Stress analysis of the Lithosphere

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STRESS ANALYSIS OF THE LITHOSPHERE

BY

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To my late father.

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ABSTRACT

Computer programs have been written to perform finite element calculations with 3-noded elements for the cases of plane strain, plane stress and axisymmetry. and to produce data for these programs. The latter is specifically orientated towards geophysical models and includes a facility for quasi-horizontal boundaries.

These have been used to investigate the stress fields associated with sedimentary basins and continental margins. From these calculations, the preferential formation of normal faults on, and to the continental side of margins, with the subsequent formation of deep sedimentary basins, is explained.

Quantitative studies on the diffusion of stresses through mathematical models of the lithosphere and asthenosphere have given time constants of 200,000 years for plates of the size of the Pacific. Simulated failure has been built into this model and leads to the prediction of a 10 yearly failure cycle with energy releases of 10^{27} ergs.

The finite element models have been extended to cope with dynamic cases and used to calculate the natural frequencies of free vibration of a sedimentary basin. For a Rayleigh type response the fundamental period has been shown to be directly proportional to the depth of the basin. The natural frequencies calculated are of the same order as those measured in surface waves generated by earthquakes and a Fraunhöfer absorption effect is suggested.

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CHAPTER 1.

Introduction to the Fundamentals of Elasticity.

Solutions of complex stress analysis problems can only be attempted with a thorough appreciation of the concepts of stress and strain and their interrelationship In particular this knowledge is essential for an understanding of the finite element method of stress analysis, of which much use will be made in later chapters. The formulation of the mathematical theory of elasticity has seen the evolution of many notations. It is the purpose of this chapter to give an introduction to the theory in the notation used throughout this thesis. namely the tensor notation (see Appendix 1 for the definition of a tensor).

The aim of the theory of elasticity is the generation of the vector field of relative displacement ($\underline{u}(x,y,z) = u_i \hat{\underline{i}}$)[1] of each point P (x,y.z) of a body, produced by the action of given forces which act

(a) on the boundaries of the body - called boundary forces,

and (b) throughout the body e.g. gravity - called body forces.

Thus the point P (x,y.z,) of the body is displaced to the point $(x+u_1, y+u_2, z+u_3)$ by the combined action of boundary and body forces. In general u_1, u_2, u_3 are functions of x,y,z.

1.1 An Analysis of Strain.

When changes take place in the relative positions of the parts of a body, that body is said to be strained.

1 - Summation convention used. See appendix 1.

Thus the strain at a point within a body is a quantity related to the relative displacement of the particle of matter at that point.

It is easily shown that the change, under a deformation $\underline{u}$ (x_1, x_2, x_3) of a small vector $\underline{A}$ joining two neighbouring points P, P′ of the body is

$$\delta A_i = \left(\frac{\partial u_i}{\partial x_j} \right)_P A_j \qquad (1.1)$$

provided the second order derivatives of $\underline{u}$ at P are small (assumption). Hence,

$$\delta A_i = u_{i,j} A_j \quad \left(u_{i,j} = \frac{\partial u_i}{\partial x_j} \right)$$

$$= (e_{ij} + \omega_{ij}) A_j \qquad (1.2)$$

where,

$$e_{ij} \equiv \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad (D1.1)$$

and $$\omega_{ij} \equiv \tfrac{1}{2}(u_{i,j} - u_{j,i}) \qquad (D1.2)$$

Note that

(i) e_{ij} is symmetric i.e. $e_{ij} = e_{ji}$ (1.3)

(ii) w_{ij} is antisymmetric i.e $w_{ij} = -w_{ji}$ (1.4)

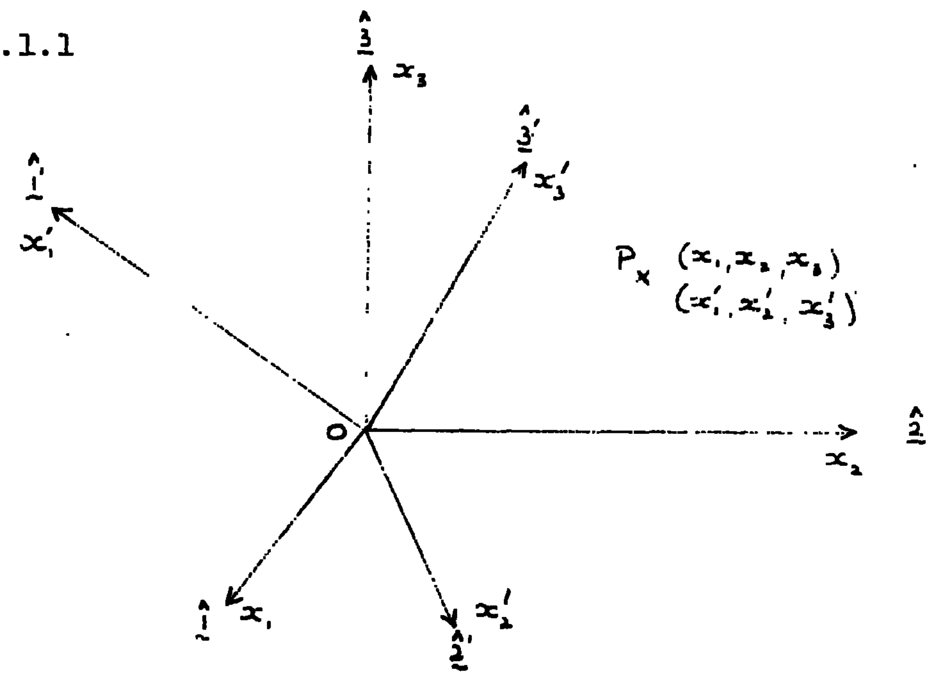
The antisymmetric component of the transformation, w_{ij} can be shown[2] to correspond to rigid body motions i.e. pure translation and rotation of the body.

2 - See appendix 2, part 3.

Obviously these motions do not produce displacement of one part of the body relative to another part. Thus they result in no strain and do not interest us.

The symmetric components of the transformation, e_{ij}, thus represent pure deformation of the body. The set of six different e_{ij} (not nine because $e_{ij} = e_{ji}$) is called the strain tensor[3] associated with the displacement field $\underline{u}$ and its components are functions of position, i.e. $e_{ij} = e_{ij}(x_1, x_2, x_3)$. These components also depend on the orientation of the axes of reference. The function $\underline{u}$ expressing the displacement of the particle at P in terms of x_1, x_2, x_3, the co-ordinates of P, is a different function $\underline{u}'(x_1', x_2', x_3')$ when expressing the same displacement, but this time in terms of the co-ordinates of P in the primed system.

Fig.1.1



3 - See appendix 2 part 1

The components of the strain tensor when referred to the two systems of co-ordinates are

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and

$$e'_{ij} = \frac{1}{2}\left(\frac{\partial u'_i}{\partial x'_j} + \frac{\partial u'_j}{\partial x'_i}\right)$$

from D1.1. and obviously in general $e_{ij} \neq e'_{ij}$. Thus the strain tensor depends on the point P and the orientation of the co-ordinate axes. The two tensors e_{ij} and e'_{ij} are related[3] and

$$e'_{ij} = l_{ki}\, l_{mj}\, e_{km} \qquad (1.5)$$

where

$$l_{ij} = \hat{\underline{i}} \cdot \hat{\underline{j}}' \qquad (D1.3)$$

The validity of (1.5) characterises the array as a tensor

For each point P there exists one set of co-ordinate axes (in general different for each P) such that the strain tensor when referred to these axes is diagonal i.e. $e_{ij} = 0\ (i \neq j)$. These axes are called the <u>principal axes of strain.</u>

The six components of strain at a point are not independent and satisfy compatibility equations 4

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \qquad (1.6)$$

3 - See appendix 2 part 1
4 - " " " " 2

So far the strain tensor has been developed purely along mathematical lines. Now we give physical significance to the elements of the tensor [5]

(a). Diagonal terms

The component e_{11} represents the extension per unit length of a small vector at P originally parallel to the x_1 axis.

(b) Off-diagonal terms

The value $2e_{ij}$ represents the decrease in the right angle between the small vectors A, B at P which were initially (i.e. before deformation u) directed along the positive x_i and x_j axes respectively.

Expressions for the components of the strain tensor in other orthogonal co-ordinate systems are given by

(a) Spherical co-ordinates

$$e_{rr} = \frac{\partial u_r}{\partial r}$$

$$e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r} \qquad (1.7)$$

$$2e_{\theta\phi} = \frac{1}{r}\left(\frac{\partial u_\phi}{\partial \theta} + u_\phi \cot\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}$$

$$2e_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

$$2e_{r\phi} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}$$

5 - See appendix 2 part 4.

(b) Cylindrical Co-ordinates.

$$
\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r} \\
e_{\phi\phi} &= \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} \\
e_{zz} &= \frac{\partial u_z}{\partial z} \\
2e_{\phi z} &= \frac{1}{r}\frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z} \\
2e_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \\
2e_{r\phi} &= \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \phi}
\end{aligned}
\qquad (1.3)
$$

1.2. An analysis of Stress

The stress at a point P within a body is associated with the forces acting at that point. Consider a small surface, ΔS, within the body with an outward normal $\hat{\nu}$. Then the material on one side

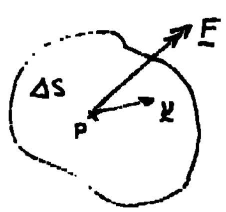


Fig 1.2

of ΔS is 'pushing' against the material on the other side of ΔS and the material on this side is 'pushing' back with an equal and opposite force. Let the force that the material on the side $\hat{\nu}$ of ΔS is exerting on the material on the other side of ΔS, be $\underline{F}$. If we now take limits as $\Delta S \to 0$ and write

$$\underset{\sim}{R}_\nu = \lim_{\Delta S \to 0}\left(\frac{F}{\Delta S}\right) \qquad (D1.4)$$

$\underline{R}_\nu$ is known as the stress vector at P for the plane $\hat{\nu}$. $\underline{R}_\nu$ can be split into components parallel to (known as the normal stress) and $\perp$ar to $\hat{\nu}$ (known as the shear stress on the plane $\underline{\nu}$ at P). Obviously the shear stress at P will obtain a maximum value for some orientation $\underline{\nu}$. (Note here that $\underline{F}$. and hence $\underline{R}_\nu$. is different for differently orientated surfaces at P). This maximum shear, and the plane on which it acts, is of great interest to geophysicists.

Note:

(1) $\underline{R}_\nu$ is independent of the orientation of the co-ordinate axes.

(11) $\underline{R}_\nu \not\parallel \underline{\nu}$ in general, but there exists three mutually perpendicular directions for which $\underline{R}_\nu \parallel \underline{\nu}$ and these are known as the principal axes. The stresses acting on these planes are known as the principal stresses.

The components of $\underline{R}_\nu$ along the x.y,z, axes (or the x_1, x_2, x_3 axes) are denoted by $\widehat{\nu x}$, $\widehat{\nu y}$, $\widehat{\nu z}$ (or $\tau_{\nu i}$, $i=1,2,3$) respectively.

i.e. $$\underset{\sim}{R}_\nu = \widehat{\nu x}\,\hat{\underline{i}} + \widehat{\nu y}\,\hat{j} + \widehat{\nu z}\,\hat{\underline{k}} \qquad \text{(D1.5)}$$

These components depend on the orientation of the co-ordinate axes.

For the complete specification of the stress at any point of a body we should require to know the force per unit area across every plane drawn through the point (i.e. for all $\underline{\nu}$). The object of an analysis of stress is to determine the nature of the quantities by which the stress at a point can be specified. It can be shown[6] that to fully define the stress at a point the nine quantities

$\widehat{xx}(\tau_{11})$, $\widehat{yy}(\tau_{22})$, $\widehat{zz}(\tau_{33})$, $\widehat{xy}(\tau_{12})$, $\widehat{xz}(\tau_{13})$, $\widehat{yx}(\tau_{21})$, $\widehat{yz}(\tau_{23})$, $\widehat{zx}(\tau_{31})$ and $\widehat{zy}(\tau_{32})$ (reduced to six because $\tau_{ij} = \tau_{ji}$[7]) are

6 – See appendix 2 part 5
7 – " " " " 6

sufficient. These stress components in fact constitute a tensor and it can be shown[8] from equilibrium considerations that if ν has direction cosines ν_i, $i=1,2,3$ in our co-ordinate system $Ox_1x_2x_3$ then

$$\tau_{\nu i} = \tau_{ji}\,\nu_j \qquad (1.9)$$

i.e. for any given surface ΔS (i.e. given $\underline{\nu}$) $\underline{R}_\nu = \tau_{\nu i}\,\hat{e}_i$ can be obtained from the stress tensor τ_{ij} $i=1,3$ $j=1,3$.

When referred to new primed axes the new stress tensor is obtained from the old by[9]

$$\tau'_{ij} = l_{ki}\,l_{mj}\,\tau_{km} \qquad (1.10)$$

where l_{ij} is defined by D1.3. As with the strain tensor for each point P a set of co-ordinate axes exists called the <u>principal axes of stress</u> for which the stress tensor when referred to these axes is diagonal i.e. $\tau_{ij} = 0$ $(i \neq j)$.

The greatest shearing stress at a point is equal to one-half the difference between the greatest and least principal stresses and acts on a plane that contains the direction of the intermediate principal stress and bisects the angle between the directions of the largest and smallest principal stresses.[10]

Equilibrium considerations between the surface stresses and body forces acting on a small volume of material within a body lead[11] to the equation of equilibrium for static deformation

$$\rho F_i + \frac{\partial \tau_{ij}}{\partial x_j} = 0 \qquad (1.11)$$

where $\underline{F}$ is the body force per unit mass and ρ the material density.

8 - See appendix 2 part 5
9 - " " " 7
10 - " " " 9
11 - " " " 6

1.3. Interrelation between stress and strain.

In order to further our theory we must make a second assumption (or axiom). that there exists a linear relationship between stress and strain (known as Hooke's law). A material which obeys Hooke's law is known as elastic. Thus there exists a set of 36 constants (C_{ijkm}) [Note· It can be shown from energy considerations that the coefficients of C are symmetric and therefore only 21 constants are needed] such that

$$\tau_{ij} = C_{ijkm} e_{km} \tag{1.12a}$$

or in matrix format

$$\begin{Bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} C \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{xy} \\ e_{xz} \\ e_{yz} \end{Bmatrix} \tag{1.12b}$$

If we assume that the elastic body we are considering is isotropic (i.e. it has the same elastic properties in all directions) then we can reduce the number of constants to two, λ and μ called lamé's constants.[12] The stress-strain relation for an isotropic material then reduces to

$$\tau_{ij} = \lambda \delta_{ij} \Delta + 2\mu e_{ij} \tag{1.13}$$

12 - See appendix 2 part 10.

where

$$\Delta \equiv e_{ii} = e_{11} + e_{22} + e_{33} . \qquad \text{(D1.6)}$$

Putting $i=j$ in (1.13) and summing

$$\Theta = (3\lambda + 2\mu)\Delta \qquad (1.14)$$

where $$\Theta = \tau_{11} + \tau_{22} + \tau_{33} . \qquad \text{(D1.7)}$$

Substitution now gives the inverse relation

$$e_{ij} = \frac{-\lambda \delta_{ij} \Theta}{2\mu(3\lambda+2\mu)} + \frac{\tau_{ij}}{2\mu} . \qquad (1.15)$$

When working with practical problems it is usual to use two different but related constants called Young's Modulus (E) and Poisson's ratio (η). These are defined as

$$E \equiv \frac{\mu(3\lambda + 2\mu)}{(\lambda+\mu)} \qquad (1.16)$$

and

$$\eta \equiv \frac{\lambda}{2(\lambda+\mu)} \qquad (1.17)$$

the inverse relations being

$$\lambda = \frac{E\eta}{(1+\eta)(1-2\eta)} \qquad (1.18)$$

and

$$\mu = \frac{E}{2(1+\eta)} \qquad (1.19)$$

The stress-strain relations 1.13 and 1.14 then assume the form

$$\tau_{ij} = \frac{E\eta\Delta}{(1+\eta)(1-2\eta)}\delta_{ij} + \frac{E}{(1+\eta)} e_{ij} \qquad (1.20)$$

and

$$e_{ij} = \frac{(1+\eta)}{E}\tau_{ij} - \frac{\eta}{E}\delta_{ij}\Theta \qquad (1.21)$$

From equation 1.13 when $i \neq j$ we have $\tau_{ij} = 2\mu e_{ij}$ and therefore, when $e_{ij} = 0\ (i \neq j)$ we have $\tau_{ij} = 0$. Thus for an isotropic material the principal axes of stress and strain coincide.

The potential strain energy per unit volume (W) in a strained body is

$$W = \tfrac{1}{2}\tau_{ij}e_{ij} \qquad (1.22)$$

(Sokolnikoff 1946, § 26)

An important principle in elasticity especially relevant to Finite Element theory is that of Saint Venant's which can be stated as follows:

If some distribution of forces acting on a portion of the surface of a body is replaced by a different distribution of forces acting on the same portion of the body then the effects of the two distributions on the parts of the body sufficiently far removed from the region of application of the forces are essentially the same, provided that the two distributions of forces are statically equivalent.

CHAPTER 2.

FINITE ELEMENT THEORY

2.1 Qualitative Analysis.

Many problems of stress analysis arising in nature are completely intractable analytically because of the complicated nature of the boundary conditions involved. Finite element methods provide a numerical means of tackling such problems irrespective of the complexity of the boundary conditions. (Zienkiewicz and Cheung 1967).

Initial discussion will be confined to the theory of the method applied to the solution of two-dimensional problems and later it will be widened to include the natural extension to three-dimensions.

The initial procedure involves a division of the model requiring analysis into a series of triangles (called elements), the corners of which are referred to as nodal points (or nodes). See Fig 2.1. Each element and node is given a reference number and values for the density, Young's modulus and Poisson's ratio are assigned to each element.

The basic unknowns in terms of which the problem is formulated are the displacements of the nodal points induced by the action of given externally applied forces. For two-dimensional problems there are two basic unknowns associated with each nodal point i e. 2N unknowns altogether, where N is the total number of nodal points.

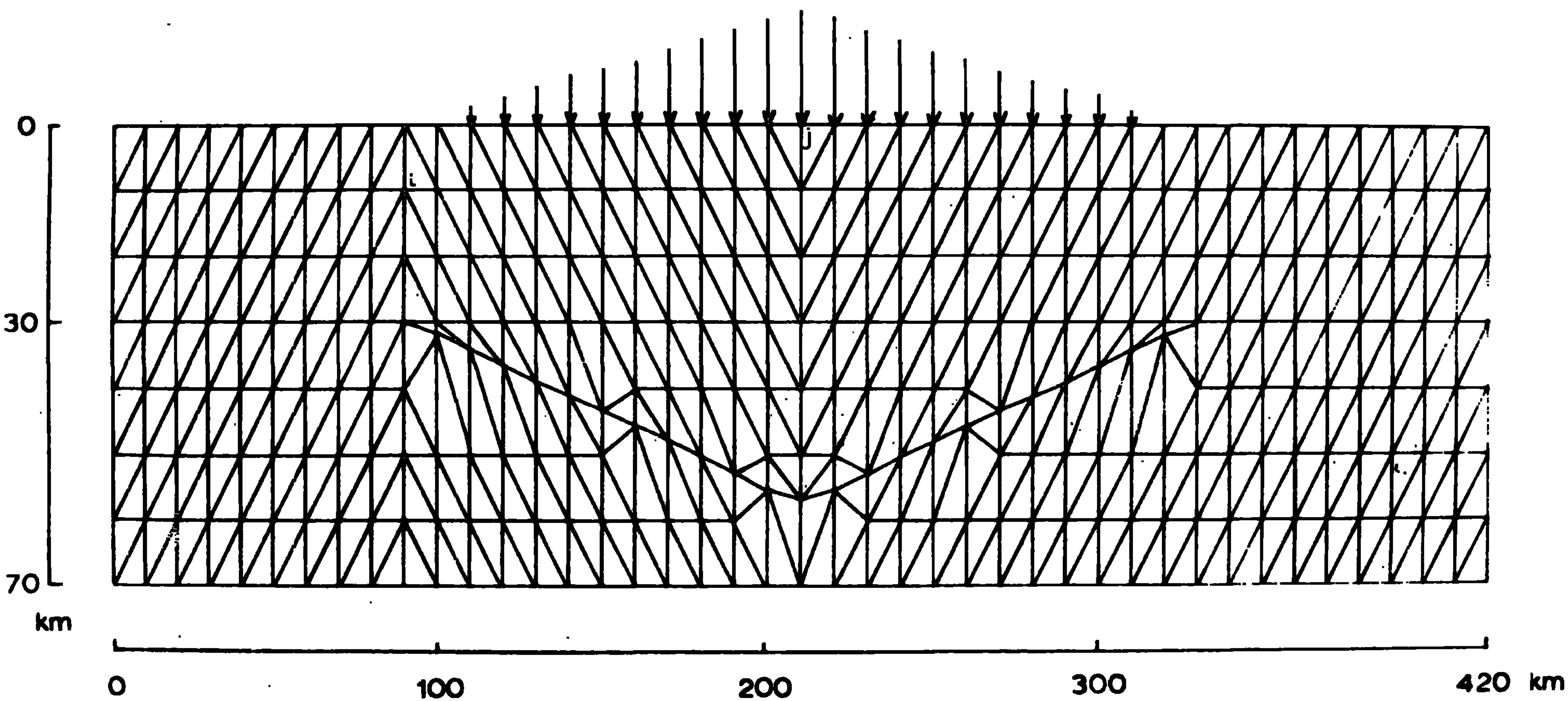


FIG. 2.1

Considering each element in turn it is possible to define a unique linear displacement throughout in terms of the six basic unknown components of nodal displacement (each element having three nodes). Displacement defines strain thus allowing the immediate determination of the strain throughout each element in terms of its six basic unknown components of nodal displacement. Hooke's law gives the stress throughout each element in terms of our basic unknowns.

The boundary stresses on each element are replaced by statically equivalent forces concentrated at its nodes. let $\underset{\sim}{F}_k^e$ denote the concentrated force introduced at the nodal point K resulting from the boundary stresses acting on element e. Note that these forces are in terms of the unknown nodal displacements.

To solve for the 2N unknowns we need to formulate 2N simultaneous equations. let us consider the conditions at:

(a) An internal node such as i in Fig.2 1

Each of the elements (six in this case) to which this node belongs contributes concentrated forces The sum of these forces must be zero. since the boundary forces to which they are equivalent are equal and opposite along adjacent faces converging at i (Newton's third law). i.e.

$$\sum_e \underset{\sim}{F}_i^e = 0 \qquad (2.1)$$

This condition gives two equations (2.1 is a vector equation) involving the basic unknowns for each internal node i.

(b) A boundary node such as j in Fig 2.1 where there is an externally applied force $\underline{P}$. Directly from the definition of stress the boundary stresses acting on the model and the externally applied forces are equivalent. Thus the sum of the concentrated forces at the nodal point j (contributed to by the boundary stresses from two elements in this case) must equal $\underline{P}$. the equivalent applied load.

i.e.

$$\sum_{e} \underline{F}_j^e = \underline{P} \qquad (2.2)$$

This condition gives two equations involving the 2N unknown displacements for every boundary node j. This completes a set of 2N simultaneous equations for the unknown nodal displacements. Solving for these unknowns completes the analysis, the displacements strains and stresses having already been defined throughout each element (and thus throughout the entire model) in terms of these basic unknowns. The following points about the method are worthy of note:

A. Compatibility of Strains

The compatibility equations governing a set of strain components (1.6) ensure that the strains represent a continuous displacement. Continuity of displacement exists throughout all elements because it is represented by a continuous function. However, for the displacement to be continuous throughout the body (i.e. for the compatibility equations to be satisfied) the displacement must be continuous across the element boundaries. Consider the three-noded triangular element discussed so far. The displacement along the face joining two nodes, i and j, varies linearly subject to the constraint at either node that it must equal the nodal displacements. Thus the nodal displacements of any two interconnected nodes uniquely determine the displacement along the interface between these nodes (i.e. the displacement along the interface is independent of the element under consideration). Therefore displacement is continuous across triangular element interfaces Such elements are said to conform.

Other conforming elements exist about which more will be said later.

B. Specified Displacements.

Sometimes the boundary conditions of a problem involve the specification of surface displacements rather than stresses. For nodal points where the displacement is given the number of basic unknown nodal displacements is correspondingly reduced and no equilibrium condition at these nodes need be formulated.

C. Rigid Body Displacements.

The set of 2N simultaneous equations for the unknown nodal displacements has a characteristic determinant of zero (i.e. the equations are either inconsistent or have an infinite number of possible solutions). This is an expression of the fact that the possibility of rigid body displacements, which do not alter the stress state, has not been eliminated in the formulation of the solution. Fixing one nodal point A say. (i.e. specifying zero displacement) cuts out rigid body translations and specifying zero displacement for any other node B in a direction perpendicular to AB stops rigid rotation. The set of (2N-3) remaining equations has a unique solution for the remaining (2N-3) unknown displacements.

D. Convergence of Solution.

For the type of element discussed so far 'exact' solutions are obtained only for uniform stress fields. This is a direct consequence of the specification of linear displacement within any element. For other stress fields the results converge to the exact solution as the size of the element is decreased.

For the type of problem tackled in this thesis it was found that elements with sides of the order of 5 Km. struck an ideal balance between convergence of solution and storage needed for computation.

E. Matrix Formulation.

When the finite element method is approached quantitatively a matrix formulation is ideal. The simultaneous equations reduce to a matrix equation of the form

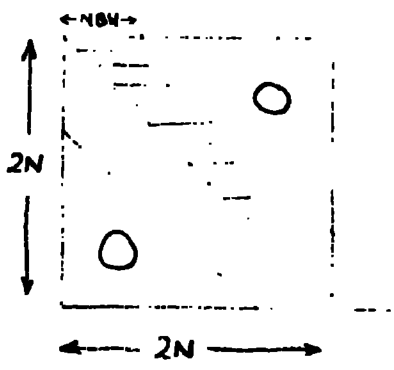
$$\underset{2N\times 2N}{[K]}\ \underset{2N\times 1}{\{\delta\}} = \underset{2N\times 1}{\{F\}} \qquad (2.3)$$

(N = Total No. of nodal points)

where $[K]$ is called the stiffness matrix, $\{\delta\}$

contains the unknown nodal displacements and $\{F\}$ the externally applied loads $[K]$ turns out to be a banded, positive definite. symmetric matrix. (Fig. 2.2).

Fig. 2.2.



The semi-band width (shaded) only is stored. With computer storage space at a premium it is obviously desirable to formulate our data in such a way as to minimize the semi-band width. With every nodal point there is associated a reference number, called the nodal number. ranging from 1 to N. The nodal points can be numbered in N! different ways. On matrix formulation it is found that the semi-band width of $[K]$ increases linearly with the greatest difference, for the whole model. between nodal point numbers belonging to the same element. It is important when numbering the nodes therefore, that we do so in a manner that keeps this difference minimal. A program for doing this can be found in appendix 5.

The Choleski method (Jenkins, 1969) is used to solve 2.3.

F. <u>Other Types of Element.</u>

Many refinements to the shape of the element are possible. Square elements (4 nodes) and triangular elements with extra nodes at the centre of each side (6 nodes altogether) are all feasible each increasing the freedom with which the displacement throughout the element can be chosen (subject to compatibility conditions). The last element allows the fitting of a full quadratic.

For three dimensional analysis the element must have volume, a tetrahedral element (4 nodes) being equivalent to the three noded triangular element allowing a linear displacement to be fitted.

2.2 Quantitative Analysis (Matrix Formulation)

let the 2N basic unknowns be represented as a column matrix

$$\{\delta\} = \begin{Bmatrix} \underline{\delta}_1 \\ \vdots \\ \vdots \\ \underline{\delta}_N \end{Bmatrix} \qquad \text{(D2.1)}$$

where $\underline{\delta}_i$ is a 2x1 matrix containing the x. y, components of displacement of the nodal point i. The nodal displacement matrix for the element c (nodes i,j.m) is defined by

$$\{\delta\}^e = \begin{Bmatrix} \underline{\delta}_i \\ \underline{\delta}_j \\ \underline{\delta}_m \end{Bmatrix} \qquad \text{(D2.2)}$$

We define the displacement $\{f\}^e \equiv \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$, throughout the element in terms of its nodal displacements by

$$\{f\}^e = [N]^e \{\delta\}^e \qquad (2.4)$$

where $[N]^e$ is a function of position within the element e subject to the constraint that $\{f\}^e$ when evaluated at the nodes is $\{\delta_i\}, \{\delta_j\}, \{\delta_m\}$ respectively.

Defining the strain matrix as

$$\{\varepsilon\} \equiv \begin{Bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{Bmatrix} \qquad \text{(D2.3)}$$

We obtain the strains throughout e from D1.1 as

$$\{\varepsilon\} = [B]^e \{\delta\}^e \qquad (2.5)$$

Hooke's law (1.13) gives the relation between Stress and Strain

$$\{\sigma\}^e = [D]^e \{\varepsilon\}^e \qquad (2.6)$$

where

$$\{\sigma\}^e \equiv \begin{Bmatrix} \widehat{xx} \\ \widehat{yy} \\ \widehat{xy} \end{Bmatrix} \qquad \text{(D2.4)}$$

We now need an expression for the concentrated nodal forces

$$\{F\}^e \equiv \begin{Bmatrix} F_i^e \\ F_j^e \\ F_m^e \end{Bmatrix}$$

equivalent statically to the stresses acting on the boundaries of e. Consider a virtual displacement $\{f^*\}^e$ imposed on the element e. The work done by the boundary stresses is

$$W = \int_S (\underset{\sim}{R}_\nu)_i \, f_i^* \, dS \qquad \text{(summed over i)}$$

where S is the surface of the element e. $\underset{\sim}{R}_\nu$ the stress resultant and ds an infinitesimal part of the surface with outward unit normal $\underset{\sim}{\nu}$. Using (1.9) we have

$$W = \int_S \tau_{ij} \nu_j f_i^* \, ds$$

$$= \int_V \frac{\partial}{\partial x_j} (\tau_{ij} f_i^*) \, dv$$

by the divergence theorem. where V is the volume of the element e.

$$= \int_V \frac{\partial \tau_{ij}}{\partial x_j} f_i^* \, dv + \int_V \tau_{ij} \frac{\partial f_i^*}{\partial x_j} \, dv$$

Using the equilibrium equation (1.11)

$$\frac{\partial \tau_{ij}}{\partial x_j} + p_i = 0$$

where p_i are the body force components per unit volume and writing

$$\frac{\partial f_i^*}{\partial x_j} = e_{ij}^* + \omega_{ij}^*$$

where by definition (c.f. D1.1 and D1.2)

$$e_{ij}^* \equiv \frac{1}{2}\left(\frac{\partial f_i^*}{\partial x_j} + \frac{\partial f_j^*}{\partial x_i} \right) = e_{ji}^*$$

and

$$\omega_{ij}^* \equiv \frac{1}{2}\left(\frac{\partial f_i^*}{\partial x_j} - \frac{\partial f_j^*}{\partial x_i} \right) = -\omega_{ji}^*$$

we have

$$W = -\int_V p_i f_i^* \, dv + \int_V \tau_{ij} e_{ij}^* \, dv .$$

$$\left[\text{Note: } \tau_{ij}\omega_{ij}^* = \tau_{ji}\omega_{ji}^* \quad \text{(dummy suffices)}\right.$$
$$= -\tau_{ij}\omega_{ij}^* \quad (\because \tau_{ij} = \tau_{ji} \text{ and } \omega_{ij}^* = -\omega_{ji}^*)$$
$$\left.= 0 \quad (\because x = -x \Rightarrow x = 0)\right] .$$

Therefore

$$W = \int_V \{\varepsilon^*\}^{e^T}\{\sigma\}^e dv - \int_V \{f^*\}_e^T\{p\} \, dv .$$

That is, the work done per unit volume by the surface stresses is

$$\{\varepsilon^*\}^{e^T}\{\sigma\}^e - \{f^*\}_e^T \{p\} \tag{2.7}$$

$$= \{\delta^*\}^{e^T}\left([B]^{e^T}\{\sigma\}^e - [N]^{e^T}\{p\}\right) \tag{2.8}$$

using (2.4) and (2.5). The work done by the equivalent nodal forces must be the same therefore

$$\{\delta^*\}^{e^T}\{F\}^e = \{\delta^*\}^{e^T}\int_V \left([B]^{e^T}\{\sigma\}^e - [N]^{e^T}\{p\}\right) dv .$$

This expression is true for all virtual displacements therefore

$$\{F\}^e = \int_V \left([B]^{e^T}\{\sigma\}^e - [N]^{e^T}\{p\}\right) dv$$

and from 2.6

$$\{F\}^e = \left(\int_V [B]^{e^T}[D]^e[B]^e dv\right)\{\delta\}^e - \int_V [N]^{e^T}\{p\} \, dv$$
$$= [K]^e\{\delta\}^e + \{F\}_p^e \tag{2.9}$$

where

$$[K]^e = \int_e [B]^{eT} [D]^e [B]^e \, dV \qquad (2.10)$$

$$\equiv \begin{bmatrix} K_{ii}^e & K_{ij}^e & K_{im}^e \\ K_{ji}^e & K_{jj}^e & K_{jm}^e \\ K_{mi}^e & K_{mj}^e & K_{mm}^e \end{bmatrix} \qquad (D2.5)$$

and

$$\{F\}_p^e = -\int_e [N]^{eT} \{p\} \, dV \qquad (2.11)$$

where i, j m are the nodal points comprising the element e.

$$\text{let} \quad \{R\} = \begin{Bmatrix} R_1 \\ \vdots \\ R_N \end{Bmatrix}$$

where R_i is a 2 x 1 matrix consisting of the x and y components of the external force applied at the nodal point i. Applying the equilibrium condition at the a^{th} node

$$\{R_a\} = \sum_{e=1}^{\text{No. of elements}} \{F_a\}^e .$$

we have from 2.9

$$\{R_a\} = \sum_e \sum_{b=1}^{N} [K_{ab}^e]\{\delta_b\} + \sum_e \{F_a\}_p^e$$

where $[K_{ab}^e]$ is defined by D2.5 when both nodal points a and b belong to e and are zero otherwise. i.e.

$$\{R\} = [K]\{\delta\} + \{F\}_p \qquad (2.12)$$

where

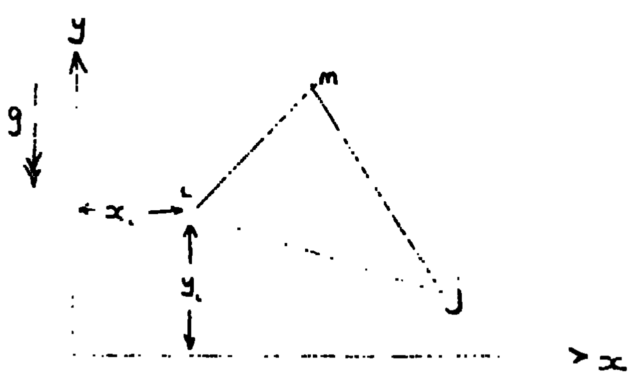
$$[K_{ab}] \equiv \sum_e [K^e_{ab}] \qquad (D2.8)$$

and

$$\{F_a\}_p \equiv \sum_e \{F_a\}^e_p \qquad (D2.9)$$

The matrix $[K]$ is known as the stiffness matrix of the model and must be inverted for the solution $\{\delta\}$ of the fundamental equation 2.12. We now look at the above formulation for the particular case of the 3-noded triangular element.

<u>The Matrix N</u>



(Fig 2.3)

$$\{\delta_i\} = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

where u_i - x displacement of node i.

and v_i = y displacement of node i.

From D2.2.

$$\{\delta\}^e = \begin{Bmatrix} \delta_i \\ \delta_j \\ \delta_m \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

In defining $[N]^e$ by 2.4 we can use 6 undetermined constants since we have two constraints on the displacement function for each node. Therefore let

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ v &= \alpha_4 + \alpha_5 x + \alpha_6 y \end{aligned} \qquad (2.13)$$

define the displacement inside e. Applying the constraints we have

$$\begin{aligned} u_i &= \alpha_1 + \alpha_2 x_i + \alpha_3 y_i \\ u_j &= \alpha_1 + \alpha_2 x_j + \alpha_3 y_j \\ u_m &= \alpha_1 + \alpha_2 x_m + \alpha_3 y_m \end{aligned} \qquad (2.14a)$$

giving $\alpha_1, \alpha_2, \alpha_3$ and

$$\begin{aligned} v_i &= \alpha_4 + \alpha_5 x_i + \alpha_6 y_i \\ v_j &= \alpha_4 + \alpha_5 x_j + \alpha_6 y_j \\ v_m &= \alpha_4 + \alpha_5 x_m + \alpha_6 y_m \end{aligned} \qquad (2.14b)$$

giving $\alpha_4, \alpha_5, \alpha_6$.

Solving 2.14 and substituting into (2.13) we obtain

$$\begin{aligned} u = \frac{1}{\Delta}\Big\{(a_i + b_i x + c_i y)u_i + (a_j + b_j x + c_j y)u_j \\ + (a_m + b_m x + c_m y)u_m\Big\} \end{aligned} \qquad (2.15)$$

$$\begin{aligned} v = \frac{1}{\Delta}\Big\{(a_i + b_i x + c_i y)v_i + (a_j + b_j x + c_j y)v_j \\ + (a_m + b_m x + c_m y)v_m\Big\} \end{aligned}$$

where

$$\Delta \equiv \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} \qquad \text{(D2.10)}$$

(note: $|\Delta|$ = 2 × area of element e)

and

$$\begin{aligned} a_i &= x_j y_m - x_m y_j \\ b_i &= y_j - y_m = y_{jm} \\ c_i &= x_m - x_j = x_{mj} \end{aligned} \qquad \text{(D2.11)}$$

a_j, b_j .... etc. are obtained by cyclic interchange of the suffices i.j.m. Rewriting 2.15 in matrix notation c f.2.4.

$$\{f\}^e = \begin{Bmatrix} u \\ v \end{Bmatrix} = [N]^e \{\delta\}^e$$

where $[N]^e = [IN_i', \, IN_j', \, IN_m']$ (2.16)

$$N_i' = \frac{1}{\Delta}(a_i + b_i x + c_i y)$$

and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The Strain Matrix B.

From D2.3 and D1.1

$$\{\varepsilon\} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{Bmatrix}$$

and then using 2.15

$$= \frac{1}{\Delta} \begin{Bmatrix} b_i u_i + b_j u_j + b_m u_m \\ c_i v_i + c_j v_j + c_m v_m \\ c_i u_i + c_j u_j + c_m u_m + b_i v_i + b_j v_j + b_m v_m \end{Bmatrix}$$

i.e.

$$\{\varepsilon\} = \frac{1}{\Delta} \begin{bmatrix} b_i & o & b_j & o & b_m & o \\ o & c_i & o & c_j & o & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \qquad (2.17)$$

and comparing with 2;5

$$[B]^e = \frac{1}{\Delta} \begin{bmatrix} b_i & o & b_j & o & b_m & o \\ o & c_i & o & c_j & o & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix} \qquad (2.18)$$

[D] Matrix for Isotropic Plane Strain.

For plane strain we have by definition $u_3 = o$ and $\frac{\partial}{\partial x_3} = o$. It follows from D1.1 that $e_{zz} = e_{z\alpha} = o$ i.e. the only non-zero components of the strain tensor are e_{xx}, e_{yy}, e_{xy}. Therefore, the only non-zero stress components are (from 1.13) $\widehat{xx}$ $\widehat{yy}$, $\widehat{zz}$, $\widehat{xy}$, and using 1.20.

$$\widehat{xx} = \frac{(1-\eta)E}{(1+\eta)(1-2\eta)} e_{xx} + \frac{\eta E}{(1+\eta)(1-2\eta)} e_{yy}$$

$$\widehat{yy} = \frac{\eta E}{(1+\eta)(1-2\eta)} e_{xx} + \frac{(1-\eta)E}{(1+\eta)(1-2\eta)} e_{yy} \qquad (2.19)$$

$$\widehat{xy} = \frac{E}{(1+\eta)} e_{xy}$$

and $\widehat{zz} = \eta(\widehat{xx} + \widehat{yy})$ (2.20)

or rewriting in matrix form

$$\begin{Bmatrix} \widehat{xx} \\ \widehat{yy} \\ \widehat{xy} \end{Bmatrix} = \frac{E(1-\eta)}{(1+\eta)(1-2\eta)} \begin{bmatrix} 1 & \eta/1-\eta & o \\ \eta/1-\eta & 1 & o \\ o & o & \frac{1-2\eta}{2(1-\eta)} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{Bmatrix}$$

and comparing with 2.6 we have

$$[D]^e = \frac{E(1-\eta)}{(1+\eta)(1-2\eta)} \begin{bmatrix} 1 & \eta/1-\eta & 0 \\ \eta/1-\eta & 1 & 0 \\ 0 & 0 & \frac{1-2\eta}{2(1-\eta)} \end{bmatrix} \quad (2.21)$$

The Element Stiffness Matrix. $[K]^e$

From 2.10

$$[K]^e = \int_e [B]^{e^T} [D]^e [B]^e \, t \, dx dy$$

where t is the thickness of element e Both $[B]^e$ and $[D]^e$ are independant of x, y (2.16 and 2.18) and therefore

$$[K]^e = t \, [B]^{e^T} [D]^e [B]^e \int_e dx dy$$

$$= \tfrac{1}{2} |\Delta| \, [B]^{e^T} [D]^e [B]^e \quad (2.22)$$

for unit thickness. Δ is defined in D2.10.

Body Force Matrix

Consider the particular case of gravitational forces. The body force p_i per unit volume is ρg in the -ve y direction (Fig.2.3). Therefore $p_1 = 0$ and $p_2 = -\rho g$ where ρ is the density of the element e and g the acceleration due to gravity. Using 2.11 and 2.16 we obtain

$$\{F\}_p^e = \frac{-\rho g}{\Delta} \int_V \begin{Bmatrix} 0, \\ N_i' \\ 0, \\ N_j' \\ 0, \\ N_m' \end{Bmatrix} dV$$

Now

$$\int_e (a_i + b_i x + c_i y) \, dx dy$$

$$= \tfrac{1}{2} |\Delta| \left(a_i + b_i \bar{x} + c_i \bar{y} \right)$$

where $(\bar{x},\bar{y})$ is the centroid of e. Using D2.11.

$$= -\frac{\Delta|\Delta|}{6} .$$

Therefore, for unit element thickness

$$\{F\}_p^e = \frac{|\Delta| \rho g}{6} \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{Bmatrix} . \qquad (2.23)$$

In Appendix 3 examples of the matrix formulation of finite element theory are worked out for axisymmetric models and for 6-noded triangular elements.

2.3 Application of Finite-element methods to the earth sciences.

Although finite element analysis has been used by engineers for many years, this powerful method has been generally ignored by earth scientists despite its obvious advantages over analytical methods with unreal boundaries and simplifying assumptions of rock homogeneity, isotropy and elasticity. Inhomogeneity and anisotropy can be introduced into the finite element method without any difficulty whatsoever and non-linear inelastic behaviour, such as plasticity, is dealt with by iterative use of the standard elastic method (Zienkiewicz and Cheung 1967).

The method is not confined to the solution of stress analysis problems and can be applied to any boundary value problem that is expressible in a variational form (Zienkiewicz and Cheung 1967). This includes the following which are of interest to the geophysicist:

(a) the behaviour of rocks under static and dynamic loads

(b) the propagation and dispersion of stress waves.

(c) heat conduction.

(d) fluid flow in porous materials.

(e) the distribution of magnetic and gravitational potential.

Voight and Sammuelson (1969) have investigated the effects of pronounced heterogeneity on the stress trajectories, the location of initial fracture surfaces and the shear stress distributions produced by the static loading of an upper crustal model.

Stephansson and Berner (1971) have adapted the method to deal with linearly viscous materials and used it to investigate the isostatic readjustment of a crustal model across the north mid-Atlantic ridge.

Beaumont and Lambert (1972) have shown that tilt near a surface load is sensitive to crustal structure and concludes that tilt measurements can be used to complement gravity and magnetic data for upper crustal modelling.

~~Attempts at~~ Fault propagation studies based on the state of stress immediately before failure (Hafner 1951) have been attempted. It is difficult to assess the relevance of these results because of the drastic effect that failure has on the initial stresses. By simulating fracture by small zero strength elements it is possible to use the finite element method in an iterative procedure so that at each stage the effect of the propagation of the fault is allowed for by the introduction of more zero strength elements. Douglas and Service (in press 1972) have been using finite elements in this way in an attempt to obtain a mechanism for rift valley formation.

In the absence of direct laboratory measurements under conditions closely approximating those encountered at depth in the earth it is impossible to predict the long term rheological behaviour of materials. Finite element methods should make it possible to model the earth at depth in an attempt to fit detailed information of surface response to relatively well known loading. Rheological models obtained in this way will be the main contribution of finite element methods to the earth sciences.

CHAPTER 3.

'Stress distributions associated with deep sedimentary basins'

Evidence for the subsidence of upper crustal material and the subsequent formation of sedimentary basins is provided by gravity, magnetic and seismic refraction surveys. Deep sedimentary basins have been proved on the continental shelves of aseismic margins (Bott and Watts, 1970) some of which extend into continental areas. (Blundell et al, 1968). Many of these basins are terminated by normal faults indicating the one time existence of regional tensions. When analysing the stresses in the region of a sedimentary basin, the interaction of the gravitational forces with local tensions, and the redistribution of these tensions by the contrast in elastic parameters between the basin and the surrounding basement rocks, are important.

Ignoring the tensions for the meantime, the small density contrast between the sediments and basement rocks means that the principal directions are more or less vertical and horizontal. The sediments are less dense than the surrounding crustal rocks and the 'vertical' principal pressure within these increases at a lesser rate with depth than in the adjacent crust. This gives conditions under which the rocks adjacent to the sediments are more susceptible to normal faulting than rocks at the same depth immediately below the centre of the basin. Because of the 'vertical-horizontal' orientation of the principal axes the imposition of a

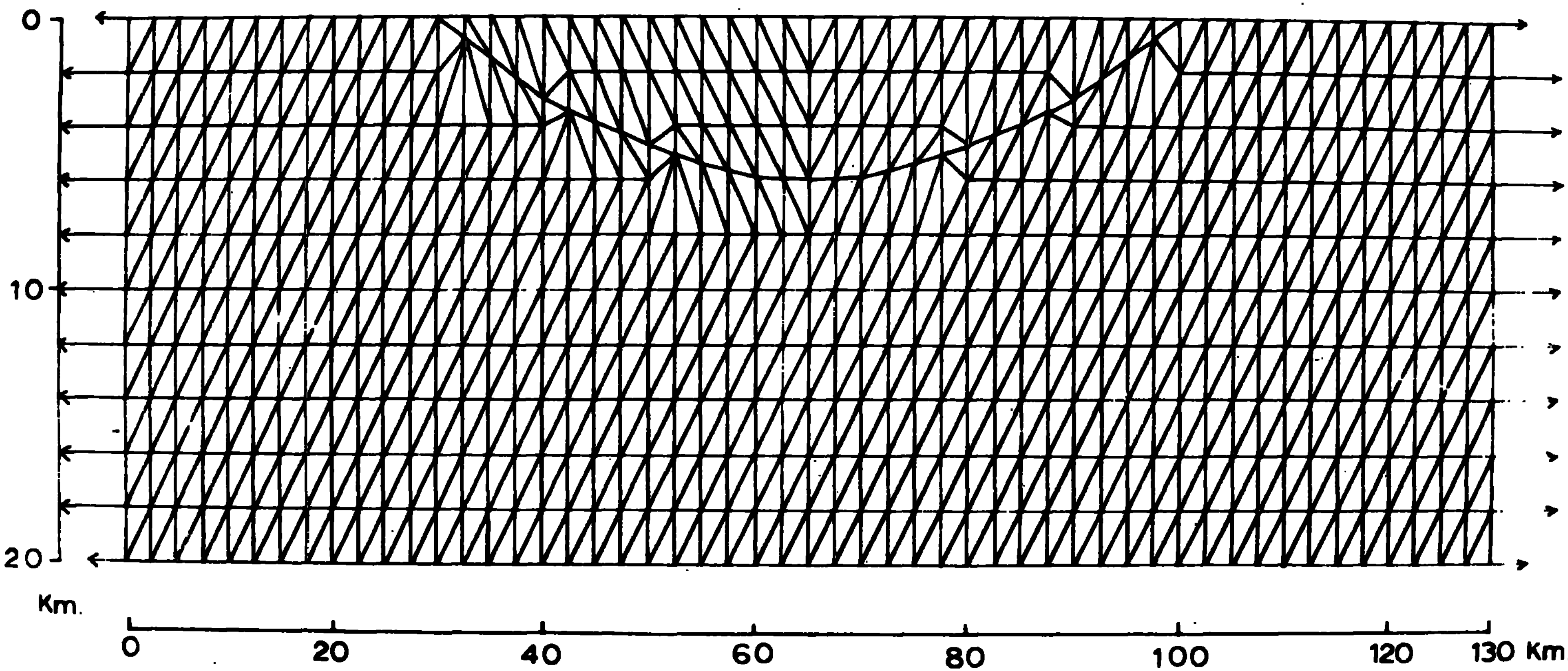


FIG. 3.1

uniform horizontal tension on this stress field will not alter this preferred area of faulting - it will just increase the likelihood of faulting. However, any uniformly applied tension will be non-uniform in the region of the basin because of its contrasting elastic parameters. This will result in a possible change in the region of preferred faulting. We use the finite element method to investigate this modification.

The model used is shown in fig.3.1. The applied tension is 1 kbar which can be scaled. The elastic parameters taken for the sediments were calculated by assuming a Poisson solid ($\lambda = \mu$) and a density of 2.45 gm/c.c. The Nafe-Drake (1963) curve gave an estimated P-wave velocity from which Young's modulus was calculated. This was

$$E = 0\cdot327 \times 10^{12} \text{ dynes/sq.cm.}$$

For the surrounding material a crustal density and P-wave velocity were used to estimate Young's modulus -

$$E_{crust} = 0\cdot928 \times 10^{12} \text{ dynes/sq.cm.}$$

The shape of the basin was based on Bott's (1965) interpretation of the East Irish Sea Basin. Its maximum depth is taken as 6 Km. and its width as 68 Km. The calculation was treated as a case of plane strain and gravitational forces ignored. The redistribution of the stress-differences in the region of the basin is represented schematically in fig.3.2. The letters represent ranges of stress difference (shown on the

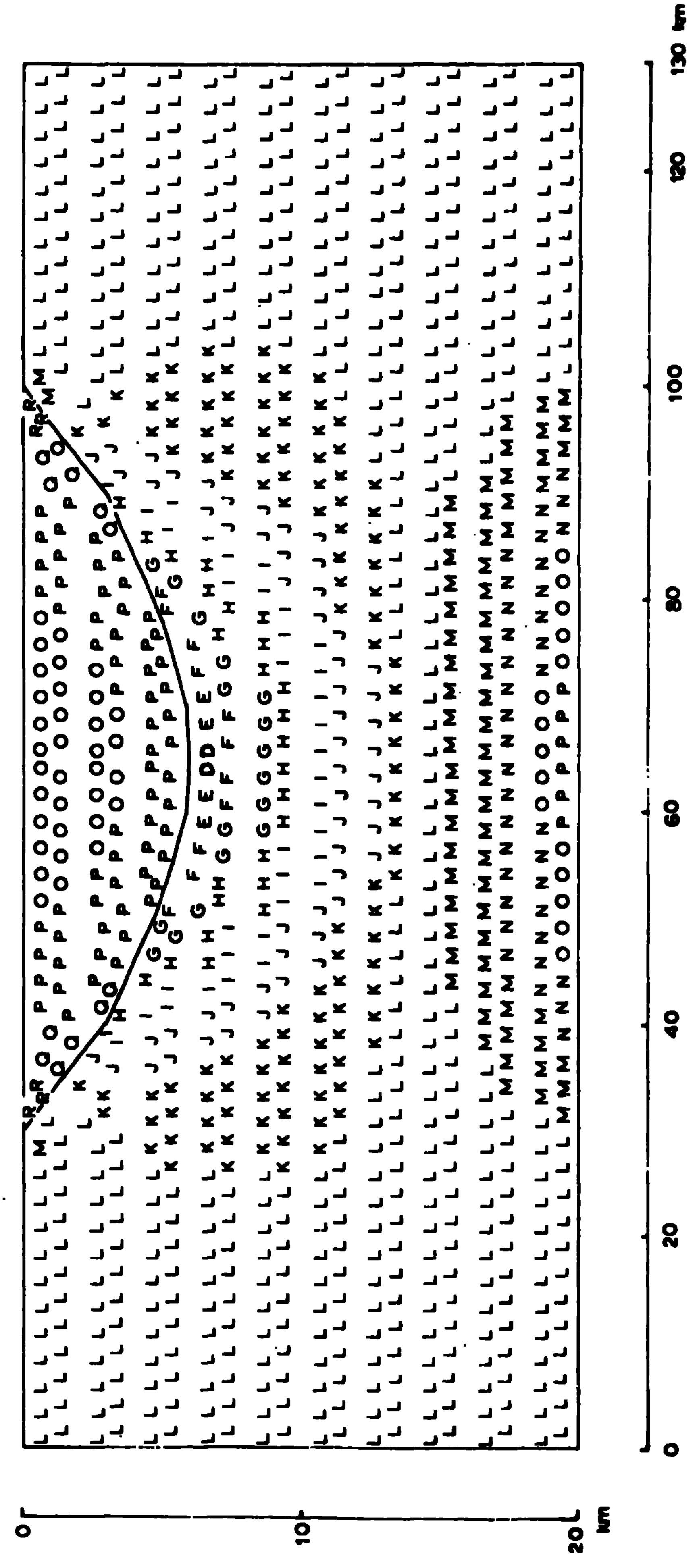


FIG. 3.2

KEY TO FIGURES 3.2, 3.4 4.4.

Symbol	Stress Range (dynes/sq.c.m.)
A	$S \geq 10^9$
B	$9.75 \times 10^8 \leq S < 10^9$
C	$9.25 \times 10^8 \leq S < 9.75 \times 10^8$
D	$8.75 \times 10^8 \leq S < 9.25 \times 10^8$
E	$8.25 \times 10^8 \leq S < 8.75 \times 10^8$
F	$7.75 \times 10^8 \leq S < 8.25 \times 10^8$
G	$7.25 \times 10^8 \leq S < 7.75 \times 10^8$
H	$6.75 \times 10^8 \leq S < 7.25 \times 10^8$
I	$6.25 \times 10^8 \leq S < 6.75 \times 10^8$
J	$5.75 \times 10^8 \leq S < 6.25 \times 10^8$
K	$5.25 \times 10^8 \leq S < 5.75 \times 10^8$
L	$4.75 \times 10^8 \leq S < 5.25 \times 10^8$
M	$4.25 \times 10^8 \leq S < 4.75 \times 10^8$
N	$3.75 \times 10^8 \leq S < 4.25 \times 10^8$
O	$3.25 \times 10^8 \leq S < 3.75 \times 10^8$
P	$2.75 \times 10^8 \leq S < 3.25 \times 10^8$
Q	$2.25 \times 10^8 \leq S < 2.75 \times 10^8$
R	$1.75 \times 10^8 \leq S < 2.25 \times 10^8$
S	$1.25 \times 10^8 \leq S < 1.75 \times 10^8$
T	$0.75 \times 10^8 \leq S < 1.25 \times 10^8$
U	$0.25 \times 10^8 \leq S < 0.75 \times 10^8$
V	$S < 0.25 \times 10^8$

figure) the higher the letter in the alphabet the greater the stress-difference it represents. L represents the uniformly applied stress-difference of 500 bar, K a stress-difference greater than this and M one slightly less.

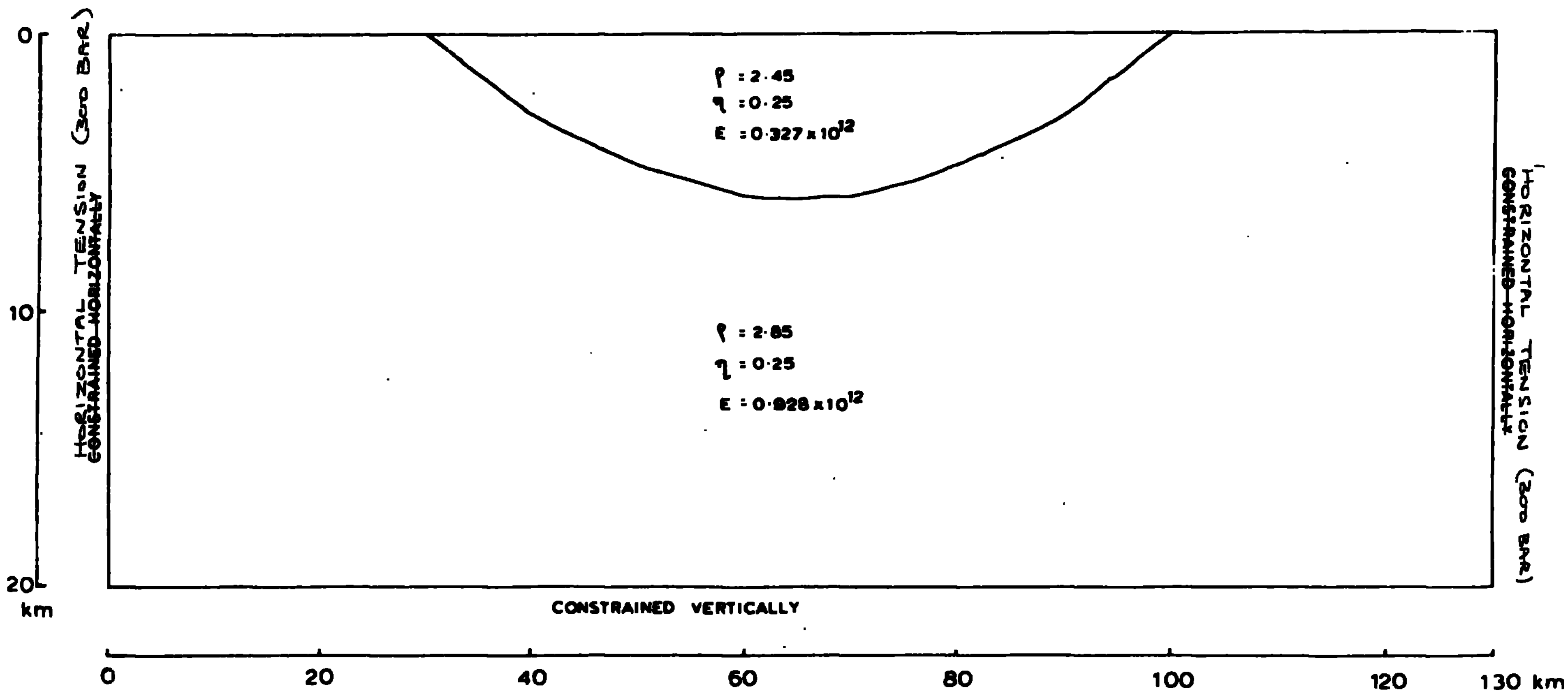
The stress-differences in the basement rocks next to the basin deviate from the uniform value seen at large distances from the basin (500 bar) In general they show an increase the exception being at the upper surface of the model where a small decrease is observed. The greatest stress differences (877.5 bar) occur in the basement rocks below the thickest sediments, this being a 75% increase in the regional value of 500 bar.

Thus the redistribution by the contrasting elastic parameters has an effect opposite to that of the low density of the basin. One leads to preferential normal faulting conditions at the sides of the basin while the other gives preferred faulting in the basement rocks immediately below the deepest part of the basin.

The actual position of initial faulting will depend on

(a) the size of the regional tension compared with the gravitational forces, and

(b) the effect of the decrease in the likelihood of faulting with depth because of (i) gradual brittle to ductile transition, (ii) higher frictional resistance across potential shear planes (Anderson,1942)



HORIZONTAL TENSION (300 BAR)
CONSTRAINED HORIZONTALLY
ρ : 2·45
η : 0·25
E : 0·327 x 10^12
ρ : 2·85
η : 0·25
E : 0·928 x 10^12
CONSTRAINED VERTICALLY
HORIZONTAL TENSION (300 BAR)
CONSTRAINED HORIZONTALLY
0
10
20
km
0
20
40
60
80
100
120
130 km

FIG. 3.3

According to Anderson the function

$$F = \frac{P-R}{2}\sin 2\theta + \mu\left(\frac{P+R}{2} + \frac{P-R}{2}\cos 2\theta\right)$$

gives a direct measure of the likelyhood of faulting, where P is the greatest principal stress and R the least principal stress (tensions + ve). Taking $\mu = 1$, as Anderson suggests likely from field observations, we have $\theta = 22\frac{1}{2}^{\circ}$. Thus

$$F = \frac{P-R}{\sqrt{2}} + \frac{P+R}{2}$$

This function does not take into consideration the gradual brittle to ductile transition.

To investigate further this interaction a finite element calculation has been performed using the model in Fig.3.3 for a basin 1 Km. deep and superimposing upon it the stresses for a horizontally applied tension of 300 bar. The distribution of F values for the combined fields is shown in Fig.3.4. The redistribution of the tension is seen to be dominant and all crustal regions in contact with the basin, apart from the surface, are the areas most susceptible to faulting. This suggests that normal faulting will initially be instigated in the crustal rocks at points below the greatest sediment thicknesses. After failure stress concentration will lead to downwards propagation of the fault to depths where ductility is sufficient for flow to reduce the stress differences to below the strength of the rocks.

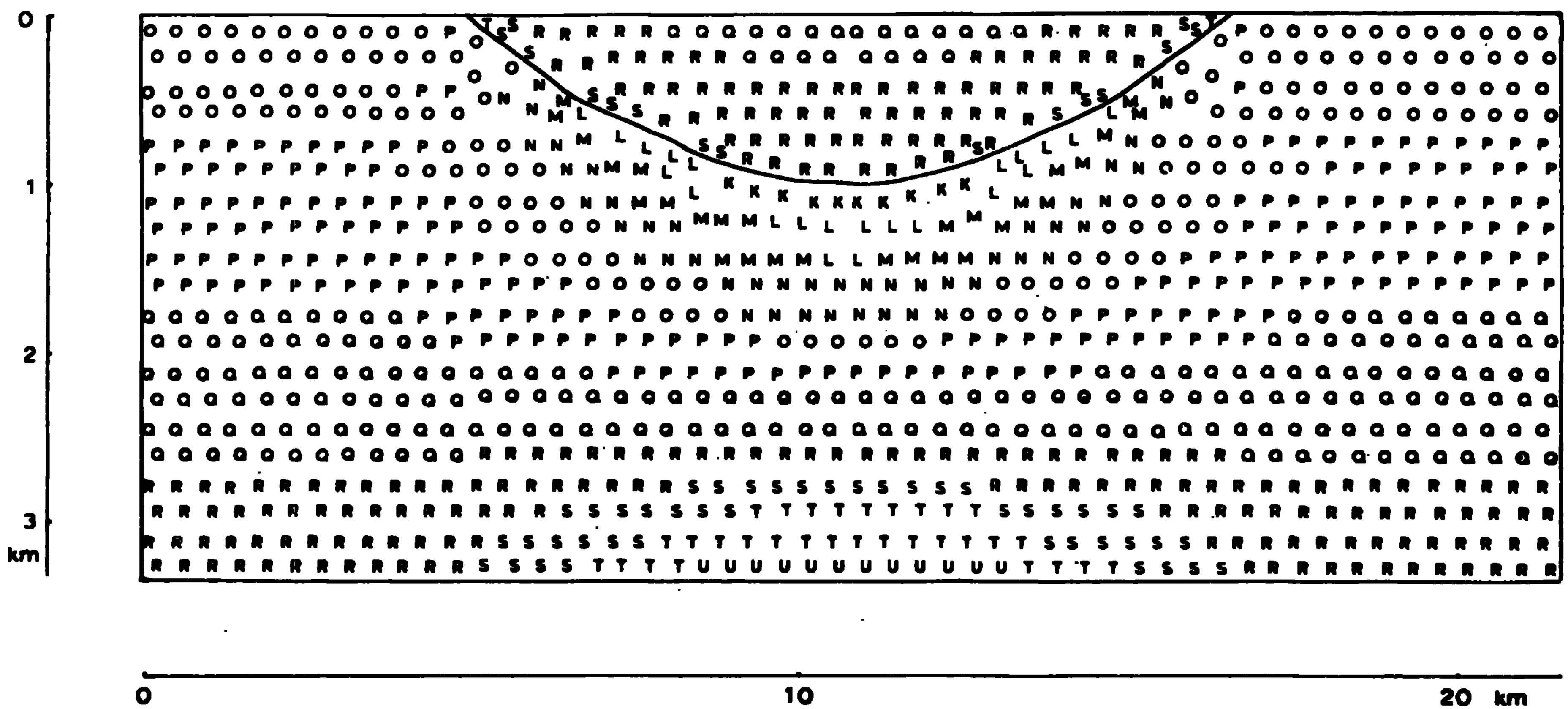


FIG. 3.4

CHAPTER 4.

