Total p-th curvature and foliations and connections

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Ph.D. Thesis

Total p-th Curvature

and

Foliations and Connections

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ABSTRACT.

This thesis is in two parts. In Part I we consider integrals of the p-th power of the total curvature of a manifold immersed in R^n and thus introduce the notions of total p-th curvature and p-convex. This generalises the ideas of total curvature(which corresponds to total 1st curvature)and tight(which corresponds to 1-convex)introduced by Chern, Lashof , and Kuiper.

We find lower bounds for the total p-th curvature in terms of the betti numbers of the immersed manifold and describe p-convex spheres. We also give some properties of 2-convex surfaces.

Finally, through a discussion of volume preserving transformations of R^n we are able to characterise those transformations which preserve the total p-th curvature (when $p>1$)as the isometries of R^n.

Part II is concerned with the theory of foliations. Three groups associated with a leaf of a foliation are described. They are all factor groups of the fundamental group of the leaf: the Ehresmann group, the holonomy group of A.G.Walker, and the "Jet group". This Jet group is introduced as the group of transformations of the fibres of a suitable bundle induced by lifting closed loops on the leaf, and also by a geometric method which gives a means of calculating them.

The relationship between these groups is discussed in a series of examples and the holonomy groups and Jet groups of each leaf are shown to be isomorphic. The holonomy group of a leaf is shown to be not a Lie group and, when the foliation is of codimension 1, it is proved that the holonomy group is a factor group of the first homology group with integer coefficients and has a torsion subgroup which is either trivial or of order 2.

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PART II. Foliations and Connections

PART I

Total p-th Curvature

SUMMARY

Our object is to study integrals of the curvature of a manifold in a way which extends the known results in the topic of "total absolute curvature". All the work is original unless otherwise stated. That is, chapters II-V and a little of chapter I is original.

In chapter I we define the total curvature of an immersion $f:M^n \to R^{n+N}$ by the integral $\tau(f) = \int_{M^n} K(m)\, \omega_n$ where K is the total curvature at $m \in M^n$ and ω_n is the riemannian volume form on M^n induced from the euclidean metric on R^{n+N}. M^n is assumed to be closed, compact, C^2 and orientable. We then state the main theorems about $\tau(f)$ and show how the theory extends to non-orientable manifolds.

We show that for any M^n there are immersions, f, for which $\tau(f)$ is arbitrarily large, i.e. there are no theorems of the type "$\tau(f) \leq$ constant for all f". On the other hand the infimum of $\tau(f)$, over all immersions f, is an interesting number; it is a differential invariant of M^n. Immersions for which inf $\tau(f)$ is attained are called "tight". In chapter II the total p-th curvature of f is defined as $\tau_p(f) = \int_{M^n} K^p(m)\, \omega_n$. As $K^p\omega_n$ is not dimension free there are no theorems for τ_p which generalise those we have stated for τ. However we show that if the immersions considered are such that the volume of $f(M^n)$ in R^{n+N} is fixed, analogues of the theorems true for τ are true for τ_p. The infimum of $\tau_p(f)$ is a differential invariant of M^n for every p and immersions for which this infimum is attained are called "p-convex". 1-convex

is equivalent to tight. We also prove that $\tau_p(f)$ is a convex function of p.

Total curvature has been generalised by, (1) immersing M^n in an arbitrary riemannian manifold, (2) integrating a curvature other than the total curvature. Where there are theorems in these more general situations, they extend to include powers of curvatures. For example we show that if M^2 is a smooth closed compact surface in R^3 with area 4π and mean curvature H, $\int_{M^2} |H|^p \omega_2 \geqslant 4\pi$ for all $p \geqslant 2$, equality being attained when M^2 is the unit sphere. (The case p = 2 is due to Willmore).

Just as sup $\tau(f)$ was unbounded we show, in chapter III, that sup $\tau_p(f)$ is unbounded for all values of $p \neq 0$. Also, it is proved that inf $\tau_p(f) = 0$ for $p < 0$ and $0 < p < 1$ and that there are no immersions for which these suprema and infima are attained.

The total p-th curvature and p-convexity of curves and surfaces is examined in chapter IV. We describe p-convex curves and spheres for all possible values of p and show that there is no 2-convex torus of revolution in R^3. We then examine 2-convex surfaces in R^3 by variational techniques, deriving equations which 2-convex immersions must satisfy. We thus prove that there is no 2-convex torus with one principal curvature constant.

In chapter V we investigate the group of transformations which preserve p-convexity. As p-convexity is defined via immersions with a fixed volume we are led to a study of transformations of a riemannian manifold which preserve the volume of every k-dimensional submanifold. These "k-volume preserving" maps are defined both locally and globally and the two definitions are shown to be equivalent. Finally, these maps are shown to be isometries and it then follows that the p-convexity preserving transformation group is the isometry group.

CHAPTER I

Total Curvature and Tight Immersions

§1. Introduction

One of the most important results of classical Differential Geometry is the Gauss-Bonnet Theorem. If M^2 is a C^2 surface (closed and compact) in R^3 with Gauss curvature K and surface volume element dA,

$$\int_{M^2} KdA = 2\pi\ \chi(M^2)$$

where $\chi(M^2)$ is the Euler characteristic of M^2.

The result is so striking that one immediately asks if there are analogues for manifolds of arbitrary dimension and other types of curvature, (the mean curvature, for example). With a suitable change in the constant 2π and a choice of the curvature K the theorem is indeed true for manifolds of even dimension greater than two, (the most comprehensive statement of the Gauss-Bonnet theorem in this generalisation is due to S. S. Chern [1]) and at present there are, broadly speaking, two "other types of curvature" that have been considered. One is the Differential Geometric approach to characteristic classes given by S. S. Chern (especially in [2] and see also Kobayashi and Nomizu [2]) in which the Gauss-Bonnet theorem appears almost as a special case.

The other type of curvature, total curvature, has its roots in the following theorem of Fenchel [1]. If γ is a closed C^2 curve in R^3, k is the Serret-Frenet curvature and ds is the line element,

$$\int_{\gamma} |k| ds \geq 2\pi$$

equality being attained when γ is a plane convex curve. This theorem, which was later generalised to curves in R^n by Borsuk [1], with its inequality and special curve in the case of equality, is the prototype for all the theorems which followed.

This chapter is concerned with the statement of these theorems in the form we shall use them later. Our ultimate purpose is to extend them to theorems about yet another "type of curvature", p-th curvature.

§2. Total Curvature

M^n will be a closed, compact, orientable, C^2, n-manifold.

Let $f: M^n \to R^{n+N}$ be a C^2 immersion of M^n into euclidean space of dimension $n + N$; that is the induced map f_* of tangent spaces is a monomorphism (or, equivalently, f has non-zero Jacobian).

Following Chern-Lashof [1], we denote by $B_f \xrightarrow{\pi} M^n$ the <u>unit normal bundle of f</u>. B_f is the subset of $T(R^{n+N}) \times M^n$ of pairs (ξ, m), where $m \in M^n$ and ξ is a unit normal vector to $f(M^n)$ in R^{n+N} at $f(m)$. π is the natural projection $T(R^{n+N}) \times M^n \to M^n$, into the second factor, restricted to B_f.

For each point (ξ, m) of B_f, the unit vector ξ at $f(m)$ may be identified, by euclidean parallel translation, with a unit vector at the origin of R^{n+N}. If we denote the unit hyper sphere at the origin of R^{n+N} by S_o^{n+N-1} we thus have a map

$$\gamma : B_f \to S_o^{n+N-1}$$

which is called the <u>Gauss map</u> of the immersion f.

We denote by g the euclidean metric on R^{n+N}. Because f is an immersion there is a natural injection of $T(M^n)$ into $T(R^{n+N})$ and hence a natural restriction of $g : T(R^{n+N}) \times T(R^{n+N}) \to R$ to a map $T(M^n) \times T(M^n) \to R$ which we denote by $g|_{M^n}$.

$g|_{M^n}$ is a riemannian metric on M^n, and it gives rise to the <u>riemannian volume element</u>, ω_n, on M^n, associated with it. Let $(x^1, \ldots, x^n)$ be a coordinate system about $m \in M$ (that is, a chart $(x^1, \ldots, x^n) : U \to R^n$, such that U is an open neighbourhood of m) and let

$g_{ij}\Big|_{(x^i)\,|\,M^n}$ be the metric $g\big|_{M^n}$ expressed with respect to these coordinates,

then

$$\omega_n = f^* \det\left(g_{ij}\Big|_{(x^i)\,|\,M^n} \right)^{1/2} dx^1 \wedge \ldots \wedge dx^n$$

ω_n is independent of the coordinate system chosen for if $(y^1,\ldots,y^n)$ is also a coordinate system about m

$$g_{ij}\Big|_{(y^i)\,|\,M^n} = g_{pq}\Big|_{(x^i)\,|\,M^n} \frac{\partial x^p}{\partial y^i} \frac{\partial x^q}{\partial y^j}, \text{ and}$$

$$dy^1 \wedge \ldots \wedge dy^n = \frac{\partial y^1}{\partial x^i} dx^i \wedge \ldots \wedge \frac{\partial y^n}{\partial x^i} dx^i$$

$$= \det\left(\frac{\partial y^i}{\partial x^j} \right) dx^1 \wedge \ldots \wedge dx^n$$

Hence,

$$\det\left(g_{ij}\Big|_{(y^i)\,|\,M^n} \right)^{1/2} dy^1 \wedge \ldots \wedge dy^n$$

$$= \det\left(g_{pq}\Big|_{(x^i)\,|\,M^n} \frac{\partial x^p}{\partial y^i} \frac{\partial x^q}{\partial y^j} \right)^{1/2} \det\left(\frac{\partial y^i}{\partial x^j} \right) dx^1 \wedge \ldots \wedge dx^n$$

$$= \det\left(g_{ij}\Big|_{(x^i)\,|\,M^n} \right) \left| \det\left(\frac{\partial x^i}{\partial y^j} \right) \right| \det\left(\frac{\partial y^i}{\partial x^j} \right) dx^1 \wedge \ldots \wedge dx^n$$

$$= \pm \det\left(g_{ij}\Big|_{(x^i)\,|\,M^n} \right) dx^1 \wedge \ldots \wedge dx^n$$

The + and - signs occurring when the Jacobian of the transformation $y^i(x^1,\ldots,x^n)$ is respectively positive or negative. If M^n is orientable there exists an atlas in which every such Jacobian is positive (Kobayashi and Nomizu [1], Volume I, page 3) and so ω_n may be defined by means of this atlas.

Now, each fibre $\pi^{-1}(m)$ of B_f is an (N-1)-sphere in R^{n+N} and so has induced on it a volume form, σ_{N-1}, whose construction is the same as ω_n for M^n. Also, B_f is a product and hence $\sigma_{N-1}\wedge\omega_n$ is a volume form for B_f. We also construct the volume form Σ_{n+N-1} on S_0^{n+N-1}.

$\sigma_{N-1}\wedge\omega_n$ and $\gamma^*\Sigma_{n+N-1}$ are both (n+N-1)-forms on the (n+N-1)-dimensional manifold B_f. These two forms must therefore differ by a real valued function on B_f. We put

$$\gamma^*\Sigma_{n+N-1} = G(\xi,m)\,\sigma_{N-1}\wedge\omega_n$$

$G:B_f \to R$ is the <u>Lipschitz-Killing Curvature</u> of $f:M^n \to R^{n+N}$ at m in the direction ξ. In co-dimension 1 G is called the <u>Gauss-Kroneker Curvature</u> and is equal to the classical Gauss curvature when n = 2 and N = 1.

The <u>Total Curvature, K(m), at m</u> is defined by

$$K(m) = \int_{\pi^{-1}(m)} |G(\xi,m)|\,\sigma_{N-1}$$

and the <u>Total Curvature of M^n</u> is defined by

$$\tau(M^n,f,R^{n+N}) = \int_{M^n} K\omega_n$$

$$= \int_{B_f} |G(\xi,m)|\,\sigma_{N-1}\wedge\omega_n$$

From the outset we have assumed that M^n is orientable, because only then is ω_n globally defined and the integrals above are meaningful. However we may deal with non-orientable manifolds as follows: (see for example Abraham [1]).

Let $\tilde{M}^n$ be the orientable 2-fold cover of M^n and for a given immersion $f:M^n \to R^{n+N}$ let $\tilde{f}$ be the map given by the following commutative diagram,

$$\begin{tikzcd}
\tilde{M}^n \arrow[d, "p"'] \arrow[dr, "\tilde{f}"] & \\
M^n \arrow[r, "f"'] & R^{n+N}
\end{tikzcd}$$

where p is the usual projection.

The volume form on $\tilde{M}^n$ is taken to be $\tilde{f}^*\omega_n$ and G, being a function $B_f \to R$ is easily "lifted" to a function $\tilde{G}$ on $B_{\tilde{f}}$ by defining $\tilde{G}(\tilde{f}^*\xi,m) = G(\xi,p(m))$. ($\tilde{f}^*$ is here the usual induced bundle map; see Husemoller [1] p.18, and $\tilde{f}^*$ is clearly a monomorphism so that $\tilde{G}$ is defined on the whole of $B_{\tilde{f}}$).

The total curvature is then defined as

$$\tau(M^n,f,R^{n+N}) = \tfrac{1}{2}\,\tau(\tilde{M}^n,\tilde{f},R^{n+N})$$

When M^n is orientable the orientable two-fold cover is $M^n \vee M^n$ (disjoint union). This explains the appearance of $\frac{1}{2}$.

§3. Main Theorems

Let ϕ be a C^2, real-valued, function on a C^2 manifold M^n. A point $m \in M^n$ is a critical point of ϕ if $d\phi = 0$ at m. Equivalently m is a critical point of ϕ if

$$\frac{\partial\phi}{\partial x^1}(m) = \ldots = \frac{\partial\phi}{\partial x^n}(m) = 0$$

with respect to a local coordinate system $(x^1,\ldots,x^n)$ about m.

A critical point m is said to be non-degenerate if the $n \times n$ matrix $\frac{\partial^2\phi}{\partial x^i\,\partial x^j}(m)$ is non-singular, and a non-degenerate critical point is said to have index k if the symmetric matrix $\frac{\partial^2\phi}{\partial x^i\,\partial x^j}(m)$ has index k. That is a coordinate system $(y^1,\ldots,y^n)$ exists in which $\frac{\partial^2\phi}{\partial y^i\,\partial y^j}$ is the matrix $\left(\begin{array}{c|c} I_{n-k} & 0 \\ \hline 0 & -I_k \end{array}\right)$ where I_p is the unit $p \times p$ matrix.

ϕ is a Morse function on M^n if all its critical points are non-degenerate. It is well known that any C^2 manifold supports Morse functions (see Milnor [1]).

Definitions

$\Phi(M^n)$ is the set of Morse functions on M^n.

$\beta_k(\phi)$ is the number of critical points of index k of ϕ, and

$\beta(\phi) = \sum_{k=0}^{n} \beta_k(\phi)$, i.e. the total number of non-degenerate critical points of ϕ.

$$\beta_k(M^n) = \min_{\phi \in \Phi(M^n)} \{\beta_k(\phi)\}$$

$$\beta(M^n) = \min_{\phi \in \Phi(M^n)} \{\beta(\phi)\}$$

Let F be a field, then the k-th betti number $b_k(M^n;F) = \dim_F H_k(M^n;F)$

where $H_k(M^n;F)$ is the k-th homology module with coefficients in F and $\dim_F$ is the dimension as an F-module.

$$b(M^n;F) = \sum_{k=0}^{n} b_k(M^n;F) \text{ and}$$

$$b(M^n) = \underset{\text{all fields } F}{\text{maximum}} \{b(M^n;F)\}.$$

We remark that $\beta(M^n) \geq \sum_{k=0}^{n} \beta_k(M^n)$ is clear and that $\sum_{k=0}^{n} \beta_k(M^n) \geq b(M^n)$ is well known as a Morse inequality (see Milnor [1]). Taken together these inequalities imply $\beta(M^n) \geq b(M^n)$.

For a given M^n, $\tau(M^n,f,R^{n+N})$ clearly varies with f. However we have the following theorem:

<u>Theorem (Kuiper)</u>. (See Kuiper [1] and Wilson [1]).

$\inf \tau(M^n,f,R^{n+N}) = \beta(M^n)\, c_{n+N-1}$, where the infimum is taken over all C^2 immersions $f:M^n \to R^{n+N}$ and c_{n+N-1} is the volume of the unit hypersphere in R^{n+N}. In the notation of §2,

$$c_n = \int_{S^n} \Sigma_n$$

For example, $c_0 = 2$, $c_1 = 2\pi$, $c_2 = 4\pi$ and in general

$$c_n = \frac{(n+1)\pi^{(n+1)/2}}{\Gamma\left(\frac{n}{2} + \frac{3}{2}\right)} \quad \text{(see Flanders [1]).}$$

<u>Definition</u>

A C^2 immersion is <u>tight</u> if $\tau(M^n,f,R^{n+N}) = \beta(M^n)$

<u>Note 1</u>.

Such immersions were originally termed "of minimal total (absolute) curvature" for obvious reasons, and "minimal" for short (e.g. in Kuiper [1]). The first of these has been dropped for reasons of euphony and the second for ambiguity. A minimal immersion is usually one for which the mean curvature is zero everywhere. Other terms which have been used are "convex" and "generalised convex" (e.g. in Kuiper [3]). See below for motivation. As we shall later introduce the notion of p-convex we have chosen to call the above immersions "tight" (as does Kuiper in [4]).

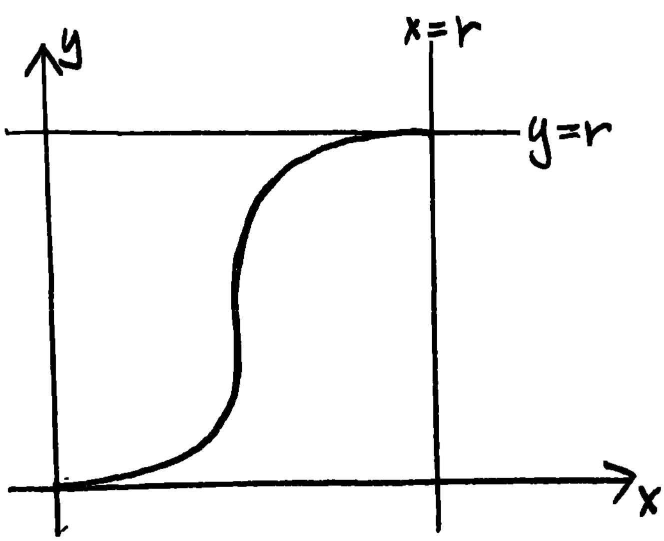
Note 2.

We remark that sup $\tau(M^n, f, R^{n+N})$ is of no interest.

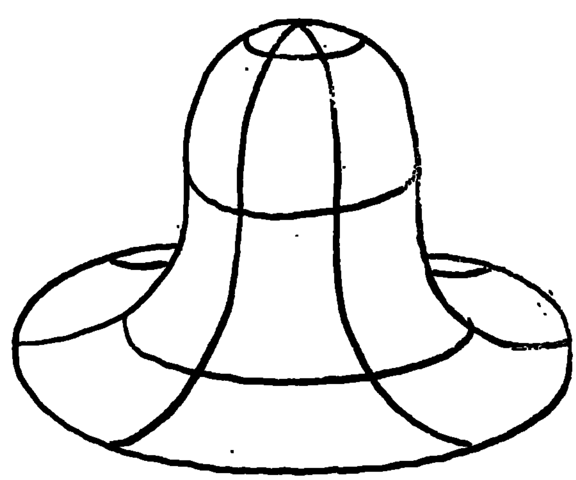
Consider the C^∞ surface in R^3 which is homeomorphic to a disc and is generated by the rotation about the "y-axis" of R^3 by the curve given by

$$y(x) = \begin{cases} re^{-(1-x)^2/x^2} & \text{when } 0 < x \leq 1 \\ 0 & \text{when } x = 0 \end{cases}$$

Picture of Curve



Picture of Surface



The total curvature of this "dome" surface is independent of r and equal to 4π. This calculation is most easily done using Kuiper's formula in [2], p.8.

Now, any 2-manifold can be immersed in R^3 so that some part of it lies in a plane. An arbitrarily large number of circular discs of radius r may then be removed from this planar region and replaced by "domes" and the differentiability class of the surface remains the same. If n "domes" are introduced the total curvature of the immersion is increased by $4\pi n$. Furthermore, an arbitrarily large number can be introduced simply by making r small. Consequently, $\sup \tau(M^2, f, R^3) = \infty$ for all 2-manifolds M^2. Similar procedures for n-manifolds will yield the same result.

We now return to the mainstream of the discussion.

M^n is compact and so $\beta_0(\phi) \geqslant 1$ and $\beta_n(\phi) \geqslant 1$ for any morse function ϕ. (ϕ is continuous and M^n is compact and so ϕ has a maximum and a minimum. Morse's lemma (see Milnor [1]) then gives $\beta_0(\phi)$ and $\beta_n(\phi) \geqslant 1$). Hence $\tau(M^n, f, R^{n+N}) \geqslant 2\, C_{n+N-1}$ by the theorem of Kuiper. The next theorem is concerned with the conditions under which equality is attained or nearly attained.

<u>Theorem</u> (Chern-Lashof [1])

(i) $\tau(M^n, f, R^{n+N}) = 2\, C_{n+N-1} \Longleftrightarrow M^n$ is embedded in sum [illegible]1) - dimensional linear subspace of R^{n+N} as a convex hypersphere.

(ii) $\tau(M^n, f, R^{n+N}) < 3\, C_{n+N-1} \Rightarrow M^n$ is homeomorphic to the n-dimensional sphere, S^n, with the usual differentiable structure.

The theorems of this section will always be quoted in the following form:

<u>Theorem 1</u>

$\tau(M^n, f, R^{n+N}) \geq 2\, c_{n+N-1}$, equality being attained when $f(M^n)$ is a convex hypersphere in some $(n+1)$-dimensional linear subspace of R^{n+N}.

<u>Theorem 2</u>

$\tau(M^n, f, R^{n+N}) < 3\, c_{n+N-1} \Rightarrow M^n$ is homeomorphic to the standard n-sphere.

§4. The Total Curvature of Surfaces

In the previous section it was pointed out that $\sum_{k=0}^{n} \beta_k(M^n) \geq b(M^n)$. If equality holds then theorem 1 is equivalent to

$$\inf \tau(M^n, f, R^{n+N}) = b(M^n)\, c_{n+N-1}$$

Thus relating the total curvature to the well known betti numbers. Manifolds for which $\sum_{k=0}^{n} \beta_k(M^n) = b(M^n)$ are therefore of interest so we shall show that all surfaces (2-manifolds which are closed, compact, etc.) have this property and derive the form of Theorem 1 which will be used in Chapter IV.

Any orientable surface T_g, where $g = 0,1,2, \ldots,$ is obtained by adjoining g orientable handles to a sphere. (See Greenberg [1]). T_0 is the sphere and T_g is the g-fold torus. g is the genus of T_g.

Any non-orientable surface U_h, where $h = 0,1,2, \ldots,$ is obtained by adjoining h ~~"twisted" handles or~~ cross caps to a sphere. U_0 is the sphere, U_1 is the real projective plane, U_2 is the klein bottle, etc.

The homology of these surfaces is

$$H_*(T_g;F) \cong F,\ F^{2g},\ F$$

$$H_*(U_h;F) \cong F,\ F^{h-1} \times F/(2),\ F_2$$

where $F^{2g} = \underbrace{F \times \ldots \times F}_{2g \text{ times}}$, (2) is the ideal generated by 1 + 1 and F_2 is the set of elements annihilated by left multiplication by 2; $F_2 = \{a \in F: (1+1)\, a = 0\}$.

Hence $b(T_g;F) = 1 + 2g + 1 = 2(g+1)$ for all fields F and $\inf (T_g, f, R^{2+N}) = 2(g+1)\, c_{N+1}$.

If F has characteristic not equal to 2 then $(2) = F$ and $F/(2) \cong 0$. Also $F_2 = 0$ so that

$$H_*(U_h;F) \cong F,\ F^{h-1},\ 0$$

and
$$b(U_h;F) = 1 + (h-1) = h$$

If F has characteristic 2 then $(2) = 0$ and $F/(2) \cong F$. Also $F_2 = F$ so that

$$H_*(U_h;F) \cong F,\ F^h,\ F$$

and
$$b(U_h;F) = 1 + h + 1 = h + 2$$

Hence $b(U_h) = b(U_h;F)$ when F has characteristic 2, for example $F = Z_2$, and

$$\inf\ (U_h, f, R^{2+N}) = (h+2)\ c_{N+1}$$

We note that

$$\chi(T_g) = b_2 - b_1 + b_o = 2 - 2g$$

and
$$\chi(U_h) = 2 - h \qquad \text{hence}$$

$$b(M^2) = 4 - \chi(M^2) \text{ for all surfaces and}$$

$$\inf\ \tau(M^2, f, R^{2+N}) = 2\pi\ (4-\chi(M^2))$$

CHAPTER II

Total p-th Curvature and p-Convex Immersions

Note

Total p-th Curvature is different from the p-th total curvature of Bang-yen Chen [1].

