interest are the Jacobi fields in $\epsilon(\gamma)$ that vanish at a point $\gamma(t_o)$. The space of such fields we denote by $\epsilon_{t_o}(\gamma)$.

$$\epsilon_{t_o}(\gamma) = \{X \in \epsilon(\gamma) | X(t_o) = 0\}.$$

Since (3) uniquely determines $X(t) \in \epsilon(\gamma)$ it follows that for a non-zero $X(t) \in \epsilon_{t_o}(\gamma)$, $\frac{\nabla X}{dt}(t_o) \neq 0$. Thus the mapping $X(t) \to \frac{\nabla X}{dt}(t_o)$ is an isomorphism of $\epsilon_{t_o}(\gamma)$ onto $T_{\gamma(t_o)}(M)$. In particular $\dim(\epsilon_{t_o}(\gamma)) = n$.

By (2) a field $X(t) \in \epsilon_{t_o}(\gamma)$ is associated with a variation $\Phi(t,\epsilon)$ having a degenerate bounding curve $\alpha(\epsilon) = \gamma(t_o)$ for all ϵ, i.e.

$$\Phi(t,\epsilon) = \exp_{\gamma(t_o)}(t\,A(\epsilon)) \tag{5}$$

In this case the variation factors through $T_{\gamma(t_o)}(M)$,

$$\Phi = \exp_{\gamma(t_o)} \circ S \tag{5'}$$

where S is a rectangle in $T_{\gamma(t_o)}(M)$. Hence $X \in \epsilon_{t_o}(\gamma)$ would arise as the image under $(\exp_{\gamma(t_o)})_*$ of a vector field $Y(t)$ along the ray ρ in $T_{\gamma(t_o)}(M)$ which goes into $\gamma(t)$ under $\exp_{\gamma(t_o)}$. If in (5) we take the field $A(\epsilon)$ to be linear in ϵ,

$$A(\epsilon) = \dot{\gamma}(t_o) + \epsilon \frac{\nabla X}{dt}(t_o),$$

then the rectangle S in (5') would be

$$S(t,\epsilon) = t(\dot{\gamma}(t_o) + \epsilon \frac{\nabla X}{dt}(t_o)) , \tag{6}$$

and $$\Phi(t,\epsilon) = \exp_{\gamma(t_o)} \circ S(t,\epsilon) \tag{6'}$$

so that

$$X(t) = (\exp_{\gamma(t_o)})_* Y(t) \tag{7}$$

Definition (3) A linear homogeneous vector field along a ray ρ in $T_m(M)$ is a curve Y above ρ in $T(T_m(M))$ - the tangent bundle to $T_m(M)$ - such that $Y(0) = 0$ and $Y''(0) = 0$ (differentiation is possible here since $T_m(M)$ is a linear space).

For every $Z_o \in T_o(T_m(M))$ there is a unique linear homogeneous vector field $Y(t)$ such that $Y'(0) = Z_o$. From the above construction of S we have:

If $X(t)$ is a Jacobi field along $\gamma(t) = \exp_m \circ \rho$ which vanishes at m, and Y is the linear homogeneous vector field along ρ such that

$$\frac{\nabla X}{dt}(0) = (\exp_m)_* \frac{\nabla Y}{dt}(0) \qquad \text{then}$$

$$X(t) = (\exp_m)_* Y(t) . \tag{7'}$$

3.3 Some Properties of Jacobi Fields

Let $\gamma(t)$ be an arbitrary geodesic in a riemannian manifold M; then there are two Jacobi fields admitted in a natural way by $\gamma(t)$: one is given by $\dot{\gamma}(t)$ and the other is $t\,\dot{\gamma}(t)$. This latter we denote by $\hat{\gamma}(t)$. It is trivial to verify that $\dot{\gamma}(t)$ and $\hat{\gamma}(t)$ satisfy Jacobi's equation.

Proposition (2) Every Jacobi field X along a geodesic γ in a riemannian manifold can be uniquely decomposed in the following form:

$$X(t) = a\,\dot{\gamma}(t) + b\,\hat{\gamma}(t) + Y(t)$$

where a, b are real numbers and $Y(t)$ is a Jacobi field along $\gamma(t)$ everywhere orhtogonal to $\gamma(t)$.

Proof Let g be the riemannian metric on M and assume that $\gamma(t)$ is parameterized by its arc length. Set

$$a = g(\dot{\gamma}(0), X(0))$$

$$b = g(\dot{\gamma}(0), \frac{\nabla X}{dt}(0))$$

$$Y = X - a\dot{\gamma} - b\hat{\gamma} \quad .$$

Since X, $\dot{\gamma}$, $\hat{\gamma}$ satisfy the Jacobi equations so does Y. From the Jacobi equations for Y we get:

$$g\left(\frac{\nabla^2}{dt^2} Y, \dot{\gamma}\right) + g(R(Y, \dot{\gamma})\dot{\gamma}, \dot{\gamma}) = 0$$

The 2nd term vanishes by the skew-symmetry of $R(Y, \dot{\gamma})$, hence the first term also vanishes. From $\frac{\nabla}{dt}\dot{\gamma} = 0$ and $\nabla g = 0$ we get

$$\frac{d^2}{dt^2} g(Y,\dot{\gamma}) = \frac{\nabla^2}{dt^2} g(Y,\dot{\gamma}) = \frac{\nabla}{dt} g(\frac{\nabla Y}{dt}, \dot{\gamma}) = g(\frac{\nabla^2}{dt^2} Y, \dot{\gamma}) = 0.$$

Thus

$$g(Y, \dot{\gamma}) = At + B, \qquad A,B \text{ are some constants.}$$

Since $\hat{\gamma}(0) = 0$, we have

$$B = g(Y(0), \dot{\gamma}(0)) = g(X(0), \dot{\gamma}(0)) - ag(\dot{\gamma}(0), \dot{\gamma}(0))$$

$$= a - a = 0.$$

Now $\frac{\nabla Y}{dt} = \frac{\nabla X}{dt} - b \frac{\nabla}{dt} \hat{\gamma}(t) = \frac{\nabla X}{dt} - b\, \dot{\gamma}(t)$ as $\gamma(t)$ is a geodesic.

Hence

$$A = \frac{d}{dt} g(Y, \dot{\gamma})|_{t=0} = g\left(\frac{\nabla X}{dt}(0), \dot{\gamma}(0)\right) - bg(\dot{\gamma}(0), \dot{\gamma}(0))$$

$$= b - b = 0.$$

Thus $g(Y, \dot{\gamma}) = 0$ and Y is orthogonal to γ everywhere.

To prove uniqueness let

$$X(t) = a_1\dot{\gamma} + b_1\hat{\gamma} + Z$$

be another decomposition of X such that Z is orthogonal to γ .

For each t we have

$$X(t) = (a + bt)\, \dot{\gamma}(t) + Y(t) = (a_1 + b_1 t)\, \dot{\gamma}(t) + Z(t).$$

Since both Y(t) and Z(t) are orthogonal to $\gamma(t)$ we have

$$a + bt = a_1 + b_1 t, \qquad Y(t) = Z(t)$$

and hence $a \equiv a_1$, $b \equiv b_1$, and $Y \equiv Z$. q.e.d.

Corollary (1) If X(t) is orthogonal to $\dot\gamma(t)$ at two points then it is orthogonal to it at all points of γ .

Proof Suppose X(t) orthogonal to $\dot\gamma(t)$ at t = s, and t = r, $r \neq s$. Write

$$X(t) = a\dot\gamma(t) + b\hat\gamma(t) + Y(t)$$

with a,b and Y as above. Then at t = r,

$$0 = g(X(r), \dot\gamma(r)) = (a + br)\, g(\dot\gamma(r), \dot\gamma(r)) = a + br$$

similarly $a + bs = 0$ and since $r \neq s$ we must have $a = b = 0$. q.e.d.

Corollary (2) If $X(t) \in \epsilon_0(\gamma)$ is orthogonal to $\dot\gamma(t)$ at any one point $\gamma(s)$, $s \neq 0$, then X(t) is orthogonal to $\dot\gamma(t)$ at all points of $\gamma(t)$.

Proof For such a field the scalar a occurring in the decomposition is necessarily zero. Also orthogonality of X(t) to $\dot\gamma(t)$ at t = s implies that

$$a + bs = 0, \qquad \text{hence} \qquad a = b = 0. \qquad \text{q.e.d.}$$

Since the behaviour of the $\dot\gamma$-component is known it is sufficient to study those Jacobi fields which are orthogonal to $\dot\gamma$.

<u>Proposition (3)</u> For $X, Y \in \epsilon(\gamma)$,

$$g\left(X, \frac{\nabla Y}{dt} \right) - g\left(\frac{\nabla X}{dt}, Y \right) = \text{constant} \tag{8}$$

<u>Proof</u>

$$\frac{d}{dt} g\left(X, \frac{\nabla Y}{dt} \right) = g\left(\frac{\nabla X}{dt}, \frac{\nabla Y}{dt} \right) + g\left(X, \frac{\nabla^2 Y}{dt^2} \right)$$

$$= g\left(\frac{\nabla X}{dt}, \frac{\nabla Y}{dt} \right) - g(X, R(Y, \dot\gamma)\dot\gamma) \quad \text{by Jacobi}$$

equations for Y. Similarly

$$\frac{d}{dt} g\left(\frac{\nabla X}{dt}, Y \right) = g\left(\frac{\nabla X}{dt}, \frac{\nabla Y}{dt} \right) - g(Y, R(X, \dot\gamma)\dot\gamma)$$

and since

$$g(X, R(Y,\dot\gamma)\dot\gamma) = g(Y, R(X, \dot\gamma)\dot\gamma), \qquad \text{we get}$$

$$\frac{d}{dt}\left(g\left(X, \frac{\nabla Y}{dt} \right) - g\left(\frac{\nabla X}{dt}, Y \right) \right) = 0$$

and hence (8). q.e.d.

<u>Corollary (1)</u> For $X, Y \in \epsilon_{t_o}(\gamma)$

$$g\left(X, \frac{\nabla Y}{dt} \right) = g\left(\frac{\nabla X}{dt}, Y \right) \tag{9}$$

3.4 <u>Conjugate Points</u>

<u>Definition (4)</u> Let $\gamma(t)$ be a geodesic in M. Two points $\gamma(t_o)$ and $\gamma(t_1)$ are said to be conjugate to each other along γ if there

exists a non-zero Jacobi field $X(t)$ which vanishes at both points, i.e. if the subspace

$$\epsilon_{t_o,t_1}(\gamma) \equiv \epsilon_{t_o}(\gamma) \cap \epsilon_{t_1}(\gamma) \subset \epsilon(\gamma)$$

is not empty.

Definition (5) The index $\lambda(\gamma;\ t_o,t_1)$ of the pair $\gamma(t_o)$ and $\gamma(t_1)$ as conjugate points is the dimension of the subspace $\epsilon_{t_o,t_1}(\gamma)$.

We now give aninterpretation of conjugate points directly in terms of the map $\exp_m : T_m(M) \to M$.

Definition (6) A point $u \in T_m(M)$ is called a conjugate point of m in $T_m(M)$ if $\exp_m$ is singular at u (i.e. if the Jacobian matrix of $\exp_m$ is singular at u).

Proposition (4) If $u \in T_m(M)$ is a conjugate point of m in $T_m(M)$, then $n = \exp_m u \in M$ is a conjugate point of m along the geodesic $\gamma(t) = \exp_m tu$, $0 \leq t \leq 1$. Conversely every conjugate point of m can be thus obtained.

Proof If X is conjugate to m in $T_m(M)$ then $(\exp_m)_*$ is singular at u. Let $Y \in T_u(T_m(M))$, $Y \neq 0$ be a vector such that $(\exp_m)_{*u} Y = 0$. Let $\bar{Y}$ be the associated constant vector field on $T_m(M)$. [$\bar{Y}$ is defined as follows:

$$v \in T_m(M) \to \bar{Y}_v = \eta_v(Y), \qquad \text{where}$$

$$\eta_v : T_m(M) \to T_v(T_m(M)) \qquad \text{is the natural}$$

identification of the two tangent spaces. If $e_1, \ldots, e_n$ is a basis of $T_m(M)$ and $w_1, \ldots, w_n$ its dual basis and if $Y = \Sigma\, a_i e_i$ then $\bar{Y} = \Sigma\, a_i (\frac{\partial}{\partial w_i})$].

Then the field $(\exp_m)_* t\bar{Y}$ is a non-trivial Jacobi field along $\gamma(t)$ which vanishes at $m(t = 0)$ and $n = \exp_m u$ $(t = 1)$. Conversely let Z be a non-trivial Jacobi field along $\gamma(t) = \exp_m tu$ such that $Z(0) = Z(1) = 0$. Let $A = \nabla_u Z \in T_m(M)$ and let $\bar{A}$ be the associated constant field on $T_m(M)$. Let

$$\bar{Z}(t) = (\exp_m)_* t\bar{A}, \qquad \text{then}$$

$$\nabla_u \bar{Z} = \nabla_u [t(\exp_m)_* \bar{A}]$$

$$= (\exp_m)_* \bar{A} + t\nabla_u [(\exp_m)_* \bar{A}] . \quad \text{At } t = 0,$$

$$\nabla_u \bar{Z}(0) = (\exp_m)_* \bar{A} = A = \nabla_u Z, \quad \text{since at } 0$$

$(\exp_m)_* \bar{A}_0 = A$. Thus by the uniqueness of Jacobi fields $Z = \bar{Z}$; hence

$\bar{Z}(1) = (\exp_m)_* \bar{A}_u = 0$. Since Z is non trivial $\bar{A} \neq 0$ and thus $\exp_m$ is singular at u.

Proposition (4) establishes the equivalence of the two definitions of conjugate points. By virtue of it we have an equivalent definition for the index of a conjugate point in the tangent space.