'Stress Systems at Young Continental Margins'

Young continental margins mark the juxtaposition of continental and oceanic crust attached together as a part of a single plate of lithosphere. Margins of this type form the border of much of the Atlantic and Indian Oceans. They lack the earthquake activity associated with the release of strain energy at plate boundaries such as are formed by the active margins around the Pacific Ocean, yet they are typically associated with a characteristic type of tectonic activity involving a substantial amount of subsidence of the shelf (Collette, 1968) including the formation of sedimentary basins (Bott 1965, Bott and Watts 1970). We now use the finite element method to investigate various stress systems associated with such a margin.

There are at least two ways in which local stress systems may be associated with a young margin of aseismic type. First the body forces associated with the variation in thickness of the low density crust and the presence of the ocean on one side give rise to a laterally varying stress system across the margin. Second, if a regional stress is applied to the lithosphere, the lateral variation of elastic parameters across the margin will cause the regional stress system to be modified in the vicinity of the margin. We use the finite element method to investigate quantitatively these types of stress system associated with an ideal margin. Stress differences may also be associated with

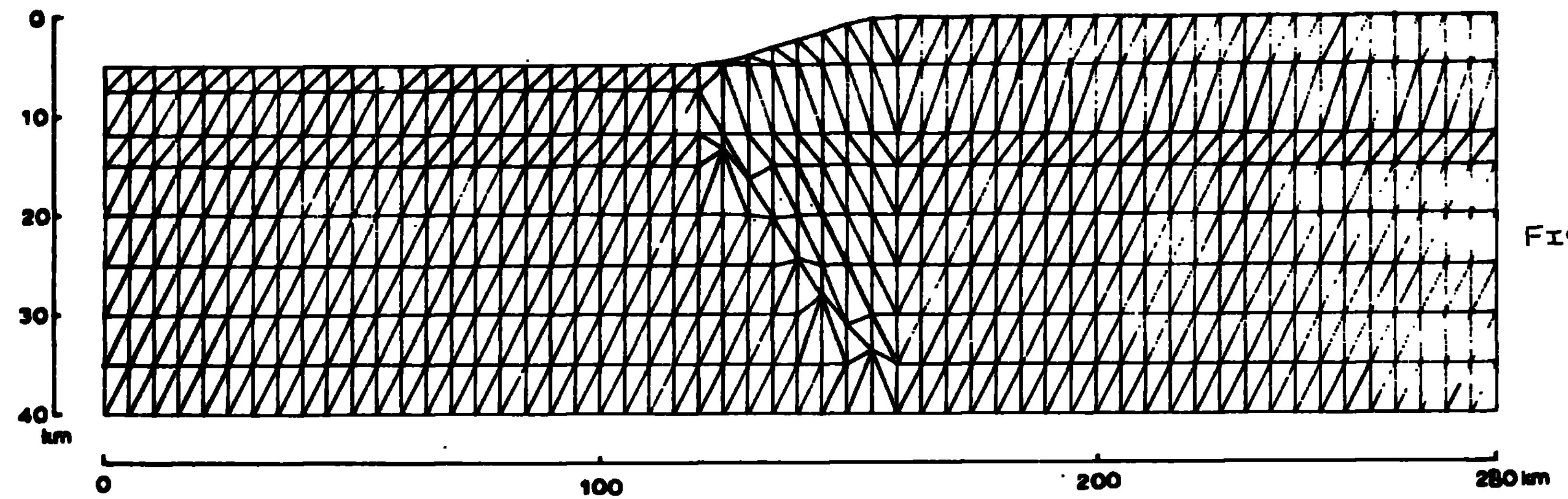

0
10
20
30
40
km
0
100
200
280 km

FIG. 4.1

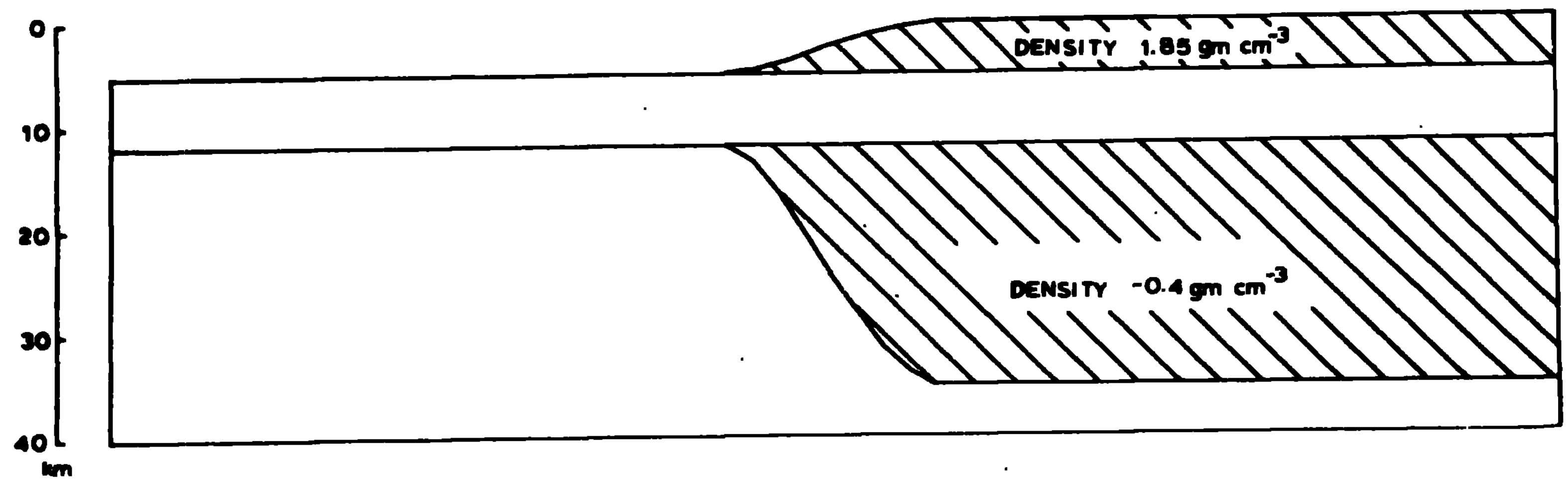

DENSITY 1.85 gm cm⁻³
DENSITY -0.4 gm cm⁻³
0
10
20
30
40
km

FIG. 4.2

a differential temperature - depth distribution, but these have not been considered.

The subdivision of the model was done in such a way as to preserve the Mohorovicic discontinuity (Fig.4.1). All the problems have been treated as cases of plane strain. In the model, the continental crust is 35 Km. thick and the oceanic crust is 6.875 Km thick; the densities of the crustal and mantle material are taken to be 2.85 and 3.25 gm/cm³ respectively. The model is thus in isostatic equilibrium across the margin. The elastic parameters used are based on measured P wave velocities assuming a Poisson solid in which Lamé 's constants $\lambda=\mu$, and Poisson's ratio = 0.25. The values of Young's modulus used were

$$E_{crust} = 0.928 \times 10^{12} \text{ dyne cm}^{-2}$$
$$E_{mantle} = 0.178 \times 10^{13} \text{ dyne cm}^{-2}$$

The transition from continental to oceanic crust takes place over a horizontal distance of about 60 km. (Worzel,1965)

The density model (Fig.4.2) was used to investigate the effect of differential loading across the margin. This was obtained by subtracting the density - depth distribution beneath the ocean from the whole structure, leaving a load at the top of the continental crust and an equal upthrust near the base. Stresses associated with the normal oceanic density - depth distribution can be superimposed on our model, both ends of which are considered to be constrained in a horizontal direction. Because the whole model is in isostatic equilibrium the

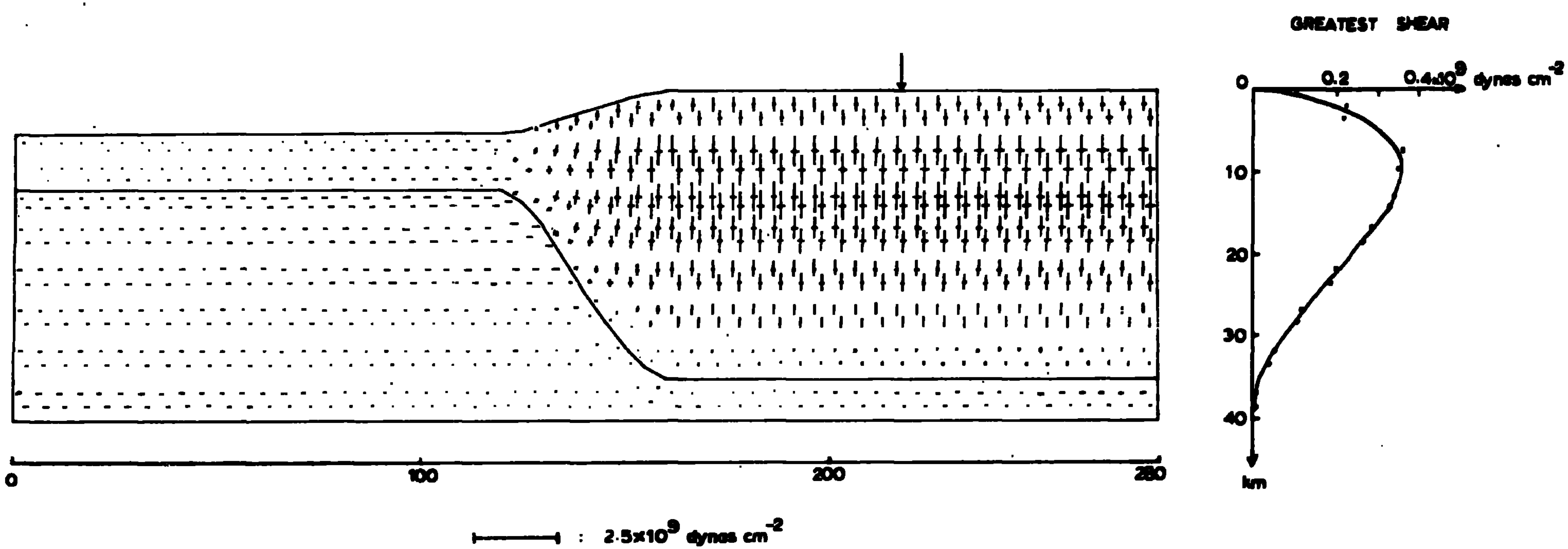

0
100
200
280
: 2.5x10⁹ dynes cm⁻²
GREATEST SHEAR
0
0.2
0.4x10⁹ dynes cm⁻²
10
20
30
40
km

FIG. 4.3

base can be left free of boundary stresses. The results of the finite element calculation are shown in Fig.4.3. The principal stresses are nearly horizontal and vertical except beneath the margin. Beneath the continental region the maximum principal compression is vertical and the minimum principal compression is horizontal. The principal stress in a direction perpendicular to the plane of the model is everywhere the intermediate one. The maximum stress difference (defined as half the difference between the greatest and least principal stress and therefore equal to the greatest shear) occurs at a depth of 10 Km and reaches about 370 bar. Also shown is the relative variation of the greatest shear with depth through a typical vertical section (marked by arrow) of the continental lithosphere. Thus the differential gravity loading across a young continental margin gives rise to a state of stress more conducive to normal faulting (Anderson,1942) parallel to the margin in the adjacent continental crust than in the oceanic crust.

When a margin is subjected to a regional tension or compression, the lateral change in elastic parameters causes local variation from the uniform stresses that would be associated with a homogeneous plate. To investigate this effect a finite element calculation was performed on a margin subjected to a horizontal tension of 500 bar. All body forces were neglected. The results of the calculation are schematically illustrated in Fig.4.4. The greatest crustal stress differences occur on the continental slope. The stress differences in the

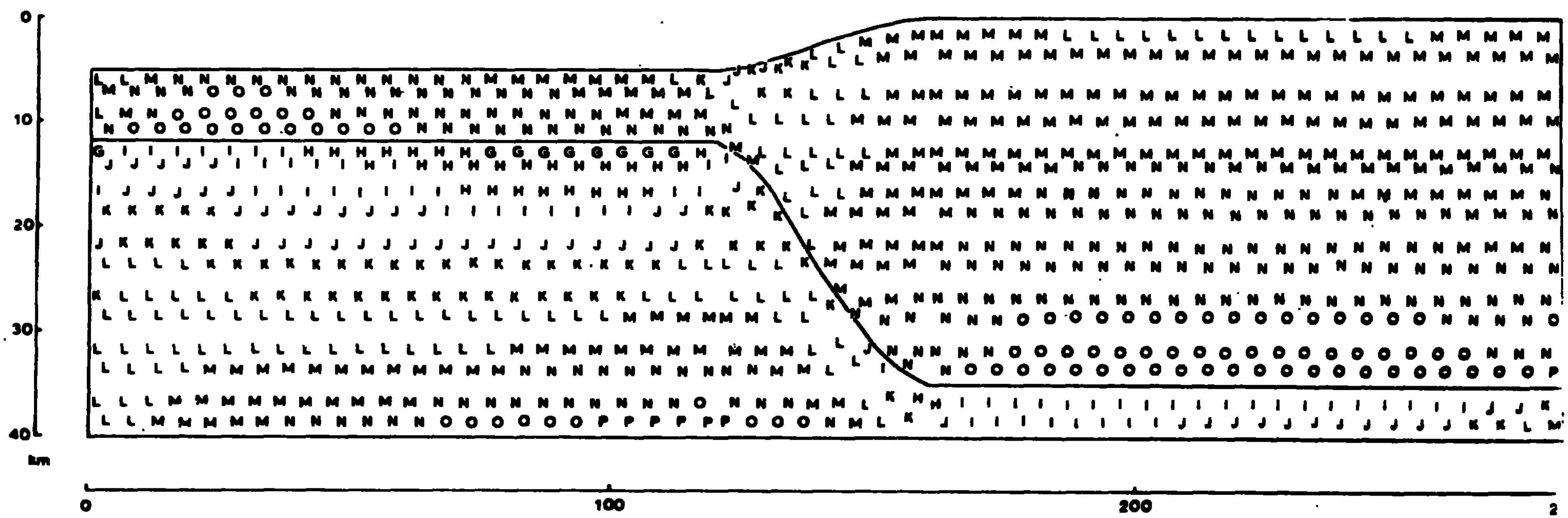


FIG. 4.4

upper most 5 Km of the continental crust are slightly greater than those in the upper 5 Km of the oceanic crust.

In general, both types of stress system will be superimposed. When the lithosphere is subjected to a local tension, both effects enhance each other in the adjacent continental crust producing a stress regime highly favourable to normal faulting particularly near the slope. When the lithosphere is subjected to regional compression, then the two effects will oppose each other in the continental crust, and the sub-oceanic region adjacent to the margin will be most susceptible to thrust faulting.

Our computations explain the prevalence of normal faulting in the continental crust on and near the continental shelf, and the corresponding lack of evidence for normal faulting in the oceanic crust.

CHAPTER 5.

'Stress-Diffusion in the lithosphere'

The theories of continental drift (Wegener, 1912), based on geological and outline similarities of continents thousands of kilometers apart, and ocean floor spreading, resulting from the study of geomagnetic anomalies of the sea floor (Vine and Matthews 1963; Pitman and Heirtzler 1966), have recently been united by the all encompassing theory of plate tectonics (McKenzie and Parker 1967, Morgan 1968).

The theory of isostasy and seismological evidence for a low velocity channel in the upper crust both suggest that the earth has a strong outer shell (lithosphere) underlain by a weak region which deforms by flow (asthenosphere) (Barrell 1914 a.b). In plate tectonics the earth's outer shell is conceived as a series of rigid blocks, referred to as plates, the surface boundaries of which are defined by the narrow regions of seismic activity. These plates are in relative motion, sliding past each other at transform faults (Wilson 1965) with the creation of oceanic crust at their ridges and its consumption at oceanic deeps. By studying the amplitude and wavelength of bending of the lithosphere in the vicinity of supercrustal loads, Walcott (1970) has obtained estimated thicknesses of the order of 100 Km for the lithosphere.

By fitting models with an elastic layer overlying viscous layers McConnell (1965) has formulated a rheological

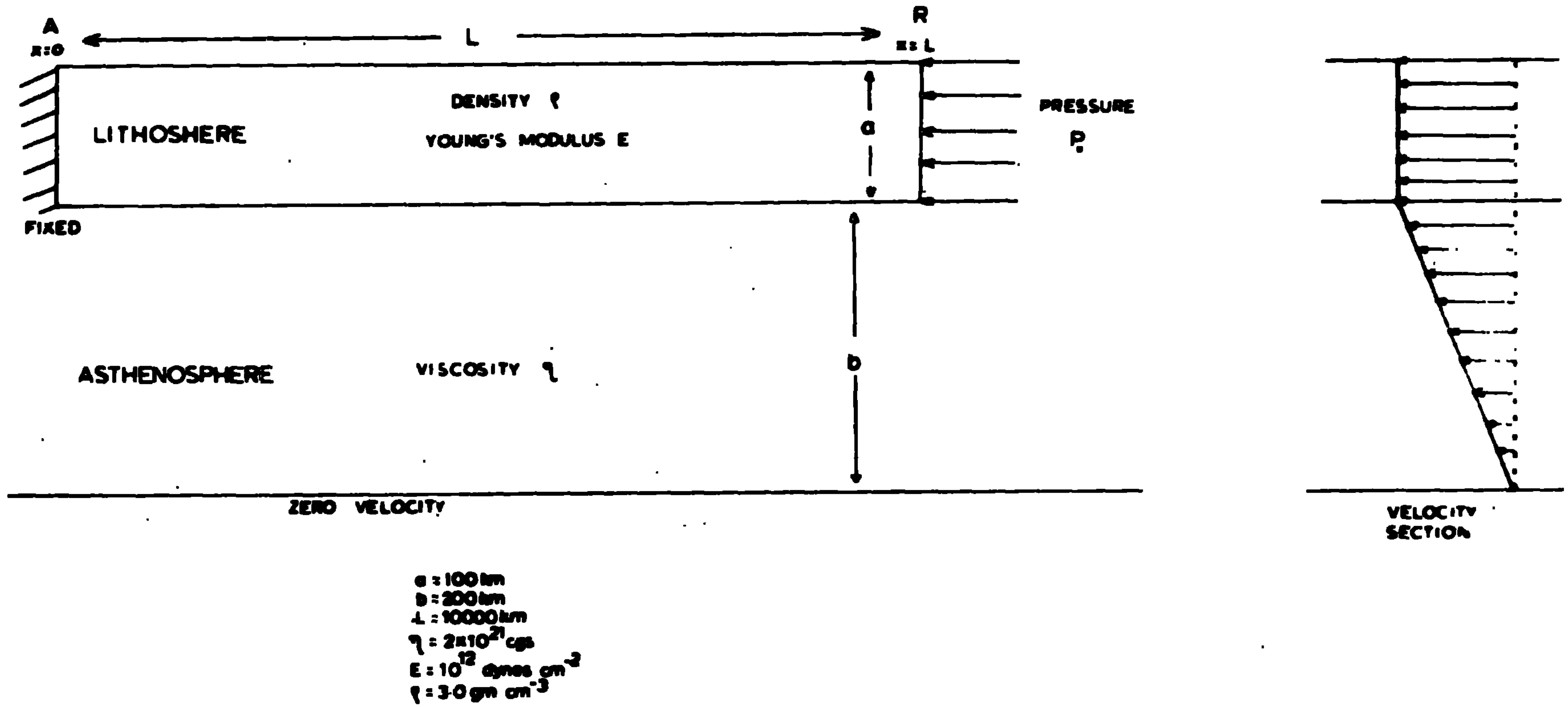

A
x=0
L
R
x=L
LITHOSHERE
DENSITY ρ
YOUNG'S MODULUS E
a
PRESSURE
P
FIXED
ASTHENOSPHERE
VISCOSITY η
b
ZERO VELOCITY
VELOCITY SECTION
a = 100 km
b = 200 km
L = 10000 km
η = 2x10^21 cgs
E = 10^12 dynes cm^-2
ρ = 3.0 gm cm^-3

FIG. 5.1

model of the outer earth to fit the observed Fennoscandian recovery after the removal of ice-age loads. He concludes that the region most susceptible to deformation lies between 100 to 400 Km depth, the resistance to deformation increasing rapidly below 400 Km. This suggests a base to the asthenosphere at a depth of some 400 Km. The apparent viscosity of this layer is of the order 3×10^{21} P.

Because of the relative rigidity of the lithosphere forces applied at one boundary are transmitted to other boundaries of the plate. Thus, for example, forces at the ridge of a lithospheric plate will be felt at island arc boundaries and it is important to have some idea of the likely effects of such forces and of the order of magnitude of time delays associated with propagation through the lithosphere. In this chapter a model has been set up to investigate further this problem.

5.1 Specification of problem.

If a sudden force is applied and maintained at the ridge of a lithospheric plate then a question of great relevance concerns the effects of this force at other plate boundaries. To obtain insight into this problem the model shown in figure (5.1) has been considered. A more sophisticated model is not warranted because the extra complexities of mathematical analysis introduced and the uncertainty in numerical values of rheological parameters would result in less progress towards an understanding of what actually takes place than that obtained from consideration of the crude model. The parameter values shown in the figure have the orders of magnitude of the corresponding values associated with a plate the size of the Pacific. If a sudden pressure, P_0, is applied and maintained at the 'ridge end', R, of the model then the propagation of this pressure is opposed by:

(1) inertial forces

and (11) viscous forces due to the underlying asthenosphere.

The question we ask is 'what is the pressure distribution along the plate, a time t after the initial application of the pressure at R?'

5.2 Formulation of the problem.

Assume that the pressure is applied at the instant in time t = o. Let u(x,t) be the horizontal displacement, at a time t, of the section of plate initially at x. Since $a \ll L$ it is possible to ignore vertical variation of this displacement. Consider the portion of the plate initially between $(x - \delta x)$ and $(x + \delta x)$. After a time t it lies between $(x - \delta x + u - \frac{\partial u}{\partial x}\delta x)$ and $(x + \delta x + u + \frac{\partial u}{\partial x}\delta x)$ and experiences forces (considering unit length perpendicular to the plane of the model) as shown in Figure(5.2), where T(x,t) is the tension at the point of the plate originally at x.

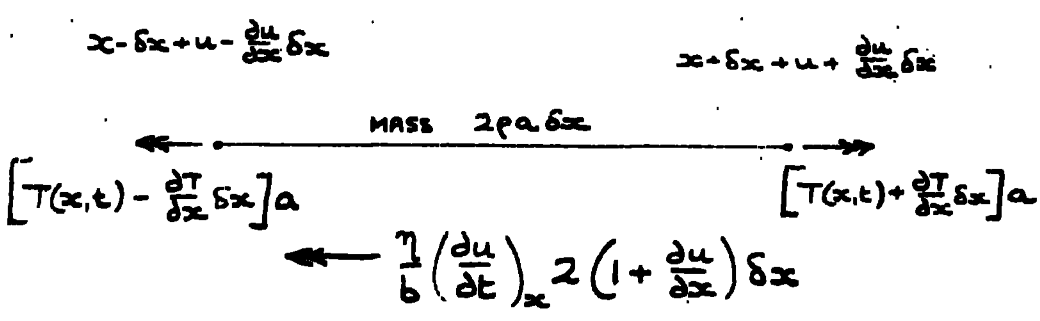


Fig.(5.2)

Applying Newton's second law we now have

$$2a\left(\frac{\partial T}{\partial x}\right)\delta x - \frac{2\eta}{b}\left(\frac{\partial u}{\partial t}\right)\left(1+\frac{\partial u}{\partial x}\right)\delta x = 2a\rho\,\delta x\,\frac{\partial^2 u}{\partial t^2}$$

or

$$\rho\frac{\partial^2 u}{\partial t^2} + \frac{\eta}{ab}\left(\frac{\partial u}{\partial t}\right)\left(1+\frac{\partial u}{\partial x}\right) = \frac{\partial T}{\partial x}$$

From the definition of Young's Modulus

$$T = E\frac{\partial u}{\partial x} \tag{5.1}$$

and it follows that

$$\rho\frac{\partial^2 u}{\partial t^2} + \frac{\eta}{ab}\frac{\partial u}{\partial t}\left(1+\frac{\partial u}{\partial x}\right) = E\frac{\partial^2 u}{\partial x^2} \tag{5.2a}$$

Because of the great strength (E) of the plate compared with the applied pressure P_o, we have $\left|\frac{\partial u}{\partial x}\right| \ll 1$ and (5.2a) reduces to

$$\rho\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = E\frac{\partial^2 u}{\partial x^2} \tag{5.2b}$$

where

$$\sigma \equiv \frac{\eta}{ab} \tag{D5.1}$$

If initially we ignore the inertial forces compared with the viscous ones equation (5.2b) reduces to the diffusion equation:

$$E\frac{\partial^2 u}{\partial x^2} = \sigma\frac{\partial u}{\partial t} \tag{5.2c}$$

To determine the pressure distribution p(x.t). in the plate after a time t we must solve (5.2c) subject to the following boundary and initial conditions,

$$
\begin{array}{llll}
u = 0 & \text{at} \quad t = 0 & 0 \leq x \leq L & \\
u = 0 & \text{at} \quad x = 0 & t > 0 & \qquad (5.3) \\
\left(\dfrac{\partial u}{\partial x}\right)_{x=L} & = \dfrac{-P_0}{E} & t > 0 &
\end{array}
$$

and then use the pressure form of (5.1) $p(x,t) = -E\left(\frac{\partial u}{\partial x}\right)$.

5.3 The Solution

The partial second order differential equation (5.2c) can be reduced to an ordinary second order differential equation by the operation of a laplace transformation, and then solved for the laplace transform of the displacement u(x,t). The problem is then reduced to one of performing an inverse laplace transformation. The solution to (5.2c) has been obtained (Appendix 4 part 1) in this way and is

$$
u(x,t) = -\frac{P_0 x}{E} - \frac{8P_0 L}{E\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin\left[\frac{(2n-1)\pi x}{2L}\right] e^{-\frac{(2n-1)^2\pi^2 E t}{4\sigma L^2}} \qquad (t>0) \qquad (5.4)
$$

This displacement distribution gives a pressure

$$
p(x,t) = P_0 + \frac{4P_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)} \cos\left[\frac{(2n-1)\pi x}{2L}\right] e^{-\frac{(2n-1)^2\pi^2 E t}{4\sigma L^2}} \qquad (t>0) \qquad (5.5)
$$

throughout the plate.

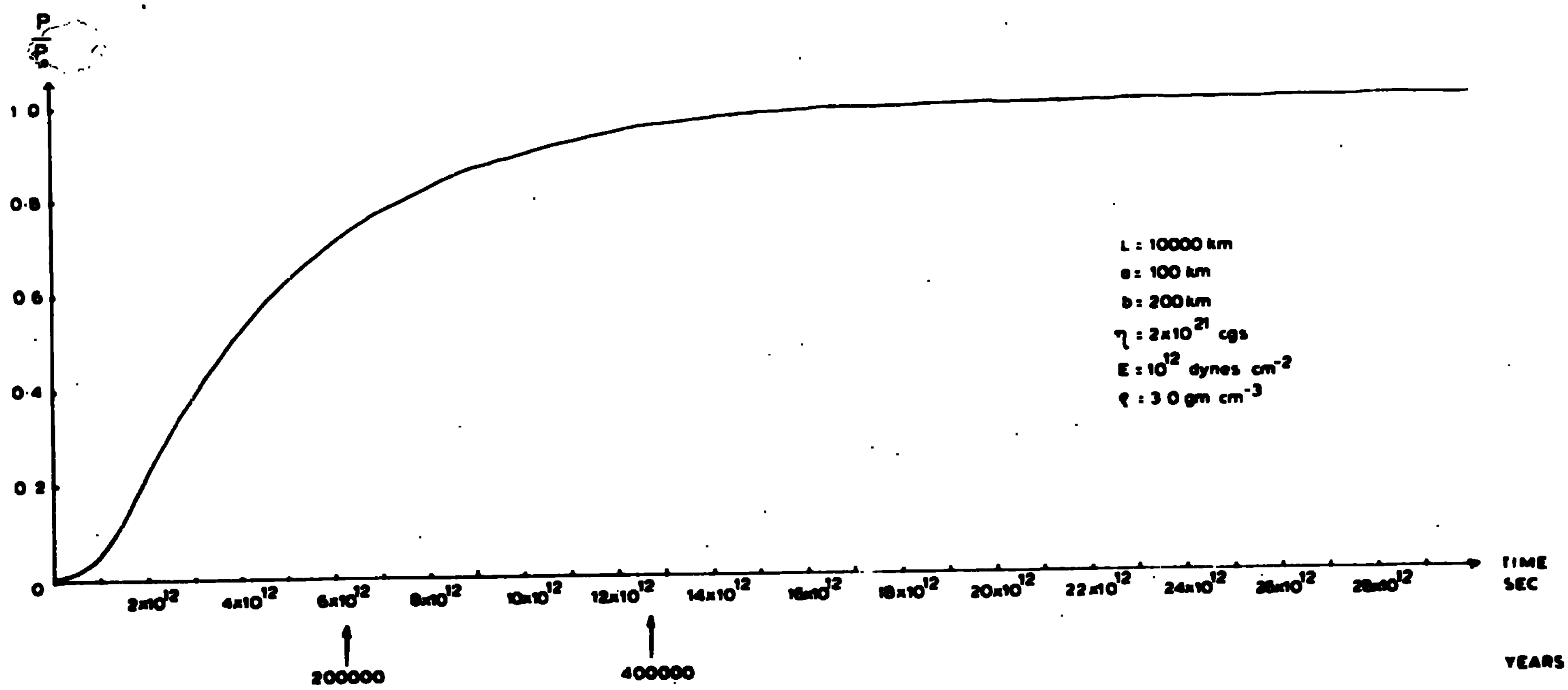

P/P
1 0
0·8
0 6
0·4
0 2
0
2x10^12
4x10^12
6x10^12
8x10^12
10x10^12
12x10^12
14x10^12
16x10^12
18x10^12
20x10^12
22x10^12
24x10^12
26x10^12
28x10^12
TIME
SEC
200000
400000
YEARS
L : 10000 km
a : 100 km
b : 200 km
η : 2x10^21 cgs
E : 10^12 dynes cm^-2
ρ : 3 0 gm cm^-3

FIG. 5.3

The series (5.5) and (5.4) have been computed and these computations are illustrated by the figures (5.3), (5.4) and (5.5). Figure (5.3) illustrates the build up of pressure with time at the fixed end, A, of the plate. The parameters used for computation are shown on the graph. The time taken for the pressure to reach one half of that applied for a plate 10,000 Km in length is of the order of 200,000 years. It should be noted from equation (5.5) that changing the values of any of the parameters does not alter the shape of the graph. What it in fact does do is change the labelling of the time axis. Because of the occurrence of the ratio $\frac{Eabt}{\eta L^2}$, increasing Young's Modulus by a factor of 2 will cause all the time labels in figure (5.3) to be halved and similarly a decrease in the length, L of the plate by a factor of 10 will cause the time labels to be decreased by a factor of 100. Figures (5.4) and (5.5) illustrate the events at the opposite end of the plate R (x = L). The first graph shows the displacement response to the applied force Po at R, and the second graph plots the gradient of this graph i.e. the velocity response to P_o. The parameters used are shown on each graph. Note that the comments regarding the effects of changes in these parameters, discussed for figure (5.3) do not apply to these last two graphs. The situation is more complicated here, involving changes in the dimensioning of both axes, the manner of which being evident on a closer examination of equation (5.4). The velocity curve in figure (5.5) shows a rapid fall off with time for 20 000 years and then a steadier period of about 300,000 years with a

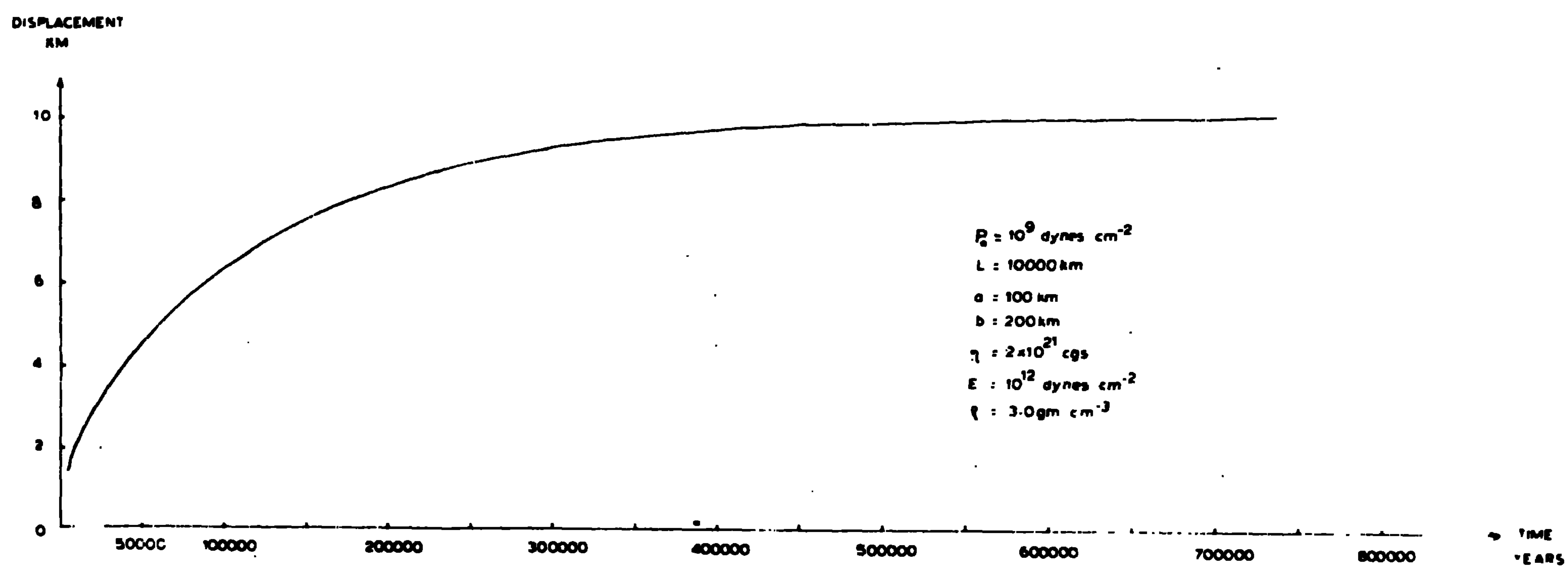

DISPLACEMENT
KM
10
8
6
4
2
0
5000C
100000
200000
300000
400000
500000
600000
700000
800000
TIME
YEARS
P_o = 10^9 dynes cm^{-2}
L : 10000 km
a : 100 km
b : 200 km
η : 2×10^{21} cgs
E : 10^{12} dynes cm^{-2}
ρ : 3.0gm cm^{-3}

FIG. 5.4

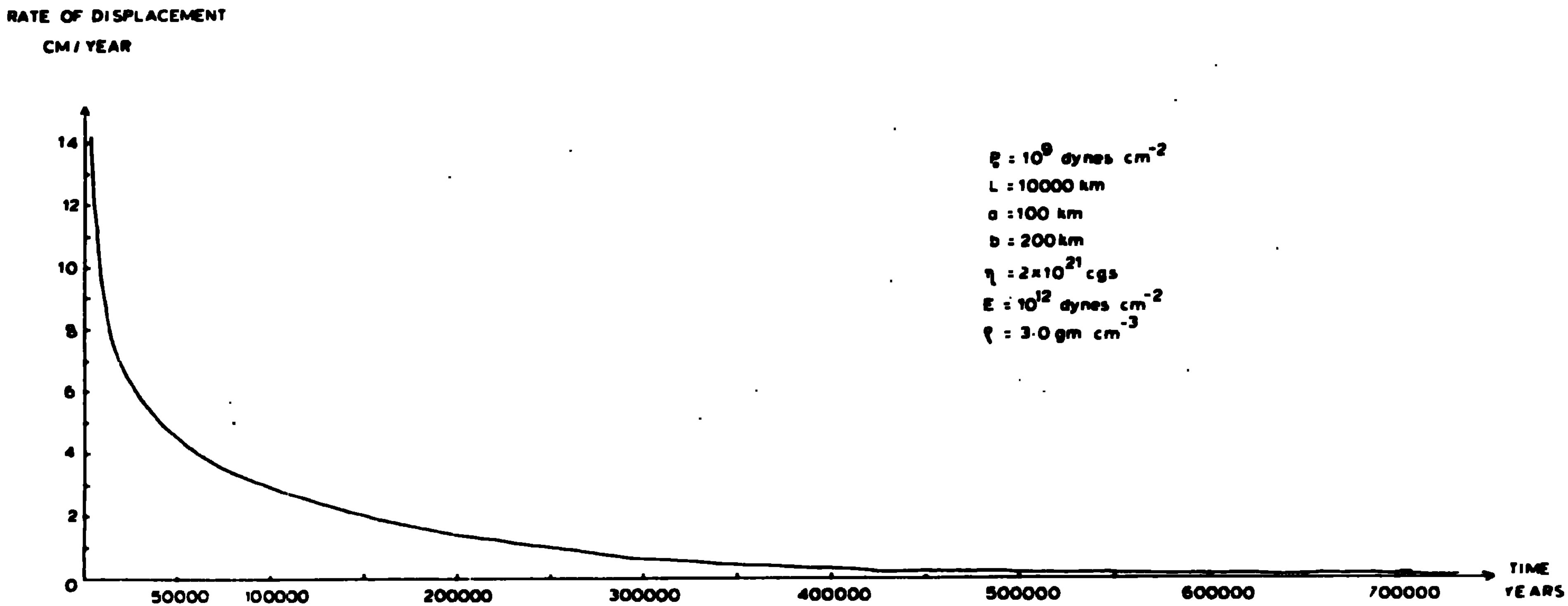

RATE OF DISPLACEMENT
CM / YEAR
14
12
10
8
6
4
2
0
50000
100000
200000
300000
400000
500000
600000
700000
TIME
YEARS
ρ : 10⁹ dynes cm⁻²
L : 10000 km
a : 100 km
b : 200 km
η : 2×10²¹ cgs
E : 10¹² dynes cm⁻²
ρ : 3·0 gm cm⁻³

FIG. 5.5

velocity between 5 cm./yr. and 1 cm./yr.

5.4 Effect of a different velocity gradient in the asthenosphere

In the model considered (figure (5.1)) a linear fall off of velocity with depth within the asthenosphere was assumed. If instead of this we apply, together with the continuity of velocity across the lithosphere - asthenosphere boundary a condition of no net flow of material through any vertical cross-section we can fit a quadratic variable velocity, v. (Jacoby,1970). Using Jacoby's notation (Figure 5.6)

$$\left(\frac{\partial v}{\partial z}\right)_{z=0} = -2Cz_0$$

where

$$C = \frac{1}{(2q-1)} \cdot \frac{v_0}{b^2}$$

$$q = \frac{(a+\frac{2}{3}b)}{(2a+b)}$$

and

$$z_0 = bq$$

For the case b = 2a, we have $q = 7/12$, $C = \frac{6u_0}{b^2}$, $z_0 = \frac{7b}{12}$ and $\left(\frac{du}{dz}\right)_{z_0} = \frac{-7u_0}{b}$. Thus, the magnitude of the velocity gradient at the top of the asthenosphere is increased by a factor of 7 (c.f. v_0/b for linear case). This is equivalent to increasing the viscosity η in the linear case by a factor of 7. which, from section 5.3 is equivalent to increasing the time axis in figures (5.3) and (5.4) by a factor of 7. The effect on

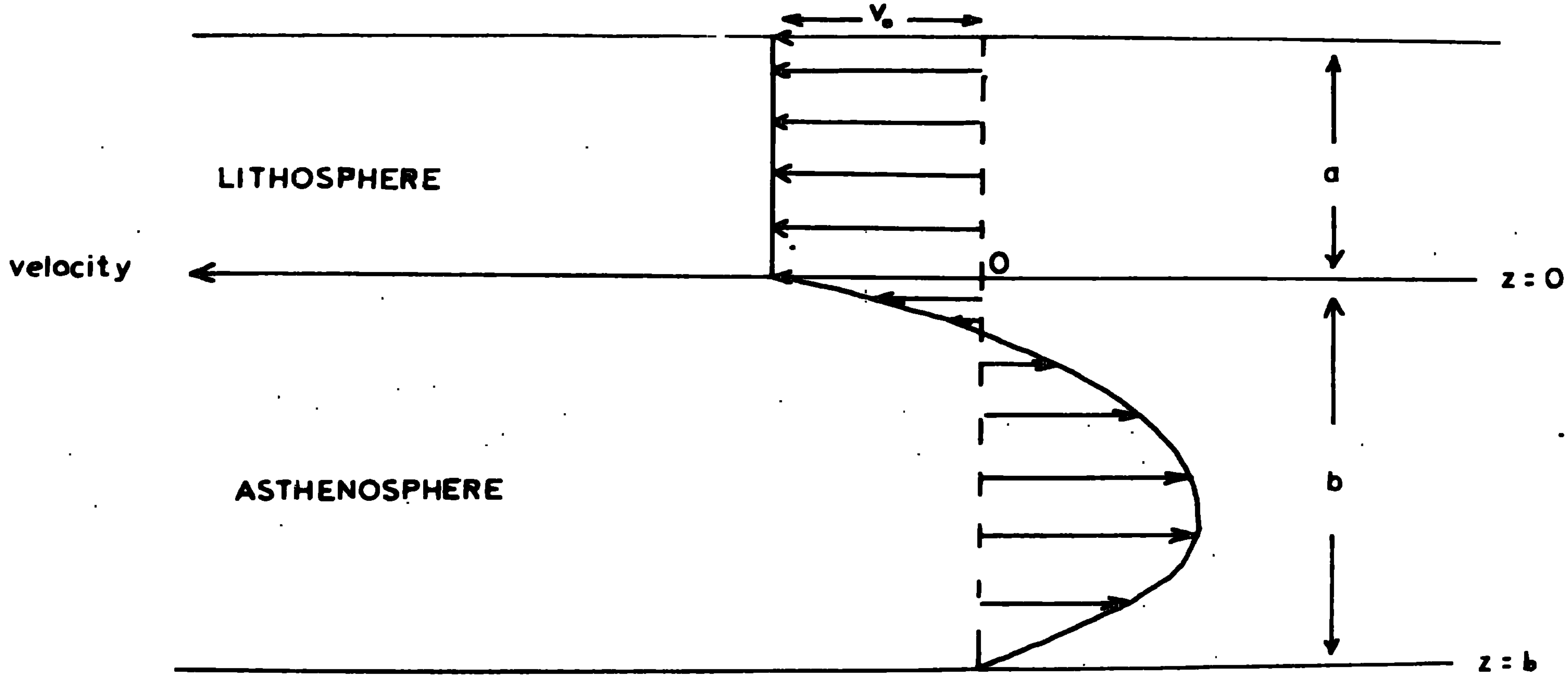


FIG. 5.6

figure (5.5) is also to reduce the velocity axis by a factor of 7.

(5.5) Inclusion of Inertial Terms

Equation (5.2b) has been solved to investigate the effect of ignoring the inertial terms. (Appendix 4. Part 2). The corresponding solutions are

$$u(x,t) = \frac{-P_o x}{E} + \frac{2P_o}{L}\sum_{n=1}^{\infty}(-1)^n \sin\left[\frac{(2n-1)\pi x}{2L}\right] \times \left\{\frac{e^{z_{n+}t}}{z_{n+}(2\rho z_{n+}+\sigma)} + \frac{e^{z_{n-}t}}{z_{n-}(2\rho z_{n-}+\sigma)}\right\} \qquad (5.6) \quad (t>0)$$

and

$$p(x,t) = P_o - \frac{P_o \pi E}{L^2}\sum_{n=1}^{\infty}(-1)^n(2n-1)\cos\left[\frac{(2n-1)\pi x}{2L}\right] \times \left\{\frac{e^{z_{n+}t}}{z_{n+}(2\rho z_{n+}+\sigma)} + \frac{e^{z_{n-}t}}{z_{n-}(2\rho z_{n-}+\sigma)}\right\} \qquad (5.7) \quad (t>0)$$

where

$$z_{n\pm} \equiv \frac{-\sigma \pm \sqrt{\sigma^2 - 4\rho\alpha_n}}{2\rho} \qquad (D5.2)$$

and

$$\alpha_n \equiv \frac{(2n-1)^2\pi^2 E}{4L^2} \qquad (D5.3)$$

For small n z_{n+} and z_{n-} are both real. However for sufficiently large n we have $\sigma^2 < 4\rho\alpha_n$ and z_{n+} and z_{n-} become complex. In this case $z_{n+} = \overline{z_{n-}}$ and in the above

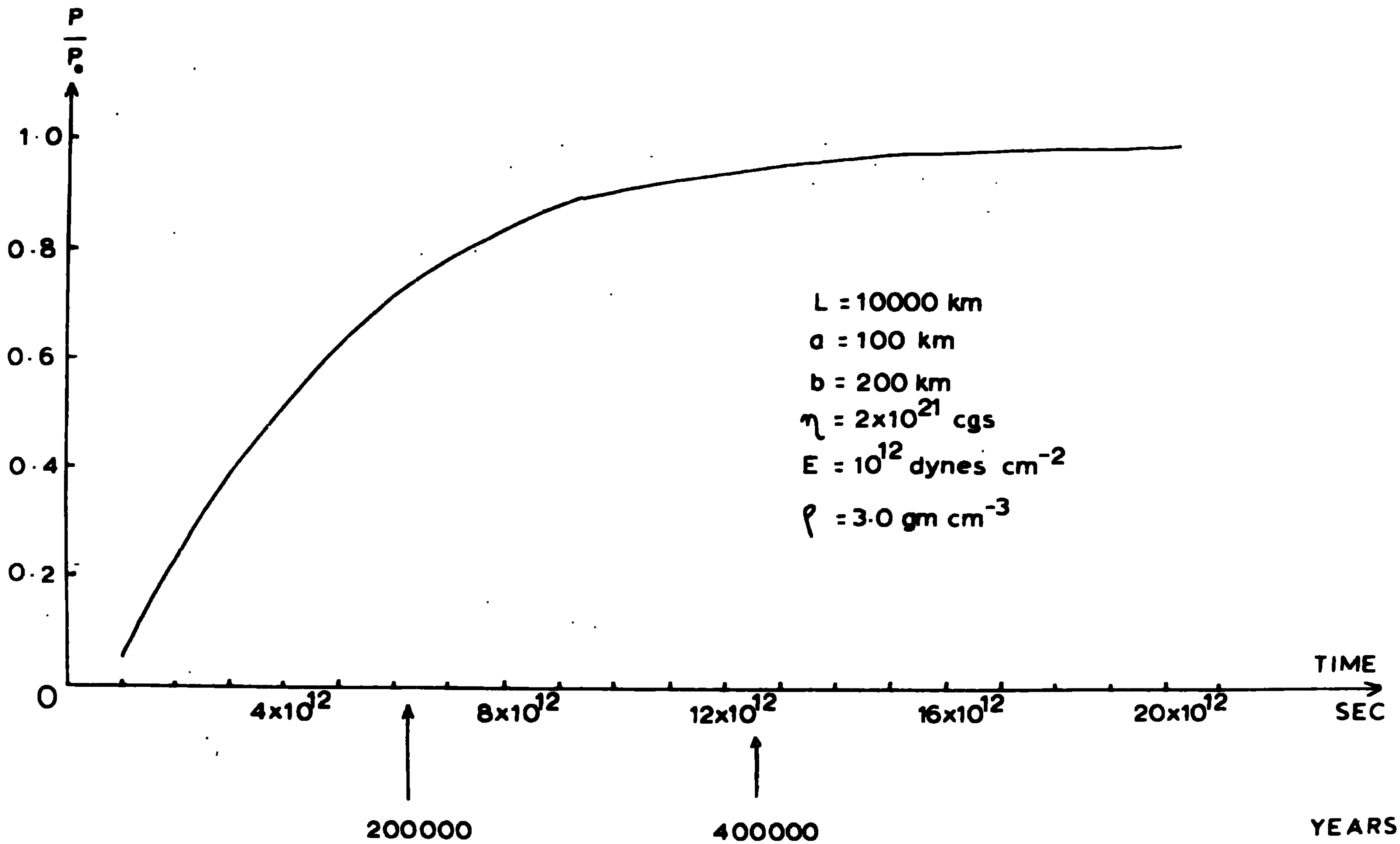

P/P₀
1·0
0·8
0·6
0·4
0·2
0
L = 10000 km
a = 100 km
b = 200 km
η = 2x10^21 cgs
E = 10^12 dynes cm^-2
ρ = 3·0 gm cm^-3
TIME
4x10^12
8x10^12
12x10^12
16x10^12
20x10^12
SEC
200000
400000
YEARS

FIG. 5.7

equations
$$\left\{ \frac{e^{z_{n_+}t}}{z_{n_+}(2\rho z_{n_+} + \sigma)} + \frac{e^{z_{n_-}t}}{z_{n_-}(2\rho z_{n_-} + \sigma)} \right\}$$
is a real quantity (since $z + \bar{z}$ is always real).

The series (5.7) has been computed for the build up of pressure at x = o. This computation presents several problems which have been discussed in appendix 4 part 3. The graph shown in fig.5.7 illustrates this computation and, on comparison with figure (5.3), it is evident that, despite considerable alteration of the analytical form of the solution (compare (5.4) and (5.6) and note that (5.6) does not reduce to (5.4) on allowing $\frac{\rho}{\alpha} \to 0$) the effect of the inertial terms for the given parameters is negligible when the two solutions are computed.

CHAPTER 6

'Stress-relaxation in the lithosphere'

For the model described in the previous chapter there is a gradual increase in pressure at the end A until it reaches the value of the applied pressure P_o (figure 5.3). If the applied pressure is sufficiently large then, after a time τ say, the pressure at A will attain a critical value P_c, at which stress relaxation will take place. It is interesting to pose the question 'How quickly will the stress at A rebuild after relaxation?'

6.1 Simulation of Stress Relaxation.

In an attempt to simulate complete relaxation at A after a time τ an instantaneous displacement u_+ has been imposed on the plate. This displacement must satisfy the following conditions: (1) it must have a decreasing effect at increasing distance from A and (11) it must give rise to a stress field that gives complete relaxation at A.

Mathematically these conditions can be written as:

(1) $u_+ \rightarrow 0$ and $\frac{du_+}{dx} \rightarrow 0$ as $x \rightarrow L$

(11) $$\left.\frac{du_+}{dx}\right|_{x=L} = \frac{P_c}{E}$$

The function $f(x) = -\frac{P_c S}{E} e^{-x/s}$ satisfies these conditions where S is a parameter that controls the range of the displacement. To simplify the boundary conditions it helps if we also impose a rigid body displacement of magnitude $\frac{P_c S}{E}$ to the plate. Because of the rigid body nature of this extra displacement it does not change the stress field within the plate and consequently does not alter the subsequent build up of stress at A. Thus we take

$$u_{+} = \frac{P_c s}{E}\left(1 - e^{-x/s}\right) \qquad (6.1)$$

as our displacement to simulate relaxation.

When attempting to solve the equation of motion of the plate for the given initial conditions it proves advantageous to express (6.1) in its equivalent Fourier series form (Appendix 4, part 3)

$$u_{+} = \frac{1}{S_N}\sum_{n=1}^{N}\left[\frac{1}{(2n-1)} - \frac{(2n-1)}{\left(\frac{4L^2}{\pi^2 s^2} + (2n-1)^2\right)}\right]\sin\left[\frac{(2n-1)\pi x}{2L}\right] \qquad (6.2)$$

where

$$S_N = \frac{E\pi}{2LP_c}\sum_{n=1}^{N}\left[1 - \frac{(2n-1)^2}{\left(\frac{4L^2}{\pi^2 s^2} + (2n-1)^2\right)}\right]. \qquad (D6.1)$$

What energy release does this displacement represent per unit length (1 cm.) perpendicular to the plane of the model? The potential strain energy before relaxation (1.22) is

$$\frac{Ea}{2}\int_{x=0}^{L}\left(\frac{du_{\tau}}{dx}\right)^2 dx$$

where u_τ is given by (A4.40), and afterwards

$$\frac{Ea}{2}\int_{x=0}^{L}\left(\frac{du_\tau}{dx}+\frac{du_+}{dx}\right)^2 dx .$$

Thus the energy released is

$$-\frac{Ea}{2}\int_{x=0}^{L}\left[\left(\frac{du_+}{dx}\right)^2+2\,\frac{du_+}{dx}\cdot\frac{du_\tau}{dx}\right]dx$$

$$=-\frac{Ea}{2}\int_{x=0}^{L}\left(\frac{P_c^2}{E^2}e^{-2x/s}-\frac{2P_cP_o}{E^2}e^{-x/s}\right.$$

$$\left.-\frac{8P_cP_o}{E^2\pi}e^{-x/s}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\cos\left[\frac{(2n-1)\pi x}{2L}\right]e^{-\frac{(2n-1)^2\pi^2E\tau}{4\sigma L^2}}\right)dx$$

using (6.1) and (A4.40).

$$=-\frac{Ea}{2}\left[\frac{P_c^2 s}{2E^2}-\frac{2P_cP_os}{E^2}\right]$$

$$+\frac{8P_cP_osa}{2E\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n e^{-\frac{(2n-1)^2\pi^2E\tau}{4\sigma L^2}}}{(2n-1)\left(1+\frac{(2n-1)^2\pi^2s^2}{4L^2}\right)}\right] \qquad (6.3)$$

where we have used the facts $s \ll L$ and

$$\int_{x=0}^{L}e^{-x/s}\cos\left[\frac{(2n-1)\pi x}{2L}\right]dx = \frac{s}{1+b^2s^2}$$

where $b = \frac{(2n-1)\pi}{2L}$. Computation for the parameter values indicated on figure (6.1) gives an energy release of 0.6 x 10^{18} ergs/cm length perpendicular to the model.

6.2 Specification of problem

We must now solve the equation of motion of the plate (5.2c)

$$E \frac{\partial^2 u}{\partial x^2} = \sigma \frac{\partial u}{\partial t} \qquad (6.4)$$

subject to the initial and boundary conditions

$$u = 0 \qquad \text{at} \qquad x = 0 \qquad t > 0 \qquad (6.5)$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=L} = \frac{-P_o}{E} \qquad t > 0 \qquad (6.6)$$

$$u = u_\tau + u_+ \qquad \text{at} \qquad t = 0 \qquad (6.7)$$

where from (5.4)

$$u_\tau = \frac{-P_o x}{E} - \frac{8P_o L}{E\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin\left[\frac{(2n-1)\pi x}{2L}\right] e^{-\frac{(2n-1)^2\pi^2 E\tau}{4\sigma L^2}}$$

and u_+ is given by (6.2)

6.3 Solution

The solution obtained (appendix 1, part 4) is

$$u(x,t) = \frac{-P_o x}{E} - \frac{8P_oL}{E\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n e^{-\frac{(2n-1)^2\pi^2E(\tau+t)}{4\sigma L^2}}}{(2n-1)^2}\,\sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

$$+\frac{1}{S_N}\sum_{n=1}^{N}\gamma_n\, e^{-\frac{(2n-1)^2\pi^2Et}{4\sigma L^2}}\,\sin\left[\frac{(2n-1)\pi x}{2L}\right] \quad (t>0)$$

(6.8)

where

$$\gamma_n = \left[\frac{1}{(2n-1)} - \frac{(2n-1)}{\left(\frac{4L^2}{\pi^2S^2}+(2n-1)^2\right)}\right] \quad (6.9)$$

This gives a pressure distribution

$$p(x,t) = P_o + \frac{4P_o}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\cos\left[\frac{(2n-1)\pi x}{2L}\right]e^{-\frac{(2n-1)^2\pi^2E(\tau+t)}{4\sigma L^2}}$$

$$-\frac{\pi E}{2LS_N}\sum_{n=1}^{N}(2n-1)\gamma_n\, e^{-\frac{(2n-1)^2\pi^2Et}{4\sigma L^2}}$$

$$(t>0) \quad (6.10)$$

The pressure build up subsequent to relaxation has been computed for the end A (x=o) and is plotted in figure (6.1) for S=3km. The parameter values used are shown on the figure. After relaxation the pressure can be seen to rebuild rapidly during the first year to within 90% of the critical value P_c which is likely to be somewhat lowered after the initial failure. Afterwards there follows a period of some 10.000 years of steadier increase. during which re-relaxation will occur - probably between one and ten years after initial relaxation.

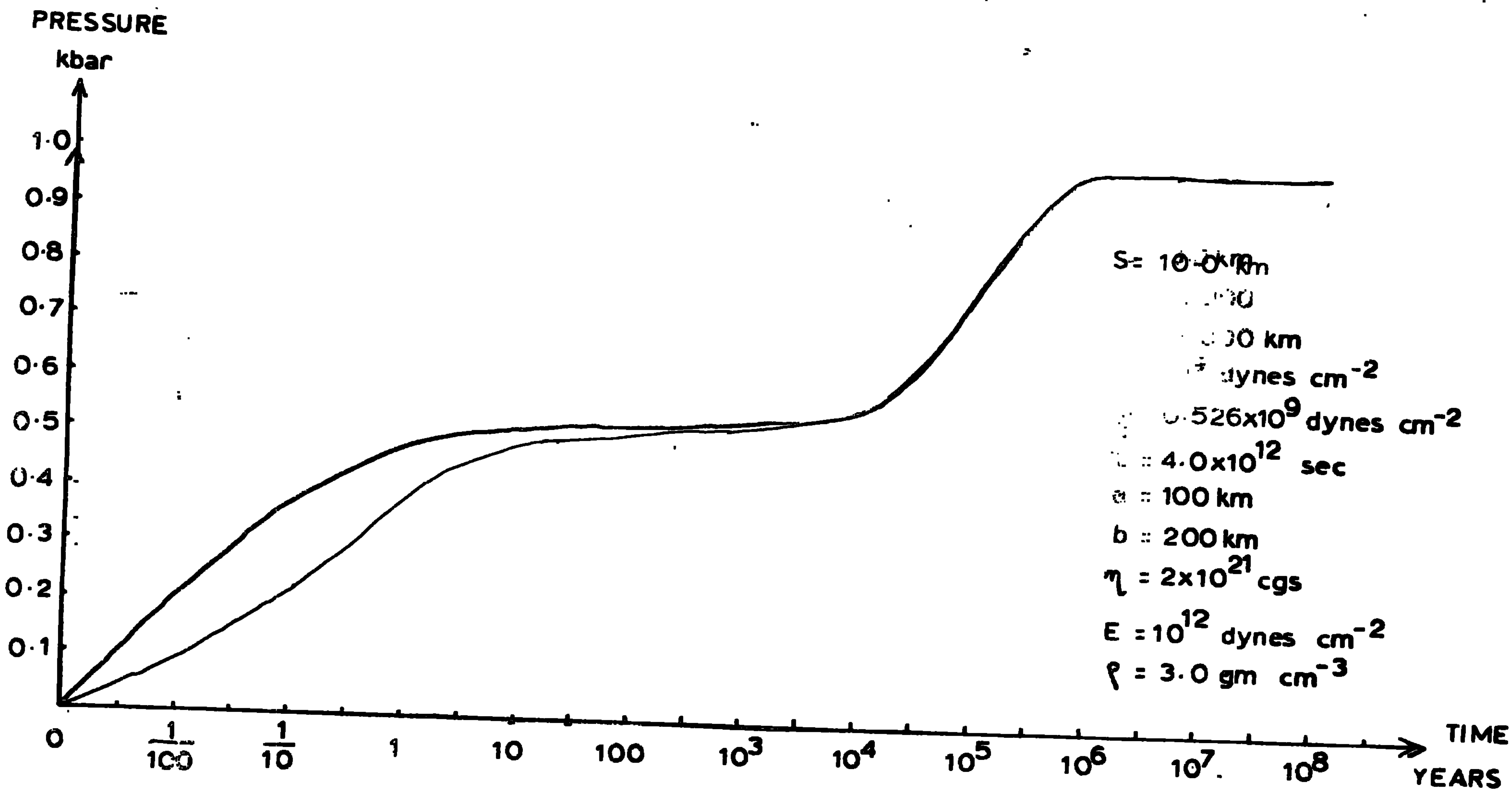

PRESSURE
kbar
1·0
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0
1/100
1/10
1
10
100
10^3
10^4
10^5
10^6
10^7
10^8
TIME
YEARS
S= 10·0 km
00 km
dynes cm^{-2}
526x10^9 dynes cm^{-2}
= 4·0x10^{12} sec
= 100 km
b = 200 km
η = 2x10^{21} cgs
E = 10^{12} dynes cm^{-2}
ρ = 3·0 gm cm^{-3}

FIG. 6.1

Because of the smallness of this time compared with τ ($\sim$ 300,000 years) the boundary and initial conditions after re-relaxation will be very similar to those after the initial failure and a cycle of pressure build-up and relaxation with a period of one to ten years will develop. This is illustrated schematically in figure (6.2)

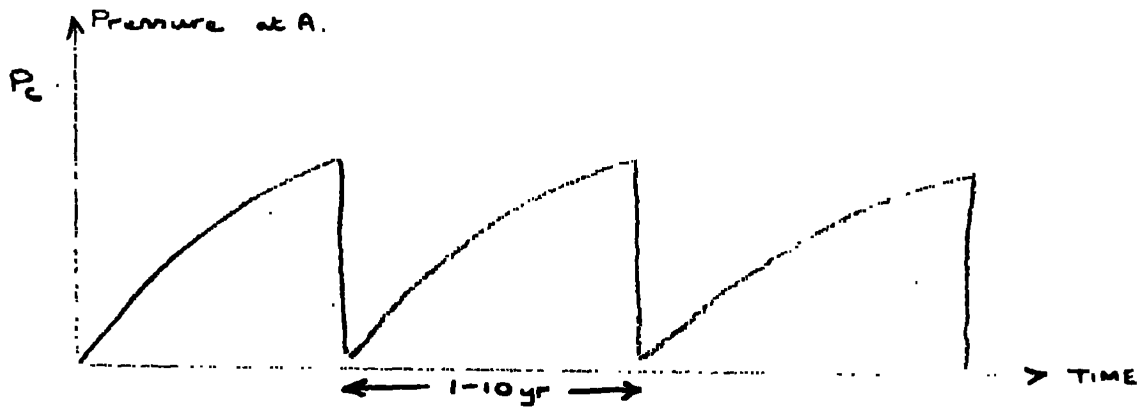


Figure 6.2

Pressure Build up - Relaxation Cycle.

The effect of increasing S i.e. the range of the instantaneous displacement is to slow down the initial build up of pressure. This will increase the period of the cycle in Fig. 6.2. The rebuilding of stress for S = 10 km. is plotted on the overlay of fig. 6.1. It should be noted that the time taken for the actual relaxation to take place has been totally ignored.

CHAPTER 7.

'DISCUSSION'

A new centre for the creation of oceanic lithosphere is formed by the splitting and forcing apart of a continent by the injection of basaltic dykes. As these dykes spread away from the source of basalt they cool to below the Curie point and are permanently magnetised in the direction of the earth's magnetic field. A new ocean basin begins to develop and the two newly formed continental margins, where oceanic and continental crust abut as part of the same lithospheric plate, begin to separate at a rate of a few centimeters per year. The continental shelves subside during this period and deep sedimentary basins are formed.

Sleep (1971) suggests that this subsidence is thermal contraction of the margin as it moves away from the centre of spreading. Bott (1971) suggests the oceanward creep of lower continental crust as a possible mechanism. The finite element calculations described in chapter 4 and illustrated in fig.4.3 show that initially the vertical pressure in the continental crust increases more rapidly with depth than in the adjacent oceanic crust. The greatest difference in pressures occurs at 10 km and persists to depths of 30 km. This is the region where ductile behaviour predominates and the layer within which the hot creep suggested by Bott occurs. The calculations of chapter four lend support to these ideas. The horizontal pressure, although it increases more rapidly with depth in the continental crust, does so at a lesser rate than the vertical pressure. Thus the conditions provided by gravitational forces give preferred normal faulting conditions on the continental side of the margin.