§1. Total p-th Curvature

We have defined the total curvature of an immersion $f:M^n \to R^{n+N}$ to be

$$\tau(M^n,f,R^{n+N}) = \int_{M^n} K\ \omega_n$$

and we wish to generalise this and consider integrals of the type $\int K^p$ where p is any real number.

Definition

The total p-th curvature of an immersion $f:M^n \to R^{n+N}$ is

$$\tau_p(M^n,f,R^{n+N}) = \int_{M^n} K^p\ \omega_n \qquad \text{where } p \in R$$

We shall refer to $\tau_p(f)$ or just τ_p when the reference to M^n, R^{n+N} or f is clear (or irrelevant, as in the next sentence).

$\tau_1 = \tau$ and so total 1st curvature is the same as total curvature.

Because $\tau = \int_{B_f} |G|\ \sigma_{N-1} \wedge \omega_n$ a definition of τ_p which has claims to being a generalisation of total curvature is

$$\tau_p = \int_{B_f} |G|^p \, \sigma_{N-1} \wedge \omega_n$$

However the generalised notion of tightness, p-convexity, turns out to be useless if we take this as our definition of τ_p. A remark which points out the exact difficulty will be made later. (p.23).

Our aim now is to prove for τ_p analogues of all the theorems we have stated for $\tau = \tau_1$. Consequently we examine the infimum and supremum of $\tau_p(f)$ over all immersions f.

Let $f:M^n \to R^{n+N}$, $\alpha \in R$ such that $\alpha > 0$ and define $\alpha f:M^n \to R^{n+N}$ by $(\alpha f)(m) = \alpha(f(m))$ where $f(m)$ is considered as a position vector in R^{n+N} with origin O. If the Lipschitz-Killing curvature for f is G, the curvature for αf is $\alpha^{-n} G$. Also, if the volume form on M^n induced by f is ω_n, the volume form induced by αf is $\alpha^n \omega_n$. Hence $\tau_p(\alpha f) = \alpha^{(1-p)n} \tau_p(f)$. For example $\tau_1(\alpha f) = \tau_1(f)$ and the total curvature thus depends on the "shape" of $f(M^n)$ and not on its "size". If p is any real number other than one, $\tau_p(\alpha f)$ can take any value in the range $0 < \tau_p < \infty$ for fixed f hence $\inf \tau_p = 0$ and $\sup \tau_p = \infty$. If meaningful results are to be obtained we must restrict the class of immersions over which we take infima and suprema. This is done by restricting the "size" of $f(M^n)$ so that τ_p depends only on the "shape". This will then be in analogy with the case $p = 1$.

<u>Definition</u>

An immersion $f:M^n \to R^{n+N}$ has the <u>standard volume</u> if

$$\int_{M^n} \omega_n = 2\, c_{n+N-1}$$

We shall always consider immersions of this type for reasons which will become clear. There is no loss of generality in doing so, for let $g:M^n \to R^{n+N}$ be an immersion for which $\int_{M^n} \omega_n = 2\, \alpha^n\, c_{n+N-1}$ where $\alpha > 0$ is a suitable real number and let $f = \alpha^{-1} g$. $\tau_o(f) = \alpha^{-n}\, \tau_o(g) = 2\, c_{n+N-1}$ and so f has the standard volume. Also, $\tau_p(g) = \alpha^{(1-p)n}\, \tau_p(f)$ and therefore any results we may have about immersions with the standard volume always imply some result we may have about immersions with any (non-standard) volume. We shall also use the terms area and length for the volume of two and one-dimensional manifolds respectively.

We now introduce the property corresponding to tight:

Definition

An immersion $f:M^n \to R^n$ with the standard volume is p-convex if

$$\tau_p(f) = \inf_{g:M^n \to R^{n+N}} \{\tau_p(g)\} \qquad \text{where g has the standard volume.}$$

There may not exist a p-convex immersion of the manifold M^n. For example if M^n is a sphere with a non-standard differentiable structure and $N = 1$, $\inf \tau_p = 2\, c_n$ by Theorem 3 and this infimum is attained when the sphere is convex. But a convex sphere is the boundary of a disc and so has the usual differentiable structure. Hence there are no p-convex immersions of exotic spheres.

However $\inf \tau_p$ is still defined and is a differential invariant of M^n. This follows from the fact that the total pth-curvature of $f:M^n \to R^{n+N}$ is calculated from the properties of the point-set $f(M^n)$ so that if $\phi:N^n \to M^n$ is a diffeomorphism, $f_o\, \phi$ is an immersion of N^n with the same total pth-curvature as $f:M^n \to R^{n+N}$ and conversely.

Note 1.

$\tau_o = 2\, c_{n+N-1}$, hence every immersion is 0-convex.

Note 2.

$\tau_1 = \tau$, hence 1-convex $\Leftrightarrow$ tight.

Note 3.

In co-dimension 1, $K = \int |G|\, \sigma_o = 2|G|$. Hence,

$$\tau_p = 2^p \int_{M^n} |G|^p\, \omega_n$$

where G is now the Gauss-Kroneker curvature and the standard volume is $2\, c_n$. For example, the sphere of radius $2^{1/n}$ in an (n+1)-dimensional linear subspace of R^{n+N} has volume $2\, c_n$ and Gauss-Kroneker curvature $\frac{1}{2}$, so that $\tau_p = \int \omega_n = 2\, c_n$. Hence $\inf \tau_p = 2\, c_n$.

Note 4.

Theorems about τ can be put in the new notation.

Theorem 1 becomes:

$\tau_1 \geqslant \tau_o$, equality being attained when $f(M^n)$ is a convex hypersphere in some (n+1)-dimensional linear subspace of R^{n+N}.

Theorem 2 becomes:

$\tau_1 < \frac{3}{2}\, \tau_o \Rightarrow M^n$ is homeomorphic to the standard n-sphere.

We are now in a position to prove the analogues of these theorems for total p-th curvature.

§2. Main Theorems

Theorem 3

If $f:M^n \to R^{n+N}$ is a C^2 immersion of a closed, compact, connected, C^2 n-manifold, M^n, $\tau_p \geqslant \tau_o$ when $p \geqslant 1$. Equality occurs when $f(M^n)$ is a convex hypersphere in some (n+1)-dimensional subspace of R^{n+N}. If also $p > 1$, the total curvature of the immersion is equal to +1.

Lemma

If f is a square integrable function on M^n with measure ω_n and p is a real number greater than 1,

$$\frac{\int f^p \omega_n}{\int \omega_n} \geqslant \left(\frac{\int f \omega_n}{\int \omega_n}\right)^p$$ equality occuring if and only if f is constant except for a set of measure zero.

Proof

If g is also square integrable we have, by Hölders inequality (see Hardy, Littlewood and Polya [1] p.140),

$$\int f\, g\, \omega_n \leqslant (\int f^p \omega_n)^{\frac{1}{p}} (\int g^q \omega_n)^{\frac{1}{q}} \quad \text{for all } q,$$

equality occurring when f and g are proportional, except possibly on a set of measure zero.

On putting $g \equiv 1$ and $q = p/(p-1)$ we find

$$\int f \omega_n \leqslant (\int f^p \omega_n)^{\frac{1}{p}} (\int \omega_n)^{\frac{p-1}{p}}$$

Hence,

$$\int f^p \omega_n \geqslant (\int f \omega_n)^p / (\int \omega_n)^{p-1}$$ which gives the required

inequality. Equality occurs when f is proportional to 1 i.e. a constant.

Proof of Theorem 3

If we put f = K in the above lemma we obtain

$$\frac{\tau_p}{\tau_o} \geqslant \left(\frac{\tau_1}{\tau_o}\right)^p$$ equality occurring when K is constant

almost everywhere. However f is C^2 and hence K is continuous so that $K \equiv$ constant. By theorem 1 $\tau_1 \geqslant \tau_o$ and so $\tau_p \geqslant \tau_o$. Then $\tau_p = \tau_o$ implies $\tau_1 = \tau_o$ which implies $K \equiv 1$ and we have the theorem.

Note 1.

We can now see why the definition $\tau_p = \int_{B_f} |G|^p \sigma_{N-1} \wedge \omega_n$, which was suggested in §1, is of no use. With a suitable choice for the standard volume we can prove the theorem; $\tau_p \geqslant \tau_o$ with equality if and only if G is constant and $f(M^n)$ lies in an (n+1)-dimensional linear subspace of R^{n+N}. When N = 1 we can proceed to an investigation of convexity in a meaningful way. It is when N > 1 that difficulty occurs.

Let $g:M^n \to R^{n+N'}$ be an immersion such that $N' < N$ and let $i:R^{n+N'} \to R^{n+N}$ be an imbedding of $R^{n+N'}$ as a linear subspace of R^{N+N}. $i_o g:M^n \to R^{n+N}$ has the property that if ν is any vector in R^{n+N} at a point of $i_o g\ (M^n)$ normal to $i_o g\ (M^n)$ then $G(\nu,m) = 0$. This calculation is most easily carried out using the geometric interpretation of G given in Chern and Lashof [1], p.311. For example, if $g:S^1 \to R^2 \hookrightarrow R^3$ and $\underline{n}$ and $\underline{b}$ are respectively the normal and binormal to $g(S^1)$ in R^3, then $G(n,p) = k$ and $G(\underline{b},p) = 0$ for all $p \in S^1$. It follows that if $f:M^n \to R^{n+N}$ is the immersion in the above theorem, $G \equiv 0$. We have

$$\int_{B_f} |G| \, \sigma_{N-1} \wedge \omega_n \geq 2 \, c_{n+N-1}$$

and we conclude that such an f does not exist. Consequently, this definition of τ_p gives us a theory of convexity in codimension 1 (which happens to be the same as we have already) and no useful theory otherwise.

Note 2.

Hölders inequality also gives information about τ_p when $p < 1$; (the inequality is reversed) but we shall see in chapter III that inequalities cannot give the strongest results for such values of p.

Theorem 4

If f is an immersion of the type specified in theorem 3 and $\tau_p < (3/2)^p \, \tau_o$ for $p \geq 1$, then M^n is homeomorphic to a sphere.

Proof

By the lemma of theorem 3 we have

$$(3/2)^p > \frac{\tau_p}{\tau_o} \geq \left(\frac{\tau_1}{\tau_o}\right)^p \quad \text{and hence } \tau_1 < 3/2 \, \tau_o \text{ and the}$$

result follows from theorem 2.

We shall also need the following theorem later.

Theorem 5

τ_p is a convex function, that is $\frac{d\tau_p}{dp}$ exists and is an increasing function of p.

Proof

M^n is closed and compact and K is a bounded continuous function of M^n because M^n is C^2.

$$\frac{d}{dp}\ \tau_p = \frac{d}{dp} \int_{M^n} K^p\ \omega_n$$

$$= \int_{M^n} K^p \log_e K\ \omega_n$$

When $K < 1$; K^p is a decreasing function of p and $\log_e K < 0$, hence $K^p \log_e K$ is an increasing function of p.

When $K \geqslant 1$; K^p is an increasing function of p and $\log_e K \geqslant 0$, hence $K^p \log_e K$ is an increasing function of p.

$\frac{d}{dp}\ \tau_p$ is therefore an increasing function of p and the proof is complete.

Corollary

$$1 \leqslant p < q \Rightarrow \inf \tau_p \leqslant \inf \tau_q$$

Proof

$$1 \leqslant p < q \Rightarrow \tau_p(f) < \tau_q(f) \text{ for all } f:M^n \to R^{n+N}$$

§3. Other Total Curvatures

It is possible to define total curvature for immersions into an arbitrary riemannian manifold. (Willmore and Saleemi [1]). When this manifold is complete, simply connected with non-positive sectional curvature, many of the theorems on total curvature in euclidean space are still true. (Willmore and Saleemi [1]; Chen [2]). Consequently it is possible to define total p-th curvature and p-convexity in this more general context and prove all the theorems of §2.

Another generalisation of total curvature is "total mean curvature". The first result is due to Willmore [2]:

$$\int_{M^2} H^2\, \omega_2 \geq 4\pi$$

where H is the mean curvature of M^2. Equality holds if and only if M^2 is a round sphere.

If we put $\sigma_p = \frac{1}{2\pi} \int_{M^2} |H|^p$ and consider only those immersions whose area is 4π we may prove, by the same method used in Theorem 3;

$$\int_{M^2} |H|^p \geq 4\pi \qquad \text{for all } p \geq 2.$$ Equality holding when M^2 is a round sphere of unit radius.

Chen has shown that not only is Willmore's theorem true for manifolds of higher dimension, but there are theorems of this type for all the principal curvatures (Chen [1]). These theorems also generalise by our method.

We make a note on the problem of minimising the integral

$\int_{M^2} (K+\lambda)^2 \omega_2$ where λ is a constant.

$$\int_{M^2} (K+\lambda)^2 \omega_2 = \int_{M^2} K^2 \omega_2 + 2\lambda \int_{M^2} K \omega_2 + \lambda^2 \int_{M^2} \omega_2$$

$\int_{M^2} K \omega_2 = 2\pi \chi(M^2)$ by the Gauss-Bonnet theorem and $\int_{M^2} \omega_2 = 4\pi$. Hence $\int_{M^2} (K+\lambda)^2 \omega_2$ and $\int_{M^2} K^2 \omega_2$ differ by a constant and we conclude that: the infimum of the integrals $\int_{M^2} (K+\lambda)^2 \omega_2$ is attained, for all values of λ, by the <u>same</u> immersion of M^2.

CHAPTER III

Exceptional Cases

§1. Introduction

So far we have considered τ_p only for values of p greater than one and have sought only lower bounds, not upper bounds. We now justify our neglect in considering the case of p less than one and in finding suprema, by describing what happens in these cases.

To obtain results we shall give immersions only for surfaces in R^3. However, exactly analogous results hold for arbitrary submanifolds in R^{n+N} with very slight modification. (An indication of how this is done is given at the end of the first lemma). It has already been noted, in chapter I, that $\sup \tau_1 = \infty$ and this result extends to the case $p \geqslant 0$ with suitable adjustment to keep the volume of the immersion fixed.

§2. Exceptional values of the total p-th curvature

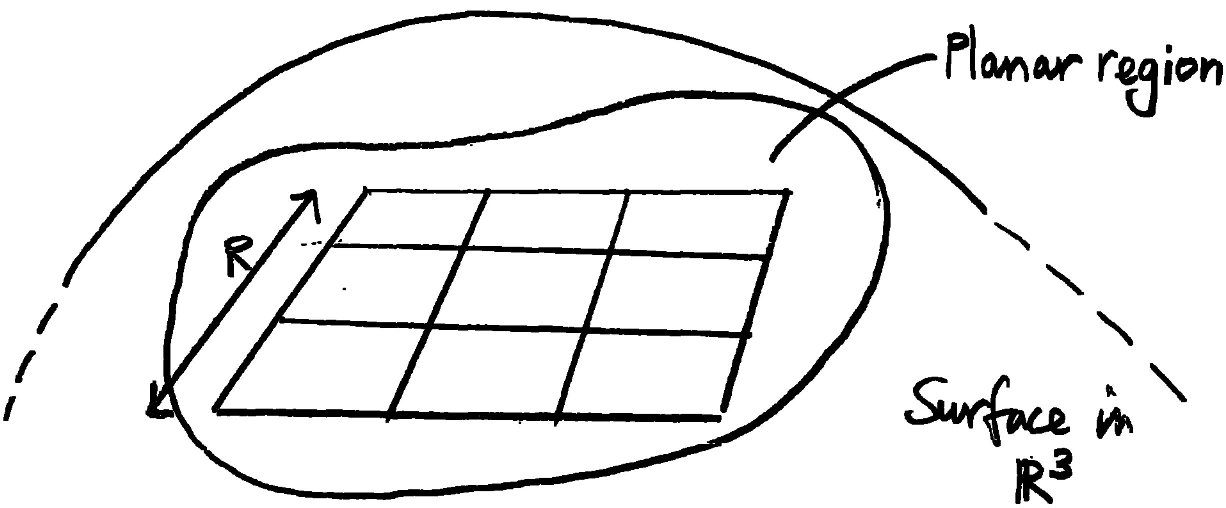
Lemma 1

$$\sup \tau_p = \infty \qquad \text{when } p \geq 0$$

Proof

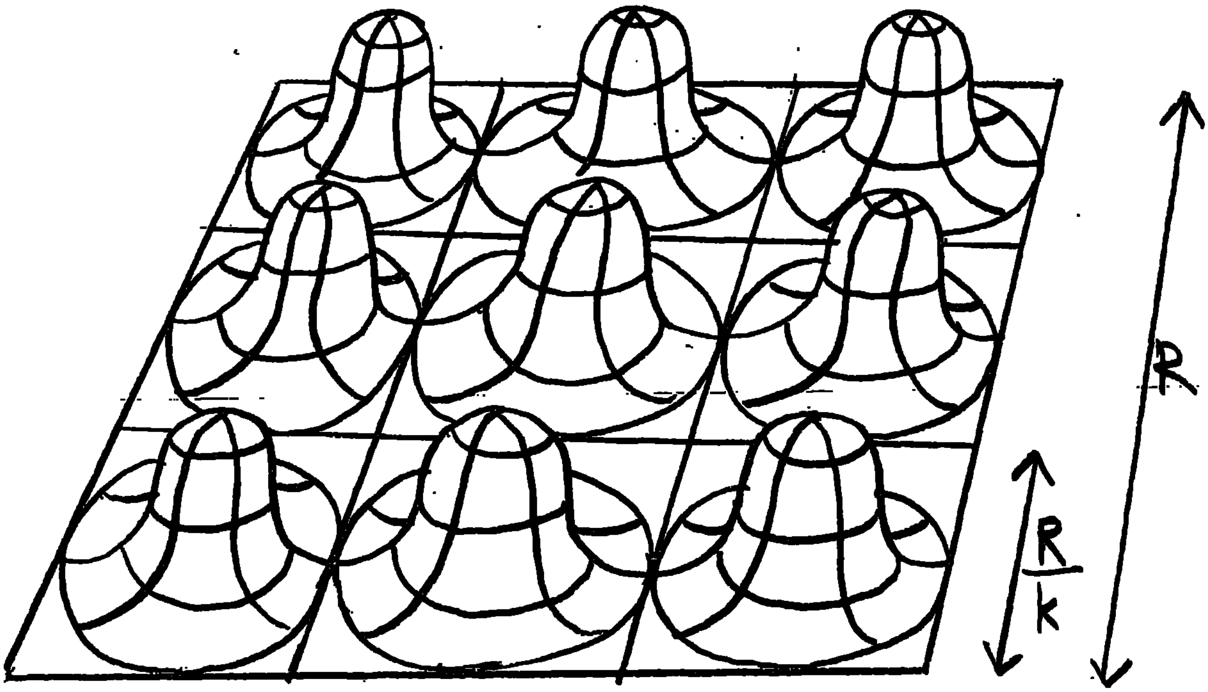
Let an immersion of any surface in R^3 be given for which there is a planar region, and consider a square, with sides of length R, in this region. Let the $R \times R$ square be divided up as a $k \times k$ "chess board". That is k^2 squares, each $\frac{R}{k} \times \frac{R}{k}$.

Picture for $k = 3$



In each $\frac{R}{k} \times \frac{R}{k}$ square remove a circular disc of diameter $\frac{R}{k}$ and replace it with a "dome" of the type described in §3. of chapter I.

Picture for $k = 3$



Let the area of a disc of radius r be Dr^2 (D is used instead of π for ease of generalisation to other dimensions) and the area of a dome on this disc be Vr^2. Then the total area of the surface contained within the boundary of the $R \times R$ square is

$$R^2 - k^2 D\left(\frac{R}{k}\right)^2 + k^2 V\left(\frac{R}{k}\right)^2 = R^2 \ (1-D+V)$$

The total area of the surface is independent of n when $n \geqslant 1$. Denoting the surface which is the original surface with the above modification by M_k we see that if M_1 is linearly expanded to give it a standard volume, the same expansion will give all the M_k the standard volume.

Now, calculation shows that the curvature at each point of a dome is of the order k^2, hence the total p-th curvature of each dome is of order $k^{2(p-1)}$ and the total curvature of all the domes is k^{2p}. When $p > 0$, k^{2p} can be made as large as we please by letting k tend to infinity. As all the M_k are homeomorphic we obtain $\sup \tau_p = \infty$ for all surfaces, as required.

To extend this result to manifolds of dimension n in R^{n+N} we first immerse M^n such that some region lies in a linear R^n in R^{n+N}. Domes are then constructed on this region in codimension 1 by rotating the generating curve about its axis in R^{n+1}. A consideration of the curvature in its geometric interpretation given in Chern and Lashof [1] shows that the curvature of a dome, constructed as above, has the same order of magnitude in all codimensions. Hence an increase in the number of domes gives an increase in the total curvature just as for the case $n = 2$, $N = 1$ given in Lemma 1.

<u>Lemma 2</u>

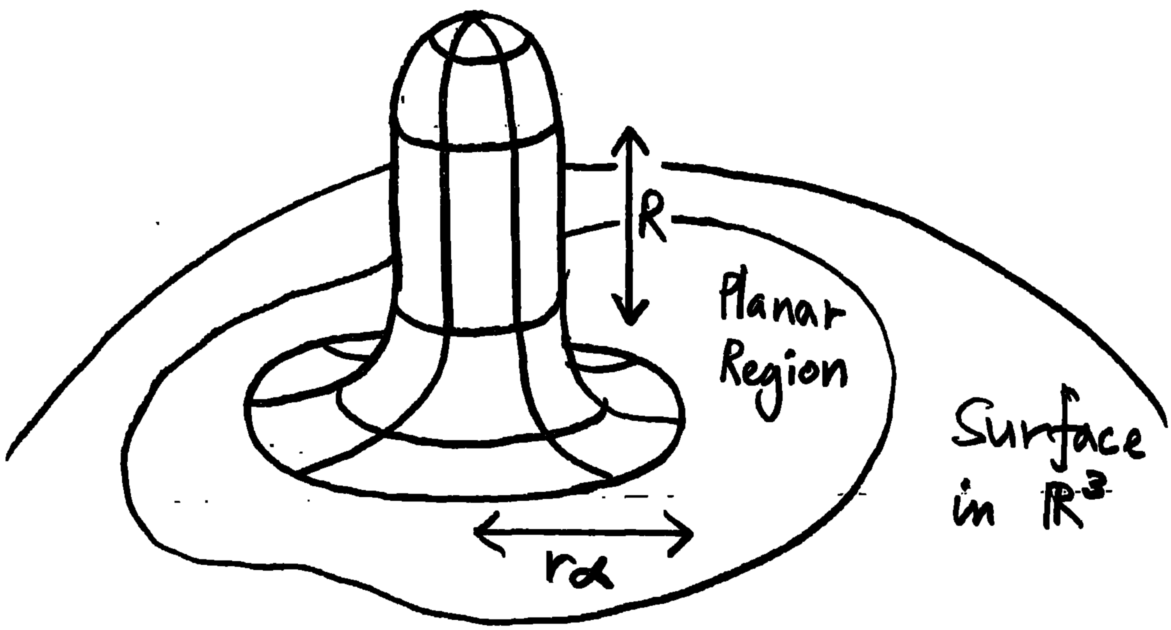
$$\inf \tau_p = 0 \quad \text{when } p < 0 \quad \text{and } 0 < p < 1.$$

<u>Proof</u>

Let $f:M^2 \to R^3$ be an immersion such that $f(M^2)$ has a planar region as in lemma 1. Define an immersion $f_\alpha:M^2 \to R^3$ by modifying αf as follows.

$\alpha f(M^2)$ will have a planar region and in this region let a disc of radius $r\alpha$ be removed and replaced by a "dome" of the type used in lemma 1 into which has been inserted a cylinder of length R.

Picture



R is chosen so that the immersed surface has the standard area and the final surface defines f_α. Clearly $f_\alpha(M^2)$ is a smooth immersion such that $f_\alpha(M^2)$ is homeomorphic to M^2 for all values of α such that $0 < \alpha < \alpha_0$ for some α_0.