<u>Definition (7)</u> The index of $u \in T_m(M)$ as conjugate to m in $T_m(M)$

is equal to the dimension of the space:

$$\theta_u \equiv \ker(\exp_m)_* \text{ at } u.$$

Going back to equation (7') we see that if $u \in T_m(M)$, where $m = \gamma(0)$ then every $Z_u \in T_u(T_m(M))$ determines a Jacobi field $X(t) \in \epsilon_o(\gamma_u(t))$. The linear mapping

$J : Z_u \rightarrow X(t)$ is easily seen to be bijective and maps θ_u onto $\epsilon_{o,1}(\gamma_u(t))$.

Define $$\bar{T}^1(u) = \left\{ \frac{\nabla X}{dt}(0), \quad X \in \epsilon_{o,1}(\gamma_u(t)) \right\} \tag{10}$$

and let $T^1(u)$ be the space generated by $\bar{T}^1(u)$ and u. Let $\bar{T}^2(u)$ [resp. $T^2(u)$] denote the orthogonal compliment of $T^1(u)$ (resp. $\bar{T}^1(u)$). Then we have (Cf Allamigeon [1])

$$\bar{T}^2(u) = \{X(0), \; X \in \epsilon_1(\gamma_u(t)) \text{ and } g(X, \dot{\gamma}) = 0\} \tag{11}$$

and $$T^2(u) = \{X(0), \; X \in \epsilon_1(\gamma_u(t))\} \tag{12}$$

To demonstrate (12) let $X \in \epsilon_1(\gamma_u(t))$ and $Y \in \epsilon_{o,1}(\gamma_u(t))$ then

$$g\left(\frac{\nabla Y}{dt}(0), X(0)\right) = \left(\frac{\nabla X}{dt}(0), Y(0)\right) = 0$$

but $\dim(\{X(0), X \in \epsilon_1(\gamma_u(t))\}) = \dim(\epsilon_1(\gamma_u(t))) - \dim(\epsilon_{o,1}(\gamma_u(t)))$

$$= n - \dim(T^1(u)) = \dim(T^2(u)).$$

Similarly for (11).

The value of the concept of conjugate points is conditioned by the following assertion. (Cf Hicks p. 147).

Proposition (5) If two points $\gamma(t_o)$ and $\gamma(t_1)$ are not conjugate along $\gamma(t)$, then for arbitrary vectors

$A \in T_{\gamma(t_o)}(M)$, $B \in T_{\gamma(t_1)}(M)$ there exists a unique Jacobi field $X(t) \in \epsilon(\gamma)$ such that

$$X(t_o) = A \text{ and } X(t_1) = B.$$

It is well known (Cf for instance Milnor [1], p. 82) that if $\gamma(t)$ is a geodesic in M then there is a neighbourhood $(-\delta, \delta)$ of 0 such that if $t \in (-\delta, \delta)$ then $\gamma(t)$ is not conjugate to $\gamma(0)$ along γ. Moreover the set of points conjugate to $\gamma(0)$ along the entire geodesic γ has no cluster points.

Now let $u \in T_m(M)$ and let $\gamma(t) = \exp_m tu$, $0 \leq t < \infty$ be a half geodesic issuing from m. Since $\exp_m$ is non-singular at the origin of $T_m(M)$, there is a positive number, say a, such that there is no conjugate point of m on $\gamma(t)$, for $0 \leq t \leq a$. If there are conjugate points of m on $\gamma(t)$, let

$S = \{r > 0,\ \gamma(r)$ is a conjugate point of m along $\gamma(t)$, $0 \leq t \leq r\}$. Let $s = \inf S$. Since $\exp_m$ is singular at ru, it is singular at su, i.e. $\gamma(s)$ is a conjugate point of m. We call $\gamma(s)$ the first conjugate point of m along γ and denote by

$L_s(\gamma)$ the distance of $\gamma(s)$ from $m = \gamma(0)$. The totality of all such points $\gamma(s)$ along all half geodesics issuing from m is called the first conjugate locus or the residual locus R_m of m; its complement CR_m is a maximal normal neighbourhood of m. If $m' \in CR_m$ then there exists only one minimal geodesic joining m to m'. The significance of the residual locus stems from the fact that:

Proposition (6) A geodesic γ issuing from m does not minimize distance from m beyond the first point conjugate to m.

A proof of this proposition (Cf Ambrose [1]) depends on considerations of the first and second variations of arc length. Let $c : [a,b] \to M$ be any piecewise smooth curve in M then the arc length of c is

$$L(c) = \int_a^b \left\|\frac{dc}{dt}\right\| dt.$$

We define the distance d on $M \times M \to \mathbb{R} \cup \{+\infty\}$ by

$$d(a,b) = \inf_{c \in \Gamma(a,b)} L(c), \quad \text{where } \Gamma(a,b) \text{ is the set of all}$$

piecewise C^∞ curves from $m = \gamma(a)$ to $n = \gamma(b)$. A geodesic segment $\gamma(t)$ $a \leq t \leq b$ is minimal if

$$L(\gamma) = d(\gamma(a), \gamma(b)).$$

Closely related to the concept of conjugate point is that of a cut point. Let $\gamma(t)$, $0 \leqslant t < \infty$ be a half geodesic issuing from $m = \gamma(0)$. Let A be the set of positive numbers s such that the segment of $\gamma(t)$ from m to $\gamma(s)$ is minimal. If $s \in A$ and $t < s$ then then $t \in A$; and if r is a positive number such that every positive number $s < r$ lies in A then $r \in A$. Thus A is either of the form $A = (0, \infty)$ and we say that there is no cut point of m along γ, or else $A = (0,r]$ for some $r > 0$ and we call $\gamma(r)$ the cut point of m along γ. The totallity of cut points to m along all half geodesics issuing from m is called the cut locus $C(m)$ of m. It is immediate from the definition that the first conjugate point along a geodesic always occurs before or simultaneously with the cut point. There are instances however where the first conjugate locus coincides with the cut locus as in the case of simply connected riemannian symmetric spaces, (Crittenden [1]).

If $m' = \gamma(r)$ is the cut point of $m = \gamma(0)$ along $\gamma(t)$ then either m' is the first point conjugate to m along γ or there exist at least two minimizing geodesics joining m to m'. (Kobayashi and Nomizu II p. 96).

Corresponding to the cut point m' of m along $\gamma(t)$ we consider the vector $X \in T_m(M)$ such that

$$\gamma(t) = \exp_m tX \quad \text{and} \quad m' = \exp_m X ;$$

we call X the cut point of m in $T_m(M)$ corresponding to m'.

3.5 Jacobi fields and Curvature

By considering the Jacobi equations it is natural to expect a close relation between the curvature of M and the distribution of conjugate points on M, if there exists any. In the case of manifolds with non-positive sectional curvature we have:

Proposition (7) Let M be a complete riemannian manifold with non positive sectional curvature and p any point in M. Then M contains no points conjugate to p.

Proof Let γ be a geodesic issuing from p and X a Jacobi field along γ, then

$$\frac{\nabla^2}{dt} X + R(X, \dot{\gamma})\dot{\gamma} = 0; \qquad \text{so that}$$

$$g\left(\frac{\nabla^2}{dt} X, X\right) = -g(R(X, \dot{\gamma})\dot{\gamma}, X) \geqslant 0.$$

Therefore

$$\frac{d}{dt} g\left(\frac{\nabla X}{dt}, X\right) = g\left(\frac{\nabla^2 X}{dt}, X\right) + g\left(\frac{\nabla X}{dt}, \frac{\nabla X}{dt}\right) \geqslant 0.$$

Thus the function $g\left(\frac{\nabla X}{dt}, X\right)$ is monotone increasing, and strictly so if $\frac{\nabla X}{dt} \neq 0$. If X vanishes both at p and $q = \gamma(r)$, $r > 0$, say, then the function $g\left(\frac{\nabla X}{dt}, X\right)$ also vanishes at p and q and hence must vanish identically throughout the interval $[0,r]$. This implies that $X(0) = \frac{\nabla X}{dt} = 0$ so that X is identically zero. q.e.d.

Theorem (2) (Cartan-Hadamard) Let M be a complete riemannian manifold with non-positive sectional curvature and let $p \in M$. Then the pair $(T_o(M), \exp_p)$ is a covering manifold of M. In particular if M is simply connected then M is diffeomorphic to $\mathbb{R}^n$.

Proof It follows from proposition (7) that M has no conjugate points and hence for each $p \in M$ the mapping $\exp_p : T_p(M) \to M$ is regular. Let us furnish $T_p(M)$ with the riemannian structure induced from M via $\exp_p$. With this structure $T_p(M)$ is complete - the geodesics through the origin in $T_p(M)$ being straight lines. The theorem then follows from the following lemma due to Ambrose ([2], p. 360).

Lemma Let V and W be two riemannian manifolds, V complete and Φ a differentiable mapping of V onto W. If $d\Phi_v$ is an isometry for each $v \in V$. Then (V, Φ) is a covering space of W. q.e.d.

Manifolds with strictly positive curvature form a subject of considerable interest. By the study of conjugate points and the cut locus on them some insight is gained into their topology as exemplified in the work of Myers, Klingenberg, Rauch and Berger. Of the earliest results is the following theorem originally due to Bonnet who proved it for surfaces:

Theorem (3) (Bonnet) Let M be a riemannian manifold and $K(\Sigma)$ the sectional curvature of a plain section Σ tangent to a geodesic $\gamma(t)$ in M. If

$$0 < k_0 \leqslant K(\Sigma) \leqslant k_1$$

for all such planes Σ, where k_0 and k_1 are positive constants, then the distance d along $\gamma(t)$ of any two consecutive conjugate points satisfies:

$$\pi/\sqrt{k_1} \leqslant d \leqslant \pi/\sqrt{k_0} \ .$$

Bonnet theorem follows immediately from the comparison theorem of Rauch (Rauch [1]) which relates the length of Jacobi fields on manifolds to their respective sectional curvatures.

Theorem (4) (Rauch) Let M and N be n-dimensional riemannian manifolds with metrics g and h respectively. Let $\sigma(t)$ (resp. $\tau(t)$), $a \leqslant t \leqslant b$ be a geodesic in M (resp. N) and X (resp. Y) a non-zero Jacobi field along σ (resp. τ) and orthogonal to it, which vanishes at $t = a$. Assume further:

(1) $\left\| \frac{\nabla X}{dt}(a) \right\| = \left\| \frac{\nabla Y}{dt}(a) \right\|$,

(2) $\sigma(a)$ (resp. $\tau(a)$) has no conjugate point on $\sigma(t)$ (resp. $\tau(t)$) for $a \leqslant t \leqslant b$,

(3) for each $t \in [a,b]$ if Σ is a plane in the tangent space at $\sigma(t)$ and π a plane in the tangent space at $\tau(t)$, then

$$K_M(\Sigma) \geqslant K_N(\pi) ,$$

where $K_M(\Sigma)$ and $K_N(\pi)$ are the sectional curvatures for Σ in M and π in N resp.

Then we have

$$g(X(t), X(t)) \leq h(Y(t), Y(t)) \quad \text{for every} \quad t \in [a,b] .$$

Theorem (3) then follows from theorem (4) if we take M to be the sphere of radius k_1 and N to be M of theorem (3), and if we then take N to be the sphere of radius k_0.

If we further assume that M is a connected complete riemannian manifold then Bonnet theorem immediately implies that M is a compact manifold whose diameter is at most $\pi/\sqrt{k_0}$. (Myers [1]).

3.6 Jacobi fields and globally harmonic manifolds

The technique of Jacobi fields was successfully used by Allamigeon ([1], [2]) in the study of harmonic manifolds. This enabled him to obtain in a harmonic manifold M certain fibrations of geodesic spheres in $T_m(M)$ by spheres S_λ and then show that λ must be equal to 1, 3 or 7, by virtue of the following:

Theorem If the sphere S^n is differentiably fibred by compact connected fibres F then F is homeomorphic to S^1, S^3 or S^7, the proof of which rests on Adam's work on maps with Hopf invariant one.

The results of Allamigeon depend upon a global definition of harmonic manifolds (not necessarily riemannian) which we give below. Let M be an affinely connected manifold on which is defined a volume element τ - an n-form on M invariant under parallelism. Let $p \in M$ and $V(p)$ a simple convex neighbourhood of p, $V(p) = \exp_p(V_o)$ where V_o is some neighbourhood of the zero vector in M. Let $q = \exp_p u$, $u \in V_o$. The n^{th} exterior power of the dual map $\exp_p{}^*$ pulls the volume element τ_q to an n-form at p. Moreover this n-form at p must be a scalar multiple of the volume element at p. Thus we have defined on V_o a real valued function R_p by

$$\exp_p{}^*(\tau_q) \;=\; \tau_p \times R_p(u) \quad . \tag{13}$$

However, there is another real-valued function defined on V_o, namely

$$L : V_o \to \mathbb{R}$$
$$u \to \tfrac{1}{2}\,||u||^2 .$$

Suppose that R_p factors by L through the reals, i.e. the following diagram commutes:

$$\begin{tikzcd} V_o \arrow[rr, "R_p"] \arrow[dr, "L"'] & & \mathbb{R} \\ & \mathbb{R} \arrow[ur, "f"'] & \end{tikzcd}$$

then the manifold is said to be locally harmonic at p.

Suppose now that M is complete, then $\exp_p$ is defined on the whole of $T_p(M)$ and is surjective. If the diagram

$$\begin{tikzcd} V_o \arrow[rr, "R"] \arrow[dr, "L"'] & & \mathbb{R} \\ & \mathbb{R} \arrow[ur, "f"'] & \end{tikzcd}$$

commutes, then M is globally harmonic at p.

The relation between R and Ruse's ρ is given by

$$\rho(p,q) = R_p(u) \tag{14}$$

where $q = \exp_p u$.

Now let M be a riemannian manifold and γ a geodesic issuing from $m \in M$ with initial tangent vector u. Let g_t be the oriented segment such that for every $s \in [0,1]$ $g_t(s) = \gamma(st)$.

