Meson phenomenology

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MESON PHENOMENOLOGY

by

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C O N T E N T S

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PREFACE

The work presented in this thesis was carried out in the Department of Mathematics in the University of Durham during the period from October 1970 to August 1973, under the supervision of Dr. R. C. Johnson.

The material in this thesis has not been submitted for any other degree in this or any other university. No claim of originality is made for either chapter one or chapter two. Chapter three is based mainly on a paper written by the author in collaboration with Dr. R. C. Johnson, and published in Phys. Letters 36B(1971) 483. The material in Chapter four is based on a paper by the author published in Nucl. Phys. B56 (1973) 136. Some of chapter five and most of chapter six are also claimed to be original, except where referenced.

My sincere thanks go to my supervisor for his guidance, patience and encouragement. I should also like to thank my research student colleagues, especially Messrs. J. B. Carrotte, J. W. Coleman and D. W. O'Hara, for stimulating discussions. A Science Research Council Studentship is gratefully acknowledged. Finally, I should like to extend my thanks to my sister, Barbara, for her skilful typing of the thesis.

ABSTRACT

A phenomenological investigation of soft meson zeros is performed. Our motivation comes from the suggestion by Lovelace of a connection between Chiral Symmetry and the Dual Resonance Model. The low energy theorems relating to the particular processes with which we are concerned are discussed in detail in chapter two. We find, in chapter three, for the Pi-Pi interaction, that the experimental phase shift data, when taken in conjunction with rigorous crossing sum rules, demand the soft meson zeros. In chapter four the soft meson zeros in τ-decay are discussed in the framework of the generalised pole model. The rate problem is re-examined and a satisfactory description of both the rate and the spectrum for the decay is obtained. Using the value of the weak vertex, $g(M^2)$, obtained in chapter four, we discuss, in chapter five models for other kaon decays, namely $K_L^0 \rightarrow 2\gamma$ and the magnetic dipole radiation in decay $K^{\pm} \rightarrow \pi^{\pm}\pi^{0}\gamma$, and obtain results in agreement with experiment. In chapter six we return to the notorious problem of the zeros in $\eta \rightarrow 3\pi$. A solution is proposed based on the variation of the coefficients of the Veneziano sum as the external pseudoscalar mesons are taken off mass shell.

CHAPTER ONE: INTRODUCTION

1.1 Lovelace's Conjecture

Some of the most interesting and far reaching consequences of Current Algebra and PCAC have been the low energy theorems for pions. These theorems relate processes in which (N+1)-pions are emitted, in the limit of one pion at zero momentum, to similar processes in which N-pions are emitted. If for some reason the N-pion process is forbidden the low energy theorems predict zeros in the (N+1)-pion amplitude in this soft pion limit. In chapter two we discuss some of the soft pion theorems which we shall be particularly concerned with, namely $K \rightarrow 3\pi$ $\eta \rightarrow 3\pi$ and $\pi\pi \rightarrow \pi\pi$.

Lovelace in a remarkable paper (1) hinted at a deep connection between the Dual theory of strong interactions and Current Algebra. He summarised his dream in terms of the following equation:

Chiral symmetry for soft mesons + Absence of exotic resonances

= Veneziano formula with no secondary terms.

But there are some serious problems associated with this conjecture, to which we shall return later, and for the time being we shall give a brief review of the work which inspired the above belief.

1.2 The Dual Resonance Model (DRM)

The DRM is an analytic expression for the scattering amplitude which is characterised by the following three properties:

a) all its singularities are due to resonance exchange,

b) Regge asymptotic behaviour,

c) exact crossing symmetry.

From the above points it is clear that such a model embodies the idea of Duality (2) since its absorptive part is given entirely by resonance contributions and yet the amplitude is Regge behaved. It is also evident that a DRM as defined above is inconsistent with the Interference model (3) in which the high energy behaviour of the amplitude is not given by the resonances.

If A(s,t) is a DRM for a binary scattering process with s- and t- but no u-channel singularities, (see appendix A for kinematics) then:

$$A(s,t) = \sum_n \frac{C_n(t)}{s-s_n} = \sum_n \frac{\tilde{C}_n(s)}{t-t_n} \tag{1.1}$$

If, furthermore, $A(s,t) = A(t,s)$ then $C_n = \tilde{C}_n$. In other words, the amplitude can be written either as sum of its s-channel resonances or as a sum of its t-channel resonances but not as a sum of both. Another ingredient which is usually added is that of planar duality, namely that the full amplitude, A(s,t,u), can be written as a linear combination of the three terms A(s,t), A(s,u) and A(t,u), where A(s,t) possesses only s- and t-channel singularities (no u-channel ones).

Lovelace (1) and Shapiro (4) used the above principle of duality, together with the idea of the absence of exotic resonances to build their model of Pi-Pi scattering. They started by considering $\pi^+\pi^-$ elastic scattering.

In this process there are no u-channel resonances, since the u-channel, being I=2, is exotic, while the s- and t-channels are identical. Veneziano's formula (5) then requires exchange-degeneracy between the f- and ρ-trajectories, and gives for the $\pi^+\pi^-$ elastic scattering amplitude the following finite sum:

$$A(s,t) = -\beta \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} + \gamma \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))} + \cdots \quad (1.2)$$

This is sufficient to completely describe Pi-Pi scattering, giving the following s-channel isospin amplitudes:

$$A^0 = \tfrac{1}{2}(3A(s,t) + 3A(s,u) - A(t,u)) \quad (1.3a)$$

$$A^1 = A(s,t) - A(s,u) \quad (1.3b)$$

$$A^2 = A(t,u). \quad (1.3c)$$

In eq. 1.2 $\alpha(s) = \alpha_0 + \alpha' s$, and is the $\rho - f^0$ trajectory function. The separate terms in eq. 1.2 may be expanded either as a sum of poles in s or in t thus satisfying duality. To show the s-channel poles at $\alpha(s) = J,\ J \geqslant 1$ explicitly for the first term, expand:

$$A(s,t) = \beta(\alpha(s)+\alpha(t)-1) \times \sum_{n=0}^{\infty} \binom{\alpha(t)+n-1}{n} \frac{1}{n+1-\alpha(s)} \quad (1.4)$$

The residue is a polynomial in t and therefore also in $\cos\theta$ Since it is not a Legendre polynomial, each resonances will be accompanied by daughters of all lower J. The factor $(\alpha(s)+\alpha(t)-1)$ gives rise to the Adler zero (6) provided that all the secondary terms are zero (γ , etc. = 0 in eq. 1.2) and that

$$\alpha(m_\pi^2) = \frac{1}{2} \quad (1.5)$$

Taking the ρ meson to be mass 765 MeV gives for the trajectory function

$$\alpha(s) = 0.483 + 0.885s, \qquad (1.6)$$

in good agreement with the phenomenological ρ trajectory. Substituting eq. 1.6 into eq.1.3 gives for the Pi-Pi scattering lengths:

$$a_0^0 = 0.395\beta \qquad \text{and} \qquad a_0^2 = -0.103\beta, \qquad (1.7)$$

the ratio of which being within 10% of Weinberg's value (7).

In discussing the asymptotic behaviour one uses the identity

$$\Gamma(z)\Gamma(1-z)\sin\pi z = \pi \qquad (1.8)$$

and the limit

$$\lim_{z\to\infty} \frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b} \qquad (1.9)$$

For fixed s, and large t, one has using eqs. 1.8 and 1.9:

$$A(s,t) \sim \Gamma(1-\alpha(s))\,[\alpha(t)]^{\alpha(s)} \sin\pi\alpha(s)\,[\cot\pi\alpha(s) + \cot\pi\alpha(t)] \qquad (1.10).$$

Apart from possible poles in $\cot\pi\alpha(t)$, this has Regge-like behaviour. We expect, for large t, that the narrow-resonance approximation will be very poor, and that in fact $\operatorname{Im}\alpha(t)$ should become large as $t\to\infty$. In which case $\cot\pi\alpha(t)\to -i$ and

$$A(s,t) \sim \Gamma(1-\alpha(s))\,e^{-i\pi\alpha(s)}\,[\alpha(t)]^{\alpha(s)} \qquad (1.11)$$

By similar arguments

$$A(s,u) \sim \Gamma(1-\alpha(s))\,[\alpha(t)]^{\alpha(s)} \qquad (1.12)$$

giving for linear trajectories

$$A^1 \sim \frac{\beta\pi}{\Gamma(\alpha(s))} \frac{(1-e^{-i\pi\alpha(s)})}{\sin\pi\alpha(s)} [\alpha' t]^{\alpha(s)} \qquad (1.13)$$

which we see is proper Regge behaviour. Comparing with the usual Regge formula

$$R \sim \gamma(s) \frac{(1-e^{-i\pi\alpha(s)})}{\sin\pi\alpha(s)} \left(\frac{t}{t_o}\right)^{\alpha(s)} \qquad (1.14)$$

shows that the ρ-residue vanishes at $\alpha(s) = 0,-1,-2,$etc., i.e. the ρ chooses nonsense.

A successful Veneziano amplitude has also been written for πK and KK scattering (8), but the troubles begin when one considers $\pi\eta$ scattering. The simplest Veneziano formula for $\pi\eta \rightarrow \pi\eta$ scattering has the form

$$A(s,t,u) \propto A(s,t) + A(s,u) + A(t,u), \qquad (1.15)$$

which ensures the correct signature for the A_2-f (exchange-degenerate with the ρ-ω) trajectory. If Adler's condition is to be satisfied one needs perfect octet masses for the pseudoscalar mesons, i.e. $M_\pi = M_\eta = 140$ MeV. This is not the only problem. The form eq.1.15 is inconsistent with phenomenology in that a large δ coupling is predicted in disagreement with the observed small δ width. So one feels forced to modify eq.1.15 by adding satellites - this is the first problem with the Lovelace conjecture.

As well as considering Pi-Pi scattering Lovelace also considered reactions of the type

$$X \rightarrow 3\pi$$

where X has the quantum numbers of the pion. In applying the Veneziano form to this type of reaction he argued that only the coefficients $\beta, \gamma,$ etc. in eq.1.2 could vary as one of the external pions is taken off mass-shell, since the Regge trajectories cannot depend kinematically on the external masses. The decays considered were $\eta \rightarrow 3\pi$ and $K \rightarrow 3\pi$ in the pole model illustrated in Fig. 1, and $\bar{p}n \rightarrow 3\pi$ at rest, where the initial state is known experimentally to be pure ${}^1S_0$ and so has the quantum numbers of the pion.

When considering τ-decay the assumption of no secondary terms gives all the required current algebra zeros (9,10), but the fit to the spectrum using the 'proper' value of 0.483 for $\alpha(0)$ suggests that something may be wrong (see Fig.3). While for the decay $\eta \rightarrow 3\pi$ the same assumption gives a good fit to the spectrum, but the most obvious current algebra zero, that when $q_{\pi^0} = 0$ (11), is not reproduced (see Fig.4). To be able to describe the process $\bar{p}n \rightarrow 3\pi$ it is necessary for $\gamma > -\beta$ the most favoured value being $\gamma \approx -2\beta$ (12,13) (see Figs. 5 and 6). Rate calculations have also been performed for $K \rightarrow 3\pi$ and $\eta \rightarrow 3\pi$, using the pole model, with some rather surprising results (14). We shall return to this problem in chapter four.

From the above points we see that if Lovelace's conjecture is rigidly obeyed with regard to the absence of secondary terms several serious problems arise, and yet it was this assumption which gave the rather nice prediction of the ρ trajectory intercept.

1.3 Programme

Our programme of work involves a phenomenological investigation of Adler zeros, particularly in those processes where the correspondence between the dual resonance model and current algebra has been suggested. We see to what extent Adler zeros are in fact demanded by experimental data, when rather general constraints are taken into consideration. In those reactions where, at first sight, some of the zeros seem to be absent we suggest modifications to the Lovelace conjecture to provide a consistent picture with all zeros present.

FIGURE CAPTIONS:

Fig.1: Pion pole dominance model in which a kaon (eta) decays weakly (electromagnetically) into a heavy pion, which in turn decays strongly into three pions.

Fig.2: Pi-Pi phase shifts given by the single channel K-matrix Veneziano model (15).

Fig.3: Single term Veneziano fits to $K^+ \to \pi^+\pi^+\pi^-$

A: fit with $\alpha(0) = 0.483$.

B: fit with $\alpha(0) = 0.528$.

The dotted line is the best linear fit. The data are from ref. (16).

Fig.4: Single term Veneziano fits to $\eta \to \pi^+\pi^-\pi^0$

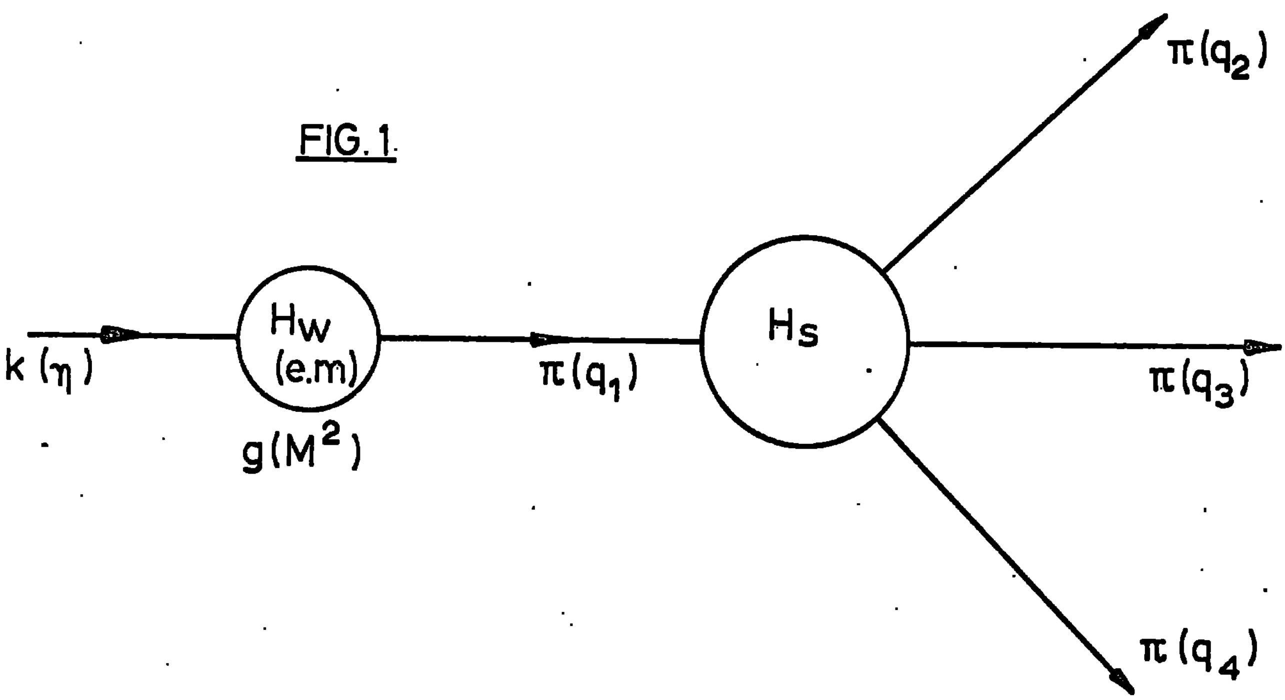
A: fit with $\alpha(0) = 0.483$.

B: fit with $\alpha(0) = 0.491$.

The data are ref. (17)

Figs.5 and 6: Comparison of the Lovelace and Berger fits to $\bar{p}n \to \pi^-\pi^-\pi^+$.

FIG. 1



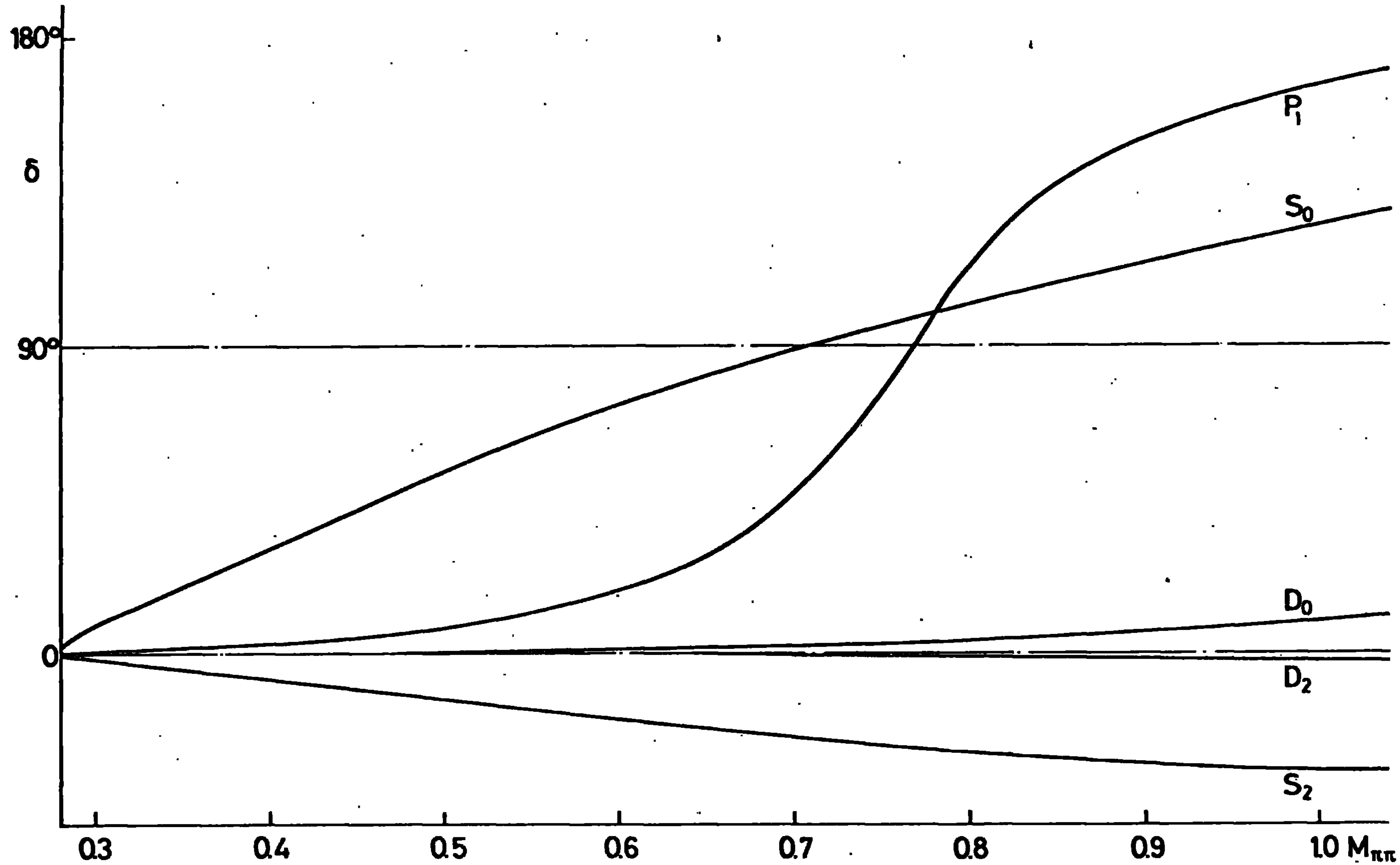
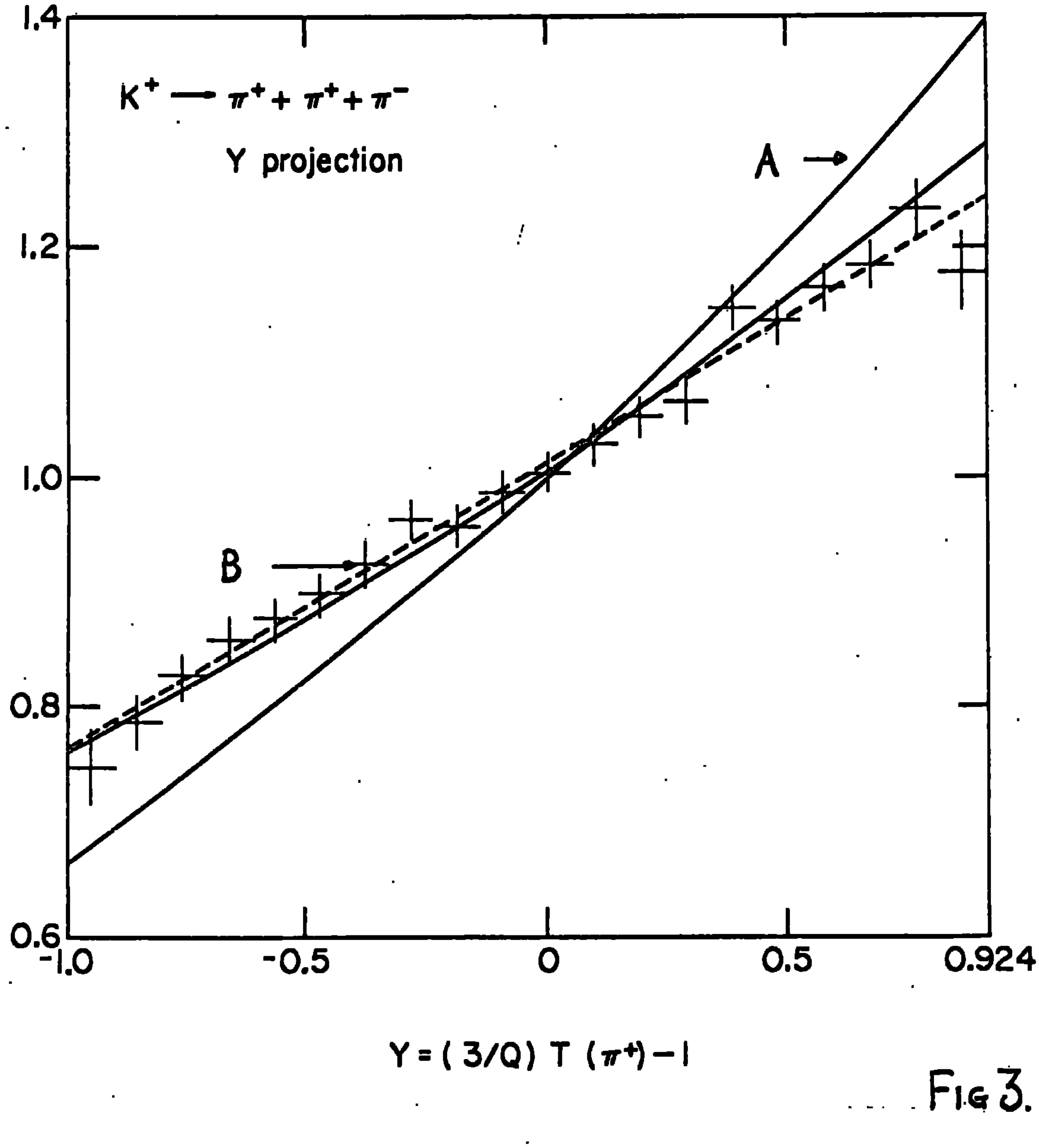


FIG. 2.



1.4
1.2
1.0
0.8
0.6
-1.0
-0.5
0
0.5
0.924
$K^+ \longrightarrow \pi^+ + \pi^+ + \pi^-$
Y projection
A
B
$Y = (3/Q)\,T(\pi^+) - 1$

FIG 3.

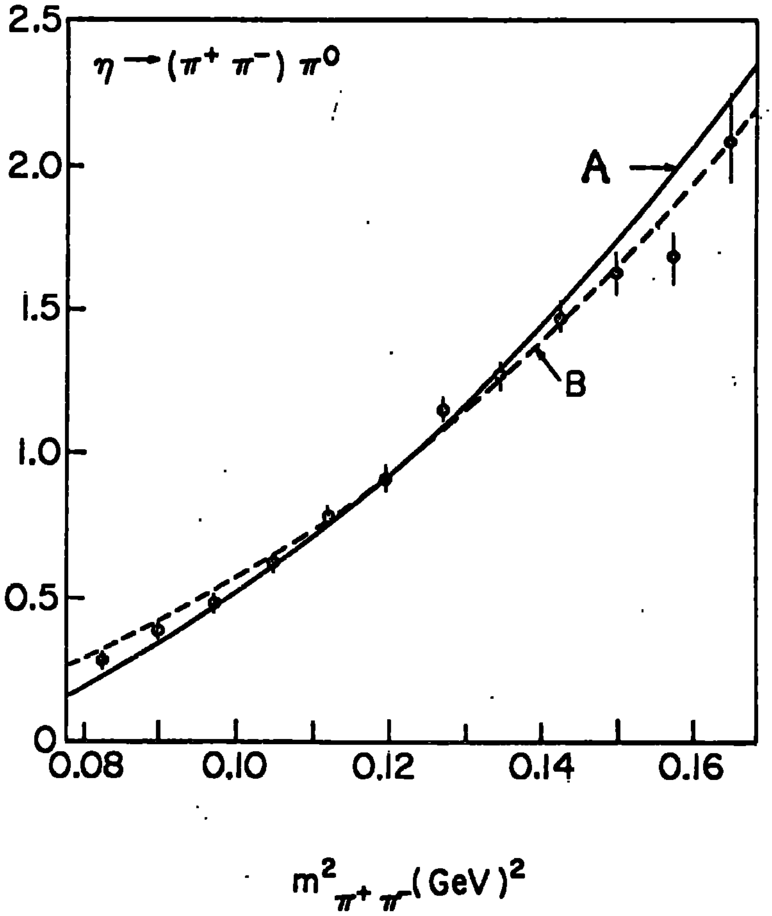
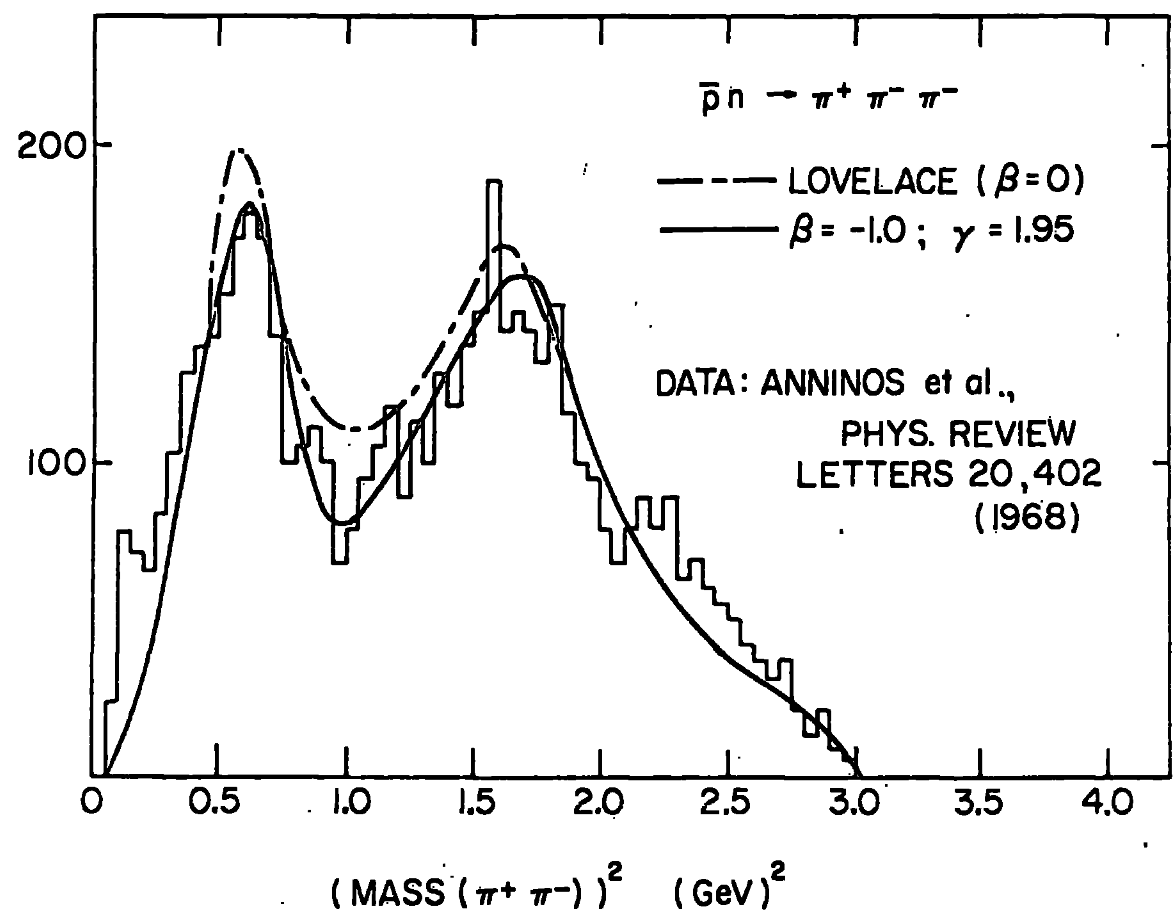


FIG 4.



$\bar{p}n \to \pi^+\pi^-\pi^-$
LOVELACE (β=0)
β= -1.0 ; γ = 1.95
DATA: ANNINOS et al., PHYS. REVIEW LETTERS 20, 402 (1968)
200
100
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
(MASS ($\pi^+\pi^-$))^2 (GeV)^2

Fig 5.

