Computer methods for the interpretation of two-dimensional gravity and magnetic anomalies

Stacey, H. A.

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COMPUTER METHODS FOR THE INTERPRETATION
of
TWO-DIMENSIONAL GRAVITY AND MAGNETIC ANOMALIES

A thesis submitted for the degree of Doctor of Philosophy in the
University of Durham.

R. A. Stacey.

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University College
December 1965.
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SUMMARY

An electronic digital computer is used to automatically adjust the dimensions and density contrast or magnetisation of two-dimensional models so as to minimise the differences between the observed and theoretical gravity and magnetic anomalies. The problem of local minima in the residual function is encountered when using a simple least squares technique or an unconstrained direct search method to minimise the residuals. A constrained gradient method, due originally to Davidon, has also been tried and first results suggest it may avoid some of the local minima in the residual function. The results obtained when applying these methods to theoretical and observed anomalies are presented and discussed.

A method of calculating the pseudo-gravity equivalent of a two-dimensional total intensity magnetic anomaly is described. A modified version of this calculation gives, subject to certain assumptions, an estimate of the direction of magnetisation of the two-dimensional body producing the magnetic anomaly. No information regarding the shape of the cross-section of the body is required, except that it is bound by a closed surface.
ACKNOWLEDGEMENTS

I wish to thank Professor K.C. Dunham for providing the facilities for this work; Dr. M.H.P. Bott for his supervision; Dr. R.A. Smith for his assistance with the two-dimensional version of the Baranov transformation (given in full in Appendix I), and W.H. Swann of the Central Instrument Laboratory, I.C.I., Wilton, for providing the two optimisation procedures used in Chapter II. The work has been financed by a Hunting Surveys Limited research grant.
CHAPTER I

INTRODUCTION

The purpose of this work has been to develop methods of interpreting gravity and magnetic anomalies which take advantage of the high speeds made possible by using an electronic digital computer. Throughout it has been assumed that the length of one axis of the gravity or magnetic anomaly is very much greater than the other (i.e. the anomaly is two-dimensional), and for magnetic anomalies, that the direction of this long axis is known. It has also been assumed that the record of the gravity or magnetic anomaly will be in digitised form.

The first task was to investigate existing methods and to decide whether any of them could be programmed, either directly or in slightly modified form, to speed up their application without affecting the accuracy of their results. Under this heading several published parameter methods have been studied and discarded because the relevant parameters of a complete set of standard curves would have to be stored in the computer all the time - introducing the usual difficulties of slow access and recovery times.

One existing approach that was brought to the author's attention by Dr. Bott and has been utilized is that of "pseudo-gravity", originally proposed by Baranov in 1957 as a way of eliminating the distortion in a magnetic anomaly produced by oblique magnetisation. The mathematical development of a two-dimensional version of the Baranov conversion of a total intensity magnetic anomaly to its pseudo-gravity equivalent has been worked out by Smith and is described in Appendix I. It was originally intended that the pseudo-gravity anomaly would be interpreted as a gravity anomaly
using an optimisation method to adjust an initial model derived from the available geological information. This scheme has never been realised because it was found that the optimisation methods could be used for the direct interpretation of magnetic as well as gravity anomalies.

However, a method of determining within limits, the direction of magnetisation of a two-dimensional body without any assumptions regarding the shape of its cross-section has been derived from the expression developed by Smith for the pseudo-gravity calculation. This is discussed in Chapter V.

Kunaratnam's thesis in 1963 appears to be the first published method of interpreting total intensity magnetic anomalies by least squares techniques. The basic model that Kunaratnam uses is a horizontal prism of infinite strike length and rectangular cross-section, and either one or a "bundle" of several of these prisms is used to approximate the geological structure that gives rise to the measured anomaly. In the present thesis, this method is extended to models with horizontal upper and lower surfaces and sloping sides, and to models with a polygonal cross-section made up of any number of faces.

Other methods for the direct interpretation of gravity anomalies which utilize electronic computers are a successive approximation technique described by Bott (1960) for the interpretation of gravity anomalies which can be attributed to sedimentary basins, and a least squares method by Corbato (1965). Corbato's method has been used to interpret gravity anomalies over glaciers and is equivalent to the least squares method using an n-sided polygonal model which is described in Chapter III. The method described in this thesis was developed independently in 1964. The methods of Kunaratnam, Bott and Corbato are discussed in Chapter II.
Two methods utilizing optimisation procedures developed by I.C.I., Wilton, are also discussed in Chapter II. The first of these is an unconstrained direct search method and the other is a gradient method due originally to Davidon (1959), but since modified by Fletcher and Powell (1963) and Swann (1965, unpublished). Both methods endeavour to minimise a function of the squared residual anomalies. These optimisation procedures have only recently come to the author's notice and a full investigation of their capabilities has not been possible in the time available.

It has been found that when applying the incremental changes, calculated by any of the optimisation methods outlined above, to the dimensions of the models and to the direction and intensity of the magnetisation, or the density contrast for gravity anomalies, that the calculation often stops at a local minimum in the residual function before the global minimum has been reached.

Several months were spend trying to interpret theoretical and observed gravity and magnetic anomalies using the basic least squares method (i.e. without using the I.C.I. optimisation procedures), before the magnitude of the local minimum problem was realised. This work has not been included in the thesis except to illustrate some of the problems encountered when using the method of least squares. The I.C.I. unconstrained direct search method is generally no better in avoiding local minima in the residual function than the basic least squares method, but the constrained gradient method, due originally to Davidon, is one of the most powerful optimisation techniques at present available and may be an improvement on the other methods. A complete investigation of the capabilities of this last method has not been possible in the time available.
Following the description of these different methods in Chapter II, their application to theoretical and observed gravity and magnetic anomalies is discussed in Chapters III and IV respectively. Outlines of the relevant computer programmes are included in these chapters and the specifications for their use are given in Appendix 2.

The calculation of the pseudo-gravity equivalent of magnetic anomalies using the equation developed by Smith, mentioned earlier, is discussed in Chapter V. If it is assumed that the magnetisation of a body has the same sign throughout the body and that its direction is constant, it is possible to adapt the pseudo-gravity calculation to provide an estimated range for this magnetisation direction. This, also, is described in Chapter V, together with a discussion on the effect of various errors that may be introduced when interpreting an observed anomaly.

In the final chapter, the methods which have been developed are summarized, together with their advantages and disadvantages.
CHAPTER II

Automatic Model Adjustment in the Interpretation of Gravity
and Magnetic Anomalies.

1. Introduction
2. The least squares process
3. The direct search method
4. The Davidon gradient method
5. Limitations to the optimisation procedures and conclusions.

1. Introduction

Some idea of the size and shape of the geological structure producing a given gravity or magnetic anomaly can be obtained by comparing the observed anomaly with those calculated over simple geometric models. This thesis is confined to the interpretation of two-dimensional anomalies and the type of model generally used is a horizontal prism of infinite strike length, with a polygonal cross-section (see figure 2.1). The usual interpretation procedure is to calculate the anomalies due to a range of models based on the known geology and to compare the results, either as complete curves or as sets of parameters, with the observed anomalies. Two methods of calculating the gravity or magnetic anomalies due to these two-dimensional models have been described elsewhere, the first by Talwani, Worzel and Landisman (1959) and the second by Morgan
and Grant (1963). The method of calculating the anomalies used in this thesis is based on the equations given by Heiland (1940) for computing the gravity and magnetic effects of a semi-infinite slab with a sloping end, and is described in Chapter III for gravity anomalies and Chapter IV for magnetic anomalies. A typical interpretation procedure using the parameter method has been described by Bruckshaw and Kunaratnam (1963).

When a theoretical curve (or set of parameters) has been found that agrees well with the observed anomaly, a possible interpretation of the data has been achieved. Due to the well-known ambiguity in the interpretation of gravity and magnetic anomalies, it is impossible to be certain that the model is realistic unless certain definite information regarding the shape and density contrast or magnetisation of the body is available.

Smith (1961) has discussed what is required in addition to a known, uniform density or magnetisation contrast, to ensure that the interpretation of the gravity or magnetic data will be unique. The additional assumptions that he requires are that the gravity or magnetic anomaly is known completely, and that any line perpendicular to the plane in which the anomaly has been measured either intersects the surface of the anomalous body twice or not at all. In practice the anomaly cannot be defined completely, but Bott (1960) has shown that, in the case of gravity anomalies, only a limited part of the anomaly needs to be known if one surface (usually the top) of the anomalous body is also defined (see below). The second condition, regarding any line perpendicular to the plane in which the anomaly has been measured, can only be fulfilled if the geology of the anomalous body is known in great detail.
When the observed anomaly can be attributed to a sedimentary basin of known lateral extent, and with a known density contrast between the basin sediments and the surrounding rocks, a unique solution is possible (Bott, 1960). The shape of such a basin could be determined by trial and error, but Bott has devised a successive approximation method that can be carried out by the computer. He divides the outcrop of the sedimentary basin into two-dimensional strips, defines the observed anomaly at the centre of each of them, and then calculates the pattern of sediment thicknesses below each strip (in the form of rectangular blocks) that will account for all the values of the observed anomaly.

A somewhat similar method has been developed by Gorbato (1965) for the interpretation of gravity anomalies due to glaciers - again the upper surface of the body and its density contrast with the surrounding rocks must be defined. Gorbato assumes an initial model with a polygonal lower surface and adjusts the vertices of this model by a least squares process to reduce the differences between the observed and calculated anomalies to a minimum. This latter method can deal with any sub-surface shape provided that the number of vertices to be varied is less than the number of observations of the actual anomaly. This is a distinct improvement on Bott's more specialised successive approximation method which cannot deal with outward sloping faces at either end of the model.

Kunaratnam (Ph.D. thesis, 1963) has used a least squares method for the interpretation of total field magnetic anomalies. The basic model he uses is a horizontal rectangular prism of infinite strike length, the dimensions and magnetisation of which are automatically adjusted to
reduce the differences between the observed and computed anomalies to a minimum. For more complicated geological structures Kunaratnam uses "bundles" of these prisms. All four sides of the rectangular prism, the direction and intensity of its magnetisation are included in the adjustment process and any dimension required to be kept constant is effectively kept so by resetting it to its initial value after each adjustment. This means that any definite geological or palaeomagnetic evidence can be incorporated into the model.

In the present thesis, computer programmes have been written for the interpretation of two-dimensional gravity anomalies which apply a least squares adjustment process similar to that of Kunaratnam and Corbato to horizontal prism models with rectangular, trapezoidal and n-sided polygonal cross-sections (Chapter III). At least one face of the last model and its density contrast must be fixed - making it equivalent to Corbato's programme. A second suite of programmes has been written for the interpretation of magnetic anomalies using horizontal prism models with rectangular and trapezoidal cross-sections (Chapter IV).

The next section of this chapter deals with the theoretical development of the least squares adjustment process, and its limitations as applied to a general two-dimensional model with a polygonal cross-section.

Following this, are two alternative methods of minimising the differences between the observed and theoretical gravity or magnetic anomalies - an unconstrained direct search optimisation procedure and a constrained gradient method of optimisation. The first of these methods is applicable to two-dimensional models with a cross-section made up of
any number of polygonal faces. The second is, at present, restricted to the interpretation of magnetic anomalies and is based on a horizontal prism model of infinite strike length with a rectangular cross-section.

The application of the processes outlined in this chapter to the interpretation of theoretical and observed gravity anomalies is described in Chapter III and to magnetic anomalies in Chapter IV.

2. The Least Squares Process.

In the following discussion, which has been adapted from Berezin and Zhidkov (1965), $A_0(x_k)$ is the measured gravity or magnetic anomaly at $(x_k)$, where $k = 1, 2, \ldots, n$, the number of points along the horizontal $x$-axis at which the anomaly is known. $A_m(x_k)$ is the anomaly (calculated by a method similar to Talwani's) at $(x_k)$ due to the model defined by $a_1, a_2, \ldots, a_m$, the co-ordinates and density contrast or magnetisation of the model (fig. 2.1). Thus, if the model is a perfect representation of the actual body, the observed anomaly can be expressed as:

\[
A_0(x_1) = f(x_1; a_1, a_2, \ldots, a_m) \\
A_0(x_2) = f(x_2; a_1, a_2, \ldots, a_m) \\
\vdots \\
A_0(x_n) = f(x_n; a_1, a_2, \ldots, a_m)
\]

In practice the chosen values of $a_1, a_2, \ldots, a_m$ will not satisfy all the equations, first, because of observational errors in $A_0(x_k)$ and secondly, because it is generally impossible to represent a geological structure exactly by a model with a polygonal cross-section. The problem
Figure 2.1. The horizontal prism model with infinite strike length (perpendicular to the page) and an n-sided polygonal cross-section. (n = (j+1)/2).
is to find a set of values, \( a_1, a_2, \ldots, a_m \), for which the sum of the squared differences between the right hand sides and the left hand sides of the set of equations 2.1 is a minimum.

Utilizing all the geological knowledge available, values \( a_1^0, a_2^0, \ldots, a_m^0 \) can be assigned to \( a_1, a_2, \ldots, a_m \) and the corrections to these terms that must be calculated are \( \alpha_i \).

\[
a_i = a_i^0 + \alpha_i \quad (i = 1, 2, \ldots, m).
\]

Assuming \( \alpha_i \) is small and the function is smooth, the expression for the anomaly \( A_0(x_k) \), can be expanded in a Taylor Series:

\[
f(x_k; a_1, a_2, \ldots, a_m) = f(x_k; a_1^0, a_2^0, \ldots, a_m^0) + \sum_{i=1}^{m} \frac{f_i'(x_k; a_1^0, a_2^0, \ldots, a_m^0)}{i!} \alpha_i \]

or

\[
f(x_k; a_1, a_2, \ldots, a_m) - f(x_k; a_1^0, a_2^0, \ldots, a_m^0) = t_k \quad (k = 1 \ldots n)
\]

\[
f_i'(x_k; a_1^0, a_2^0, \ldots, a_m^0) = b_{ki} \quad (k = 1 \ldots n, \ i = 1 \ldots m).
\]

This gives us a set of conditional equations:

\[
b_{11} \alpha_1 + b_{12} \alpha_2 + \ldots + b_{1m} \alpha_m = t_1
\]

\[
b_{21} \alpha_1 + b_{22} \alpha_2 + \ldots + b_{2m} \alpha_m = t_2
\]

\[
\vdots 
\]

\[
b_{n1} \alpha_1 + b_{n2} \alpha_2 + \ldots + b_{nm} \alpha_m = t_n
\]

in which \( b_{ki} \) and \( t_k \) are known and \( \alpha_i \) has to be found. This set contains more equations than unknowns and is normally insoluble for \( \alpha_1^0, \ldots, \alpha_m^0 \).

However, we must find \( \alpha_1 \) such that all the equations in the set are satisfied with minimum error.
The least squares solution of such a set chooses values of the unknown that minimise the sum of the squared differences \( S \) between the right and left hand sides:

\[
S = \sum_{k=1}^{n} \left[ t_k - \sum_{i=1}^{m} b_{ki} \alpha_i \right]^2
\]

It follows from the existence of a minimum \( S \) that \( \alpha_1, \alpha_2, \ldots, \alpha_m \) satisfy the following set of linear equations:

\[
\frac{\partial S}{\partial \alpha_1} = 0; \quad \frac{\partial S}{\partial \alpha_2} = 0; \quad \ldots \quad \frac{\partial S}{\partial \alpha_m} = 0. 
\]

These are the normal equations of the least squares solution and there are now an equal number of equations and unknowns. In explicit form we have:

\[
\frac{\partial S}{\partial \alpha_j} = -2 \sum_{k=1}^{n} \left[ t_k - \sum_{i=1}^{m} b_{ki} \alpha_i \right] b_{kj} = 0, 
\]

or

\[
\sum_{k=1}^{n} \left\{ \sum_{i=1}^{m} b_{ki} b_{kj} \right\} \alpha_i = \sum_{k=1}^{n} b_{kj} t_k \quad (j = 1, 2, \ldots, m). 
\]

This set of equations is solved for \( \alpha_1 \) using the form of Gaussian elimination which is employed by the standard Elliott 803 Algol matrix programme.

The \( \alpha_1 \) obtained from this first evaluation are rarely the best possible solution as the original Taylor series approximation may not be strictly valid, i.e. the second and higher \( \rho_{\alpha_1} \alpha_1^2 \ldots \alpha_1 \) are not negligible. Successive iterations have to be made to improve on this first solution and the calculation will stop when one of the following criteria are satisfied.
where $e_1$ is a small, positive fraction expressing the accuracy required.

\[
(i) \quad \sum_{k=1}^{n} \left[ A_0(x_k) - A_m(x_k) \right]^2 \leq e_1 \leq \sum_{k=1}^{n} A_0(x_k)^2,
\]

where $e_2$ is again a small, positive fraction and $q$ is the number of changes made to the initial estimate of $\alpha_1$.

In practice, this standard iterative method often fails to improve on the initial solution due to the neglect of second and higher powers in the Taylor approximation. One way of improving the method that has been suggested by Levenberg (1944) is to limit the absolute values of the increments $\alpha_1$ and solve the set of equations 2.4 subject to these limiting conditions. An alternative method described by Curry (1944), which is similar to that used by Kunaratnam (1963), is to modify the calculated incremental changes $\alpha_1^q$:

\[
a_{i}^{q+1} = a_{i}^{q} + v. \alpha_{i}^{q}, \quad (i = 1, 2, \ldots, m)
\]

$a_{i}^{q}$ and $a_{i}^{q+1}$ are the calculated sets of values for $a_i$ after $q$ and $q+1$ iterations and $v (0 < v < 1)$ is chosen so that $S^{q+1} \leq S^q$, where $S^q$ and $S^{q+1}$ are the sums of the squared residuals after $q$ and $q+1$ iterations.

3. **The Direct Search Method**.

The function to be minimised is the sum of the squared differences

*The majority of this section has been taken from a paper by W.H. Swann (1964) and so his terminology has been retained to avoid confusion.*
between the observed and theoretical gravity or magnetic anomalies and can be expressed as $f(x_1,x_2,\ldots,x_n) = f(\mathbf{x})$, where $x_1,x_2,\ldots,x_n$ are the independent variables representing the co-ordinates and density or magnetisation of the model. The minimisation is done by an iterative process which defines a new point $\mathbf{x}^{q+1}$ from the present point $\mathbf{x}^q$ in such a way that $f(\mathbf{x}^{q+1}) \leq f(\mathbf{x}^q)$. The standard form of the iterative equation is:

$$\mathbf{x}^{q+1} = \mathbf{x}^q + h^q \mathbf{u}^q.$$  

$f(\mathbf{x}^q)$ and $f(\mathbf{x}^{q+1})$ are the values of the residual function after $q$ and $q+1$ iterations, $f(\mathbf{x}^q) \geq f(\mathbf{x}^{q+1})$, $h^q$ is a constant and $\mathbf{u}^q$ is an $n$-dimensional vector which determines the direction to be taken at the $q$th point. The magnitude of $h^q \mathbf{u}^q$ determines how large a step is made in that direction.

The method of direct search optimisation used is due to W.H. Swann (1964), and the optimisation programme as described in Swann's report is as follows. $n$ is the number of variables and $\mathbf{x}$ is defined above.

"Choose $n$ mutually orthogonal directions $\mathbf{u}^0_1, \mathbf{u}^0_2, \ldots, \mathbf{u}^0_n$. (For simplicity, these are always chosen to be the co-ordinate directions).

Given a starting point $\mathbf{x}^0_0$, make a univariate search in the direction $\mathbf{u}^0_1$ to locate the minimum $\mathbf{x}^0_1$.

Starting at $\mathbf{x}^0_1$, make a univariate search in the direction $\mathbf{u}^0_2$ to locate the minimum $\mathbf{x}^0_2$.

$$\ldots$$

Starting at $\mathbf{x}^0_{n-1}$, make a univariate search in the direction $\mathbf{u}^0_n$ to locate the minimum $\mathbf{x}^0_n$ ($= \mathbf{x}^1_0$).
Starting at \( x_0 \), make a univariate search in the direction given by

\[ (x_n^0 - x_0^0) \]

to locate the minimum \( x_1^1 \).

If the actual distance between \( x_0^0 \) and \( x_1^1 \) is less than the current value of the step-length, then reduce the latter and begin again, starting at \( x_1^1 \), by searching in the direction \( x_1^1 \) etc.

Otherwise, place any direction in which the actual distance moved was less than the accuracy required to the end of the list of directions. If the number of such directions is \( n-1 \) or \( n \), reduce the step-length and continue from there.

Recompute the new directions by the process described above with \( (x_n^0 - x_0^0) \) as the first direction, without altering any direction in which the actual distance moved was less than the accuracy required.

Starting at \( x_1^1 \), make a univariate search in the direction \( x_2^1 \) to locate the minimum \( x_2^1 \).

And so on until the step-length is reduced to a value less than the required accuracy (or when \( n \cdot k \) function evaluations have been made, whichever occurs first, \( n \) being the number of variables and \( k \), a specified number of function evaluations).

The use of this procedure and the results obtained are discussed in Chapters III and IV.

The first of these examples - the interpretation of the gravity high over the Cape Smith belt in Northern Quebec - indicates that constraints limiting the changes to the independent variables are required. One method of achieving this is illustrated in figure 2.2, where the constraint on the variable \( a_i \) is \( a_i \leq c_i \) (where \( a_i \) represents a co-ordinate of the model and \( c_i \) is the constraint on \( a_i \).) \( F_0 \) is the initial value of the
residual function and \( F_1, F_2 \) and \( F_3 \) are succeeding values as the univariate search proceeds. In figure 2.2, \( a_i \) is greater than \( c_i \) for \( F_3 \), therefore, \( \alpha_i \) is made equal to \( 2^{-n} \alpha_i \), where \( n = 1, 2, \ldots \) until \( a_i = a_i' \leq c_i \). The value of the residual function is now calculated for this value of \( a_i' \), giving \( F_3' \). The calculation now moves on to the next variable \( a_{i+1} \) and begins another univariate search, retaining the value of \( a_i \) or \( a_i' \) corresponding to \( F_2 \) or \( F_3' \), whichever is smaller.

An alternative method is to modify the residual function by a penalty function which is very large whenever a constraint is violated. In this way a minimum will be produced in the residual function if the variable leaves the feasible area, and the solution for \( a_i \), corresponding to this minimum, can then be determined.

Neither of these methods has yet been written into the SWANOPT procedure.

4. The Gradient Method

The direct search optimisation technique used above requires only the value of the residual function at any given value of the independent variables. If, in addition, it is possible to calculate the partial derivatives of the residual function with respect to the variables, as it is in this case, then the optimisation process can be very much more efficient. As before, the new point is defined from the current one by the equation

\[
\vec{x}^{q+1} = \vec{x}^q + h^q \cdot D^q,
\]
and for minimisation $f(x^{q+1}) \leq f(x^q)$. $h^q$ is a positive constant and $p^q$ is an n-dimensional direction vector evaluated at the $q^{th}$ iteration, determining the direction to be followed from $x^q$. The magnitude of the term $h^q$. $p^q$ determines the size of the step to be taken in that direction. The direction vector is chosen so that

$$p^q = -H^{-1} g^q,$$

where

$$g^q = \left[ \frac{\partial f}{\partial a_1}, \frac{\partial f}{\partial a_2}, \ldots, \frac{\partial f}{\partial a_n} \right],$$

and $H$ is a positive definite $n \times n$ weighting matrix. In its simplest form this is known as the Method of Steepest Descent, and all other gradient methods are derived from it.

A method due to Davidon (1959) assumes an initial positive definite value for $H$ and modifies the assumed value after each iteration, always keeping it positive definite. Davidon's original method has been modified by Fletcher and Powell (1963) and again by Swann (personal communication, 1965). Swann's computer programme was written in KDF9 Autocode (using a compiler developed by I.C.I.) and had first to be rewritten in Elliott 803 Algol for the Durham University Computer. The main features of this method have been described by Fletcher and Powell, and will not be repeated here.

The programme incorporates explicit, implicit and equality constraints, of which only explicit constraints of the form

$$a_i \leq a_i \leq f_i,$$

where $a_i (i = 1, 2, \ldots, n)$ defines the model, are used in the interpretation of gravity and magnetic anomalies. Swann uses the created response surface technique (Carroll, 1961,
(referred to by Swann)) to keep the variables within their respective feasible zones whilst reducing the residual function. A further modification introduced by Swann to improve the stability of the method, scales the explicit constraints on the independent variables to $a_i \pm 5$ units. This means that the upper surface ($z_1$ for a rectangular prism) may be constrained to lie between $z_1 + 10$ units and the lower surface ($z_2$) may be constrained to lie between $z_2 - 1000$ units, but both will be automatically scaled to have a feasible range of $\pm 5$ units.

The results obtained by this method in the interpretation of magnetic anomalies using a horizontal prism model of infinite length with a rectangular cross-section, are discussed in Chapter IV.

5. Limitations to the Optimisation Procedures and Conclusions

The most serious limitation is imposed by the presence of local minima in the residual function and applies particularly to the basic least squares and the unconstrained direct search methods. The latter method will move towards the nearest minimum in the residuals, but the basic least squares method cannot be guaranteed to do even this unless the chosen model is a very good representation of the actual body, and the observed anomaly is not distorted by anomalies from other sources. These conditions can be rarely satisfied in practice.

The modified version of Davidon's gradient method, with scaled constraints, which has been described briefly in the last section, is believed to be one of the most powerful optimisation techniques
available at present. However, the relative merits of different minimisation techniques depend on the type of problem to be optimised, and can only be judged by the results they produce. At present, Davidon's method is only used for magnetic anomalies and is restricted to a simple rectangular model, so its full capabilities remain unknown.

In the next two chapters the computer programmes based on the methods outlined in this chapter are described and their use in the interpretation of theoretical and observed anomalies is discussed.
CHAPTER III

Computer Programmes for the Interpretation of Two-dimensional Gravity Anomalies

1. Introduction.

2. Automatic adjustment of a rectangular prism model (utilizing second derivatives).

3. Automatic adjustment of a trapezoidal model.

4. Automatic adjustment of an n-sided polygonal model.
   (i) Least squares method.
   (ii) Direct search method.

5. Summary and conclusions.

1. Introduction

In this chapter the application of the least squares and direct search methods to the interpretation of gravity anomalies is discussed and some examples of the process are given. The anomalies are assumed to be two-dimensional and the shape and density contrast of horizontal prisms of infinite length are adjusted automatically until the differences between the observed and theoretical anomalies are reduced to a minimum.

The first computer programme (LSGR) uses the simplest model with a rectangular cross-section, and the positions of its vertical sides are assumed to correspond to the positions of the zeros of the second derivatives of the measured gravity anomaly. The depth of the top and bottom surfaces and the magnitude of the density contrast of the model
are adjusted by least squares until the residuals reach a minimum. The second derivatives have been used to cut down the time required for each adjustment of the model, but they are only applicable to bodies with a top surface wider than it is deep. Tests to fix this depth/width have been carried out using anomalies computed over different rectangular models. This programme has also been used to assess the errors introduced by using a rectangular model to approximate a trapezoidal body.

The second programme (LSGT) uses a model with a trapezoidal cross-section and specified dimensions, including the density contrast, are adjusted by least squares. This model has been used in the interpretation of the Rookhope gravity low (Weardale) and the gravity high over the Cape Smith belt in Northern Quebec.

The remaining programme using the basic least squares technique (LSGN) is similar to that described by Corbato (1965) and uses a model with an n-sided polygonal cross-section, the density contrast and upper surface of which are fixed. This model has been used to try to discover whether the Cape Smith volcanic sequence has a discernable "root".