Let $X_i(t)$ $1 \leqslant i \leqslant n$ be a basis for $\epsilon_o(\gamma(t))$. Thus

$$X_i(t) = (\exp_m)_* Y_i(t) \qquad \text{where}$$

$$Y_i(t) = t\frac{\nabla X_i}{dt}(0)$$

where as usual we identify $T_m(M)$ with $T_o(T_m(M))$. If τ is the element of volume on M then

$$\tau_{\gamma(t)}(X_1(t),\dots,X_n(t)) = \tau_{\gamma(t)}((\exp_m)_* t\frac{\nabla X_1}{dt}(0),\dots,(\exp_m)_* t\frac{\nabla X_n}{dt}(0))$$

$$= t^n((\exp_m)^* \tau_{\gamma(t)})(\frac{\nabla X_1}{dt}(0),\dots,\frac{\nabla X_n}{dt}(0))$$

$$= t^n R(u)\, \tau_m(\frac{\nabla X_1}{dt}(0),\dots,\frac{\nabla X_n}{dt}(0))$$

by virtue of (13),

$$= t^n \rho(g_t)\, \tau_m(\frac{\nabla X_1}{dt}(0),\dots,\frac{\nabla X_n}{dt}(0)) \tag{15}$$

Theorem (8) (Allamigeon) Let M be a complete riemannian manifold globally harmonic at m. Then there exists a real number $L \geqslant 0$ (possibly infinite) and an integer $\lambda \geqslant 0$ such that the first point conjugate to m along any half geodesic issuing from m is at the same distance L and have the same index λ .

Proof Let $X_i(t)$ and $g(t)$ be as in (15).
Since M is harmonic at m then for any geodesic segment $\gamma(t)$ issuing from m we have Ruse's invariant satisfying

$$\rho(m, \gamma(t)) = \rho(g_t) = f(L(g_t))$$

where f is an arbitrary function. We may assume that the geodesic is parameterized by its arc length so that

$$\rho(g_t) = f(t)$$

Thus (15) becomes:

$$\tau_{\gamma(t)}(X_1(t),\ldots,X_n(t)) = t^n f(t)\tau_{\gamma(0)}\left(\frac{\nabla X_1}{dt}(0),\ldots,\frac{\nabla X_n}{dt}(0)\right) \tag{16}$$

The left hand side is zero iff $t = 0$ or $\gamma(t)$ is conjugate to $\gamma(0)$. Consequently, $L_s(\gamma)$ is equal to the smallest positive zero of the function $f(t)$ or to $+\infty$ if $f(t)$ has no zeros. In the latter case the manifold is free from conjugate points and the result trivially follows. So let us consider the first case and we take L to be the smallest positive zero of f.

Define

$$\bar{X}_i(t) = \tau_{t,L} X_i(t) \in T_{\gamma(L)}(M)$$

where $\tau_{t,L}$ denotes parallel transport along $\gamma|[t,L]$. Choose the $\{X_i\}$ so that $(X_1,\ldots,X_\lambda)$ form a basis for $\epsilon_{o,L}(\gamma)$, then

$$\bar{X}_i(t) = (t - L)\frac{\nabla X_i}{dt}(L) + o(t - L) \qquad (1 \leq i \leq \lambda)$$

$$\bar{X}_i(t) = X_i(L) + o(1) \qquad (\lambda < i \leq n).$$

Hence

$$\tau_{\gamma(t)}(X_1(t),\ldots,X_n(t)) = \tau_{\gamma(L)}(\bar{X}_1(t),\ldots,\bar{X}_n(t))$$

$$= (t-L)^\lambda \times \tau_{\gamma(L)}\left(\frac{\nabla X_1}{dt}(L),\ldots,\frac{\nabla X_\lambda}{dt}(L),X_{\lambda+1}(L),\ldots,X_n(L)\right)$$

$$+ o((t - L)^\lambda).$$

This implies that

$$f(t) = A \times (t - L)^\lambda + o((t - L)^\lambda),$$

where

$$A = \frac{\tau_{\gamma(L)}\left(\frac{\nabla X_1}{dt}(L),\ldots,\frac{\nabla X_\lambda}{dt}(L), X_{\lambda+1}(L),\ldots,X_n(L)\right)}{L^n \times \tau_{\gamma(0)}\left(\frac{\nabla X_1}{dt}(0),\ldots,\frac{\nabla X_n}{dt}(0)\right)}$$

In order that λ does not depend on γ it is sufficient to show that A is finite and $\neq 0$. But

(a) $\tau_{\gamma(0)}\left(\frac{\nabla X_1}{dt}(0),\ldots,\frac{\nabla X_n}{dt}(0)\right) \neq 0$ because the $\frac{\nabla X_i}{dt}(0)$ are linearly independent, and $X_i(0) = 0$.

(b) $\tau_{\gamma(L)}\left(\frac{\nabla X_1}{dt}(L),\ldots,\frac{\nabla X_\lambda}{dt}(L), X_{\lambda+1}(L),\ldots,X_n(L)\right) \neq 0$ (*)

because

(1) $X_{\lambda+1}(L),\ldots,X_n(L)$ are non-zero independent vectors, being part of the transported basis of $T_{\gamma(t)}(M)$.

(2) $\frac{\nabla X_1}{dt}(L),\ldots,\frac{\nabla X_\lambda}{dt}(L)$ are also linearly independent for if $\sum_{i=1}^{\lambda} C_i \frac{\nabla X_i}{dt}(L) = 0$, let $W = \sum_{i=1}^{\lambda} C_i X_i$ be a Jacobi field which vanishes at L. Also $\frac{\nabla W}{dt}(L) = 0$ which would imply that $W \equiv 0$ by the uniqueness theorem for Jacobi fields. Thus $C_i = 0$ and $\frac{\nabla X_i}{dt}(L)$ $1 \leqslant i \leqslant \lambda$ are independent.

(3) for $1 \leqslant i \leqslant \lambda$ and $\lambda < j \leqslant n$

$$g\left(\frac{\nabla X_i}{dt}(L), X_j(L)\right) = 0 \; (**) .$$

From (8) :

$$g\left(\frac{\nabla X_i}{dt}, X_j\right) - g\left(X_i, \frac{\nabla X_j}{dt}\right) = \text{const}$$

along $\gamma(t)$. Since X_i and X_j both vanish at $\gamma(t)$ and X_i vanishes at $\gamma(L)$ we get (**).

Therefore the set $\frac{\nabla X_i}{dt}(L)$, $X_j(L)$, $1 \leq i \leq \lambda$, $\lambda < j \leq n$ form a basis for $T_{\gamma(L)}(M)$ and (*) follows, thus completing proof of the theorem. q.e.d.

Corollary (1) Under the same hypothesis of theorem (8) geodesics issuing from m are either:

(1) all without conjugate points to m, or

(2) all simply closed and with the same length 2L.

Proof (1) If f has no zeros then $L = L_s(\gamma) = +\infty$ and $\exp_m$ is everywhere of maximal rank.

(2) So let $\gamma(t)$ be a geodesic segment of length L with extremities $m = \gamma(0)$, $m' = \gamma(L)$. Since m and m' are conjugate, there exists another geodesic $\gamma'(t)$ with the same length and extremities. Assume that their tangent vectors coincide at m' so that γ' prolongs γ. Let $m_1 = \gamma(L/2)$ and $m_2 = \gamma(3L/2)$; then the geodesic arc $m_1\, m'\, m_2$ is of length L and hence realizes a minimum. Also the distance of m_1 to m_2 along the broken arc $m_1\, m\, m_2$ being equal to L implies that $m_1\, m\, m_2$ is an arc of a geodesic

and so the tangent vectors to γ and γ' also coincide at m. This proves the assertion. q.e.d.

<u>Corollary (2)</u> Under the same hypothesis M is either

(a) diffeomorphic to $\mathbb{R}^n$, or

(b) compact.

<u>Proof</u> (a) is a consequence of theorem (2) (Cartan-Hadamard), since $\exp_m : T_m(M) \to M$ is regular.

(b) any point m' could be joined to m by a geodesic arc of length $\leq L$, hence by Myer's result M is compact. q.e.d.

<u>Corollary (3)</u> The integral cohomology ring of M is that of a symmetric space of rank one.

This follows from Bott's results on manifolds all of whose geodesics are closed, Bott [1].

This restricts considerably the class of riemannian manifolds that admit harmonic metrics. It does not appear, however, sufficient to settle the conjecture (Lichnerowicz [2]) that such manifolds are necessarily locally symmetric. For manifolds of dimension up to 4 it is known (RWW [1], p. 142) that every locally harmonic manifold with positive definite metric is locally symmetric. For higher dimensions Avez [1] proved that every simply connected globally harmonic riemannian manifold is locally symmetric (hence globally symmetric) under the additional assumption of compactness.

So let M be a compact simply connected manifold of dimension n equipped with a positive definite riemannian metric g. Denote by Δ the laplacian (of g) on $C^{\infty}(M)$ [for the validity of Avez's theorem we need only consider C^2 functions on M]. We define for $C^{\infty}(M)$ a global scalar product by

$$< f, g > = \int_M f(x)\ g(x)\ \tau_x$$

where τ_x is the element of volume of M.

Theorem (9) (Avez [1]) Let M be a compact simply connected globally harmonic riemannian manifold; then M is locally symmetric.

Proof Since M is compact it follows from corollary (1) of theorem (8) (Allamigeon) that all geodesics issuing from a point $x_o \in M$ are closed and all have the same length 2L, and that all geodesics of arc length less than L are free from conjugate points. Hence we can define the symmetry s_{x_o} w.r.t. x_o globally over M. In fact if x belongs to the residual locus of x_o then $s_{x_o} x = x$ and if x does not belong to the residual locus of x_o then $s_{x_o} x$ is the symmetric point of x w.r.t. x_o on the geodesic $(x_o x)$.

Let f be a characteristic function of Δ , i.e.

$$(\Delta f)(x) = \lambda f(x) \tag{17}$$

for some scalar λ and all $x \in M$. Let $N(x,y)$ be the elementary solution of Δ, so that

$$\Delta_x N(x,y) = \delta_x(y) \, ,$$

where in the above equation y is regarded as a (fixed) pole, δ is the Dirac's delta function and Δ_x is the Laplacian when y is kept fixed.

Now since M is compact the operator Δ is self-adjoint w.r.t. the scalar product $<,>$ (Cf de Rham [1], p. 126). So we have

$$\begin{aligned} < N(x,y), (\Delta f)(x) > &= < \Delta_x N(x,y), f(x) > \\ &= \int_M \Delta_x N(x,y) \, f(x) \, \tau_x \\ &= \int_M \delta_x(y) \, f(x) \, \tau_x \\ &= f(y). \end{aligned} \tag{18}$$

(17) and (18) then imply that

$$\lambda < N(x,y), f(x) > = f(y) \quad . \tag{19}$$

In (18) put $x = s_y z$ and note that τ is invariant under the symmetries (Lichnerowicz [2]) i.e. $\tau_{s_y z} = \tau_z$ and note also that $N(x,y)$ depends only on the distance of x from y so,

$$N(s_y z, y) = N(z,y) \, .$$

Hence,

$$f(y) = \lambda \int_M N(s_y z, y)\, f(s_y z)\, \tau_{s_y z}$$

$$= \lambda \int_M N(z,y)\, f(s_y z)\, \tau_z ,$$

that is

$$\lambda < N(x,y),\ f(s_y x) > \ = \ f(y). \tag{20}$$

Let $h(y)$ be another characteristic function of Δ ,

$$(\Delta h)(y) = \mu h(y) ;$$

if $\mu \neq \lambda$ then $< f,h > \ = 0.$

Then we deduce from (20) that

$$\lambda \ll N(x,y),\ f(s_y x) >,\ h(y) > \ = \ < f(y),\ h(y) > \ = 0.$$

But in the neighbourhood of y, $N(x,y) = O(s^{2-n})$, where s is the distance from x to y. The integral $< N(x,y),\ f(s_y x) >$ is therefore absolutely convergent in y on M and Fubini's theorem gives:

$$\ll N(x,y), f(s_y x) >,\ h(y) > \ = \ \ll N(x,y), h(y) >, \qquad f(s_y x) >$$

Hence,

$$\ll N(x,y),\ h(y) >,\ f(s_y x) > \ = \ 0.$$

Noticing that $N(x,y) = N(y,x)$ and applying relation (19) to $h(y)$ we get

$$\mu < N(x,y),\ h(y) > \ = \ h(x),$$

and thus

$$< h(x),\ f(s_y x) > \ = \ 0.$$

The function $f(s_y x)$ is therefore orthogonal to the set E of eigenfunctions corresponding to eigenvalues distinct from λ. Because the metric is elliptic the set

$$E \oplus \{f, \Delta f = \lambda f\}$$

is dense in the space of functions defined over M. Therefore

$$f(s_y x) \in \{f, \Delta f = \lambda f\}, \qquad \text{and}$$

$$(\Delta f)(s_y x) = \lambda f(s_y x) .$$

We now define another metric $\bar{g}$ on M obtained from the harmonic metric g by:

$$\bar{g}(x) = g(s_y x) \qquad \text{for all } x.$$

We denote by $\bar{\Delta}$ the Laplacian corresponding to $\bar{g}$. This can be expressed as

$$(\bar{\Delta} f)(x) = \lambda f(x).$$

We therefore deduce from (17) that for all eigenfunctions

$$(\bar{\Delta} - \Delta)f = 0.$$

But, to every function F(x) of class C^2 we can make correspond a sequence of finite linear combinations of eigenfunctions of Δ which converge uniformly to F(x), [Kolomogroff as cited by Avez [1]), the same property being valid for the partial derivatives of order $\leqslant 2$.

Hence, for every function $F(x)$ of class C^2,

$$(\bar{\Delta} - \Delta) = 0 \quad . \tag{21}$$

Let x be any point of M and let (x^α) be a system of normal coordinates at x w.r.t. g. Let X be an arbitrary vector at x. If we take

$$F = \tfrac{1}{2}(X_\alpha x^\alpha)^2, \qquad \text{we get}$$

$$(\partial_\alpha F)(x) = 0, \quad (\partial_{\alpha\beta} F)(x) = X_\alpha X_\beta \, .$$

At the point x, (21) gives

$$\bar{g}(X, X) = g(X, X).$$

Since x and X are arbitrary, it follows that $\bar{g} = g$, i.e. $g(x) = g(s_y x)$ for every $y \in M$, and M is therefore locally symmetric.