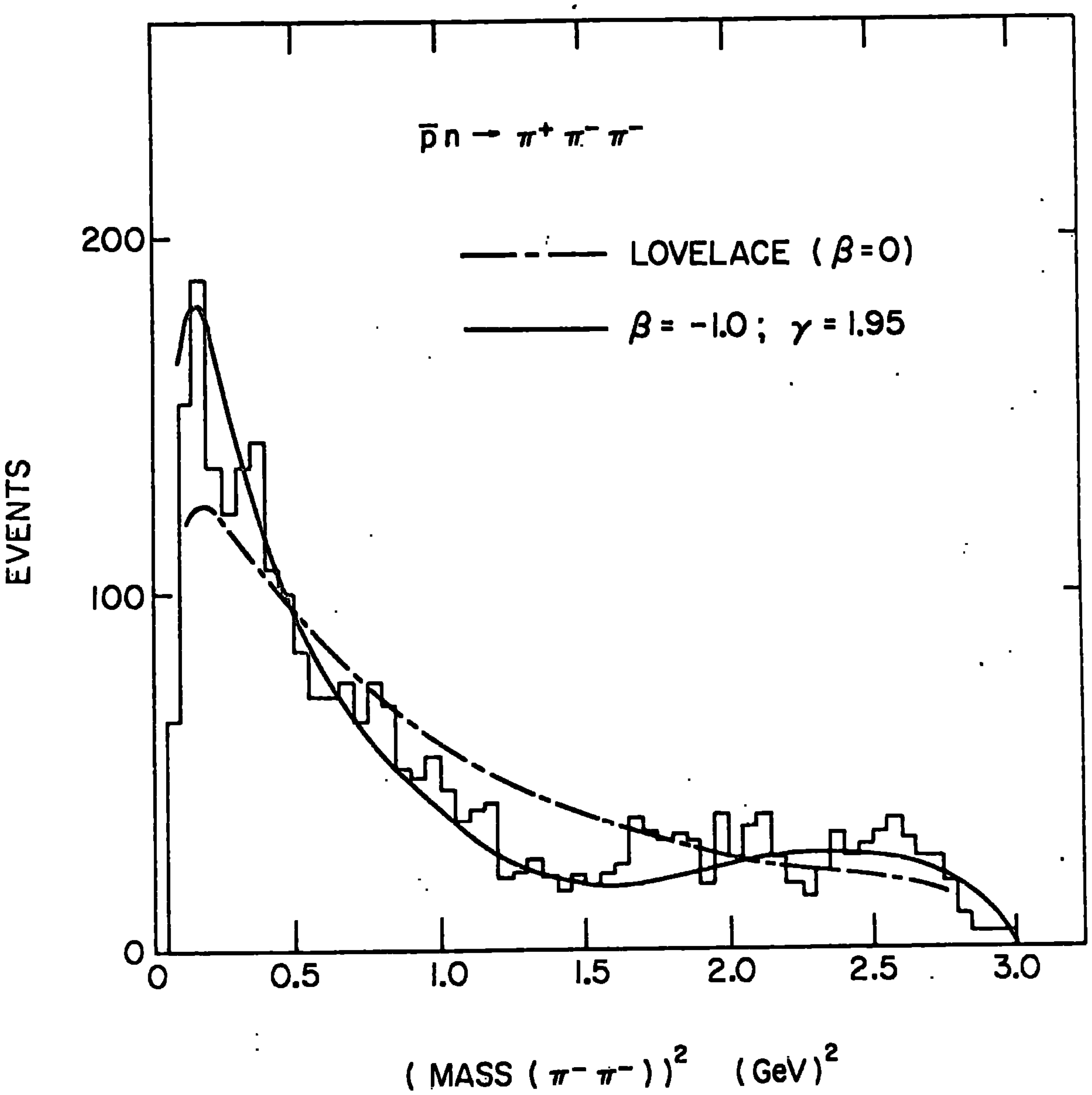

p̄ n → π⁺ π⁻ π⁻
LOVELACE (β=0)
β = -1.0 ; γ = 1.95
EVENTS
200
100
0
0
0.5
1.0
1.5
2.0
2.5
3.0
(MASS (π⁻ π⁻))² (GeV)²

FIG 6.

CHAPTER TWO : LOW ENERGY THEOREMS FOR PIONS

2.1 Introduction:

Nambu and Lurié (18) were the first to use the PCAC hypothesis to study processes involving "soft" pions, that is, pions with zero or small four-momentum. By combining PCAC and the algebra of currents, Callan and Treiman (9) were able to relate the processes $K\rightarrow 3\pi$ and $K\rightarrow 2\pi$ in the limit of one soft pion. They found that the result depended on which pion was taken to zero momentum. Hara and Nambu, and Elias and Taylor (10) continued this work, and by assuming a linear dependence on the energy of the odd pion for the matrix element of the decay $K\rightarrow 3\pi$ were able to obtain a very good description of the rates and slopeparameters for the decay by relating it to the decay $K\rightarrow 2\pi$. Bell (19) then showed that the results of Callan and Treiman were still true in the intermediate vector boson theory. This tempted Sutherland (11) to apply the same analysis to the phenomenologically similar decay $\eta\rightarrow 3\pi$. The result was rather surprising; he found that if one assumed a linear matrix element, which worked so well for τ decay, then the decay $\eta\rightarrow 3\pi$ should be forbidden! This led to many questions: (a) was current algebra wrong? (b) were the linearity assumptions wrong? (c) was the similarity of K and η decay within their Dalitz plots purely accidental? It seems that one is forced, with great reluctance, to take the point of view that the linearity assumption is wrong, which in turn means that the results obtained for τ decay, beautiful though they are, must be accidental.

Weinberg (7) applied these ideas of PCAC and current algebra to the problem of Pi-Pi scattering and obtained some very interesting and rather surprising results.

In this chapter we rederive the low energy theorems for the particular processes which we will be considering later. At each stage in the derivation we will mention the approximations which have been made. The plan of the chapter is as follows: in section two we briefly review the current algebra ingredients of the low energy theorems, in section three we derive an identity relating the process $\alpha \to \beta + \pi$, in the soft pion limit, to the "easier" process of $\alpha \to \beta$. The remaining sections consist of special cases of α and β.

2.2 SU(3) X SU(3) Current Algebra

In the SU(3) scheme of things, the weak and electromagnetic interactions of hadrons are described by a set of Vector and Axial-vector currents denoted by $V_{i\mu}(x)$ and $A_{i\mu}(x)$ respectively, where the unitary spin index, i, runs from 1 to 8 (20). The Vector and Axial-vector charges, defined by

$$F_j\ (t) = -i\int V_{j4}(x)\ d\underline{x} \text{ and } F_j^5 = -i\int A_{j4}(x)\ d\underline{x} \qquad (2.1)$$

respectively, satisfy the commutation relations

$$\left[F_k(t),\ F_\ell(t)\right] = if_{k\ell m}F_m\ (t), \quad \left[F_k^5(t),\ F_\ell^5(t)\right] = if_{k\ell m}F_m(t)$$

and

$$\left[F_k(t),\ F_\ell^5(t)\right] = if_{k\ell m}\ F_m^5\ (t) \qquad (2.2)$$

where the $f_{k\ell m}$ are the structure constants of SU(3). The isotopic spin and hypercharge operators are given by:

$$I_j = F_j \ (j = 1,2,3) \text{ and } Y = \frac{2}{\sqrt{3}} F_8. \qquad (2.3)$$

For the electromagnetic current, J_μ^γ , one has the following form:

$$J_\mu^\gamma = V_{3\mu} + \frac{V_{8\mu}}{\sqrt{3}}, \qquad (2.4)$$

giving the well-known Gell-Mann Nishijima relation

$$Q = I_3 + \frac{Y}{2} \qquad (2.5)$$

Since it was natural to expect the currents $V_{i\mu}$ and $A_{i\mu}$ themselves to be octets, the following equal time commutation relations were postulated:

$$[F_k(t), V_{\ell\mu}(x)] = i f_{k\ell m} V_{m\mu}(x), \quad [F_k(t), A_{\ell\mu}(x)] = i f_{k\ell m} A_{m\mu}(x)$$
$$[F_k^5(t), V_{\ell\mu}(x)] = i f_{k\ell m} A_{m\mu}(x), \quad [F_k^5(t), A_{\ell\mu}(x)] = i f_{k\ell m} V_{m\mu}(x). \qquad (2.6)$$

The interaction Hamiltonian density which describes the coupling of the electromagnetic (e.m.) field, $A_\mu(X)$ to hadrons is:

$$\mathcal{H}_{em}(x) = -J_\mu^\gamma(x) A_\mu(X), \qquad (2.7)$$

which describes the e.m. properties of hadrons to all orders in the e.m. coupling.

For weak interactions the best that can be done is to use an effective Hamiltonian, the matrix elements of which directly describe the lowest order weak effects. All experimental evidence, apart from the "very weak" decay $K_L^0 \rightarrow 2\pi$, is consistent with a current x current effective Hamiltonian:

$$\mathcal{H}_W^{eff.} = \mathcal{H}_W^{\ell} + \mathcal{H}_W^{s\ell} + \mathcal{H}_W^{h} = -\frac{G}{\sqrt{2}}\left[J_\mu \bar{J}_\mu\right] \tag{2.8}$$

where

$$J_\mu = L_\mu + J_\mu^{0}\cos\theta + J_\mu^{1}\sin\theta. \tag{2.9}$$

Here L_μ is the leptonic current, θ - the Cabibbo angle ($\simeq 15^0$) and $J_\mu^{0,1}$ are the strangeness-conserving and strangeness changing hadronic currents respectively, given by

$$J_\mu^{0} = V_{1\mu} + A_{1\mu} + i\left(V_{2\mu} + A_{2\mu}\right) \tag{2.10}$$

and

$$J_\mu^{1} = V_{4\mu} + A_{4\mu} + i\left(V_{5\mu} + A_{5\mu}\right). \tag{2.11}$$

If one writes the hadronic weak Hamiltonian, $\mathcal{H}_W^{h}$, as a sum of parity conserving and parity violating parts, that is

$$\mathcal{H}_W^{h} = \mathcal{H}_W^{P.C.} + \mathcal{H}_W^{P.V.} \tag{2.12}$$

then $\mathcal{H}_W^{P.C.}$ will contain terms of the type $V_{i\mu}V_{j\mu}$ or $A_{i\mu}A_{j\mu}$ while $\mathcal{H}_W^{P.V.}$ will have terms like $V_{i\mu}A_{j\mu}$ or $A_{i\mu}V_{j\mu}$. Using the commutation relations eq. 2.6 one obtains:

$$\left[F_i^5, \mathcal{H}_W^{P.C.}\right] = \left[F_i, \mathcal{H}_W^{P.V.}\right], \quad \left[F_i^5, \mathcal{H}_W^{P.V.}\right] = \left[F_i, \mathcal{H}_W^{P.C.}\right] \tag{2.13}$$

In particular one is interested in the case when $i = (1 \pm i2)/\sqrt{2}$, or 3, that is when the F's are just the usual isospin operators $I_\pm$ or I_3.

The usual assumption withregard to the strong interaction Hamiltonian is that it is of the form:

$$H = H_0 + g H_8 = H_0 + g\int \mathcal{H}_8 \, d\underline{x} \tag{2.14}$$

here H_0 is SU(3) invariant and H_8 is the eighth member of an octet. This form gives for the divergence of the vector currents the following

$$\partial_\mu V_{k\mu}(x) = g f_{k8m} \mathcal{H}_m(x). \tag{2.15}$$

That is, with this type of SU(3) breaking, only the isotopic spin and hypercharge currents are conserved, and the time dependence of vector charges is given by

$$\dot{F}_k(t) = g f_{k8m} H_m \tag{2.16}$$

The hypothesis for the divergence of the axial-vector currents had its origin in the work of Goldberger and Treiman on the decay of the charged pion (21), and is the hypothesis of Partially-Conserved Axial-vector Current (PCAC) given by

$$\partial_\mu A_{i\mu} = C_i \phi_i \tag{2.17}$$

In particular for i = 1,2,3 one has

$$\partial_\mu(A_{1\mu} \pm i A_{2\mu}) = \sqrt{2}\, F_\pi m_\pi^2\, \pi^{\mp}(x), \quad \partial_\mu A_{3\mu} = F_\pi m_\pi^2 \pi^0(x), \tag{2.18}$$

where $\pi^{\pm}_{0}(x)$ are the pion fields, m_π the pion mass and F_π is the pion decay constant defined by

$$(2\pi)^{3/2} \sqrt{2P_0}\, \langle 0 | A_{1\mu} + i A_{2\mu} | \pi^-(p) \rangle = i\sqrt{2}\, F_\pi P_\mu \tag{2.19}$$

By looking at the decay $\pi \to \mu + \nu_\mu$, one can obtain an "experimental" value for F_π = 93 Mev.

This empirical value differs by only 12% from the Goldberger-Treiman value given by

$$F_\pi = \frac{g_A M_N}{g_{\pi NN}} \simeq 85 \text{ MeV} \tag{2.20}$$

where g_A is the nucleon axial-vector coupling constant ($g_A = g_A(0) \doteq 1.2$), M_N is the nucleon mass and $g_{\pi NN}$ is the pion-nucleon coupling constant. $(g_{\pi NN}^2/4\pi \simeq 14.6)$

2.3 An identity (22)

Let $J_{k\mu}(X)$ be any of the currents $V_{k\mu}(x)$ or $A_{k\mu}(x)$ and let O(X) be any Heisenberg operator; now consider the amplitude

$$M_{k\mu} = i\int dx\, e^{-iq\cdot x}\,\theta(x_0)\langle\beta|[J_{k\mu}(x), O(0)]|\alpha\rangle \tag{2.21}$$

where α and β are physical hadron states. Then

$$\begin{aligned} iq_\mu M_{k\mu} &= i\int dx\,(-\partial_\mu e^{-iq\cdot x})\,\theta(x_0)\langle\beta|[J_{k\mu}(x), O(0)]|\alpha\rangle \\ &= i\int dx\, e^{-iq\cdot x}\,\partial_\mu\left(\theta(x_0)\langle\beta|[J_{k\mu}(x), O(0)]|\alpha\rangle\right) \\ &\quad - i\int dx\,\partial_\mu\left(e^{-iq\cdot x}\theta(x_0)\langle\beta|[J_{k\mu}(x), O(0)]|\alpha\rangle\right). \end{aligned} \tag{2.22}$$

By Gauss's Theorem the second integral on the r.h.s. of eq. 2.22 can be converted into a surface integral which is set equal to zero. This is because the spacial surface terms give no contribution if wave packets are used; for positive time the integral vanishes if a small positive imaginary part is given to q_0 and for negative time because of the $\theta(x_0)$. Using $\partial_0\theta(x_0) = \delta(x_0)$ one obtains:

$$\begin{aligned} iq_\mu M_{k\mu} &= i\int dx\, e^{-iq\cdot x}\,\theta(x_0)\langle\beta|[\partial_\mu J_{k\mu}(x), O(0)]|\alpha\rangle \\ &\quad + i\int dx\, e^{-iq\cdot x}\,\delta(x_0)\langle\beta|[J_{k0}(x), O(0)]|\alpha\rangle \end{aligned} \tag{2.23}$$

setting $J_{k\mu} = A_{k\mu}$ and using the PCAC hypothesis:

$$\partial_\mu A_{k\mu}(x) = F_\pi m_\pi^2 \pi_k(x), \quad k = 1,2,3 \tag{2.18}$$

and taking the limit $q_\mu \to 0$ one has

$$\lim_{q_\mu \to 0} -iF_\pi m_\pi^2 \int dx\, e^{-iq\cdot x}\theta(x_0)\langle\beta|[\pi_k(x),O(0)]|\alpha\rangle$$
$$= i\langle\beta|[F_k^5(0),O(0)]|\alpha\rangle - \lim_{q_\mu\to 0} iq_\mu M_{k\mu}. \quad (2.24)$$

One now considers the matrix element

$$\mathcal{R} = (2\pi)^{3/2}\sqrt{2q_0}\,\langle\beta\,\pi_k(q)|O(0)|\alpha\rangle \quad (2.25)$$

By the L.S.Z. reduction technique the matrix element can be written as

$$\mathcal{R} = i\int dx\, e^{-iq\cdot x}(m_\pi^2 - \Box)\theta(x_0)\langle\beta|[\pi_k(x),O(0)]|\alpha\rangle \quad (2.26)$$

with an integration by parts one obtains

$$\mathcal{R} = i\int dx\, e^{-iq\cdot x}(m_\pi^2 + q^2)\theta(x_0)\langle\beta|[\pi_k(x),O(0)]|\alpha\rangle. \quad (2.27)$$

By comparing eq. 2.24 and 2.27 one notices that

$$\lim_{q_\mu\to 0} (2\pi)^{3/2}\sqrt{2q_0}\; F_\pi\langle\beta\,\pi_k(q)|O(0)|\alpha\rangle$$
$$= -i\langle\beta|[F_k^5(0),\,O(0)]|\alpha\rangle + \lim_{q_\mu\to 0} iq_\mu M_{k\mu} \quad (2.28)$$

Here the second term on the r.h.s. vanishes unless $M_{k\mu}$ has a singularity at $q_\mu = 0$. This can happen only if a single particle state, degenerate in mass with either α or β contributes to $M_{k\mu}$. In the cases were are interested in no such singularities occur. The first term on the r.h.s. of eq. 2.28 is an equal-time commutator that is known from current algebra considerations.

Hence, by the relation eq.2.28 a process involving matrix elements of O(o) between two states $(\beta+\pi_k)$ and α is connected, in the soft pion limit, to a process that involves the matrix element of a related operator between β and α. In other words, one is able to connect a process in which N pions are involved to others in which N-1, N-2, ---, 0 pions are involved. It should be noted that if the matrix element for a process in which n soft pions are emitted is, for kinematic reasons, of order $q_1 q_2 \cdots\cdots q_n$, straight forward application of current algebra gives no information about it, because one is neglecting the second term on the r.h.s. of eq. 2.28.

2.4 Kaon Decays

(i) $K \to 2\pi$

To first order in $\mathcal{H}_w^h$ the reduced T-matrix element for the decay

$$K^\alpha(p) \longrightarrow \pi_i(k_1) + \pi_j(k_2) \tag{2.29}$$

where α is the charge state of the kaon and i and j are the isospin indices of the pions, is:

$$T_{ij}^\alpha = -\langle \pi_i(k_1)\pi_j(k_2) | \mathcal{H}_W^{P.V.}(0) \; | K^\alpha(p)\rangle \tag{2.30}$$

T_{ij}^α is related to the Feynman amplitude $\mathcal{M}_{ij}^\alpha$ via

$$T_{ij}^\alpha = -\frac{1}{(2\pi)^{3/2}} \frac{R_{ij}^\alpha}{\sqrt{2k_{10}}} \tag{2.31}$$

with

$$R_{ij}^\alpha = \frac{i}{(2\pi)^3} \frac{\mathcal{M}_{ij}^\alpha}{\sqrt{4k_{20}P_0}}$$

Applying the identity of §2.4, eq. 2.28, one obtains

$$\lim_{K_1\to 0} R^{\alpha}_{ij}(k_1,k_2) = -\frac{i}{F_\pi}\langle \pi_j(k_2)|[F^5_i(0),\mathcal{H}^{P.V.}_W(0)]|K^\alpha(p)\rangle \tag{2.32}$$

which becomes, after using the commutation relations in eq. 2.13

$$\lim_{K_1\to 0} R^{\alpha}_{ij}(k_1,k_2) = \frac{-i}{F_\pi}\langle \pi_j(k_2)|[F_i(0),\mathcal{H}^{P.C.}_W(0)]|K^\alpha(p)\rangle \tag{2.33}$$

The road is now clear since the Fi's are just the isospin operators $I_\pm$ and I_3. Writing the matrix element for $K\to\pi$ as:

$$\langle \pi(k_2)|\mathcal{H}^{P.C.}_W(0)|k(p)\rangle = \frac{1}{(2\pi)^3\sqrt{4k_{20}P_0}}\,\mathcal{M}_{K\pi} \tag{2.34}$$

one arrives at the following relation:

$$\begin{aligned}\mathcal{M}(K^+\to\tilde{\pi}^+\pi^0) &= (\sqrt{2}F_\pi)^{-1}\langle \pi^0(k_2)|[I_-,\mathcal{H}^{P.C.}_W(0)]|K^+\rangle(2\pi)^3\sqrt{4k_{20}P_0}\\ &= (\sqrt{2}F_\pi)^{-1}\{\sqrt{2}\,\mathcal{M}_{K^+\pi^+}+\mathcal{M}_{K^0\pi^0}\}\end{aligned} \tag{2.35}$$

taking this together with the other charge states we have the following:

$$\begin{aligned}
\mathcal{M}(K^+\to\tilde{\pi}^+\pi^0) &= \frac{1}{2F_\pi}\{2\mathcal{M}_{K^+\pi^+}+\mathcal{M}_{K^0_L\pi^0}\}\\
\mathcal{M}(K^+\to\pi^+\tilde{\pi}^0) &= \frac{-1}{2F_\pi}\,\mathcal{M}_{K^+\pi^+}\\
\mathcal{M}(K^0_1\to\tilde{\pi}^+\pi^-) &= \frac{1}{2F_\pi}\,\mathcal{M}_{K^+\pi^+} = \mathcal{M}(K^0_1\to\pi^+\tilde{\pi}^-)\\
\mathcal{M}(K^0_1\to\tilde{\pi}^0\pi^0) &= \frac{-1}{2F_\pi}\,\mathcal{M}_{K^0_L\pi^0} = \mathcal{M}(K^0_1\to\pi^0\tilde{\pi}^0)
\end{aligned} \tag{2.36}$$

If one now puts in the $\Delta I = \frac{1}{2}$ rule "by hand", that is setting

$$\mathcal{M}_{K^+\pi^+} = -\mathcal{M}_{K_L^0\pi^0} \tag{2.37}$$

eqs. 2.36 simply to the following

$$\begin{aligned} \frac{1}{2F_\pi}\mathcal{M}_{K^+\pi^+} &= \mathcal{M}(K^+\to\tilde{\pi}^+\pi^0) \\ &= -\mathcal{M}(K^+\to\pi^+\tilde{\pi}^0) \\ &= \mathcal{M}(K_1^0\to\tilde{\pi}^+\pi^-) \\ &= \mathcal{M}(K_1^0\to\tilde{\pi}^0\pi^0) \end{aligned} \tag{2.38}$$

(ii) $K\to 3\pi$

The procedure for $K\to 3\pi$ is now clear. Applying eq. 2.28 as before one has

$$\begin{aligned} &\lim_{k_3\to 0}(2\pi)^{3/2}(2k_{30})^{1/2}\langle\pi_1\pi_2\pi_3|\mathcal{H}_W^{P.C.}(0)|K\rangle \\ &\quad = \frac{-i}{F_\pi}\,\underline{\pi}_3\cdot\langle\pi_1\pi_2|[\underline{F}^5,\mathcal{H}_W^{P.C.}(0)]|K\rangle, \end{aligned} \tag{2.39}$$

where $\underline{\pi}_3$ is the isospin wave function of π_3. One now replaces the commutator $[\underline{F}^5,\mathcal{H}_W^{P.C.}]$ by $[\underline{F},\mathcal{H}_W^{P.V.}]$ as before and obtains:

$$\begin{aligned} 2F_\pi\mathcal{M}(K^+\to\tilde{\pi}^+\pi^+\pi^-) &= -\mathcal{M}(K_S^0\to\pi^0\pi^0) \\ 2F_\pi\mathcal{M}(K_L^0\to\pi^+\pi^-\tilde{\pi}^0) &= \mathcal{M}(K_S^0\to\pi^+\pi^-) \\ 2F_\pi\mathcal{M}(K^+\to\tilde{\pi}^0\pi^0\pi^+) &= \mathcal{M}(K^+\to\pi^+\pi^0) \\ 2F_\pi\mathcal{M}(K^+\to\pi^0\pi^0\tilde{\pi}^+) &= -\mathcal{M}(K_S^0\to\pi^0\pi^0)-4\mathcal{M}(K^+\to\pi^+\pi^0) \\ 2F_\pi\mathcal{M}(K_L^0\to\tilde{\pi}^0\pi^0\pi^0) &= \mathcal{M}(K_S^0\to\pi^0\pi^0) \end{aligned} \tag{2.40}$$

Again using the $\Delta I = \frac{1}{2}$ rule and neglecting the pion mass differences, one finally arrives at

$$\begin{aligned}
\frac{1}{2F_\pi}\,\mathcal{M}(K^0_S \to \pi^0\pi^0) &= \frac{1}{2F_\pi}\,\mathcal{M}(K^0_S \to \pi^+\pi^-) \\
&= -\mathcal{M}(K^+ \to \tilde{\pi}^+\pi^+\pi^-) \\
&= -\mathcal{M}(K^+ \to \pi^0\pi^0\tilde{\pi}^+) \\
&= \mathcal{M}(K^0_L \to \pi^+\pi^-\tilde{\pi}^0) \\
&= \mathcal{M}(K^0_L \to \tilde{\pi}^0\pi^0\pi^0)
\end{aligned} \tag{2.41}$$

2.5 Eta Decay (11)

The G-parity violating decay $\eta \to \pi_a\pi_b\pi_c$ proceeds via a second order e.m. interaction.

$$\mathcal{M}(\eta \to \pi_a\pi_b\pi_c) \propto \langle \pi_a\pi_b\pi_c | \int dy\, D_{\mu\nu}(y)\, T\left(J^\gamma_\mu(y)\, J^\gamma_\nu(0)\right) | \eta \rangle \tag{2.42}$$

where J^γ_μ is the e.m. current and $D_{\mu\nu}$ is the photon propagator. From eq. 2.4 one writes J^γ_μ as the sum of isovector and isoscalar pieces

$$J^\gamma_\mu = J^3_\mu + J^8_\mu \tag{2.43}$$

here J^3_μ is the neutral component of an isovector and J^8_μ is an isoscalar. Rewriting eq. 2.42 bearing in mind that $J^8_\mu J^8_\nu$ and $J^3_\mu J^3_\nu$ terms do not contribute because of G-parity, one gets

$$\mathcal{M}(\eta \to \pi_a\pi_b\pi_c) \propto \langle \pi_a\pi_b\pi_c | \int dy\, D_{\mu\nu}(y)\, T\left(J^3_\mu(y)\, J^8_\nu(0)\right) | \eta \rangle \tag{2.44}$$

together with a similar term but with '3' and '8' interchanged, which can be treated in the same way.

The next step involves taking the matrix element of the expansion:

$$\frac{\partial}{\partial x_\lambda} T\left(A_{c\lambda}(x)\, J^3_\mu(y)\, J^8_\nu(0)\right) = T\left(\frac{\partial A_{c\lambda}}{\partial x_\lambda}(x)\, J^3_\mu(y) J^8_\nu(0)\right) +$$

$$T\left\{\delta(x_0 - y_0)\left[A_{c0}(x), J^3_\mu(y)\right] J^8_\nu(0) + \delta(x_0) J^3_\mu(y)\left[A_{c0}(x), J^8_\nu(0)\right]\right\}, \tag{2.45}$$

between $\langle \pi_a \pi_b | \int dy\, D_{\mu\nu}(y) \int dx\, e^{-iq_c x}$ and $|\eta\rangle$. The term arising from the l.h.s. of eq. 2.45 is integrated by parts over X, giving, with the usual neglect of surface terms, a term proportional to $q_{c\mu}$, which vanishes in the limit $q_{c\mu} \to 0$. In this limit the term coming from the first piece on the r.h.s. of the equation is, using PCAC and L.S.Z. reduction, proportional to $\mathcal{M}(\eta \to \pi_a \pi_b \pi_c(0))$ so we have:

$$\mathcal{M}(\eta \to \pi_a \pi_b \tilde{\pi}_c) \propto$$

$$\langle \pi_a \pi_b | \int dx\, e^{-iq_c \cdot x} \int dy\, D_{\mu\nu}(y)\, T\left(\pi_c(x)\, J^3_\mu(y)\, J^8_\nu(0)\right) |\eta\rangle$$

$$\propto \langle \pi_a \pi_b | \int dy\, D_{\mu\nu}(y)\, T\left(\left[F^5_c(y_0), J^3_\mu(y)\right] J^8_\nu(0) + J^3_\mu(y)\left[F^5_c(0), J^8_\nu(0)\right]\right) |\eta\rangle \tag{2.46}$$

The two commutators in the above are given directly by eqs. 2.6, and in particular when C = 3, corresponding to the neutral pion, the two commutators vanish; so one has Sutherland's result

$$\mathcal{M}(\eta \to \pi^+ \pi^- \tilde{\pi}^0) = \mathcal{M}(\eta \to \pi^0 \pi^0 \tilde{\pi}^0) = 0 \tag{2.47}$$

If one now makes the usual assumption that the three pion final state has isospin 1, then using Bose symmetry, one is forced to conclude that the amplitude must vanish when any of the three pions is taken to zero four momentum.

