Curvature, singularities and projections of smooth maps

Smith, Michael Ian Nimmo

CURVATURE, SINGULARITIES

AND PROJECTIONS OF SMOOTH MAPS

A THESIS

BY

MICHAEL IAN NIMMO SMITH

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CONTENTS

ABSTRACT

This work is an initial attempt to extend to many-parameter families of smooth functions on a smooth manifold, and projections of smooth maps into subspaces of higher dimension, the well-known interrelations, between the space of Morse function on a smooth manifold and the space of immersions of the manifold in a cartesian space, which are given by the Gauss-maps of the immersions, and the orthogonal projections of the immersions onto lines in the cartesian spaces. Results, both local and global, are obtained.

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INTRODUCTION

This work is concerned with the category of smooth finite-dimensional manifolds and smooth maps. In particular, it examines the geometrical and topological properties of certain classes of maps at singular points.

In §1 are assembled the conventions and, in particular, precise statements of the principal methods of the type of local differential analysis that dominates this work.

In §2, a review is made of the classical theory concerning smooth immersions, curvature, gauss-maps, and 'height' functions. Much of this matter can be found in the references [6], [9], [11], [12] in connection with the theory of Total Absolute Curvature.

Coherent measures on smooth bundles are defined in §3, and are used as a technique to generalise the global conclusions of §2. The result of §2 states that a certain subset of an n-sphere has measure zero. The result of §3 shows that the same subset is nowhere dense in 'most' great subspheres of the n-sphere.

By Thom's Transversality Theorems, the classical Theorem of Morse on the approximation of smooth functions is re-established in §4. By the same techniques a local description is obtained of the 'generic' one parameter family of smooth functions on a compact manifold. The presentation of §4 is modelled on [4] which treats the case $n = 2$.

In §5 the important paper of Whitney, [32] on mapping of a plane to a plane, is generalised. The problem is to describe the 'generic' smooth maps of a smooth compact manifold into the plane. Connections with the theory of §4 are indicated.

The work of §4 is generalised in §6 to describe the 'generic' form of a many-parameter family of smooth functions on a compact smooth manifold.

In §7 a general Morse-type Lemma is established and is applied, together with the Weierstrass-Malgrange Preparation Theorem of §1, to give canonical forms for the singularities classified in §§4 - 6. Those for Whitney maps generalise those of [32], those for the many-parameter family of smooth function realise some of the forms obtained by Levine in [14], but without remainder terms. The technique of applying the Weierstrass-Malgrange in this way is suggested in [27].

Various global properties of the maps of §5 are illustrated in §8. In particular connections are established with §§2 - 4, and with the theory of 'minimal' maps [12].

In §9 a start is made on the problem of generalising the results of §2 to the study of singularities of projections of immersion into planes. Precise local differential geometric descriptions are obtained.

Finally §10 indicates further directions for attention and more connections between the present work and other researches.

§1 PRELIMINARIES

(A): Assumptions and Conventions

Throughout, smooth means infinitely differentiable. For the definitions and elementary properties of smooth manifolds, maps, coordinate systems consult [3], [13], [11]. If M, N are smooth manifolds, and $f : M \to N$ is a smooth map, then TM, TN will denote the tangent bundles of M, N, $Df : TM \to TN$ will denote the derivative of f, $T_m M$ denotes the tangent space to M at m and $Df(m): T_m M \to T_{f(m)} N$ the restriction of Df to $T_m M$.

Cartesian n-space, $\mathbb{R}^n$, is taken with its natural structures as a smooth manifold, a vector space, an inner product (euclidean) space, and a riemannian manifold. Throughout this work there is a casual, but systematic, identification between different tangent vectors and between tangent vectors and points of $\mathbb{R}^n$.

The Grassmann manifold of p-planes through the origin of $\mathbb{R}^n$ is denoted by $G(n,p)$. For its structure see [11], [8]. If $\mathbb{R}^n$ is taken with its standard inner product, then $O(n) = O(\mathbb{R}^n)$ will denote the orthogonal group of $\mathbb{R}^n$.

If M^n is a smooth manifold, with $x_i : M \to \mathbb{R}$ for $i = 1,\ldots,s$ a subsystem of coordinate functions at a point $m \in M$, and if $y_i : M \to \mathbb{R}$ for $i = 1,\ldots,s$ then the Jacobian matrix of the y_i with respect to the $\{x_j\}$ at m,

$$\left(\frac{\partial y_i}{\partial x_j}(m) : i = 1,\ldots,r; \quad j = 1,\ldots,s \right)$$

will be denoted by

$$J(y_1, \ldots, y_r / x_1, \ldots, x_s)(m).$$

When $r = s$, the Jacobian determinant of the y_i with respect to the x_j

at m,

$$\frac{\partial(y_1, \ldots, y_r)}{\partial(x_1, \ldots, x_r)}(m)$$

will be denoted by

$$D(y_1, \ldots, y_r | x_1, \ldots, x_r)(m) .$$

If $\{x_i\}$, $\{s_{ij}\}$, $\{t_{ijk}\}$ are the components of a vector, a matrix a tensor, for some ranges of the indices i, j, k, then they will be represented occasionally by the symbols $\underline{x}$, $\underline{\underline{s}}$, $\underline{\underline{\underline{t}}}$.

For the definitions and elementary properties of jet-spaces see [8], [2]. If M^n, N^p are smooth manifolds, then $J^r(M,N)$ will denote the space of r-jets with source in M and target in N. If $f : M \to N$ is smooth, then the r-prolongation of f will be denoted by $f^{(r)}: M \to J^r(M,N)$.

If M,N are smooth manifolds, then $\mathcal{L}(M,N)$ will denote the space of smooth maps from M to N. $\mathcal{L}(M,N)$ is endowed with the topology of uniform convergence of all partial derivatives on compact subsets of M. See [2], [3], [29] for details. The cases of interest here are when M is compact. If L is a smooth manifold a smooth map $f : L \to \mathcal{L}(M,N)$ is one such that the associated map $\hat{f} : M \times L \to N$ is smooth.

Beyond the classical tools, the Implicit Function Theorem and Taylor's Expansion Formula, there are now certain other results of great importance for local differential analysis. They are outlined in the next section.

(B): The Principal Tools of Local Differential Analysis

Sard's Theorem [24] and [8, p.316]

If M^n *is a smooth manifold and* $f : M^n \to \mathbb{R}^p$ *is a map of class* C^k, *and* $S \subseteq M^n$ *is a set of points such that* rank $Df(s) \leq r$, *for a given* $r < n$; *then* $f(S)$ has $[r + (p-r)/k]$-*Hausdorff measure zero*.

<u>Consequences</u> (i) If f is smooth with the conditions as above, then $f(S)$ has $[r+\epsilon]$-Hausdorff measure zero for any positive ϵ.

(ii) If $n = p$, $r = n-1$, S the singular locus of f, then $f(S)$ has n-Hausdorff measure zero.

One of the most important applications of Sard's Theorem is to prove Thom's Source Transversality Theorem, [26], which is a major tool in the study of smooth maps and their singularities.

Let M^n, N^p be smooth manifolds, K a subset of M, and $S = \{S_1, \ldots, S_k\}$ a collection of disjoint submanifolds of N which are '<u>stratified</u>' in the following sense:

(i) dimension $S_i >$ dimension S_{i-1}, $i = 2, \ldots, k$

(ii) $S_1 \cup \ldots \cup S_i$ is closed in N, $i = 1, \ldots, k$.

$f \in \mathcal{L}(M^n, N^p)$ is transversal to S in K if f is transversal to each of the S_i at each point of K. Then one may state

<u>Thom's Source Transversality Theorem</u>

<u>Let</u> M^n <u>be a smooth manifold</u>, K <u>a compact subset of</u> M, <u>and</u> $S = \{S_1, \ldots, S_k\}$ <u>a stratified submanifold of</u> $J^r(M^n, \mathbb{R}^p)$. <u>Then the set of</u> $f \in \mathcal{L}(M^n, \mathbb{R}^p)$ <u>such that</u> $f^{(r)} : M^n \to J^r(M^n\ \mathbb{R}^p)$ <u>is transversal to</u> S <u>on</u> K <u>forms an open dense subspace of</u> $\mathcal{L}(M^n, \mathbb{R}^p)$.

For completeness, one may state the companion:

<u>Thom's Target Transversality Theorem</u>

<u>Let</u> M^n <u>be a smooth manifold</u>, K <u>a compact subset of</u> M <u>and</u> $S = \{S_1, \ldots, S_k\}$ <u>a stratified submanifold of</u> $(J^r(M^n, \mathbb{R}^p))^t$, <u>and let</u> $M_{(t)}$ <u>denote the subset of elements of</u> $(M)^t$ <u>whose components are distinct</u>. <u>Then the set of</u> $f \in \mathcal{L}(M^n, \mathbb{R}^p)$ <u>such that</u> $f^{(r)} \times \ldots \times f^{(r)} : M_{(t)} \to (J^r(M^n, \mathbb{R}^p))^t$ <u>is transversal to</u> S <u>on</u> $(K)^t \cap M_{(t)}$ <u>forms a dense subspace of</u> $\mathcal{L}(M^n, \mathbb{R}^p)$.

For proofs see [2]. The case that concerns one here is when M is compact, K = M.

The Weierstrass-Malgrange Preparation Theorem

Let M^n, V^p be smooth manifolds, $f : M \to V$ a smooth map and let $(U, \theta, \{x_i : i = 1,\dots,n\})$ and $(W, \phi, \{y_j : j = 1,\dots,p\})$ be coordinate systems centred on $m \in M$ and $v = f(m) \in V$. Let E_m, E_v be the $\mathbb{R}$-algebras of germs of smooth real valued functions at $m \in M$, $v \in V$. Let $f^* : E_v \to E_m$ be the homomorphism induced by composition with f. Then let $f_j = f^*(y_j) = y_j \circ f$. E_m may be regarded as an E_v-module via f^*.

Let $\mathbb{R}[[\hat{x}_1, \dots, \hat{x}_n]]$ denote the $\mathbb{R}$-algebra of formal power series in the indeterminates, and for $g \in E_m$ let $\hat{g} \in \mathbb{R}[[\hat{x}_1, \dots, \hat{x}_n]]$ denote the Taylor expansion of g at m with respect to the x_i. f^* induces a map $\hat{f}^* : \mathbb{R}[[\hat{y}_1, \dots, \hat{y}_p]] \to \mathbb{R}[[\hat{x}_1, \dots, \hat{x}_n]]$ such that $\hat{f}^*(\hat{y}_j) = (y_j \hat{\circ} f) = f^*(y_j) = \hat{f}_j$. Let $\hat{I}(f)$ denote the ideal in $\mathbb{R}[[\hat{x}_1, \dots, \hat{x}_n]]$ generated by $\hat{f}_1, \dots, \hat{f}_p$. Then the following statement comes directly from the Corollary to Theorem 1 of [17, p.205]:

Let $g_1, \dots, g_r \in E_m$; then $g_1, \dots, g_r$ generate E_m as an f^*-E_v-module if and only if the cosets of $g_1, \dots, g_r$ in the quotient space $\mathbb{R}[[\hat{x}_1, \dots, \hat{x}_n]]/\hat{I}(f)$ generate $\mathbb{R}[[\hat{x}_1, \dots, \hat{x}_n]]/\hat{I}(f)$ as a vector $\mathbb{R}$-space.

Consequences Let $F \in E_m$ be regular of order r in x_n, thus

$$\frac{\partial^i}{\partial x_n^{\,i}} F(m) = 0, \qquad 0 \leq i \leq r-1$$

$$\frac{\partial^r}{\partial x_n^{\,r}} F(m) \neq 0.$$

Let $p = n$, $V^n = \mathbb{R}^n$. Let $v = \underline{0} \in \mathbb{R}^n$, and let $(\mathbb{R}^n, 1_{\mathbb{R}^n}, y_j)$ be the standard coordinate system on $\mathbb{R}^n$. Define $f : M \to \mathbb{R}^n$ by $f_i = x_i$, $i \neq n$, $f_n = F$. Now in $\hat{F}$, the first power of $\hat{x}_n$ to appear with nonzero coefficient in x_n^r, hence the ideal $\hat{I}(f) \subseteq \mathbb{R}[[\hat{x}_1, \ldots, \hat{x}_n]]$ generated by $\hat{x}_1, \ldots, \hat{x}_{n-1}, \hat{F}$ is equal to the ideal generated by $\hat{x}_1, \ldots, \hat{x}_{n-1}, \hat{x}_n^r$. Thus the cosets of $\hat{1}, \hat{x}_n, \ldots, \hat{x}_n^{r-1}$ in $\mathbb{R}[[\hat{x}_1, \ldots, \hat{x}_n]]/\hat{I}(f)$ generate (and in fact form a basis of) the vector $\mathbb{R}$-space $\mathbb{R}[[\hat{x}_1, \ldots, \hat{x}_n]]/\hat{I}(f)$. Thus by the result above

$1, x_n, \ldots, x_n^{r-1}$ <u>generate</u> E_m <u>as an</u> f^*-E_v<u>-module</u>, or

(i) <u>Given any</u> $g \in E_m$ <u>there exist</u> $a_i \in E_v$, $0 \leqslant i \leqslant r-1$, <u>such that</u>

$$g = \sum_{i=0}^{r-1} f^*(a_i).x_n^i .$$

(ii) <u>Let</u> E'_m <u>denote the subalgebra of</u> E_m <u>of smooth functions in the</u> x_i, $1 \leqslant i \leqslant n-1$. <u>Then given any</u> $g \in E_m$ <u>there exist</u> $q \in E_m$, $b_i \in E'_m$ <u>such that</u>

$$g = \sum_{i=0}^{r-1} b_i \cdot x_n^i + q \cdot F .$$

(iii) <u>There exist</u> $c_i \in E_v$, $0 \leqslant i \leqslant r-1$, <u>such that</u>

$$f^*(c_o) = x_n^r + \sum_{i=1}^{r-1} f^*(c_i) \cdot x_n^i .$$

(iv) <u>There exist</u> $d_i \in E'_m$, $q \in E_m$, $0 \leqslant i \leqslant r-1$, <u>such that</u>

$$x_n^r = q \cdot F + \sum_{i=0}^{r-1} d_i \cdot x_n^i .$$

Now (iii), (iv) come direct from (i), (ii) by specialisation and rearrangement. The derivation of (ii) from (i) is fairly direct and can be seen in [17, p. 206] ; or established by a different method in [16, p.08]. The following facts about the special cases (iii) and (iv) are important; they are established by differentiating with respect to x_n repeatedly and evaluating at $m \in M$.

$$\text{(iii)}' \quad c_o(v) = \dots = c_{r-1}(v) = 0$$

$$\frac{\partial c_o}{\partial y_n}(v) \neq 0, \quad \frac{\partial c_o}{\partial y_i}(v) = 0 \quad \text{for } i \neq n .$$

$$\text{(iv)}' \quad d_o(m) = \dots = d_{r-1}(m) = 0$$

$q(m) \neq 0$, and is the inverse of the coefficient of $\hat{x}_n^r$ in $\hat{F}$.

Applications of (iv), (iv)' are made to make coordinate changes at source; (iii), (iii)' are used to make simultaneous coordinate changes at source and target.

(C) Adapted coordinate systems

Lemma Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function, $f(\underline{0}) = \underline{0}$, and rank $Df(\underline{0}) = k$. Then there exist

(i) a coordinate system (U, θ), with coordinate functions $\{x_i : 1 \leq i \leq n\}$, centred on $\underline{0} \in \mathbb{R}^n$,

(ii) an orthogonal matrix $A \in O(\mathbb{R}^m)$

such that, if $\{y_i : 1 \leq i \leq m\}$ are the standard coordinate functions on $\mathbb{R}^m$, then

(iii) $y_i \circ A^{-1} \circ f = x_i$, $\qquad 1 \leq i \leq k$,

(iv) $\dfrac{\partial(y_{k+\alpha} \circ A^{-1} \circ f)}{\partial x_i}(\underline{0}) = 0, \qquad 1 \leq \alpha \leq m-k, \quad 1 \leq i \leq n.$

Such a coordinate system is said to be linearly adapted to f at $\underline{0}$.

Proof Let L^k be the linear subspace of $\mathbb{R}^m$ which identifies with the image of $Df(\underline{0})$. Let $P : \mathbb{R}^m \to L$ be orthogonal projection onto L and let $\{z_i : 1 \leq i \leq k\}$ be a system of orthonormal linear coordinate functions on L. The map $P \circ f : \mathbb{R}^n \to L$ has rank k at $\underline{0} \in \mathbb{R}^n$. Hence, by the Implicit Function Theorem, there exists a coordinate system (U, θ), with coordinate functions $\{x_i : 1 \leq i \leq n\}$, centred on $\underline{0} \in \mathbb{R}^n$, such that

$$z_i \circ P \circ f = x_i , \qquad 1 \leq i \leq k.$$

The functions $w_i = z_i \circ P : \mathbb{R}^m \to \mathbb{R}$ may be extended to a system of orthonormal linear coordinate functions $\{w_j : 1 \leq j \leq m\}$ on $\mathbb{R}^m$. Let $A \in O(\mathbb{R}^m)$ be the orthogonal matrix such that

$$y_j = w_j \circ A, \qquad 1 \leq j \leq m,$$

where $\{y_j : 1 \leq j \leq m\}$ are the standard coordinate functions on $\mathbb{R}^m$. Then U, θ, x_i, A are the required objects.

Corollary Let V^n, W^m be smooth manifolds, $v \in V$, $w \in W$, and $f : V^n \to W^m$ a smooth map with $f(v) = w$, and rank $Df(v) = k$. Then there exist

(i) a coordinate system (U, θ), with coordinate functions $\{x_i : 1 \leq i \leq n\}$, centred on $v \in V$,

(ii) a coordinate system (N, ϕ), with coordinate functions $\{y_i : 1 \leq j \leq m\}$ centred on $w \in W$,

such that

(iii) $y_i \circ f = x_i$, $\qquad 1 \leq i \leq k$

(iv) $\dfrac{\partial(y_{k+\alpha} \circ f)}{\partial x_i}(v) = 0$, $\qquad 1 \leq \alpha \leq m-k,\ 1 \leq i \leq n.$

<u>Moreover the</u> $\{N, \phi, y_j\}$ <u>may be obtained from an orthogonal transformation of a given coordinate system centred on</u> $w \in W$.

Such a pair of coordinate systems is said to be <u>adapted to</u> f <u>at</u> V.

Furthermore, given (V, W, f, v, w) as in this corollary, let $(U, \theta, \{x_i\}, (N, \phi, \{y_j\})$ be adapted to f at v. Let $(U', \theta', \{X_i\})$, $(N', \phi', \{Y_j\})$ be a pair of coordinate systems centred on $v \in V$, $w \in W$. Consider the following three types of conditions:

$$\text{(I)} \quad \left.\begin{array}{ll} D(Y_1, \ldots, Y_k | y_1, \ldots, y_k)\ (w) \neq 0 & \\ X_i = Y_i \circ f, & 1 \leq i \leq k \\ Y_{k+\beta} = y_{k+\beta} & 1 \leq \beta \leq m-k \\ X_{k+\alpha} = x_{k+\alpha} & 1 \leq \alpha \leq n-k \end{array}\right\}$$

$$\text{(II)} \quad \left.\begin{array}{ll} D(X_{k+1}, \ldots, X_n | x_{k+1}, \ldots, x_n)(v) \neq 0 & \\ X_i = x_i & 1 \leq i \leq k \\ Y_j = y_j & 1 \leq j \leq m \end{array}\right\}$$

$$\text{(III)} \quad \left.\begin{array}{ll} D(Y_{k+1}, \ldots, Y_m | y_{k+1}, \ldots, y_m)(w) \neq 0 & \\ J(Y_{k+1}, \ldots, Y_m | y_1, \ldots, y_k)(W) = \underline{\underline{0}} & \\ X_i = x_i & 1 \leq i \leq n \\ Y_j = y_j & 1 \leq j \leq k \end{array}\right\}$$

It is a simple matrix calculation to verify that each of these three types of source-target coordinate changes produces a pair of coordinate systems adapted to f at v, and that every adapted system may be obtained from a given one by a sequence (possibly trivial at some stage) of coordinate changes of these types. By this means, when one attempts to find canonical forms for a class of smooth maps, where one has an adapted system as a starting point, to ensure that a final pair of coordinate systems is adapted it suffices to restrict all changes of coordinates at source and target to the three types, I, II and III.

Adapted systems are used, thought not named, throughout the papers of Whitney, Thom, and others. Linearly adapted systems, though a very restrictive class, are an essential tool in studying the geometry of smooth maps into a rigid space such as $\mathbb{R}^m$; for regular submanifolds, linearly adapted system constitute <u>Monge presentations</u>.

§2 GAUSS-MAPS OF IMMERSIONS

Let M^n be a smooth n-dimensional manifold and $f : M^n \to \mathbb{R}^{n+k}$ a smooth immersion of M. Denote by $< , >$ the standard inner product and let $z \in \mathbb{R}^{n+k}$ have unit norm. Define the map $< z,f > : M \to \mathbb{R}$ by

$$< z,f >(m) = < z,f(m) >$$

for $m \in M$.

There is a simple geometric description of the points of M where $< z,f >$ is singular, or equivalently where $D < z,f >$ has rank zero. For let $(U, \theta, \{x_i : 1 \leq i \leq n\})$ be a coordinate system centred on $m \in M$. Then m is a singular point of $< z,f >$ if and only if, for each i, $1 \leq i \leq n$

$$D < z,f > \left(\frac{\partial}{\partial x_i} \right)(m) = 0.$$

Now $D < z,f > = < z,Df >$, and the $Df\left(\frac{\partial}{\partial x_i}(m) \right)$ span the immersed tangent space $Df(T_m M)$ of f at m. Thus

<u>Proposition 2.1</u> $< z,f >$ <u>is singular at</u> $m \in M$ <u>if and only if</u> z <u>is perpendicular to the immersed tangent plane of</u> f <u>at</u> m.

This last criterion may be rephrased as follows, by considering the unit normal bundle of f and its associated gauss-map.

<u>Definitions 2.2</u> Let S^{n+k-1} denote the sphere of unit vectors of $\mathbb{R}^{n+k}$ with its standard smooth structure. The <u>unit normal bundle of</u> M <u>and</u> f, denoted by UNM(f) or by UNM when there is no doubt about the identity of f, is a smooth compact (n+k-1)-dimensional manifold whose underlying space is the set of pairs $(m,z) \in M \times S^{n+k-1}$ such that z is perpendicular to the immersed tangent space $Df(T_m M)$ of f at m. Let

$p : UNM \to M$, and $\Gamma : UNM \to S^{n+k-1}$ denote the projections onto first and second factors of $UNM \subseteq M \times S^{n+k-1}$. The manifold structure on UNM can be obtained either from bundle-theoretic considerations, or from identifying it with the boundary of a tubular neighbourhood of M in $\mathbb{R}^{n+k}$, or, as will be seen later, by specifying local coordinate systems in $M \times S^{n+k-1}$ which reveal UNM as a regular submanifold. The map $p : UNM \to M$ has the structure of a smooth (k-1)-sphere bundle, whose fibre over $m \in M$ is the (k-1)-sphere of unit vectors of $\mathbb{R}^{n+k}$ perpendicular to $Df(T_m M)$. The map $\Gamma : UNM \to S^{m+k-1}$ is called the <u>gauss-map of</u> f.

In these terms

<u>Proposition 2.3</u> <u>the singular set of</u> $< z,f >$ <u>is the set</u> $p(\Gamma^{-1}(z))$. The advantages of this approach are revealed when one comes to analyse the second-order structure of $< z,f >$ at its singular points.

First, at a point $m \in M$, with $(U, \theta, \{x_i : 1 \leq i \leq n\})$ a coordinate system centred on m, the second differential $D^2 < z,f > \; = \; < z, D^2 f >$ is characterised by the matrix

$$\left(< z, \frac{\partial^2 f}{\partial x_i \partial x_j} (m) > \; : \; 1 \leq i, \; j \leq n \right)$$

which is the hessian matrix of $< z, f^\theta > = < z, f \circ \theta^{-1} >$ at $\underline{0} \in \mathbb{R}^n$.

Secondly one may analyse the local geometry of f at m, and interpret $D^2 < z,f >$, when m is a singular point of $< z,f >$, variously as a second fundamental form, a curvature matrix of a hypersurface, and the differential of the gauss map Γ.