As the new ocean widens with further injection of material the relative abrupt transition from oceanic to continental crust becomes gradational with crustal thinning beneath the continent and thickening below the ocean. The accompanying subsidence results in the accumulation of shallow sediments on the continental shelf.

The finite element calculation (fig.4.4) shows how a horizontal regional tension is concentrated on the continental slope. It has already been argued that gravitational forces give conditions preferential to normal faulting on the continental as opposed to the oceanic side of the margin. Thus any normal faulting produced by regional tensions will be concentrated on the continental slope and in the adjacent continental crust. This is exactly in accordance with observation. Basins already formed as a result of subsidence followed by sedimentation will further refine the stress field locally. The finite element calculation performed in chapter 3 shows that this further refinement increases the likelihood of normal faulting in the crustal rocks immediately below the thickest sediments.

The normal faults will terminate at depths less than 10 km because rocks below this depth will deform by plastic flow as opposed to brittle fracture. The narrowing wedges of crust thus formed can sink into the ductile layer in isostatic response to the sediments above and deep basins are formed. Normal faulting in the oceanic crust adjacent to the margin is unlikely.

The process of subsidence, normal faulting in response to local tension and the formation of deep sedimentary basins may go on for 300 - 500 m.y. after which the ocean can expand no more. Ten thousand kilometers or so of new ocean will have been created in this time. New oceanic material continues to be added at the spreading centre and pressure begins to build up throughout the lithospheric plate resulting in the ultimate formation of a Beniof zone at the continental margin, where the oceanic plate underthrusts the continental plate. The model considered in chapter 5 was set up to investigate the characteristics of this build up in pressure. A time delay of some hundreds of thousands of years exists between the ocean ceasing to expand and the pressure at the continental margin reaching the same order of magnitude of that at which the dykes are being intruded at the spreading centre. The redistribution of this horizontal pressure by lateral changes in the elastic parameters across the margin can increase the stress differences on the slope by as much as 75% as discussed in chapter 4. This explains why thrusting is likely to be initiated in the region of the continental slope, with the subsequent underthrusting of the continental lithosphere by the oceanic lithosphere. If the pressure at which the dykes are intruded is large then thrusting could take place in the oceanic crust before the pressure has had time to propagate as far as the margin. In this case an island arc would develop.

Measurements of the energy released and the periodicity of earthquakes at regions of underthrusting (Beniof, 1955) give stress releases of around 10^{27} ergs at periods of 1 - 10 years. An explanation of these results can be obtained from the model considered in chapter 6. Dyke injection continues after thrusting and the pressure in the region of the thrust will start to rebuild. The discussion in chapter 6 gives an estimate in agreement with Beniof's observed time for the pressure to attain a value sufficient for further underthrusting. The calculations of chapter 6 give energy releases of 0.6×10^{27} ergs for each underthrusting for a length of 10,000 km. The earthquakes continue periodically until such a time as the source of oceanic material is depleted and injection ceases.

CHAPTER 8

Energy Absorbtion by Sediments.

Free vibration takes place when an elastic system vibrates under the action of forces inherent in the system itself, and in the absence of external forces. A system under free vibration will vibrate at one or more of its natural frequencies. Vibrations that take place under the excitation of external forces do so at the frequency of the exciting force. When this frequency coincides with one of the natural frequencies of the system dangerously high amplitudes may result. Thus the calculation of natural frequencies is of great interest.

8.1 General extension of finite element theory to dynamic problems.

(Zienkiewicz and Cheung, 1967 - note different approach)

For static elastic problems the equilibrium equation reduced to

$$\frac{\partial \tau_{ij}}{\partial x_j} + \rho p_i = 0 \qquad (1.11)$$

where p is the body force/unit mass. For dynamic problems the inertial terms must be included and the equation of motion

$$\frac{\partial \tau_{ij}}{\partial x_j} + \rho p_i = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad (8.1)$$

is the equivalent equation. The general arguments put forward in chapter 2 still hold good but when replacing the boundary forces acting on each element by statically equivalent nodal forces one must be careful to use (8.1) instead of (1.11).

Using the same notation as in section 2.2, we have

$$\{\delta^*\}^T\{F\} = \int_V \frac{\partial \tau_{ij}}{\partial x_j} f_i^* \, dv + \int_V \tau_{ij} e_{ij}^* \, dv$$

$$= -\int_V \rho p_i f_i^* \, dv + \int_V \rho f_i^* \frac{\partial^2 u_i}{\partial t^2} dv + \int_V \{\delta^*\}^T [B]^T [D][B]\{\delta\} dv$$

$$= \{\delta^*\}^T \left[-\rho \int_V [N]^T\{p\} dv + \rho \frac{\partial^2}{\partial t^2} \int_V [N]^T[N]\{\delta\} dv + \int_V [B]^T[D][B]\{\delta\} dv \right]$$

$$= [K]^e\{\delta\}^e + \frac{\partial^2}{\partial t^2}[M]^e\{\delta\}^e + \{F\}_p^e$$

where $[K]^e$, $\{F\}_p^e$ are definted in (2.10) (2.11) and

$$[M]^e = \int_e \rho [N]^T[N] \, dv \quad . \qquad (8.2)$$

Constructing 'overall' matrices $[K]$, $[M]$ and $\{F\}_p$ as in chapter 2 we have

$$[K]\{\delta\} + [M]\{\ddot{\delta}\} = \{F\} + \{F\}_p \quad . \qquad (8.3)$$

For a system under the influence of no external forces

$$[K]\{\delta\} = -[M]\{\ddot{\delta}\} \quad . \qquad (8.4)$$

When the model vibrates at a natural frequency all points vibrate in phase with a frequency $f = \omega/2\pi$

i.e. $$\{\delta\} = \{\delta_0\} e^{i\omega t} \quad . \qquad (8.5)$$

Equation (8.4) now reduces to

$$\underset{n\times n}{[K]}\ \underset{n\times 1}{\{\delta_o\}} = \omega^2\ \underset{n\times n}{[M]}\ \underset{n\times 1}{\{\delta_o\}} \qquad (8.6)$$

(8.6) has non trivial solutions $\{\delta\}_o^{(n)}$ for n values of $\omega = \omega_n$. These give the natural frequencies of free vibration of the system and the normal modes, $\{\delta\}_o^{(n)}$, give the relative nodal amplitudes of vibration at that frequency.

8.2 $[M]_e$ for the 3-noded triangular element with Rayleigh type response.

The only non zero displacement components are in the plane of the element i.e. u(x,y) and v(x,y). $[K]_e$ and $[N]$ are the same as in chapter 2, equations 2.16 and 2.22.

$$\int_e N_i N_j \, dx\, dy = \begin{cases} \dfrac{\Delta}{24} & i \neq j \\ \dfrac{\Delta}{12} & i = j \end{cases} \qquad (8.7)$$

(see Zienkiewicz & Cheung, 1967, p. 174).

Thus from the definition of $[M]_e$ (8.2) we have

$$[M]_e = \frac{\rho\Delta}{12} \begin{bmatrix} 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 1 \end{bmatrix} \qquad (8.8)$$

8.3 $[M]_e$ for Love type response

The only displacement is perpendicular to the plane of the model W(x,y).

Thus we define

$$\{\delta\}_e \equiv \begin{Bmatrix} \omega_i \\ \omega_j \\ \omega_k \end{Bmatrix} \qquad \text{(D8.1).}$$

$$\{\varepsilon\} \equiv \begin{Bmatrix} \frac{\partial \omega}{\partial x} \\ \frac{\partial \omega}{\partial y} \end{Bmatrix} \equiv \begin{Bmatrix} 2e_{xz} \\ 2e_{yz} \end{Bmatrix} \qquad \text{(D8.2)}$$

$$\{\sigma\} \equiv \begin{Bmatrix} \widehat{xz} \\ \widehat{yz} \end{Bmatrix} . \qquad \text{(D8.3)}$$

Therefore from 2.6 and 2.4

$$[D] = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} \qquad (8.9)$$

and

$$[N] = [N_i \ , \ N_j \ , \ N_k] \qquad (8.10)$$

where N_i etc. is defined in 2.16. Using 8.2. 8.10 and 8.7 we have

$$[M]_e = \frac{\rho\Delta}{12} \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{bmatrix} . \qquad (8.11)$$

From D8.1, D8.2 and D2.5

$$[B] = \frac{1}{\Delta}\begin{bmatrix}(y_j-y_k) & (y_k-y_i) & (y_i-y_j)\\ (x_k-x_j) & (x_i-x_k) & (x_j-x_i)\end{bmatrix} \qquad (8.12)$$

8.4 Imposition of constraints

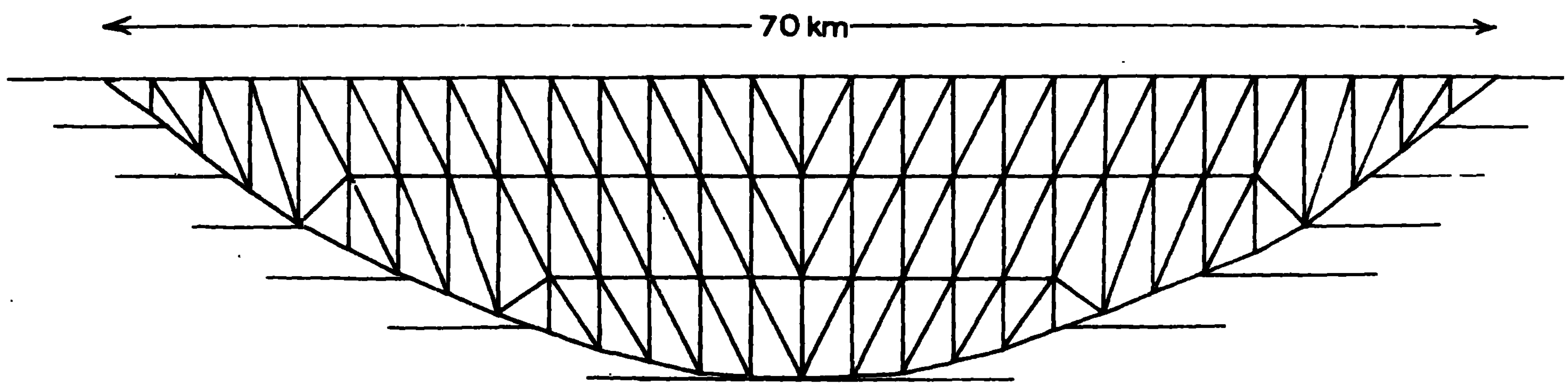
If a particular nodal point is fixed then the equations defining the equilibrium conditions there are eliminated. The number of normal modes of vibration is correspondingly reduced.

Consider as an example the i^{th} node being completely constrained for Rayleigh type excitation. Then both the $(2i)^{th}$ and $(2i-1)^{th}$ rows and columns of $[K]$ and $[M]$ are removed together with the $(2i)^{th}$ and $(2i-1)^{th}$ elements of $\{\delta\}_0$. The reduced set of equations give (2N-2) natural frequencies.

8.5 Natural frequencies of sedimentary basins

Computer programs have been written to make finite element calculations for the natural frequencies and normal modes of two-dimensional elastic systems for both Rayleigh and Love type responses. (Appendix 5 Programs NOMOFV and LOVE respectively).

These programs have been used to calculate the natural frequencies of sedimentary basins of various depths. The model used is shown in fig.8.1. The basin is considered constrained along its surface of contact with the basement rock. This assumption is valid when the contrast in rigidity between the sediments and the basement is large. In general there will be coupling between the two types of rock and energy will be lost



YOUNG'S MODULUS 0.327×10^{12} dynes cm^{-2}

POISSON'S RATIO 0.25

FIG. 8.1

from the basin. The width of the basin was fixed at 70 km. and its depth varied between 0 and 6 km. The lowest natural frequency or the fundamental frequency as it is often called, is the most important as this requires less energy to excite it than the higher natural frequencies. The variation of this frequency with the depth of basin is plotted in fig. 8.2 for both types of response For the Rayleigh response the fundamental frequency increases rapidly with decreasing basin depth to reach periods of 1 second at depths less than 1 km. For a deep basin of some 6 km. the fundamental frequency is of the order of 0.1 cycles/sec. The shape of the graph suggests a linear period-depth relation for a fixed basin width. This is verified by the plot shown in fig 8.3. In contrast the response to love waves shows only a small increase in frequency with decreasing basin depth. The fundamental periods range from 6.25 seconds to 8.7 seconds.

8.6 Discussion.

A Rayleigh wave of period T seconds travelling along the surface of a Poisson solid ($\lambda=\mu$) with a shear wave velocity β Km/sec. suffers a vertical (z direction) decrease in

(a) its horizontal amplitude proportional to

$$\exp\left(\frac{-2\pi \, . \, 0\cdot 85 z}{0\cdot 92 \beta T}\right) \qquad (8.13a)$$

and (b) its SV amplitude proportional to

$$\exp\left(\frac{-2\pi \, . \, 0\cdot 39 z}{0\cdot 92 \beta T}\right) \qquad (8.13b)$$

(Bullen, 1947)

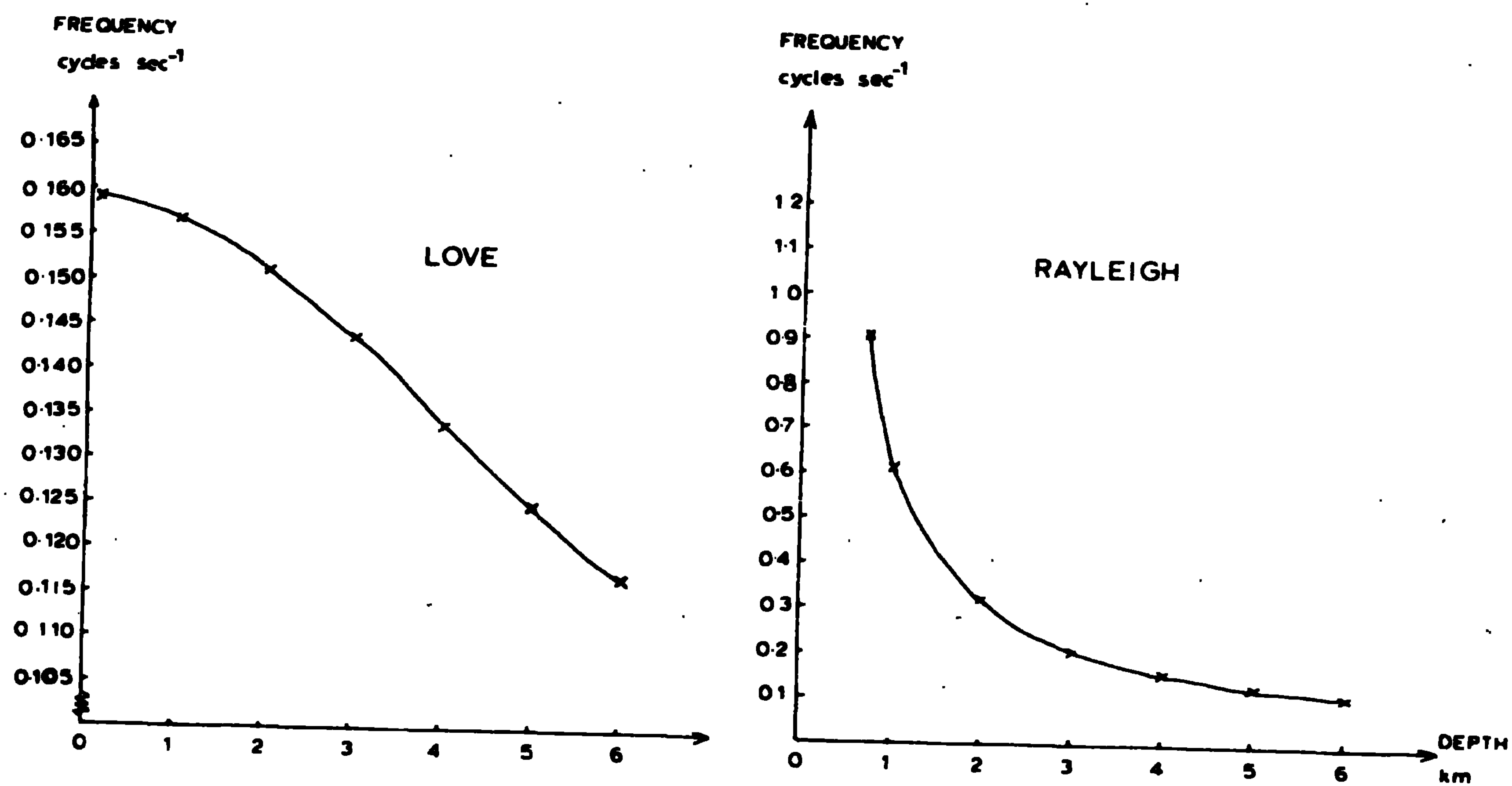

FREQUENCY
cycles sec⁻¹
0·165
0 160
0 155
0·150
0·145
0·140
0·135
0·130
0·125
0·120
0·115
0 110
0·105
LOVE
0
1
2
3
4
5
6
FREQUENCY
cycles sec⁻¹
1 2
1·1
1 0
0·9
0·8
0·7
0·6
0·5
0·4
0 3
0·2
0 1
RAYLEIGH
DEPTH
km

FIG. 8.2

Scaling the model of fig 8.1 by a factor K leaves the matrix [K] unchanged and the matrix [M] scaled by a factor K^2. Thus the fundamental period is scaled by a factor K (See equation 8.6). This suggests that while scaling the vertical dimensions to obtain Fig. 8.3 for Rayleigh response I have also scaled horizontal distances (inadvertently). D.S.D.

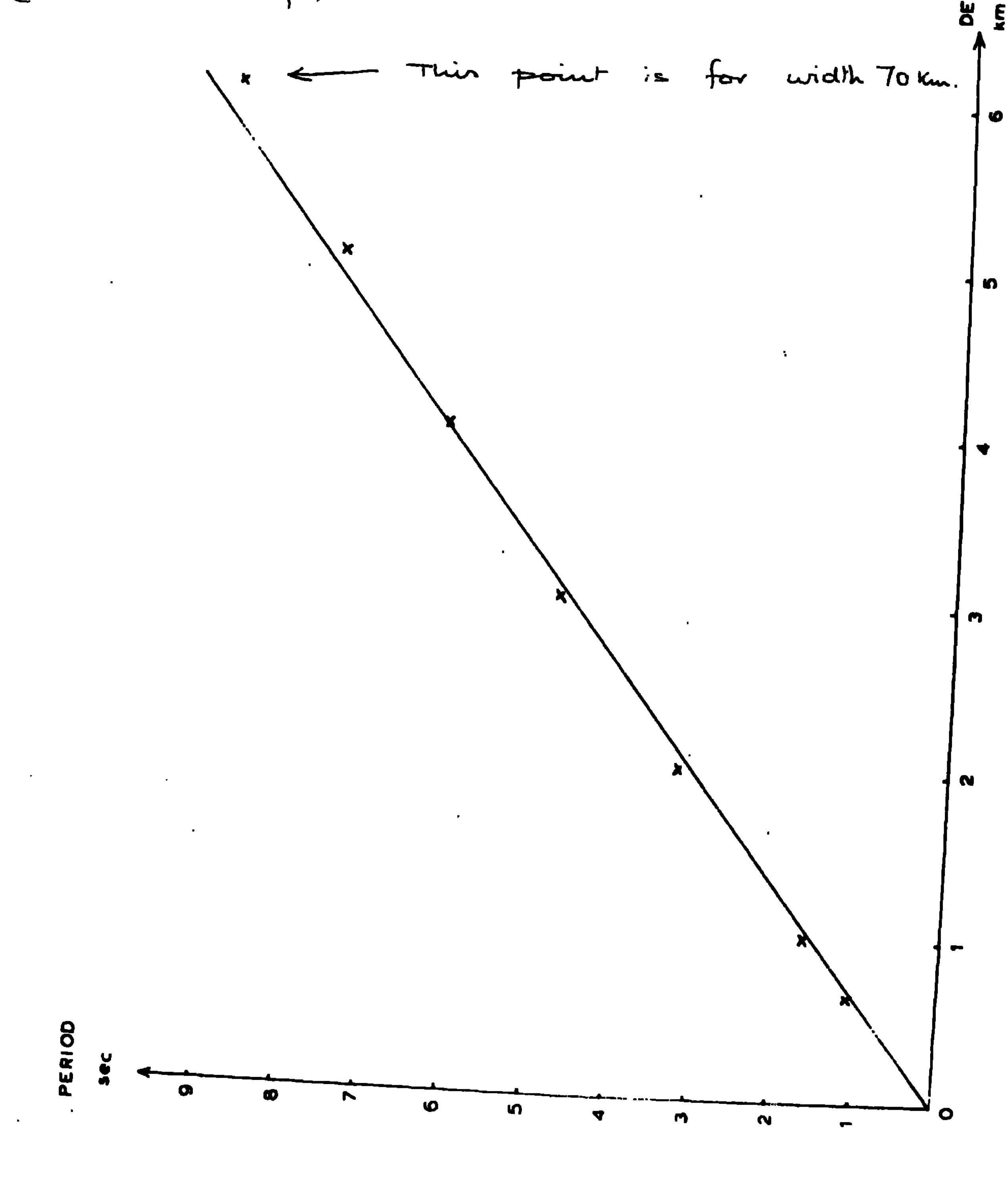


FIG. 8.3

For T = 10 and $\beta \sim 3.5$ the horizontal amplitude is decreased by a factor ($1/e$) by an increase in depth of 6 km. Thus the vertical extent of Rayleigh waves at this frequency is of the same order as the depth of the sedimentary basin that will resonate at this frequency (see fig. 8.2). The depth at which the Rayleigh wave amplitude reaches ($1/e$) of its surface value decreases linearly with period (equation 8.13). Fig. 8.3 shows a linear fall off of the fundamental period with decreasing depth of basin. Thus there is a direct linear correlation between the characteristic depth of a Rayleigh wave of given frequency and the depth of the sedimentary basin that resonates at that frequency.

The fundamental frequencies calculated are of the same order as those observed for surface waves generated by earthquakes. A deep sedimentary basin lying between the point of observation and the origin of surface waves will absorb energy at its fundamental frequencies. Thus 'holes' in the observed energy spectrum would be expected at these frequencies in the same manner that the Fraunhöfer lines in the sun's spectrum coincide with the resonant frequencies of the solar gases.

Given the right conditions (i.e. high energy content at the fundamental frequency) the resonant vibration of a sedimentary basin in the vicinity of the epicentre of an earthquake will concentrate damage within its boundaries.

CHAPTER 9.

'Dragging of the Lithosphere'

During a final year research seminar it was suggested that the problem of a plate being dragged along the surface of a viscous layer by tensional forces might be worth considering. This chapter presents a solution to this problem but makes no attempt to discuss its relevance.

The model considered is shown in fig.9.1. The tension, T_o, is applied at time t = o and maintained thereafter. Using the same notation as developed in chapter 5 and assuming a linear drop off of velocity with depth in the asthenosphere we wish to solve (see D5.1 and 5.2c)

$$E \frac{\partial^2 u}{\partial x^2} = \sigma \frac{\partial u}{\partial t} \qquad (9.1)$$

subject to

$$u = o \quad \text{at} \quad t = o \quad o \leq x \leq L \qquad (9.2)$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=o} = o \qquad t > o \qquad (9.3)$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=L} = \frac{T_o}{E} \qquad t > o \qquad (9.4)$$

Taking the laplace transform of the equation and its boundary conditions we have

$$E \frac{\partial^2 \bar{u}}{\partial x^2} = \sigma z \bar{u} \qquad (9.5)$$

subject to

$$\left(\frac{\partial \bar{u}}{\partial x}\right)_{x=o} = 0 \qquad (9.6)$$

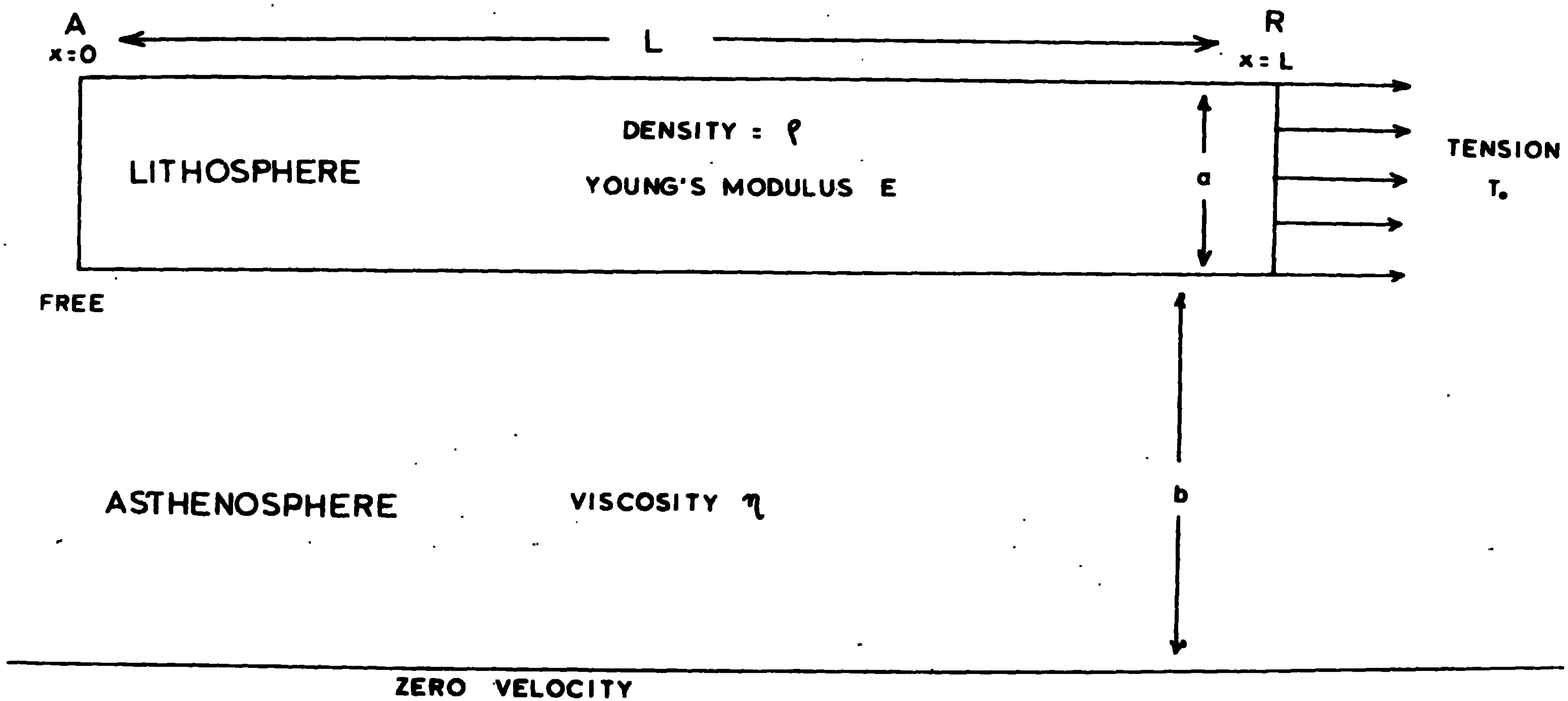

A
x=0
L
R
x=L
LITHOSPHERE
DENSITY = ρ
YOUNG'S MODULUS E
a
TENSION
T_o
FREE
ASTHENOSPHERE
VISCOSITY η
b
ZERO VELOCITY

FIG. 9.1

$$\left(\frac{\partial \bar{u}}{\partial x}\right)_{x=L} = \frac{T_o}{Ez} \quad (9.7)$$

The general solution of (9.5) is

$$\bar{u} = a \cosh\sqrt{\frac{\sigma z}{E}}\, x + b \sinh\sqrt{\frac{\sigma z}{E}}\, x$$

(9.6) $\Longrightarrow$ $b = o$

(9.7) $\Longrightarrow$ $a = T_o \Big/ \left(\sqrt{\sigma E}\; z^{3/2} \sinh\sqrt{\frac{\sigma z}{E}}\, L\right)$.

Thus

$$\bar{u} = \frac{T_o \cosh\sqrt{\frac{\sigma z}{E}}\, x}{\sqrt{E\sigma}\; z^{3/2} \sinh\sqrt{\frac{\sigma z}{E}}\, L} \quad (9.8)$$

Taking the inverse Laplace transform

$$u(x,t) = \frac{T_o}{2\pi i\sqrt{E\sigma}} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{zt} \cosh\sqrt{\frac{\sigma z}{E}}\, x}{z^{3/2} \sinh\sqrt{\frac{\sigma z}{E}}\, L}\, dz \quad (9.9)$$

Considering the integral around the contour in Fig.A4.1 we have poles at $z = o$ and where $\sinh\sqrt{\frac{\sigma z}{E}}\, L = 0$ i.e.

$$z = \frac{-n^2\pi^2 E}{\sigma L^2} \qquad n = 1, 2, \ldots$$

<u>Note:</u> 1. As before $z = o$ is not a branch point of the integrand.

2. The pole at the origin is <u>not simple.</u>

<u>Residue at $z = o$</u>

$$e^{zt}\bar{u} = \frac{f(z)}{z^2} \qquad \text{where}$$

$$f(z) = \frac{T_o e^{zt}}{\sigma} \left\{ \frac{1 + \frac{1}{\underline{|2}} \frac{\sigma x^2}{E} z + \frac{1}{\underline{|4}} \left(\frac{\sigma x^2}{E}\right)^2 z^2 + \cdots}{L + \frac{1}{\underline{|3}} \left(\frac{\sigma}{E}\right) L^3 z + \frac{1}{\underline{|5}} \left(\frac{\sigma}{E}\right)^2 L^5 z^2 + \cdots} \right\}.$$

Thus $\bar{u}e^{zt}$ has a second order pole at z = o with residue (Dennery and Krzywicki,1969) $\frac{1}{L}\left(\frac{df}{dz}\right)_{z=0}$

$$= \frac{T_0(3x^2-L^2)}{6LE} + \frac{T_0 t}{\sigma L} \qquad (9.10)$$

<u>Residue at $z = z_n = \frac{-n^2\pi^2 E}{\sigma L^2}$</u>.

Put $g(z) = e^{zt} z^{-3/2} \cosh\sqrt{\frac{\sigma z}{E}}\, x$

and $h(z) = \sinh\sqrt{\frac{\sigma z}{E}}\, L$.

Then

$$h'(z_n) = (-1)^n L^2 \sigma / 2iEn\pi \neq 0 .$$

Therefore pole at $z = z_n$ is simple for n = 1,2 3,. . ...
and has residue

$$= \frac{g(z_n)}{h'(z_n)}$$

$$= (-1)^{n+1} \frac{2L\sqrt{\frac{\sigma}{E}}}{n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2 Et}{\sigma L^2}} . \qquad (9.11)$$

If the integral around ABC of fig.A4.1 goes to zero as $r_m \to \infty$ then we have a solution

$$u(x,t) = \frac{T_0(3x^2-L^2)}{6LE} + \frac{T_0 t}{\sigma L} - \frac{2T_0 L}{\pi^2 E}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2 Et}{\sigma L^2}} \quad (t>0) \qquad (9.12)$$

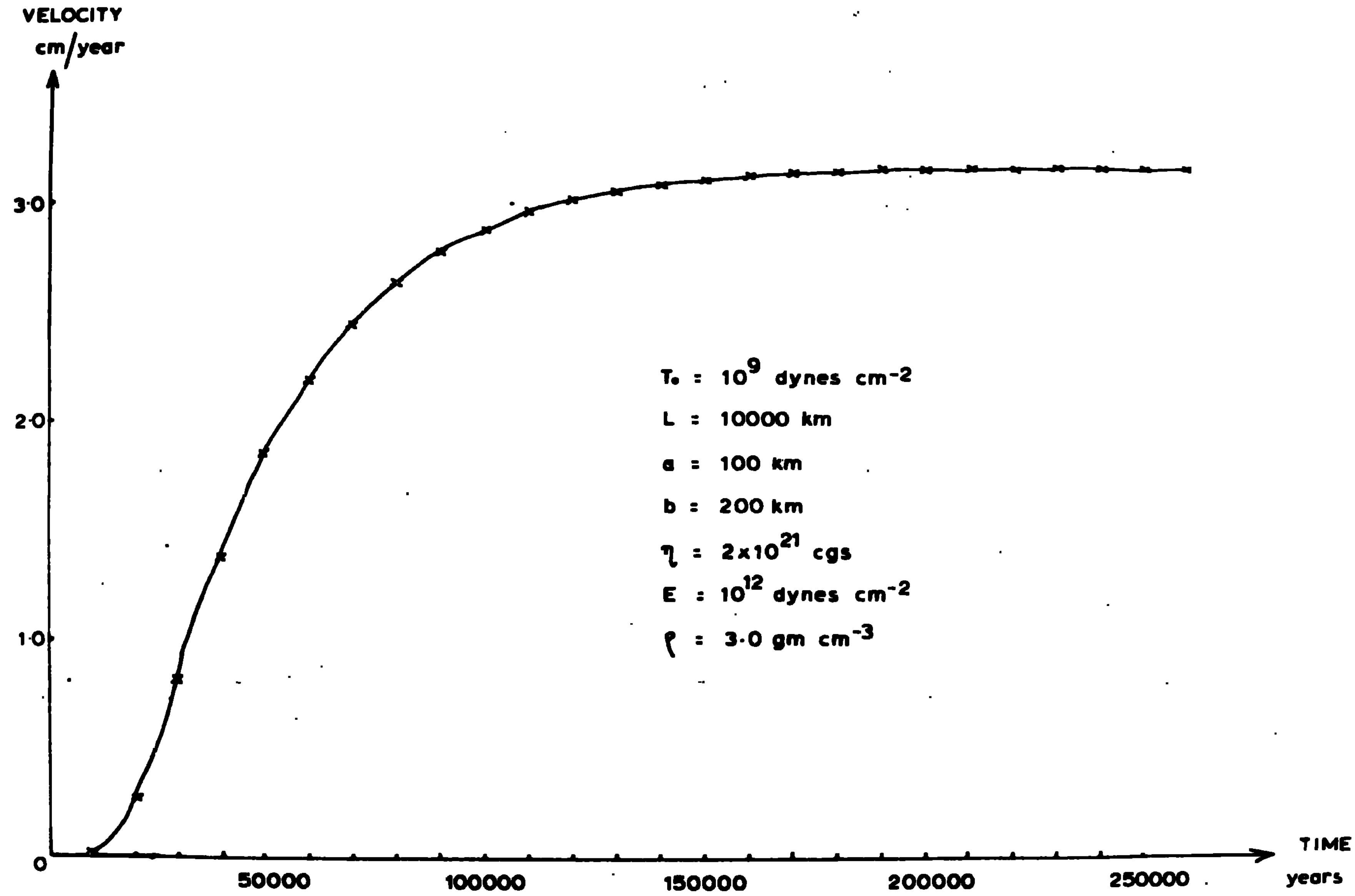

VELOCITY
cm/year
3·0
2·0
1·0
0
50000
100000
150000
200000
250000
TIME
years
T₀ : 10^9 dynes cm^-2
L : 10000 km
a : 100 km
b : 200 km
η : 2x10^21 cgs
E : 10^12 dynes cm^-2
ρ : 3·0 gm cm^-3

FIG. 9.2

giving a velocity distribution

$$v(x,t) = \frac{T_o}{\sigma L} + \frac{2T_o}{\sigma L}\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2 E t}{\sigma L^2}} \qquad (t>0) \tag{9.13}$$

These solutions have been verified by substitution into (9.1),(9.2),(9.3) and (9.4) this being somewhat simpler than proving the necessary condition on the integral around ABC. The velocity of the end A (x=o) is plotted in fig.9.2 for the parameter values shown. Some 10,000 years are seen to elapse before this end starts to move appreciably. This is followed by a period of some 50,000 years during which the velocity accelerates uniformly to over 2 cm/year. The acceleration then slows and the velocity tends asymptotically to a value of 3.15 cm/year. The plate now moves with this terminal velocity without further deformation and the tension drops off uniformly along the plate i.e. $T(x) \longrightarrow \frac{T_o x}{L}$.

Altering the parameters changes the axes and not the shape of the graph in fig.9.2. By examining the form of equation (9.13) it is seen that a return flow in the asthenosphere equivalent to an increment in viscosity by a factor 7 would increase the time axes by a factor of seven and divide the velocity axis by the same factor.

APPENDIX 1.

Cartesian Tensors.

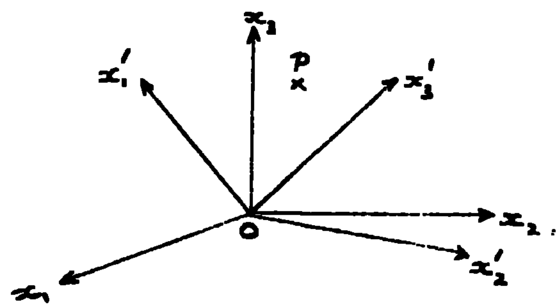
1. Reason

When the equations of mathematical physics are written out in full cartesian form the structural simplicity is often hidden by the mechanical labour of writing every term explicitly. The tensor notation, with the implied summation convention, is a form of shorthand leading to a great simplification of the writing. We use cartesian co-ordinates so that the distinction between covariant and contravariant tensors disappears.

2. Cartesian tensors of the first order (vectors)

If we have two sets of rectangular axes (Ox_1, Ox_2, Ox_3), (Ox_1', Ox_2', Ox_3') at the same origin.

Fig.A1.1



Let the co-ordinates of a point P be (x_1, x_2, x_3), (x_1', x_2', x_3') in each system respectively. If we denote the cosine of the angle between Ox_i and Ox_j' by l_{ij} then we can write

$$\begin{aligned} x_1' &= l_{11}x_1 + l_{21}x_2 + l_{31}x_3 \\ x_2' &= l_{12}x_1 + l_{22}x_2 + l_{32}x_3 \\ x_3' &= l_{13}x_1 + l_{23}x_2 + l_{33}x_3 \end{aligned} \qquad \text{(A1.1a)}$$

or

$$x_j' = \sum_{i=1}^{3} l_{ij}x_i \qquad \text{(A1.1b)}$$

let the co-ordinates of a point P be (x_1, x_2, x_3). $(x_1, x_2 \cdot x_3)$ in each system respectively. If we denote the cosine of the angle between Ox and Ox by then we can write

(A1.1a)

or (A1.1b)

Summation Convention: We make it a regular convention that whenever a suffix in a single term is repeated then the term is to be summed over that suffix from 1 to 3. Using this convention we can now write the last equation as

$$x_j' = l_{ij} x_i \tag{A1.1c}$$

The matrix $L = \{l_{ij}\}$ is in fact unitary i.e. $L^T = L^{-1}$ as can be seen by considering the length OP which equals $x_i^2 = x_j'^2$ (note implied sum).

$\therefore \; x_i^2 = l_{ij} x_i l_{kj} x_k$ using A1.1c. This is true for all points P

$$\therefore \; x_i^2 \equiv l_{ij} l_{kj} x_i x_k$$

and comparing coefficients

$$l_{ij} l_{kj} = \delta_{ik} \tag{A1.2a}$$

or

$$L L^T = I \tag{A1.2b}$$

in matrix notation where I is the unit matrix.

From A1.1c

$$l_{kj} x'_j = l_{kj} l_{ij} x_i$$

$$= \delta_{ik} x_i \qquad \text{from A1.2a}$$

$$= x_k \, .$$

Therefore the inverse of A1.1c is

$$x_k = l_{kj} x'_j \, . \qquad (A1.3)$$

In mathematical-physics one often has to deal with sets of three quantities (u_1, u_2, u_3) such that in relation to a different set of axes the corresponding quantities are (u'_1, u'_2, u'_3) and are related by

$$u'_j = l_{ij} \; u_i \qquad j = 1,2,3$$

and

$$u_i = l_{ij} \; u'_j \qquad i = 1,2,3 \, .$$

Such a set of three quantities is called a tensor of the first order (or commonly a vector). The individual u_1, u_2, u_3 are called the components of the first order tensor.

3. Cartesian tensors of the second order.

Suppose we have two first order tensors u_i and v_k. We can construct nine quantities $\varepsilon_{ik} = u_i v_k$. let us investigate how these quantities transform when expressed in our new axes $\varepsilon'_{ik} = u'_i v'_k = l_{ji} l_{mk} u_j v_m$

i.e.

$$\varepsilon'_{ik} = l_{ji} l_{mk} \varepsilon_{jm} \, . \qquad (A1.4)$$

A set of nine quantities that transform in this way on rotation of the co-ordinate axes is called a tensor of the second order (or just tensor). The individual ε_{ij} are called components. To obtain the inverse transformation we use Al.2a and Al.4 giving

$$\varepsilon_{np} = l_{pk} l_{ni} \varepsilon'_{ik} \quad . \qquad (A1.5)$$

APPENDIX 2.

1. Strain Tensor?

To prove that the components of strain do in fact constitute a tensor we must prove that they transform according to (A1.4) when we rotate our frame of reference. In the unprimed co-ordinate system the strain components are defined by D1.1

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

In the primed system we have from A1.3

$$\frac{\partial x_i}{\partial x'_k} = l_{ik}$$

and

$$u'_i = l_{ji}\, u_j$$

$$e'_{ij} = \frac{1}{2}\left(\frac{\partial u'_i}{\partial x'_j} + \frac{\partial u'_j}{\partial x'_i}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u'_i}{\partial x_k}\frac{\partial x_k}{\partial x'_j} + \frac{\partial u'_j}{\partial x_k}\frac{\partial x_k}{\partial x'_i}\right)$$

$$= \frac{1}{2}\left(l_{pi}\, l_{kj}\frac{\partial u_p}{\partial x_k} + l_{pj}\, l_{ki}\frac{\partial u_p}{\partial x_k}\right)$$

i.e.

$$e'_{ij} = l_{pi}\, l_{kj}\, e_{pk} \qquad \text{(A2.1)}$$

Comparing with (A1.4) we see that $\{e_{ij}\}$ are in fact the components of a second order tensor.

2. Compatibility of Strains

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$\therefore \quad e_{ij,kl} = \frac{1}{2}\left(u_{i,jkl} + u_{j,ikl}\right)$$

and

$$e_{kl,ij} = \frac{1}{2}\left(u_{k,lij} + u_{l,kij}\right)$$

Adding

$$e_{ij,kl} + e_{kl,ij} = \tfrac{1}{2}\left(u_{i,jkl} + u_{j,ikl} + u_{k,lij} + u_{l,kij}\right).$$

Interchanging j and k we have

$$e_{ik,jl} + e_{jl,ik} = \tfrac{1}{2}\left(u_{i,jkl} + u_{j,ikl}\right) + \tfrac{1}{2}\left(u_{k,lij} + u_{l,kij}\right).$$

Thus

$$e_{ik,jl} + e_{jl,ik} = e_{ij,kl} + e_{kl,ij} \qquad \text{(A2.2a)}$$

(A2.2a) are the compatibility equations and number 81. Some are identically satisfied and others repeated. Only six are independent these being:

$$\frac{\partial^2 e_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z} - \frac{\partial e_{yz}}{\partial x}\right)$$

$$\frac{\partial^2 e_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x} - \frac{\partial e_{zx}}{\partial y}\right)$$

$$\frac{\partial^2 e_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} - \frac{\partial e_{xy}}{\partial z}\right) \qquad \text{(A2.2b)}$$

$$2\frac{\partial^2 e_{yx}}{\partial x \partial y} = \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2}$$

$$2\frac{\partial^2 e_{yz}}{\partial z \partial y} = \frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2}$$

$$2\frac{\partial^2 e_{zx}}{\partial z \partial x} = \frac{\partial^2 e_{zz}}{\partial x^2} + \frac{\partial^2 e_{xx}}{\partial z^2}.$$

These restrictions on the strain tensor ensure that it gives single-valued continuous displacements.

3. Rigid Body Displacements.

The change in length δA, of a vector $\underline{A}$ (length A) under the deformation (1.1) is given by

$$(A+\delta A)^2 = (A_i + \delta A_i)^2$$

or

$$A\,\delta A = A_i\,\delta A_i \qquad \text{(A2.3)}$$

ignoring small terms of second order. For rigid body displacements $\delta A = 0$ for all vectors $\underline{A}$. Therefore, from (1.1)

$$A_i\,\delta A_i = u_{i,j}\,A_i\,A_j = 0$$

Taking $\underline{A}$ in the xy-plane i.e. $(A_3 = 0,\ A_1 \neq 0,\ A_2 \neq 0)$ we have

$$u_{1,2} = -u_{2,1}\,.$$

Similarly $u_{1,3} = -u_{3,1}$ and $u_{2,3} = -u_{3,2}$.

Thus a necessary and sufficient condition for the transformation (1.1,1.2) to represent a rigid body movement is

$$u_{i,j} = -u_{j,i} \quad (\text{i.e. } e_{ij} = 0).$$

Thus $\{\omega_{ij}\}$ represents the rigid body part of the transformation. Thus for pure deformation (1.1) becomes

$$\delta A_i = e_{ij}\,A_j\,. \qquad \text{(A2.4)}$$

4. Physical Significance of the Strain Components

(a) Diagonal Terms:

From (A2.4) and (1.2) we have

$$A\delta A = A_i \delta A_i = (e_{ij} + \omega_{ij}) A_i A_j .$$

For pure deformation $\omega_{ij} = 0$ and therefore $A\delta A = e_{ij} A_i A_j$.

Now consider a vector $\underline{A}$ initially along the x_κ axis.

We have

$$A\delta A = e_{\kappa\kappa} A_\kappa^2 \qquad \text{(Not summed)}$$

or

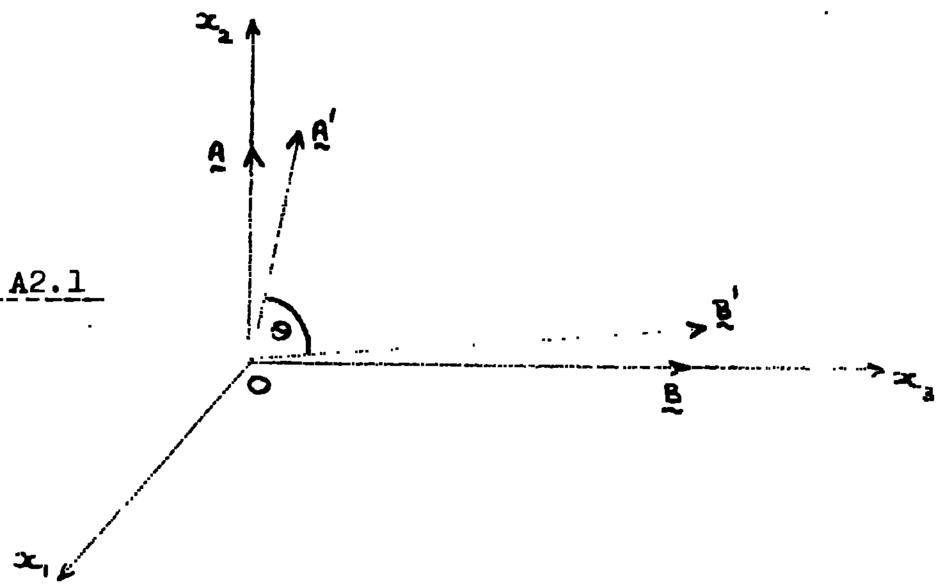
$$e_{\kappa\kappa} = \frac{\delta A}{A_\kappa} .$$

That is $e_{\kappa\kappa}$ represents the change in length per unit length of a vector originally parallel to the x_κ axis.

(b) Off-diagonal Terms:

Consider e_{23} . Take two vectors $\underline{A} = A_2 \hat{\underline{e}}_2$ and $\underline{B} = B_3 \hat{\underline{e}}_3$ initially directed along the x_2 and x_3, axes respectively. Upon deformation these become $\underline{A}'$ and $\underline{B}'$. (Fig. A2.1).

Fig. A2.1



$$\underline{A}' \doteq \delta A_1 \hat{\underline{e}}_1 + (A_2 + \delta A_2)\hat{\underline{e}}_2 + \delta A_3 \hat{\underline{e}}_3$$

$$\underline{B}' = \delta B_1 \hat{\underline{e}}_1 + \delta B_2 \hat{\underline{e}}_2 + (B_3 + \delta B_3)\hat{\underline{e}}_3 .$$

The angle $A'OB' = \theta$ is given by

$$A'B' \cos\theta = \underline{A}'.\underline{B}' = A_2 \delta B_2 + B_3 \delta A_3$$

i.e.

$$\cos\theta = \frac{A_2 \delta B_2 + B_3 \delta A_3}{A_2 B_2}$$

(to 1st order).

$$= \frac{\delta B_2}{B_3} + \frac{\delta A_3}{A_2} .$$

From (A2.4)

$$\delta A_3 = e_{23} A_2$$

$$\delta B_2 = e_{23} B_3 .$$

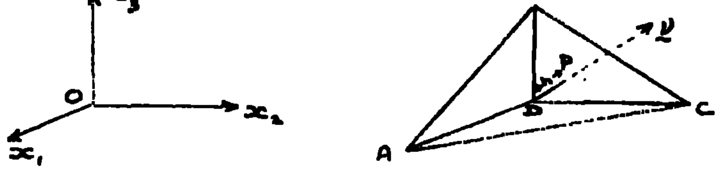
Therefore

$$\cos\theta = \sin\left(\tfrac{\pi}{2} - \theta\right) \doteq \tfrac{\pi}{2} - \theta = 2e_{23} .$$

Thus $2e_{23}$ is equal to the change in angle between two vectors originally parallel to the axes. (x_2, x_3).

5. <u>To show that $\{\tau_{ij}\}$ i,j = 1,2,3 fully defines the stress at a point</u>.

To do this we must show how the stress vector $\underline{R}_\nu$ (or its components) at a point P can be expressed in terms of the stress components $\{\tau_{ij}\}$ at P and the direction cosines ν_i of the surface $\underline{\nu}$. We consider the equilibrium of a small tetrahedron under the action of surface tractions and body forces F_i per unit mass. (Fig.A2.2)



(Fig.A2.2)

Force on face ABC // el to i = $\tau_{\nu i}\,\delta s$

Force on face BDC // el to i = $-\tau_{1i}\,\nu_1\,\delta s$

Force on face BDA // el to i = $-\tau_{2i}\,\nu_2\,\delta s$

Force on face ADC // el to i = $-\tau_{3i}\,\nu_3\,\delta s$

Body force // el to i = $\frac{1}{3}\rho h F_i\,\delta s$.

Therefore for equilibrium

$$(\tau_{\nu i} - \tau_{ji}\nu_j + \tfrac{1}{3}\rho h F_i)\,\delta s = 0.$$

Letting $h \to 0$ we have

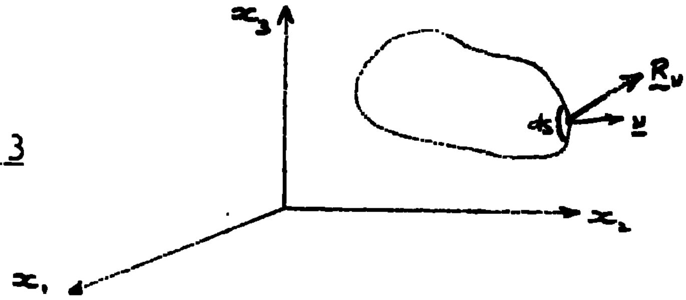
$$\tau_{\nu i} = \tau_{ji}\,\nu_j. \qquad \text{(A2.5)}$$

Thus $\{\tau_{ij}\}$ fully defines the stress at P.

6. Equilibrium of Stresses.

Consider an arbitrary volume V fixed in space (note: not a fixed volume of material). (Fig.A2.3) Let $\rho(x)$ be its density and $\underline{F}$ the body force per unit mass

Fig.A2.3



The total force in i-direction is

$$\int_V F_i \rho\, dv + \int_S \underline{R}_\nu \cdot \hat{\underline{i}}\, ds$$

$$= \int_V \rho F_i\, dv + \int_S \tau_{ji}\,\nu_j\, ds \qquad \text{(from A2.5)}$$

$$= \int_V \rho F_i\, dv + \int_S (\tau_{ji}\,\hat{j}) \cdot \underline{ds}$$

$$= \int_V \rho F_i \, dV + \int_V \frac{\partial \tau_{ji}}{\partial x_j} \, dV .$$

(Divergence Theorem)

Therefore for equilibrium

$$\int_V \left(\rho F_i + \frac{\partial \tau_{ji}}{\partial x_j} \right) dV = 0 .$$

The volume V was chosen arbitrarily, thus

$$\rho F_i + \frac{\partial \tau_{ji}}{\partial x_j} = 0 . \qquad \text{(A2.6)}$$

The moment of these forces the x_i axis must also vanish

i.e.

$$\int_V \varepsilon_{ijk} x_j F_k \rho \, dV + \int_S \varepsilon_{ijk} (\underline{P}_\nu)_k x_j \, dS = 0$$

or from (A2.5)

$$\int_V \varepsilon_{ijk} x_j F_k \rho \, dV + \int_S \varepsilon_{ijk} \tau_{lk} \nu_l x_j \, dS = 0$$

or using Divergence theorem

$$\int_V \varepsilon_{ijk} x_j F_k \rho \, dV + \int_V \frac{\partial (\varepsilon_{ijk} x_k \tau_{lk})}{\partial x_l} \, dV = 0$$

or

$$\int_V \varepsilon_{ijk} x_j F_k \rho \, dV + \int_V \varepsilon_{ijk} \left[x_j \frac{\partial \tau_{lk}}{\partial x_l} + \delta_{jl} \tau_{lk} \right] dV$$

or using A2.6

$$\int_V \varepsilon_{ijk} \delta_{jl} \tau_{lk} \, dV = 0 .$$

The volume V was chosen arbitrarily therefore,

$$\varepsilon_{ijk} \tau_{jk} = 0$$

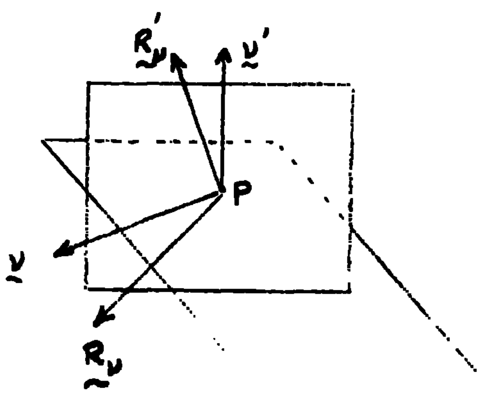
or $$\tau_{jk} = \tau_{kj} \, . \qquad \text{(A2.7)}$$

Hence the stress components are symmetric.

7. $\{\tau_{ij}\}$ a tensor?

Consider two planes with respective normals both containing the point P (Fig.A2.4)

Fig.A2.4



$\underline{R}_\nu$ and $\underline{R}_{\nu'}$ are the respective stress vectors at P. The component of $\underline{R}_\nu$ in the direction $\underline{\nu}'$ is

$$\underline{R}_\nu . \underline{\nu}' = (\underline{R}_\nu)_j \, \nu'_j$$

$$= \tau_{ij}\, \nu_i\, \nu_j' \qquad \text{from (A2.5)}$$

$$= \tau_{ji}\, \nu_j'\, \nu_i \qquad \text{from (A2.7)}$$

$$= (\underline{R}_{\nu'})_i\, \nu_i \qquad \text{from (A2.5)}$$

i.e.

$$\underline{R}_{\nu} \,.\, \underline{\nu}' = \underline{R}_{\nu'} \,.\, \underline{\nu} \,. \qquad \text{(A2.8)}$$

Now consider effect of rotation of axes on $\{\tau_{ij}\}$.

In primed axes

$$\tau_{ij}' = \text{comp. on surface} \perp \hat{\underline{x}}_i' \text{ along } \hat{\underline{x}}_j'$$

$$= (\underline{R}_{x_i'}) \,.\, \hat{\underline{x}}_j'$$

$$= (\underline{R}_{x_i'})_k\, l_{kj} \qquad \text{from (A1.1c)}$$

$$= \tau_{mk}\, l_{mi}\, l_{kj}. \qquad \text{from (A2.5)}$$

Thus

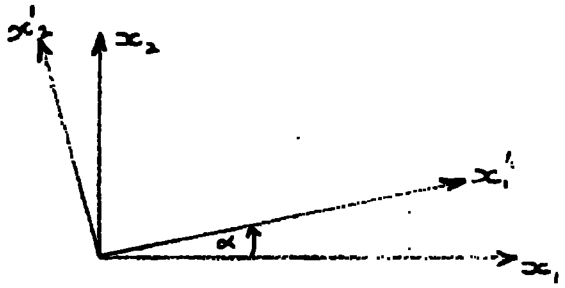
$$\tau_{ij}' = l_{mi}\, l_{kj}\, \tau_{mk} \qquad \text{(A2.9)}$$

and $\{\tau_{ij}\}$ constitute the components of a second order tensor.

8. Principal Axes.

Consider the case where $\tau_{13} = \tau_{23} = 0$. Rotate the axes through an angle α. (Fig.A2.5)

Fig.A2.5.



We have $l_{ij} = \cos(x_j' O x_i)$ therefore $l_{13} = l_{23} = l_{31} = l_{32} = 0$, $l_{12} = -\sin\alpha$, $l_{11} = \cos\alpha$, $l_{21} = \sin\alpha$, $l_{22} = \cos\alpha$, $l_{33} = 1$.

$$\therefore \quad \tau_{11}' = l_{m1} l_{k1} \tau_{mk} = \tau_{11}\cos^2\alpha + \tau_{22}\sin^2\alpha + 2\sin\alpha\cos\alpha\,\tau_{12}$$

$$\tau_{12}' = l_{m1} l_{k2} \tau_{mk} = -\tau_{11}\cos\alpha\sin\alpha + \tau_{22}\cos\alpha\sin\alpha + \tau_{12}(\cos^2\alpha - \sin^2\alpha)$$

$$\tau_{13}' = 0$$

$$\tau_{22}' = \tau_{11}\sin^2\alpha + \tau_{22}\cos^2\alpha - 2\tau_{12}\cos\alpha\sin\alpha$$

$$\tau_{23}' = 0$$

$$\tau_{33}' = \tau_{33} .$$

Thus if the axes are rotated through an angle α given by

$$(\tau_{22} - \tau_{11})\cos\alpha\sin\alpha + \tau_{12}(\cos^2\alpha - \sin^2\alpha) = 0$$

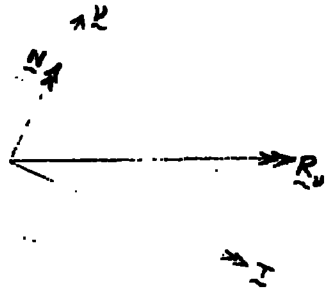
we have the principal axes since $\tau_{ij}' = 0 \quad i \neq j$.

i.e. $$\alpha = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{12}}{\tau_{11} - \tau_{22}}\right). \qquad \text{(A2.10)}$$

9. Greatest Shearing Stress.

Suppose we have found the principal axes and that the stress tensor referred to these axes are τ_{11}, τ_{22} and τ_{33}. We wish to determine the plane $\underline{\nu}$ which has the greatest shear stress across it (Fig.A2.6). The stress vector $\underline{R}_\nu$

Fig.A2.6



can be split into a normal component ($\underline{N}$) and tagential component($\underline{T}$.). (A2.5) gives $(\underline{R}_\nu)_1 = \tau_{11}\nu_1$, $(\underline{R}_\nu)_2 = \tau_{22}\nu_2$ and $(\underline{R}_\nu)_3 = \tau_{33}\nu_3$.

Hence $$|\underline{R}_\nu|^2 = \tau_{11}^2\nu_1^2 + \tau_{22}^2\nu_2^2 + \tau_{33}\nu_3^2$$

and $$N = \underline{R}_\nu . \hat{\underline{\nu}} = \tau_{11}\nu_1^2 + \tau_{22}\nu_2^2 + \tau_{33}\nu_3^2 .$$

Therefore

$$T^2 = \tau_{11}^2\nu_1^2 + \tau_{22}^2\nu_2^2 + \tau_{33}^2\nu_3^2 - (\tau_{11}\nu_1^2 + \tau_{22}\nu_2^2 + \tau_{33}\nu_3^2)^2.$$

We wish to maximise T subject to $\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$

i.e.

$$T^2 = \tau_{11}^2\nu_1^2 + \tau_{22}^2\nu_2^2 + \tau_{33}^2(1 - \nu_1^2 - \nu_2^2)$$

$$- (\tau_{11}\nu_1^2 + \tau_{22}\nu_2^2 + \tau_{33}[1 - \nu_1^2 - \nu_2^2])^2.$$

Putting $\frac{\partial T^2}{\partial \nu_1} = \frac{\partial T^2}{\partial \nu_2} = 0$ we have

$$\nu_1 (\tau_{11} - \tau_{33}) \left[(\tau_{11} - \tau_{33}) - 2\nu_1^2 (\tau_{11} - \tau_{33}) - 2\nu_2^2 (\tau_{22} - \tau_{33}) \right] = 0$$

and

$$\nu_2 (\tau_{22} - \tau_{33}) \left[(\tau_{22} - \tau_{33}) - 2\nu_1^2 (\tau_{11} - \tau_{33}) - 2\nu_2^2 (\tau_{22} - \tau_{33}) \right] = 0 .$$

Solutions

$$\nu_1 = \nu_2 = 0 , \quad \nu_3 = \pm 1$$

$$\nu_1 = \nu_3 = 0 , \quad \nu_2 = \pm 1$$

$$\nu_2 = \nu_3 = 0 , \quad \nu_1 = \pm 1$$

give planes of minimum (zero) shear.

Solutions

$$\nu_1 = 0 , \quad \nu_2 = \pm \frac{1}{\sqrt{2}} , \quad \nu_3 = \pm \frac{1}{\sqrt{2}}$$

$$\nu_2 = 0 , \quad \nu_1 = \pm \frac{1}{\sqrt{2}} , \quad \nu_3 = \pm \frac{1}{\sqrt{2}}$$

$$\nu_3 = 0 , \quad \nu_1 = \pm \frac{1}{\sqrt{2}} , \quad \nu_2 = \pm \frac{1}{\sqrt{2}}$$

give planes with respective shears

$$\tfrac{1}{2}(\tau_{22} - \tau_{33}) , \quad \tfrac{1}{2}(\tau_{11} - \tau_{33}) , \quad \tfrac{1}{2}(\tau_{11} - \tau_{22}) .$$

Thus the greatest shearing stress occurs across a plane that bisects the angle between the greatest and least principal stresses and in which the intermediate stress lies. The magnitude of the shear is equal to one half the difference between the maximum and minimum principal stresses.

10. The Lamé Constants

The most general linear relationship between stress and strain involves 21 constants. For an isotropic material i.e. one which has the same elastic properties in all directions these constants are inter-dependant and can be reduced to 2 independant constants λ and μ known as the Lamé constants.

For an isotropic body we can write the linear relation as

$$\tau_{11} = Ae_{11} + A'(e_{22}+e_{33}) + Ce_{23} + C'(e_{12}+e_{13})$$

$$\tau_{22} = Ae_{22} + A'(e_{11}+e_{33}) + Ce_{13} + C'(e_{21}+e_{23})$$

$$\tau_{33} = Ae_{33} + A'(e_{11}+e_{22}) + Ce_{12} + C'(e_{31}+e_{32})$$

$$\tau_{12} = De_{33} + D'(e_{11}+e_{22}) + Be_{12} + B'(e_{13}+e_{23})$$

$$\tau_{13} = De_{22} + D'(e_{11}+e_{33}) + Be_{13} + B'(e_{12}+e_{23})$$

$$\tau_{23} = De_{11} + D'(e_{22}+e_{33}) + Be_{23} + B'(e_{12}+e_{13}).$$

If we reverse the x_1 axis then e_{12}, e_{13}, τ_{12}, τ_{13} change their sign, while e_{11} and τ_{11} stay the same. Therefore $C = C' = 0$ and $B' = D = D' = 0$. Thus we have

$$\tau_{11} = Ae_{11} + A'(e_{22}+e_{33}) \qquad \text{etc.}$$

and $\tau_{12} = Be_{12}$ etc.