Now the curvature of the cylinder is zero and so

$$\tau_p(f_\alpha) = \tau_p(\alpha f) + \text{p-th total curvature of a "dome"}.$$

But,

$$\tau_p(\alpha f) = \alpha^{(1-p)n}\, \tau_p(f) \quad \text{when } p \neq 0$$

and the p-th total curvature of a dome is of order $\alpha^{2(1-p)}$, as in lemma 1. When $p < 0$, $1-p > 0$ and so $\lim_{\alpha\to 0} \tau_p(f_\alpha) = 0$. The reasoning breaks down at $p = 1$ for then we have

$$\tau_p(f_\alpha) = \tau_p(\alpha f) + \text{volume of cylinder}$$

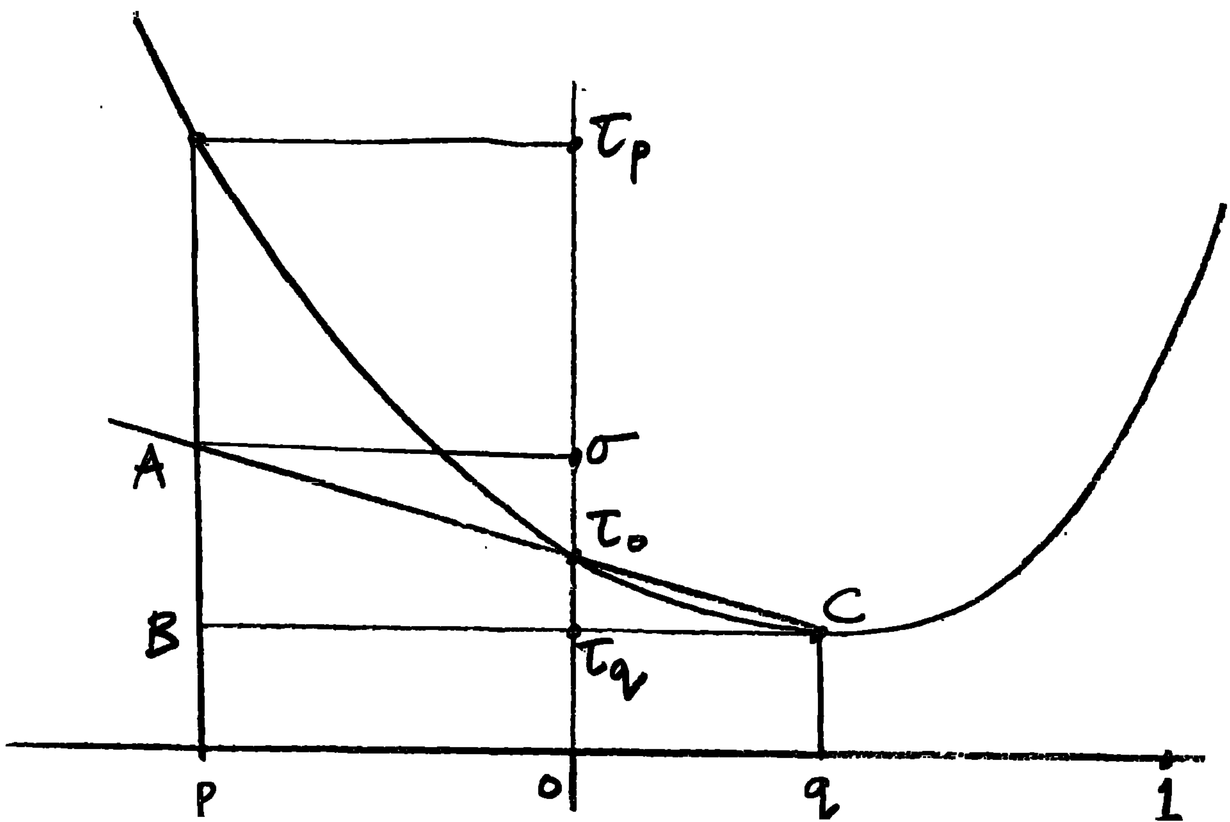
and the volume of the cylinder tends to the standard volume as α tends to zero. Hence $\inf \tau_p = 0$ over the desired range.

Lemma 3

$\operatorname{Sup} \tau_p = \infty$ when $p < 0$

Proof

Let q be a real number such that $0 < q < 1$. Then if $\phi < 0$, $\tau_p \geqslant \sigma$ where σ is given by the following diagram and using the convexity of τ_p as a function of p. (Chapter II, Theorem 5)



By considering the triangle ABC we see that

$$\frac{\sigma-\tau_q}{\tau_o-\tau_q} = \frac{q-p}{q}$$

Hence,

$$\sigma = \tau_q + (\tau_o-\tau_q)(q-p)/q \text{ , } \quad \text{and}$$

$$\tau_p \geqslant \tau_q + (\tau_o-\tau_q)(q-p)/q$$

By lemma 2 it is possible to choose an immersion such that $\tau_q \leqslant \frac{\tau_o}{2}$ for all q in the given range. Hence,

$$\tau_p > (\tau_o-\tau_q)(q-p)/q \geqslant \tau_o(q-p)/2q$$

Now, $\lim\limits_{q\to 0}\left(\frac{q-p}{q}\right) = \infty$ and so $\sup \tau_p = \infty$ for $p < 0$ as required.

We collect the results of the previous three lemmas into:

Theorem 6

For any closed compact C^2 manifold immersed in euclidean space,

$$\sup \tau_p = \infty \quad \text{for } p \neq 0$$

$$\inf \tau_p = 0 \quad \text{for } p < 1 \text{ and } p \neq 0$$

§3. Non-existence of immersions

We have the following corollaries to theorem 6.

Corollary 1

There are no p-convex immersions for $p < 1$ and $p \neq 0$.

Proof

Suppose $f: M^n \to R^{n+N}$ is a p-convex immersion where $p < 1$ and $p \neq 0$. By theorem b,

$$\int_{M^n} K^p \, \omega_n = 0 \quad \text{and so } K \equiv 0 \text{ on } M^n$$

But by Theorem 2 of Chapter 1 we have

$$\int_{M^n} K \, \omega_n \geqslant 2 \, c_{n+N-1} \quad \text{hence there does not exist such an } f.$$

Corollary 2

There are no immersions for which $\sup \tau_p$ is attained when $p \neq 0$.

Proof

By Theorem 1 $\sup \tau_p = \infty$ when $p \neq 0$. However f is C^2 and M^n is compact; hence K is bounded on a compact manifold and cannot attain values for which $\tau_p = \infty$.

CHAPTER IV

Curves and Surfaces

In this chapter we completely classify p-convex immersions of curves and spheres and strengthen the results of Chapter II in the case when M^n is a surface.

By surface we mean a closed, compact, C^2, two-dimensional manifold.

§1. Curves in R^{N+1}

Let $f:S^1 \to R^{1+N}$ be a C^2 immersion, then by the theorem of Borsuk

$\frac{1}{\pi} \int_{f(S^1)} |k| \, ds \geq 2$ where k is the curvature of $f(S^1)$ given by the Serret-Frenet formulae as usual.

If we consider immersions with length 2π and define

$$\tau_p = \frac{1}{\pi} \int_{f(S^1)} |k|^p \, ds$$

we have, by the methods of theorem 3,

Corollary 1

If $f:S^1 \to R^{1+N}$ is a C^2 immersion with length 2π

$\frac{1}{\pi} \int_{f(S^1)} |k|^p \, ds \geq 2$, equality being attained when $f(S^1)$ is $f(S^1)$

a plane unit circle for $p > 1$, and a plane convex circle for $p = 1$.

Proof

One only needs to check that if the total curvature of a plane curve is constant, then so is the "Serret-Frenet" curvature. This follows easily from Chern and Lashof's geometric interpretation of the Lipschitz-Killing curvature in [1].

Analogues of the immersions given in the previous chapter show that

$$\inf \tau_p = 0 \quad \text{for} \quad p < 1 \text{ and } p \neq 1$$

and $$\sup \tau_p = \infty \quad \text{for} \quad p \neq 0 \text{ as we should expect.}$$

We can now give the complete set of p-convex immersions of S^1 in R^{1+N} with length 2π.

	$\inf \tau_p$	p-convex immersion
$p < 0$	0	None
$p = 0$	2π	Any immersion
$0 < p < 1$	0	None
$p = 1$	2π	Plane convex circle
$p > 1$	2π	Plane unit circle

§2. Surfaces in R^3

Let $f:M^2 \to R^3$ be a C^2 immersion. We have, from Theorem 2,

$$\frac{1}{2\pi} \int_{M^2} |K| \, dA \geq b(M^2, Z_2) = 4 - \chi(M^2) \geq 2$$ where K is now the classical Gauss curvature.

If we define

$$\tau_p = \frac{1}{2\pi} \int_{M^2} |K|^p \, dA$$

we have for immersions with area 4π,

$$\tau_o = 2$$

$$\tau_1 = \tau \geq 2$$

By the lemma of theorem 3 we have

$$\frac{\tau_p}{\tau_o} \geq \left(\frac{\tau_1}{\tau_o}\right)^p = \frac{(4-\chi)^p}{2^p}$$ and hence

Corollary 2

If $f:M^2 \to R^3$ is a C^2 immersion with area 4π ,

$$\frac{1}{2\pi} \int_{M^2} |K|^p \, dA \geq \frac{(4-\chi(M^2))^p}{2^{p-1}} \geq 2$$ for $p \geq 1$, the equality $\frac{1}{2\pi} \int_{M^2} |K|^p \, dA \geq 2$ being attained when M^2 is homeomorphic to S^2. Also, S^2 is embedded as a unit sphere (respectively convex sphere) for $p > 1$ (respectively $p = 1$).

Proof

One only needs to check that a sphere with constant curvature in R^3 and with area 4π is the unit sphere. That this is so follows from the theorem of Liebmann that a closed orientable star-shaped surface with constant Gauss curvature is a sphere. (Guggenheimer [1] p.252).

W can now give the complete set of p-convex immersions of S^2 in R^3 with area 4π.

	$\inf_p$	p-convex immersion
$p < 0$	0	None
$p = 0$	4π	Any immersion
$0 < p < 1$	0	None
$p = 1$	4π	Convex sphere in some $R^3 \subset R^{2+N}$
$p > 1$	4π	Unit sphere in some $R^3 \subset R^{2+N}$

When M^2 is the torus, $\chi(M^2) = 0$ and corollary 2 gives $\frac{1}{2\pi} \int_{\text{Torus}} |K|^p \, dA > 2^{p+1}$. The inequality is strict because K can never be constant on a torus.

When $p = 2$ we have

$$\frac{1}{2\pi} \int_{\text{Torus}} K^2 > 8$$

This estimate can be improved by considering special cases of immersions.

Lemma 1

$$\frac{1}{2\pi} \int_{\text{Torus}} K^2 \, dA > \pi^2 \qquad (\pi^2 = 9.869 \ldots > 8)$$

Proof

Consider the torus of revolution in R^3 given by coordinates

$$S^1 \times S^1 \to R^3$$

$$(\theta,\phi) \to ((R+r \operatorname{Cos}\phi) \operatorname{Sin}\theta, (R+r \operatorname{Cos}\phi) \operatorname{Cos}\theta, r \operatorname{Sin}\phi)$$

(This torus is embedded when $r < R \Rightarrow r < \frac{1}{\sqrt{\pi}}$).

The Gauss curvature of this torus is

$$K = \frac{\operatorname{Cos}\phi}{r(R+r \operatorname{Cos}\phi)}$$

and an element of area is

$$dA = r(R+r \operatorname{Cos}\phi) \, d\phi \, d\theta$$

Hence

$$\int dA = \int_0^{2\pi} \int_0^{2\pi} r(R+r \operatorname{Cos}\phi) \, d\phi \, d\theta = 4\pi^2 r R$$

so that if the torus has area 4π, $R = \frac{1}{\pi r}$.

Denote by T_r the torus of revolution with $R = \frac{1}{\pi r}$, then

$$\int_{T_r} K^2 \, dA = \int_0^{2\pi} \int^{2\pi} \frac{\operatorname{Cos}^2\phi \, d\phi \, d\theta}{r(R+r \operatorname{Cos}\phi)}$$

$$= \frac{2}{r^4} \int_0^{2\pi} \frac{d\phi}{1+\pi r^2 \operatorname{Cos}\phi} - \frac{4\pi}{r^4}$$

This is well known to be equal to

$$\frac{2}{r^4}\left(\frac{2\pi}{\sqrt{1-\pi^2 r^4}}\right) - \frac{4\pi}{r^4}$$

and can be shown to take its minimum value of $2\pi^3$ at $r = 0$. Hence,

$$\int_{T_r} K^2 \, dA \geq 2\pi^3$$

That is,

$$\frac{1}{2\pi} \int_{\substack{\text{Torus of} \\ \text{revolution}}} K^2 \, dA \geq \pi^2$$

~~Also,~~ there is no value of r for which equality is attained and we obtained the required inequality.

Corollary

There is no 2-convex torus of revolution.

§3. 2-Convex Immersions in R^3

Let a closed orientable C^2 surface M^2 in R^3 with area 4π be given by the position vector r and denote by M^2_a the surface whose position vector is $r_a = r - aN$ where $a \in R$ and N is one of the two unit normal vector fields on M^2. The area 2-form and Gauss curvature on M^2_a are then given by

$$dA_a = (1+2\,H_a+K\,a^2)\;dA$$

$$K_a = K/(1+2\,H_a+K\,a^2)$$

where dA is the area 2-form on M^2 and H and K are the mean curvature and Gauss curvature of M^2 (see Willmore [1] p.117).

If the surface $M_{a,c}$ is given by the position vector cr_a, the area form and the mean and Gauss curvatures are given by

$$dA_{a,c} = c^2\;dA_a$$

$$K_{a,c} = K_a/c^2$$

$$H_{a,c} = H_a/c$$

The total 2nd-curvature of $M_{a,c}$ is then

$$I(a,c) = \int K^2_{a,c}\;dA_{a,c} = \frac{1}{c^2}\int K_a^2\;dA$$

$$= \frac{1}{c^2}\int \frac{K^2 dA}{1+2\,H_a+K_a^2}$$

Now, $1 + 2\,H_a + K_a^2 = (1+a\,k_1)(1+a\,k_2)$ where k_1 and k_2 are the principal curvatures of M^2, hence $1 + 2\,H_a + K_a^2 = 0$ if and only if k_1 or $k_2 = -\frac{1}{a}$.

As M^2 is C^2, k_1 and k_2 will be continuous bounded functions on M^2 and so $1 + 2\,H_a + K_a{}^2 \neq 0$ where a is in some open interval containing 0. We shall be interested in the derivative of $I(a,c)$ when $a = 0$ and hence for our purposes there will be no singularities in the integrals.

We choose c such that

$$\int dA_{a,c} = \int dA$$

i.e.

$$c^2 \int (1+2\,H_a+K_a{}^2)\,dA = \int dA = 4\pi$$

Then

$$I(a,c) = \frac{1}{4\pi} \int (1+2\,H_a+K_a{}^2)\,dA \int \frac{K^2\,dA}{(1+2\,H_a+K_a{}^2)}$$

$$\frac{dI}{da} = \frac{1}{4\pi} \int (2H+2\,K_a)\,dA \int \frac{K^2\,dA}{1+2\,H_a+K_a{}^2}$$

$$- \frac{1}{4\pi} \int (1+2\,H_a+K_a{}^2)\,dA \int \frac{(2H+2\,K_a)\,K^2\,dA}{(1+2\,H_a+K_a{}^2)^2}$$

Hence,

$$\left.\frac{dI}{da}\right|_{a=0} = \frac{1}{2\pi} \int H\,dA \int K^2\,dA - 2 \int H\,K^2\,dA$$

and we obtain

$$\left.\frac{dI}{da}\right|_{a=0} = 0 \Leftrightarrow \int H\,dA \int K^2\,dA = 4\pi \int H\,K^2\,dA \qquad (1)$$

On taking the second derivative of I we find

$$\left.\frac{d^2I}{da^2}\right|_{a=0} = \frac{1}{2\pi} \int K\,dA \int K^2\,dA - \frac{1}{\pi} \int H\,dA \int H\,K^2\,dA$$

$$- \frac{1}{\pi} \int H\,dA \int H\,K^2\,dA - \int 2K^2\,dA$$

Thus, if I has a local minimum at a = 0

$$\frac{1}{2\pi} \int K\, dA \int K^2\, dA - \frac{2}{\pi} \int H\, dA \int H K^2\, dA - 2 \int K^3\, dA \geq 0 \tag{2}$$

If M^2 is the torus

$\frac{1}{2\pi} \int K\, dA = 0$ by the Gauss-Bonnet theorem so that (2) becomes

$$\int H\, dA \int H K^2\, dA + \pi \int K^3\, dA \leq 0 \tag{2'}$$

But by (1)

$$\int H\, dA \int H K^2\, dA = \frac{1}{4\pi} \left(\int H\, dA\right)^2 \int K^2\, dA$$

and hence

$$\frac{1}{4\pi} \left(\int H\, dA\right)^2 \int K^2\, dA + \pi \int K^3 \leq 0 \tag{3}$$

Any 2 convex torus in R^3 must satisfy equation (3).

<u>Lemma 2</u>

A torus with one principal curvature constant cannot be 2-convex.

<u>Proof</u>

As before we denote the principal curvatures by k_1 and k_2 and let k_1 be constant

$2H = k_1 + k_2$ and $K = k_1 k_2$ hence $H = \frac{k_1}{2} + \frac{K}{2k_1}$ and

$\int H\, dA = \frac{k_1}{2} \int dA$ because $\int K\, dA = 0$ by the Gauss-Bonnet theorem and $\int H\, dA = 2\pi\, k_1$.

Similarly

$$\int H K^2\, dA = \frac{k_1}{2} \int K^2\, dA + \frac{1}{2k_1} \int K^3\, dA$$

and
$$\int H\, dA \int K^2\, dA = 2\pi\, k_1 \int K^2\, dA$$

Hence (1) becomes

$$2\pi\, k_1 \int K^2\, dA = 2\pi\, k_1 \int K^2\, dA + \frac{2\pi}{k_1} \int K^3\, dA$$

and
$$\int K^3\, dA = 0$$

Hence (3) becomes

$$\left(\int H\, dA\right)^2 \int K^2\, dA \leq 0$$

This is only possible if either $H = 0$ or $K = 0$, both of which is impossible.

Because of this lemma and the corollary to lemma 1 we make the following conjecture.

<u>Conjecture</u>

$$\inf \frac{1}{2\pi} \int_{\text{Torus}} K^2\, dA = \pi^2$$ where the infinum is taken over all tori in R^3 with area 4π. Also, there is no immersion for which this infimum is attained.

CHAPTER V

Volume Preserving Maps

§1. Introduction

We have been examining immersions $f:M^n \to R^{n+N}$ for which $f(M^n)$ has the geometric property of p-convexity. In such a situation it is natural to consider the set of all transformations of R^{n+N} which preserve this property. (It is, in fact, in the spirit of the "Erlanger Programm"). This set of transformations will always form a group (because by transformation we mean at least bijection) and we shall not only want to find what the transformations are but also specify the group structure.

For example, "straightness" of lines is preserved by homothetic transformations of R^n , 2-convexity of spheres by isometries, and the property of being a conic by projective transformations. Moreover, in each case the transformation group given is the largest group which preserves the given property for all the geometric objects possessing the property. N. H. Kuiper (in [2],p.13) has shown that the projective group of R^3 is a subgroup of the group which preserves 1-convexity of surfaces. However, this is clearly not true for p-convexity when $p > 1$ and the theorem has no analogue when the ambient space is anything but R^3.

The first criterion for a transformation which preserves p-convexity is that it preserves volume: p-convexity is defined for a set of immersions all having the same volume. Furthermore, the volume of <u>all</u> immersed sub-manifolds must be preserved, otherwise the transformations obtained will only preserve a particular p-convex immersion.

Our first objective is then to calculate the group of transformations of a riemannian manifold which preserve the volume of every n-dimensional sub-manifold. As we shall want to calculate the total pth curvature of the submanifolds we shall assume that the transformations are differentiable. The required transformation group will be shown to be the isometry group of the manifold and because an isometry preserves all the riemannian structure, the isometry group will be the group which preserves p-convexity.

§2. Locally k-Volume Preserving Maps

We have seen in §1. of Chapter I that on an n-dimensional manifold M^n with metric g, there is a locally defined n-form $g^{1/2}\, dx^1 \wedge \ldots \wedge dx^n$ where $g^{1/2}$ is the square root of the determinant of g as a symmetric matrix with respect to the coordinate system $(x^1,\ldots,x^n)$. If V^k is a k-dimensional submanifold of M^n there is a naturally induced metric on V^k given by i^*g where $i: V^k \to M^n$ is the natural inclusion. Hence g gives rise to a k form on V^k, namely $(i^*g)^{1/2}\, dy^1 \wedge \ldots \wedge dy^k$, where $(y^1,\ldots,y^n)$ is a coordinate system on M^n in which V^k is given by y^i = constant for $i = k+1,\ldots,n$. However, we should like a k-form on M^n which measures the volume of every k-dimensional submanifold. This will be the form $\omega\big|_{V^k_m}$ of definition 1. It will be defined locally and, for the most part, used locally; but when a global extension is needed an integration is always involved and we shall assume the procedure in §1. of Chapter I.

Let M^n be a C^1 riemannian n-manifold with positive definite metric tensor g and let V^k_m be a k-dimensional subspace of the tangent space of M^n at m. Denote by $g\big|_{V^k_m}$ the restriction of g to V^k_m and chose a coordinate system $(x^1,\ldots,x^n)$ on a neighbourhood of m such that V^k_m is spanned by $\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^k}$.

Definition 1

$$\omega\big|_{V^k_m} = \left(\det_{x^i} g\big|_{V^k_m}\right)^{1/2} dx^1 \wedge \ldots \wedge dx^k$$

where $\det_{x^i}$ is the operation of taking the determinant of $g\big|_{V^k_m}$ as a $k \times k$ matrix with respect to coordinates $(x^1,\ldots,x^n)$. The form $\omega\big|_{V^k_m}$ is differentiable as a map $G_{k,n} \to R$ where $G_{k,n}$ is the Grassmann bundle of k-planes. Also, $\omega\Big|_{V^k_m}$ is invariant under a change of coordinates which respect V^k_m; that is, if we choose another coordinate system $(y^1,\ldots,y^n)$ about m such that V^k_m is spanned by $\frac{\partial}{\partial y^1},\ldots,\frac{\partial}{\partial y^k}$, a straightforward calculation shows that

$$\left(\det_{x^i} g\Big|_{V^k_m}\right)^{1/2} dx^1\wedge \ldots \wedge dx^k = \left(\det_{y^i} g\Big|_{V^k_m}\right)^{1/2} dy^1\wedge \ldots \wedge dy^k$$

(This calculation was carried out, mutatis mutandis, in Chapter 1 of part I). Hence $\omega\Big|_{V^k_m}$ is a local k-form on M^n.

Definition 2

A C^1 map $f:M^n \to M^n$ of a C^1 reimannian n-manifold, M^n, is locally k-volume preserving if

$$f^*\left(\omega\Big|_{V^k_{f(m)}}\right) = \omega\Big|_{V^k_m}$$

where f^* is the usual "pull-back" of forms, and $1 \leq k \leq n$.

The set of all locally k-volume preserving maps forms a group by considering it as a subset of Diff (M^n,M^n).

Note 1

If $(x^1,\ldots,x^n)$ is a coordinate system, chosen as above and $y^i = x^i \circ f$, then if f is locally k-volume preserving

$$\omega\Big|_{V^k_m}\left(\frac{\partial}{\partial y^1},\ldots,\frac{\partial}{\partial y^k}\right) = f^*\left(\omega\Big|_{V^k_{f(m)}}\right)\left(\frac{\partial}{\partial y^1},\ldots,\frac{\partial}{\partial y^k}\right)$$

$$= \omega\Big|_{V_m^k}\left(f_*\frac{\partial}{\partial y^1},\ldots, f_*\frac{\partial}{\partial y^k}\right)$$

and using $f_*\dfrac{\partial}{\partial y^i} = \dfrac{\partial}{\partial x^i}$,

$$= \omega\Big|_{V_m^k}\left(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^k}\right)$$

but

$$\omega\Big|_{V_m^k}\left(\frac{\partial}{\partial y^1},\ldots,\frac{\partial}{\partial y^k}\right) = \omega\Big|_{V_m^k}\left(\frac{\partial f^i}{\partial y^1}\frac{\partial}{\partial x^i},\ldots,\frac{\partial f^i}{\partial y^k}\frac{\partial}{\partial x^i}\right)$$

$$= \left(\det \frac{\partial f^i}{\partial y^j}\right)\omega\Big|_{V_m^k}\left(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^k}\right)$$

$$\Rightarrow \det \frac{\partial f^i}{\partial y^j} = 1$$

We see that an equivalent definition for locally k-volume preserving is: Jacobian of f equals 1 on every k dimensional subspace of the tangent space.

<u>Note 2</u>

If V_m^k is a k-dimensional submanifold of M^n and f is k-volume preserving, then

$$\int_{f(V_m^k)} \omega\Big|_{V_m^k} = \int_{V_m^k} f^*\left(\omega\Big|_{V_m^k}\right)$$

by usual integration theory, (see, for example, Flanders [1] Ch.V).

$$= \int_{V_m^k} \omega\Big|_{V_m^k}$$

Hence the volume of V_m^k and $f(V_m^k)$ is the same and we can say that f preserves the volume of every k-dimensional submanifold of M^n. The volume of V^k may, of course, be infinite.