Let $m \in M$, $(U, \theta, \{x_i : 1 \leq i \leq n\})$ be as above, $z \in S^{n+k-1}$ perpendicular to the immersed tangent space of f at m. Let $\{\underline{e}_j : 1 \leq j \leq n+k\}$ be an orthonormal basis of $\mathbb{R}^{n+k}$ such that $\{\underline{e}_i : 1 \leq i \leq n\}$ span the linear subspace of $\mathbb{R}^{n+k}$ parallel to $Df(T_m M)$

and hence $\{\underline{e}_s : n+1 \leq s \leq n+k\}$ are perpendicular to $Df(T_m M)$, and let $\underline{e}_{n+k} = z$. Let $\{y_j : 1 \leq j \leq n+k\}$ be the corresponding coordinate functions on $\mathbb{R}^{n+k}$, and let $\{f_j : 1 \leq j \leq n+k\}$ denote their compositions with f.

If ∇ denotes the Levi-Civita connection in $\mathbb{R}^{n+k}$, then

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \nabla_{\frac{\partial}{\partial x_i}} \left(\sum_{r=1}^{n+k} \frac{\partial f_r}{\partial x_j} \frac{\partial}{\partial y_r} \right)$$

$$= \sum_{r,s=1}^{n+k} \frac{\partial f_r}{\partial x_j} \frac{\partial f_s}{\partial x_i} \nabla_{\frac{\partial}{\partial y_s}} \frac{\partial}{\partial y_r} + \sum_{r=1}^{n+k} \frac{\partial^2 f_r}{\partial x_i \partial x_j} \frac{\partial}{\partial y_r}$$

$$= \sum_{r=1}^{n+k} \frac{\partial^2 f_r}{\partial x_i \partial x_j} \frac{\partial}{\partial y_r} .$$

Now, evaluating at m, and taking the normal component one obtains the following formula for the <u>second fundamental form</u> α_m <u>of</u> f <u>at</u> $m \in M$:

$$\alpha_m \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \sum_{\sigma=n+1}^{n+k} \frac{\partial^2 f_\sigma}{\partial x_i \partial x_j} (m) \cdot \underline{e}_\sigma$$

The <u>second fundamental form</u> h_z <u>of</u> f <u>at</u> $m \in M$ <u>in the normal direction</u> z is given by:

$$h_z \left(\frac{\partial}{\partial x_i} (m), \frac{\partial}{\partial x_j} (m) \right) = < \alpha_m \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right), z >$$

$$= < \underline{e}_{n+k}, \sum_{\sigma=n+1}^{n+k} \frac{\partial^2 f_\sigma}{\partial x_i \partial x_j} (m) \cdot \underline{e}_\sigma >$$

$$= \frac{\partial^2 f_{n+k}}{\partial x_i \partial x_j} (m) .$$

But $f_{n+k} = \langle \underline{e}_{n+k}, f \rangle = \langle z,f \rangle$. Hence

$$h_z \left(\frac{\partial}{\partial x_i}(m), \frac{\partial}{\partial x_j}(m) \right) = \frac{\partial^2}{\partial x_i \partial x_j} \langle z,f \rangle (m) .$$ Thus

Proposition 2.4. At a singular point of $\langle z,f \rangle$, the second fundamental form of f in the direction z is equivalent to the hessian matrix, or second differential of $\langle z,f \rangle$.

Next, by the same considerations as in the preceding paragraph, consider the map f_z obtained by projecting $\mathbb{R}^{n+k}$ into the (n+1)-dimensional linear subspace spanned by z and $Df(T_mM)$, and composing with f. f_z has rank n at $m \in M$ and hence in some neighbourhood U of $m \in M$ immerses U as a hypersurface in an $\mathbb{R}^{n+1}$. Let $(U, \theta, \{x_i : 1 \leq i \leq n\})$ be a coordinate system linearly adapted to f_z at m, i.e. such that $\{Df_z(m)(\frac{\partial}{\partial x_i}) : 1 \leq i \leq n\}$ is an orthonormal basis of $Df_z(T_mM)$. The second fundamental form, or curvature matrix, of f_z at m is given by the hessian of $\langle z,f \rangle$ at $m \in U$. In particular the principal curvatures and Gauss-Kronecker curvature of (U, f_z) at $m \in U$ are given by the eigenvalues and determinant of the hessian of $\langle z,f \rangle$ at m, up to sign. Thus

Proposition 2.5 At a singular point of $\langle z,f \rangle$, the curvature matrix, and in particular the principal and Gauss-Kronecker curvatures, of the projection of f(M) into the space spanned by the tangent space of f(M) at f(m) and the normal z, are determined by the hessian matrix of $\langle z,f \rangle$ at m and the riemannian metric induced on M by f.

Finally, to calculate the differential of the gauss-map Γ at $(m,z) \in UNM$, one may introduce local coordinates in UNM in a neighbourhood of (m,z) as follows. Let $\{\underline{e}_j : 1 \leq j \leq n+k\}$ be as before, with $\underline{e}_{n+k} = z$, and let $(U, \theta, \{x_i\})$ be the coordinate system centred on $m \in M$

obtained by first applying f, then projecting orthogonally into $(Df)(T_m M)$, then by parallel translation into the $\{\underline{e}_1, \ldots, \underline{e}_n\}$-plane and finally applying the canonical map into $\mathbb{R}^n$ given by the basis $\{\underline{e}_1, \ldots, \underline{e}_n\}$. In some neighbourhood U of m, this map is a diffeomorphism, θ. For $\underline{x} = (x_1, \ldots, x_n) \in \theta(U)$

$$f^\theta(\underline{x}) = f \circ \theta^{-1}(\underline{x}) = f(m) + \sum_{i=1}^{n} x_i \underline{e}_i + \sum_{\sigma=1}^{k} \lambda^\theta_{n+\sigma}(\underline{x})\underline{e}_{n+\sigma}$$

where

$$\left.\begin{aligned} &\lambda^\theta_{n+\sigma} = \langle \underline{e}_{n+\sigma}, f^\theta(\underline{x}) \rangle - \langle \underline{e}_{n+\sigma}, f(m) \rangle \\ \text{and} \\ &\frac{\partial \lambda_{n+\sigma}}{\partial x_i}(m) = 0, \qquad 1 \leq \sigma \leq k, \quad 1 \leq i \leq n . \end{aligned}\right\}$$

Now consider $z = \underline{e}_{n+k} \in S^{n+k-1}$ as the north pole, and let $S^{n+k-1}_+ = \{v \in S^{n+k-1} : \langle \underline{e}_{n+k}, v \rangle > 0\}$ be the upper hemisphere. The central projection β from S^{n+k-1}_+ onto the (n+k-1)-plane of $\mathbb{R}^{n+k-1}$ tangent to S^{n+k-1} at $\underline{e}_{n+k}$ is a diffeomorphism, which by composing with the coordinate functions $\{y_\lambda : 1 \leq \lambda \leq n+k-1\}$ gives coordinate functions $\{w_\lambda : 1 \leq \lambda \leq n+k-1\}$. If $v \in S^{n+k-1}_+$ then, if $v = \sum_{i=1}^{n+k} v_i \underline{e}_i$,

$$w_\lambda(v) = v_\lambda / v_{n+k}. \qquad \text{Note that } v_{n+k}\cdot\left(\sum_{\lambda=1}^{n+k-1} w_\lambda(v)\cdot\underline{e}_\lambda\right) + v_{n+k}\cdot\underline{e}_{n+k} = v.$$

Let $\mathcal{N} = U \times S^{n+k-1}$ and let $\mathcal{N}' = \mathcal{N} \cap UNM$. The functions $\{x_i, w_\lambda : 1 \leq i \leq n, \; 1 \leq \lambda \leq n+k-1\}$ for a system of coordinate functions on $\mathcal{N}$, centred on (m,z).

Now let $(m^*, z^*) \in \mathcal{N}'$. z^* is perpendicular to $Df(T_{m^*}M)$ and hence for $1 \leq i \leq n$

$$0 = \left\langle z^*, \frac{\partial f}{\partial x_i}(m^*) \right\rangle \quad ,$$

$$\text{so} \quad 0 = \left\langle \sum_{\lambda=1}^{n+k-1} w_\lambda(z^*)\underline{e}_\lambda + \underline{e}_{n+k}, \ \underline{e}_i + \sum_{\sigma=1}^{k} \frac{\partial\lambda_{n+\sigma}}{\partial x_i}(m^*)\underline{e}_{n+\sigma} \right\rangle \quad ,$$

$$\text{so} \quad 0 = w_i(z^*) + \sum_{\sigma=1}^{k-1} w_{n+\sigma}(z^*) \frac{\partial\lambda_{n+\sigma}}{\partial x_i}(m^*) + \frac{\partial\lambda_{n+k}}{\partial x_i}(m^*) \ .$$

These equations display $\mathcal{N}'$ as a submanifold of $\mathcal{N}$ of codimension n, and show that in $\mathcal{N}'$ the functions $\{x_i, w_{n+\sigma} : 1 \leq i \leq n, \ 1 \leq \sigma \leq k-1\}$ define a coordinate system $(\mathcal{N}', \phi)$ centred on (m,z). Let $\Gamma_\lambda = w_\lambda \circ \Gamma$, $1 \leq \lambda \leq n+k-1$. Then for $1 \leq i \leq n$, $1 \leq \sigma \leq k-1$,

$$\left.\begin{aligned} \Gamma_i(m^*, z^*) &= -\sum_{\sigma=1}^{k-1} w_{n+\sigma}(z^*) \frac{\partial\lambda_{n+\sigma}}{\partial x_i}(m^*) - \frac{\partial\lambda_{n+k}}{\partial x_i}(m^*) \\ \Gamma_{n+\sigma}(m^*, z^*) &= w_{n+\sigma}(z^*) \ . \end{aligned}\right\}$$

Hence the Jacobian matrix of Γ in the coordinate systems $(\mathcal{N}', \phi)$ and (S_+^{n+k-1}, β) at $(m^*, z^*) \in \mathcal{N}'$ is given by

$$\left.\begin{aligned} \frac{\partial\Gamma_i}{\partial x_j}(m^*,z^*) &= -\sum_{\sigma=1}^{k-1} w_{n+\sigma}(z^*) \frac{\partial^2\lambda_{n+\sigma}}{\partial x_i \partial x_j}(m^*) - \frac{\partial^2\lambda_{n+k}}{\partial x_i \partial x_j}(m^*) \\ \frac{\partial\Gamma_i}{\partial y_{n+\tau}}(m^*,z^*) &= -\frac{\partial\lambda_{n+\tau}}{\partial x_i}(m^*) \\ \frac{\partial\Gamma_{n+\sigma}}{\partial x_j}(m^*, z^*) &= 0 \\ \frac{\partial\Gamma_{n+\sigma}}{\partial y_{n+\tau}}(m^*, z^*) &= \delta_{\sigma\tau} \end{aligned}\right\}$$

where $1 \leqslant i,\ j \leqslant n,\ 1 \leqslant \sigma,\ \tau \leqslant k-1$. Evaluating these equations at (m,z) which is the origin of the coordinate system $(\mathcal{N}', \phi)$ one obtains, using the fact that $\dfrac{\partial\lambda_{n+\sigma}}{\partial x_i}(m) = 0$,

$$\left.\begin{aligned}
&\frac{\partial\Gamma_i}{\partial x_j}(m,v) = -\frac{\partial^2\lambda_{n+k}}{\partial x_i\partial x_j}(m)\\
&\frac{\partial\Gamma_i}{\partial y_{n+\tau}}(m,v) = \frac{\partial\Gamma_{n+\sigma}}{\partial x_j} = 0\\
&\frac{\partial\Gamma_{n+\sigma}}{\partial y_{n+\tau}} = \delta_{\sigma\tau} \quad .
\end{aligned}\right\}$$

But $\lambda_{n+k} = \langle \underline{e}_{n+k}, f \rangle = \langle z,f \rangle$, hence the jacobian matrix of the gauss-map Γ of UNM at (m,z) is given by

$$(-\text{ hessian matrix of } \langle z,f \rangle) \oplus \text{ identity matrix.}$$

If, as before, the $\left\{\dfrac{\partial}{\partial x_i}(m) : i = 1,\ldots,n\right\}$ form an orthonormal basis of T_mM with respect to the f-induced metric, then the determinant of this jacobian matrix, multiplied by $(-1)^n$, determines a curvature density $G(m,z)$ on the unit normal bundle of (f,M) called the <u>Lipschitz-Killing curvature</u>.

In this manner one may state

<u>Proposition 2.6.</u> <u>At a singular point of</u> $\langle z,f \rangle$ <u>the differential of the gauss map at the corresponding point of the unit normal bundle of</u> f <u>is characterised by the second differential of</u> $\langle z,f \rangle$. In particular <u>the nullity of the gauss map is equal to the nullity of the second differential of</u> $\langle z,f \rangle$.

Definition 2.7 One describes a singular point of a real-valued function as degenerate or non-degenerate according as the nullity of the second differential of the function is positive or zero, or equivalently as the hessian matrix of the function in some coordinate system is singular or non-singular, at the point in question.

By an application of Sard's Theorem,

Proposition 2.8 The set of singular values of Γ has (n+k-1)-hausdorff measure zero in S^{n+k-1}. Combining Props. 2.3,6 and Defn. 2.7 and this last fact one obtains:

Proposition 2.9 For almost every $z \in S^{n+k-1}$, each singular point of $< z,f >$ is non-degenerate.

Definition 2.10 Define a Morse function on M^n to be a smooth real valued function all of whose singular points are non-degenerate; then one may restate

Proposition 2.11 If $f \in \mathcal{L}(M^n, \mathbb{R}^{n+k})$ is an immersion, then for almost every $z \in S^{n+k-1}$, $< z,f >$ is a Morse function.

If $\pi : S^{n+k-1} \to S^{n+k-1}$, $\pi(z) = -z$, denotes the antipodal involution of S^{n+k-1}, then it is simple to show that $UNM \subseteq M \times S^{n+k-1}$ is π-invariant, and that $\Gamma : UNM \to S^{n+k-1}$ is a π-invariant map. One may identify S^{n+k-1}/π with $G(n+k, 1)$, the projective space of lines through the origin of $\mathbb{R}^{n+k}$, UNM/π with the space NLM of normal lines of the immersion f of M. If $L \in G(n+k,1)$ corresponds to $\pm z \in S^{n+k-1}$, then P_L, the orthogonal projection of $\mathbb{R}^{n+k}$ onto P_L corresponds to $< z, >/\pi$ and $P_L \circ f : M^n \to L$ corresponds to $< z,f >/\pi$. The terminology 'singular', 'degenerate', 'Morse', etc. is invariant under the action of π. One has a gauss-map

$$\Gamma/\pi : NLM \to G(n+k, 1).$$

In these terms Proposition 2.9 becomes

<u>Proposition 2.12</u> <u>If</u> $f \in \mathcal{L}(M^n, \mathbb{R}^{n+k})$ <u>is an immersion, then for almost all</u> $L \in G(n+k-1)$, $P_L \circ f$ <u>is a Morse map</u>.

§3 COHERENT MEASURES AND 1-GOOD MAPS

Definition 3.1 Let $F \rightarrow E \xrightarrow{\pi} X$ be a fibre bundle with structure group G, and let μ_F, μ_E, μ_X be positive measures on the corresponding spaces. The measures $\{\mu_F, \mu_E, \mu_X\}$ will be said to be π-coherent if conditions (i) and (ii) below hold.

(i) μ_F and μ_E are invariant under the action of G.

(ii) For each open set $U \subseteq X$ over which there is a π-trivialising G-map $\theta : \pi^{-1}(U) \rightarrow U \times F$ the measure $\mu_{E|U}$ induced from μ_E by restriction to $\pi^{-1}(U)$ is equal, via θ, to $\mu_{X|U} \times \mu_F$ on $U \times F$, where $\mu_{X|U}$ is the restriction of μ_X to U.

The particular application of coherent measures which concerns one here is to an extension of Fubini's Theorem [8, p.115].

Proposition 3.2 Fubini's Theorem *If* $F \rightarrow E \xrightarrow{\pi} X$ *is a fibre bundle with structure group* G *and* π-*coherent measures* $\{\mu_F, \mu_E, \mu_X\}$ *and if* $N \subseteq E$ *is a* μ_E-*measurable set then*

(i) $\pi^{-1}(x) \cap N$ *is* μ_F-*measurable, via a* π-G-*isomorphism between* $\pi^{-1}(x)$ *and* F, *for* μ_X-*almost-every* $x \in X$.

(ii) $\mu_E(N) = \int \mu_F(\pi^{-1}(x) \cap N)d\mu_X(x)$, *where* μ_F *is interpreted in the sense of (i).*

Corollary 3.3 *If* $N^* \subseteq E$ *has outer*-μ_E-*measure zero, then for* μ_X-*almost-every* $x \in X$ $\pi^{-1}(x) \cap N^*$ *has outer*-μ_F-*measure zero.*

Consider now the fibre bundles

$$O(N-1)/O(s-1)\times O(N-s) \rightarrow O(N)/O(1)\times O(s-1)\times O(N-s) \xrightarrow{\pi_1} O(N)/O(1)\times O(N-1)$$

$$O(s)/O(1)\times O(s-1) \rightarrow O(N)/O(1)\times O(s-1)\times O(N-s) \xrightarrow{\pi_2} O(N)/O(s)\times O(N-s),$$

with structure groups $O(N-1)$ and $O(s)$, fibres $G(N-1, s-1)$ and $G(s,1)$, and base spaces $G(N,1)$ and $G(N,s)$ respectively. Denote $O(N)/O(1)\times O(s-1)\times O(N-s)$ by E. Then, in $G(N-1, s-1) \to E \xrightarrow{\pi_1} G(N,1)$, E may be interpreted as the space of ordered pairs (L,P) where L is a line through the origin of $\mathbb{R}^N$ and P is an $(s-1)$-plane through the origin of $\mathbb{R}^N$ orthogonal to L, $\pi_1(L,P) = L$, and the fibre over $L \in G(N,1)$ is the space of $(s-1)$-planes through the origin in the $(N-1)$-dimensional orthogonal complement of L. Similarly, in $G(s,1) \to E \xrightarrow{\pi_2} G(N,s)$, E may be interpreted as the space of ordered pairs (L,Q) where Q is an s-plane through the origin of $\mathbb{R}^N$ and L is a line lying in Q and containing the origin, $\pi_2(L,Q) = Q$, and the fibre over $Q \in G(N,s)$ is the space of lines in Q through the origin. The identification between these two interpretations of E is performed by the maps

$$(L, P) \mapsto (L, L + P)$$

$$(L, Q) \mapsto (L, \perp(Q,L))$$

where $\perp(Q,L)$ denotes the orthogonal complement of L in Q.

Now the Haar measures on $O(N)$, $O(N-1)$ and $O(s)$ induce measures on E, $G(N,1)$, $G(N,s)$ which are $O(N)$ invariant, on $G(N-1, s-1)$ which is $O(N-1)$ invariant, and on $G(s,1)$ which is $O(s)$ invariant. In particular, the measures on $G(N-1, s-1)$ and E are $O(N-1)$ invariant and the measures on $G(s,1)$ and E are $O(s)$ invariant. Moreover, from the fibre-wise (coset-wise) manner in which these measures are constructed from homogeneous principle bundles, it follows that the measures on the two fibre bundles are coherent in the above sense. See [8, §2.7] .

Now let $f : M^n \to E^N$ be a smooth immersion of a smooth manifold. Let $\mathcal{A} \subseteq G(N,1)$ be the set of lines $L \in G(N,1)$ such that $P_L \circ f$ is not Morse, or more explicitly has a degenerate singularity. (If $PN(M,f)$ denotes the projective bundle of normal lines and $P\Gamma : PN(M,f) \to G(N,1)$ denotes the associated projective gauss map then $\mathcal{A}$ is the image under $P\Gamma$ of the singular point set of $P\Gamma$). By Proposition 2.12, $\mathcal{A}$ has outer measure zero in $G(N,1)$, and hence $\pi_1^{-1}(\mathcal{A})$ has outer-μ_E-measure zero in E. By direct application of the Corollary 3.3 of the Fubini theorem for coherent measures to $\pi_1^{-1}(\mathcal{A}) = \mathcal{B} \subseteq E$ in the bundle $G(s,1) \to E \xrightarrow{\pi_2} G(N,s)$, one concludes that for almost every $Q \in G(N,s)$ $\pi_2^{-1}(Q) \cap \mathcal{B}$ has outer measure zero in $\pi_2^{-1}(Q) \approx G(s,1)$. But $\mathcal{B} = \pi_2^{-1}(\mathcal{A})$, so $\pi_2^{-1}(Q) \cap \mathcal{B} = \pi_2^{-1}(Q) \cap \pi_1^{-1}(\mathcal{A})$ is precisely the set of members of $\mathcal{A}$ which lie in Q. Hence

<u>Proposition 3.4</u> <u>If</u> $f : M^n \to E^N$ <u>is a smooth immersion then for almost every s-plane</u> Q <u>in</u> E^N <u>the following property holds:</u>

> <u>for almost every line</u> L <u>through the origin of</u> Q <u>and lying in</u> Q <u>the map</u> $P_L \circ f : M \to L$ <u>is Morse.</u>

<u>Definition 3.5</u> Now define a map $\phi : M \to E^s$ to be <u>1-good</u> if for almost all $L \in G(s,1)$ $P_L \circ \phi$ is a Morse map. Then observe that if $Q \subseteq Q' \subseteq E^N$ are linear subspaces and if P_Q and $P_{Q'}$ are the corresponding orthogonal projection then $P_Q = P_Q|Q' \circ P_{Q'}$. Then Proposition 3.4 becomes:

<u>Proposition 3.6</u> <u>If</u> $f : M^n \to E^N$ <u>is a smooth immersion then for almost every s-plane</u> $Q \in G(N,s)$, $1 \leq s \leq N$, $P_Q \circ f : M^N \to Q$ <u>is 1-good.</u>

Indeed this is a synthesis of the following two statements, the first of which has been already stated in three separate forms, and the second of which is a direct consequence of the bundle-measure arguments.

<u>Proposition 3.7</u> <u>A smooth immersion is 1-good</u>

<u>Proposition 3.8</u> <u>If</u> $f : M^n \to E^N$ <u>is 1-good, then for almost every s-plane</u> $Q \in G(N,s)$, $1 \leq s \leq N$, $P_Q \circ f : M^n \to Q$ <u>is 1-good.</u>

§4 MORSE FUNCTIONS AND CERF PATHS

A now classical application of Thom's Source Transversality Theorem is to a proof of Morse's Theorem [18, p.178], on the approximation of smooth functions. See [25, p.61] and [33, §16].

Let M^n be a compact smooth manifold of dimension n, and $\mathcal{L}(M,\mathbb{R})$ be the space of smooth real-valued function on M, and let $J^1(M,\mathbb{R})$ be the corresponding space of 1-jets. Let $\Sigma \subseteq J^1(M,\mathbb{R})$ be the set of 1-jets of elements of $\mathcal{L}(M,\mathbb{R})$ at points where their differentials are zero. Σ turns out to be a regular submanifold of $J^1(M,\mathbb{R})$ of codimension n. Define $f \in J^1(M,\mathbb{R})$ to be <u>Morse</u> if $f^{(1)} : M \to J^1(M,\mathbb{R})$ is transversal on Σ. Then the transversality theorem yields

<u>Proposition 4.1</u> <u>Morse functions form an open dense subspace of</u> $\mathcal{L}(M,\mathbb{R})$.

To describe a Morse function in classical language, let (U, θ) be a coordinate system in M with coordinate functions $\{x_i : i = 1, \ldots, n\}$. Let $J^1(U,\mathbb{R})$ be the part of $J^1(M,\mathbb{R})$ lying over U; then one may cover $J^1(U,\mathbb{R})$ with a coordinate chart and coordinate functions $\{\bar{x}_i, \bar{y}, \bar{p}_i : 1 \leq i \leq n\}$ where if $f \in \mathcal{L}(M,\mathbb{R})$ and $m \in U$ then

$$\left.\begin{aligned} \bar{x}_i \circ f^{(1)}(m) &= x_i(m) \\ \bar{y} \circ f^{(1)}(m) &= f(m) \\ \bar{p}_i \circ f^{(1)}(m) &= \frac{\partial f}{\partial x_i}(m) \end{aligned}\right\}$$

In this coordinate system the map

$$A : \mathbb{R}^{2n+2} \to \mathbb{R}^n, \quad (\underline{\bar{x}}, \bar{y}, \underline{\bar{p}}) \to (\underline{\bar{p}})$$

defines the part $\Sigma(U)$ of Σ which lies in $J^1(U,\mathbb{R})$; i.e.

$\Sigma(U) = A^{-1}(\underline{0})$. A has maximal rank everywhere and hence Σ is a submanifold of $J^1(M,\mathbb{R})$ of codimension n. The transversality condition for a Morse $f \in \mathcal{L}(M,\mathbb{R})$ becomes

<u>at every point</u> $m \in U$ <u>where</u> $A \circ f^{(1)}(m) = \underline{0}$

<u>the map</u> $A \circ f^{(1)} : M^n \to \mathbb{R}^n$ <u>has maximal rank</u>.