The final programme (GRAVN SWANOPT) uses the unconstrained direct search developed by Swann and is also based on an n-sided polygonal model - the density contrast of which is fixed. The programme has been used for the Cape Smith anomaly and the results from this and the previous method are compared.
2. Automatic Adjustment of a Rectangular Prism Model

(Utilizing Second Derivatives)

Programme: LSGR

Specification: Appendix 2.2

The vertical sides of this model are assumed to correspond to the positions of the zeros of the second derivative of the observed anomaly. The top, bottom and density contrast are then adjusted by least squares until the residuals are minimised. The resulting model is intended to give a rough idea of the dimensions and density contrast of the anomalous mass, which can then serve as a starting point for the adjustment of a more complex model.

The observed gravity anomaly $A_0(x_i)$ is specified at equal intervals ($dx$) along a profile perpendicular to the strike of the anomaly. These symbols, together with those introduced below are defined in figures 3.1 and 3.2. The number of values of the anomaly is $n$, where $n$ is an odd number. The second derivatives are calculated in terms of the central differences, using five values of the measured anomaly ("Interpolation and Allied Tables", 1956):

$$
A''_0(x_i) = \left[ -A_0(x_{i-2}) + 6A_0(x_{i-1}) - 30A_0(x_i) + 16A_0(x_{i+1}) - A_0(x_{i+2}) \right] / 12 (d x)^2.
$$

Starting from the central value of $A''_0(x_1) (< 0)$ with $i = (n+1)/2$, the value of $i$ is decreased until $A''_0(x_1) \geq 0$.

Then assuming the second derivative to be linear between $x_{i-1}$ and $x_i$, by interpolation we obtain:
Figure 3.1. Explanation of the symbols used in section 2.

Figure 3.2. Form of the second derivative curve for a simple gravity anomaly.
\[ x_1 = -\alpha \cdot A_0''(x_i) \left/ \left[ A_0''(x_{i-1}) + A_0''(x_{i+1}) \right] \right. + x_i \]

We then reset \( i = (n+1)/2 \) and increase \( i \) until \( A_0''(x_1) \geq 0 \), and obtain,

\[ x_2 = -\alpha \cdot A_0''(x_{i-1}) \left/ \left[ A_0''(x_i) + A_0''(x_{i+1}) \right] \right. + x_{i-1} \]

Using these values of \( X_1 \) and \( X_2 \), together with the initial values of \( Z_1, Z_2 \) and the density contrast \( \rho \), the anomaly \( A_m(x_1) \) over the model is computed for \( i = 1 \ldots n \).

\[ A_m(x_i) = 2 \xi \rho \left[ z_2 (u_2 - u_{i+1}) - z_2 (u_i - u_2) - \frac{(x_i - u_2)y_2}{r_2} - \frac{(x_2 - u_i)y_i}{r_i} \right] \]

where \( u_1, \ldots, u_n \), \( u_i, \ldots, u_n \) are defined in figure 3.1 and figure 3.2.

The partial derivatives of this expression with respect to \( Z_1 (a_1) \), \( Z_2 (a_2) \) and \( \rho (a_3) \) required for the least squares process are:

\[ \frac{\partial A}{\partial a_1} = 2 \xi \rho (u_3 - u_1) \]

\[ \frac{\partial A}{\partial a_2} = 2 \xi \rho (u_2 - u_4) \]

\[ \frac{\partial A}{\partial a_3} = \frac{A_m(x_i)}{\rho} \]

As explained in Chapter II, the increments \( \alpha_k \) (\( k=1,2,3 \)) are now found by solving:

\[ \sum_{k=1}^{3} \left[ \sum_{i=1}^{n} \frac{\partial A_m(x_i)}{\partial a_k} \cdot \frac{\partial A_m(x_i)}{\partial a_s} \right] \alpha_k = \sum_{i=1}^{n} R(x_i) \cdot \frac{\partial A_m(x_i)}{\partial a_s} \]

where \( s = 1,2,3 \).
<table>
<thead>
<tr>
<th>Depth: Width</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$\rho$</th>
<th>Number of Estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Calc.</td>
<td>True</td>
<td>Calc.</td>
<td>True</td>
<td>Calc.</td>
<td>True</td>
</tr>
<tr>
<td>1:10</td>
<td>1992</td>
<td>2000</td>
<td>4008</td>
<td>4000</td>
<td>195.6</td>
<td>200</td>
</tr>
<tr>
<td>1:5</td>
<td>965</td>
<td>1000</td>
<td>2035</td>
<td>2000</td>
<td>153.4</td>
<td>200</td>
</tr>
<tr>
<td>1:2.5</td>
<td>463</td>
<td>500</td>
<td>1037</td>
<td>1000</td>
<td>148.6</td>
<td>200</td>
</tr>
<tr>
<td>1:1</td>
<td>482</td>
<td>500</td>
<td>718</td>
<td>700</td>
<td>187.7</td>
<td>200</td>
</tr>
<tr>
<td>1:0.5</td>
<td>3930</td>
<td>5500</td>
<td>8070</td>
<td>6500</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1:0.1</td>
<td>4614</td>
<td>5900</td>
<td>7386</td>
<td>6100</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1. Errors introduced by assuming the vertical sides of the anomalous structure ($X_1$ and $X_2$) correspond to the position of the zeros of the second derivatives of the gravity anomaly. $z_1$, $z_2$ and $\rho$ have been calculated by the basic least squares process.
<table>
<thead>
<tr>
<th>$D^0$</th>
<th>Number of adjustments</th>
<th>$R(x_1)_{\text{max}}$</th>
<th>Percentage error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$x_2 - x_1$</td>
<td>$z_1$</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>0.02</td>
<td>0.11</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0.03</td>
<td>0.43</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>0.05</td>
<td>0.93</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>0.05</td>
<td>1.71</td>
</tr>
<tr>
<td>20</td>
<td>3</td>
<td>0.09</td>
<td>3.21</td>
</tr>
<tr>
<td>30</td>
<td>(5)</td>
<td>-</td>
<td>-16.40</td>
</tr>
</tbody>
</table>

Table 3.2 Errors introduced when a rectangular model is used to approximate a trapezoidal body. The co-ordinates of the true model are $z_1=50$ m., $z_2=300$ m. and the width of its top surface $x_2 - x_1=2,000$ m., the density contrast $\rho = 1$ gm/cm$^3$. $D^0$ is the deviation of the sides of this model from the vertical (the width of the model increases downwards).
The new model is defined by \( a'_k = a_k + \alpha_k \) for \( k = 1, 2, 3 \) and the process is continued until either the residuals are insignificant or the rate of change of the sum of the squared residuals is very small. From tests using gravity anomalies computed over models with different width/depth, it appears that the use of the second derivative to estimate \( X_1 \) and \( X_2 \) is only justified if the depth to the top surface of the body is less than its width. These results are summarized in Table 3.1.

The LSGR programme has also been used to estimate the errors introduced when approximating a body with a trapezoidal cross-section by a rectangular prism. Gravity anomalies have been computed for a series of symmetrical trapezoidal models with a constant depth/width of \( 1/40 \) for the upper surface. The horizontal top of the basic model was 50m. from the surface and 2,000m. wide, the bottom was 300m. from the surface and the density contrast was in all cases \( 1 \text{ gm/cm}^3 \). The results are summarized in Table 3.2 and from them it is apparent that the sides of this particular body may deviate approximately \( 20^\circ \) from the vertical before a completely different model is attained - representing a second minimum in the residual function. In Table 3.2 the final calculation with \( D = 30^\circ \) was stopped after five adjustments.

3. **Automatic Adjustment of a Trapezoidal Prism Model**

Programmes: LSGT and LSGT/1

Specification: Appendix 2.2.

The automatically adjusted rectangular model discussed in the previous section is only applicable to gravity anomalies measured over very simple geological structures. Also, using second derivatives
to position the sides means that the depth/width must be less than one and that the sides must be almost vertical. Obviously, the next stage is a model with automatically adjusted sloping sides as well as variable top and bottom surfaces and density contrast. This model, with a trapezoidal cross-section, can be used to approximate any flat-topped geological structure, for example, igneous intrusions, sedimentary basins, and even anticlines or synclines if the limits of the top or bottom surfaces are allowed to converge.

The gravity anomaly over this trapezoidal model is found by a method similar to that described by Talwani, Worzel and Landisman (1959), for calculating the effect of a two-dimensional model with an n-sided polygonal cross-section. Each face defined by \( x_j z_j, x_{j+1} z_{j+1} \) as in figure 3.3, is assumed to be the end of a horizontal, semi-infinite slab, and the anomaly due to the n-sided polygon is found by summing the effects of these slabs when \( z_{j+1} > z_j \) and subtracting their effect when \( z_{j+1} < z_j \). The expression for the gravity anomaly \( \Delta A(x_i) \) over one such slab \( x_1 z_1, x_2 z_2 \) at the origin is (Heiland, 1940):

\[
\Delta A(x_i) = 2 \rho G z_0 \sin \left\{ \sum_{j=1}^{n-1} (u_j - z_i) + \sum_{j=1}^{n-1} \cos \left( u_j - z_i \right) \right\}.
\]

3.1

\( G \) is the universal gravity constant and \( \rho \) is the density contrast \((g/cm^3)\) between the model and its surroundings.

The trapezoidal model is a special case of the n-sided polygonal model and its gravity anomaly can be found simply by computing the
Figure 3.3a. The calculation of the gravity anomaly $A_m(x_1,0)$ at $(x_1,0)$ due to a semi-infinite horizontal slab bounded by a sloping face $(x_1z_1,x_2z_2)$. This, together with the second face $(x_3z_1,x_4z_2)$, defines the trapezoidal model.

Figure 3.3b. The co-ordinate system for the least squares adjustment of a trapezoidal prism model.
effect of the slab terminated by the face \( x_1 z_1 \), \( x_2 z_2 \) and subtracting the effect of the slab terminated by \( x_3 z_1 \), \( x_4 z_2 \) as in figure 3.3(a).

Initial values \( a_k \) (\( k = 1 \ldots 7 \)) for the dimensions and density contrast of the model (figure 3.3(b)) are based on the available geological evidence (or on the form of the curve if no such information exists) and the gravity anomaly \( A_m(x_i) \) due to this model is computed for \( i = 1 \ldots n \), where \( n \) is the number of points at which the anomaly has been measured on the profile. The general expression for the partial derivatives of equation 3.1 for the anomaly over the end of a single slab with respect to \( a_1 \), \( a_2 \), \( a_3 \), \( a_4 \), \( a_5 \), \( a_6 \) \( a_7 \) is:

\[
\frac{\partial A_m(x_i)}{\partial a_k} = 2A \left[ \rho \left[ \sin i \left[ \frac{z_1}{\partial a_k} \frac{\partial y_1}{\partial x_1} + \frac{z_2}{\partial a_k} \frac{\partial y_2}{\partial x_1} \right] - x_1 \frac{\partial y_1}{\partial a_k} - \frac{\partial x_1}{\partial a_k} y_1 \right]
\right.

\[
- \left\{ \sin i \left[ \frac{z_1}{\partial a_k} \frac{\partial y_1}{\partial x_1} + \frac{z_2}{\partial a_k} \frac{\partial y_2}{\partial x_1} \right] - x_1 \frac{\partial y_1}{\partial a_k} - \frac{\partial x_1}{\partial a_k} y_1 \right\}
\]

\[
+ \left\{ \sin i \left[ \frac{z_1}{\partial a_k} \frac{\partial y_1}{\partial x_1} + \frac{z_2}{\partial a_k} \frac{\partial y_2}{\partial x_1} \right] - x_1 \frac{\partial y_1}{\partial a_k} - \frac{\partial x_1}{\partial a_k} y_1 \right\}
\]

\[
+ \cos i \left[ z_1 y_2 - x_1 y_1 - \left\{ \sin i \sin i + 2 \cos i \right\} \left\{ \sin i \frac{z_1}{\partial a_k} + \cos i (y_2 - y_1) \right\} \right]
\]

\[
+ \left[ \sin i \left[ \frac{z_1}{\partial a_k} \frac{\partial y_1}{\partial x_1} + \frac{z_2}{\partial a_k} \frac{\partial y_2}{\partial x_1} \right] - x_1 \frac{\partial y_1}{\partial a_k} - \frac{\partial x_1}{\partial a_k} y_1 \right]
\]

\[b = 1, 2 \ldots 7.\]
The partial derivative with respect to \( a_k \) for the trapezoidal model is:

\[
\frac{\Delta A_m(x_i)^{\text{trap.}}}{\Delta a_k} = \frac{\Delta A_m(x_i)^{\text{rect.1}}}{\Delta a_k} - \frac{\Delta A_m(x_i)^{\text{rect.2}}}{\Delta a_k},
\]

\( k = 1, 2, \ldots, 7. \)

The incremental adjustments \( \alpha_k \) to be added to the dimensions and density contrast of the previous model \( a_k \), are found by solving (see Chapter II, equation 2.4):

\[
\sum_{k=1}^{7} \left[ \sum_{i=1}^{n} \frac{\Delta A_m(x_i)}{\Delta a_k} \cdot \frac{\Delta A_m(x_i)}{\Delta a_s} \right] \alpha_k = \sum_{i=1}^{n} C(x_i) \cdot \frac{\Delta A_m(x_i)}{\Delta a_s},
\]

\( s = 1, 2, \ldots, 7. \)

The gravity anomaly due to the new model defined by \( a_k' = a_k + \alpha_k \) (\( k = 1, 2, \ldots, 7 \)), is calculated and further adjustments are made until the residuals reach a minimum or decrease very slowly (conditions 1 and 2, Chapter II, section 2). Specified dimensions of the model or its density contrast can be kept effectively constant by resetting them to their original values after each set of adjustments. In a modified version of the programme (LSGT/1) the depth to the upper surface of the model is excluded from the adjustment process.

The LSGT programmes have been used to interpret the almost circular gravity low of about 35 mgals. over the Alston Block, centred on Rookhope village. A detailed gravity survey has been made by Bott and Masson-Smith (1957) and recently, a borehole sunk at Rookhope penetrated a granitic mass overlain unconformably by Carboniferous sediments at 430 m. from the surface (Dunham, Johnson, Bott and Hodge, 1961). The density measurements made on Lower Palaeozoic and Carboniferous volcanic and sedimentary rocks by Bott and Masson-Smith, together with those made on core samples of the granite itself, suggest that the density of the granite
Figure 3.4. N-S. section across the Rookhope gravity anomaly

Model 1. Table 3.3.

$z_i = 3.41$ m.

West

Stanhope

Mg1s.

-20

-1

0

1

2

3

4

5

6

7

8

9

10

Mg1s.

(Residuals)

Mg1s.

Mg1s.

Mg1s.

Mg1s.

Mg1s.

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is about 0.15 $\text{gm/cm}^3$ less than its surroundings.

The trapezoidal models used in the interpretation are two-dimensional and it is normally assumed that the measured profile will be perpendicular to the longest axis of the anomaly. In this case the anomaly has no well defined "long axis" and so the results will inevitably give an underestimate of the true depth. However, the interpretation is discussed at some length because the surface geology is well known and there is also the borehole data which can be taken into consideration. No attempt has been made to integrate the observed anomaly into its two-dimensional equivalent using the method suggested by Bruckshaw and Kunaratnam (1963).

The regional Bouguer anomaly over the area is between 5 mgals. and 12 mgals., (Bott and Masson-Smith, 1957) and the usual practice is to measure the anomaly relative to these limiting values and an intermediate value, which is considered to be the most likely. In the interpretation discussed below only the most likely value (11 mgals.) is used. The profile across the anomaly (figure 3.4) has been taken from the Bouguer anomaly map published by Bott and Masson-Smith (Plate XI) and corresponds to their north-south section $E-E'$. 

The dimensions of the initial model and its density contrast are based on the borehole and surface geological information, together with the approximate positions of the zeros of the second derivatives of the anomaly, which were assumed to correspond to the limits of the upper surface of the body. The difference in elevation between the top of the borehole at Rookhope, where the granitic mass is known to
The figures underlined were re-set to their initial value after each adjustment. Programme LSGT/1 was used for model 2 with $z_1$ excluded from the adjustment process. The units are: dimensions - km., density - gm/cm$^3$. NT is the number of values of the observed anomaly.
be 430 m. from the surface, and where the profile passes through Stanhope is 160 m. If the surface of the granitic mass is assumed to be level (both Rookhope and Stanhope are within the flat central region of the anomaly), this gives a depth to the anomalous mass to 270 m. at Stanhope.

The results are summarized in Table 3.3 and model 1 is shown in figure 3.4. Models 1, 2 and 3 are virtually indistinguishable from their residuals, but the residuals for models 4 and 5, using a fixed density contrast of -0.15 gm/cm³, are considerably higher - suggesting that the chosen density contrast is too large. The dimensions of models 1, 2 and 3 are very similar, as they are for models 4 and 5 (where the higher density contrast is reflected in the shallower bottom surfaces for both these models), but neither set can be discarded on the basis of the known geology.

The second example of the use of the LSGT programme is in the interpretation of the gravity high (figure 3.6) associated with the sediments and volcanics of the Cape Smith belt in Northern Quebec (this anomaly is also used to illustrate the LSGN and GRAVN SWANOPT programmes described in the next section). The belt crosses the Ungava Peninsula in an East-West direction and is 250 miles long and between 10 and 60 miles wide, and the maximum gravity anomaly of 75 mgals., is associated with the widest part of the belt. The geology has been summarized by Stam (1962) and the gravitational field of the Ungava Region, together with an interpretation of the Cape Smith anomaly, using a graticule method, has been published by the Dominion Observatory (Tanner and McConnell, 1964).

The upper surface of the initial model is assumed to be at a
Table 3.4. Possible interpretations of the Cape Smith gravity anomaly. The figures underlined were re-set to their initial values (left hand columns) after each adjustment. The units are: dimensions - Kms., density - g/cm^3. NT is the number of values of the observed anomaly.
uniform depth of 3 m., in models 1 and 2, and 5 m. in model 3, and
to have a width corresponding to that of the belt of sedimentary and
volcanic rocks shown on the geological map accompanying Tanner and
McConnell's report. The density contrast is taken to be 0.3 gm/cm$^3$
the same as that used by Tanner and McConnell. The results are
summarized in Table 3.4, and Figure 3.6.

In model 1 only the lower surface was varied to get some idea
of the thickness of the anomalous mass. This information is
incorporated into model 2 and the width of the upper surface and the
density contrast are also allowed to change. The lower surface of the
resulting model is shallower than the average depth found by the
graticule method, but this is due to the higher density contrast
found by the LSGT method. The outward slope of the northern limit of
the mass and its shallow inward slope in the south, both indicated by
the graticule method, are confirmed. The difference between the
adjusted limits of the upper surface and those taken from the geological
map is probably insignificant as most of the mapping in the area has been
on a regional basis. Model 3, with the top of the mass 5 m. from the
surface, is essentially the same as Model 2 - suggesting that an error
of a few metres in the depth of the top surface will have little effect
on the dimensions of the rest of the model.

General conclusions to be drawn from these examples regarding the
method, are discussed in the last section of this chapter.
4. **Automatic Adjustment of an n-sided Polygonal Model**

(i) Least Squares Method.

Programme: LSGN

Specification: Appendix 2.2

The computer programme (LSGN) is similar to that described by Corbato (1965) although it was independently developed. It calculates the gravity anomaly over a horizontal, two-dimensional prism with an n-sided polygonal cross-section by a method similar to that of Talwani, Worzel and Landisman (1959) and then adjusts the vertices of the prism by the least squares process developed in Chapter II. Assuming the measured anomaly is due to a single body it is possible to calculate a unique mass distribution to account for the anomaly, provided that at least one surface of the body and the density contrast between the body and its surroundings can be defined. This is the basis of Bott's (1960) successive approximation method for determining the bottom of sedimentary basins, and Corbato's (1965) least squares process for determining the sub-surface shape of glaciers. In contrast to Bott's method the LSGN model can have outward sloping faces at each end, and so can be used to approximate igneous intrusions as well as sedimentary basins. Also, any number of polygonal faces can be used to make the top of the model as good an approximation to the upper surface of the actual structure as possible. From Corbato's description, his programme does not have this facility. In the LSGN programme neither the faces making up the top surface of the model nor the density contrast are involved in the least squares adjustment process.

The expression for the gravity anomaly $A_m(x_i)$ at the point $(x_i,0)$ due
to a semi-infinite slab defined by \((x_j, z_j), (x_{j+1}, z_{j+1})\), is given in equation 3.1 and the generalised partial derivative of this expression with respect to the co-ordinates and the density contrast of the slab is given in equation 3.2. In the case of LSGN each partial derivative involves no more than three vertices, representing two adjacent slabs A and B with one co-ordinate in common, and the partial differentiation is always with respect to the common co-ordinate (figure 3.5).

\[
\frac{\partial A_m(x_i)}{\partial \alpha_k} = \frac{\partial A_m(x_i)}{\partial \alpha_k} \frac{\partial \alpha_k}{\partial \alpha_k} A + \frac{\partial A_m(x_i)}{\partial \alpha_k} \frac{\partial \alpha_k}{\partial \alpha_k} B
\]

\[\alpha_1 \equiv x_j, \quad \alpha_2 \equiv z_j\]

This means that the anomaly due to a single slab is differentiated in one of two ways, depending whether \((x_j, z_j)\) is the upper or lower co-ordinate of the face \((j\) increases in an anti-clockwise direction round the model). Four possible situations are illustrated in figure 3.5 (i) to (iv) and the derivatives in each case are found by differentiating with respect to:

(i) The lower co-ordinate of A plus the upper co-ordinate of B,

(ii) The lower co-ordinate of A minus the lower co-ordinate of B,

(iii) The upper co-ordinate of A minus the upper co-ordinate of B,

(iv) The upper co-ordinate of A plus the lower co-ordinate of B.

The reverse of situations (ii) and (iii) have not been included in the computer programme up to now, as they would imply that the body was approaching the surface — this is always possible and they should eventually be incorporated into the programme.

The main difficulty when adjusting a large number of variables is
Figure 3.5. The co-ordinate system for the LSGN model and the four different conditions encountered in the partial differentiation of the residual function.
to ensure that the residuals do decrease with successive applications of the least squares process. This can be a very serious problem when using a complicated polygonal model and the various ways of minimising it are discussed in Chapter II.

(ii) Direct Search Method.

Programme: GRAVN SWANOPT.

Specification: Appendix 2.3.

An alternative method of adjusting the n-sided polygonal model is to use a standard unconstrained direct search optimisation procedure (modified so that the initial step length can be set by the user) to minimise the differences between the observed and calculated anomalies. Only the density contrast is fixed, but the upper surface of the model is generally excluded from the adjustment process. This method is simple and will always approach a minimum in the residual function (which the basic least squares method cannot be guaranteed to do), but requires considerably more computing time. The problem of computing time is particularly critical when using a small machine such as the Elliott 803, as several hours are often needed to reach the minimum in the residual function. If it is then apparent that this is only a local minimum, a new initial model must be used and the calculation repeated, possibly more than once.

Both methods of adjusting the n-sided polygonal model have been used in an attempt to discover whether the volcanic sequence of the Cape Smith belt in Northern Quebec has a "root". The co-ordinates and density contrast of model 2, derived from the LSGT programme, have been used as a starting point and a root has been added to the lower
Figure 3.6 Model interpretation of the N-S profile across the Cape Smith gravity anomaly in Northern Quebec. The geological section is modified from Stam (1961).
surface of the trapezoidal model. Attempts have been made to adjust the dimensions of this root using the basic least squares and the direct search methods.

As can be seen from figure 3.6 there are considerable differences between the observed anomaly and that calculated over the trapezoidal model. It is apparent that the presence of a root, such as that shown below the model, will reduce the residuals, but the LSGN programme could not improve on this initial estimate and the residual function did not decrease with successive adjustments. The large differences between the observed and calculated anomalies north and south of the position of the root probably account for this instability.

Similar problems were encountered when the direct search method was used and, although the residual function decreased, the x coordinate on the south side of the root became greater than that on the north side. This example of the use of the GRAVN SWANOPT programme indicates that constraints are needed to limit the changes to the independent variables. Possible ways of incorporating these into the programme are discussed in Chapter II.

5. Summary and Conclusions

Computer programmes have been developed for the calculation of gravity anomalies over horizontal two-dimensional prisms with polygonal cross-sections. The size, shape and density contrast of an initial model can then be altered by a least squares process to reduce the differences between the observed and calculated anomalies to a minimum. The basic models used are:
(i) **Rectangular Prism.**

The positions of the sides of this model are assumed to correspond to that of the zeros of the second derivative of the observed anomaly and the top, bottom and density contrast of the model are adjusted by least squares. Any combination of these last three variables can be kept effectively constant by re-setting them to their initial value after each adjustment. It has been found that the second derivative method of fixing the sides of the model is only valid for bodies that are shallower than they are wide. The LSGR programme has also been used to assess the errors introduced when a rectangular prism model is used to approximate a trapezoidal body.

(ii) **Trapezoidal Prism.**

This model can be used to represent sedimentary basins or igneous intrusions. In one version of the programme (LSGT), all the dimensions and the density contrast of the model are involved in the adjustment process, and any variable required to be kept constant is effectively kept so by re-setting it to its initial value after each adjustment. In a second version of the programme (LSGT/1), the level of the top of the model is pre-determined and is not involved in the least squares process. The first of these programmes has been used to interpret the Rookhope (Weardale) gravity low and the gravity high associated with the Cape Smith belt in Northern Quebec.

(iii) **N-sided Polygonal Prism.**

The upper surface and the density contrast of this model are fixed.
and take no part in the adjustment process. The top of the model can be made up of any number of polygonal faces in order to approximate the upper surface of the anomalous body and the sub-surface shape of the model is adjusted until the differences between the calculated and observed anomalies are reduced to a minimum.

Two methods have been developed to improve the agreement between the observed and theoretical anomalies - the first is the basic least squares process used in the adjustment of the rectangular and trapezoidal prisms and the second is a direct search technique. Both methods have been used in an attempt to gain further information regarding the sub-surface shape of the body producing the Cape Smith gravity high (discussed earlier in connection with the trapezoidal prism model). No improvement on the initial model was obtained using the basic least squares programme (LSGN) and although the residual function did approach a minimum using the direct search method (GRAVN SWANOPT), the adjusted co-ordinates were not realistic. These results indicate that basic least squares calculation is unstable when the theoretical model is not a good representation of the actual body and that the direct search method requires some form of constraints to limit the changes to the individual variables.
CHAPTER IV

Computer Programmes for the Interpretation of Two-dimensional Magnetic Anomalies

1. Introduction.
2. General programme for calculating the magnetic effects of two-dimensional models.
4. Automatic adjustment of a rectangular prism model.
   (i) Least squares method.
   (ii) Davidon's gradient optimisation method.
5. Automatic adjustment of a trapezoidal prism model.
6. Automatic adjustment of an n-sided polygonal model.
7. Examples of the interpretation of two-dimensional anomalies using methods based on parameters, standardized curves, the basic least squares process and Davidon's gradient optimisation method.
8. Summary and conclusions.

1. Introduction

As explained at the beginning of Chapter II, some idea of the size and shape of the geological structure producing a given gravity or magnetic anomaly can be obtained by comparing the observed anomaly with those calculated over simple geometric models. Five methods of defining satisfactory theoretical models using high-speed digital computers are described in this chapter.
The first programme (MAGN) is described in section 2 and calculates the horizontal, vertical or total intensity magnetic anomaly at specified points along a profile perpendicular to the strike of a horizontal prism of infinite length with a polygonal cross-section and any strike direction. The method is similar to that used to calculate gravity anomalies, but is described here because it forms the basis of later calculations.