If we now rotate our axes through an angle θ about the x_3 axis (Fig.A2.5) then from section 8 we have

$$\tau'_{12} = \sin\theta\cos\theta\,(\tau_{22}-\tau_{11}) + (\cos^2\theta - \sin^2\theta)\,\tau_{12}$$

i.e. $$\tau'_{12} = \sin\theta\cos\theta\,(A-A')(e_{22}-e_{11}) + (\cos^2\theta-\sin^2\theta)\,B e_{12}. \quad \text{——(A)}$$

Using (A2.1)

$$e'_{12} = \sin\theta\cos\theta\,(e_{22}-e_{11}) + e_{12}(\cos^2\theta - \sin^2\theta).$$

Therefore

$$\tau'_{12} = B\left[\sin\theta\cos\theta\,(e_{22}-e_{11}) + e_{12}(\cos^2\theta-\sin^2\theta)\right]. \quad \text{——(B)}$$

True for all values of θ, therefore, comparing coefficients in A and B

$$A - A' = B.$$

If we write $B = 2\mu$ and $A' = \lambda$ we have $A = 2\mu + \lambda$ and the stress-strain relation becomes

$$\tau_{ij} = \lambda\,\delta_{ij}\,\Delta + 2\mu\,e_{ij}.$$

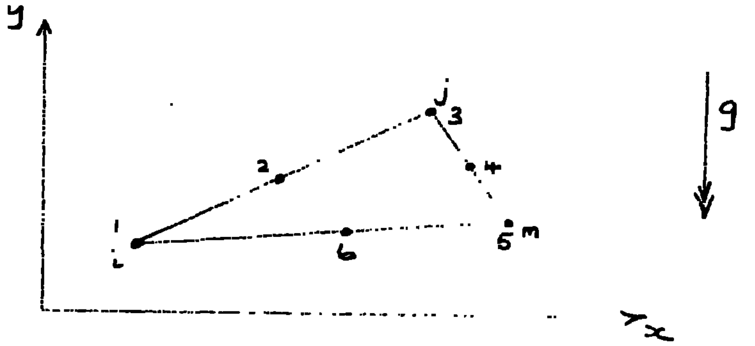
where

$$\Delta \equiv e_{11} + e_{22} + e_{33} = e_{kk}.$$

APPENDIX 3.

1. 6-Noded Triangular Elements.

Fig. A3.1



[N] Matrix.

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = [N]^e \{\delta\}^e.$$

There are two constraints at each of the six nodes. Therefore we can use twelve constants to define the displacement throughout e.

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$$
$$v(x,y) = \alpha_7 + \alpha_8 x + \alpha_9 y + \alpha_{10} x^2 + \alpha_{11} xy + \alpha_{12} y^2$$

or in matrix form

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{12} \end{Bmatrix}.$$

2 x 1   2 x 12   12 x 1

Define

$$[P] = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{bmatrix}. \qquad (A3.1)$$

We also have

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 + \alpha_4 x_1^2 + \alpha_5 x_1 y_1 + \alpha_6 y_1^2$$

$$v_1 = \alpha_7 + \alpha_8 x_1 + \alpha_9 y_1 + \alpha_{10} x_1^2 + \alpha_{11} x_1 y_1 + \alpha_{12} y_1^2$$

etc.

or

$$\{\delta\}^e = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_6 \\ v_6 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 \\ & & & & & \text{etc.} & & & & & & \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{12} \end{Bmatrix}.$$

Define

$$[C] \equiv \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 \\ & & & & & \text{etc.} & & & & & & \end{bmatrix}. \qquad (A3.2)$$

Thus

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = [P][C]^{-1}\{\delta\}^e \qquad (A3.3)$$

or

$$[N] = [P][C]^{-1}. \qquad (A3.4)$$

B Matrix.

$$\{\varepsilon\} = [B]\{\delta\}^e.$$

From D2.3 and D1.1 and using A3.3 we have

$$\{\varepsilon\} = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix} [C]^{-1} \{\delta\}^e$$

or

$$[B] = [B'] [C]^{-1} \qquad \text{(A3.5)}$$

where

$$[B'] \equiv \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix}. \qquad \text{(A3.6)}$$

<u>[K]^e Matrix</u>

from 2.10

$$[K]^e = \int_e \left([C]^{-1}\right)^T [B']^T [D] [B'] [C]^{-1} dx\,dy$$

or

$$[K]^e = \left([C]^{-1}\right)^T \left\{ \int_e [B']^T [D] [B'] dx\,dy \right\} [C]^{-1}.$$

(Note: In program (Appendix 5) $[S] = [C]^{-1}$)

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & D_{11} & 0 & 2xD_{11} & yD_{11} & 0 & 0 & 0 & D_{12} & 0 & xD_{12} & 2yD_{12} \\
0 & 0 & D_{33} & 0 & xD_{33} & 2yD_{33} & 0 & D_{33} & 0 & 2xD_{33} & yD_{33} & 0 \\
0 & 2xD_{11} & 0 & 4x^2D_{11} & 2xyD_{11} & 0 & 0 & 0 & 2xD_{12} & 0 & 2x^2D_{12} & 4xyD_{12} \\
0 & yD_{11} & xD_{33} & 2xyD_{11} & \begin{matrix} y^2D_{11} \\ +x^2D_{33} \end{matrix} & 2xyD_{33} & 0 & xD_{33} & yD_{12} & 2x^2D_{33} & xy(D_{33}+D_{12}) & 2y^2D_{12} \\
0 & 0 & 2yD_{33} & 0 & 2xyD_{33} & 4y^2D_{33} & 0 & 2yD_{33} & 0 & 4xyD_{33} & 2y^2D_{33} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & D_{33} & 0 & xD_{33} & 2yD_{33} & 0 & D_{33} & 0 & 2xD_{33} & 2yD_{33} & 0 \\
0 & D_{21} & 0 & 2xD_{21} & yD_{21} & 0 & 0 & 0 & D_{22} & 0 & xD_{22} & 2yD_{22} \\
0 & 0 & 2xD_{33} & 0 & 2x^2D_{33} & 4xyD_{33} & 0 & 2xD_{33} & 0 & 4x^2D_{33} & 2xyD_{33} & 0 \\
0 & xD_{21} & yD_{33} & 2x^2D_{21} & xy(D_{21}+D_{33}) & 2y^2D_{33} & 0 & yD_{33} & xD_{22} & 2xyD_{33} & \begin{matrix} x^2D_{22}+ \\ y^2D_{33} \end{matrix} & 2xyD_{22} \\
0 & 2yD_{21} & 0 & 4xyD_{21} & 2y^2D_{21} & 0 & 0 & 0 & 2yD_{22} & 0 & 2xyD_{22} & 4y^2D_{22}
\end{bmatrix}
\qquad (A3.7)
$$

Fig A3.2 $[B']^T[D][B']$

Thus to evaluate $[K]^e$ we must evaluate

$$\int_e x\,dx\,dy = \bar{x}\,\frac{|\Delta|}{2}$$

$$\int_e y\,dx\,dy = \bar{y}\,\frac{|\Delta|}{2}$$

$$I_1 = \int_e x^2\,dx\,dy \qquad (A3.8)$$

$$I_2 = \int_e xy\,dx\,dy$$

$$I_3 = \int_e y^2\,dx\,dy$$

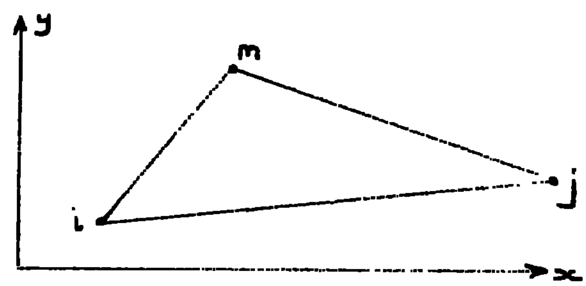
where Δ is given by D2.10. Using these definitions $[K]^e$ becomes

$$[K]^e = \left([C]^{-1}\right)^T [KL]^e [C]^{-1} \qquad (A3.9)$$

where $[KL]^e$ is given by Fig.A3.3. (see over)

<u>Calculation of I_1, I_2, I_3</u>

Fig.A3.4



The equation of the line joining nodes i and m is

$y = A_{im}x + B_{im}$ provided $x_i \neq x_m$

where

$$A_{im} = \frac{y_m - y_i}{x_m - x_i}$$

and

$$B_{im} = \frac{y_i x_m - y_m x_i}{x_m - x_i} \qquad x_i \neq x_m \qquad (A3.10a)$$

$$
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{D_{11}|\Delta|}{2} & 0 & \bar{x}|\Delta|D_{11} & \bar{y}\frac{|\Delta|}{2}D_{11} & 0 & 0 & 0 & \frac{|\Delta|}{2}D_{12} & 0 & \bar{x}\frac{|\Delta|}{2}D_{12} & \bar{y}|\Delta|D_{12} \\
0 & 0 & \frac{|\Delta|}{2}D_{33} & 0 & \bar{x}\frac{|\Delta|}{2}D_{33} & \bar{y}|\Delta|D_{33} & 0 & \frac{|\Delta|}{2}D_{33} & 0 & \bar{x}|\Delta|D_{33} & \bar{y}\frac{|\Delta|}{2}D_{33} & 0 \\
0 & \bar{x}|\Delta|D_{11} & 0 & 4I_1D_{11} & 2I_2D_{11} & 0 & 0 & 0 & \bar{x}|\Delta|D_{12} & 0 & 2I_1D_{12} & 4I_2D_{12} \\
0 & \bar{y}\frac{|\Delta|}{2}D_{11} & \bar{x}\frac{|\Delta|}{2}D_{33} & 2I_2D_{11} & \begin{matrix} I_3D_{11} \\ +I_1D_{33} \end{matrix} & 2I_2D_{33} & 0 & \bar{x}\frac{|\Delta|}{2}D_{33} & \bar{y}\frac{|\Delta|}{2}D_{12} & 2I_1D_{33} & I_2(D_{33}+D_{12}) & 2I_3D_{12} \\
0 & 0 & \bar{y}|\Delta|D_{33} & 0 & 2I_2D_{33} & 4I_3D_{33} & 0 & \bar{y}|\Delta|D_{33} & 0 & 4I_2D_{33} & 2I_3D_{33} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{|\Delta|}{2}D_{33} & 0 & \bar{x}\frac{|\Delta|}{2}D_{33} & \bar{y}|\Delta|D_{33} & 0 & \frac{|\Delta|}{2}D_{33} & 0 & \bar{x}|\Delta|D_{33} & \bar{y}\frac{|\Delta|}{2}D_{33} & 0 \\
0 & \frac{|\Delta|}{2}D_{21} & 0 & \bar{x}|\Delta|D_{21} & \bar{y}\frac{|\Delta|}{2}D_{21} & 0 & 0 & 0 & \frac{|\Delta|}{2}D_{22} & 0 & \bar{x}\frac{|\Delta|}{2}D_{22} & \bar{y}|\Delta|D_{22} \\
0 & 0 & \bar{x}|\Delta|D_{33} & 0 & 2I_1D_{33} & 4I_2D_{33} & 0 & \bar{x}|\Delta|D_{33} & 0 & 4I_1D_{33} & 2I_2D_{33} & 0 \\
0 & \bar{x}\frac{|\Delta|}{2}D_{21} & \bar{y}\frac{|\Delta|}{2}D_{33} & 2I_1D_{21} & I_2(D_{21}+D_{33}) & 2I_3D_{33} & 0 & \bar{y}\frac{|\Delta|}{2}D_{33} & \bar{x}\frac{|\Delta|}{2}D_{22} & 2I_2D_{33} & \begin{matrix} I_1D_{22}+ \\ I_3D_{33} \end{matrix} & 2I_2D_{22} \\
0 & \bar{y}|\Delta|D_{21} & 0 & 4I_2D_{21} & 2I_3D_{21} & 0 & 0 & 0 & \bar{y}|\Delta|D_{22} & 0 & 2I_2D_{22} & 4I_3D_{22}
\end{matrix}
$$

Fig. A3.3

Therefore,

$$\iint_i^m x^2\,dx\,dy = \int_{x_i}^{x_m} x^2(A_{im}x + B_{im})\,dx = \frac{A_{im}}{4}\left(x_m^4 - x_i^4\right) + \frac{B_{im}}{3}\left(x_m^3 - x_i^3\right)$$

$$\iint_i^m xy\,dx\,dy = \int_{x_i}^{x_m} \frac{x(A_{im}x + B_{im})^2}{2}\,dx$$

$$= \frac{A_{im}^2}{8}\left(x_m^4 - x_i^4\right) + \frac{B_{im}^2}{4}\left(x_m^2 - x_i^2\right) + \frac{B_{im}A_{im}}{3}\left(x_m^3 - x_i^3\right)$$

and

$$\iint_i^m y^2\,dx\,dy = \int_{x_i}^{x_m} \frac{(A_{im}x + B_{im})^3\,dx}{3}$$

$$= \frac{A_{im}}{12}\left(x_m^4 - x_i^4\right) + \frac{A_{im}^2 B_{im}}{3}\left(x_m^3 - x_i^3\right) + \frac{A_{im}B_{im}^2}{2}\left(x_m^3 - x_i^3\right) + \frac{B_{im}}{3}\left(x_m - x_i\right).$$

When $x_i = x_m$ these integrals are zero and we can therefore define

$$A_{im} \equiv B_{im} \equiv 0 \qquad x_i = x_m . \qquad \text{(A3.10b)}$$

Therefore, provided $i \rightarrow j \rightarrow m$ goes anticlockwise around the element e in Fig.A3.4 (i.e. $\Delta > 0$) we have

$$I_1 = \frac{1}{4}\left\{A_{im}(x_m^4 - x_i^4) + A_{ji}(x_i^4 - x_j^4) + A_{mj}(x_j^4 - x_m^4)\right\}$$

$$+ \frac{1}{3}\left\{B_{im}(x_m^3 - x_i^3) + B_{ji}(x_i^3 - x_j^3) + B_{mj}(x_j^3 - x_m^3)\right\}.$$

If $i \rightarrow j \rightarrow m$ clockwise (i.e. $\Delta < 0$) then I_1 is the negative of this answer. Thus in all cases we have

$$I_1 = \left(\frac{\Delta}{|\Delta|}\right) \times \Big[\frac{1}{4}\left\{A_{im}(x_m^4 - x_i^4) + A_{ji}(x_i^4 - x_j^4) + A_{mj}(x_j^4 - x_m^4)\right\}$$

$$+ \frac{1}{3}\left\{B_{im}(x_m^3 - x_i^3) + B_{ji}(x_i^3 - x_j^3) + B_{mj}(x_j^3 - x_m^3)\right\}\Big] . \qquad \text{(A3.11)}$$

Similarly,

$$I_2 = \left(\frac{\Delta}{|\Delta|}\right)\left[\frac{1}{8}\left\{A_{im}^2(x_m^4-x_i^4) + A_{ji}^2(x_i^4-x_j^4) + A_{mj}(x_j^4-x_m^4)\right\}\right.$$

$$+ \frac{1}{4}\left\{B_{im}^2(x_m^2-x_i^2) + B_{ji}^2(x_i^2-x_j^2) + B_{mj}^2(x_j^2-x_m^2)\right\}$$

$$\left. + \frac{1}{3}\left\{B_{im}A_{im}(x_m^3-x_i^3) + B_{ji}A_{ji}(x_i^3-x_j^3) + B_{mj}A_{mj}(x_j^3-x_m^3)\right\}\right] \quad \text{(A3.12)}$$

$$I_3 = \left(\frac{\Delta}{|\Delta|}\right)\left[\frac{1}{12}\left\{A_{im}^3(x_m^4-x_i^4) + A_{ji}^3(x_i^4-x_j^4) + A_{mj}(x_j^4-x_m^4)\right\}\right.$$

$$+ \frac{1}{3}\left\{A_{im}^2B_{im}(x_m^3-x_i^3) + A_{ji}^2B_{ji}(x_i^3-x_j^3) + B_{mj}A_{mj}^2(x_j^3-x_m^3)\right\}$$

$$+ \frac{1}{2}\left\{A_{im}B_{im}^2(x_m^2-x_i^2) + A_{ji}B_{ji}^2(x_i^2-x_j^2) + A_{mj}B_{mj}^2(x_j^2-x_m^2)\right\}$$

$$\left. + \frac{1}{3}\left\{B_{im}^3(x_m-x_i) + B_{ji}^3(x_i-x_j) + B_{mj}^3(x_j-x_m)\right\}\right] \quad \text{(A3.13)}$$

$\{F\}_p^e$ due to gravity.

From 2.11 and A3.4 we have

$$\{F\}_p^e = -\rho g \int_e \left([C]^{-1}\right)^T [P]^T \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} dx\,dy$$

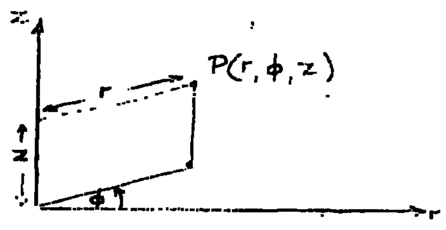
which reduces to using A3.1

$$\{F\}_p^e = -\rho g \left([C]^{-1}\right)^T \int_e \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{Bmatrix} dx\,dy$$

$$= -g\rho\left([C]^{-1}\right)^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ |\Delta|/2 \\ \bar{x}\,|\Delta|/2 \\ \bar{y}\,|\Delta|/2 \\ I_1 \\ I_2 \\ I_3 \end{bmatrix} \qquad (A3.14)$$

2. Axi-symmetric Analysis.

Fig. A3.5

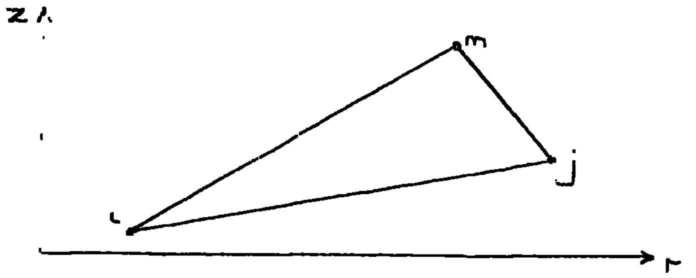


We take the Z-axis as the axis of symmetry i.e. the problem is independent of ϕ .

Matrix $[N]^e$

Using the 3-noded triangular element, the analysis is the same as in Chapter 2 with Z replacing y and r replacing y.

Fig. A3.6



$$[N]^e = [I N_i', \; I N_j', \; I N_m']$$

where

$$N_i' = \frac{1}{\Delta}(a_i + b_i r + c_i z)$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$a_i = r_j z_m - r_m z_j \qquad \text{(A3.15)}$$

$$b_i = z_j - z_m$$

$$c_i = r_m - r_j$$

$$\Delta = \begin{vmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{vmatrix}.$$

Matrix $[B]^e$

The expressions for the components of strain in cylindrical co-ordinates are given by 1.8. For axi-symmetrical problems $\frac{\partial}{\partial\phi} = u_\phi = 0$ and it follows that the non zero components are $e_{rr} = \frac{\partial u_r}{\partial r}$, $e_{\phi\phi} = \frac{u_r}{r}$, $e_{zz} = \frac{\partial u_z}{\partial z}$ and $2e_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$. We thus define

$$\{\varepsilon\} \equiv \begin{Bmatrix} e_{zz} \\ e_{rr} \\ e_{\phi\phi} \\ 2e_{rz} \end{Bmatrix} = \begin{Bmatrix} \partial v/\partial z \\ \partial u/\partial r \\ u/r \\ \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \end{Bmatrix} \qquad \text{(A3.16)}$$

where we have written $u_r = u$ and $u_z = v$. Using A3.15, A3.16, and 2.4 we have

$$\{\varepsilon\} = \frac{1}{\Delta}\left[A_i, A_j, A_m\right]\{\delta\}^e \qquad \text{(A3.17)}$$

where

$$A_i = \begin{bmatrix} 0 & c_i \\ b_i & 0 \\ \frac{a_i}{r} + b_i + \frac{zc_i}{r} & 0 \\ c_i & b_i \end{bmatrix} \qquad \text{(A3.18)}$$

etc.

Comparing with 2.5 we have

$$[B]^e = \frac{1}{\Delta}\left[A_i, A_j, A_m\right] \qquad \text{(A3.19)}$$

Matrix $[D]^e$

From Hooke's Law 1.13 we have

$$\widehat{zz} = \lambda\,(e_{rr} + e_{zz} + e_{\phi\phi}) + 2\mu e_{zz}$$
$$\widehat{rr} = \lambda\,(e_{rr} + e_{zz} + e_{\phi\phi}) + 2\mu e_{rr}$$
$$\widehat{\phi\phi} = \lambda\,(e_{rr} + e_{zz} + e_{\phi\phi}) + 2\mu e_{\phi\phi}$$
$$\widehat{rz} = 2\mu e_{rz}$$

or in matrix form

$$\{\sigma\}^e = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix}\{\varepsilon\}^e$$

where

$$\{\sigma\}^e \equiv \begin{Bmatrix} \widehat{zz} \\ \widehat{rr} \\ \widehat{\phi\phi} \\ \widehat{rz} \end{Bmatrix}. \qquad (A3.20)$$

Comparing with 2.6 we have

$$[D]^e = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \qquad (A3.21a)$$

or using 1.18 and 1.19

$$[D]^e = \frac{E(1-\eta)}{(1+\eta)(1-2\eta)} \begin{bmatrix} 1 & \eta/1-\eta & \eta/1-\eta & 0 \\ \eta/1-\eta & 1 & \eta/1-\eta & 0 \\ \eta/1-\eta & \eta/1-\eta & 1 & 0 \\ 0 & 0 & 0 & \frac{1-2\eta}{2(1-\eta)} \end{bmatrix}. \qquad (A3.21b)$$

<u>Matrix $[K]^e$</u>

From 2.10

$$[K]^e = 2\pi \int_e [B]^{e^T} [D]^e [B]^e \, r \, dr \, dz .$$

If we write $[B] = [\bar{B}] + [B']$ where

$$[\bar{B}] = \frac{1}{\Delta} [\bar{A}_i , \bar{A}_j , \bar{A}_m] \qquad (A3.22)$$

$$[B'] = \frac{1}{\Delta} [A'_i , A'_j , A'_m] \qquad (A3.23)$$

$$\bar{A}_i = \begin{bmatrix} 0 & c_i \\ b_i & 0 \\ \frac{a_i}{\bar{r}} + b_i + \frac{\bar{z}}{\bar{r}} c_i & 0 \\ c_i & b_i \end{bmatrix} \qquad (A3.24)$$

etc.

$$A_i' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{a_i}{r} + \frac{z}{r}c_i - \frac{a_i}{\bar{r}} - \frac{\bar{z}}{\bar{r}}c_i & 0 \\ 0 & 0 \end{bmatrix} \qquad (A3.25)$$

$$\bar{r} = \frac{1}{3}(r_i + r_j + r_m)$$

$$\bar{z} = \frac{1}{3}(z_i + z_j + z_m)$$

then we have

$$[K]^e = 2\pi[\bar{B}]^T[D][\bar{B}]\int_e r\,dr\,dz + 2\pi\int_e [B']^T[D][B']r\,dr\,dz$$

$$+ 2\pi[\bar{B}]^T[D]\int_e [B']r\,dr\,dz + 2\pi\int_e [B']^T r\,dr\,dz\,[D][\bar{B}].$$

Using the fact that $\int_e r\,dr\,dz = \frac{\bar{r}|\Delta|}{2}$ and $\int_e z\,dr\,dz = \frac{\bar{z}|\Delta|}{2}$

we have

$$\int_e [B']r\,dr\,dz = 0 = \int_e [B']^T r\,dr\,dz.$$

Therefore $[K]^e$ reduces to

$$[K]^e = \pi\bar{r}|\Delta|[\bar{B}]^T[D][\bar{B}] + 2\pi\int_e [B']^T[D][B']r\,dr\,dz. \qquad (A3.26)$$

If we let

$$2\pi\int_e [B']^T[D][B']r\,dr\,dz \equiv \begin{bmatrix} K_{ii}' & K_{ij}' & K_{im}' \\ K_{ji}' & K_{jj}' & K_{jm}' \\ K_{mi}' & K_{mj}' & K_{mm}' \end{bmatrix}$$

then

$$[K_{rs}'] = \frac{2\pi}{\Delta^2}\begin{bmatrix} D_{33} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\times \int_e r\left(\frac{a_r}{r} + \frac{z}{r}c_r - \frac{a_r}{\bar{r}} - \frac{\bar{z}}{\bar{r}}c_r\right)\left(\frac{a_s}{r} + \frac{z}{r}c_s - \frac{a_s}{\bar{r}} - \frac{\bar{z}}{\bar{r}}c_s\right)dr\,dz$$

Putting

$$I_1 = \frac{1}{|\Delta|} \int_e \frac{1}{r}\, dr\, dz$$

$$I_2 = \frac{1}{|\Delta|} \int_e \frac{z}{r}\, dr\, dz \qquad (A3.27)$$

$$I_3 = \frac{1}{|\Delta|} \int_e \frac{z^2}{r}\, dr\, dz$$

and noting that $\int_e dr\,dz = \frac{1}{2}|\Delta|$ we have

$$\left[K'_{rs}\right] = \frac{\pi}{|\Delta|} \begin{bmatrix} D_{33} & 0 \\ 0 & 0 \end{bmatrix} \times \left\{ a_r a_s \left(2I_1 - \frac{1}{\bar{r}}\right) + (a_s c_r + a_r c_s)\left(2I_2 - \frac{\bar{z}}{\bar{r}}\right) + c_r c_s \left(2I_3 - \frac{\bar{z}^2}{\bar{r}}\right)\right\}.$$

(A3.28)

Calculation of I_1, I_2, I_3

Referring to Fig.A3.6 the equation of the line joining node i to node m is

$$z = A_{im}\, r + B_{im}$$

where

$$A_{im} = \frac{z_m - z_i}{r_m - r_i} \qquad (A3.29a)$$

and

$$B_{im} = \frac{(z_i r_m - r_i z_m)}{r_m - r_i}$$

provided $r_m \neq r_i$.

$\underline{I_1}$

$$\int_{r=r_i}^{r_m}\int_{z=0}^{A_{im}r+B_{im}} \frac{1}{r}\,dr\,dz = \int_{r_i}^{r_m} \frac{B_{im}+A_{im}r}{r}\,dr$$

$$= B_{im}\log_e\frac{r_m}{r_i} + (z_m - z_i) \quad . \qquad \text{(A3.30)}$$

Therefore, provided $i \rightarrow m \rightarrow j$ is clockwise around the element e (i.e. $\Delta > 0$)

$$\int_e \frac{1}{r}\,dr\,dz = (B_{ji} - B_{im})\log_e r_i + (B_{mj} - B_{ji})\log_e r_j + (B_{im} - B_{mj})\log_e r_m$$

(and the negative of this result if $i \rightarrow m \rightarrow j$ anticlockwise i.e. $\Delta < 0$).

Therefore in general ($\Delta \lessgtr 0$)

$$I_1 = \frac{1}{\Delta}\Big[(B_{ji} - B_{im})\log_e r_i + (B_{mj} - B_{ji})\log_e r_j$$

$$+ (B_{im} - B_{mj})\log_e r_m\Big] \quad . \qquad \text{(A3.31)}$$

Note 1 If $r_i = r_m$ then the integral in A3.30 is zero, the equivalent of which is obtained by putting

$$A_{im} = B_{im} = 0 \quad , \quad r_m = r_i \quad . \qquad \text{(A3.29b)}$$

Note 2 When $r_i = 0$ I is not defined by A3.3. Zero X infinity term. But

$$\lim_{r_i \rightarrow 0}\Big[(B_{ji} - B_{im})\log r_i\Big] = 0 \qquad \text{by}$$

L'Hopital's rule.

$\underline{I_2}$

$$\int_{r_i}^{r_m}\int_{z=0}^{A_{im}r+B_{im}} \frac{z}{r}\,dr\,dz = \frac{1}{2}\int_{r_i}^{r_m} \frac{1}{r}\left(A_{im}r + B_{im}\right)^2 dr$$

$$= \frac{1}{2}\left[B_{im}^2 \log_e \frac{r_m}{r_i} + \frac{1}{2}A_{im}^2\left(r_m^2 - r_i^2\right) + 2A_{im}B_{im}\left(r_m - r_i\right)\right]$$

and therefore generally

$$I_2 = \Big[\frac{1}{2}\left\{(B_{im}^2 - B_{mj}^2)\log_e r_m + (B_{mj}^2 - B_{ji}^2)\log_e r_j + (B_{ji}^2 - B_{im}^2)\log_e r_i\right\}$$
$$+ A_{im}B_{im}(r_m - r_i) + A_{mj}B_{mj}(r_j - r_m) + A_{ji}B_{ji}(r_i - r_j) \qquad \text{(A3.32)}$$
$$+ \frac{1}{4}\left\{A_{im}^2(r_m^2 - r_i^2) + A_{mj}^2(r_j^2 - r_m^2) + A_{ji}^2(r_i^2 - r_j^2)\right\}\Big].$$

<u>Note:</u> By L'Hopital's rule the term containing log r_i tends to zero as r_i goes to zero.

$\underline{I_3}$

$$\int_{r=r_i}^{r_m}\int_{z=0}^{A_{im}r+B_{im}} \frac{z^2}{r}\,dr\,dz = \frac{1}{3}\int_{r_i}^{r_m} \frac{\left(A_{im}r + B_{im}\right)^3}{r}\,dr$$

$$= \frac{1}{3}B_{im}^3 \log_e \frac{r_m}{r_i} + B_{im}^2 A_{im}(r_m - r_i) + \frac{1}{2}B_{im}A_{im}^2(r_m^2 - r_i^2) + \frac{1}{9}A_{im}^3(r_m^3 - r_i^3)$$

and I_3 becomes

$$I_3 = \frac{1}{\Delta}\Big[\frac{1}{3}\left\{(B_{ji}^3 - B_{im}^3)\log_e r_i + (B_{mj}^3 - B_{ji}^3)\log_e r_j + (B_{im}^3 - B_{mj}^3)\log_e r_m\right\}$$
$$+ A_{im}B_{im}^2(r_m - r_i) + A_{mj}B_{mj}^2(r_j - r_m) + A_{ji}B_{ji}^2(r_i - r_j)$$
$$+ \frac{1}{2}\left\{A_{im}^2 B_{im}(r_m^2 - r_i^2) + A_{mj}^2 B_{mj}(r_j^2 - r_m^2) + A_{ji}^2 B_{ji}(r_i^2 - r_j^2)\right\}$$
$$+ \frac{1}{9}\left\{A_{im}^3(r_m^3 - r_i^3) + A_{mj}^3(r_j^3 - r_m^3) + A_{ji}^3(r_i^3 - r_j^3)\right\}\Big].$$

(A3.33)

<u>$\{F\}_p^e$ due to gravity.</u>

From 2.11 and A3.15

$$\{F\}_p^e = \frac{-2\pi\rho g}{\Delta} \int_e \begin{bmatrix} 0 \\ N_i \\ 0 \\ N_j \\ 0 \\ N_m \end{bmatrix} r\,dr\,dz$$

$$= \frac{-|\Delta|\pi\rho g}{\Delta} \begin{bmatrix} 0 \\ a_i\bar{r} + b_i I_4 + c_i I_5 \\ 0 \\ a_j\bar{r} + b_j I_4 + c_j I_5 \\ 0 \\ a_m\bar{r} + b_m I_4 + c_m I_5 \end{bmatrix} \qquad (A3.34)$$

where

$$I_4 = \frac{2}{|\Delta|} \int_e r^2\,dr\,dz \qquad (A3.35)$$

$$I_5 = \frac{2}{|\Delta|} \int_e zr\,dr\,dz \qquad (A3.36)$$

the rest of the quantities being defined by A3.15.

From A3.8, A3.11, A3.12

$$I_4 = \frac{2}{\Delta}\left[\frac{1}{4}\left\{A_{im}(r_m^4 - r_i^4) + A_{ji}(r_i^4 - r_j^4) + A_{mj}(r_j^4 - r_m^4)\right\} + \frac{1}{3}\left\{B_{im}(r_m^3 - r_i^3) + B_{ji}(r_i^3 - r_j^3) + B_{mj}(r_j^3 - r_m^3)\right\}\right] \qquad (A3.37)$$

and

$$I_5 = \frac{2}{\Delta}\left[\frac{1}{8}\left\{A_{im}^2(r_m^4 - r_i^4) + A_{ji}^2(r_i^4 - r_j^4) + A_{mj}^2(r_j^4 - r_m^4)\right\} + \frac{1}{4}\left\{B_{im}^2(r_m^2 - r_i^2) + B_{ji}^2(r_i^2 - r_j^2) + B_{mj}^2(r_j^2 - r_m^2)\right\} + \frac{1}{3}\left\{B_{im}A_{im}(r_m^3 - r_i^3) + B_{ji}A_{ji}(r_i^3 - r_j^3) + B_{mj}A_{mj}(r_j^3 - r_m^3)\right\}\right]. \qquad (A3.38)$$

Appendix 4.

PART 1

We wish to solve

$$E \frac{\partial^2 u}{\partial x^2} = \sigma \frac{\partial u}{\partial t} \qquad (A4.1)$$

subject to

$$u = 0 \quad \text{at} \quad t = 0 \quad , \quad 0 \le x \le L \qquad (A4.2)$$

$$u = 0 \quad \text{at} \quad x = 0 \quad , \quad t > 0 \qquad (A4.3)$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=L} = \frac{-P_0}{E} \quad , \quad t > 0 . \qquad (A4.4)$$

Applying a Laplace transformation to the equation and its boundary conditions and denoting the laplace transform of the function u(x,t) by $\bar{u}(x) = \int_0^\infty u(x,t)e^{-zt}dt$ we have

$$\left(\overline{\frac{\partial^2 u}{\partial x^2}}\right) = \int_0^\infty \frac{\partial^2 u}{\partial x^2} e^{-zt} dt = \frac{\partial^2}{\partial x^2}\int_0^\infty u e^{-zt} dt = \frac{\partial^2 \bar{u}}{\partial x^2} .$$

and

$$\left(\overline{\frac{\partial u}{\partial t}}\right) = \int_0^\infty \frac{\partial u}{\partial t} e^{-zt} dt$$

$$= \left[u e^{-zt}\right]_{t=0}^{\infty} + z\int_0^\infty u e^{-zt} dt \qquad \text{(by parts)}$$

$= z\,\bar{u}$. from A4.2. Therefore the Laplace transform of A4.1 is

$$E \frac{\partial^2 \bar{u}}{\partial x^2} = \sigma z \bar{u} . \qquad (A4.5)$$

Boundary condition A4.3 becomes

$$\bar{u} = o \quad \text{at} \quad x = o \tag{A4.6}$$

and A4.4 becomes

$$\left(\frac{\partial \bar{u}}{\partial x}\right) = \frac{-P_o}{Ez} \quad \text{at} \quad x = L \tag{A4.7}$$

noting that

$$\overline{\left(\frac{\partial u}{\partial x}\right)} = \left(\frac{\partial \bar{u}}{\partial x}\right)$$

and

$$\int_o^\infty \frac{-P_o}{E} e^{-zt}\, dt = \frac{-P_o}{Ez} .$$

The general solution of A4.5 is

$$\bar{u} = a \operatorname{Cosh}\sqrt{\frac{\sigma z}{E}}\, x + b \operatorname{Sinh}\sqrt{\frac{\sigma z}{E}}\, x \tag{A4.8}$$

where a and b are arbitrary functions of z.

$$(A4.6) \implies a = o$$

and (A4.7) gives

$$b = \frac{-P_o}{\sqrt{E\sigma}\; z^{3/2} \operatorname{Cosh}\sqrt{\frac{\sigma z}{E}}\, L} .$$

Substituting into A4.8.

$$\bar{u} = \frac{-P_o \operatorname{Sinh}\sqrt{\frac{\sigma z}{E}}\, x}{\sqrt{E\sigma}\; z^{3/2} \operatorname{Cosh}\sqrt{\frac{\sigma z}{E}}\, L} . \tag{A4.9}$$

Our required solution is given by taking the inverse Laplace transform of A4.9 (had to be done this way because the inverse of A4.9 could not be found in standard tables)

$$u(x,t) = \frac{-1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{P_o}{\sqrt{E\sigma}} \cdot \frac{e^{zt}\,\mathrm{Sinh}\sqrt{\frac{\sigma z}{E}}\,x}{z^{3/2}\,\mathrm{Cosh}\sqrt{\frac{\sigma z}{E}}\,L} \cdot dz \qquad (t>0) \qquad \text{(A4.10)}$$

(Spiegel, 1965) where γ is real and greater than the real parts of all the singularities of the integrand in the complex z-plane. At first sight it seems that the integrand has a branch point at the origin due to the $\sqrt{z}$ in the functions. However, on expansion

$$\frac{\mathrm{Sinh}\sqrt{\frac{\sigma z}{E}}\,x}{z^{3/2}\,\mathrm{Cosh}\sqrt{\frac{\sigma z}{E}}\,L} = \frac{\left(\sqrt{\frac{\sigma z}{E}}\,x + \frac{\left(\sqrt{\frac{\sigma z}{E}}\,x\right)^3}{\underline{|3}} + \frac{\left(\sqrt{\frac{\sigma z}{E}}\,x\right)^5}{\underline{|5}} + \cdots\right)}{z.z^{1/2}\left(1 + \frac{\left(\sqrt{\frac{\sigma z}{E}}\,L\right)^2}{\underline{|2}} + \frac{\left(\sqrt{\frac{\sigma z}{E}}\,L\right)^4}{\underline{|4}} + \cdots\right)}$$

$$= \left[\frac{\left(\sqrt{\frac{\sigma}{E}}\,x + \frac{z\left(\sqrt{\frac{\sigma}{E}}\,x\right)^3}{\underline{|3}} + \frac{z^2\left(\sqrt{\frac{\sigma}{E}}\,x\right)^5}{\underline{|5}} + \cdots\right)}{z\left(1 + \frac{\sigma z L^2}{E\underline{|2}} + \frac{\sigma^2 z^2 L^4}{E^2\underline{|4}} + \cdots\right)}\right]$$

which obviously has no branch point in the complex z-plane. However, there exists a simple pole at the origin with residue

$$\lim_{z\to 0} z \times \left[\quad\right] = \sqrt{\frac{\sigma}{E}}\,x \qquad \text{(A4.11)}$$

Poles also exist where $\cosh\sqrt{\frac{\sigma z}{E}}\, L = 0$ i.e. at

$$z = z_n = \frac{-(2n-1)^2\pi^2 E}{4\sigma L^2} \qquad n = 1, 2, 3, \ldots \qquad \text{(A4.12)}$$

What is the residue of $f(z) = \dfrac{e^{zt}\sinh\sqrt{\frac{\sigma z}{E}}\, x}{z^{3/2}\cosh\sqrt{\frac{\sigma z}{E}}\, L}$ at $z = z_n$?

Let

$$g(z) = e^{zt} z^{-3/2} \sinh\sqrt{\frac{\sigma z}{E}}\, x$$

and $h(z) = \cosh\sqrt{\frac{\sigma z}{E}}\, L$.

Both $g(z)$ and $h(z)$ are regular at $z = z_n$ and $h(z_n) = 0$.

$$h'(z_n) = \frac{1}{2} L \sqrt{\frac{\sigma}{E}}\, z_n^{-1/2} \sinh\sqrt{\frac{\sigma z_n}{E}}\, L$$

$$= \frac{1}{2} L \sqrt{\frac{\sigma}{E}}\, 2L \sqrt{\frac{\sigma}{E}}\, \frac{1}{(2n-1) i\pi} (-1)^{n+1} i$$

$$= \frac{L^2 \sigma (-1)^{n+1}}{E\pi(2n-1)} \neq 0 .$$

Therefore, there is a simple pole at $z = z_n$ with residue $R_n = g(z_n)/h'(z_n)$.

$$R_n = (-1)^{n+1} \frac{E\pi(2n-1)}{\sigma L^2} \cdot \frac{i 8 L^3 \sigma^{3/2}}{(2n-1)^3 \pi^3 E^{3/2}} \sinh\left[\frac{(2n-1) i\pi}{2L} x\right] \times e^{-\frac{(2n-1)^2\pi^2 E t}{4\sigma L^2}}$$

$$= (-1)^n \frac{8\sigma^{1/2} L}{E^{1/2}\pi^2(2n-1)^2} \sin\left[\frac{(2n-1)\pi x}{2L}\right] e^{-\frac{(2n-1)^2\pi^2 E t}{4\sigma L^2}} \qquad \text{(A4.13)}$$

To perform the inverse Laplace transform A4.10 we consider the integral of

$$\phi(z) = \frac{-P_o}{2\pi i\sqrt{E\sigma}} \cdot \frac{e^{zt}\,\mathrm{Sinh}\sqrt{\frac{\sigma z}{E}}\,x}{z^{3/2}\,\mathrm{Cosh}\sqrt{\frac{\sigma z}{E}}\,L}$$

around the contour C_m in Fig.A4.1.

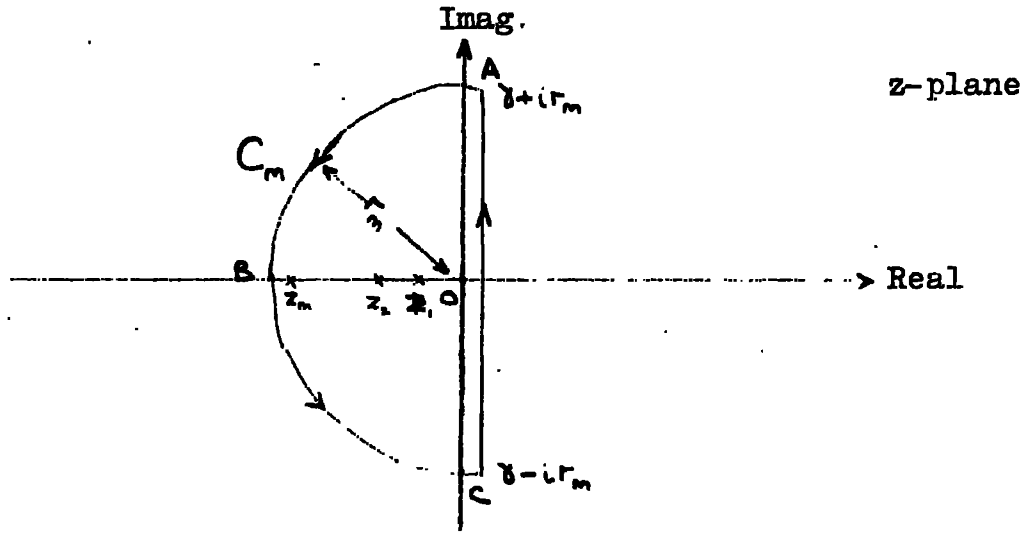


Fig.A4.1

$\phi(z)$ is regular on and within the contour except at the simple poles at the origin and at $z = z_n$ $n = 1,2,3,$ ... ...m. provided we take the radius of the semi-circle ABC to be r_m where r_m is chosen so that the contour contains only a finite number (M+1) of poles and $r_m \neq z_n$ for any n i.e. take $r_m = m^2\pi^2E/\sigma L^2$. Note that inkeeping with the earlier condition γ can be an arbitrarily small real positive number.

Now,

$$\oint_{C_m} \phi(z)\,dz = \int_{\gamma-ir_m}^{\gamma+ir_m} \phi(z)\,dz + \int_{ABC} \phi(z)\,dz$$

$$= 2\pi i \sum \text{Residues of } \phi(z) \text{ inside } C_m$$

(Cauchy's Theorem - Dennery & Krzywicki, 1967)

Therefore,

$$\int_{\gamma - ir_m}^{\gamma + ir_m} \phi(z)\,dz + \int_{ABC} \phi(z)\,dz = 2\pi i\left(\frac{-P_0}{2\pi i\sqrt{E\sigma}}\right)\left\{\sqrt{\frac{\sigma}{E}}\,x + \sum_{n=1}^{M} R_n\right\}.$$

Thus provided we can show (later) that

$$\lim_{r_m \to \infty} \int_{ABC} \phi(z)\,dz \rightarrow 0$$

we have from A4.10 and A4.13.

$$u(x,t) = \frac{-P_0 x}{E} - \frac{8P_0 L}{E\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\,\mathrm{Sin}\left[\frac{(2n-1)\pi x}{2L}\right] e^{-\frac{(2n-1)^2\pi^2 E t}{4\sigma L^2}}$$

$$(t > 0)$$

(A4.14)

We must now show that in fact

$$\int_{ABC} \phi(z)\,dz \rightarrow 0$$

as $r_m \to \infty$.

i.e. $$\int_{\Gamma} \frac{e^{zt}\,\mathrm{Sinh}\,K_1\sqrt{z}}{z^{3/2}\,\mathrm{Cosh}\,K_2\sqrt{z}}\,dz \rightarrow 0$$ as

$r_m \to \infty$ (i.e. as $m \to \infty$) where

$$r_m = \frac{m^2\pi^2}{K_2^2} \qquad m = 1,2,3,\ldots\ldots .$$

$$K_1 = \sqrt{\frac{\sigma}{E}}\,x$$

and $$K_2 = \sqrt{\frac{\sigma}{E}}\,L \quad .$$

Γ is the contour shown in Fig.A4.2

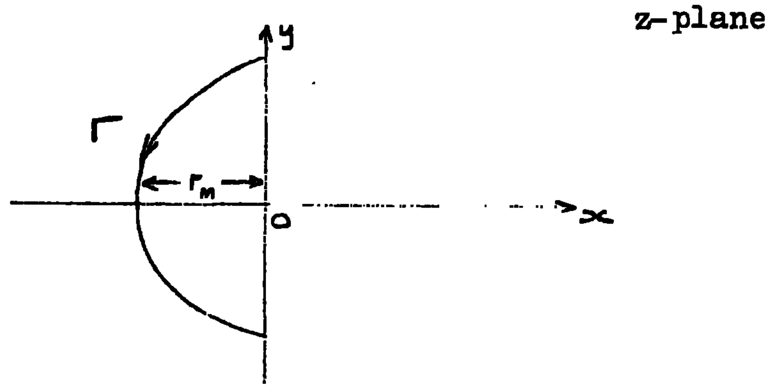


Fig.A4.2

Consider the value of $\left|\dfrac{\sinh K_1\sqrt{z}}{\cosh K_2\sqrt{z}}\right|$ on the contour Γ. Apply the transformation $\omega^2 = z$ $(\omega = \xi + i\eta)$ to the w-plane. $\Gamma \to \Gamma'$ where Γ' is part of a circle radius $r_m' = \frac{m\pi}{K_2}$ centre O (fig.A4.3). Now consider the value of $\left|\dfrac{\sinh K_1\omega}{\cosh K_2\omega}\right|$ on $A'B'$. (Note: $A'C' = C'B' = O'C' = \frac{m\pi}{K_2}$)

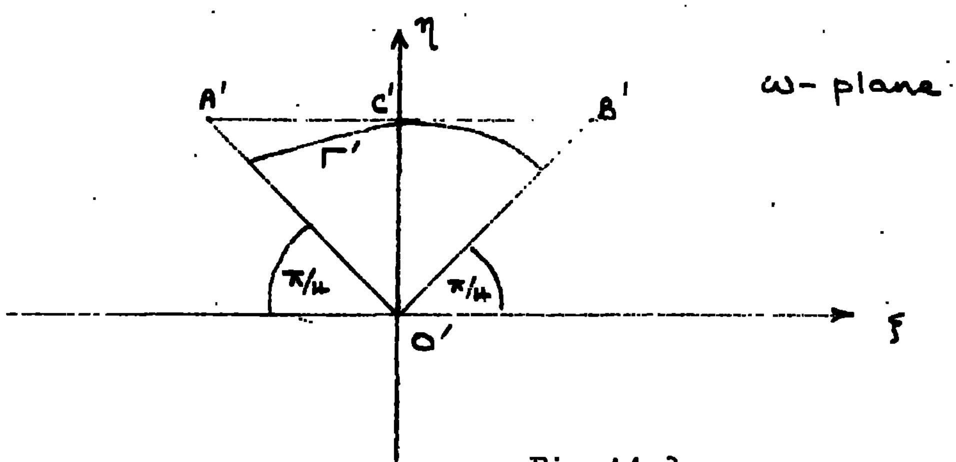


Fig.A4.3

On $A'B'$

$$\left|\sinh K_1\omega\right| = \left|\frac{e^{K_1\omega} - e^{-K_1\omega}}{2}\right| \geq \frac{1}{2}\left|\,\left|e^{K_1\omega}\right| - \left|e^{-K_1\omega}\right|\,\right|$$

$$= \frac{1}{2}\left|e^{K_1\xi} - e^{-K_1\xi}\right| = \left|\sinh K_1\xi\right|$$

and

$$|\cosh K_2 w| = \left|\frac{e^{K_2 w} + e^{-K_2 w}}{2}\right| = \left|\frac{e^{K_2\xi}e^{im\pi} + e^{-K_2\xi}e^{-im\pi}}{2}\right|$$

$$= \left|\frac{(-1)^m(e^{K_2\xi} + e^{-K_2\xi})}{2}\right| = |\cosh K_2\xi| .$$

Thus on $A'B'$

$$\left|\frac{\sinh K_1 w}{\cosh K_2 w}\right| = \frac{|\sinh K_1 w|}{|\cosh K_2 w|} \leq \frac{|\sinh K_1\xi|}{\cosh K_2\xi} \leq 1 .$$

Since $K_1 \leq K_2$ and both $K_1\xi$ and $K_2\xi$ are real ..

In the z plane the line $A'B'$ is the parabola γ (fig.A4.4)

$$x = \xi^2 - \frac{m^2\pi^2}{K_2} \quad , \quad y = \frac{2m\pi}{K_2}\xi \qquad \left(\frac{-m\pi}{K_2} \leq \xi \leq \frac{m\pi}{K_2}\right)$$

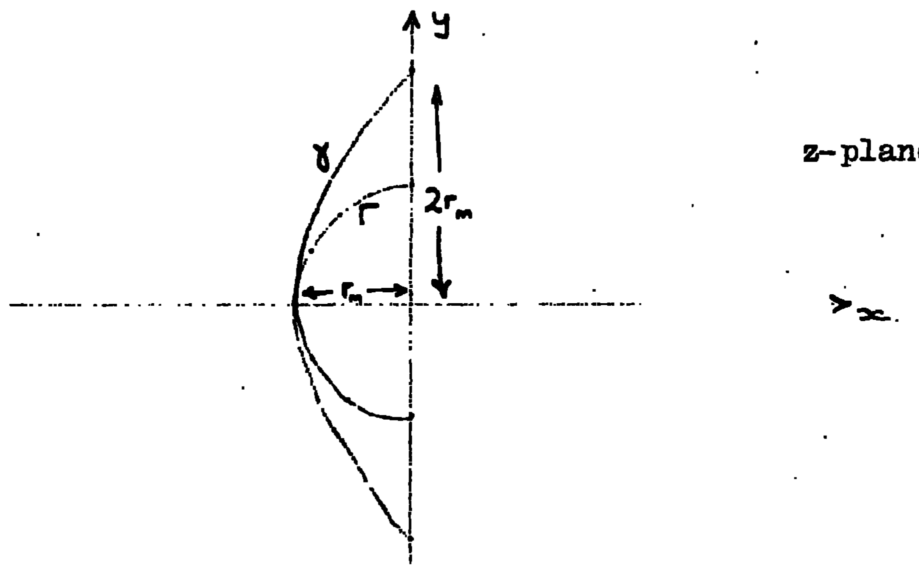


Fig.A4.4

Revising our original contour C_m to γ_m leaves the previous analysis unaltered and on γ

$$\left|\frac{\sinh K_1\sqrt{z}}{\cosh K_2\sqrt{z}}\right| \leq 1 \quad . \qquad |e^{zt}| \leq 1$$

since $\text{Re}\, z \leq 0$ and $t > 0$ and

$$|z^{3/2}| \geq r_m^{3/2} = \frac{m^3\pi^3}{K_2^3} .$$

The length of $\gamma < 6r_m = \frac{6m^2\pi^2}{K_2^2}$.

Therefore

$$\left| \int_\gamma \frac{e^{zt} \operatorname{Sinh} K_1\sqrt{z}}{z^{3/2} \operatorname{Cosh} K_2\sqrt{z}} \, dz \right| \le \int_\gamma \left| \frac{e^{zt} \operatorname{Sinh} K_1\sqrt{z}}{z^{3/2} \operatorname{Cosh} K_2\sqrt{z}} \, dz \right|$$

$$= \int_\gamma \frac{|e^{zt}| \, |\operatorname{Sinh} K_1\sqrt{z}|}{|z^{3/2}| \, |\operatorname{Cosh} K_2\sqrt{z}|} |dz|$$

$$\le \int_\gamma \frac{K_2^3}{m^3\pi^3} |dz|$$

$$= \frac{K_2^3}{m^3\pi^3} \times \text{length of } \gamma$$

$$\le \frac{K_2^3}{m^3\pi^3} \cdot \frac{6m^2\pi^2}{K_2^2} \longrightarrow 0 \quad \text{as } m \to \infty .$$

Thus

$$\lim_{r_m \to \infty} \int_\gamma \frac{e^{zt} \operatorname{Sinh} K_1\sqrt{z}}{z^{3/2} \operatorname{Cosh} K_2\sqrt{z}} \, dz \longrightarrow 0$$

and A4.14 is in fact the solution of A4.1 subject to A4.2, A4.3 and A4.4.

PART 2.

We wish to solve

$$\rho \frac{\partial^2 u}{\partial t^2} + \sigma \frac{\partial u}{\partial t} = E \frac{\partial^2 u}{\partial x^2} \qquad \text{(A4.15)}$$

with boundary and initial conditions

$$u = o \quad \text{at } t = o \qquad o \leq x \leq L \qquad \text{(A4.16)}$$

$$u = o \quad \text{at } x = o \qquad t > o \qquad \text{(A4.17)}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=L} = \frac{-P_o}{E} \quad \text{at } x = L \qquad \text{(A4.18)}$$

and $\frac{\partial u}{\partial t} = o$ at $t = o$ $\quad o \leq x \leq L$. (A4.19)

Note that the inclusion of the inertial term necessitates the inclusion of an extra initial condition (A4.19) Applying a Laplace transform to the equation and its boundary conditions.

$$\overline{\left(\frac{\partial^2 u}{\partial t^2}\right)} = \int_0^\infty e^{-zt} \frac{\partial^2 u}{\partial t^2} dt$$

$$= \left[e^{-zt} \frac{\partial u}{\partial t}\right]_0^\infty + z \int_0^\infty \frac{\partial u}{\partial t} e^{-zt} dt$$

$$= z \overline{\left(\frac{\partial u}{\partial t}\right)}$$

$$= z^2 \bar{u} .$$

(from previous)

Thus (A4.15) transforms to

$$\rho z^2 \bar{u} + \sigma z \bar{u} = E \frac{\partial^2 \bar{u}}{\partial x^2}$$

i.e.

$$\frac{\partial^2 \bar{u}}{\partial x^2} = \left(\frac{\rho z^2 + \sigma z}{E} \right) \bar{u} \qquad \text{(A4.20)}$$

with boundary conditions

$$\bar{u} = 0 \quad \text{at} \quad x = 0 \quad t > 0 \qquad \text{(A4.21)}$$

and

$$\left(\frac{\partial \bar{u}}{\partial x} \right) = \frac{-P_0}{Ez} \quad \text{at} \quad x = L, \; t > 0. \qquad \text{(A4.22)}$$

The general solution of (D.20) is

$$\bar{u} = a \operatorname{Cosh}\left(\sqrt{\frac{\rho z^2 + \sigma z}{E}}\, x \right) + b \operatorname{Sinh}\left(\sqrt{\frac{\rho z^2 + \sigma z}{E}}\, x \right) \qquad \text{(A4.23)}$$

where a and b are arbitrary functions of z.

(A4.21) $\Longrightarrow$ a = o and (A4.22) gives

$$b = -P_0 \Big/ \left[z \sqrt{E(\rho z^2 + \sigma z)} \operatorname{Cosh}\left(\sqrt{\frac{\rho z^2 + \sigma z}{E}}\, L \right) \right]$$

Substituting back into (A4.23) gives

$$\bar{u} = \frac{-P_0 \operatorname{Sinh}\left(\sqrt{\frac{\rho z^2 + \sigma z}{E}}\, x \right)}{z \sqrt{E(\rho z^2 + \sigma z)} \operatorname{Cosh}\left(\sqrt{\frac{\rho z^2 + \sigma z}{E}}\, L \right)} . \qquad \text{(A4.24)}$$

Performing the inverse Laplace transform

$$u(x,t) = \frac{-1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{P_o e^{zt} \operatorname{Sinh}\left(\sqrt{\frac{\rho z^2 + \sigma z}{E}}\, x\right)}{z\sqrt{E(\rho z^2 + \sigma z)}\, \operatorname{Cosh}\left(\sqrt{\frac{\rho z^2 + \sigma z}{E}}\right)}\, dz \qquad (t > 0) \tag{A4.25}$$

with the same conditions on γ as previous. As before the integrand does not have a branch point at the origin. There is a pole (simple) at the origin with residue

$$\frac{-1}{2\pi i} \cdot \frac{P_o x}{E}.$$

Poles also occur where $\operatorname{Cosh}\left(\sqrt{\frac{\rho z^2 + \sigma z}{E}}\, L\right) = o$ i.e. where

$$\rho z^2 + \sigma z = \frac{-(2n-1)^2 \pi^2 E}{4L^2} \qquad n = 1, 2, 3 \ldots\ldots \tag{A4.26a}$$

Put
$$\alpha_n \equiv \frac{(2n-1)^2 \pi^2 E}{4L^2} \tag{A4.26b}$$

and we have poles at

$$z_{n_\pm} = \frac{-\sigma \pm \sqrt{\sigma^2 - 4\rho\alpha_n}}{2\rho}. \tag{A4.27}$$

<u>Residue at z_n:</u>

Put $g(z) = \dfrac{-P_o \operatorname{Sinh}\left(\sqrt{\frac{\rho z^2 + \sigma z}{E}}\, x\right) e^{zt}}{zE\sqrt{\frac{\rho z^2 + \sigma z}{E}}}$

and $h(z) = \operatorname{Cosh}\left(\sqrt{\frac{\rho z^2 + \sigma z}{E}}\, L\right)$.

$g(z)$ and $h(z)$ are both regular at z_n and $h(z_n) = o$.

$$h'(z_n) = \frac{1}{2}L\left[\sqrt{\frac{\rho z_n^2+\sigma z_n}{E}}\right]^{-1}\left(\frac{2\rho z_n+\sigma}{E}\right)\operatorname{Sinh}\left(\sqrt{\frac{\rho z_n^2+\sigma z_n}{E}}\,L\right)$$

$$= \frac{L}{2}\left(\frac{2\rho z_n+\sigma}{E}\right)\cdot\frac{2L}{i(2n-1)\pi}\cdot\operatorname{Sinh}\left(\frac{i(2n-1)\pi}{2}\right)$$

$$= \frac{L^2}{(2n-1)\pi}\left(\frac{2\rho z_n+\sigma}{E}\right)\operatorname{Sin}\left(\frac{(2n-1)\pi}{2}\right)$$

$$= \frac{L^2(2\rho z_n+\sigma)}{(2n-1)E\pi}(-1)^{n+1} \neq 0 .$$

Thus we have a simple pole at $z = z_n$ with residue $\frac{1}{2\pi i}\frac{g(z_n)}{h'(z_n)}$.

$$g(z_n) = \frac{-P_0 2L}{iz_n(2n-1)\pi}\operatorname{Sinh}\left[\frac{i(2n-1)\pi x}{2L}\right]e^{z_n t}$$

$$= \frac{-2LP_0 e^{z_n t}}{z_n\pi E(2n-1)}\operatorname{Sin}\left[\frac{(2n-1)\pi x}{2L}\right].$$

Hence the residue at z_n is

$$\frac{-1}{2\pi i}\cdot\frac{2P_0(-1)^{n+1}e^{z_n t}}{z_n E(2\rho z_n+\sigma)}\operatorname{Sin}\left[\frac{(2n-1)\pi x}{2L}\right].$$

As before provided

$$\lim_{r\to\infty}\int_{ABC}\phi(z)\,dz \to 0$$

we have

$$u(x,t) = \frac{-P_o x}{E} + \frac{2P_o}{L}\sum_{n=1}^{\infty}(-1)^n \sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

$$\times \left\{\frac{e^{z_{n+}t}}{z_{n+}(2\rho z_{n+}+\sigma)} + \frac{e^{z_{n-}t}}{z_{n-}(2\rho z_{n-}+\sigma)}\right\}.$$

$$(t>0) \qquad \text{(A4.28)}$$

To prove that this is a solution rather than attempt to prove $\lim_{r\to\infty}\int_{ABC}\phi(z)\,dz$ we shall verify it by substitution back into A4.15 and its boundary conditions. A4.28 obviously satisfies A4.17 and A4.18. From A4.28

$$\sigma\frac{\partial u}{\partial t} = \frac{2P_o\sigma}{L}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\frac{e^{z_n t}}{(2\rho z_n+\sigma)}$$

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{2P_o\rho}{L}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\frac{z_n e^{z_n t}}{(2\rho z_n+\sigma)}$$

$$-E\frac{\partial^2 u}{\partial x^2} = \frac{2(2n-1)^2\pi^2 P_o E}{4L^3}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\frac{e^{z_n t}}{z_n(2\rho z_n+\sigma)}.$$

Adding we have

$$\sigma\frac{\partial u}{\partial t} + \rho\frac{\partial^2 u}{\partial t^2} - E\frac{\partial^2 u}{\partial x^2}$$

$$= \frac{2P_o}{Lz_n}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\frac{e^{z_n t}}{(2\rho z_n+\sigma)}\left\{z_n\sigma + z_n^2\rho + \frac{(2n-1)^2\pi^2 E}{4L^2}\right\}$$

$= 0$ from A4.26a.

Thus A4.28 is the solution and it follows immediately that the pressure variation at the origin is

$$p(t) = P_o - \frac{P_o \pi E}{L^2} \sum_{n=1}^{\infty} (-1)^n (2n-1) \left\{ \frac{e^{z_{n+} t}}{z_{n+}(2\rho z_{n+} + \sigma)} + \frac{e^{z_{n-} t}}{z_{n-}(2\rho z_{n-} + \sigma)} \right\}.$$

$$(t > o) \qquad (A4.29)$$

Computation of (A4.29)

Suppose that for $n > N$ $\sigma^2 < 4\rho\alpha_n$. Consider the terms in A4.29 with $n > N$. Put $\beta_n = \sqrt{4\rho\alpha_n - \sigma^2}$.

Then we have from A4.27 terms of the type

$$-\frac{2\rho P_o \pi E}{L^2} e^{-\sigma t/2\rho} \sum_{n=N+1}^{\infty} (-1)^n (2n-1) \left\{ \frac{e^{i\beta_n t/2\rho}}{(i\beta_n - \sigma) i\beta_n} + \frac{e^{-i\beta_n t/2\rho}}{(i\beta_n + \sigma) i\beta_n} \right\}$$

$$= \frac{4\rho P_o \pi E}{L^2} e^{-\frac{\sigma t}{2\rho}} \sum_{n=N+1}^{\infty} \frac{(-1)^n (2n-1)}{\beta_n (\beta_n^2 + \sigma^2)} \left\{ \beta_n \cos\left(\frac{\beta_n t}{2\rho}\right) + \sigma \sin\left(\frac{\beta_n t}{2\rho}\right) \right\}.$$

Since $t > o$ and $\sigma t \gg 2\rho$ for times in which we are interested these terms are negligibly small.