Now $A \circ f^{(1)}(m) = \left(\frac{\partial f}{\partial x_1}(m), \ldots, \frac{\partial f}{\partial x_n}(m)\right)$ and

so the jacobian matrix of $A \circ f^{(1)}$ at $m \in U$ is

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(m) : 1 \leq i,\ j \leq n\right)$$

which is the hessian of f at m. Hence $f \in \mathcal{L}(M,\mathbb{R})$ is a Morse function if and only if whenever $Df(m)$ is zero, the hessian matrix of f at m is non-singular. Thus there is agreement between previous and present uses of the adjective 'Morse'. Then Proposition 4.1 becomes the classical theorem of Morse:

<u>Proposition 4.2</u> <u>Any</u> $f \in \mathcal{L}(M,\mathbb{R})$ <u>may be arbitrarily closely approximated by an</u> $f' \in \mathcal{L}(M,\mathbb{R})$ <u>all of whose singular points are nondegenerate</u>.

Taking account of the effect of a change of coordinates on a hessian, one sees that the only algebraic invariants of a hessian are those of the orbits of the space of symmetric matrices under the similarity action $(g,s) \mapsto g^t s g$ of the general linear group, namely <u>rank</u> and <u>index</u>. Thus the algebraic invariants of the singular points of a Morse function are their indices. The canonical forms and topological relationships for the singular points of the various possible indices are well-known [19, p.25] and [22]. It is immediate from the transversality type definition, that a Morse function has isolated critical points, and hence, for compact M, only a finite number.

Let M^n be compact, and let $\mathcal{ML}(M,\mathbb{R})$ be the space of Morse functions on M. $\mathcal{ML}(M,\mathbb{R})$ is an open dense subspace of $\mathcal{L}(M,\mathbb{R})$. However the stability of non-degenerate critical points of each index-type implies that two elements in the same path-component of $\mathcal{ML}(M,\mathbb{R})$ have the same number of critical points of each index-type; but $\mathcal{ML}(M,\mathbb{R})$ contains elements for which these numbers differ, as may be shown by reversing the procedures of [20] to obtain two extra singular points, one of index zero and one of index 1. Thus the space of smooth paths in $\mathcal{ML}(M,\mathbb{R})$ is very far from dense in the space of smooth paths in $\mathcal{L}(M,\mathbb{R})$.

Thus the question arises: <u>what is the class of singularities that a dense subspace of the space of smooth paths in $\mathcal{L}(M,\mathbb{R})$ must exhibit</u>?

A smooth path $f : I \to \mathcal{L}(M,\mathbb{R})$, $t \mapsto f_t$, is identified with a smooth map $f \in \mathcal{L}(M \times I, \mathbb{R})$. Let $J^2(M \times I, \mathbb{R})$ be the corresponding bundle of 2-jets, and let $\Sigma \subseteq J^2(M \times I, \mathbb{R})$ be the set of 2-jets, $f^{(2)}(m,t)$, of smooth maps $f : M \times I \to \mathbb{R}$ where m is a degenerate singular point of the partial map $f_t \in \mathcal{L}(M,\mathbb{R})$.

In fact, let $(U, \theta, \{x_i : i = 1, \ldots, n\})$ be a coordinate system in M. Then one may introduce on $J^2(U \times I, \mathbb{R})$, the open subspace of $J^2(M \times I, \mathbb{R})$ lying over $U \times I$, a coordinate system θ', with coordinate functions

$$\{ \bar{x}_i, \bar{t}, \bar{y}, \bar{p}_i, \bar{\Phi}, \bar{s}_{ij}, \bar{\eta}_i, \bar{\Psi} : 1 \leq i \leq j \leq n \}$$

giving maps

$$\underline{\bar{x}}, \underline{\bar{p}}, \underline{\bar{\eta}} : J^2(U \times I, \mathbb{R}) \to \mathbb{R}^n$$

$$\bar{t} : J^2(U \times I, \mathbb{R}) \to I$$

$$\bar{y},\ \bar{\phi},\ \bar{\psi} : J^2(U \times I,\ \mathbb{R}) \rightarrow \mathbb{R}$$

$$\bar{\underline{\underline{s}}} : J^2(U \times I,\ \mathbb{R}) \rightarrow \mathbb{R}^{\frac{1}{2}n(n+1)} = \text{symmetric Matrices of order } n,$$

where if $f \in \mathcal{L}(M \times I, \mathbb{R})$, $m \in U$, $t \in I$, then for $1 \leqslant i \leqslant j \leqslant n$

$$\left.\begin{aligned}
\bar{x}_i \circ f^{(2)}(m,t) &= x_i(m) \\
\bar{t} \circ f^{(2)}(m,t) &= t \\
\bar{p}_i \circ f^{(2)}(m,t) &= \frac{\partial f}{\partial x_i}(m,t) \\
\bar{y} \circ f^{(2)}(m,t) &= f(m,t) \\
\phi \circ f^{(2)}(m,t) &= \frac{\partial f}{\partial t}(m,t) \\
\bar{s}_{ij} \circ f^{(2)}(m,t) &= \frac{\partial^2 f}{\partial x_i \partial x_j}(m,t) \\
\bar{\eta}_i \circ f^{(2)}(m,t) &= \frac{\partial^2 f}{\partial x_i \partial t}(m,t) \\
\bar{\psi} \circ f^{(2)}(m,t) &= \frac{\partial^2 f}{\partial t^2}(m,t)
\end{aligned}\right\}$$

Let $A : J^2(U \times I, \mathbb{R}) \rightarrow \mathbb{R}^{n+1}$ be defined by

$$A^{\theta'}(\underline{\bar{x}},\ \bar{t},\ \bar{y},\ \underline{\bar{p}},\ \bar{\underline{\underline{s}}},\ \underline{\bar{\eta}},\ \bar{\psi}) = (\underline{\bar{p}},\ \det(\bar{\underline{\underline{s}}})\ ) \in \mathbb{R}^n \times \mathbb{R}. \qquad (4.1)$$

Then, if $\Sigma(U)$ denotes the part of Σ that lies over $U \times I$, $\Sigma(U)$ is defined by the map A, i.e. $\Sigma(U) = A^{-1}(\underline{0},\ 0)$ where $(\underline{0}, 0) \in \mathbb{R}^{n+1}$ is the origin. Now $\Sigma\ (U)$ possesses a natural 'stratification' according to the rank of $\bar{\underline{\underline{s}}}$; in particular one may decompose $\Sigma(U)$ as $\Sigma^{n-1}(U) \mathring{\cup} \Sigma^*(U)$, where a point of $\Sigma(U)$ is in $\Sigma^{n-1}(u)$ or $\Sigma^*(u)$ according as rank $\bar{\underline{\underline{s}}} = n-1$ or rank $\underline{\underline{s}} < n-1$.

This decomposition may be interpreted, via A, in the following way: $\Sigma^{n-1}(U)$ is the set of points of $\Sigma(U)$ where DA has rank $(n+1)$, and $\Sigma^*(U)$ is the set of points of $Z(u)$ where DA has rank less than $(n+1)$. This arises from the observation that the cofactors, and in general all the minors, of the 'general' determinant of order n may be got from appropriate differentiations of the corresponding polynomial function of degree n in n^2 variables. $\Sigma^{n-1}(U)$ is thus a regular submanifold of $J^2(U \times I, \mathbb{R})$ of codimension $(n+1)$, and $\Sigma^*(u)$ is a 'stratified' collection of submanifolds of codimensions greater than $(n+1)$.

Evidently the requirement that $f \in \mathcal{L}(M \times I, \mathbb{R})$ should contain only essential or 'generic' degeneracies can be described by insisting that $f^{(2)}$ be 'transversal' on Σ. In view of the above data concerning codimensions one may define:

Definition 4.3 $f \in \mathcal{L}(M \times I, \mathbb{R})$ is a Cerf path if $f^{(2)}(M \times I) \cap \Sigma^* = \emptyset$ and $f^{(2)}$ is transversal on Σ^{n-1}.

Thom's Source Transversality Theorem gives the result

Proposition 4.4 Cerf paths form an open dense subspace of $\mathcal{L}(M \times I, \mathbb{R})$.

To determine a coordinate description of a Cerf path, let $\bar{\delta} : J^2(U \times I, \mathbb{R}) \to \mathbb{R}$ denote the function $\det(\underline{\underline{\bar{s}}})$. Then for $f \in \mathcal{L}(U \times I, \mathbb{R})$,

$$\bar{\delta} \circ f^{(2)} = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j} : 1 \leq i,\ j \leq n\right)$$

$$= D\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \,\middle|\, x_1, \ldots, x_n\right)$$

$$= D(\underline{\bar{p}} \circ f^{(2)} \mid \underline{x}) .$$

Since $\Sigma^{n-1}(u)$ is the set of regular points of the map A, given by (4.1), which lie in $\Sigma(u)$, and hence where DA has rank $(n+1)$, the condition of transversality of $f^{(2)}$ when f is a Cerf path is that

$$D(A \circ f^{(2)}) \text{ \underline{has rank} } (n+1) \text{ \underline{at all points} }$$
$$(m,t) \in U \times I \text{ \underline{where} } \underline{\bar{p}} \circ f^{(2)}(m,t) = \underline{0}$$
$$\text{\underline{and} } \bar{\delta} \circ f^{(2)}(m,t) = 0.$$

Now $A \circ f^{(2)} = (\underline{\bar{p}} \circ f^{(2)}, \ \bar{\delta} \circ f^{(2)})$, thus the jacobian matrix of $A \circ f^{(2)}$ at $(m,t) \in U \times I$ is

$$J(\underline{\bar{p}} \circ f^{(2)}, \ \bar{\delta} \circ f^{(2)} \mid \underline{x}, t)(m,t)$$

with determinant

$$D(\underline{p} \circ f^{(2)}, \ \delta \circ f^{(2)} \mid \underline{x}, t)(m,t) = \mathcal{D}(m,t).$$

Observing that one could define a function $\bar{\mathcal{D}}$ on $J^3(U \times I, \mathbb{R})$, by appropriate extension of the coordinate system θ' to take in the third order derivatives, such that $\mathcal{D} = \bar{\mathcal{D}} \circ f^{(3)}$, one may state

<u>Proposition 4.5</u> $f \in \mathcal{L}(M \times I, \mathbb{R})$ <u>is a Cerf path if and only if in a system of coordinate charts</u> $(U, \theta, \{x_i : 1 \leq i \leq n\})$ <u>which covers</u> M, <u>and hence in all coordinate systems, the</u> $(n+2)$ <u>functions</u> $\underline{\bar{p}} \circ f^{(2)}$, $\bar{\delta} \circ f^{(2)}, \mathcal{D}$ <u>on</u> $U \times I$ <u>are never simultaneously zero</u>. Moreover, <u>every</u> $g \in \mathcal{L}(M \times I, \mathbb{R})$ <u>can be arbitrarily closely approximated by an</u> $f \in \mathcal{L}(M \times I, \mathbb{R})$ <u>which satisfies these local conditions everywhere</u>.

<u>Definition 4.6</u> Henceforth $f \in \mathcal{L}(M \times I, \mathbb{R})$ will denote a Cerf path. The next stage is to analyse the geometry of f, or more precisely of the singularities that it encounters. For this purpose one defines the <u>characteristic curve of</u> f, denoted by C(f), to be the set of points of $M \times I$ such that the point of M is a singular point of the corresponding partial

function:

$$C(f) = \{(m,t) \in M \times I : D(f_t)(m) = 0\}$$

The <u>tracking-map</u> τ <u>of</u> f, $\tau : M \times I \to I \times \mathbb{R}$ is defined by

$$\tau(m,t) = (t, f(m,t));$$

The image of $C(f)$ under τ is called the <u>track of</u> f and is denoted by $T(f)$.

Return now to local coordinates on U and $J^2(U \times I, \mathbb{R})$. On U, $C(f)$ is defined by the equation $\bar{\underline{p}} \circ f^{(2)} = \underline{0}$. On $C(f)$, $\bar{\delta} \circ f^{(2)}$ and are never simultaneously zero, since f is Cerf (Prop. 4.5), and hence the jacobian matrix $J(\bar{\underline{p}} \circ f^{(2)} \mid \underline{x}, t)$, obtained from $J(\bar{\underline{p}} \circ f^{(2)}, \bar{\delta} \circ f^{(2)} \mid \underline{x}, t)$ by omitting the last row, has rank n at every point of $C(f)$. Hence

<u>Proposition 4.7</u> <u>The characteristic curve of</u> f <u>is a singularity-free curve, given locally in</u> $U \times I$ <u>by the transversal intersections of the hyper-surfaces</u>

$$\frac{\partial f}{\partial x_i} = 0, \qquad 1 \leqslant i \leqslant n.$$

Consider the set $H^o(f)$ of points of $U \times I$ where $\bar{\delta} \circ f^{(2)}$ is zero. At their points of intersection $H^o(f)$ is transversal to $C(f)$, since $\bar{\underline{p}} \circ f^{(2)}$ defines $C(f)$ and $\delta \circ f^{(2)}$ defines $H^o(f)$, and where $H^o(f)$ meet $C(f)$ $\mathcal{D} = D(\bar{\underline{p}} \circ f^{(2)}, \bar{\delta} \circ f^{(2)} \mid \underline{x}, t)$ is non-zero. Thus, as could also be deduced from the transversal jet definition and the codimension of Σ^{n-1},

<u>Proposition 4.8</u> $\bar{\delta} \circ f^{(2)}$ <u>is zero at only a finite set of points of the characteristic curve of</u> f.

Let $\mathcal{D}_i$ be the cofactor of $\dfrac{\partial(\bar{\delta} \circ f^{(2)})}{\partial x_i}$ in the matrix $J(\bar{\underline{p}} \circ f^{(2)}, \bar{\delta} \circ f^{(2)} \mid \underline{x}, t)$. Together with $\bar{\delta} \circ f^{(2)}$, the $\mathcal{D}_i$ are the signed n-by-n minors of the matrix $J(\bar{\underline{p}} \circ f^{(2)} \mid \underline{x}, t)$, which is known by previous considerations to have rank n at every point of $C(f)$. Hence at a point of $C(f)$ where $\bar{\delta} \circ f^{(2)}$ is zero, i.e. in $H^0(f) \cap C(f)$, one of the $\mathcal{D}_i$ must be non-zero.

$C(f)$ is defined by the equation $\underline{p} \circ f^{(2)} = 0$, and so, along $C(f)$

$$\frac{dx_1}{\mathcal{D}_1} = \dots = \frac{dx_n}{\mathcal{D}_n} = \frac{dt}{\delta \circ f^{(2)}} = \frac{d(\bar{\delta} \circ f^{(2)})}{\mathcal{D}} = d\sigma \qquad (4.2)$$

for some parameter σ. With respect to the basis $\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial t} \; ; \; 1 \leq i \leq n \right\}$ the direction coefficients of the tangent to $C(f)$ in $U \times I$ are proportional to $(\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n, \delta \circ f^{(2)})$. Hence the points of $C(f)$ where $\bar{\delta} \circ f^{(2)}$ is zero are precisely those points where the tangent to $C(f)$ is <u>horizontal</u>, that is - annihilated by the lateral projection $M \times I \to I$.

Let $(m,t) \in C(f)$ be a point where the tangent is horizontal. Assume, without loss of generality, that $\mathcal{D}_n \neq 0$. Using Prop. 4.5, one may define a local parameter σ of $C(f)$ by the equation

$$\frac{d\sigma}{dx_n} = 1/\mathcal{D}_n \,.$$

One has

$$\left.\begin{aligned} \frac{dt}{d\sigma} &= \bar{\delta} \circ f^{(2)} \\ \frac{d^2t}{d\sigma^2} &= \frac{d(\bar{\delta} \circ f^{(2)})}{d\sigma} = \mathcal{D}\,. \end{aligned}\right\}$$

Now $\mathcal{D}(m,t) \neq 0$, since $(m,t) \in C(f)$ and $\bar{\delta} \circ f^{(2)}(m,t) = 0$. Thus

$$\frac{dt}{d\sigma}(m,t) = 0, \qquad \frac{d^2t}{d\sigma^2}(m,t) \neq 0$$

Hence

Proposition 4.9 At a point of the characteristic curve of f where the tangent is horizontal, C(f) has a simple maximum or simple minimum under the projection $M \times I \to I$ and in particular C(f) lies locally on one side of the horizontal hypersurface there.

In other words, lateral projection restricted to C(f) is a Morse map. If $\mathcal{D}(m,t) < 0$, then, as $t' \in I$ passes through t from below, $f(\ \ ,t')$ has two non-degenerate singular points in a neighbourhood of m which move together towards m, coalesce when $t' = t$ at m where $f(\ \ ,t)$ has a degenerate singularity, and then vanishes when $t' > t$ (death-point). If $\mathcal{D}(m,t) > 0$ then this process occurs in reverse (birth-point).

To complete the present local description of a Cerf path, one will now turn to the tracking map. Let $f \in \mathcal{L}(M^n \times I, \mathbb{R})$ be a Cerf path. then

$$\tau : M^n \times I \to I \times \mathbb{R};\ (m,t) \mapsto (t, f(m,t))$$

defines the tracking map. First one must discover the singularities of τ. The jacobian matrix of τ at (m,t) in a coordinate neighbourhood $(U \times I, \theta \times 1_I, \{x_i, t\})$ of (m,t) is

$$\begin{pmatrix} 0 & \ldots & 0 & 1 \\ \dfrac{\partial f}{\partial x_1} & \ldots & \dfrac{\partial f}{\partial x_n} & \dfrac{\partial f}{\partial t} \end{pmatrix} (m,t).$$

Hence (i) rank $D\tau \geqslant 1$

(ii) rank $D\tau(m,t) = 1$ if and only if m is a singular point of $f_t = f(\ \ ,t)$.

Thus τ is regular at points of $M \times I$ which do not lie on C(f). Now consider the effect of τ restricted to C(f). By equations (4.2), at a point of C(f) where $\bar{\delta} \circ f^{(2)}$ is non-zero, the coordinate t may be taken as a local parameter. Hence, at points of $C(f)$ where the tangent is not horizontal τ restricted to $C(f)$ is locally a regular map.

Finally, one must consider the effect of τ restricted to $C(f)$ at points where the tangent is horizontal. At such points $\bar{\delta} \circ f^{(2)} = D\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \,\middle|\, x_1, \ldots, x_n\right)$ is zero, but

$$\mathcal{D} = D\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, \bar{\delta} \circ f^{(2)} \,\middle|\, x_1, \ldots, x_n, t\right)$$ is non-zero. As has been seen, $J\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \,\middle|\, x_1, \ldots, x_n\right)$ has rank $(n-1)$ at such a point. Now, considering this as the hessian of the function $f(\ \ , t)$ where $(m,t) \in C(f)$ is the point in question, one may choose a coordinate system centred on $m \in M$, $(U, \theta\ \{x_i\})$, such that

$$\left.\begin{aligned} &J\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n-1}} \,\middle|\, x_1, \ldots, x_{n-1}\right)(m,t) \quad \text{is non-singular} \\ &\frac{\partial^2 f}{\partial x_i \partial x_n}(m,t) = 0, \qquad i = 1, \ldots, n \end{aligned}\right\}$$

(see [21, Theorem 4.2] and §7 of this work). Such a coordinate system is said to be hessian-adapted to f at (m,t).

In such a coordinate system, now make the simplification $\bar{c} \circ f^{(2)} = c$ where $\bar{c}$ is a function on $J^2(U \times I, \mathbb{R})$. Expand $\delta = \bar{\delta} \circ f^{(2)} = D(p_1, \ldots, p_n | x_1, \ldots, x_n)$ along the last row to get

$$\delta = \sum_{i=1}^{n} \delta_i \frac{\partial^2 f}{\partial x_i \partial x_n}, \quad \text{where}$$

$$\left.\begin{aligned} \delta_1(m,t) = \ldots = \delta_{n-1}(m,t) = 0, \quad \delta_n(m,t) \neq 0, \\ \delta_n = D(p_1, \ldots, p_{n-1} | x_1, \ldots, x_{n-1}). \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{Now}\quad \frac{\partial\delta}{\partial x_1}(m,t) &= \dots = \frac{\partial\delta}{\partial x_{n-1}}(m,t) = 0 \\ \frac{\partial\delta}{\partial x_n}(m,t) &= \delta_n(m,t)\cdot\frac{\partial^3 f}{\partial x_n^{\,3}}(m,t) \end{aligned}\right\}$$

and $J(p_1, \dots, p_n, \delta | x_1, \dots, x_n, t) =$

$$\left(\begin{array}{ccc|c|c} & & & \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_1 \partial t} \\ & J(p_1, \dots, p_{n-1} | x_1, \dots, x_{n-1}) & & \vdots & \vdots \\ & & & \frac{\partial^2 f}{\partial x_{n-1} \partial x_n} & \frac{\partial^2 f}{\partial x_{n-1} \partial t} \\ \hline \frac{\partial^2 f}{\partial x_1 \partial x_n} & \dots & \frac{\partial^2 f}{\partial x_{n-1} \partial x_n} & \frac{\partial^2 f}{\partial x_n^{\,2}} & \frac{\partial^2 f}{\partial x_n \partial t} \\ \hline \frac{\partial\delta}{\partial x_1} & \dots & \frac{\partial\delta}{\partial x_{n-1}} & \frac{\partial\delta}{\partial x_n} & \frac{\partial\delta}{\partial t} \end{array}\right)$$

$$\text{Whence}\quad \mathcal{D}(m,t) = \left(D(p_1, \dots, p_{n-1} | x_1, \dots, x_{n-1}) \cdot \frac{\partial^2 f}{\partial x_n \partial t} \cdot \frac{\partial\delta}{\partial x_n}\right)(m,t).$$

$$= k \cdot \left(\frac{\partial^2 f}{\partial x_n \partial t} \cdot \frac{\partial^3 f}{\partial x_n^{\,3}}\right)(m,t).$$

Now $\mathcal{D}\ (m,t) \neq 0$, hence $\dfrac{\partial^2 f}{\partial x_n \partial t}(m,t) \neq 0 \neq \dfrac{\partial^3 f}{\partial x_n^{\,3}}(m,t)$.

Also $\mathcal{D}_1(m,t) = \dots = \mathcal{D}_{n-1}(m,t) = 0$, $\mathcal{D}_n(m,t) \neq 0$.

Introduce therefore, the parameter σ on $C(f)$, centred on (m,t):

$$\left.\begin{aligned} \sigma(m,t) &= 0 \\ \frac{d\sigma}{dx_n} &= 1/\mathcal{D}_n \end{aligned}\right\}$$

Now regard x_i, t, y, ..., as functions of σ on $C(f)$. One has the track map given by $\sigma \mapsto (t(\sigma), y(\sigma))$, and

$$\left.\begin{aligned} \delta(0) &= 0 \\ p_i &\equiv 0 \\ \frac{dp_i}{d\sigma} &\equiv 0 \end{aligned}\right\} \qquad (4.3)$$

Now

$$\left.\begin{aligned} \frac{dy}{d\sigma} &= \sum_{i=1}^{n} p_i \frac{dx_i}{d\sigma} + \phi\delta \\ &= \sum_{i=1}^{n} p_i \, \mathcal{D}_i + \phi\delta \\ \frac{d^2y}{d\sigma^2} &= \sum_{i=1}^{n} \left(\frac{dp_i}{d\sigma} \quad {}_i + p_i \frac{d \quad {}_i}{d\sigma} \right) + \frac{d\phi}{d\sigma}\delta + \phi\frac{d\delta}{d\sigma} \end{aligned}\right\}$$

using (4.3) and remembering $\phi = \bar{\Phi} \circ f^{(2)} = \dfrac{\partial f}{\partial t}$.

Thus

$$\left.\begin{aligned} \frac{dy}{d\sigma}(0) &= 0 \\ \frac{d^2y}{d\sigma^2}(0) &= \phi(0)\frac{d\delta}{d\sigma}(0) \\ \frac{d^3y}{d\sigma^3}(0) &= 2\frac{d\phi}{d\sigma}(0)\frac{d\delta}{d\sigma}(0) + \phi(0)\frac{d^2\delta}{d\sigma^2}(0) \end{aligned}\right\}$$

Also

$$\left.\begin{aligned} \frac{dt}{d\sigma}(0) &= \delta(0) = 0 \\ \frac{d^2t}{d\sigma^2}(0) &= \frac{d\delta}{d\sigma}(0) \\ \frac{d^3t}{d\sigma^3}(0) &= \frac{d^2\delta}{d\sigma^2}(0)\,. \end{aligned}\right\}$$

Hence

$$\det \begin{vmatrix} \frac{d^2t}{d\sigma^2}(0) & \frac{d^3t}{d\sigma^3}(0) \\ \frac{d^2y}{d\sigma^2}(0) & \frac{d^3y}{d\sigma^3} \end{vmatrix} = 2\left(\frac{d\delta}{d\sigma}(0)\right)^2 \frac{d\phi}{ds}(0)\,.$$

Now $\frac{d\delta}{d\sigma}(0) = \mathcal{D}(m,t)$, and

$$\frac{d\phi}{d\sigma} = \frac{d}{d\sigma}\left(\frac{\partial f}{\partial t}\right)$$

$$= \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial t} \frac{dx_i}{d\sigma} + \frac{\partial^2 f}{\partial t^2} \cdot \frac{dt}{ds}$$

$$= \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial t} \mathcal{D}_i + \frac{\partial^2 f}{\partial t^2} \cdot \delta$$

$$= D(\underline{p}, \left.\frac{\partial f}{\partial t}\right| \underline{x}, t)$$

$=$ determinant of the hessian of f.