A modified version of this programme (MGSC, described in section J) calculates the total field intensity at equal intervals along a line perpendicular to the strike of any given two-dimensional model. The programme then scales the amplitude and the distance between the mean (or half-mean) values of the maximum and minimum of the anomaly to specified dimensions. By scaling the observed anomaly to the same dimensions two variables can be eliminated from the interpretation problem: the scale and the intensity of magnetisation of the anomalous body. Following the description of the computer programme, a scheme for the interpretation of total field magnetic anomalies due to infinite dykes is discussed.

The first least squares programme (LSMR) uses a single rectangular prism model and is similar to that developed by Kunaratnam (1963). A modified version (LSMR-T) takes into account topographic variations along the line of the observed anomaly. Both programmes have provision for specified co-ordinates of the model or its magnetisation to be kept constant by resetting them to their original values after each adjustment. A third version of the rectangular prism programme (LSMR/2) calculates the adjustments to be made by altering specified parameters only, and
so avoids any complications introduced by the re-setting procedure. The LSMR programme has been used to determine the errors introduced when a rectangular prism model is used to approximate a body with sloping sides.

An alternative method of minimising the differences between observed anomalies and those calculated over rectangular prism models is to use Davidon's gradient method of optimisation. This programme (MAGR DAVOPTC) incorporates constraints limiting the changes to the dimensions and magnetisation of the initial model, and scaling to improve the stability of the calculation. The LSMR and MAGR DAVOPTC programmes are both described in section 4 of this chapter.

The second least squares programme (LSMr) uses a trapezoidal model with horizontal upper and lower surfaces. This programme has the same provision as LSMR for keeping specified co-ordinates of the model or its magnetisation constant throughout the calculation. The final programme described in this chapter (MAGN SWANOPT) is based on an n-sided polygonal model and a direct search process (unconstrained) is used to minimise the residual function.

In section 7 several two-dimensional magnetic anomalies are interpreted and a comparison is made between the results from a typical parameter method, and the scaling, basic least squares and Davidon's methods. All of these examples are based on dyke models (extending to infinite depth in the case of the parameter and scaling methods).

In the final section the various methods are compared and summarized.
2. General Programme for Calculating the Magnetic Effects of Two-dimensional Models

Programmes: MAGN and MAGN/T

Specifications: Appendix 2.4

The method of computing the total intensity magnetic anomaly due to a horizontal, two-dimensional prism with an n-sided polygonal cross-section described here provides the basic calculation needed for the scaling and least squares methods discussed later in this Chapter. Each polygonal face of the two-dimensional model (figure 4.1) forms the end of a semi-infinite horizontal slab and the effect of the complete model is found by summing the effects of such slabs. The form of the expressions for the horizontal ($\Delta H$) and vertical ($\Delta Z$) anomalies at the point ($x_i, 0$), due to each face of the model is derived from those given by Heiland (p. 397, 1940) for the effect of a horizontal semi-infinite slab with a sloping end:

$$\Delta H (x_i, 0) = 2 \sin i \left[ J_x E_1 (x_i, 0) + J_z E_2 (x_i, 0) \right], \quad 4.1$$

$$\Delta Z (x_i, 0) = 2 \sin i \left[ J_x E_2 (x_i, 0) - J_z E_1 (x_i, 0) \right]. \quad 4.2$$

Where $E_i (x_i, 0) = \sin i (\nu_2 - \nu_1) - \cos i \ln \frac{r_2}{r_1}$,
\[ J = k_r H + R \] (c.g.s. units - see below)
\[ D^O = \text{the angle between magnetic North and the strike of the anomaly, measured clockwise.} \]

Figure 4.1. The co-ordinate system for the calculation of \( \Delta H \) and \( \Delta Z \) over an n-sided polygonal prism (the strike is perpendicular to the page). The vectors \( \mathbf{J}, \mathbf{H} \) and \( \mathbf{R} \) represent the resultant magnetization of the anomalous body, the Earth's field and the remanent magnetization of the body. \( k_r \) is the susceptibility contrast between the body and its surroundings.
The co-ordinate system is defined in Figure 4.1.

As the magnitude of the Earth's field is generally large compared with the actual anomaly, a good approximation to $\Delta F$ is given by the projection of $\Delta H$ and $\Delta Z$ in the direction of the Earth's field:

$$\Delta F(\xi, 0) = 2 \sin i \left[ \sin R \cot IE \left\{ J_x E_1 + J_z E_1 \right\} + \sin IE \left\{ J_x E_2 - J_z E_1 \right\} \right]. \tag{4.3}$$

The first part of the computer programme reads in the co-ordinates and strike of the model from the data tape and then computes $\sin i(j)$, $E_1(x_{1i}, 0; j)$ and $E_2(x_{1i}, 0; j)$ for all the faces of the model ($j = 1 \ldots k$) at each point $(x_{1i}, 0)$ in turn ($i = 1 \ldots n$). The second part of the programme reads in from the data tape, the inclination of the Earth's field in the magnetic meridian and the direction of magnetisation of the model in the plane perpendicular to the strike of the model and then calculates:

$$\sum_{i=1}^{k} \Delta H(\xi, 0); \quad \sum_{j=1}^{k} \Delta Z(\xi, 0); \quad \sum_{j=1}^{k} \Delta F(\xi, 0);$$

at $(x_{1i}, 0)$ for $i = 1 \ldots n$. The required components of the magnetic anomaly are specified at the beginning of the data tape and can now be output onto punched tape or teletypeprinter if so required.

The programme returns to the beginning of the second part and the anomaly can be recalculated using a different direction of magnetisation.
As \( \sin i(j) \), \( E_1(x_1,0;j) \) and \( E_2(x_1,0;j) \) do not depend on the direction of magnetisation, they need not be recalculated. If there are no more magnetisation directions for this particular model (signified by \(-1\ -1\) on the data tape), the first part of the programme is re-entered and a new model is read in.

Using this programme, observed anomalies can only be compared with those calculated over prismatic models on a trial and error basis. In sections 4, 5 and 6 of this chapter various ways of automatically adjusting the dimensions and magnetisation of different types of prismatic model are discussed.

3. **Scaling Method**

Programme: MGSC.

Specification: Appendix 2.5.

The magnetic anomaly at any given point is dependent on the magnetisation and shape of the body causing it, but is independent of the scale of the anomalous body in relation to the point of observation. This is true for any structure, but if the body can be approximately represented by a vertical dyke with its lower surface at infinite depth, it should be possible to calculate unambiguous values for the intensity and direction of magnetisation of the body in the plane perpendicular to the strike, and the depth and width of its upper surface. This is the basis of all interpretation methods which use an infinite dyke model and match parameters measured from an actual anomaly with a similar
set of parameters measured from theoretical curves (the Bruckshaw and Kunaratnam (1963) method is typical of this approach). Whereas a parameter method uses selected parts of the observed anomaly for comparison with the corresponding parts of a set of theoretical curves for infinite dykes with different depth/width and magnetisation directions, the scaling method described here utilizes the whole of the observed anomaly (Stacey, 1961).

The MGSC computer programme is an extension of the MAGN programme described in the previous section. After calculating the total field magnetic anomaly over the given model it normalizes the amplitude and the distance between two values of the calculated anomaly to specified limits. The first stage in the interpretation procedure is to use MGSC to prepare a library of theoretical anomalies over infinite dykes for a range of depth/width and magnetisation directions, normalized to, say, 100 gamma amplitude and a distance of 100 units between the mean values of the maximum and the minimum of the anomaly (or, if the maximum/minimum is between 0.5 and 1.5, the half-mean value). If a different model is used, for instance a dyke with its lower surface at a finite depth, the library of normalized anomalies must be extended to cover a range of different depths to this surface in addition to the ranges for different depth/width for the upper surface and for the direction of magnetisation of the model.

In the interpretation procedure described below, the observed anomalies are scaled to the same dimensions as the theoretical curves and when one of the latter has been found that matches the observed anomaly,
the true intensity of magnetisation (\(|J|\)) of the disturbing body is given by:

\[ |J| = \frac{|J|_{\text{model}}}{|J|_{\text{Amplitude scale factor for the theoretical curve}}}, \]

Amplitude scale factor for the field curve

\(|J|\) is in c.g.s. units.

Similarly, the true depth and width of the top of the body are:

\[ \text{Depth} = \frac{\text{Depth to the top of the model}}{\text{Horizontal scale factor for the field curve}}, \]

\[ \text{Width} = \frac{\text{Width of the top of the model}}{\text{Horizontal scale factor for the field curve}}. \]

The 803 Algol computer programme (MGSC) for calculating the theoretical curves is an improved version of an earlier Pegasus machine code programme (Stacey, 1961). The programme first reads in the co-ordinates of a model and calculates the total intensity magnetic anomaly over it at equally spaced intervals. It then scales the amplitude of the anomaly to the size specified on the data tape, selects the mean (or the half-mean as explained earlier) values of the maximum and the minimum of the anomaly and scales the distance between them to the amount specified on the data tape. The normalized anomaly and the scale factors used are then output on the teleprinter or tape-punch, as required.

The library of standardised anomalies is usually calculated on the assumption that the strike of the observed anomaly will be magnetic East-West, and that the profile will lie in the magnetic meridian. If this is not so, the apparent inclination of the Earth's field (IE*) in the plane
of the profile must be used instead.

\[
\text{IE}^* = \tan^{-1} \left[ \frac{\tan(\text{IE})}{\sin(D)} \right],
\]

where D is the strike of the anomaly measured clockwise from magnetic North.

4. **Automatic Adjustment of a Rectangular Prism Model**

(i) Least squares method.

Programmes: LSMR, LSMR/2 and LSMR-T

Specification: Appendix 2.6

These programmes calculate the total intensity magnetic anomaly over a horizontal rectangular prism of infinite strike length, and then adjust the dimensions and magnetisation of the model by a least squares process until the differences between the observed and calculated anomalies are reduced to a minimum. The expression for the total intensity magnetic anomaly over a horizontal rectangular prism at the point \((x_i, 0)\) is derived from equations 4.1 and 4.2 for the horizontal \(\Delta H\) and vertical \(\Delta Z\) magnetic anomalies over a semi-infinite slab with a sloping end (see figure 4.1). The anomaly due to the slab defined by \(x_{1z_1}, x_{1z_2}\) is derived from equations 4.1 and 4.2 by putting \(i = 90^\circ\):

\[
\Delta H(x_i, 0) = 2 |T| \left[ \omega s (I\theta) (x_{2} - x_{1}) + \sin (I\theta) \ln \frac{x_{2}}{x_{1}} \right],
\]

\[
\Delta Z(x_i, 0) = 2 |T| \left[ \omega s (I\theta) \ln \frac{x_{2}}{x_{1}} - \sin (I\theta) (x_{2} - x_{1}) \right].
\]
The effect of the horizontal prism is found by removing the effect of a second slab defined by \(x_2^a, y_2^a\) and the resulting expressions are:

\[
\Delta H(x_i, o) = 2\left| J \right| \left[ \cos(I\theta) \left( q_2 - q_1 - q_4 + q_3 \right) + \sin(I\theta) \left( \frac{r_3}{\eta_i} \right) \right]
\]

\[
= 2 \left[ J_x E_1(x_i, o) + J_z E_2(x_i, o) \right], \quad 4.7
\]

\[
\Delta Z(x_i, o) = 2\left| J \right| \left[ \cos(I\theta) \left( \frac{r_3}{\eta_i} \right) - \sin(I\theta) \left( q_2 - q_1 - q_4 + q_3 \right) \right]
\]

\[
= 2 \left[ J_x E_2(x_i, o) - J_z E_1(x_i, o) \right]. \quad 4.8
\]

The projection of \(\Delta H\) and \(\Delta Z\) in the direction of the Earth's field gives the total intensity anomaly due to the prism:

\[
\Delta F(x_i, o) = \Delta H(x_i, o) = \Delta Z(x_i, o) = \frac{2}{J} \left[ \sin D \cos I\theta \left( J_x E_1(x_i, o) + J_z E_2(x_i, o) \right) + \sin I\theta \left( J_x E_2(x_i, o) - J_z E_1(x_i, o) \right) \right]. \quad 4.9
\]

The least squares adjustment process requires the partial derivatives of this expression with respect to \(z_1 (a_1), z_2 (a_2), x_1 (a_3), x_2 (a_4), IB (a_5)\) and \(J (a_6)\) with \(x_1, 0\) as the origin:

\[
\delta \frac{\Delta H(x_i, o)}{\delta a_1} = \frac{\partial}{\partial a_1} \left( \frac{r_3}{\eta_i} - \frac{x_1}{r_i} \right) + \frac{\partial}{\partial z_1} \left( \frac{1}{r_3} - \frac{1}{r_i} \right),
\]
\[
\frac{\delta A_m(\xi_i, 0)}{\delta a_1} = A_1 \left( \frac{x_1}{r_1} - \frac{x_2}{r_2} \right) + A_2 z_2 \left( \frac{1}{r_2} - \frac{1}{r_4} \right),
\]
\[
\frac{\delta A_m(\xi_i, 0)}{\delta a_2} = A_1 \left( \frac{z_1}{r_1} - \frac{z_2}{r_2} \right) + A_2 z_1 \left( \frac{1}{r_2} - \frac{1}{r_1} \right),
\]
\[
\frac{\delta A_m(\xi_i, 0)}{\delta a_4} = A_1 \left( \frac{z_2}{r_4} - \frac{z_3}{r_2} \right) + A_2 z_3 \left( \frac{1}{r_2} - \frac{1}{r_4} \right),
\]
\[
\frac{\delta A_m(\xi_i, 0)}{\delta a_5} = A_1 E_1(\xi_i, 0) + A_2 E_2(\xi_i, 0),
\]
\[
\frac{\delta A_m(\xi_i, 0)}{\delta a_6} = \frac{\partial A_1}{\partial (\xi_i)} E_1(\xi_i, 0) + \frac{\partial A_2}{\partial (\xi_i)} E_2(\xi_i, 0).
\]

where
\[
A_1 = 2 |J| \left[ \sin \varphi \cos \Omega E \cos IB - \sin IE \sin IE \right],
\]
\[
A_2 = 2 |J| \left[ \sin \varphi \cos \Omega E \sin IB + \sin IE \cos IE \right],
\]
\[
\frac{\partial A_1}{\partial (\xi_i)} = 2 |J| \left[ -\sin \varphi \cos \Omega E \sin IB - \cos IB \sin IE \right],
\]
\[
\frac{\partial A_2}{\partial (\xi_i)} = 2 |J| \left[ \sin \varphi \cos \Omega E \cos IB \cos IE \sin IE \right].
\]

The remaining symbols are defined in section 2 of this Chapter and in figure 4.1.

The incremental changes to the model defined by \( a_k \) \((k = 1 \ldots 6)\) are
obtained by solving the following equation for \( \alpha_k \) \((k = 1 \ldots 6)\).

(The derivation of this equation is explained in Chapter II.)

\[
\sum_{k=1}^{6} \left[ \sum_{i=1}^{n} \frac{\partial A_m(x_i, \omega)}{\partial a_k} \right] \alpha_k = \sum_{i=1}^{n} R(x_i, \omega) \frac{\partial A_m(x_i, \omega)}{\partial a_s},
\]

\(s = 1, 2 \ldots 6\).

The new model is defined by \( a_k' = a_k + \alpha_k \) \((k = 1 \ldots 6)\), and the least squares adjustment process is continued until either the residuals are insignificant or the rate of change of the sum of the squares of the residuals is very slow. Specified dimensions of the model or its magnetisation can be reset to their initial values after each adjustment, keeping them effectively constant (see Chapter II).

This programme is written in 803 Algol and is similar to a programme in Mercury Autocode written by Kunaratnam (1963), for the same purpose. The time for each adjustment using the 803 Algol programme is 4 min. using 41 values of the observed anomaly. This is very much slower than the Mercury Autocode programme, but the higher operating speeds of the Mercury computer account for the difference.

A modified version of the 803 programme \((\text{LSMR}/2)\) permits any dimension of the model or the direction or intensity of its magnetisation to be excluded from the least squares process. The relative merits of this and the resetting procedure used in \(\text{LSMR}/\) are discussed in Chapter II. A second modification to the original programme \((\text{LSMR}/\)T\) allows topographic changes along the line of the observed profile and unequal intervals between specified values of the anomaly to be taken into account.
In order to assess the errors introduced when a rectangular model is used to approximate a trapezoidal body, the LSMR programme has been used to calculate the total intensity magnetic anomalies over a number of trapezoidal models with depth:width ratios of 1:2 and 1:10 and for sides deviating between $0^\circ$ and $20^\circ$ from the vertical. The intensity of magnetisation for the trapezoidal model was $J = 10^{-3}$ c.g.s. units and the same value was assumed for the initial rectangular model and kept effectively constant by re-setting $J = 10^{-3}$ c.g.s. units after each least squares adjustment. For all models the strike was East-West and the inclination of the Earth's field (IE) and that of the magnetisation of the model (IB) are parallel. As the anomaly for $IE = IB = 0^\circ$ has the same form as that for $IE = IB = 90^\circ$, only the results for $IE = IB = 0^\circ$ are tabulated (similarly for $IE = IB = 30^\circ$ and $IE = IB = 60^\circ$). The results are presented in Tables 4.1(a), (b) and 4.2(a), (b).

In all cases it can be seen that a good estimate of the horizontal extent of the upper surface of the trapezoidal body is obtained after the least squares adjustment of the initial rectangular prism model. The estimated inclination of the trapezoidal model's magnetisation is also good, with negligible errors when $IB = 0^\circ$ or $90^\circ$ and only very small errors when $IB = 30^\circ$ or $60^\circ$. The error in the estimated depth to the top surface of the trapezoidal model increases rapidly in all cases, being approximately 50 percent too great when the sides of the model deviate from the vertical by $20^\circ$. This over-estimate of the depth of the top surface is reflected in the calculated depth of the bottom of the model, although the percentage error is not quite so great in this case.
Tables 4.1a and b. Errors in approximating a rectangular prism model to a trapezoidal body. The true co-ordinates are in brackets and the co-ordinates of the initial model were $z_1=25, z_2=100, x_1=125$ and $x_2=175$. The intensity of magnetisation was fixed at $10^{-3}$ c.g.s. units.

$$\text{Fit} = \frac{\sum [A_o(x_i) - A_m(x_i)]^2}{\sum A_o(x_i)^2}$$

$i = 1, 2, \ldots, n$
Tables 4.2 a and b. Errors in approximating a rectangular prism model to a trapezoidal body. The true co-ordinates are in brackets and the co-ordinates of the initial model were $z_1=5$, $z_2=100$, $x_1=125$ and $x_2=175$. The intensity of magnetisation was fixed at $10^{-3}$ c.g.s. units.

$$\text{Fit} = \frac{\sum [A_o(x_i) - A_m(x_i)]^2}{\sum A_o(x_i)^2} \quad i = 1, 2... n$$
The degree of fit between the original anomaly over the trapezoidal model and that over the final rectangular prism model is expressed as the ratio of the sum of the squares of the residuals to the sum of the squares of the original anomaly for each point \( x(i) \) (\( i = 1, 2, \ldots, n \), the number of values of the original anomaly). As might be expected, this ratio increases as the difference between the dimensions of the trapezoidal model and those of the rectangular prism increase.

From this ratio alone it would appear that the agreement between the two models is better when the depth:width is 1:10 then when it is 1:2, although the actual errors in the dimensions of the rectangular prism are approximately the same in both cases. However, the larger residuals when the depth:width is 1:2 are due to a small lateral displacement of the steep part of the anomaly between its maximum and minimum, and the residuals for the same lateral displacement when this gradient is gentler, when the depth:width is 1:10, for instance, are considerably smaller. It would seem that a more realistic way of expressing the agreement between the curves would be as a function of the gradients of the two anomalies.

The LSMR programme has also been used in the comparison between various methods of interpreting magnetic anomalies using infinite dyke or rectangular prism models and the results are discussed in section 7 of this chapter.

(ii) Davidon's Gradient Optimisation Method.

Programme: MAGR DAVOPTC

Specification: Appendix 2.8
The modified version of Davidon's gradient method of optimisation, with constraints and scaling, can be used as an alternative to the basic least squares method for adjusting a rectangular prism model. The residual function to be minimised and its partial derivatives with respect to \(a_k\), the dimensions and magnetisation of the model are:

\[
F = \frac{1}{NT} \sum_{i=1}^{NT} [A_0(\chi_i) - A_m(\chi_i)]^2,
\]

\[
\frac{\delta F}{\delta a_k} = -\frac{2}{NT} \sum_{i=1}^{NT} [A_0(\chi_i) - A_m(\chi_i)] \frac{\partial A_m(\chi_i)}{\partial a_k}.
\]

Where \(i = 1,2...NT\), the number of values of the observed anomaly and \(k = 1,2...m\), the number of independent variables. The partial derivatives \(\frac{\partial A_m(\chi_i)}{\partial a_k}\) are the same as those used in the basic least squares method described above.

The method has only recently been programmed and its capabilities are, at present, uncertain, but it is believed to be capable of avoiding some of the local minima in the residual function. Even if it does not live up to expectations in this respect, it will generally reduce the value of the residual function with successive adjustments quicker than the direct search method described later.

The interpretation of two magnetic anomalies using the MAGR DAVOPTC programme is discussed, in conjunction with several other methods, in section 7 of this chapter. From the results it appears that the constant which relates the response surface to the actual residual function, set
at 50 at present, could be improved upon. This is evident from the eight "surface weight reduced" messages in succession in example HB 6. Similarly, the present factor of 0.13 for the rate of change of the constant may be improved upon. These factors depend on the character of the function being optimised and can only be chosen by trial and error. A range of values should be tried to establish the best combination for the magnetic residual function (F) defined above.

5. **Automatic Adjustment of a Trapezoidal Prism Model**

Programme: LSMT

Specification: Appendix 2.6

This programme calculates the total intensity magnetic anomaly over a horizontal prism with a trapezoidal cross-section and infinite strike length, it then adjusts the dimensions and magnetisation of this prism by a least squares process until the differences between the observed and theoretical curves are reduced to a minimum. The expression for the horizontal (ΔH) and vertical (ΔZ) magnetic anomalies at a point (x1,0) due to a horizontal semi-infinite slab with a sloping end defined by x1z1, x2z2 are given in equations 4.1 and 4.2. The total intensity anomaly (ΔF) is assumed to be the projection of ΔH and ΔZ in the direction of the Earth's field and is given by equation 4.3. The effect of a trapezoidal prism at the point (x1,0) is found by removing the effect of a second slab defined by x3z1, x4z2 and the total intensity anomaly due to the complete
model at \((x_1,0)\) is:

\[
F(x_1,0) = A_{\omega}(x_1,0) = \Delta F(x_1,0)^{\text{meas.1}} - \Delta F(x_1,0)^{\text{meas.2}} = \\
I_x \left[ \sin i_1 \left\{ J_1 E_1(x_1,0) + J_2 E_2(x_1,0) \right\}^2 - \sin i_2 \left\{ J_3 E_3(x_1,0) + J_4 E_4(x_1,0) \right\}^2 \right] \\
+ I_2 \left[ \sin i_1 \left\{ J_1 E_2(x_1,0) - J_2 E_1(x_1,0) \right\}^2 - \sin i_2 \left\{ J_3 E_4(x_1,0) - J_4 E_3(x_1,0) \right\}^2 \right].
\]

Where \(i_1\) and \(i_2\) are the inclinations of the first and second faces respectively. \(I_x = \sin(D) \cdot \cos(IE)\) and \(I_2 = \sin(IE)\). The remaining symbols have been defined earlier.

The least squares adjustment process requires the partial derivatives of this expression with respect to \(z_1(a_1), z_2(a_2), x_1(a_3), x_2(a_4), x_3(a_5), x_4(a_6), I_B(a_7)\) and \(J(a_8)\) with \((x_1,0)\) as the origin. The general form of the derivative of equation 4.10 with respect to \(a_k\) \((k = 1 \ldots 8)\) is:

\[
\frac{\partial A_{\omega}(x_1,0)}{\partial a_k} = \left[ I_x \left\{ \sin i_1 \left\{ J_1 \frac{\partial E_1}{\partial a_k} + \frac{\partial (J_1 E_1)}{\partial a_k} \right\} + \sin i_2 \left\{ J_2 \frac{\partial E_2}{\partial a_k} + \frac{\partial (J_2 E_2)}{\partial a_k} \right\} \right\} \\
+ \frac{\partial (\sin i_1)}{\partial a_k} \left\{ J_1 E_1 + J_2 E_2 \right\} \\
- \sin i_2 \left\{ J_3 \frac{\partial E_3}{\partial a_k} + \frac{\partial (J_3 E_3)}{\partial a_k} \right\} \}
\]
The incremental changes to the model defined by $\mathbf{K}_k$ ($k = 1 \ldots 8$) are obtained by solving the following equation for $\mathbf{K}_k$ ($k = 1 \ldots 8$). (The
The new model is defined by \( a_k^i = a_k + \kappa_k \) \((k = 1..8)\), and the least squares adjustment process is continued until either the residuals are insignificant or the rate of change of the sum of the squares of the residuals is very slow. Specified parameters of the model can be reset to their initial value after each adjustment, keeping them effectively constant (see Chapter II). If the residuals increase with successive adjustments to the model, then the increments are changed: \( \kappa_k^i = 2^{-s} \kappa_k \) \(s = 1...t\). The anomaly and residuals are then recalculated. This process continues until the sum of the squared residuals is less than it was after the previous full least squares adjustment to the model.

6. **Automatic Adjustment of an n-sided Polygonal Model**

Programme: MAGN SWANOPT

Specification: Appendix 2.7.

This programme adjusts specified co-ordinates of an n-sided polygonal model until the differences between the observed and calculated anomalies have been reduced to a minimum. The unconstrained direct search technique developed by Swann and described in Chapter 2 of this
thesis is used to optimise the initial model. The residual function to be minimised is

\[ F = \sum_{i=1}^{N_t} \left[ \frac{A_0(x_i) - A_m(x_i)}{A_0(x_i)} \right]^2, \]

where \( A_0(x_i) \) and \( A_m(x_i) \) are the observed and calculated anomalies at \( x_i \) and \( N_t \) is the number of values of the observed anomaly.

The SWANOPT procedure has been modified to permit the user to specify the initial step length for the univariate search. Care must be used in selecting the initial step-length as it will be used for all the variables and although a particular change may be realistic for the lower surface of the model, the anticipated change in the co-ordinates nearer the top of the model may be considerably less. Therefore, it is best to choose an average step-length for all the co-ordinates, and to allow the step to increase or decrease in the usual way.

Although the programme has only been used sufficiently to establish that it is working, it is anticipated that some kind of constraint on the variables will be required. Ways of incorporating these constraints into the SWANOPT procedure are discussed in Chapter 2. A second change which could be made is provision for different initial step-lengths for different variables, permitting for instance, an initial step-length of 500 units for co-ordinates defining the lower surface of the model and 5 units for co-ordinates defining the model's upper surface. This change would be expected to reduce the number of function evaluations required to reach a minimum.
Examples of the Interpretation of Two-dimensional Anomalies Using Methods Based on Parameters, Standardised Curves, the Basic Least Squares Process and Davidon's Gradient Optimisation Method.

The total intensity magnetic anomalies over two Permo-Carboniferous dykes in North-East England have been measured and are used to illustrate various interpretation methods. The measured profiles lie in the plane of the magnetic meridian and the strike of the dykes is approximately magnetic east-west. The amplitude of the first, HB 6, is approximately 900 gamma and of the second, CP 2+10, about 1,800 gamma. In both cases the geology suggests that it is a vertical dolerite dyke intruded into a limestone, sandstone, shale sequence which causes the magnetic anomaly.

Four methods have been used to interpret the anomalies:


2. The normalised curve technique described in section 3 of this chapter - again the model used is a dyke with its lower surface at great depth.