Consider now the series

$$\sum_{n=1}^{N} (-1)^n (2n-1) \frac{e^{z_{n-} t}}{[z_{n-} (2\rho z_{n-} + \sigma)]}$$

$$= \sum_{n=1}^{N} (-1)^n (2n-1) \frac{e^{-\sigma t/2\rho} \, e^{-\sqrt{\sigma^2 - 4\rho\alpha_n} \cdot \frac{t}{2\rho}}}{[z_{n-} (2\rho z_{n-} + \sigma)]}$$

using A4.27. These terms are also negligible. This just leaves

$$\sum_{n=1}^{N} (-1)^n (2n-1) \frac{e^{z_{n+}t}}{\left[z_{n+}(2\rho z_{n+} + \sigma)\right]}$$

$$= 2\rho \sum_{n=1}^{N} (-1)^n (2n-1) \frac{e^{-\sigma t/2\rho}\; e^{\sqrt{\left[\sigma^2 - \frac{\rho E \pi^2 (2n-1)^2}{L^2}\right]}\frac{t}{2\rho}}}{\left[\sigma^2 - \frac{\rho E \pi^2 (2n-1)^2}{L^2} - \sigma\sqrt{\sigma^2 - \frac{\rho E \pi^2 (2n-1)^2}{L^2}}\right]}.$$

Computation of this series as it stands proves difficult because it consists of very large numbers (too large in fact to be stored by the computer) of alternating sign. Thus we must consider the sum of pairs of terms. Let n be odd, then combining the nth and (n+1)th terms we have

$$-\frac{2\rho P_0 \pi E}{L^2} e^{-\frac{\sigma t}{2\rho}} \left[(-1)^n (2n-1) \frac{e^{\frac{t}{2\rho}\sqrt{\left[\sigma^2 - \frac{\rho E \pi^2 (2n-1)^2}{L^2}\right]}}}{\left[\sigma^2 - \frac{\rho E \pi^2 (2n-1)^2}{L^2} - \sigma\sqrt{\sigma^2 - \frac{\rho E \pi^2 (2n-1)^2}{L^2}}\right]} \right.$$

$$\left. + (-1)^{n+1} (2n+1) \frac{e^{\frac{t}{2\rho}\sqrt{\left[\sigma^2 - \frac{\rho E \pi^2 (2n+1)^2}{L^2}\right]}}}{\left[\sigma^2 - \frac{\rho E \pi^2 (2n+1)^2}{L^2} - \sigma\sqrt{\sigma^2 - \frac{\rho E \pi^2 (2n+1)^2}{L^2}}\right]} \right].$$

For $\sigma^2 \gg \frac{\rho E \pi^2 (2n+1)^2}{L^2}$ we have

$$\sqrt{\sigma^2 - \frac{\rho E \pi^2 (2n+1)^2}{L^2}} \doteqdot \sigma^2 - \frac{\rho E \pi^2 (2n+1)^2}{2\sigma L^2}$$

and the combined term reduces to

$$-\frac{4P_0}{\pi}\left[\frac{e^{-\frac{E\pi^2(2n-1)^2 t}{4\sigma L^2}}}{(2n-1)} - \frac{e^{-\frac{E\pi^2(2n+1)^2 t}{4\sigma L^2}}}{(2n+1)}\right]. \qquad (A4.30)$$

If $\sigma^2 \not\gg \frac{\rho E \pi^2 (2n+1)^2}{L^2}$ then the exponential terms make the term negligible.

PART 3

Fourier Series for $f(x)=1-e^{-x/s}$ $\quad 0 \leq x \leq L$.

Extend the range by defining

$$f(x) = \begin{cases} -\left(1-e^{-(2L+x)/s}\right) & -2L \leq x \leq -L \\ -\left(1-e^{x/s}\right) & -L \leq x \leq 0 \\ 1-e^{-x/s} & 0 \leq x \leq L \\ 1-e^{-(2L-x)/s} & L \leq x \leq 2L \end{cases} \qquad \text{(A4.31)}$$

Let $y = \frac{\pi x}{2L}$ and put $g(y)=f(x)$.

If $|g\rangle \sim g(y)$ and $|e_m\rangle \sim \frac{e^{imy}}{\sqrt{2\pi}}$ then $\{|e_m\rangle\}$ is a complete orthonormal set in $L^2_1(-\pi,\pi)$ (Dennery & Krzywicki, 1967) Therefore

$$|g\rangle = \sum_{n=-\infty}^{\infty} a_n |e_n\rangle$$

and

$$a_m = \langle e_m|g\rangle = \int_{-\pi}^{\pi} \frac{e^{-imy}}{\sqrt{2\pi}} g(y)\, dy$$

i.e.

$$\sqrt{2\pi}\; a_m = -\int_{-\pi}^{-\pi/2} e^{-imy}\left(1-e^{-(2L+\frac{2Ly}{\pi})/s}\right)dy$$

$$-\int_{-\frac{\pi}{2}}^{0} e^{-imy}\left(1-e^{2Ly/\pi s}\right)dy$$

$$+\int_{0}^{\pi/2} e^{-imy}\left(1-e^{-2Ly/\pi s}\right)dy$$

$$+\int_{\pi/2}^{\pi} e^{-imy}\left(1-e^{-(2L-\frac{2Ly}{\pi})/s}\right)dy$$

$$= \frac{-ie^{\frac{im\pi}{2}}}{m} + \frac{i(-1)^m}{m} - \frac{e^{\frac{im\pi}{2}}e^{-L/s}}{\left(im + \frac{2L}{\pi s}\right)} + \frac{(-1)^m}{\left(im + \frac{2L}{\pi s}\right)} - \frac{i}{m}\left(1 - e^{\frac{im\pi}{2}}\right)$$

$$+ \frac{1}{\left(\frac{2L}{\pi s} - im\right)}\left[1 - e^{-L/s}e^{\frac{im\pi}{2}}\right] + \frac{i}{m}e^{-\frac{im\pi}{2}} - \frac{i}{m} + \frac{1}{\left(\frac{2L}{\pi s} + im\right)}\left[e^{-\frac{im\pi}{2}}e^{-L/s} - 1\right]$$

$$+ \frac{i}{m}\left[(-1)^m - e^{-\frac{im\pi}{2}}\right] + \frac{e^{-L/s}e^{-\frac{im\pi}{2}}}{\left(\frac{2L}{\pi s} - im\right)} - \frac{(-1)^m}{\left(\frac{2L}{\pi s} - im\right)}$$

Since $L \gg s$ we can ignore the terms with $e^{-L/s}$ giving

$$a_m = \frac{2i}{\sqrt{2\pi}}\left[\frac{(-1)^m - 1}{m} + \frac{m - (-1)^m m}{\left(\frac{4L^2}{\pi^2 s^2} + m^2\right)}\right] \qquad \text{(A4.32)}$$

Noting $a_m = -a_{-m}$ and $a_o = o$ we have

$$|g\rangle = \sum_{n=1}^{\infty} a_n \left\{ |e_n\rangle - |e_{-n}\rangle \right\}$$

$$= \sum_{m=1}^{\infty} -\frac{2}{\pi}\left[\frac{1}{m} - \frac{m}{\left(\frac{4L^2}{\pi^2 s^2} + m^2\right)}\right]\left[(-1)^m - 1\right] \frac{e^{imy} - e^{-imy}}{2i}$$

or

$$f(x) = \sum_{m=1}^{\infty} \frac{2}{\pi}\left[\frac{1}{m} - \frac{m}{\left(\frac{4L^2}{\pi^2 s^2} + m^2\right)}\right]\left[1 - (-1)^m\right] \sin\left(\frac{m\pi x}{2L}\right)$$

Putting $m = 2n-1$

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)} - \frac{(2n-1)}{\left[\frac{4L^2}{\pi^2 s^2} + m^2\right]}\right] \sin\left[\frac{(2n-1)\pi x}{2L}\right] \qquad \text{(A4.33)}$$

u_+ must have gradient $\frac{P_c}{E}$ at $x = o$ so

$$u_+ = \frac{1}{S_N}\sum_{n=1}^{N}\left[\frac{1}{(2n-1)} - \frac{(2n-1)}{\left(\frac{4L^2}{\pi^2 s^2} + (2n-1)^2\right)}\right] \sin\left[\frac{(2n-1)\pi x}{2L}\right] \qquad \text{(A4.34)}$$

where

$$S_N = \sum_{n=1}^{N} \left[\frac{1}{(2n-1)} - \frac{(2n-1)}{\left(\frac{4L^2}{\pi^2 s^2} + (2n-1)^2 \right)} \right] \sin \left[\frac{(2n-1)\pi x}{2L} \right] . \quad (A4.35)$$

PART 4

Solution of

$$E \frac{\partial^2 u}{\partial x^2} = \sigma \frac{\partial u}{\partial t} \tag{A4.36}$$

subject to

$$u = o \quad \text{at} \quad x = o \quad t > o \tag{A4.37}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=L} = \frac{-P_o}{E} \quad t > o \tag{A4.38}$$

$$u = u_\tau + u_+ \quad \text{at} \quad t = o \tag{A4.39}$$

where

$$u_\tau = \frac{-P_o x}{E} - \frac{8P_oL}{E\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\left[\frac{(2n-1)\pi x}{2L}\right] e^{-\frac{(2n-1)^2\pi^2 E\tau}{4\sigma L^2}} \tag{A4.40}$$

and u_+ is given by A4.34.

Taking the Laplace transformation we obtain

$$E\frac{\partial^2\bar{u}}{\partial x^2} = \sigma z\bar{u} + \frac{\sigma P_o x}{E} + \frac{8P_o\sigma L}{E\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\left[\frac{(2n-1)\pi x}{2L}\right] e^{-\frac{(2n-1)^2\pi^2E\tau}{4\sigma L^2}} - \frac{\sigma}{S_N}\sum_{n=1}^{N}\left[\frac{1}{(2n-1)} - \frac{(2n-1)}{\left(\frac{4L^2}{\pi^2 S^2}+(2n-1)^2\right)}\right]\sin\left[\frac{(2n-1)\pi x}{2L}\right] \tag{A4.41}$$

with boundary conditions

$$\bar{u} = o \quad \text{at} \quad x = o \tag{A4.42}$$

and

$$\left(\frac{\partial\bar{u}}{\partial x}\right)_{x=L} = \frac{-P_o}{Ez} \tag{A4.43}$$

If we look for a particular solution of A4.41 of the form

$$\bar{u} = e_1 x + \sum_{n=1}^{\infty} f_n \sin\left[\frac{(2n-1)\pi x}{2L}\right] + \sum_{n=1}^{N} g_n \sin\left[\frac{(2n-1)\pi x}{2L}\right] \qquad \text{(A4.44)}$$

we have on substitution into the equation

$$- \sum_{n=1}^{\infty} \frac{E(2n-1)^2\pi^2}{4L^2} f_n \sin\left[\frac{(2n-1)\pi x}{2L}\right] - \sum_{n=1}^{N} \frac{E(2n-1)^2\pi^2}{4L^2} g_n \sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

$$= \sigma z e_1 x + \sigma z \sum_{n=1}^{\infty} f_n \sin\left[\frac{(2n-1)\pi x}{2L}\right] + \sigma z \sum_{n=1}^{N} g_n \sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

$$+ \frac{\sigma P_o x}{E} + \frac{8 P_o \sigma L}{E\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin\left[\frac{(2n-1)\pi x}{2L}\right] e^{-(2n-1)^2\pi^2 E\tau/4\sigma L^2}$$

$$- \frac{\sigma}{S_N} \sum_{n=1}^{N} \left[\frac{1}{(2n-1)} - \frac{(2n-1)}{\left(\frac{4L^2}{\pi^2 S^2} + (2n-1)^2\right)}\right] \sin\left[\frac{(2n-1)\pi x}{2L}\right].$$

Comparing coefficients we have

$$e_1 = \frac{-P_o}{Ez} \qquad \text{(A4.45)}$$

$$g_n = \frac{4L^2\sigma \sum_{n=1}^{N} \gamma_n}{S_N\left(4L^2\sigma z + E(2n-1)^2\pi^2\right)} \qquad \text{(A4.46)}$$

$$\gamma_n = \left[\frac{1}{(2n-1)} - \frac{(2n-1)}{\left(\frac{4L^2}{\pi^2 S^2} + (2n-1)^2\right)}\right] \qquad \text{(A4.47)}$$

and

$$f_n = \frac{-32 P_o \sigma L^3 (-1)^n e^{\frac{-(2n-1)^2\pi^2 E \tau}{4\sigma L^2}}}{E\pi^2 \left(E(2n-1)^2\pi^2 + 4\sigma L^2 z\right)(2n-1)^2} \qquad (A4.48)$$

Therefore the general solution of A4.41 is

$$\bar{u} = d_1 \cosh\sqrt{\frac{\sigma z}{E}}\, x + d_2 \sinh\sqrt{\frac{\sigma z}{E}}\, x - \frac{P_o x}{E z}$$

$$- \frac{8 P_o L}{E\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n e^{\frac{-(2n-1)^2\pi^2 E\tau}{4\sigma L^2}}}{(2n-1)^2 \left(z + \frac{E(2n-1)^2\pi^2}{4\sigma L^2}\right)} \sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

$$+ \frac{1}{S_N} \sum_{n=1}^{N} \frac{\gamma_n}{\left(z + \frac{E(2n-1)^2\pi^2}{4\sigma L^2}\right)} \sin\left[\frac{(2n-1)\pi x}{2L}\right] \qquad (A4.49)$$

where d_1 and d_2 are arbitrary functions of z.

A4.42 gives $d_1 = 0$

A4.43 gives $d_2 = 0$

Thus our solution for the laplace transform of u(x,t) is

$$\bar{u} = \frac{-P_o x}{E z} - \frac{8 P_o L}{E\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n e^{\frac{-(2n-1)^2\pi^2 E\tau}{4\sigma L^2}}}{(2n-1)^2 \left(z + \frac{E(2n-1)^2\pi^2}{4\sigma L^2}\right)} \sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

$$+ \frac{1}{S_N} \sum_{n=1}^{N} \frac{\gamma_n}{\left(z + \frac{E(2n-1)^2\pi^2}{4\sigma L^2}\right)} \sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

(A4.50)

All the terms in this expression have standard inverse Laplace transforms (see tables - Spiegel 1965). Thus

$$u(x,t) = \frac{-P_o x}{E} - \frac{8P_o L}{E\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n e^{-\frac{(2n-1)^2\pi^2 E(\tau+t)}{4\sigma L^2}}}{(2n-1)^2}\operatorname{Sin}\left[\frac{(2n-1)\pi x}{2L}\right]$$

$$+\frac{1}{S_N}\sum_{n=1}^{N}\gamma_n e^{-\frac{(2n-1)^2\pi^2 E t}{4\sigma L^2}}\operatorname{Sin}\left[\frac{(2n-1)\pi x}{2L}\right]. \qquad (A4.51)$$

APPENDIX 5.

SPECIFICATION OF PROGRAMS.

A. FINEL

Purpose. FINEL is a fortran program written to perform two-dimensional finite element calculations for the cases of plane strain and plane stress, using a 3-noded triangular element.

Data Input: is by punched card

Card 1:

1 – 10
NJOB

NJOB is an integer contained in columns 1 - 10 (right hand justified) specifying the number of finite element calculations to be done during this run.

Card 2. Title card for first set of data

Card 3.

1 – 10	11 – 20	30	40	50	60
NPOIN	NELEM	A	B	C	D

where

NPOIN is number of nodes.

NELEM is number of elements.

A = 0 plane strain
    1 plane stress

B = 0 isostacy forces not to be included.
    1 isostacy forces to be included.

C = 0 No visual display of stress differences.
    1 Visual display of stress differences required.

D = 0 No punched output
    1 punched output for graphplot.

Card 4: (NPOIN of them) Nodal point data

1 – 10	11 – 20	30	32	40	42	50	60	70
NREF	X	Y	N	DX	M	DY	R(1)	R(2)
I10	F10.0	F10.0	I1	F8.0	I1	F8.0	E10.0	E10.0

NREF - reference number of nodal point (range 1 to NPOIN)

X - X co-ordinate of nodal point in Km.

Y - Y co-ordinate of nodal point in Km.

N = 0 - displacement in x direction not specified.
1 - displacement in x direction is specified.

DX - (if N = 1) specified displacement in x direction in metres.

M = 0 - displacement in y direction not specified.
1 - displacement in y direction is specified.

DY - (if M = 1) specified displacement in direction in metres.

R (1) - x component of externally applied force acting at node (in dynes)

R (2) - y-component of externally applied force acting at node (in dynes)

Card 5: (Only included if B = 1 on card 3)

1–10	11–20
YSUB	ROSUB
F10.0	F10.0

where

YSUB - Y co-ordinate of base of model (in Km.)

ROSUB - density of substratum (gm/cc.)

Card 6: Element data (NELEM cards)

1–10	11–20	21–30	31–40	41–50	51–60	61–70
IEREF	NA	NB	NC	E	V	RO
I10	I10	I10	I10	E10.0	F10.0	F10.0

IEREF - Reference number of element.

NA,NB,NC - Reference numbers of the three nodes comprising the element (order irrelevant)

E - Young's modulus (dynes/sq.cm.)

V - Poisson's ratio

RO - density (gm/cc.)

Card 7: (only included if c = 1 on card 3)

1 2 … 23	24 25	26 27	28 29
bABC. . . . . .UV	NR		NC
	I2		I2

b - blank

ABC. . . . . . UV - literally

NR - Number of rows in model

NC - Number of columns in model.

Storage:

NPOIN - No. of nodal points.

NELEM - No. of elements.

RNODAR (NPOIN,4) - Nodal point data

1 - x co-ord
2 - y co-ord
3 - x displacement if specified.
4 - y displacement if specified.

ISDISP (NPOIN,2) - displacement markers

ISDISP (NPOIN,1) = 0 - No displacement specified in x direction
1 - displacement specified in x direction.

ISDISP (NPOIN,2) = 0 - No displacement specified in y direction
1 - displacement specified in y direction.

IREFEL (NELEM,3) - Nodal points comprising the elements.

ELACON (NELEM,2) - Elastic constants

1 - Young's Modulus.

2 - Poisson's ratio.

RO (NELEM) - density of elements

RFORCE (2NPOIN) - applied nodal forces.

After CHOLS it contains the displacements of the nodal points.

MARK (1) - Isostacy marker (1 - yes; 0 - No)

MARK (2)	-	Visual display marker (1 - yes; 0 - no)
MARK (3)	-	Punched output marker (1 - yes; 0 - no)
IMK	-	plane strain or plane stress marker.
NBW	-	computed semi-band width of stiffness matrix
B (3,6)	-	B - matrix c.f. 2.5
D (3,3)	-	D - matrix c.f. 2.6.
ST( )	-	Stiffness matrix (semi-band width only stored)
YSUB	-	co-ordinate of base of model.
ROSUB	-	density of underlying material.
NBASEP	-	number of nodes on the base of model.

IBASEP (NBASEP) - the reference numbers of these nodes.

Choleski Routine

Used for solving the stiffness equation $[k]\{x\}=\{\delta\}$ for the unknown column matrix $\{x\}$, where det $k \neq 0$ and $[k]$ is symmetric. For such a $[k]$ we can find an upper triangular matrix U such that

$$[k] = U^T . U \qquad (A5.1)$$

Thus

$$U^T U\{x\} = \{\delta\}$$

and writing

$$U\{x\} = \{y\} \qquad (A5.2)$$

we have

$$U^T\{y\} = \{\delta\} \qquad (A5.3)$$

Having found the triangular matrix U the process consists of solving A5.3 for $\{y\}$ and then solving A5.2 for $\{x\}$.

U is computed using

$$u_{11} = \sqrt{k_{11}}$$

$$u_{1i} = k_{1i}/u_{11}$$

$$u_{ii} = \sqrt{\left(k_{ii} - \sum_{m=1}^{(i-1)} u_{mi}^2\right)} \quad (i>1) \qquad (A5.4)$$

$$u_{ij} = \left(k_{ij} - \sum_{m=1}^{(i-1)} u_{mi}u_{mj}\right)/u_{ii} \quad (j>i)$$

$$u_{ij} = 0 \quad (i>j)$$

Note that U can have purely complex components but if one element of a row is complex then the whole row is complex. $\{y\}$ is evaluated from A5.3 using

$$y_1 = \delta_1 / u_{11}$$
$$y_i = \left(\delta_i - \sum_{m=1}^{(i-1)} u_{mi}\, y_m\right) \Big/ u_{ii} \qquad \text{(A5.5)}$$

Note that if row i of U is purely complex then so is y_i.
$\{x\}$ is evaluated from A5.2 using

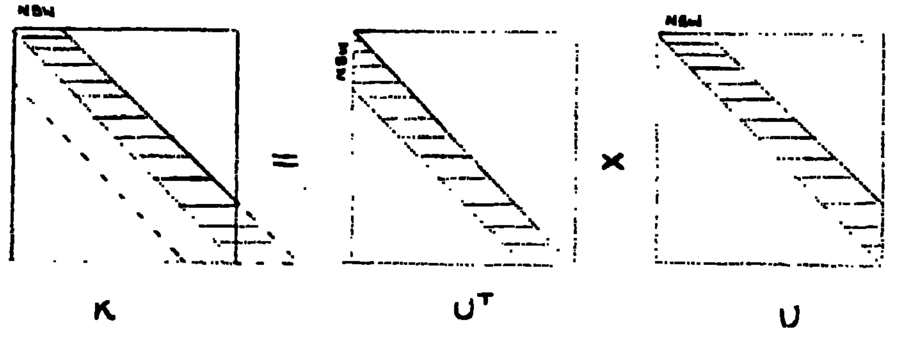
$$x_n = y_n / u_{nn} \qquad \text{(A5.6)}$$
$$x_i = \left(y_i - \sum_{m=i+1}^{n} u_{im} x_m\right) \Big/ u_{ii} \quad (i<n)$$

<u>Advantages of the method for finite element calculations.</u>

1. In all finite element calculations $[k]$ is positive de definite as well as being symmetric. This means that U and $\{y\}$ are <u>real</u> matrices. The routine presented here is general and includes the complex case as this can be done with little increase in storage requirements.

2. The matrix $[k]$ is banded for finite element calculations. It is easily seen that both U and $[k]$ have the same band width.

Fig.A5.1



To save storage only the semi-band width (shaded in (Fig.A5.1) of $[k]$ is stored. With storage space at a premium the great advantage of the Choleski routine is that when calculating U from k it can be stored by overwriting k (see A5.4). $\{\delta\}$ can be overwritten by $\{y\}$ (see A5.5) which in turn is overwritten by $\{x\}$ (see A5.6). Thus no extra storage space is required for the matrices $[U]$, $\{y\}$ and $\{x\}$.

<u>Notes on routine:</u>

Solving $[A]\{x\} = \{c\}$ Role of MAR (1)

(a) = 0 then row 1 of U and y_i are purely real.

(b) = 1 then row 1 of U and y_i are purely imaginary.

The real or imaginary part is stored for any component of U together with MAR. Thus if u_{kk} is <u>real</u> then we store as follows (A5.4,A5.5,A5.6,)

$$u_{kk} = \sqrt{a_{kk} - \sum_{\substack{m=1\\ \text{real } m}}^{k-1} u_{mk}^2 + \sum_{\substack{m=1\\ \text{imag } m}}^{k-1} u_{mk}^2}$$

$$\underset{k<j}{u_{kj}} = \frac{\left[a_{kj} - \sum_{\substack{m=1\\ \text{real } m}}^{k-1} u_{mk}u_{mj} + \sum_{\substack{m=1\\ \text{imag } m}}^{k-1} u_{mk}u_{mj}\right]}{u_{kk}} \qquad \text{(A5.7)}$$

$$y_k = \frac{\left[c_k - \sum_{\substack{m=1\\ \text{real } m}}^{k-1} u_{mk}y_m + \sum_{\substack{m=1\\ \text{imag } m}}^{k-1} u_{mk}y_m\right]}{u_{kk}}$$

$$x_k = \frac{\left[y_k - \sum_{m=k+1}^{n} u_{km}x_k\right]}{u_{kk}}$$

and if u_{kk} is imaginary we store

$$u_{kk} = \sqrt{\sum_{\substack{m=1 \\ \text{real } m}}^{k-1} u_{mk}^2 - \sum_{\substack{m=1 \\ \text{imag } m}}^{k-1} u_{mk}^2 - a_{kk}}$$

$$\underset{k<j}{u_{kj}} = \frac{\left[\sum\limits_{\substack{m=1 \\ \text{real } m}}^{k-1} u_{mk} u_{mj} - \sum\limits_{\substack{m=1 \\ \text{real } m}}^{k-1} u_{mk} u_{mj} - a_{kj} \right]}{u_{kk}}$$

(A5.8)

$$y_k = \frac{\left[\sum\limits_{\substack{m=1 \\ \text{real } m}}^{k-1} u_{mk} y_{mj} - \sum\limits_{\substack{m=1 \\ \text{imag } m}}^{k-1} u_{mk} y_{mj} - c_k \right]}{u_{kk}}$$

$$x_k = \frac{\left[y_k - \sum\limits_{m=k+1}^{n} u_{km} x_m \right]}{u_{kk}}$$

Note: The stiffness element k_{ij} is stored in ST (IN (I,J)).

Function IN

The stiffness matrix is symmetric and banded with a semi-band width NBW. To ensure economy of storage space the semi-band sidth only is stored in the linear array ST, i.e. the shaded area of k only is stored (Fig.A5.2). This part is stored row by row in ST.

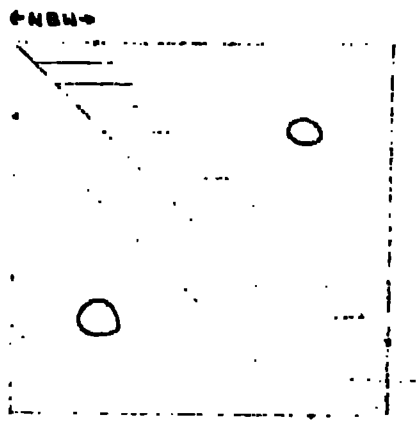


Fig.A5.2

Thus the $(ij)^{th}$ element of $[k]$ is stored in

$$ST\left(\left\{(i-1) \times NBW + (j-i+1)\right\}\right)$$

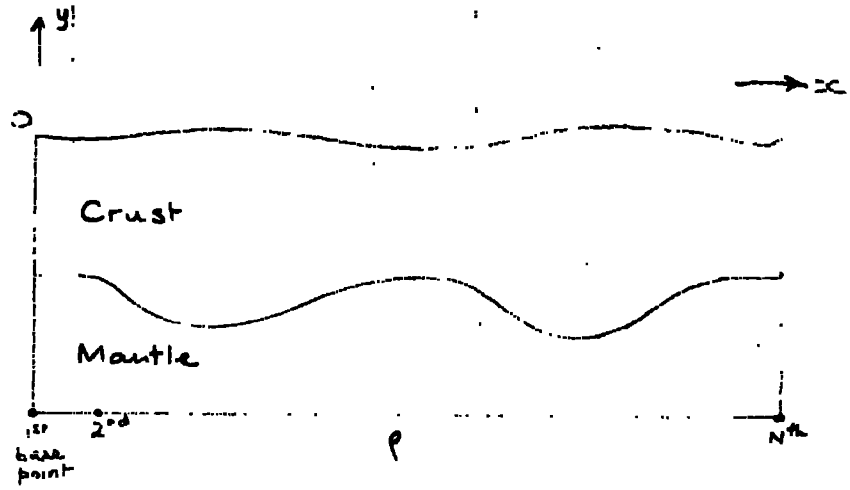
or ST (IN (I,J))

where IN(I,J) = (I - 1) x NBW + (J-I+1) (A5.9)

(A5.9)

Subroutine Isos:

Fig. A5.3



If isostatic forces are introduced on the base of the model then these are proportional to the displacement of the base. If the density of the underlying material is ρ and the displacement of the i^{th} base point (node b_i) in the y - direction is $\delta_{y_{b_i}}$, then the isostatic force exerted on this node is

$$-\rho g \delta_{y_{b_i}} \left\{ \frac{|x_{b_{i+1}} - x_{b_i}|}{2} + \frac{|x_{b_i} - x_{b_{i-1}}|}{2} \right\} \qquad \text{(A5.10a)}$$

unless it is the first or last base point, in which case it is

$$-\rho g \delta_{y_{b_1}} \left\{ \frac{|x_{b_1} - x_{b_2}|}{2} \right\} \qquad \text{(A5.10b)}$$

and

$$-\rho g \delta_{y_{b_N}} \left\{ \frac{|x_{b_N} - x_{b_{N-1}}|}{2} \right\} \qquad \text{(A5.10c)}$$

respectively.

Writing the isostatic force on the nodal point i as $-A_i \delta_{y_{bi}}$ where $A_i = o$ if i is not a base point, we have by adding these extra loads on to the applied force matrix

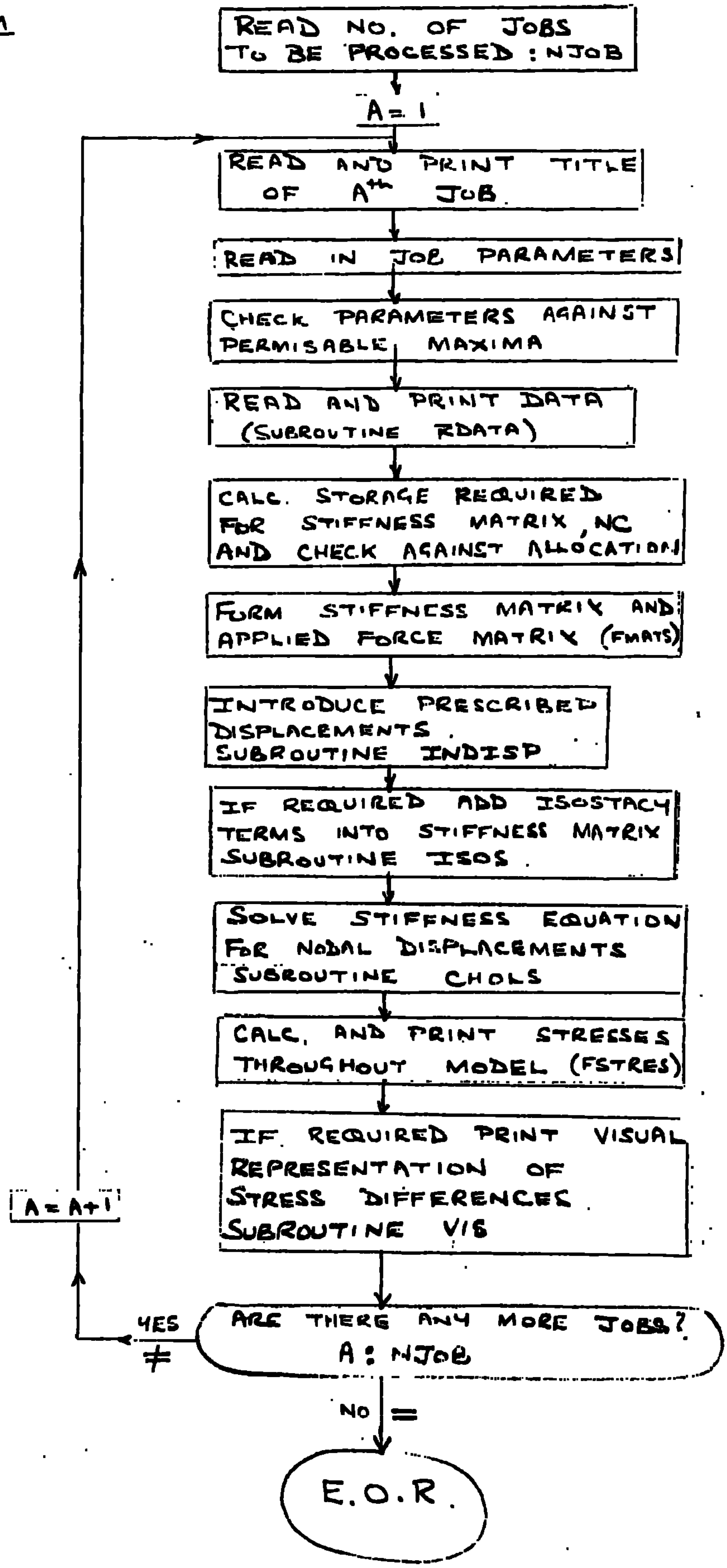
$$[K]\{\delta\} = \{R\} - [A]\{\delta\} \qquad (A5.11)$$

where

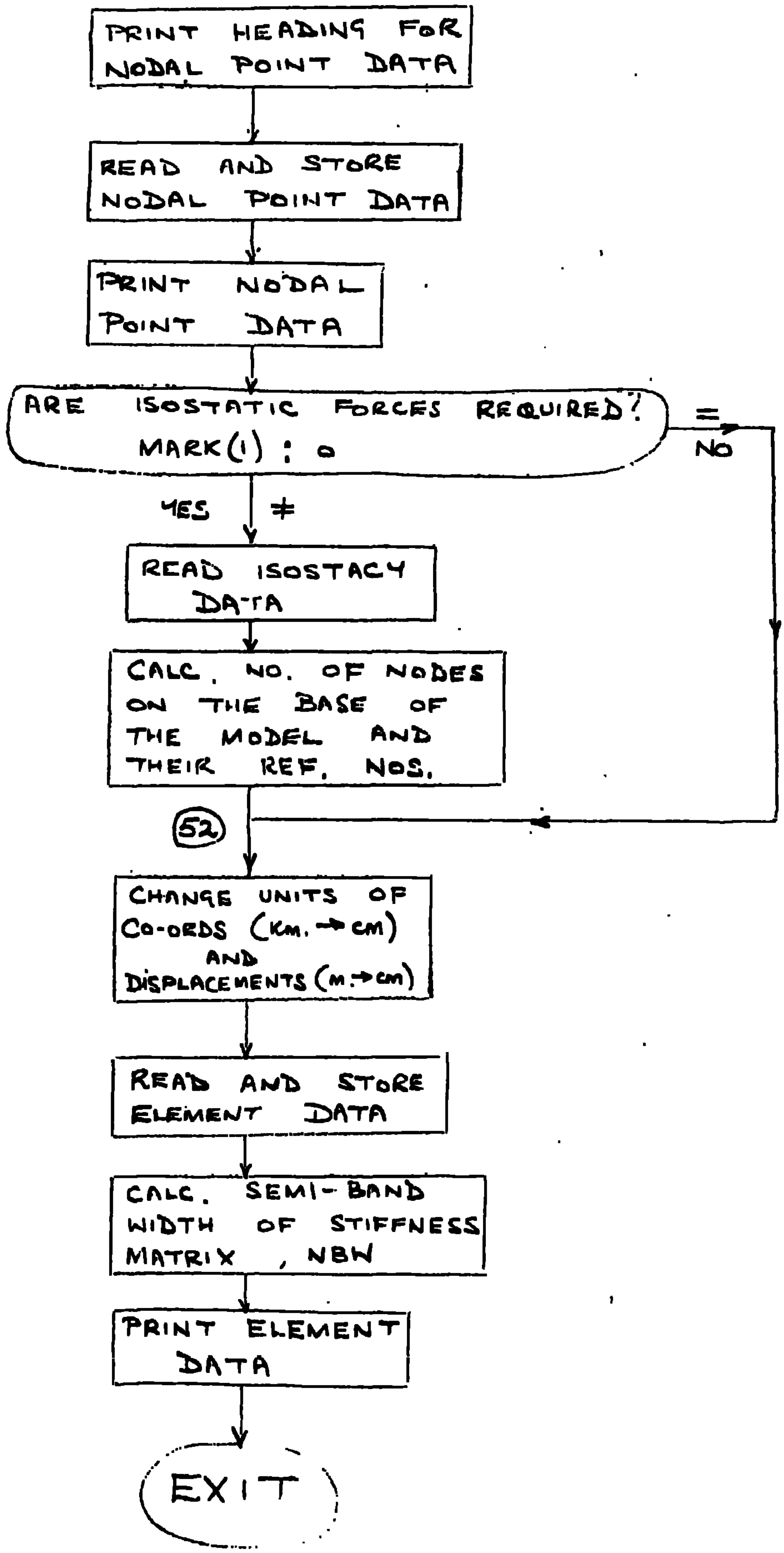
$$[A] = \begin{bmatrix} o & & & \\ & A_1 & & \\ & & o & \\ & & & A_2 \\ & & & & \ddots \end{bmatrix} \qquad (A5.12)$$

Thus replacing $[K]$ by $[K] + [A]$ introduces the required isostatic forces.

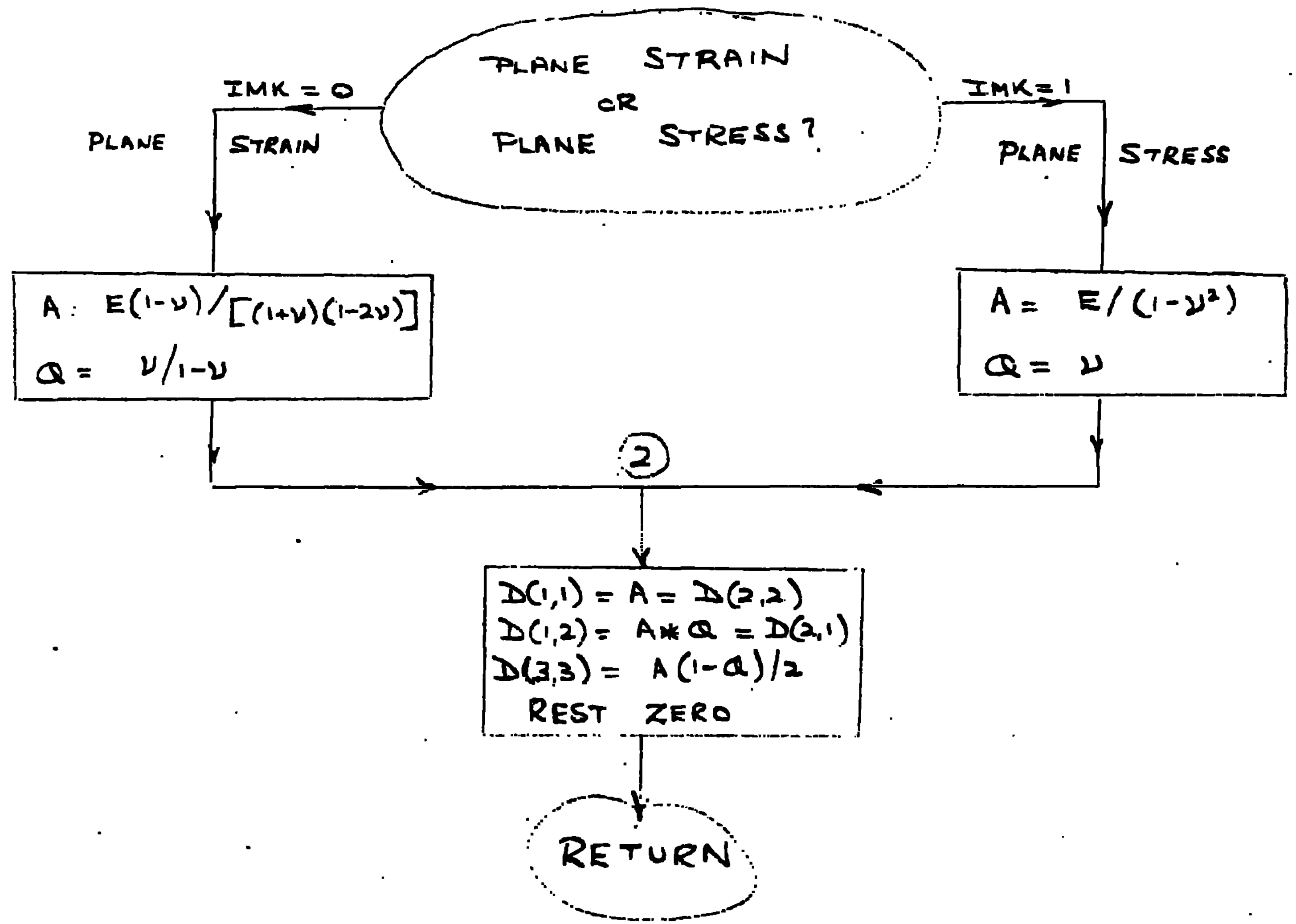
MAIN PROGRAM



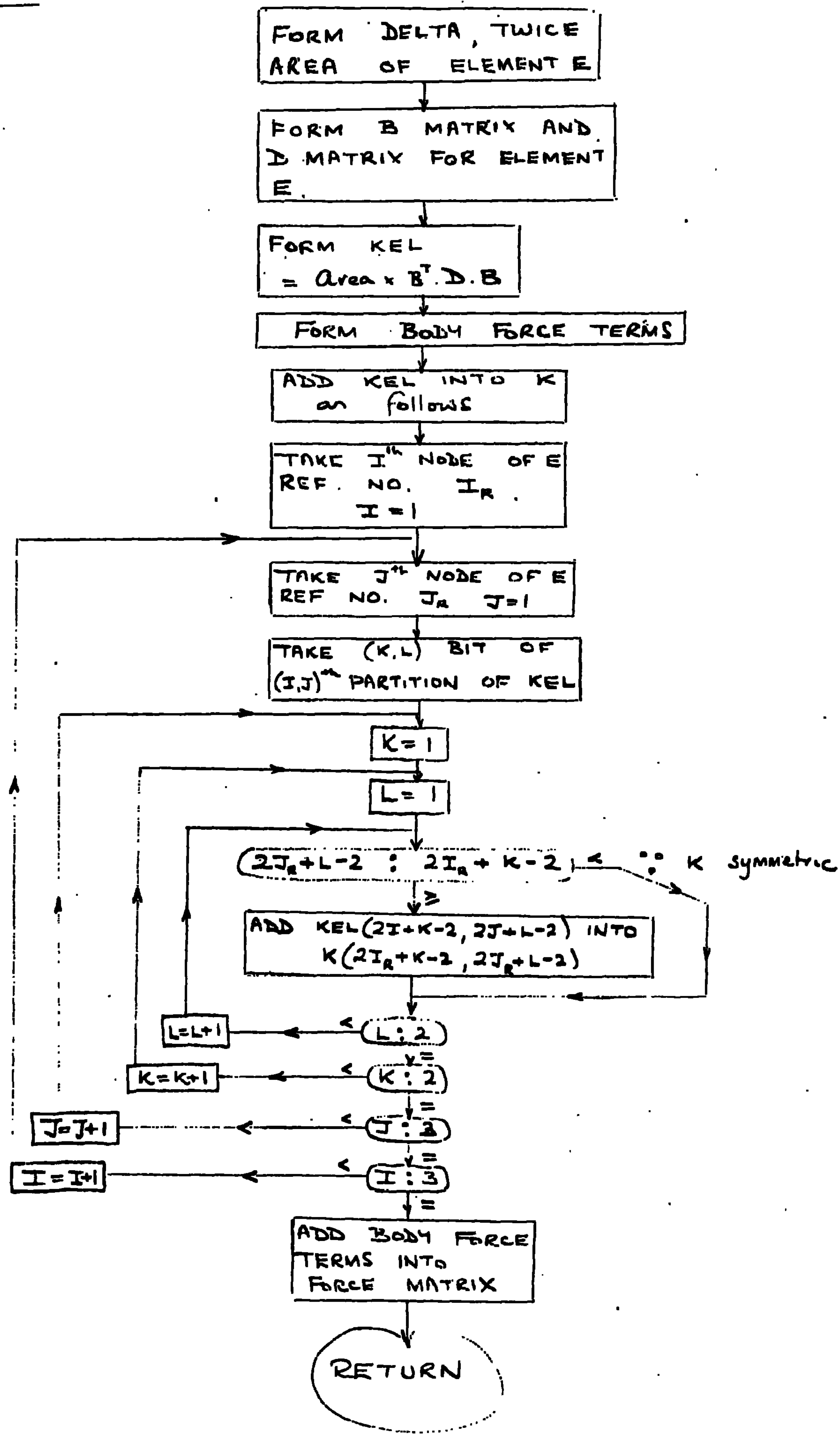
RDATA


PRINT HEADING FOR NODAL POINT DATA
READ AND STORE NODAL POINT DATA
PRINT NODAL POINT DATA
ARE ISOSTATIC FORCES REQUIRED? MARK(1) : 0
=
NO
YES
≠
READ ISOSTACY DATA
CALC. NO. OF NODES ON THE BASE OF THE MODEL AND THEIR REF. NOS.
52
CHANGE UNITS OF CO-ORDS (KM. → CM) AND DISPLACEMENTS (M → CM)
READ AND STORE ELEMENT DATA
CALC. SEMI-BAND WIDTH OF STIFFNESS MATRIX, NBW
PRINT ELEMENT DATA
EXIT

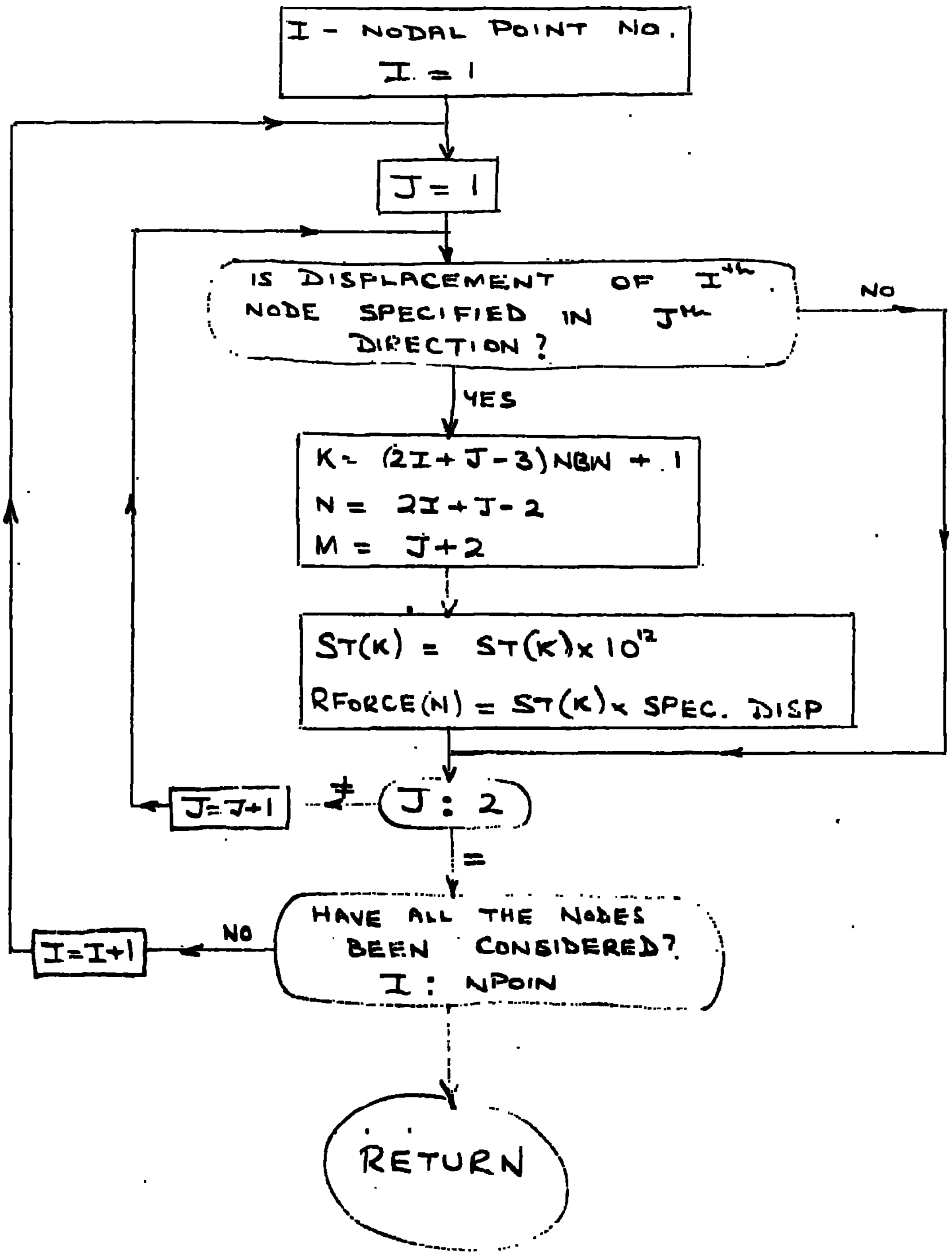
FORMD


PLANE STRAIN
OR
PLANE STRESS ?
IMK = 0
PLANE STRAIN
IMK = 1
PLANE STRESS
A : E(1-ν)/[(1+ν)(1-2ν)]
Q = ν/1-ν
A = E/(1-ν²)
Q = ν
2
D(1,1) = A = D(2,2)
D(1,2) = A*Q = D(2,1)
D(3,3) = A(1-Q)/2
REST ZERO
RETURN

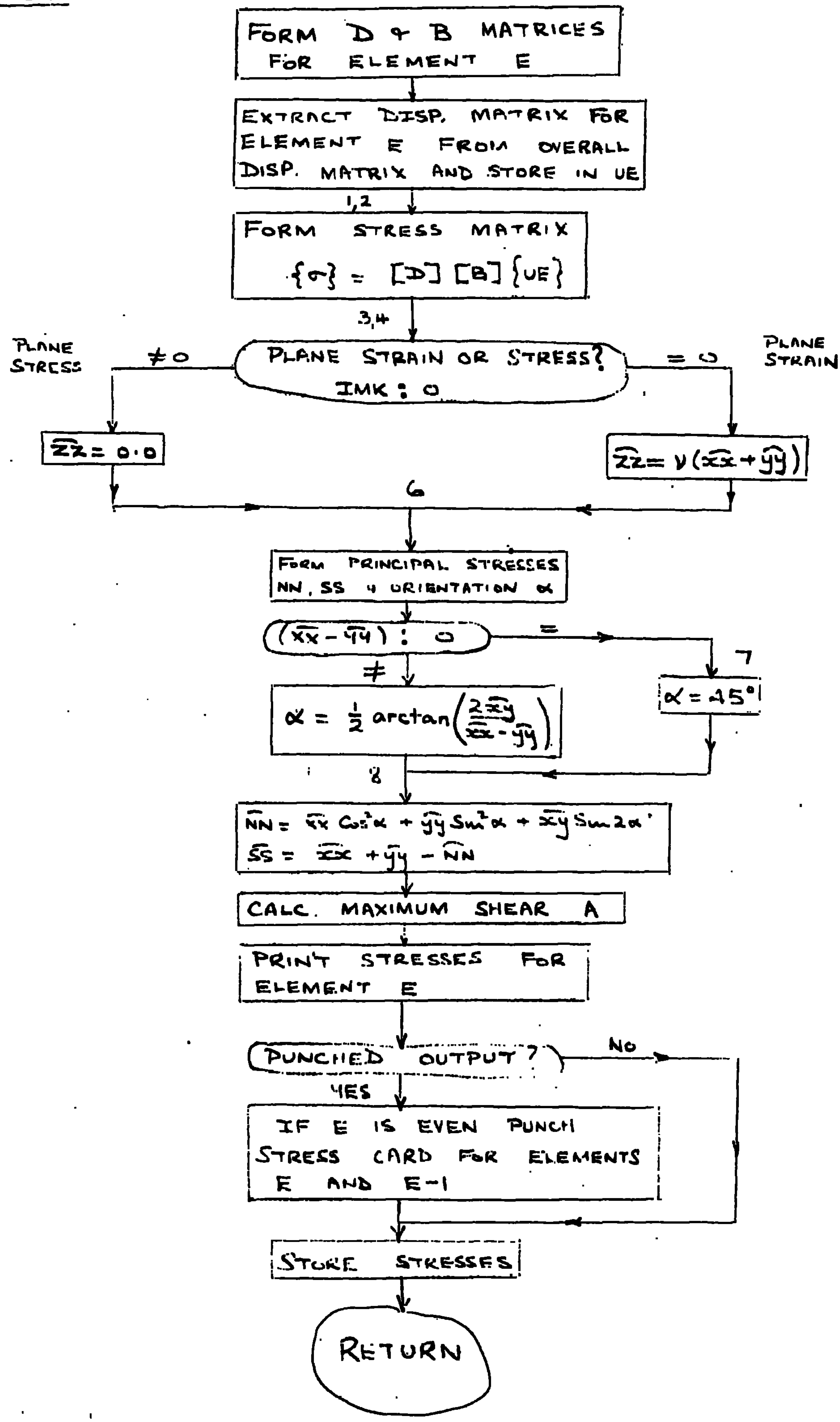
FMATS

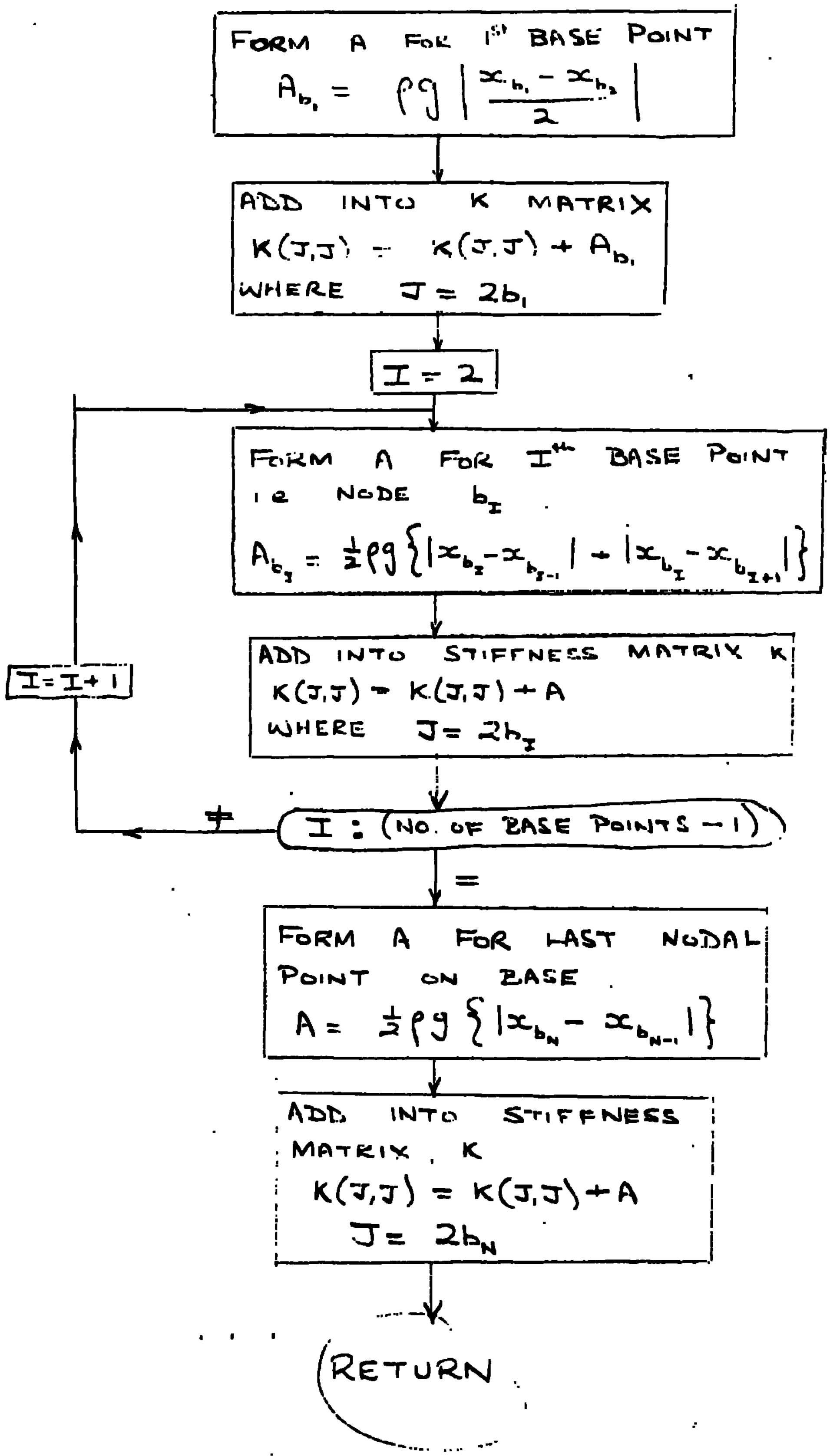

FORM DELTA, TWICE AREA OF ELEMENT E
FORM B MATRIX AND D MATRIX FOR ELEMENT E.
FORM KEL = area × B^T.D.B
FORM BODY FORCE TERMS
ADD KEL INTO K as follows
TAKE I^th NODE OF E REF. NO. I_R. I = 1
TAKE J^th NODE OF E REF NO. J_R J=1
TAKE (K,L) BIT OF (I,J)^th PARTITION OF KEL
K=1
L=1
(2J_R+L-2 : 2I_R+K-2)
<
∴ K symmetric
≥
ADD KEL(2I+K-2, 2J+L-2) INTO K(2I_R+K-2, 2J_R+L-2)
L=L+1
L : 2
K=K+1
K : 2
J=J+1
J : 3
I=I+1
I : 3
=
ADD BODY FORCE TERMS INTO FORCE MATRIX
RETURN

INDISP

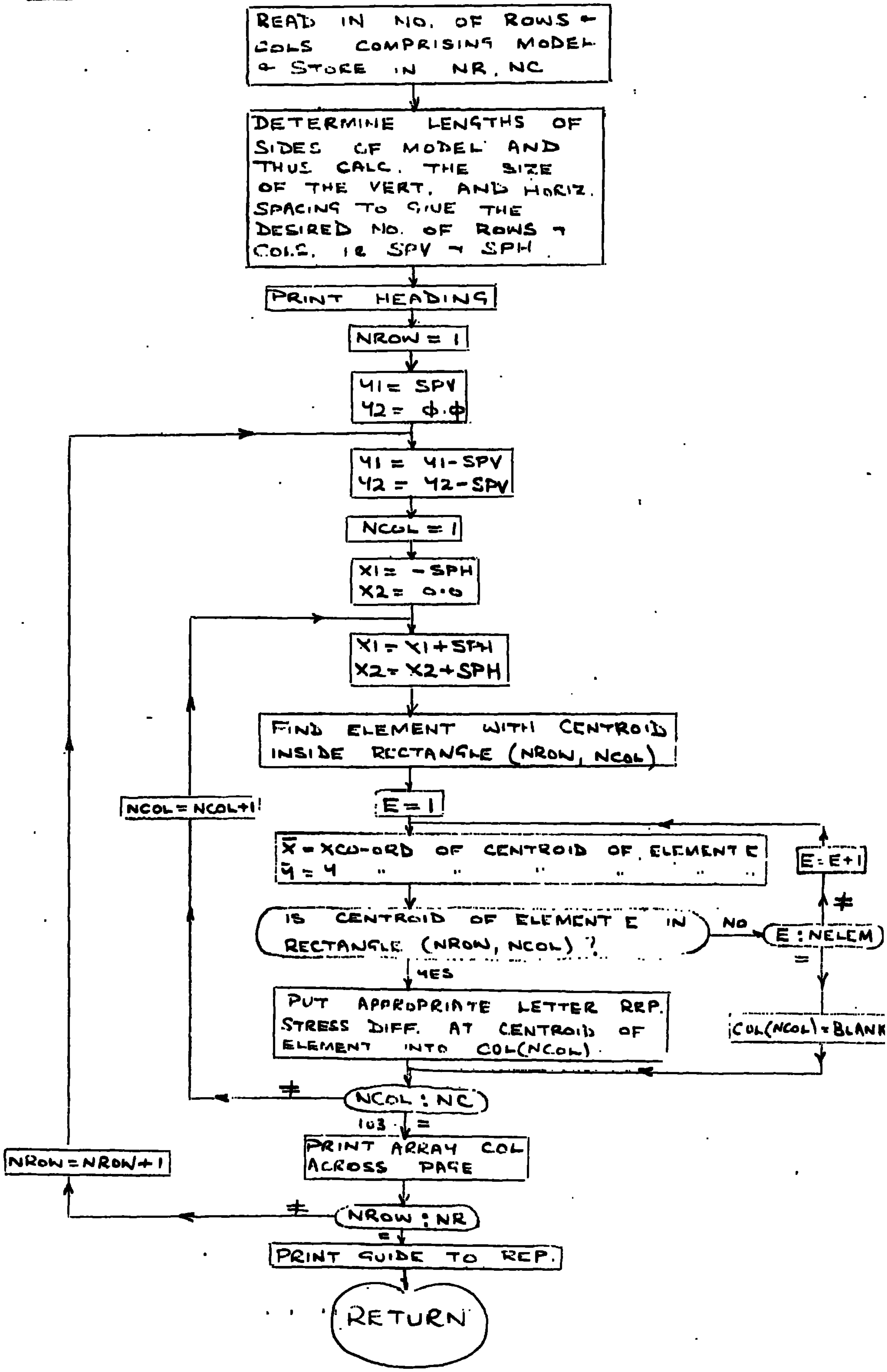

I - NODAL POINT NO.
I = 1
J = 1
IS DISPLACEMENT OF I^th NODE SPECIFIED IN J^th DIRECTION ?
NO
YES
K = (2I + J - 3) NBW + 1
N = 2I + J - 2
M = J + 2
ST(K) = ST(K) x 10^12
RFORCE(N) = ST(K) x SPEC. DISP
J = J+1
≠
J : 2
=
HAVE ALL THE NODES BEEN CONSIDERED?
I : NPOIN
NO
I = I+1
RETURN

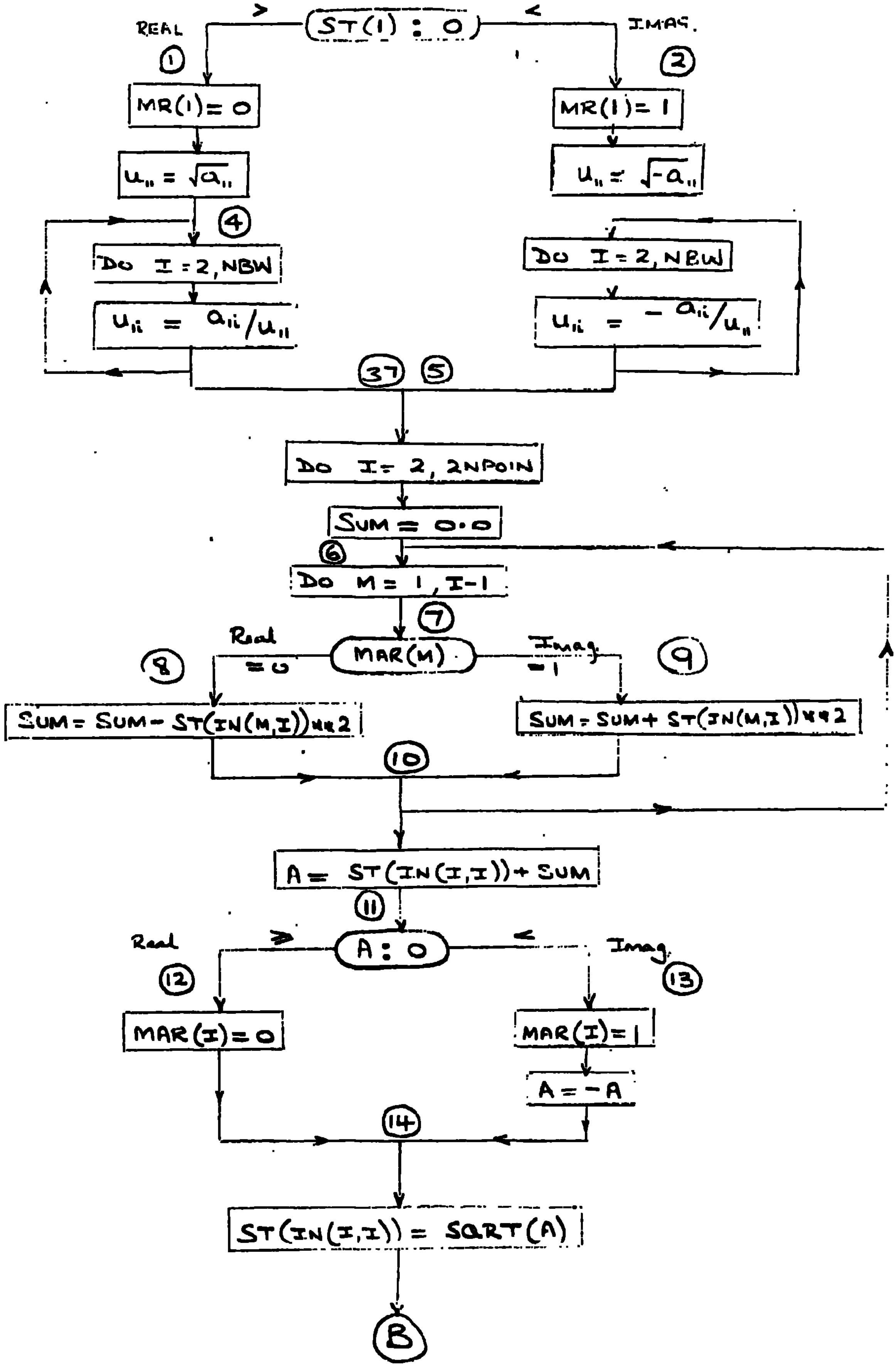
FSTRES

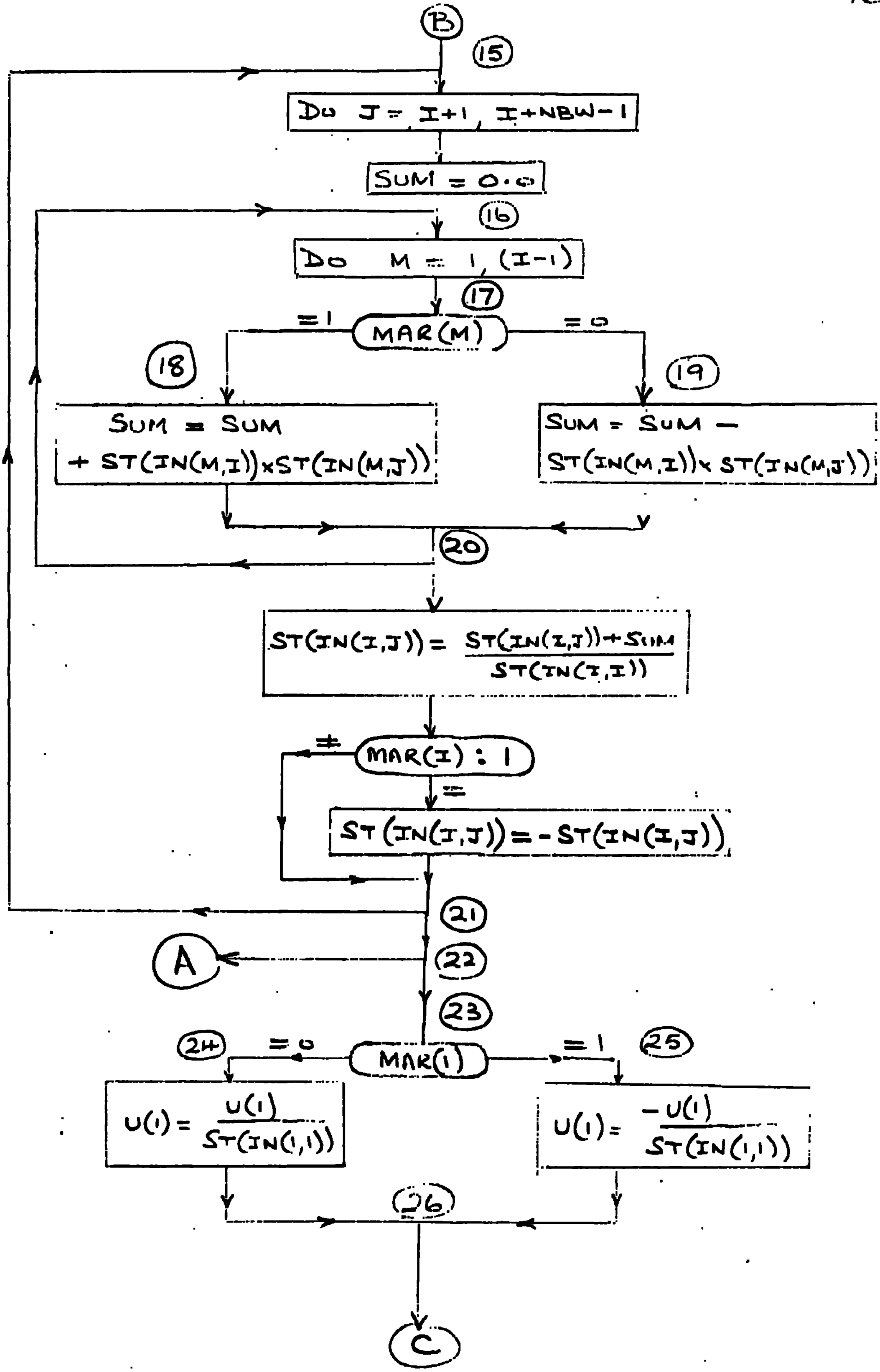

FORM D & B MATRICES FOR ELEMENT E
EXTRACT DISP. MATRIX FOR ELEMENT E FROM OVERALL DISP. MATRIX AND STORE IN UE
1,2
FORM STRESS MATRIX
{σ} = [D] [B] {UE}
3,4
PLANE STRESS
≠ 0
PLANE STRAIN OR STRESS? IMK : 0
= 0
PLANE STRAIN
ZZ = 0.0
ZZ = ν (XX + YY)
6
FORM PRINCIPAL STRESSES NN, SS & ORIENTATION α
(XX − YY) : 0
=
7
≠
α = ½ arctan(2XY / (XX − YY))
α = 45°
8
NN = XX Cos²α + YY Sin²α + XY Sin 2α
SS = XX + YY − NN
CALC. MAXIMUM SHEAR A
PRINT STRESSES FOR ELEMENT E
PUNCHED OUTPUT ?
NO
YES
IF E IS EVEN PUNCH STRESS CARD FOR ELEMENTS E AND E−1
STORE STRESSES
RETURN