CHAPTER IV

k-HARMONIC MANIFOLDS

4.1 Algebraic preliminaries

Let V be an n-dimensional vector space with a symmetric bilinear functional $G : V \times V \to \mathbb{R}$ on it. If $\{e_i\}$ is any basis, then G is completely determined by the values:

$$g_{ij} = G(e_i, e_j) \qquad 1 \leq i,j \leq n.$$

If we change the basis with a matrix $T = (t^i_j)$ such that

$$\bar{e} = Te$$

then the matrix $G = (g_{ij})$ is changed according to

$$TG\,{}^tT \ , \text{ where } {}^tT \text{ denotes transpose.}$$

If H is a second symmetric bilinear functional and

$$h_{ij} = H(e_i, e_j)$$

we may consider the determinant:

$$\det(\lambda g_{ij} - h_{ij}) = \sum_{k=0}^{n} (-1)^k \lambda^{n-k} S_k (g,h).$$

Note that under a change of basis the ratio of any two coefficients in this polynomial will be multiplied by $(\det T)^2$ and is thus an invariant independent on the choice of basis. If moreover g_{ij} is non-singular one then has

$$\frac{\det(\lambda g_{ij} - h_{ij})}{\det (g_{ij})} = \sum_{k=0}^{n} (-1)^k \lambda^{n-k} \sigma_k (g,h) \qquad (1)$$

where the coefficients

$$\sigma_k(g,h) = S_k(g,h)/\det(g_{ij})$$

are all invariants.

The equation $\det(\lambda g_{ij} - h_{ij}) = 0$ gives the characteristic polynomial of h_{ij} w.r.t. g_{ij} and its roots are the eigenvalues of h_{ij} w.r.t. g_{ij}. Apart from their order they are intrinsically associated with H and G.

The invariants occurring in (1) are explicitly the elementary symmetric polynomials of h_{ij} w.r.t. g_{ij} and are given by:

$$\sigma_1(g,h) = g^{ij} h_{ij}$$

$$\sigma_n(g,h) = \det(h_{ij})/\det(g_{ij})$$

and in general

$$\sigma_k(g,h) = \frac{1}{\det(g_{ij})} \sum_{i_1<\dots< i_k} \begin{vmatrix} g_{11} & \dots & g_{1n} \\ h_{i_1 1} & \dots & h_{i_1 n} \\ h_{i_k 1} & \dots & h_{i_k n} \\ g_{n1} & \dots & g_{nn} \end{vmatrix} .$$

We recall that for a square matrix A the k^{th} elementary symmetric polynomial of the eigenvalues of A is equal to the sum of all k-rowed principal minors of A - i.e. those minors whose diagonal is part of the diagonal of A.

Now given any two symmetric bilinear functionals of V, G and H with G non-degenerate there is a natural endomorphism

$$\Phi : V \to V$$

associated with G and H and given by

$$G(\Phi X, Y) = H(X,Y) \tag{2}$$

for all $X,Y \in V$. Φ, thus defined, is a symmetric endomorphism w.r.t. G. This follows from the symmetry of G and H:

$$G(\Phi X,Y) = H(X,Y) = H(Y,X) = G(\Phi Y,X) = G(X,\Phi Y).$$

If w.r.t. a basis $\{e_i\}$ X and Y are given by

$$X = X^i e_i, \quad Y = Y^i e_i \quad \text{and} \quad \Phi = (\phi^i_j)$$

then (2) becomes

$$g_{kj} \phi^k_i X^i Y^j = h_{ij} X^i Y^j, \qquad \text{for all } X,Y \in V,$$

where $\qquad h_{ij} = H(e_i, e_j), \; G_{ij} = G(e_i, e_j)$.

Since this is true for all X and Y we must have

$$h_{ij} = g_{kj} \phi^k_i \qquad \text{or explicitly}$$

$$\phi^i_j = g^{ik} h_{kj} . \tag{3}$$

If we form the characteristic polynmial of Φ,

$$\det(\lambda\delta^i_j - \phi^i_j) = \det(\lambda g^{ik} g_{kj} - g^{ik} h_{kj}) = \det(g^{ik}(\lambda g_{kj} - h_{kj}))$$

$$= \det(\lambda g_{kj} - h_{kj})/\det(g_{ij}) = \sum_{k=0}^{n} (-1)^k \lambda^{n-k} \sigma_k (g,h)$$

by (1); so that the symmetric polynomials of Φ are exactly those of H relative to G. In particular

$$\mathrm{trace}\,(\Phi) = \sigma_1(g,h) = g^{ik}\,h_{ki}$$

and $$\det(\Phi) = \sigma_n(g,h) = \det(h_{ij})/\det(g_{ij}).$$

4.2 Geometric preliminaries

Let M be a complete riemannian manifold with metric tensor g and let $p_o \in M$. Let u be a unit tangent vector at p_o and $\gamma_u(t)$ the geodesic issuing from p_o tangential to u,

$$\gamma_u(t) = \exp_{p_o} tu, \qquad\qquad 0 \leqslant t < \infty .$$

The differential of $\exp_{p_o}$ at tu is a linear map:

$$(\exp_{p_o})_{*tu} : T_{tu}\,(T_{p_o}(M)) \to T_{\exp_{p_o} tu}(M).$$

We identify $T_{tu}(T_{p_o}(M))$ with $T_{p_o}(M)$ by the usual procedure of parallel transport $(\tau_{o,tu})^{-1}$ along the curve $t \to tu$ in $T(M)$ (which is euclidean) from tu to the origin, then followed by the identification of $T_o(T_{p_o}(M))$ with $T_{p_o}(M)$ by the isomorphism η_o^{-1} which sends each vector in $T_o(T_{p_o}(M))$ into its end point in $T_{p_o}(M)$. Let us set

$$f_{tu} = (\exp_{p_o})_{*tu} \circ \tau_{o,t} \circ \eta_o \tag{4}$$

then f_{tu} is a linear map : $T_{p_o}(M) \rightarrow T_p(M)$ where $p = \exp_{p_o} tu$.

The riemannian structure on M defines an inner product g_p on each tangent space $T_p(M)$. Let us denote by h the pull-back of g_p to $T_{p_o}(M)$ via f_{tu} i.e.

$$h(X,Y) = (f^*_{tu}\, g_p)(X,Y)$$

$$= g_p(f_{tu*}X,\ f_{tu*}Y) \quad \text{for all } X,Y \in T_{p_o}(M).$$

In the manner of (2) we can define an endomorphism:

$$\phi_{tu} : T_{p_o}(M) \rightarrow T_{p_o}(M)\ , \qquad \text{by}$$

$$g_{p_o}(\phi_{tu}X,Y) = h(X,Y) \tag{5}$$

for all $X,Y \in T_{p_o}(M)$. We can also form the various elementary symmetric polynomials of ϕ_{tu} which are precisely those of h relative to g_{p_o} as given by (1).

In terms of local coordinates at p_o we have by (3)

$$(\phi_{tu})^i_j = g^{ik}_{p_o}\, h_{kj} \ . \tag{6}$$

If we have a system of normal coordinates (y^i) centred at p_o, then the exponential mapping is essentially represented by the identity transformation and h_{kj} is then equal to $(g_{ij})_p$ relative to y^i, i.e.

$$(\phi_{tu})^i_j = ({}^*g^{ik})_{p_o}\, ({}^*g_{kj})_p \tag{7}$$

where as before * denotes that components are w.r.t. a normal coordinate system. In particular

$$\det(\Phi_{tu}) = \det(*g_p)/\det(*g_{p_o})$$

and we thus see that det (Φ_{tu}) is non other than the square of Ruse's invariant $\rho(t,u)$ given in Chapter I.

4.3 k-harmonic manifolds

The concept of k-harmonic manifolds was introduced by T.J.Willmore [2] as a natural generalization of harmonic manifolds discussed earlier. We recall that one of the definitions for a manifold M to be harmonic at a point $p_o \in M$ is that there is a normal neighbourhood W of p_o on which Ruse's invariant ρ factors through the reals by the distance function Ω . This is equivalent to the commutativity of the diagram

$$\begin{tikzcd} T(W) \arrow[rr, "\rho"] \arrow[dr, "\Omega"'] & & \mathbb{R} \\ & \mathbb{R} \arrow[ur, "f"'] & \end{tikzcd}$$

where now we regard both ρ and Ω as being defined on $T(W)$, the restriction of the tangent bundle of M to W.

We may similarly regard the elementary symmetric polynomials of Φ_{tu} as real valued functions defined on $T(W)$. For each $(p_o, tu) \in T(W)$ there corresponds by (5) an endomorphism of the fibre through p_o

$$\Phi_{tu} : \pi^{-1}(p_o) \to \pi^{-1}(p_o) .$$

The observation that:

$$\rho^2(t,u) = \det(\phi_{tu}) = \sigma_n(\phi_{tu})$$

leads to the following:

Definition (1) A riemannian manifold is k-harmonic at p_o if $\sigma_k(\phi_{tu})$ is a function of Ω only, $\sigma_k(\phi_{tu}) = f(\Omega)$, i.e. if the following diagram commutes:

$$\begin{tikzcd} T(W) \arrow[rr, "\sigma_k(\phi_{tu})"] \arrow[dr, "\Omega"'] & & \mathbb{R} \\ & \mathbb{R} \arrow[ur, "f"'] & \end{tikzcd}$$

Otherwise said: $\sigma_k(\phi_{tu})$ is constant on geodesic spheres centred at p_o.

Note that with this definition the n-harmonic manifolds are non other than the usual harmonic manifolds.

Definition (2) A riemannian manifold is simply k-harmonic if $\sigma_k(\phi_{tu})$ is constant.

Willmore's original definition of k-harmonic manifolds (Willmore [2]) was given in a different but equivalent way: let p_o, p be any two points in M and let (x^α), (x^i) denote coordinate systems in a neighbourhood of p_o, p respectively. If $T(p_o)$ (resp $T(p)$) is the vector space of tensors at p_o (resp. p) we denote by

$$T(p_o, p) = T(p_o) \otimes T(p)$$

the space of bi-tensors over (p_o, p).

An example of a bi-tensor is

$$\Omega_{i\alpha} = \frac{\partial^2\Omega}{\partial x^\alpha \partial x^i} = \Omega_{\alpha i} \qquad .$$

We can raise or lower Greek suffixes by means of $(g^{\alpha\beta})$ and $(g_{\alpha\beta})$ at p_0 and similarly raise or lower Roman suffixes by means of g^{ij} and g_{ij} at p. Thus we have

$$\Omega^\alpha_i = g^{\alpha\gamma}\,\Omega_{\gamma i}$$

and
$$\Omega^i_\beta = g^{ij}\,\Omega_{\beta j} \qquad .$$

Consider then the pure bi-tensor:

$$\omega^\alpha_\beta = \Omega^\alpha_i\,\Omega^i_\beta$$

and similarly ω^i_j. Its determinant is given by

$$\det(\omega^\alpha_\gamma) = \det(g^{\alpha\beta}\,\Omega_{\beta i}\,g^{ij}\,\Omega_{j\gamma})$$

$$= (\det(\Omega_{i\alpha}))^2/[(\det g_{p_0})\det(g_p)]$$

$$= 1/\rho^2 \ .$$

The definition originally given by Willmore was in terms of the elementary symmetric polynomials of ω^i_j. We maintain that the two definitions are equivalent. In fact the linear transformation of $T_{p_0}(M) \to T_{p_0}(M)$ represented by $(\phi_{tu})^\alpha_\beta$ w.r.t. a coordinate system (x^α) is the inverse of the transformation represented by ω^α_β - and thus have reciprocal eigenvalues - as could be seen from the following sequence of maps:

$$T_{p_o}(M) \xrightarrow{t^i_\alpha} T_p(M) \xrightarrow{g_{ij}} {}^*T_p(M) \xrightarrow{t^i_\alpha} {}^*T_{p_o}(M) \xrightarrow{g^{\alpha\beta}} T_{p_o}(M) ,$$

given by:

$$\lambda^\alpha \to t^i_\alpha \lambda^\alpha \to g_{ij}\, t^i_\alpha \lambda^\alpha \to t^j_\beta\, g_{ij}\, t^i_\alpha \lambda^\alpha \to (g^{\beta\gamma}\, t^j_\beta\, g_{ij}\, t^i_\alpha)\lambda^\alpha$$

$$= a^\gamma_\alpha \lambda^\alpha , \tag{8}$$

where $\lambda^\alpha \in T_{p_o}(M)$ and t^i_α represents the linear transformation:

$$(\exp_{p_o})_{*tu} : T_{p_o}(M) \to T_{p_o}(M)$$

with the usual identification. It is obvious that the matrix (a^γ_α) in (8) is in fact ϕ^γ_α as given by (6).

We now assume that the local coordinates (x^i) are chosen so that they are normal centred at p_o. In this special system $t^i_\alpha = \delta^i_\alpha$. Moreover using equation (17) p. 17 of RWW we get in the present notation

$$y^\alpha = -g^{\alpha\beta}\,\Omega_\beta = -\Omega^\alpha$$

and $$\omega^\gamma_\alpha = \Omega^\gamma_i \Omega^i_\alpha = g^{ik} g_{\alpha\beta}\, \Omega^\beta_k\, \Omega^\gamma_i = g^{ik}\, g_{\alpha\beta}\, \delta^\beta_k\, \delta^\gamma_i$$

whereas $$\phi^\gamma_\alpha = g^{\beta\gamma}\, g_{ij}\, \delta^j_\beta\, \delta^i_\alpha$$

which proves our assertion.

The above analysis was given by Willmore to indicate the geometrical significance of ω^i_j and ω^α_β .

4.4 Some properties of k-harmonic manifolds

Theorem (1) (Willmore [2]) A two-point homogeneous riemannian manifold with positive definite metric is k-harmonic for all k.