Note 3

If k = n is a locally n-volume preserving map it is volume preserving in the usual sense. Such maps occur naturally, e.g. as motions of incompressible fluids or as the flows of Hamiltonian vector fields in "phase space" (Liouville's Theorem). The set of all such maps forms a Lie group and can be given a faithful unitary representation into $L^2(M^n,\omega)$, the set of complex valued functions on M^n with ω-summable square (Arnold and Avez [1] p.23 with a bit more work to show that the given map is a faithful representation).

Note 4

When k = 1, V_m^1 is a one dimensional distribution on M^n : we may choose coordinates $(x^1,\dots,x^n)$ such that V_m^1 is spanned by $\frac{\partial}{\partial x^1}$ in a neighbourhood of m.

$$\omega\Big|_{V_m^1} = \left[g\Big|_{V_m^1}\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}\right)\right]^{1/2} dx^1$$

and with the previous definition of $(y^1,\dots,y^n)$ as $y^i = x^i$ of

$$g\Big|_{V_m^1}\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^1}\right) = g\Big|_{V_{f(m)}}\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}\right)$$

and hence a locally 1-volume preserving map preserves lengths. It follows (by Myers and Steenrod [1]) that f is an isometry. Clearly then, locally 1-volume preserving $\Leftrightarrow$ isometry.

Another property of k-volume preserving maps is given in the following

Lemma

A k-volume preserving map is a local diffeomorphism.

Proof

Let $\xi_1,\ldots,\xi_k$ be k linearly independent vectors at $m \in M^n$ and let V_m^k be spanned by $\xi_1,\ldots,\xi_k$. Choose a coordinate system $(x^1,\ldots,x^n)$ about m such that g is the identity $n \times n$ matrix with respect to these coordinates and V_m^k is spanned by $\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^k}$.

Let $\xi_i = \xi_i^j \frac{\partial}{\partial x^j}$. Then

$$\omega\Big|_{V_m^k}(\xi_1,\ldots,\xi_k) = \left(\det_{x^i} g\Big|_{V_m^k}\right)^{1/2} (dx^1\wedge \ldots \wedge dx^k)(\xi_1,\ldots,\xi_k)$$

$$= (dx^1\wedge \ldots \wedge dx^k)\left(\xi_1^i \frac{\partial}{\partial x^i},\ldots,\xi_k^i \frac{\partial}{\partial x^i}\right)$$

$$= \det \xi_j^i$$

It follows that $\xi_1,\ldots,\xi_k$ are linearly independent if and only if $\omega\Big|_{V_m^k}(\xi_1,\ldots,\xi_k) \neq 0$.

If f is k-volume preserving

$$f^*\,\omega\Big|_{V_{f(m)}^k}(\xi_1,\ldots,\xi_k) = \omega\Big|_{V_m^k}(\xi_1,\ldots,\xi_k)$$

$$= \omega\Big|_{V_{f(m)}^k}(f_*\,\xi_1,\ldots,f_*\,\xi_k)$$

Hence $f_* \xi_1, \ldots, f_* \xi_k$ are linearly independent if and only if $\xi_1, \ldots, \xi_n$ are linearly independent.

If f_* is not of rank n at m, there exists a tangent vector at m, ν_1 say, such that $f_* \nu_1 = 0$. If $\nu_2, \ldots, \nu_n$ are then any other tangent vectors at m such that $\nu_1, \ldots, \nu_k$ are linearly independent, $f_* \nu_1, \ldots, f_* \nu_k$ are not linearly independent. This contradicts the fact that f is k-volume preserving and we deduce that f_* is of rank n at all points $m \in M^n$. Hence f is a local diffeomorphism (see for example Flett [1]).

§3. Globally k-Volume Preserving Maps

We have defined volume preserving (definition 2) in terms of local forms and this is the most useful definition of the concept of volume preserving when calculations are to be made. However, another formulation, that given in the introduction, requires that the total volume of each submanifold be preserved. This appears below as definition 2'. We have seen that definition 2 implies definition 2' but it is the converse which will be of most use, and this we prove.

Definition 2'

A C^1 map $f:M^n \to M^n$ of a C^1 riemannian n-manifold M^n is globally k-volume preserving if for all C^1 k-dimensional submanifolds V^k of M^n,

$$\int_{V^k} \omega\Big|_{V^k} = \int_{f(V^k)} \omega\Big|_{f(V^k)} \;, \quad 1 \leqslant k \leqslant n.$$

We have denoted the tangent space of V^k by "V^k" but there is no confusion in doing this.

Both sides of the equality may of course be infinite.

Theorem 1

Definitions 2 and 2' are equivalent.

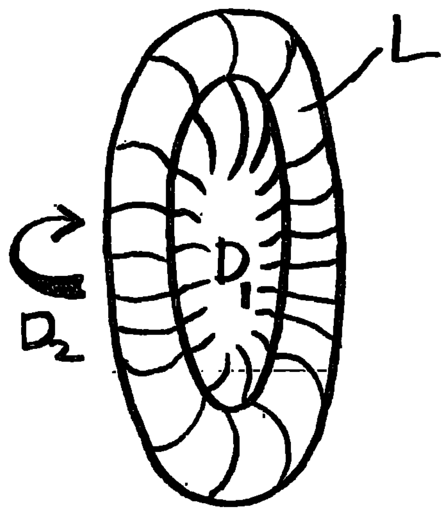
Proof

By note 3 of §1. locally k-volume preserving implies globally k-volume preserving.

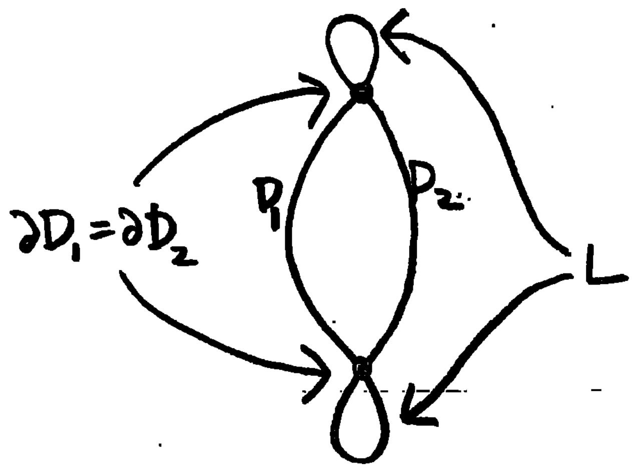
As for the converse, let D_1 be a closed k-dimensional C^1 disc in M^n with C^1 boundary, and let D_2 D_3 be two other such discs such that $\partial D_1 = \partial D_2 = \partial D_3$ and the three surfaces $D_i \cup D_j$; $i,j = 1,2,3$; $i \neq j$ are

C^1 smoothable. For example, if we consider $D_1 \cup D_2$, which is homeomorphic to S^k, there exists a C^1 immersed submanifold $L = [0,1] \times S^1$ such that $D_1 \cup L \cup D_2$ is a C^1 sphere.

Picture:



Cross-section



If f is globally k-volume preserving

$$\text{Volume of } (D_1 \cup L \cup D_2) = \text{Volume of } (f(D_1 \cup L \cup D_2))$$
$$= \text{Volume of } (f(D_1) \cup f(l) \cup f(D_2))$$

Hence,

$$\text{Volume of } (D_1 \cup D_2) + \text{Volume of } L$$
$$= \text{Volume of } f(D_1 \cup D_2) + \text{Volume of } f(L)$$

Now, the volume of L can be made arbitrarily small and so the volume of $D_1 \cup D_2$ is preserved although it may not be a C^1 manifold.

Let the volume of D_i be a_i and let the volume of $f(D_i)$ be b_i, then,

$$a_1 + a_2 = b_1 + b_2$$

$$a_2 + a_3 = b_2 + b_3$$

$$a_3 + a_1 = b_3 + b_1$$

Hence $a_i = b_i$; $i = 1,2,3$, $\therefore$ the volume of D_1 is preserved and D_1 is arbitrary.

Let $(x^1,\ldots,x^n)$ be any coordinate system about $m \in M^n$ and let

$$D_\varepsilon = \{p \in M^n : \sum_{i=1}^{k} (x^i(p))^2 \leqslant \varepsilon \ ; \ x^{k+1} = \ldots = x^n = 0\}$$

There certainly exists $\varepsilon_0 > 0$ such that D_ε is homeomorphic to a closed C^1 k-dimensional disc for all $\varepsilon \leqslant \varepsilon_0$. Then we have

$$\int_{D_\varepsilon} \omega\Big|_{D_\varepsilon} = \int_{f(D_\varepsilon)} \omega\Big|_{f(D_\varepsilon)} = \int_{D_\varepsilon} f^* \omega\Big|_{f(D_\varepsilon)}$$

Hence,

$$\int(\omega\Big|_{D_\varepsilon} - f^* \omega\Big|_{f(D_\varepsilon)}) = 0 \quad \text{for all } \varepsilon \leqslant \varepsilon_0 \qquad (1)$$

If $\omega\big|_{D_\varepsilon} \neq f^* \omega\big|_{f(D_\varepsilon)}$ at some $m \in M^n$, then $\omega\big|_{D_\varepsilon} \neq f^* \omega\big|_{f(D_\varepsilon)}$ on some neighbourhood of m because ω and f are both C^1 and the integral in (1) would be non-zero. We conclude that $\omega = f^*\omega$ and f is locally k-volume preserving.

From now on <u>k-volume preserving</u> will mean either locally of globally k-volume preserving according to the context.

We come now to the main theorem on k-volume preserving maps.

§4. Characterisation of k-volume Preserving Maps.

By note 5 of the previous section, the group of transformations which preserves 1-volume is a lie group whose continuous part (i.e. except for "isolated" transformations) is of dimension $\frac{1}{2}n(n+1)$. (See Kobayashi and Nomizu [1]). On the other hand, a consideration of note 3 p51 shows that the group of transformations which preserve n-volume is infinite dimensional. We might expect, then, that the transformation groups which preserve k-volume have dimensions which increase as k increases. This is not the case.

Theorem 2

On a C^2 manifold with positive definite riemannian metric, a k-volume preserving map is an isometry for $0<k<n$.

Proof

Let $f: M^n \to M^n$ be a k-volume preserving map of a C^2 riemannian n-manifold M^n with metric g_m, $m \in M^n$. g_m and $f^* g_{f(m)}$ are both positive definite metrics on M^n and positive definite symmetric matrices with respect to a coordinate system. Hence, there exists a coordinate system $(x^1, \ldots, x^n)$ on an open neighbourhood U of p such that g_p is the identity matrix and $f^* g_{f(m)}$ is diagonal. This is the classical theorem of Weierstrass on symmetric forms (see, for example, Van der Waerden [1], p.27).

Put $$g_m\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij}$$

and $$f^* g_{f(m)}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = a_i\, \delta_{ij} \quad \text{(no summation)}$$

Let V^k_m be the subspace of the tangent space of M^n at m which is spanned by $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}\right)_p$. Then, as f is k-volume preserving

$$f^* \omega\Big|_{V^k_{f(m)}}\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}\right) = \omega\Big|_{V^k_m}\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}\right) \tag{1}$$

Let V be an open neighbourhood of m such that f is a local diffeomorphism, and choose coordinates $y^i = x^i \circ f^{-1}$ on the open neighbourhood $f(U \cap V)$ of f(m).

(1) becomes

$$\left(\det_{y^i} g\Big|_{V^k_{f(m)}}\right)^{1/2} dy^1 \wedge \ldots \wedge dy^k \left(f_* \frac{\partial}{\partial x^1}, \ldots, f_* \frac{\partial}{\partial x^k}\right)$$

$$= \left(\det_{x^i} g\Big|_{V^k_m}\right)^{1/2} dx^1 \wedge \ldots \wedge dx^k \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}\right)$$

Now,

$$f_* \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i} \qquad \text{for all } i,$$

hence,

$$\det_{y^i} g\Big|_{V^k_{f(m)}} = \det_{x^i} g\Big|_{V^k_m}$$

By the choice of $(x^1,\ldots,x^n)$, $\det\limits_{x^i} g\Big|_{V_m^k} = 1$ and

$$f^* g_{f(m)}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{f(m)}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)$$

$$= a_i\,\delta_{ij}$$

and we obtain

$$\det_{y^i} g\Big|_{V^k_{f(m)}} = \prod_{i=1}^{k} a_i = 1$$

We chose V_m^k to be the subspace spanned by the first k of the set of vectors, $\left(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\right)$ but we could equally well have chosen any set of k. Hence

$$\prod_{i\in K} a_i = 1$$

where K is a set containing k of the integers, $1,\ldots,n$. To complete the proof of the theorem we need the following.

<u>Lemma</u>

If $(a_1,\ldots,a_n)$ is an n-tuple of real numbers such that the product of any k is 1, when $k < n$, then

$$k \text{ odd} \Rightarrow a_i = 1 \quad \text{for all } i$$

and $\qquad k \text{ even} \Rightarrow a_i = 1$ for all i, and the a_i's are all positive or all negative).

Proof of Lemma

The lemma is trivial for $k = 1$.

Let $k \geq 2$ and choose any $k + 1$ of the a_i, say the first $k + 1$. We define

$$P = \prod_{i=1}^{k+1} a_i \quad \text{and}$$

$$P_\lambda = \prod_{\substack{i=1 \\ i \neq \lambda}}^{k+1} a_i \quad \text{for } 1 \leq \lambda \leq k+1$$

Then,

$$P = a_\lambda P_\lambda \text{ and } P_\lambda = 1 \text{ for all } \lambda \text{ by hypothesis.}$$

Hence $a_\lambda = P$ for all λ, and

$$P = \prod_{i=1}^{k+1} a_i = P^{k+1}$$

and,

$$P^k = 1$$

If k is odd, $P = 1$ and so $a_i = 1$ for all i; $1 \leq i \leq k + 1$

If k is even, $P = \pm 1$ and so $a_i = \pm 1$ for all i; $1 \leq i \leq k + 1$

Finally we note that we chose the first $k + 1$ of the a_i's but could have chosen any set of $k + 1$. The result then follows.

Conclusion of the Proof to Theorem 2

$$g_{f(m)}\left(f_* \frac{\partial}{\partial x^i}, f_* \frac{\partial}{\partial x^j}\right) = \pm \delta_{ij} \text{ (by the lemma)}$$

$$= \pm g_m\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

However, f^*g is positive definite because

$$(f^*g_{f(m)})\left(u^i \frac{\partial}{\partial x^i}, u^i \frac{\partial}{\partial x^i}\right) = g_{f(m)}\left(f_*\left(u^i \frac{\partial}{\partial x^i}\right), f_*\left(u^i \frac{\partial}{\partial x^i}\right)\right) \geq 0$$

Hence $f^*g_{f(m)} = g_m$ and f is an isometry as required.

Note 1

By the previous "note 5", the above theorem for $k = 1$ is the theorem of Myers and Steenrod [1].

Note 2

On a manifold with non-positive-definite metric, the definition of volume preserving makes sense. It is therefore possible to ask if theorem 2 remains true if the "positive-definite" condition on the metric is relaxed. The proof of theorem 2 given breaks down in the non-positive definite case at a crucial point and we do not have a counter-example.

§5. p-Convexity Preserving Transformations

We have seen in the introduction that the most general transformation which preserves total p-th curvature must be k-volume preserving. It so happens that this necessary condition is also sufficient.

Theorem

A C^2 map of a reimannian manifold M^n onto itself preserves p-convexity if and only if it is an isometry.

Proof

When M^n is euclidean n-space the group of isometries is the group of euclidean rigid motions and the result is clear. In the general case it follows from the fact that isometry preserves all the riemannian structure (see Hicks), but we shall not go into the details because we should need, as a prerequisite, a full account of the definition of total p-th curvature in a general riemannian manifold.

Note

When M^n is n-dimensional euclidean space we can consider p-convex maps for all fixed volumes because the p-convex immersions for different volumes differ only by a linear multiple or rigid motion of R^n (see §2. of Chapter II). Hence the group of transformations of R^n which preserves p-convexity for all fixed volumes is the group of homothetic transformations.

PART II

Foliations and Connections

SUMMARY

The theory of foliations, initiated by C. Ehresmann and G. Reeb, is studied mainly as a branch of topology or differential topology. However, the work of A. G. Walker, R. Bott and others has shown that the methods of differential geometry are also effective in the study of foliations. Our work is very much in the spirit of differential geometry and is especially indebted to the work of A. G. Walker.

All results are original unless otherwise stated but some of the material was developed with P. M. D. Furness (see Furness [1] for an alternative proof of the main theorem in Chapter V).

In Chapter I we give the definition of a foliation on a manifold M^n (following A. Haefliger) and describe its leaves, leaf topology and special maps. We introduce the germs and jets of special maps, using them, in Chapter II, to construct covering spaces of M^n. The classical Ehresmann group is then obtained by "lifting" the fundamental group of a leaf into the covering space constructed from germs. The same procedure performed in the covering space constructed from jets gives a group for each leaf which we all the "Jet group". Both groups are factor groups of the fundamental group of the leaf for which they are defined and the Jet group is a factor group of the Ehresmann group. We then give a geometric method for calculating these groups.

The examples in Chapter III, as well as illustrating the concepts

introduced in Chapters I and II, are used to make observations on the relation between the Ehresmann and Jet groups of leaves in topologically equivalent foliations. Also, as an application of differential techniques, we consider C^2 foliations in codimension 1 on a riemannian manifold and derive a differential equation with the property that a unit vector field will satisfy the equation if and only if it is perpendicular to a foliation. C^2 foliations in codimension 1 are thus classified by vector fields satisfying a differential equation and we deduce a decomposition theorem for a manifold M^n with such a foliation.

Chapter IV describes the D-connections, and Chapter V the holonomy groups, introduced by A. G. Walker. In analogy with the classical holonomy theory we define these groups for piecewise C^k curves on a leaf with basepoint and show that, up to isomorphism, the groups are the same for all values of k and for all base points. After showing the existence of a special coordinate chart, whose use greatly simplifies the analysis, we show that Walker's holonomy group and the Jet group of a leaf are isomorphic.

Some properties of the holonomy group are given in Chapter VI. It is shown that the holonomy group cannot be a lie group* (this time contrasting with the classical holonomy theory) and that when the foliation has codimension 1 it is a factor group of the first homology group of the leaf, with integer coefficients, and has torsion subgroup of order 0 or 2. Finally, we give all the possible isomorphism classes of holonomy groups of compact leaves in foliations of two- and three-manifolds.

*Our convention is that lie groups have dimension > 0.

CHAPTER I

Definitions

§1. Foliations

The definition of a foliated structure (or foliation) to be given here will follow A. Haefliger [1].

A foliated structure is put on a manifold in much the same was as a differentiable or Piecewise linear structure is put on a topological space, i.e. we begin with a standard example of a foliation which will be mapped on to the manifold, locally, in several "patches". Maps of the standard example onto itself will then be used to "glue" these patches together. We therefore begin with.

The Standard Foliation of R^n in codimension p.

Let $(x^1,\dots,x^n)$ be the rectangular cartesian coordinate system of R^n. Denote by early greek letters (e.g. α,β,γ) any suffix taking values from 1 to n-p and by late greek letters (e.g. λ,μ,ν) any suffix taking values from n-p+1 to n. Let roman letters denote suffixes taking values from 1 to n. The leaves of the standard foliation are the (n-p)-dimensional planes defined by x^λ = constant and the leaf through an arbitrary point $(x_o{}^1,\dots,x_o{}^n)$ of R^n will be $\{(x^1,\dots,x^{n-p},x_c^{n-p+1},\dots x_o{}^n):$ $x^\alpha \in R\}$.

N.B. We are here following A. G. Walker's convention of defining the leaves by x^λ = constant. A Haefliger's convention is x^α = constant.

Leaf-preserving Local Maps

A local C^r homeomorphism, h, of R^n is a homeomorphism between two open sets of R^n such that both h and h^{-1} are C^r; $r = 0,1,2,\ldots,\infty$ or ω. By C^0 we mean continuous, C^r means differentiable r times and C^ω means analytic. A leaf perserving local C^r homeomorphism $h:R^n \to R^n$, where R^n has the standard foliation in codimension p, is a local C^r homeomorphism such that if (x^α, x^λ) is any point of R^n about which h is defined,

$$h^\alpha = h^\alpha(x^\beta, x^\mu)$$

$$h^\lambda = h^\lambda(x^\mu)$$

i.e. h maps leaves to leaves. h^{-1} will also have this representation in coordinates.

Foliation

A foliation, $\mathcal{F}$ of class r and codimension p on a C^k n-manifold M^n $(r \leq k)$ is defined by a maximal collection of charts

$h_i : u_i \to M^n$, called "leaf charts of $\mathcal{F}$", where i is in some indexing set, I say.

Each h_i is a C^r homeomorphism of an open set u_i of R^n into M^n. The h_i must satisfy

(1) $\{h_i(u_i)\}_{i\varepsilon I}$ is an open cover for M^n.

(2) Any map of the form $h_j^{-1} \circ h_i \big|_{u_{ij}}$ is a leaf preserving local C^r homeomorphism of R^n where $i,j \varepsilon I$ and $u_{ij} = h_i^{-1}(h_i(u_i) \cap h_j(u_j))$.

$\mathcal{F}$ is called topological, differentiable of class r or analytic according to whether $r = 0$, $0 \leq r \leq \infty$ or $r = \omega$ respectively.

By a "maximal collection of charts" we mean the following:

If u is an open subset of R^n and $h:u \to M^n$ is a C^r local homeomorphism such that $h_i^{-1} \circ h\Big|_{h^{-1}(h_i(u_i)\cap h(u))}$ is a local leaf preserving map for all $i \in I$, then $h:u \to M^n$ is in the collection of charts, $\mathfrak{F}$.

The Leaf Topology

We now describe a new topology on M^n which is finer than its usual topology. In this topology the leaves of $\mathfrak{F}$ are connected components and consequently $\pi_1(M^n,m) = \pi_1(L,m)$ where L is the leaf through $m \in M^n$.

Let M^n be a C^k n-manifold with C^r foliation $\mathfrak{F}$. On $R^n = R^{n-p} \times R^p$ put the topology T_0 which is the product of the usual topology on R^{n-p} and the discrete topology on R^p. Relative to T_0 the leaves of the standard foliation on R^n are connected components.

A local leaf preserving homeomorphism h of R^n, with the usual topology, is also a local leaf preserving homeomorphism of R^n with the topology T_0 because $h^\alpha(x^\beta,x^\mu)$ is continuous in x^β with the usual topology and hence for T_0 also and $h^\alpha(x^\beta,x^\mu)$ is automatically continuous in x^μ because the topology in R^p is discrete. The charts of $\mathfrak{F}$ will therefore induce a topology on M^n which will also be denoted by T_0. It is easily checked that T_0 is finer than the usual topology (using the definition of "finer" in Kelly [1] p.38).

Special Maps

Let π be the projection $R^{n-p} \times R^p \to R^p$ given by projection on to the second factor and let $h_i:u_i \to R^n$ be the charts of a foliation, $\mathfrak{F}$, of a manifold M^n. A continuous map f of an open set V of M^n into R^p is called a <u>special map of $\mathfrak{F}$</u> if for all $m \in V$ there exists a chart $h_i : u_i \to \mathcal{M}^n$ such that $f = \pi_o h_i^{-1}$ on $h_i(u_i) \cap V$. The terminology "special map" is not standard because there seems to be no universally accepted translation of A. Haefliger's "applications distinguées". (A. Haefliger [1], p.369).

If we denote the special maps by $f_i : V_i \to R^p$ where i is indexed by some set, I' say, then we have immediately

(1) $\{V_i\}_{i \in I'}$ is an open cover for M^n because $\pi_o h_i^{-1} : f_i(u_i) \to R^p$ is a special map and so $\{f_i(u_i)\}_{i \in I}$ is an open cover for M^n and $f_i(u_i) = V_j$ for some $j \in I'$ by definition.

(2) If $f_i : V_i \to R^p$, $f_j : V_j \to R^p$ and $m \in V_i \cap V_j$ then there exists a local C^r homeomorphism of R^p, h, such that $f_i = h_o f_j$. For if $V \subset V_i \cap V_j$ is an open set for which $f_i = \pi_o h_k^{-1}$ and $f_j = \pi_o h_\ell^{-1}$ then

$$(h_k^{-1})^\alpha = (h_k^{-1})^\alpha ((h_\ell^{-1})^\beta, (h_\ell^{-1})^\mu)$$

$$(h_k^{-1})^\lambda = (h_k^{-1})^\lambda ((h_\ell^{-1})^\mu)$$ using h_ℓ^{-1} as a coordinate chart on

V. $(\pi_o h_i^{-1})^\lambda$ is a C^r homeomorphism in the variables $(h_\ell^{-1})^\mu = (\pi_o h_\ell^{-1})^\mu$. Hence $\pi_o h_k^{-1}$ and $\pi_o h_\ell^{-1}$ differ by a local C^r homeomorphism.

Given a foliation $\mathfrak{F}$, the set of special maps may be constructed. Conversely, if a manifold M^n supports a set of charts $f_i : V_i \to R^p$ with the above conditions (1) and (2), a foliation is defined on M^n by letting the leaves be locally defined by $f_i^{-1}(x)$ where $x \in R^p$. In other words the special maps completely characterise the foliation and we may expect that properties of $\mathfrak{F}$ such as the Ehresmann groups, will be described in terms of them alone.