Hence $$\frac{d\phi}{d\sigma}(0) = \frac{\partial^2 f}{\partial x_n \partial t}(m,t)\,.\,\mathcal{D}_n(m,t) \neq 0\,.$$

Hence

$$\det \begin{vmatrix} t'' & t''' \\ y'' & y''' \end{vmatrix} \neq 0$$

at (m,t). Thus the track of f traces out a cusp of the first species (ramphoid, simple, [10, §410] and [23, p.46]) at such a point.

One may summarise:

Proposition 4.10 The tracking map of a Cerf path is regular except on the characteristic curves which it immerses regularly except for a finite number of simple cusps. The tangent to the track is never vertical.

Moreover, in any hessian adapted coordinate system at a point where the tracking map has a cusp point

$$\frac{\partial f}{\partial x_n} = \frac{\partial^2 f}{\partial x_n^2} = 0, \qquad \frac{\partial^2 f}{\partial x_n \partial t} \neq 0 \neq \frac{\partial^3 f}{\partial x_n^3} .$$

§5 WHITNEY MAPS

Let M^n be a compact smooth n-dimensional manifold. The problem now under consideration is to determine the types of singularities that a map $f : M^n \to \mathbb{R}^2$ need exhibit generically, or to find a criterion of 'goodness' in $\mathcal{L}(M^n, \mathbb{R}^2)$.

As in the previous section, the determination is best performed in the jet-spaces of $\mathcal{L}(M, \mathbb{R}^2)$ by designating an hierarchy of 'natural' transversality conditions. Let $\Sigma \subseteq J^1(M, \mathbb{R}^2)$ be the 'stratified' pair of manifolds given by the 1-jets of maps in $\mathcal{L}(M, \mathbb{R}^2)$ at points where they are singular; excluding the case $n = 1$ hence-forth, being catered for by Whitney's immersion theorem, singular prints occur precisely when the rank is not equal to 2. In order that the singular rank structure of $f \in \mathcal{L}(M, \mathbb{R}^2)$ be as simple as possible, one defines

<u>Definition 5.1</u> $f \in \mathcal{L}(M, \mathbb{R}^2)$ is said to be <u>rank-good</u> if $f^{(1)} : M \to J^1(M, \mathbb{R}^2)$ is transversal on Σ.

Having confirmed that Σ is the 'stratified' union of two regular submanifolds of $J^1(M, \mathbb{R}^2)$, one has by source transversality

<u>Proposition 5.2</u> <u>Rank-good maps form an open dense subspace of</u> $\mathcal{L}(M, \mathbb{R}^2)$.

In order to describe a rank-good map in local terms, let $(U, \theta, \{x_i : 1 \leq i \leq n\})$ be a coordinate system in M, and let $(\mathbb{R}^2, 1_{\mathbb{R}^2}, \{y_\alpha : \alpha = 1,2\})$ be the standard coordinate system on $\mathbb{R}^2$. Then a coordinate system $(J^1(U, \mathbb{R}^2), \Theta)$ may be introduced on the open subspace $J^1(U, \mathbb{R}^2)$ of $J^1(M, \mathbb{R}^2)$ with coordinate functions $\{\bar{x}_i, \bar{y}_\alpha, \bar{p}_{\alpha i} : 1 \leq i \leq n, \ \alpha = 1,2\}$, where, if $f \in \mathcal{L}(M, \mathbb{R}^2)$ and $m \in U$,

$$\left.\begin{aligned} \bar{x}_i \circ f^{(2)} &= x_i \\ \bar{y}_\alpha \circ f^{(2)} &= y_\alpha \circ f = f_\alpha \\ \bar{p}_{\alpha i} \circ f^{(2)} &= \frac{\partial(y_\alpha \circ f)}{\partial x_i} = \frac{\partial f_\alpha}{\partial x_i} \end{aligned}\right\} \tag{5.1}$$

In this coordinate system $\Sigma^0(U)$, the part of Σ lying over U containing jets of rank 0, is defined by map $A_0 : J^1(U, \mathbb{R}^2) \to \mathbb{R}^{2n}$ given by

$$A_0^\Theta(\underline{x}, \underline{y}, \underline{\underline{p}}) = \underline{\underline{p}},$$

that is to say, $\Sigma^0(U) = A_0^{-1}(\underline{\underline{0}})$. Now A_0 has (maximal) rank 2n at every point, thus $\Sigma^0(U)$ is a submanifold of $J^1(U, \mathbb{R}^2)$ of codimension 2n. So Σ^0 is a submanifold of $J^1(M, \mathbb{R}^2)$ of codimension 2n. Thus $f^{(1)} : M \to J^1(M, \mathbb{R}^2)$ is transversal on Σ^0 if and only if $f^{(1)}(M) \cap \Sigma^0 = \emptyset$. Hence

<u>Proposition 5.3</u> <u>A rank-good map in</u> $(M, \mathbb{R}^2)$ <u>has rank everywhere equal to 1 or 2.</u>

Suppose now that $f \in \mathcal{L}(M, \mathbb{R}^2)$ has rank Df equal to 1 at $m \in M$. Let $(U, \theta, \{x_i\})$, $(\mathbb{R}^2, \pm e^{i\phi}, \{y_\alpha\})$ be systems linearly adapted to f at m. Using the conventions (5.1) for the coordinate functions on $J^1(U, \mathbb{R}^2)$ one has

$$\left.\begin{aligned} \bar{p}_{11} \circ f^{(1)}(m) &= 1 \\ \bar{p}_{\alpha i} \circ f^{(1)}(m) &= 0, \quad \text{otherwise.} \end{aligned}\right\} \tag{5.2}$$

If Σ^1 denotes the part of Σ consisting of jets of rank 1, and $\Sigma^1(U)$ denotes the part of Σ^1 lying over U, then in an open neighbourhood $\mathcal{U}$ of $f^{(1)}(m) \in U$ in $J^1(U, \mathbb{R}^2)$ in which the functions $\bar{p}_{11}$ and $\bar{p}_{12}$ do not simultaneously vanish $\mathcal{U} \cap \Sigma^1(U)$ is defined by the map

$A_1 : J^1(U, \mathbb{R}^2) \cap \mathcal{U} \to \mathbb{R}^{n-1}$,

$$A_1^{\Theta}(\bar{\underline{x}}, \bar{\underline{y}}, \bar{\underline{\underline{p}}}) = (\bar{p}_{11}\,\bar{p}_{2r} - \bar{p}_{21}\,\bar{p}_{1r} : 2 \leqslant r \leqslant n).$$

A_1 has (maximal) rank (n-1) at every point of $J^1(U, \mathbb{R}^2) \cap \mathcal{U}$. Thus $\Sigma^1(U) \cap \mathcal{U}$ is a regular submanifold of $J^1(U, \mathbb{R}^2) \cap \mathcal{U}$ of codimension (n-1), and so Σ^1 is a regular submanifold of $J^1(M, \mathbb{R}^2)$ of codimension (n-1).

Now $f^{(1)}$ is transversal on Σ^1 at $m \in U$ if and only if $A_1 \circ f^{(1)} : U \to \mathbb{R}^{n-1}$ has maximal rank at m. If $m' \in U$

$$A_1 \circ f^{(1)}(m') = \left(\frac{\partial f_1}{\partial x_1}(m')\,\frac{\partial f_2}{\partial x_r}(m') - \frac{\partial f_2}{\partial x_1}(m')\,\frac{\partial f_1}{\partial x_r}(m') \;:\; 2 \leqslant r \leqslant n\right)$$

Hence the jacobian matrix of $A_1 \circ f^{(1)}$ at $m' \in U$ is given by

$$\left(\begin{array}{cc} \dfrac{\partial^2 f_1}{\partial x_1 \partial x_i}(m')\,\dfrac{\partial f_2}{\partial x_r}(m') + \dfrac{\partial f_1}{\partial x_1}(m')\,\dfrac{\partial^2 f_2}{\partial x_r \partial x_i}(m') & \\ & : \begin{array}{c} 2 \leqslant r \leqslant n \\ 1 \leqslant i \leqslant n \end{array} \\ -\dfrac{\partial^2 f_2}{\partial x_1 \partial x_i}(m')\,\dfrac{\partial f_1}{\partial x_r}(m') + \dfrac{\partial f_2}{\partial x_1}(m')\,\dfrac{\partial^2 f_1}{\partial x_r \partial x_i}(m') & \end{array}\right).$$

Using (5.2,1), the jacobian matrix of $A_1 \circ f^{(1)}$ at m is thus

$$\left(\frac{\partial^2 f_2}{\partial x_r \partial x_i}(m) \;:\; 2 \leqslant r \leqslant n, \quad 1 \leqslant i \leqslant n\right), \qquad (5.3)$$

which has rank (n-1) if $f^{(1)}$ is transversal on Σ^1 at m.

Moreover, in the linearly adapted systems, $\bar{y}_1 \circ f^{(2)} = f_1 = x_1$. Hence $A_1 \circ f^{(1)}(m') = \underline{0} \in \mathbb{R}^{n-1}$ if and only if

$$\bar{p}_{2r} \circ f^{(2)}(m') = \frac{\partial f_2}{\partial x_r}(m') = 0.$$

Thus the points of U where Df has rank 1 are given by the equations

$$\bar{p}_{2r} \circ f^{(2)} = 0, \quad 2 \leq r \leq n . \qquad (5.4)$$

The jacobian matrix (5.3) is just

$$J(\bar{p}_{22} \circ f^{(2)}, \ldots, \bar{p}_{2r} \circ f^{(2)} \mid x_1, \ldots, x_n)(m) ;$$

the statement that this matrix has rank $(n-1)$ at m and the fact that $\bar{p}_{2r} \circ f^{(2)}(m) = 0$, $(2 \leq r \leq n)$, imply that the points of U where Df has rank 1 are given by the transversal intersections of the hypersurfaces (5.4). Hence, by compactness, as could also be deduced from transversality and the codimension of Σ^1 ,

Proposition 5.4 If $f \in \mathcal{L}(M, \mathbb{R}^2)$ is rank-good, then the points of M where Df has rank 1 form a finite collection of smooth regular closed curves which do not intersect one another.

Furthermore one has the following local criterion for rank-good maps:

Proposition 5.5 $f \in \mathcal{L}(M, \mathbb{R}^2)$ is rank-good if and only if

(i) rank Df is greater than zero everywhere

(ii) at any point $m \in M$ where rank $Df(m) = 1$, and in any coordinate system adapted to f at m, the matrix

$$\left(\frac{\partial^2 f_2}{\partial x_r \partial x_i}(m) : 1 \leq i \leq n, \quad 2 \leq r \leq n \right)$$

has rank $(n-1)$.

One will denote by C(f) the curves of a rank-good map $f \in \mathcal{L}(M, \mathbb{R}^2)$ described in Prop. 5.4, and will call them the crease-curves of f.

There is one further transversality condition that it is natural to impose on $f \in \mathcal{L}(M, \mathbb{R}^2)$, namely that f, when restricted to its crease curves, should be as regular as possible. At each point $m \in C(f)$, $Df(m)$ has rank 1 and hence kernel rank $(n-1)$; f restricted to $C(f)$ is regular so long as the tangent to $C(f)$ at m does not lie in the kernel of $Df(m)$. Thus the condition to be imposed on f is that this type of singularity should happen as cleanly as possible.

If $f \in \mathcal{L}(M, \mathbb{R}^2)$ is rank-good, and if $m, m' \in C(f)$, then, in coordinate systems adapted to f at m, the kernel of $Df(m')$ is spanned by $\dfrac{\partial}{\partial x_2}, \ldots, \dfrac{\partial}{\partial x_n}$, and $C(f)$ is given by the intersections of the surfaces $\dfrac{\partial f_2}{\partial x_r} = 0$, $2 \leq r \leq n$. The condition that the tangent to $C(f)$ at m' lies in the kernel of $Df(m')$ is thus given by the equivalent conditions

$$\left.\begin{aligned}
&\det\left(\frac{\partial^2 f_2}{\partial x_r \partial x_s}(m') : 2 \leq r, \quad s \leq n \right) = 0 \\
&D(\bar{p}_{22} \circ f^{(2)}, \ldots, \bar{p}_{2n} \circ f^{(2)} \mid x_2, \ldots, x_n)(m') = 0 \\
&J(\bar{p}_{22} \circ f^{(2)}, \ldots, \bar{p}_{2n} \circ f^{(2)} \mid x_2, \ldots, x_n)(m') \quad \text{is singular}
\end{aligned}\right\}$$

One observes in passing that for f rank-good and $m \in C(f)$ the condition that $J(\bar{p}_{22} \circ f^{(2)}, \ldots, \bar{p}_{2n} \circ f^{(2)} \mid x_1, x_2, \ldots, x_n)(m)$ has rank $(n-1)$ implies that $J(\bar{p}_{22} \circ f^{(2)}, \ldots, \bar{p}_{2n} \circ f^{(2)} \mid x_2, \ldots, x_n)(m)$ cannot have rank less than $(n-2)$.

Now consider $J^2(M, \mathbb{R}^2)$, which may be considered in a natural way as a bundle over $J^1(M, \mathbb{R}^2)$. If $(U, \theta, \{x_i\})$ is a coordinate system in M then composing the coordinate functions on $J^1(U, \mathbb{R}^2)$ with the projection from $J^2(U, \mathbb{R}^2)$ and using the same symbols for the objects corresponding one obtains coordinate functions

$$\{\bar{x}_i\ ,\ \bar{y}_\alpha,\ \bar{p}_{\alpha i},\ \bar{s}_{\alpha ij}\ :\ 1 \leq j \leq i \leq n,\quad \alpha = 1,2\}$$

on $J^2(U, \mathbb{R}^2)$, where

$$\left.\begin{aligned} \bar{x}_i \circ f^{(2)} &= x_i \\ \bar{y}_\alpha \circ f^{(2)} &= y_\alpha \circ f = f_\alpha \\ \bar{p}_{\alpha i} \circ f^{(2)} &= \frac{\partial f_\alpha}{\partial x_i} \\ \bar{s}_{\alpha ij} \circ f^{(2)} &= \frac{\partial^2 f_\alpha}{\partial x_i \partial x_j} \end{aligned}\right\}$$

Now let $f \in \mathcal{L}(M, \mathbb{R}^2)$, $m \in M$, rank $Df(m) = 1$ and let the coordinates above be linearly adapted to f at m. Assume further that $J(\bar{p}_{22} \circ f^{(2)}, \ldots, \bar{p}_{2n} \circ f^{(n)} \mid x_2, \ldots, x_n)(m)$ is singular. Now $\Sigma^1 \subseteq J^2(M, \mathbb{R}^2)$ is defined, in a neighbourhood $\mathcal{U}$ of $f^{(2)}(m) \in \Sigma^1$ by the map A_1, where

$$A_1^\Theta(\underline{\bar{x}}, \underline{\bar{y}}, \underline{\underline{\bar{p}}}, \underline{\underline{\bar{s}}}) = (\bar{p}_{11}\, \bar{p}_{2r} - \bar{p}_{21}\, \bar{p}_{1r} : 2 \leq r \leq n)$$

and in the neighbourhood $\mathcal{U}$ of $f^{(2)}(m)$ the members of Σ^1 which display the second order singularities of the type which have been described, and which f has at m, are defined by the map $B_{1*} : J^2(U, \mathbb{R}^2) \to \mathbb{R}^{n-1} \times \mathbb{R}$ where

$$B_{1*}^\Theta(\underline{\bar{x}}, \underline{\bar{y}}, \underline{\underline{p}}, \underline{\underline{s}}) = (\bar{p}_{11}\, \bar{p}_{2r} - \bar{p}_{21}\, \bar{p}_{1r},\ \det(\bar{s}_{2rs}) : 2 \leq r,\quad s \leq n).$$

Let Σ^{1*} be the corresponding jets. $\Sigma^{1*} = \Sigma^{1,n-2} \mathring{\cup} \ldots \mathring{\cup} \Sigma^{1,0}$ $= \Sigma^{1,n-2} \mathring{\cup} \Sigma^{1,\deg}$, where locally $\Sigma^{1,n-2}$ are the regular points of B_{1*} lying in Σ^{1*}, and $\Sigma^{1,\deg}$ are the singular points of B_{1*} in Σ^{1*}. The stratification of Σ^{1*} is given locally by the rank of the matrix

$(\bar{S}_{2rs} : 2 \leq r,\ s \leq n)$. $\Sigma^{1,n-2}$ is thus a submanifold of $J^2(M, \mathbb{R}^2)$ of codimension n, and $\Sigma^{1,\deg}$ is a collection of submanifolds of $J^2(M, \mathbb{R}^2)$ of codimensions greater than n.

The natural condition to impose on $f \in \mathcal{L}(M, \mathbb{R}^2)$ is that $f^{(2)}$ be transversal to $\Sigma^{1,n-2}$, and that $f^{(2)}$ meets no points of $\Sigma^{1,\deg}$. So define

Definition 5.6 $f \in \mathcal{L}(M,\mathbb{R}^2)$ is crease-good if $f^{(2)} : M \to J^2(M, \mathbb{R}^2)$ is transversal to $\Sigma^{1,n-2}$ and $f^{(2)}(M) \cap \Sigma^{1,\deg} = \emptyset$.

Source transversality yields:

Proposition 5.7 Crease-good maps from an open dense subspace of $\mathcal{L}(M,\mathbb{R}^2)$

By observation above, the condition $f^{(2)}(M) \cap \Sigma^{1,\deg} = \emptyset$ is redundant for rank-good maps.

For a local coordinate description of crease-good maps, let $f \in \mathcal{L}(M, \mathbb{R}^2)$ be crease-good, $m \in M$, rank $Df(m) = 1$, and $D(\bar{p}_{22} \circ f^{(2)}, \ldots, \bar{p}_{2n} \circ f^{(2)} \mid x_2, \ldots, x_n)(m) = 0$, let $(U, \theta, \{x_i\})$ be adapted to f at m and take the usual coordinate system on $J^2(U, \mathbb{R}^2)$. Let $\bar{\delta}' : J^2(U, \mathbb{R}^2) \to \mathbb{R}$ be defined by $\bar{\delta}'^{\Theta}(\underline{x}, \underline{y}, \underline{p}, \underline{\underline{s}}) = \det(\bar{S}_{2rs} : 2 \leq r,\ s \leq n$ Then $B_{1*} = A_1 \times \bar{\delta}' : J^2(U, \mathbb{R}^2) \to \mathbb{R}^{n-1} \times \mathbb{R}$, $\bar{\delta}' \circ f^{(2)} = D(\bar{p}_{22} \circ f^{(2)}, \ldots, \bar{p}_{2n} \circ f^{(2)} \mid x_1, \ldots, x_n)$. Then the transversality condition is that $B_{1*} \circ f^{(2)} : U \to \mathbb{R}^{n-1} \times \mathbb{R}$ has maximal rank n at $m \in U$. Now

$$B_{1*} \circ f^{(2)} = \left(\frac{\partial f_2}{\partial x_2}, \ldots, \frac{\partial f_2}{\partial x_n}, \bar{\delta}' \circ f^{(2)} \right),$$

hence $B_{1*} \circ f^{(2)}$ has jacobian matrix at m:

$$J\left(\frac{\partial f_2}{\partial x_2}, \ldots, \frac{\partial f_2}{\partial x_n}, \bar{\delta}' \circ f^{(2)} \mid x_1, \ldots, x_n \right)(m)$$

and the transversality condition implies that at m, This matrix has (maximal) rank n. Observe that at m, the n surfaces $\dfrac{\partial f_2}{\partial x_2} = 0, \ldots, \dfrac{\partial f_2}{\partial x_n} = 0$, $\bar{\delta}' \circ f^{(2)} = 0$ have transversal intersection. Thus the local criterion for crease-goodness is

<u>Proposition 5.8</u> $f \in \mathcal{L}(M, \mathbb{R}^2)$ <u>is crease-good if and only if at any point of</u> M <u>where rank</u> $Df(m) = 1$ <u>and in any coordinate system adapted to</u> f <u>at</u> m <u>either</u>

$$\text{(i)} \quad \bar{\delta}' \circ f^{(2)}(m) = \det\left(\frac{\partial^2 f_2}{\partial x_r \partial x_s} : 2 \leqslant r, \quad s \leqslant n\right)(m) \neq 0$$

<u>or</u>

$$\text{(ii)} \quad \bar{\delta}' \circ f^{(2)}(m) = 0 \quad \text{and} \quad D\left(\frac{\partial f_2}{\partial x_2}, \ldots, \frac{\partial f_2}{\partial x_n}, \bar{\delta}' \circ f^{(2)} \,|\, x_1, \ldots, x_n\right) \times (m) \neq 0.$$

Points of M where a crease-good map has rank 1 are thus characterised by whether $\left(\dfrac{\partial^2 f_2}{\partial x_r \partial x_s}\right)$ has rank $(n-1)$, (<u>type I</u>), or rank $(n-2)$, (<u>type II</u>).

By the fact that points of type II are given by the transversal intersection of n hypersurfaces in an n-dimensional space and by compactness, or from the transversality definition and the codimension of $\Sigma^{1,n-1}$, a crease-good map has only a finite number of points of type II.

<u>Definition 5.9</u> Now defining $f \in \mathcal{L}(M^n, \mathbb{R}^2)$ to be a <u>Whitney map</u> if f is rank-good and crease-good, Props. 5.2,7 give

<u>Proposition 5.10</u> <u>Whitney maps form an open dense subspace of</u> $\mathcal{L}(M, \mathbb{R}^2)$.

Propositions 5.5 and 8 give

<u>Proposition 5.11</u> $f \in \mathcal{L}(M, \mathbb{R}^2)$ is a Whitney map if and only if

(i) <u>rank</u> $Df(m) \geqslant 1$ <u>for all</u> $m \in M$

(ii) <u>if</u> $m \in M$ and $Df(m)$ <u>has rank 1, then in any coordinate systems adapted to</u> f <u>at</u> m <u>the functions</u> $\delta' = D\left(\frac{\partial f_2}{\partial x_2}, \ldots, \frac{\partial f_2}{\partial x_n} \middle| x_2, \ldots, x_n\right)$ <u>and</u>

$$\mathcal{D}' = D\left(\frac{\partial f_2}{\partial x_2}, \ldots, \frac{\partial f_2}{\partial x_n}, \frac{\partial \delta'}{\partial x_n} \middle| x_1, x_2, \ldots, x_n\right)$$ <u>do not both vanish at</u> m.

Moreover, the set of points, $C(f)$ the crease curve of f, where Df has rank 1 is a finite collection of smooth closed nonintersecting curves. The points $m \in C(f)$ where $\delta'(m) \neq 0$ are called the <u>fold points</u> of f, the points $m \in C(f)$ where $\delta'(m) = 0$ are called the <u>cusp points</u> of f. There are a finite number of cusp points. $\mathcal{D}'$ is non-zero at cusp points.

Whitney originally defined Whitney maps as in Prop. 5.11 for the case $n = 2$ in [32, §4]. It is a simple matter to translate the present Prop. 5.11 into [32, §4. 1-7]. The general case is mentioned in [33, §18].

The parallels with the analysis of the previous section are now clear. By substituting n+1 for n in Prop. 5.11 then t for x_n, comparing with Prop. 4.5, and extending the definitions of a Whitney map in the obvious way to manifolds with boundary, one may state:

<u>Proposition 5.12</u> <u>The tracking map of a Cerf path is a Whitney map.</u>

§6 WHITNEY CELLS

The problems and procedures of §4 admit a natural extension. Let $\mathcal{M}(M)$ denote the space of Morse functions on the compact smooth n-dimensional manifold M. Then one constructed in $\mathcal{L}(I, \mathcal{L}(M, \mathbb{R}))$ $= \mathcal{L}(M \times I, \mathbb{R})$ the open dense subspace $\mathcal{C}(M)$ of Cerf paths of M. Elements of $\mathcal{C}(M)$ are paths in $\mathcal{L}(M, \mathbb{R})$ which pass through just a finite number of points not in $\mathcal{M}(M)$, where they exhibit non-Morse singularities transversally.