Proof Let p and q be any two points lying on a geodesic sphere of radius r centred at $p_o \in M$. From the definition of two-point homogeneity it follows that there exists an isometry of M which carries p into q and leaves p_o fixed. In terms of normal coordinates centred at p_o we have by (7)

$$\phi^i_j = (*g^{ik})_{p_o} (*g_{kj})_p .$$

It follows that the matrix ϕ^i_j has the same eigenvalues at all points on the same geodesic sphere. Thus $\sigma_k(\phi)$ depends only upon Ω and the space is k-harmonic for all k. q.e.d.

Corollary The complex projective plane is a compact riemannian manifold, k-harmonic for all k but does not have constant curvature. Thus the apparently plausible conjecture that k-harmonic for all k implies constant curvature is false.

Theorem (2) A k-harmonic manifold is Einstein.

Proof Let (y^i) be a system of normal coordinates at p_o. The coordinates of a point p distant s from p_o are

$$y^i = sa^i ,$$

where a^i are components of the unit tangent vector at p_o to the geodesic arc $(p_o\ p)$,

$$(*g_{ij})_{p_o}\ a^i\ a^j\ =\ e\ .$$

Consider the Taylor expansion of the metric $*g_{ij}$ about p_o :

$$*g_{ij}(y)\ =\ *g_{ij}(0) + \left(\frac{\partial *g_{ij}}{\partial y^{m_1}}\right)_{p_o} y^{m_1} + \left(\frac{\partial^2\ *g_{ij}}{\partial y^{m_1}\ \partial y^{m_2}}\right)_{p_o} \frac{y^{m_1}\ y^{m_2}}{2!} + \ldots$$

where the coefficients are the metric normal tensors (Cf Veblen [1] p. 97). The first normal tensor

$$\left(\frac{\partial *g_{ij}}{\partial y^{m_1}}\right)_{p_o}\ =\ 0$$

since the Christoffel symbols at the pole of the normal coordinate system are zero. The second normal tensor satisfies:

$$\left(\frac{\partial^2\ *g_{ij}}{\partial y^{m_1}\ \partial y^{m_2}}\right)_{p_o}\ =\ \frac{1}{3}\ (R_{im_1jm_2} + R_{jm_1im_2}\ ) \qquad (9)$$

substituting $y^i\ = a^i s$ we get

$$*g_{ij}(a^i s)\ =\ *g_{ij}(0) + \frac{1}{6}(R_{im_1jm_2} + R_{jm_1im_2})\ a^{m_1}\ a^{m_2}\ s^2 + O(s^3).$$

Now

$$\phi^k_j\ =\ (*g^{ik})_{p_o}\ (*g_{ij})_p\ =$$

$$*g^{ik}(0)[*g_{ij}(0) + \frac{1}{6}\ (R_{im_1jm_2} + R_{jm_1im_2})\ a^{m_1}a^{m_2}\ s^2 + O(s^3)]$$

i.e. $$\phi^k_j = \delta^k_j + \frac{1}{6}s^2\ [R^k_{m_1jm_2} + R^k_{m_2jm_1}]\ a^{m_1}\ a^{m_2} + O(s^3) \qquad (11)$$

Also we have $\sigma_k(\phi)$ satisfies:

$$\det(\lambda\delta^k_j - \phi^k_j) = \sum_{k=0}^{n} (-1)^k \lambda^{n-k} \sigma_k(\phi).$$

So by (1) we have

$$\sum_{k=0}^{n} (-1)^k \lambda^{n-k}\sigma_k(\phi) = \det(x\delta^k_j - s^2 m^k_j)$$

where

$$x = (\lambda - 1), \qquad \text{and}$$

$$m^k_j = \frac{1}{6}(R^k_{m_1 j m_2} + R^k_{m_2 j m_1})\, a^{m_1} a^{m_2} + O(s)\ .$$

Now

$$\det(x\delta^k_j - s^2 m^k_j) = \det(s^2(\frac{x}{s^2}\delta^k_j - m^k_j)) =$$

$$s^{2n}\left[\left(\frac{x}{s^2}\right)^{n} + q_1\left(\frac{x}{s^2}\right)^{n-1} + q_2\left(\frac{x}{s^2}\right)^{n-2} + \dots + q_n\right]$$

where q_1 = sum of all i-valued minors of (m^k_j)

= $p_i + O(s)$, where

p_i = sum of all i-valued minors of

$$\bar{R}^k_j = \frac{1}{6}(R^k_{m_1 j m_2} + R^k_{m_2 j m_1})\, a^{m_1} a^{m_2} \quad .$$

Therefore

$$\sum_{k=0}^{n} (-1)^k \lambda^{n-k} \sigma_k(\phi) = x^n + q_1 s^2 x^{n-1} + q_2 s^4 x^{n-2} + \dots + q_n s^{2n} =$$

$$= x^n + (p_1 + 0(s))s^2x^{n-1} + (p_2 + 0(s))s^4x^{n-2} + \ldots + (p_n + 0(s))s^{2n}$$

$$= x^n + p_1 s^2 x^{n-1} + 0(s^3).$$

Comparing coefficients of λ^k on both sides:

$$(-1)^{n-k}\, \sigma_{n-k}(\phi) = (-1)^k\left[\binom{n}{k} + p_1 s^2 \binom{n-1}{k} \right] + 0(s^3)$$

$$= (-1)^k \binom{n-1}{k} \left[\frac{n}{n-k} + p_1 s^2 \right] + 0(s^3)$$

which yields,

$$\sigma_k(\phi) = (-1)^n \binom{n-1}{k-1} \left[\frac{n}{k} + p_1 s^2 \right] + 0(s^3)$$

where

$$p_1 = \frac{1}{6}\, \text{trace}\, (R^k_{m_1 j m_2} + R^k_{m_2 j m_1})\, a^{m_1}\, a^{m_2}$$

$$= \frac{1}{3} R_{m_1 m_2}\, a^{m_1}\, a^{m_2}$$

where $R_{m_1 m_2}$ is the Ricci tensor. So that

$$\sigma_k(\phi) = (-1)^n \binom{n-1}{k-1} \left[\frac{n}{k} + \frac{1}{3} R_{m_1 m_2}\, a^{m_1}\, a^{m_2}\, s^2 \right] + 0(s^3) \tag{12}$$

If the manifold is k-harmonic then

$$\sigma_k(\phi) = f(\Omega).$$

Using the Taylor expansion for $f(\Omega)$:

$$f(\Omega) = f(0) + \Omega\, f'(0) + \frac{\Omega^2}{2!}\, f''(0) + \ldots$$

and substituting

$$\Omega \;=\; \tfrac{1}{2} e\, s^2 , \qquad\qquad \text{we get}$$

$$f(\Omega) \;=\; f(0) + \tfrac{1}{2}\, e s^2 f'(0) + \frac{1}{4.2!}\, e^2\, s^4\, f''(0) + \ldots \tag{13}$$

(12) and (13) must be identical if M is to be k-harmonic. Hence comparing coefficients:

$$f(0) \;=\; (-1)^n \binom{n}{k}$$

and $\tfrac{1}{2} e\, f'(0) \;=\; (-1)^n \binom{n-1}{k-1} \tfrac{1}{3} R_{m_1 m_2}\, a^{m_1}\, a^{m_2}$ for all a^i.

Hence,

$$R_{m_1 m_2} \;=\; k\; g_{m_1 m_2} , \tag{14}$$

where $\qquad k_1 \;=\; (-1)^n \tfrac{3}{2}\, f'(0) \Big/ \binom{n-1}{k-1}$

which shows that M must be Einstein. q.e.d.

<u>Theorem (3) [Willmore]</u> An n-harmonic manifold is 1-harmonic.

<u>Proof</u> We first use the identity (9) of Chapter I,

$$\Omega^{\alpha}\, \Omega_{\alpha} \;=\; 2\,\Omega$$

which we differentiate covariantly w.r.t. x^i and raise the suffix to get

$$\Omega^{\alpha}\, \Omega^{i}_{\alpha} \;=\; \Omega^{i} .$$

We differentiate again w.r.t. x^j to get

$$\Omega^{\alpha}_{j}\,\Omega^{i}_{\alpha} + \Omega^{\alpha}\,\Omega^{i}_{\alpha,j} \;=\; \Omega^{i}_{j} \;, \qquad \text{i.e.}$$

$$\omega^{i}_{j} \;=\; \Omega^{i}_{,j} - \Omega^{\alpha}\,\Omega^{i}_{j,\alpha}$$

where in the last term we have interchanged the order of covariant differentiation w.r.t. x^j and x^α. Thus

$$\operatorname{trace}(\omega^{i}_{j}) \;=\; \Delta_2\Omega - \Omega^{\alpha}(\Delta_2\Omega)_{\alpha} \;. \tag{15}$$

Now if M is n-harmonic then

$$\Delta_2\Omega \;=\; \chi(\Omega) \;.$$

Hence

$$\operatorname{trace}(\omega^{i}_{j}) \;=\; \chi(\Omega) - \Omega^{\alpha}\,\chi'(\Omega)\Omega_{\alpha}$$

which is a function of Ω alone. q.e.d.

Corollary A simply n-harmonic riemannian manifold is simply 1-harmonic.

Proof If M is simply n-harmonic then $\Delta_2\Omega = n$ and hence by (15)

$$\operatorname{trace}(\omega^{i}_{j}) \;=\; n. \qquad \text{q.e.d.}$$

The converse to theorem (3) is by no means evident and we suspect it is false. However we have the following converse to the corollary:

Theorem (4) A simply 1-harmonic riemannian manifold is simply n-harmonic.

<u>Proof</u> If the space is simply 1-harmonic then

$$\text{trace } \omega^i_j = n.$$

(15) would then become

$$n = \Delta_2\Omega - \Omega^\alpha(\Delta_2\Omega)_\alpha .$$

If we use normal coordinates (y^α) centred at p_0 then

$$\Omega^\alpha = -y^\alpha ,$$ and we would have

$$n = \Delta_2\Omega + y^\alpha \frac{\partial}{\partial y^\alpha}(\Delta_2\Omega) .$$

Putting

$$f = \Delta_2\Omega - n$$

this could be written as

$$f + y^\alpha \frac{\partial}{\partial y^\alpha} f = 0 \tag{16}$$

and $f \to 0$ as $y^\alpha \to 0$ i.e. as $p \to p_0$.

To solve (16) we are going to apply the following theorem in linear partial differential equations (Cf Kells [1] p. 352).

<u>Theorem</u> If $u_i(x_1, \ldots, x_n, z) = c_i$, $i = 1, \ldots, n$ are independent solutions of the ordinary differential equations:

$$\frac{dx_1}{P_1(x_1,\ldots,x_n)} = \frac{dx_2}{P_2(x_1,\ldots,x_n)} = \cdots = \frac{dx_n}{P_n(x_1,\ldots,x_n)} = \frac{dz}{R(x_1,\ldots,x_n,z)}$$

then

$$\phi(u_1, \ldots, u_n) = 0$$

is the general solution of

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + \ldots + P_n \frac{\partial z}{\partial x_n} = R$$

where ϕ represents an arbitrary function.

So in our case we find solutions to the equations

$$\frac{dy^\alpha}{y^\alpha} = \frac{df}{-f} \qquad \alpha = 1,\ldots,n \tag{17}$$

of the form

$$u_\alpha(y^1, \ldots, y^n, f) = c_\alpha .$$

Then the general solution of (16) would be

$$\phi(u_1, \ldots, u_n) = 0 ,$$

where ϕ is an arbitrary function.

Rewrite (17) as:

$$\frac{dy^1}{y^1} = \frac{dy^\alpha}{y^\alpha} \qquad \alpha = 2,\ldots,n$$

$$\frac{dy^1}{y^1} = \frac{df}{-f} ,$$

solutions of which are

$$\frac{y^\alpha}{y^1} = c_\alpha \qquad \alpha = 2,\ldots,n \tag{18}$$

$$fy^1 = c_1$$

where the c's are arbitrary constants.

So the general solution of (16) is

$$\phi\left(\frac{y^2}{y^1}, \frac{y^3}{y^1}, \ldots, \frac{y^n}{y^1}\right) = 0. \tag{19}$$

This solved for fy^1 gives:

$$fy^1 = \psi\left(\frac{y^2}{y^1}, \frac{y^3}{y^1}, \ldots, \frac{y^n}{y^1}\right) \tag{19'}$$

for some arbitrary ψ .

Now for the point p with coordinates (y^α) the geodesic $(p_o\ p)$ is given by

$$y^\alpha = a^\alpha s, \qquad \text{where}$$

$$({}^*g_{\alpha\beta})_o\, a^\alpha a^\beta = e ,$$

hence along this geodesic we have

$$fa^1 s = \psi\left(\frac{a^2}{a^1}, \frac{a^3}{a^1}, \ldots, \frac{a^n}{a^1}\right) \tag{20}$$

is a constant. Hence taking the limit as $s \to 0$ we have

$$0 = \lim_{s\to 0} fa^1 s = \psi\left(\frac{a^2}{a^1}, \ldots, \frac{a^n}{a^1}\right)$$

This implies that $f \equiv 0$. Hence

$$\Delta_2 \Omega = n \quad \text{and the space is simply n-harmonic.}$$

Remark: that $f = 0$ really follows from (18) where we have fy^1 = constant. Letting $p \to p_o$, y^1 becomes zero and thus the constant must be zero. q.e.d.