§2. Germs and Jets

Let X,Y be topological spaces. Two continuous functions f and g from X to Y are said to be in the same germ at $x \in X$ if there is an open neighbourhood of x on which f and g are equal. We define the germ of f at x, G(f,x) to be the set of continuous functions $g:X \to Y$ such that f and g are in the same germ at x.

If X = Y we may consider the set of all G(f,x) such that f(x) = x and f is a local C^r homeomorphism. We denote this set by $G^r(X,x)$, and its elements by $G^r(f,x)$, abbreviating these symbols to G(X) and G(f) respectively when the reference to x and r is clear. There is a natural product "o" which makes G(X) into a group. We let

$$G(f) \circ G(g) = G(f \circ g)$$

This product is well defined, for if G(f) = G(f') and G(g) = G(g') then $f|_u = f'|_u$ for some open set u and $g|_V = g'|_V$ for some open set V. Hence $f \circ g|_{g^{-1}(u) \cap V} = f' \circ g'|_{g^{-1}(u) \cap V}$ and hence $G(f \circ g) = G(f' \circ g')$. Associativity is immediate. If id_X is the identity map on X, $G(id_X)$ is clearly an identity for G(X) and as each f considered is a local C^r homeomorphism there exists an open neighbourhood u of x such that $f^{-1} \circ f|_u = id_X|_u$. Hence the inverse of G(f) is $G(f^{-1})$ and "o" puts a group structure on G(X).

Jets of C^1 maps

The jets we are about to describe are usually called 1-jets but as we shall have no occasion to use n-jets when n > 1 we shall call these 1-jets simply "jets".

Let X^n, Y^n be C^1 manifolds. Two local C^1 homeomorphisms f and g from X^n to Y^n are said to be in the same jet at $x \in X^n$ if $f_* = g_*$ at x. We define the Jet of f at x, $J(f,x)$ to be the set of local C^1 homeomorphisms $g:X^n \to Y^n$ about x which have the property $g_*(x) = f_*(x)$. If $(x^1,\ldots,x^n)$ is a C^1 coordinate chart x and $(y^1,\ldots,y^n)$ is a C^1 coordinate chart about $f(x)$ then the condition $g_*(x) = f_*(x)$ is equivalent to

$$\frac{\partial g^i}{\partial x^j} = \frac{\partial f^i}{\partial x^j} \quad \text{for } i,j = 1,\ldots,n$$

This condition is, as may be expected from its intrinsic definition, independent of the coordinate charts chosen. If $X^n = Y^n$ we may consider the set of all $J(f,x)$ such that $f(x) = x$. We denote this set by $J(X^n,x)$; its elements are $J(f,x)$ and we abbreviate these symbols to $J(X^n)$ and $J(f)$ when the reference to x or X^n is clear.

There is a product "o" which makes $J(X^n)$ into a group. We let

$$J(f)_o\, J(g) = J(f_o g)$$

This product is well defined because

$$J(f) = J(f') \Longleftrightarrow f_*(x) = f_*'(x)$$

$$J(g) = J(g) \Longleftrightarrow g_*(x) = g_*'(x) \qquad \text{hence}$$

$(f_o g)_*(x) = (f_* \,_o\, g_*)(x) = (f_*' \,_o\, g_*')(x) = (f' \,_o\, g')_*(x)$ and $J(f_o g) = J(f' \,_o\, g')$. Associativity is immediate. As with germs, $J(id_{X^n})$ is the identity and the inverse of $J(f)$ is $J(f^{-1})$.

There is a natural map $\alpha: G^r(X^n) \to J(X^n)$ when $r \geq 1$ defined by

$\alpha(G^r(f)) = J(f)$. α is well defined because if $G^r(f) = G^r(g)$, $f = g$ on some open neighbourhood of x and hence $f_*(x) = g_*(x)$. α is also a group homeomorphism, for

$$\alpha(G(f) \circ G(g)) = \alpha(G(f \circ g)) = J(f \circ g) = J(f) \circ J(g) = \alpha(G(f)) \circ \alpha(G(g))$$

If we write $G_o(X^n) = \ker \alpha = \{G^r(f) : J(f) = J(1_{X^n})\}$ then

$$J(X^n) \cong G(X^n) \,/\, G_o(X^n)$$

Finally, we note that as f_* is an invertible linear map of $T_x(X^n)$, the tangent space of X^n at x, $J(X^n)$ is naturally identified with $GL(n,R)$, the general linear group on R^n.

CHAPTER II

The Ehresmann and Jet Groups

§1. Covering Spaces of Germs and Jets

Let $\mathcal{F}$ be a foliation of class r on a C^k manifold M^n. We shall assume in this section that M^n has the leaf topology, T_o.

We define the covering space of germs of special maps $\mathcal{G}$ by $\mathcal{G} = \{(m, G^r(f,m)) : m \in M^n$ and f is a special map defined on some neighbourhood of $m\}$.

The r which appears in this definition is the same r as the class of the foliation and will in future be omitted.

The projection for this covering space, $\alpha_G : \mathcal{G} \to M^n$ is defined by

$$\alpha_G(m, G(f,m)) = m$$

If $r \geqslant 1$ we also define the covering space of jets of special maps, $\mathcal{J}$ by

$\mathcal{J} = \{(m, J(f,m)) : m \in M^n$ and f is a special map defined on some neighbourhood of $m\}$.

The projection, $\alpha_J : \mathcal{J} \to M^n$ is defined by

$$\alpha_J(m, J(f,m)) = m$$

In the rest of this section we shall justify the above terminology by proving that $\mathcal{G}$ and $\mathcal{J}$, with the projections α_G and α_J, are covering spaces of M^n.

Let $f_i:u_i \to R^p$ be a special map of $\mathfrak{F}$ with $m \varepsilon u_i$, and let $u_{f_i,m} = f_i^{-1} \circ f_i(m)$. i.e. $u_{f_i,m}$ is that part of the leaf through m which intersects u_i. $u_{f_i,m}$ is an open set because we have the leaf topology on M^n. Let

$<u_{f_i,m}>_G = \{(x, G(f_{i,x})) \varepsilon \mathfrak{G} : x \varepsilon u_{f_i,m}$ and $f_i:u_i \to R^p$ is a special map$\}$.

$<u_{f_i,m}>_J$ is defined similarly be replacing G and $\mathfrak{G}$ by J and $\mathfrak{J}$.

<u>Lemma 1</u>

$\{<u_{f_i,m}>_G\}_{i\varepsilon I}$ is a base for a topology on $\mathfrak{G}$ and

$\{<u_{f_i,m}>_J\}_{i\varepsilon I'}$ is a base for a topology on $\mathfrak{J}$.

<u>Proof</u>

We need only show that for given $f_i:u_i \to R^p$, $f_j:u_j \to R^p$, $m \varepsilon u_i$ and $q \varepsilon u_k$, there exists a special map $f_k:u_k \to R^p$ and an $x \varepsilon u_k$ such that

$<u_{f_i,m}>_G \cap <u_{f_j,q}>_G = <u_{f_k,x}>_G$ and similarly for J. (Kelley [1], p.47).

Let $(x, G(f,x)) \varepsilon <u_{f_i,m}>_G \cap <u_{f_j,q}>_G$

(If this intersection is empty the proof is complete).

Then $\qquad x \varepsilon u_{f_i,m} \cap u_{f_j,q}$

and $\qquad G(f_{i,}x) = G(f_{j,}x)$

Hence if $u_i \cap u_j = u_k$ and $f_k = f_i|_{u_k}$,

$$<u_{f_i,m}>_G \cap <u_{f_j,q}>_G = <u_{f_k,x}>$$

The proof for jets is obtained by replacing $\mathcal{G}$ by $\mathcal{J}$ and G by J throughout.

Theorem

$\pi_G: \mathcal{G} \to M^n$ and $\pi_J: \mathcal{J} \to M^n$ are covering spaces when M^n is given the leaf topology and $\mathcal{G}$ and $\mathcal{J}$ are given the topologies whose bases are $\{\langle u_{f_i,m}\rangle_G\}_{i\varepsilon I}$ and $\{\langle u_{f_i,m}\rangle_J\}_{i\varepsilon I'}$ respectively.

Proof

Let $m \varepsilon M^n$ and $f_i: u_i \to M^n$ be a special map such that $m \varepsilon u_i$. $\pi_G^{-1}(u_{f_i,m}) = \{\langle u_{f_j,m}\rangle : f_j = h \circ f_i\}$ where h is a local homeomorphism of R^p. $\pi_G^{-1}(u_{f_i,m})$ is thus the union of open sets of $\mathcal{G}$ and thus open. Also, $\pi_G |\langle u_{f_i,m}\rangle| = u_{f_i,m}$ which is also open in M^n. Hence π_G is a local homeomorphism. Similarly we prove that π_J is a local homeomorphism.

We now show that about every $m \varepsilon M^n$ there is an open set whose inverse image under π_G or π_j is a disjoint union of homeomorphic copies. (See Spanier [1], p.62).

Consider the open neighbourhood $u_{f_i,m}$ of m.

$\pi_G(\langle u_{f_i,m}\rangle_G) = \pi_G(\langle u_{h_o f_i,m}\rangle_G)$ so that the inverse image of $u_{f_i,m}$ is $\{\langle u_{h_o f_i,m}\rangle_G$: h is a local homeomorphism of $R^p\}$. Let $(x, G(g,x)) \varepsilon \langle u_{f_i,x}\rangle \cap \langle u_{h\circ f_i,x}\rangle$. (If the intersection is empty the proof is complete).

$x \varepsilon u_i$ and $G(g,x) = G(f_i,x) = G(h_o f_i,x)$ and hence h = id and $f_i = h_o f_i$.

Therefore $\langle u_{f_i,m}\rangle_G = \langle u_{h_o f_i,m}\rangle_G$ and we conclude that the $\langle u_{h_o f_i,m}\rangle_G$ are disjoint.

For jets we obtain $J(g,x) = J(f_i,x) = J(h_o f_i,x)$ and so $h_* = id_*$ and $f_{i_*} = h_* \circ f_{i_*}$.

Therefore $\langle u_{f_i}, m\rangle J = \langle u_{h_o f_i}, m\rangle J$.

§2. The Ehresmann and Jet Groups

Covering projections over connected bases have the homotopy lifting property, so if $\gamma:[0,1] \to M^n$ is a continuous loop with base point $m = \gamma(0) = \gamma(1)$, there exists a "lift", $\tilde{\gamma}:[0,1] \to \mathcal{G}$ which is a continuous map such that $\pi_G \circ \tilde{\gamma} = \gamma$ and the homotopy class of $\tilde{\gamma}$ depends only on the homotopy class of γ. $\tilde{\gamma}$ then determines a map $\pi_G^{-1}(m) \to \pi_G^{-1}(m)$ given by $\tilde{\gamma}(0) \to \tilde{\gamma}(1)$ and hence a map $E:\pi_1(M^n,m) \to G^r(R^p,0)$. E is a homomorphism of groups (see Spanier [1] p.86) and the group $E(\pi_1(M^n,m))$ is unique up to isomorphism for all m in the same path component - that is leaves of a foliation $\mathcal{F}$ on M^n.

The group $E(\pi_1(M^n,m))$ thus defined for all leaves L of a foliation is the Ehresmann Group and is denoted by E(L). $E(\pi_1(M^n,m))$ is a representation of E(L) in $G^r(R^p,0)$.

The same considerations apply to jets. Instead of E we obtain the map $J:\pi_1(M^n,m) \to J(R^p,0)$ and the group $J(\pi_1(M^n,m))$, defined up to isomorphism for each leaf L, is the Jet Group of L and will be denoted by J(L). $J(\pi_1(M^n,n))$ is a representation of J(L) in GL(p,R). (See p71).

We now give a geometric way of calculating these groups.

Let $\gamma:[0,1] \to L$ a continuous map (necessarily into a leaf L of $\mathcal{F}$). $[0,1]$ is compact and there exists a finite cover $\{V_i\}_{i\varepsilon s}$ where $f_i:V_i \to R^p$ are special maps of $\mathcal{F}$. Let t_k, $k = 0,\ldots,r$ be points of $[0,1]$ such that $t_o = 0$, $t_r = 1$ and $\gamma([t_k,t_{k+1}]) \varepsilon\ V_i$ for some i.

Let $h:u \to R^n$ be a leaf chart such that $h \circ \gamma(t_k) = 0 \varepsilon R^n$ and let P be the p-plane perpendicular to the standard leaves in R^n through 0. $T_k = h^{-1}(P) \cap \{u_i\}_{i\varepsilon s}$ is the transverse disc to L at $\gamma(t_k)$ induced by

h. Let $\pi : u_i \to T_k$ be given by $\pi(x) = T_k \cap \{\text{leaf through } x\}$. (This π is thus the projection $R^{n-p} \times R^p \to R^p$, on the standard foliation, mapped on to the manifold by a leaf chart).

If W is any subset of M^n, the leaf topology on M^n induces a topology on W. An equivalence relation is then defined on W by $x \sim y$ if and only if x and y are in the same path component of W. The set of equivalence classes of W is denoted by $\bar{W}$. Any transverse disc about m is locally homeomorphic to $\bar{u}$ with the quotient topology (and quotient differential structure when it exists) so that all transverse discs are naturally locally homeomorphic about m.

Let $\gamma(t_k), \gamma(t_{k+1}) \in V_i$ where $f_i : V_i \to R^p$ is a special map. $f_i|T_k$ and $f_i|T_{k+1}$ are local C^r homeomorphisms into R^p such that $f_i|T_k\, \gamma(t_k) = f_i|T_{k+1}\, \gamma(t_{k+1})$, hence $\Phi_{k,k+1} = f_i|T_{k+1}^{-1} \circ f_i|T_k$ is a local C^r homeomorphism from T_k to T_{k+1}. If γ is lifted to a curve $\tilde{\gamma}$ in the space of germs of local homeomorphisms of R^p, $\tilde{\gamma}(t_k)$ and $\tilde{\gamma}(t_{k+1})$ are both induced by the same special map f_i. Hence

$$\tilde{\gamma}(t_{k+1}) = G(f_i|T_{k+1} \circ \pi) = G(f_i|T_k \circ \Phi_{k,k+1} \circ \pi)$$

If we choose the special map $f_{i+1} : V_{i+1} \to R^p$, say, such that $\gamma(t_{k+1})$ and $\gamma(t_{k+2}) \in V_{i+1}$ and $G(f_i|T_{k+1}) = G(f_{i+1}|T_{k+1})$ we may choose special maps inductively beginning at $k = 1$ such that

$$\begin{tikzcd}
 & & & \mathbb{R}^p & & \\
T_1 \arrow[urrr, "f_1|_{T_1}"] \arrow[r, "\Phi_{1,2}"'] & \cdots \arrow[r] & T_k \arrow[ur, "f_k|_{T_k}"] \arrow[r, "\Phi_{k,k+1}"'] & T_{k+1} \arrow[u, "f_{k+1}|_{T_{k+1}}"'] \arrow[r] & \cdots \arrow[r, "\Phi_{r-1,r}"'] & T_r \arrow[ull, "f_r|_{T_r}"']
\end{tikzcd}$$

commutes when germs of the maps are taken. We then have

$$\tilde{\gamma}(t_r) = G(f_r|_{T_r}) = G(f_1|_{T_1} \circ \Phi_{1,2}^{-1} \circ \cdots \circ \Phi_{r-1,r}^{-1} \circ \pi) .$$

The maps of the form $\Phi_{r-1,r} \circ \cdots \circ \Phi_{1,2}$ thus generate the elements of the Ehresmann group of a leaf, when all homotopy classes of curves γ are taken, by taking their germs. The Jet group is generated by taking jets.

CHAPTER III

Examples

§1. Simple Foliations

On an n-manifold with C^r foliation $\mathcal{F}$ there is an equivalence relation $\sim$ (introduced in the previous section) given by $x \sim y \Leftrightarrow x$ and y are in the same leaf. $M^n/\sim$ is called the space of leaves and a foliation is <u>simple</u> when $M^n/\sim$ is a manifold, i.e. there exists a map $\phi : M^n \to W^p$, where W^p is a C^r p-manifold, such that for every homeomorphism of class r, $g : u \to W^p$, where u is an open set of R^p, $g^{-1} \circ \phi$ is a special map of $\mathcal{F}$. The leaves are thus the inverse images under f of points of W^p and the space of leaves is given a manifold structure. The Ehresmann and Jet groups for each leaf are trivial by the geometric discussion of Chapter II §2.

If $M^n \simeq X^n \times Y^n$ where X^n and Y^n are manifolds, the projection $f : X^n \times Y^n \to Y^n$ defines a simple foliation on M^n which is a <u>product foliation</u> whose leaves are all homeomorphic to Y^n. Not every simple foliation is a product. Consider the twisted S^1-bundle over S^1 whose total space is the Klein bottle, K. The projection $\pi : K \to S^1$ defines a simple foliation on K whose leaves are all homeomorphic to S^1; but K is not a product.

§2. Integrable Distributions

Denote by $T(M^n) \xrightarrow{\pi} M^n$ the tangent bundle of a C^k manifold M^n. A p-dimensional distribution, D, on M^n is a map $D: M^n \hookrightarrow T(M^n)$ such that $\pi_o D$ is the identity on M^n and D(m) is a p dimensional linear subspace of $\pi^{-1}(m)$ for all $m \in M^n$. If, on a neighbourhood of $m \in M^n$, there are linearly independent vectors V_i which are all in $D(M^n)$, we say that D(m) is spanned by the V_i at m. $D(M^n)$ is a C^r distribution if the vectors V_i can be chosen to be C^r vector fields about all points $m \in M^n$.

A p-dimensional distribution $D(M^n)$ is integrable if there is a co-ordinate system $(x^1, \ldots, x^n)$ on a neighbourhood of each $m \in M^n$ such that $D(M^n)$ is spanned by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^p}$ on that neighbourhood. In other words, D is given by the tangent spaces of the submanifolds of u defined by x^i = constant, $i = p + 1, \ldots, n$. Hence the coordinate system $(x, \ldots, x^n)$ is a leaf chart of the same differentiability class as $D(M^n)$ and so the C^r integrable distribution gives rise to a C^r foliation.

Conversely, if $h_i : u_i \to M^n$ is a leaf chart of a C^r foliation on M^n, h_{i*} will map the tangent spaces of the standard leaves on R^n onto tangent spaces of C^r submanifolds on M^n thus giving an integrable distribution on M^n.

If $r \geqslant 2$ there is a useful characterisation of integrable distributions.

Frobenius' Theorem (see Dieudonne [1], p.308).

A C^2 distribution on a C^k manifold, with $k \geqslant 2$, is integrable if and only if for any two C^2 vector fields u,v in the distribution, $[u,v]$ is also in the distribution. $[\ ,\ ]$ is the "lie bracket" of vector fields defined by $[u,v] = uv - vu$.

We now have a one-to-one correspondence between foliations and distributions satisfying an algebraic condition. This will be used extensively in Chapters IV-V.

§3. Foliations in Co-dimension 1

As an application of §2. we shall study foliations in co-dimension 1.

Let M^n be a differentiable manifold with a differentiable foliation $\mathfrak{F}$ of co-dimension 1 and let g be any riemannian metric on M^n. $\mathfrak{F}$ defines an (n-1)-dimensional distribution on M^n for which there is a unique orthogonal one-dimensional distribution. We shall denote this distribution by $\pm v$ where v is a locally defined vector field. (If $\mathfrak{F}$ is orientable in the sense of Haefliger [1], V will be globally defined). From $\mathfrak{F}$ we have constructed a local vector field v; however, not every such v gives rise to a foliation. For example, consider the vector fields in R^4, with cartesian coordinates (x,y,z,w), given by

$$v_1 = - y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - w \frac{\partial}{\partial z} + z \frac{\partial}{\partial w}$$

$$v_2 = + x \frac{\partial}{\partial x} - w \frac{\partial}{\partial y} - x \frac{\partial}{\partial z} + y \frac{\partial}{\partial w}$$

$$v_3 = + w \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} - x \frac{\partial}{\partial w}$$

On $S^3 = \{(x,y,z,w) \in R^4 : x^2 + y^2 + z^2 + w^2 = 1\}$, v_1, v_2 and v_3 are orthonormal tangent vectors. That is, the v_i give a parallelisation of S^3 which is orthonormal. Furthermore the v_i satisfy

$$[v_1,v_2] = -2\, v_3, \quad [v_2,v_3] = -2\, v_1, \quad [v_3,v_1] = -2\, v_2$$

Hence $V_1^{\perp}$ (the orthogonal compliment to v_1) is spanned by v_2 and v_3 but is not integrable because $[v_2,v_3] \notin v_1^{\perp}$.

We shall now seek algebraic conditions on v which are necessary and sufficient for $v^{\perp}$ to be integrable.

Let $(x^1,\ldots,x^n)$ be a coordinate chart of M^n on a neighbourhood of $m \in M$. Put $e_i = \frac{\partial}{\partial x^i}$ and let $v = v^i e_i$.

Define $p_i = e_i - g(e_i,v)v = e_i - g_{ij}\, v^j\, v^k\, e_k$, for $i = 1,2,\ldots,n$.

Then $g(p_i,v) = g_{ip}\, v^p - g_{ij}\, v^j\, g_{kp}\, v^k\, v^p$, but $g(v,v) = 1$ hence $g(p_i,v) = 0$ and $p_i \in v^{\perp}$ for all i. Also $p_i = 0 \Rightarrow e_i - g(e_i,v)\, v = 0$ $\Rightarrow v = \frac{e_i}{g(e_i,v)}$. Hence if $p_j = 0$ also, $\frac{e_i}{g(e_i,v)} = \frac{e_j}{g(e_j,v)}$ which is impossible, hence at most one p_i can vanish at any point of M^n.

If $\alpha_i \in R$, $i = 1,\ldots,n$ we note that

$$\Sigma\, \alpha_i p_i + [\Sigma\, \alpha_i\, g(e_i,v)]v = \Sigma\, \alpha_i e_i$$

The e_i span the tangent space at m, hence so do $p_1,\ldots,p_n$, v. Therefore $p_1,\ldots,p_n$ span $V^{\perp}$.

$V^{\perp}$ is integrable if and only if $[p_i,p_j] \in V^{\perp}$ for all i,j. A routine calculation gives

$$g([p_i,p_j],v) = -(g_{jr}\, v^r\, v^s)_{|i}\, g_{su}\, v^u + (g_{ip}\, v^p\, v^q)_{|j}\, g_{qu}\, v^u$$

$$+ g_{ip}\, v^p\, v^q\, (g_{jr}\, v^r\, v^s)_{|q}\, g_{su}\, v^u - g_{jr}\, v^r\, v^s\, (g_{ip}\, v^p\, v^q)_{|s}\, g_{qu}\, v^u$$

where "$|i$" denotes covariant differentiation in the direction $\frac{\partial}{\partial x^i}$ with respect to the riemannian connection of g. $g(v,v) = 1$ so that $g_{ij}\, v^i\, v^j = 1$ and $g_{ij}\, v^i\, v^j{}_{|k} = 0$. Hence

$$- g_{jr}\, v^r{}_{|i} + g_{ip}\, v^p{}_{|j} + g_{ip}\, v^p\, v^q\, g_{jr}\, v^r{}_{|q} - g_{jr}\, v^r\, v^s\, g_{ip}\, v^p{}_{|s} = 0$$

is a necessary and sufficient condition for $v^{\perp}$ to be integrable.

We denote by ∇ the riemannian connection of g and define the unit vector field n by $\nabla_v\, v = kn$ where k is a real valued function on M^n. At any point $m \in M^n$, k and n are the curvature and normal of the integral curve of v, through m, parametrised by arc length.

$$\nabla_v\, v = v^i\, v_{|i}{}^j\, e_j = k\, n^j\, e_j$$

$$v^i\, v_{|i}{}^j = k\, n^j \text{ , hence}$$

$$- g_{jr}\, v^r{}_{|i} + g_{ip}\, v^p{}_{|j} + g_{ip}\, v^p\, g_{jr}\, k\, n^r - g_{jr}\, v^r\, g_{ip}\, k\, n^p = 0$$

$$v_{j|i} - v_{i|j} = k\,(v_j n_i - v_i n_j)$$

Finally, because the connection is symmetric,

$$v_{i.j} - v_{j.i} = k\,(v_i n_j - v_j n_i) \qquad (1)$$

where "$_{.j}$" indicates the partial derivative $\frac{\partial}{\partial x^j}$.

We define $(\mathrm{Curl}\, v)_{ij} = v_{i|j} - v_{j|i}$. (This is the standard definition of curl in a riemannian manifold, see for example Willmore [1] p.231.)

Also, for vectors $a = a^i \frac{\partial}{\partial x^i}$, $b = b^i \frac{\partial}{\partial x^i}$, define

$(a\times b)_{ij} = a_i b_j - a_j b_i$. $(a\times b)_{ij}$ is a local 2-form on M^n which reduces to the usual "cross-product" of vectors when M^n is three dimensional euclidean space. (After the usual tangent/co-tangent identification).