2.6 $\pi\pi \rightarrow \pi\pi$

A very interesting application of the above ideas was made by Weinberg (7) for the case of Pi-Pi scattering. Let the reaction be denoted by

$$k(a) + p(c) \longrightarrow q(b) + \ell(d), \tag{2.48}$$

Where k,p,q,l represent the momenta of the pions and a,b,c,d are their isospin labels. Bose statistics, crossing symmetry, and isospin invariance dictate the following structure for the Pi-Pi amplitude, to second order in the pion momenta,

$$T^{ba}_{cd} = \delta_{ab}\delta_{cd}[\alpha+\beta(s+u)+\gamma t]+\delta_{ad}\delta_{bc}[\alpha+\beta(s+t)+\gamma u]$$

$$+\delta_{ac}\delta_{bd}[\alpha+\beta(u+t)+\gamma s], \tag{2.49}$$

where α, β and γ are constants, independent of the pion momenta, and

$$s=(p+k)^2\ ,\ t=(k-q)^2 \text{ and } u=(p-q)^2. \tag{2.50}$$

To check the Bose requirement observe that the amplitude is indeed even, e.g., under $a \leftrightarrow c$ and $k \leftrightarrow p$ (hence $s \leftrightarrow s$, $t \leftrightarrow u$). Similarly, crossing symmetry requires that the amplitude be even under $c \leftrightarrow b$, $p \leftrightarrow -q$ (hence $s \leftrightarrow t$, $u \leftrightarrow u$); and we see that this is satisfied too. What is remarkable about eq.248 is that the amplitude does not depend explicitly on the mass variables k^2, p^2, q^2 and l^2, except for their appearance in the relation $s+t+u = k^2+p^2+q^2+l^2$. Expressed in terms of the parameters α, β and γ the S-wave scattering lengths a_I (I is the total isospin) are given by $(m_\pi^2=1)$

$$a_0 = (32\pi)^{-1}[5\alpha + 8\beta + 12\gamma] \tag{2.51a}$$

and

$$a_2 = (32\pi)^{-1}[2\alpha + 8\beta]. \tag{2.51b}$$

To determine to coefficients α, β and γ one again considers the amplitude

$$M^{ba}_{\nu\mu} = i\int dx\, e^{-iq\cdot x}\theta(x_0)\langle \ell(d)|[A_{b\nu}(x), A_{a\mu}(0)]|p(c)\rangle \tag{2.21}$$

performing this time a double contraction gives

$$\begin{aligned} q_\nu M^{ba}_{\nu\mu} k_\mu = {} & i\int dx\, e^{-iq\cdot x}\theta(x_0)\langle \ell(d)|[\partial_\nu A_{b\nu}(x), \partial_\mu A_{a\mu}(0)]|p(c)\rangle \\ & + k_\mu \int dx\, e^{-iq\cdot x}\delta(x_0)\langle \ell(d)|[A_{b0}(x), A_{a\mu}(0)]|p(c)\rangle \\ & + i\int dx\, e^{-iq\cdot x}\delta(x_0)\langle \ell(d)|[A_{a0}(0), \partial_\nu A_{b\nu}(x)]|p(c)\rangle \end{aligned} \tag{2.52}$$

Using PCAC, the first term on the right in the above equation is, up to factors, the Pi-Pi amplitude T^{ba}_{cd}. The third term on the right, the "σ-term", σ^{ba}_{cd} is symmetric in the indices b and a, when $k = q \to 0$. Let $q = k \to 0$, but keep $p = 1$ on mass shell. To first order in q and k the left hand side of eq. 2.52 can be neglected - there are no pole terms in the Pi-Pi problem. The second term on the right is a familiar commutator given by current algebra. Thus, for $k = q \to 0$, to first order in $k = q$, one has

$$T^{ba}_{cd} = \frac{1}{F_\pi^2}[\delta_{bc}\delta_{da} - \delta_{ba}\delta_{dc}]\,2p\cdot k + \sigma^{ba}_{cd}. \tag{2.53}$$

But to this order 4 p.k $\to$ u-s. From eq. 2.53 form the quantity $(T^{ba}_{cd} - T^{ab}_{cd})$, noting that σ^{ba}_{cd} is symmetric under the interchange a $\leftrightarrow$ b and comparing with the same quantity obtained from eq. 2.49 gives

$$\gamma - \beta = \frac{1}{F_\pi^2} \tag{2.54}$$

From eqs. 2.51 it then follows that

$$2a_0 - 5a_2 = 6L\,, \quad L = (8\pi F_\pi^2)^{-1} \approx 0.10 \tag{2.55}$$

Next, consider the implications of the Adler PCAC consistency condition (6). It asserts that the amplitude must vanish when the momentum of any one pion goes to zero, all the other pions being held on mass shell. Kinematically this corresponds to the point s = t = u = 1; and the PCAC condition yields the result

$$\alpha + 2\beta + \gamma = 0 \tag{2.56}$$

To complete the analysis one more condition is needed, and here Weinberg introduces an extra physical assumption concerning the σ-term in eq. 2.52. With p = 1 on mass shell, and to lowest order in k = q, the σ-terms is symmetric in the indices a and b. Weinberg now made the assumption that it is, in fact, proportional to $\delta_{ab}\delta_{cd}$ This property is at any rate true in the σ-model (23.) To lowest order in k = q one then has

$$T^{ba}_{cd} \propto \delta_{ab}\delta_{cd}\,, \quad s = u = 1\,, \; t = 0 \tag{2.57}$$

From eq. 2.49 it then follows that

$$\alpha+\beta+\gamma=0$$

Altogether one finds

$$\beta=0\,,\quad \alpha=-\gamma\,,\quad \gamma=\frac{1}{F_\pi^2}\,, \tag{2.58}$$

and for the scattering lengths

$$a_0=\frac{7L}{4}\,,\qquad a_2=-\frac{L}{2}\,, \tag{2.59}$$

giving in terms of the conventional amplitudes for Pi-Pi scattering (24) the following

$$A(s,t,u)=\frac{1}{F_\pi^2}\left(s-\mu^2\right)$$

$$B(s,t,u)=A(t,s,u)\,;\quad C(s,t,u)=A(u,t,s).$$

CHAPTER 3 : STRUCTURE IN S-WAVE $\pi\pi$ SCATTERING

3.1 Introduction

Weinberg's current algebra amplitude, described in the previous chapter, embodies one of the most interesting predictions for the Pi-Pi interaction, namely the presence of sub-threshold S-wave zeros, and consequently rather small S-wave scattering lengths. This property came as rather a surprise at first, since it was completely at variance with the well-known S-wave dominance theory of Chew and Mandelstam (24).

In this chapter we analyse a representative set of S-wave Pi-Pi phase shifts; taking these in conjunction with rigorous crossing sum rules (25), we show how it is possible to investigate the occurrence, or otherwise, of S-wave zeros.

To take into account the threshold branch point we use a dispersion relation, written not for the S-wave amplitude itself, but for its inverse (26). This is because, firstly, in inverse partial wave amplitude satisfies a very simple elastic unitarity equation, which can be solved in closed form; and secondly, because for the inverse amplitude the effects of possible zeros are potentially large - they appear as poles.

In section two we derive from the inverse amplitude partial wave dispersion relation, a generalised effective range representation for both S-wave inverse amplitudes (I = 0,2). In section three we describe in detail the model dependent parts of the input used in the analysis. The quantities which are model dependent are (i) the real part of the S-wave amplitude, $\mathrm{Re}\,A^{I}(s)$, for $s < 0$, i.e. on the left-hand cut, and (ii) the ratio of the inelastic to the elastic S-wave partial cross sections. In section four we describe how experimental information is obtained on the Pi-Pi interaction, and in particular discuss the Pi-Pi phase shifts which are used in this analysis. The rigorous constraints, which follow from crossing, are set out in section five,where we also define quantities by which we measure the violation of the constraints. In section six we describe various plausible models for the S-wave Pi-Pi interaction. For instance, the data suggest a simple scattering length model, which gives a reasonable description of the phase shifts. However, since both $I = 0$ and $I = 2$ amplitudes are available we can construct the S-wave amplitude for the symmetric process $\pi^0\pi^0 \to \pi^0\pi^0$ and can then immediately rule out the model because of its violation of an elementary crossing test. In a similar way other models giving good fits to the phase shifts can be eliminated, and we conclude that the simplest account of the data consistent with the rigorous constraints is given by a model containing a sub-threshold zero in both S-waves. The favoured parameter values (scattering lengths, etc.) are similar to those predicted by current algebra, and the amplitude zeros can be identified with the PCAC Adler zeros.

In this chapter we use units in which the pion mass is one and amplitude normalisation such that $N_e = 1$ (see appendix A).

3.2 Dispersion Relations

Let A^0 and A^2 be respectively the S-wave $I_s = 0$ and $I_s = 2$ partial wave amplitudes. Unitarity for $s \geqslant 4$ reads:

$$\mathrm{Im}\, A^I(s) = \rho(s)\left(1 + r^I(s)\right)\left|A^I(s)\right|^2, \tag{3.1}$$

where

$$\rho(s) = \sqrt{\frac{s-4}{s}} \tag{3.2}$$

and where $r^I(s)$ is the ratio of the inelastic to the elastic S-wave cross-section. Let B^I denote the inverse of A^I, then eq. 3.1 becomes in terms of B^I:

$$\mathrm{Im}\, B^I(s) = -\rho(s)\left(1 + r^I(s)\right). \tag{3.3}$$

In the scattering region below the first inelastic threshold, s_{in}, one has in terms of a real phase shift, $\delta^I(s)$, that:

$$\mathrm{Re}\, B^I(s) = \rho(s) \cot \delta^I(s). \tag{3.4}$$

For $s < 0$, on the left-hand cut, the discontinuity of B^I is given by:

$$\mathrm{Im}\, B^I(s) = -\mathrm{Im}\, A^I(s) / \left|A^I\right|^2. \tag{3.5}$$

To calculate this quantity one needs a knowledge of both Re A^I and Im A^I for $s < 0$.

One now writes for both S-wave inverse amplitudes a dispersion relation subtracted once at threshold (26).

For $4 \leqslant s < s_{in}$, one obtains the following generalised effective range formula:

$$\rho \cot \delta = \frac{1}{a_I} + \frac{\rho}{\pi} \ln\left(\frac{1+\rho}{1-\rho}\right) + (s-4)\left(L^I + R^I + P^I\right). \qquad (3.6)$$

Here a_I is the scattering length, $(=\mathrm{Re}A^I(4))$, and the logarithmic function comes from the integral over the elastic region of the right-hand cut. The terms L^I, R^I and P^I arise respectively from the left-hand cut integration, the inelastic part of the right-hand cut integration, and any possible poles, that is zeros of $A^I(s)$, so one has:

$$L^I = -\frac{1}{\pi}\int_{-\infty}^{0} \frac{\mathrm{Im}A^I(s')}{|A^I(s')|^2} \frac{ds'}{(s'-s)(s'-4)} \qquad (3.7)$$

$$R^I = -\frac{1}{\pi}\int_{s_{in}}^{\infty} \frac{\rho(s')\, r^I(s')\, ds'}{(s'-s)(s'-4)} \qquad (3.8)$$

and

$$P^I = \sum_p \frac{C_p^I}{(s-s_p^I)(s_p^I-4)}, \qquad (3.9)$$

where in eq.3.9 if $A^I(s_p) = 0$ then

$$C_p^I = \left[\left. \frac{dA^I(s)}{ds} \right|_{s=s_p^I} \right]^{-1} \qquad (3.10)$$

It should be noted in passing that since $r^I(s) \geqslant 0$, then with $4 \leqslant s < s_{in}$, one has

$$R^I \leqslant 0 \tag{3.11}$$

We shall assume that a representation of the form of eq. 3.6 is valid, with all the integrals existing, for both S-waves up to $\sqrt{S} \approx 1\,\mathrm{GeV}$. Experimentally this marks the onset of inelasticity (27-29) (e.g. 4π and $k\bar{k}$ production), so we take s_{in} to be about 50. In particular eq. 3.6 should be valid in the region of the rho meson, $\sqrt{S} \approx 760$ MeV.

3.3 Models

(i) Left-hand cut Integrals

One of the consequences of axiomatic field theory is that there are two subtractions required in fixed-s dispersion relations for $-28 \leqslant s \leqslant 4$. If one assumes Mandelstam analyticity this region is extended to $-32 \leqslant s \leqslant 4$. With this result one has that the Froissart-Gribov projection converges in the above region for $\ell \geqslant 2$. That is for $\ell \geqslant 2$ one has:

$$A_\ell^I(s) = \frac{2}{\pi(s-4)} \left\{ \int_4^\infty D_t^I(s,t_1)\, Q_\ell\left(1+\frac{2t_1}{s-4}\right) dt_1 - \int_4^\infty D_u^I(s,u_1)\, Q_\ell\left(-1-\frac{2u_1}{s-4}\right) du_1 \right\} \tag{3.12}$$

by noting that $\mathrm{Im}Q_\ell(Z) = -\frac{\pi}{2}P_\ell(Z)$ for $-1 \leqslant Z \leqslant 1$ one can write:

$$\mathrm{Im}A_\ell^I(s) = \frac{1}{s-4} \left\{ \int_4^{4-s} D_t^I(s,t_1)\, P_\ell\left(1+\frac{2t_1}{s-4}\right) dt_1 + \int_4^{4-s} D_u^I(s,u_1)\, P_\ell\left(-1-\frac{2u_1}{s-4}\right) du_1 \right\} \tag{3.13}$$

eq. 3.13 exists for all $\ell \geqslant 0$ and we will use this representation to calculate $\mathrm{Im}A_0^I(s)$ on the left-hand cut. In principle this can only be used for $S \geqslant -32$, since it is at this point that one meets the (t,u)-double spectral function; nevertheless, we follow the usual practice of ignoring the divergence, and will calculate the left-hand cut for $S \geqslant -50$ as if the third double spectral function were absent. Using crossing one can write:

$$D_t^I(s,t_1) = \sum_{I'} C_{st}^{II'} \, D_t^{I'}(s,t_1) \tag{3.14}$$

and

$$D_u^I(s,u_1) = \sum_{I'} (-)^{I+I'} C_{st}^{II'} \, D_t^{I'}(s,u_1). \tag{3.15}$$

Also one can expand the absorptive parts $D_t^{I'}$ and $D_u^{I'}$ in partial wave series

$$D_t^{I'}(s,t_1) = \sum_{(I'+\ell')\text{- even}} (2\ell'+1)\, \mathrm{Im}\, A_{\ell'}^{I'}(t_1)\, P_{\ell'}(z_{t_1}). \tag{3.16}$$

together with the same expression for $D_u^{I'}(S,U_1)$, but with t and u interchanged. Combining the above three equations and substituting in eq. 3.13 one obtains:

$$\mathrm{Im}A_\ell^I(s) = (s-4)^{-1} \times$$

$$\Bigg\{ \int_4^{4-s} dt_1\, P_\ell\Big(1+\frac{2t_1}{s-4}\Big) \sum_{I'} C_{st}^{II'} \sum_{(I'+\ell')\text{-even}} (2\ell'+1)\, \mathrm{Im}\, A_{\ell'}^{I'}(t_1)\, P_{\ell'}\Big(1+\frac{2s}{t_1-4}\Big) +$$

$$\int_4^{4-s} du_1\, P_\ell\Big(-1-\frac{2u_1}{s-4}\Big) \sum_{I'} (-)^{I+I'} C_{st}^{II'} \sum_{(I'+\ell')\text{-even}} (2\ell'+1)\, \mathrm{Im}\, A_{\ell'}^{I'}(u_1)\, P_{\ell'}\Big(-1-\frac{2s}{u_1-4}\Big) \Bigg\} \tag{3.17}$$

One can now combine the two terms on the r.h.s. of the equation by noting that $P_\ell(-Z) = (-1)^\ell P_\ell(Z)$, and so the second expression is equal to the first apart from the factor $(-1)^{I+I'+\ell+\ell'}$, but this is equal to 1 because of Bose statistics and hence one has:

$$\mathrm{Im}A_\ell^I(s) = \frac{2}{s-4}\int_4^{4-s} P_\ell\left(1+\frac{2t_1}{s-4}\right)\times$$

$$\times\sum_{I'} C_{st}^{II'} \sum_{(I'+\ell')\text{-even}} (2\ell'+1)\,\mathrm{Im}A_{\ell'}^{I'}(t_1)P_{\ell'}\left(1+\frac{2s}{t_1-4}\right)dt_1. \qquad (3.18)$$

For $\mathrm{Im}A_{\ell'}^{I'}(t_1)$ we take the narrow resonance approximation, namely:

$$\mathrm{Im}A(t_1) = \frac{\pi M\Gamma}{\rho(M)}\,\delta(M^2-t_1) \qquad (3.19)$$

Where M and Γ are respectively the mass and width of the particular resonance involved. Taking a Breit-Wigner form rather than the δ-function form in eq. 3.19. makes very little overall difference. We find that D- and higher spin exchanges (f(1260) and g(1650)) have a negligible effect.

For $s<0$ the real part of A^I cannot be calculated directly from crossing because of divergence difficulties with the Q_ℓ functions. For instance, taking a P-wave exchange of mass M, then, in the narrow resonance approximation, the ReQo(Z) gives rise to a term of the form

$$\mathrm{Re}\,A^I(s) \propto \mathrm{Re}\,Q_0\left(1+\frac{2M^2}{s-4}\right) = \frac{1}{2}\ln\left|\frac{s+M^2-4}{M^2}\right|, \qquad (3.20)$$

this of course diverges at $S = 4-M^2$, which is in the range of interest. One has, therefore, to resort to model values for Re $A^I(s<o)$. From a recent semi-phenomenological calculation of low energy Pi-Pi phase shifts using partial wave dispersion relations, crossing symmetry, and rigorous sum rules (30), one finds that to a good approximation the required quantities are slowing varying, with typical values:

$$\mathrm{Re}A^o = -1.2 \quad \text{and} \quad \mathrm{Re}A^2 = 0.5 \tag{3.21}$$

in the region of interest. The results we quote use these values, to which we may attach nominal errors of $\pm$ 25% to take account of the neglected S-dependence.

(ii) Inelasticity

To evaluate the absorption terms $R^I(S)$, one needs some knowledge of the ratio of the S-wave partial cross-sections. All that can be said at present is that ratio must be non-negative and significantly different from zero in the region of the $k\bar{k}$ threshold. Experiments with various possibilities for $r^I(S)$ lead to the conclusion that for present purposes it is quite adequate to use a simple model, where for $S > S_1$ we have

$$r^I(S) = \bar{r}^I \text{ (constant)} \tag{3.22a}$$

and for $S_{in} \leqslant S \leqslant S_1$

$$r^I(S) = \bar{r}^I(S-S_{in})/(S_1-S_{in}). \tag{3.22b}$$

That is, $r^I(S)$ rises linearly from threshold at $S=S_{in}$ to a constant value $\bar{r}^I(S)$ at and beyond $S = S_1$. We take $S_{in} = 50$, and choose $S_1 = 51$. So for each partial wave the inelastic effects are characterised by the single strength parameter, $\bar{r}^I$. Since $\bar{r}^I \geqslant 0$, all models give, after integration, very similar s-dependence between $S = 0$ and $\sqrt{S} = 1$ GeV, and the data are not good enough to resolve them. In fact, simply because $r^I(s)$ appears inside an integral, any detailed structure will be "washed-out". With this model for $r(s)$, the integral over the inelastic part of the r.h.c. is easily evaluated, giving the following:

$$-\frac{(s-4)}{\pi}\int_{S_1}^{\infty}\frac{\rho(s')r(s')ds'}{(s'-s)(s'-4)} = \bar{r}\,\frac{\rho}{\pi}\,\ln\left(\frac{1+\rho}{1-\rho}\,\frac{\rho_1-\rho}{\rho_1+\rho}\right)$$

where $\qquad \rho_1 \equiv \rho(s_1) \qquad$ (3.8a)

3.4. Data

We look forward to the day when we have pions in the ISR, but for the time being we have to be content with experimental information obtained by indirect means. Most of our knowledge of Pi-Pi scattering comes from the investigation of peripheral pion production; the reactions in question being

$$\begin{array}{llll}
\pi^- p \to \pi^-\pi^0 p & (a) & \pi^+ n \to \pi^+\pi^- p & (e) \\
\pi^- p \to \pi^-\pi^+ n & (b) & \pi^+ p \to \pi^+\pi^+ n & (f) \\
\pi^- p \to \pi^0\pi^0 n & (c) & \pi^+ p \to \pi^+\pi^- \Delta^{++} & (g) \\
\pi^+ n \to \pi^0\pi^0 p & (d) & \pi^- p \to \pi^-\pi^- \Delta^{++} & (h)
\end{array}$$

(3.23).

which all allow one pion exchange (OPE, see fig. 1) Apart from (a), where a clear ω-like contribution is seen (31), the above reactions are all charge exchange, so forbidding any I = 0 exchange.
Goebel (32) and Chew and Low (33) suggested that cross-sections for on-shell Pi-Pi scattering could be extracted from the observed differential cross-sections of the above reactions by an extrapolation to the pion pole at $\Delta^2 = -\mu^2$, the physical region being Δ^2 positive.
Fig. 2 shows the phase shift data we consider. They are taken from the paper of Baton, Laurens and Reignier (28), who using the reaction (3.23a) first obtained δ_1^1 and δ_0^2; the advantage in that being the absence of any Pi-Pi isospin zero contribution. The Chew-Low extrapolation using conformal mapping techniques was applied, which is supposed to deal automatically with the ω. The δ_0^2 is clearly negative, slowly decreasing in the region 600 MeV $\lesssim \sqrt{s} \lesssim$ 850 MeV. At the rho mass a value of $\delta_0^2 = -15^\circ \pm 5^\circ$ seems likely. Information on δ_0^0 in the rho region comes from the study of S-P interference in the reaction (3.23b). Fig. 2 exhibits the well-known "up-down" ambiguity, which is inherent to the study of S-P interference only, in which, neglecting δ_0^2, one measures $\sin\delta_1^1 \sin\delta_0^0 \times \cos(\delta_1^1 - \delta_0^0)$ so the ambiguity $\delta_0^0 \to \frac{\pi}{2} + (\delta_1^1 - \delta_0^0)$ results. The two solutions differ above $\sqrt{s} \approx$ 700 MeV, a resonant one going up smoothly through 90° at $\sqrt{s} \approx$ 740 MeV, and a non-resonant one hanging below this value.

3.5 Constraints

As we have said, the advantage of the inverse amplitude dispersion relation is that analyticity and unitarity are readily incorporated. Unfortunately it seems impossible to build crossing symmetry into this framework; but a necessary and sufficient condition for a candidate set of partial wave amplitudes to belong to a crossing symmetric amplitude, is that they satisfy the "Roskies constraints" (25).

Introducing

$$g\ (s) = 2A^0 - 5A^2$$

and

$$f\ (s) = A^0 + 2A^2 \tag{3.24}$$

the sum rules involving S-waves only are:

$$I_1 = \int_0^4 (s-4)(3s-4)f(s)\,ds = 0 \tag{3.25a}$$

and

$$I_2 = \int_0^4 (s-4)g(s)\,ds = 0. \tag{3.25b}$$

The above two sum rules are used to test the various possibilities for the S-wave amplitudes after the data have been fitted.

The inverses of the amplitudes required are calculated for $o<s<4$ directly from eq. 3.6 by continuing counter-clockwise around the threshold branchpoint:

$$\rho(s) \rightarrow |\rho(s)|\, e^{i\pi/2} \tag{3.26}$$

$$\rho(s)\ ln\left(\frac{\rho(s)+1}{\rho(s)-1}\right) \rightarrow 2|\rho(s)|\arctan\left(\frac{1}{|\rho(s)|}\right) \tag{3.27}$$

$$\rho(s)\ ln\left(\frac{\rho_1+\rho}{\rho_1-\rho}\right) \rightarrow 2|\rho(s)|\arctan\left(\left|\frac{\rho_1}{\rho}\right|\right) \tag{3.28}$$

We note that there are some simple implications of the sum rules, eq. 3.25, which may be easily tested without having to resort to detailed numerical integration:

(i) From eq. 3.25b, $g(s)$ must have at least one sign-change in the range $o < s < 4$.

(ii) From eq. 3.25a, integrating by parts one has (34)

$$\int_0^4 s(s-4)^2 f'(s)ds = 0 \tag{3.29}$$

implying that $f(s)$ must have at least one turning point for $o < s < 4$. In fact, it can be shown, by also taking into account analyticity and unitarity, that $f(s)$, which is the S-wave amplitude for the process $\pi^o\pi^o \rightarrow \pi^o\pi^o$ must have a unique minimum in the range $1.127 < s < 1.697$ (35)

Introducing

$$\hat{g}(s) = 2A^0 + 5A^2$$

$$\hat{f}(s) = A^0 - 2A^2 \tag{3.24a}$$

we check the "Roskies Constraints" in terms of the quantities

$$X_1 = \left|\frac{I_1}{\hat{I}_1}\right| \quad \text{and } X_2 = \left|\frac{I_2}{\hat{I}_2}\right|$$

where $\hat{I}$ is the same as I, but with $g(f)$ replaced by $\hat{g}(\hat{f})$.

3.6 Fits

With the experimental data for δ^I, plus the relevant crossed-channel exchanges, we can calculate the quantities:

$$Q^I(s) = \rho\left\{\cot\delta^I - \frac{1}{\pi}\ln\left(\frac{1+\rho}{1-\rho}\right)\right\} - (s-4)\,L^I(s), \tag{3.30}$$

subject to the conditions of §3.3 (i). From eq. 3.6 we also have that

$$Q^{I}(S) = \frac{1}{a_I} + (S-4)\left(R^{I}(S) + P^{I}(S)\right). \qquad (3.31)$$

That is, the phenomenological quantities $Q^I(s)$ can be described by a model containing, as free parameters, a scattering length, a_I, an absorption "strength" parameter, $\bar{r}^I$, and contributions, P^I, from amplitude zeros, determined by their position, Sp, and residue, $C_p^I/(S_p^I-4)$.

In Fig. 3a we have plotted Q(s) for the two I=o phase-shift solutions of Barton et al, the upper one being the non-resonant branch, the lower the resonant one. Fig. 3b shows the plot of $Q^{-1}(s)$ for the I=2 phase shifts - the inverse function being more convenient here because δ is small. Inspection of these figures suggests the following simple model for the S-waves:

Model I

(i) I = o resonant

$$Q^o(s) = \frac{1}{a_o} + (S-4)\, R^o(S) \qquad (3.32)$$

(ii) I = o non-resonant

$$Q^o(s) = \frac{1}{a_o} \qquad (3.33)$$

That is, for the resonant solution, the downward curvature of Q^o from the constant value of $\frac{1}{a_o}$ is caused entirely by absorption effects, ($R^o < o$). The non-resonant branch is reasonably described by the simple scattering length without any inelasticity, but this seems hardly consistent with the strong $k\bar{k}$ threshold in this channel.

(iii) $I = 2$, the approximation

$$Q^2(s) = \frac{1}{a_2} \tag{3.34}$$

agrees reasonably well with the data. Zero or small inelasticity may be expected since several prominent channels (e.g. $k\bar{k}$, $\pi\omega$ $N\bar{N}$) do not couple to $I=2$.

When Model I is subjected to the tests outlined in §3.5 one finds, in terms of the quantities $X_{1,2}$, that the "Roskies Contraints" are badly violated ($\sim 60\%$). In fact, $f(s)$ has no turning points at all for $o < s < 4$ (much less a minimum in the required place), and $g(s)$ has no sign change below threshold either.

To take account of possible inaccuracies in the l.h.c. contributions, we note that, even allowing 100% changes in L^0 and L^2, makes only very slight difference in the S-wave structure below threshold and has negligible effect on the sum rule values. Therefore, we can conclude that the S-wave crossing sum rules alone are sufficient to eliminate this model for the Pi-Pi S-waves, characterised by large scattering lengths and no amplitude zeros, i.e. the Chew Mandelstam S-wave dominant solution.

Several other plausible models which give reasonable fits to the data can be ruled out on similar grounds, i.e. that they violate one or both crossing constraints by large amounts. We find, in fact, that the only simple models which describe the data adequately and approach satisfaction of eqs. 3.25, all involve a pole below threshold in each channel. In fact, looking closely at Fig. 3, we see, especially for $I=2$, a noticeable trend, which suggests a zero in Q^{-1} in the threshold region.