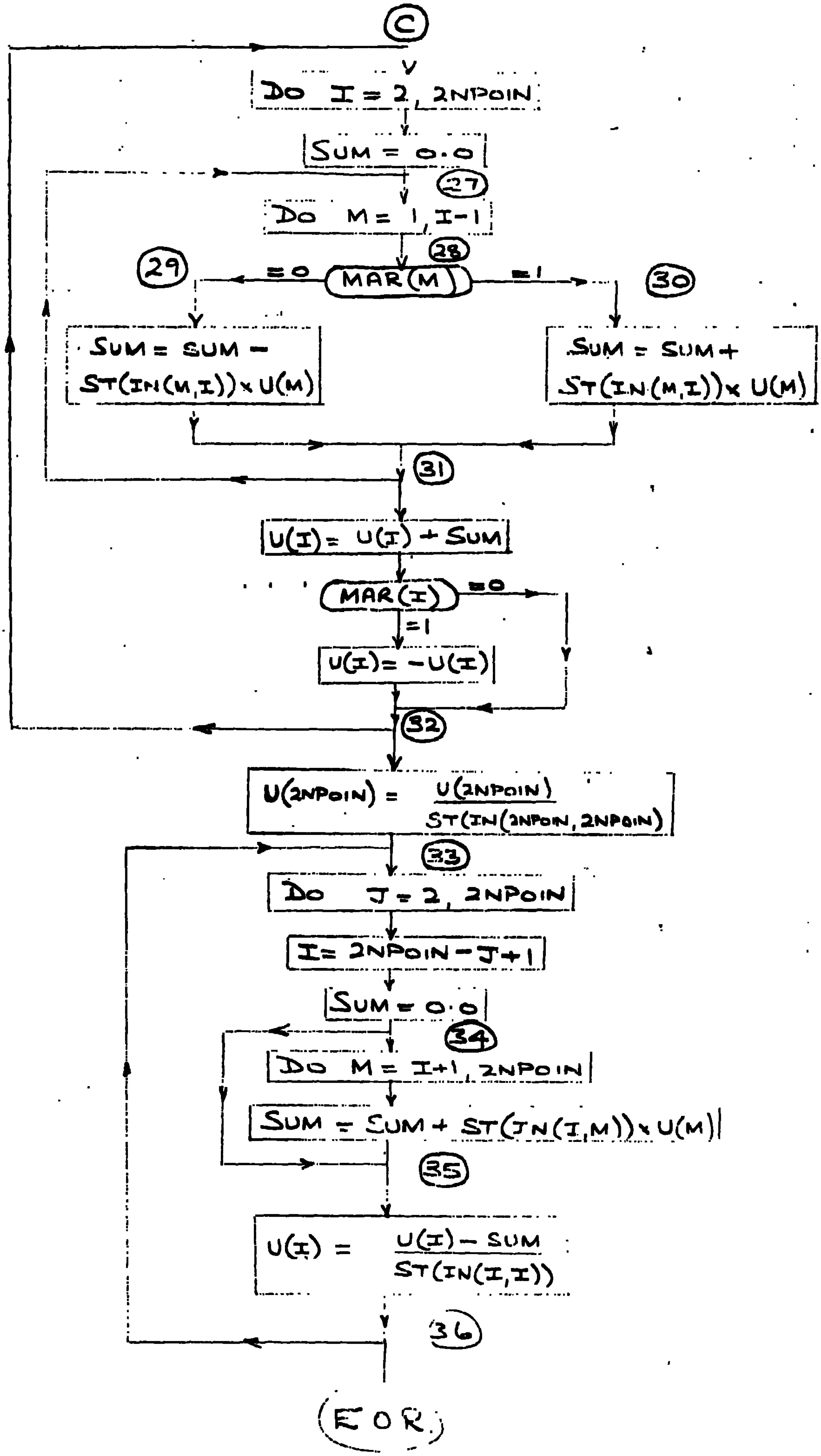

FORM A FOR 1ST BASE POINT
$A_{b_1} = \rho g \left| \frac{x_{b_1} - x_{b_2}}{2} \right|$
ADD INTO K MATRIX
$K(J,J) = K(J,J) + A_{b_1}$
WHERE $J = 2b_1$
I = 2
FORM A FOR Ith BASE POINT
i.e NODE b_I
$A_{b_I} = \frac{1}{2}\rho g \left\{ |x_{b_I} - x_{b_{I-1}}| + |x_{b_I} - x_{b_{I+1}}| \right\}$
ADD INTO STIFFNESS MATRIX K
$K(J,J) = K(J,J) + A$
WHERE $J = 2b_I$
I = I + 1
≠
I : (NO. OF BASE POINTS − 1)
=
FORM A FOR LAST NODAL POINT ON BASE
$A = \frac{1}{2}\rho g \left\{ |x_{b_N} - x_{b_{N-1}}| \right\}$
ADD INTO STIFFNESS MATRIX, K
$K(J,J) = K(J,J) + A$
$J = 2b_N$
RETURN

VIS


READ IN NO. OF ROWS + COLS COMPRISING MODEL + STORE IN NR, NC
DETERMINE LENGTHS OF SIDES OF MODEL AND THUS CALC. THE SIZE OF THE VERT. AND HORIZ. SPACING TO GIVE THE DESIRED NO. OF ROWS + COLS. IE SPV + SPH.
PRINT HEADING
NROW = 1
Y1 = SPV
Y2 = φ.φ
Y1 = Y1 - SPV
Y2 = Y2 - SPV
NCOL = 1
X1 = -SPH
X2 = 0.0
X1 = X1 + SPH
X2 = X2 + SPH
FIND ELEMENT WITH CENTROID INSIDE RECTANGLE (NROW, NCOL)
E = 1
X̄ = XCO-ORD OF CENTROID OF ELEMENT E
ȳ = Y " " " " "
IS CENTROID OF ELEMENT E IN RECTANGLE (NROW, NCOL)?
NO
E : NELEM
≠
E = E+1
=
COL(NCOL) = BLANK
YES
PUT APPROPRIATE LETTER REP. STRESS DIFF. AT CENTROID OF ELEMENT INTO COL(NCOL).
NCOL = NCOL+1
NCOL : NC
≠
103
=
PRINT ARRAY COL ACROSS PAGE
NROW = NROW+1
NROW : NR
≠
=
PRINT GUIDE TO REP.
RETURN


ST(1) : 0
>
<
REAL
IMAG.
1
2
MR(1) = 0
MR(1) = 1
$u_{11} = \sqrt{a_{11}}$
$u_{11} = \sqrt{-a_{11}}$
4
DO I = 2, NBW
DO I = 2, NBW
$u_{1i} = a_{1i}/u_{11}$
$u_{1i} = -a_{1i}/u_{11}$
37
5
DO I = 2, 2NPOIN
SUM = 0.0
6
DO M = 1, I-1
7
MAR(M)
Real
= 0
Imag
= 1
8
9
SUM = SUM - ST(IN(M,I))**2
SUM = SUM + ST(IN(M,I))**2
10
A = ST(IN(I,I)) + SUM
11
A : 0
≥
<
Real
Imag.
12
13
MAR(I) = 0
MAR(I) = 1
A = -A
14
ST(IN(I,I)) = SQRT(A)
B


B
15
DO J = I+1, I+NBW-1
SUM = 0.0
16
DO M = 1, (I-1)
17
=1
MAR(M)
=0
18
19
SUM = SUM + ST(IN(M,I))×ST(IN(M,J))
SUM = SUM − ST(IN(M,I))×ST(IN(M,J))
20
ST(IN(I,J)) = ST(IN(I,J))+SUM / ST(IN(I,I))
≠
MAR(I) : 1
=
ST(IN(I,J)) = -ST(IN(I,J))
21
A
22
23
24
=0
MAR(1)
=1
25
U(1) = U(1) / ST(IN(1,1))
U(1) = −U(1) / ST(IN(1,1))
26
C


C
DO I = 2, 2NPOIN
SUM = 0.0
27
DO M = 1, I-1
28
29
=0
MAR(M)
=1
30
SUM = SUM − ST(IN(M,I)) × U(M)
SUM = SUM + ST(IN(M,I)) × U(M)
31
U(I) = U(I) + SUM
MAR(I)
=0
=1
U(I) = −U(I)
32
U(2NPOIN) = U(2NPOIN) / ST(IN(2NPOIN, 2NPOIN))
33
DO J = 2, 2NPOIN
I = 2NPOIN − J + 1
SUM = 0.0
34
DO M = I+1, 2NPOIN
SUM = SUM + ST(IN(I,M)) × U(M)
35
U(I) = (U(I) − SUM) / ST(IN(I,I))
36
EOR

```
      DOUBLE PRECISION ST,REFORCE
      DOUBLE PRECISION SUMX,SUMY
      COMMON REFORCE(2000),ST(51500),RNODAR(1000,4),IREFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,B(3,6),D(3,3),IMK,RO(1500),
     2ISDISP(1000,2),MARK(3)
      COMMON/A/ YSUB,ROSUB,IBASEP(200),NBASEP

      REAL NN
      INTEGER E
      READ (5,6) NJOB
    6 FORMAT (I10)
      DO 1000 JN= 1,NJOB
      DO 9 I=1,51500
    9 ST(I)=0.0
C READ AND PRINT TITLE
      READ (5,1)
    1 FORMAT  (1H1,'
     1                                                  ')
      WRITE (6,1)
      READ(5,2) NPOIN,NELEM,IMK,(MARK(I),I=1,3)
    2 FORMAT(7I10)
      IF (NPOIN.GT.1000) GO TO 11
      IF (NELEM .GT. 1500) GO TO 12
      CALL RDATA
      NC = 2*NBW*NPOIN
      SUMX=0.0
      SUMY=0.0
C*******    BODY FORCES
      DO 80 E=1,NELEM
      DELTA = (RNODAR(IREFEL(E,3),2) * RNODAR(IREFEL(E,2),1))
     1        -(RNODAR(IREFEL(E,2),2) * RNODAR(IREFEL(E,3),1))
     2        -(RNODAR(IREFEL(E,1),1) * RNODAR(IREFEL(E,3),2))
     3        +(RNODAR(IREFEL(E,1),1) * RNODAR(IREFEL(E,2),2))
     4        +(RNODAR(IREFEL(E,1),2) * RNODAR(IREFEL(E,3),1))
     5        -(RNODAR(IREFEL(E,1),2) * RNODAR(IREFEL(E,2),1))
      DELTA = 0.5*ABS(DELTA)
      SUMY = SUMY-DELTA*RO(E)*981.0
   80 CONTINUE
      WRITE(6,88) SUMY
   88 FORMAT(10X,E15.4)
C*********       APPLIED FORCES
```

```
      DO 81 I=1,NPOIN
      J=2*I-1
      K=2*I
      SUMX = SUMX+RFORCE(J)
      SUMY = SUMY+RFORCE(K)
   81 CONTINUE
      WRITE(5,83) SUMX,SUMY
   83 FORMAT (/5HSUMX=,F15.4,5X,5HSUMY=,F15.4)
      IF (NC .GT. 51500) GO TO 13
C******  FORM STIFFNESS MATRIX K AND APPLIED FORCE MATRIX F
      DO 3 I=1,NELEM
      CALL FMATS(I)
    3 CONTINUE
C******  FORM R.H.S. OF STIFFNESS EQN. AND INTRODUCE PRESCRIBED DISP.
      CALL INDISP
C******  ADD ISOSTACY EFFECT INTO STIFFNESS MATRIX AND APPLIED FORCE MATRIX
      IF (MARK(1) .EQ. 1)CALL ISOS
C******  SOLVE STIFFNESS EQN. AND PRINT DISP. FOR EACH NODAL PNT.
      CALL CHOLS
      WRITE (5,5)
    5 FORMAT(1H1,//3X,7HELEMENT,12X,5HALPHA,20X,2HNN,21X,2HSS,21X,2HZZ,
     110X,6HSTRESS,/20X,9H(DEGREES),12X,3(13H(DYNES/SQ CM),10X),4HDIFF)
      DO 4 I=1,NELEM
      CALL FSTRES(I)
    4 CONTINUE
C*********      VISUAL DISPLAY OF STRESS DIFFERENCES
      IF (MARK(2) .EQ. 1)CALL VIS
 1000 CONTINUE
      GO TO 10
   11 I=1
      GO TO 999
   12 I=2
      GO TO 999
   13 I = 3
      GO TO 999
  999 CALL ERROR(I)
   10 STOP
      END
```

```
      SUBROUTINE RDATA
      DOUBLE PRECISION ST,REFORCE
      COMMON REFORCE(2000),ST(51500),RNODAR(1000,4),IREFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,P(3,6),D(3,3),IMK,RO(1500),
     2ISDISP(1000,2),MARK(3)
      COMMON/A/ YSUB,ROSUB,IBASER(200),NBASEP

      DIMENSION R(2)
      NBW=0
C     PRINT HEADING FOR NODAL POINT DATA
      WRITE (6,1) NPOIN,NELEM
    1 FORMAT (//32X,19H NO. OF NODAL PNTS.,22X,16H NO. OF ELEMENTS//38X,
     1I4,34X,I4)
      WRITE (6,3)
    3 FORMAT (////4X,12H REF. NO. OF,18X,8H CO-ORDS,22X,16H SPECIFIED DI
     1SP.,19X,14H APPLIED FORCE/5X,10H NODAL PNT,15X,2H X,12X,2H Y,18X,
     22H X,12X,2H Y,18X,2H X,12X,2H Y//)
C     READ AND STORE NODAL POINT DATA
      DO 2 I = 1,NPOIN
      READ (5,4) NREF,(RNODAR(NREF,J),J=1,2),ISDISP(NREF,1),RNODAR(NREF,
     13),ISDISP(NREF,2),RNODAR(NREF,4),R(1),R(2)
    4 FORMAT (I10,2F10.0,I2,F8.0,I2,F8.0,2E10.0)
C     FORM APPLIED FORCE MATRIX R
      M = NREF*2-1
      DO 11 J=1,2
      REFORCE(M) = R(J)
      M = M+1
   11 CONTINUE
    2 CONTINUE
C     PRINT NODAL POINT DATA
      DO 12 I=1,NPOIN
      K=2*I-1
      L=2*I
C*******  IS X DISP. SPECIFIED.
      IF (ISDISP(I,1) .EQ. 0) GO TO 13
C*******   Y DISP SPECIFIED.  IS Y DISP. SPECIFIED.
   16 IF (ISDISP(I,2) .EQ. 0) GO TO 18
C*******  BOTH  X AND Y DISP. SPECIFIED.
   20 WRITE (6,21) I,(RNODAR(I,J),J=1,4),REFORCE(K),REFORCE(L)
      GO TO 12
C*******  X DISP. ONLY SPECIFIED.
```

```
   18 WRITE (6,32) I,(RNODAR(I,J),J=1,3),REFORCE(K),REFORCE(L)
      GO TO 12
C*******  X DISP. NOT SPECIFIED.  IS Y DISP. SPECIFIED
   13 IF (ISDISP(I,2) .EQ. 0) GO TO 15
C*******  Y DISP. ONLY SPECIFIED.
   17 WRITE (6,33) I,(RNODAR(I,J),J=1,2),RNODAR(I,4),REFORCE(K),REFORCE(L)
      GO TO 12
C*******  [illegible] DISP. SPECIFIED
   15 WRITE (6,34) I,(RNODAR(I,J),J=1,2),REFORCE(K),REFORCE(L)
   31 FORMAT (7X,I4,15X,2(F9.3,5X),2(6X,F9.3),11X,E10.3,3X,E10.3)
   32 FORMAT (7X,I4,15X,2(F9.3,5X),6X,F9.3,26X,E10.3,3X,E10.3)
   33 FORMAT (7X,I4,15X,2(F9.3,5X),20X,F9.3,12X,E10.3,3X,E10.3)
   34 FORMAT (7X,I4,15X,2(F9.3,5X),41X,E10.3,3X,E10.3)
   12 CONTINUE                                                          39
      IF (MARK(1) .EQ. 0)GO TO 52
      READ(5,44)YSUB,POSUB
   44 FORMAT(2F10.0)
      NRASEP=0
      DO 45 I=1,NPOIN
      IF (RNODAR(I,2) .NE. YSUB)GO TO 45
      NRASEP=NRASEP+1
      IRASEP(NRASEP)=I
   45 CONTINUE
C*******  CHANGE UNITS OF CO-ORDS AND DISPLACEMENTS
   52 DO 4  I=1,NPOIN
      DO 39  JY = 1,2
      MY = JY+2
      RNODAR(I,JY) = RNODAR(I,JY)*1.0E5
      RNODAR(I,MY) = RNODAR(I,MY)*1.0E2
   39 CONTINUE
    4 CONTINUE
C     READ AND STORE ELEMENT DATA                                       40
      WRITE (6,14)                                                      41
   14 FORMAT (1H1,//9X,4H REF NO.,16X,23H NODAL PNTS. OF ELEMENT,19X,19H
     1 ELASTIC CONSTANTS,15X,3H PR,/7X,11H OF ELEMENT,16X,4H (1),4X,4H (
     22),4X,4H (3),2 X,7H E(CGS),7X,7H V,15X,9H (GM/CC))
[illegible]
      DO 25 I=1,[illegible]                                             43
      READ (5,27)IEREF   ,(IREFEL(IEREF   ,J),J=1,3),(ELACON(IEREF   ,K)46
     1,K=1,2),PR(IEREF)
   27 FORMAT (4I10,E10.0,2F10.0)
      IA =IABS(IREFEL(IEREF,1) - IREFEL(IEREF,2))
```

```
      IB =IABS(IREFEL(IEREF,1) - IREFEL(IEREF,3))
      IC =IABS(IREFEL(IEREF,2) - IREFEL(IEREF,3))
      NBW = MAX0(IA,IB,IC,NBW)
   25 CONTINUE
      NBW = 2 + 2*NBW
C     PRINT ELEMENT DATA
      DO 30 I=1,NELEM
      WRITE (6,19) I,(IREFEL(I,J),J=1,3),(ELACON(I,K),K=1,2),RO(I)
   19 FORMAT (11X,I4,19X,3(I4,4X),16X,E9.3,4X,F5.3,15X,F5.3)
   30 CONTINUE
C*********     READ LAYER DATA
      END


      SUBROUTINE FORMD(N)
      DOUBLE PRECISION ST,REFORCE
      COMMON RLFORCE(3000),ST(51500),RNODAS(1000,4),IREFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,B(3,6),D(3,3),IMK,RO(1500),
     2ISDISP(1000,2),MARK(3)
C FORM D MATRIX FOR ELEMENT N
C  IF IMK=1 PLANE STRESS . IF IMK=0 THEN PLANE STRAIN
      IF (IMK .EQ. 1) GO TO 1
C  PLANE STRAIN
      A= ELACON(N,1)*(1.0 -ELACON(N,2))/((1.0 + ELACON(N,2))*(1.0 - 2*EL
     1ACON(N,2)))
      B= ELACON(N,2)/(1.0 - ELACON(N,2))
      GO TO 2
C     PLANE STRESS
    1 A= ELACON (N,1)/(1.0 - ELACON(N,2)**2)
      B= ELACON (N,2)
C     FORM MATRIX D
    2 D(1,1) = A
      D(1,2) = A*B
      D(1,3) = 0.0
      D(2,1) = A*B
      D(2,2) = A
      D(2,3) = 0.0
      D(3,1) = 0.0
      D(3,2) = 0.0
      D(3,3) = A*(1.0 - B)/2.0
```

```
      RETURN
      END
```

```
      SUBROUTINE FORMB (N)
C     FORM (B) THE STAIN - DISP. MATRIX FOR ELEMENT N
      DOUBLE PRECISION ST,RFORCE
      COMMON RFORCE(2000),ST(51500),RNODAR(1000,4),IREFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,B(3,6),D(3,3),IMK,RP(1500),
     2ISDISP(1000,2),MARK(3)
      DIMENSION Q(3),C(3)
      DELTA = (RNODAR(IREFEL(N,3),2) * RNODAR(IREFEL(N,2),1))
     1        -(RNODAR(IREFEL(N,2),2) * RNODAR(IREFEL(N,3),1))
     2        -(RNODAR(IREFEL(N,1),1) * RNODAR(IREFEL(N,2),2))
     3        +(RNODAR(IREFEL(N,1),1) * RNODAR(IREFEL(N,3),2))
     4        +(RNODAR(IREFEL(N,1),2) * RNODAR(IREFEL(N,2),1))
     5        -(RNODAR(IREFEL(N,1),2) * RNODAR(IREFEL(N,3),1))
      Q(1)=(RNODAR(IREFEL(N,2),2)-RNODAR(IREFEL(N,3),2))/DELTA
      C(1)=(RNODAR(IREFEL(N,3),1)-RNODAR(IREFEL(N,2),1))/DELTA
      Q(2)=(RNODAR(IREFEL(N,3),2)-RNODAR(IREFEL(N,1),2))/DELTA
      C(2)=(RNODAR(IREFEL(N,1),1)-RNODAR(IREFEL(N,3),1))/DELTA
      Q(3)=(RNODAR(IREFEL(N,1),2)-RNODAR(IREFEL(N,2),2))/DELTA
      C(3)=(RNODAR(IREFEL(N,2),1)-RNODAR(IREFEL(N,1),1))/DELTA
      B(1,1)=Q(1)
      B(1,2)=0.0
      B(1,3)=Q(2)
      B(1,4)=0.0
      B(1,5)=Q(3)
      B(1,6)=0.0
      B(2,1)=0.0
      B(2,2)=C(1)
      B(2,3)=0.0
      B(2,4)=C(2)
      B(2,5)=0.0
      B(2,6)=C(3)
      B(3,1)=C(1)
      B(3,2)=Q(1)
      B(3,3)=C(2)
      B(3,4)=Q(2)
      B(3,5)=C(3)
```

```
      B(3,4)=D(3)
      RETURN
      END

      SUBROUTINE FMATS(E)
      DOUBLE PRECISION ST,RFORCE
      COMMON RFORCE(2000),ST(51510),RNODAR(1060,4),IREFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,B(3,4),D(3,3),TMK,PD(1500),
     2ISDISP(1000,2),MARK(3)
      DIMENSION KEL(6,6),FEL(6)
      INTEGER E
      REAL KEL
      DELTA = (RNODAR(IREFEL(E,3),2) * RNODAR(IREFEL(E,2),1))
     1        -(RNODAR(IREFEL(E,2),2) * RNODAR(IREFEL(E,3),1))
     2        -(RNODAR(IREFEL(E,1),1) * RNODAR(IREFEL(E,3),2))
     3        +(RNODAR(IREFEL(E,1),1) * RNODAR(IREFEL(E,2),2))
     4        +(RNODAR(IREFEL(E,1),2) * RNODAR(IREFEL(E,3),1))
     5        -(RNODAR(IREFEL(E,1),2) * RNODAR(IREFEL(E,2),1))
C*********        DELTA = 2*AREA IN THIS ROUTINE
      DELTA = ABS(DELTA)
      C=981.
C      FORM (B) MATRIX FOR ELEMENT (E)
      CALL FORMB(E)
C      FORM (D) MATRIX FOR ELEMENT (E)
      CALL FORMD(E)
C      FORM (K) MATRIX FOR ELEMENT (E)
      DO 1 I=1,6
      DO 1 J=1,6
      KEL(I,J) = 0.0
      IF (I .GT. J) GO TO 5
      DO 2 L = 1,3
      DO 2 M = 1,3
      KEL(I,J) = KEL(I,J)+(B(L,I)*D(L,M)*B(M,J))
    2 CONTINUE
      KEL(I,J) = (DELTA*KEL(I,J))/2.0
      GO TO 1
    5 KEL(I,J) = KEL(J,I)
    1 CONTINUE
C      FORM (F) MATRIX FOR ELEMENT (E)
```

```
      A = -(PD(E)*G*DELTA)/6.0
      DO 3 I=1,5,2
      J=I+1
      FEL(I)=0.0
      FEL(J)=A
    3 CONTINUE
C     FORM STIFFNESS MATRIX (K) AND STORE IN ST( ).
      DO 4 I=1,3
      KA = (IREFEL(E,I)-1)*2
      KB = (I-1)*2
      DO 4 J=1,3
      LA = (IREFEL(E,J)-1)*2
      LB = (J-1)*2
      DO 4 K=1,2
      KC = KA+K
      KD = KB+K
      DO 4 L=1,2
      LC = LA+L
      LD = LB+L
      IF (KC .GT. LC) GO TO 4
      MIN = (KC - 1)*NBW + LC - KC + 1
      ST(MIN)= ST(MIN) + KEL(KD,LD)
    4 CONTINUE
C     FORM  MATRIX  F
      DO 7 I=1,3
      DO 7 J=1,2
      K = 2*I - 2 + J
      M = 2*IREFEL(E,I) + J - 2
      RFORCE(M)=RFORCE(M)+FEL(K)
    7 CONTINUE
      RETURN
      END


      SUBROUTINE INDISP
      DOUBLE PRECISION ST,RFORCE
      COMMON RFORCE(2000),ST(51500),NODAR(1000,4),IREFEL(1500,3),
     1FLACON(1500,2),NELEM,NPOIN,NBW,B(3,6),D(3,3),INK,RD(1500),
     2ISDISP(1000,2),MARK(3)
C     INTRODUCE  PRESCRIBED DISP.
```

```
      DO 1 I=1,NPOIN
      DO 2 J=1,2
      IF (ISDISP(I,J).EQ.0) GO TO 2
C     DISP SPECIFIED
      K = (2*I+J-3)*NBW+1
      N = 2*I+J-2
      M=J+2
      ST(K) = ST(K)*1.0E12
      REFORCE(N) = ST(K)*RNODAR(I,M)
    2 CONTINUE
    1 CONTINUE
      RETURN
      END


      SUBROUTINE CHOLS
      DOUBLE PRECISION ST,REFORCE,U,SUM,A
      COMMON REFORCE(2000),ST(51500),RNODAR(1000,4),IREFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,B(3,6),D(3,3),IMK,PO(1500),
     2ISDISP(1000,2),MARK(3)
      DIMENSION MAR(2000),U(2000)
      DIMENSION RA(3),PP(3)
      EQUIVALENCE (ISDISP,MAR)
      EQUIVALENCE (REFORCE,U)
      NL = 2*NPOIN
C ******     FORM V AND STORE IN ST
      IF (ST(1) .GE. 0.0) GO TO 1
    2 MAR(1) = 1
      ST(1) = -ST(1)
      ST(1) = DSQRT(ST(1))
    3 DO 5  I = 2,NBW
      ST(I) =  -ST(I)/ST(1)
    5 CONTINUE
      GO TO 37
    1 MAR(1) = 0
      ST(1) = DSQRT(ST(1))
    4 DO 37  I = 2,NBW
      ST(I) = ST(I)/ST(1)
   37 CONTINUE
C *******  I - ROW NO.        J - COLUMN NO.
```

```
      DO 22 I = 2,NA
      SUM = 0.0
      ND = I-1
    4 DO 17 M = 1,ND
      NE = M+NBW-1
      IF (I .GT. NE) GO TO 10
    7 IF (MAR(M) .EQ. 0) GO TO 8
    9 SUM = SUM + ST(IN(M,I))**2
      GO TO 10
    8 SUM = SUM - ST(IN(M,I))**2
   10 CONTINUE
      A = ST(IN(I,I)) + SUM
   11 IF (A .GE. 0.0) GO TO 12
   13 MAR(I) = 1
      A = -A
      GO TO 14
   12 MAR(I) = 0
   14 ST(IN(I,I)) = DSQRT(A)
      NC = I+1
      NG = I+NBW-1
   15 DO 21  J = NC,NP
      SUM = 0.0
   16 DO 20  M = 1,ND
      NE = M+NBW-1
      IF (J .GT. NE) GO TO 20
   17 IF (MAR(M) .EQ. 0) GO TO 19
   18 SUM = SUM + ST(IN(M,I))*ST(IN(M,J))
      GO TO 20
   19 SUM = SUM - ST(IN(M,I))*ST(IN(M,J))
   20 CONTINUE
      ST(IN(I,J)) = (ST(IN(I,J))+SUM)/ST(IN(I,I))
   18 (MAR(I) .EQ. 1) ST(IN(I,J)) = -ST(IN(I,J))
   21 CONTINUE
   22 CONTINUE
C *******     FORM Y AND STORE IN U
   23 IF (MAR(1) .EQ. 0) GO TO 24
   25 U(1) = -U(1)/ST(1)
      GO TO 26
   24 U(1) = +U(1)/ST(1)
   26 DO 32 J = 2,NA
      ND = J-1
```

```
      SUM = 0.0
   27 DO 31 M = 1,ND
      NI = M+NBW-1
      IF (I .GT. NE) GO TO 31
   28 IF (MAR(M) .EQ. 0) GO TO 29
   30 SUM = SUM + ST(IN(M,I))*U(M)
      GO TO 31
   29 SUM = SUM - ST(IN(M,I))*U(M)
   31 CONTINUE
      U(I) = (U(I)+SUM)/ST(IN(I,I))
      IF (MAS(I) .EQ. 1) U(I) = -U(I)
   32 CONTINUE
C *******       FORM X AND STORE IN U
      U(NA) = U(NA)/ST(IN(NA,NA))
   33 DO 36 J=2,NA
      I=NA+1-J
      NC = I+1
      SUM = 0.0
   34 DO 35 M = NC,NA
      NE = I+NBW-1
      IF (M .GT. NE) GO TO 35
      SUM = SUM + ST(IN(I,M))*U(M)
   35 CONTINUE
      U(I) =(U(I)-SUM)/ST(IN(I,I))
   36 CONTINUE
C *******    PRINT NODAL DISPLACEMENTS
      WRITE (6,38)
   38 FORMAT (1H1,///,41X,11H NODAL PNT.,13X,18H DISPLACEMENT (M) ,/65X,
     12H X,12X,2H Y)
      DO 42  I = 1,NPOIN
      N = 2*I-1
      M = 2*I
C******* CHANGE UNITS OF DISPLACEMENTS
      DUMMYA = U(N)*1.0E-2
      DUMMYB = U(M)*1.0E-2
      WRITE(6,39) I,DUMMYA,DUMMYB
   39 FORMAT (43X,I4,16X,E10.3,6X,E10.3)
   42 CONTINUE
   50 RETURN
      END
```

```
      FUNCTION IN(I,J)
C ******  TO CALCULATE  THE ARRAY NO. FOR THE (I,J)TH ELEMENT OF A
C ******  SYMMETRIC MATRIX OF SEMI-BAND WIDTH NBW
      DOUBLE PRECISION ST,RFORCE
      COMMON RFORCE(2000),ST(51500),RNODAR(1000,4),IREFEL(1500,3),
     1PLACON(1500,2),NELEM,NPOIN,NBW,B(3,6),D(3,3),IMK,RO(1500),
     2ISDISP(1000,2),MARK(3)
      IN = (I-1)*NBW + J - I + 1
      RETURN
      END


      SUBROUTINE FSTRES(E)
      DOUBLE PRECISION ST,RFORCE,XX,XY,YY
      COMMON RFORCE(2000),ST(51500),RNODAR(1000,4),IREFEL(1500,3),
     1PLACON(1500,2),NELEM,NPOIN,NBW,B(3,6),D(3,3),IMK,RO(1500),
     2ISDISP(1000,2),MARK(3)
      INTEGER E
      DIMENSION UE(6)
      DIMENSION SA(3)
      REAL NM
      IF (E .EQ. 1)K=1
      CALL FORMD(E)
      CALL FORMB(E)
C******  EXTRACT DISP. MATRIX FOR ELEMENT E FROM DISP. MATRIX U
      DO 1 I=1,3
      DO 2 J=1,2
      M = 2*I-2+J
      L = 2*IREFEL(E,I)-2+J
      UE(M) = RFORCE(L)
    2 CONTINUE
    1 CONTINUE
C*******  FORM STRESSES (CONST. THROUGHOUT ELEMENT)
      XX=0.0
      YY=0.0
      XY=0.0
      DO 3 I=1,3
      DO 4 J=1,6
      XX= XX + D(1,I)*B(I,J)*UE(J)
      YY= YY + D(2,I)*B(I,J)*UE(J)
```

```
      YY= YY + D(3,I)*P(I,J)*UE(J)
    4 CONTINUE
    3 CONTINUE
C*******    FORM  ZZ
      IF (IMK .EQ. 2) GO TO 5
C*******    PLANE STRESS
      ZZ=0.0
      GO TO 6
C*******    PLANE STRAIN
    5 ZZ= FLAGON(E,2)*(XX+YY)
C*******    CALC. PRINCIPAL STRESSES AND AXES
    6 IF ((XX-YY) .EQ. 0.0) GO TO 7
      ALPHA = (2.0*XY)/(XX-YY)
      IF (ALPHA .GE. 0.0) GO TO 10
      ALPHA = 0.5*(ATAN(ALPHA)*57.29+180.0)
      GO TO 8
   10 ALPHA = 0.5*ATAN(ALPHA)*57.29
      GO TO 8
    7 ALPHA = 45.0
    8 NN = (XX*(COS(ALPHA/57.29)**2))+(YY*(SIN(ALPHA/57.29)**2))
     1 +(XY*SIN(2.0*ALPHA/57.29))
      SS = XX+YY-NN
C*******    PRINT STRESSES
      A=AMAX1(SS,NN,ZZ)
      C = AMIN1(SS,NN,ZZ)
      A = 0.5*(A-C)
      WRITE(6,9)E,ALPHA,NN,SS,ZZ,A
    9 FORMAT( 4X,I4,13X,F7.2,10X,F10.4,13X,F10.4,13X,F10.4,13X,F10.4)
      IF (MARK(3) .EQ. 0)GO TO 13
      IF (K .EQ. 1)GO TO 12
      K=K-1
      WRITE(7,11)K,(IREFEL(K,J),J=1,3),RA(1),RA(2),RA(3),E,
     1(IREFEL(E,J),J=1,3),ALPHA,NN,SS
   11 FORMAT(2(4I4,F5.2,2F9.3))
      K=1
      GO TO 13
   12 RA(1) = ALPHA
      RA(2) = NN
      RA(3) = SS
      K=K+1
   13 CONTINUE
```

```
      ST(F) = A
      RETURN
      END


      SUBROUTINE ERROR(I)
      DOUBLE PRECISION ST,RFORCE
      COMMON RFORCE(2000),ST(51500),RNODAR(1000,4),IREFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,B(3,6),D(3,3),IMK,RD(1500),
     2ISDISP(1500,2),MARK(3)
      GO TO (1,2,3),I
    1 WRITE (6,101)
  101 FORMAT(/11CHNO.OF NODAL PNTS. TOO LARGE. MUST ENLARGE STORAGE ALLO
     1CATED IN COMMON STATEMENTS AND ERROR CHECK IN MAIN PROG.)
      GO TO 201
    2 WRITE (6,102)
  102 FORMAT(/1 8HNO. OF ELEMENTS TOO LARGE. MUST ENLARGE STORAGE ALLOCA
     1TED IN COMMON STATEMENTS AND ERROR CHECK IN MAIN PROG.)
      GO TO 201
    3 WRITE (6,103)NBW
  103 FORMAT(/116CHSTORAGE SPACE ALLOCATED FOR STIFFNESS MATRIX NOT ENOUG
     1H AND THE STORAGE ALLOCATED TO ST IN COMMON MUST BE INCREASED TO A
     2T LEAST 2XNPOINXNBW WHERE NBW EQUALS     ,I4)
  201 RETURN
      END


      SUBROUTINE ISOS
      DOUBLE PRECISION ST,RFORCE
      COMMON RFORCE(2000),ST(51500),RNODAR(1000,4),IREFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,B(3,6),D(3,3),IMK,RD(1500),
     2ISDISP(1000,2),MARK(3)
      COMMON/A/ YSUB,RDSUB,IBASEP(300),NBASEP

      A = ABS((RNODAR(IBASEP(1),1)-RNODAR(IBASEP(2),1))/2.0)*981.0
      J = 2*IBASEP(1)
      ST(IN(J,J)) = ST(IN(J,J)) + A*RDSUB
      K = NBASEP - 1
      DO 30 I = 2,K
```

```
      J = I+1
      L = I-1
      M=2*IBASEP(I)
      A =(ABS(RNODAP(IBASEP(J),1)-RNODAP(IBASEP(I),1))+ABS(RNODAP(IBASEP
     1(L),1)-RNODAP(IBASEP(I),1)))*0.5*0.1
      ST(IN(M,M)) = ST(IN(M,M)) + A*ROSUB
   20 CONTINUE
      A = ABS(RNODAP(IBASEP(NBASEP),1)-RNODAP(IBASEP(K),1))*0.5*0.1
      J = 2*IBASEP(NBASEP)
      ST(IN(J,J)) = ST(IN(J,J)) + A*ROSUB
      RETURN
      END



      SUBROUTINE VIS
      DOUBLE PRECISION ST,RFORCE
      COMMON RFORCE(2000),ST(51500),RNODAP(1000,4),IPEEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,B(3,6),D(3,3),IMK,RO(1500),
     2ISOISP(1000,2),MARK(3)
      COMMON/A/ YSUB,ROSUB,IBASEP(200),NBASEP

      DIMENSION AS(23),COL(100),AT(80)
      INTEGER B
      READ(5,102)BLANK,(AS(I),I=1,22),NR,NC
  102 FORMAT(23A1,I2,2X,I2)
      SCLSTR = 1.0E-9
      DO 111 I=1,100
  111 COL(I) = BLANK
      XM = 0.0
      YM = 0.0
      DO 100 I=1,NPOIN
      IF (RNODAP(I,1) .GT. XM) XM=RNODAP(I,1)
  100 IF (RNODAP(I,2) .LT. YM) YM=RNODAP(I,2)
      SPV = -YM/NR
      SPH = XM/NC
      WRITE(6,101)
  101 FORMAT(1H1,10X,26HSTRESS DIFFERENCES*1.0E-9,//)
      YCNE=0.0
      YTWO=0.0
      DO 103 NROW=1,NR
```

```
      YONE=0.0
      XTWO=0.0
C********      SET UP ROW
      YONE=YTWO
      YTWO=YTWO-SPV
      DO 105 NCOL=1,NC
      XONE=XTWO
      XTWO=XTWO+SPH
      DO 104 E=1,NELEM
      XBAR = (RNODAR(IREFEL(E,1),1)+RNODAR(IREFEL(E,2),1)+
     1RNODAR(IREFEL(E,3),1))/3.0
      YBAR = (RNODAR(IREFEL(E,1),2)+RNODAR(IREFEL(E,2),2)+
     1RNODAR(IREFEL(E,3),2))/3.0
      IF (YBAR .LT. YTWO)GO TO 104
      IF (YBAR .GE. YONE)GO TO 104
      IF (XBAR .GT. XTWO)GO TO 104
      IF (XBAR .LE. XONE)GO TO 104
C********      PUT STRESS DIFFERENCE IN
      S=ST(E)*SCLSTD
      IF (S .LT. 0.25)GO TO 107
      R=0.75
      DO 108 I=1,19
      IF (S .LT. R)GO TO 109
  108 R=R+0.5
      IF (S .LT. 10.0)GO TO 110
      COL(NCOL)=AS(1)
      GO TO 105
  110 COL(NCOL)=AS(2)
      GO TO 105
  107 COL(NCOL) = AS(22)
      GO TO 105
  109 I=22-I
      COL(NCOL)=AS(I)
      GO TO 105
  104 CONTINUE
      COL(NCOL)=BLANK
  105 CONTINUE
C********      PRINT THE ROW THAT HAS JUST BEEN SET UP
  106 FORMAT(15X,100A1)
  103 WRITE(6,106)COL
      WRITE(6,112)
```

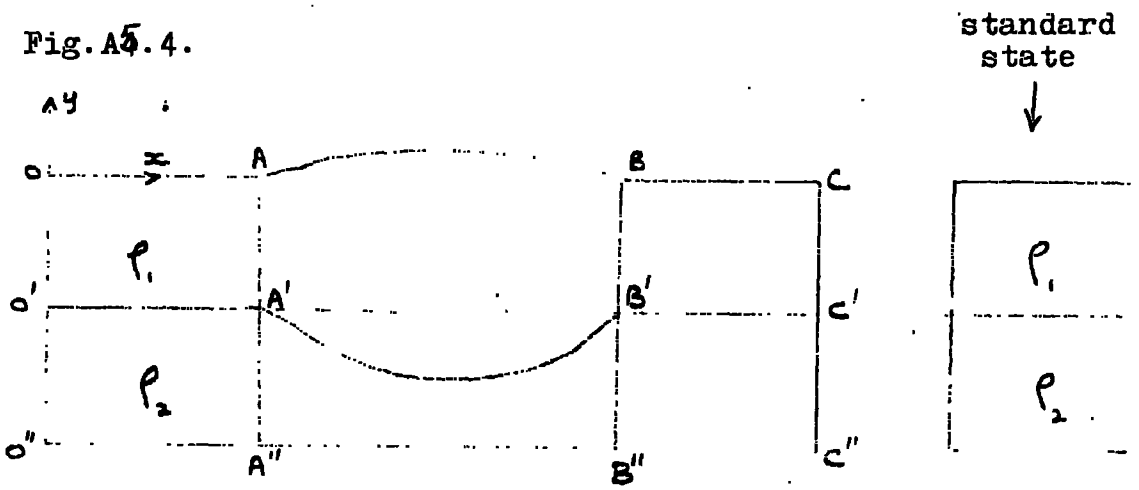
```
112 FORMAT(//10X,5HA----,14X,12HS   .GE.  1 .C)
113 FORMAT(10X,A1,7H---     ,F4.2,15H .LE.   S   .LT. .F4.2)
    DO 114 I=2,22
    IF (I .EQ. 2) GO TO 115
    IF (I .EQ. 22) GO TO 116
    F1=F1-0.5
    F2=F1+0.5
    WRITE(6,113)AS(I),F1,F2
    GO TO 114
115 F1=9.75
    F2=10.0
    WRITE(6,117)AS(2),F1,F2
117 FORMAT(10X,A1,7H---     ,F4.2,15H .LE.   S   .LT. .F4.1)
    GO TO 114
116 F1=0.0
    F2=0.25
    WRITE(6,113)AS(22),F1,F2
114 CONTINUE
C**********      PUNCH STRESS DIFFERENCES
    DO 1 I=1,NELEM,80
    N=I+79
    M=I
    DO 2 J=M,N
    L=J-I+1
    S=ST(J)*SCLSTD
    IF(S .LT. 0.25)GO TO 7
    [illegible]
    [illegible]
    IF (S .LT. [illegible])GO TO 9
    [illegible]
    IF(S .LT.10.0)GO TO [illegible]
    AT(L)=AS(1)
    [illegible]
 [illegible] AT(L)=AS(2)
    GO TO 5
  7 AT(L)=AS(22)
    GO TO 5
  9 K=22-K
    AT(L)=AS(K)
  5 CONTINUE
  2 CONTINUE
```

```
      WRITE(7,2)AT
1     CONTINUE
2     FORMAT(8[illegible]))
      RETURN
      END
```

B. GENFIN

Function: GENFIN generates the mesh data for the two-dimensional finite element program FINEL. FINEL requires a cross-section of the model to be divided into a network of triangles (called elements). The corners of the triangles are numbered so that the greatest difference for all elements between the nodal numbers of any one element is kept as small as possible.

Fig.A5.4.



GENFIN assumes that the cross-section is rectangular with a small perturbation on the top surface (OABC) allowable. This surface and any internal boundary such as (O'A'B'C') are defined at 5 km. intervals in the input data. The co-ordinates of C" give the overall size of the section. For optimum band-width we require OC > OO". The density and elastic constants of each layer must also be given as data.

A facility exists for the variation of the size of the mesh generated as one proceeds along the model in a horizontal direction. (Note that it is not possible to vary the size of the mesh in a vertical direction with the present program). To do this one must split the cross-section into a series of blocks (OO"A"A, AA"B"B, BB"C"C in fig.A5.4) by defining the x co-ordinate of the lines AA", BB" and then specify the horizontal and vertical dimensions of the elements in each block (SPH, SPV, respectively). Note here that because of the discrete way the boundaries are specified (every 5 km) the lines AA", BB" must coincide with one of these 5 km lines and also the horizontal dimension of any element (SPH) must be an integral devisor or multiple of 5 km. A constraint also exists on the possible transition values of the vertical dimension of the element size. It can 'change' in one of three ways as a new block is entered:

1. Double
2. Halve
3. Stay the same (i.e. SPH changes alone)

Thus a drastic change in element size must be achieved by using a series of small blocks, the vertical dimension of the element being halved at each block, if the size is being decreased, and doubled if being increased. Note that no similar constraint exists for changes in SPV.

Yet another facility allows the subtraction of a standard state (fig.A5.4). The standard state is defined by horizontal layers of given density. The density given to the element is then equal to the density of the actual layer in which it lies minus the density of the standard state layer in which it lies. This is useful if the effect of the density contrast of the root in FigA5.4 is required. The density given to the root is $\rho_1 - \rho_2$.

Data Input: is by card.

Card 1:

1–10
NJOB
I10

NJOB - No. of jobs to be processed.

Card 2:

1–10	20	30	40
MX	NY	NB	NL
I10	I10	I10	I10

where

(MX,NY) - Co-ordinates of bottom right hand corner C" of rectangle in Km. (Note: NY usually -ve and MX is multiple of 5)

NB - No. of boundaries (including top surface although not base)

NL - No. of layers in standard state (Note: NL ≥ 1, therefore, if no standard state wanted we have NL=1 and give layer zero density).

Card 3:

Boundary data (MX/5 + 1 cards)

1–10	20	
Y CO-ORD OF 1ST BOUNDARY AT X=0 Km.	Y CO-ORD OF 2nd BOUNDARY AT X=0 Km	→ etc.
F10.0	F10.0	

1–10	20	
Y CO-ORD OF 1ST BOUNDARY AT X=5 Km	Y CO-ORD OF 2nd BOUNDARY AT X=5 Km	→ etc.

↓ etc.

Card 4:

1–10	20	30	
ROB1	ROB2	ROB3	etc.
F10.0	F10.0	F10.0	

where,

ROBI is the density of the material immediately below boundary I (gm./c.c.)

Card 5:

1	10	20	30	
	E1	E2	E3	etc.
	E10.0	E10.0	E10.0	

where,

EI is the Young's Modulus of the material immediately below boundary I (dynes/sq.cm.)

Card 6:

1	10	20	30	
	V1	V2	V3	etc.
	F10.0	F10.0	F10.0	

where VI is the value of Poisson's ratio for the material immediately below boundary I (dimensionless).

Card 7:

Layer data for standard state (NL cards)

1	10	20
	D(1)	ROSS(1)
	F10.0	F10.0

where D(I) is the y co-ordinate (usually - ve) of the base of layer I and ROSS(I) is the density of layer I. Units km. and gm/cc.

Card 8:

Spacing data.

1	10	20	30
	XS(I)	SPV(I)	SPH(I)
	F10.0	F10.0	F10.0

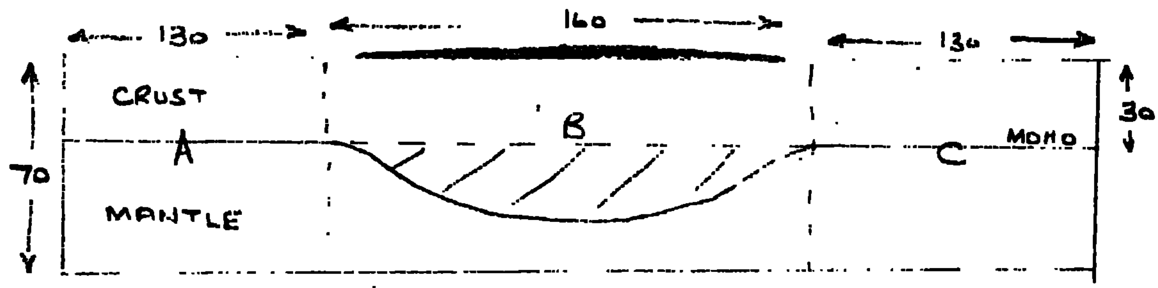
where XS(I) is the x co-ordinate (km) of the end of block I and the elements within this block are to have vertical dimensions SPV(I) and horizontal dimensions SPH(I). The data input is terminated as soon as a card with

XS(I) = MX is read.

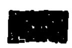
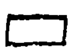
Sample data and run.

We wish to produce the model shown in Fig.A5.5 of a mountain and root

Fig.A5 5.



with

 part		ρ = 2.85
 part		ρ = −0.4
 part		ρ = 0.0 .

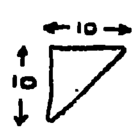
Elastic parameters

E_{crust} = 0.928×10^{12} dynes/sq.cm.

η_{crust} = 0.25

E_{mantle} = 0.178×10^{13} dynes/sq.cm.

η_{mantle} = 0.25

Element size: A and C 

B 

To achieve this we choose boundaries and standard state as in fig.A5.6.

Real parameters.

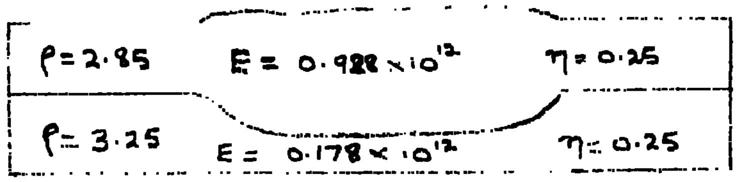
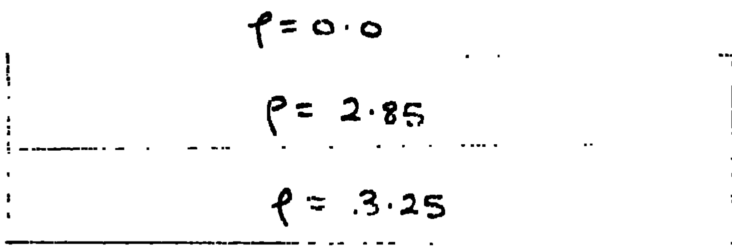


Fig. A5.6.

Standard state



The output data from GENFIN is plotted in Fig. A5.7.

73-80	Optional Sequence	6	Statement Continuation
		7-72	Statement
		73-80	Optional Sequence

PROGRAM Sample data for GENFIN — PAGE 1 OF 4

PROGRAMMER D.S. Dean. — DATE

```
         1                                          Only one job.
       420         -70         2         3          Note -ve sign.
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
       0.0       -30.0
     0.125     -30.891
     0.250     -31.782
     0.425     -33.030
     0.600     -34.278
```

Boundary Data (applies to all rows from 0.0 -30.0 down to 0.600 -34.278)

WRITTEN	Ø	O	I	1
PUNCH	zero	letter O	letter	number 1

73-80	Optional Sequence	6	Statement Continuation	73-80	Optional Sequence	statem... in program
		7-72	Statement			
		73-80	Optional Sequence			

196

PROGRAM: Sample data for GENFIN — PAGE 2 OF 4

PROGRAMMER: D.S. Dean. — DATE

0.733	-35.565
0.945	-36.733
1.123	-38.061
1.301	-39.269
1.438	-40.249
1.576	-41.230
1.726	-42.299
1.876	-43.369
2.039	-44.527
2.202	-45.686
2.352	-46.755
2.502	-47.825
2.677	-49.072
2.852	-50.320
3.027	-51.568
3.202	-52.816
3.364	-53.974
3.527	-55.133
3.639	-55.935
3.751	-56.737
3.639	-55.935
3.527	-55.133
3.364	-53.974
3.202	-52.816
3.027	-51.568

Boundary Data.

196

0 zero | O letter O | I letter | 1 number 1

73-80	Optional Sequence	6	Statement Continuation	73-80	Optional Sequence	statement in program
		7-72	Statement			
		73-80	Optional Sequence			

197

PROGRAM Sample data for GENFIN — PAGE 3 OF 4

PROGRAMMER D.S. Dean. — DATE

2.852	-5ϕ.32ϕ
2.677	-49.ϕ72
2.5ϕ2	-47.825
2.352	-46.755
2.2ϕ2	-45.686
2.ϕ39	-44.527
1.876	-43.369
1.726	-42.299
1.576	-41.23ϕ
1.438	-4ϕ.249
1.3ϕ1	-39.269
1.123	-38.ϕϕ1
ϕ.945	-36.733
ϕ.733	-35.5ϕ5
ϕ.6ϕϕ	-34.278
ϕ.425	-33.ϕ3ϕ
ϕ.25ϕ	-31.782
ϕ.125	-3ϕ.891
ϕ.ϕ	-3ϕ.ϕ
ϕ.ϕ	-3ϕ.ϕ
ϕ.ϕ	-3ϕ.ϕ
ϕ.ϕ	-3ϕ.ϕ
ϕ.ϕ	-3ϕ.ϕ
ϕ.ϕ	-3ϕ.ϕ
ϕ.ϕ	-3ϕ.ϕ

} Boundary Data.