(1) can now be written,

$$\mathrm{Curl}\ v = kv \times n \quad \text{or} \quad \mathrm{Curl}\ v = v \times \nabla_v v \qquad (2)$$

We note that if v satisfies (2) then so will -v. Hence we may speak of a unit distribution of one-dimension satisfying (2). We now have the theorem:

Theorem

On a riemannian manifold, C^2 foliations of co-dimension 1 are in one-to-one correspondence with one-dimensional unit distributions, $\pm v$, satisfying Curl $v = v \times \nabla_v v$.

In regions where Curl $v = 0$

(1) $v = \mathrm{grad}\ \phi$ for some $\phi : M^n \to R$, and the leaves are given by $\phi^{-1}(x)$ where $x \in R$.

(2) $v \times \nabla_v v = 0 \Rightarrow \nabla_v v = \alpha v$ for some $\alpha : M^n \to R$. Also, $g(v,v) = 1$, hence $g(\nabla_v v, v) = 0$, $\alpha = 0$ and v is a geodesic vector field. If Curl $v \neq 0$ then $v \times \nabla_v v \neq 0$ and $\nabla_v v \neq \alpha v$. $\nabla_v v$ therefore has a non-vanishing component in $v^{\perp}$. i.e. in the tangent space of the leaves.

We now have,

Corollary

On a riemannian manifold M^n with C^2 foliation $\mathcal{F}$ of codimension 1, there is a closed set* u on which the foliation is simple and complementary to a geodesic vector field, and such that, on $M^n \setminus u$ there is a non vanishing vector field which lies in the tangent space to the leaves.

* u may be empty.

Note

On any subset of u which is an n-dimensional open submanifold of M^n, g is bundle-like in the sense of Reinhart [1].

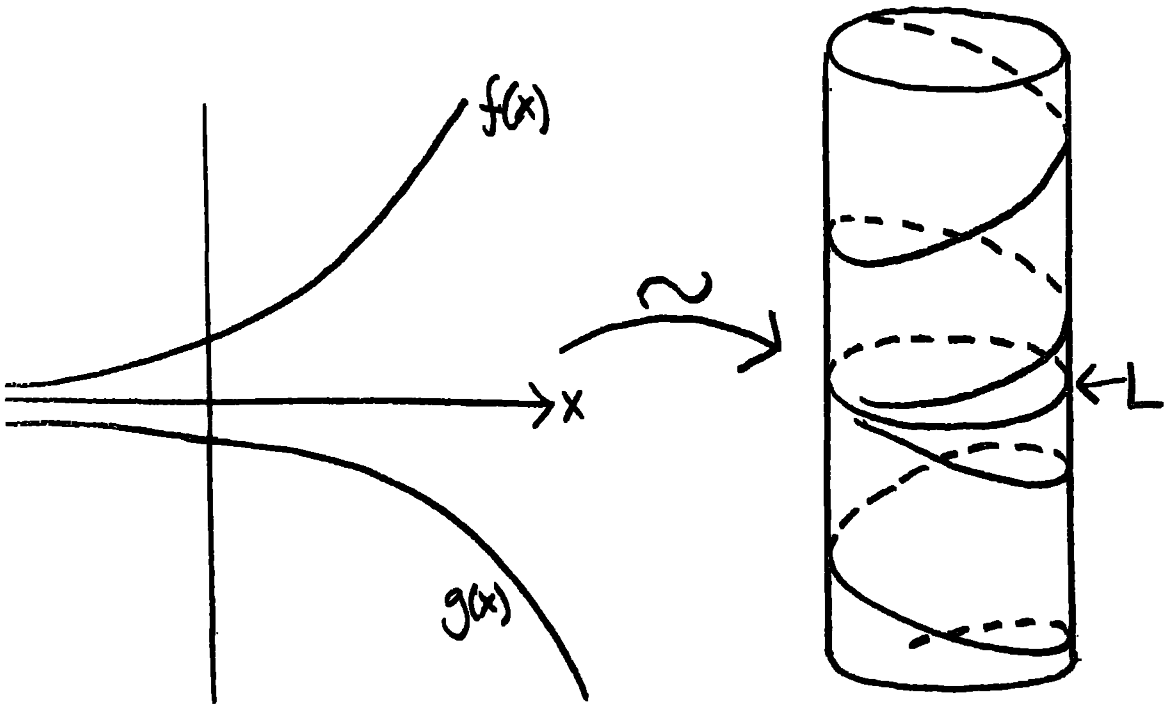
In the special case $u = \phi$, $\mathcal{F}$ is simple (see §1.) and when $u = M^n$ each leaf has euler class zero.

§4. Foliations of $S^1 \times R$

The purpose of this section is to present examples which illustrate the relation between the topological and differentiable properties of foliations. We therefore calculate the Ehresmann and Jet groups, and differentiability class of the foliations in some detail. Statements of a general nature which may be deduced from these examples will be made under the heading "observations" at the end.

Let $f:R \to R^{>0}$ be a C^1 function of one real variable taking values which are strictly positive real numbers. Also let $\frac{df}{dx} > 0$ and $\lim_{x\to-\infty} f(x) = 0$. The set of functions $f(x+c)$ where $c \in R$ have the property that their graphs foliate the open upper half plane, $R \times R^{>0}$. Foliate the lower half plane similarly with a function g and add the leaf $R \times \{0\}$ to give a foliation of R^2. Let $\sim$ be the equivalence relation $(x,g) \sim (u,v) \Longleftrightarrow x-u \in Z$ and $y = v$. $R^2/\sim$ is homeomorphic to $S^1 \times R$ and the foliation on R^2 is induced onto $S^1 \times R$ by $\sim$. Call the leaf $\frac{R\times\{0\}}{\sim}$, L.

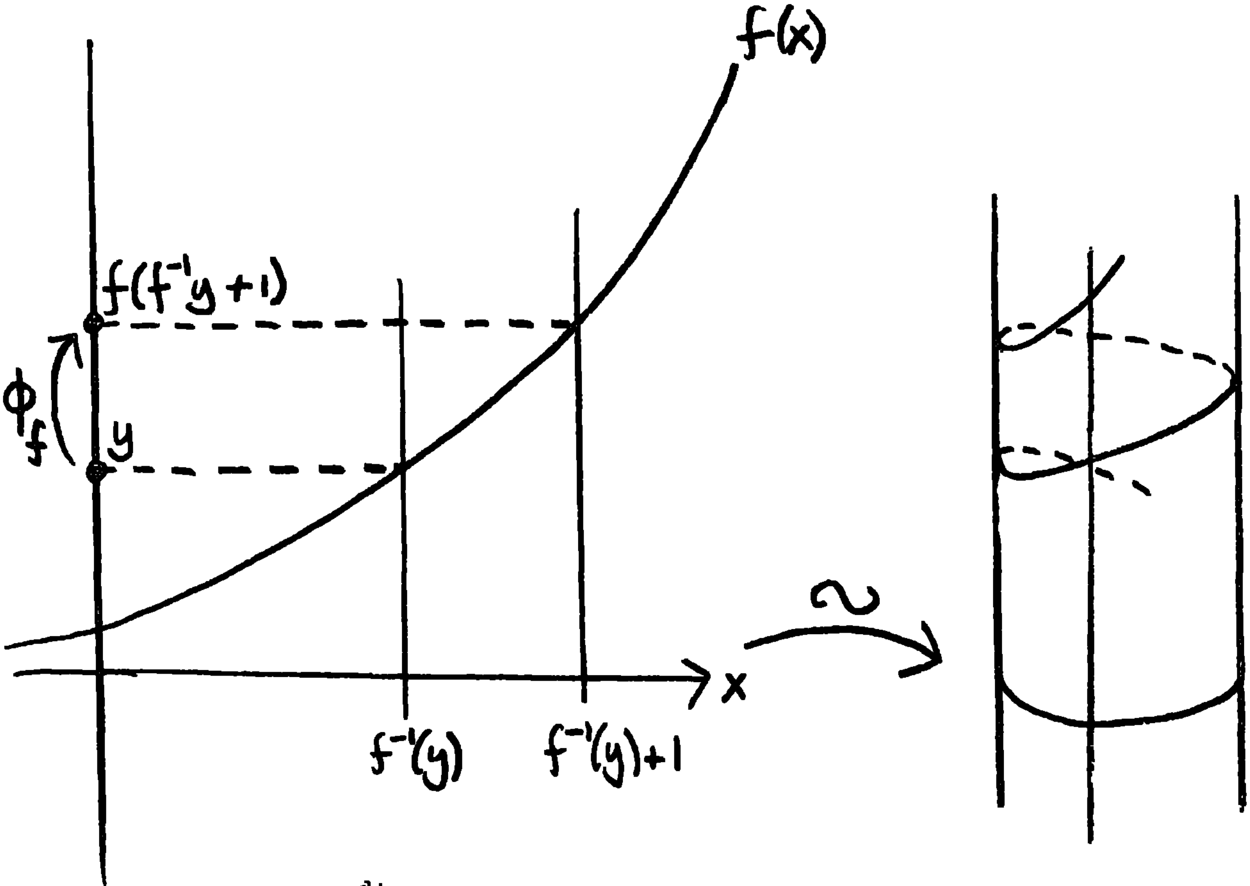
Picture



The Ehresmann group is zero for every leaf except L when it is Z. The Jet groups for all leaves except L are also zero but the Jet group of L depends on properties of the functions f and g.

The local homeomorphism ϕ of R corresponding to any loop on L which "goes round once" is

$$\phi_f : y \to f(f^{-1}y+1)$$



Now, $\dfrac{d\phi_f}{dy} = \dfrac{df}{dy}(f^{-1}y+1)\,\dfrac{df^{-1}}{dy}$.

If the Jet group exists we must have $\left.\dfrac{d\phi_f}{dy}\right|_0 = \left.\dfrac{d\phi_g}{dy}\right|_0$ and if this is the case, a necessary and sufficient condition for the Jet group to be zero is

$$\left.\frac{df}{dy}(f^{-1}y+1)\,\frac{df^{-1}}{dy}\right|_0 = 1 \; . \tag{1}$$

That is, ϕ_f has the same derivative at 0 as the identity map. Otherwise the Jet group of L is Z.

<u>Example A</u>

$$f(x) = e^x,\; g(x) = -e^x.$$

$$\frac{df}{dx} = e^x,\ f^{-1}(x) = \log_e x,\ \frac{df^{-1}}{dx} = \frac{1}{x}.$$ Hence condition (1) is

$$\left.\frac{d\phi_f}{dy}\right|_0 = e^{\log_e y+1} \left.\frac{1}{y}\right|_{y=0} = ye \left.\frac{1}{y}\right|_{y=0} = e.$$ Similarly $\left.\frac{d\phi_g}{dy}\right|_0 = e.$

The Jet group of L is Z. As the leaves may be given by the distribution $\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, this foliation is C^∞.

Example B

$$f(x) = -\frac{1}{x},\ g(x) = \frac{1}{x}$$

$$\frac{df}{dx} = \frac{1}{x^2},\ f^{-1}(x) = -\frac{1}{x},\ \frac{df^{-1}}{dx} = \frac{1}{x^2}$$

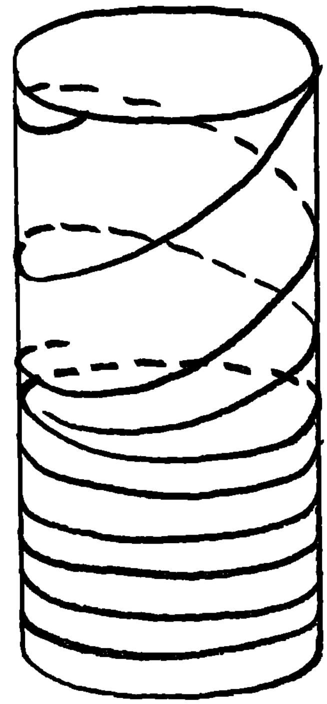
$$\left.\frac{d\phi_f}{dy}\right|_0 = \frac{1}{\left(-\frac{1}{y}+1\right)^2}\left.\frac{1}{y^2}\right|_{y=0} = \left.\frac{1}{(y-1)^2}\right|_{y=0} = 1.$$ Similarly

$\left.\frac{d\phi_g}{dy}\right|_0 = 1.$ Hence the Jet group of L is zero. Also the leaves may be given by the distribution $\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$ and the foliation is C^∞.

Example C

Let $f(x) = e^x$ and foliate $S^1 \times R^{<0}$ by circles parallel to $S^1 \times \{0\}$.

Picture



As in Example A, $\left.\frac{d\phi_f}{dy}\right|_0 = e$, but the first derivative of the local homeomorphisms of R for $y < 0$ is 1. Hence the Jet group of L does not exist. The foliation can be given by the distribution

$$\begin{cases} \frac{\partial}{\partial x} + y\frac{\partial}{\partial y} & \text{for } y \geq 0 \\ \frac{\partial}{\partial x} & \text{for } y \leq 0 \end{cases}$$

which is C^0 but not C^1.

Example D

$f(x) = -\frac{1}{x}$ and foliate $S^1 \times R^{<0}$ as in Example C. As in Example B $\left.\frac{d\phi_f}{dy}\right|_0 = 1$, and the Jet group of L is zero. As the distribution is given by

$$\begin{cases} \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} & \text{for } y \geq 0 \\ \frac{\partial}{\partial x} & \text{for } y \leq 0 \end{cases}$$, the foliation

is C^1 but not C^2.

Observations

Examples A and B are topologically equivalent in the sense that there exists a homeomorphism of $S^1 \times R$ taking one foliation onto the other. However this homeomorphism does not preserve the Jet groups, i.e. Jet groups are not topological invariants.

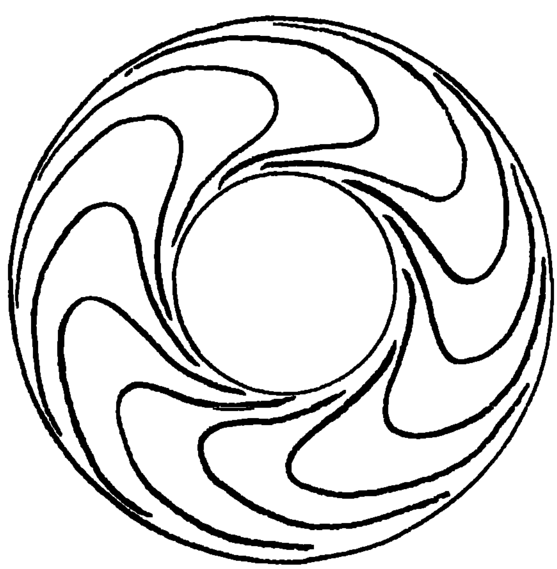
The Ehresmann and Jet groups in Example B were not equal, but there existed a topologically equivalent foliation, namely Example A, for which they were. We ask then, "Given a foliation with different Ehresmann and Jet groups, does there always exist a topologically equivalent foliation for which they are the same?" The answer is no because in Example D the jets of local homeomorphisms in the lower part of $S^1 \times R$ are jets of the identity. Hence the Jet group, if it exists, must be zero. However, the Ehresmann group of Example D is Z.

§5. Three-manifolds

Example A The "Reeb Component"

Let (x,y,z) be cartesian coordinates in R^3. The curves $y = (1-x^2)^{-1} + c$, $z = 0$, foliate the strip $S = \{(x,y,z) : |x| < 1, z = 0\}$. If S is rotated about the y-axis in R^3 we obtain a foliation of the open solid cylinder $\{(x,y,z) : x^2 + z^2 < 1\}$. By adding the leaf given by $x^2 + z^2 = 1$ and factoring out by the equivalence relation $\sim$ given by $(x,y,z) \sim (u,v,w) \Leftrightarrow y - v \in Z$, $x = u$ and $z = w$, we obtain a foliation of a closed solid torus. Each leaf, except the boundary of the solid torus, is homeomorphic to R^2.

Picture



Example B Two Reeb Components

By identifying the boundaries of two Reeb components we can foliate certain closed three-manifolds e.g.

(i) The "ordinary" S^2 bundle over S^1, $S^2 \times S^1$:

Let the boundary of a Reeb component be given by coordinates (θ,ϕ) where $\theta, \phi \in R$ (mod 1) and let two such components be identified by the relation $\sim$ given by $(\theta,\phi) \sim (\theta',\phi') \Leftrightarrow \theta = \theta'$ and $\phi' = \phi + 1/2$ (mod 1).

(ii) The "twisted" S^2 bundle over S^1:

As in (i) but let the relation be $(\theta,\phi) \sim (\theta',\phi') \Leftrightarrow \theta = -\theta'$ (mod 1) and $\phi' = \phi + 1/2$ (mod 1).

(iii) The sphere S^3. (The "Reeb Foliation" of S^3)

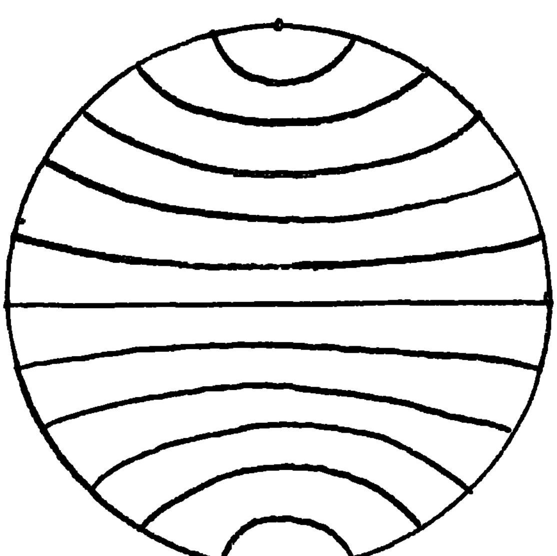
Let the relation be $(\theta,\phi) \sim (\theta',\phi') \Leftrightarrow \theta = \phi'$ and $\phi = \theta'$.

This foliation can be described as follows.

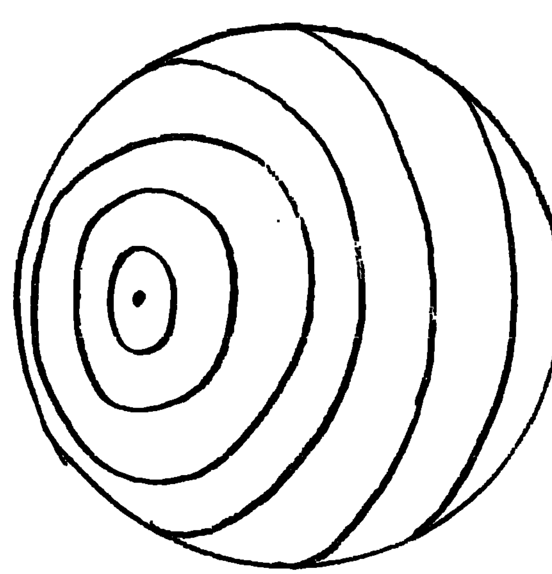
Let D be a closed three-ball foliated by cylinders in its interior, and by circles, with two singular points, on its boundary.

Picture

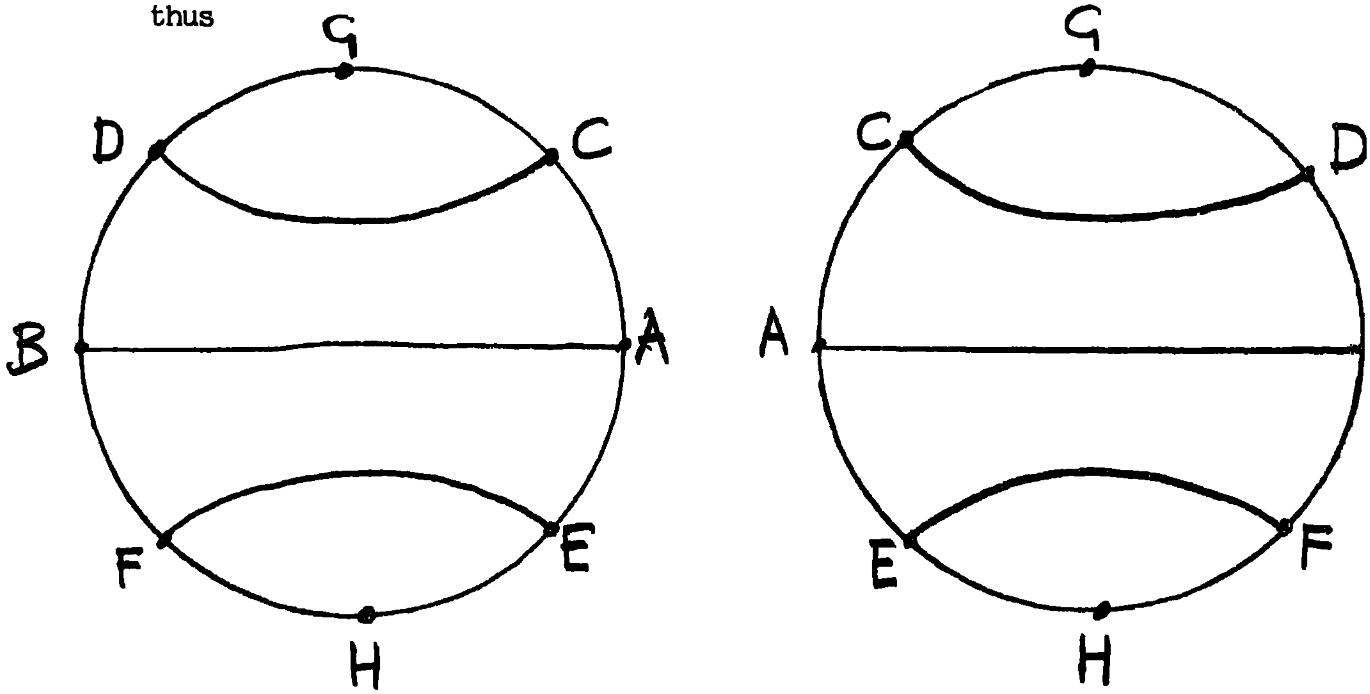
Interior



Boundary



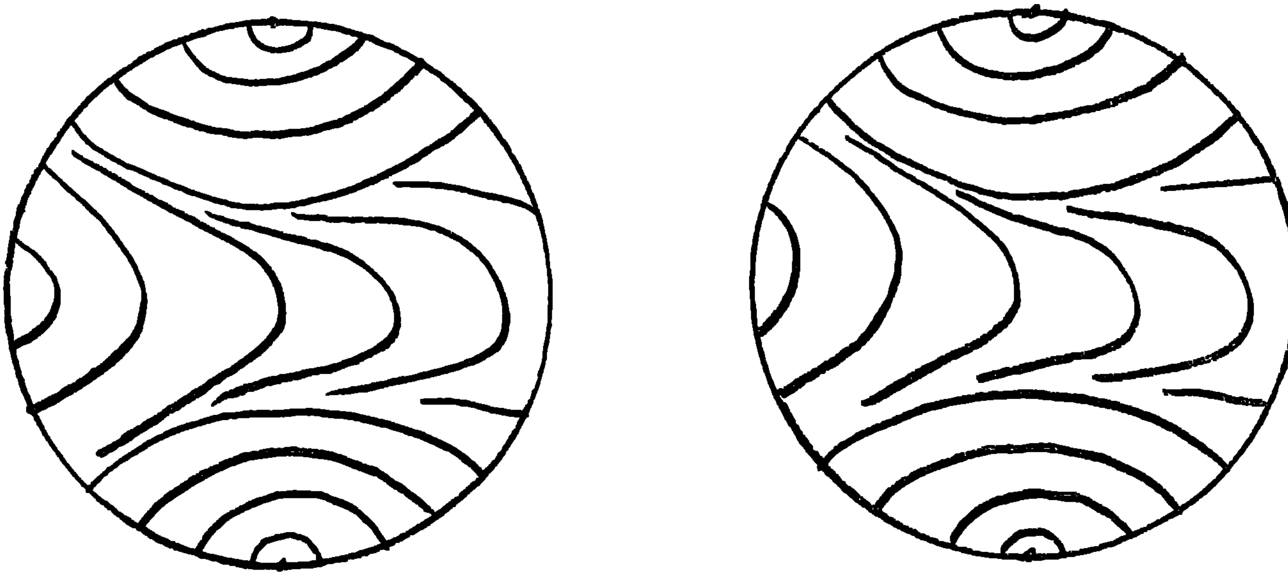
If two such balls are identified by their boundaries to give a sphere, thus



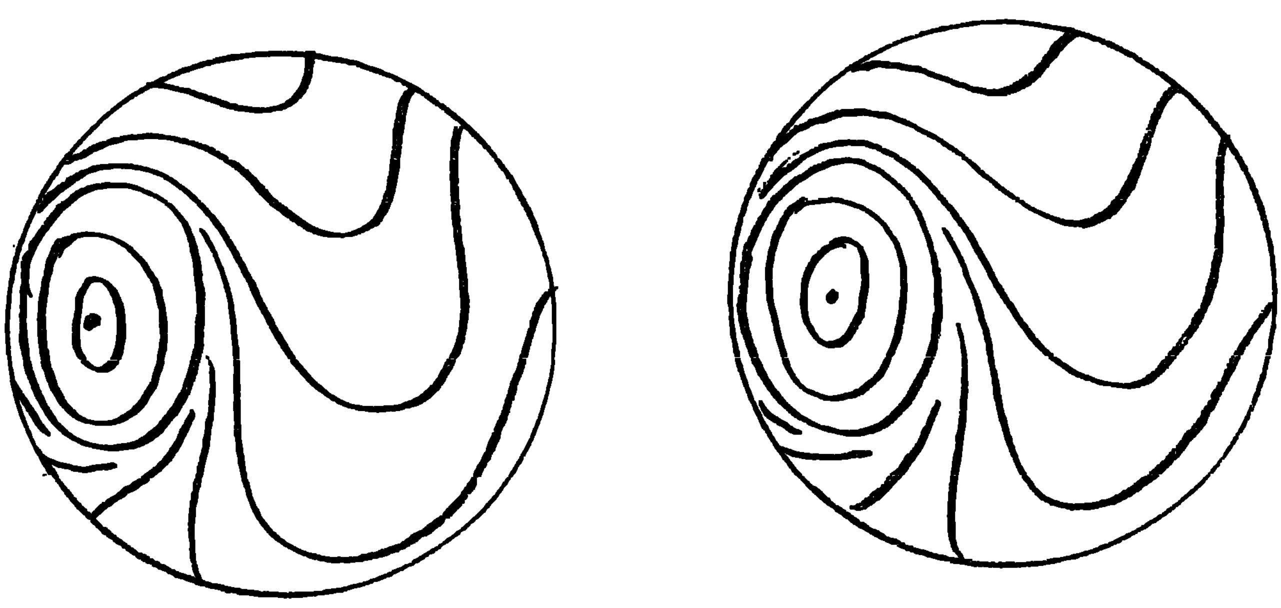
we have the sphere foliated by tori, like CDEF, and two singular circles, AB and GH. The torus CDEF divides the sphere into two solid tori because on both sides of CDEF the tori are "nested" about the circles AB and GH. These two solid tori can then be re-foliated by Reeb components to give the Reeb foliation of the sphere.