So we have the following model:

<u>Model II</u>

(i) I = o resonant

$$Q^o(s) = \frac{1}{a_o} + \frac{(S-4)C_p^o}{(S-S_p^o)(S_p^o-4)} + (S-4)R_1^o(S) \tag{3.35}$$

(ii) I = o non-resonant

$$Q^o(s) = \frac{1}{a_o} + \frac{(S-4)C_p^o}{(S-S_p^o)(S_p^o-4)} + (S-4)R_2^o(S) \tag{3.36}$$

(iii) I = 2

$$Q^2(s) = \frac{1}{a_2} + \frac{(S-4)C_p^2}{(S-S_p^2)(S_p^2-4)} + (S-4)R^2(S) \tag{3.37}$$

Here we have added absorption terms to Q^o (non-resonant) and Q^2 to see what "strength" of inelasticity the data suggest. One now performs a least squares fit to each Q^I taking as free parameters: a_I, S_p^I, C_p^I, and $\bar{r}^I$. This, of course, gives an excellent description of the phase shifts. One interesting point emerges at this stage, which is that the best fit to Q^o(non-resonant) requires $\bar{r}_2^o$ to be negative. This is clearly unphysical, but what we can say in this case, is that the best physical fit is obtained with the least absorption (i.e. $\bar{r}_2^o = o$). As we have said before, this is inconsistent with the strong $k\bar{k}$ threshold in this channel. In table 1 we summarize the values of the parameters for the three cases, together with the corresponding current algebra values, taken from Weinberg's amplitude with $F_\pi = 93$ MeV.

The solid lines in figs. 2 and 3 are the results of this kind of model. For comparison (36) fig. 3 also shows (as dashed lines) the predictions, with $R^I = o$, of the simple current algebra model. With regard to the constraints we find, with the resonant solution, that eqs. 3.25 are satisfied to within 3%, while with the non-resonant solution they are satisfied to within 5%.

3.7 Conclusion

Our conclusions may be summarized as follows:

a) phenomenological phase shift data and rigorous crossing sum rules clearly indicate the existence of a zero below threshold in both the S-wave Pi-Pi amplitudes;

b) the position of the zeros, and the slopes of the amplitudes as they change sign, are in good agreement with simple current algebra predictions (see table 1);

c) partly on the basis of crossing sum rules and partly in the knowledge that the $k\bar{k}$ threshold is likely to be a strong influence in the I = o S-wave channel, this approach tends to discriminate between the alternative phase shift solutions of Baton et al. in favour of the one containing the resonance.

As a postscript we should mention something about the rho meson and how the sub-threshold zeros are related to the physical requirement of a resonanting P-wave (37). One knows that the P- and higher partial waves have kinematic zeros at threshold, whereas the S-waves are in principle only bounded by unitarity, which gives

$$|A^I(s)| \leq \sqrt{\frac{s}{s-4}} \qquad (3.38)$$

So if the S-waves do not have zeros near threshold, one might expect them to dominate the whole Pi-Pi scattering amplitude in the low energy region. However, the work of Chew, Mandelstam and Noyes (38) has shown that if the S-waves do dominate there can be no rho meson. A careful and detailed analysis along the lines of that discussed in the present chapter, but including the P-wave, has recently been performed (39), and the conclusions are substantially similar to the above.

Table 1

Parameter values for the fits to Q(s) shown in the figures and described in the text. The bracketed numbers are those expected on the basis of current algebra, ref. (7).

	a	s_p	C	$\bar{r}$
I = 0	0.19	0.59	18.2	8.2
(resonant)	(0.16)	(0.5)	(22)	
I = 0	0.20	0.50	21.5	0.0
(non-resonant)				
I = 2	-0.05	1.85	-39.2	0.16
	(-0.05)	(2.0)	(-44)	

Figure Captions

Fig. 1: Pion production by exchange pion.

Fig. 2: Phase shift solutions of Barton et al. (28), showing the best fits as in Fig. 3 and Table 1.

Fig. 3: (a) Plots of $Q(s)$ for the two $I = o$ phase shift solutions of Barton et al., the upper one being the non-resonant branch, the lower the resonant one.
(b) Plot of $Q^{-1}(s)$ for the $I = 2$ phase shifts of Barton et al.

In both graphs the continuous curves are the best fits described in the text and summarized in Table 1, and the dashed lines are the current algebra curves, (with $F_{\pi} = 93$ MeV).

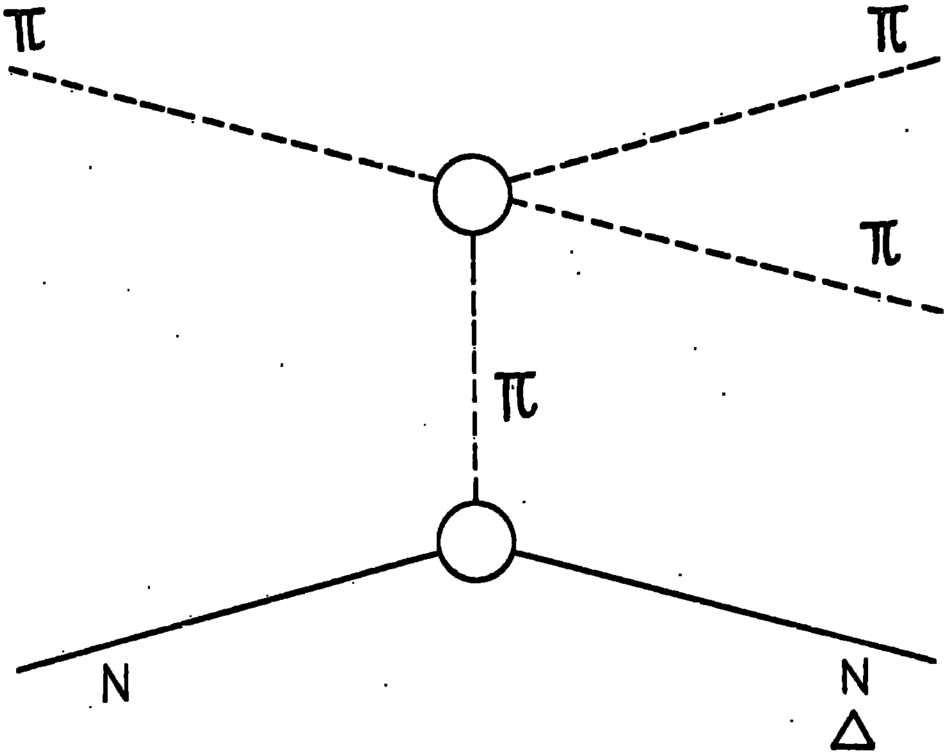


FIG.1

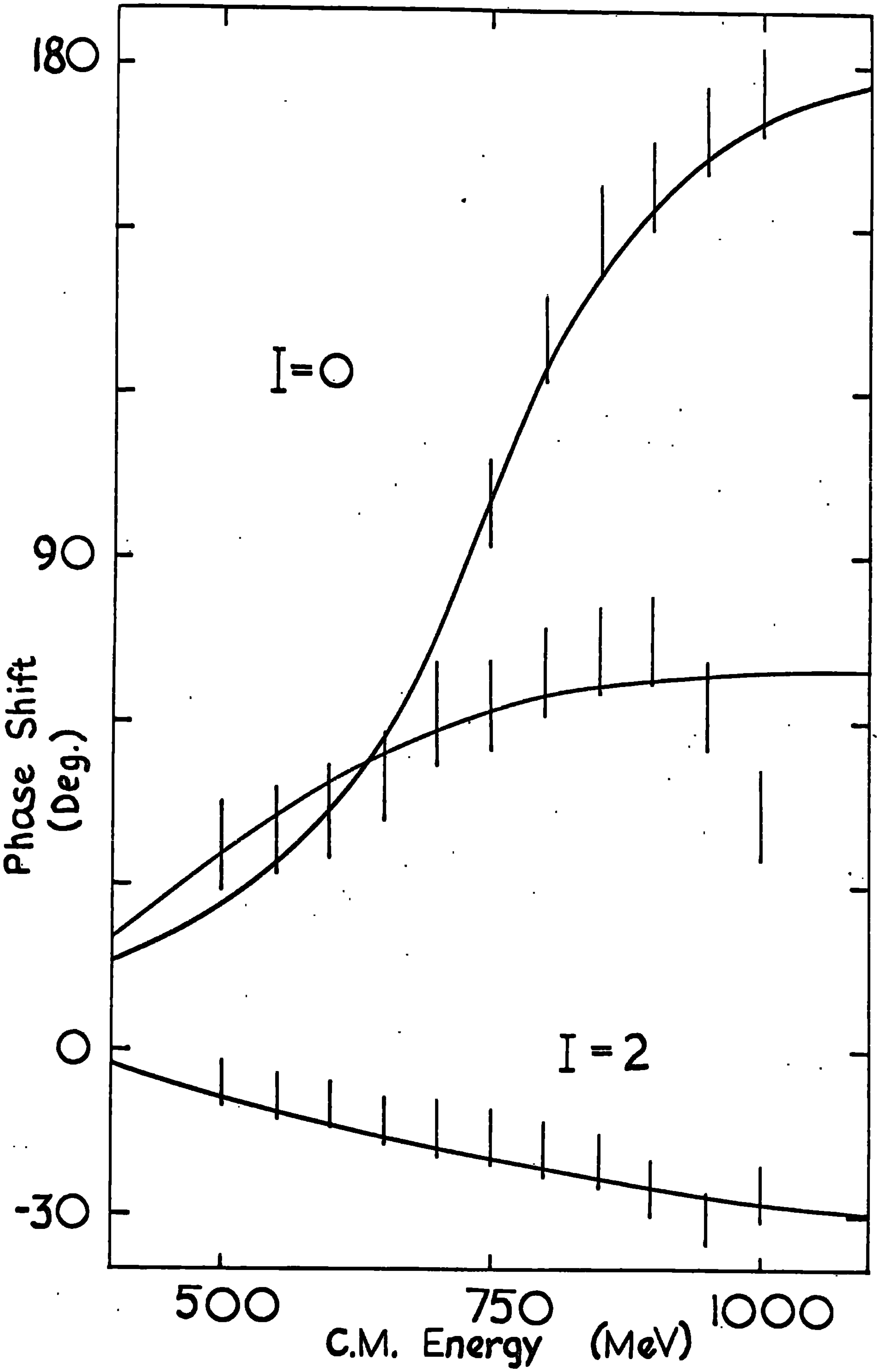


FIG. 2

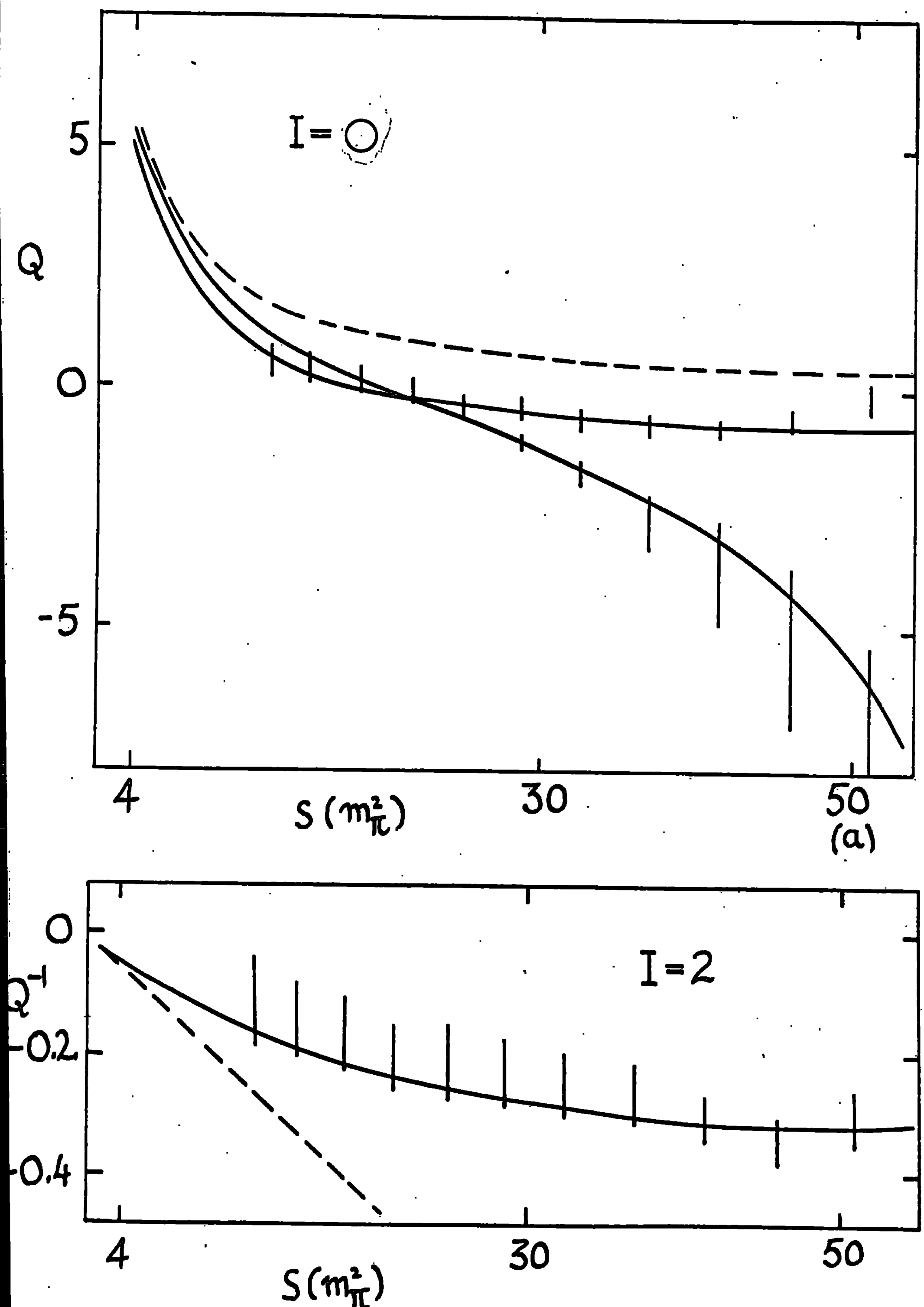


Fig 3(b)

CHAPTER FOUR : $K \to 3\pi$ AND THE GENERALISED POLE MODEL

4.1 Introduction

From the previous chapter we have seen that the prediction of amplitude zeros, as embodied in Weinberg's amplitude, is in fact, borne out in nature. These zeros, as we have said, are also a feature of the single term Veneziano model provided one takes

$$\alpha(m_\pi^2) = 1/2.$$

The next line of investigation, with the hope of gaining more insight into Lovelace's conjecture, is to consider the decays $K \to 3\pi$ and $\eta \to 3\pi$ in the framework of the generalised pole model. The idea that the decays $K \to 3\pi$ and $\eta \to 3\pi$ are dominated by the pole diagram in Fig. 1a is not a new one, and has been the cause of much discussion (40, 41). This assumption immediately explains the similarity of the spectra observed in these decays. When using any model for Pi-Pi scattering which incorporates the zeros required by Adler's self-consistency condition, the amplitude corresponding to Fig. 1a gives all the zeros required by current algebra in the decay $K \to 3\pi$. Unfortunately the same cannot be said for $\eta \to 3\pi$, and this problem will be considered separately in a later chapter.

If one makes some assumption with regard to the behaviour of the amplitude for $K \to 2\pi$ as one of the pions is taken to zero momentum, then it is possible using the above pole model, together with the current algebra results relating $K \to (N+1)\pi$ to $K \to N\pi$, to calculate the rate for $K \to 3\pi$.

The assumption that $M(K \to 2\pi)$ remains constant during this extrapolation leads to an estimate of the rate which is far too large, and it is this result which has caused the pole dominance model to be seriously criticized. (14)

Using the current algebra relations derived in chapter two, together with the extrapolation properties of the pion four-point function,we show how to set up a self consistency condition for the pole dominance model, which leads to a relation between the on-shell amplitude for $K \to 2\pi$ and the same amplitude but with one of the pions at zero momentum. This internal consistency is explained schematically in Fig. 2. With this information, it is possible to make a more realistic estimate of the rate, and one finds that it is indeed possible to get a satisfactory description of both the rate and the spectrum for the decay

The experimental data on the three-pion decay of the kaon show that the probability density for the process has the simple structure of a constant term, plus a small linear dependence on the energy of the "odd" pion. This deviation from linearity is conventionally parameterized as

$$|M|^2 = |M_c|^2 \left\{ 1 + \frac{g(s-s_0)}{m_{\pi^+}^2} \right\}. \qquad (4.1)$$

Here $|Mc|$ is the magnitude of the amplitude at the centre of the Dalitz plot, s = t = u = So. We neglect the pion and kaon mass differences, g is the slope of the decay spectrum and s is the odd pion variable in charged kaon decay and the neutral pion variable in neutral kaon decay. (See Fig. 1.)

The linear form for the amplitude itself was first suggested by Weinberg (42). This idea, which is an essential ingredient in the work of Hara and Nambu and Elias and Taylor (10), could be in serious trouble (43). The non-linearity problem for the matrix element itself comes with the experiments of Albrow et al. (44) and Buchanan et al.(45) on the decay $K_L^0 \to \pi^+\pi^-\pi^0$. In their experiments they fit $|M|^2$ as:

$$|M|^2 = |M_c|^2 \left\{1 + \alpha_1 (QY/M_K) + \alpha_2 (QY/M_K)^2 + \cdots \right\} \quad (4.2)$$

where Q is the total kinetic energy available in the decay and

$$Y = (3T_4 - Q)/Q \quad (4.3)$$

with T_4 the kinetic energy of the neutral pion. (See Appendix B for details). The experiment of Albrow et al. gives for the best quadratic fit

$$\alpha_1 = -5.11 \pm 0.09 \quad \text{and} \quad \alpha_2 = -0.7 \pm 1.2 \quad (4.4a)$$

while for the experiment of Buchanan et al. one has

$$\alpha_1 = -4.36 \pm 0.08 \quad \text{and} \quad \alpha_2 = -1.6 \pm 1.0 \quad (4.4b)$$

First of all we note that there is no strong evidence that a second order term is needed in eq. 4.1, since a linear fit to the spectrum of Albrow et al gives $\alpha_1 = -5.14 \pm 0.09$ with the same χ^2 per D.F. as the quadratic fit. But with a linear expression for the matrix element itself,

$$|M| = |M_c| \left\{1 + \alpha YQ/M_K\right\} \quad (4.5)$$

one obtains:

$$|M|^2 = |M_c|^2 \left\{1 + 2\alpha (YQ/M_K) + \alpha^2 (YQ/M_K)^2\right\} \quad (4.6)$$

showing that in this case

$$\alpha_2 = (\alpha_1/2)^2 \tag{4.7}$$

So if M were linear, and $\alpha_1 = -5.11 \pm 0.09$ as in eq. 4.4a., α_2 would be 6.53 ± 0.24. So taking the data of Albrow et al. at its face value implies a Y^2 term in $|M|$ of magnitude $-(3.6 \pm 0.6)\ (QY/Mk)^2$. At the soft pion point, $E_4 = 0$, this terms will contribute -3.5 ± 0.6 compared with a value 1 from the constant part of $|M|$. But it must be emphasised that the effect of the quadratic term over the Dalitz plot region is less than 15%, and that systematic errors in the experiments will probably be more important at the ends of the spectrum , where the higher order effects seem to reveal themselves, so perhaps the implied non-linearity could be entirely spurious. In Fig. 6 we show the data of Albrow et al. with the best linear and cubic fits to the spectrum. The linear fit gives a χ^2 per D.F. of 1.1 and the cubic fit a χ^2 per D.F. of 0.96, whereas the best fit with a linear matrix element as in eq. 4.5 gives a χ^2 per D.F. of 6.5. Some recent experimental results on the rates are summarized in table 1 (46). We see from these results that the relations obtained by assuming the $\Delta I = \frac{1}{2}$ rule are violated by about 10%. The pole dominance model is, of course, a $\Delta I = \frac{1}{2}$ model, and so we will be aiming for a description of the decays which is good to 90%.

It has been suggested that the diagram shown in Fig. 1b should also be included (41). But we are of the opinion that this diagram is relatively small, and our reasons for this belief are given in section two. In section three, using the Weinberg (7) and Veneziano (5) amplitudes, we show how the consistency condition leads to an estimate of the weak vertex $g(M^2)$. These estimates of $g(M^2)$ are much smaller than those obtained from the simplest soft pion calculation, which gives

$$g(M^2) = 2F_\pi \left| \langle \pi^0 \pi^0 | H_W | K_1^0 \rangle \right| = 7 \times 10^{-2}\ \mathrm{MeV}^2 \qquad (4.8)$$

For the Weinberg amplitude one gets within 17% of the experimental amplitude at the centre of the Dalitz plot, while for the Veneziano model the agreement is to within 28%. In Section four we use a less restrictive model for the pion four-point function, which contains a term depending explicitly on the external masses and which hopefully gives a more reliable off shell extrapolation. Using this we obtain a description of the decay which is within our prescribed limit of ±10%. The results are presented in section five and our conclusions in section six. The details of the kinematics and normalised projections for τ-decay are given separately in Appendix B.

4.2 Other diagrams

In order to estimate the relative contributions of the two diagrams shown in Fig. 1a and 1b it is necessary to have some idea of the q^2 dependence of $g(q^2)$. To this end we start by following Hara and Nambu (10). We have from current algebra, eqs. 2.38, that

$$\begin{aligned} \frac{1}{2F_\pi} \mathcal{M}_{K^+\pi^+} &= \mathcal{M}(K^+ \to \tilde{\pi}^+ \pi^0) \\ &= -\mathcal{M}(K^+ \to \pi^+ \tilde{\pi}^0) \\ &= \mathcal{M}(K_1^0 \to \tilde{\pi}^+ \pi^-) \\ &= \mathcal{M}(K_1^0 \to \pi^+ \tilde{\pi}^-). \end{aligned} \qquad (2.38)$$

If one now assumes that the $K\to 2\pi$ decay vertex is a quadratic function of the meson four-momenta, the most general form of CP-invariant vertex that satisfies eqs. 2.38 is:

$$\mathcal{M}(K^+\to\pi^+\pi^0) = A(q^2_{\pi^0} - q^2_{\pi^+}) \tag{4.9a}$$

and

$$\mathcal{M}(K^0_1\to\pi^+\pi^-) = A(2q^2_K - q^2_{\pi^+} - q^2_{\pi^-}) + B(q^2_{\pi^+} + q^2_{\pi^-} - q^2_K) \tag{4.9b}$$

From eqs. 2.38, implementing energy momentum conservation, one has

$$\begin{aligned} g(q^2) &= \mathcal{M}(K^+(q)\to\pi^+(q)) \\ &= 2F_\pi\,\mathcal{M}(K^+(q)\to\pi^+(0)\pi^0(q)), \end{aligned} \tag{4.10}$$

which, taken together with eqs. 4.9, gives

$$g(q^2) = 2F_\pi A q^2 \tag{4.11}$$

Consider now the particular reaction $K^+\to\pi^0\pi^0\pi^+$; the diagrams which are supposed to contribute are shown in Fig. 1c. summing these diagrams gives us

$$\mathcal{M}(K^+\to\pi^0\pi^0\pi^+) = \mathcal{M}^\pi(K^+\to\pi^0\pi^0\pi^+) + \mathcal{M}^K(K^+\to\pi^0\pi^0\pi^+), \tag{4.12}$$

where

$$\mathcal{M}^\pi(K^+\to\pi^0\pi^0\pi^+) = g(M^2)\frac{1}{M^2-\mu^2}\,\frac{1}{F^2}\,(s-\mu^2) \tag{4.13a}$$

and

$$\mathcal{M}^k(K^+\to\pi^0\pi^0\pi^+) = g(\mu^2)\,\frac{1}{M^2-\mu^2}\,\frac{1}{2F^2}\,(s-2\mu^2). \tag{4.13b}$$

Here for illustration and convenience, we have taken the current algebra amplitudes for the Pi-Pi and Pi-K interactions (47), and $F_\pi^{\,2} = F_k^{\,2} = F^2$. Comparing eqs. 4.13a and 4.13b, gives the relative contributions for Figs. 1a and 1b, and one has

$$\frac{\mathcal{M}^K(00+)}{\mathcal{M}^\pi(00+)} = \frac{1}{2}\,\frac{g(\mu^2)\,(s-2\mu^2)}{g(M^2)(s-\mu^2)}\,, \tag{4.14}$$

where s is in the range $4\mu^2 \leq s \leq (M-\mu)^2$. The maximum value of this ratio in this range is

$$\left. \frac{\mathcal{M}^{k}(00+)}{\mathcal{M}^{\pi}(00+)} \right|_{max} = 0.033. \qquad (4.14a)$$

So, with this point of view, we are justified in neglecting the contributions from the diagram in Fig. 1b.

The opposite point of view is to suppose that $g(q^2)$ is independent of q^2, but this leads to two rather unpleasant results. Firstly, the $K \to 3\pi$ spectra one obtains using the Weinberg or Veneziano amplitudes bear very little resemblance to what is actually the case, in particular including both diagrams in the Veneziano framework leads to a slight positive slope for the decay $K^+ \to \pi^+\pi^+\pi^-$ in contrast with a substantial negative observed experimentally; and, secondly, the rather nice zero structure obtained for τ-decay from Fig. 1a alone is spoilt when Fig. 1b is included (14).

4.3 The Weinberg and Veneziano Amplitudes

By assuming the $\Delta I = \frac{1}{2}$ rule, in other words assuming that H_W is a tensor operator of rank $\frac{1}{2}$, allows us to use the "Wigner-Eckart theorem" which gives us the relation:

$$g(q^2) = \left| \langle \pi^0(q) | H_W | K_L^0(q) \rangle \right| = \left| \langle \pi^+(q) | H_W | K^+(q) \rangle \right| \qquad (4.15)$$

Using this result, we write the amplitudes corresponding to Fig. 1a for the different charge states of $K \to 3\pi$ as

$$M(++-) = g(M^2) \frac{1}{M^2 - \mu^2} \left(B(s,t,u) + C(s,t,u) \right) \qquad (4.16a)$$

$$M(00+) = g(M^2) \frac{1}{M^2 - \mu^2} A(s,t,u) \qquad (4.16b)$$

$$M(+-0) = g(M^2)\frac{1}{M^2-\mu^2}A(s,t,u) \tag{4.16c}$$

Here we have set the pion masses to be equal to μ, and the kaon masses to be equal to M. $A(s,t,u) = B(s,u,t) = C(u,t,s)$ are the conventional amplitudes for Pi-Pi scattering as introduced by Chew and Mandelstam (24). For Weinberg's amplitude one has:

$$A(s,t,u) = \frac{1}{F_\pi^2}(s-\mu^2), \tag{4.17}$$

where F_π (= 93MeV) is defined by eq, 2.19. Using the Veneziano model one has:

$$A(s,t,u) = \frac{(M_g^2-\mu^2)}{\pi F_\pi^2}\left\{ V_{st} + V_{su} - V_{tu} \right\} \tag{4.18}$$

with

$$V_{st} = -\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}, \tag{4.19}$$

and

$$\alpha(s) = \frac{1}{2} + (s-\mu^2)(2M_g^2-\mu^2)^{-1} \tag{4.20}$$

For definiteness we will confine our attention to the process $K^+\to\pi^0\pi^0\pi^+$. Substituting Weinberg's amplitude into eq. 4.16b gives:

$$M(K^+\to\pi^0\pi^0\pi^+) = g(M^2)\frac{1}{M^2-\mu^2}\frac{1}{F_\pi^2}(s-\mu^2) \tag{4.21}$$

Taking the π^+ to zero momentum one obtains:

$$M(K^+\to\pi^0\pi^0\pi^+(0)) = g(M^2)\frac{1}{F_\pi^2}. \tag{4.22}$$

From the current algebra results of chapter two one has

$$M(K^+\to\pi^0\pi^0\pi^+(0)) = \frac{1}{2F_\pi}\left|M(K_s^0\to\pi^+\pi^-)\right|, \tag{4.23}$$

and from eq. 4.15 that

$$g(M^2) = 2F_\pi \left| M(K_S^0 \to \pi^0\pi^0(0)) \right| \tag{4.24}$$

On substituting the above two equations into eq. 4.22 and using the $\Delta I=\frac{1}{2}$ rule, one obtains the following:

$$\left|M(K_S^0 \to \pi^+\pi^-)\right| = \left|M(K_S^0 \to \pi^0\pi^0)\right| = 4\left|M(K_S^0 \to \pi^0\pi^0(0))\right|. \tag{4.25}$$

If instead one uses the Veneziano model one obtains:

$$\left|M(K_S^0 \to \pi^+\pi^-)\right| = \left|M(K_S^0 \to \pi^0\pi^0)\right| = 58\left|M(K_S^0 \to \pi^0\pi^0(0))\right|. \tag{4.26}$$

One is now in a position to calculate $g(M^2)$ since it is given in terms of the physical amplitude for $K \to 2\pi$ together with the appropriate factor from whatever Pi-Pi amplitude one believes.

Why should one believe such a large variation in $M(K \to 2\pi)$ as one pion is taken to zero momentum? Firstly, the $\Delta I=\frac{1}{2}$ rule implies that:

$$\left|\frac{M(K_S^0 \to \pi^0\pi^0(0))}{M(K^+ \to \pi^+\pi^0(0))}\right| = 1, \tag{4.27}$$

whereas in nature, with both pions on shell, the ratio is about 22. This indicates that there must be a large variation in either or both amplitudes as the pion's momentum is taken to zero. Secondly, since

$$g(M^2) = \left|\langle \pi^+(M^2) | H_W | K^+(M^2) \rangle\right| = 2F_\pi \left|M(K^+(M^2) \to \pi^+(M^2)\pi^0(0))\right|, \tag{4.28}$$

We see that as the pion is taken to zero momentum the other pion acquires the kaon mass (i.e. energy-momentum conservation is imposed to give $g(q^2 = M^2)$), so that the extrapolation is not as innocent as it seems.