197

PUNCH	ϕ zero	O letter O	I letter	1 number 1

Columns		
7-72	Statement	
73-80	Optional Sequence	

PROGRAM: Sample data for GENFIN
PROGRAMMER: D. S. Dean.
PAGE 4 OF 4
DATE

Cols 1–10	Cols 11–20	Cols 21–30	Comment
ϕ.ϕ	-3ϕ.ϕ		Boundary Data.
ϕ.ϕ	-3ϕ.ϕ		
ϕ.ϕ	-3ϕ.ϕ		
ϕ.ϕ	-3ϕ.ϕ		
ϕ.ϕ	-3ϕ.ϕ		
ϕ.ϕ	-3ϕ.ϕ		
ϕ.ϕ	-3ϕ.ϕ		
ϕ.ϕ	-3ϕ.ϕ		
ϕ.ϕ	-3ϕ.ϕ		
ϕ.ϕ	-3ϕ.ϕ		
ϕ.ϕ	-3ϕ.ϕ		
ϕ.ϕ	-3ϕ.ϕ		
2.85	3.25		Elastic parameters
ϕ.928E12	ϕ.178E13		
ϕ.25	ϕ.25		
ϕ.ϕ	ϕ.ϕ		Layer data
-3ϕ.ϕ	2.85		
-7ϕ.ϕ	3.25		
13ϕ.ϕ	1ϕ.ϕ	1ϕ.ϕ	Spacing data.
29ϕ.ϕ	5.ϕ	1ϕ.ϕ	
42ϕ.ϕ	1ϕ.ϕ	1ϕ.ϕ	

WRITTEN	ϕ	O	I	1
PUNCH	zero	letter O	letter	number 1

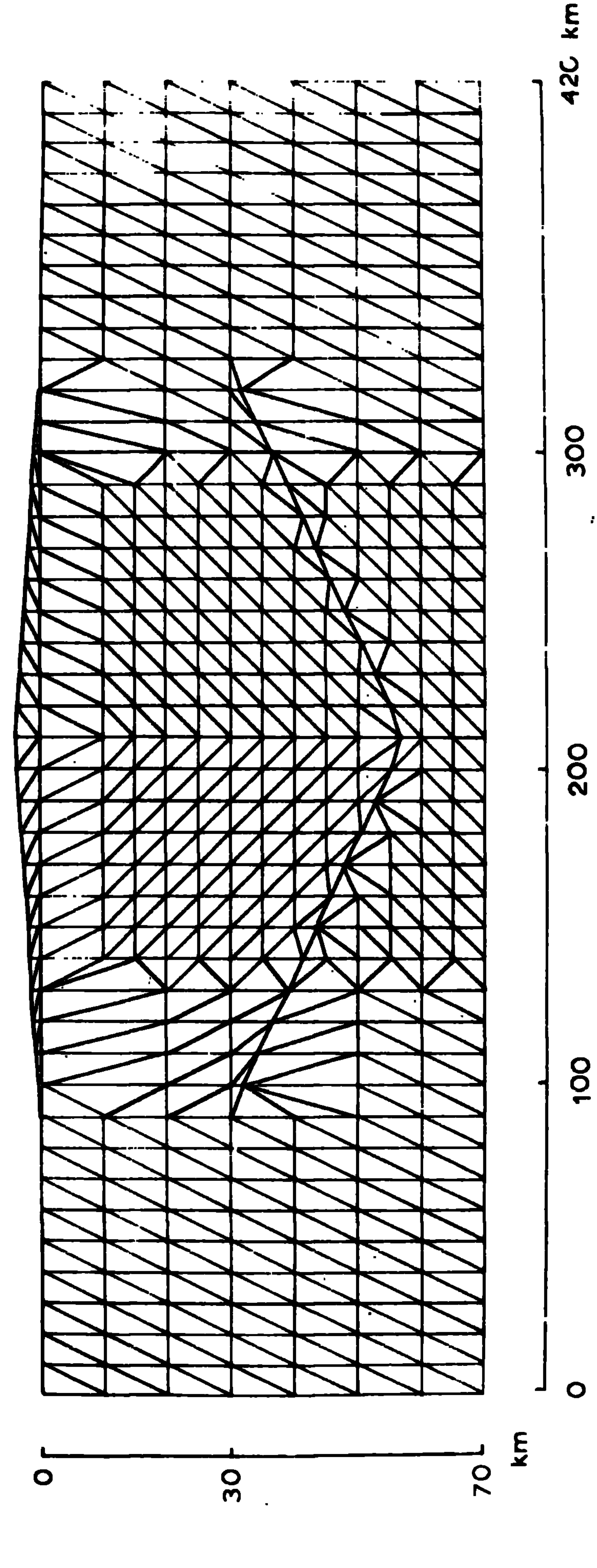
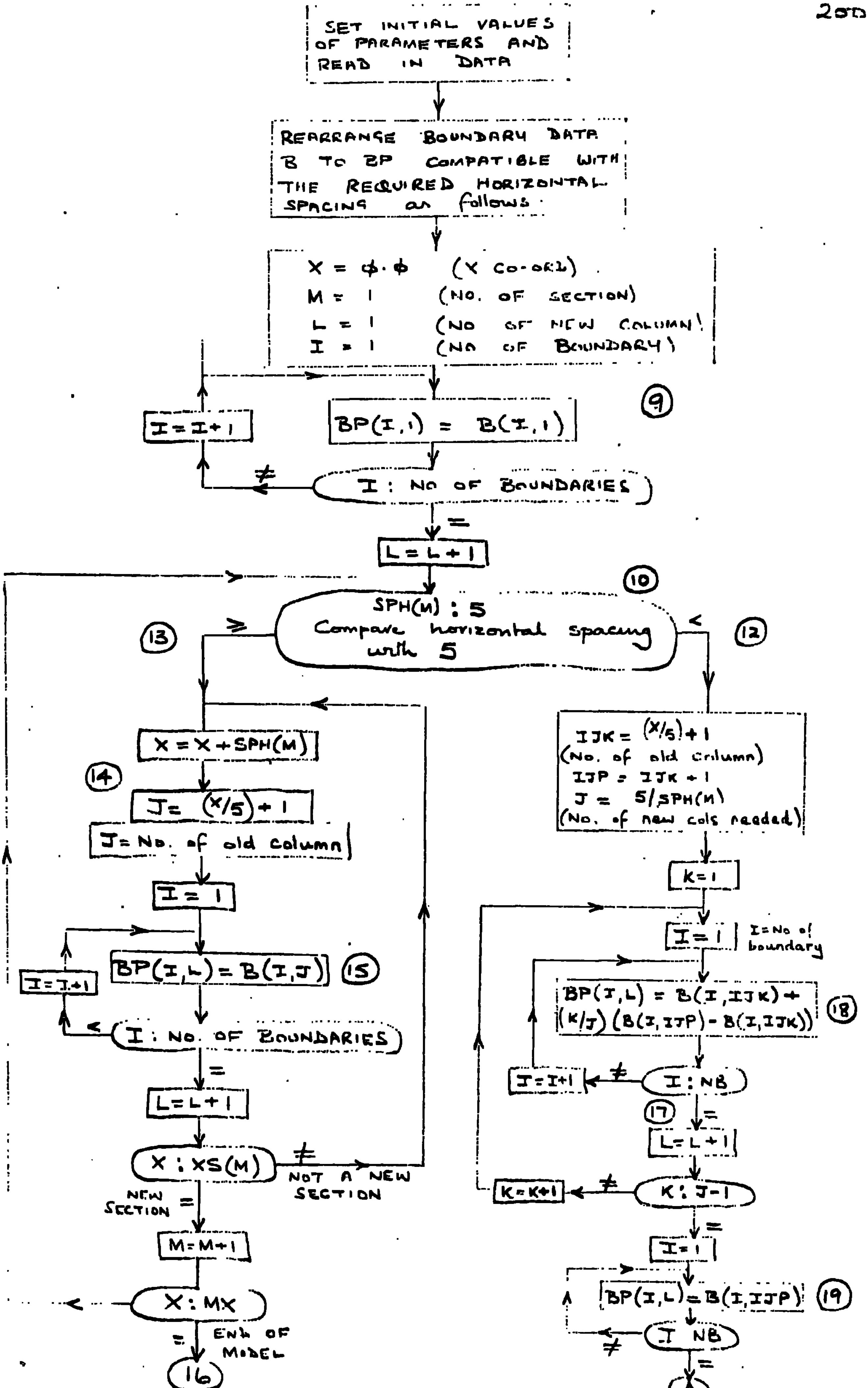
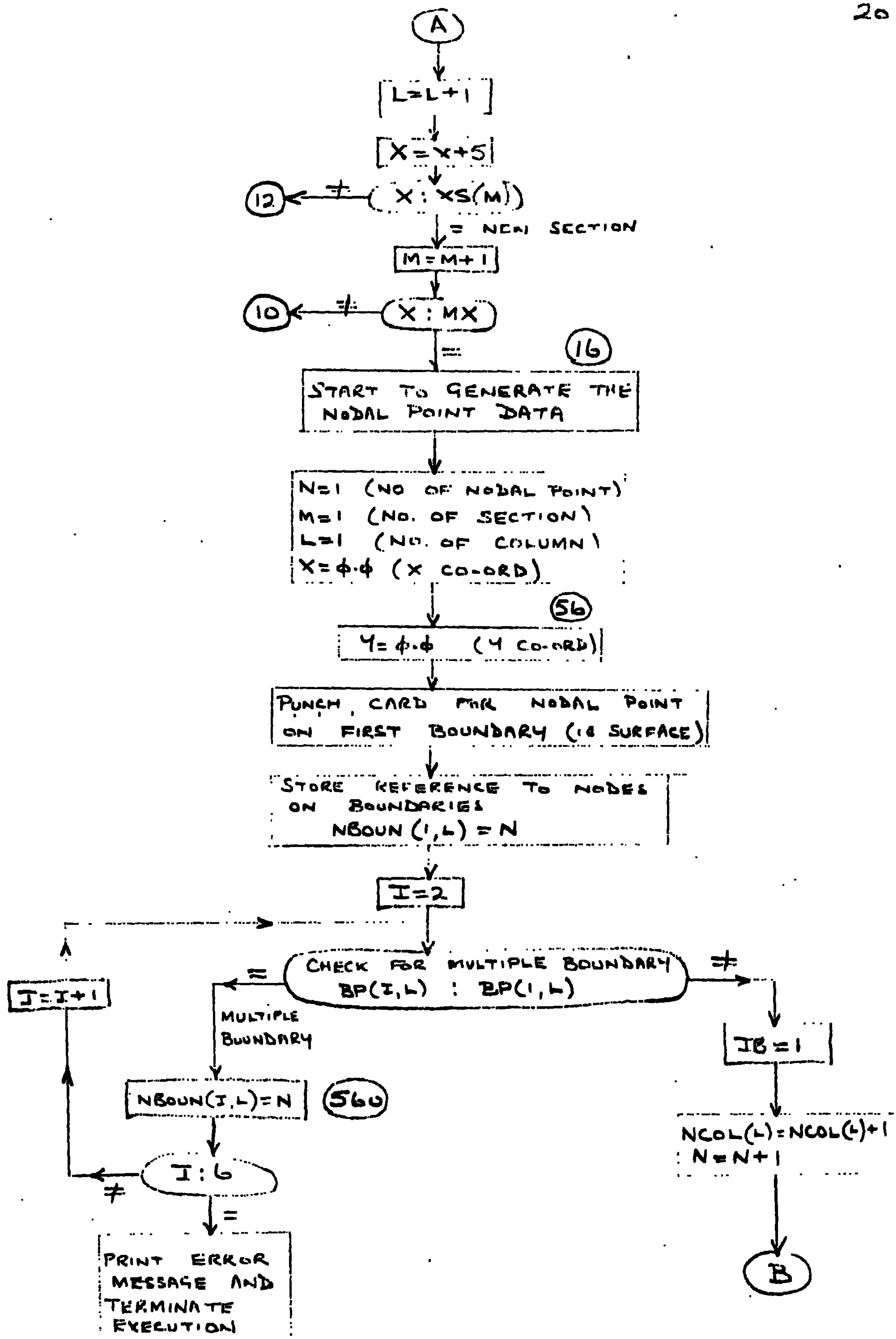
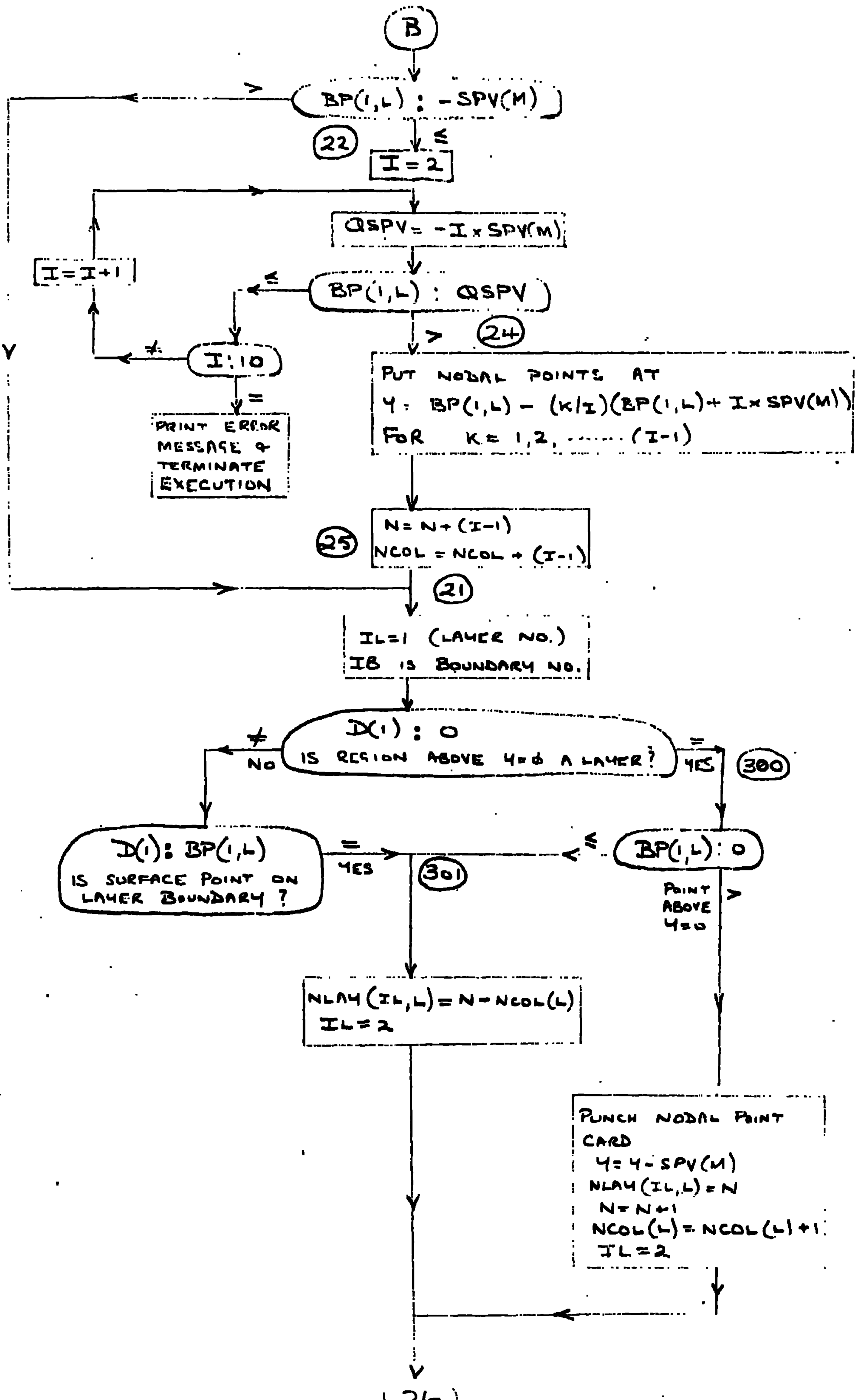
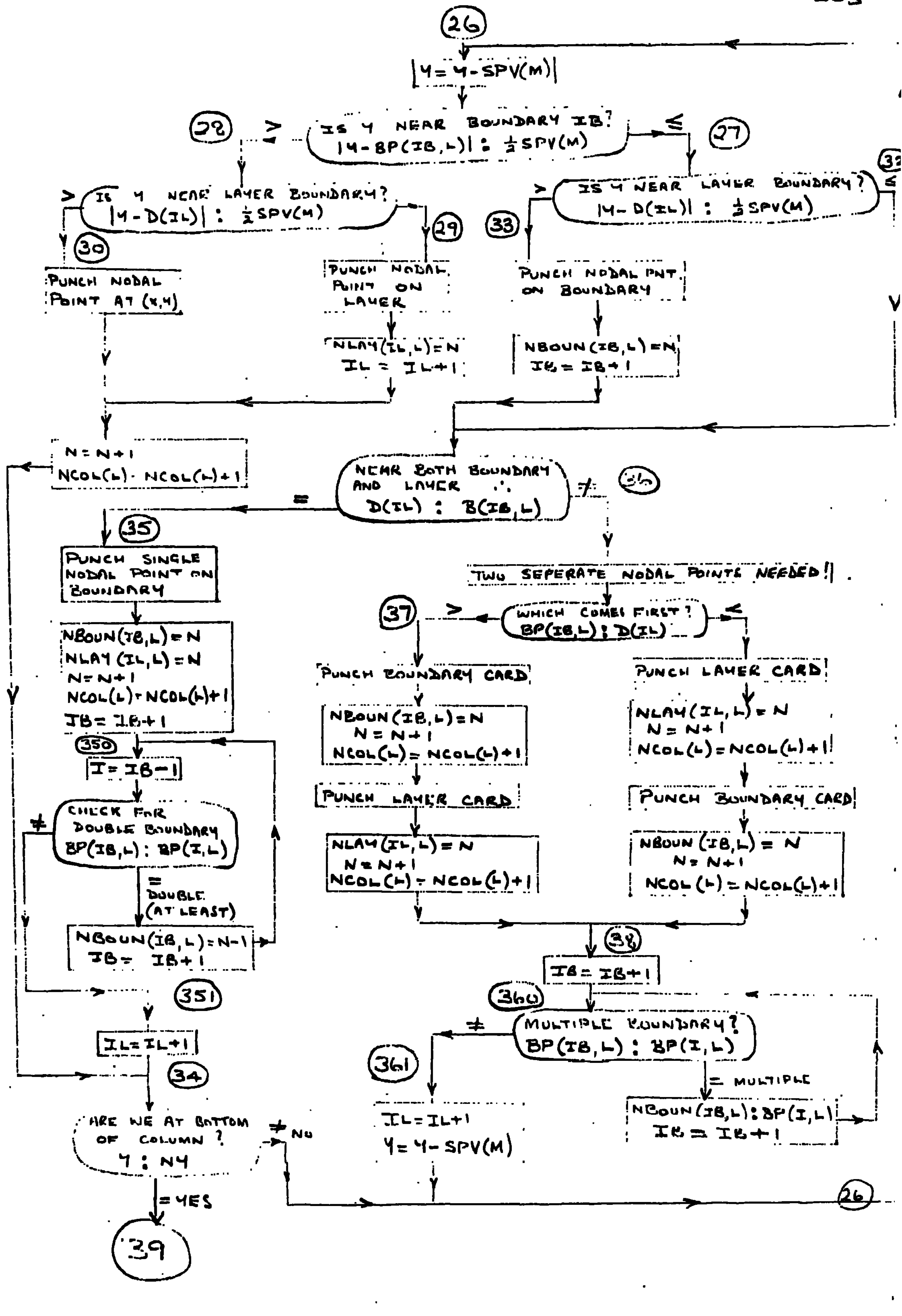


FIG. AS.7

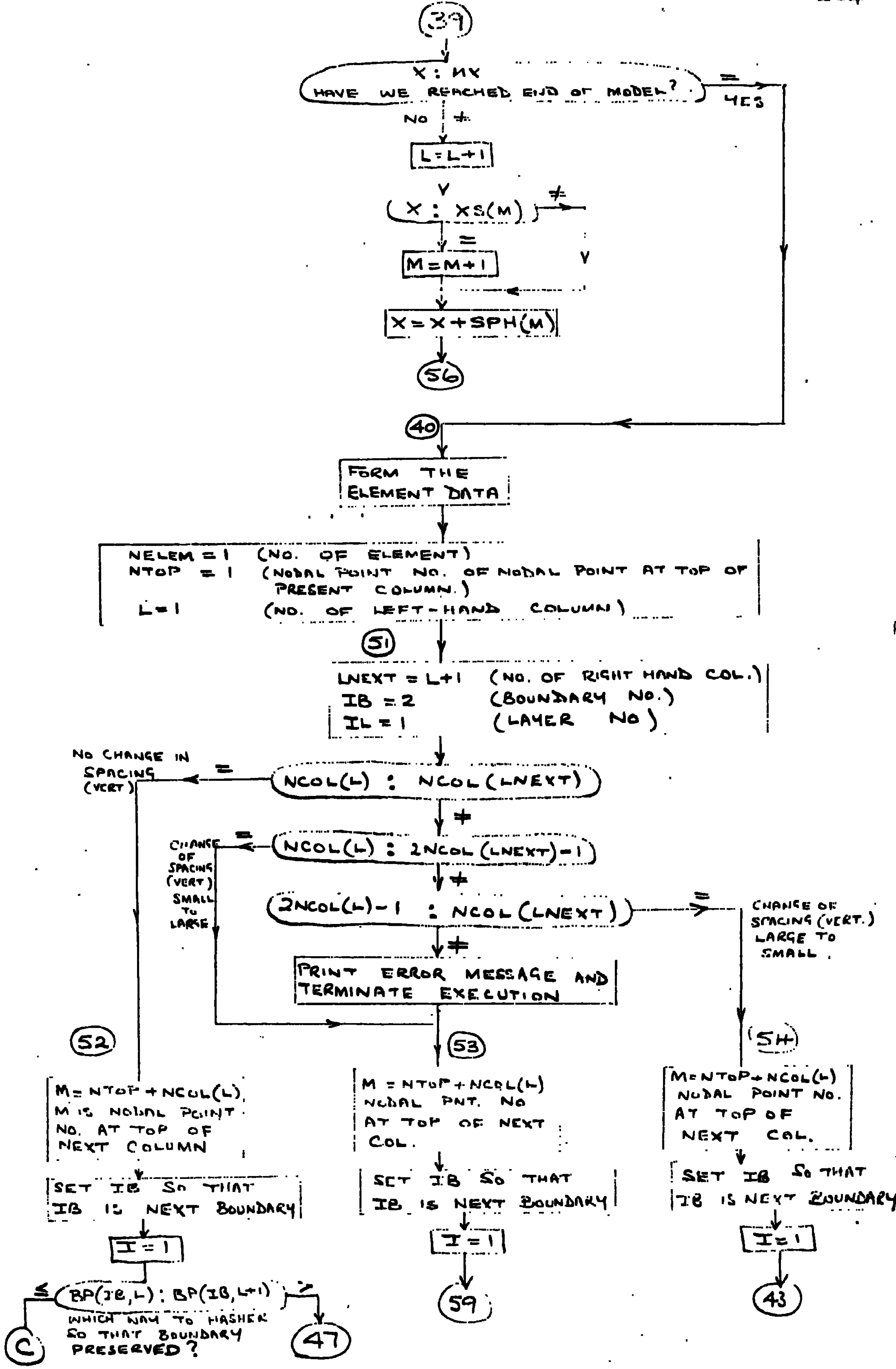

SET INITIAL VALUES OF PARAMETERS AND READ IN DATA
REARRANGE BOUNDARY DATA B TO BP COMPATIBLE WITH THE REQUIRED HORIZONTAL SPACING as follows
X = 0.0 (X CO-ORD)
M = 1 (NO. OF SECTION)
L = 1 (NO OF NEW COLUMN)
I = 1 (NO OF BOUNDARY)
9
BP(I,1) = B(I,1)
I = I+1
I : NO OF BOUNDARIES
≠
=
L = L+1
10
SPH(M) : 5
Compare horizontal spacing with 5
13
≥
12
<
X = X + SPH(M)
14
J = (X/5) + 1
J = No. of old column
I = 1
BP(I,L) = B(I,J)
15
I = I+1
I : NO. OF BOUNDARIES
L = L+1
X : XS(M)
NOT A NEW SECTION
NEW SECTION
M = M+1
X : MX
END OF MODEL
16
IJK = (X/5) + 1
(No. of old column)
IJP = IJK + 1
J = 5/SPH(M)
(No. of new cols needed)
K = 1
I = 1
I = No of boundary
BP(I,L) = B(I,IJK) + (K/J) (B(I,IJP) - B(I,IJK))
18
I = I+1
I : NB
17
L = L+1
K = K+1
K : J-1
I = 1
BP(I,L) = B(I,IJP)
19
I NB
A

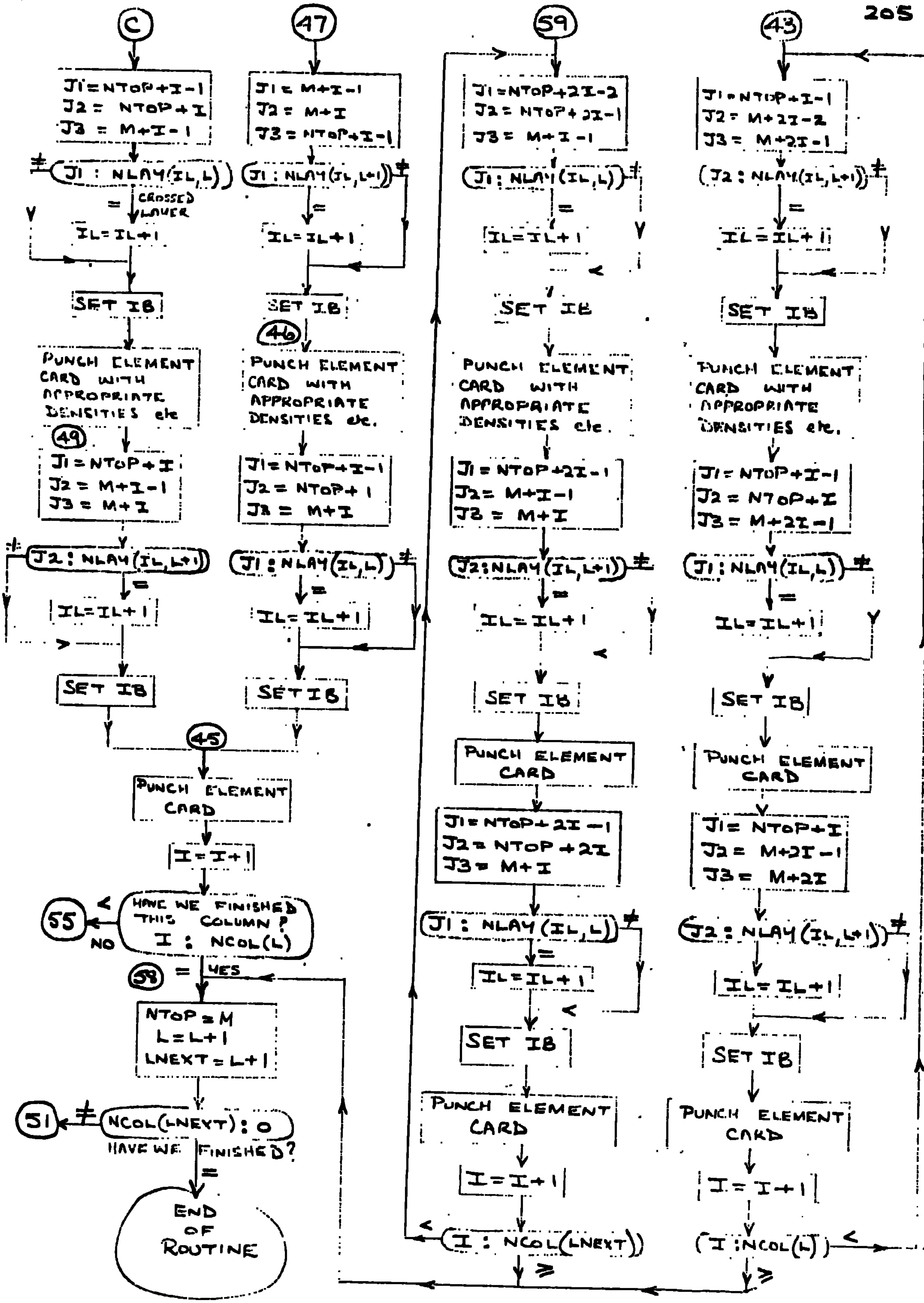

A
L=L+1
X=X+5
12
≠
X : XS(M)
= NEW SECTION
M=M+1
10
≠
X : MX
=
16
START TO GENERATE THE NODAL POINT DATA
N=1 (NO OF NODAL POINT)
M=1 (NO. OF SECTION)
L=1 (NO. OF COLUMN)
X=φ.φ (X CO-ORD)
56
Y=φ.φ (Y CO-ORD)
PUNCH CARD FOR NODAL POINT ON FIRST BOUNDARY (IE SURFACE)
STORE REFERENCE TO NODES ON BOUNDARIES
NBOUN (1,L) = N
I=2
CHECK FOR MULTIPLE BOUNDARY
BP(I,L) : BP(1,L)
=
MULTIPLE BOUNDARY
≠
I=I+1
NBOUN(I,L)=N
560
I : 6
≠
=
PRINT ERROR MESSAGE AND TERMINATE EXECUTION
IB=1
NCOL(L)=NCOL(L)+1
N=N+1
B


B
BP(1,L) : -SPV(M)
>
≤
22
I = 2
QSPV = -I × SPV(M)
I = I+1
BP(1,L) : QSPV
≤
>
24
I : 10
≠
=
PRINT ERROR MESSAGE & TERMINATE EXECUTION
PUT NODAL POINTS AT
Y = BP(1,L) - (K/I)(BP(1,L) + I × SPV(M))
FOR K = 1,2, ... (I-1)
25
N = N + (I-1)
NCOL = NCOL + (I-1)
21
IL=1 (LAYER NO.)
IB IS BOUNDARY NO.
D(1) : 0
IS REGION ABOVE Y=0 A LAYER?
≠
NO
=
YES
300
D(1): BP(1,L)
IS SURFACE POINT ON LAYER BOUNDARY ?
=
YES
301
≤
BP(1,L) : 0
POINT ABOVE Y=0
>
NLAY(IL,L) = N - NCOL(L)
IL=2
PUNCH NODAL POINT CARD
Y = Y - SPV(M)
NLAY(IL,L) = N
N = N+1
NCOL(L) = NCOL(L)+1
IL=2
26



26
|Y = Y - SPV(M)|
28
IS Y NEAR BOUNDARY IB?
|Y - BP(IB,L)| : ½SPV(M)
27
IS Y NEAR LAYER BOUNDARY?
|Y - D(IL)| : ½SPV(M)
29
30
33
IS Y NEAR LAYER BOUNDARY?
|Y - D(IL)| : ½SPV(M)
PUNCH NODAL POINT AT (X,Y)
PUNCH NODAL POINT ON LAYER
NLAY(IL,L) = N
IL = IL+1
PUNCH NODAL PNT. ON BOUNDARY
NBOUN(IB,L) = N
IB = IB+1
N = N+1
NCOL(L) = NCOL(L)+1
NEAR BOTH BOUNDARY AND LAYER ∴
D(IL) : B(IB,L)
35
36
PUNCH SINGLE NODAL POINT ON BOUNDARY
NBOUN(IB,L) = N
NLAY(IL,L) = N
N = N+1
NCOL(L) = NCOL(L)+1
IB = IB+1
350
I = IB-1
CHECK FOR DOUBLE BOUNDARY
BP(IB,L) : BP(I,L)
DOUBLE (AT LEAST)
NBOUN(IB,L) = N-1
IB = IB+1
351
IL = IL+1
34
ARE WE AT BOTTOM OF COLUMN ?
Y : NY
NO
YES
39
TWO SEPERATE NODAL POINTS NEEDED!
37
WHICH COMES FIRST ?
BP(IB,L) : D(IL)
PUNCH BOUNDARY CARD
NBOUN(IB,L) = N
N = N+1
NCOL(L) = NCOL(L)+1
PUNCH LAYER CARD
NLAY(IL,L) = N
N = N+1
NCOL(L) = NCOL(L)+1
PUNCH LAYER CARD
NLAY(IL,L) = N
N = N+1
NCOL(L) = NCOL(L)+1
PUNCH BOUNDARY CARD
NBOUN(IB,L) = N
N = N+1
NCOL(L) = NCOL(L)+1
38
IB = IB+1
360
MULTIPLE BOUNDARY?
BP(IB,L) : BP(I,L)
361
MULTIPLE
IL = IL+1
Y = Y - SPV(M)
NBOUN(IB,L) : BP(I,L)
IB = IB+1
26


39
X : MX
HAVE WE REACHED END OF MODEL?
=
YES
No ≠
L=L+1
X : XS(M)
≠
=
M=M+1
X=X+SPH(M)
56
40
FORM THE ELEMENT DATA
NELEM = 1 (NO. OF ELEMENT)
NTOP = 1 (NODAL POINT NO. OF NODAL POINT AT TOP OF PRESENT COLUMN.)
L=1 (NO. OF LEFT-HAND COLUMN)
51
LNEXT = L+1 (NO. OF RIGHT HAND COL.)
IB = 2 (BOUNDARY NO.)
IL = 1 (LAYER NO)
NO CHANGE IN SPACING (VERT)
=
NCOL(L) : NCOL(LNEXT)
≠
CHANGE OF SPACING (VERT) SMALL TO LARGE
=
NCOL(L) : 2NCOL(LNEXT)-1
≠
2NCOL(L)-1 : NCOL(LNEXT)
=
CHANGE OF SPACING (VERT.) LARGE TO SMALL.
≠
PRINT ERROR MESSAGE AND TERMINATE EXECUTION
52
M= NTOP + NCOL(L), M IS NODAL POINT NO. AT TOP OF NEXT COLUMN
SET IB SO THAT IB IS NEXT BOUNDARY
I = 1
≤
BP(IB,L) : BP(IB,L+1)
>
WHICH WAY TO HASHER SO THAT BOUNDARY PRESERVED?
C
47
53
M = NTOP + NCOL(L) NODAL PNT. NO AT TOP OF NEXT COL.
SET IB SO THAT IB IS NEXT BOUNDARY
I = 1
59
54
M=NTOP+NCOL(L) NODAL POINT NO. AT TOP OF NEXT COL.
SET IB SO THAT IB IS NEXT BOUNDARY
I = 1
43



C
J1 = NTOP+I-1
J2 = NTOP+I
J3 = M+I-1
J1 : NLAY(IL,L)
CROSSED LAYER
IL=IL+1
SET IB
PUNCH ELEMENT CARD WITH APPROPRIATE DENSITIES etc
49
J1 = NTOP+I
J2 = M+I-1
J3 = M+I
J2 : NLAY(IL,L+1)
IL=IL+1
SET IB
47
J1 = M+I-1
J2 = M+I
J3 = NTOP+I-1
J1 : NLAY(IL,L+1)
IL=IL+1
SET IB
46
PUNCH ELEMENT CARD WITH APPROPRIATE DENSITIES etc.
J1 = NTOP+I-1
J2 = NTOP+1
J3 = M+I
J1 : NLAY(IL,L)
IL=IL+1
SET IB
45
PUNCH ELEMENT CARD
I=I+1
HAVE WE FINISHED THIS COLUMN ?
I : NCOL(L)
55
NO
58
YES
NTOP = M
L=L+1
LNEXT = L+1
51
NCOL(LNEXT) : 0
HAVE WE FINISHED?
END OF ROUTINE
59
J1 = NTOP+2I-2
J2 = NTOP+2I-1
J3 = M+I-1
J1 : NLAY(IL,L)
IL=IL+1
SET IB
PUNCH ELEMENT CARD WITH APPROPRIATE DENSITIES etc.
J1 = NTOP+2I-1
J2 = M+I-1
J3 = M+I
J2 : NLAY(IL,L+1)
IL=IL+1
SET IB
PUNCH ELEMENT CARD
J1 = NTOP+2I-1
J2 = NTOP+2I
J3 = M+I
J1 : NLAY(IL,L)
IL=IL+1
SET IB
PUNCH ELEMENT CARD
I=I+1
I : NCOL(LNEXT)
43
J1 = NTOP+I-1
J2 = M+2I-2
J3 = M+2I-1
J2 : NLAY(IL,L+1)
IL=IL+1
SET IB
PUNCH ELEMENT CARD WITH APPROPRIATE DENSITIES etc.
J1 = NTOP+I-1
J2 = NTOP+I
J3 = M+2I-1
J1 : NLAY(IL,L)
IL=IL+1
SET IB
PUNCH ELEMENT CARD
J1 = NTOP+I
J2 = M+2I-1
J3 = M+2I
J2 : NLAY(IL,L+1)
IL=IL+1
SET IB
PUNCH ELEMENT CARD
I=I+1
I : NCOL(L)

```
C******    GENEIN
      COMMON JENE,JTWO,JTHR,IR,IL,POR(6),F(6),V(6),DELTM,POSS(6),
     1M,NTOP,NBOUN(6,1000),L,LNEXT
      DIMENSION P(6,500),XS(10),SPV(10),SPH(10),BP(6,1000),NCOL(500),
     1NLAY(6,1000),R(6)
      DO 111 I=1,5
      DO 111 J=1,1000
      BP(I,J)=732.676
  111 NBOUN(I,J)=0
C******    READ AND STORE DATA
      READ(5,342)NJOB
  342 FORMAT(I10)
      DO 343 JN=1,NJOB
      DO 1 I=1,500
    1 NCOL(I)=0.0
      READ(5,2) MX,NY,NB,NL
    2 FORMAT(4I10)
      M = MX/5 +1
C******   BOUNDARY DATA
      DO 3 J=1,M
    3 READ(5,4)(P(I,J),I=1,NB)
    4 FORMAT(6F10.5)
      READ(5,4) (POR(I),I=1,NB)
      READ(5,73)(F(I),I=1,NB)
   73 FORMAT(6F10.0)
      READ(5,4)(V(I),I=1,NB)
C******    LAYER DATA
      DO 5  I=1,NL
    5 READ(5,6) R(I),POSS(I)
    6 FORMAT(2F10.5)
C******   .SPACING DATA
      I=1
    7 READ(5,11) XS(I),SPV(I),SPH(I)
   11 FORMAT(3F10.5)
      IA = IFIX(XS(I))
      IF (IA .EQ. MX) GO TO 8
      I = I+1
      IF (I .GT. 10) GO TO 1001
      GO TO 7
C******     MAKE THE BOUNDARY SPECIFICATIONS COMPATIBLE WITH THE REQUIRED
C           HORIZONTAL SPACING
```

```
C******     M= NO. OF SECTION
C           L= NO. OF COLUMN
      8 X =0.0
        M=1
        L=1
C******     FIRST COLUMN
        DO 9  I=1,NB
      9 BP(I,1) = B(I,1)
        L=L+1
     10 IF (SPH(M) .LT. 5) GO TO 12
     13 X = X+SPH(M)
     14 IX = IFIX(X)
        J=IX/5 + 1
        DO 15 I=1,NB
     15 BP(I,L) = B(I,J)
        L=L+1
        IXS = IFIX(XS(M))
        IF (IXS .NE. IX) GO TO 13
        M=M+1
        IF (IX .NE. MX) GO TO 10
        GO TO 16
     12 IX = IFIX(X)
        IJK= IX/5   +1
        IJP = IJK + 1
        A =   5.0/SPH(M)
        J =   IFIX(A)
        JP = J-1
        DO 17 K=1,JP
        DO 18 I=1,NB
     18 BP(I,L) = B(I,IJK) + (K*(B(I,IJP)-B(I,IJK))/J)
     17 L=L+1
        DO 19 I=1,NB
     19 BP(I,L) = B(I,IJP)
        L=L+1
        X=X+5.0
        IX = IFIX(X)
        IXS = IFIX(XS(M))
        IF (IXS .NE. IX) GO TO 12
        M = M+1
        IF (IX .EQ. MX) GO TO 16
        GO TO 10
```

```
C******   GENERATE NODAL POINTS
C******   N = NO. OF NODAL POINT
C         M = NO. OF SECTION
C         L = NO. OF COLUMN
C         NCOL(L) = NO. OF NODAL POINTS IN COLUMN L.
C         NBOUN(IB,L)= NODAL PT. NO. OF POINT IN COL L ON BOUNDARY IB
C         NLAY(IL,L) = NODAL PT. NO. OF POINT IN COL L ON LAYER IL
   16 L=1
      N=1
      M=1
      X=0.0
C******   PUNCH SURFACE CARD I.E. BOUNDARY ONE
   56 Y=0.0
      WRITE(6,20) N,X,BP(1,L)
      WRITE(7,20) N,X,BP(1,L)
   20 FORMAT (I10,2F10.3)
      NBOUN(1,L)=N
      DO 560 I=2,6
      IF (BP(I,L) .EQ. BP(1,L))GO TO 560
      IB=I
      GO TO561
  560 NBOUN(I,L)=N
      GO TO 1007
  561 NCOL(L)=NCOL(L)+1
      N=N+1
      IF (BP(1,L) .GT. -SPV(M)) GO TO 21
   22 DO 23 I=2,10
      DSPV = -I*SPV(M)
      IF (BP(1,L) .GT. DSPV) GO TO 24
   23 CONTINUE
      GO TO 1005
   24 Y = -(I-1)*SPV(M)
      J = I-1
      DO 25 K=1,J
      YP = BP(1,L)-(K*(BP(1,L)+I*SPV(M))/I)
      WRITE(6,20) N,X,YP
      WRITE(7,20) N,X,YP
      N=N+1
   25 NCOL(L)=NCOL(L)+1
C******   PUNCH NODAL POINTS FOR REST OF COLUMN
   21 IL=1
```

```
      IF (D(IL) .EQ. 0.0)GO TO 26
      IF (Y(I) .EQ. RP(I,L))GO TO 301
      GO TO 26
 300  IF (RP(I,L) .LE. 0.0)GO TO 301
      GO TO 302
 301  NLAY(IL,L)=N-NCOL(L)
      IL=2
      GO TO 26
 302  WRITE(6,20)N,X,D(IL)
      WRITE(7,20)N,X,D(IL)
      Y=Y-SPV(M)
      NLAY(IL,L)=N
      N=N+1
      NCOL(L)=NCOL(L)+1
      IL=2
  26  Y = Y-SPV(M)
      A = ABS(Y-RP(IR,L))
      S = 0.5*SPV(M)
      C = ABS(Y-D(IL))
      IF (A .LE. S) GO TO 27
  28  IF (C .LE. S) GO TO 29
C*******    NODAL POINT NOT ON BOUNDARY OR LAYER
  30  WRITE(6,20) N,X,Y
      WRITE(7,20) N,X,Y
      GO TO 31
C*******    NODAL POINT ON LAYER MARGIN
  29  WRITE(6,20) N,X,D(IL)
      WRITE(7,20) N,X,D(IL)
      NLAY(IL,L) = N
      IL=IL+1
      GO TO 31
  27  IF (C .LE. S) GO TO 32
C*******    NODAL POINT ON BOUNDARY
  32  WRITE(6,20)N,X,RP(IR,L)
      WRITE(7,20)N,X,RP(IR,L)
      NBOUN(IR,L)=N
      IR=IR+1
 330  I=IR-1
      IF (RP(IR,L) .NE. RP(I,L))GO TO 31
      NBOUN(IR,L)=N
      IR=IR+1
```

```
      GO TO 330
   31 N=N+1
      NCOL(L)=NCOL(L)+1
      GO TO 34
C******  PUT BOTH LAYER AND BOUNDARY POINTS IN
   32 A=ABS(D(IL) - BP(IB,L))
      IF (A .LT. 0.1) GO TO 35
C******    PUNCH TWO SEPERATE CARDS
   36 IF (BP(IB,L) .GT. D(IL))GO TO 37
      WRITE(6,20) N,X,D(IL)
      WRITE(7,20) N,X,D(IL)
      NLAY(IL,L) = N
      N=N+1
      NCOL(L)=NCOL(L)+1
      WRITE(6,20)N,X,BP(IB,L)
      WRITE(7,20)N,X,BP(IB,L)
      NBOUN(IB,L) = N
      N=N+1
      NCOL(L)=NCOL(L)+1
      GO TO 38
   37 NBOUN(IB,L)=N
      WRITE(6,20) N,X,BP(IB,L)
      WRITE(7,20) N,X,BP(IB,L)
      N=N+1
      NCOL(L)=NCOL(L)+1
      WRITE(6,20) N,X,D(IL)
      WRITE(7,20) N,X,D(IL)
      NLAY(IL,L)=N
      N=N+1
      NCOL(L)=NCOL(L)+1
   38 IB=IB+1
  360 I=IB-1
      IF (BP(IB,L) .NE. BP(I,L))GO TO 361
      NBOUN(IB,L) = N-1
      IB=IB+1
      GO TO 360
  361 IL=IL+1
      Y=Y-SPV(M)
      GO TO 26
C**********       PUNCH SINGLE CARD
   35 WRITE(6,20)N,X,D(IL)
```

```
      WRITE(7,20)N,X,R(IL)
      NBOUN(IB,L)=N
      NLAY(IL,L)=N
      N=N+1
      NCOL(L)=NCOL(L)+1
      IB=IB+1
 350  I=IB-1
      IF (BR(IB,L) .NE. BR(I,L))GO TO 351
      NBOUN(IB,L)=N-1
      IB = IB+1
      GO TO 350
 351  IL=IL+1
  34  IY = IFIX(Y)
      IF (IY .EQ. NY) GO TO 39
      GO TO 26
  39  IX=IFIX(X)
      IF (IX .EQ. AX) GO TO 40
      L=L+1
      IXS = IFIX(XS(M))
      IF (IXS .EQ. IX) M=M+1
      X=X + SPH(M)
      GO TO 56
 1001 WRITE (6,1002)
 1002 FORMAT (1 CH DATA ERROR-NO. OF CHANGES OF SPACING TOO GREAT OR LIST
     1T XS(I) DOES NOT AGREE WITH WIDTH OF STRUCTURE )
      GO TO 6
  40  CONTINUE
C*******    FORM  ELEMENT DATA
      NELEM = 1
      NTOP  = 1
      L = 1
  51  LNEXT = L+1
      IP=2
      IL=1
      IF (NCOL(L) .EQ. NCOL(LNEXT)) GO TO 52
      I = 2*NCOL(LNEXT) - 1
      IF (NCOL(L) .EQ. I) GO TO 53
      I = 2*NCOL(L) - 1
      IF (I .EQ. NCOL(LNEXT)) GO TO 54
      GO TO 1003
  52  M = NTOP + NCOL(L)
```

```
      CALL SETIB
      IF (IB .EQ. 0)GO TO 1007
      I = 1                                                           133
   55 IF (RP(IB,L) .GT. RP(IB,LNEXT))GO TO 47
      JONE = NTOP + I - 1
      JTWO = NTOP + I
      JTHR = M + I - 1
      IF (JONE .EQ. NLAY(IL,L)) IL=IL+1
      CALL SET(JONE,L)
      IF (IB .EQ. 0)GO TO 1007
      CALL PUNCH
      GO TO 49
   47 JONE = M+I-1
      JTWO = M+I
      JTHR = NTOP+I-1
      IF (JONE .EQ. NLAY(IL,LNEXT)) IL=IL+1
      CALL SET(JONE,LNEXT)
      IF (IB .EQ. 0)GO TO 1007
   46 CALL PUNCH
      JONE = NTOP+I-1
      JTWO = NTOP+I
      JTHR = M+I
      IF (JONE .EQ. NLAY(IL,L))IL=IL+1
      CALL SET(JONE,L)
      IF (IB .EQ. 0)GO TO 1007
      GO TO 45
   49 JONE = NTOP+I
      JTWO = M+I-1
      JTHR = M+I
      IF (JTWO .EQ. NLAY(IL,LNEXT)) IL=IL+1
      CALL SET(JTWO,LNEXT)
      IF (IB .EQ. 0)GO TO 1007
   45 CALL PUNCH
      I = I+1                                                         134
      IF (I .LT. NCOL(L)) GO TO 55
   59 NTOP = M
      L = L+1                                                         137
      LNEXT = L+1
      IF (NCOL(LNEXT) .EQ. 0) GO TO 61
      GO TO 51
C*******     CHANGE OF SPACING.    SMALL TO LARGE                     135
```

```
   53 M = NTOP + NCOL(L)
      CALL SETIB
      IF (IB .EQ. 0)GO TO 1007
      I = 1                                                     151
   59 JONE = NTOP + 2*I -2
      JTWO = NTOP + 2*I -1
      JTHR = M+I-1
      IF (JONE .EQ. NLAY(IL,L)) IL=IL+1
      CALL SET(JONE,L)
      IF (IB .EQ. 0)GO TO 1007
      CALL PUNCH
      JONE = NTOP + 2*I -1
      JTWO = M+I-1
      JTHR = M+I
      IF (JTWO .EQ. NLAY(IL,LNEXT)) IL=IL+1
      CALL SET(JTWO,LNEXT)
      IF (IB .EQ. 0)GO TO 1007
      CALL PUNCH
      JONE = NTOP + 2*I -1
      JTWO = NTOP + 2*I
      JTHR = M+I
      IF (JONE .EQ. NLAY(IL,L))IL=IL+1
      CALL SET(JONE,L)
      IF (IB .EQ. 0)GO TO 1007
      CALL PUNCH
      I = I+1                                                   142
      IF (I .LT. NCOL(LNEXT)) GO TO 59
      GO TO 58
C*******      CHANGE OF SPACING.   LARGE TO SMALL              14
   54 M = NTOP + NCOL(L)
      CALL SETIB
      IF (IB .EQ. 0)GO TO 1007
      I = 1                                                     13
   43 JONE = NTOP + I - 1
      JTWO = M + 2*I -2
      JTHR = M + 2*I -1
      IF (JTWO .EQ. NLAY(IL,LNEXT)) IL=IL+1
      CALL SET(JTWO,LNEXT)
      IF (IB .EQ. 0)GO TO 1007
      CALL PUNCH
      JONE = NTOP + I -1
```

```
      JTWO = NTOP+I
      JTHR = M + 2*I -1
      IF (JONE .EQ. NLAY(IL,L)) IL=IL+1
      CALL SET(JONE,L)
      IF (IB .EQ. 7)GO TO 1007
      CALL PUNCH
      JONE = NTOP + I
      JTWO = M + 2*I -1
      JTHR = M + 2*I
      IF (JTWO .EQ. NLAY(IL,LNEXT)) IL=IL+1
      CALL SET(JTWO,LNEXT)
      IF (IB .EQ. 7)GO TO 1007
      CALL PUNCH
      I = I+1                                                           227
      IF (I .LT. NCOL(L)) GO TO 43                                      228
      GO TO 58
 1003 WRITE (6,1004)
 1004 FORMAT (5X,53H DATA ERROR - NO. OF NODAL POINTS IN ROW INCOMPATIBL231
     1E)                                                                232
      GO TO 60
 1005 WRITE(6,1006)BP(1,L)
 1006 FORMAT(5X,29HSPACING TOO SMALL FOR SURFACE,2X,F10.4,2X,38HSO FAR B
     1ELOW Y=0 - INCREASE DO LOOP 23)
      GO TO 60
 1007 WRITE(6,1008)
 1008 FORMAT(28H NO. OF BOUNDARIES EXCEEDS 6)
      GO TO 61
   61 CONTINUE
  343 CONTINUE
   60 STOP
      END


      SUBROUTINE PUNCH
      COMMON JONE,JTWO,JTHR,IB,IL,ROB(6),E(6),V(6),NELEM,ROSS(6),
     1M,NTOP,NBOUN(6,1000),L,LNEXT
      I=IB-1
      ROJ = ROB(I)-ROSS(IL)
      EJ = E(I)
      VJ = V(I)
```

```
      WRITE(6,1) NELEM,JONE,JTWO,JTHR,EJ,VJ,ROJ
      WRITE(7,1) NELEM,JONE,JTWO,JTHR,EJ,VJ,ROJ
    1 FORMAT(4I10,F10.3,2F10.4)
      NELEM = NELEM+1
      RETURN
      END


      SUBROUTINE SETIB
      COMMON JONE,JTWO,JTHR,IB,IL,ROO(6),E(6),V(6),NELEM,ROSS(6),
     1M,NTOP,NBOUN(6,1000),L,LNEXT
      DO 1 I=2,6
      IF (M .NE. NBOUN(I,LNEXT))GO TO 2
      IF (NTOP .NE. NBOUN(I,L))GO TO 2
      GO TO 1
    2 IB=I
      RETURN
    1 CONTINUE
      IB=0
      RETURN
      END


      SUBROUTINE SET(J,LP)
      COMMON JONE,JTWO,JTHR,IB,IL,ROO(6),E(6),V(6),NELEM,ROSS(6),
     1M,NTOP,NBOUN(6,1000),L,LNEXT
      DO 1 I=IB,6
      IF (J .EQ. NBOUN(I,LP))GO TO2
      RETURN
    2 IB=IB+1
    1 CONTINUE
      IB=0
      RETURN
      END
```

C. Program to improve band-width.

Purpose

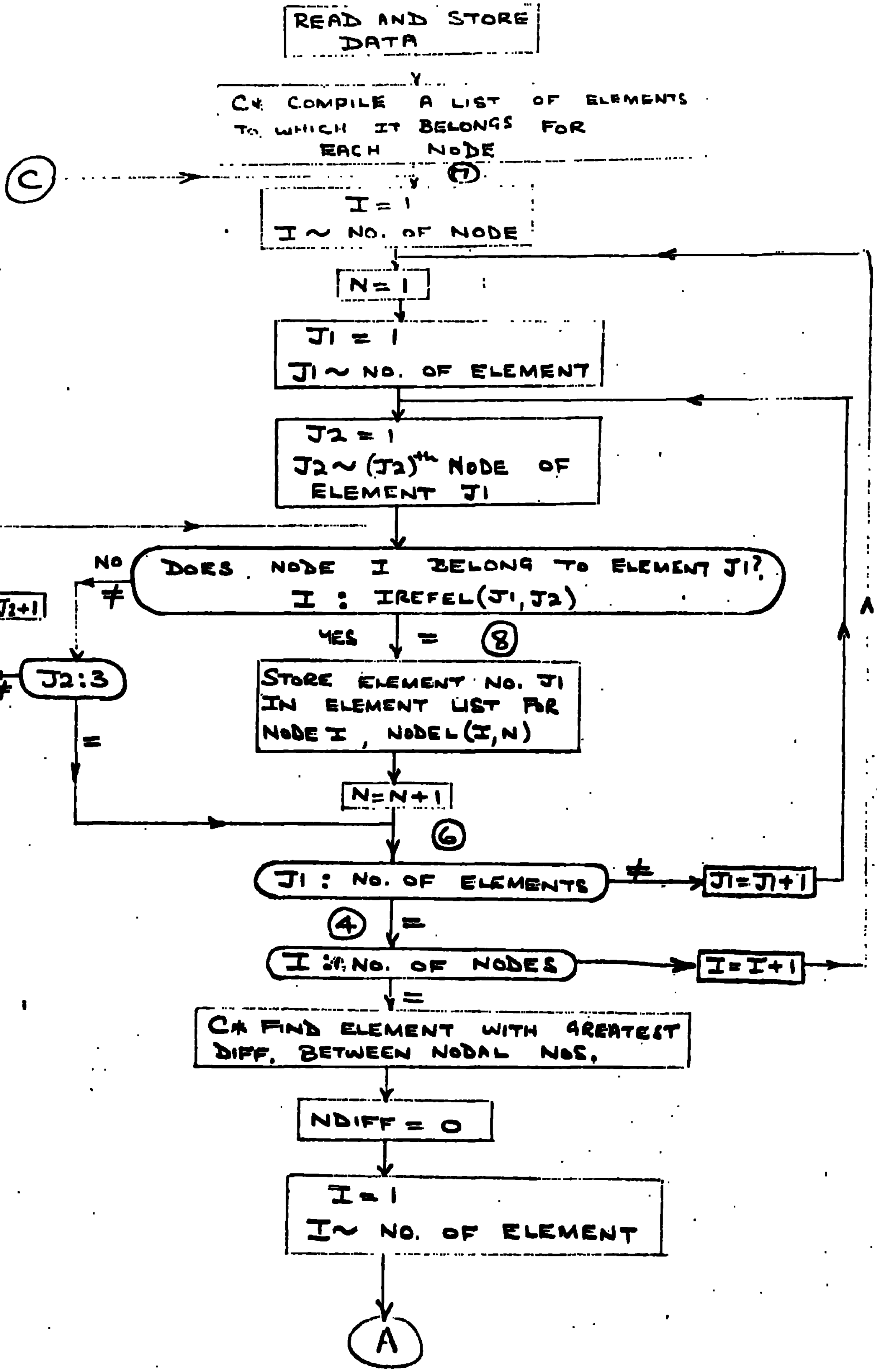
To reduce the band width of a given set of finite element data.

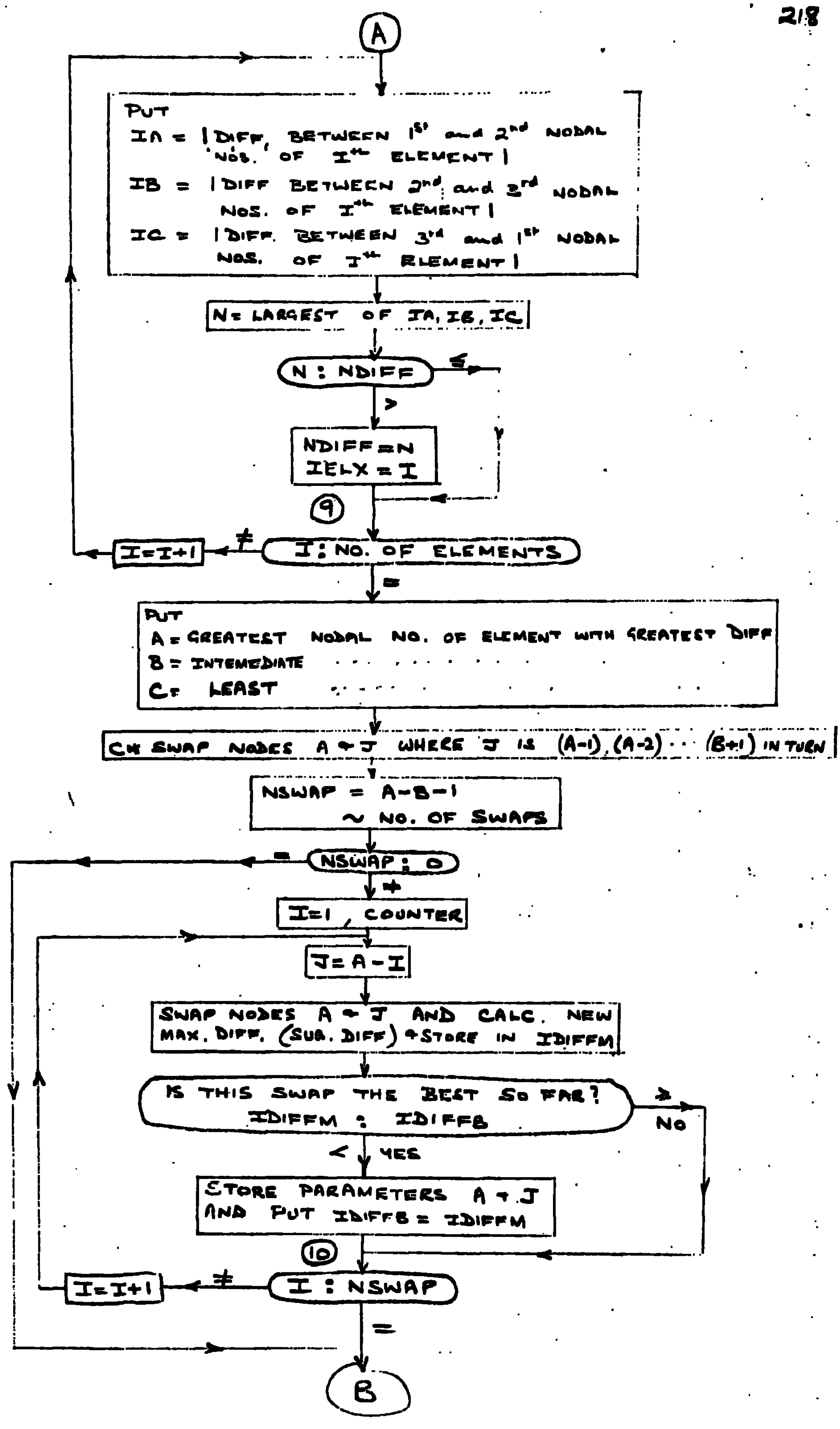
Data Input

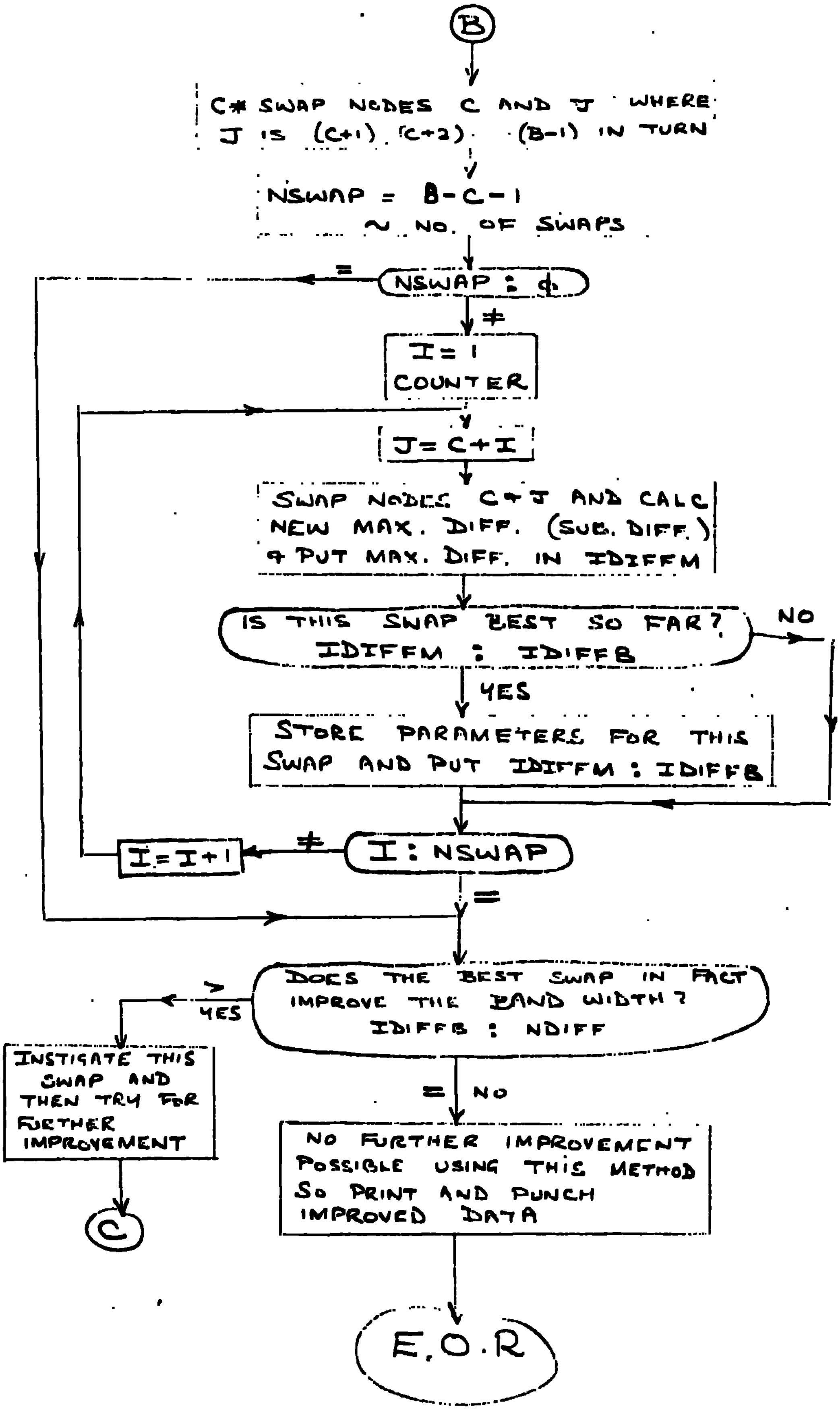
The mesh data as for FINEl.

Data Output

The mesh data as for FINEL with a reduced band width.


READ AND STORE DATA
C* COMPILE A LIST OF ELEMENTS TO WHICH IT BELONGS FOR EACH NODE
7
C
I = 1
I ~ NO. OF NODE
N = 1
J1 = 1
J1 ~ NO. OF ELEMENT
J2 = 1
J2 ~ (J2)th NODE OF ELEMENT J1
DOES NODE I BELONG TO ELEMENT J1?
I : IREFEL(J1, J2)
NO
≠
YES
=
8
J2=J2+1
J2:3
≠
=
STORE ELEMENT NO. J1 IN ELEMENT LIST FOR NODE I, NODEL(I,N)
N=N+1
6
J1 : NO. OF ELEMENTS
≠
J1=J1+1
4
=
I : NO. OF NODES
I=I+1
=
C* FIND ELEMENT WITH GREATEST DIFF. BETWEEN NODAL NOS.
NDIFF = 0
I = 1
I ~ NO. OF ELEMENT
A


A
PUT
IA = | DIFF. BETWEEN 1st and 2nd NODAL NOS. OF Ith ELEMENT |
IB = | DIFF BETWEEN 2nd and 3rd NODAL NOS. OF Ith ELEMENT |
IC = | DIFF. BETWEEN 3rd and 1st NODAL NOS. OF Ith ELEMENT |
N = LARGEST OF IA, IB, IC
N : NDIFF
≤
>
NDIFF = N
IELX = I
9
I : NO. OF ELEMENTS
≠
I = I + 1
=
PUT
A = GREATEST NODAL NO. OF ELEMENT WITH GREATEST DIFF
B = INTEMEDIATE
C = LEAST
CK SWAP NODES A + J WHERE J IS (A-1), (A-2) · · · (B+1) IN TURN
NSWAP = A - B - 1
~ NO. OF SWAPS
NSWAP : 0
=
≠
I = 1 , COUNTER
J = A - I
SWAP NODES A + J AND CALC. NEW MAX. DIFF. (SUB. DIFF) + STORE IN IDIFFM
IS THIS SWAP THE BEST SO FAR?
IDIFFM : IDIFFB
>
NO
<
YES
STORE PARAMETERS A + J
AND PUT IDIFFB = IDIFFM
10
I : NSWAP
≠
I = I + 1
=
B



B
C* SWAP NODES C AND J WHERE J IS (C+1) (C+2) (B-1) IN TURN
NSWAP = B-C-1 ~ NO. OF SWAPS
NSWAP : 0
=
≠
I = 1 COUNTER
J = C+I
SWAP NODES C + J AND CALC NEW MAX. DIFF. (SUB. DIFF.) + PUT MAX. DIFF. IN IDIFFM
IS THIS SWAP BEST SO FAR? IDIFFM : IDIFFB
NO
YES
STORE PARAMETERS FOR THIS SWAP AND PUT IDIFFM : IDIFFB
I = I+1
≠
I : NSWAP
=
DOES THE BEST SWAP IN FACT IMPROVE THE BAND WIDTH? IDIFFB : NDIFF
>
YES
INSTIGATE THIS SWAP AND THEN TRY FOR FURTHER IMPROVEMENT
C
=
NO
NO FURTHER IMPROVEMENT POSSIBLE USING THIS METHOD SO PRINT AND PUNCH IMPROVED DATA
E.O.R

```
C*********     PROGRAM TO IMPROVE BAND-WIDTH OF FINITE ELEMENT DATA
      COMMON IDIFFM,NODEL(1000,10),IREFEL(1500,3)
      DIMENSION RNODAR(1000,6),ISDISP(1000,2),ELASTO(1500,3)
      DIMENSION DUM(6)
C*********     READ AND STORE DATA
      READ(5,1)NPOIN,NELEM
      IF (NPOIN .GT. 1000)STOP1
      IF (NELEM .GT. 1500)STOP2
      DO 35 J=1,NPOIN
   35 READ(5,2) N,(RNODAR(N,I),I=1,2),ISDISP(N,1),RNODAR(N,3),ISDISP(N,2
     1),RNODAR(N,4),RNODAR(N,5),RNODAR(N,6)
      DO 36 J=1,NELEM
   36 READ(5,3) N,(IREFEL(N,I),I=1,3),(ELASTO(N,I),I=1,3)
C*********     COMPILE LIST OF ALL THE ELEMENTS TO WHICH EACH NODE BELONGS
   17 DO 4 I=1,NPOIN
      N=1
      DO 5 J=1,10
    5 NODEL(I,J)=0
      DO 6  J1=1,NELEM
C**        DOES  NODE I BELONG TO ELEMENT J1
      DO 7  J2=1,3
      IF (I .EQ. IREFEL(J1,J2))GO TO 8
    7 CONTINUE
      GO TO 6
    8 NODEL(I,N)=J1
      N=N+1
      IF (N .GT. 10)STOP3
    6 CONTINUE
    4 CONTINUE
C*********       FIND ELEMENT WITH GREATEST NODAL DIFFERENCE
      NDIFF=0
      DO 9  I=1,NELEM
      IA = IABS(IREFEL(I,1)-IREFEL(I,2))
      IB = IABS(IREFEL(I,2)-IREFEL(I,3))
      IC = IABS(IREFEL(I,1)-IREFEL(I,3))
      N  = MAX0(IA,IB,IC)
      IF (N .LE. NDIFF)GO TO 9
      IELX = I
      NDIFF=N
    9 CONTINUE
      WRITE(6,1) IELX,NDIFF
```

```
C*********       ORDER THE NODES OF THE ELEMENT WITH GREATEST DIFFERENCE
C                IN NODAL NOS. IN DESCENDING ORDER  A,B,C
      IA = MAX0(IREFEL(IELX,1),IREFEL(IELX,2),IREFEL(IELX,3))
      IC = MIN0(IREFEL(IELX,1),IREFEL(IELX,2),IREFEL(IELX,3))
      IB =      (IREFEL(IELX,1)+IREFEL(IELX,2)+IREFEL(IELX,3)-IA-IC)
C*********       SWAP NODES A AND J WHERE J IS (A-1),(A-2).....(B+1) IN TURN
      IDIFFB = NDIFF
      NSWAP = IA-IB-1
      IF (NSWAP .EQ. 0)GO TO 10
      DO 10 I=1,NSWAP
      J=IA-I
C*********       CALC. NEW MAXIMUM DIFF. AFTER THIS SWAP
      CALL DIFF(IA,J)
C*********        WE ARE AFTER THE SWAP THAT GIVES THE LEAST MAXIMUM DIFF.
      IF (IDIFFM .GE. IDIFFB)GO TO 10
C*********        STORE  PARAMETERS FOR BEST SWAP SO FAR
      IDIFFB = IDIFFM
      IAB=IA
      JB=J
   10 CONTINUE
C*********        NOW SWAP NODE C WITH J WHERE J IS (C+1),(C+2),....(B-1) IN TURN
      NSWAP = IB-IC-1
      IF (NSWAP .EQ. 0)GO TO 11
      DO 11 I=1,NSWAP
      J = IC+I
C*********        CALC. NEW MAX. DIFF.AFTER THIS SWAP
      CALL DIFF(IC,J)
C*********        WHICH SWAP GIVES THE LEAST MAXIMUM DIFF.
      IF (IDIFFM .GE. IDIFFB)GO TO 11
C*********        STORE  PARAMETERS FOR BEST SWAP
      IDIFFB = IDIFFM
      IAB=IC
      JB=J
   11 CONTINUE
C*********        DOES THE BEST SWAP IMPROVE THE BAND-WIDTH
      IF (IDIFFB .EQ. NDIFF)GO TO 25
C*********       IT DOES THEREFORE INSTIGATE SWAP AND START AGAIN
      DO 12 I=1,6
      DUM(I)= RNODAR(IAB,I)
      RNODAR(IAB,I) = RNODAR(JB,I)
   12 RNODAR(JB,I) = DUM(I)
```

```
      IDUM1 = ISDISP(IAB,1)
      IDUM2 = ISDISP(IAB,2)
      ISDISP(IAB,1)=ISDISP(JB,1)
      ISDISP(IAB,2)=ISDISP(JB,2)
      ISDISP(JB,1)=IDUM1
      ISDISP(JB,2)=IDUM2
      DO 14 I=1,NELEM
      DO 14 J=1,3
      IF (IREFEL(I,J) .EQ. IAB)GO TO 15
      IF (IREFEL(I,J) .EQ.  JB)GO TO 16
      GO TO 14
   15 IREFEL(I,J) = JB
      GO TO 14
   16 IREFEL(I,J) = IAB
   14 CONTINUE
C*********       HAVE A FURTHER GO AT IMPROVEMENT
C               PRINT SWAP
      WRITE(6,1)IAB,JB
      GO TO 17
C*********       NO FURTHER IMPROVEMENT POSSIBLE USING THIS METHOD
C               SO PRINT AND PUNCH IMPROVED DATA
   25 DO 18 N=1,NPOIN
      WRITE(6,2) N,(RNODAR(N,I),I=1,2),ISDISP(N,1),RNODAR(N,3),ISDISP(N,
     12),(RNODAR(N,I),I=4,6)
   18 WRITE(7,2) N,(RNODAR(N,I),I=1,2),ISDISP(N,1),RNODAR(N,3),ISDISP(N,
     12),(RNODAR(N,I),I=4,6)
      DO 19  N=1,NELEM
      WRITE(6,3) N,(IREFEL(N,I),I=1,3),(ELASTO(N,I),I=1,3)
   19 WRITE(7,3) N,(IREFEL(N,I),I=1,3),(ELASTO(N,I),I=1,3)
    1 FORMAT(2I10)
    2 FORMAT(I10,2F10.3,I2,F10.3,I2,3F10.3)
    3 FORMAT(4I10,E10.3,2F10.3)
      STOP
      END


      SUBROUTINE DIFF(N1,N2)
      COMMON IDIFFM,NODEL(1000,13),IREFEL(1500,3)
C*********     CALCULATE THE NEW GREATEST NODAL NO. DIFF. OVER ALL THE ELEMENTS
C     CONTAINING NODES N1 AND N2 WHEN THESE NODAL NOS. ARE INTERCHANGED.
```

```
      NMD=0
      J=N2
      IA=N1
      I=0
      N=1
    5 IF (IREFEL(NODEL(J,N),1) .EQ. J)GO TO 1
      IF (IREFEL(NODEL(J,N),2) .EQ. J)GO TO 2
      JA = IREFEL(NODEL(J,N),1)
      JB = IREFEL(NODEL(J,N),2)
      GO TO 3
    1 JA = IREFEL(NODEL(J,N),2)
      JB = IREFEL(NODEL(J,N),3)
      GO TO 3
    2 JA = IREFEL(NODEL(J,N),1)
      JB = IREFEL(NODEL(J,N),3)
    3 I1 = IABS(JA-IA)
      I2 = IABS(JB-IA)
      I3 = IABS(JA-JB)
      M = MAX0(I1,I2,I3)
      IF (M .GT. NMD)NMD=M
      N=N+1
      IF(NODEL(J,N) .EQ. 0)GO TO 4
      GO TO 5
    4 IF (I .EQ. 1)GO TO 6
      N=1
      J=N1
      IA=N2
      I=1
      GO TO 5
    6 IDIFFM = NMD
      RETURN
      END
```

D. 3TO6

Purpose

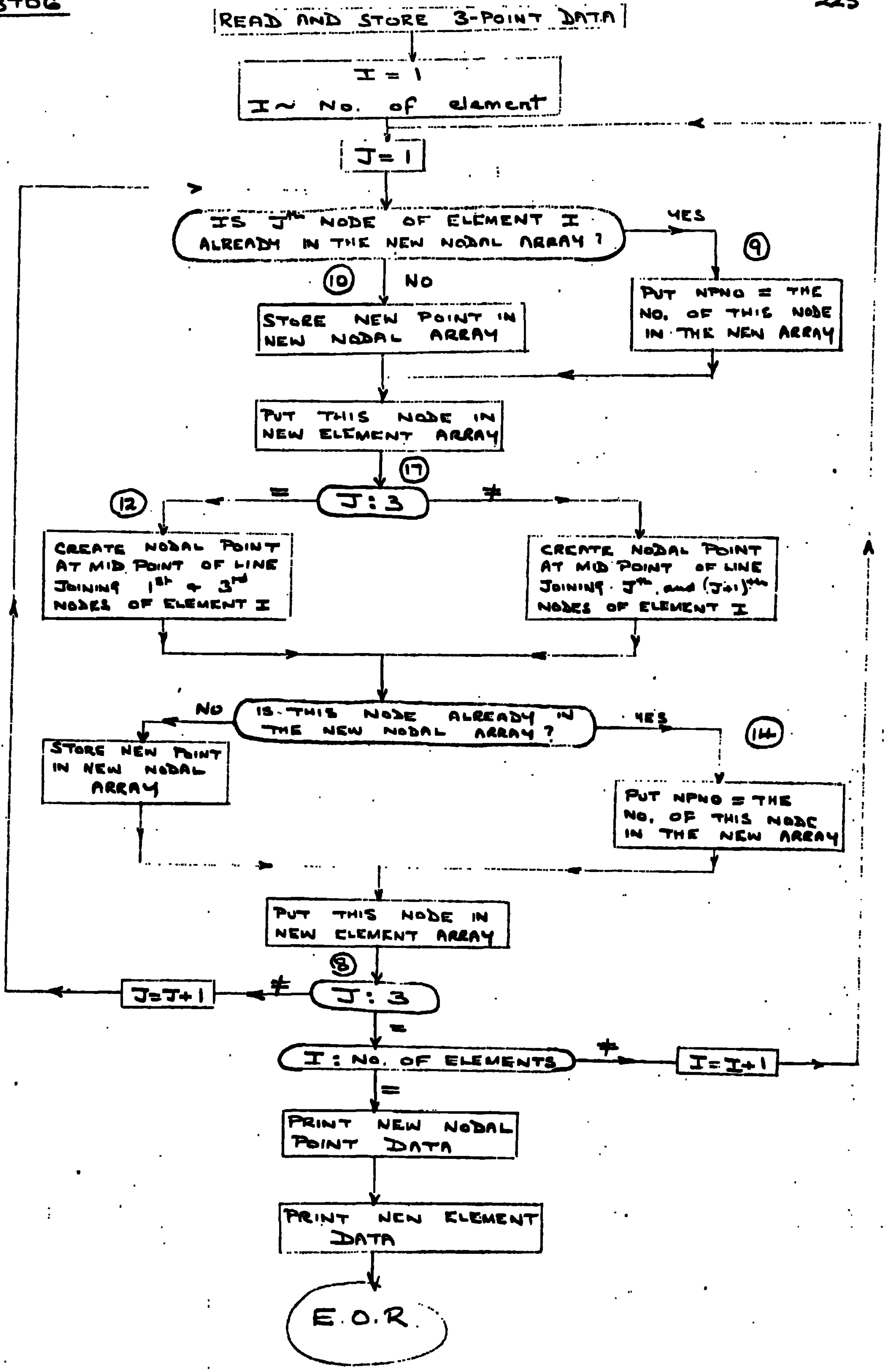
To produce data compatible with FINEL6 using data for FINEL.