Picture

Interiors

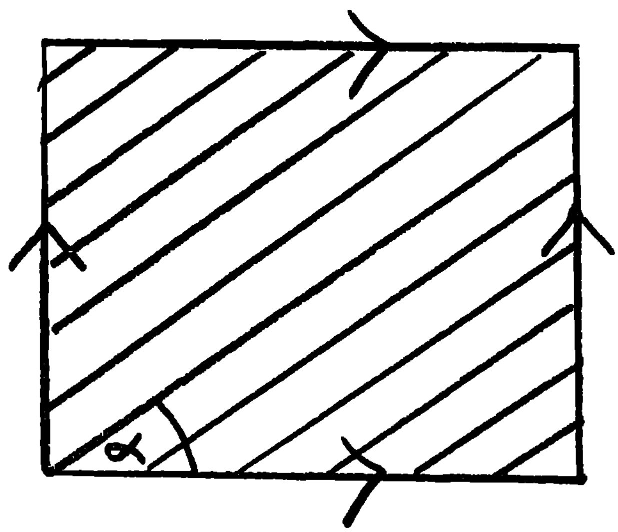


Boundaries



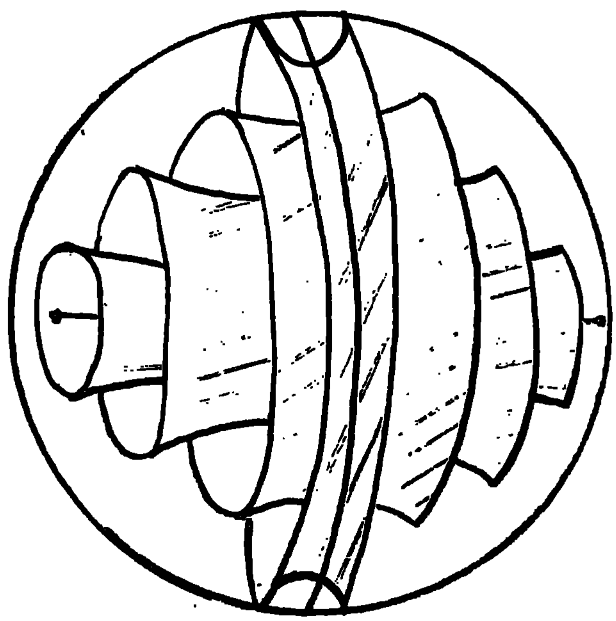
Example C

We can give a pictorial description of S^3 foliated in dimension 1 using the nested tori and singular circles of the previous example. Each torus is foliated by one of the well known "rational and irrational flows".



Tori near AB have values of α near zero and tori near GH have values of α near $\pi/2$. CDEF has $\alpha = \pi/4$ and α is an increasing function for all the other tori.

Picture



Example D

Let D^2 be the closed disc in R^2, with polar coordinates (r,θ), given by $r^2 \leq 1$ and let $D^2 \times [0,1]$ be foliated by the lines $\{p\} \times [0,1]$ where $p \in D^2$. When the discs $D^2 \times \{0\}$ and $D^2 \times \{1\}$ are identified by the relation $(r,\theta) \sim (r,\theta+\alpha)$ we obtain a foliation of $D^2 \times S^1$ in which the leaf $\{0\} \times S^1$ is always a circle but the other leaves have homeomorphism types depending on α.

Case 1 $\frac{\alpha}{2\pi}$ irrational.

Each leaf other than $\{0\} \times S^1$ is homeomorphic to R and has trivial Ehresmann and Jet groups. $\{0\} \times S^1$ has Ehresmann and Jet groups Z when $\alpha \neq 0$ and trivial groups when $\alpha = 0$.

<u>Case 2</u> $\frac{\alpha}{2\pi} = \frac{a}{b}$ where a,b are coprime integers.

Every leaf is homeomorphic to S^1. The Ehresmann and Jet groups for $\{0\} \times S^1$ are Z_b and trivial for all other leaves.

Note

Because the Ehresmann and Jet groups are factor groups of the fundamental group of the leaf, the only possible isomorphism classes for these groups when the leaf is S^1 are $Z_2, Z_3, \ldots, Z$ and the trivial group. We have, in Example D, Foliations in which an S^1 leaf has all possible Ehresmann and Jet groups. The lowest dimension for a manifold in which this can be done is three. (See Chapter 6, Theorem 2, Example 1).

CHAPTER IV

D-connections

§1. Introduction

In Example 2 we saw that a differentiable foliation gives rise to an integrable distribution, and vice-versa. We now study foliations from this point of view by ignoring the C^1 distributions and applying Frobenius' theorem. In place of special maps, which are projections in the standard foliation, we have a projection tensor on the manifold, and instead of maps, Φ, between transverse discs we have parallel translation between tangent spaces. Just as the maps Φ were defined by special maps, the parallel translation will be defined by the projection tensor. This approach to foliations, which from a geometric point of view is very similar to that of Haefliger [1], is due to A. G. Walker; [1], [2] and [3].

§2. Distributions

Let D' be a differentiable distribution on a C^k n-manifold, M^n, and let g be a riemannian metric on M^n. We define D'' to be the orthogonal distribution to D'. If D' has dimension n-p, D'' will have dimension p. If λ is a local vector field on M^n, λ decomposes uniquely as $\lambda = \lambda' + \lambda''$ where $\lambda' \in D'$ and $\lambda'' \in D''$. We define the tensor a' by $a'(\lambda) = \lambda'$ for all such λ. If I is the identity on $T(M^n)$ (i.e. the identity 1-covariant, 1-contravariant tensor) we put $a'' = I - a'$ so that $a''(\lambda) = \lambda''$.

In terms of a coordinate system $(x^1,\ldots,x^n)$ we write $a'\left(\frac{\partial}{\partial x^i}\right) = a^i_j \frac{\partial}{\partial x^i}$. Let $a^i_{j|k}(\Gamma)$ denote covariant differentiation with respect to a connection Γ on M^n (not necessarily the metric connection).

Contractions with a'^i_j and $a''^i_j = \delta^i_j - a'^i_j$ are written

$$C^i_{jk}\, a'^j_p = C^i_{p'k} \ ; \quad C^i_{jk}\, a''^p_i = C^{p''}_{jk} \ ; \quad C^i_{jk}\, a'^j_k\, a''^k_q = C^i_{p'q''} \qquad \text{etc.}$$

We use the convention that contraction follows differentiation, thus

$$C^i_{p|k}\, a'^p_j = C^i_{j'|k} \ ; \quad C^p_{j|q}\, a'^i_p\, a''^q_k = C^{i'}_{j|k''} \qquad \text{etc.}$$

We begin by deriving some fundamental identities satisfied by a' and a''.

$$a' + a'' = I \Rightarrow a'^i_{j|k} + a''^i_{j|k} = 0 \qquad (1.$$

$a'a'' = 0$ because $a'a''(\lambda) = a'(\lambda'') = 0$ for all local vector fields λ, hence

$$(a'a'')^{i}_{j|k} = a'^{i}_{p|k}\, a''^{p}_{j} + a'^{i}_{p}\, a''^{p}_{j|k} = 0$$

$$\Rightarrow a'^{i}_{p|k}\, a''^{p}_{j} - a'^{i}_{p}\, a'^{p}_{j|k} = 0 \qquad \text{by (1)}$$

$$\left.\begin{aligned} &\Rightarrow a'^{i}_{j''|k} = a'^{i'}_{j|k} \\ &\text{Similarly} \qquad a''a' = 0 \Rightarrow a'^{i''}_{j|k} = a'^{i}_{j'|k} \end{aligned}\right\} \qquad (2)$$

$$a'^{i'}_{j'|k} = a'^{i'}_{p|k}\, a'^{p}_{j} = a'^{i}_{p''|k}\, a'^{p}_{j} = a'^{i}_{q|k}\, a''^{q}_{p}\, a'^{p}_{j} = 0$$

Similarly

$$a'^{i''}_{j''|k} = 0$$

Finally

$$a'^{i'}_{j''|k} = a'^{i'}_{p|k}\, a''^{p}_{j} = a'^{i}_{p''|k}\, a''^{p}_{j} = a'^{i}_{q|k}\, a''^{q}_{p}\, a''^{p}_{j} = a'^{i}_{q|k}\, a''^{q}_{j} = a'^{i}_{j''|k}$$

and similarly

$$a'^{i''}_{j'|k} = a'^{i''}_{j|k}$$

Summarising, we have

$$\left.\begin{aligned} a'^{i'}_{j|k} &= a'^{i}_{j''|k} = a'^{i'}_{j''|k} \\ a'^{i''}_{j|k} &= a'^{i}_{j'|k} = a'^{i''}_{j'|k} \\ a'^{i'}_{j'|k} &= a'^{i''}_{j''|k} = 0 \end{aligned}\right\} \qquad (3)$$

(These rules are in fact true for any derivation locally defined, e.g. the partial derivative in the direction $\frac{\partial}{\partial x^k}$.)

It is now convenient to define

$$a^i_{jk} = a'^{i'}_{j|k} + a''^{i''}_{j|k}$$

From (3) it is straightforward to prove

$$\left.\begin{aligned} a^{i'}_{jk} &= a^{i}_{j''k} = a^{i'}_{j''k} = a'^{i'}_{j|k} \\ a^{i''}_{jk} &= a^{i}_{j'k} = a^{i''}_{j'k} = -\,a'^{i''}_{j|k} \\ a^{i'}_{j'k} &= a^{i''}_{j''k} = 0 \end{aligned}\right\} \qquad (4)$$

Let $\gamma: R \to M^n$ be a C' curve on M^n. We say that <u>D' is parallel along γ</u> if vectors in D' remain in D' when parallel translated along γ. Let the tangent vector to γ be $b^i \frac{\partial}{\partial x^i}$; i.e. $\gamma_*\left(\frac{d}{dt}\right) = b^i \frac{\partial}{\partial x^i}$.

Let λ be a parallel vector field along γ. We have $\lambda^i_{|k}\, b^k = 0$.

$a'(\lambda)$ is parallel along γ if and only if

$$(a'^i_j\, \lambda^j)_{|k}\, b^k = a'^i_{j|k}\, \lambda^j\, b^k + a'^i_j\, \lambda^j_{|k}\, b^k = 0$$

Which, because $\lambda^i_{|k}\, b^k = 0$, is equivalent to $a'^i_{j|k}\, \lambda^j\, b^k = 0$.

Let $\lambda \in D'$ at one point of γ say $\gamma(t)$. Parallel translation is unique, hence $\lambda \in D'$ along γ if and only if $a'(\lambda)$ is parallel along γ. Hence λ remains in D' if and only if $a'^i_{j|k}\, \lambda^j\, b^k = 0$.

If λ is an arbitrary vector field, along γ, in D' it is of the form $\lambda^i\, a'^j_i$. D' is then parallel along γ if and only if $a'^i_{j|k}\, a'^j_p\, \lambda^p\, b^k = 0$ for all λ^p. Hence $a'^i_{j|k}\, a'^j_p\, b^k = a'^i_{p'|k}\, b^k = 0$.

Similarly D" is parallel along γ if and only if $a^{'i}_{p''|k}\, b^k = 0$.

If D" is parallel along all C' paths γ in D' we say that <u>D" is parallel relative to D'</u>. We then have $a^{'i}_{p''|k}\, b^k = 0$ for all b^k such that $b^k \frac{\partial}{\partial x^k} \in D'$. Hence $a^{'i}_{p''|k'} = 0$. Similarly D' is parallel relative to D" if and only if $a^{'i}_{p'|k''} = 0$.

When D' is parallel along all C' paths γ in D' we say that <u>D' is self parallel</u>. The necessary and sufficient condition for this is $a^{'i}_{p'|k'} = 0$. Similarly for D". When written in terms of a^i_{jk} all these conditions become,

$$\left.\begin{array}{lll} \text{D' parallel relative to D"} & \Leftrightarrow & a^i_{j'k''} = 0 \\ \text{D" parallel relative to D'} & \Leftrightarrow & a^i_{j''k'} = 0 \\ \text{D' self parallel} & \Leftrightarrow & a^i_{j'k'} = 0 \\ \text{D" self parallel} & \Leftrightarrow & a^i_{j''k''} = 0 \end{array}\right\} \qquad (5)$$

We shall see later that if the connection is symmetric, self parallel implies integrable. D" is not in general integrable so we seek a weaker condition than self parallel.

<u>D' is path-parallel</u> if for any $m \in M^n$ and any $\lambda \in D_m'$, the geodesic determined by λ remains in D'. This condition is equivalent to

$$a^{'i}_{j|k}\, \lambda^j \lambda^k = 0 \qquad \text{for all } \lambda \in D_m'$$

Hence $$a^{'i}_{j|k}\, a^{'j}_p\, a^{'k}_q\, \lambda^p \lambda^q = a^{'i}_{p'|q'}\, \lambda^p \lambda^q = 0 \qquad \text{and}$$

$$(a^{'i}_{p'|q'} + a^{'i}_{q'|p'})\, \lambda^p \lambda^q = 0$$

Therefore D' is path parallel if and only if $a'^{i}_{p'|q'} + a'^{i}_{q'|p'} = 0$. It is convenient to denote symmetric alternation of suffices by ( ) and skew-symmetric alternation by $[\ ,\ ]$. e.g.

D' is path parallel if and only if $a'^{i}_{(p'|q')} = 0$.

Finally we seek the conditions on a^{i}_{jk} which are equivalent to integrability.

Let λ,μ be local vector fields in D. D' is integrable if and only if $[\lambda,\mu] \in D'$ for all such λ,μ (see Example 2). Hence D' is integrable if and only if $a''[a'\lambda,a'\mu] = 0$ for all λ,μ (not necessarily in D').

If $e_i = \dfrac{\partial}{\partial x^i}$ on some coordinate chart $(x^1,\ldots,x^n)$ and $\lambda = \lambda^i e_i$, $\mu = \mu^i e_i$

$$a''[a'\lambda,a'\mu] = a''[a'^{i}_{p}\,\lambda^p\,e_i,\ a'^{j}_{q}\,\mu^q\,e_j]$$

$$= a''\{a'^{i}_{p}\,\lambda^p\,(a'^{j}_{q}\,\mu^q)_{\circ i}\,e_j - a'^{j}_{q}\,\mu^q\,(a'^{i}_{p}\,\lambda^p)_{\circ j}\,e_i\}$$

(where "$\circ i$" denotes partial differentiation by x^i,)

$$= a''^{m}_{i}\{a'^{j}_{p}\,\lambda^p(a'^{i}_{q}\,\mu^q)_{\circ j} - a'^{j}_{q}\,\mu^q(a'^{i}_{p}\,\lambda^p)_{\circ j}\}\,e_m$$

$$= (a''^{m}_{i}\,a'^{j}_{p}\,a'^{i}_{q\circ j} - a''^{m}_{i}\,a'^{j}_{q}\,a'^{i}_{p\circ j})\,\lambda^p\,\mu^q = 0$$

for all λ,μ. Hence $a''^{'m}_{q\circ p'} - a''^{'m}_{p\circ q'} = 0$ and by (3), $a'^{i}_{q'\circ p'} - a'^{i}_{p'\circ q'} = 0$. If Γ is symmetric this condition is easily shown to be equivalent to $a'^{i}_{p'|q'} - a'^{i}_{q'|p'}$. Then using (4) we may say

D' is integrable if and only if $a^{i}_{[p'|q']} = 0$. Similarly for D".

Finally, we note that when Γ is symmetric we have the following interdependence of terms:

$$\text{Self parallel} \begin{array}{l} \Rightarrow \text{path parallel} \\ \Rightarrow \text{integrable} \end{array} \qquad \text{Self parallel} \Leftarrow \begin{array}{c} \text{path parallel} \\ + \\ \text{integrable} \end{array}$$

§3. D-Connections

A D-connection, D, on a manifold with the distributions D' and D" of §2. is a symmetric connection with respect to which D' and D" are both path parallel and parallel relative to each other. The algebraic conditions for a D-connection are thus

$$a^i_{(jk')} = a^i_{(j''k'')} = a^i_{j''k'} = a^i_{j'k''} = 0$$

Now,
$$a^i_{jk} + a^i_{kj} = a^i_{j'k'} + a^i_{j'k''} + a^i_{j''k'} + a^i_{j''k''}$$
$$+ a^i_{k'j'} + a^i_{k'j''} + a^i_{k''j'} + a^i_{k''j''}$$

Hence $a^i_{(jk)} = a^i_{(j'k')} + a^i_{(j''k'')} = 0$ so that $a^i_{(jk)} = 0$ for a D-connection. Conversely, $a^i_{(jk)} = 0 \Rightarrow a^i_{(pq)}\, a'^p_i\, a'^q_j = 0$

$$\Rightarrow a^i_{(p'q')}\, a'^p_i\, a'^q_j + a^i_{(p''q'')}\, a'^p_i\, a'^q_j = a^i_{(p'q')} = 0$$

Similarly $\quad a^i_{(p''q'')} = 0$

Also $a^i_{(jk)} = 0 \Rightarrow a^{i'}_{j''k'} + a^{i'}_{k'j''} = 0 \Rightarrow a^{i'}_{j''k'} = 0 \Rightarrow a^i_{j''k'} = 0.$

Similarly $a^i_{(jk)} = 0 \Rightarrow a^i_{j'k''} = 0$ and the necessary and sufficient conditions for a D-connection become $a^i_{(jk)} = 0$.

If Γ^i_{jk} is an arbitrary connection on M^n, any other connection, D^i_{jk} say, is given by $\Gamma^i_{jk} + V^i_{jk}$ where V^i_{jk} is symmetric in j and k. We distinguish between covariant differentiation with respect to different connections thus: $a^i_{jk}(\Gamma)$ and $a^i_{jk}\ (\Gamma+V)$.

Now, $a^{'i}_{j|k}(D) = a^{'i}_{j|k}\ (\Gamma+V)$

$$= a^{'i}_{j\cdot k} + (\Gamma^i_{hk} + V^i_{hk})\, a^h_j - (\Gamma^h_{jk} + V^h_{jk})\, a^i_h$$

$$= a^{'i}_{j|k}\ (\Gamma) + V^i_{j'k} - V^{i'}_{jk}$$

Similarly

$$a^{''i}_{j|k}(D) = a^{''i}_{j|k}(\Gamma) + V^i_{j''k} - V^{i''}_{jk}$$

Hence

$$a^i_{jk}(D) = a^{'i'}_{j|k}(D) + a^{''i''}_{j|k}(D)$$

$$= a^{'i'}_{j|k}(\Gamma) + V^{i'}_{j'k} - V^{i'}_{jk} + a^{''i''}_{j|k}(\Gamma) + V^{i''}_{j''k} - V^{i''}_{jk}$$

$$= a^i_{jk}(\Gamma) - V^{i'}_{j''k} - V^{i''}_{j'k}$$

Lastly,

$$2\,a^i_{(jk)}\ (D) = a^i_{jk}\ (D) + a^i_{kj}\ (D)$$

$$= 2\,a^i_{(jk)}\ (\Gamma) - (V^{i'}_{j''k} + V^{i''}_{j'k} + V^{i''}_{jk''} + V^{i''}_{jk'})$$

so that the most general D-connection is given by

$$D^i_{jk} = \Gamma^i_{jk} + V^i_{jk} \quad \text{where } V^i_{jk} \text{ satisfies}$$

$$2\,a^i_{(jk)}\ (\Gamma) = V^i_{j''k} + V^{i'}_{jk''} + V^{i''}_{j'k} + V^{i''}_{jk'}$$

Let $$V^i_{jk} = 2\,a^i_{(jk)}(\Gamma) - a^i_{(j'k')}(\Gamma) - a^i_{(j''k'')}(\Gamma) + C^i_{jk}$$

Then a routine calculation gives

$$V^i_{j''k} + V^{i'}_{jk''} + V^{i''}_{j'k} + V^{i''}_{jk'} = 2\,a^i_{(jk)} + C^{i'}_{j''k} + C^{i'}_{jk''} + C^{i''}_{j'k} + C^i_{jk''}$$

Hence $\Gamma^i_{jk} + V^i_{jk}$ is a D-connection if and only if

$$C^{i'}_{j''k} + C^{i'}_{jk''} + C^{i''}_{j'k} + C^{i'}_{jk''} = 0, \qquad \text{that is}$$

$$C^{i'}_{j''k'} + C^{i'}_{j''k''} + C^{i'}_{j''k''} + C^{i''}_{j'k'} + C^{i''}_{j'k''} + C^{i'}_{j'k''} + C^{i'}_{j''k''} = 0$$

All combinations of primed and double-primed suffices appear except $C^{i'}_{j'k'}$ and $C^{i''}_{j''k''}$; hence a general solution is given by

$C^i_{jk} = E^{i'}_{j'k'} + F^{i''}_{j''k''}$ where E,F are symmetric but otherwise arbitrary.

We deduce that the most general D-connection is given by

$$D^i_{jk} = \Gamma^i_{jk} + 2\,a^i_{(jk)}(\Gamma) - a^i_{(j'k')}(\Gamma) - a^i_{(j''k'')}(\Gamma) + E^{i'}_{j'k'} + F^{i''}_{j''k''}$$

CHAPTER V

Holonomy Groups

§1. Introduction

Let M^n be a differentiable manifold with a differentiable foliation $\mathcal{F}$. Let D' denote the integrable distribution given by the tangent spaces of the leaves of $\mathcal{F}$ and let D'' be any complimentary distribution. If a connection Γ on M has the property that D'' is parallel relative to D' we may define holonomy groups on each leaf L as follows:

Let $\nu(L) \xrightarrow{\pi} L$ be the bundle of p-planes D'' restricted to $L \subset M^n$. $\nu(L)$ is the set of pairs (m,w) where $m \in L$, $w \in D''_m$ and $\pi(m,w) = m$. $\nu(L)$ is naturally identified with the normal bundle of L in M^n and so has the structure of a principal $GL(p,R)$ bundle. Let γ be a closed piecewise C^t curve (that is, C^t except for a finite number of points) such that $\gamma(0) = \gamma(1) = m$ and let $\pi_\gamma : D''_m \to D''_m$ be parallel translation, with respect to Γ, along γ from $\gamma(0)$ to $\gamma(1)$. π_γ is an element of the structure group of $\nu(L)$. If $\Lambda_t(L,m)$ is the loop space of piecewise C^t curves at m then we define a map $\pi : \Lambda_t(L,m) \to GL(p,R)$ by $\pi(\gamma) = \pi_\gamma$. The image of $\Lambda_t(L,m)$ under π is a group (Kobayashi and Nomizu [1], p.71). When Γ is a D-connection, this group is denoted by $\Psi_t(L,m)$ and is called the Walker Holonomy Group of class t on L at m.

§2. Two lemmas

Lemma 1

On a C^k manifold M^n, every continuous loop $\gamma : [0,1] \to M^n$ is homotopic to some piecewise C^k loop.