Again, from their transversality definition, the numbers of characteristic curves and death/birth points of two elements in the same path component of $\mathcal{C}(M) \subseteq \mathcal{L}(M \times I, \mathbb{R})$ are the same. But, again, $\mathcal{C}(M)$ contains elements for which these numbers differ. More specifically, given $f \in \mathcal{C}(M)$, then in any open region of $M \times I$ not containing characteristic points of f one may make an arbitrarily C^0-close approximation f' of f, which agrees with f outside the open set, such that $f' \in \mathcal{C}(M)$ and the characteristic set of f' contains precisely one more curve, with one more death point and one more birth point on it, besides the characteristic set of f. This one may do by constructing a smooth 'dynamic' 1-parameter version of the introduction of two singular points with consecutive indices followed by their removal. A description of the analogous procedure for Whitney maps is found in [32, §21]. In handle-body language the process corresponds to making the trivial addition of an n-cell, which has a standard decomposition as the sum of a λ-handle and a $(\lambda+1)$-handle, followed by the identification of $(M \cup (n\text{-cell}))$ with M.

Thus the problem arises to discover what points of $\mathcal{L}(M \times I, \mathbb{R})$ other than those of $\mathcal{C}(M)$ a smooth path in $\mathcal{L}(M \times I, \mathbb{R})$ must contain.

Evidently, having discovered this class of maps in $\mathcal{L}(I, \mathcal{L}(M \times I, \mathbb{R})$ $= \mathcal{L}(M \times I^2, \mathbb{R})$, one might raise the whole problem again for this new class. Thus one is led to considering the general case of k-parameter families of functions on M, i.e. the space $\mathcal{L}(M \times I^k, \mathbb{R})$, but keeping rigid one's regard for the order of the parameters $\{t_j : 1 \leqslant j \leqslant k\}$ in I^k. One proceeds by the following inductive scheme.

Let $(U, \theta, \{x_i : i = 1, \ldots, n\})$ be a coordinate system on M, and let k be any non-negative integer. Consider the map space $\mathcal{L}(M \times I^k, \mathbb{R})$ and the associated jet space $J^{k+2}(M \times I^k, \mathbb{R})$, which contains $J^{k+2}(U \times I^k, \mathbb{R})$ as an open subspace. Then one may define on $J^{k+2}(U \times I^k, \mathbb{R})$ functions $\{\bar{p}_i, \bar{\mathcal{D}}_j : i = 1, \ldots, n, \; j = 0, \ldots, k\}$ such that for $f \in \mathcal{L}(M \times I^k, \mathbb{R})$

(i) $\underline{p} = \bar{\underline{p}} \circ f^{(k+2)} = J(f|\underline{x}) = \left(\dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_n}\right)$

(ii) $\bar{\mathcal{D}}_j$ is an algebraic function of the standard coordinate functions on $J^{j+2}(M \times I^i, \mathbb{R})$ lifted to $J^{k+2}(M \times I^k, \mathbb{R})$ by the projection of $J^{k+2}(M \times I^k, \mathbb{R}) \to J^{j+2}(M \times I^k, \mathbb{R}) \to J^{j+2}(M \times I^j, \mathbb{R})$ where the first map is the standard jet-bundle projection and the second map is induced by the canonical projection of I^k on I^j given by $(t_1, \ldots, t_k) \mapsto (t_1, \ldots, t_j)$.

(iii) Defining $\mathcal{D}_j = \bar{\mathcal{D}}_j \circ f^{(k+2)}$ one has

$$\begin{aligned}
\mathcal{D}_0 &= D(\underline{p}|\underline{x}) \\
\mathcal{D}_1 &= D(\underline{p}, \mathcal{D}_0|\underline{x}, t_1) \\
\mathcal{D}_2 &= D(\underline{p}, \mathcal{D}_0, \mathcal{D}_1|\underline{x}, t_1, t_2) \\
&\vdots \\
\mathcal{D}_k &= D(\underline{p}, \mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_{k-1}|\underline{x}, t_1, t_2, \ldots, t_k).
\end{aligned}$$

<u>Definition 6.1</u> $f \in \mathcal{L}(M \times I^k, \mathbb{R})$ is a <u>Whitney k-cell</u> if and only if for each coordinate system $(U, \theta, \{x_i : i = 1, \ldots, n\})$ on M

(i) $\underline{p}, \mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_k$ are never simultaneously zero.

(ii) for each $j = 0,\ldots,k$ and for each point $(m,\underline{t}) \in U \times I^k$ where $\underline{p} = \underline{0}$ and $\mathcal{D}_0, \ldots, \mathcal{D}_{j-1}$ are zero, $J(\underline{p},\mathcal{D}_0,\ldots,\mathcal{D}_{j-1}|x,t_1,\ldots,t_k)$ has rank $(n+j)$ at $(m,\underline{t})$.

<u>Proposition 6.2</u> <u>Whitney k-cells form an open dense subspace of</u> $\mathcal{L}(M \times I^k, \mathbb{R})$.

Indeed condition (i) is implied by condition (ii). And condition (ii) states that when $f^{(k+2)}(m, \underline{t})$ lies in the subspace of $J^{k+2}(M \times I^k, \mathbb{R})$ defined by the zeros of the map $(\bar{\underline{p}}, \bar{\mathcal{D}}_0, \ldots, \bar{\mathcal{D}}_{j-1})$ then $f^{(k+2)}(m,\underline{t})$ is a regular point of this map and $f^{(k+2)}$ is transversal to the stratified space given by the zeros of this map, at $(m,\underline{t})$. Thus by applying Thom's Source Transversality Theorem (k+1) times the space of Whitney k-cells is the intersection of (k+1) open dense subspaces of $\mathcal{L}(M \times I^k, \mathbb{R})$, hence the proposition.

Denote by $C(f)$, $f \in \mathcal{L}(M \times I^k, \mathbb{R})$ the points $(m,\underline{t}) \in M \times I^k$ where $D(f(\ ,\underline{t}))$ is singular. Denote by $C_j(f)$ the points of $C(f)$ where, locally, $\mathcal{D}_0 = \mathcal{D}_1 = \ldots = \mathcal{D}_{j-1} = 0$, and where $\mathcal{D}_j \neq 0$. Then $C_k(f)$ consists of a finite collection of points where $f(\ ,\underline{t})$ displays a <u>singularity of codimension k</u>. At points of $C_j(f)$, $f(\ , t_1, \ldots, t_j, \ ): M \times I^{k-j} \to \mathbb{R}$ displays a (k-j)-parameter family of singularities of codimension j.

From the rank conditions one knows that $C(f)$ is a submanifold of $M \times I^k$ of dimension k, and that $C_j(f) \cup C_{j+1}(f) \cup \ldots \cup C_k(f)$ is a submanifold of $C(f)$ of dimension (k-j). The points of $C_k(f)$ are precisely the singular points of the horizontal projection of

$C(f) \subseteq M \times I^k = (M \times I^{k-1}) \times I$ into I. More generally, the points of $C_k(f) \cup \ldots \cup C_j(f)$, being defined by $\underline{p} = \underline{0}$, $\mathcal{D}_0 = \ldots = \mathcal{D}_{j-1} = 0$, are precisely the singular points of the projection of $C(f) \subseteq M \times I^k = (M \times I^j) \times I^{k-j}$ into I^{k-j}.

Given $f \in \mathcal{L}(M \times I^k, \mathbb{R}) = \mathcal{L}(I, \mathcal{L}(M \times I^{k-1}, \mathbb{R}))$ a Whitney k-cell, except at the finite set $C_k(f)$, the map $f(\ \ , t_k)$ is a Whitney (k-1)-cell. $C(f)$ is thus generated by the motion of the singular set of $f(\ \ , t_k)$ as t_k runs through I, the points of $C_k(f)$ being the places where the singular set locally contracts to a point.

By matrix manipulations, one may show that at every point of $C(f)$, when f is a Whitney k-cell, $J(\underline{p}|\underline{x})$ has rank $(n-1)$, and hence, by using local hessian-adapted coordinates, that at a point of $C_j(f)$

(i) $$\frac{\partial f}{\partial x_n} = \frac{\partial^2 f}{\partial x_n^2} = \ldots = \frac{\partial^{j-1} f}{\partial x_n^{j-1}} = 0, \quad \frac{\partial^j f}{\partial x_n^j} \neq 0$$

(ii) the map $\left(\frac{\partial f}{\partial x_n}, \frac{\partial^2 f}{\partial x_n^2}, \ldots, \frac{\partial^{j-1} f}{\partial x_n^{j-1}}\right) : M \times I^k \to \mathbb{R}^{j-1}$ has rank j-1.

Clearly the Whitney 0-cells are the Morse functions, and the Whitney 1-cells are the Cerf paths. Moreover the singularities presented by the tracking map of a Whitney k-cell are of precisely the same type as are presented by the projection $C(f) \subseteq M \times I^{k+1} \to I^{k+1}$ of a Whitney (k+1)-cell, neglecting the regular part of the hessian.

§7 CANONICAL FORMS

The broad purpose of the study of canonical forms is twofold: first the attempt to characterise a smooth map locally by algebraic invariants derived from its differential coefficients, and second to produce simple forms which display the geometric and topological properties of the map. Clearly there can be no hope of being comprehensive in this respect: there is no way in which a study of the differential coefficients at the origin of $\mathbb{R}$ can distinguish between the constant zero function and the flat function $\exp(-1/x^2)$.

After the Implicit Function Theorem [7, p.265], the Lemma of Morse [19, p.44, Lemma 10.1] is the classic example of a canonical form. As one has seen, the generic singularity of a smooth function of n real variables is a point where the first derivatives of the function vanish, and the matrix of second derivatives is non-singular. The equivalence class of this matrix under regular coordinate changes at source is characterised by its index, or number of negative eigenvalues. Hence for two functions of this type to differ by a source diffeomorphism, and a constant at targent, it is necessary that their indices be the same. If one's criterion of 'equivalence' permits order reversing diffeomorphisms at targets in $\mathbb{R}$, then for two such functions to be equivalent in this sense, it is necessary that their indices at the points in question be the same or add up to n. The Lemma of Morse, by producing a canonical form whose dependence on the index/coindex is patent, shows that these indicial criteria are sufficient.

In fact the methods that establish the Lemma of Morse are the source of a family of results that have independent interest, and applications beyond the scope of the original result. They are all of a 'preparatory' nature. The <u>Arbitrary Rank Morse Lemma</u> [21, Theorem 4.2, p.27] is a prime example. By suitable choice of coordinates one may separate the 'regular' and 'nonregular' parts of a function at a singularity.

Let M^{n+m} be a smooth manifold, and $V^n \subseteq M$ a smooth submanifold. Let $v \in V$ and let $f : M \to \mathbb{R}$ be a smooth function. Let $(W, \phi, \{x_1, \ldots, x_{n+m}\})$ be a smooth coordinate system centred on $v \in M$, such that $\{y_1, \ldots, y_n\}$ which are the restrictions of $\{x_{m+1}, \ldots, x_{m+n}\}$ to $U = V \cap W$ are the coordinate functions of a smooth coordinate system $(U, \theta, \{y_1, \ldots, y_n\})$ centred on $v \in V$. Let $\pi = \pi_\phi$ be the projection $\theta^{-1} \circ \pi_2 \circ \phi : W \to U$, where $\pi_2 : \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection on the second factor.

Now for $t \in \mathbb{R}$, $w \in W$, denote by $t * w$ the point in W w[illegible] $x_i(t * w) = t.x_i(w)$ for $i = 1, \ldots, m$ and $x_{m+j}(t * w) = x_{m+j}(w)$ for $j = 1, \ldots, n$. Then $0 * w = \pi(w)$, $s * (t * w) = st * w$, $1 * w = w$, and

$$f(w) = f(0 * w) + \int_0^1 \frac{d}{dt} f(t * w)dt$$

$$= f(0 * w) + \sum_{i=1}^{m} x_i \int_0^1 \frac{\partial f}{\partial x_i} (t * w)dt$$

$$= f(0 * w) + \sum_{i=1}^{m} x_i \int_0^1 \frac{\partial f}{\partial x_i} (0 * w)dt + \sum_{i,j=1}^{m} x_i x_j \int_0^1 \frac{\partial}{\partial x_j}$$

$$\times \int_0^1 \frac{\partial f}{\partial x_i} (st * w)dt\, ds$$

$$= (f \circ \pi)(w) + \sum_{i=1}^{m} x_i \left(\frac{\partial f}{\partial x_i} \circ \pi\right)(w) + \sum_{i,j=1}^{m} x_i\, x_j\, h_{ij}(w).$$

Thus, by symmetrising, there exist smooth functions $h_{ij} = h_{ji} : W \to \mathbb{R}$ such that

$$f = f\circ\pi + \sum_{i=1}^{m} x_i \cdot \frac{\partial f}{\partial x_i}\circ\pi + \sum_{i,j=1}^{m} x_i\, x_j\, h_{ij} \tag{7.1}$$

A case of particular interest is that when $\dfrac{\partial f}{\partial x_i}$ is zero on V for $i = 1, \ldots, m$, then

$$f = f\circ\pi + \sum_{i,j=1}^{m} x_i\, x_j\, h_{ij} \,. \tag{7.2}$$

Definition 7.1 In order to go further in this direction at this level of generality, one must describe the coordinate changes of the given system of $v \in M$ which do not affect the spirit of the decomposition (7.2). For the present one is interested only in the alteration of the transverse coordinate functions, $\{x_i : i = 1, \ldots, m\}$. A new system of coordinates $(W', \phi', \{z_1, \ldots, z_{n+m}\})$ centred on $v \in M$ will be said to be rigidly adapted to (θ, f) at v if

$$z_{m+j} = x_{m+j} \quad \text{for } j = 1, \ldots, n \,.$$

Then the transverse m-sheets of the two systems are the same. The projections π_ϕ and $\pi_{\phi'}$ coincide where both are defined.

A linear change in the $x_1, \ldots, x_m$ is rigidly adapted. Assume that $(h_{ij}(v) : i,j = 1,\ldots,m)$ has rank r and index λ. Then one may generalise the Arbitrary Rank Morse Lemma in the following

Proposition 7.2 Under the foregoing hypotheses, one may find a coordinate system $(W', \phi', \{z_1, \ldots, z_{n+m}\})$ rigidly adapted to (θ, f) at $v \in M$ such that

$$f = \sum_{i=1}^{r} \epsilon_i z_i^2 + \sum_{j,k=r+1}^{m} z_j z_k g_{jk} ,$$

<u>where</u>

(i) $\epsilon_i = \pm 1$, <u>and</u> $= -1$ <u>for</u> λ <u>values of</u> $i = 1,\ldots,r$,

(ii) $g_{jk} = g_{kj} : W' \to \mathbb{R}$ are smooth and $g_{jk}(v) = 0$.

The proof is by induction on m and uses the standard 'algebraic' model. All that is required to find 'standard' transverse coordinates at each point of $U' = W' \cap V$ which can be pieced together globally.

Since $f = \sum_{i,j=1}^{m} x_i x_j h_{ij} + f \circ \pi$, and $(h_{ij}(v) : i,j = 1,\ldots,m)$ has rank r and index λ, one may assume that, so long as $r \neq 0$ in which case one has a coordinate system of the desired type, $h_{11}(v) \neq 0$. All that is required to achieve this is a linear change of the $\{x_1, \ldots, x_m\}$.

Let W^1 be an open neighbourhood of $v \in M$ on which $h_{11} \neq 0$. Set

$$\xi_1 = |h_{11}|^{\frac{1}{2}} (x_1 + \sum_{j=2}^{m} x_j h_{1j} / h_{11})$$

$$\xi_j = x_j , \qquad j = 2,\ldots,m+n.$$

Then in W^1 the $\{\xi_i : i = 1,\ldots,m+n\}$ are the coordinate functions of a system (W^1, ϕ^1) rigidly adapted to (θ, f) at v. In this system f takes the form

$$f = \epsilon_1 \xi_1^2 + \sum_{j,k=2}^{m} \xi_j \xi_k h_{jk}^1 + f \circ \pi$$

where $\epsilon_1 = \pm 1$ according as $h_{11}(v) \quad 0$. The h_{jk}^1 can be chosen to be symmetric. Now replace V by the manifold V^1 where $\xi_2, \ldots, \xi_m$ are zero. Let $\pi^1 : W^1 \to V^1$ be the corresponding projection. Then $f \circ \pi^1 = \epsilon_1 \xi_1^2 + f \circ \pi$ and $(h^1_{jk}(v) : j,k = 2,\ldots,m)$ has rank $(r-1)$ and index λ, $\lambda-1$ according as $\epsilon_1 = \pm 1$. Now apply the inductive hypothesis for $(m-1)$ to the configuration $(M,V^1,f,v,\xi_1,\xi_{m+1},\ldots,\xi_{m+n})$. The proposition establishes itself.

The Morse Lemma and the Arbitrary Rank Morse Lemma follow from Proposition 7.2 by specialising to the case when $V = \{v\}$.

Cerf paths: 'static' case

Let M^n be compact and let $f : M \times I \to \mathbb{R}$ be a Cerf path, (§ 4). Let (m,t) be a point of the characteristic curve of f. Denote $f(\ ,t) = f_t$ by f'. Then either f' has a nondegenerate singularity at m, and canonical forms are known in this case, or f' has a degenerate singularity at m. In the latter case, (m,t) is a cusp point of the tracking map of f, and the characteristic curve of f has horizontal tangent at (m,t). It is desired to find a simple model for the singularity of f' at m.

By the analysis of §4, there exists a (hessian adapted) coordinate system $(U, \theta, \{x_1, \ldots, x_n\})$ centred on $m \in M$ such that

$$\text{(i)} \qquad \frac{\partial f'}{\partial x_1}(m) = \cdots = \frac{\partial f'}{\partial x_n}(m) = 0$$

$$\text{(ii)} \qquad J\left(\frac{\partial f'}{\partial x_1}, \ldots, \frac{\partial f'}{\partial x_{n-1}} \,\middle|\, x_1, \ldots, x_{n-1}\right)(m) \ \text{ is nonsingular}$$

$$\text{(iii)} \qquad \frac{\partial^2 f'}{\partial x_1 \partial x_n}(m) = \cdots = \frac{\partial^2 f'}{\partial x_{n-1}\partial x_n}(m) = 0$$

(iv) $\dfrac{\partial^2 f'}{\partial x_n^2}(m) = 0, \quad \dfrac{\partial^3 f'}{\partial x_n^3}(m) \neq 0 .$

Then in a neighbourhood U' of m the hypersurfaces $\dfrac{\partial f'}{\partial x_1} = 0, \ldots, \dfrac{\partial f'}{\partial x_{n-1}} = 0$ intersect transversally to give a smooth curve V passing through m, on which x_n may be taken as a parameter. Let the coordinate presentation of V be

$$x_1 = a_1(x_n), \ldots, x_{n-1} = a_{n-1}(x_n) ,$$

where $a_1(0) = \ldots = a_{n-1}(0) = 0$. Then define on U' the functions

$$\left.\begin{aligned} w_1 &= x_1 - a_1(x_n) , \\ &\vdots \\ w_{n-1} &= x_{n-1} - a_{n-1}(x_n) , \\ w_n &= x_n . \end{aligned}\right\}$$

They define a coordinate system $(U', \theta', \{w_i : i = 1,\ldots,n\})$.
Let $\pi = \pi_{\theta'}$. Then by the preceding methods, using the analogues of (i) to (iv)

$$f' = f' \circ \pi + \sum_{i=1}^{n-1} \epsilon_i y_i^2 ,$$

in some coordinate system $(U'', \theta'', \{y_1, \ldots, y_n\})$ rigidly adapted to (ϕ, f) at m where $(V, \phi, \{x_n = w_n = y_n\})$ is the system on V. Here, setting $f' \circ \pi = g$ as a function of y_n one has

$$\frac{dg}{dy_n}(m) = \frac{d^2 g}{dy_n^2}(m) = 0, \quad \frac{d^3 g}{dy_n^3}(m) \neq 0 .$$

Hence $g = y_n^3 . k(y_n)$ where $k(0) \neq 0$. Now set $X_1 = y_1, \ldots, X_{n-1} = y_{n-1}$,

$X_n = y_n \cdot k^{\frac{1}{3}}$. In a neighbourhood N of $v \in M$, the $\{X_i : i = 1,\dots,n\}$ define a coordinate map X and a system in which f' takes the form

$$f' = f'(m) + \sum_{i=1}^{n-1} \epsilon_i X_i^2 + X_n^3 .$$

This is the desired canonical form.

Whitney cells: 'static case'.

By the same token, if $f : M \times I^k \to \mathbb{R}$ is a Whitney cell (§6) which exhibits at $(m,\underline{t})$ a singularity of codimension j, then there exists a coordinate system $(U, \theta, \{x_1, \dots, x_n\})$ centred on $m \in M$ such that

$$f' = f(\ , \underline{t}) = f'(m) + \sum_{i=1}^{n-1} \epsilon_i x_i^2 \pm x_n^{j+2} .$$

Cerf paths: 'dynamic' case

Let $f : M^n \times I \to \mathbb{R}$ be a Cerf path. Let $(m^*,t^*) \in C(f)$ be a point of the characteristic curve of f. One is now going to find canonical forms for f in a neighbourhood of (m^*, t^*). As has been seen, one can find canonical forms for the partial function of f at level t^* in a neighbourhood of m^*. However, as t^* varies, and m^* varies one may not be able to choose a coordinate system around $m^* \in M$ which presents f simply. The coordinate systems for neighbouring points of C(f) will differ. One must therefore break one's previous regard for the separate identities of M and I in the coordinate systems chosen on $M \times I$. Initially however, let $(U, \theta, \{x_1, \dots, x_n\})$ be a coordinate system centred on $m^* \in M$. There are, as ever, two cases.

(i) Codimension zero. Here $J\left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\middle| x_1,\dots,x_n\right)(m^*,t^*)$ is non-singular and

$\dfrac{\partial f}{\partial x_1}(m^*,t^*) = \ldots = \dfrac{\partial f}{\partial x_n}(m^*,t^*) = 0.$ $C(f)$ is given by the transversal intersections of the hypersurfaces $\dfrac{\partial f}{\partial x_1} = 0, \ldots, \dfrac{\partial f}{\partial x_n} = 0$. As has been seen, in a neighbourhood U' of (m^*,t^*) $C(f)$ can be parametrised by t. Let the coordinate presentation of $C(f)$ be

$$x_1 = b_1(t), \ldots, x_n = b_n(t),$$

where $b_1(t^*) = \ldots = b_n(t^*) = 0$. Then define on U' the functions

$$\left.\begin{aligned} \eta_1 &= x_1 - b_1(t) , \\ &\vdots \\ \eta_n &= x_n - b_n(t) , \\ \tau &= t - t^* . \end{aligned}\right\}$$

They define a coordinate system $(U', \theta', \{\eta_1, \ldots, \eta_n, \tau\})$ centred on $(m^*,t^*) \in M \times I$. By using the Proposition 7.2, one may assert the existence of a coordinate system $(U'', \theta'', \{y_1, \ldots, y_n, \tau\})$ centred on $(m^*, t^*) \in M \times I$ in which f takes the canonical form

$$f = f\circ\pi + \sum_{i=1}^{n} \epsilon_i\, y_i^2 ,$$

where π is the transversal projection onto $C(f)$ given by the initial system $\theta \times 1_I$. The functions $\{ y_1, \ldots, y_n \}$ restricted to each transversal sheet form a coordinate system.