In fact, it is not difficult to write down an amplitude for $K\to 2\pi$ exhibiting the above behaviour. Taking the amplitudes in eqs. 4.9 and setting

$$B = -2A\left(\frac{M^2+\mu^2}{M^2-2\mu^2}\right), \qquad (4.29)$$

gives on shell

$$|M(K_S^0\to\pi^+\pi^-)| = 4AM^2 \quad , \quad |M(K^+\to\pi^+\pi^0)| = 0.$$

while off shell one obtains:

$$|M(K_S^0\to\pi^+\pi^-(0))| = AM^2 \quad , \quad |M(K^+\to\pi^+\pi^0(0))| = AM^2.$$

4.4 Quadratic Amplitude*

On shell one knows that the Weinberg amplitude does not satisfy the rigorous constraints of Martin (48); so one might expect a somewhat unrealistic off shell extrapolation. In the case of the Veneziano model one has to suppress the rho to be able to describe the process $\bar{p}n\to 3\pi$ (1,12,13) this also suggests a questionable off shell behaviour. We shall be returning to this particular point later when discussing the problem of the zeros in $\eta\to 3\pi$.

We now consider a less restrictive model for the Pi-Pi interaction, to see whether it is possible to obtain the correct rates and spectra for $K\to 3\pi$ while still preserving the many features of the Pi-Pi interaction. To this end we consider the most general quadratic amplitude for the pion four-point function: (49)

$$A(s,t,u) = a+bs + c\,(t+u) + d\,(t^2+u^2) + es\,(t+u) + fs^2 + gtu + h\,\Big(\sum_{i>j} q_i^2\, q_j^2 + 6 - 3\,(s+t+u)\Big) \qquad (4.30)$$

---

* For convenience, in this section we use amplitude normalization with $N\epsilon = 1$, see Appendix A.

Here we have resorted to units of $\mu = 1$. The q_i refer to the pion momenta, and the last term in the expansion is non-zero only when more than one pion is off shell.

To determinethe coefficients we proceed as follows:

(i) The soft pion conditions

(a) Adler consistency condition $A(s=t=u=1) = 0$

(b) Non-exoticity of the σ-term $A(s=t=1, u=0) = 0$

(c) $K \rightarrow 3\pi$ zero $A(s=t=1, u=M^2) = 0$

(d) Adler-Weisberger condition $\frac{\partial A}{\partial x}(1+x,0,1-x) = \frac{1}{32\pi F_\pi^2}$

We combine conditions (a), (b) and (c) into the slightly stronger condition of $A(s=t=1,\ u=M^2) = 0$ for all M^2, with one pion at zero momentum and another at mass M. This gives us the following:

$$a + b + c + d + e + f + h = 0 \qquad \text{(i)}$$

$$c + e + g - h = 0 \qquad \text{(ii)}$$

$$d = 0 \qquad \text{(iii)}$$

condition (d) gives us that

$$b - c - 2d + 2f = (32\pi F_\pi^2)^{-1} \qquad \text{(iv)}$$

(ii) The on shell conditions

In terms of the S-wave scattering lengths, a_0^0 and a_0^2, one has that

$$5a + 12b + 8c + 32d + 48f = a_0^0 \qquad \text{(v)}$$

$$2a + 8c + 32d = a_0^2 \qquad \text{(vi)}$$

while for the P-wave one has

$$b - c + 4e - 4g = \tfrac{3}{4}\, a_1^1 \qquad \text{(vii)}$$

If one now supposes that the P-wave effective range is given by a Breit-Wigner resonance of mass 765 MeV and width 125 MeV one obtains the relation

$$d + e - f - g = \frac{3}{16}\ \frac{a_1^1}{q_m^2} \qquad \text{(viii)}$$

where $\quad 4q_m^2 = M_S^2 - 4.$

In terms of the D-wave scattering lengths, one has that

$$d - e + f = \frac{15}{16}\, a_2^2 \qquad \text{(ix)}$$

$$4d - e + f - \frac{3g}{2} = \frac{15}{16}\, a_2^0 \qquad \text{(x)}$$

Since we are dealing with an amplitude quadratic in s,t, and u, the scattering lengths satisfy exactly the sum rule:

$$2a_0^0 - 5a_0^2 = 18a_1^1 - 30\,(2a_2^0 - 5a_2^2). \qquad (4.31)$$

To obtain the correct rates, in the spirit of section 4.3, we see, taking the $\pi^0\pi^0\pi^+$ decay, that

$$g(M^2) = \frac{(M^2-\mu^2)}{2F_\pi}\,\frac{|M(K_S^0\to\pi^+\pi^-)|}{A(s=M^2, t=u=1)} \qquad (4.32)$$

So one needs that

$$\frac{|M_c(K^+\to\pi^0\pi^0\pi^+)|}{|M(K_S^0\to\pi^+\pi^-)|} = \frac{1}{2F_\pi}\,\frac{A(s=t=u=M^2/3+1)}{A(s=M^2; t=u=1)} \qquad (4.33)$$

For the l.h.s. of the above equation we take an averaged value from experiment. We see from the data (50) that

$$M\,(k_S^0 \to \pi^+\pi^-) = 0.389 \times 10^{-6}\ \text{GeV}$$

and

$$M\,(k_S^0 \to \pi^0\pi^0) = 0.363 \times 10^{-6}\ \text{GeV}$$

Taking these results together with those in Table 1, we obtain

$$\left|\frac{Mc\,(k \to 3\pi)}{M\,(k_S^0 \to 2\pi)}\right| \doteqdot 2.51 \qquad (4.34)$$

So we have that

$$A\,(s = t = u = SO\,) = 5.02 \times F_{\pi}\; A(S = M^2,\; t = u = 1) \quad \text{(xi)}$$

which gives another relation between the eight coefficients in A. Taking eqs. (i) - (v), (vii), (viii) and (xi), one can determine the eight unknowns in terms of the scattering lengths a_0^0 and a_1^1, and F_{π}

4.5 Results

Taking $a_0^0 = 0.19$, $a_1^1 = 0.0365$ and the experimental value of 93 MeV for F_{π} we obtain a quadratic amplitude for Pi-Pi scattering, whose scattering lengths are the following:

$a_0^0 = 0.19$ $\qquad a_0^2 = -0.035$ $\qquad a_1^1 = 0.0365$

$a_2^0 = 0.0022$ and $a_2^2 = 0.0002$.

In Figs. 3 and 4 we have plotted the normalised projections obtained from the above amplitude. From Fig. 5 we see that the amplitude also satisfies all the S-wave $\pi^0\pi^0$ constraints of Martin (48) namely:

$$f\,(4) > f(0)$$

$$f'(s) < 0 \qquad 0 \leqslant s \leqslant 1.12$$

$$f'(s) > 0 \qquad 1.7 \leqslant s \leqslant 4$$

Unique minimum in $1.12 \leqslant s \leqslant 1.7$

$$f(0) > f(3.19)$$

$$f(3.205) > f(0.213) > f(2.986)$$

$$4f'(0) < -(2f(4) - f(2) - f(0)),$$

Where f(s) is the S-wave $\pi^0\pi^0$ partial wave amplitude. The values of the coefficients are as follows:

$a = -32.11$, $b = 25.35$, $c = 3.65$, $d = 0$,

$e = 0.17$, $f = 0.36$, $g = -1.25$, $h = 2.57$.

The rigorous sum rules which follow from crossing (25) will, of course, be satisfied exactly since our form is explicitly crossing symmetric.

4.6 Conclusion

We see that it is possible to set up a realistic model for Pi-Pi scattering, having a mass dependent term when more than one pion is off shell ($h \neq 0$) and which when taken together with the entirely plausible variation of the amplitude for $K \to 2\pi$ can give a satisfactory description of the decay $K \to 3\pi$.

Since the Veneziano model gives a value for the amplitude at the centre of the Dalitz plot which is about 30% too small and from the fact that a non-negligible mass-dependent term is required to obtain the correct rates we can conclude that even by the time one has reached the physical region for τ-decay significant departures from the Lovelace conjecture have occurred, and that satellite terms are probably present. This should also explain the slight discrepancy obtained by Lovelace in his fit to the $K^+ \to \pi^+\pi^+\pi^-$ spectrum.

TABLE 1

Process	Experiment	Weinberg*	Veneziano*	Quadratic
$\frac{1}{2}\left\|M_c(K^+\rightarrow\pi^+\pi^+\pi^-)\right\|$	0.96 ± 0.01	0.80	0.69	0.93
$\left\|M_c(K^+\rightarrow\pi^0\pi^0\pi^+)\right\|$	0.97 ± 0.02	0.80	0.69	0.93
$\left\|M_c(K_L^0\rightarrow\pi^+\pi^-\pi^0)\right\|$	0.86 ± 0.02	0.80	0.69	0.93

*Here we have used the Goldberger-Treiman value for F_π = 85 MeV.

Figure captions:

Fig. 1a: Pion pole dominance diagram in which a kaon (eta) decays weakly (electromagnetically) into a heavy pion, which in turn decays strongly into three pions.

Fig. 1b: The pole diagram in which the kaon (eta) decays strongly into two pions and a light kaon (eta) which in turn decays weakly (electromagnetically) into a pion.

Fig. 1c: The diagrams of the above types which contribute to $K^+ \rightarrow \pi^0\pi^0\pi^+$.

Fig. 2: Schematic representation of the self-consistency of the pole model.

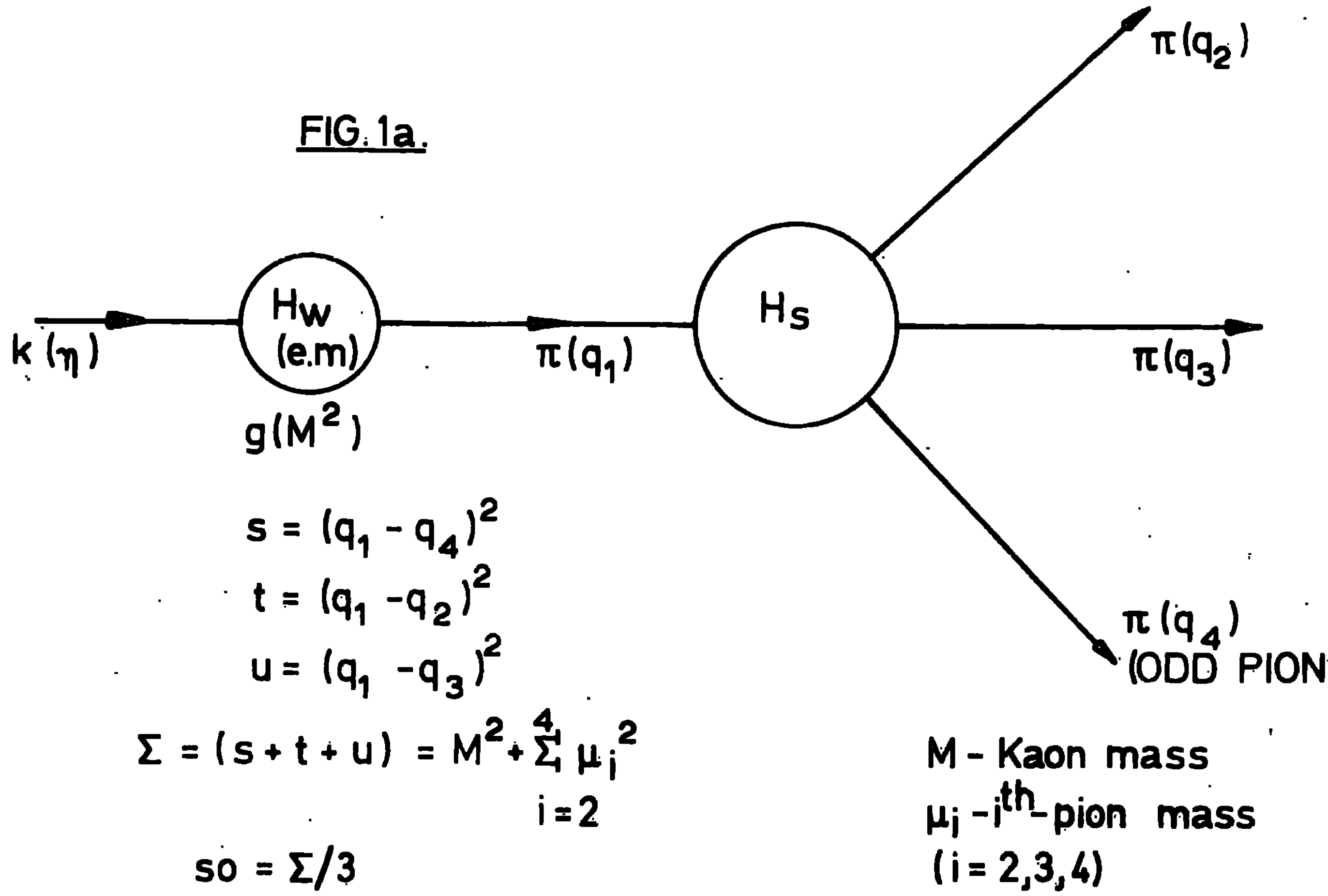
Fig. 3: Plots of the normalized projections for $K^\pm \rightarrow \pi^\pm\pi^\pm\pi^\mp$. The data are from Mast et al. (16)

Fig. 4: (a) Plot of the normalised Y projection for $K^\pm \rightarrow \pi^0\pi^0\pi^\pm$. The data are from Davison et al.(51).
(b) Plots of $A(s = t = u = \frac{M^2}{3} + \mu^2)$ as a function of M^2.
(i) Weinberg
(ii) Veneziano
(iii) Quadratic
(iv) Variation required to obtain the correct rates assuming no variation of $M(K \rightarrow 2\pi)$

Fig. 5: Plot of the S-wave $\pi^0\pi^0$ partial wave amplitude, f(s), for $0 < s < 4$.

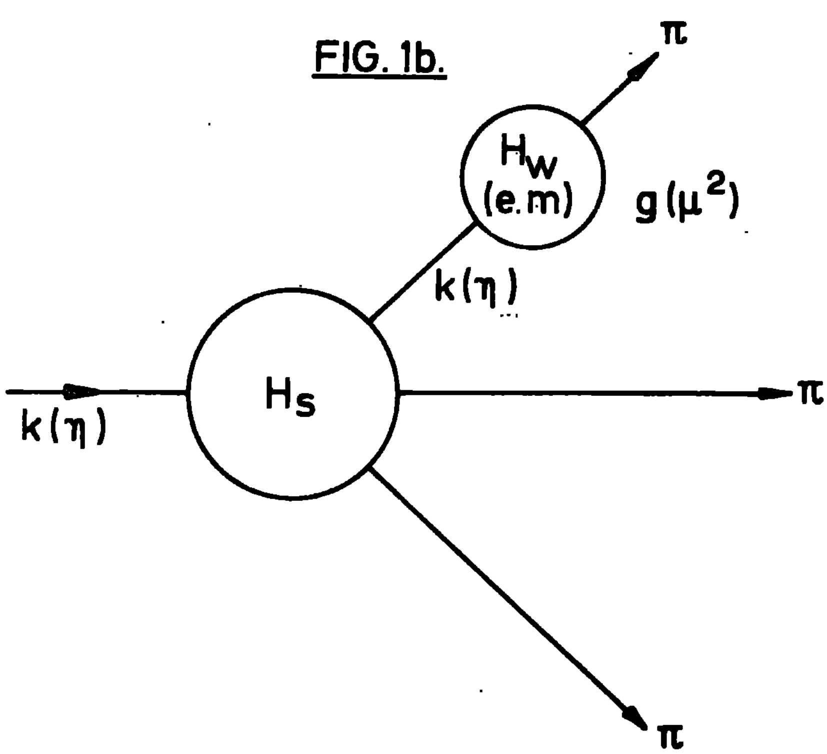
Fig. 6: The normalised Y projection for $K_L^0 \rightarrow \pi^+\pi^-\pi^0$, showing the best linear and cubic fits to the spectrum. The data are from Albrow et al. (44).

FIG. 1a.



H_W
(e.m)
$g(M^2)$
$k(\eta)$
$\pi(q_1)$
H_S
$\pi(q_2)$
$\pi(q_3)$
$\pi(q_4)$
(ODD PION)
$s = (q_1 - q_4)^2$
$t = (q_1 - q_2)^2$
$u = (q_1 - q_3)^2$
$\Sigma = (s + t + u) = M^2 + \sum_{i=2}^{4} \mu_i^2$
$so = \Sigma/3$
M - Kaon mass
μ_i - i^{th} - pion mass
$(i = 2,3,4)$

FIG. 1b.



π
H_W
(e.m)
$g(\mu^2)$
$k(\eta)$
H_S
$k(\eta)$
π
π

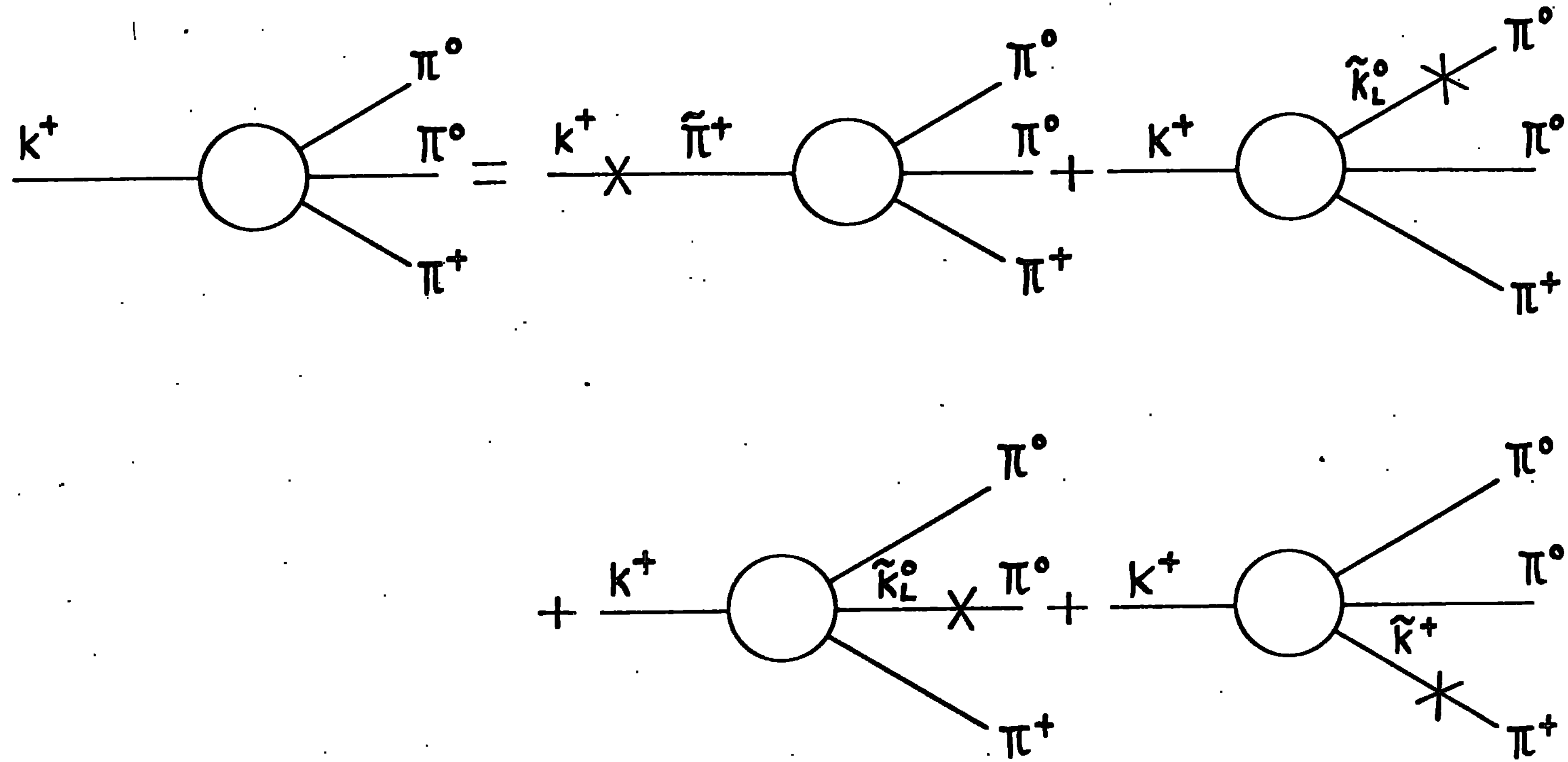
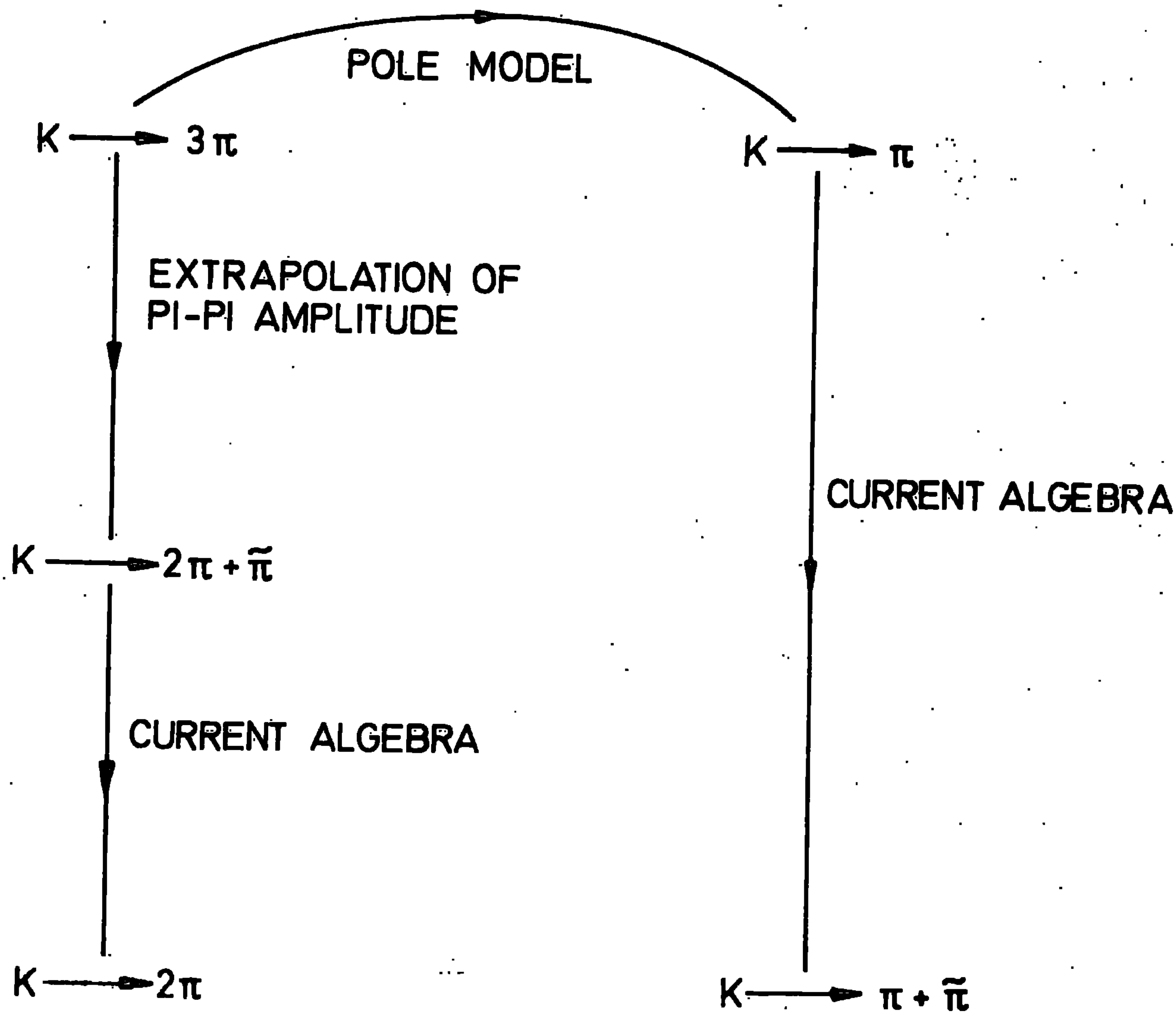


FIG. 1c.

FIG.2.



π ~ ON MASS SHELL PION.
π̃ ~ OFF MASS SHELL PION.

FIG. 3.

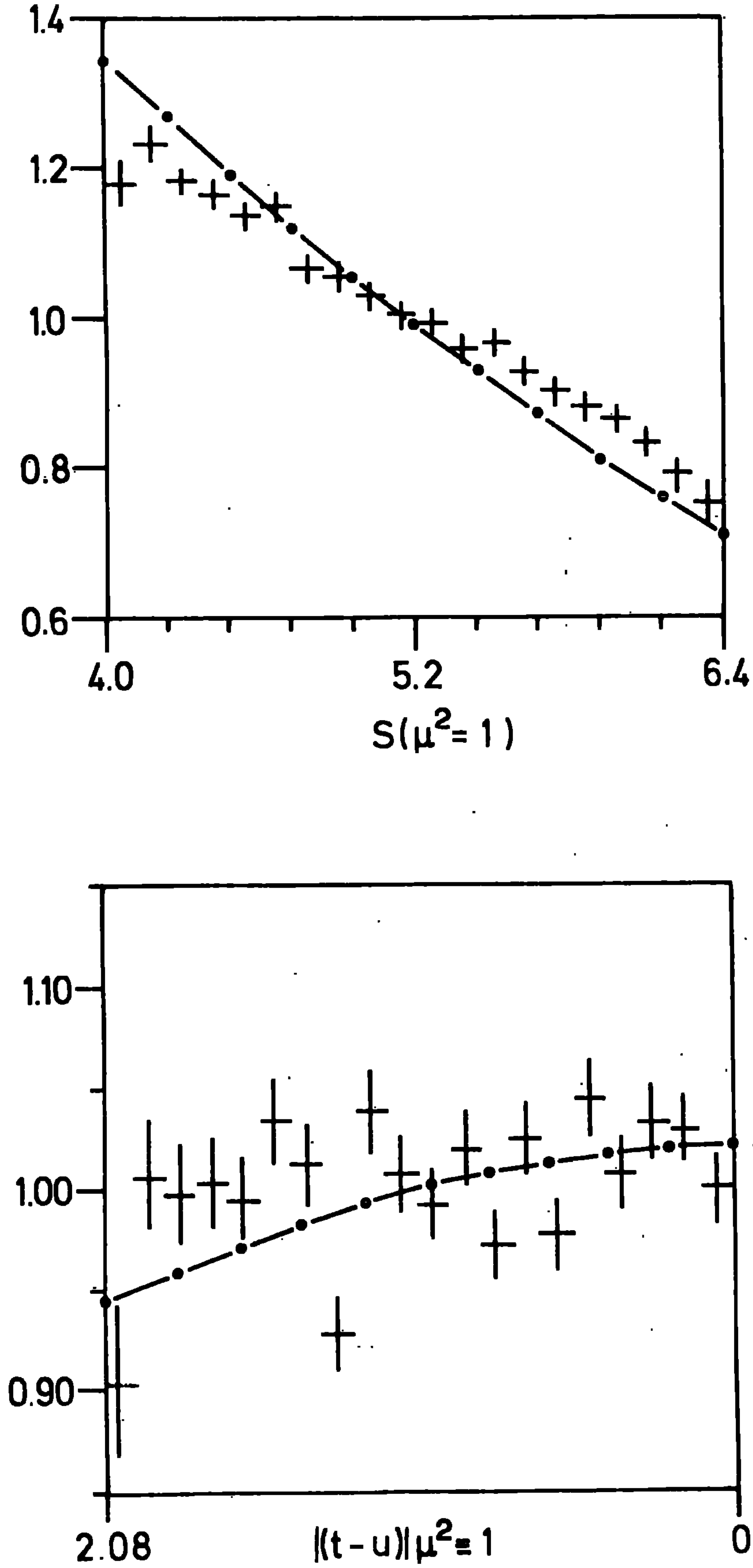


FIG.4a.