3. The basic least squares process using a model with a rectangular cross-section and with the size of the incremental changes to the variables limited in the manner described in Chapter 2.

4. The final method uses the modified Davidon gradient technique with upper and lower limits on all the variables. The model used in this case is also a rectangular prism.

The results are summarized in Table 4.3. The first three methods (Bruckshaw, Scaling and LSMR) agree well with one another, except that
Table 4.3  Comparison of various interpretation methods.

Bruckshaw: Bruckshaw and Kunaratnam parameter method (infinite dyke model).

Scaling: Comparison of normalised observed and theoretical curves (infinite dyke model).

LSMR: Basic least squares adjustment of a rectangular prism.

M. DAVOPTC: Gradient optimisation of a rectangular prism model.

All dimensions are in feet, J is in cgs units and IB in degrees. $x_1, x_2$ are defined in figure 4.1.
the Bruckshaw and Kunaratnam parameter method gives a shallower depth to the top of the body in both cases. The MAGR DAVOPTC programme arrives at a model for HB 6 which is shallower but otherwise in good agreement with those arrived at by the other methods, but the MAGR DAVOPTC model for CP 2+10 is deeper and considerably narrower than the other models. However, there was not enough computer time available to complete the calculation and so this last model may not be realistic. The differences between the MAGR DAVOPTC and the LSMR models for both anomalies may be partly due to the fact that only twelve values of the observed anomaly were used in the former calculations, whereas 49 were used for HB 6 and 41 for CP 2+10 in the LSMR calculation. Also, it should be pointed out that it may be possible to improve the rate of convergence when using MAGR DAVOPTC by changing the initial position and the subsequent rate of change of the response surface as explained in section 4 (ii) of this chapter.

Tables 4.4 and 4.5 show the intermediate results in computing the models for HB 6 and CP 2+10 using the LSMR programme. In both cases considerable improvement in the residuals has been achieved. Unfortunately, this steady improvement of the residuals is not always possible using the basic least squares method when the initial model is not very good. Table 4.6 shows the corresponding results for MAGR DAVOPTC and it is clear that the improvement in the residuals is not so great, nor is the convergence as fast as for the LSMR programme. However, the full capabilities of this programme are unknown at present.
<table>
<thead>
<tr>
<th>Estimate</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$J \cdot 10^3$</th>
<th>IB</th>
<th>$\frac{1}{NT} \sum_{i=1}^{NT} R(x_i)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.0</td>
<td>500</td>
<td>110</td>
<td>125</td>
<td>2.00</td>
<td>46</td>
<td>4.754</td>
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<tr>
<td>1</td>
<td>6.6</td>
<td>450</td>
<td>113</td>
<td>126</td>
<td>2.30</td>
<td>52</td>
<td>2.295</td>
</tr>
<tr>
<td>2</td>
<td>7.3</td>
<td>460</td>
<td>113</td>
<td>128</td>
<td>2.60</td>
<td>50</td>
<td>330</td>
</tr>
<tr>
<td>3</td>
<td>7.6</td>
<td>506</td>
<td>112</td>
<td>128</td>
<td>2.50</td>
<td>50</td>
<td>199</td>
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<tr>
<td>4</td>
<td>7.4</td>
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<td>112</td>
<td>129</td>
<td>2.42</td>
<td>49</td>
<td>196</td>
</tr>
<tr>
<td>5</td>
<td>7.4</td>
<td>613</td>
<td>112</td>
<td>129</td>
<td>2.41</td>
<td>49</td>
<td>195</td>
</tr>
<tr>
<td>6</td>
<td>7.4</td>
<td>658</td>
<td>112</td>
<td>129</td>
<td>2.40</td>
<td>49</td>
<td>195</td>
</tr>
</tbody>
</table>

Table 4.4  HB 6 - successive estimates for the dimensions and magnetisation of the dyke using programme LSMR. Units: for the dimensions in feet multiply by 2.24, $J$ is in c.g.s. units and IB in degrees. $NT = 49.$
<table>
<thead>
<tr>
<th>Estimate</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$J \times 10^3$</th>
<th>IB</th>
<th>$\frac{\sqrt{NT}}{\sum \text{c} R(x_1)^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>85</td>
<td>105</td>
<td>2.00</td>
<td>57</td>
<td>138,209</td>
</tr>
<tr>
<td>1</td>
<td>11.0</td>
<td>450</td>
<td>87</td>
<td>116</td>
<td>2.30</td>
<td>63</td>
<td>65,155</td>
</tr>
<tr>
<td>2</td>
<td>9.9</td>
<td>495</td>
<td>83</td>
<td>109</td>
<td>2.60</td>
<td>69</td>
<td>55,315</td>
</tr>
<tr>
<td>3</td>
<td>8.9</td>
<td>545</td>
<td>87</td>
<td>116</td>
<td>2.90</td>
<td>67</td>
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</tr>
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<td>4</td>
<td>8.3</td>
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<td>69</td>
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<tr>
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<td>85</td>
<td>113</td>
<td>3.80</td>
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<td>8.0</td>
<td>784</td>
<td>85</td>
<td>113</td>
<td>3.81</td>
<td>68</td>
<td>138</td>
</tr>
</tbody>
</table>

Table 4.5  CP 2×10 - successive estimates for the dimensions and magnetisation of the dyke using programme LSMR. Units: for the dimensions in feet multiply by 1.18, $J$ is in c.g.s. units and IB in degrees. $NT = 41$. 
### HB 6

<table>
<thead>
<tr>
<th>Estimate</th>
<th>$z_1$</th>
<th>$z_2$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$J \cdot 10^3$</th>
<th>IB</th>
<th>$\frac{\sum N T}{N T} \cdot R(x_1)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.0</td>
<td>500</td>
<td>110</td>
<td>125</td>
<td>2.00</td>
<td>46</td>
<td>5640</td>
</tr>
<tr>
<td>15</td>
<td>5.99</td>
<td>500</td>
<td>109</td>
<td>126</td>
<td>2.24</td>
<td>46.5</td>
<td>3206</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Surface weight reduced eight times</td>
</tr>
<tr>
<td>23</td>
<td>5.99</td>
<td>500</td>
<td>109</td>
<td>126</td>
<td>2.24</td>
<td>46.5</td>
<td>3112</td>
</tr>
</tbody>
</table>

### CP 2+10

<table>
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<th>Estimate</th>
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<th>$z_2$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$J \cdot 10^3$</th>
<th>IB</th>
<th>$\frac{\sum N T}{N T} \cdot R(x_1)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10.0</td>
<td>500</td>
<td>85</td>
<td>105</td>
<td>2.00</td>
<td>57</td>
<td>165,254</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Surface weight reduced</td>
</tr>
<tr>
<td>21</td>
<td>9.95</td>
<td>496</td>
<td>84.5</td>
<td>106.4</td>
<td>4.12</td>
<td>50.6</td>
<td>81,391</td>
</tr>
<tr>
<td>23</td>
<td>9.95</td>
<td>496</td>
<td>84.5</td>
<td>106.4</td>
<td>4.12</td>
<td>50.6</td>
<td>81,391</td>
</tr>
</tbody>
</table>

Table 4.6 Successive estimates for the dimensions and magnetisation of the dyke from profiles HB 6 and CP 2+10 using programme MAGR DAVOPTC. Units: for dimensions in feet multiply HB 6 by 2.24 and CP 2+10 by 1.18, J is in c.g.s. units and IB in degrees. MT = 12 for both anomalies.
8. Summary and Conclusions

Computer programmes (MAGN and MAGN/T) have been described for calculating the horizontal, vertical and total field magnetic anomalies at specified points along a profile perpendicular to the strike of a horizontal prism of infinite length. The model has a polygonal cross-section and each face represents the end of a semi-infinite slab, and the anomaly due to the whole model is found by summing the effects of these slabs at the point concerned. This method is similar to that described by Talwani, Worzel and Landisman (1959) for the calculation of gravity anomalies over two-dimensional models.

A modified version of the MAGN programme (MGSC) calculates the total intensity magnetic anomaly in the same way and then scales the amplitude and the distance between the mean, or half-mean, values of the maximum and the minimum of the anomaly to specified limits. This programme is the basis of the "scaling method" of interpretation and is used to complete the library of normalised curves which are then compared with similarly scaled observed anomalies. The library of curves can be computed either for the strike of the anomaly or for a magnetic east-west strike, in which case, the direction and intensity of the magnetisation established by matching the observed and theoretical curves must be corrected in the manner outlined by Bruckshaw and Kunaratnam (1963).

The remaining programmes calculate the total intensity magnetic anomaly over a given model and then modify the magnetisation and co-ordinates of the model by various optimisation processes until the
differences between the observed and theoretical curves have been reduced to a minimum. The first optimisation method is an iterative least squares process and the basic models are a rectangular prism (LSMR) and a prism with a trapezoidal cross-section (LSMT). If the initial model is not a good representation of the actual body, this method will not always improve the residual function with successive iterations. Also, it has been found that the changes to the independent variables must be limited to prevent unrealistic results.

A more sophisticated gradient method for reducing the sum of the squared residuals is a modified version of Davidon's (1959) variable metric minimisation process. This has been used to optimise the dimensions and magnetisation of a prism model with a rectangular cross-section (MAGR DAVOPTC), but the method has not yet been fully developed. Even so, Davidon's method will always reduce the value of the residual function and the programme incorporates constraints on the variables, which limit each one to a feasible zone.

The final technique which has been used is a direct search method developed by Swann (unpublished) which will optimise the co-ordinates of a model with an n-sided polygonal cross-section. This process will always reduce the value of the residual function, but the changes to the variables need to be limited to prevent the development of an unrealistic model.

As the final result of an interpretation depends largely on the choice of the initial model, the model must incorporate all the geological information available. If no such information exists, the suggested
interpretation procedure is to use a simple parameter method, such as that described by Bruckshaw and Kunaratnam (1963), and to optimise the model this gives. In addition, the calculation of the pseudo-gravity equivalent of the observed magnetic anomaly can provide, subject to certain assumptions which are discussed in the next chapter, a permissible range for the direction of magnetisation of the anomalous body.
CHAPTER V

Pseudo-gravity

1. Introduction.

2. Evaluation of the two-dimensional pseudo-gravity integral.
   (i) The pseudo-gravity calculation when $\eta > 0$.
   (ii) The pseudo-gravity calculation when $\eta = 0$.
   (iii) The pseudo-gravity calculation of the permissible range of values for $\beta$.

3. The computer programmes and theoretical examples.

4. Sources of error in the calculation of $\beta$.

5. Applications and conclusions.

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1. Introduction

Baranov (1957) has published a method of transforming three-dimensional total intensity magnetic anomalies, which are generally asymmetric due to an oblique direction of magnetisation, into pseudo-gravity anomalies using Poisson's relationship connecting gravity and magnetic potentials:

$$ J \cdot \mathbf{U} = G \rho \mathbf{V} \quad (\text{see note at foot of page}). $$

Where $\mathbf{U}$ is the gravitational potential and $\mathbf{V}$ the magnetic potential. $\mathbf{J}$ is the magnetisation vector and can vary in magnitude within the magnetised body, but not in direction. $G$ is the gravitational constant and $\rho$ the density contrast, which must vary in the body such that $\frac{|\mathbf{J}|}{\rho}$ is constant for all parts of the body. The Baranov transformation assumes a conventional density of $G \rho = |\mathbf{J}|$ and that the magnetisation of the body is parallel to the Earth's field, but it assumes nothing else. Note $J_\mathbf{U}$ is the derivative of the gravitational potential in the direction of the magnetisation vector $\mathbf{J}$.
regarding the shape of the disturbing body. The pseudo-gravity anomaly resulting from the Baranov transformation is due to a magnetised body and is related to a fictitious density distribution defined as above, but otherwise it has all the characteristics of a gravity anomaly and can be interpreted as such. This aspect is discussed in the last section of this chapter.

If the additional assumption is made that the intensity of magnetisation has the same sign throughout the body, then it follows that the computed pseudo-gravity anomaly must always have this same sign. For any given total field magnetic anomaly there is only a limited range of magnetisation directions that satisfy this condition. The practical details of setting these limits to the range of possible directions of magnetisation for any given two-dimensional total field magnetic anomaly are discussed in section 2(iii) below.

Baranov's original expression for the transformation is:

$$ g(M) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(M,P) T(P) \, dS , $$

where $T(P)$ is the measured value of the magnetic field at each point $P$ on the datum plane, and $H(M,P)$ is the kernel of the transformation allowing the direct computation of $g(M)$, the pseudo-gravity anomaly. For the kernel as given by Baranov the magnetisation of the body must be parallel to the Earth's field and so the assumption that the pseudo-gravity anomaly will always be one sign if the magnetisation is the same sign throughout the body is not applicable.

A two-dimensional version of the Baranov transformation, which does not require the magnetisation of the body to be parallel to the Earth's
field, has been developed by Dr. R.A. Smith (Appendix I) for the purpose of this thesis. Smith's expression for the pseudo-gravity equivalent of a total field magnetic anomaly over a closed two-dimensional body is:

\[ P_G(\xi, \eta) = \frac{\mu_0}{3\pi} \int_0^\infty T(\kappa, \omega) K(\xi - \kappa, \eta) \, d\kappa, \]

for \( \eta > 0 \).

(The coordinate system is explained in Figure 5.1.)

This gives the pseudo-gravity \( P_G(\xi, \eta) \) at the point \((\xi, \eta)\) in terms of the magnetic anomaly \( T(x, 0) \) at all points on the \( x \)-axis.

The kernel function, as defined by Smith is:

\[ K(\xi - \kappa, \eta) = \cos \beta \cdot \ln(\kappa) - \sin \beta \cdot \theta. \]

Where \((r, \theta)\) are the polar co-ordinates of \((\xi, \eta)\) from \((x, 0)\) and \(\beta = \frac{\pi}{2} + \mu \cdot \sigma - \alpha \cdot (\cos \alpha, \sin \alpha)\) are the direction cosines of the measured gravity field and if we take the vertical component of this field \((\kappa = \frac{\pi}{2})\), \(\beta = \mu \cdot \sigma \cdot (\cos \mu, \sin \mu)\) and \((\cos \sigma, \sin \sigma)\) are the direction cosines of the total magnetisation vector within the body and the direction of the Earth's field respectively, in the plane of the profile perpendicular to the strike of the anomaly. If this plane is not that of the magnetic meridian, then the apparent inclination of the Earth's field \((\sigma' \cdot \chi)\) in the plane of the profile must be used instead.

\[ \sigma' \cdot \chi = \tan^{-1} \left[ \tan(\sigma') \right]. \]

Where \(D\) is the angle between magnetic north and the strike, measured clockwise. Smith's symbols \(\sigma'\) (or \(\sigma' \cdot \chi\)) and \(\mu\) are used in section 2 of this chapter to define the dip of the Earth's field and the inclination
of the magnetisation of the body in the plane perpendicular to the strike of the anomaly. Elsewhere in this chapter the symbols IE (or IE*) and IB are used. These have been defined earlier in the thesis to represent the inclination of the Earth's field and that of the magnetisation of the body respectively.

2. Evaluation of the Two-dimensional Pseudo-gravity Integral

The method of evaluating integral 5.1 is described in this section, first when \( \gamma > 0 \) and then when \( \gamma = 0 \). The method of calculating the permissible range of \( \gamma \) when the pseudo-gravity equivalent of the magnetic anomaly is assumed to be always positive is also described.

The integral 5.1 giving the pseudo-gravity anomaly at the point \((x, \gamma)\) requires the magnetic field to be known completely between \( x = \pm \infty \), which, in practice, is impossible. The best that can be done is to define the anomaly at discrete points over a limited range \((-R \leq x \leq +R)\) and to assume the form of the curve outside this range. Taking the origin \((x = 0, y = 0)\) to be approximately over the centre of the body, the integration can be carried out numerically using Simpson's rule between \( x = -R \) and \( x = R \). For \( x \geq |R| \) the anomaly is approximated by the function:

\[
\tau(x, \gamma) = \sum_{n=-1}^{N} a_n \psi^{-n} \quad (x \geq |R|) \tag{5.14}
\]

which is a solution of Laplaces equation valid as \( |x| \to \infty \) and can be integrated from \(-\infty\) to \(-R\) and from \( R \) to \( \infty \). Also we have for any magnetic anomaly:
The magnetic anomaly \( T(x, 0) \) is measured relative to a background anomaly which has been fixed by inspection, but if the value of this background is in error, that is if

\[
\int_{-\infty}^{+\infty} T(x, 0) \, dx \neq 0,
\]

it can be corrected when the constants \( a_n \) are calculated for the polynomial approximation for the anomaly when \( x \geq |R| \). The values of the anomaly that must satisfy the function 5.4 are taken to be \( [T(-R, 0) + k] \) and \( [T(R, 0) + k] \), where \( k \) is the number of gamma that the background is in error.

The equations to be solved are:

\[
\int_{-\infty}^{+\infty} T(x, 0) \, dx = \int_{-\infty}^{+\infty} \sum_{n=1}^{N} a_n \, x^{-(n+1)} \, dx + \int_{-\infty}^{+\infty} a_n \, x^{-(n+1)} \, dx = 0.
\]
\[ T(-R,0) + k = \sum_{n=1}^{N} a_n (R)^{-(n+1)} \]

\[ T(R,0) + k = \sum_{n=1}^{N} a_n (R)^{-(n+1)} \]

\[ \sigma = \int_{-R}^{R} (T(x,0) + k) \, dx + \int_{-\infty}^{-R} a_n x^{-(n+1)} \, dx + \int_{R}^{\infty} a_n x^{-(n+1)} \, dx. \]

If the series is truncated at \( N=2 \) these equations give \( a_1 \), \( a_2 \) and \( k \). This was incorporated into one of the computer programmes, but the accuracy of the polynomial approximation for the anomaly when \( x \geq |R| \) was not satisfactory – the results are discussed in section 4 of this chapter.

To improve the accuracy the function,

\[ T(x,0) = \sum_{n=1}^{N} a_n x^{-(n+1)} \]

must be evaluated to \( N = 4 \), and the additional equations required are:

\[ T(R - \bar{x}) + k = \sum_{n=1}^{N} a_n (R - \bar{x})^{-(n+1)} \]

\[ T(\bar{x} - R) + k = \sum_{n=1}^{N} a_n (\bar{x} - R)^{-(n+1)} \]

Where \( \bar{x} \) is the interval between successive values of the magnetic anomaly \( T(x, 0) \).
The values of $a_1$, $a_2$ and $k$ from the first set of equations ($N = 2$) are:

$$a_1 = \frac{R}{4} \left[ R \left\{ \tau (-R, 0) + \tau (R, 0) \right\} - \int_{-R}^{+R} \tau (u, 0) \, du \right],$$

$$a_2 = \frac{R^2}{2} \left[ \tau (R, 0) - \tau (-R, 0) \right],$$

$$k = -\frac{1}{4} \left[ \frac{1}{R} \int_{-R}^{R} \tau (x, 0) \, dx + \tau (-R, 0) + \tau (R, 0) \right],$$

and when $N = 4$, the values of $a_1$, $a_2$, $a_3$, $a_4$ and $k$ using two more values of the magnetic anomaly are:

$$a_1 = e_1 k + e_2,$$

$$a_2 = \frac{R^2 \sum A \mathbf{M} - (R - \pi)^2 A_2 \mathbf{M}}{2 \left[ R^2 - (R - \pi)^2 \right]}$$

$$a_3 = f_1 k + f_2,$$

$$a_4 = -\frac{R^2 (R - \pi)^2 \left[ R^2 A \mathbf{M} - (R - \pi)^2 A_2 \mathbf{M} \right]}{2 \left[ R^2 - (R - \pi)^2 \right]}$$

$$k = -\frac{\left[ \frac{R^2 A + 2 e_2 \sum A^2 - \frac{3}{2} f_2}{R^4 + 2 e_2 R^2 + \frac{3}{2} f_1} \right]}{\left[ \frac{R^2 A + 2 e_2 \sum A^2 - \frac{3}{2} f_2}{R^4 + 2 e_2 R^2 + \frac{3}{2} f_1} \right]}.$$
where \( A_1 \rho = \mathcal{T}(R, 0) + \mathcal{T}(-R, 0) \),

\[ A_1 \mu = \mathcal{T}(R, 0) - \mathcal{T}(-R, 0) , \]

\[ A_2 \rho = \mathcal{T}(R-R_x, 0) + \mathcal{T}(R_x-R, 0) , \]

\[ A_2 \mu = \mathcal{T}(R-R_x, 0) - \mathcal{T}(R_x-R, 0) , \]

\[
e_1 = \frac{1}{2} \left[ R^2 + (R-R_x)^2 \right],
\]

\[
e_2 = \frac{R^4 A_1 \rho - (R-R_x)^4 A_2 \rho}{2 \left[ R^2 - (R-R_x)^2 \right]}
\]

\[
f_1 = -\frac{R^2 (R-R_x)^2}{2}
\]

\[
f_2 = \frac{-R^2 (R-R_x)^2 \left[ R^2 A_1 \rho - (R-R_x)^2 A_2 \rho \right]}{2 \left[ R^2 - (R-R_x)^2 \right]}
\]

\[ A = \int_{-K}^{K} \mathcal{T}(\mu, 0) \, d\mu. \]

(1) The pseudo-gravity calculation when \( \gamma > 0 \).

Expand \( I_1(\xi, \eta) \), \( I_2(\xi, \eta) \) and \( I_3(\xi, \eta) \) with \( N = 2 \) and substitute for \( K(\xi - x, \eta) \):
\[ \phi > 0 \]

\[ r = \left[ (x - \xi)^2 + \gamma^2 \right]^{\frac{1}{2}} \]

\[ \theta = \arctan \left( \frac{\gamma}{\xi - x} \right) \]

\( x = 0 \) is approximately over the centre of the body.

\[ \phi = 0 \]

\[ r = \left[ (x - \xi)^2 + \gamma^2 \right]^{\frac{1}{2}} \]

For \( x < \xi \), \( \theta = 0 \)

For \( x > \xi \), \( \theta = \pi \)

Figure 5.1 The definition of the symbols used in the pseudo-gravity calculation.
\[ I_1(\xi, \eta) = \cos \beta \int_{-\xi}^{\xi} \frac{a_1}{x^3} \, x \, d\xi + \sin \beta \int_{-\eta}^{\eta} \frac{a_2}{x^3} \, \theta \, d\eta \]

\[ = I_{1c}(\xi, \eta) + I_{1s}(\xi, \eta) \]

\[ I_2(\xi, \eta) = \cos \beta \int_{-\xi}^{\xi} \left[ \frac{a_1}{x^3} + \frac{a_2}{x^3} \right] \, x \, d\xi + \sin \beta \int_{-\eta}^{\eta} \left[ \frac{a_1}{x^3} + \frac{a_2}{x^3} \right] \, \theta \, d\eta \]

\[ = I_{2c}(\xi, \eta) + I_{2s}(\xi, \eta) \]

\[ I_3(\xi, \eta) = \cos \beta \int_{-\xi}^{\xi} \left[ \frac{a_1}{x^3} + \frac{a_2}{x^3} \right] \, x \, d\xi + \sin \beta \int_{-\eta}^{\eta} \left[ \frac{a_1}{x^3} + \frac{a_2}{x^3} \right] \, \theta \, d\eta \]

\[ = I_{3c}(\xi, \eta) + I_{3s}(\xi, \eta) \]

For computation purpose these expressions are now regrouped into \( \cos \beta \) and \( \sin \beta \) terms and \( I_{2,3c}(\xi, \eta) \) and \( I_{2,3s}(\xi, \eta) \) are evaluated (the definitions of the symbols used are given in figure 5.1).

\[ I_{3c}(\xi, \eta) + I_{3c}(\xi, \eta) = \]

\[ \cos \beta \left[ a_1 \left( \ln \left( \frac{1}{\xi} \right) + \frac{1}{(\xi^2 + \eta^2)} \right) \right] + \sin \beta \left( \frac{1}{\theta} \right) \frac{1}{(\xi^2 + \eta^2)} \]

\[ + \frac{a_2}{2} \left[ \ln \left( \frac{1}{\eta} \right) - \frac{1}{(\xi^2 + \eta^2)} \right] + 2 \frac{\pi \eta (\xi + \theta \xi)}{(\xi^2 + \eta^2)^{2/3}} - \frac{2 \pi}{2} \frac{(\xi^2 + \eta^2)^{2/3}}{\xi (\xi^2 + \eta^2)^{2/3}} \].
\[ I_{25}(\xi, \gamma) + I_{35}(\xi, \gamma) = \]

\[ - \sin \beta \left[ a_1 \left[ \frac{1}{(\xi^2 + \gamma^2)} \ln \frac{r - a}{r_k} + \alpha \left\{ \frac{1}{r} - \frac{3}{(\xi^2 + \gamma^2)} \right\} + \theta \left( \frac{1}{r} - \frac{3}{(\xi^2 + \gamma^2)} \right) \right] \right] \]

\[ + \frac{a_2}{2} \left[ \frac{2 \xi \gamma}{(\xi^2 + \gamma^2)^2} \ln \frac{r - a}{r_k} + (\alpha + \ell \gamma) \left\{ \frac{1}{r^2} - \frac{(\xi^2 - \gamma^2)}{(\xi^2 + \gamma^2)^2} \right\} - \frac{2 \xi \gamma}{(r_k^2 - \xi^2 + \gamma^2)} \right]. \]

\[ I_1(\xi, \gamma) \] is evaluated by Simpson's rule and the expression for the pseudo-gravity at \((\xi, \gamma)\) can now be evaluated as:

\[ P_{\xi}(\xi, \gamma) = \frac{6 \phi}{5 \pi} \left[ \cos \beta \left\{ I_{1c}(\xi, \gamma) + I_{2c}(\xi, \gamma) + I_{3c}(\xi, \gamma) \right\} \right] \]

\[ + \sin \beta \left\{ I_{1s}(\xi, \gamma) + I_{2s}(\xi, \gamma) + I_{3s}(\xi, \gamma) \right\}. \]

\[ \text{for } x' < \xi \]

(11) \text{ The pseudo-gravity calculation when } \gamma = 0.