4.5 Equations of Walker

Formal series of linear operators:

Let $\mathfrak{a}$ be an associative algebra with unit over the reals, and denote by $\mathfrak{a}([X])$ the algebra of formal power series in one indeterminate X with coefficients in $\mathfrak{a}$,

$$A[X] = \sum_{k=0}^{\infty} a_k X^k, \qquad a_k \in \mathfrak{a} .$$

If $F \in \mathbb{R}([X])$,

$$F[X] = \sum_{k=0}^{\infty} f_k X^k, \qquad f_k \in \mathbb{R}$$

we may write formally

$$F(A) = \sum_{k=0}^{\infty} f_k \times A^k .$$

If the constant term a_o of A is the unit I, we define

$$\log A = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (A - I)^k .$$

If A is invertible we write

$$A^{-1} = \sum_{k=0}^{\infty} (I - A)^k .$$

Finally we write DA for the derived series

$$DA[X] = \sum_{k=1}^{\infty} k\, a_k\, X^{k-1} .$$

Now suppose that $\mathfrak{a}$ is the algebra of linear endomorphisms of a finite dimensional vector space V and put

$$\text{trace } A[X] = \sum_{k=0}^{\infty} \text{trace } (a_k)\, X^k, \qquad \text{trace } A[X] \in \mathbb{R}([X]) .$$

If $a_0 = I$ let us write

$$A = I + U, \qquad \text{then}$$

$$D \log A = DU + \ldots + \frac{(-1)^{k-1}}{k}(U^{k-1}.DU + U^{k-2}.DU.U + \ldots + DU.U^{k-1}) + \ldots$$

Using the formula

$$\text{trace } (AB) = \text{trace } (BA), \qquad \text{we get}$$

$$\text{trace } (D \log A) = \text{trace}\left(\sum_{k=0}^{\infty} (-U)^k . DU \right)$$

$$= \text{trace } ((I + U)^{-1} . DU) .$$

Hence

$$\text{trace } (D \log A) = \text{trace } (A^{-1} . DA).$$

Now if $f : \mathbb{R} \to \mathfrak{a}$ is a C^∞ mapping of a neighbourhood of zero we denote by Tay $(f[t])$ the Taylor expansion:

$$\text{Tay } (f[t]) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0)\, t^k .$$

Clearly $f \to \mathrm{Tay}(f)$ is an algebra homomorphism and moreover

$$\mathrm{Tay}(f') = D(\mathrm{Tay}(f)).$$

Lemma (1) Let $f\colon \mathbb{R} \to \mathrm{Hom}(V, V)$ be a C^∞ mapping of a neighbourhood of zero satisfying $f(0) = I$. Then

$$\mathrm{Tay}(\log \det(f)) = \mathrm{trace}(\log(\mathrm{Tay}(f))) .$$

Proof It is sufficient to compare the derived series since the constant term of each side is zero. We note first that $f(t)$ is invertible in a neighbourhood of zero, and

$$D(\mathrm{Tay}(\log \det(f))) = \mathrm{Tay}\left(\frac{d}{dt} (\det(f)) \times (\det(f))^{-1}\right),$$

but

$$\frac{d}{dt} (\det(f)) = \mathrm{trace}(f^{-1} . f') \times \det (f).$$

Consequently

$$D(\mathrm{Tay}(\log \det(f))) = \mathrm{Tay}(\mathrm{trace}(f^{-1} . f')) =$$

$$\mathrm{trace}\{ [\mathrm{Tay}(f)]^{-1} \times D[\mathrm{Tay}(f)]\} = D(\mathrm{trace}(\log \mathrm{Tay}(f))). \qquad \text{q.e.d.}$$

Now let M be a riemannian manifold and let $m \in M$. For each $u \in T_m(M)$ let us define the endomorphism:

$$\Gamma(u) : T_m(M) \to T_m(M)$$

given by

$$\Gamma(u)X = R(X,u)u \tag{21}$$

for $X \in T_m(M)$, where R is the curvature tensor. Let $\gamma(t) = \exp_m tu$

be a geodesic through m and $\dot{\gamma}(t)$ its tangent field. We define the field of endomorphisms:

$$\Gamma(\dot{\gamma}) : T_{\gamma(t)}(M) \to T_{\gamma(t)}(M)$$

along $\gamma(t)$ as in (21).

$$\Gamma(\dot{\gamma}(t))\, X(t) = R(X(t), \dot{\gamma}(t))\dot{\gamma}(t), \quad X(t) \in T_{\gamma(t)}(M).$$

For any field of endomorphisms $S(t)$ along $\gamma(t)$ let us denote $\frac{\nabla}{dt} S(\dot{\gamma}(t))$ by $\dot{S}(t)$. So in particular

$$\dot{\Gamma}(t) = \frac{\nabla}{dt} \Gamma(\dot{\gamma}(t)) .$$

With this set up we now state:

<u>Theorem (5) (Walker [1])</u> Let M be a riemannian manifold and $m \in M$. For $u \in T_m(M)$ let $\gamma(t)$ be the geodesic through m tangential to u,

$$\gamma(t) = \exp_m tu.$$

Let $C_k(t)$ be a field of endomorphisms along $\gamma(t)$ defined recursively by

$$C_0(t) = 0, \qquad C_1(t) = I$$
$$C_{k+2}(t) = 2\dot{C}_{k+1}(t) - \ddot{C}_k(t) - C_k(t).\Gamma(t) \quad . \tag{22}$$

If $\rho(t,u)$ is Ruse's invariant along $\gamma(t)$, then

$$\text{Tay}(\log \rho(t,u) = \text{trace}\left(\log\left(\sum_{k=1}^{\infty} \frac{C_k(0)}{k!}\, t^{k-1}\right)\right) \tag{23}$$

We postpone the proof of Walker's equations till section (7) where we obtain a generalization thereof.

Let us now write

$$\operatorname{trace}\left(\log\left(\sum_{k=1}^{\infty}\frac{C_k(0)}{k!}t^{k-1}\right)\right) = \sum_{k=1}^{\infty}\frac{W_k(0)}{k!}t^k .$$

Then we have:

<u>Corollary (1)</u> A necessary and sufficient condition for M to be harmonic at m is that there exist constants k_r, $r = 1,2,\ldots$ such that

$$W_{2r}(0) = k_r(g(u,u))^r$$

$$W_{2r+1}(0) = 0.$$

<u>Proof</u> For M to be n-harmonic we must have $\rho(t,u) = F(\Omega(t))$ for some function F, where

$$\Omega(t) = \tfrac{1}{2}g(\dot{\gamma}(t), \dot{\gamma}(t))t^2.$$

Let us write the Taylor series for log F as

$$\operatorname{Tay}(\log F(t)) = \sum_{r=0}^{\infty}\frac{2^r}{(2r)!}k_r t^r .$$

Then

$$\operatorname{Tay}(\log \rho(t,u)) = \operatorname{Tay}\{\log F(\tfrac{1}{2}g(\dot{\gamma}, \dot{\gamma})t^2\}$$

$$= \sum_{r=0}^{\infty}\frac{1}{(2r)!}k_r(g(\dot{\gamma}, \dot{\gamma}))^r t^{2r} .$$

But by Walker's equations this equals

$$\sum_{r=0}^{\infty} W_r(0)\, t^r ,$$

hence by comparing coefficients and evaluating at m the result follows. q.e.d.

This method of obtaining necessary and sufficient conditions for n-harmonicity is equivalent to Ledger's method. In fact (Cf RWW p. 66) the W's of Walker are related to the λ's of Ledger's recurrence formulae by

$$\lambda_r = \frac{r}{r+1} W_r .$$

Thus all properties of n-harmonic manifolds stated in Chapter I could be obtained using Walker's equations.

A considerable simplification of the recurrence formulae (22) is achieved when we further assume that the manifold is riemannian symmetric (locally). In this case $\dot{\Gamma}(t) = 0$ and thus

$$C_{k+2}(t) = 2\dot{C}_{k+1}(t) - \ddot{C}_k(t) - C_k(t) . \Gamma(t) \qquad (24)$$

which could be solved in terms of Γ only to yield:

$$\left.\begin{aligned} C_{2m}(t) &= 0 \\ C_{2m+1}(t) &= (-\Gamma(t))^m \end{aligned}\right\} \quad m = 0,1,2,\ldots \qquad (25)$$

We can demonstrate (25) by induction on m. From (24) it is easily seen to hold for m = 0,1. Assume its validity for all m < r, say, then

$$C_{2r} = 2\dot{C}_{2r-1} - \ddot{C}_{2r-2} - C_{2r-2}\,\Gamma$$

the last two terms being zero by the induction hypothesis.

Also

$$2\dot{C}_{2r-1} = 2\frac{\nabla}{dt}\left[(-\Gamma)^{r-1} \right] = 0 \quad \text{as} \quad \frac{\nabla}{dt}\Gamma(\dot{\gamma}) = 0$$

this proves $C_{2r}(t) = 0.$ (26)

In

$$C_{2r+1} = 2\dot{C}_{2r} - \ddot{C}_{2r-1} - C_{2r-1} \cdot \Gamma \quad ,$$

the first term vanishes because of (26), the second term

$$\ddot{C}_{2r-1} = \frac{\nabla^2}{dt^2}\left((-\Gamma)^{r-1} \right) = 0 \quad \text{because} \quad \frac{\nabla}{dt}\Gamma = 0.$$

Thus

$$\begin{aligned} C_{2r+1} &= -C_{2r-1} \cdot \Gamma = C_{2r-1} \cdot (-\Gamma) \\ &= (-\Gamma)^{r-1} \cdot (-\Gamma) \quad \text{by induction} \\ &= (-\Gamma)^r \qquad \text{as was to be demonstrated.} \end{aligned}$$

Hence in a symmetric space (23) becomes

$$\mathrm{Tay}(\log \rho(t,u)) = \mathrm{trace}\left(\log\left(\sum_{k=0}^{\infty} \frac{(-\Gamma(u))^k}{(2k+1)!} t^{2k} \right)\right) \tag{27}$$

and we have the following proposition:

<u>Proposition (Walker [4])</u> Let M be a riemannian locally symmetric space and $m \in M$, then there exists a neighbourhood of m on which

$$\mathrm{Tay}(\log \rho(t,u)) = \sum_{k=0}^{\infty} \alpha_{2k}\, \mathrm{trace}\,(-\Gamma(u))^k\, t^{2k}$$

where α_k is the coefficients of t^k in the expansion

$$\mathrm{Tay}\left(\log \frac{\sin t}{t} \right) = \sum_{k=1}^{\infty} \alpha_{2k}\, t^{2k} . \qquad \text{q.e.d.}$$

Corollary (Walker [4]) A riemannian locally symmetric space is n-harmonic at $m \in M$ if and only if for any $u \in T_m(M)$ the eigenvalues of $\Gamma(u) : T_m(M) \to T_m(M)$ are constant multiples of $g(u,u)$. It is simply n-harmonic if and only if all eigenvalues of $\Gamma(u)$ are zero.

4.6 Application of a Formula of Helgason

Let $M = G/H$ be a riemannian symmetric space with a (G-invariant) riemannian metric g, where G is the connected component of the group of isometries of M and H the isotropy group at the origin $p_o \in M$. As in Chapter II let

$$\underline{g} = \underline{h} + \underline{m}$$

be the cannonical decomposition of the Lie algebra of G. For each $h \in G$ denote by $L(h)$ the diffeomorphism

$$L(h) : xH \to hxH \quad \text{of } G/H \text{ onto itself.}$$

Then by the invariance of g under the action of G we have at each point $p \in M$,

$$g_p(dL(h)X_o,\ dL(h)Y_o) = g_{p_o}(X_o,\ Y_o) \tag{28}$$

$$X_o,\ Y_o \in T_{p_o}(M).$$

In section (2) we defined for each unit vector $u \in T_{p_o}(M)$ the endomorphism Φ_{tu} of $T_{p_o}(M)$ by

$$g_p(f_{tu}X_o, f_{tu}Y_o) = g_{p_o}(\Phi_{tu}X_o, Y_o) \tag{29}$$

$X_o, Y_o \in T_{p_o}(M)$, where $p = \exp_{p_o} tu$ and f_{tu} is given by (4). We are now going to obtain an explicit formula for Φ_{tu} using an expression given by Helgason (Helgason [1], p. 180) for the differential of the exponential mapping at tu as

$$(\exp_{p_o})_{*tu} = dL(\exp_{p_o} tu) \circ \sum_{k=0}^{\infty} \frac{(\theta_{tu})^k}{(2k+1)!} \tag{30}$$

where θ_{tu} denotes the restriction of $(\mathrm{ad}\,tu)^2$ to $\underline{m}$

Let us write

$$A_{tu} = \sum_{k=0}^{\infty} \frac{(\theta_{tu})^k}{(2k+1)!} .$$

Using (4) and (30), (29) would become

$$\begin{aligned} g_{p_o}(\Phi_{tu} X_o, Y_o) &= g_p(dL(\exp_{p_o} tu) \circ A_{tu}X_o,\ dL(\exp_{p_o} tu) \circ A_{tu}Y_o) \\ &= g_{p_o}(A_{tu}X_o, A_{tu}Y_o) \qquad \text{by (28)} \\ &= g_{p_o}((A_{tu})^* A_{tu} X_o, Y_o) \end{aligned}$$

where $(A_{tu})^*$ denotes the adjoint of the operator A_{tu}.

Thus we have

$$\Phi_{tu} = (A_{tu})^* (A_{tu}).$$

Now since ad(tu) is a skew symmetric operator with respect to the riemannian inner product, we have θ_{tu} as a symmetric operator and hence A_{tu} is symmetric too.

Therefore

$$\Phi_{tu} = (A_{tu})^2 = \left(\sum_{k=0}^{\infty} \frac{(\mathrm{ad}\ tu)^{2k}}{(2k+1)!} \right)^2$$

$$= \left(\sum_{k=0}^{\infty} \frac{(\mathrm{adu})^{2k}}{(2k+1)!} t^{2k} \right)^2 \quad . \tag{31}$$

In section (5) we have $\Gamma(u)$ defined by $\Gamma(u) = R(X,u)u$, but in a riemannian symmetric space we saw in Chapter II, that

$$R(X,u)u = -[[X,u],u] = [u,[X,u]]$$

$$= (\mathrm{ad}\ u)([X,u]) = (\mathrm{ad}\ u)(-\mathrm{adu}(X))$$

$$= -(\mathrm{adu})^2(X).$$

Hence we can rewrite (31) as

$$\Phi_{tu} = \left(\sum_{k=0}^{\infty} \left(\frac{(-\Gamma(u))^k}{(2k+1)!} t^{2k} \right)^2 \right. \tag{32}$$

We have Φ_{tu} related to Ruse's invariant by

$$\rho(t,u)^2 = \det(\Phi_{tu}).$$

Hence (32) would give:

$$\rho(t,u) = \det\left(\sum_{k=0}^{\infty} \frac{(-\Gamma(u))^k \, t^{2k}}{(2k+1)!} \right) .$$

Taking $\log \rho(t,u)$ and using the fact that

$$\log(\det(A)) = \operatorname{trace}(\log A) \qquad \text{we get}$$

equation (27) and this establishes Walker's equations in the symmetric case.