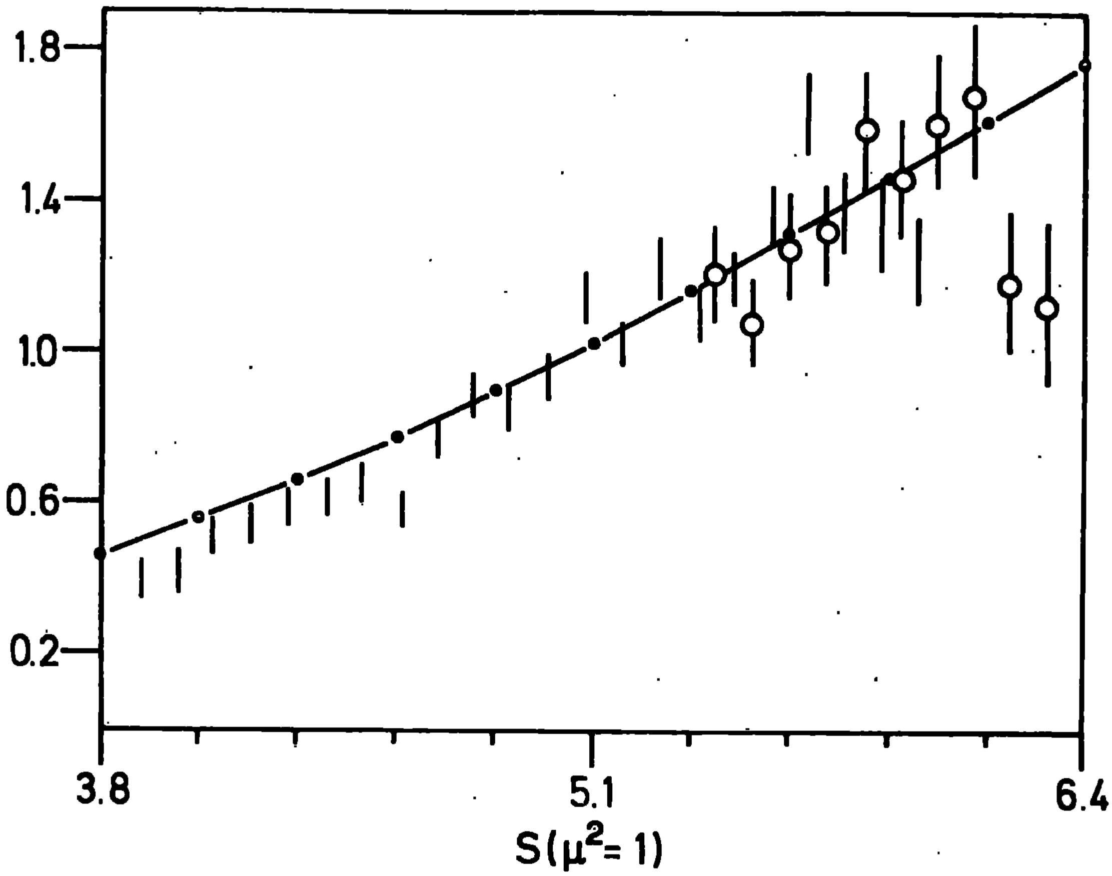
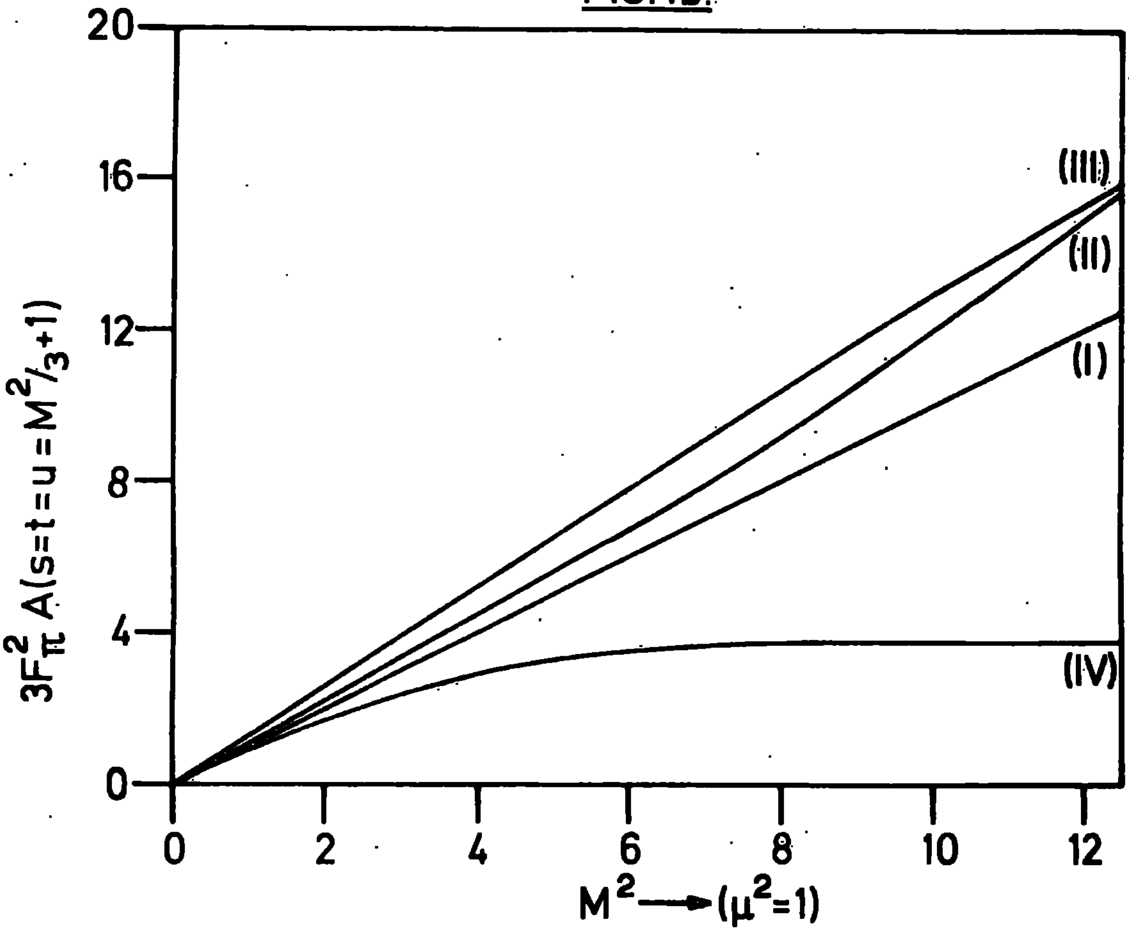

1.8
1.4
1.0
0.6
0.2
3.8
5.1
6.4
S(μ^2= 1)

FIG.4b.


20
16
12
8
4
0
0
2
4
6
8
10
12
$3F_\pi^2 A(s=t=u=M^2/3+1)$
$M^2 \longrightarrow (\mu^2=1)$
(III)
(II)
(I)
(IV)

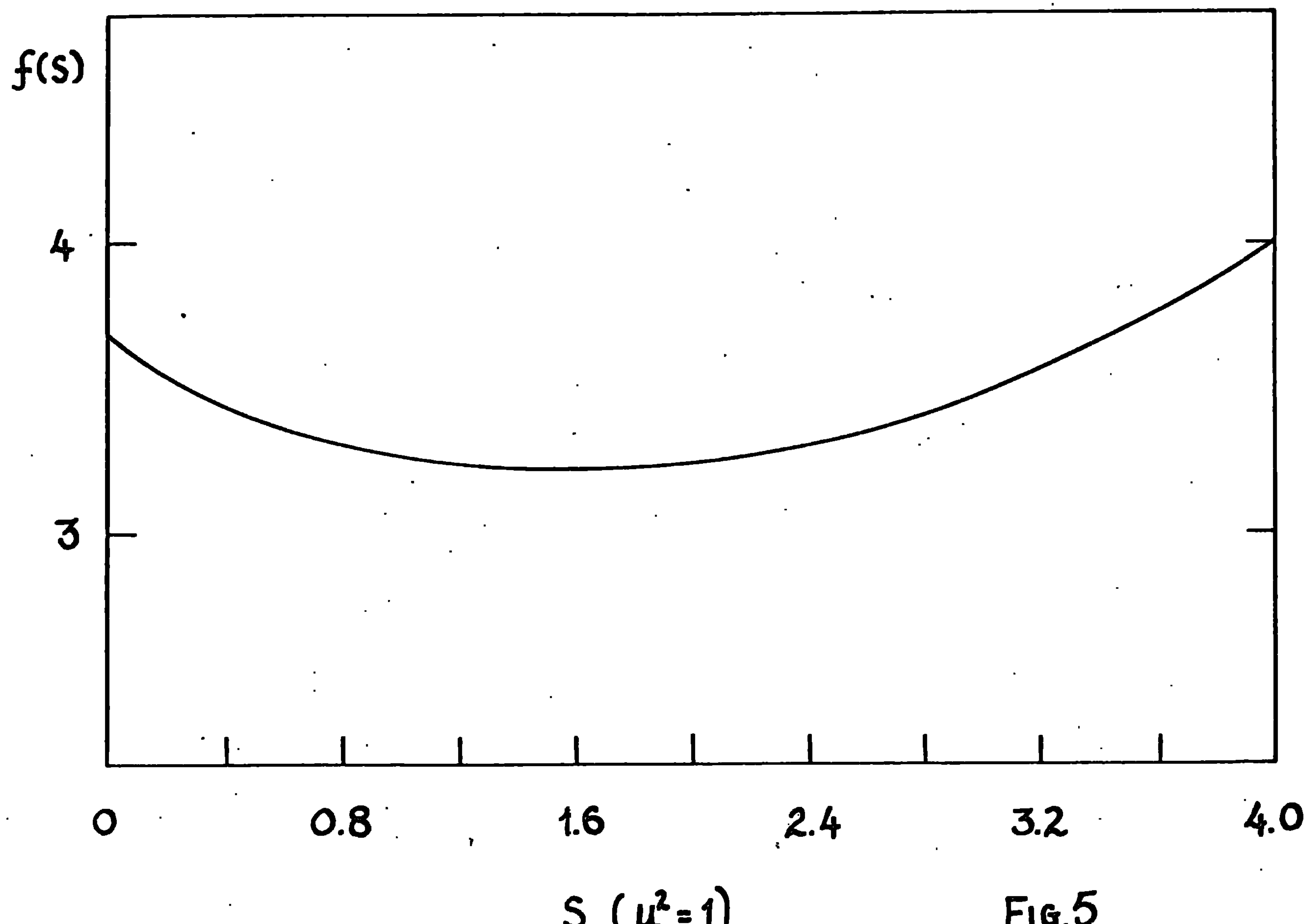


FIG.5.

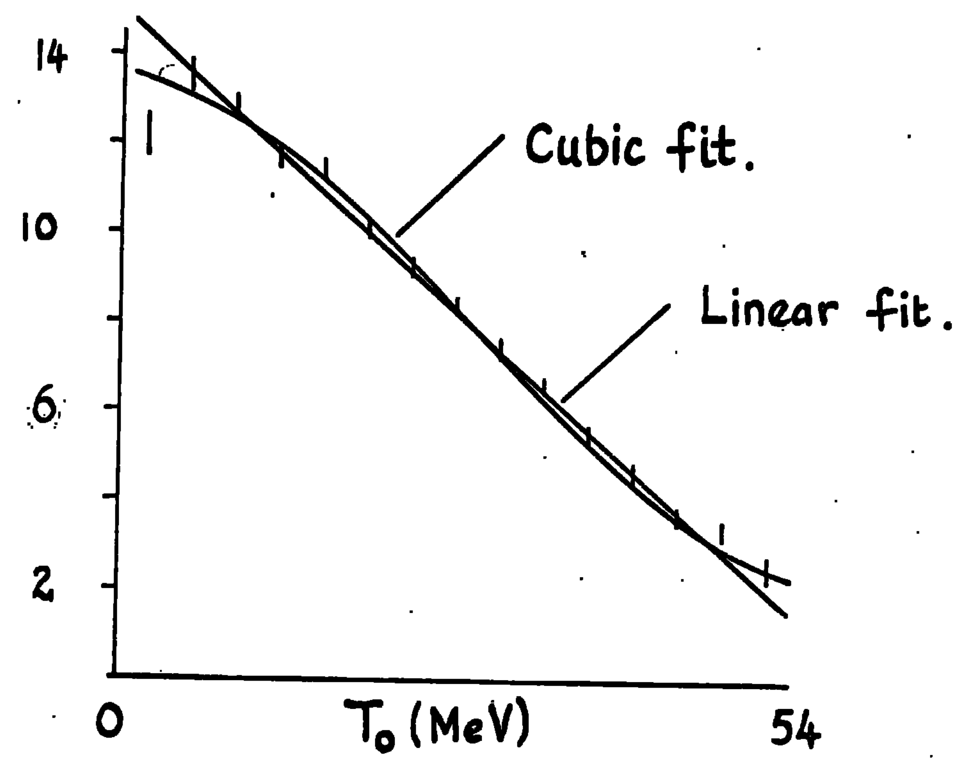


FIG. 6.

CHAPTER FIVE : OTHER KAON DECAYS

5.1 Introduction

In the previous chapter we were able to obtain a satisfactory description of τ-decay. The analysis indicated a value for the coupling, $g_{K\pi}$, rather smaller than was previously supposed. Taking, for example, the quadratic amplitude one obtains

$$g_{K\pi} = \left| \left\langle \pi^{o}(M_K^2) \right| H_W \left| K_L^0 \ (M_K^2) \right\rangle \right| = 1.73\text{x}10^{-2}\text{MeV}^2, \tag{5.1}$$

while for the Weinberg amplitude one has

$$g_{K\pi} = 1.81 \times 10^{-2} \text{ MeV}^2 \tag{5.2}$$

Here we have taken F_π = 93 MeV. The question now arises of how this value of $g_{K\pi}$ fits in with the analysis of other kaon decays where a knowledge of $g_{K\pi}$ is required. To try and answer this question we look at the radiative decay $K_L^o \rightarrow 2\gamma$ and the magnetic radiation term in the decay $K^\pm \rightarrow \pi^\pm \pi^o \gamma$.

In section two we consider two models for $K_L^o \rightarrow 2\gamma$, one proposed by Wong (52), the other by Matsuda and Oppo (53). Both models involve rather strong assumptions, and while it is possible to obtain the correct rate for $K_L^o \rightarrow 2\gamma$ using either model, it is rather difficult to reconcile them. To obtain the correct rate from Wong's model, the η and X^0 poles have to be neglected, whereas they form a significant contribution in the model of Matsuda and Oppo. To avoid the ambiguities caused by the η and X^0 poles, we consider, in section three, the charged decay $K^\pm \rightarrow \pi^\pm \pi^o \gamma$ in the framework of the generalised pole model as discussed by Dass and Kamal (54).

5.2 The radiative decay $K_L^0 \to 2\gamma$.

To start we consider what is possibly the simplest model for this decay. The amplitude for the process is supposed to be given by the sum of the two similar terms shown in Fig. 1 (52). That is,

$$A(K_L^0 \to 2\gamma) = 2 g_{K\pi}(M_K^2) \frac{1}{M^2-\mu^2} g_{\rho\pi\gamma} \frac{(4\pi\alpha)^{1/2}}{f_\rho} \epsilon_{\alpha\beta\gamma\delta} k_1^\alpha \epsilon_{\lambda_1}^\beta k_2^\gamma \epsilon_{\lambda_2}^\delta, \tag{5.3}$$

where α is the fine-structure constant, and k_1 and k_2 are the four-momenta and ϵ_{λ_1} and ϵ_{λ_2} the polarization four-vectors of the photons γ_1 and γ_2, respectively. The coupling $g_{\rho\pi\gamma}$ is assumed to be the same as with all particles on their mass shell, and defined via the vertex function.

$$A(\rho(P) \to \pi + \gamma(k)) = g_{\rho\pi\gamma} \epsilon_{\mu\nu\sigma\lambda} P^\mu \epsilon_\rho^\nu k^\sigma \epsilon_\gamma^\lambda, \tag{5.4}$$

which is in turn related to the partial width via:

$$\Gamma(\rho \to \pi\gamma) = |g_{\rho\pi\gamma}|^2 \frac{m_\rho^3}{96\pi} \left(1 - \left(\frac{m_\pi}{m_\rho}\right)^2\right)^3. \tag{5.5}$$

The coupling $(4\pi\alpha)^{1/2}/f_\rho$ of the rho to the photon is calculated by assuming the one-photon exchange model for the decay $\rho \to e^+e^-$. Taking the Orsay result (55) of

$$\Gamma(\rho \to e^+e^-) = 7.4 \pm 0.7 \text{ keV}, \tag{5.6}$$

gives for f_ρ the following:

$$\frac{f_\rho^2}{4\pi} = 1.85 \pm 0.17. \tag{5.7}$$

From eq. 5.3 we see that the coupling $g_{K_L^0\gamma\gamma}$ is given by

$$g_{K_L^0\gamma\gamma} = 2\, g_{K\pi} \frac{1}{M^2-\mu^2}\, g_{\rho\pi\gamma} \frac{(4\pi\alpha)^{1/2}}{f_\rho} \tag{5.8}$$

But $g_{K_L^0\gamma\gamma}$ may also be calculated from the experimental width $\Gamma(K_L^0\to 2\gamma)$ via

$$|g_{K_L^0\gamma\gamma}| = \sqrt{\frac{64\pi\,\Gamma(K_L^0\to 2\gamma)}{M_K^3}}\,. \tag{5.9}$$

Taking the averaged branching ratio (50) to be 4.9 x 10^{-4} gives from eq. 5.9 that

$$|g_{K_L^0\gamma\gamma}| = 3.182 \times 10^{-12}\ \mathrm{MeV}^{-1}.$$

Taking the quadratic amplitude value for $g_{K\pi}$ i.e. eq. 5.1, we see that eq. 5.8 gives the correct value for $|g_{K_L^0\gamma\gamma}|$ provided that one takes

$$|g_{\rho\pi\gamma}| = 3.345 \times 10^{-4}\ \mathrm{MeV}^{-1}. \tag{5.10}$$

This corresponds to the following partial width:

$$\Gamma(\rho\to\pi\gamma) = 0.15\ \mathrm{MeV}, \tag{5.11}$$

that is, a branching ratio of approximately 0.12%. This is in good agreement with the most recent experiment (56) which has set an upper bound on the branching ratio of 0.2%. This result is very good, but there is no reason why one should not also include the η and X^0 poles, the η going via $\omega\gamma$ and the X^0 via $\rho\gamma$. Both of these contributions are expected to be large since $\frac{f_\omega^2}{4\pi}$ is large ($\doteqdot$ 14) and the decay $X^0\to\rho\gamma$ has a branching ratio of approximately 29%.

Another model which has been used to describe the transition $K_L^0 \to 2\gamma$ is shown in Fig. 2 (53). Here the η and X^0 poles are included and, in fact, turn out to be significant. From Fig. 2 one sees that there are a lot of coupling constants to be determined, and a certain amount of ambiguity is unavoidable. The couplings $g_{P\gamma\gamma}$, where $P \equiv \pi^0, \eta, X^0$, are assumed to be the same as with all particles on their mass shell, and defined via

$$A(P \to 2\gamma) = g_{P\gamma\gamma}\, \epsilon_{\alpha\beta\gamma\delta}\, K_1^{\alpha} \epsilon_{\lambda_1}^{\beta}\, K_2^{\gamma} \epsilon_{\lambda_2}^{\delta} \tag{5.12}$$

Taking the following partial widths:

$$\Gamma(\pi^0 \to 2\gamma) = 7.8 \pm 0.9\ \text{eV} \tag{5.13a}$$

$$\Gamma(\eta \to 2\gamma) = 1.00 \pm 0.08\ \text{keV} \tag{5.13b}$$

$$\Gamma(X^0 \to 2\gamma) \leq 38.0 \pm 6.0\ \text{keV} \tag{5.13c}$$

gives for the respective coupling constants the following

$$|g_{\pi^0\gamma\gamma}| = 2.53 \times 10^{-5}\ \text{MeV}^{-1} \tag{5.14a}$$

$$|g_{\eta\gamma\gamma}| = 3.48 \times 10^{-5}\ \text{MeV}^{-1} \tag{5.14b}$$

$$|g_{X^0\gamma\gamma}| \leq 9.32 \times 10^{-5}\ \text{MeV}^{-1}. \tag{5.14c}$$

It is interesting to compare these values for the coupling constants, with the following broken SU(3) sum rule

$$g_{\pi\gamma\gamma} - \sqrt{3}\cos\alpha\, g_{\eta\gamma\gamma} + \sqrt{3}\sin\alpha\, g_{X^0\gamma\gamma} = 0 \tag{5.15}$$

where α is the η-X^0 SU(3) mixing angle, ($=\pm\ 10.24^0$). Rearranging eq. 5.15 gives

$$g_{X^0\gamma\gamma} = \frac{\sqrt{3}\cos\alpha\, g_{\eta\gamma\gamma} - g_{\pi\gamma\gamma}}{\sqrt{3}\sin\alpha}. \tag{5.15a}$$

Suppose now that $g_{\pi\gamma\gamma}$ and $g_{\eta\gamma\gamma}$ are of the same sign, then, using eqs. 5.14a and 5.14b in eq. 5.15a, gives the following

$$g_{X^o\gamma\gamma} = \pm 1.105 \times 10^{-4}\ \mathrm{MeV}^{-1}, \tag{5.16a}$$

or

$$\Gamma(X^o \to 2\gamma) = 53\ \mathrm{keV}. \tag{5.16b}$$

Alternatively taking $g_{\pi\gamma\gamma}$ and $g_{\eta\gamma\gamma}$ to be of opposite sign, gives

$$g_{X^o\gamma\gamma} = \pm 2.75 \times 10^{-4}\ \mathrm{MeV}^{-1} \tag{5.17a}$$

or

$$\Gamma(X^o \to 2\gamma) = 331\ \mathrm{keV}^{-1}. \tag{5.17b}$$

So, taking $g_{\pi\gamma\gamma}$ and $g_{\eta\gamma\gamma}$ to be of the same sign, one obtains from SU(3) a value for $g_{X^o\gamma\gamma}$ in reasonable agreement with experiment, eq. 5.14c. One is, therefore, tempted to use broken SU(3) to relate the couplings $g_{K_L^o P}$

$$g_{K_L^o\pi} - \sqrt{3}\cos\alpha\, g_{K_L^o\eta} + \sqrt{3}\sin\alpha\, g_{K_L^o X^o} = 0 \tag{5.18}$$

Summing the diagrams in Fig. 2 gives for $g_{K_L^o\gamma\gamma}$ the expression

$$g_{K_L^o\gamma\gamma} = \sum_{P=\pi^o\eta X^o} g_{K_L^o P} \frac{1}{M_K^2 - M_P^2}\, g_{P\gamma\gamma}. \tag{5.19}$$

Taking the l.h.s. of the above equation from experiment allows one to calculate the only unknown quantity $g_{\eta\gamma\gamma}$ Because of the sign ambiguities in $g_{K_L^o P}$, two sets of solutions are obtained.

Set (i) up to an overall sign:

$$g_{K_L^o\pi^o} = 1.73\times10^{-2},\ g_{K_L^o\eta} = 0.465\times10^{-2},\ g_{K_L^o X^o} = -3.044\times10^{-2}$$

Set (ii) up to an overall sign:

$$g_{K_L^o\pi^o} = 1.73\times10^{-2},\ g_{K_L^o\eta} = -0.917\times10^{-2},\ g_{K_L^o X^o} = 0.543\times10^{-2}$$

In terms of the contributions from the separate diagrams in Fig. 2 we have:

Set (i) up to an overall sign:

A (Pion-pole) $= 1.92 \times 10^{-12}$

A (eta-pole) $= -2.97 \times 10^{-12}$

A (X^0-pole) $= 4.24 \times 10^{-12}$

A (total) $= 3.19 \times 10^{-12}$

Set (ii) up to an overall sign:

A (Pion-pole) $= 1.92 \times 10^{-12}$

A (eta-pole) $= -5.86 \times 10^{-12}$

A (X^0 - pole) $= 0.75 \times 10^{-12}$

A (total) $= -3.19 \times 10^{-12}$

It is rather difficult to see what can be concluded from these results. If one believes that Fig. 2 indeed represents the mechanism for the decay $K_L^0 \to 2\gamma$ and that the use of broken SU(3) sum-rules is valid, bearing in mind that $K \to 2\pi$ is forbidden under SU(3) (57) and also if one believes the rather optimistic assumptions regarding $g_{\rho\gamma\gamma}$, then one is forced to conclude that the η and X^0 pole diagrams are indeed important. It is then difficult to believe the rather nice result, eq. 5.11, based on pion pion-pole dominance alone.

5.3 Direct radiation in $K^+ \to \pi^+\pi^0\gamma$.

In order to avoid the uncertainties of the previous section because of the possible presence of eta and X^0 pole terms, we consider now the charged decay $K^+ \to \pi^+\pi^0\gamma$, where neutral poles cannot contribute. This transition can proceed by two distinct mechanisms. One arises from the emission of the photon from the ingoing or outgoing charged particle and is known as inner bremsstrahlung, Fig. 3a.

The amplitude for this process is directly related to the amplitude for $K^+ \to \pi^+\pi^o$, and so must proceed via a $\Delta I \geq 3/2$ transition. The second contribution arises from the direct radiative decay, Fig. 3b; here the presence of the photon at the decay vertex allows the two pions to be in a relative P-wave state with I = 1, which means that the decay may proceed via $\Delta I = 1/2$.

It is of great interest to know the relative contributions of these two mechanisms, since it is this knowledge which determines whether or not it is feasible to study CP violation in this decay (58). On the one hand one might expect the direct radiative processes to be small relative to the inner bremsstrahlung since the former depends on the finiteness of the spacial region over which the virtual processes leading to the decay extend, whereas contributions to inner bremsstrahlung for a photon of momentum k come from regions with linear dimensions 1/k. On the other hand the operation of the $\Delta I = 1/2$ rule, which makes $K_1^o \to \pi^+\pi^-$ approximately 500 times faster than $K^+ \to \pi^+\pi^o$, leads one to expect a relatively prominent direct process (59).

The inner bremsstrahlung matrix element for $K^+ \to \pi^+\pi^o\gamma$ is:

$$M_B = (4\pi\alpha)^{1/2}\, M(K^+ \to \pi^+\pi^o) \left(\frac{p \cdot \epsilon}{p \cdot k} - \frac{P \cdot \epsilon}{P \cdot k} \right), \tag{5.20}$$

where P, p and k represent the K^+, π^+ and photon four-momenta, respectively and ϵ is the photon polarization. For the direct transitions the electric and magnetic dipole matrix elements, leading to P-wave $\pi^+\pi^o$ states, are given by (58)

$$M_E = (4\pi\alpha)^{1/2}\, M(K^+ \to \pi^+\pi^0) X_e\, \mu^{-4}\, p^\alpha q^\beta (K_\alpha \epsilon_\beta - k_\beta \epsilon_\alpha)\, e^{i(\delta_1 - \delta_0)} \tag{5.21}$$

$$M_M = (4\pi\alpha)^{1/2}\, M(K^+ \to \pi^+\pi^0) X_m\, \mu^{-4} \epsilon_{\alpha\beta\rho\sigma}\, p^\alpha q^\beta k^\rho \epsilon^\sigma\, e^{i(\delta_1 - \delta_0)} \tag{5.22}$$

respectively. Here q = P - p - k, the factor μ^{-4} is to make the strength parameters X_e and X_m dimensionless and δ_1 and δ_0 are the P- and S- wave $\pi^+\pi^0$ phase shifts, at the energy $\sqrt{(p+q)^2}$. It is the magnetic (parity conserving) part of the direct transition which is relevant to the pion pole dominance idea (54). The diagrams which contribute to this part of the decay are shown in Figs. 4a and 4b. The diagrams of the type in Fig. 4b, are expected to be relatively small for the same reasons as in τ-decay, and are neglected. For the diagram in Fig. 4a one obtains for the magnetic part of the direct amplitude the following

$$M_M(K^+ \to \pi^+\pi^0\gamma) = g(M^2)\, \frac{1}{M^2 - \mu^2}\, A(s,t,u)\, \epsilon_{\mu\nu\rho\sigma}\, p^\mu q^\nu k^\rho \epsilon^\sigma. \tag{5.23}$$

A (s,t,u) generates the dynamical structure of the amplitude:

$$A(s,t,u) = \beta_\gamma (A_{st} + A_{su} + A_{tu}) \tag{5.24}$$

with

$$A_{st} = \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))}. \tag{5.25}$$

$\alpha(s)$ is the ρ trajectory and s,t, and u are the Mandelstam invariants for $\pi\tilde{\pi} \to \pi\gamma$. Normalising the amplitude at the ρ pole gives

$$\beta_\gamma = \alpha' g_{\rho\pi\gamma}\, g_{\rho\pi\tilde{\pi}} \tag{5.26}$$

with $g_{\rho\pi\gamma}$ defined by eq. 5.4, and $g_{\rho\pi\tilde{\pi}}$ is assumed to be $g_{\rho\pi\pi}$, that is with all the particles on shell. With Γ_ρ = 120 MeV, one has

$$g_{\rho\pi\pi} = 5.5. \tag{5.27}$$

Comparing eqs. 5.22 and 5.23, one can write the strength of the magnetic term X_m as

$$X_M = \frac{g(M_K^2)}{(4\pi\alpha)^{1/2}} \frac{\mu^4}{M_K^2-\mu^2} \frac{\alpha' g_{\rho\pi\gamma} g_{\rho\pi\pi}}{|M(K^+\to\pi^+\pi^o)|} (A_{st}+A_{su}+A_{tu}) \qquad (5.28)$$

which becomes, on using eq. 5.1 for $g(M_k^2)$ and taking a $\rho\to\pi\gamma$ branching ratio of $\lesssim$ 0.2%(56), the following

$$|X_M| \le 1.2\times 10^{-2}(A_{st}+A_{su}+A_{tu}), \qquad (5.29)$$

which gives, for instance, with a charged pion kinetic energy between 55 and 90 MeV

$$|X_M| \le 0.14 \qquad (5.30)$$

corresponding to a magnetic part of the direct transition of $\lesssim$ 4% of the inner bremsstrahlung contribution. The restriction on the charge pion kinetic energy comes from experiment. The $\pi\pi\gamma$ events are selected only in the above region, corresponding to a centre of mass momentum of the charged pion of between 135 and 183 MeV, because on the one hand the lower limit prevents overwhelming contamination from the decay $K^+\to\pi^o\pi^o\pi^+$, while, on the other hand, the upper limit excludes the decay $K^+\to\pi^+\pi^o$.