As \( \gamma \to 0, \theta \to 0 \) or \( \pi \) and when \( \gamma = 0, \theta = 0 \) for \( x < \xi \) and \( \theta = \pi \) for \( x > \xi \). There is a singularity in the expression for the pseudo-gravity when \( x = \xi \).
This is overcome by expressing the magnetic anomaly in the region \( x < \xi \) as a quartic and the integral 5.1 is re-expressed as:

\[
\rho \xi (\xi, \gamma) = \frac{g \rho}{\pi^2} \left[ \int_{-\pi}^{\pi} \sum_{n=1}^{N} a_n \xi^{-n+1} K\left(\xi^{-\nu}, 0\right) d\nu + \int_{-\pi}^{\pi} \frac{1}{T(\xi, 0)} K\left(\xi^{-\nu}, 0\right) d\nu 
\right.
\]

\[
+ \int_{\pi}^{0} \sum_{n=1}^{N} b_n \xi^{-n+1} K\left(\xi^{-\nu}, 0\right) d\nu
\]

\[
+ \int_{-\pi}^{\pi} \frac{1}{T(\xi, 0)} K\left(\xi^{-\nu}, 0\right) d\nu + \int_{-\pi}^{0} \sum_{n=1}^{N} a_n \xi^{-n+1} K\left(\xi^{-\nu}, 0\right) d\nu.
\]

where \( N = 2 \) and \( \Pi = 5 \).

\[
= \frac{g \rho}{\pi^2} \left[ I_4(\xi, 0) + I_5(\xi, 0) + I_6(\xi, 0) + I_7(\xi, 0) + I_8(\xi, 0) \right] \tag{5.7}
\]

In this expression \( I_5(\xi, 0) \) and \( I_7(\xi, 0) \) are integrated numerically using Simpson's rule and \( I_4(\xi, 0) \) and \( I_8(\xi, 0) \) are obtained by putting \( \theta = 0 \) and \( \theta = \pi \) in \( I_4(\xi, \gamma) \) and \( I_8(\xi, \gamma) \) respectively and putting \( \gamma = 0 \) in both. The values of the constants in the quartic equation expressing the magnetic anomaly over the range \( x = \xi^{-\nu} \) are:
\[ b_1 = \tau(n, 0), \]
\[ b_2 = \frac{1}{12 \pi} \left[ -6 \left\{ \tau(n - \pi, 0) - 2\pi, 0 \right\} + \left\{ \tau(n + \pi, 0) - 2\pi, 0 \right\} \right], \]
\[ b_3 = \frac{1}{24 \pi} \left[ -30 \tau(n, 0) + 10 \left\{ \tau(n - \pi, 0) + \tau(n + \pi, 0) \right\} - \left\{ \tau(n - 2\pi, 0) + \tau(n + 2\pi, 0) \right\} \right], \]
\[ b_4 = \frac{1}{12 \pi} \left[ 2 \left\{ \tau(n - \pi, 0) - \tau(n + \pi, 0) \right\} - \left\{ \tau(n - 2\pi, 0) - \tau(n + 2\pi, 0) \right\} \right], \]
\[ b_5 = \frac{1}{24 \pi} \left[ 6 \tau(n, 0) - 4 \left\{ \tau(n - \pi, 0) + \tau(n + \pi, 0) \right\} + \frac{1}{2} \left\{ \tau(n - 2\pi, 0) + \tau(n + 2\pi, 0) \right\} \right]. \]

Substituting for \( K(\frac{\pi}{2} - x, 0) \), integral \( I_6(\frac{\pi}{2}, 0) \) becomes:

\[ I_6(\frac{\pi}{2}, 0) = \cos \beta \leq \frac{2b_n \pi^m}{m} \left[ \ln \pi - \frac{1}{m} \right] + \sin \beta \leq \frac{b_p \pi (p-1)}{\rho}, \]

\[ m = 1, 2, 3, \]
\[ \rho = 1, 2, 3, 4, 5. \]

(cos \beta terms for even values of \( m \) are zero, and values of the terms for \( m > 5 \) and \( \rho > 4 \) are negligible.

(iii) The pseudo-gravity calculation of the permissible range of values for \( \beta \)

Equation 5.6 can be re-expressed as:
If the intensity of the magnetisation is assumed to be positive throughout the body, the pseudo-gravity equivalent of the magnetic anomaly will always be positive and the following inequality will be true,

\[-\frac{\pi}{2} < [\beta - \lambda(\xi, \eta)] < \frac{\pi}{2},\]

or

\[\lambda(\xi, \eta) - \frac{\pi}{2} - \sigma^{(\text{mod } 2\pi)} < \mu < \lambda(\xi, \eta) + \frac{\pi}{2} - \sigma^{(\text{mod } 2\pi)}.\]

This gives a permissible range for \(\mu\), \((\mu = \beta - \sigma)\) of \(\pi\) in the plane of the profile perpendicular to the strike of the anomaly, assuming this plane lies in the magnetic meridian. This range decreases as \(\lambda(\xi, \eta)\) is calculated for increasing values of \(|\xi|\).

3. The Computer Programmes and Theoretical Examples

Programmes: \(\text{PG}(\eta > 0), \text{PG}(\eta = 0), \text{PGI}\) and \(\text{PGI-B}\).
Specification: Appendix 2.9 PG(γ > 0) and PG(γ = 0)
Appendix 2.10 PGI and PGI-B

For each programme the limits between which the magnetic anomaly can be defined (2 |R|, see figure 5.1) are read in on the data tape together with the co-ordinates of the first point at which the pseudo-gravity is to be calculated ((δ₁, γ) for PG(γ > 0) and just (δ₁) for PG(γ = 0)), the interval required between successive values of the pseudo-gravity (dδ), the interval between successive values of the magnetic field (dx), the number of values of the magnetic field (NT) and the conventional density to be used (ρ/J) (this is not required for PGI or PGI-B). These figures are followed by the actual values of the magnetic anomaly in gamma.

Programmes PG(γ > 0) and PG(γ = 0) also require the inclination of the total magnetisation of the body (IB) and the direction of measurement of the anomaly in the plane perpendicular to the strike of the anomaly. As explained in section 2 of this chapter, unless this happens to lie in the magnetic meridian, the apparent inclination of the Earth's field (IE≪) in the plane of the profile must be used instead.

\[ IE ≪ = \tan^{-1} \left( \frac{\tan IE}{\sin D} \right) \]

where D is the angle between magnetic north and the strike measured clockwise and IE the inclination of the Earth's field in the magnetic meridian.

The values of the pseudo-gravity anomaly at (δ₁, γ) or (δ₁, 0) calculated by PG(γ > 0) or PG(γ = 0), and the value of λ(δ, γ) for PGI or PGI-B, are output on teleprinter or tape-punch as required.
\[ x_{i+1} - x_i = 200 \text{ m.} \quad \rho = 1 \text{ g/m}^3 \quad J = 10^{-3} \text{c.g.s.units} \]

(a) IE = IB = 0° 
(b) IE = IB = 315°

IE and IB are measured anti-clockwise from the positive x-axis

Figure 5.2. The model used to obtain the results in Tables 5.1 and 5.2.
### Table 5.1  Magnetic, gravity and pseudo-gravity anomalies over a horizontally magnetised rectangular prism

(IE = IB = 0°).

<table>
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<tr>
<th>Magnetic $z_1=200$ m.</th>
<th>Gravity $z_1=300$ m.</th>
<th>PG($\eta &gt; 0$) $\eta =100$ m.</th>
<th>Gravity $z_1=200$ m.</th>
<th>PG($\eta =0$)</th>
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Table 5.2 Magnetic, gravity and pseudo-gravity anomalies over a rectangular prism with oblique magnetisation ($\text{IE} = \text{IB} = 315^\circ$).
\[ \lambda_1 = \lambda(-R+dx, \eta) - \text{IE} \quad \quad \lambda_2 = \lambda(R-dx, \eta) - \text{IE} \]

(IE = 0°)

Figure 5.3 Each value of \( \lambda(s, \eta) \) gives a permissible range of \( \lambda(s, \eta) \pm \frac{\pi}{2} \) for beta. In the diagram IB = \( \beta - \text{IE} \) (or \( \mu = \beta - \sigma \) from section 2 of this chapter) and the estimated range and mean value of IB are 340°-380°(20°) and 360°(0°). The true value of IB is 360°(0°).
\[ \lambda_1 = \lambda(-R+dx, \gamma) - IE \quad \lambda_2 = \lambda(R-dx, \gamma) - IE \] 

(IE = 315°)

Figure 5.4  Each value of \( \lambda(\xi, \gamma) \) gives a permissible range of \( \lambda(\xi, \gamma) \pm \frac{\pi}{2} \) for beta. In the diagram \( IB = \beta - IE \) (or \( \mu = \beta - \sigma \) from section 2 of this chapter) and the estimated range and mean value of beta are 295°-335° and 315°. The true value of beta is 315°.
Tables 5.1 and 5.2 show the pseudo-gravity anomaly (calculated for \( \eta > 0 \) and \( \eta = 0 \)) resulting from the transformation of the total intensity magnetic anomaly calculated over a horizontal rectangular prism with an east-west strike and magnetised at \( IE = IB = 0^\circ \) and \( IE = IB = 315^\circ \) (figure 5.2). The true gravity anomaly over the same model, corrected for \( \eta \) when \( \eta \neq 0 \), is included for comparison with the calculated pseudo-gravity anomaly.

Figures 5.3 and 5.4 demonstrate the accuracy of the PGI calculation for the permissible range of beta, using the same theoretical anomaly - the effect of errors likely to be encountered when interpreting observed anomalies are dealt with in the next section. A modified version of the PGI programme (PGI-B) allows the value of the background anomaly to be adjusted at the same time as the constants for the polynomial approximation for the anomaly when \( x \geq |R| \) are calculated. The use of this programme is discussed with the effect of errors in the background on the permissible range of values for beta in the next section of this chapter.

Other uses of the pseudo-gravity programmes in conjunction with the least squares adjustment of two-dimensional models for gravity anomalies are discussed in the last section of this chapter.

4. Sources of Error in the Calculation of Beta

The method of calculating the permissible range of beta, and hence IB (or \( \mu \)), described in section 2(iii) of this Chapter has proved extremely valuable. Therefore, an attempt has been made to assess the influence of various errors on the computed values of \( \lambda (\xi, \eta) \), and the results are discussed in this section. For this purpose the total
field anomaly was calculated along a north-south profile over an east-west linear dipole at unit depth, for different magnetisation directions ranging from $IE = IB = 0^\circ$ to $IE = IB = -45^\circ$ at $7.5^\circ$ intervals (this is equivalent to a range of beta from $0^\circ$ to $-90^\circ$). Errors of known magnitude were then introduced into these anomalies.

The range of the numerical integration and errors introduced in the position of the origin are expressed in terms of the depth to the dipole, and the background value has been adjusted first by a constant number of gamma and then on the assumption that the background is inclined across the anomaly. The effect of these errors and of random errors, on the permissible range of values for beta (and hence for IB) and the various ways of eliminating them are discussed below and the results are summarized in Tables 5.3, 5.4, 5.5, 5.6 and 5.7.

(i) **The range of the numerical integration**

The dipole anomalies were integrated between different limits ($\pm R$) and it has been found that the range of integration ($2R$) must be such that

$$\frac{2\, R}{|x_m - x_M|} \geq 4,$$

for the errors to be insignificant. ($|x_m - x_M|$ is the modulus of the distance between the maximum and minimum values of the anomaly.) The results of this test are summarized in Table 5.3.

(ii) **The value of $\eta$**

It has been found that for negligible errors,

$$\frac{\eta}{x_i - x_{i-1}} \leq 1,$$
<table>
<thead>
<tr>
<th>True value of $\Theta$ (degrees)</th>
<th>Range of numerical integration (the depth to the linear dipole is 1.0 units)</th>
<th>Estimated range and mean value of $\Theta$ (to the nearest degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>270</td>
<td>1.6: 217 - 323 (270) 3.2: 234 - 305 (270) 4.0: 240 - 300 (270) 5.6: 247 - 293 (270)</td>
<td>270: 217 - 323 (270) 3.2: 234 - 305 (270) 4.0: 240 - 300 (270) 5.6: 247 - 293 (270)</td>
</tr>
<tr>
<td>300</td>
<td>1.6: 256 - 366 (311) 3.2: 266 - 339 (302) 4.0: 270 - 333 (301) 5.6: 277 - 324 (300)</td>
<td>300: 256 - 366 (311) 3.2: 266 - 339 (302) 4.0: 270 - 333 (301) 5.6: 277 - 324 (300)</td>
</tr>
<tr>
<td>360</td>
<td>1.6: 302 - 418 (360) 3.2: 321 - 399 (360) 4.0: 327 - 393 (360) 5.6: 335 - 386 (360)</td>
<td>360: 302 - 418 (360) 3.2: 321 - 399 (360) 4.0: 327 - 393 (360) 5.6: 335 - 386 (360)</td>
</tr>
</tbody>
</table>

Table 5.3 Pseudo-gravity estimate of the range and mean (in parenthesis) directions of magnetisation for a linear dipole situated at unit depth, with varying extents for the numerical integration either side of the dipole. $\Theta = (IE + IB)$.  

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Table 5.4. Pseudo-gravity estimate of the range and mean (in parenthesis) directions of magnetisation for a linear dipole situated at unit depth, for different errors in the position of the origin. For displacements to the south, the errors are equivalent in magnitude but in the opposite sense. The range of the numerical integration is 2.0 units either side of the assumed position of the dipole. \( \beta = (IE + IB) \).
is a suitable condition. \((x_i - x_{i-1})\) is the interval between successive values of the magnetic anomaly \(T(x, 0)\) and \(y = \gamma (\gamma > 0)\) is the plane for which the pseudo-gravity is to be calculated, see figure 5.1).

(iii) The position of the origin

Theoretically, the position of the origin is unimportant, but the method of evaluating the end correction polynomials requires that it should be approximately over the centre of the body. This position can be critical because as the range of the numerical integration decreases, so the effect of errors in the position of the origin becomes larger. In practice, its position is difficult to fix without knowledge of the direction of magnetisation, but it can be assumed to be mid-way between the positions of the maximum and the minimum of the magnetic anomaly if they are equal, and nearer the larger feature if they are not.

For testing purposes, the origin for the linear dipole anomalies was moved either side of its true position by distances equal to 0.2, 0.4 and 0.6 times the depth to the dipole.

If it is assumed that the depth to the body is approximately equal to the distance between the maximum and minimum of the anomaly, it means that the origin can be positioned to better than 0.1 times the depth. The distortion this produces in the limits on beta can be estimated from Table 5.4. Although it has not actually been incorporated into the programme, it is reasonable to assume that the origin can be fixed by adjusting the position and magnetisation of a linear dipole by least squares to fit the observed anomaly, and then to take the \(x\) co-ordinate of its position to be the origin for the pseudo-gravity calculation.
An alternative procedure for estimating the centre of a body that approximates to infinite dyke has been given by Hutchison (1958):

\[ \Delta T_{\text{max}} - \Delta T_{\text{min}} = f (\phi_{\text{max}}). \]

Where \( \Delta T_{\text{max}} \) and \( \Delta T_{\text{min}} \) are the maximum and minimum values of the anomaly and \( f (\phi) \) is the angle subtended by the top of the body at the point of measurement of the anomaly. This angle is a maximum over the centre of the body \( f (\phi_{\text{max}}) \). Although the PGI calculation is not strictly valid for an infinite dyke, the errors introduced by Hutchison's approximation will generally be small for any two-dimensional total field magnetic anomaly.

(iv) The background anomaly

(a) A uniform background anomaly.

As explained in section 2, the background anomaly is normally fixed by inspection and then the local anomaly is measured relative to this estimate. Any error in this estimated background anomaly, provided it is constant, can be largely eliminated when calculating the constants for the end correction polynomial as explained in section 2. This calculation has been incorporated into the PGI-B programme and Table 5.5 summarizes the results of tests using a dipole anomaly with an amplitude of 2,500 gamma and \( IE = IB = -15^\circ \).

For the first set of results in Table 5.5, the end correction polynomial has been evaluated to \( N = 2 \) and the range of the numerical integration was 4.0 times the depth to the dipole and the errors are considerable, even when the true background is used. The results improve
Table 5.5  Accuracy of the background correction using different values of N, different values of the magnetic anomaly to calculate the constants, and different ranges for the numerical integration (2R equals 4.0 and 6.0 times the depth of the dipole).

<table>
<thead>
<tr>
<th>N</th>
<th>Background error</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0 gamma</td>
<td>+100 gamma</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.0</td>
<td>6.0</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>69</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>(last values at each end)</td>
<td>169</td>
<td>117</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>18</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(last two values at each end)</td>
<td>118</td>
<td>101</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>28</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>(last and fourth from last at each end)</td>
<td>128</td>
<td>102.5</td>
</tr>
<tr>
<td>Change in the background anomaly ( gamma )</td>
<td>True value of $\beta$ ( degrees )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-----------------------------------------</td>
<td>-------------------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>270</td>
<td>315</td>
<td>360</td>
</tr>
<tr>
<td></td>
<td>Estimated range and mean value of $\beta$ ( to the nearest degree ) followed by the background error after correction ( gamma )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>+50</td>
<td>235 - 304 ( 270 ) 0</td>
<td>280 - 368 ( 324 ) -14</td>
<td>311 - 418 ( 365 ) -20</td>
</tr>
<tr>
<td>+200</td>
<td>235 - 304 ( 270 ) 0</td>
<td>280 - 368 ( 324 ) -14</td>
<td>311 - 418 ( 365 ) -20</td>
</tr>
<tr>
<td>+500</td>
<td>235 - 304 ( 270 ) 0</td>
<td>280 - 368 ( 324 ) -14</td>
<td>311 - 418 ( 365 ) -20</td>
</tr>
</tbody>
</table>

Table 5.6. Pseudo-gravity estimate of the range and mean ( in parenthesis ) directions of magnetisation for a linear dipole situated at unit depth, with different constant errors in the background anomaly. These figures are followed by the error in the background anomaly after correction. For negative changes in the background anomaly, the errors are equivalent in magnitude but in the opposite sense. The range of the numerical integration is 2.0 units either side of the dipole. $\beta = ( IE + IB )$. 

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considerably when the numerical integration is extended to 6.0 times the
depth to the dipole, but it is apparent that the polynomial must be
evaluated to $N = 4$ before a reasonable accuracy can be achieved. From
the results in Table 5.5 it appears that the last two values at each
end of the defined anomaly, rather than two values separated by an
appreciable distance, give the most accurate results.

The linear dipole anomalies used in the earlier tests have been
used to illustrate the effectiveness of the background adjustment
incorporated into programme PGI-B. This programme evaluates the end
correction polynomial to $N = 4$ using the last two values at each end
of the defined anomaly and estimates the permissible range of beta from
the given magnetic anomaly. The background values of the dipole
anomalies have been adjusted by +50, +200 and +500 gamma and −50, −200
and −500 gamma (the amplitude of the dipole anomalies is approximately
2,000 gamma) and the effects of these changes on the permissible range
and mean value of beta, and the errors in the corrected background
anomaly are summarized in Table 5.6. From these results it appears
that the errors are largest for symmetric anomalies ($\beta = 360^\circ$).

(b) An inclined background anomaly.

The effect of errors in the background gradient of 25 and 50 gamma/
unit distance, increasing first to the north and then to the south, on
the permissible range of beta has been calculated for the dipole anomalies
and the results are summarized in Table 5.7. From these figures, it would
appear that the errors are greatest when one peak of the anomaly is
approximately half the magnitude of the other and that the distortion
<table>
<thead>
<tr>
<th>Error in gradient gamma/unit distance</th>
<th>Direction of increasing gradient</th>
<th>True value of $\beta$ (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>270</td>
</tr>
<tr>
<td></td>
<td></td>
<td>315</td>
</tr>
<tr>
<td></td>
<td></td>
<td>360</td>
</tr>
<tr>
<td><strong>Estimated range and mean value of $\beta$ (to the nearest degree)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>north</td>
<td>233 - 307 (270)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>274 - 362 (318)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>304 - 411 (357)</td>
</tr>
<tr>
<td>south</td>
<td></td>
<td>239 - 301 (270)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>286 - 377 (331)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>309 - 416 (363)</td>
</tr>
<tr>
<td>50</td>
<td>north</td>
<td>231 - 309 (270)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>269 - 357 (313)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>298 - 403 (350)</td>
</tr>
<tr>
<td>south</td>
<td></td>
<td>244 - 296 (270)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>293 - 388 (340)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>317 - 422 (370)</td>
</tr>
</tbody>
</table>

Table 5.7. The pseudo-gravity estimate of the range and mean (in parenthesis) directions of magnetisation for a linear dipole situated at unit depth when a uniform gradient has been super-imposed upon the background anomaly. The range of the numerical integration is 2.0 units either side of the dipole. $\beta = (IB + IB)$. 
produced in an anti-symmetric anomaly is negligible. These tests assume the background gradient is constant between the limits of the numerical integration ($\pm R$) - the effect on the values of beta when this is not true have not been assessed.

(v) **The Influence of Super-imposed Anomalies.**

This is difficult to assess, but changes in the anomaly obviously due to bodies other than that under consideration can be eliminated by smoothing. However, distortions which are not apparent on visual inspection are almost impossible to eliminate.

As a test, random errors ranging between $\pm 32$ gamma were added to the linear dipole anomaly for $\beta = 330^\circ$ and the results are summarized below. This result suggests that genuine random errors have little effect on the permissible range of values for beta.

<table>
<thead>
<tr>
<th></th>
<th>True dipole anomaly</th>
<th>Anomaly plus random errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limits on beta</td>
<td>$299^\circ - 364^\circ$</td>
<td>$295^\circ - 364^\circ$</td>
</tr>
</tbody>
</table>

The range of the numerical integration was $2R = 4.0$ times the depth to the dipole.

(vi) **Conclusions to be drawn from the results of the tests**

From the results of these tests using anomalies calculated over a
linear dipole, it would appear the pseudo-gravity method can give a good estimate of the range of values for beta or IB. Also, if the background anomaly and origin are correct then the mean value of the range is an accurate estimate of the true direction of magnetisation.

5. Applications and Conclusions

Dr. Smith's two-dimensional version of the Baranov transformation provides a method of calculating the pseudo-gravity equivalent of the total magnetic field anomaly over a two-dimensional body with an arbitrary, but closed, cross-section. The intensity of magnetisation of the body can vary from point to point within it and the direction of magnetisation, although it must be uniform, need not be parallel to the Earth's field. If the additional assumption is made that the magnetisation of the body is all one sign, then the pseudo-gravity equivalent of the magnetic anomaly will always be the same sign. This provides a means of limiting the range of possible values for IB, the direction of magnetisation of the body in the plane perpendicular to the strike of the anomaly and has been used in all versions of the PGI programme for this purpose.

The pseudo-gravity calculation is particularly susceptible to errors in the choice of the background anomaly and in the positioning of the origin over the centre of the body. Methods of overcoming these difficulties have been suggested - the use of the theorem stating that the integral of the anomaly between minus infinity and plus infinity must be zero to adjust the background, and a least squares process or Hutchison's approximation to fix the origin. It has not been possible
### Table 5.8 Pseudo-gravity estimate of the range and mean (in parenthesis..) directions of magnetisation for a two-dimensional body with a triangular cross-section as illustrated above. 27 values of the magnetic anomaly, with unit distance between each, were used to calculate the pseudo-gravity. $\beta = (IE + IB)$.
to eliminate errors in the gradient of the background anomaly and the distorting effects of adjacent anomalies.

As an example of the PGI calculation estimates of the range and mean values of beta have been made from the magnetic anomaly over a two-dimensional body magnetised at $\text{IE} = \text{IB} = 0^\circ$, $\text{IE} = \text{IB} = -22.5^\circ$ and $\text{IE} = \text{IB} = 45^\circ$. The results (summarized in Table 5.8) indicate that the pseudo-gravity method can give a good estimate for the inclination of the magnetisation of a two-dimensional body in the plane perpendicular to the strike of the anomaly - assuming that the cross-section of the body is bounded by a closed surface.

Although a pseudo-gravity anomaly is not related to actual density changes, it otherwise has all the characteristics of a gravity anomaly and can be interpreted as such in conjunction with one of the optimisation techniques described in Chapter III.
CHAPTER VI

Summary and Conclusions

1. Interpretation of gravity anomalies.
2. Interpretation of magnetic anomalies.

1. Interpretation of Gravity Anomalies.

The observed gravity anomaly is assumed to be two-dimensional and the model used to represent the disturbing mass is a horizontal prism of infinite length with a polygonal cross-section. Each face of the model is the sloping end of a horizontal slab, infinite in both the positive x direction and parallel to the strike of the anomaly. The strike is assumed to be perpendicular to the plane of the profile. The total effect of the polygonal model at any point is found by summing the effect of each of the slabs at the point in question. This calculation is performed by the GRAVN programme which computes the gravity anomaly at a series of points over a specified polygonal model of given density contrast.

Three computer programmes (LSGR, LSGT and LSGN) have been developed to optimise an initial model which approximates the disturbing mass. In each case the function to be minimised is the sum of the squared differences between the observed and calculated anomalies at each point for which the
former has been defined. The basic models are two-dimensional horizontal prisms with rectangular (LSGR), trapezoidal (LSGT) and n-sided polygonal (LSGN) cross-sections and the optimisation is done by an iterative least squares process. The main disadvantage with the method is that it is unstable if the initial model is not a good representation of the actual disturbing body, and successive iterations may not reduce the residual function.

An alternative method based on a model with an n-sided polygonal cross-section uses a direct search technique to optimise the initial model (GRAVN SWANOPT). This method will always reduce the value of the residual function, but if the initial model is not a good representation of the actual body, then the adjusted co-ordinates of the model may not be realistic. This disadvantage could be overcome by putting upper and lower limits on the magnitude of the individual variables and optimising the model within the feasible zones of each variable. Ways of incorporating these constraints into the programme have been discussed in Chapter 2, and examples of the iterative least squares and direct search methods are described in Chapter 3.

2. Interpretation of Magnetic Anomalies

It is assumed that the observed magnetic anomaly is two-dimensional and that its strike is known. A horizontal prism model with infinite length parallel to the strike and a polygonal cross-section in the plane of the profile, is used to represent the disturbing body. Each face of the model is the end of a semi-infinite horizontal slab and the horizontal
and vertical anomalies due to the slab are projected in the direction of the Earth's field to give the total intensity magnetic anomaly. The magnetic anomaly due to the whole model at any point is then found by summing the effects of each face in the same way as in the calculation of gravity anomalies. The MAGN computer programme calculates the horizontal, vertical and total field anomalies (or any combination of these anomalies) at a series of points over a model with a polygonal cross-section, the co-ordinates, strike, direction and intensity of magnetisation of which are specified by the user.

A modified version of this programme (MGSC) can be used to prepare normalised anomalies over a range of models for comparison with similarly scaled anomalies measured in the field. This process eliminates the intensity of magnetisation and one dimension of the model as variables in the interpretation problem and a scheme for the interpretation of total intensity anomalies attributable to dykes is described in Chapter 4.

Two computer programmes (LSMR and LSMT) have been developed to optimise an initial model representing the disturbing body by an iterative least squares process. In each case the function to be minimised is the sum of the squared differences between the observed and calculated anomalies at each point for which the former has been defined. The basic models are two-dimensional horizontal prisms with rectangular (LSMT) and trapezoidal (LSMT) cross-sections. The main disadvantage of this method of optimisation is that it may not always reduce the residual function with successive iterations if the initial model is not a good representation of the actual body.
An alternative method of optimising the dimensions and magnetisation of a rectangular prism model uses a modified version of Davidon's gradient optimisation process (MAGR DAVOPTC). This programme incorporates constraints limiting the individual variables to feasible zones and a scaling process which adjusts each variable to lie between the same limits, thus ensuring greater stability in the calculation. At present the number of iterations required to reach a minimum in the residual function is considerably more than is needed using the basic least squares process, but it is possible to improve MAGR DAVOPTC by adjusting the initial level of the response surface and its subsequent rate of change.

The final programme for the interpretation of total intensity magnetic anomalies (MAGN SWANOPT) is based on an n-sided polygonal model and uses a direct search process to minimise the residual function. Users of this programme may find that it is necessary to put limits on the changes to the individual variables and ways of doing this are discussed in Chapter 2.

After the description of the programmes in Chapter 4, two total intensity anomalies attributable to dykes have been interpreted using a parameter method, the normalised curve technique, the basic least squares and Davidon's optimisation methods.

3. **Pseudo-gravity**

A two-dimensional version of the pseudo-gravity calculation, first proposed by Baranov in 1957, has been developed by Smith and is presented in full in Appendix I. It was originally intended to calculate the pseudo-
gravity equivalent of a given magnetic anomaly and then to optimise the co-ordinates of a two-dimensional model to account for the pseudo-gravity anomaly. The magnetic effect of this model would then have been calculated and compared with the observed magnetic anomaly. Later, it was decided that the optimisation process could be applied directly to the observed magnetic anomaly and although programmes have been written for the calculation of pseudo-gravity \((\text{PG}(\eta > 0)\text{ and PG}(\eta = 0))\), they have never been used.