(ii) <u>Codimension one</u>. Here one may assume that the initial system θ is hessian adapted:

$$\left.\begin{aligned}
&J\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n-1}} \,\middle|\, x_1, \ldots, x_{n-1}\right)(m^*,t^*) \quad \text{is non-singular} \\
&\frac{\partial f}{\partial x_1}(m^*,t^*) = \ldots = \frac{\partial f}{\partial x_n}(m^*,t^*) = 0 \\
&\frac{\partial^2 f}{\partial x_1 \partial x_n}(m^*,t^*) = \ldots = \frac{\partial^2 f}{\partial x_n{}^2}(m^*,t^*) = 0 \\
&\frac{\partial^3 f}{\partial x_n{}^3}(m^*,t^*) \neq 0 \neq \frac{\partial^2 f}{\partial x_n \partial t}(m^*,t^*)
\end{aligned}\right\}.$$

Then the hypersurfaces $\frac{\partial f}{\partial x_1} = 0, \ldots, \frac{\partial f}{\partial x_{n-1}} = 0$ intersect transversally in a neighbourhood of (m^*,t^*) in a 2-manifold on which (x_n, t) may be taken as coordinates. Let the 2-manifold be called V and let it be given in coordinate form by

$$\left.\begin{aligned}
x_1 &= c_1(x_n, t)\,, \\
&\vdots \\
x_{n-1} &= c_{n-1}(x_n,t)\,,
\end{aligned}\right\}$$

where $c_1(0, t^*) = \ldots = c_{n-1}(0, t^*) = 0$. Now define functions

$$\left.\begin{aligned}
w_1 &= x_1 - c_1(x_n, t) \\
&\vdots \\
w_{n-1} &= x_{n-1} - c_{n-1}(x_n, t) \\
w_n &= x_n \\
t &= t
\end{aligned}\right\}$$

In a neighbourhood of $(m^*, t^*) \in M \times I$, they form a coordinate system rigidly adapted to $(\{x_n, t\}, f)$ at (m^*, t^*). By Proposition 7.2, one may find another coordinate system $(U'', \theta'', \{y_1,\dots,y_{n-1}, x_n, t\})$ at $(m^*, t^*) \in M \times I$ so that

$$f = f \circ \pi + \sum_{i=1}^{n-1} \epsilon_i y_i^2 .$$

One may consider $f \circ \pi$ as a function $g : V \to \mathbb{R}$ such that

$$\frac{\partial g}{\partial x_n}(m^*,t^*) = \frac{\partial^2 g}{\partial x_n^2}(m^*,t^*) = 0$$

$$\frac{\partial^2 g}{\partial x_n \partial t}(m^*,t^*) \neq 0 \neq \frac{\partial^3 g}{\partial x_n^3}(m^*,t^*)$$

Then $g - g(m^*, t^*) = g'$ is regular of order 3 in x_n at (m^*,t^*). So by the Weierstrass-Malgrange Preparation Theorem (iv), (iv'), (§1), there exist functions $d_o, d_1, d_2 : \mathbb{R} \to \mathbb{R}$, $q : V \to \mathbb{R}$ such that

$$g' = q \cdot (x_n^3 + 3d_2(t) \cdot x_n^2 + 3d_1(t) \cdot x_n + d_o(t))$$

where $q(m^*,t^*) \neq 0$, $d_o(t^*) = d_1(t^*) = d_2(t^*) = 0$. Now set

$$y_n = q^{\frac{1}{3}} \cdot (x_n + d_2(t)).$$

Then in a neighbourhood V' of $(m^*,t^*) \in V$ the functions (y_n, t) are the functions of a coordinate system in which g' takes the form

$$g' = y_n^3 + e_1(t) \cdot y_n + e_2(t) .$$

Now $e_1(t^*) = 0$, and since $\dfrac{\partial^3 g'}{\partial x_n \partial t}(m^*,t^*) \neq 0$, $De_1(t^*) \neq 0$. So put

$\tau = e_1(t)$ to get a coordinate system $\{y_n, \tau\}$ in a neighbourhood V'' of $(m^*, t^*) \in V$ in which g' takes the form

$$g' = y_n^3 + \tau y_n + h(\tau).$$

Thus one may assemble a coordinate system $\{y_1, \ldots, y_n, \tau\}$ in which τ differs from t by a diffeomorphism, which presents f in the form

$$f = k(\tau) + \sum_{i=1}^{n-1} \epsilon_i y_i^2 + y_n^3 + \tau y_n ,$$

where k is some smooth function.

This is the desired canonical form; according as τ is positively or negatively related to t, f presents a death or birth point at (m^*, t^*).

Whitney cells: codimension k singularities: 'dynamic' case

Let $f : M^n \times I^k \to \mathbb{R}$ have a singularity of codimension k at $(m^*, \underline{t}^*)$. Then there exists a coordinate system $(U', \theta', \{x_1, \ldots, x_n, \tau_1, \ldots, \tau_k\})$ centred on $(m^*, \underline{t}^*)$ in $M^n \times I^k$ such that

$$f = \ell(\tau_1, \ldots, \tau_k) + \sum_{i=1}^{n-1} \epsilon_i x_i^2 + \tau_1 x_n + \tau_2 x_n^2 + \ldots + \tau_k x_n^k + x_n^{k+2} .$$

This, though technically more troublesome, is obtained from Proposition 7.2 and the properties in §6 of Whitney cells in hessian-adapted systems, by application of the Weierstrass-Malgrange Preparation Theorem. At the last stage simultaneous substitutions for the $t_1, \ldots, t_k$ give the $\tau_1, \ldots, \tau_k$.

Whitney maps: reduction to the case n = 2

Let $n > 2$ and let $f : M^n \to \mathbb{R}^2$ be a Whitney map. Let $m \in M$ be a singular point of f. Then by §5 there exist coordinate systems $(U, \theta, \{x_1, \ldots, x_n\})$, $(V, \phi, \{y_1, y_2\})$ adapted to f at m such that, denoting $y_\alpha \circ f$ by f_α ,

$$\left.\begin{aligned} f_1 &= x_1 , \\ \frac{\partial f_2}{\partial x_1}(m) &= \frac{\partial f_2}{\partial x_2}(m) = \dots = \frac{\partial f_2}{\partial x_n}(m) = 0, \end{aligned}\right\}$$

and, denoting $D\left(\frac{\partial f_2}{\partial x_2}, \dots, \frac{\partial f_2}{\partial x_n} \,\middle|\, x_2, \dots, x_n\right)$ by δ' ,

$$\left.\begin{aligned} &\text{either} \quad \delta'(m) \neq 0, \\ &\text{or} \quad D\left(\frac{\partial f_2}{\partial x_2}, \dots, \frac{\partial f_2}{\partial x_n}, \delta' \,\middle|\, x_1, x_2, \dots, x_n\right)(m) \neq 0 . \end{aligned}\right\}$$

But, in either case $J\left(\frac{\partial f_2}{\partial x_2}, \dots, \frac{\partial f_2}{\partial x_n} \,\middle|\, x_2, \dots, x_n\right)(m)$ has rank at least n-2. Thus, by a linear change in the $x_2, \dots, x_n$ it may be assumed that $J\left(\frac{\partial f_2}{\partial x_2}, \dots, \frac{\partial f_2}{\partial x_{n-1}} \,\middle|\, x_2, \dots, x_{n-1}\right)(m)$ has rank n-2, and that $\frac{\partial^2 f_2}{\partial x_2 \partial x_n}(m) = \dots = \frac{\partial^2 f_2}{\partial x_{n-1}\partial x_n}(m) = 0$.

Then the hypersurfaces $\frac{\partial f_2}{\partial x_2} = 0, \dots, \frac{\partial f_2}{\partial x_{n-1}} = 0$ intersect transversally at m in a surface V^2 on which $\{x_1, x_n\}$ may be taken as parameters:

$$\begin{aligned} x_2 &= \ell_2(x_1, x_n) \\ &\vdots \\ x_{n-1} &= \ell_{n-1}(x_1, x_n) \end{aligned}$$

Proceeding as before to a new coordinate system which presents V^2 as a plane, and applying Proposition 7.2, one gets a system of coordinate functions $\{x_1, w_2, \ldots, w_{n-1}, x_n\}$ centred on $m \in M$ such that

$$\left.\begin{aligned} y_1 \circ f &= x_1 \\ y_2 \circ f &= g(x_1, x_n) + \sum_{i=2}^{n-1} \epsilon_i w_i^2 \ . \end{aligned}\right\}$$

Now the restrictions on g are

$$\left.\begin{aligned} &\text{either} \quad \frac{\partial^2 g}{\partial x_n^2}(m) \neq 0, \text{ [fold point] }, \\ &\text{or} \quad \frac{\partial^2 g}{\partial x_1 \partial x_n}(m) \neq 0 \neq \frac{\partial^3 g}{\partial x_n^3}(m), \quad \text{[cusp point]} \end{aligned}\right\}$$

Thus one is led to the consideration of

<u>Whitney maps: case n = 2.</u>

(i) <u>Fold points</u> Let $f \in \mathcal{L}(M^2, \mathbb{R}^2)$ be a Whitney map and let $m \in M$ be a fold point of f. Then, in any coordinate systems $(U, \theta, \{x_1, x_2\})$, $(V, \phi, \{y_1, y_2\})$, adapted to f at m

$$\left.\begin{aligned} &y_1 \circ f = x_1 \\ &\frac{\partial(y_2 \circ f)}{\partial x_1}(m) = \frac{\partial(y_2 \circ f)}{\partial x_2}(m) = 0 \\ &\frac{\partial^2(y_2 \circ f)}{\partial x_2^2}(m) \neq 0 \end{aligned}\right\}$$

Hence $y_2 \circ f$ is regular of order 2 in x_2 at m. Thus by (iii), (iii)' of the Malgrange-Weierstrass Preparation Theorem there exist smooth functions c_0, c_1 at $m' = f(m) \in \mathbb{R}^2$ such that

$$\left.\begin{aligned} c_0 \circ f &= x_2^{\,2} + (c_1 \circ f) \,.\, x_2 \\ c_0(m') &= c_1(m') = 0 \\ \frac{\partial c_0}{\partial y_1}(m') &= 0, \quad \frac{\partial c_0}{\partial y_2}(m') \neq 0 \,. \end{aligned}\right\}$$

Now define

$$\left.\begin{aligned} Y_1 &= y_1 \\ Y_2 &= c_0 \end{aligned}\right\}$$

In a neighbourhood of $m' = f(m)$ they are the coordinate functions of a coordinate system (V', ϕ') centred on $m' \in \mathbb{R}^2$. Together with $(U, \theta, \{x_1, x_2\})$ they constitute a Type III change of adapted coordinates (§1); in the new system

$$\left.\begin{aligned} Y_1 \circ f &= x_1 \\ Y_2 \circ f &= x_2^{\,2} + (c_1 \circ f) \,.\, x_2 \,. \end{aligned}\right\}$$

Defining

$$\left.\begin{aligned} X_1 &= x_1 \\ X_2 &= x_2 + \tfrac{1}{2}(c_1 \circ f) \,, \end{aligned}\right\}$$

one gets a new coordinate system (U', θ') centred on $m \in M^2$ which with (V', ϕ') constitutes a Type II change of adapted coordinates. In this new system

$$\left.\begin{aligned} Y_1 \circ f &= X_1 \\ Y_2 \circ f &= X_2^{\,2} - (\tfrac{1}{4} c_1^{\,2}) \circ f \,. \end{aligned}\right\}$$

Now $c_1(m') = 0$, hence $\dfrac{\partial(c_1^{\,2})}{\partial Y_1}(m') = \dfrac{\partial(c_1^{\,2})}{\partial Y_2}(m') = 0,$

Thus defining

$$\left.\begin{aligned} Z_1 &= Y_1 \\ Z_2 &= Y_2 + \tfrac{1}{4} c_1^{\,2} \end{aligned}\right\}$$

gives a Type III change of adapted coordinates, and one finishes with coordinate systems $(U', \theta', \{X_1, X_2\})$, $(V', \phi'', \{Z_1, Z_2\})$ in which f takes the canonical form

$$\left.\begin{aligned} Z_1 \circ f &= X_1 \\ Z_2 \circ f &= X_2^{\,2} \end{aligned}\right\} .$$

(ii) <u>Cusp points</u> Let $f \in \mathcal{L}(M^2, \mathbb{R}^2)$ be a Whitney map, and let $m \in M$ be a cusp point of f. Then in any coordinate systems $(U, \theta, \{x_1, x_2\})$, $(V, \phi, \{y_1, y_2\})$ adapted to f at m,

$$y_1 \circ f = x_1,$$

$$\frac{\partial(y_2 \circ f)}{\partial x_1}(m) = \frac{\partial(y_2 \circ f)}{\partial x_2}(m) = \frac{\partial^2(y_2 \circ f)}{\partial x_2^2}(m) = 0,$$

and $$D\left(\frac{\partial(y_2 \circ f)}{\partial x_2}, \frac{\partial^2(y_2 \circ f)}{\partial x_2^{\,2}} \middle| x_1, x_2 \right)(m) \neq 0,$$

or equivalently $$\frac{\partial^3(y_2 \circ f)}{\partial x_2^{\,3}}(m) \neq 0 \neq \frac{\partial^2(y_2 \circ f)}{\partial x_1 \partial x_2}(m) .$$

Then $y_2 \circ f$ is regular of order 3 in x_2 at m, so by (iii), (iii)' of the Weierstrass-Malgrange Preparation Theorem there exist smooth functions c_0, c_1, c_2 at $m' = f(m) \in \mathbb{R}^2$ such that

$$\left.\begin{aligned} & c_0 \circ f = x_2^{\,3} + 3(c_2 \circ f) \cdot x_2^{\,2} + 3(c_1 \circ f) \cdot x_2 \\ & c_0(m') = c_1(m') = c_2(m') = 0 \\ & \frac{\partial c_0}{\partial y_1}(m') = 0, \quad \frac{\partial c_0}{\partial y_2}(m') \neq 0 \quad . \end{aligned}\right\}$$

Thus defining $Y_1 = y_1$, $Y_2 = c_0$ gives a coordinate system $(V', \phi', \{Y_1, Y_2\})$ which with (U, θ) is related to the previous pair of systems by a coordinate change of Type III. In the new pair of systems

$$\left.\begin{aligned} Y_1 \circ f &= x_1 \\ Y_2 \circ f &= x_2^3 + 3(c_2 \circ f) . x_2^2 + 3(c_1 \circ f) . x_2 \end{aligned}\right\} .$$

Now defining

$$\left.\begin{aligned} X_1 &= x_1 \\ X_2 &= x_2 + c_2 \circ f \end{aligned}\right\}$$

gives a coordinate system $(U', \theta', \{X_1, X_2\})$ centred on $m \in M$ which with (V', ϕ') is related to (U, θ), (V', ϕ') by a change of Type II.

$$\left.\begin{aligned} Y_1 \circ f &= X_1 \\ Y_2 \circ f &= X_2^3 + 3(c_1 - c_2^2) \circ f . X_2 + (3c_1c_2 - 2c_2^3) \circ f \\ &= X_2^3 + (d_1 \circ f) . X_2 + (d_2 \circ f) \end{aligned}\right\}$$

Now $\dfrac{\partial^2 (Y_2 \circ f)}{\partial X_1 \partial X_2}(m) \neq 0$, hence $\dfrac{\partial d_1}{\partial Y_1}(m) \neq 0$. So defining

$$\left.\begin{aligned} W_1 &= d_1 \\ W_2 &= Y_2 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} V_1 &= d_1 \circ f \\ V_2 &= X_2 \end{aligned}\right\}$$

gives coordinate systems $(U'', \theta'', \{V_1, V_2\})$, $(V'', \phi'', \{W_1, W_2\})$ related to (U', θ'), (V', ϕ') by a change of adapted coordinates of Type I.

$$\left.\begin{aligned} W_1 \circ f &= V_1 \\ W_2 \circ f &= V_2^3 + V_1 V_2 + d_2 \circ f . \end{aligned}\right\}$$

Now $d_2 = 3c_1c_2 - 2c_2^3$, hence $\dfrac{\partial d_2}{\partial W_1}(m) = \dfrac{\partial d_2}{\partial W_2}(m)$; hence defining

$$\left.\begin{aligned} Z_1 &= W_1 \\ Z_2 &= W_2 - d_2 \end{aligned}\right\}$$

gives a coordinate system $(V''', \phi''', \{Z_1, Z_2\})$ at $m' \in \mathbb{R}^2$ which with $(U'', \theta'', \{V_1, V_2\})$ forms an adapted pair which presents f at m in the canonical form:

$$\left.\begin{aligned} Z_1 \circ f &= V_1 \\ Z_2 \circ f &= V_2^3 + V_1 V_2 \end{aligned}\right\} .$$

Whitney maps: general case

Putting together the results of the two previous sections, one may state

(i) If $f : M^n \to \mathbb{R}^2$ is a Whitney map which has a fold point at $m \in M$, then there exist coordinate systems $(U, \theta, \{x_1, \ldots, x_n\})$, $(V, \phi, \{y_1, y_2\})$ adapted to f at m such that

$$\left.\begin{aligned} y_1 \circ f &= x_1 \\ y_2 \circ f &= \sum_{i=1}^{n} \epsilon_i x_i^2 . \end{aligned}\right\}$$

(ii) If $f : M^n \to \mathbb{R}^2$ is a Whitney map which has a cusp point at $m \in M$, then there exist coordinate systems $(U, \theta, \{x_1, \ldots, x_n\})$, $(V, \phi, \{y_1, y_2\})$ adapted to f at m such that

$$\left.\begin{aligned} y_1 \circ f &= x_1 \\ y_2 \circ f &= \sum_{i=1}^{n-1} \epsilon_i x_i^2 + x_n^3 + x_1 x_n . \end{aligned}\right\}$$

These are the desired canonical forms.

§8 GLOBAL PROPERTIES OF WHITNEY MAPS

1-goodness

Let $f : M^n \to \mathbb{R}^2$ be a Whitney map, and let $\sigma : \mathbb{R}^2 \to \mathbb{R}$ be a function which has no singularities. One will proceed to investigate the singularities of the composed function $\sigma \circ f : M^n \to \mathbb{R}$.

First, if $m \in M$ is a regular point of f, then m is a regular point of $\sigma \circ f$.

Next, if $m \in M$ is a fold point of f, then one may by §7 take coordinate systems $(U, \theta, \{x_i\})$, $(V, \phi, \{y_\alpha\})$ adapted to f at m such that f assumes the canonical form

$$\left.\begin{aligned} y_1 \circ f &= x_1 \\ y_2 \circ f &= -\sum_{i=2}^{\lambda} x_i^2 + \sum_{j=\lambda+1}^{n} x_j^2 \end{aligned}\right\} .$$

Moreover, let the Taylor expansion of σ at f(m) be

$$\sigma = a_0 + a_1 y_1 + a_2 y_2 + a_{11} y_1^2 + a_{12} y_1 y_2 + a_{22} y_2^2 + A(y_1, y_2)$$

where $a_0, a_1, a_2, a_{11}, a_{12}, a_{22} \in \mathbb{R}$ and $A : \mathbb{R}^2 \to \mathbb{R}$ is smooth and vanishes with its first and second order partial derivatives at $(0,0) \in \mathbb{R}^2$. Then

$$\sigma \circ f = a_0 + a_1 x_1 - \sum_{i=2}^{\lambda} a_2 x_i^2 + \sum_{j=\lambda+1}^{n} a_2 x_j^2 + a_{11} x_1^2 + B(x_1, \ldots, x_n) ,$$

where $B : \mathbb{R}^n \to \mathbb{R}$ is smooth, and vanishes with its first and second derivatives at $\underline{0} \in \mathbb{R}^n$. Now, $\sigma \circ f$ is singular at $m \in M$ if and only if $a_1 = 0$; and when $a_1 = 0$, $\sigma \circ f$ has a non-degenerate singularity at $m \in M$ if and only if the hessian matrix

$$\begin{pmatrix} 2a_{11} & 0 & 0 \\ 0 & -2I_{\lambda-1}\cdot a_2 & 0 \\ 0 & 0 & 2I_{n-\lambda}\cdot a_2 \end{pmatrix}$$

is non-singular, hence if and only if both a_2 and a_{11} are non-zero. Now σ is nowhere singular, hence a_1 and a_2 are not both zero. Hence $\sigma \circ f$ has a singular point at m if and only if $a_1 = 0$, and the singularity is degenerate if and only if $a_{11} = 0$.

Third, let $m \in M$ be a cusp point in f, and assume that the coordinate systems chosen above present f at m in the canonical form

$$\left.\begin{aligned} y_1 \circ f &= x_1 \\ y_2 \circ f &= x_1 x_n + x_n^3 - \sum_{i=2}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n-1} x_i^2 \end{aligned}\right\}$$

and assume that σ takes the form given above. Then

$$\sigma \circ f = a_0 + a_1 x_1 + a_2 x_1 x_n + a_2 x_n^3 - \sum_{i=2}^{\lambda} a_2 x_i^2 + \sum_{i=\lambda+1}^{n-1} a_2 x_i^2 + a_{11} x_1^2$$

$$+ c(x_1, \ldots, x_n)$$

where $c : \mathbb{R}^n \to \mathbb{R}$ is smooth and vanishes with its first and second derivatives at $\underline{0} \in \mathbb{R}^n$. Now $\sigma \circ f$ is singular at $m \in M$ if and only if $a_1 = 0$; and when $a_1 = 0$, $\sigma \circ f$ has a non-degenerate singularity at $m \in M$ if and only if the hessian matrix

$$\begin{pmatrix} 2a_{11} & 0 & a_2 \\ \hline 0 & \begin{matrix} -2a_2 I_{\lambda-1} & 0 \\ 0 & 2a_2 I_{n-\lambda} \end{matrix} & 0 \\ \hline a_2 & 0 & 0 \end{pmatrix}$$

is non-singular, hence if and only if a_2 is non-zero. As was remarked above, when $a_1 = 0$, a_2 is non-zero and so $\sigma \circ f$ can only have a non-degenerate singularity at a cusp point of f.

In order to classify the singularities of $\sigma \circ f$ one must interpret the conditions $a_1 = 0$ and $a_{11} = 0$. Let $m \in M$ be a fold point, then in canonical coordinates the x_1-axis gives the crease curve of f locally. The tangent to the image of the crease curve through m is given by linear multiples of $\frac{\partial}{\partial y_1}(v)$, where $f(m) = v$. The tangent to the level curve of σ through v is given by linear multiples of $a_2 \frac{\partial}{\partial y_1}(v) - a_1 \frac{\partial}{\partial y_2}(v)$. Hence $a_1 = 0$ if and only if the image of the crease line of f at m is tangent to the level curve of σ through v, or, in other words, if and only if σ composed with the restriction of f to the crease curve is singular at m. Moreover $a_{11} = 0$ is precisely the condition that $\sigma \circ f$ restricted to the crease curve of f will have a degenerate singularity at m, or equivalently that the image of the crease curve and the level curve of σ have second order tangency at v.

In the case that m is a cusp point, the crease curve is given locally by the equations $x_1 + 3x_n^2 = x_2 = \ldots = x_{n-1} = 0$ and so the crease curve is locally parametrised by x_n. Restricting f to the crease curve via the parameter $\gamma : x_n \mapsto (-3x_n^2, 0, \ldots, 0, x_n)^\theta$, the tangent to the image curve at $f(\gamma(x_n))$, $x_n \neq 0$, is given by linear multiples of $-6x_n \frac{\partial}{\partial y_1} (f \circ \gamma(x_n)) - 6x_n^2 \frac{\partial}{\partial y_2} (f \circ \gamma(x_n))$, or equivalently by linear multiples of $\frac{\partial}{\partial y_1} (f \circ \gamma)) + x_n \frac{\partial}{\partial y_2} (f \circ \gamma(x_n))$. The limit of these tangent lines as $x_n \to 0$ is given by linear multiples of $\frac{\partial}{\partial y_1} (f(m))$, which will be called <u>the tangent line to the cusp</u> at v. Again, the tangent to the level curve of σ at v is given by linear multiples of $a_2 \frac{\partial}{\partial y_1} (v) - a_1 \frac{\partial}{\partial y_2} (v)$. Hence $a_1 = 0$ if and only if the tangent to the cusp at v is equal to the tangent to the level curve of σ through v, or equivalently if and only if the composition of σ with the restriction of f to the crease curve has a point of inflexion at m.

In summary :

<u>Proposition 8.1</u> <u>If</u> $f : M^n \to \mathbb{R}^2$ <u>is a Whitney map and</u> $\sigma : \mathbb{R}^2 \to \mathbb{R}$ <u>is a function which is nowhere singular, then</u> $\sigma \circ f$ <u>is singular at</u> $m \in M$ <u>if and only if</u>

(i) m <u>lies on the crease curve of</u> f <u>and</u>

(ii) <u>the level curve of</u> σ <u>and the image of the crease curve are tangent at</u> f(m). Moreover

(iii) m <u>is a degenerate singular point of</u> $\sigma \circ f$ <u>if</u> m <u>is a fold point of</u> f <u>and the tangency in</u> (ii) <u>is of the second or higher order.</u>

Now let z be a unit vector in $\mathbb{R}^2$ and let $< z, \ > : \mathbb{R}^2 \to \mathbb{R}$ be scalar product with z. Then $< z, \ >$ satisfied the conditions of this Proposition, and the level curves of $< z, \ >$ are just the lines of $\mathbb{R}^2$ perpendicular to z.

<u>Corollary 8.2</u> <u>If</u> $f : M^n \to \mathbb{R}^2$ <u>is a Whitney map and</u> z <u>is a unit vector in</u> $\mathbb{R}^2$, <u>then</u> $< z,f > : M \to \mathbb{R}$ <u>is singular at</u> $m \in M$ <u>if and only if</u>

(i) m <u>lies on the crease curve of</u> f <u>and</u>

(ii) <u>the tangent line to the image of the crease curve at</u> m <u>is perpendicular to</u> z. <u>Moreover</u>

(iii) m <u>is adegenerate singular point of</u> $< z,f >$ <u>if and only if</u> m <u>is a fold point and the image of the crease curve has zero curvature at</u> $f(m)$.