The most recent experiment on this process (60) gives an inner bremsstrahlung branching ratio of $(2.55\pm 0.18)\times 10^{-4}$, which compares favourably with the theoretical value of 2.50×10^{-4} for a charged pion K.E. in the region considered (55 → 90 MeV). The various projections obtained in this experiment were well described assuming no electric dipole component (X_e = o), and, normalizing to the theoretical inner bremsstrahlung rate, gave a branching ratio for the magnetic dipole radiation of $(1.56 \pm 0.35) \times 10^{-5}$, with a systematic uncertainty of $\pm 0.5 \times 10^{-5}$. In other words assuming $X_e = 0$ one has a magnetic contribution of $(6.24 \pm 3.40)\%$ of the inner bremsstrahlung component; our value of $\lesssim 4\%$ is certainly consistent with this figure.

5.4 Conclusion

We have seen that taking eq. 5.1 as an estimate for $g_{K\pi}$ gives for the magnetic dipole transition in $K^+ \to \pi^+\pi^0\gamma$ a value consistent with experiment. Also by taking this value for $g_{K\pi}$ we have removed the ambiguity in the model of Dass and Kamal (54), in the case where they identified the system "A" with the pion and proceeded to use current algebra to calculate $g_{K\pi}$. The same value for $g_{K\pi}$ used in the model of Wong (52) gives a value for $\Gamma(K_L^0 \to 2\gamma)$ also in agreement with experiment, but this result must be viewed in the light of the results obtained from the model of Matsuda and Oppo (53).

Figure Captions:

Fig. 1: Diagrams contributing to the pion pole-dominance model of Wong (52).

Fig. 2: Diagrams contributing to the pole-dominance model of Matsuda and Oppo (53).

Fig. 3: a) Internal bremsstrahlung contribution to $K^+ \to \pi^+\pi^0\gamma$.

b) Direct transition contribution to $K^+ \to \pi^+\pi^0\gamma$.

Fig. 4a) and b) Pion- and kaon-pole contributions to the generalised pole model of Dass and Kamal (54) for the decay $K^+ \to \pi^+\pi^0\gamma$.

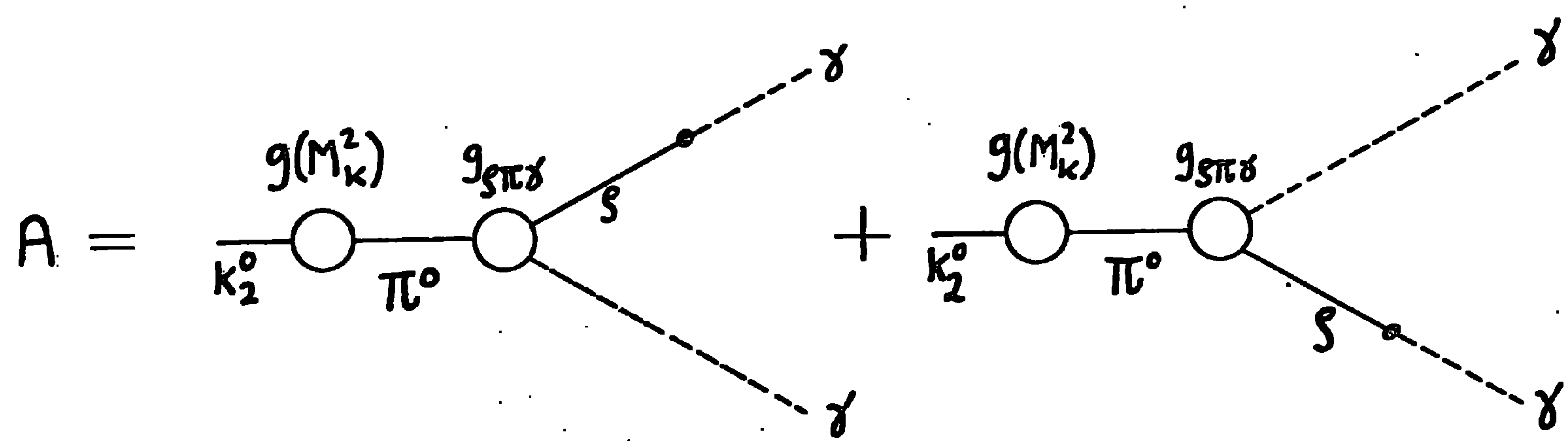
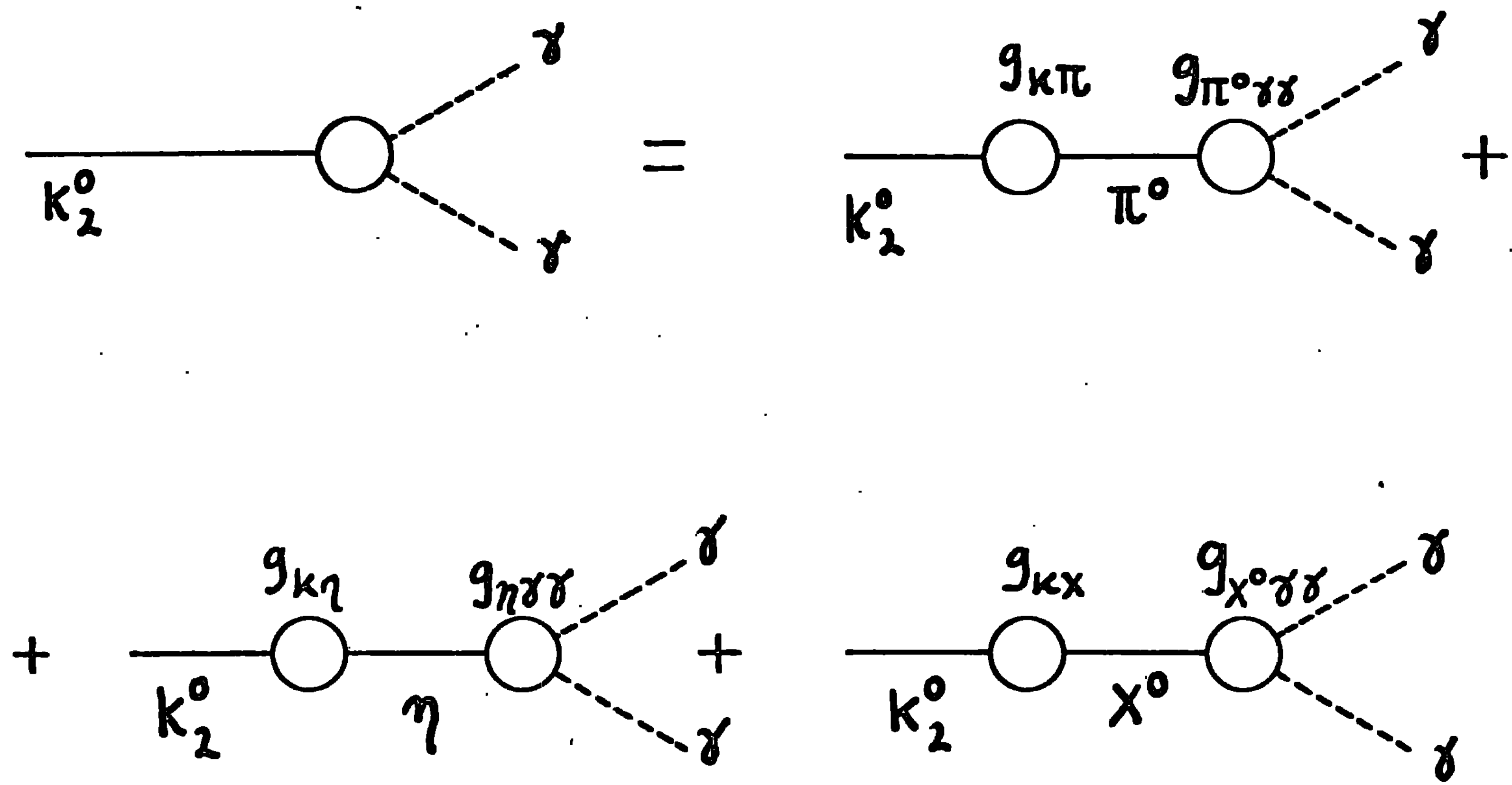

A =
$g(M_K^2)$
$g_{\rho\pi\gamma}$
K_2^0
π^0
ρ
γ
γ
+
$g(M_K^2)$
$g_{\rho\pi\gamma}$
K_2^0
π^0
ρ
γ
γ

FIG. 1.



K_2^0
γ
γ
=
$g_{K\pi}$
$g_{\pi^0\gamma\gamma}$
K_2^0
π^0
+
+
$g_{K\eta}$
$g_{\eta\gamma\gamma}$
K_2^0
η
+
g_{KX}
$g_{X^0\gamma\gamma}$
K_2^0
X^0

FIG. 2.

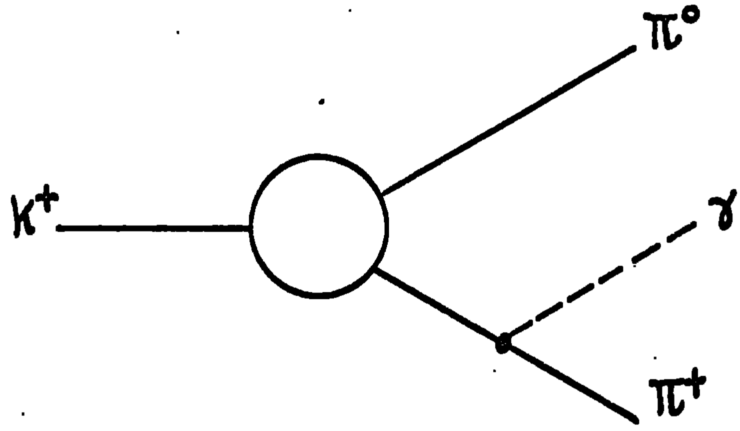


FIG. 3a.

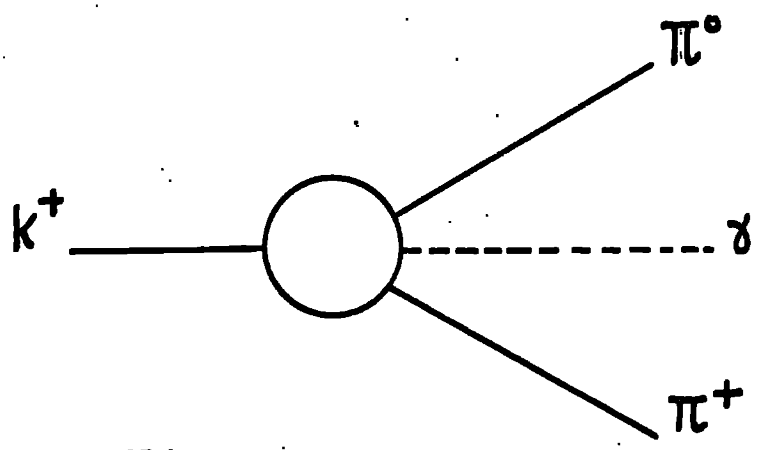


FIG. 3b.

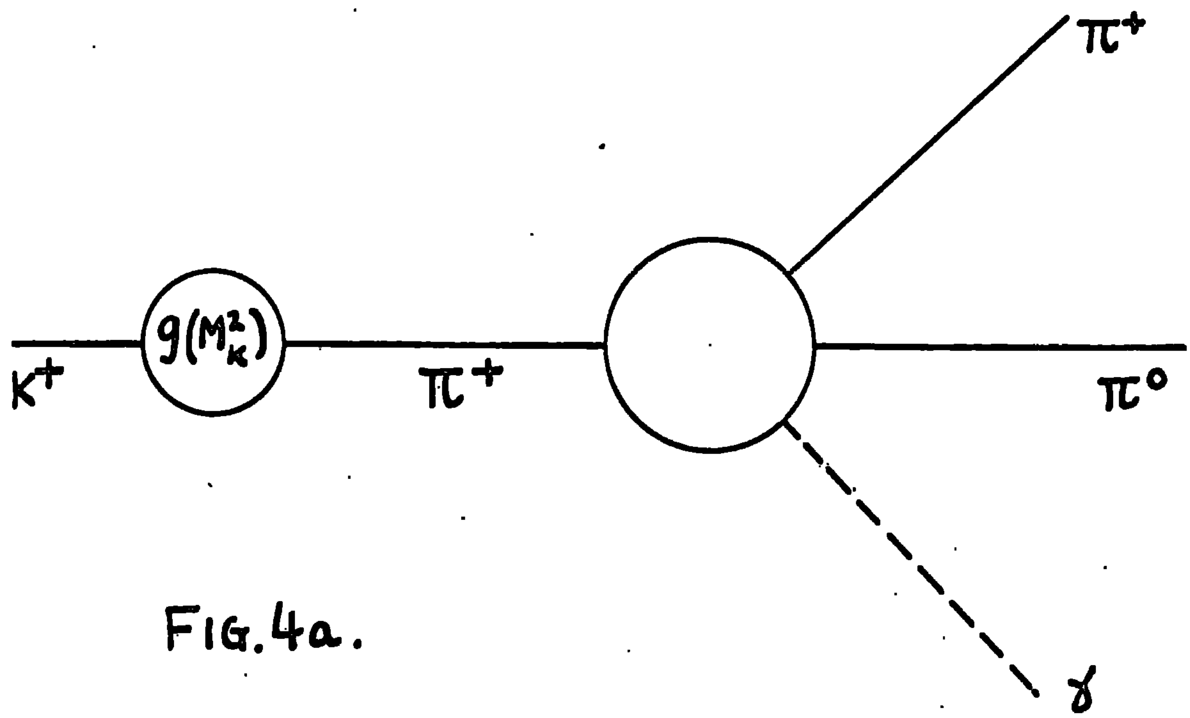


FIG. 4a.

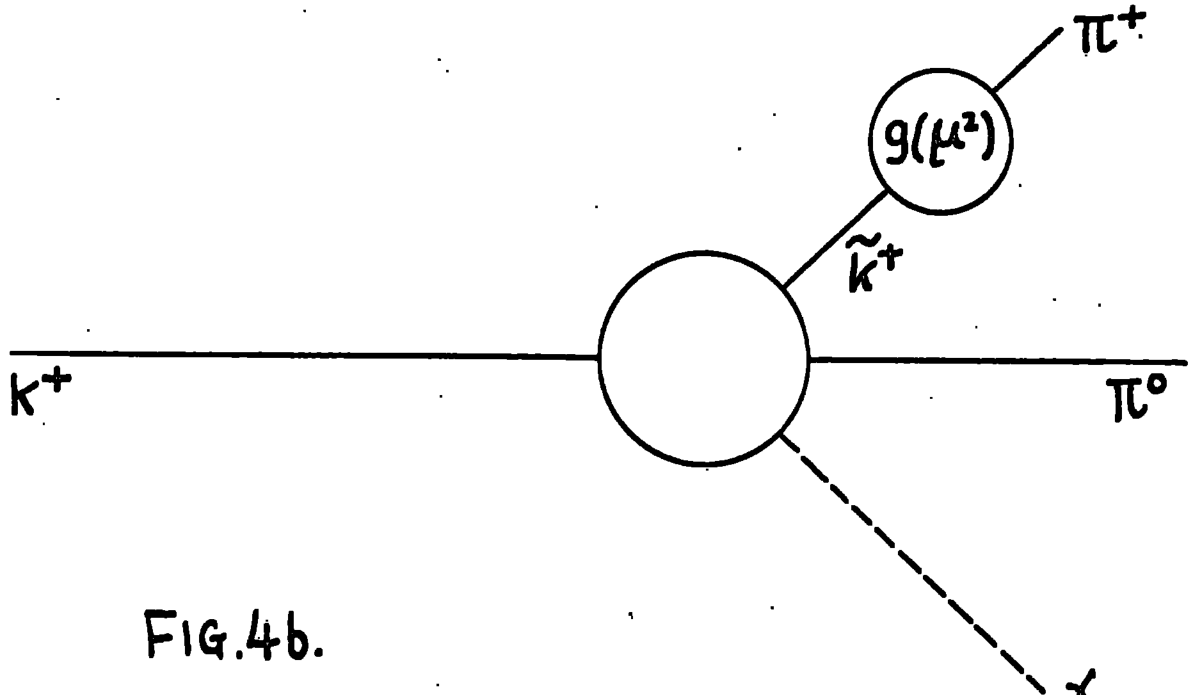


FIG. 4b.

CHAPTER SIX : ZEROS IN $\eta\to 3\pi$, AND GENERALISED POLE DOMINANCE

6.1 Introduction

We now return to the long standing problem of the three pion decay of the eta. There are two points of view which can be adopted; one either believes Sutherland's result (11) of the zero, when $q_{\pi^0}=0$, or one does not. If one takes the former point of view, that is, if one believes that current algebra is correct in $\eta\to 3\pi$ and that the decay is an electromagnetic one, then one is immediately confronted with the problem of explaining the beautiful results of Hara and Nambu and Elias and Taylor (10), based on the linearity assumption. As we have said in chapter four, the linearity assumption may be in serious trouble, in which case the results of Hara and Nambu and Elias and Taylor for τ-decay would no longer stand. With the other point of view, that Sutherland's result is wrong, which seems, at first sight, to be supported by experiment, one has either to explain why current algebra is wrong in this case, or abandon electromagnetism as the main origin of the decay of the eta, or seriously modify it (61).

We shall suppose that the third zero in $\eta\to\pi^+\pi^-\pi^0$ is in fact present. We shall also suppose that the similarity of $\eta\to 3\pi$ and $K\to 3\pi$ within their Dalitz plots is more than a coincidence. If this is the case, one has to look for ways of reconciling the generalised pole model description of $K\to 3\pi$ and $\eta\to 3\pi$ with the requirements of current algebra. There are several clues, which are just the problems arising from the Lovelace conjecture. They are as follows:

(i) The suppression of the rho in the description of $\bar{p}n \rightarrow 3\pi$ (1, 12, 13).

We regard this as significant, since the ability of the model to describe this process as well as it does is remarkable and no plausible alternative description exists.

(ii) The problem of Adler zeros in $\pi\eta$ scattering coupled with the resulting unrealistic δ width.

(iii) The failure of the single Veneziano term, in chapter four, to give a good description of the τ-decay rate, and also the problem of the $\tau^{\pm}$-decay spectrum fit with $\alpha(0) = 0.483$.

In this chapter we return to the Veneziano amplitude for $\pi^+\pi^-$ elastic scattering,

$$M(s,t)=\beta\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}+\gamma\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))}+\ldots \quad (1.2)$$

and work on the assumption that the variation of β and γ, as the external pions are taken off mass shell, is non-negligible. In section two, we find, by making an SU(3) type assumption, relating the Pi and the Eta, just that form of variation required to produce all the current algebra zeros in $\eta\rightarrow 3\pi$. In section three we discuss the $\eta\rightarrow 3\pi$ spectrum produced by this model. In section four we investigate other manifestations of this type of variation.

6.2. Mass dependence and Adler zeros

To start we rewrite eg. (1.2), considering only the two leading terms, in the following form:

$$M(s,t)=\beta\left(1-\alpha(s)-\alpha(t)+\delta(12,34)\right)V_{st}\,, \quad (6.1)$$

where $V_{st} = \Gamma(1-\alpha(s))\Gamma(1-\alpha(t))/\Gamma(2-\alpha(s)-\alpha(t))$, (6.2)

$$\delta(12,34) = \delta(P_1^2P_2^2, P_3^2P_4^2) \quad (6.3)$$

Here δ depends explicitly on the external masses, and describes the relative variation of the two leading terms in eq. (1.2) as the external masses are changed. s $\leftrightarrow$ t crossing implies a certain symmetry for δ which we shall assume remains as the pions are taken off mass shell. That is, one has:

$$\delta(12,34) = \delta(42,31) = \delta(13,24) \tag{6.4}$$

with $s = (P_1 + P_2)^2$, $t = (P_1 + P_3)^2$, $u = (P_1 + P_4)^2$

(6.5)

Consider now $\pi\eta$ elastic scattering, where the amplitude has the form

$$A(s,t,u,) \propto M(s,t) + M(s,u) + M(t,u), \tag{6.6}$$

which ensures positive signature for the A_2-f trajectory, exchange degenerate with the ρ-ω trajectory. We now make an assumption which is of central importance to the discussion. We suppose that the δ's which appear in eq. 6.6 for $\pi\eta$ scattering are the same as those which appear in eq. 6.1 for $\pi\pi$ scattering, but with the relevant masses inserted. This is not an unreasonable idea since the same trajectories are exchanged, and in the SU(3) scheme the π and the η are the "same" apart from their masses.

The next step is to impose the Adler zeros in the processes $\pi\pi\to\pi\pi$, $\pi\eta\to\pi\eta$ and $\eta\eta\to\eta\eta$. With s, t and u defined as in eq. 6.5 one has the following:

$$M(s,t) \propto \left(1-\alpha(s)-\alpha(t)+\delta(12,34)\right)V_{st} \tag{6.7a}$$

$$M(s,u) \propto \left(1-\alpha(s)-\alpha(u)+\delta(12,43)\right)V_{su} \tag{6.7b}$$

$$M(t,u) \propto \left(1-\alpha(t)-\alpha(u)+\delta(14,32)\right)V_{tu} \tag{6.7c}$$

(i) Adler zeros in $\pi\pi$. $s=t=u=\mu^2$. $P_1 = 0, P_2^2=P_3^2=P_4^2=\mu^2$.

$$M(\mu^2,\mu^2) \propto (1-2\alpha(\mu^2)+\delta(0\mu^2,\mu^2\mu^2))V(\mu^2\mu^2) \qquad (6.8)$$

Therefore $M(\mu^2,\mu^2) = 0$ providing one takes

$$\delta(0\mu^2,\mu^2\mu^2) = 2\alpha(\mu^2)-1 \qquad (6.9)$$

or taking $\alpha(s) = \alpha_0 + \alpha' s$

$$\frac{1}{\alpha'}\delta(0\mu^2,\mu^2\mu^2) + \frac{1-2\alpha_0}{\alpha'} = 2\mu^2. \qquad (6.10)$$

(ii) Adler zeros in $\pi\eta$

a) soft π. $s = u = M^2$, $t = \mu^2$ $P_2=0$, $P_1^2=P_3^2=M^2$, $P_4^2=\mu^2$.

Imposing the zeros term by term in eq. 6.6 gives the following:

$$0 = (1-\alpha(M^2)-\alpha(\mu^2)+\delta(M^2 0,M^2\mu^2)) \qquad (6.11a)$$

$$0 = (1-2\alpha(M^2)+\delta(M^2 0,\mu^2 M^2)) \qquad (6.11b)$$

$$0 = (1-\alpha(\mu^2)-\alpha(M^2)+\delta(M^2\mu^2,M^2 0)) \qquad (6.11c)$$

giving

$$\frac{1}{\alpha'}\delta(M^2\mu^2,M^2 0) + \frac{1-2\alpha_0}{\alpha'} = M^2+\mu^2$$

$$\frac{1}{\alpha'}\delta(M^2\mu^2,0M^2) + \frac{1-2\alpha_0}{\alpha'} = 2M^2. \qquad (6.12)$$

b) soft η. $s = u = \mu^2$, $t = M^2$, $P_1 = 0$, $P_2^2=P_4^2=\mu^2$, $P_3^2=M^2$.

Again imposing the zeros term by term in eq. 6.6 gives

$$0 = (1-\alpha(\mu^2)-\alpha(M^2)+\delta(0\mu^2,M^2\mu^2)) \qquad (6.13a)$$

$$0 = (1-2\alpha(\mu^2)+\delta(0\mu^2,\mu^2M^2)) \qquad (6.13b)$$

$$0 = (1-\alpha(M^2)-\alpha(\mu^2)+\delta(0\mu^2,M^2\mu^2)) \qquad (6.13c)$$

giving

$$\frac{1}{\alpha'}\delta(M^2\mu^2,0\mu^2) + \frac{1-2\alpha_0}{\alpha'} = M^2+\mu^2$$

$$\frac{1}{\alpha'}\delta(M^2\mu^2,\mu^2 0) + \frac{1-2\alpha_0}{\alpha'} = 2\mu^2 \qquad (6.14)$$

(iii) Adler zeros in $\eta\eta$. $s = t = u = M^2$. $P_1=0$, $P_2^2=P_3^2=P_4^2=M^2$.

$$0 = \left(1-2\alpha(M^2)+\delta(0M^2,M^2M^2)\right) \tag{6.15}$$

or

$$\frac{1}{\alpha'}\,\delta(M^2M^2,M^2 0) + \frac{1-2\alpha_0}{\alpha'} = 2M^2 \tag{6.16}$$

Consider now eqs. 6.14 and looking at fig. 1 we see that the value of δ depends on which pions are taken off mass shell. If the two pions which are left on mass shell are the same, i.e. both π^+'s or both π^-'s, then δ takes one particular value; if on the other hand the two pions which are left on mass shell are of different charge then δ takes some other value. This is a rather strange and complex behaviour.

6.3 Zeros in $\eta\to 3\pi$ and generalised pole dominance

One observes that eqs. 6.14 give just those values of δ which are required when checking the current algebra zeros in the generalised pole dominance description of $\eta\to 3\pi$. Consider, in particular $\eta\to\pi^+\pi^-\pi^0$, the amplitude for this process is, in the pole model, given by the sum of the two diagrams shown in fig. 2. That is:

$$A(s,t,u) = g(M^2)\frac{1}{M^2-\mu^2}\,A(\tilde{\pi}^0\pi^0\to\pi^+\pi^-) + g(\mu^2)\frac{1}{\mu^2-M^2}\,A(\eta\tilde{\eta}\to\pi^+\pi^-) \tag{6.17}$$

where

$$A(\tilde{\pi}^0\pi^0\to\pi^+\pi^-) = f^2(M_{st}+M_{su}-M_{tu}) \tag{6.18a}$$

and

$$A(\eta\tilde{\eta}\to\pi^+\pi^-) = C_{ts}\frac{f^2}{3}(M_{st}+M_{su}+M_{tu}). \tag{6.18b}$$

Here

$$M_{st} = \left(1-\alpha(s)-\alpha(t)+\delta(M^2\mu^2,\mu^2\mu^2)\right)V_{st}, \tag{6.19}$$

and $C_{ts} = -\sqrt{3}$ is the (t,s) - crossing matrix for $\pi\eta$ scattering. It is straight forward to check that the amplitude, given in eq. 6.17, has all the required current algebra zeros, when those values of δ given in eq. 6.14 are used.

To proceed to calculate the spectrum for $\eta \to 3\pi$ one needs to know two things; firstly the relative contribution of the two terms in eq. 6.17, i.e. the q^2 dependence of $g(q^2)$, and secondly the value of $\delta(M^2\mu^2, \mu^2\mu^2)$. With regard to the former, nothing conclusive has emerged from the literature. There are arguments (62), which have been questioned (63), which suggest that

$$g(M^2)/g(\mu^2) = M^2/\mu^2. \tag{6.20}$$

While, at the same time, there are cases in the literature which consider a constant form-factor dependence for the electromagnetic vertex, $g(q^2)$, (64), that is

$$g(M^2) = g(\mu^2) = \bar{g}. \tag{6.21}$$

We shall consider both cases. Taking eq. 6.21 for the form factor dependence, and substituting into eq. 6.17, gives for the $\eta \to \pi^+\pi^-\pi^0$ amplitude the following:

$$A(s,t,u) = \frac{\bar{g}f^2}{M^2-\mu^2}\left\{M_{st}+M_{su}-M_{tu}+\frac{1}{\sqrt{3}}\left(M_{st}+M_{su}+M_{tu}\right)\right\}. \tag{6.22}$$

Also using $\alpha(\mu^2)=\frac{1}{2}$ in eqs. 6.14 gives

$$\delta(M^2\mu^2; 0\mu^2) = 0.25$$

and

$$\delta(M^2\mu^2; \mu^2 0) = 0 \tag{6.14a}$$

The above two equations suggest a value for $\delta(M^2\mu^2, \mu^2\mu^2)$ somewhere near the middle of the range 0 to 0.25. Taking, for instance,

$$\delta(M^2\mu^2; \mu^2\mu^2) = 0.12 \tag{6.23}$$

gives the fit to the spectrum shown in fig. 3; the agreement is excellent.