However, the pseudo-gravity calculation does give a method of limiting the range of possible directions for the magnetisation of the body in the plane of the profile, provided that the body is bounded by a closed surface and the intensity of magnetisation is one sign throughout the body. An attempt has been made to assess the influence of various errors in the measured anomaly on the limits of the feasible range for the direction of magnetisation and the results are discussed in Chapter 5.

4. Conclusions

From the interpretation of various anomalies presented in this thesis it is clear that the basic least squares process is a very efficient method for adjusting the co-ordinates and density or magnetisation of an initial model to reduce a function of the residual anomalies. However, unless the initial model is a good representation of the actual anomalous body, there may be no reduction of the residual function with successive iterations. A second disadvantage is that unrealistic co-ordinates may result from the adjustment process, but this can be overcome by incorporating
constraints on the individual variables into the calculation. This has been done in an elementary way by limiting the size of the changes to each variable.

The direct search method of optimising the initial model ensures that the residual function decreases with successive iterations but can still produce unrealistic values for the variables. As with the basic least squares process, this can be overcome by constraining the variables to lie within certain feasible zones. These constraints have not yet been incorporated into the computer programmes.

The modified version of Davidon's method does not suffer from any of the disadvantages of the above methods, but it requires more development before it can be considered an efficient means of optimising an initial model.

None of the three optimisation methods used in this thesis can prevent the residual function converging on a local minimum as opposed to the global minimum. All optimisation methods known to the author suffer from this drawback and the only way to overcome it is, at present, to use a variety of initial models, and to take that which agrees best with the observed anomaly as the solution.
REFERENCES


"Interpolation and Allied Tables" H.M.S.O. 1956.


APPENDIX I

The Baranov Transformation in Two Dimensions (R.A. Smith)

Notation  We shall suppose throughout that \( \mathbf{a} = (\cos \alpha, \sin \alpha) \), \( \mathbf{m} = (\cos \mu, \sin \mu) \), \( \mathbf{s} = (\cos \sigma, \sin \sigma) \) are constant unit vectors. We shall write

\[
\nabla \mathbf{u}(x, y) = \left( \frac{\partial \mathbf{u}}{\partial x}, \frac{\partial \mathbf{u}}{\partial y} \right), \quad \nabla \mathbf{u}(\xi, \eta) = \left( \frac{\partial \mathbf{u}}{\partial \xi}, \frac{\partial \mathbf{u}}{\partial \eta} \right).
\]

Then \( (\mathbf{a} \cdot \nabla) \mathbf{u} = \omega \mathbf{a} \frac{\partial \mathbf{u}}{\partial x} + \mathbf{m} \omega \frac{\partial \mathbf{u}}{\partial y} \), etc.

Also \( r = \left[ (\xi - \xi')^2 + (\eta - \eta')^2 \right]^{\frac{1}{2}} \) throughout. Observe that

\[
\mathbf{a} \cdot \nabla \log \frac{1}{r} = -\mathbf{a} \cdot \nabla \log \frac{1}{r}.
\]

Lemma 1  If \( (\mathbf{a} \cdot \nabla) \mathbf{u}(\xi, \eta) = 0 \), for all \((\xi, \eta)\) with \( \eta > 0 \), and \( \mathbf{u}(\xi, \eta) \to 0 \) as \( \xi^2 + \eta^2 \to \infty \) through the half plane \( \eta > 0 \), then \( \mathbf{u}(\xi, \eta) = 0 \) for all \((\xi, \eta)\) in this half plane.

Proof  Take new axes \( O, x, y \) so that the \( x \)-axis is parallel to the vector \( \mathbf{a} \) and points into the half plane \( \eta > 0 \) (to do this \( Ox \) and \( \mathbf{a} \) might have to point in opposite senses).
Then \( (e, \vec{r}) \nu(S, \gamma) = \pm \frac{dy}{dx} \) (minus if \( a \) and \( Ox \) are of opposite sense).

Therefore \( o = \frac{dy}{dx} \) at all points in the half plane \( \gamma > 0 \).

Therefore \( \gamma \) is constant along all rays parallel to \( Ox \) which lie in the half plane \( \gamma > 0 \). Since \( \gamma \to 0 \) as \( x^2 + y^2 \to \infty \) along each of the rays, the constant value of \( \gamma \) along the ray must be zero.

Hence \( \gamma = 0 \) at all points of the half plane \( \gamma > 0 \).

**Lemma 2**

Suppose that \((r, \theta)\) are the polar co-ordinates of the point whose cartesian co-ordinates are \((x, y)\).

If \( \beta = \frac{x}{r} \cdot \frac{x}{r} - \alpha \) and \( K(x, \gamma, \gamma, \gamma) = \cos \beta \log(r) + \sin \beta \theta \),

then \((e, \vec{r})(e, \vec{r})K = \frac{2}{3} \frac{2}{3} \cos \beta \log(r) + \frac{1}{3} \frac{1}{3} \).

**Proof**

If \( u = \log(r) \) and \( v = \theta \) then the Cauchy-Riemann equations hold, namely

\[
\frac{\partial u}{\partial \gamma} = \frac{\partial v}{\partial \gamma}, \quad \frac{\partial v}{\partial \gamma} = -\frac{\partial u}{\partial \gamma}.
\]

Also

\[
\frac{\partial^2 u}{\partial \gamma^2} + \frac{\partial^2 u}{\partial x^2} = 0.
\]

\((e, \vec{r})K = K \gamma \cos \sigma + K \gamma \sin \sigma,

= [u_x \cos \beta + u_x \sin \beta] \cos \sigma + [u_y \cos \beta + u_y \sin \beta] \sin \sigma,

= [u_x \cos \beta - u_y \sin \beta] \cos \sigma + [u_y \cos \beta + u_y \sin \beta] \sin \sigma,

by equation 1.
\[(\vec{w} \cdot \vec{\nabla})(\vec{w} \cdot \vec{\nabla})K = [\vec{w} \cdot \vec{\nabla} u_y] \cos(\beta - \sigma) - [\vec{w} \cdot \vec{\nabla} u_y] \sin(\beta - \sigma),\]

\[= \left[ u_{55} \cos \mu + u_{57} \sin \mu \right] \cos(\beta - \sigma) - \left[ u_{57} \cos \mu + u_{77} \sin \mu \right] \sin(\beta - \sigma),\]

Since \(u_{55} = -u_{77}\), by equation 2, this gives

\[(\vec{w} \cdot \vec{\nabla})(\vec{w} \cdot \vec{\nabla})K = -u_{77} \cos(\beta - \sigma - \mu) - u_{57} \sin(\beta - \sigma - \mu),\]

\[= -u_{77} \sin \alpha - u_{57} \cos \alpha, \quad (\because \beta = \frac{\pi}{2} + \mu + \sigma - \alpha)\]

\[= -\frac{\partial}{\partial \gamma} \left[ \cos \alpha \frac{\partial u}{\partial \xi} + \sin \alpha \frac{\partial u}{\partial \eta} \right],\]

\[= \frac{\partial}{\partial \gamma} \alpha \cdot \vec{\nabla} (-u),\]

\[= \frac{\partial}{\partial \gamma} \alpha \cdot \vec{\nabla} \log(\frac{1}{r}).\]

This proves Lemma 2.

**Lemma 3** Suppose that

(i) \(U(x,y)\) is harmonic at all points \((x,y)\) with \(y \geq 0\),

(ii) \(U(x,y) \to 0\) as \(x^2 + y^2 \to \infty\) through the half plane \(y \geq 0\).

Then for all \((s, \gamma)\) with \(\gamma > 0\), we have

\[u(s, \gamma) = \frac{1}{\pi} \int_{-\infty}^{\infty} u(x, \gamma) \left\{ \frac{1}{\partial x} \right\}^2 \log(\frac{1}{r}) \frac{1}{52} dx,\]

which is the upward continuation formula.
The physical problem

The region $B$ in the $(x,y)$ plane is the cross-section of a two-dimensional body extended at right angles to this plane. If $G(\bar{x}, \bar{y})$ denotes the gravity potential produced by $B$ at the point $(\bar{x}, \bar{y})$ when its density function is $J(x,y)$, then

$$G(\bar{x}, \bar{y}) = \frac{k}{2} \int \int_B J(x,y) \log(\frac{1}{r}) \, dx \, dy.$$  

Where $k$ is the universal gravity constant. If $\Omega(\bar{x}, \bar{y})$ denotes the magnetic potential produced by $B$ at $(\bar{x}, \bar{y})$ when its magnetic polarisation function is $J(x,y)m$, then

$$\Omega(\bar{x}, \bar{y}) = \frac{1}{2} \int \int_B J(x,y) m \cdot \nabla \log(\frac{1}{r}) \, dx \, dy.$$  

Since

$$m \cdot \nabla \log(\frac{1}{r}) = -\frac{(x-\bar{x}) \cos \mu + (y-\bar{y}) \sin \mu}{r^2} = -\mu \cdot \nabla \log(\frac{1}{r}),$$

this gives

$$\Omega(\bar{x}, \bar{y}) = -\frac{1}{2} \int \int_B J(x,y) m \cdot \nabla \log(\frac{1}{r}) \, dx \, dy = -\frac{1}{k} \mu \cdot \nabla G(\bar{x}, \bar{y}).$$
If \( T(\xi, \eta) \) denotes the resolved part parallel to \( \xi \) of the magnetic field at \((\xi, \eta)\) due to \( B \), then

\[
T(\xi, \eta) = \xi \cdot \nabla \Omega(\xi, \eta) - \frac{1}{k} (\xi \cdot \nabla)(\omega \cdot \nabla) \zeta(\xi, \eta).
\]

\[
: (\xi \cdot \nabla) T(\xi, \eta) = (\xi \cdot \nabla)(\omega \cdot \nabla) \left[ -\frac{1}{k} \xi \cdot \nabla \zeta(\xi, \eta) \right] \tag{3}.
\]

If we take \( U(\xi, \eta) = T(\xi, \eta) \) in Lemma 3, it gives

\[
T(\xi, \eta) = \frac{1}{\pi} \int_{-\infty}^{+\infty} T(\xi, 0) \left\{ \frac{\partial}{\partial y} \log(\frac{\eta}{y}) \right\} \bigg|_{y=0} \, d\eta.
\]

Since \( \frac{\partial}{\partial y} \log(\frac{\eta}{y}) = -\frac{\partial}{\partial y} \log(\frac{\eta}{y}) \), this can be written as

\[
T(\xi, \eta) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} T(\xi, 0) \left\{ \frac{\partial}{\partial y} \log(\frac{\eta}{y}) \right\} \bigg|_{y=0} \, d\eta.
\]

\[
: (\xi \cdot \nabla) T(\xi, \eta) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} T(\xi, 0) \left\{ \frac{\partial}{\partial y} \log(\frac{\eta}{y}) \right\} \bigg|_{y=0} \, d\eta
\]

\[
= -\frac{1}{\pi} \int_{-\infty}^{+\infty} T(\xi, 0) \left\{ (\omega \cdot \nabla)(\xi \cdot \nabla) \zeta \right\} \bigg|_{y=0} \, d\eta
\]

\[(\text{Lemma 2})\]

\[
= (\omega \cdot \nabla)(\xi \cdot \nabla) \left[ -\frac{1}{k} \xi \cdot \nabla \zeta(\xi, \eta) \right] - \frac{1}{\pi} \int_{-\infty}^{+\infty} T(\xi, 0) \left( \zeta(\xi - \eta, \eta) \right) \, d\eta \bigg|_{y=0}.
\]

If we subtract 3 from this, we get

\[
0 = (\omega \cdot \nabla)(\xi \cdot \nabla) \left[ \frac{1}{k} \xi \cdot \nabla \zeta - \frac{1}{\pi} \int_{-\infty}^{+\infty} T(\xi, 0) \left( \zeta(\xi - \eta, \eta) \right) \, d\eta \right].
\]
Since this holds for all \((\xi, \gamma)\) with \(\gamma > 0\), a repeated application of Lemma 1 gives

\[
o = \left[ -\frac{1}{k} \cdot \frac{\partial}{\partial \xi} \cdot \zeta - \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} T(\zeta, \gamma) \cdot K(\zeta - \xi, \gamma) \, d\zeta \right],
\]

for all \((\xi, \gamma)\) with \(\gamma > 0\). This can be written:

\[
a \cdot \frac{\partial}{\partial \xi} \zeta(\xi, \gamma) = \frac{k}{\pi} \cdot \int_{-\infty}^{\infty} \frac{T(\zeta, \gamma)}{\zeta - \xi} \cdot K(\zeta - \xi, \gamma) \, d\zeta,
\]

which is the Baranov formula expressing the resolved part parallel to \(a\) of the gravity field at \((\xi, \gamma)\) in terms of the magnetic component \(T(x, 0)\) at all points of the \(x\)-axis. The kernel function \(K(\xi - \zeta, \gamma - \zeta)\) is defined in Lemma 2. We have derived this formula assuming that \(\gamma > 0\).

We could now make \(\gamma \to 0\) on both sides and obtain the expression for \(a \cdot \frac{\partial}{\partial \xi} \zeta\) at the point \((\xi, 0)\) on the \(x\)-axis. If we take \(a = (0, 1)\) this would be the vertical component of \(\zeta\).
APPENDIX II

Specifications for the computer programmes used in the interpretation of two-dimensional gravity and magnetic anomalies.

1. GRAVN
2. LSCR, LSGT and LSGN
3. GRAVN SWANOPT
4. MAGN
5. MGSC
6. LSMR and LSMT
7. MAGN SWANOPT
8. MAGR DAVOPTC
9. PG(γ >0) and PG(γ =0)
10. PGI and PGI - B
TITLE: GRAVN (5-hole Elliott 803 Algol)

PURPOSE: Given a two-dimensional model with a polygonal cross-section, the programme calculates the gravity anomaly at equal intervals along a line perpendicular to the strike of the model.

METHOD: The method used is similar to that described by Talwani, Worzel and Landisman (1959). The anomalous body is assumed to be two-dimensional and is represented by a two-dimensional model with an n-sided polygonal cross-section, and its long axis parallel to the strike of the anomaly. The anomaly at \((x, z)\), usually on a plane equivalent to that containing the observed anomaly, due to the model is

\[ A(x, z) = \sum_{j=1}^{m} \Delta F_j, \]

where \( \Delta F_j \) is the gravity anomaly at \((x, z)\) produced by the \(j\)th face of the model \((j = 1, 2, \ldots, m)\). Each face is the inclined end of a semi-infinite slab and the anomaly it produces is given by Heiland as

\[ \Delta F = 2G\rho \sin i [z \sin i + (x \cos i + z \sin i)] \left[ \sin i \frac{h}{r_i} + \cos i (\psi_2 - \psi_1) \right] \]

The co-ordinate system is explained in the accompanying diagram.

Caution 1. The programme contains no facilities for dealing with models infinite in the negative \(x\) direction.

DATA FORMAT: E Model X Z

\[
\begin{array}{ccc}
X_1 & dX & NT \\
x_1 & z_1 & x_2 & z_2 \\
\cdots & \cdots & \cdots \\
x_m & z_m & x_{m+1} & z_{m+1} \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

Units: metres

where \(x_{m+1} = x_1\)

\(z_{m+1} = z_1\)

\(z_i > 0\)
The co-ordinate system for the calculation of gravity anomalies over two-dimensional models.
Line 1. Title required for each model.

2. $X_1$ is the co-ordinate of the first point at which the anomaly is required, $dX$, the interval between successive points and $NT$, the number of points plus one.

3. $(x_{1z_1}),(x_{2z_2})$ are the co-ordinates of the first face of the model (in anti-clockwise sequence) and $\rho$ is the density contrast between the model and its surroundings ($\rho$ c.g.s. units). The first $x$ co-ordinate of each face must not be zero.

4. Successive faces of the model, working in an anti-clockwise direction. Faces for which $z_j = z_{j+1}$ have no effect and can be ignored.

5. The five zeros are used to signify that the last face of the model has been read in.

Conditions:

1. $NT < 100$

2. Less than 20 faces.

These conditions can be changed by altering the array declarations.

RESULTS: The value of each x co-ordinate is followed by the gravity anomaly at that point in milligals.

TIME: 1.5 secs./face/point.


R. A. Stacey

PURPOSE: Given a measured gravity anomaly and a two-dimensional model with a rectangular (LSGR), a trapezoidal (LSGT) or an n-sided polygonal (LSGN) cross-section which approximates the disturbing body, the programme adjusts specified dimensions and/or the density contrast of the initial model to reduce the differences between the calculated and observed anomalies to a minimum.

METHOD: The initial model is based on the geological information available and is built up in the same way as that for programme GRAVN. Ideally, the following relationship should hold:

\[ A_0(x_i) = f(x_i; a_1, a_2, \ldots, a_m), \]

where \( A_0(x_i) \) is the observed anomaly at \( x_i \) \((i = 1, 2, \ldots, n)\) and the right hand side represents the theoretical anomaly at \( x_i \) caused by the model defined by \( a_1, a_2, \ldots, a_m \). If \( a^0_1, a^0_2, \ldots, a^0_m \) represents the initial model and the correction terms to be evaluated are \( \alpha^r \) \((r = 1, 2, \ldots, m)\)

\[ a^r = a^0 + \alpha^r \]

where \( a^r \) is the best set of dimensions for the model. The set of normal equations to be solved for \( \alpha^r \) are

\[ \sum_{r=1}^{m} \left\{ \sum_{k=1}^{n} b_{kr} \frac{\partial f}{\partial a_k} \right\} \alpha^r = \sum_{k=1}^{n} b_{kj} l_k \quad (j = 1, 2, \ldots, m) \]

The development of this set of equations is described in the thesis, Chapter II. The process is repeated using the new value of \( a^r \) and continues until either
The co-ordinate systems for programmes LSGR, LSGJ, and LSGN.
\[
\sum_{k=1}^{n} \left[ l_k \right]_q^2 \leq e_1 \sum_{k=1}^{n} \left[ A_0(x_k) \right]_q^2
\]

or
\[
\sum_{k=1}^{n} \left[ l_k \right]_q^{-1} - \sum_{k=1}^{n} \left[ l_k \right]_q^{-2} \leq e_2 \sum_{k=1}^{n} \left[ l_k \right]_q^{-1}
\]

where \( l_k \) is the residual at \( x_k \) and \( e_1 \) and \( e_2 \) are small, positive fractions.

Variations:

1. **LSGR.** The depths to the upper and lower surfaces and the density contrast of the model are adjusted by the process outlined above. The vertical sides of the model are assumed to coincide with the positions of the zeros of the second derivatives \( (A''_o(x_i)) \) of the observed anomaly.

\[
A''_o(x_i) = \left[ -A_o(x_{i-2}) + 16 A_o(x_{i-1}) - 30 A_o(x_i) + 16 A_o(x_{i+1}) - A_o(x_{i+2}) \right] / 12(\ dx)^2
\]

From tests, it appears that this approximation is valid when the width of the top surface of the model is greater than its depth. The details of this method are given in Chapter III of the thesis.

2. **LSGT.** The depth and lateral extent of the horizontal upper and lower surfaces of the trapezoidal model can be adjusted by the process outlined above. In a modified version, the depth to the top surface of the model is excluded from the adjustment process.

3. **LSGN.** One surface, usually the top, and the density contrast are not included in the adjustment process. The conditions for terminating the calculations are \( e_1 \) satisfied or

\[
\sum_{k=1}^{n} \left[ l_k \right]_q^{-1} < \sum_{k=1}^{n} \left[ l_k \right]_q^{-2} < \sum_{k=1}^{n} \left[ l_k \right]_q^{-1}
\]
No improvement is printed if the latter condition is encountered.

Cautions:

1. The calculated adjustments to the initial model cause the residual function to increase if the initial model is inadequate. If this occurs, or if it is suspected that the minimum obtained is not the global minimum, then a different initial model must be used.

2. If the residual function increases, conditions 1 and 2 may not be satisfied and the calculation must be stopped manually (except in the case of LSGN, as explained earlier).

3. The variables are unconstrained and $x_1 > x_2$ (i.e. for the trapezoidal model) or $z < 0$ can occur.

In programmes LSGR and LSGW the size of the correction terms is limited, but the method of constraint has no mathematical justification and should be used with care.

DATA FORMAT:

1. LSGR £ Model X-§

   $z_1 a_1 z_2 a_2 \sqrt{a_3}$

   $M \; dX \; NT$

   $e_1 \; e_2$

   $T_1, T_2, \ldots , T_{NT}$

   Units: metres

Conditions:

1. Maximum number of iterations is 20

2. $NT < 50$

3. $a_n = 1$, parameter constant

   $a_n = 0$, parameter variable

4. Limits: new value $z_1 = z_1 (1 - 0.1)$

   new value $\rho = \rho + 0.05$

   if $z_2 < z_1$ then $M \cdot z_1 (M > 1)$
2. LSGT £ Model " ?

\[ z_1 / a_1 z_2 a_2 \]
\[ x_1 a_3 x_2 a_4 \]
\[ x_3 a_5 x_4 a_5 \]

\( \rho \)
\[ M dX NT \]
\[ T_1, T_2 \ldots T_{NT} \]

Conditions:
1. Maximum number of iterations is 20
2. NT < 50
3. \( a_n = 1 \), parameter constant.
   \( a_n = 0 \), parameter variable.
4. Limits: new value \( z_1 = z_1 (1 \pm 0.1) \)
   \( \rho = \rho \pm 0.03 \)
   if \( z_2 < z_1 \) then \( z_2 = M z_1 \) \((M 1)\)
   if \( x_1 > x_3 \) then \( x_1 = x_3, x_3 = x_1 \)
   if \( x_2 > x_4 \) then \( x_2 = x_4, x_4 = x_2 \)

3. LSGN \( X_1 dX NT \)
\[ NC \ b_1 \ b_2 \]
\( x_1 z_1 \)
\[ ........ \]
\[ x_{NC} z_{NC} \]
\[ T_1, T_2 \ldots T_{NT} \]

Conditions:
1. Maximum number of iterations is 20
2. NC < 20
3. NT < 50
4. \( b_1 \) and \( b_2 \) are the first and last co-ordinates to be involved in the adjustment process. For \( x_1 \), \( b = 1 \); for \( z_1 \), \( b = 2 \)....
   for \( x_2 \), \( b = 3 \)....
RESULTS: 1. LSGR. The value of the following quantities is printed after each iteration.

\[ x_1, x_2 (\text{not involved in the adjustment process}) \]

\[ \sum_{k=1}^{n} \left( \frac{l_k}{\ell_k} \right)^2 \]

\[ z_1, z_2 \]

The values of the individual residuals may be printed if required.

2. LSGT. The value of the following quantities is printed after each iteration.

\[ \sum_{k=1}^{n} \left( \frac{l_k}{\ell_k} \right)^2 \]

\[ x_1, x_2, x_3, x_4 \]

\[ z_1, z_2 \]

The values of the individual residuals may be printed if required.

3. LSGN. The value of the following quantities is printed after each iteration.

\[ \sum_{k=1}^{n} \left( \frac{l_k}{\ell_k} \right)^2 \]

The values of the x and/or z co-ordinates specified by \( b_1 \) and \( b_2 \).

The values of the individual residuals may be printed if required.

TIME:

1. LSGR. 0.3 secs/value of the anomaly for one iteration.

2. LSGT. 20 secs/value of the anomaly for one iteration.

3. LSGN. 15 secs/vertex/value of the anomaly for one iteration.

R. A. Stacey

PURPOSE: Given a measured gravity anomaly and a two-dimensional model with an n-sided polygonal cross-section which approximates the disturbing body, the programme adjusts specified vertices of the initial model by a direct search process, to reduce the differences between the calculated and observed anomalies to a minimum. The function to be minimised is

\[ F = \sum_{i=1}^{N} \frac{[A_o(x_i) - A_m(x_i)]^2}{\sum_{i=1}^{N} |A_o(x_i)|} \]

where

\[ A_m(x_i) = \sum_{j=1}^{n} \gamma_j \sin \left( z_j - z_i \right) + \cos \left( z_j \right) \]

The co-ordinate system is explained in the GRAVN specification and \( A_o(x_i) \) and \( A_m(x_i) \) are the observed and calculated anomalies at \( x_i \), which lies on the line of the measured profile.

METHOD: The Swanopt procedure is due to W.H. Swann of I.C.I., Wilton (May 1964) and the description that follows has been taken from his specification for Programme - 589 Direct Search Optimisation (Unconstrained).

"The method used is to set up n mutually orthogonal directions. A univariate search is made to locate the optimum along each of these directions in turn and then a new set of directions is calculated. As the optimum is approached, the step length used in the univariate search is reduced. The programme terminates either when the step length is reduced to a value less than the accuracy specified by the user, or after nk function evaluations, whichever occurs first. k is the maximum number of function evaluations permissible per independent variable and is specified by the user.

\( k = 150 \) is usually sufficient."
Caution:

1. Cautions 1 and 3 in the LSGR, LSGT and LSGN specification apply to this programme.

DATA FORMAT: £ Title ·?

```
e n J A K
NC .x_1 dX NT \rho
x_1 z_1
x_2 z_2........
x_j z_j
x_1 z_1
a_1 a_2 .... a_n
T_1 T_2 .......... T_{NT}
```

- \( \rho \) accuracy factor. (try .0001)
- \( n \) the number of co-ordinates to be adjusted.
- \( J = -1 \) for minimisation.
- \( A = 0 \) for printing after each completed adjustment process.
- \( K \) maximum number of function determinations (try \( K = n \cdot 100 \))
- \( NC \) the number of co-ordinate pairs \((x_j, z_j)\), finishing with the first co-ordinate, \( NC = j + 1 \).
- \( x_1 \) co-ordinate of the first value of the observed gravity anomaly.
- \( dX \) the interval between successive values of the anomaly.
- \( NT \) the number of values of the observed anomaly, which must be equally spaced.
- \( x_{i1} \) the co-ordinates of the initial model.
- \( \rho \) the density contrast (c.g.s. units)
- \( a_1 \ldots a_n \) the co-ordinates to be adjusted. For \( x_1 \), \( a = 1 \); for \( z_1 \), \( a = 2 \); for \( x_2 \), \( a = 3 \) ........
- \( T_1 \ldots T_{NT} \) the values of the observed anomaly at equal intervals \( dX \) \( (T_1 \neq 0) \).
Conditions:
1. \( n < 16 \)
2. \( N T < 50 \)
3. \( NC < 12 \)

These conditions can be changed by altering the array declarations.

RESULTS:

\[
\begin{array}{cccc}
N & F & P_1 & P_2 & \cdots & P_n \\
\end{array}
\]

- \( N \) \( (\leq nK) \): the number of times the residual function has been evaluated.
- \( F \): the value of the residual function at the end of the calculation (either when \( N = nK \) or when \( e \), the accuracy factor has been satisfied).
- \( P_1, \ldots, P_n \): the final values of the co-ordinates involved in the adjustment process.

TIME: Depends mainly on the time required to evaluate the residual function.

For \( NT = 11 \) and \( NC = 5 \), the time required for each evaluation was one minute.

R. A. Stacey
TITLE: MAGN (5-hole Elliott 803 Algol)

PURPOSE: Given a two-dimensional model with a polygonal cross-section, the programme calculates the horizontal, vertical and total intensity magnetic anomalies at equal intervals along a line perpendicular to the strike of the model. The magnetisation is assumed to be perpendicular to the strike and need not be parallel to the inducing field.

Modifications:

1. Allowance can be made for topography along the line of the profile (MAGN-T).

2. The points at which the anomaly is to be calculated can be specified individually, permitting unequal spacing between them (MAGN-T).