4.7 Generalization of Walker's equations

As another application of Jacobi fields we are going to obtain generalized Walker-type equations satisfied by the endomorphism Φ_{tu} similar to those satisfied by $\rho(t,u)$. Let $m \in M$ and $u \in T_m(M)$ and let

$$\gamma(t) = \exp_m tu.$$

Let $Y \in T_m(M)$. Then as we had in Chapter III Y determines a unique Jacobi field $X(t)$ such that

$$X(0) = 0, \quad Y = \frac{\nabla X}{dt}(0).$$

We recall the construction of $X(t)$. Let $\bar{Y}$ be that vector in $T_o(T_m(M))$ obtained from Y by the identification

$$\eta_o : T_m(M) \to T_o(T_m(M)).$$

Let $Y(t)$ be the vector field in $T(M)$ along the curve

$$\rho : t \to tu \qquad \text{given by}$$

$$Y(t) = t \times \tau_{0,t} \bar{Y} \tag{33}$$

$\tau_{0,t}$ being parallel translation from 0 to tu along ρ .

Then we put

$$X(t) = (\exp_m)_{*tu} Y(t)$$

and $X(t)$ is the required Jacobi field.

Now consider along $\gamma(t)$ a set of n linearly independent Jacobi fields $X_1(t), \ldots, X_n(t)$ that vanish at $t = 0$. From above each X_i is the image under $\exp_m$ of some linear fields Y_i along ρ .

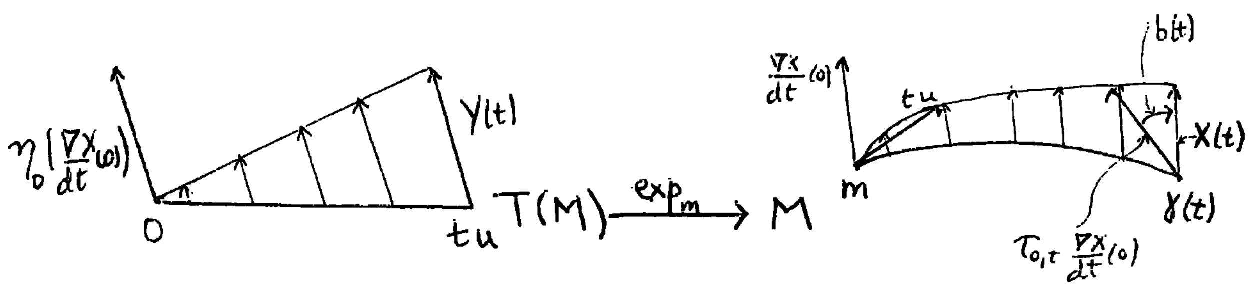
$$X_i(t) = (\exp_m)_{*tu} Y_i(t)$$

$$Y_i(t) = t \cdot \tau_{0,t} \circ \omega_i \tag{34}$$

$$\omega_i = \eta_o \left(\frac{\nabla X_i}{dt} (0) \right)$$

For each $t \in$ domain of definition of γ we let $b(t)$ be the endomorphism of $T_{\gamma(t)}(M)$ such that for $X_i(t) \in \epsilon_o(\gamma)$,

$$X_i(t) = b(t) \circ \tau_{0,t} \circ \frac{\nabla X_i}{dt} (0) \tag{35}$$



From (34)

$$X_i(t) = (\exp_m)_{*tu}\left(t \times \tau_{0,t} \circ \eta_o \frac{\nabla X_i}{dt}(0)\right) \tag{36}$$

Hence

$$b(t) \circ \tau_{0,t} = (\exp_m)_{*tu} \circ t \circ \tau_{0,t} \circ \eta_o \tag{37}$$

i.e. we have the commutative diagram

$$\begin{tikzcd}
T_m(M) \arrow[r, "\eta_o"] \arrow[d, "\tau_{0,t}"] & T_o(T_m(M)) \arrow[r, "t \times \tau_{0,t}"] & T_{tu}(T_m(M)) \arrow[d, "(\exp_m)_{*tu}"] \\
T_{\gamma(t)}(M) \arrow[rr, "b(t)"] & & T_{\gamma(t)}(M)
\end{tikzcd}$$

Note that $\tau_{0,t}$ in the right side of (37) is translation in the tangent bundle along ρ, whereas on the left side $\tau_{0,t}$ is transaltion in M along γ.

Now since $X_i(t)$ is a Jacobi field,

$$\frac{\nabla^2}{dt} X_i(t) + \Gamma(\dot{\gamma}(t))\, X_i(t) = 0 \tag{38}$$

where $\Gamma(\dot{\gamma}(t))$ is the operator defined by (21).

From (35)

$$\frac{\nabla X_i}{dt} = \frac{\nabla}{dt} b(t) \circ \tau_{0,t} \frac{\nabla X_i}{dt}(0) + b(t) \frac{\nabla}{dt}\left(\tau_{0,t} \frac{\nabla X_i}{dt}(0)\right)$$

$$= \frac{\nabla}{dt} b(t) \circ \tau_{0,t} \frac{\nabla X_i}{dt}(0) \tag{39}$$

Hence

$$\frac{\nabla^2}{dt^2} X_i(t) = \frac{\nabla^2}{dt} b(t) \circ \tau_{0,t} \frac{\nabla X_i}{dt}(0)$$

but

$$\frac{\nabla^2}{dt^2} X_i(t) = -\Gamma(\dot{\gamma}) X_i(t),$$

$$= -\Gamma(\dot{\gamma})\, \tau_{0,t} \frac{\nabla X_i}{dt}(0)$$

therefore

$$\frac{\nabla^2}{dt} b(t) + \Gamma(\dot{\gamma})\, b(t) = 0. \tag{40}$$

Also from (35) $b(0) = 0$ and by (34)

$$\frac{\nabla b}{dt}(0) = I_m, \quad \text{the identity on } T_m(M).$$

So we see that the field of endomorphisms $b(t)$ behaves like the Jacobi fields $X_i(t)$ (in that it satisfies the same differential equation).

From (40) it is evident that $b(t)$ maps $\epsilon_o(\gamma(t))$ onto itself. We should note here that b also depends on the geodesic $\gamma(t)$ or what amounts to the same thing on u. To make this dependence explicit we write $b(tu)$ for b. Also for each $u \in T_m(M)$ and each $0 \leqslant t < \infty$ we have defined by (5) the endomorphism ϕ_{tu} :

$$g_m(\phi_{tu} X, Y) = g_{\gamma(t)}(fX, fY), \qquad X,Y \in T_m(M)$$

where f is given by (4) as

$$f = (\exp_m)_{*tu} \circ \tau_{0,t} \circ \eta_o$$

$$= t \circ b(t) \circ \eta_o \qquad \text{by (37)}$$

By the invariance of the inner product under parallel translation we get

$$g_{\gamma(t)}(fX, fY) = g_m(\tau_{0,t}^{-1} \circ fX, \tau_{0,t}^{-1} \circ fY)$$

$$= \frac{1}{t^2} g_m(\bar{b}(t)X, \bar{b}(t)Y)$$

where

$$\bar{b}(t) = \tau_{0,t}^{-1} \circ b(t) \circ \tau_{0,t} : T_m(M) \to T_m(M).$$

Hence we have

$$\phi_{tu} = \frac{1}{t^2} \bar{b}^*(t) \circ \bar{b}(t) \tag{41}$$

where as before * denotes adjoint.

We now set out to get a formal Taylor's expansion for $\bar{b}(t)$ as

$$\text{Tay } (\bar{b}(t)) = \sum_{k=0}^{\infty} \left(\frac{\nabla}{dt} \right)^k \bar{b}(0) \frac{t^k}{k!} \quad .$$

From (40) the second order covariant derivative of b(t) is a function of b(t) only, so it is natural to expect that all higher order derivatives of b(t) be expressible in terms of b(t) and $\frac{\nabla}{dt}$ b(t) only, i.e. that

$$\left(\frac{\nabla}{dt}\right)^k b(t) = A_k(t)\, b(t) + C_k(t)\, \frac{\nabla b}{dt}(t) \qquad (42)$$

where $A_k(t)$ and $C_k(t)$ are operators, functions of t and u to be determined by induction.

For k = 0, 1, 2 we have

$$\begin{aligned} A_o(t) &= I, & C_o(t) &= 0 \\ A_1(t) &= 0, & C_1(t) &= I \\ A_2(t) &= -\Gamma(\dot\gamma(t)), & C_2(t) &= 0\,. \end{aligned}$$

Assuming (42) to hold for all $j \leq m$, we have

$$\left(\frac{\nabla}{dt}\right)^{m+1} b(t) = \frac{\nabla}{dt}\left(\left(\frac{\nabla}{dt}\right)^m b(t)\right)$$

$$= \frac{\nabla}{dt}\left(A_m(t)\, b(t) + C_m(t)\, \frac{\nabla b}{dt}(t)\right)$$

$$= \frac{\nabla}{dt} A_m(t).b(t) + A_m(t).\, \frac{\nabla}{dt} b(t) +$$

$$+ \frac{\nabla}{dt} C_m(t).\, \frac{\nabla b}{dt}(t) + C_m(t) \circ \frac{\nabla^2}{dt} b(t) =$$

$$\left[\frac{\nabla}{dt} A_m(t) - C_m(t).\Gamma(\dot\gamma)\right] \circ b(t) + \left[A_m(t) + \frac{\nabla}{dt} C_m(t)\right] \circ \frac{\nabla}{dt} b(t).$$

Thus (42) is also valid for j = m + 1 with

$$A_{m+1}(t) = \frac{\nabla}{dt} A_m(t) - C_m(t)\, \Gamma(\dot{\gamma}(t)) \qquad (a)$$

$$C_{m+1}(t) = A_m(t) + \frac{\nabla}{dt} C_m(t) \qquad (b) \qquad (43)$$

We now solve (43) to obtain recurrence relations for $C_m(t)$ as follows:

$$C_{m+2}(t) = A_{m+1}(t) + \frac{\nabla}{dt} C_{m+1}(t)$$

$$= \frac{\nabla}{dt} C_{m+1}(t) - C_m(t)\, \Gamma(\dot{\gamma}(t)) + \frac{\nabla}{dt} A_m(t),$$

but

$$\frac{\nabla}{dt} A_m(t) = \frac{\nabla}{dt} C_{m+1}(t) - \frac{\nabla^2}{dt} C_m(t).$$ Therefore

$$C_{m+2}(t) = 2 \frac{\nabla}{dt} C_{m+1}(t) - C_m(t)\, \Gamma(\dot{\gamma}(t)) - \frac{\nabla^2}{dt} C_m(t) \,. \qquad (44)$$

The first few C's are:

$$C_0 = 0, \quad C_1 = I, \quad C_2 = 0, \quad C_3 = -\Gamma(\dot{\gamma})$$

$$C_4 = -2\frac{\nabla}{dt} \Gamma(\dot{\gamma}), \qquad C_5 = \Gamma^2(\dot{\gamma}) - 3\left(\frac{\nabla}{dt}\right)^2 \Gamma(\dot{\gamma})$$

$$C_6 = -4\left(\frac{\nabla}{dt}\right)^3 \Gamma(\dot{\gamma}) + 2\Gamma \circ \frac{\nabla}{dt} \Gamma(\dot{\gamma}) + 4 \frac{\nabla}{dt} \Gamma(\dot{\gamma}) \circ \Gamma(\dot{\gamma})$$

etc.

Now from (42)

$$\frac{\nabla^k}{dt} \bar{b}(0) = \frac{\nabla^k}{dt} b(0) = C_k(0)$$

and therefore

$$\mathrm{Tay}(\bar{b}(t)) = \sum_{k=0}^{\infty} \frac{c_k(0)}{k!} t^k \tag{45}$$

q.e.d.

We know that Ruse's invariant satisfies

$$\rho(t,u)^2 = \det \phi_{tu} ,$$

and by (41)

$$\phi_{tu} = \frac{1}{t^2} \bar{b}^*(t) \bar{b}(t)$$

so that

$$\rho(t,u)^2 = \frac{1}{t^{2n}} \det \left[\bar{b}^*(t) \circ \bar{b}(t)\right] = \left[\det(\frac{1}{t} \bar{b}(t))\right]^2$$

and hence,

$$\mathrm{Tay}(\log \rho(t,u)) = \log\left(\det \left(\sum_{k=1}^{\infty} \frac{c_k(0)\ t^{k-1}}{k!} \right)\right)$$

$$= \mathrm{trace} \left(\log \left(\sum_{k=1}^{\infty} \frac{c_k(0)}{k!} t^{k-1} \right)\right)$$

Which proves Walker's equation (5).

We can write (41) explicitly as

$$\mathrm{Tay}(\phi_{tu}) = \sum_{k=0}^{\infty} \left(\sum_{r=0}^{k} \frac{c_r^*(0) \circ c_{k-r}(0)}{r!\ (k-r)!} \right) t^{k-2} .$$

<u>Lemma</u> $\quad c_r^*(0)\ c_{k-r}(0) = c_{k-r}(0)\ c_r^*(0) \qquad (46)$

Proof Use induction on k and $r \leq k$. For $k = 1$, $r = 0,1$

$$C_1 = C_1^* = I, \quad C_o = C_o^* = 0.$$

For $k = 2$, $r = 0,1,2$, $C_2 = C_2^* = 0$ and hence (46) holds for $r \leq k = 1,2$.

Assume (46) to hold for all $r \leq k \leq m$.