Taking the other point of view, embodied in eq. 6.20, gives for the $\eta \to \pi^+\pi^-\pi^0$ amplitude:

$$A(s,t,u) = \frac{g(M^2)f^2}{M^2-\mu^2}\left\{M_{st}+M_{su}-M_{tu}\right\}. \tag{6.24}$$

This is, of course, the original Lovelace amplitude for the decay (1), and from his work we know that a good description of the spectrum is obtained provided one takes

$$\alpha(\mu^2) = 1/2$$

and

$$\delta(M^2\mu^2; \mu^2\mu^2) = 0 \tag{6.23a}$$

6.4 $\pi\eta \rightarrow \pi\eta$ scattering

First of all we consider Osborn's current algebra amplitude for $\pi\eta \rightarrow \pi\eta$ (65). This amplitude has all the correct Adler zeros and depends explicitly on the momenta of the η's:

$$A(s,t,u) = \frac{1}{3F^2}\left(s+t+u-3\mu^2 - \frac{M^2-\mu^2}{M^2}(k_1^2+k_2^2)\right) \tag{6.25}$$

Comparing this with our modified Veneziano form, with both pions on mass-shell,

$$\begin{aligned} A(s,t,u) \propto\ & \left(1-\alpha(s)-\alpha(t)+\delta(k_1^2\mu^2, k_2^2\mu^2)\right)V_{st} \\ & + \left(1-\alpha(s)-\alpha(u)+\delta(k_1^2\mu^2, \mu^2 k_2^2)\right)V_{su} \\ & + \left(1-\alpha(t)-\alpha(u)+\delta(k_1^2\mu^2, k_2^2\mu^2)\right)V_{tu} \end{aligned} \tag{6.27}$$

at the point $\left(k_2^2 = \mu^2,\ s=t=u=(k_1^2+3\mu^2)/3\right)$ gives

$$s+t+u-(3-6\alpha_0)/2\alpha' - 3\delta(k_1^2\mu^2\mu^2\mu^2)/2\alpha' = s+t+u-3\mu^2 - \frac{M^2-\mu^2}{M^2}(k_1^2+\mu^2),$$

or taking $\alpha(\mu^2) = 1/2$ gives

$$\delta(k_1^2\mu^2; \mu^2\mu^2) = \frac{2\alpha'}{3M^2}(M^2-\mu^2)(k_1^2+\mu^2). \tag{6.28}$$

This is not going to give the wholestory, because we are supposing that the momentum dependence on the pion is the same as on the Eta, whereas in eq. 6.25 one has dependence only on the momenta of the Etas, but eq. 6.28 should give some idea of the variation, especially for $K_1^2 \gg \mu^2$. Setting $K_1{}^2 = 4M_N{}^2$ gives the δ required for the process $\bar{p}n \to 3\pi$ and in fact one finds

$$\delta(4M_N^2\mu^2, \mu^2\mu^2) = 1.95. \tag{6.29}$$

This is in remarkable agreement with the values obtained by Altarelli and Rubinstein (12) and Berger (13).

We now investigate how the δ's effect the width of the δ-meson. The residue of eq. 6.27 at the δ-pole, $\alpha(s) = 1$, is given by:

$$\lim_{s \to M_\delta^2} (s - M_\delta^2) A(s,t,u) = \frac{\beta}{\alpha'} \left(1.02 - \delta(M^2\mu^2, M^2\mu^2) - \delta(M^2\mu^2, \mu^2M^2)\right)$$
$$\propto \Gamma_\delta \tag{6.30}$$

If we call Γ_c the width of the δ-meson obtained from the conventional Veneziano expression, eq. 1.15, and assuming β to be the same in both cases we obtain:

$$\frac{\Gamma_\delta}{\Gamma_c} = \frac{1.02 - \delta(M^2\mu^2, M^2\mu^2) - \delta(M^2\mu^2, \mu^2M^2)}{1.02} \tag{6.31}$$

Using eqs. 6.12 as estimates for $\delta(M^2\mu^2 M^2\mu^2)$ and $\delta(M^2\mu^2\mu^2M^2)$ with $\alpha(\mu^2) = 1/2$ gives

$$\Gamma_\delta / \Gamma_c \simeq 0.25 \tag{6.31a}$$

That is the effect of the δ's is to change the δ-meson width by about a factor 4 in the right direction.

6.5 Conclusion

By making the "pi = eta" assumption for the δ's and imposing the Adler zeros in $\pi\eta \to \pi\eta$, produces the correct zero structure in $\eta \to 3\pi$. The Adler zeros in $\pi\eta \to \pi\eta$ have been imposed by effectively adding satellites, which at the same time solves the problem of the δ-meson width. We consider it more than a coincidence that the $\delta(K^2\mu^2, \mu^2\mu^2)$ based on Osborn's amplitude gives a value of 1.95 for $\delta(4M_N^2\mu^2, \mu^2\mu^2)$, when Altarelli and Rubinstein obtained a value of 1.86, and when Berger needed exactly 1.95 for his fits to the $\bar{p}n \to 3\pi$ spectra.

The problem remains, of course, to find the exact functional form of δ, looking at the values required to impose the Adler zeros in $\pi\pi$, $\pi\eta$ and $\eta\eta$ elastic scattering, we see that δ must have quite a complex structure, which seems to have its origin in the electromagnetic and weak structure of the pseudoscalar mesons.

For τ-decay one cannot play the same game, since the similarities which exist between $\pi\pi$ and $\pi\eta$, (symmetry and Regge exchanges), do not exist to the same extent between $\pi\pi$ and πK (because of the strangeness).

Figure Captions:

Fig. 1 : Four line connected part for $\pi^+\pi^-$ elastic scattering showing momentum labelling.

Fig. 2 : Pole diagrams for the decay $\eta \rightarrow \pi^+\pi^-\pi^o$.

Fig. 3 : $\eta \rightarrow \pi^+\pi^-\pi^o$ spectrum from the modified Veneziano model, as described in the text. The data are from ref. (17).

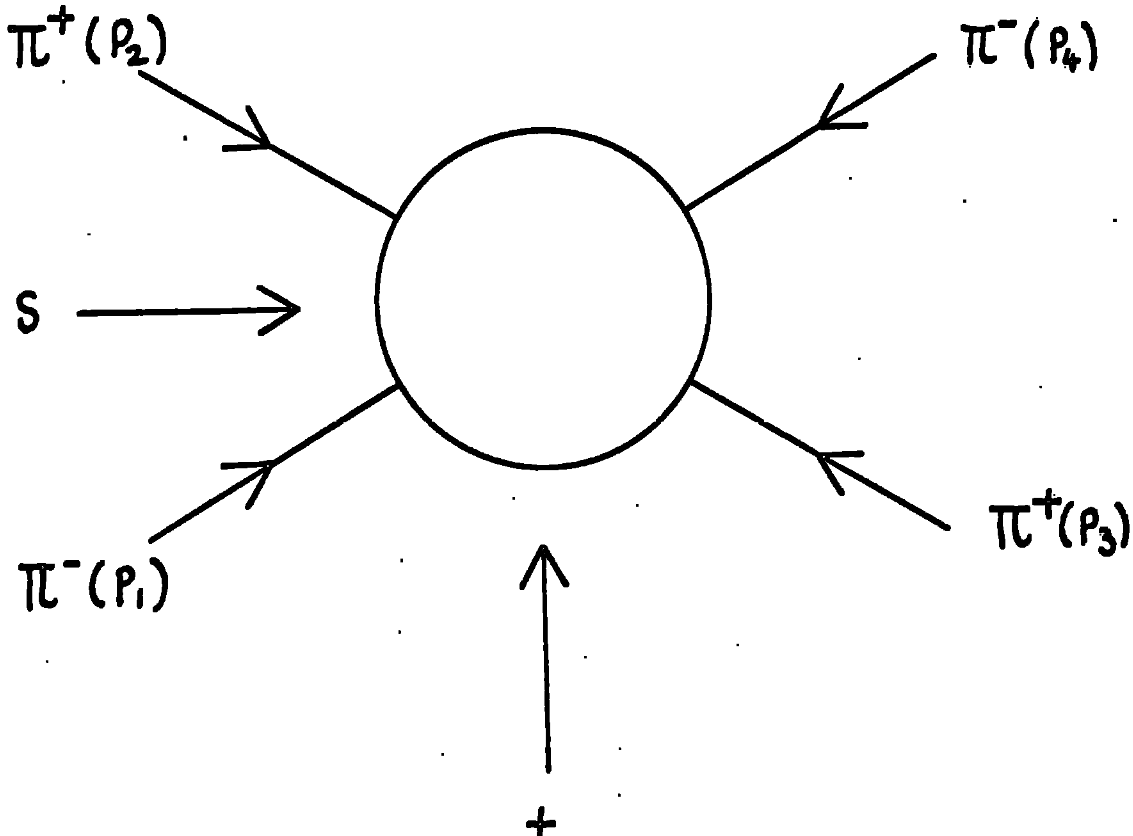


Fig. 1.

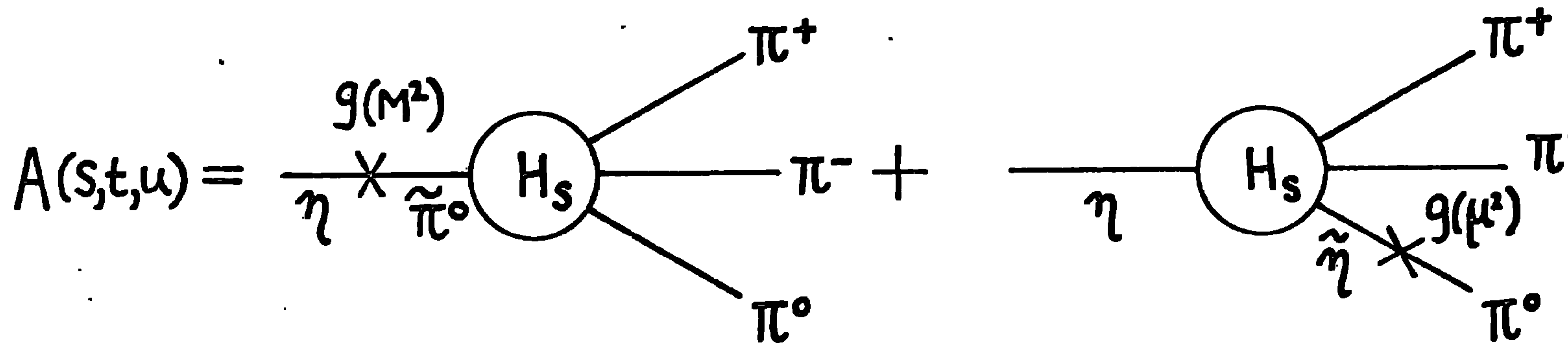
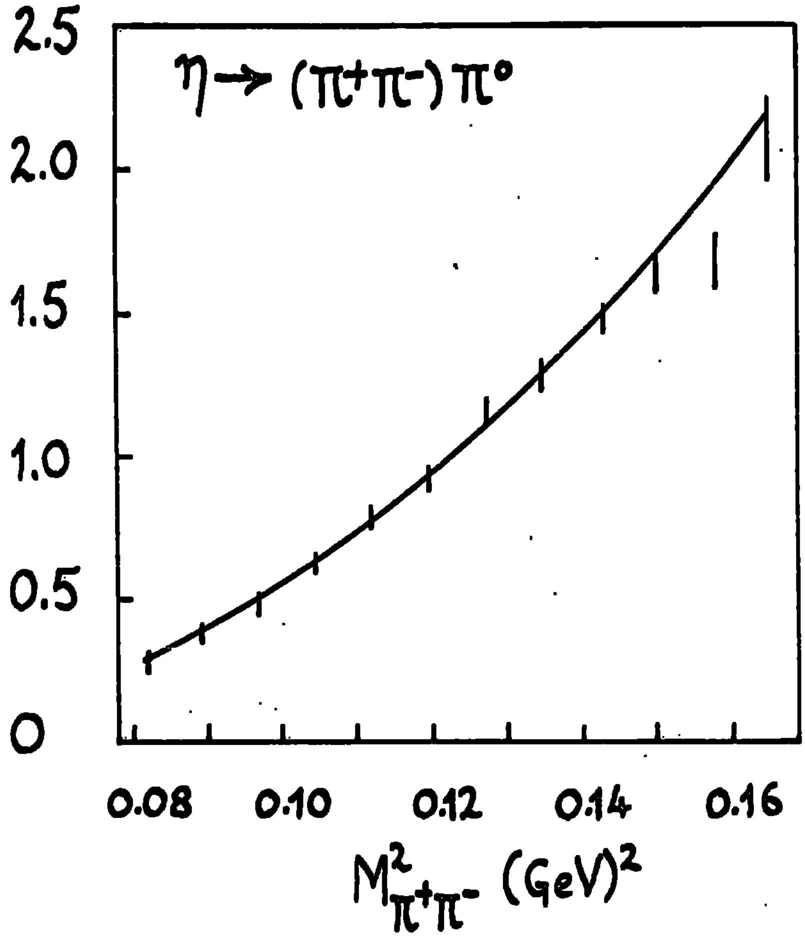


FIG. 2.

FIG. 3.



CHAPTER SEVEN : CONCLUSION

Our overall conclusions may be summarised as follows:

a) The positions of the sub-threshold zeros in Pi-Pi scattering and the slopes of the amplitudes as they change sign are in good agreement with Weinberg's simple current algebra amplitude. In terms of the Veneziano model, this means that the variation of the coefficients, β, γ, etc., during the extrapolation $\mu^2 \rightarrow 0$ is negligible, and so the prediction $\alpha(\mu^2) = 1/2$ remains good.

b) The failure of the single term Veneziano amplitude to describe adequately the τ-decay rate, the amplitude being 30% too small at the centre of the Dalitz plot, coupled with the fact that a non-negligible mass dependent term is required to produce a satisfactory description of both the rates and spectra, leads one to conclude that, even by the time one has reached the τ decay physical region, significant changes in the coefficients, β, γ, etc., have occurred, that is Satellites are present. This should also explain the slight discrepancy obtained by Lovelace in his fit to τ^+-decay with $\alpha(\mu^2) = 1/2$.

c) By relating the Pi and Eta as in chapter six and imposing the Adler zeros in $\pi\eta \rightarrow \pi\eta$ solves the problem of the δ-width and gives all the current algebra zeros in the generalised pole model description of the decay $\eta \rightarrow 3\pi$. By drawing a comparison with Osborn's current algebra amplitude for $\pi\eta \rightarrow \pi\eta$ and extrapolating to the physical region of $\bar{p}n \rightarrow 3\pi$ gives an amplitude very similar to that of Altarelli and Rubinstein and Berger. The variation of the coefficients is rather strange and complex.

APPENDIX A

A.1 Amplitude normalisation

For the transition $|i\rangle \rightarrow |f\rangle$ the S-matrix element S_{fi} is related to the invariant Feynman amplitude M_{fi} by

$$S_{fi} = \delta_{fi} - i(2\pi)^4 \delta^4 (P_f - P_i) \frac{M_{fi}}{(\prod_j 2E_j)}$$

We use units $\hbar = c = 1$, and P_i (P_f) is the sum of all four-momenta in the initial (final) state and E_j is the energy of the j-th particle, j running over both initial and final state particles.

For the binary reaction $1 + 2 \rightarrow 3 + 4$ we define the usual Mandelstam invariants, s,t,u, by

$$s = (P_1 + P_2)^2 = (P_3 + P_4)^2 \tag{A2a}$$

$$t = (P_1 - P_3)^2 = (P_2 - P_4)^2 \tag{A2b}$$

$$u = (P_1 - P_4)^2 = (P_2 - P_3)^2 \tag{A2c}$$

with

$$s + t + u = \sum_{i=1}^{4} M_i^2 \tag{A3}$$

For Pi-Pi elastic scattering we have in terms of the s-channel centre of mass three-momentum, $\underline{q}$, and scattering angle Θs,

$$s = 4(q^2 + \mu^2) \tag{A4a}$$

$$t = -2q^2(1-\cos\Theta s) \tag{A4b}$$

$$u = -2q^2(1+\cos\Theta s) \tag{A4c}$$

We define for each normalisation (N) the invariant amplitude

$$N_A = - \frac{M}{16\pi N} \tag{A5}$$

A.2 Unitarity and Partial wave expansions

For $|i\rangle = |f\rangle$ the unitarity condition $SS^{+} = 1$ gives the optical theorem

$$\mathrm{Im}\,{}^{N}A\;(i = f) = \frac{q\sqrt{s}}{8\pi N}\,\sigma_{tot} \tag{A6}$$

We define partial wave amplitudes ${}^{N}A^{I}_{l}$ (s) for spinless bosons in a state of well-defined isospin, I, by

$${}^{N}A^{I}(s,Z_s) = \sum_{l\,+\,I\ \mathrm{even}} (2l+1)\,{}^{N}A^{I}_{l}(s)P_l(Zs) \tag{A7}$$

where $Zs = \mathrm{Cos}\Theta s$. The sum over $(l + I)$-even is a consequence of Bose statistics. Unitarity expressed in terms of partial wave amplitudes becomes

$$\mathrm{Im}\,{}^{N}A^{I}_{l}(s) = N\epsilon\,\rho(S)\;R^{I}_{l}(s)\;\left|{}^{N}A^{I}_{l}(s)\right|^{2} \tag{A8}$$

where $$\epsilon = \begin{cases} 1 & \text{for two different particles in } |i\rangle \\ \tfrac{1}{2} & \text{for two identical particles in } |i\rangle \end{cases} \tag{A9}$$

$$\rho(s) = \frac{2q}{\sqrt{s}} \tag{A10}$$

and

$$R^{I}_{l}\;(s) = \frac{\sigma^{I,l}_{tot}\;(s)}{\sigma^{I,l}_{el}\;(s)} \tag{A11}$$

In terms of the inverse amplitude this becomes

$$\mathrm{Im}\left[{}^{N}A^{I}_{l}(s)\right]^{-1} = -N\epsilon\,\rho(s)\;R^{I}_{l}\;(s) \tag{A12}$$

The above equations may be solved to give

$${}^{N}A^{I}_{l}(s) = \frac{1}{N\epsilon\,\rho(s)}\;\frac{\eta^{I}_{l}e^{2i\delta^{I}_{l}} - 1}{2i}, \tag{A13}$$

Where $\delta^{I}_{l}(s)$ is the real phase shift and $\eta^{I}_{l}(s)$ is the real inelasticity,

$$0 \leqslant \eta^{I}_{l} \leqslant 1 \tag{A14}$$

and $\eta^{I}_{l} = 1$ below the first inelastic threshold.

A.3 Isospin crossing matrices

The s, t and u channels are defined by

s-channel : $1 + 2 \rightarrow 3 + 4$ (A15a)

t-channel : $\bar{4} + 2 \rightarrow 3 + \bar{1}$ (A15b)

u-channel : $\bar{3} + 2 \rightarrow \bar{1} + 4$ (A15c)

The s-channel amplitude $A_s^I(12 \rightarrow 34)$ is assumed to be an analytic function of (s,t,u) having only those singularities demanded by unitarity. When continued to the t- or u- channel physical regions it describes a linear combination of t- and u- channel isospin amplitudes, the coefficients of which are given by the crossing matrices. Thus:

$$A_s^{I_1} = \sum_{I_2} C_{st}(I_1 I_2)\, A_t^{I_2} \qquad \text{(A16)}$$

together with similar expressions relating the other channels and where

$$C_{ts} = C_{st}^{-1} \quad \text{and} \quad C_{ut} = C_{us}.C_{st}$$

In addition to isospin amplitudes, covariant amplitudes are often used (24). These are defined in terms of the covariant states. Thus for pions, $|\pi_a\rangle$, a = 1,2,3 are defined by

$$|\pi^{\pm}\rangle = 2^{½}(|\pi_1\rangle \pm i\,|\pi_2\rangle)\,,\ |\pi^0\rangle = |\pi_3\rangle.$$

$\pi\pi \rightarrow \pi\pi$

$$C_{st} = \frac{1}{6}\begin{bmatrix} 2 & 6 & 10 \\ 2 & 3 & -5 \\ 2 & -3 & 1 \end{bmatrix}, \qquad C_{su} = \frac{1}{6}\begin{bmatrix} 2 & -6 & 10 \\ -2 & 3 & 5 \\ 2 & 3 & 1 \end{bmatrix}.$$

$$\langle d, c \mid T \mid a,b \rangle = A\,\delta_{ab}\,\delta_{cd} + B\,\delta_{ac}\,\delta_{bd} + C\,\delta_{ad}\,\delta_{bc}$$

$$\begin{pmatrix} A_s^0 \\ A_s^1 \\ A_s^2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$A(s,t,u) = B(t,s,u) = C(u,t,s)$$

and

$$A(s,t,u) = A(s,u,t)$$

APPENDIX B

B.1 Dalitz plot for $K \to 3\pi$ decays

The small deviation from uniformity of the Dalitz plot for the 3π decay of the K meson is usually described by a "slope parameter" (66). The Dalitz plot distribution is parametrized by the expression.

$$|M|^2 \propto 1 + g\,(s - so)/m^2_{\pi+} + h\left[(s-so)/m^2_{\pi+}\right]^2 + \text{----}, \quad \text{(B1)}$$

where $m^2_{\pi+}$ is introduced to make the parameters g,h, etc., dimensionless,

$$s = (q_1 - q_4)^2 = (M_1 - M_4)^2 - 2M_1T_4, \quad \text{(B2a)}$$

$$t = (q_1 - q_2)^2 = (M_1 - M_2)^2 - 2M_1T_2, \quad \text{(B2b)}$$

$$u = (q_1 - q_3)^2 = (M_1 - M_3)^2 - 2M_1T_3 \quad \text{(B2c)}$$

$$so = \frac{1}{3}\,(s+t+u) = \frac{1}{3}\sum_{i=1}^{4} M_i^2, \quad \text{(B3)}$$

and q_1 refers to the four-momentum of the kaon, while q_i ($i = 2,3,4$) refer to the four-momentum of the $i^{\underline{th}}$ pion. The index 4 refers to the odd pion. The possible charged stated for $K \to 3\pi$ are called τ, τ' and τ_o:

$$\tau^{\pm} \;:\; K^{\pm} \to \pi^+\pi^-\pi^{\pm}$$
$$\tau'^{\pm} \;:\; K^{\pm} \to \pi^0\pi^0\pi^{\pm}$$
$$\tau_o \;:\; K^0_L \to \pi^+\pi^-\pi^0$$

where in τ_o-decay the odd pion is taken to be the neutral one.

For the charged K decay the Dalitz plot variables X and Y are defined by:

$$X = \frac{\sqrt{3}}{Q}\,(T_2 - T_3) \text{ and } Y = \frac{3T_4}{Q} - 1 \quad \text{(B4)}$$

where Q is the total kinetic energy available

$$Q = M_1 - M_2 - M_3 - M_4 \tag{B5}$$

In our analysis we neglect the pion mass differences, in which case the Dalitz plot variables reduce to

$$X = \frac{\sqrt{3}}{2MQ}(u - t) \quad \text{and } Y = \frac{3}{2MQ}(s_0 - s), \tag{B6}$$

where $M = M_1$ and $M_2 = M_3 = M_4 = \mu$. Instead of analysing the spectra in terms of X and Y, we shall use the variable s and $z = (t - u)$.

B.2 Normalised Projections

(i) S-projections:

$$\frac{d\sigma}{ds} = \frac{\int_{t_1(s)}^{t_2(s)} |M|^2 \, dt}{\int_{t_1(s)}^{t_2(s)} dt} \tag{B7}$$

where $t_{1,2}$ (s) are the roots of:

$$st(3\mu^2 + M^2 - s - t) = \mu^2(M^2 - \mu^2)^2 \tag{B8}$$

setting $\Sigma = M^2 + 3\mu^2$ and $c = \mu^2(M^2 - \mu^2)^2$ then:

$$2t_{1,2}(s) = (\Sigma - s \pm \sqrt{(s - \Sigma)^2 - 4\,c/s}) \tag{B9}$$

(ii) Z - projection:

$$\frac{d\sigma}{dz} = \frac{\int_{s_1(z)}^{s_2(z)} |M|^2 \, ds}{\int_{s_1(z)}^{s_2(z)} ds} \tag{B10}$$

Boundary of the Dalitz plot is given by

$$stu = c \tag{B11}$$

or, using $s + t + u = \Sigma$ and $z = t - u$,

$$s^3 - 2\Sigma s^2 + s(\Sigma^2 - z^2) = 4c \tag{B12}$$

$S_{1,2}$ (z) are the two relevant roots of the above equation, B12, and are given by:

$$s_1(z) = \frac{2}{3}\Sigma - \sqrt[3]{\rho}\,(\cos\theta/3 + 3\sin\theta/3) \tag{B13a}$$

$$s_2(z) = \frac{2}{3}\Sigma - \sqrt[3]{\rho}\,(\cos\theta/3 - 3\sin\theta/3) \tag{B13b}$$

with $$\rho = \sqrt{\left(r^2 + \left|q^3 + r^2\right|\right)} \tag{B14a}$$

and $$\theta = \tan^{-1}\left(\frac{\sqrt{\left|q^3 + r^2\right|}}{r}\right) \tag{B14b}$$

where $$q = \frac{a_1}{3} \quad \text{and} \quad r = \frac{-a_0}{2} \tag{B15}$$

with

$$a_0 = \frac{2}{27}\Sigma^3 - \frac{2}{3}z^2\Sigma - 4c \tag{B16a}$$

and $$a_1 = -\left(\frac{\Sigma^2}{3} + z^2\right) \tag{B16b}$$

Fig. B1 shows the Dalitz plot region for $K \rightarrow 3\pi$ and the limits of integration for the two normalized projections.

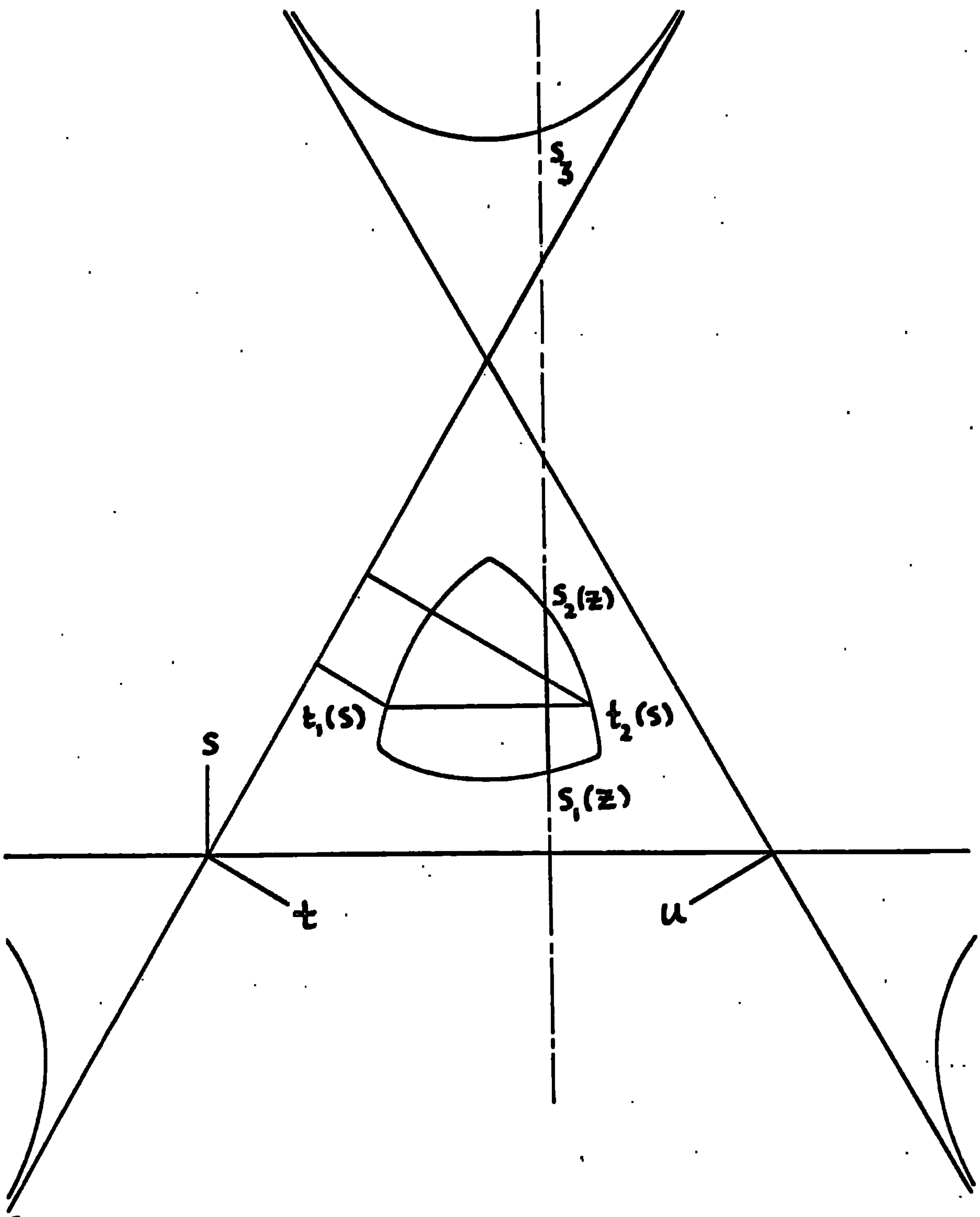


FIG. B1.

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