METHOD: The method used is similar to that described by Talwani, Worzel and Landisman (1959). The anomalous body is assumed to be two-dimensional and is represented by a two-dimensional model with an n-sided polygonal cross-section, and its long axis parallel to the strike of the anomaly (see figure overleaf). The anomaly at \((x,z)\), usually on a plane equivalent to that containing the observed anomaly, due to the model is

\[
A(x,z) = \sum_{j=1}^{m} \Delta F_j
\]

the model having \(m\) faces, each causing a disturbance \(\Delta F\) at \((x,z)\), and

\[
\Delta F = \Delta H \cos (\theta) \sin (\phi) + \Delta Z \sin (\theta)
\]

assuming the direction in which the anomaly has been measured is parallel to the inducing fields. \(\Delta H\) and \(\Delta Z\) are the horizontal and vertical magnetic anomalies at \((x,z)\) produced by a semi-infinite slab with a sloping end facing south (Heiland, 1940),

\[
\Delta H = 2 \sin i \left[ J_x E_1(x,z) + J_z E_2(x,z) \right]
\]

\[
\Delta Z = 2 \sin i \left[ J_x E_2(x,z) - J_z E_1(x,z) \right]
\]
\( \alpha \) is the angle between magnetic north and the strike of the anomaly measured clockwise.

The co-ordinate system for the calculation of \( \Delta H \) and \( \Delta Z \) over a horizontal, two-dimensional prism. The strike of the model is perpendicular to the page.
where \( E_1(x,z) = \sin i (\psi_2 - \psi_1) - \cos i \ln \left( \frac{r_2}{r_1} \right) \)

\( E_2(x,z) = \cos i (\psi_2 - \psi_1) + \sin i \ln \left( \frac{r_2}{r_1} \right) \)

\[ J_x = |J| \cos (1\theta) \quad J_z = |J| \sin (1\theta) \]

**DATA FORMAT:**

```
Model X?
H Z F
1
X1 DX NT
x1 z1 x2 z2 J
......
Xm zm Xm+1 zm+1 J
0 0 0 0 0
IE0 IB0
...-
1 -1
```

Units: arbitrary

1. Title required for each model.
2. The three positions indicate whether the horizontal (H), vertical (Z) or total (F) intensity anomalies are required (0 = not required, 1 = required).
3. \( \alpha \) is the strike of the model in degrees, measured clockwise from magnetic north.
4. \( x_1 \) is the co-ordinate of the first point at which the anomaly is required, \( dx \) the interval between successive points and \( NT \) the number of points.
5. \((x_1 z_1), (x_2 z_2)\) are the co-ordinates of the first face (in anti-clockwise sequence) and \( J \) is the magnetisation contrast between the model and its surroundings (\( J \) in c.g.s. units). The first x co-ordinate of each face must not be zero.
6. The next face of the model working an an anti-clockwise direction. Faces for which \( z_j = z_{j+1} \) have no effect and can be omitted.
7. The five zeros are used to signify that the last face of the model has been read.
8. IE, the inclination of the Earth's field in the magnetic meridian and IB, the inclination of the magnetisation of the model in the plane perpendicular to its strike. There can be a list of such combinations, permitting the effect of different magnetisation directions to be calculated for the same model. The angles (in degrees) are measured clockwise from the positive x-axis.

9. -1 -1 indicates the end of the list of magnetisation directions. The next model is then automatically read in.

Conditions:
1. NT < 40
2. Maximum number of faces is 9.

These conditions can be altered by changing the array declarations.

RESULT: The value of each x co-ordinate is followed by the magnetic anomaly or anomalies at that point in gamma.

TIME: Part I - 1.05 secs./face/point.
Part II - 1.05 secs./face/point/magnetisation direction


R. A. Stacey
7th November, 1965.
TITLE: MGSC (5-hole Elliott 803 Algol)

PURPOSE: Given the co-ordinates and magnetisation of a two-dimensional model with a polygonal cross-section, this programme calculates the total intensity magnetic anomaly at equal intervals along a profile perpendicular to the strike of the model. The programme then scales the amplitude of the computed anomaly to a specified number of gamma and the distance between the mean or half-mean values of the maximum and minimum of the anomaly to a specified distance. In this way a library of standardized curves can be prepared.

Modifications:

1. Another version incorporates a procedure for an off-line digital plotter which can produce the curves automatically.

METHOD: The total intensity magnetic anomaly is calculated by the MAGN method (see MAGN specification). A quadratic function is then fitted to the three values either side of each optimum of the anomaly and the results are differentiated to find the turning points. The difference between the maximum and minimum obtained in this way is scaled to the required number of gamma and the remaining values of the anomaly are scaled correspondingly. The positions of the values of the anomaly above and below the mean (or half-mean) of the maximum and the minimum are determined and the positions corresponding to the mean (or half-mean) values are found by linear interpolation over the relevant interval (see figure overleaf). The distance between these positions is then scaled to the required amount. Finally, the amplitude and model scale factors are printed, together with the scaled anomaly.

A set of such standardized anomaly curves can be computed for a range of rectangular models with different depth/width and magnetisation directions, with the bottom surface very deep (effectively at infinity). The field anomalies are then scaled to the same dimensions and when a theoretical curve has been found
Case 1. The definitions of the amplitude (AMP) and spread (SPR) of the observed and calculated anomalies.

Case 2. Mean value of anomaly indeterminate.
that matches the observed anomaly, the true intensity of magnetisation
(|J|) of the disturbing body is given by:

\[ |J| = \frac{|J|_{\text{model}}}{\text{Amplitude scale factor for the field curve}} \]

(Amplitude scale factor for the theoretical curve)

(|J| is in c.g.s. units)

Similarly, the true depth and width of the top of the body are:

Depth = \frac{(\text{Depth to the top of the model.})}{(\text{Scale factor for model})}

(Horizontal scale factor for the field curve)

Width = \frac{(\text{Width of the top of the model.})}{(\text{Scale factor for model})}

(Horizontal scale factor for the field curve)

Caution:

1. If the maximum/minimum is between 0.5 and 1.5, the mean values of
   the anomaly are often indeterminate and the points corresponding to
   the half-mean are used instead for this range.

2. The library of standardized anomalies is usually computed for models
   with magnetic East-West strike. If the strike of the observed anomaly
   is not parallel to this direction, then the apparent inclination
   of the Earth's field \( IE^* \) in the plane perpendicular to the strike
   of the observed anomaly must be used instead.

\[ IE^* = \tan^{-1} \left[ \frac{\tan(IE)}{\sin(\alpha)} \right] \]

**DATA FORMAT:** 

\[ \text{IE}^0 \quad \text{IB}^0 \quad \alpha^0 \]

\[ \text{dX} \quad \text{NT} \quad \text{NC} \quad \text{AMP} \quad \text{SPR} \]

\[ x_1 \quad z_1 \quad J \]

\[ \ldots \ldots \ldots \]

\[ x_{\text{NC}} \quad z_{\text{NC}} \quad J \]
AMP and SPR are explained in the diagram overleaf and the co-ordinate system is given in the MGN specification.

Conditions:

1. NC<15
2. Nr<50

These may be changed by altering the array declaration.

RESULTS: The AMP and SPR scale factors and the maximum and minimum values of the anomaly are printed together with the values of IE, IB and the depth/width. These are followed by the scaled anomaly.

TIME: 7.5 minutes for 60 points over a rectangular model.

R. A. Stacey.
7th November, 1965.
TITLES: LSMR (matrix exc.) and LSMT (5-hole Elliott 803 Algol)

PURPOSE: Given a measured total intensity magnetic anomaly and a two-dimensional model with either a rectangular (LSMR) or a trapezoidal (LSMT) cross-section which approximates the disturbing body, the programme adjusts specified dimensions and/or the magnetisation of the initial model by a least squares process to reduce the differences between the calculated and observed anomalies to a minimum.

Modifications:

1. LSMR-T permits unequal intervals between the specified values of the anomaly and topographic changes along the line of the profile to be taken into consideration.

METHOD: The initial model is based on the geological information available and is built up in the same way as that for programme MAGN. Ideally, the following relationship should hold:

\[ A_0(x_i) = f(x_i; a_1, a_2, \ldots, a_m) \]

where \( A_0(x_i) \) is the observed anomaly at \( x_i \) \( (i = 1, 2, \ldots, n) \) and the right hand side represents the theoretical anomaly at \( x_i \) caused by the model defined by \( a_1, a_2, \ldots, a_m \). If \( a_1^0, a_2^0, \ldots, a_m^0 \) represents the initial model and the correction terms should be evaluated are \( \alpha_r \) \( (r = 1, 2, \ldots, m) \)

\[ a_r = a_r^0 + \alpha_r \]

where \( a_r \) is the best set of dimensions for the model. The set of normal equations to be solved for \( \alpha_r \) are

\[ \sum_{r=1}^{m} \left( \sum_{k=1}^{n} b_{kr} \right) \alpha_r = \sum_{k=1}^{n} b_{kj} \ell_k \quad (j = 1, 2, \ldots, m) \]
Explanation of the symbols used in programmes LSMR and LSMT.
The development of this set of equations is described in the thesis, Chapter II. The process is repeated using the new value of $a_r$ and continues until either

$$\sum_{k=1}^{n} [t_k]^2 \leq e_1 \sum_{k=1}^{n} [A_o(x_k)]^2$$

1.

or

$$\sum_{k=1}^{n} [t_k]^2 - \sum_{k=1}^{n} [t_{k-1}]^2 \leq e_2 \sum_{k=1}^{n} [t_k]^2$$

2.

where $t_k$ is the residual at $x_k$ and $e_1$ and $e_2$ are small, positive fractions.

Cautions:

1. The calculated adjustments to the initial model cause the residual function to increase if the initial model is inadequate. If this occurs, or if it is suspected that the minimum obtained is not the global minimum, then a different initial model must be used.

2. If the residual function increases, conditions 1 and 2 may not be satisfied and the calculation must be stopped manually. LSMR prints "no improvement" if this occurs.

3. The variables are unconstrained and $x_1 > x_2$ or $z < 0$ can occur (see data format). Modified versions (LSMR(limited) and LSMT(limited)), limit the size of the correction terms, but the method of constraint has no mathematical justification and should be used with care.

DATA FORMAT: & Rectangular model (LSMR) ?

IE$^{rad}$ IB$^{rad}$ Al$^{rad}$ $\alpha$ $^{rad}$
dX NT
\[ z_1 \quad A_2 \quad z_2 \quad A_3 \]
\[ x_1 \quad A_4 \quad x_2 \quad A_5 \quad (A_n = 0 \text{ parameter varied, } A_n = 1 \text{ parameter unchanged}) \]
\[ J \quad A_6 \]

\[ T_1, T_2 \ldots T_{NT} \]

**Trapezoidal model (LSMT)?**

\[ IB^0 \quad A^0 \quad dX \quad NT \]
\[ z_1 \quad A_1 \quad z_2 \quad A_2 \]
\[ x_1 \quad A_3 \quad x_2 \quad A_4 \quad (A_n = 1 \text{ parameter varied, } A_n = 0 \text{ parameter unchanged}) \]
\[ x_3 \quad A_5 \quad x_4 \quad A_6 \]
\[ J \quad A_7 \]
\[ IB^0 \quad A_8 \]
\[ e_1 \quad e_2 \]

\[ T_1, T_2 \ldots T_{NT} \]

\( T_1, T_2 \ldots T_{NT} \) is the observed magnetic anomaly at equal intervals \( dX \) along the line of the profile and \( e_1 \) and \( e_2 \) are the small, positive fractions defined earlier. The remaining symbols are explained in the MAGN specification.

**Conditions:**

1. \( NT < 50 \) for both programmes.

**TIME:**

Variable, depending on the number of values of the anomaly used and the number of iterations required to reach a minimum.

**RESULTS:**

For LSMR and LSMT, \[ \sum_{k=1}^{n} t_k^2 \] and the values of the co-ordinates and magnetisation of the model are printed after each iteration. The values of the individual residuals may be printed if required.

R. A. Stacey

7th November, 1965.
TITLE: MAGN SWANOPT (unconstrained) (8-hole Elliott 803 Algol)

PURPOSE: Given a measured total intensity magnetic anomaly and a two-dimensional model with an n-sided polygonal cross-section which approximates the disturbing body, the programme adjusts specified vertices and/or the magnetisation of the initial model by a direct search process, to reduce the differences between the calculated and observed anomalies to a minimum. The function to be minimised is

$$ F = \frac{1}{nT} \sum_{i=1}^{nT} \left[ \frac{A_o(x_i) - H_m(x_i)}{NT|A_o(x_i)||H_m(x_i)|} \right]^2 $$

where

$$ A_m(x_i) = \sum_{j=1}^{nE-1} \left[ \Delta H E\, \cos(E) \xi_\pm(\alpha) + \Delta Z \, \xi_\pm(E) \right] $$

and $A_o(x_i)$ and $A_m(x_i)$ are the observed and calculated anomalies at $x_i$, which lies on the line of the measured profile. The remaining symbols are defined in the MAGN specification.

METHOD: The Swanopt procedure is due to W.H. Swann of I.C.I., Wilton (May 1964) and the description that follows has been taken from his specification for Programme-589 Direct Search Optimisation (Unconstrained).

"The method used is to set up n mutually orthogonal directions. A univariate search is made to locate the optimum along each of these directions in turn and then a new set of directions is calculated. As the optimum is approached, the step length used in the univariate search is reduced. The programme terminates either when the step length is reduced to a value less than the accuracy specified by the user, or after nk function evaluations, whichever occurs first. k is the maximum number of function evaluations permissable per independent variable and is specified by the user. (k = 150 is usually sufficient.)"
Caution:

1. Cautions 1 and 3 in the LSMR and LSMT specifications apply to this programme.

DATA FORMAT; 

<table>
<thead>
<tr>
<th>s</th>
<th>n</th>
<th>J</th>
<th>A</th>
<th>K</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>NC</td>
<td>$X_1$</td>
<td>$dX$</td>
<td>NT</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>$z_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_2$</td>
<td>$z_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\ldots\ldots$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_j$</td>
<td>$z_j$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_1$</td>
<td>$z_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>JJ</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$IB^0$</td>
<td>$IE^0$</td>
<td>$\alpha^0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_1$</td>
<td>$a_2$</td>
<td>$\ldots\ldots$</td>
<td>$a_n$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_1$</td>
<td>$T_2$</td>
<td>$\ldots\ldots$</td>
<td>$T_{NT}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

s accuracy factor (try .0001)

n the number of co-ordinates to be adjusted.

J = -1 for minimisation.

A = 0 for printing after each completed adjustment process.

K maximum number of function determinations (try K=n.100)

s initial step length

NC the number of co-ordinate pairs $(x_j,z_j)$, finishing with the first co-ordinate, $NC = j + 1$.

$X_1$ co-ordinate of the first value of the observed magnetic anomaly.

dX the interval between successive values of the anomaly.

NT the number of values of the observed anomaly, which must be equally spaced.

$x_i,z_i$ the co-ordinates of the initial model.

JJ the intensity of magnetisation (c.g.s. units).

IB the inclination of the magnetisation of the model (degrees).

IE the inclination of the Earth's field (degrees).
the strike of the model (degrees).

\( a_1 \ldots a_n \) the co-ordinates to be adjusted: e.g. for \( x_1 \ a_1 = 1 \); for \( z_1 \ a_1 = 2 \), etc.

The magnetisation of the model can be treated in the same way (\( J_1 = 2 \cdot NC + 1 \) and \( IB = 2 \cdot NC + 2 \)).

\( T_1 \ldots T_{NT} \) the values of the observed anomaly at equal intervals \( dX (T_i \neq 0) \).

Conditions:
1. \( n < 16 \)
2. \( NT < 50 \)
3. \( NC < 12 \)

These conditions can be changed by altering the array declarations.

\[ \text{RESULTS:} \quad N \quad F \quad P_1 \quad P_2 \ldots \ldots \ldots P_n \]

\( N \) (\( \leq nk \)) the number of times the residual function has been evaluated.

\( F \) the value of the residual function at the end of the calculation (either when \( N = nk \) or when \( e, \) the accuracy factor has been satisfied).

\( P_1 \ldots P_n \) the final value of the co-ordinates involved in the adjustment process.

\[ \text{TIME:} \quad \text{Depends mainly on the time required to evaluate the residual function.} \]

For \( NT = 21 \) and \( NC = 5 \), the time required for each evaluation was two minutes.

R. A. Stacey

7th November, 1965.
TITLE: MAGR DAVOPTC (8-hole Elliott 803 Algol)

PURPOSE: Given a measured total intensity magnetic anomaly and a two-dimensional rectangular prism model which approximates the disturbing body, the programme adjusts specified co-ordinates and the magnetisation of this initial model until the differences between the observed and calculated anomalies have been reduced to a minimum. A gradient method is used to optimise the model and the function to be minimised is

\[ f = \sum_{i=1}^{N_T} \left[ A_o(x_i) - A_m(x_i) \right]^2 \]

where

\[ A_m(x_i) = 2 \left[ \left( \varphi_2 - \varphi_3 - \varphi_4 + \varphi_5 \right) \cos(I_E + I_B) + \left( \frac{\varphi_7}{\varphi_2} \right) \cos(I_E + I_B) \right] \]

The co-ordinate system is explained in the diagram and \( A_o(x_i) \) and \( A_m(x_i) \) are the observed and calculated anomalies at \( x_i \), which lies on the line of the measured profile. It is assumed that the measured profile lies in the magnetic meridian and that the strike of the anomaly is magnetic east-west.

METHOD: The gradient optimisation method used was first developed by Davidon \ldots\ldots\ldots then modified by Fletcher and Powell (1963). Swann (unpublished) has recently added constraints to the variables using Carrol's (1961) created response surface technique. The procedure has been translated by the author into 8-hole Elliott 803 Algol from the KDF 9 Autocode programme written by Swann at I.C.I. Wilton using an I.C.I. developed Autocode.
IE is the inclination of the Earth's field in the plane of the profile measured clockwise from the positive x-axis in degrees.

IB is the inclination of the magnetisation of the body in the plane of the profile measured clockwise from the positive x-axis in degrees.

J is the intensity of magnetisation of the model (c.g.s. units).

The co-ordinate system for the MAGR DAVOPTC programme.
DATA FORMAT: & Title ?

\[
\begin{align*}
n & \text{ the number of parameters to be varied.} \\
ea & \text{ the accuracy factor (try .0001).} \\
X_1 & \text{ the position of the first value of the observed magnetic anomaly} \\
dX & \text{ the interval between successive values of the anomaly.} \\
N'r & \text{ the number of values of the anomaly.} \\
a, z_2, x_1, x_2, J, IB \text{ and IE are defined in the diagram.} \\
\text{Units are as for the observed anomaly and IB and IE are measured clockwise from the positive x-axis in degrees.} \\
CA_1 \ldots CA_6 & CA_j = \frac{1}{2} \text{ for variable parameters, otherwise zero.} \\
e_1 \ldots e_6 & \text{ the lower limit on the variable.} \\
f_1 \ldots f_6 & \text{ the upper limit on the variable.} \\
& \text{ (when } CA_j = 0, e_j = f_j = 0) \\
AC_1 \ldots AC_n & \text{ the subscript values of the variable parameters.} \\
\text{e.g. for } z_1 \text{ variable, } AC_1 = 1; \text{ for } x_2 \text{ and IB variables, } AC_1 = 4 \text{ and } AC_2 = 6.
\end{align*}
\]
\( T_1 \ldots T_{N_T} \) the values of the observed anomaly in gamma.

**Note:** Owing to an omission in the programme \( e_6 \) and \( f_6 \) must be defined in radians.

**Conditions:**

1. \( N_T < 50 \)
2. \( n < N_T \)

**RESULTS:**

\( N \quad F \quad P_1 \ldots P_n \)

- \( N \) the number of function evaluations.
- \( F \) the current value of the function.
- \( P_1 \ldots P_n \) the current values of the variables.

Optional printing: the value of the function, the values of the variables and the number of function evaluations may be printed after each evaluation if required.

**TIME:** Approximately 0.5 secs./variable/value of the anomaly.

**REFERENCES:**


R. A. Stacey

20th December, 1965.
PURPOSE: Given a two-dimensional total intensity magnetic anomaly, this programme calculates its pseudo-gravity equivalent in the plane \( y = \eta \) (PG(\( \eta > 0 \))) and \( y = 0 \) (PG(\( \eta = 0 \))).

METHOD: The anomalous body is assumed to have a conventional density \( \rho = |J|/G \), where \( G \) is the universal gravity constant and \( |J| \) is the magnitude of the magnetisation vector. The expression for the pseudo-gravity equivalent of a two-dimensional total intensity magnetic anomaly is

\[
P_{\mathcal{G}}(\xi, \eta) = \frac{\xi \rho}{J \pi} \int_{-\infty}^{\infty} A_0(\eta', 0) K(\sqrt{\xi^2 + (\eta - \eta')^2}) \, d\eta'
\]

(See thesis Appendix 1 for the derivation of this equation and the diagram overleaf for the explanation of the symbols.) In evaluating this integral at \((\xi, \eta)\), the observed anomaly \( A_0(x, 0) \) is defined over the range \(-R \leq x \leq R\) and can be integrated numerically by Simpson's rule. For \( x \geq |R| \) the anomaly is approximated by the function

\[
A_0(\eta', 0) = \sum_{n=1}^{N} a_n \frac{\eta}{(n+1)^2}
\]

which is a solution of Laplace's equation as \( x \to \pm \infty \) and can be integrated from \(-\infty\) to \(-R\) and from \( R \) to \( \infty \). The kernel function is

\[
K(\sqrt{\xi^2 + (\eta - \eta')^2}) = \cos \beta \mathbf{h}(r) + \sin \beta \, \theta
\]

where \( \beta = \frac{\pi}{2} \, \, \eta + \sigma - \alpha \) and \( \alpha \), the direction of measurement of the gravity field is \( \pi/2 \). \( \sigma \) and \( \mu \) are the inclination of the ambient field and the magnetisation of the body in the plane of the profile.
Explanation of the symbols used in the PGI(\(\eta > 0\)) programme.

All angles \((\lambda, \beta, \theta)\) are measured anti-clockwise from the positive x-axis. The origin \((0,0)\) is positioned over the linear dipole corresponding to the anomalous body.

\[
r = \left[ (x - s)^2 + \gamma^2 \right]^{\frac{1}{2}}
\]

\[
\theta = \arctan \left[ \frac{\gamma}{s - x} \right]
\]
When $f=0$ there is a singularity in the kernel function when $x=f$.

To overcome this the magnetic anomaly in the region $x \neq f$ is expressed as

$$A_o(x; f) = \sum_{n=1}^{M} b_n \chi^{(\kappa - 1)}$$

which, when substituted into equation 1 can be integrated.

Caution:

1. If the plane of the profile perpendicular to the strike of the anomaly is not also that of the magnetic meridian, then the apparent inclination of the Earth's field ($\sigma^\Sigma$) in the plane of the profile must be used instead

$$\sigma^\Sigma = \tan^{-1} \left( \frac{\tan(\sigma)}{\sin(D)} \right)$$

where $D$ is the angle between magnetic north and the strike, measured clockwise.

DATA FORMAT:

1. $PG(\gamma > 0)$

   \[ \mu \sigma \int_{T_1}^{T_2} \left| R \right| \ dx \ dz \ \rho \ \frac{\rho}{J} \ NT \]

2. $PG(\gamma = 0)$

   \[ \mu \sigma \int_{T_1}^{T_2} \left| R \right| \ dx \ \int_{f_1}^{f} \ dz \ NT \ \rho \ \frac{\rho}{J} \]

   inclination of the magnetisation of the body in the plane of the profile (radians)

   $\sigma$ inclination of the ambient field in the plane of the profile (radians)

   $\left| R \right|$ the extent of the numerical integration either side of the origin (metres).
the x co-ordinate of the first point at which the pseudo-gravity is to be calculated. (metres)

(\(\eta > 0\)) the ordinate of the points at which the pseudo-gravity is to be calculated. (metres)

d\(\xi\) the interval between successive values of the pseudo-gravity anomaly (metres)

d\(\xi\) the number of values of the magnetic anomaly.

\(\rho\) the anticipated density contrast (gm/cm\(^3\))

\(J\) the magnetisation contrast (c.g.s. units)

\(T_{1}, T_{NT}\) the magnetic anomaly. (gamma)

Conditions:

1. \(NT < 100\)

2. \(\xi \neq 0\) (PG(\(\eta = 0\)) only)

3. \(\rho\) and \(J\) are only approximate estimates in practice.

RESULTS: In both cases the values of \(\xi\) and the corresponding pseudo-gravity anomalies are printed. If the above system of units has been used for \(\rho\) and \(J\), the result is in milligals.

TIME:

1. PG(\(\eta > 0\)) 2 secs./point using 33 values of the magnetic anomaly.

2. PG(\(\eta = 0\)) 2 secs./point using 33 values of the magnetic anomaly.


R. A. Stacey

20th November, 1965
TITLE: PGI (5-hole Elliott 803 Algol)

PURPOSE: Given a two-dimensional total intensity magnetic anomaly, this programme calculates the range of possible directions of magnetisation for the body, assuming the pseudo-gravity equivalent of the observed magnetic anomaly will always be one sign.

Modification:

1. An alternative version, (PGI-B) includes facility for correcting the background value of the anomaly assuming

   $\phi = \int_{-\infty}^{+\infty} A_0(\nu, 0) d\nu$

2. $\gamma = 0$ (PGI($\gamma = 0$)).

METHOD: The anomalous body is assumed to have a conventional density $\rho = |J|/G$, where $G$ is the universal gravity constant and $|J|$ is the magnitude of the magnetisation vector. The expression for the pseudo-gravity equivalent of a two-dimensional total intensity magnetic anomaly is

   $\rho\xi(\delta, \gamma) = \frac{G\rho}{\pi} \int_{-\infty}^{+\infty} A_0(\nu, 0) K(\delta - \nu, \gamma) d\nu \quad \gamma > 0$

(See thesis Appendix 1 for the derivation of this equation and the diagram in PGI($\gamma = 0$) specification for the explanation of the symbols.) In evaluating this integral at $(\delta, \gamma)$ the observed anomaly $A_0(x_1, 0)$ is defined over the range $-R < x_1 < R$ and can be integrated numerically by Simpson's rule. For $x > |R|$ the anomaly is approximated by the function

   $A_0(\nu, 0) = \sum_{n=1}^{N} a_n \nu^{-(n+1)}$

which is a solution of Laplace's equation as $x \to \pm \infty$ and can be integrated from $-\infty$ to $-R$ and from $R$ to $\infty$. The kernel function is
\[ K(x, y) = \cos \beta \ln(r) + \sin \beta \theta \]

where \( \beta = \frac{\pi}{2} + \mu + \sigma - \alpha \) and \( \alpha \), the direction of measurement of the gravity field is \( \Pi/2 \). \( \sigma \) and \( \mu \) are the inclination of the ambient field and the magnetisation of the body in the plane of the profile.