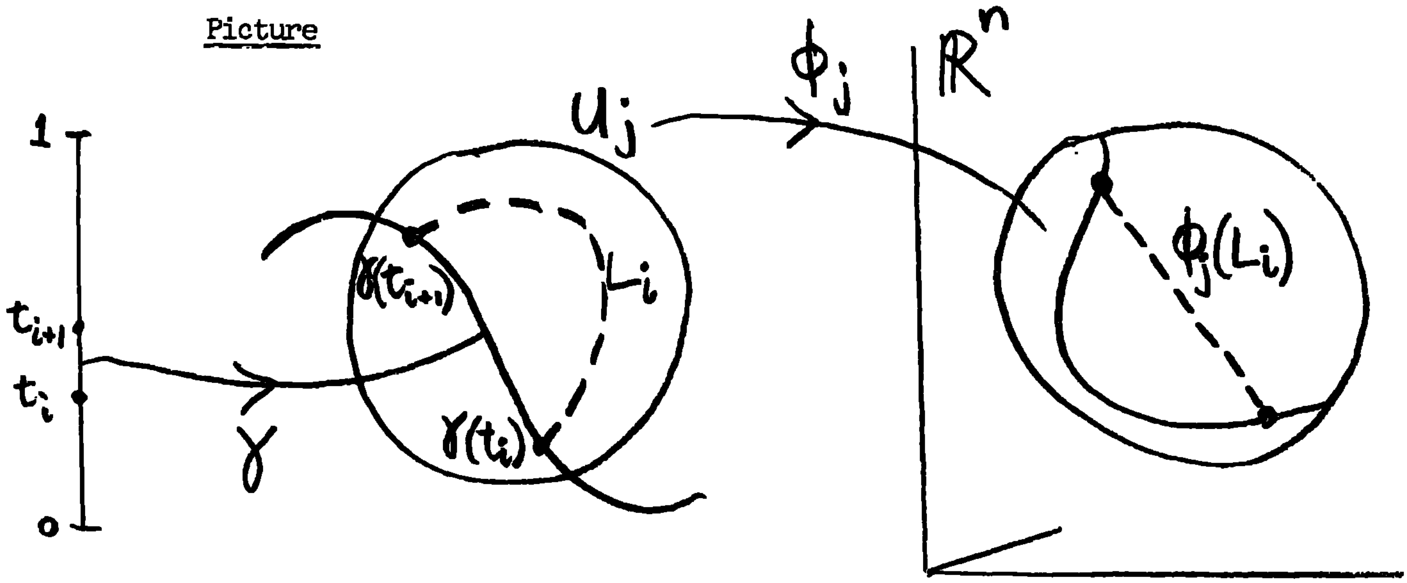
Proof

Let $\gamma(0) = \gamma(1) = m \in M^n$ and let $\{\phi_i : V_i \to R^n\}$ be a C^k atlas on M^n. Let $m \in V_j$ for some j. $\phi_j(V_j)$ is an open neighbourhood of $\phi_j \circ \gamma(0)$ in R^n and there exists a convex open set C_j such that $\phi_j \circ \gamma(0) \in C_j \subset \phi_j(V_j)$. We shall consider only those coordinate charts which are of the form $\phi_i : u_i \to R^n$ where $\phi_i(u_i)$ is convex.

Let $\{u_i\}$ be a cover for $\gamma[0,1]$. Because γ is continuous and $[0,1]$ is compact, there is a finite subcover of $\{u_i\}$. Let $\{u_i\}_{i=1,\dots,q}$ be such that $\{\phi_i^{-1}(u_i)\}_{i=1,\dots,q}$ covers $[0,1]$.

Subdivide $[0,1]$ by $[0,1] = \bigcup_{i=1}^{q} [t_i, t_{i+1}]$ such that $t_1 = 0$, $t_{q+1} = 1$, $t_1 < t_2 < \dots < t_{q+1}$ and $[t_i, t_{i+1}] \subset \phi_j^{-1}(u_j)$ for some j. $\phi_j \circ \gamma[t_i, t_{i+1}]$ is a curve in a convex subset of R^n. If $\phi_j(L_i)$ is the straight line between $\phi_j \circ \gamma(t_i)$ and $\phi_j \circ \gamma(t_{i+1})$, L_i is a C^k curve joining $\gamma(t_i)$ and $\gamma(t_{i+1})$ which is homotopic to $\gamma[t_i, t_{i+1}]$ because a convex subset of R^n is homotopic to a point. $\cup L_i$ is then homotopic to $\gamma[0,1]$ and C^k except possibly at the points $\gamma(t_i)$.

Picture



Lemma 2

On an n-manifold M^n with a differentiable foliation, given by an integrable distribution D' of dimension n-p, and a complimentary distribution D'', there exists a coordinate chart $(y^1,\ldots,y^n)$ about each point $m \in M^n$ such that

(1) The leaves of the foliation are given by y^i = constant; $i = n-p+1,\ldots,n$.

(2) On the leaf through m, D'' is spanned by $\frac{\partial}{\partial y^i}$; $i = n-p+1,\ldots,n$.

Proof

Denote suffices taking values $1,\ldots,n-p$ by α or β and suffices taking values $n-p+1,\ldots,n$ by λ or μ. By definition of a foliation there exists a chart $(x^1,\ldots,x^n)$ such that the leaves are given by x^i = constant; $i = n-p+1,\ldots,n$. Let D'' be spanned by $b^\alpha_\lambda \frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial x^\lambda}$ for some b^α_λ.

Let D'^* and D''^* be the duals of D' and D'', respectively. That is the images under the usual identification $T(M) \to T^*(M)$ of the tangent bundle with the co-tangent bundle. In $T^*(M)$, D'^* is spanned by $dx^\alpha - b^\alpha_\lambda\, dx^\lambda$ and D''^* is spanned by dx^λ, because if we put

$$\omega^\alpha = dx^\alpha - b^\alpha_\lambda\, dx^\lambda \quad \text{we find}$$

$$\omega^\alpha\left(\frac{\partial}{\partial x^\beta}\right) = dx^\alpha\left(\frac{\partial}{\partial x^\beta}\right) - b^\alpha_\lambda\, dx^\lambda\left(\frac{\partial}{\partial x^\beta}\right) = \delta^\alpha_\beta$$

$$\omega^\alpha\left(b^\beta_\lambda \frac{\partial}{\partial x^\beta} + \frac{\partial}{\partial x^\lambda}\right) = (dx^\alpha - b^\alpha_\mu\, dx^\mu)\left(b^\beta_\lambda \frac{\partial}{\partial x^\beta} + \frac{\partial}{\partial x^\lambda}\right)$$

$$= b^\alpha_\lambda - b^\alpha_\lambda = 0$$

Let $(y^1,\ldots,y^n)$ be the coordinate chart given by

$$y^\alpha = x^\alpha - b^\alpha_\lambda\, (x^\mu_o)\, x^\lambda$$

$$y^\lambda = x^\lambda$$

where x^μ_o is the value of x^μ on the leaf through m.

$$dy^\alpha = dx^\alpha - b^\alpha_\lambda\, (x^\mu_o)\, dx^\lambda$$

$$dy^\lambda = dx^\lambda$$

hence $dy^\alpha = \omega^\alpha$ on the leaf through m, D'^* is spanned by dy^α and D''^* is spanned by dy^λ. i.e. D'' is spanned by $\frac{\partial}{\partial y^i}$; $i = n-p+1,\ldots,n$ as required.

Finally, if we denote by $J = \det\left(\frac{\partial y^i}{\partial x^j}\right)$ the jacobian of the coordinate transformation $y^i = y^i\,(x^1,\ldots,x^n)$,

$$\frac{\partial y^\alpha}{\partial x^\beta} = \frac{\partial}{\partial x^\beta}\,(x^\alpha - b^\alpha_\lambda(x^\mu_i)x^\lambda) = \delta^\alpha_\beta$$

$$\frac{\partial y^\alpha}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\,(x^\alpha - b^\alpha_\lambda(x^\mu_o)x^\lambda) = -b^\alpha_\mu$$

$$\frac{\partial y^\mu}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}\,(x^\mu) = 0$$

$$\frac{\partial y^\mu}{\partial x^\lambda} = \frac{\partial}{\partial x^\lambda}\,(x^\mu) = \delta^\mu_\lambda$$

Hence $\dfrac{\partial y^i}{\partial x^j} = \left(\begin{array}{c|c} \delta^\alpha_\beta & -b^\alpha_\mu \\ \hline 0 & \delta^\lambda_\mu \end{array}\right)$ and $J = 1$. Hence $(y^1,\ldots,y^n)$ is a coordinate chart about m with properties (1) and (2).

§3. Main Theorem

On a C^k manifold M^n with differentiable foliation ($C^{\geq 2}$) of which L is a leaf,

(1) $m' \in L$ and $m'' \in L \Rightarrow \Psi_t(L,m') \cong \Psi_t(L,m'')$

(2) $\Psi_1(L,m) \cong \ldots \cong \Psi_k(L,m)$. If M^n is C^∞ k may be ∞.

(3) $\Psi_k(L,m) \cong J(L)$, the Jet group of L.

Proof

We use the notation of Lemma 2. The projection tensor a' has components,

$$a'^\beta_\alpha = \delta^\beta_\alpha,\ a'^\lambda_i = 0,\ a'^\alpha_\lambda = b^\alpha_\lambda \quad \text{where } b^\alpha_\lambda(x^\mu_o) = 0.$$

If u^λ is a parallel vector field along a path of the form y^i = constant; $i \neq \alpha$,

$$u^\lambda_{|\alpha} = u^\lambda_{,\alpha} + D^\lambda_{\mu\alpha}\, u^\mu = 0$$

From the expression for an arbitrary D-connection given in Chapter IV §2.,

$$D^\lambda_{\mu\alpha} = \Gamma^\lambda_{\mu\alpha} + 2\, a^{\ \lambda}_{(\mu\alpha)}\ (\Gamma) - a^{\ \lambda}_{(\mu'\alpha')}\ (\Gamma) - a^{\ \lambda}_{(\mu''\alpha'')}\ (\Gamma)$$

$$+ E^{\lambda'}_{\mu'\alpha'} + F^{\lambda''}_{\mu''\alpha''}\ .$$

Let $(x^1, \dots, x^n)$ be a coordinate chart such that on the leaf defined by x^μ_0 = constant, D' is spanned by $\frac{\partial}{\partial x^\alpha}$ and D" by $\frac{\partial}{\partial x^\lambda}$.

We have,

$$\begin{aligned}
D^\lambda_{\mu\alpha} &= \Gamma^\lambda_{\mu\alpha} + 2\, a^{\ \lambda}_{(\mu\alpha)}\ (\Gamma) \\
&= \Gamma^\lambda_{\mu\alpha} + a^{\ \lambda}_{\mu\alpha}\ (\Gamma) + a^{\ \lambda}_{\alpha\mu}\ (\Gamma) \\
&= \Gamma^\lambda_{\mu\alpha} + a'^{\lambda'}_{\mu|\alpha}\ (\Gamma) + a''^{\lambda''}_{\mu|\alpha}\ (\Gamma) \\
&\qquad + a'^{\lambda'}_{\alpha|\mu}\ (\Gamma) + a''^{\lambda''}_{\alpha|\mu}\ (\Gamma) \\
&= \Gamma^\lambda_{\mu\alpha} - a'^{\lambda}_{\mu|\alpha}\ (\Gamma) - a'^{\lambda}_{\alpha|\mu}\ (\Gamma) \\
&= \Gamma^\lambda_{\mu\alpha} - a'^{\lambda}_{\alpha|\mu}\ (\text{since } a'^{\lambda}_{\mu} = 0) \\
&= \Gamma^\lambda_{\mu\alpha} - a'^{\lambda}_{\alpha,\mu} - \Gamma^\lambda_{i\mu}\, a'^{i}_{\alpha} + \Gamma^i_{\alpha\mu}\, a'^{\lambda}_{i} \\
&= \Gamma^\lambda_{\mu\alpha} - \Gamma^\lambda_{\alpha\mu}\quad (\text{since } a'^{\lambda}_{i} = 0) \\
&= 0 \text{ because } \Gamma \text{ is symmetric.}
\end{aligned}$$

Hence, u^λ is parallel if and only if $u^\lambda_{,\alpha} = 0$, i.e. u^λ = constant. Hence $u^\lambda_{|\alpha} V^\alpha = u^\lambda_{,\alpha} V^\alpha = 0$ and u^λ = constant is parallel along all differentiable curves through m and parallel translation along a curve depends only on the homotopy ~~type~~ class of the curve. This result is originally due to Walker, [3].

L is connected and locally euclidean and hence path connected (Spanier [1] p.65), and if $m', m'' \varepsilon L$ there is a path α such that $\alpha(0) = m'$ and $\alpha(1) = m''$. Then by lemma 1 there is a C^k path, β, homotopic to α and therefore a map

$\tilde{\beta} : \underline{\Psi}_t(L,m') \to \underline{\Psi}_t(L,m'')$ given by

$\tilde{\beta}\,(\Pi(\gamma)) = \Pi(\alpha\,\gamma\,\alpha^{-1}) = \Pi(\alpha)\,\Pi(\gamma)\,\Pi(\alpha)^{-1}$. $\tilde{\beta}$ is a conjugacy

and $\underline{\Psi}_t(L,m') \cong \underline{\Psi}_t(L,m'')$. This proves (1). As for (2) we note that

$$\underline{\Psi}_1(L,m) \supseteq \underline{\Psi}_2(L,m) \supseteq \ldots \supseteq \underline{\Psi}_k(L,m) \text{ as subgroups,}$$

and as every piecewise C^1 loop is C^1 and every C^1 loop is homotopic to a piecewise C^k loop, $\underline{\Psi}_1(L,m) \subseteq \underline{\Psi}_k(L,m)$. Hence the $\underline{\Psi}_t(L,m)$ are all isomorphic.

Finally we prove (3).

Let $\gamma : [0,1]$ be a piecewise C^k loop at m. As in Lemma 1 $[0,1] = \bigcup_{i=1}^{q} [t_i, t_{i+1}]$ but this time with the condition that each $\gamma([t_i, t_{i+1}])$ lies in a coordinate chart $(y^1, \ldots, y^n)$ with the properties (1) and (2) of lemma 2. Let T_r and T_{r+1} be transverse discs at $\gamma(t_r)$ and $\gamma(t_{r+1})$ given by y^λ = constant. The tangent space of T_r at $\gamma(t_r)$ coincides with D" at $\gamma(t_r)$ and similarly for T_{r+1}. The local homeomorphism $\Phi : T_r \to T_{r+1}$

introduced in Chapter II §2., has the property that

$$\phi^{\lambda}(y^1,\ldots,y^n) = y^{\lambda}$$

hence Φ_* maps $D''_{(t_r)}$ to $D''_{(t_{r+1})}$ by $u^{\lambda} \to u^{\lambda}$. But we have shown that this is the map of $D''_{(t_r)}$ to $D''_{(t_{r+1})}$ given by parallel translation relative to a D-connection. Because the maps like Φ_* generate the Jet group of L we deduce that parallel translation relative to a D-connection also generates the Jet group. i.e. $\Psi_k(L,m) \cong J(L)$ as required.

<u>Note</u>

It has been thought that Walker's holonomy group is the Ehresmann group. A counter example is example B of Chapter III §4. in which the Ehresmann group is Z and the Jet group zero. The fact that the Walker holonomy group is the Jet group and not the Ehresmann group can be partially explained by continuing our introduction to Chapter IV: The projection a" is given by the first derivative of a special map so that the holonomy group will be given by the first derivative of the maps between transverse discs. This is just the Jet group.

CHAPTER VI

Properties of Holonomy Groups

§1. The Holonomy Group is not a Lie Group

Holonomy groups are, in general, lie groups (Kobayashi and Nomizu [1] p.73) and have no manifold structure only in special cases, for example, a flat connection gives rise to a zero holonomy group. We show that a D-connection on a foliated manifold is one of these special cases.

Lemma

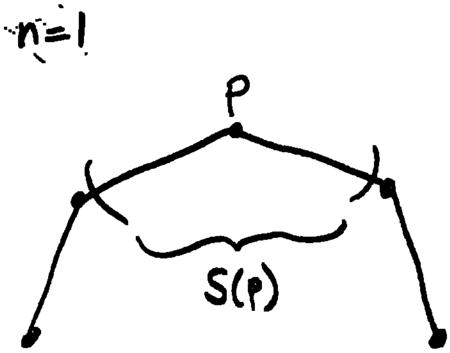
If M^n is a paracompact, connected, differentiable manifold, $\pi_1(M^n)$ is countable.

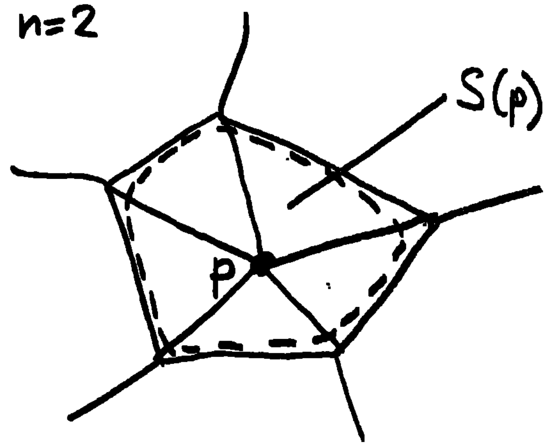
Proof

M^n supports a Piecewise Linear structure (Munkres [1] Chapter II) and for this lemma we shall assume that M^n has a triangulation. Any continuous map $\gamma : [0,1] \to M^n$ is, by the simplicial approximation theorem (Hilton and Wylie [1], p.37) homotopic to a simplicial map. Hence γ is homotopic to an edge path and can be specified up to homotopy by an ordered collection of 0-simplexes. As $[0,1]$ is compact and γ is continuous, this collection of 0-simplexes is finite.

Consider the open cover of M^n given by the open stars of the 0-simplexes. That is, if p is a 0-simplex on M^n, the star of p, denoted by u(p), is the union of all closed n-simplexes containing p. The open star of p, S(p) is then given by $S(p) = u(p) - \partial u(p)$.

Picture





The collection of sets $\{S(p) : p$ is a 0-simplex$\}$ is an open cover for M^n and each open star contains one and only one 0-simplex. Hence there is no subcover of M^n. M^n is paracompact and the topology of M^n has a countable base; hence every open cover has a countable subcover by the Lindelöf theorem (Kelley [1] p.49). In particular $\{S(p)\}$ has a countable subcover. But $\{S(p)\}$ has no subcover and hence the number of 0-simplexes is countable.

Finally, as each path on M^n is specified by a finite number of 0-simplexes we deduce that the number of paths, up to homotopy, is countable, and hence $\pi_1(M^n)$ is countable.

Corollary

If M^n is a compact, connected, differentiable manifold, $\pi_1(M^n)$ is finitely generated.

Proof

As in the lemma we find that the paths on M^n are, up to homotopy, given by a finite sequence of 0-simplexes where the number of 0-simplexes is finite. $\pi_1(M^n)$ is therefore finitely generated.

Theorem

$\Psi_t(L,m)$ is not a lie group. N.B. Our convention is that the dimension of a lie group is >0. Otherwise we may restate the theorem as "$\dim \Psi_t(L,m) = 0$."

Proof

L is a submanifold of M^n and therefore paracompact. (M^n paracompact $\Rightarrow$ existence of a metric $\Rightarrow$ existence of a metric on L $\Rightarrow$ L paracompact. See Hicks [1] p.87). $\Psi_t(L,m)$ is a factor group of $\pi_1(L)$, which is countable, hence $\Psi_t(L,m)$ is countable and cannot admit a manifold structure (of dimension >0).

We conclude this chapter with a discussion of the properties of $\Psi_t(L,m)$ when the foliation has codimension 1.

§2. Holonomy Groups of Foliations of Codimension 1

Theorem

If a foliation on a differentiable manifold has codimension 1, $\Psi_t(L,m)$ is a factor group of $H_1(L;Z)$, the first homology group with integer coefficients, and has a torsion subgroup which is either trivial or Z_2.

Proof

$\Psi_t(L,m)$ has a faithful representation in $GL(1,R) \cong R \setminus \{0\}$ under multiplication. Hence the holonomy group is abelian.

Let α be the homomorphism $\alpha: \pi_1(L) \to \Psi_t(L,m)$. Since $\Psi_t(L,m)$ is abelian, the commutator subgroup $\pi_1'(L)$ of $\pi_1(L)$ is contained in ker α. (Hall [1] p.138). Let β be the map $\beta : \frac{\pi_1(L)}{\pi_1'(L)} \to \frac{\pi_1(L)}{\ker \alpha}$ defined by $\beta: a\,\pi_1'(L) \to a \ker \alpha$ where $a \in \pi_1(L)$. If $a\,\pi_1'(L) = b\,\pi_1'(L)$, $b^{-1}a \in \pi_1'(L)$ and so $b^{-1}a \in \ker \alpha$, $a \ker \alpha = b \ker \alpha$ and β is well defined.

Also, $\beta(a\,\pi_1'(L)\; b\,\pi_1'(L)) = \beta(a\, b\, \pi_1'(L))$

$= a\, b \ker\alpha$

$= (a \ker\alpha)(b \ker\alpha)$

since $\pi_1'(L)$ and $\ker\alpha$ are both normal in $\pi_1(L)$. Hence β is a homomorphism. We define ψ by the commutative diagram:

$$\begin{tikzcd} \pi_1(L) \arrow[r, "\text{canonical}"] & \dfrac{\pi_1(L)}{\pi_1'(L)} \arrow[r, "\beta"] \arrow[rr, bend right, "\psi"'] & \dfrac{\pi_1(L)}{\ker\alpha} \arrow[r, "\cong"] & \bar{\Psi}_t(L,m) \end{tikzcd}$$

and we have $\bar{\Psi}_t(L,m) \cong \dfrac{\pi_1(L)/\pi_1'(L)}{\ker\psi}$. Now, $\dfrac{\pi_1(L)}{\pi_1'(L)} \cong H_1(L;Z)$; (Greenberg [1], p.48) and so the holonomy group is a factor of the first homology group as required.

Finally, $\bar{\Psi}_t(L,m)$ is isomrophic to a subgroup of $R \setminus \{0\}$ and the only elements of finite order are ± 1. Hence the torsion subgroup of $\bar{\Psi}_t(L,m)$ is trivial or Z_2.

<u>Corollary</u>

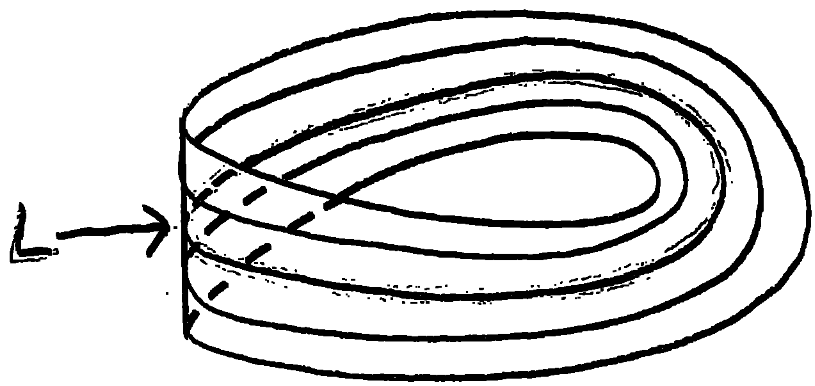
If L is compact, the isomorphism classes of $\bar{\Psi}_t(L,m)$ are Z^q and $Z_2 \times Z^q$ where q is an integer $\geqslant 0$.

<u>Proof</u>

By the corollary of §1., $\bar{\Psi}_t(L,m)$ is finitely generated. Hence $\bar{\Psi}_t(L,m) \cong \mathop{\mathsf{X}}_{i=1}^{n} Z_{p_i} \times Z^q$ (Ledermann [1] p.151). By the theorem above $\mathop{\mathsf{X}}_{i=1}^{n} Z_{p_i} \cong 0$ or Z_2 and the corollary is proved.

Example 1

If the dimension of L is 1, $\pi_1(L) \cong 0$ or Z and $\psi_t(L,m) \cong 0$, Z_2 or Z. The examples of Chapter III §4. exhibit the isomorphism classes 0 and Z. The following foliation of the moebius band exhibits the class Z_2.



Example 2

If the dimension of L is 2, $H_1(T_g;Z) \cong Z^{2g}$ where T_g is the sphere with g handles and $H_1(u_h;Z) \cong Z_2 \times Z^{h-1}$ where u_h is the sphere with h cross caps. Hence

$$\Psi_t(T_g,m) \cong Z^q \text{ or } Z_2 \times Z^{q-1} \quad \text{where } q \leq 2g$$

and $\Psi_t(u_h,m) \cong Z^q$ or $Z_2 \times Z^q$ where $q \leq h-1$.

These examples give the isomorphism classes of the Holonomy group for all compact leaves of foliations of 3-manifolds.

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