The last remark is precisely the statement that $< z,f >$ restricted to the crease curve of f has a point of inflexion at m. Now by Proposition 2.11 the set of $z \in S^1$ such that z is perpendicular to a tangent line of $f(C(f) - C^*(f))$ at a point of inflexion has measure zero in S^1. Hence for almost every $z \in S^1$, $< z,f >$ is a Morse function, or in the terminology of §3,

<u>Proposition 8.3</u> <u>A Whitney map is 1-good</u>

<u>Liftings</u>

Let M^n be a smooth compact manifold and let $f \in \mathcal{L}(M, \mathbb{R}^2)$ be a Whitney map. Let C be a crease curve of f and let $c : [0,1] \to M$, $c(0) = c(1)$, be a smooth parametrisation of C with $c(0)$ a fold point. Denote by C^* the cusp points of f that lie on C. Let $\{t_1, \ldots, t_s\}$ be the values of the parameter which correspond to C^*.

For $t \in [0,1]$, let $L(t)$, $P(t)$ be respectively the tangent spaces of C at $c(t)$, and the kernel of Df at $c(t)$. Imposing an auxiliary riemannian metric on M, let $N(t)$ be the normal space to C in M at $c(t)$.

Orient the field L, and the spaces $N(0)$, $P(0)$ so that the ordered pairs $(L(0), P(0))$ and $(L(0), N(0))$ of complementary subspaces of

$T_{c(0)}M$ induce the same orientation on $T_{c(0)}M$. Now extend the orientations on $N(0)$ and $P(0)$ continuously round C to $c(1) = c(0)$.

Since, for $i = 1,\ldots,s$, at t passes through t_i the line $L(t)$ passes through $P(t)$ transversally, the orientations on $T_{c(t_i)}M$ given by

$$\tilde{o}(t_s +) = \lim_{\epsilon \to 0+} \tilde{o}(t_s + \epsilon)$$

$$\tilde{o}(t_s -) = \lim_{\epsilon \to 0+} \tilde{o}(t_s - \epsilon)$$

will be incompatible. Here $\tilde{o}(t)$, $t \notin \{t_1, \ldots, t_s\}$ denotes the orientation induced on $T_{c(t)}M$ by the orientations on the ordered pair $(L(t), P(t))$. Let $\hat{o}(t)$, $t \in [0,1]$, denote the orientation induced on $T_{c(t)}M$ by the orientations on the ordered pair $(L(t), N(t))$. The following results are immediate.

(i) $\hat{o}(0)$ is compatible with $\tilde{o}(0)$.

(ii) $\hat{o}(0)$ is compatible with $\hat{o}(1)$ if and only if C has an orientable tubular neighbourhood.

(iii) $\tilde{o}(0)$ is compatible with $\tilde{o}(1)$ if and only if the field P is orientable.

(iv) $\tilde{o}(1)$ is compatible with $\hat{o}(1)$ if and only if s is even.

Hence

<u>Proposition 8.4</u> <u>A tubular neighbourhood of C and the field of kernels of Df along C are simultaneously orientable or non-orientable if and only if C contains an even number of cusp points.</u>

Now specialise to the case $n = 2$. Let $f \in \mathcal{L}(M^2, \mathbb{R}^2)$ be a Whitney map of a smooth compact surface M.

<u>Definitions 8.5</u> A function $g \in \mathcal{L}(M^2, \mathbb{R})$ is said to <u>lift</u> f if $f \times g : M^2 \to \mathbb{R}^3$ is an immersion. In this case one says that f <u>admits the lifting function</u> g. If f admits a lifting function, then f is the projection of an immersion into a plane.

Let $g \in \mathcal{L}(M^2, \mathbb{R})$ lift f and let C be an arbitrary crease curve of f. Now in the notation of the last discussion, g lifts f if and only if at each point $c(t) \in C$, the kernel of Df and the kernel of Dg have trivial intersection. Or equivalently if and only if g is non-singular at $c(t)$ and $P(t)$ is transverse to the tangent to the level curve of g at $c(t)$.

Now impose an auxilliary riemannian metric of M, denoted by $< , >_M$. then grad(g) is a nowhere vanishing vector field along C such that P(t) is never orthogonal to grad(g) (c(t)). Choose a unit vector p(0) in P(0) and extend it to a field p of unit vectors round C, with $p(0) = \pm p(1)$ according as P is orientable or not. Then the function $\lambda : [0,1] \to \mathbb{R}$ defined by

$$\lambda(t) = < p(t), \mathrm{grad}(g)\ (c(t)\ ) >_M$$

is continuous and is never zero. Hence $p(1) = p(0)$, and in consequence the field of kernels of Df along C is orientable.

Conversely, let the field P along C be orientable. Using the riemannian metric on M, let $\ell \subseteq L$ be a field of unit tangent vectors along C and let $n \subseteq N$ be a field of unit normals along C. Now $\ell(0) = \ell(1)$, and $n(1) = \pm n(0)$ according as C has an orientable or non-orientable neighbourhood in M. Let $p \subseteq P$ be a field of unit vectors in P. Since P is orientable, $p(0) = p(1)$.

Define functions $\alpha, \beta : [0,1] \to \mathbb{R}$ by

$$\alpha(t) = < p(t), \ell(t) >_M, \qquad \beta(t) = < p(t), n(t) >_M .$$

α is periodic of period 1, β is periodic or antiperiodic according as $N(1) = \pm n(0)$. (Here the words 'periodic' and 'antiperiodic' are used in the following sense. Let $\gamma : [0,1] \to \mathbb{R}$ be a smooth function and let $\bar{\gamma} : (-\epsilon, 1+\epsilon) \to \mathbb{R}$ be a smooth extension of γ. Then γ is periodic or antiperiodic according as $\gamma(0) = \pm\gamma(0)$, $\frac{d^i}{dt^i}\gamma(1) = \pm \frac{d^i}{dt^i}\gamma(0)$ for $i = 1,2,\ldots,$).

Now parametrise a closed tubular neighbourhood N' of C with parameters $(t,s) \in I \times [-1,1]$, by identifying N' with the unit normal disc bundle N' of C and by mapping $(t,s) \in I \times [-1,1]$ onto $s.n(t) \in N(t)$.

A function g' on N' which is linear on the normal rays can be expressed in the form

$$g'(t,s) = A(t) + sB(t)$$

where A is periodic, and B is periodic or antiperiodic according as N' is orientable or not.

Now, such a function g' lifts f on N' if and only if, for each $t \in I$,

$$\alpha(t) . \frac{dA}{dt}(t) + \beta(t)\, B(t) \neq 0 \qquad (8.1)$$

Let $\alpha' : [0,1] \to \mathbb{R}$ be a smooth function such that there exist positive real numbers ϵ, t_1, t_2, t_3, t_4 with $0 < t_1 - \epsilon < t_1 < t_2 < t_2 + \epsilon < t_3 - \epsilon < t_3 < t_4 < t_4 + \epsilon < 1$ such that α' equals α on $[0, t_1 - \epsilon] \cup [t_2 + \epsilon, t_3 - \epsilon] \cup [t_4 + \epsilon, 1]$, such that α' is non-zero and takes opposite signs on $[t_1, t_2]$ and $[t_3, t_4]$ and such that for each $t \in [0,1]$

$$\alpha(t).\alpha'(t) + \beta(t).\beta'(t) \neq 0.$$

Let $\gamma : [0,1] \to \mathbb{R}$ be a smooth function taking positive values, and taking value 1 except possibly on $[t_1, t_2] \cup [t_3, t_4]$ such that

$$\int_0^1 \gamma(t)\, \alpha'(t)\, dt = 0.$$

This is made possible by exploiting the fact that α' has intervals where it takes positive and negative values.

Now define

$$A(t) = \int_0^t \gamma(x)\, \alpha'(x)\, dx$$

$$B(t) = \gamma(t)\, \beta(t)$$

$$g'(t,s) = A(t) + s\,B(t).$$

Then g', satisfying the requirements of smoothness, periodicity and the inequality (8.1), is a function which lifts f on the tubular neighbourhood N'. Perform this construction for each crease curve of f and use the Whitney extension theorem [28] to obtain a function $g: M \to \mathbb{R}$ which lifts f. Indeed, by making an arbitrarily small change in g one will not disturb the transversal property of the level curves of g and the kernels of f, hence g may even be chosen to be a Morse function which lifts f.

Proposition 8.6 If $f \in \mathcal{L}(M^2, \mathbb{R}^2)$ is a Whitney map, then f admits a lifting function if and only if the field of kernels of Df along each crease curve is orientable.

Combining this result with Proposition 8.4 one obtains

Proposition 8.7 A Whitney map of a compact smooth manifold admits a lifting function, or equivalently, is the projection of an immersion in

$\mathbb{R}^3$, if and only if each crease curve contains an even or odd number of cusp points according as its tubular neighbourhood is orientable or non-orientable.

Proposition 8.6 generalises in the following way. Let $n \geqslant 3$ and let M^n be a compact smooth manifold. Consider the map space $\mathcal{L}(M, \mathbb{R}^{2n-2})$ and the associated jet-space $J^1(M, \mathbb{R}^{2n-2})$. By [33, §21], or by the methods of §5, there is an open dense subspace of $(M, \mathbb{R}^{2n-2})$ whose members are called good maps, with the following

Definition 8.8 $f \in (M, \mathbb{R}^{2n-2})$ is good if

(i) rank $Df \geqslant (n-1)$ everywhere,

(ii) $f^{(1)} : M \to J^1(M, \mathbb{R}^{2n-2})$ is transverse to the space of 1-jets of rank $(n-1)$. The singular locus of f is thus a finite collection of smooth closed curves,

(iii) the kernel of Df and the tangent line to a singular curve never coincide at a singular point.

Let $f : M \to \mathbb{R}^{2n-2}$ be good, then

Definition 8.9 $g : M \to \mathbb{R}$ lifts f if and only if $f \times g : M \to \mathbb{R}^{2n-1}$ is an immersion.

Using an auxilliary riemannian metric on M, one determines that $g : M \to \mathbb{R}$ lifts f if and only if

$$< \text{grad}\,(g)(m),\ \ker Df(m) >_M \neq \{0\}$$

for each singular point m of f.

Let K be any singular curve of f, let $k : [0,1] \to K$ be a smooth parametrisation, and let $P(t)$ be the kernel of Df at $k(t)$. If g lifts f, then projecting grad (g) $(k(t))$ orthogonally into $P(t)$ gives a nowhere zero smooth section of P, hence an orientation of P.

Conversely let $p \subseteq P$ be a smooth periodic field of unit vectors. Let N be the normal bundle of K in M and let $N(t)$ be the normal space to K at $k(t)$. Let $m_1, \ldots, m_{n-1} \subseteq N$ be a smooth orthonormal frame field with $m_2, \ldots, m_{n-1}$ periodic, and with $m_1(1) = \pm m_1(0)$ according as N is orientable or not.

Let N' be a closed tubular neighbourhood of K in M. Denoting by D^{n-1} the closed unit disc in $\mathbb{R}^{n-1}$, one parametrises N' by parameters $(t,\underline{s}) \in I \times D^{n-1}$, where $\underline{s} = (s_1, \ldots, s_{n-1})$, by identifying N' with the unit normal disc bundle of K in M and mapping $(t,\underline{s}) \in I \times D^{n-1}$ onto $s_1 \cdot m_1(t) + \ldots + s_{n-1} \cdot m_{n-1}(t) \in N(t)$. A function g' on N' which is linear on the normal spaces can be expressed in the form

$$g'(t,\underline{s}) = A(t) + \sum_{i=1}^{n-1} B_i(t) \cdot s_i \,.$$

Define $\alpha(t) = \langle p(t), \ell(t) \rangle$, $\beta_i(t) = \langle p(t), m_i(t) \rangle$. Approximate α by α' in the above fashion, so that $\alpha(t) \cdot \alpha'(t) + \sum_{i=1}^{n-1} \beta_i^2(t)$ is never zero, and proceed as before. Thus

<u>Proposition 8.10</u> <u>A good map</u> $f \in \mathcal{L}(M, \mathbb{R}^{2n-2})$ <u>lifts to an immersion in</u> $\mathbb{R}^{2n-1}$ <u>if and only if the field of kernels of</u> f <u>along each singular curve is orientable.</u>

The Number of Cusps

The following proposition, due to Thom [25, Th.9, p.84] will now be established. Let M^n be a smooth compact manifold.

<u>Proposition 8.11</u> <u>The number of cusp points of a Whitney map of</u> M <u>has the same parity as the Euler characteristic of</u> M.

First, letting $f \in \mathcal{L}(M, \mathbb{R}^2)$ be a Whitney map, $C(f)$ the singular set of f and $C^*(f)$ the cusp points of f, one observes, from Morse's Euler formula [22], that the Euler characteristic of M and the number of singular points of a Morse function on M have the same parity.

Next, returning to the analysis that precedes Proposition 8.3, let $z \in \mathbb{R}^2$ be a unit vector such that $< z,f >$ is a Morse function, and z is not perpendicular to any of the tangent lines to the cusps of f. The number of singular points of $< z,f >$ is precisely the number of points of $C(f) - C^*(f)$ where the tangent to the image crease curve is perpendicular to z. By the choice of z, the restriction of $< z,f >$ to $C(f)$ has simple maxima and minima at just these points.

Let C be one of the components of $C(f)$ and let C^* be the cusp set of f on C. Orient C, and define smoothly on $C^* - C$ an oriented tangent to the image of $C^* - C$ so that Df preserves the orientation. In the limit, at a point of C^* the oriented tangent to the image makes an antipodal jump. Thus the stationary points of $< z,f >$ on $C - C^*$ are alternately maxima and minima, except that consecutive stationary points which are separated by just one cusp point are of the same type. Consequently there are $2k + s$ singular points of $< z,f >$ on $C - C^*$ where k is a non-negative integer and s is the number of cusp points on C. Summing over all the components of f, one obtains: the number of critical points of $< z,f >$ on M has the same parity as the number of cusp points of f. This, with the first observation, yields the proposition.

Minimality and Lifting

Given a smooth compact manifold M of n dimensions, there is a well known invariant of M, known as the Morse number of M.

<u>Definition 8.12</u> If $\mathcal{ML}(M, \mathbb{R})$ denotes the space of Morse functions on M, and if $\#: \mathcal{ML}(M, \mathbb{R}) \to \mathbb{Z}$ denotes the function which assigns to each Morse function the number of its singular points, then <u>the Morse number of M</u>, $\mu(M)$, is defined by

$$\mu(M) = \inf\{\#f : f \in \mathcal{ML}(M, \mathbb{R})\}.$$

If $f \in \mathcal{L}(M^n, \mathbb{R}^p)$ is 1-good (§3), then for almost all $L \in G(p,1)$, $\#(P_L \circ f)$ is defined. One defines <u>the total curvature of</u> f, $\tau(f)$, by

$$\int_{G(p,1)} (P_L \circ f)\, d\mu_{G(p,1)}(L),$$

where $\mu_{G(p,1)}$ is a normalised invariant measure.

If $f \in \mathcal{L}(M^n, \mathbb{R}^p)$ is 1-good, then f is said to be <u>minimal</u> if $\tau(f) = \mu(M)$.

There is an extensive literature concerning minimal immersions (see [12] and [9] for bibliographies). The connection between total curvature and the classical definition in [6] is quickly recovered by applying Fubini's Theorem [8, p.115] to the considerations of §2.

Trivially there is a minimal map in $\mathcal{L}(M, \mathbb{R})$.

<u>Proposition 8.13</u> <u>Any compact smooth surface admits a Whitney map which is minimal.</u>

The proof is by example, divided into the following three cases.

(i) If the surface is the sphere S^2, then any planar projection of the standard embedding of S^2 in $\mathbb{R}^3$ is a Whitney map which is minimal. The singular locus is a great circle which is mapped diffeomorphically into a circle in the plane. $\tau = 2 = \mu(S^2)$.

(ii) If the surface is the Klein bottle K^2, then a minimal Whitney map of K^2 into a plane may be constructed as follows. Let A, A' be congruent closed annuli. (Fig. I).

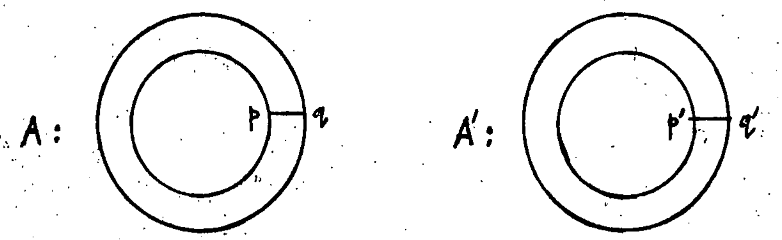

Fig. I

Let pq, p'q' be radial segments, and cut A, A' along these segments. (Fig. II).

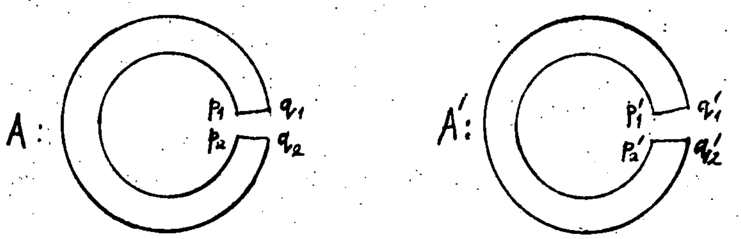

Fig. II

Superimpose A upon A', identify the circular parts of the boundaries and identify p_1q_1 with $p_2'\,q_2'$ and $p_1'q_1'$ with p_2q_2. (Fig. III)

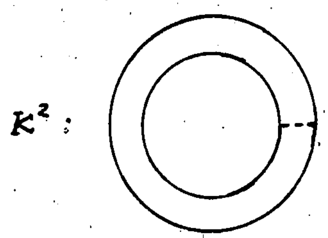

Fig. III

These identifications give a smooth map of K^2 into $\mathbb{R}^2$ which is Whitney and minimal. The singular locus is two closed curves which are mapped (embedding) into concentric circles in the plane. $\tau = 4 = \mu(K^2)$.

(iii) If the surface is the projective plane P^2 then a minimal Whitney map of P^2 into a plane may be constructed as follows. Consider the map $f : D^2 \to \mathbb{R}^2$, where D^2 is the disc of radius 3 in $\mathbb{R}^2$, given by

$$f(z) = z^2 + 2\alpha(|z|)\bar{z}$$

using the Argand representation of $\mathbb{R}^2$, where $\alpha : [0,3] \to \mathbb{R}$ is a smooth function such that

$$\left.\begin{aligned} \alpha(t) &= \tfrac{1}{2}, & 0 \leq t \leq 1 \\ \alpha(t) &= 0, & 2 \leq t \leq 3 \\ -1 \leq \frac{d\alpha}{dt}(t) &\leq 0, & 1 \leq t \leq 2 \end{aligned}\right\} .$$

Then f is a Whitney map with singular locus the circle of radius $\frac{1}{2}$ and cusp-points at $\frac{1}{2}$, $\frac{1}{2}\exp(\pm 2\pi i/3)$. The image of the singular locus of f is a tricuspid curve (Steiner's hypocycloid). (Fig. IV).

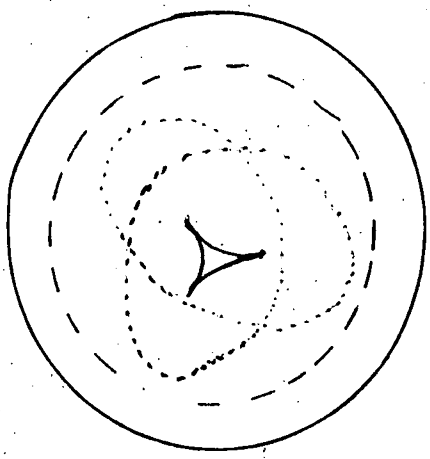

Fig. IV

The hatched curve represents the image of the circle of radius 2, the dotted curve the circle of radius 1.5, the outer heavy curve the image of the circle of radius 3. Note that f on $\{z : 2 \leq |z| \leq 3\}$ is a double covering of $\{z : 4 \leq |z| \leq 9\}$. In particular antipodal points of the boundary of D^2 are mapped to the same point of $\mathbb{R}^2$ in such a way that f and all its derivatives can be identified.

The compact surface obtained by identifying the antipodal boundary points of D^2 is just P^2. Since f respects this identification, f may be regarded as a member of $\mathcal{L}(P^2, \mathbb{R}^2)$. Thus regarded, f is easily seen to be a Whitney map whose crease set contains two closed curves. The first, which is null-homotopic, contains three cusp points and is mapped bijectively onto the tricuspid curve as above. The second, which is essential, contains no cusps and is embedded as the circle of radius 9.

Moreover f is minimal: the tricuspid contributes one singular point, the circle two. $\tau(f) = 3 = \mu(P^2)$.

(iv) Handles may be added to the surfaces in (i), (ii), (iii) so that the respective Whitney map extends to a Whitney map with one extra crease curve for each handle, corresponding to the 'waist' of a handle of negative curvature, (Fig. V), which is mapped to a circle in the plane.

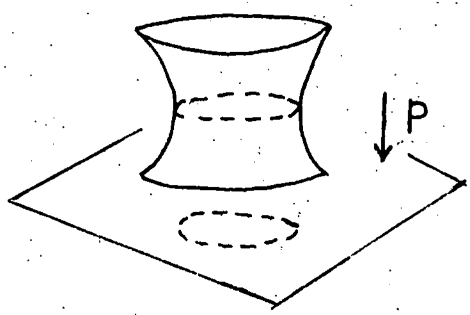

Fig. V

Each handle contributes 2 to the total curvature. Every compact surface is of one of the types (M^2 plus g handles), where $M^2 = S^2, P^2, K^2$. $\mu(S^2 + g \text{ handles}) = 2(1 + g)$, $\mu(P^2 + g \text{ handles}) = 1 + 2(1 + g)$, $\mu(K^2 + g)$ handles) $= 2 + 2(1 + g)$. Each of the corresponding Whitney maps is minimal; the image of the crease set has the form shown in Fig. VI, where the general figure has g dotted circles.

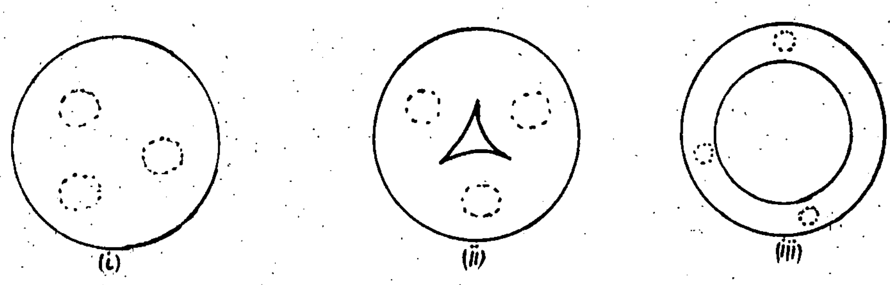

Fig. VI
(g = 3)

There are two aspects of this theorem, and its examples, that are worth commenting on. First, by the criterion of this section, the orientable surfaces and their maps admit a lifting to immersions on $\mathbb{R}^3$; the corresponding minimal Whitney maps are projections (vertical) of the 'standard' pictures of these surfaces as regular submanifolds of $\mathbb{R}^3$. The non-orientable surfaces, on the other hand, have been given minimal

Whitney maps which do not admit liftings to immersions in $\mathbb{R}^3$. They can be lifted to immersions at all but a finite number of points, two for ($\mathbb{P}^2$ + g handles), one for ($\mathbb{K}^2$ + g handles), where the map into $\mathbb{R}^3$ exhibits singularities of the 'cuspidal' type described in [30], [31] and [33, §20].

The second point is that for $\mathbb{P}^2$ there is a gap in the dimensions of the euclidean spaces into which there exist minimal maps. For it is known that there exist minimal immersions of $\mathbb{P}^2$ and $\mathbb{R}^4$, and that there do not exist minimal immersions of $\mathbb{P}^2$ in $\mathbb{R}^3$, [12].

A Whitney map of $\mathbb{P}^2$ into $\mathbb{R}^2$ which does admit lifting can be obtained by a suitable pro jection of its immersion in $\mathbb{R}^3$ as Boy's Surface. An excellent photograph illustrating such a Whitney map may be seen in 'Geometry and the Imagination' by Hilbert and Cohn-Vassen, Chelsea, New York, 1952. The total curvature of this Whitney map is greater than 3 (non-minimal) and less than 5. Fig. VII represents the image of the unique crease curve of this map.

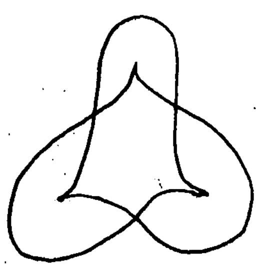

Fig. VII

The fact that this map lifts derives directly from the criterion above, or alternatively from the following easily proved result

Proposition 8.14 A Whitney map of a compact surface which has just one crease curve can be lifted to an immersion in $\mathbb{R}^3$.