Equation 1 can be re-expressed as

\[ \text{PG}(s, \gamma) = \frac{G \rho}{J \pi} \left[ \cos \beta \int_{-\infty}^{+\infty} A_0(x_1, 0) \ln(r) \, dx + \sin \beta \int_{-\infty}^{+\infty} A_0(x_1, 0) \theta \, dx \right] \]

\[ = \frac{G \rho}{J \pi} \left[ \cos \beta \cdot I_1(s, \gamma) + \sin \beta \cdot I_2(s, \gamma) \right] \]

\[ = \frac{G \rho}{J \pi} \left[ \cos(\beta - \lambda(s, \gamma)) \right] \left[ I_1(s, \gamma) + I_2(s, \gamma) \right] \]

where \( \tan \lambda(s, \gamma) = \frac{I_2(s, \gamma)}{I_1(s, \gamma)} \)

If the intensity of magnetisation is assumed to be positive throughout the body, the pseudo-gravity equivalent of the magnetic anomaly will always be positive and the following inequality will be true:

\[-\frac{\pi}{2} < [\beta - \lambda(s, \gamma)] < +\frac{\pi}{2} \]

or

\[ -\left[ \lambda(s, \gamma) - \frac{\pi}{2} \right] < \beta < \left[ \lambda(s, \gamma) + \frac{\pi}{2} \right] \]

This gives a permissible range for \( \beta \), and hence for \( \mu \) (\( \mu = \beta - \sigma \)) of pi in the plane of the profile perpendicular to the strike of the anomaly, assuming this plane lies in the magnetic meridian. This range decreases as \( \lambda(s, \gamma) \) is calculated for increasing values of \( |s| \).
Caution:

1. If the plane of the profile perpendicular to the strike of the anomaly is not also that of the magnetic meridian, then the apparent inclination of the Earth's field (\(\boldsymbol{\sigma}^{*}\)) in the plane of the profile must be used instead

\[
\sigma^{*} = \tan^{-1} \left[ \frac{\tan (\sigma)}{\sin (D)} \right]
\]

where \(D\) is the angle between magnetic north and the strike, measured clockwise.

DATA FORMAT: (Applicable to PGI and PGI-B)

\[ S \gamma \ R \ dX \ d\gamma \ NT \]
\[ T_{1} \ T_{2} \ldots \ldots \ldots \ T_{NT} \]

- \(S\) is the x co-ordinate of the first point at which \(\lambda(\xi, \gamma)\) is to be calculated (usually \(S = -R + dX\)).
- \(\gamma\) is the ordinate of the points at which \(\lambda(\xi, \gamma)\) is to be calculated (\(\gamma = dX\)).
- \(R\) is the half-range over which the anomaly can be defined.
- \(dX\) is the interval between successive values of the magnetic anomaly.
- \(d\gamma\) is the interval between successive values of \(\lambda(\xi, \gamma)\) (\(d\gamma = |R - dX|\)).
- \(NT\) is the number of values of the magnetic anomaly.

\[ T_{1} \ldots T_{NT} \] the magnetic anomaly.

Conditions:

1. \(NT < 100\)
2. Should it be required to calculate more than the usual three values of \(\lambda(\xi, \gamma)\), the limit is also less than 100 with \(-R < \xi < R\)
3. \(\gamma > 0\)

RESULTS: The values of \(\xi\) and the corresponding values of \(\lambda(\xi, \gamma)\) are printed out,
and assuming the recommended values of $\xi$ have been used, the limits on beta are:

$$[\lambda(\xi_1) - \frac{\pi}{2}] < \beta < [\lambda(\xi_1) + \frac{\pi}{2}]$$

From which it follows that the limits on $\mu$ are

$$[\lambda(\xi_2) + \varphi - \frac{\pi}{2}] < \mu < [\lambda(\xi_1) + \varphi + \frac{\pi}{2}]$$

TIME: $\frac{1}{2}$ sec./point/value of the magnetic anomaly.

R. A. Stacey

7th November, 1965.
BEGIN
ARRAY TL, TH, TIK, P(1:150), AA(1:100)
REAL XI, ETA, R, LL, X, XI, TL, IT, TH, LPR, LMR,
    AC, BC, CC, AS, BS, CS, GC, RR, LL, S1, S2, XE, LM, ETA,
    INTEGER NT, I, H, TT
SWITCH S=LL0, LL1, LL3, LL4, LL5

LL0: H:=1
READ INSTRINCAA, NT
READ XI, ETA, R, X, XI
FOR I:=1 STEP 1 UNTIL NT DO READ TCI)
ST:=S2:=0
FOR I:=2 STEP 2 UNTIL (NT-2) DO
BEGIN ST:=ST+TC1) S2:=S2+TC(I+1)' END
IT:=X*CTC1)+CTC(TNT)+4*CTC(TNT-1))/2*S2/3
AR:=R*C3+IT*R*(CTCNT)+TC1)'/4
BR:=R*C3*CTCNT+TCl'/2
CI:=+5*RR+3*(R*CTCNT+TC1))'/7

LL1: TH:=ARCTAN(ETA/XI+R)) AL:=ARCTAN(ETA/CR-XI))
LPR:=+5*LNC(R-XI)*ETA/XE+LMR*C1/R+XIfXE+AL+TH))/XE
BC:=B*C(2*XI*ETA/(R-LPR)2+AL+TH)2
CC:=C*CLPR-LPR)/XE+AL*CR-XI/XE+TH*CR-XI/XE)
RR:=+R
FOR I:=1 STEP 1 UNTIL NT DO
BEGIN TL(I):=+5*TC1)+LNC(XI-RR)+2*ETA(2)
RR:=RR+X
END
ST:=S2:=0
FOR I:=2 STEP 2 UNTIL (NT-2) DO
BEGIN ST:=ST+TTH(I) S2:=S2+TTH(I+1)' END
ITTH:=X*CTTH(I)+CTHT(TNT)+4*CTHT(TNT-1)+2*S2/3
QS:=c-AS-BS-CS+ITTH)

IF AR GR 0 THEN BEGIN LM:=ARCTANGS/GR)
    IF LM LESS 0 THEN LM:=6.28316+LM END
ELSE LM:=3.14159*ARCTANGS/GR)
PRINT FREELN(L6), XI/100, SAMELINE, ALIGNED(4.6), LM
XI:=XI+XIXI
IF XI LESS R THEN GOTO LL1
PRINT EXIT???
GOTO LL0
END
MAQC

BEGIN ARRAY X, Z, JC, E1, E2
REAL DH, DZ, DF, JX, JZ, JXE, JZE, DX, IE, IE, IE1, IE2
INTEGER NT, H, L, I, J, K, L, LL
SWITCH SS := L1, L2

PROCEDURE MAQCXTT, X1, Z1, X2, Z2, JJ, EE1, EE2

REAL XT, X1, X2, Z1, Z2, JJ, EE1, EE2
BEGIN REAL S, C, L, PP, LL
L := SQRT((X1 - X2)**2 + (Z2 - Z1)**2)
S := (Z2 - Z1) / L
C := (X1 - X2) / L
PP := ATAN((X1 - XT) / 2L) - ATAN((X2 - XT) / 2L)
LL := 5 * LN((X2 - XT)**2 + Z2**2) / ((X1 - XT)**2 + Z1**2)
EE1 := JJ * S * C * PP - C * LL
EE2 := JJ * S * C * PP + C * LL
END

LL0: N := 1, INSTRING(T, N)
N := 1, OUTSTRING(T, N)
READ H, ZZ, F, IE, IE, IE1, IE2
IF H = 1 THEN PRINT ESSZ?
IF ZZ = 1 THEN PRINT ESSF?
FOR N := 0 STEP 1 UNTIL NT - 1 DO BEGIN XT := XTO + N * DX
FOR I := 1 STEP 2 UNTIL NI - 2 DO BEGIN DH := DH + JX * IE(I, N + 1) + JZ * E2(I, N + 1)
DZ := DZ + JX * E2(I, N + 1) + JZ * IE(I, N + 1)
END
DF := DF + JXE * DH + JZE * DZ
PRINT ALIGNC7, 23, XT
IF H = 1 THEN PRINT SAMELINE, ALIGNC6, 23, HH
IF ZZ = 1 THEN PRINT SAMELINE, ALIGNC6, 23, ZZ
IF F = 1 THEN PRINT SAMELINE, ALIGNC6, 23, FF
END
PRINT EEL6??" GOTO LL2 END

END
LSER

LSQND D1, D2, RO
BEGIN Array T, Q2, TAC(1:100), ADC(1:3), RC(0:20), CC, BBC(1:3, 1:1), AAC(1:3, 1:3)
REAL D1, D2, DX, E1, E2, T1, T0, X01, X02, X1, X3, R1, R2, R3, R4, N, A, B, C,
F1, F2, F3, F4, RO, X0
INTEGER NT, N1, N11, N111, SWITCH S=LL0, LL1, LL2

LLO:N=1' PRINT C1, C2, X01, X02, RO
BEGIN INSTRUC T, 111, TT, X01, X02, X1, X3, RO, C, BBC1, J, 1: J, AAC1, J, 1: J
REAL D1, D2, DX, E1, E2, T1, T0, X01, X02, X1, X3, R1, R2, R3, R4, N, A, B, C,
F1, F2, F3, F4, RO, X0

N=1: N=N+1' END
FOR N=1 STEP 1 UNTIL NT DO BEGIN READ D1, A, D2, B, RO, C, M, DX, NT, E1, E2
BEGIN TT=0' N111=1' RC(0):=1000000
FOR N=1 STEP 1 UNTIL NT DO BEQIN
READ TC(N)
TT:=TT+TC(N)
END
FOR N:=1 STEP 1 UNTIL NT-2 DO
Cl2(N):=e-re N-2:+16•<T<N-1 :>+Tc N+1).:>-JO•T<N:> -TcN+2:>:>/< 12•DX••2:> \nN:=(NT+1)/2 ।
FOR Q2(N):=Q2(N)
WHILE Cl2(N) LESS 0 DO N:=N-1'·
X01:=DX•CABSCQ2cN+1))•(N-1)+ABS(Cl2CN))•N)/CABSCQ2(N+1)) +ABSCQ2CN)))
N:=(NT+1)/2+1' . .
FOR Q2(N):=Q2(N)
WHILE ~2(N) LESS 0 DO N:=N+1'
X02:=DX•CABSCCl2cN-1):>•cN-1)+ABSC~2CN))•CN-2))/CABSC~2CN-1)) +ABSCQ2CN)))
PRINT ££L2??,X01, X02, ££L??

LL1: FOR I:=1 STEP 1 UNTIL 3 DO BEGIN BBC1, 1):=0'
FOR II:=1 STEP 1 UNTIL 3 DO AAC1, II):=0'
BEGIN X0:=D0-D1' RCI1):=0'
FOR N=0 STEP 1 UNTIL NT-1 DO BEGIN
X1:=X0-N=DX' X3:=X02-N=DX'
R1:=X1•+2=DI•+2' R2:=X1•+2=DI•+2'
ADC1):=013334*RO*C-F1) ADC2):=013334*RO*C-F1-F4)
ADC3):=TAC(N+1)):=013334*RO*C-F1) ADC4):=TAC(N+1)):=013334*RO*C-F1-F4)
FOR I:=1 STEP 1 UNTIL 3 DO BEGIN
BBC1, 1):=BBC1, 1)+TCTCI+RO*TAC(N+1)+ADC1)
FOR II:=1 STEP 1 UNTIL 3 DO
A AC1, II):=AAC1, II)+ADC1)+ADC11)
END END
FOR N=1 STEP 1 UNTIL NT-1 DO RC111):=RC111)+CHECKRC(TC(N)-RO*TAC(N))+++2
PRINT RC111)

IF RC111)/TT LESS E1 THEN BEGIN PRINT ££L??, £E1?' GOTO LL2 END
IF ABS(RC111)-R1) LESS E2 THEN GOTO LL2'

XQUOT CC, AA, BB)
IF A=1 THEN CCC1, 1):=0'
IF B=1 THEN CCC2, 1):=0'
IF C=1 THEN CCC3, 1):=0'
IF CCC1, 1):=0' THEN CCC1, 1)+1=SIGNCCC1, 1)+1)
ELSE D1:=D1+CCC1, 1)
ELSE D2:=CCC2, 1)
ELSE RO:=RO+SIGNCCC3, 1)+5
ELSE RO:=RO+CCC3, 1)'
PRINT ALIGNEDC7, 3), D1, SAMELINE, D2, RO'
PRINT ££L12??' GOTO LL0' END END'

LL2: FOR N=1 STEP 1 UNTIL NT DO PRINT ALIGNEDC3, 2), TC(N)-RO*TAC(N)'
PRINT ££L112??' GOTO LL0' END END'
BEGIN
ARRAY CX,CZ,JCC(1:20),TAC(1:100)
REAL IE, IB, AL, AA1, AA2, DX, XXS, TX, TMX, XM, TNN, TSC, TH, 
XM, XSC, XMP, CMS, SPI, AA, AAA, UD
INTEGER NT, NC, I, J
SWITCH SS:=LLO

PROCEDURE FAN0!, CXS, X1, Z1, X2, Z2, J, A1, A2, FA
REAL XS, X1, X2, Z1, Z2, J, A1, A2, FA
BEGIN
REAL S, C, L, FI
S:=CZ2-Z1)/SQRT(CX1-X2)^2+CZ1-Z2)^2
C:=(CX1-X2)/SQRT(CX1-X2)^2+CZ1-Z2)^2
L: =.5
LNC ((CX2-XS)^2+Z2)^2)
FI:=ARCTAN(CX1-XS)/(Z1)-ARCTAN(CX2-XS)/(Z2)
END
PROCEDURE SOLVES, CA, XT, T1, T2, TJ, TX1, TX2, TXJ, XX
REAL A, XT, T1, T2, TJ, TX1, TX2, TXJ, XX
BEGIN
ARRAY CF, TSC1:J, 1:1), TXC1:1, 1:J)
REAL XX1, XX2, TSC1, 1)=T1 TSC2, 1)=T2
TXC1, 1)=TX1, 1)=TX1, 2)=TX1, 2)=TX1, 3)=TX1, 3)=TX1, 3)=TX1
TXC2, 1)=TX2, 1)=TX2, 2)=TX2, 2)=TX2, 3)=TX2, 3)=TX2, 3)=TX2
TXCJ, 1)=TXJ, 1)=TXJ, 1)=TXJ, 1)=TXJ, 1)=TXJ, 1)=TXJ
MXQUOT (CF, TX, TS)
TS C 3, 1) : =T
TXC1, J)=1 TXC2, J)=1 TXCJ, 3)=1
IF A=1 THEN
XT:=CFCJ, 1)-CFC2, 1)^2/IC4.CFC1, 1))
ELSE BEGIN
XX1: =C-CFC2, 1)+SQRT(CFC2, 1)^2-4.CFC1, 1)(CFCJ, 1)-XT)))/C2.CFC1, 1))
IF XX1 GREQ TX1 AND XX1 LESSEQ TX3 THEN
XX:=XX1 ELSE XX:=XX2
END END

N:=1 IF TAC3\.LESS TAC3+1 THEN BEGIN
FOR N:=N WHILE TACN LESS TACN+3 DO N:=N+1
SOLVES, (1, TMX, TACN-1), TACN, TACN-1), (CN-2)*DX, (CN-1)*DX, N=DX, UD)
FOR N:=N WHILE TACN OR TACN+1 DO N:=N+1
SOLVES, (1, TMX, TACN-1), TACN, TACN-1), (CN-2)*DX, (CN-1)*DX, N=DX, UD)
END ELSE BEGIN
SOLVES, (1, TMX, TACN-1), TACN, TACN-1), (CN-2)*DX, (CN-1)*DX, N=DX, UD)
FOR N:=N WHILE TACN LESS TACN+1 DO N:=N+1
SOLVES, (1, TMX, TACN-1), TACN, TACN-1), (CN-2)*DX, (CN-1)*DX, N=DX, UD)
END
TSC:=AMP/CABS(TM+X)*AABS(TM+Y))
IF N:=1 STEP 1 UNTIL NT DO TACN:=TACN+1
TMX:=TX+TSC TMN:=TM*TSC
IF .5 LESSSEQ AABS(TM+X)/TMN AND 1.5 GREQ AABS(TM+Y)/TMN
THEN AAA:=.25 ELSE AAA:=.5
TM:=TMN+AA*LMP
N:=1 IF TACN \R TM THEN BEGIN
FOR N:= N WHILE TACN GREQ TM DO N:=N+1
SOLVES, CO, TM, TACN-2), TACN, TACN-1), CN-2)*DX, CN-1)*DX, N=DX, UD)
FOR N:=N WHILE TACN OR TACN+1 DO N:=N+1
SOLVES, CO, TM, TACN-2), TACN, TACN-1), CN-2)*DX, CN-1)*DX, N=DX, UD)
END ELSE BEGIN
SOLVES, CO, TM, TACN-2), TACN, TACN-1), CN-2)*DX, CN-1)*DX, N=DX, UD)
FOR N:=N WHILE TACN LESS TACN+1 DO N:=N+1
SOLVES, CO, TM, TACN-2), TACN, TACN-1), CN-2)*DX, CN-1)*DX, N=DX, UD)
END
END
PRINT SELZ??, ALIGNEDC2, 1), TSC, XSC, XEL2??
PRINT SELZ??, ALIGNEDC4, 2), IE!/01754, SAMELINE, lB!/01754,
XCNDCO, XCC10, XCC1, XEL2??, XSC, XEL2??, TMX, TNN, XEL2??
FOR N:=0 STEP 1 UNTIL NT-1 DO
PRINT ALIGNEDC4, 2), 1:DX, SAMELINE, ELE3??, TACN+1)
GOTO LLO END END
REAL MU, PSI, X1, ETA, R, RR, X, XI, TL, IT, ITTH, LPR, LMR, A, B, C, AC, BC, CS, AS, BS, CS, GC, GS, PG, RO, J, S1, S2, XE, LN, BETA
INTEGER NT, I, I1, J, S
SWITCH S := L00, L01, L03, L04, L05
L00: M := 1
READ INSTRCAA, M)
H := 1
PRINT OUTSTRINGCAA, M)
READ X1, ETA, R, X, XI; NT
XI := XI•100
ETA := ETA•100
R := R•100
X := X•100
FOR I := 1 STEP 1 UNTIL NT DO READ TC;)
S1 := S2 := 0
FOR I := 2 STEP 2 UNTIL (NT-3) DO
BEGIN S1 := S1+TC1
S2 := S2+TC1+1
END
IT := X•CTC1+TCNT)+4•CS1+TCNT-1)+2•S2)/3
A := R•CR•CTC1+TCNT-1)/4
B := R•3•CTCNT+TC1)/2
C := CHECKRC<CT/R+TCNT+TC1>)/4
FOR I := 1 STEP 1 UNTIL NT DO TC; := TC;+C
END
LPR := +.5•LNCCR-X1)••2+ETA••2)
UIR := +.5•LNCCR+X1)••2+ETA••2)
XE := X1••2+ETA••2
AC := A•CLPR•C1/R-X1/XE)+LMR•C1/R-X1/XE)+CAL+TH)•ETA/XE)
BC := B•C2•X1•ETA•CLMR-LPR)/XE••2
+CAL+TH)•C1/R-X1/XE)
RR := R
FOR I := 1 STEP 1 UNTIL NT DO BEGIN TLCl) := +.5•TC;+TCNT)+4•CS1+TCNT-1)+2•S2)/3
END
S1 := S2 := 0
FOR I := 2 STEP 2 UNTIL (NT-3) DO
BEGIN S1 := S1+TTLCl)
S2 := S2+TTLCl+1)
END
ITTH := X•CTTHCl+TTHCNT)+4•CS1+TTHCNT-1)+2•S2)/3
LM := ARCTANCETA/CR-X1)/4
AS := A•CETA•CLMR-LPR)/XE+AL•C1/R-X1/XE)+TH•C-1/R-X1/XE)
BS := B•(C1-X1•ETA•CLMR-LPR)/XE+2+CAL+TH)•C1/R-X1/XE+3.141593/R••2)
RR := R
FOR I := 1 STEP 1 UNTIL NT DO BEGIN IF XI = RR THEN BEGIN TTHCl) := TC;+1.5708
GO TO L05 END IF RR (GR XI THEN TTHCl) := TC;+3.141593-ARCTANCETA/CR-X1)/4 ELSE TTHCl) := TC;+ARCTANCETA/X1-RD)/4
L05: RR := RR+X
END
S1 := S2 := 0
FOR I := 2 STEP 2 UNTIL (NT-3) DO BEGIN S1 := S1+ITTHCl)
S2 := S2+ITTHCl+1)
END
ITTH := X•CTTHCl+TTHCNT)+4•CS1+TTHCNT-1)+2•S2)/3
GS := AS-BS+ITTH)
IF GC GR 0 THEN BEGIN LH := ARCTANGS/GC)
IF LH LESS 0 THEN LH := 6.283186+LH END ELSE LH := 3.14159+ARCTANGS/GC)
PRINT FREEPOINT(63), XI•100, SAMELINE, ALIGNED(4, 6), LH/.01745
XI := XI•11X1)
IF XI LESS R THEN GOTO L1
PRINT CEL12??
GOTO L01
END
P(_\eta=0_)

P_\eta (\eta=0)

BEGIN
BEGIN ABC(1:150), TL, TH(1:150)
REAL S1, S2, A, B, C, D, E, DX, XI, X11, XI, L, MU, SIG, X, TL, PITL, CB, SB, RO, J, IT, TH, PG
INTEGER I, N, NT
SWITCH SI := LLO, LL1, LL2, LL3

LL0 := READ MU, SIG, R, DX, XI, X11, NT, RO, J
FOR I := 1 STEP 1 UNTIL NT DO READ TL(I)
S1 := S2 := 0
FOR I := 2 STEP 2 UNTIL NT - 3 DO
BEGIN S1 := S1 + TL(I)
S2 := S2 + TL(I + 1)
END
IT := DX * TL(1) + TL(*) + 4 * (S1 + TL(NT - 2)) + 2 * (S2)
E := CHECKRCR * 3 * (CT(NT - 1) / 2)

LL1 := N := ABS((XI + R) / DX + 1)
A := CHECKRCDX * (A * (XI + R))
C := CHECKRCDX * (A * (XI + R))

IF MU + SIG = 1.5708 THEN
BEGIN CB := 0 GOTO LL2
X10 := CHECKRC2 * DX * (A * XI + LN(DX) - 1)
X := R
END
FOR I := 1 STEP 1 UNTIL N - 1 DO
BEGIN TL(i) := TL(i) + LN(ABS((XI + R))]
X := X + R * DX
END
FOR I := 1 UNTIL N - 2 DO
BEGIN TL(I) := TL(I) + LN(ABS((XI + R))]
X := X + R * DX
END
S1 := S2 := 0
FOR I := 2 STEP 2 UNTIL N - 4 DO
BEGIN S1 := S1 + TL(I)
S2 := S2 + TL(I + 1)
END
PTL := CHECKRCX * (C * (XI + R))

IF XI LESS R THEN GOTO LL1
PRINT XL
END

IF XI LESS R THEN GOTO LL0

END
```plaintext
2D GRAVITY
BEGIN INTEGER ARRAY XSC(1:100)
   REAL ARRAY TC(1:30), AC(1:100), X, Y, P, DAC(1:50)
   REAL XXS, DX
   INTEGER 1, J, NS
   SWITCH SS = LL0, LL1, LL2, LL3, LL4, LL5
   PROCEDURE GRAV (XA, XX1, YY1, YY2, PP, AA)
   REAL XA, XX1, YY1, XX2, YY2, PP, AA
   BEGIN REAL L, R1, R2, FI1, FI2, S, C
   L := SQRT((YY2 - YY1)^2 + (XX2 - XX1)^2)
   S := (YY2 - YY1)/L
   C := (XX1 - XX2)/L
   R1 := (XA - XX1)^2 + YY1^2
   R2 := (XA - XX2)^2 + YY2^2
   Lf := 5 * LN(C/R1)
   FI1 := 1.5708 - ARCTAN((XX1 - XA)/YY1)
   FI2 := 1.5708 - ARCTAN((XX2 - XA)/YY2)
   AA := 0.133334 * PP * C * YY2 * FI2 - YY1 * FI1 - (-XX1 - XA) * S + YY1 * C
   * (S + L + C * (FI2 - FI1))
   END
   PRINT ECL1, X, AEL2??
   LL0: WAIT
       i := J = 1
       READ INSTRING(T, I)
       i := i + 1
       PRINT ECL5??, OUTSTRING(T, I), EEL2??
       READ XXS, DX, NS
       LL2: READ XS(J), YC(J), Xd(J+1), Yd(J+1), PC(J)
       IF Xd(J) NOTEQ 0 THEN BEGIN J := J + 2 GOTO LL2 END
       i := J = 1
       XSC(J) := XXS - AC(J)
       AC(J) := 0
       LL3: IF 1 = NS THEN BEGIN i := 1 GOTO LL5 END
       LL4: IF Xd(J) = 0 THEN BEGIN i := i + 1
       XSC(i) := XSC(i-1) + DX
       AC(i) := 0 J := J + 1 GOTO LL3 END
       IF Yd(J+1) = YC(J) THEN GRAV(XS(J), Xd(J), YC(J), Xd(J+1), Yd(J+1), PC(J), DAC(i))
       ELSE GRAV(XSC(J), Xd(J+1), Yd(J+1), Xd(J), Yd(J), PC(J), DAC(i))
       J := J + 2
       AC(i) := AC(i) + DAC(i)
       GOTO LL4
       LL5: PRINT FREEPOINTC6), XSC(J), ALIGNED(4, 5),
       SAMELINE, AC(J)
       i := i + 1
       IF i LESS NS+1 THEN GOTO LL5 ELSE GOTO LL0
   END
```
PSEUDO-GRAVITY UNITS: METRES

BEGIN ARRAY T, TH, TL, THH, ITL, ITTH, LPP, LMR,
A, B, C, AC, BC, CC, AS, BS, CS, SC, PC, PO, RO, J, S1, S2, X
INTEGER NT, N
SWITCH S:=L0, L1, L1, L2, L3, L4, L5

...
H(K+1,0) := F_1
H(K+1,1) := 0
H(K+1,2) := 0
PRINT $L_2??

C.XC(N):=C.XC N) + C.CC(N)

END

FOR

A(K+1,1) := -C(Y) - Y_2 J • C.X2 - X_2

X_2 := C.XC(N) - (X_0 + N • D_X)

Y_2 := C.X(N)

F(K+2,0) := Y_2
F(K+2,1) := 0
F(K+2,2) := 1

A(K+2,1) := -C(Y) - Y_2 J • C.X2 - X_2

X_2 := C.X(N) - (X_0 + N • D_X)

N := N + 2

D(K+2,0) := F_1
D(K+2,1) := C.X2 - X_2 • C.Y_2 - Y_2

E(K+1,0) := X_1 - X_2
E(K+1,1) := 0
E(K+1,2) := 0

D(K+1,0) := F_2
D(K+1,1) := -Y_2 / R_2
D(K+1,2) := X_2 / R_2

BECAUSE ARRAY BB, CCC1: NC, 1:1), AA(1:NC, 1:NC)

X_1 := C.XC(N) - 1 • C.XO + N • D_X

FOR N := 0 STEP

IF

C.CK+1,0) := F_2 - F_1
C.CK+1,1) := 0
C.CK+1,2) := 0

D(K+1,0) := F_2
D(K+1,1) := -Y_2 / R_2
D(K+1,2) := X_2 / R_2

END

E N D

T A C N+1) := CHECKRC

FOR

IF

CZC(N) G,R CZ(I) • THEN

T := O
N := 2 • C.IW - L+1

END

END

END

FOR

IF

CZC(N) G,R CZ(I) • THEN

X_2 := C.X(N) - (X_0 + N • D_X)

Y_1 := C.X(N)

FOR N := 0 STEP

END

END

END

END

END

END
for procedure SWANOPT(n,e,x,J,A,K,TN,F,T1)

FP: FP+((PT-FI)*A-FI)/(a*abs(PT-FI)*abs(A-FI));

else

PX[2*PNC-1] := PX[1];

EE1 := JJ*S*(S*PP-C*LL); EE2 := JJ*S*(S*LL+C*PP);

MAG. SW_NOPT;

for begin

procedure L90: FUNCTION(xtn,)

EXIT!

L73:

L10:

L5a:

IA8:

IA7: 

f[O] := f[2];

end.;

=cos(IE) *sin(u[1]:::ex);