For $k = m + 1$, $r = k$ (46) follows trivially so need only verify (46) for $k = m + 1$, $r < k$.

$$\begin{aligned} C_r^* C_{m+1-r} &= C_r^*(2 \frac{\nabla}{dt} C_{m-r} - C_{m-r-1} \circ \Gamma - \frac{\nabla^2}{dt} C_{m-r-1}) \\ &= 2 \frac{\nabla}{dt} C_{m-r} \circ C_r^* - C_{m-r-1} \circ C_r^* \circ \Gamma - \frac{\nabla^2}{dt} C_{m-r-1} \circ C_r^* \end{aligned}$$

by induction hypothesis and assuming that

$$\frac{\nabla}{dt} C_r^* = \left(\frac{\nabla}{dt} C_r \right)^* \tag{46'}$$

Moreover if

$$C_r^* \circ \Gamma = \Gamma \circ C_r^* \tag{47}$$

we would get

$$\begin{aligned} C_r^* \circ C_{m+1-r} &= \left(2 \frac{\nabla}{dt} C_{m-r} - C_{m-r-1} \circ \Gamma - \frac{\nabla^2}{dt} C_{m-r-1} \right) \circ C_r^* \\ &= C_{m+1-r} \circ C_r^* \end{aligned}$$

and this proves (46) modulo (46') and (47).

For (46') we have by definition

$$\left(\frac{\nabla}{dt} C^*\right) X = \frac{\nabla}{dt}(C^* X) - C^*\left(\frac{\nabla X}{dt}\right), \qquad X \in \mathfrak{X}(M).$$

Now let $Y \in \mathfrak{X}(M)$,

$$g\left(\left(\frac{\nabla C^*}{dt}\right)X, Y\right) = g\left(\frac{\nabla}{dt}(C^*X), Y\right) - g\left(C^*\left(\frac{\nabla X}{dt}\right), Y\right)$$

$$= \frac{d}{dt} g(C^*X, Y) - g\left(C^*X, \frac{\nabla Y}{dt}\right) - g\left(C^*\left(\frac{\nabla X}{dt}\right), Y\right)$$

$$= \frac{d}{dt} g(X, CY) - g\left(X, C\left(\frac{\nabla Y}{dt}\right)\right) - g\left(\frac{\nabla X}{dt}, CY\right)$$

$$= g\left(X, \frac{\nabla}{dt}(CY)\right) - g\left(X, C\left(\frac{\nabla Y}{dt}\right)\right)$$

$$= g\left(X, \left(\frac{\nabla C}{dt}\right)Y\right) = g\left(\left(\frac{\nabla C}{dt}\right)^* X, Y\right) \Rightarrow \frac{\nabla C^*}{dt} = \left(\frac{\nabla C}{dt}\right)^*$$

<u>Proof of (47)</u> Again use induction on r, let $r = 0,1,2$, $C_o = 0$, $C_1 = I$, $C_2 = 0$ and (47) holds trivially so assume (47) to hold for all $r \leqslant p$, say, then

$$C^*_{p+1} \circ \Gamma = \left(2\frac{\nabla}{dt} C^*_p - \Gamma^* \circ C^*_{p-1} - \frac{\nabla^2}{dt} C^*_{p-1}\right) \circ \Gamma$$

$$= \Gamma \circ 2\frac{\nabla}{dt} C^*_p - \Gamma^* \circ \Gamma \circ C^*_{p-1} - \Gamma \circ \frac{\nabla^2}{dt} C^*_{p-1}$$

$$= \Gamma \circ C^*_{p+1} \quad \text{as } \Gamma \text{ is symmetric.}$$

This proves (47) and completes proof of lemma. q.e.d.

From (46) it follows that

$$\bar{b}^*(t) \circ b(t) = b(t) \circ \bar{b}^*(t)$$

i.e. $\bar{b}(t)$ is a normal operator. From the properties of normal operators it follows that if λ is an eigenvalue of $\bar{b}(t)$ then λ is also an eigenvalue of $\bar{b}^*(t)$ with same eigenvector, and thus by (41) it would follow that $t^2\lambda^2$ is an eigenvalue of ϕ_{tu}.

4.8 Case of Symmetric space

If we now assume that the manifold M is locally symmetric then (44) would simplify considerably and as in (25) we get

$$C_{2m}(0) = 0, \quad C_{2m+1}(0) = (-\Gamma(u))^m, \quad m = 0,1,2,..$$

and hence

$$\mathrm{Tay}(\bar{b}(t)) = \sum_{k=0}^{\infty} \frac{(-\Gamma(u))^k \, t^{2k+1}}{(2k+1)!} . \tag{48}$$

From which it follows that $\bar{b}(t)$ is an operator symmetric w.r.t. the inner product g,

i.e. $$\bar{b}^*(t) = \bar{b}(t).$$

Thus for a symmetric space,

$$\mathrm{Tay}(\phi_{tu}) = \left(\sum_{k=0}^{\infty} \frac{(-\Gamma(u)^k \, t^{2k}}{(2k+1)!} \right)^2 \tag{49}$$

which generalizes (27) in a natural way.

Now if λ is an eigenvalue of $\Gamma(u)$ with eigenvector X, i.e. $\Gamma(u)X = \lambda X$, then $(-\lambda)^k$ is an eigenvalue of $(-\Gamma(u))^k$ with same eigenvector,

$$(-\Gamma(u))^k X = (-\lambda)^k X .$$

Thus by (49)

$$\phi_{tu} X = A^2 X, \qquad \text{where}$$

$$A = \sum_{k=0}^{\infty} \frac{(-\Gamma(u))^k t^{2k}}{(2k+1)!} .$$

$$AX = \left(\sum_{k=0}^{\infty} \frac{(-\Gamma(u))^k t^{2k}}{(2k+1)!} \right) X = \sum_{k=0}^{\infty} \frac{(-\Gamma(u))^k X t^{2k}}{(2k+1)!}$$

$$= \left(\sum_{k=0}^{\infty} \frac{(-\lambda)^k t^{2k}}{(2k+1)!} \right) X ,$$

so that

$$\phi_{tu} X = A^2 X = A(AX) = A \left(\sum_{k=0}^{\infty} \frac{(-\lambda)^k t^{2k}}{(2k+1)!} \right) X$$

$$= \left(\sum_{k=0}^{\infty} \frac{(-\lambda)^k t^{2k}}{(2k+1)!} \right) AX = \left(\sum_{k=0}^{\infty} \frac{(-\lambda)^k t^{2k}}{(2k+1)} \right)^2 X$$

$$= \frac{\sin t\sqrt{\lambda}}{t\sqrt{\lambda}} X .$$

Thus if λ is an eigenvalue of $\Gamma(u)$ then $\frac{\sin t\sqrt{\lambda}}{t\sqrt{\lambda}}$ is the corresponding eigenvalue of ϕ_{tu} .

We recall Walker's criterion for a locally symmetric space to be n-harmonic at m – that the eigenvalues λ_i of $\Gamma(u)$ be constant on geodesic spheres centred at m for all $u \in T_m(M)$. By above if λ_i is an eigenvalue of $\Gamma(u)$ then $\dfrac{\sin t\sqrt{\lambda_i}}{t\sqrt{\lambda_i}}$ is an eigenvalue of Φ_{tu}. Thus if M is n-harmonic all eigenvalues of Φ_{tu} would be constant on geodesic spheres centred at m and we thus have the following:

Theorem (6) Let M be a locally symmetric riemannian manifold. If M is n-harmonic at m then M is k-harmonic at m for all k. q.e.d.

Corollary If M is locally symmetric and is

(1) simply n-harmonic, then it is simply k-harmonic for all k.

(2) simply 1-harmonic, then it is simply k-harmonic for all k.

4.9 k-harmonic symmetric manifolds

Riemannian locally symmetric spaces which admit positive definite n-harmonic metrics are fully characterized. Those with decomposable metrics have been shown (Lichnerowicz [2]) to be necessarily flat (Cf RWW p. 216). Those with indecomposable metrics were shown by Ledger [2] to be precisely the symmetric spaces of rank one. In this section we are going to obtain similar results for k-harmonic metrics under the additional assumption of compactness. So our results are necessarily more restrictive.

Theorem (7) (Willmore and El-Hadi [1]) Let M be a compact, simply-connected n-dimensional C^∞ manifold equipped with an irreducible positive definite riemannian symmetric metric. Let M be k-harmonic for some k. Then M must be a symmetric space of rank one.

Note it is sufficient to assume M only locally symmetric since the hypothesis of simple-connectedness and compactness imply global symmetry [Helgason [1], p. 187] .

Proof In proving theorem (7) we follow closely the treatment given by Rauch [2] of the Jacobi equations on symmetric spaces. So let $p_o \in M$ and $u \in T_{p_o}(M)$ a unit vector. Let $\gamma(t) = \exp_{p_o} tu$, be a geodesic issuing from p_o tangential to u. Let Z be a unit vector perpendicular to $\rho(t) = tu$ in $T_{p_o}(M)$. Then the set of tangent vectors $t\mathbf{Z}$ defined at points distant t along $\rho(t)$ will map under $(\exp_{p_o})_*$ into a Jacobi field $X(t)$ along $\gamma(t)$. It is proved (Rauch [2], p. 117) that by a suitable choice of orthonormal basis $(u, \mathbf{Z}_1, \ldots, \mathbf{Z}_n)$ of $T_{p_o}(M)$ that the components of X satisfy equations with constant coefficients, namely

$$X''_\alpha + K_\alpha X_\alpha = 0, \qquad \alpha \text{ (not summed)} = 1,2,\ldots,n-1. \qquad (50)$$

Here K_α is the sectional curvature at p_o of the plane specified by u and $\mathbf{Z}_\alpha$. In fact it follows from section (1) of Chapter II and theorems (2) and (4) there, that

$$K_\alpha = \| [u, Z_\alpha] \|^2 \geq 0$$

when the norm is taken w.r.t. the (definite) Killing form of G when M is considered as the coset space

$$M = G/H.$$

We are interested in solutions of (50) such that

$$X_\alpha(0) = 0, \qquad \text{and}$$

$$\frac{\nabla X_\alpha}{dt}(0) = Z_\alpha .$$

Thus we have

$$\begin{aligned} X_\alpha &= t Z_\alpha , \quad \text{for } K_\alpha = 0, \quad \text{and} \\ X_\alpha &= \frac{1}{\sqrt{K_\alpha}} (\sin t\sqrt{K_\alpha})\, Z_\alpha, \quad \text{for } K_\alpha \neq 0 \end{aligned} \tag{51}$$

Suppose now that of the set $\{K_\alpha\}$, exactly ℓ are zeros - the non-zero K_α's may, of course, occur with various multiplicities. Then this gives a cannonical form (diagonal) for the pull-back metric $h = (\exp_{p_0})^*(g_{\gamma(t)})$. The symmetric polynomials σ_K of the eigenvalues of this form are exactly the symmetric polynomials of Φ_{tu} as was shown in (5). Thus we have in particular:

$$\begin{aligned} \sigma_1 &= \ell t + \frac{1}{\sqrt{K_1}} \sin t\sqrt{K_1} + \dots + \frac{1}{\sqrt{K_{n-\ell}}} \sin t\sqrt{K_{n-\ell}} \\ &\dots\dots \\ \sigma_n &= t^\ell \cdot \frac{\sin t\sqrt{K_1}}{\sqrt{K_1}} \times \dots \times \frac{\sin t\sqrt{K_{n-\ell}}}{\sqrt{K_{n-\ell}}} \end{aligned} \tag{52}$$

Now we assume M to be k-harmonic, i.e. we impose the condition that σ_k is a function of t alone, i.e. σ_k is independent of u. It follows that in particular ℓ must be independent of the choice of u.

Consider now the group theoretic picture of M as a homogeneous coset space G/H. Here G is the connected component of the group of isometries of M, and H is the compact isotropy group at p_o. The assumptions of compactness and simple-connectedness of M imply that G must be compact and simple. According to the corollary to theorem (2) of Chapter II we can identify the linear holonomy group H_o' at p_o with the linear isotropy group H' there. An important implication of this identification is given by the following

Lemma (Rauch [2]) Let $p_o \in M$ and γ a geodesic issuing from p_o with unit tangent vector u, and let Z_α be a unit eigenvector at p_o belonging to the eigenvalue $K_\alpha > 0$. Then there exists a one-parameter subgroup of H whose image in the holonomy group H_o' is transitive on the unit vectors issuing from p_o which lie in the two plane determined by u and Z_α. We have as a consequence of the Lemma:

Corollary (1) An eigensolution of (50) belonging to the positive K_α and such that $X_\alpha(0) = 0$ is obtained as follows: Take the one-parameter group of the lemma, apply it to a point of γ and differentiate the coordinates of the orbit w.r.t. the group parameter and set the latter equal to zero.

Indeed the group takes γ into geodesics which also emanate from p_o, and the group parameter plays the role of the coordinate ϵ in the variation defining the Jacobi field. Moreover, the eigenvalues and their multiplicities are not necessarily the same for all geodesics issuing from p_o but only for those which are transformable into γ by the holonomy group at p_o.

Now, to require ℓ to be independent of the choice of u, we require that the holonomy group acts transitively on the unit sphere S^{n-1} in $T_{p_o}(M)$. For this we refer to the results of Berger [1] and Simons [1]. In particular theorems (8) and (9) of Simons [1] give:

<u>Theorem (8) (Berger-Simons)</u> Let M be a compact, simply connected, irreducible riemannian symmetric manifold. Let $p_o \in M$ and let H_o' be the connected component of the linear holonomy group, acting transitively on S^{n-1} in $T_{p_o}(M)$. Then M is a symmetric space of rank one.

From this theorem (7) immediately follows. q.e.d.

<u>Theorem (9) (Willmore - El Hadi [1])</u> Let M be a compact simply-connected, n-dimensional C^∞ manifold equipped with a positive definite irreducible riemannian symmetric metric. Let M be k-harmonic for some k. Then M is k-harmonic for all k.

Proof By theorem (7) M is a symmetric space of rank one. Theorem (9) of Chapter II shows M then to be two-point homogeneous. But theorem (1) asserts that a two point homogeneous manifold with a positive definite metric is k-harmonic for all k. q.e.d.

The result of Avez [1] stated as theorem (9) of Chapter III shows that a compact n-harmonic manifold is locally (and hence globally) symmetric. Combining this with theorem (9) we get:

Theorem (10) Let M be a compact simply connected manifold with a positive definite riemannian n-harmonic metric. Then M is k-harmonic for all k. q.e.d.

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