§9 PROJECTIONS OF SMOOTH MAPS

Definitions 9.1 Let M^n be a smooth manifold, $m \in M$, k and r positive integers and $f : M^n \to \mathbb{R}^k$ a smooth map. Then f is r^{th} order non-degenerate at m if in some, and hence any, coordinate system containing m, regarding f and its partial derivatives of all orders as vector valued functions, the space spanned by the partial derivatives of f at m, up to the r^{th} order derivatives, has maximal dimension, [15]. One says that f is r^{th} order non-degenerate, if f is r^{th} order non-degenerate at every point of M.

The following consequences of this definition are immediate:

(i) If $k < n$, and $f \in \mathcal{L}(M, \mathbb{R}^k)$ is first order non-degenerate then Df has maximal rank k everywhere, or equivalently is a submersion.

(ii) If $k \geqslant n$ and $f \in \mathcal{L}(M, \mathbb{R}^k)$ is first order non-degenerate then Df has maximal rank n everywhere, or equivalently is an immersion.

(iii) If $n = 1$, $k = 3$, and $f \in \mathcal{L}(M^1, \mathbb{R}^3)$ is first and second order non-degenerate then equivalently f is an immersion of a curve with nowhere zero curvature.

(vi) If $n = 1$, $k = 3$ and $f \in \mathcal{L}(M^1, \mathbb{R}^3)$ is first, second and third order non-degenerate then equivalently f is an immersion of a curve with curvature and torsion nowhere zero.

(v) If $k = n+1$ and $f \in \mathcal{L}(M^n, \mathbb{R}^{n+1})$ is first and second order non-degenerate then equivalently f is an immersion such that at no point are all the principal curvatures zero.

(vi) The set of r^{th} order non-degenerate maps in $\mathcal{L}(M^n, \mathbb{R}^k)$ is open.

Note that in general they are non-generic, being defined by the condition that the jet prolongation does not encounter a jet-submanifold of possibly low codimension.

Closed curves

Let $\mathcal{I}(S^1, \mathbb{R}^3)$ be the space of smooth closed curves immersed in $\mathbb{R}^3$. $\mathcal{I}(S^1, \mathbb{R}^3)$ is open and dense in $\mathcal{L}(S^1, \mathbb{R}^3)$. In $J^2(S^1, \mathbb{R}^3)$ the set of jets derived from elements of $\mathcal{I}(S^1, \mathbb{R}^3)$ at points where curvature is zero forms a submanifold of codimension 3. Thus the space $\mathcal{N}_2(S^1, \mathbb{R}^3)$ of first and second order nondegenerate curves is open and dense in $\mathcal{I}(S^1, \mathbb{R}^3)$. In $J^3(S^1, \mathbb{R}^3)$ the set of jets derived from elements of $\mathcal{N}_2(S^1, \mathbb{R}^3)$ at points where torsion is zero form a submanifold of codimension 1. Define $f \in \mathcal{N}_2(S^1, \mathbb{R}^3)$ to be <u>3-regular</u> if $f^{(3)}$ is transversal to this submanifold. 3-regular curves are thus open and dense in $\mathcal{N}_2(S^1, \mathbb{R}^3)$ and are characterised by the properties: (a) curvature is never zero, and (b) when torsion is zero its derivative is non-zero, (and consequently torsion is zero at just a finite collection of points).

Let $f = \mathcal{I}(S^1, \mathbb{R}^3)$, $m \in S^1$ and let f be parametrised by arc length in a neighbourhood of m as base point. Using Taylor's expansion formula and the Serret-Frenet formulae [34], one obtains

$$\begin{aligned} f(s) \;=\; f(m) &+ [s - \kappa^2 s^3/3! - \kappa\kappa' s^4/8]\underline{t} \\ &+ [\kappa s^2/2 + \kappa' s^3/3! + (\kappa'' - \kappa\tau^2 - \kappa^3)s^4/4!]\underline{n} \\ &+ [\kappa\tau s^3/3! + (2\kappa'\tau + \kappa\tau')s^4/6!]\underline{b} + \underline{O}(s^4) , \end{aligned}$$

where $(\underline{t}, \underline{n}, \underline{b})$ is the Frenet frame of f at m and $\kappa, \tau, \kappa', \tau', \kappa''$ denote the values of the curvature and torsion and their derivatives at m.

Composing f with a projection P onto a line L in $\mathbb{R}^3$ one sees that $P \circ f$ has a singularity at m if and only if L is parallel to the normal plane of f at m, that the singularity is non-degenerate if and only if κ is not zero; if however κ is zero, the singularity is of codimension one if κ' is non-zero, and in general if $\kappa, \kappa', \ldots, \kappa^{(r)}$ are zero, then the singularity is of codimension $(r+1)$ if $\kappa^{(r+1)}$ is non-zero. Hence

<u>Proposition 9.2</u> <u>If $f \in \mathcal{L}(S^1, \mathbb{R}^3)$ is a second order non-degenerate immersion, then every orthogonal projection of f onto a line of $\mathbb{R}^3$ is a Morse map.</u>

Next, composing $f \in \mathcal{L}(S^1, \mathbb{R}^3)$ with a projection P onto a 2-plane π in $\mathbb{R}^3$ one sees that $P \circ f$ has a singularity at m if and only if π is parallel to the normal plane of f at m. Moreover, when this is the case:

(i) If $\kappa \neq 0$, $\tau \neq 0$ then the singularity is a cusp of the first species with model $y^2 = x^3 + 0(x^3)$

(ii) If $\kappa \neq 0$, $\tau = 0$, $\tau' \neq 0$ then the singularity is a cusp of the second species (keratoid) with model $y^2 = x^4 + 0(x^4)$.

Singularities of higher codimension are determined from further terms of the Taylor expansion under the dual conditions that κ, τ are r^{th}, s^{th} order regular (§1) at m. As has been shown, the cases (i), (ii) above cover the generic configurations.

<u>Proposition 9.3</u> <u>If $f \in \mathcal{L}(S^1, \mathbb{R}^3)$ is a 3-regular immersion then</u>

(i) <u>projection of f into a plane has singularities if and only if the unit normal of the plane lies on the tangent indicatrix of f ;</u>

(ii) <u>the singularities are a finite number of cusps which (except when the normal of the plane is a point of the tangent indicatrix which has unit curvature) are all of semi-cubical-paraboloid type;</u>

(iii) <u>projections along the finite number of exceptional directions have at least one cusp of the second species</u>.

Surfaces

Let M^2 be a smooth compact surface and let $f : M^2 \to \mathbb{R}^3$ be a smooth immersion. Let $\underline{z}$ be a fixed unit vector in $\mathbb{R}^3$, let $\Pi^2_{\underline{z}} \subseteq \mathbb{R}^3$ be a plane orthogonal to $\underline{z}$ and let $P_{\underline{z}} : \mathbb{R}^3 \to \Pi_{\underline{z}}$ denote orthogonal projection onto $\Pi_{\underline{z}}$. The question arises: what is the relation between the geometry of f and the geometry of the singular locus of the composed map $P_{\underline{z}} \circ f : M^2 \to \Pi^2_{\underline{z}}$?

Let $m \in M$, and let $(U, \theta, \{X_1, X_2\})$ be a coordinate system centred on $m \in M$. Denote $\Pi_{\underline{z}}$, $P_{\underline{z}}$ ambiguously by Π, P, keeping $\underline{z}$ fixed throughout. $Df(m)$ is a monomorphism; the kernel of $DP(f(m))$ is the line L through $f(m)$ parallel to $\underline{z}$. Either $Df(T_mM)$ has trivial intersection with L and $D(P \circ f)(m)$ has rank 2 and m is a regular point of $P \circ f$, or $Df(T_mM)$ contains L and $D(P \circ f)(m)$ has rank 1 and m is a singular point of $P \circ f$. Now $Df(T_mM)$ contains L if and only if the line N normal to $Df(T_mM)$ through $f(m)$ is normal to L, if and only if the (two) unit normals of f at m lie on the great circle $S^1(\underline{z})$ in S^2 orthogonal to $\underline{z}$. If $p : UNM \to M$, $\Gamma : UNM \to S^2$ denote the unit normal bundle and normal gauss-map of f one may summarise this paragraph by

<u>Proposition 9.4</u> $P \circ f : M^2 \to \Pi^2$ <u>has rank everywhere equal to</u> 1 or 2; <u>the set of points where</u> $P \circ f$ <u>has rank</u> 1 <u>is the set</u> $p(\Gamma^{-1}(S^1(\underline{z})))$.

Hence the singular locus of $P \circ f$ is determined by the gauss-map of the immersion f. Let $m \in M$ be a singular point of $P \circ f$. Let $(U, \theta, \{X_1, X_2\})$ be a coordinate system centred on m. Let $\underline{n} : U \to S^2$ be a selected unit normal vector field of f over U. Locally the singular set of $P \circ f$ is given by $\underline{n}^{-1}(S^1(\underline{z}))$. Note that $\underline{n}(m) \in S^1(\underline{z})$

If $\underline{n}$ is transversal to $S^1(\underline{z})$ at m, then locally $\underline{n}^{-1}(S^1(\underline{z}))$ is a smooth curve. This condition is exhaused by the following two cases.

(i) $D\underline{n}(m)$ has rank 2, i.e. the Gauss curvature at $m \in M$ of the immersion f is non-zero, (see § 2). Then $\underline{n}$ is locally a diffeomorphism and $\underline{n}^{-1}(S^1(\underline{z}))$ is locally a smooth curve passing through m.

(ii) $D\underline{n}(m)$ has rank 1, i.e. the Gauss curvature at $m \in M$ is zero, but one of the principal curvatures is non-zero, and $D\underline{n}(T_m M)$ is transverse to $T_{\underline{n}(m)} S^1(\underline{z})$. Then $\underline{n}^{-1}(S^1(\underline{z}))$ is locally a smooth curve passing through m.

The following two cases describe the occasions when $\underline{n}$ is not transversal to $S^1(\underline{z})$ at m.

(iii) $D\underline{n}$ has rank 1, and $D\underline{n}(T_m M) = T_{\underline{n}(m)} S^1(\underline{z})$.

(iv) $D\underline{n}$ has rank 0.

The case (i) covers the elliptic and hyperbolic points, (ii) and (iii) the parabolic points and (iv) the flat points of the immersion f of M.

To elaborate these cases and to discuss the restriction of $P \circ f$ to its singular locus, one will describe these cases in local coordinates.

Let $m \in M$, and let $(U, \theta, \{x_1, x_2\}, (\mathbb{R}^3, A, \{y_1, y_2, y_3\})$ be coordinate systems linearly adapted to f at m. Then

$$\left.\begin{aligned}
x_1 &= y_1 \circ f - (y_1 \circ f)(m) \\
x_2 &= y_2 \circ f - (y_2 \circ f)(m) \\
\frac{\partial f}{\partial x_1}(m) &= \frac{\partial}{\partial y_1} \circ f(m) \\
\frac{\partial}{\partial x_2}(m) &= \frac{\partial}{\partial y_2} \circ f(m)
\end{aligned}\right\}$$

Now m is a singular point of $P \circ f$ if and only if there exist real numbers λ_1, λ_2 such that $\lambda_1^2 + \lambda_2^2 = 1$ and

$\underline{z} = \lambda_1 \dfrac{\partial f}{\partial x_1}(m) + \lambda_2 \dfrac{\partial f}{\partial x_2}(m)$. Note that $\dfrac{\partial f}{\partial x_1}(m)$ and $\dfrac{\partial f}{\partial x_2}(m)$ are orthogonal vectors of unit length. Let $\underline{n}^*$ denote a unit normal of f at m. Then the plane through the origin of $\mathbb{R}^3$, perpendicular to $\underline{z}$, has the vector $\underline{n}^*$ and $\underline{z}^* = -\lambda_2 \dfrac{\partial f}{\partial x_1}(m) + \lambda_1 \dfrac{\partial f}{\partial x_2}(m)$ as an orthogonal basis.

Hence

$$P \circ f = \langle \underline{z}^*, f \rangle \underline{z}^* + \langle \underline{n}^*, f \rangle \underline{n}^* .$$

Thus

$$\frac{\partial (P \circ f)}{\partial x_\alpha} = \langle \underline{z}^*, \frac{\partial f}{\partial x_\alpha} \rangle \underline{z}^* + \langle \underline{n}^*, \frac{\partial f}{\partial x_\alpha} \rangle \underline{n}^* ,$$

where $\alpha = 1,2$. Hence the singularities of $P \circ f$ in $U \subseteq M$ are given by the zeros of the map $A : U \to \mathbb{R}$, where

$$A = \langle \underline{z}^*, \frac{\partial f}{\partial x_1} \rangle \langle \underline{n}^*, \frac{\partial f}{\partial x_2} \rangle - \langle \underline{n}^*, \frac{\partial f}{\partial x_1} \rangle \langle \underline{z}^*, \frac{\partial f}{\partial x_2} \rangle .$$

Now, for $\alpha = 1, 2$, $\langle \underline{n}^*, \dfrac{\partial f}{\partial x_\alpha}(m) \rangle = 0$. Hence

$$\frac{\partial A}{\partial x_1}(m) = \langle \underline{z}^*, \frac{\partial f}{\partial x_1}(m) \rangle \langle \underline{n}^*, \frac{\partial^2 f}{\partial x_1 \partial x_2}(m) \rangle - \langle \underline{n}^*, \frac{\partial^2 f}{\partial x_1^2}(m) \rangle \langle \underline{z}^*, \frac{\partial f}{\partial x_2}(m) \rangle$$

$$\frac{\partial A}{\partial x_2}(m) = \langle \underline{z}^*, \frac{\partial f}{\partial x_1}(m) \rangle \langle \underline{n}^*, \frac{\partial^2 f}{\partial x_2^2}(m) \rangle - \langle \underline{n}^*, \frac{\partial^2 f}{\partial x_1 \partial x_2}(m) \rangle \langle \underline{z}^*, \frac{\partial f}{\partial x_2}(m) \rangle$$

Now $\underline{z}^* = -\lambda_2 \dfrac{\partial f}{\partial x_1}(m) + \lambda_1 \dfrac{\partial f}{\partial x_2}(m)$, hence

$$\left.\begin{aligned}\frac{\partial A}{\partial x_1}(m) &= -\lambda_2 \frac{\partial^2 h^*}{\partial x_1 \partial x_2}(m) - \lambda_1 \frac{\partial^2 h^*}{\partial x_1^2}(m) \\ \frac{\partial A}{\partial x_2}(m) &= -\lambda_2 \frac{\partial^2 h^*}{\partial x_2^2}(m) - \lambda_1 \frac{\partial^2 h^*}{\partial x_1 \partial x_2}(m)\end{aligned}\right\}$$

where $h^* = \langle \underline{n}^*, f \rangle : M \to \mathbb{R}$ is the 'height function' of f in the direction $\underline{n}^*$. Denote the partial derivatives of h^* at m by suffices. Then the kernel of DA(m) is given by the space of

$\mu_1 \dfrac{\partial}{\partial x_1}(m) + \mu \dfrac{\partial}{\partial x_2}(m) \in T_m M$ such that

$$\lambda_1\mu_1 h^*_{11} + (\lambda_1\mu_2 + \lambda_2\mu_1)h^*_{12} + \lambda_2\mu_2 h^*_{22} = 0.$$

Note that the $h^*_{\alpha\beta}$ are by §2 the components of the second fundamental form of f at m in the direction $\underline{n}^*$. The kernel of DA(m) is the 'tangent space' to the singular locus of P o f at m.

DA(m) has rank 0, A is not transversal to 0 at m, if and only if

$$\left.\begin{aligned}\lambda_1 h^*_{11} + \lambda_2 h^*_{12} &= 0 \\ \lambda_1 h^*_{12} + \lambda_2 h^*_{22} &= 0\end{aligned}\right\}.$$

Now, if by choice $\dfrac{\partial}{\partial x_1}(m)$, $\dfrac{\partial}{\partial x_2}(m)$ are chosen to be orthogonal principal axes [34] of f at $m \in M$, then h^*_{11}, h^*_{22} are the principal curvatures of f at m, assuming that $\left\{ \dfrac{\partial f}{\partial x_1}(m), \dfrac{\partial f}{\partial x_2}(m), \underline{n}^* \right\}$ is a frame coherent with the standard orientation of $\mathbb{R}^3$; and $h^*_{12} = 0$. Thus, if

(i) m is an elliptic or hyperbolic point of f, then A is transversal to 0 at m,

(ii) m is a parabolic point of f, then A is not transversal to O at m if and only if $\underline{z}$ is perpendicular to the Df-image of the direction of principal non-zero curvature at m, and

(iii) m is a flat umbilic point of f, then A is not transversal to O at m, and T_mM is the kernel of DA(m).

Moreover, it follows directly from the above that if T_mM contains a vector, 'tangent' to the singular locus of P o f, whose Df-image is parallel to $\underline{z}^*$, then this vector is asymptotic [34] with respect to f at m.

Projections of surfaces and Whitney maps

In order to compare the singularities of P o f with those of a Whitney map, one must introduct coordinate systems which present P o f at the singular point $m \in M$ in a suitably adapted form.

Therefore define $\eta_i : \mathbb{R}^3 \to \mathbb{R}$, $\xi_\alpha : U \to \mathbb{R}$ by

$$\left.\begin{aligned} \eta_1 &= -\lambda_2 y_1 + \lambda_1 y_2 \\ \eta_2 &= y_3 \\ \eta_3 &= \lambda_1 y_1 + \lambda_2 y_2 \end{aligned}\right\}$$

$$\left.\begin{aligned} \xi_1 &= -\lambda_2 x_1 + \lambda_1 x_2 \\ \xi_2 &= \lambda_1 x_1 + \lambda_2 x_2 \,. \end{aligned}\right\}$$

Then

$$\left.\begin{aligned} \eta_1 \circ f &= \langle \underline{z}^*, f \rangle \\ \eta_2 \circ f &= \langle \underline{n}^*, f \rangle \\ \eta_3 \circ f &= \langle \underline{z}, f \rangle \end{aligned}\right\}$$

$$\left.\begin{aligned} \xi_1 &= \langle \underline{z}^*, f - f(m) \rangle \\ \xi_2 &= \langle \underline{z}, f - f(m) \rangle \,. \end{aligned}\right\}$$

Hence

$$\left.\begin{aligned} \eta_1 \circ f &= \xi_1 + < \underline{z}^*, f(m) > \\ \eta_2 \circ f &= < \underline{n}^*, f > = h^* . \end{aligned}\right\}$$

But $$P \circ f = (\eta_1 \circ f)\underline{z}^* + (\eta_2 \circ f)\underline{n}^*$$

and $$\frac{\partial(\eta_2 \circ f)}{\partial \xi_1}(m) = \frac{\partial(\eta_2 \circ f)}{\partial \xi_2}(m) = 0 .$$

Thus $(U, \theta', \{\xi_1, \xi_2\})$, $(\mathbb{R}^3, A', \{\eta_1, \eta_2, \eta_3\})$ are coordinate systems linearly adapted to f at m (modulo transposition of η_2, η_3) such that $(U, \theta', \{\xi_1, \xi_2\})$, $(\Pi_{\underline{z}}, A'', \{\eta_1, \eta_2\})$ are linearly adapted to $P \circ f$ at m.

Applying the definitions of §5 ,

(I) m is a <u>fold point</u> of $P \circ f$ if and only if

$$\frac{\partial^2(\eta_2 \circ f)}{\partial \xi_2^{\,2}}(m) \neq 0 .$$

Now

$$x_1 = -\lambda_2 \xi_1 + \lambda_1 \xi_2$$

$$x_2 = \lambda_1 \xi_1 + \lambda_2 \xi_2$$

and $$\eta_2 \circ f = h^*$$

Hence

$$\frac{\partial(\eta_2 \circ f)}{\partial \xi_2} = \lambda_1 \frac{\partial h^*}{\partial x_1} + \lambda_2 \frac{\partial h^*}{\partial x_2}$$

Thus

$$\frac{\partial^2(\eta_2 \circ f)}{\partial \xi_2^{\,2}}(m) = \lambda_1^2 h^*_{11} + 2\lambda_1\lambda_2 h^*_{12} + \lambda_2^2 h^*_{22} .$$

Hence m <u>is a fold point of</u> P o f <u>if and only if</u>

$$\lambda_1^2 h^*_{11} + 2\lambda_1\lambda_2 h^*_{12} + \lambda_2^2 h^*_{22} \neq 0.$$

(II) Similarly, m is a cusp point of P o f if and only if

$$\frac{\partial^2(\eta_2 \circ f)}{\partial \xi_2^2}(m) = 0$$

$$\frac{\partial^2(\eta_2 \circ f)}{\partial \xi_1 \partial \xi_2}(m) \neq 0$$

$$\frac{\partial^3(\eta_2 \circ f)}{\partial \xi_2^3}(m) \neq 0$$

Now

$$\frac{\partial^2(\eta_2 \circ f)}{\partial \xi_1 \partial \xi_2}(m) = -\lambda_1\lambda_2 h^*_{11} + (\lambda_1^2 - \lambda_2^2) h^*_{12} + \lambda_1\lambda_2 h^*_{22},$$

and

$$\frac{\partial^3(\eta_2 \circ f)}{\partial \xi_2^3}(m) = \lambda_1^3 h^*_{111} + 3\lambda_1^2 \lambda_2 h^*_{112} + 3\lambda_1\lambda_2^2 h^*_{122} + \lambda_2^3 h^*_{222}.$$

Hence m is a cusp point of P o f if and only if

$$\lambda_1^2 h^*_{11} + 2\lambda_1\lambda_2 h^*_{12} + \lambda_2^2 h^*_{22} = 0$$

$$\lambda_1^2 h^*_{12} + \lambda_1\lambda_2 (h^*_{22} - h^*_{11}) - \lambda_2^2 h^*_{12} \neq 0$$

$$\lambda_1^3 h^*_{111} + 3\lambda_1^2 \lambda_2 h^*_{112} + 3\lambda_1\lambda_2^2 h^*_{122} + \lambda_2^3 h^*_{222} \neq 0$$

(III) By the same token, P o f <u>is rank good at</u> m if and only if

<u>either</u>

$$\lambda_1^2 h^*_{11} + 2\lambda_1\lambda_2 h^*_{12} + \lambda_2^2 h^*_{22} \neq 0$$

<u>or</u>

$$\lambda_1^2 h^*_{12} + \lambda_1\lambda_2 (h^*_{22} - h^*_{11}) - \lambda_2^2 h^*_{12} \neq 0.$$

Now, adopting as above, a coordinate system $\{x_1, x_2\}$ on M such that $\frac{\partial}{\partial x_1}(m)$, $\frac{\partial}{\partial x_2}(m)$ are principal directions and denoting h^*_{11}, h^*_{22} in this system by k_1, k_2, the principal curvatures, one may reinterpret (I, II, III) by the following geometrical forms.

$\lambda_1^2 k_1 + \lambda_2^2 k_2 = 0$ if and only if $\lambda_1 \frac{\partial}{\partial x_1}(m) + \lambda_2 \frac{\partial}{\partial x_2}(m)$ is an asymptotic direction at m of f.

$\lambda_1\lambda_2(k_2 - k_1) = 0$ if and only if <u>either</u> m is an umbilic point of f, <u>or</u> $\lambda_1 \frac{\partial}{\partial x_1}(m) + \lambda_2 \frac{\partial}{\partial x_2}(m)$ is a principal direction of f at m.

Note that an elliptic point has no asymptotic directions, and a hyperbolic point has two asymptotic directions. A parabolic point has one asymptotic direction which is also principal, with principal curvature zero. At a flat umbilic point all directions are both principal and asymptotic.

<u>Proposition 9.5</u> <u>If</u> $P \circ f : M^2 \to \Pi^2$ <u>has</u> $m \in M$ <u>as a singular point, then</u>

(i) m <u>is a fold point of</u> $P \circ f$ <u>if and only if the</u> Df-<u>preimage of</u> <u>z</u> <u>in</u> T_mM <u>is not an asymptotic direction of</u> f <u>at</u> m.

(ii) m <u>is a cusp point of</u> $P \circ f$ <u>if and only if the</u> Df-<u>preimage of</u> <u>z</u> <u>in</u> T_mM <u>is a nonprincipal, asymptotic direction of</u> f <u>at</u> m, <u>which is not a zero of the cubic form</u> $D^3(\langle \underline{n}^*, f \rangle)(m)$.

be made to orientable manifolds of even dimension. The formula would be one which related the 'winding numbers' of the images of the crease curves with the Euler number of the manifold. The methods of Proposition 8.11 do this modulo 2.

Finally, it would be satisfactory to describe, if possible, a reasonable class of immersions of surfaces in $\mathbb{R}^3$ which are 2-good in the sense that almost all their planar projections are Whitney maps, and hopefully to extend such a result and the analysis of §9 to immersions and more general smooth maps of manifolds of arbitrary dimension.

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