Data Input

The mesh data for FINEL.

Data Output

The mesh data for FINEL6.


READ AND STORE 3-POINT DATA
I = 1
I ~ No. of element
J = 1
IS Jth NODE OF ELEMENT I ALREADY IN THE NEW NODAL ARRAY ?
YES
9
PUT NPNO = THE NO. OF THIS NODE IN THE NEW ARRAY
10
NO
STORE NEW POINT IN NEW NODAL ARRAY
PUT THIS NODE IN NEW ELEMENT ARRAY
17
J : 3
=
≠
12
CREATE NODAL POINT AT MID POINT OF LINE JOINING 1st & 3rd NODES OF ELEMENT I
CREATE NODAL POINT AT MID POINT OF LINE JOINING Jth and (J+1)th NODES OF ELEMENT I
IS THIS NODE ALREADY IN THE NEW NODAL ARRAY ?
NO
YES
14
STORE NEW POINT IN NEW NODAL ARRAY
PUT NPNO = THE NO. OF THIS NODE IN THE NEW ARRAY
PUT THIS NODE IN NEW ELEMENT ARRAY
8
J : 3
≠
J=J+1
=
I : No. OF ELEMENTS
≠
I=I+1
=
PRINT NEW NODAL POINT DATA
PRINT NEW ELEMENT DATA
E.O.R.

```
C******     3T06
      COMMON NNEWP,RNEWNP(6000,4)
      DIMENSION RNODAR(1000,4),ISDISP(1000,2),R(2),IREFEL(1500,3),
     1ELACON(1500,2),RO(1500),RFORCE(2000),NISDIS(6000,2),NREFEL(1500,6)
     2,A(2),RFORCN(1200)
C******     READ AND STORE 3-POINT DATA
      READ(5,1) NPOIN,NELEM
    1 FORMAT(2I10)
      DO 2 I=1,NPOIN
      READ(5,3)NREF,(RNODAR(NREF,J),J=1,2),ISDISP(NREF,1),RNODAR(NREF,3)
     1,ISDISP(NREF,2),RNODAR(NREF,4),R(1),R(2)
    3 FORMAT(I10,2F10.0,2(I2,F8.0),2E10.0)
      M = NREF*2-1
      DO 4 J=1,2
      RFORCE(M)=R(J)
    4 M=M+1
    2 CONTINUE
      DO 5 I=1,NELEM
      READ(5,6)IEREF,(IREFEL(IEREF,J),J=1,3),(ELACON(IEREF,K),K=1,2),RO(
     1IEREF)
    6 FORMAT(4I10,E10.0,2F10.0)
    5 CONTINUE
      NNEWP = 1
      DO 7 I=1,NELEM
      DO 8 J=1,3
      CALL   ISITNU(RNODAR(IREFEL(I,J),1),RNODAR(IREFEL(I,J),2),MARK,NPNO
     1)
      IF (MARK .EQ. 0) GO TO 9
C*******  NEW POINT TO BE STORED IN NODAL POINT ARRAY
      DO 10  K=1,4
   10 RNEWNP(NNEWP,K) = RNODAR(IREFEL(I,J),K)
      DO 11 K=1,2
   11 NISDIS(NNEWP,K) = ISDISP(IREFEL(I,J),K)
      IA=2*NNEWP
      IB=IA-1
      IC=2*IREFEL(I,J)
      ID=IC-1
      RFORCN(IB)=RFORCE(ID)
      RFORCN(IA)=RFORCE(IC)
      NNEWP = NNEWP + 1
    9 K= 2*J-1
```

```
      IF (MARK .EQ. 0) GO TO 16
      NREFEL(I,K) = NNEWP - 1
      GO TO 17
   16 NREFEL(I,K) = NPNO
C*******  CREATE  NEW POINT AT MIDPOINT OF SIDE
   17 IF (J .EQ. 3) GO TO 12
      K=J+1
      GO TO 13
   12 K=1
   13 A(1)=(RNODAR(IREFEL(I,K),1)+RNODAR(IREFEL(I,J),1))/2.0
      A(2)=(RNODAR(IREFEL(I,K),2)+RNODAR(IREFEL(I,J),2))/2.0
      CALL ISITNU(A(1),A(2),MARK,NPNO)
      IF (MARK .EQ. 0) GO TO 14
C*******  NEW POINT TO BE STORED IN NODAL POINT ARRAY
      DO 15 K=1,2
      RNEWNP(NNEWP,K) = A(K)
      KD=K+2
      RNEWNP(NNEWP,KD) = 0.0
   15 NISDIS(NNEWP,K) = 0
      IA=2*NNEWP
      IB=IA-1
      RFORCN(IB)=0.0
      RFORCN(IA)=0.0
      NNEWP = NNEWP + 1
   14 K=2*J
      IF (MARK .EQ. 0) GO TO 18
      NREFEL(I,K) = NNEWP-1
      GO TO 19
   18 NREFEL(I,K)=NPNO
   19 CONTINUE
    8 CONTINUE
    7 CONTINUE
C**********    PUNCH NEW DATA
      N=NNEWP-1
      WRITE(7,20)N,NELEM
   20 FORMAT(2I10)
      DO 21 I=1,N
      K=2*I-1
      L=2*I
   21 WRITE(7,22)I,(RNEWNP(I,J),J=1,2),NISDIS(I,1),RNEWNP(I,3),NISDIS(I,
     12),RNEWNP(I,4),RFORCN(K),RFORCN(L)
```

```
   22 FORMAT(I10,2F10.4,I2,F8.4,I2,F8.4,2E10.4)
      DO 24 I=1,NELEM
   24 WRITE(7,23)I,(NREFEL(I,J),J=1,6),ELACON(I,1),ELACON(I,2),RO(I)
   23 FORMAT(I10,6I5,E10.4,F10.4,F10.4)
      STOP
      END



      SUBROUTINE ISITNU(X,Y,MARK,NPNO)
      COMMON  NNEWP,RNEWNP(6000,4)
      N=NNEWP-1
      IF (N .EQ. 0)GO TO 1
      DO 3 I=1,N
      IF (X .EQ. RNEWNP(I,1))GO TO 4
      GO TO 3
    4 IF (Y .EQ. RNEWNP(I,2))GO TO 5
    3 CONTINUE
      GO TO 1
C******* NOT A NEW POINT
    5 NPNO=I
      MARK=0
      GO TO 2
C*******  A NEW POINT
    1 MARK=1
    2 RETURN
      END
```

E. FINEL6

Purpose

To perform a finite element calculation with a 6-noded element for the cases of plane stress and strain.

Data Input

As obtained from 3T06.

For data storage and theory see Appendix 3. For flow charts see FINEL.

```
C*****************        FINEL6
C                        D.S.DEAN
      DOUBLE PRECISION ST
      DOUBLE PRECISION RFORCE
      DOUBLE PRECISION SUMX,SUMY
      COMMON ST(37000),RFORCE(6000),RNODAR(3000,4),IREFEL(1500,6),
     1NELEM,NPOIN,NBW,B(3,12),S(12,12),D(3,3),IMK,RO(1500),ISDISP(3000,2
     2),FLACON(3000,2),MARK(3)

      COMMON/A/ YSUB,ROSUB,IBASEP(200),NBASEP
      REAL NN
      INTEGER S
      READ (5,6) NJOB
    6 FORMAT (I10)
      DO 1000 JN=1,NJOB
      DO 9 I=1,37000
    9 ST(I)=0.0
C READ AND PRINT TITLE
      READ (5,1)
    1 FORMAT  (1H1,'
     1                                      ')
      WRITE (6,1)
      READ(5,2) NPOIN,NELEM,IMK,(MARK(I),I=1,3)
    2 FORMAT(6I10)
      IF (NPOIN .GT. 3000) GO TO 11
      IF (NELEM .GT. 1500) GO TO 12
      CALL RDATA
      NC = 2*NBW*NPOIN
      IF (NC .GT. 37000) GO TO 13
C*******  FORM STIFFENESS MATRIX K AND APPLIED FORCE MATRIX F
      DO 3 I=1,NELEM
      CALL FMATS(I)
    3 CONTINUE
C******  FORM R.H.S. OF STIFFENESS EQN. AND INTRODUCE PRESCRIBED DISP.
      CALL INDISP
C***********          ADD IN ISOSTATIC FORCES
      IF (MARK(1) .EQ. 1)CALL ISOS
C******  SOLVE STIFFENESS EQN. AND PRINT DISP. FOR EACH NODAL PNT.
      CALL CHOLS
      WRITE (6,5)
    5 FORMAT(1H1,//3X,7HELEMENT,12X,5HALPHA,20X,2HNN,21X,2HSS,21X,2HZZ,
```

```
     110X,6HSTRESS,/20X,9H(DEGREES),12X,3(13H(DYNES/SQ CM),10X),4HDIFF)
      DO 4 I=1,NELEM
      CALL FSTRES(I)
    4 CONTINUE
  100 CONTINUE
      GO TO 10
   11 I=1
      GO TO 999
   12 I=2
      GO TO 999
   13 I = 3
      GO TO 999
  999 CALL ERROR(I)
   10 STOP
      END


      SUBROUTINE RDATA
      DOUBLE PRECISION ST
      DOUBLE PRECISION RFORCE
      COMMON ST(37000),RFORCE(6000),RNODAR(3000,4),IREFEL(1500,6),
     1NELEM,NPOIN,NBN,B(3,12),S(12,12),D(3,3),IMK,PD(1500),ISDISP(3000,2
     2),ELACON(3000,2),MARK(3)
      COMMON/A/ YSUB,POSUB,IBASER(200),NBASER
      DIMENSION R(2)
      NRJ=0
C     PRINT HEADING FOR NODAL POINT DATA                                 5
      WRITE (6,1) NPOIN,NELEM                                            6
    1 FORMAT (//32X,19H NO. OF NODAL PNTS.,20X,16H NO. OF ELEMENTS//38X,
     1I4,34X,I4)
      WRITE (6,3)
    3 FORMAT (////4X,12H REF. NO. OF,18X,8H CO-ORDS,22X,16H SPECIFIED DI
     1SP.,19X,14H APPLIED FORCE/5X,10H NODAL PNT,15X,2H X,12X,2H Y,18X,
     22H X,12X,2H Y,15X,2H X,12X,2H Y///)
C     READ AND STORE NODAL POINT DATA                                   11
      DO 2 I = 1,NPOIN                                                  12
      READ (5,4) NREF,(RNODAR(NREF,J),J=1,2),ISDISP(NREF,1),RNODAR(NREF,
     13),ISDISP(NREF,2),RNODAR(NREF,4),R(1),R(2)
    4 FORMAT (I10,2F10.0,I2,F8.0,I2,F8.0,2F10.0)
```

```
      IF (NREF .GT. NPOIN)GO TO 50
C     FORM APPLIED FORCE MATRIX R                                      25
      M = NREF*2-1
      DO 11  J=1,2                                                     27
      RFORCE(M) = R(J)
      M = M+1                                                          30
   11 CONTINUE
    2 CONTINUE
C     PRINT NODAL POINT DATA                                           33
      DO 12 I=1,NPOIN                                                  34
      K=2*I-1                                                          35
      L=2*I                                                            36
C*******  IS X DISP. SPECIFIED.
      IF (ISDISP(I,1) .EQ. 0) GO TO 13
C*******    X DISP SPECIFIED.  IS Y DISP. SPECIFIED.
   16 IF (ISDISP(I,2) .EQ. 0) GO TO 18
C*******  BOTH  X AND Y DISP. SPECIFIED.
   20 WRITE (6,31) I,(RNODAR(I,J),J=1,4),RFORCE(K),RFORCE(L)
      GO TO 12
C*******  X DISP. ONLY SPECIFIED.
   18 WRITE (6,32) I,(RNODAR(I,J),J=1,3),RFORCE(K),RFORCE(L)
      GO TO 12
C*******  X DISP. NOT SPECIFIED.  IS Y DISP. SPECIFIED
   13 IF (ISDISP(I,2) .EQ. 0) GO TO 15
C*******  Y DISP. ONLY SPECIFIED.
   17 WRITE (6,33) I,(RNODAR(I,J),J=1,2),RNODAR(I,4),RFORCE(K),RFORCE(L)
      GO TO 12
C*******    NO DISP. SPECIFIED
   15 WRITE (6,34) I,(RNODAR(I,J),J=1,2),RFORCE(K),RFORCE(L)
   31 FORMAT (7X,I4,15X,2(F9.3,5X),2(6X,F9.3),11X,E10.3,3X,E10.3)
   32 FORMAT (7X,I4,15X,2(F9.3,5X),6X,F9.3,26X,E10.3,3X,E10.3)
   33 FORMAT (7X,I4,15X,2(F9.3,5X),19X,F9.3,11X,E10.3,3X,E10.3)
   34 FORMAT (7X,I4,15X,2(F9.3,5X),41X,E10.3,3X,E10.3)
   12 CONTINUE                                                         39
      IF (MARK(1) .EQ. 0)GO TO 52
      READ(5,44)YSUB,FSUB
   44 FORMAT(2F10.0)
      NBASEP=0
      DO 45 I=1,NPOIN
      IF (RNODAR(I,2) .NE. YSUB)GO TO 45
      NBASEP=NBASEP+1
```

```
      ITASER(NTASER)=I
   47 CONTINUE
CAKWYSIA    CHANGE UNITS OF CO-ORDS AND DISPLACEMENTS
      DO 4  I=1,NPOIN
      DO 3C   JY = 1,2
      MY = JY+2
      RNODAR(I,JY) = RNODAR(I,JY)*1.0E5
      RNODAR(I,MY) = RNODAR(I,MY)*1.0E2
   3C CONTINUE
   4  CONTINUE
C     READ AND STORE ELEMENT DATA                                      4
      WRITE (6,16)                                                     41
   16 FORMAT (1H1,//2X,8H REF NO.,22X,23H NODAL PNTS. OF ELEMENT,24X,19H
     1 ELASTIC CONSTANTS,15X,2H NO./11H,OF ELEMENT,14X,4H (1),
     24X,4H (2),4X,4H (3),4X,4H (4),4X,4H (5),4X,4H (6),20X,7H E(CGS),7X
     3,2H V,15X,2H (GM/CC))
      DO 25 I=1,NELEM                                                  45
      READ(5,27)IREFE,(IREFEL(IREFE,J),J=1,6),(ELACON(IREFE,K)
     1,K=1,2),RO(IREFE)
   27 FORMAT(I10,6I5,E10.2,2F10.C)
      IA = IABS(IREFEL(IREFE,1) - IREFEL(IREFE,2))
      IB = IABS(IREFEL(IREFE,1) - IREFEL(IREFE,3))
      IC = IABS(IREFEL(IREFE,1) - IREFEL(IREFE,4))
      ID = IABS(IREFEL(IREFE,1) - IREFEL(IREFE,5))
      IE = IABS(IREFEL(IREFE,1) - IREFEL(IREFE,6))
      IF = IABS(IREFEL(IREFE,2) - IREFEL(IREFE,3))
      IG = IABS(IREFEL(IREFE,2) - IREFEL(IREFE,4))
      IH = IABS(IREFEL(IREFE,2) - IREFEL(IREFE,5))
      II = IABS(IREFEL(IREFE,2) - IREFEL(IREFE,6))
      IJ = IABS(IREFEL(IREFE,3) - IREFEL(IREFE,4))
      IK = IABS(IREFEL(IREFE,3) - IREFEL(IREFE,5))
      IL = IABS(IREFEL(IREFE,3) - IREFEL(IREFE,6))
      IM = IABS(IREFEL(IREFE,4) - IREFEL(IREFE,5))
      IN = IABS(IREFEL(IREFE,4) - IREFEL(IREFE,6))
      IO = IABS(IREFEL(IREFE,5) - IREFEL(IREFE,6))
      NBW = MAX0(IA,IB,IC,ID,IE,IF,IG,IH,II,IJ,IK,IL,IM,IN,IO,NBW)
   25 CONTINUE
      MAXN = 2 + 2*NBW
C     PRINT ELEMENT DATA                                               40
      DO 37 I=1,NELEM
      WRITE (6,19) I,(IREFEL(I,J),J=1,6),(ELACON(I,K),K=1,2),RO(I)
```

```
10 FORMAT(4X,I4,14X,A(I4,4X),11X,F9.2,4X,F5.2,12X,F5.2)
 8  CONTINUE
51 RETURN
52 WRITE(6,53)
53 FORMAT(5 H REF. NO. OF NODAL POINT GREATER THAN NO. OF NODES)
   STOP
   END


   SUBROUTINE FORMD(N)
   DOUBLE PRECISION ST
   DOUBLE PRECISION RFORCE
   COMMON ST(27000),RFORCE(6000),RNODAR(3000,4),IPEFEL(1500,6),
  1NELEM,NPOIN,NBW,B(3,12),S(12,12),D(3,3),IMK,PR(1500),ISDISP(3000,2
  2),ELACON(3000,2),MASK(3)
C FORM D MATRIX FOR ELEMENT N
C  IF IMK=1 PLANE STRESS , IF IMK=2 THEN PLANE STRAIN
   IF (IMK .EQ. 1) GO TO 1
C  PLANE STRAIN
   A= ELACON(N,1)*(1.0 -ELACON(N,2))/((1.0 + ELACON(N,2))*(1.0 - 2*EL
  1ACON(N,2)))
   B= ELACON(N,2)/(1.0 - ELACON(N,2))
   GO TO 2
C     PLANE STRESS
 1 A= ELACON (N,1)/(1.0 - ELACON(N,2)**2)
   B= ELACON (N,2)
C     FORM MATRIX D
 2 D(1,1) = A
   D(1,2) = A*B
   D(1,3) = 0.0
   D(2,1) = A*B
   D(2,2) = A
   D(2,3) = 0.0
   D(3,1) = 0.
   D(3,2) = 0.0
   D(3,3) = A*(1.0 - B)/2.0
   RETURN
   END
```

```
      SUBROUTINE [illegible](I)
      [illegible] ST,[illegible]
      COMMON ST(37[illegible]),[illegible](A[illegible]),[illegible]ODAR(3,[illegible],4),IREFEL(15[illegible],6),
     1 [illegible],S([illegible],12),[illegible](3,2),[illegible],IS [illegible]
     2),[illegible](2[illegible],2),[illegible](2)
      INTEGER [illegible]
C[illegible]  [illegible] MATRIX  [illegible] AT CENTROID
      [illegible] = ([illegible]DAR(I[illegible]EL(I,1),1)+[illegible]DAR(I[illegible]EL(I,2),1)+[illegible]DAR(IREFEL(
     1I,4),1))/3.
      [illegible] = [illegible]DAR(I[illegible]EL(I,1),2)+[illegible]DAR(I[illegible]EL(I,2),2)+[illegible]DAR(IREFEL(
     1I,3),2))/3.[illegible]
      DO 1 I=1,3
      DO 1 J=[illegible],12
    1 [illegible](I,J) = [illegible].
      [illegible](1,2)=[illegible]
      [illegible](1,4)=[illegible]
      [illegible](1,5)=[illegible]
      [illegible](2,6)=1.[illegible]
      [illegible](2,11)=[illegible]
      [illegible](2,12)=[illegible]
      [illegible](3,[illegible])=1.
      [illegible](3,5)=[illegible]
      [illegible](3,[illegible])=[illegible]
      [illegible]
      [illegible](3,[illegible])=[illegible]
      [illegible]
      [illegible]
      [illegible]

      [illegible] (TI[illegible])
      [illegible] ST,[illegible]
      [illegible] ST([illegible]),[illegible](A[illegible]),[illegible]DAR(3[illegible],4),IREFEL(15[illegible],6),
     1[illegible],[illegible],S([illegible],12),[illegible](12,12),D(3,3),[illegible],[illegible](2[illegible]),[illegible](3[illegible],2
     2),[illegible](2)
      DIMENSION [illegible](12,12),[illegible](12),[illegible](12,12),F(12)
      INT[illegible]
      [illegible],I[illegible],K[illegible]
      [illegible]
      [illegible](I[illegible],[illegible],5),[illegible]) [illegible] ([illegible],3),11)
```

```
     1         -(RNODAR(IREFEL(E,3),2) * RNODAR(IREFEL(E,5),1))
     2         -(RNODAR(IREFEL(E,1),1) * RNODAR(IREFEL(E,5),2))
     3         +(RNODAR(IREFEL(E,1),1) * RNODAR(IREFEL(E,3),2))
     4         +(RNODAR(IREFEL(E,1),2) * RNODAR(IREFEL(E,5),1))
     5         -(RNODAR(IREFEL(E,1),2) * RNODAR(IREFEL(E,3),1))
      CALL FORMC(E)
      CALL FORMD(E)
      XI = RNODAR(IREFEL(E,1),1)
      XJ = RNODAR(IREFEL(E,3),1)
      XM = RNODAR(IREFEL(E,5),1)
      YI = RNODAR(IREFEL(E,1),2)
      YJ = RNODAR(IREFEL(E,3),2)
      YM = RNODAR(IREFEL(E,5),2)
      XBAR = (XI+XJ+XM)/3.0
      YBAR = (YI+YJ+YM)/3.0
      IF (XI .EQ. XM) GO TO 1
      AIM = (YM-YI)/(XM-XI)
      BIM = (YI*XM-YM*XI)/(XM-XI)
      GO TO 2
    1 AIM = 0.0
      BIM = 0.
    2 IF (XI .EQ. XJ) GO TO 3
      AIJ = (YI-YJ)/(XI-XJ)
      BIJ = (YJ*XI-YI*XJ)/(XI-XJ)
      GO TO 4
    3 AIJ=0.0
      BIJ=0.
    4 IF (XM .EQ. XJ) GO TO 5
      AMJ = (YJ-YM)/(XJ-XM)
      BMJ = (YM*XJ-YJ*XM)/(XJ-XM)
      GO TO 6
    5 AMJ = 0.0
      BMJ = 0.
    6 II = (DELTA/ABS(DELTA))*((AIM*(XM**4-XI**4)+AJI*(XI**4-XJ**4)+AMJ*
     1(XJ**4-XM**4))/4.    + (BIM*(XM**3-XI**3)+BIJ*(XI**3-XJ**3)+BMJ*(XJ
     2**3-XM**3))/3.0)
      IJ = (DELTA/ABS(DELTA))*((AIM**2*(XM**4-XI**4)+AIJ**2*(XI**4-XJ**4
     1)+AMJ**2*(XJ**4-XM**4))/8.0   +  (BIM**2*(XM**2-XI**2)+BIJ**2*(XI**
     22-XJ**2)+BMJ**2*(XJ**2-XM**2))/6.0   +  (AIM*BIM*(XM**3-XI**3)+AIJ*
     3BIJ*(XI**3-XJ**3)+AMJ*BMJ*(XJ**3-XM**3))/3.0)
      IX = (DELTA/ABS(DELTA))*((AIM**3*(XM**4-XI**4)+AIJ**3*(XI**4-XJ**4
```

```
1)+AMJ**3*(YJ**4-YM**4))/12.0    +   (AIM**2*BIM*(YM**2-XI**2)+AJI**2*
2BJI*(XI**2-XJ**2)+AMJ**2*BMJ*(XJ**2-XM**2))/2.0   +   (AIM*BIM**2*(X
3I**2-XI**2)+AJI*BJI**2*(XI**2-XJ**2)+AMJ*BMJ**2*(XJ**2-XM**2))/2.0
4+   (BIM**3*(YM-YI)+BJI**3*(XI-XJ)+BMJ**3*(XJ-XM))/3.0)
 DO 7  I=1,12
 DO 7  J=1,12
 KFL(I,J) = 0.0
7 KL(I,J) =0.0
 KL(2,2)=ABS(DELTA)*D(1,1)/2.0
 KL(2,4)=ABS(DELTA)*XBAR*D(1,1)
 KL(2,5)=ABS(DELTA)*YBAR*D(1,1)/2.0
 KL(2,9)=ABS(DELTA)*D(1,2)/2.0
 KL(2,11)=ABS(DELTA)*XBAR*D(1,2)/2.0
 KL(2,12)=ABS(DELTA)*YBAR*D(1,2)
 KL(3,3)=ABS(DELTA)*D(3,3)/2.0
 KL(3,4)=ABS(DELTA)*XBAR*D(3,3)/2.0
 KL(3,6)=ABS(DELTA)*YBAR*D(3,3)
 KL(3,8)=ABS(DELTA)*D(3,3)/2.0
 KL(3,10)=ABS(DELTA)*YBAR*D(3,3)
 KL(3,11)=ABS(DELTA)*YBAR*D(3,3)/2.0
 KL(4,4)=4.0*I1*D(1,1)
 KL(4,5)=3.0*I2*D(1,1)
 KL(4,9)=ABS(DELTA)*XBAR*D(1,2)
 KL(4,11)=2.0*I1*D(1,2)
 KL(4,12)=4.0*I2*D(1,2)
 KL(5,5)=I2*D(1,1) + I1*D(3,3)
 KL(5,6)=4.0*I1*D(3,3)
 KL(5,8)=ABS(DELTA)*YBAR*D(3,3)/2.0
 KL(5,9)=ABS(DELTA)*YBAR*D(1,2)/2.0
 KL(5,10)=2.0*I1*D(3,3)
 KL(5,11)=I2*(D(3,3)+D(1,2))
 KL(5,12)=2.0*I3*D(1,2)
 KL(6,6)=4.0*I3*D(3,3)
 KL(6,8)=ABS(DELTA)*YBAR*D(3,3)
 KL(6,10)=4.0*I2*D(3,3)
 KL(6,11)=2.0*I3*D(3,3)
 KL(8,8)=ABS(DELTA)*D(3,3)/2.0
 KL(8,10)=ABS(DELTA)*XBAR*D(3,3)
 KL(8,11)=ABS(DELTA)*YBAR*D(3,3)/2.0
 KL(9,9)=ABS(DELTA)*D(2,2)/2.0
 KL(9,11)=ABS(DELTA)*XBAR*D(2,2)/2.0
```

```
      KL(4,1?)=ABS(DELTA)*Y[illegible]*D(?,?)
      [illegible](1?,1?)=4.*I[illegible](3,3)
      KL(1?,11)=2.*I2*D(4,?)
      KL(1?,1?)=[illegible](2,?) + I2*[illegible](?,3)
      KL(1?,1?)=?.*I[illegible](?,?)
      KL(1?,1?)=4.?*I2*D(2,2)
      DO 14 I=1,12
      DO 14 J=1,12
      IF(I.LE.J)GO TO 14
      KL(I,J) = KL(J,I)
   14 CONTINUE
      DO 8 I=1,12
      DO 8 J=1,12
      DO 8 L=1,12
      DO 8 M=1,12
    8 KEL(I,J) = KEL(I,J) + S(L,I)*KL(L,M)*S(M,J)
C**** [illegible] [illegible](E) FOR ELEMENT
      DO 9 I=1,6
    9 F(I)=0.0
      F(7)= -QAB[illegible](E)*ABS(DELTA)/2.0
      F([illegible])= -[illegible](E)*A[illegible](DELTA)*[illegible]/2.
      F(?)= -[illegible](E)*ABS(DELTA)*YBAR/2.0
      F([illegible])= -[illegible]*D(E)*[illegible]
      F([illegible])= -[illegible](E)*[illegible]
      F([illegible])= -[illegible](E)*I[illegible]
      DO [illegible] I=1,12
      FE(I)=0.
      DO 11 I=1,12
      DO 11 J=1,[illegible]
   11 FE(I) = FE(I)+S(J,I)*F(J)
C [illegible] [illegible] [illegible]([illegible]) [illegible] [illegible] [illegible]).
      DO [illegible] I=1,4
      [illegible](I[illegible]-1(E,I)-1)*[illegible]
      [illegible] = (I-1)*3
      DO [illegible] J=1,4
      [illegible]=(I[illegible](E,J)-1)*3
      L[illegible] = (J-1)*3
      DO [illegible] K=1,3
      [illegible] = [illegible]+K
      [illegible] = K[illegible]+K
      DO [illegible] [illegible]=1,3
```

```
      [illegible] = [illegible]+1
      [illegible] = [illegible]+1
      IF ([illegible] .GT. [illegible]) GO TO 12
      [illegible] = (KC-1)[illegible] - KC + 1
      [illegible]( [illegible] ) = [illegible] + [illegible]
   1[illegible] CONTINUE
C [illegible] FORM [illegible]
      DO [illegible] I=1,[illegible]
      DO 13 J=1,2
      [illegible] = [illegible]+J
      [illegible] = [illegible](K,I)+J-2
   13 [illegible](M) = [illegible](M) + EEL(K)
      RETURN
      END

      SUBROUTINE [illegible]
      DOUBLE PRECISION ST
      [illegible]
      [illegible] ST(37[illegible] ),[illegible](4[illegible] ), [illegible]([illegible]),I[illegible](1[illegible],6),
     1[illegible],NB[illegible],B(3,1[illegible]),S([illegible],1[illegible]),D(3,3),I[illegible],[illegible]( ),ISOISP(3[illegible],2
     [illegible]), [illegible]([illegible]),[illegible](3)
      [illegible] DISC[illegible] [illegible].
      [illegible] I=1,[illegible]
      [illegible] I=1,2
      IF (ISOISP(I,J).[illegible].[illegible]) GO TO 2
      [illegible]
      [illegible] = ([illegible]+J-[illegible])[illegible]
      [illegible] = [illegible]+J-2
      [illegible] = [illegible]+2
      [illegible](M) = ST(K)*[illegible]
   [illegible] [illegible](M) = ST(K)*[illegible](I,M)
      [illegible]
      [illegible]
      RETURN
      END
```

```
      SUBROUTINE CHOLS
      DOUBLE PRECISION U,ST,SUM,A
      DOUBLE PRECISION REDRCE
      COMMON ST(37000),REDRCE(6000),RAODAR(3000,4),IREFEL(1500,6),
     1NELEM,NPOIN,NRW,T(3,12),S(12,12),D(3,3),IMK,RO(1500),ISDISP(3000,2
     2),ELACON(3000,2),MARK(3)
      DIMENSION U(2000)
      DIMENSION MAR(3000)
      EQUIVALENCE (ISDISP,MAR)
      EQUIVALENCE (REDRCE,U)
      NA = 2*NPOIN
C *******     FORM V AND STORE IN ST
      IF (ST(1) .GE. 0.0) GO TO 1
    2 MAR(1) = 1
      ST(1) = -ST(1)
      ST(1) = DSQRT(ST(1))
    3 DO 5  I = 2,NRW
      ST(I) =  -ST(I)/ST(1)
    5 CONTINUE
      GO TO 37
    1 MAR(1) = 0
      ST(1) = DSQRT(ST(1))
    4 DO 37  I = 2,NRW
      ST(I) = ST(I)/ST(1)
   37 CONTINUE
C *******  I - ROW NO.         J - COLUMN NO.
      DO 22 I = 2,NA
      SUM = 0.0
      NO = I-1
    6 DO 10 M = 1,NO
      NE = M+NRW-1
      IF (I .GT. NE) GO TO 10
    7 IF (MAR(M) .EQ. 1) GO TO 9
    8 SUM = SUM + ST(IN(M,I))**2
      GO TO 10
    9 SUM = SUM - ST(IN(M,I))**2
   10 CONTINUE
      A = ST(IN(I,I)) + SUM
   11 IF (A .GE. 0.0) GO TO 12
   13 MAR(I) = 1
      A = -A
```

```
      GO TO 14
   [illegible] MAP(I) = [illegible]
   [illegible] ST(IN(I,I)) = DSQRT(A)
      NC = I+1
      NB = I+[illegible]W-1
   [illegible] DO [illegible]  J = NC,NB
      SUM = 0.0
   [illegible] DO [illegible]  [illegible] = 1,[illegible]
      NI = [illegible]+N[illegible]W-1
      IF (J .GT. NI) GO TO [illegible]
   [illegible] IF (MAP(M) .EQ. [illegible]) GO TO [illegible]
   [illegible] SUM = SUM + ST(IN(M,I))*ST(IN(M,J))
      GO TO [illegible]
   [illegible] SUM = SUM - ST(IN(M,I))*ST(IN(M,J))
   [illegible] CONTINUE
      ST(IN(I,J)) = (ST(IN(I,J))+SUM)/ST(IN(I,I))
      IF (MAP(I) .[illegible]. 1) ST(IN(I,J)) = -ST(IN(I,J))
   [illegible] CONTINUE
   [illegible] CONTINUE
C ******    [illegible] Y AND STORE IN U
   [illegible] IF (MAP(I) .EQ. [illegible]) GO TO [illegible]
   [illegible] U(I) = -U(I)/ST(I)
      GO TO [illegible]
   [illegible] U(I) = +U(I)/ST(I)
   [illegible] [illegible] I = 2,[illegible]
      NO = I-1
      [illegible]
   [illegible] [illegible] = 1,[illegible]
      NE = M+NBW-1
      IF (I .GT. NE) GO TO [illegible]
   [illegible] IF (MAP( ) .[illegible]. [illegible]) GO TO [illegible]
      SUM = SUM + ST(IN(M,I))*U(M)
      [illegible]
   [illegible] SUM = SUM - ST(IN(M,I))*U(M)
   [illegible] CONTINUE
      U(I) = (U(I)+SUM)/ST(IN(I,I))
      IF (MAP(I) .[illegible]. 1) U(I) = -U(I)
   [illegible] CONTINUE
C ******    [illegible] X AND STORE IN U
      U(NA) = [illegible](NA)/ST(IN(NA,NA))
   [illegible] [illegible] J=2,NA
```

```
      I=NA+1-J
      NC = I+1
      SUM = 0.0
   34 DO 35 M = NC,NA
      NE = I+NBW-1
      IF (M .GT. NE) GO TO 35
      SUM = SUM + ST(IN(I,M))*U(M)
   35 CONTINUE
      U(I) =(U(I)-SUM)/ST(IN(I,I))
   36 CONTINUE
C ******   PRINT NODAL DISPLACEMENTS
      WRITE (6,38)
   38 FORMAT (1H1,///,41X,11H NODAL PNT.,13X,18H DISPLACEMENT (M) ,/65X,
     12H X,12X,2H Y)
      DO 42  I = 1,NPOIN
      N = 2*I-1
      M = 2*I
C******* CHANGE UNITS OF DISPLACEMENTS
      DUMMYX = U(N)*1.0E-2
      DUMMYY = U(M)*1.0E-2
      WRITE(6,39) I,DUMMYX,DUMMYY
   39 FORMAT (43X,I4,16X,F10.3,6X,F10.3)
   42 CONTINUE
   50 RETURN
      END


      FUNCTION IN(I,J)
C ******  TO CALCULATE  THE ARRAY NO. FOR THE (IJ)TH ELEMENT OF A
C ******  SYMMETRIC MATRIX OF SEMI-BAND WIDTH NBW
      DOUBLE PRECISION ST
      DOUBLE PRECISION RAPPCF
      COMMON ST(27000),RFORCE(6000),RNODAR(3000,4),IREFEL(1500,6),
     1NELEM,NPOIN,NBW,B(3,12),S(12,12),D(3,3),INK,RO(1500),ISDISP(3000,2
     2),FLAGIN(3000,2),MARK(3)
      IN = (I-1)*NBW + J - I + 1
      RETURN
      END
```

```
      SUBROUTINE FORMC(E)
      DOUBLE PRECISION ST,RFORCE
      COMMON ST(37..,.),RFORCE(6000),RNODAR(3000,4),IREFEL(1500,6),
     1NELEM,NPOIN,NBW,B(3,12),S(12,12),D(3,3),IMK,RO(1500),ISDISP(3000,2
     2),ELACON(3...,3),MARK(.)
      DIMENSION C(6,6),LW(6),MW(6)
      INTEGER E
      DIMENSION CI(12,12),CJ(12,12)
      DO 10 I=1,12
      DO 10 J=1,12
      CI(I,J) = 0.0
   10 CJ(I,J) = 0.0
      DO 1 I=1,6
      C(I,1) = 1.0
      C(I,2) = RNODAR(IREFEL(E,I),1)
      C(I,3) = RNODAR(IREFEL(E,I),2)
      C(I,4) = C(I,2)**2
      C(I,5) = C(I,3)*C(I,2)
    1 C(I,6) = C(I,3)**2
      DO 11 I=1,6
      J=2*I
      K=2*I-1
      DO 11 L=1,6
      CI(K,L)=C(I,L)
      M=L+6
   11 CI(J,M)=C(I,L)
      N=6
      CALL MINV(C,N,T,LW,MW)
      IF(T .EQ. 0.0)GO TO 4
C****** S = (C) INVERSE
      DO 3 I=1,12
      DO 3 J=1,12
    3 S(I,J)=0.0
      DO 2 I=1,6
      J=2*I
      K=2*I-1
      DO 2 L=1,6
      S(L,K) = C(L,I)
      M=L+6
    2 S(M,J) = C(L,I)
      DO 12 I=1,12
```

```
      DO 12 J=1,12
      DO 12 K=1,12
   12 CJ(I,J) = CJ(I,J) + CI(I,K)*S(K,J)
   14 FORMAT(10X,F15.5)
   12 CONTINUE
      RETURN
    4 WRITE(6,5)
    5 FORMAT(31H MATRIX C HAS ZERO DETERMINANT)
      STOP3
      END


      SUBROUTINE ESTRES(E)
      DOUBLE PRECISION ST
      DOUBLE PRECISION XX,YY,XY
      DOUBLE PRECISION PFORCE
      COMMON ST(37000),PFORCE(4000),RNODAR(2000,4),IPEFEL(1500,6),
     1NELEM,NPOIN,NEW,H(3,12),S(12,12),D(3,3),IMK,RO(1500),ISDISP(3000,2
     2),FLAGON(2000,3),MARK(2)
      INTEGER E
      DIMENSION UE(12),RA(3)
      REAL NN
      IF(E .EQ. 1)K=1
      CALL FORMD(E)
      CALL FORMB(E)
      CALL FORMC(E)
C****** EXTRACT DISP. MATRIX FOR ELEMENT E FROM DISP. MATRIX U
      DO 1  I=1,6
      DO 2  J=1,2
      M = 2*I-2+J
      L = 2*IPEFEL(E,I)-2+J
      UE(M) = PFORCE(L)
    2 CONTINUE
    1 CONTINUE
C****** FORM STRESSES
      XX=0.0
      YY=0.0
      XY=0.0
      DO 3  I=1,3
      DO 3  J=1,12
```

```
      DO 3 K=1,12
      XX = XX + D(1,I)*B(I,J)*S(J,K)*UF(K)
      YY = YY + D(2,I)*B(I,J)*S(J,K)*UF(K)
      XY = XY + D(3,I)*B(I,J)*S(J,K)*UF(K)
    3 CONTINUE                                                          24
C*******    FORM  ZZ                                                    26
      IF (IMK .EQ. 0) GO TO 5                                           27
C*******    PLANE STRESS                                                28
      ZZ=0.0                                                            29
      GO TO 6                                                           30
C*******    PLANE STRAIN                                                31
    5 ZZ= ELACON(E,2)*(XX+YY)                                           32
C*******    CALC. PRINCIPAL STRESSES AND AXES                           33
    6 IF ((XX-YY) .EQ. 0.0) GO TO 7                                     34
      ALPHA = (2.0*XY)/(XX-YY)
      IF (ALPHA .GE. 0.0) GO TO 10
      ALPHA = 0.5*(ATAN(ALPHA)*57.29+180.0)
      GO TO 8
   10 ALPHA = 0.5*ATAN(ALPHA)*57.29
      GO TO 8                                                           36
    7 ALPHA = 45.0
    8 NN = (XX*(COS(ALPHA/57.29)**2))+(YY*(SIN(ALPHA/57.29)**2))
     1 +(XY*SIN(2.0*ALPHA/57.29))
      SS = XX+YY-NN
C*******    PRINT STRESSES                                              39
      A=AMAX1(SS,NN,ZZ)
      C = AMIN1(SS,NN,ZZ)
      A = 0.5*(A-C)
      WRITE(6,9)E,ALPHA,NN,SS,ZZ,A
    9 FORMAT( 5X,I3,13X,F7.2,10X,F10.4,13X,E10.4,13X,F10.4,13X,F10.4)
      IF (MARK(3) .EQ. 0)GO TO 13
      IF (K .EQ. 1)GO TO 12
      K=K-1
      WRITE(7,11)K,((REFEL(K,J),J=1,5,2),PA(1),PA(2),PA(3),E,
     1((REFEL(E,J),J=1,5,2),ALPHA,NN,SS
   11 FORMAT(2(4I4,F5.2,F9.3))
      K=1
      GO TO 13
   12 PA(1)=ALPHA
      PA(2)=NN
      PA(3)=SS
```

```
      K=K+1
   10 CONTINUE
      RETURN                                                              42
      END                                                                 43


      SUBROUTINE ERROR(I)
      DOUBLE PRECISION ST
      DOUBLE PRECISION FFORCE
      COMMON ST(37000),FFORCE(6000),RNODAR(3000,4),IREFEL(1500,6),
     1NELEM,NPOIN,NPW,B(3,12),S(12,12),D(3,3),IMK,PO(1500),ISDISP(3000,2
     2),ELACON(3000,2),MARK(3)
      GO TO (1,2,3),I
    1 WRITE (6,101)
  101 FORMAT(/31HNO.OF NODAL PNTS. TOO LARGE. MUST ENLARGE STORAGE ALLO
     1CATED IN COMMON STATEMENTS AND ERROR CHECK IN MAIN PROG.)
      GO TO 201
    2 WRITE (6,102)
  102 FORMAT(/31HNO. OF ELEMENTS TOO LARGE. MUST ENLARGE STORAGE ALLOCA
     1TED IN COMMON STATEMENTS AND ERROR CHECK IN MAIN PROG.)
      GO TO 201
    3 WRITE (6,103)MAW
  103 FORMAT(/16(HSTORAGE SPACE ALLOCATED FOR STIFFNESS MATRIX NOT ENOUG
     1H AND THE STORAGE ALLOCATED TO ST IN COMMON MUST BE INCREASED TO A
     2T LEAST 2XNPOINXNPW WHERE NPW EQUALS      ,I4)
  201 RETURN
      END


      SUBROUTINE ISOS
      DOUBLE PRECISION ST,FFORCE
      COMMON ST(37000),FFORCE(6000),RNODAR(3000,4),IREFEL(1500,6),
     1NELEM,NPOIN,NPW,B(3,12),S(12,12),D(3,3),IMK,PO(1500),ISDISP(3000,2
     2),ELACON(3000,2),MARK(3)
      COMMON/A/ YSUB,ROSUB,IEASER(200),NEASER

      A = ABS((RNODAR(IEASER(1),1)-RNODAR(IEASER(2),1))/2.0)*981.0
      J = 2*IEASER(1)
      ST(IN(J,J)) = ST(IN(J,J)) + A*ROSUB
```

```
      K = NBASEP - 1
      DO 39 I = 2,K
      J = I+1
      L = I-1
      M=2*IBASEP(I)
      A =(ABS(RNODAR(IBASEP(J),1)-RNODAR(IBASEP(I),1))+ABS(RNODAR(IBASEP
     1(L),1)-RNODAR(IBASEP(I),1)))*0.5*981.0
      ST(IN(M,M)) = ST(IN(M,M)) + A*ROSUB
   39 CONTINUE
      A = ABS(RNODAR(IBASEP(NBASEP),1)-RNODAR(IBASEP(K),1))*0.5*981.0
      J = 2*IBASEP(NBASEP)
      ST(IN(J,J)) = ST(IN(J,J)) + A*ROSUB
      RETURN
      END


      SUBROUTINE VIS
      DOUBLE PRECISION ST,RFORCE
      COMMON ST(27000),RFORCE(4000),RNODAR(3000,4),IPREEL(1500,6),
     1NELEM,NPOIN,NBW,B(3,12),Q(12,12),D(3,3),IMK,RO(1500),ISDISP(3000,2
     2),FLACON(3000,2),MARK(3)
      COMMON /A/ YSUB,ROSUB,IBASEP(200),NBASEP

      DIMENSION AS(23),COL(100),AT(90)
      INTEGER F
      READ(5,102)BLANK,(AS(I),I=1,23),NR,NC
  102 FORMAT(23A1,I2,2X,I2)
      SCLSTR = 1.0E-9
      DO 111 I=1,100
  111 COL(I) = BLANK
      XM = 0.0
      YM = 0.0
      DO 100 I=1,NPOIN
      IF (RNODAR(I,1) .GT. XM) XM=RNODAR(I,1)
  100 IF (RNODAR(I,2) .LT. YM) YM=RNODAR(I,2)
      SPV = -YM/NR
      SPH = XM/NC
      WRITE(6,101)
  101 FORMAT(1H1,10X,25HSTRESS DIFFERENCES*1.0E-8,//)
      YINF=0.0
```

```
      YTWO=0.0
      DO 103 NROW=1,NR
      XONE=0.0
      XTWO=0.0
C*******        SET UP ROW
      YONE=YTWO
      YTWO=YTWO-SPV
      DO 105 NCOL=1,NC
      XONE=XTWO
      XTWO=XTWO+SPH
      DO 104 E=1,NELEM
      XBAR =  (RNODAR(IREFEL(E,1),1)+RNODAR(IREFEL(E,3),1)+
     1RNODAR(IREFEL(E,5),1))/3.0
      YBAR = (RNODAR(IREFEL(E,1),2)+RNODAR(IREFEL(E,3),2)+
     1RNODAR(IREFEL(E,5),2))/3.0
      IF (YBAR .LT. YTWO)GO TO 104
      IF (YBAR .GE. YONE)GO TO 104
      IF (XBAR .GT. XTWO)GO TO 104
      IF (XBAR .LE. XONE)GO TO 104
C********       PUT STRESS DIFFERENCE IN
      S = S1(E)*SCLSTD
      IF (S .LT. 0.25)GO TO 107
      R=0.75
      DO 108 I=1,10
      IF (S .LT. R)GO TO 109
  108 R=R+0.5
      IF (S .LT. 10.0)GO TO 110
      COL(NCOL)=AS(1)
      GO TO 105
  11  COL(NCOL)=AS(2)
      GO TO 105
  10  COL(NCOL) = AS(22)
      GO TO 105
  115 I=22-I
      COL(NCOL)=AS(I)
      GO TO 105
  104 CONTINUE
      COL(NCOL)=BLANK
  105 CONTINUE
C*******        PRINT THE ROW THAT HAS JUST BEEN SET UP
  196 FORMAT(15X,100A1)
```

```
103 WRITE(6,106)COL
    WRITE(6,112)
112 FORMAT(//10X,5HA----,14X,12HS  .GE. 10.0)
113 FORMAT(10X,A1,7H----    ,F4.2,15H .LE.  S  .LT. ,F4.2)
    DO 114 I=2,22
    IF (I .EQ. 2) GO TO 115
    IF (I .EQ. 22) GO TO 116
    F1=F1-0.5
    F2=F1+0.5
    WRITE(6,113)AS(I),F1,F2
    GO TO 114
115 F1=9.75
    F2=10.0
    WRITE(6,117)AS(2),F1,F2
117 FORMAT(10X,A1,7H----    ,F4.2,15H .LE.  S  .LT. ,F4.1)
    GO TO 114
116 F1=0.0
    F2=0.25
    WRITE(6,113)AS(22),F1,F2
114 CONTINUE
    DO 1 I=1,NELEM,80
    N=I+79
    M=I
    DO 2 J=M,N
    L = J-I+1
    S=ST(J)*SCLSTD
    IF(S .LT. 0.25)GO TO 7
    B=0.75
    DO 8 K=1,19
    IF (S .LT. B)GO TO 9
  8 B=B+0.5
    IF(S .LT.10.0)GO TO 10
    AT(L) =AS(1)
    GO TO 5
 10 AT(L)=AS(2)
    GO TO 5
  7 AT(L)=AS(22)
    GO TO 5
  9 K=22-K
    AT(L)=AS(K)
  5 CONTINUE
```

```
2 CONTINUE
  WRITE(7,3)AT
1 CONTINUE
3 FORMAT(8(A1))
  RETURN
  END
```

F. AFINEL

Purpose

To perform an axisymmetric finite element calculation.

Theory See appendix 3.

Data Input

As for FINEL with y-axis taken as axis of symmetry.

```
C*****************          AFINEL
C                        D.S.DEAN
      DOUBLE PRECISION SUMR,SUMZ
      DOUBLE PRECISION RFORCE,ST
      COMMON RFORCE(2000),ST(24572),RNODAR(1000,2),IREFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,BB(4,6),D(4,4),RO(1500)
     2,MARK(2)
      COMMON/A/YSUB,FOSUB,IBASEP(200),NBASEP
      REAL NN
      INTEGER E
      READ (5,6) NJOB
    6 FORMAT (I10)
      DO 1000 IN=1,NJOB
      DO 9 I=1,24752
    9 ST(I)=0.0
C READ AND PRINT TITLE
      READ (5,1)
    1 FORMAT  (1H1,'
     1                                     ')
      WRITE (6,1)
      READ(5,2)NPOIN,NELEM,(MARK(I),I=1,2)
    2 FORMAT(2I10,10X,3I10)
      IF (NPOIN.GT.1000) GO TO 11
      IF (NELEM .GT. 1500) GO TO 12
      CALL RDATA
      NC=2*NBW*NPOIN
      SUMR= .0
      SUMZ= .0
C*******     BODY FORCES
      PI=3.141416
      DO 3 E=1,NELEM
      RBAR =(RNODAR(IREFEL(E,1),1)+RNODAR(IREFEL(E,2),1)+RNODAR(IREFEL(E
     1,3),1))/3.0
      DELTA = (RNODAR(IREFEL(E,3),2) * RNODAR(IREFEL(E,2),1))
     1        -(RNODAR(IREFEL(E,2),2) * RNODAR(IREFEL(E,3),1))
     2        -(RNODAR(IREFEL(E,1),1) * RNODAR(IREFEL(E,3),2))
     3        +(RNODAR(IREFEL(E,1),1) * RNODAR(IREFEL(E,2),2))
     4        +(RNODAR(IREFEL(E,1),2) * RNODAR(IREFEL(E,3),1))
     5        -(RNODAR(IREFEL(E,1),2) * RNODAR(IREFEL(E,2),1))
      DELTA = 0.5*ABS(DELTA)
      SUMZ =SUMZ - DELTA*RO(E)*981.0*2.0*PI*RBAR
```

```
 80 CONTINUE
      WRITE(6,90) SUMZ
 90 FORMAT(1X,E15.4)
C*********     APPLIED FORCES
      DO 81 I=1,NPTIN
      J=2*I-1
      K=2*I
      SUMR = SUMR + RFORCE(J)
      SUMZ = SUMZ + RFORCE(K)
 81 CONTINUE
      WRITE(6,82) SUMR,SUMZ
 82 FORMAT (7H SUMR=,E15.4,5HSUMZ=,E15.4)
      IF (NO .GT. 24572)GO TO 12
C****** FORM STIFFNESS MATRIX K AND APPLIED FORCE MATRIX F
      DO 3 I=1,NELEM
      CALL FMATS(I)
  3 CONTINUE
C****** FORM R.H.S. OF STIFFNESS EQN. AND INTRODUCE PRESCRIBED DISP.
      CALL INDISP
C***********     ADD IN ISOSTACY EFFECTS
      IF (MASK(1) .EQ. 1)CALL ISOS
C****** SOLVE STIFFNESS EQN. AND PRINT DISP. FOR EACH NODAL PNT.
      CALL CHOLS
      WRITE (6,5)
  5 FORMAT(1H1,//2X,7HELEMENT,11X,5HALPHA,10X,2HRR,21X,2HSS,21X,2HTT,
     110X,6HSTRESS,/20X,9H(DEGREES),12X,3(13H(DYNES/SQ CM),10X),
     [illegible]
      DO 4 I=1,NELEM
      CALL STRESS(I)
  4 CONTINUE
C**********     VISUAL DISPLAY OF STRESS [illegible]
      IF (MASK(2) .EQ. 1)CALL VIS
 1    [illegible]
      GO TO 13
 11 I=1
      GO TO 900
 12 I=2
      GO TO 900
 13 I = [illegible]
      GO T[illegible]
 900 CALL TREND(I)
```

```
10 STOP
   END

      SUBROUTINE PDATA
      DOUBLE PRECISION RFORCE,ST
      COMMON RFORCE(2000),ST(24572),RNODAR(1000,2),IOPREL(1500,3),
     1ILACON(1500,3),NELEM,NPOIN,NRW,PR(4,4),P(4,4),PD(1500)
     2,MARK(3)
      COMMON/A/YSUP,ROSUP,IRASPR(200),NRASPR
      DIMENSION F(2)
      NRW=1
C     PRINT HEADING FOR NODAL POINT DATA
      WRITE (6,1) NPOIN,NELEM
    1 FORMAT (//32X,19H NO. OF NODAL PNTS.,20X,14H NO. OF ELEMENTS//38X,
     1I4,34X,I4)
      WRITE (6,3)
    3 FORMAT (////15X,12H REF. NO. OF,18X,9H CO-ORDS,19X,14H APPLIED FOR
     1CE/14X,11H NODAL PNT,15X,2H X,12X,2H Y,20X,2H X,12X,2H Y//)
C     READ AND STORE NODAL POINT DATA
      DO 2 I = 1,NPOIN
      READ(5,4) NREF,(RNODAR(NREF,J),J=1,2),F(1),F(2)
    4 FORMAT (I10,2F10.5,20X,2F10.3)
C     FORM APPLIED FORCE MATRIX R
      M = 2*NREF-1
      DO 11 J=1,2
      RFORCE(M) = F(J)
      M = M+1
   11 CONTINUE
    2 CONTINUE
C     PRINT NODAL POINT DATA
      DO 33 I=1,NPOIN
      K=2*I-1
      L=2*I
      WRITE(6,31) I,(RNODAR(I,J),J=1,2),RFORCE(K),RFORCE(L)
   31 FORMAT (18X,I4,15X,2(F9.3,5X),10X,F10.3,10X,F10.3)
   33 CONTINUE
      IF (MARK(1) .EQ. 0)GO TO 52
      READ(5,44)YSUP,ROSUP
   44 FORMAT(2F10.3)
```

```
      NBASEP=0
      DO 45 I=1,NPOIN
      IF (RNODAP(I,2) .NE. YSUB)GO TO 45
      NBASEP = NBASEP+1
      IBASEP(NBASEP) = I
   45 CONTINUE
C*******   CHANGE UNITS OF CO-ORDS AND DISPLACEMENTS
   50 DO 40 I=1,NPOIN
      DO 30  JY = 1,2
      MY = JY+2
      RNODAP(I,JY) = RNODAP(I,JY)*1.0E5
      RNODAP(I,MY) = RNODAP(I,MY)*1.0E2
   30 CONTINUE
   40 CONTINUE
C     READ AND STORE ELEMENT DATA
      WRITE (6,14)
   14 FORMAT (1H1,//2X,8H REF NO.,15X,23H NODAL PNTS. OF ELEMENT,19X,19H
     1 ELASTIC CONSTANTS,15X,3H SG,/7X,11H OF ELEMENT,16X,4H (1),4X,4H (
     22),4X,4H (3),20X,7H E(CGS),7X,2H V,15X,8H (GM/CC))
      DO 25 I=1,NELEM
      READ (5,27)IEREF    ,(IREFEL(IEREF    ,J),J=1,3),(ELACON(IEREF    ,K)4
     1,K=1,2),RO(IEREF)
   27 FORMAT (4I10,E10.0,2F10.3)
      IA =IABS(IREFEL(IEREF,1) - IREFEL(IEREF,2))
      IB =IABS(IREFEL(IEREF,1) - IREFEL(IEREF,3))
      IC =IABS(IREFEL(IEREF,2) - IREFEL(IEREF,3))
      NBW = MAX0(IA,IB,IC,NBW)
   25 CONTINUE
      NBW = 2 + 2*NBW
C     PRINT ELEMENT DATA
      DO 35 I=1,NELEM
      WRITE (6,19) I,(IREFEL(I,J),J=1,3),(ELACON(I,K),K=1,2),RO(I)
   19 FORMAT (1X,I4,19X,3(I4,4X),16X,F9.3,4X,F6.3,15X,F5.3)
   35 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE BBBAR(E)
      DOUBLE PRECISION BFORCE,ST
      COMMON BFORCE(2000),ST(24572),RNODAR(1000,2),IREFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,BB(4,6),D(4,4),BD(1500)
     2,MARK(3)
C*******   FORM B-MATRIX AT CENTROID OF ELEMENT E   FOR AXII-SYMMETRIC CASE
      INTEGER E
      DELTA = (RNODAR(IREFEL(E,3),2) * RNODAR(IREFEL(E,2),1))
     1        -(RNODAR(IREFEL(E,2),2) * RNODAR(IREFEL(E,3),1))
     2        -(RNODAR(IREFEL(E,1),1) * RNODAR(IREFEL(E,3),2))
     3        +(RNODAR(IREFEL(E,1),1) * RNODAR(IREFEL(E,2),2))
     4        +(RNODAR(IREFEL(E,1),2) * RNODAR(IREFEL(E,3),1))
     5        -(RNODAR(IREFEL(E,1),2) * RNODAR(IREFEL(E,2),1))
      BB(1,1) = 0.0
      BB(1,2) = RNODAR(IREFEL(E,3),1) - RNODAR(IREFEL(E,2),1)
      BB(1,3) = 0.0
      BB(1,4) = RNODAR(IREFEL(E,1),1) - RNODAR(IREFEL(E,3),1)
      BB(1,5) = 0.0
      BB(1,6) = RNODAR(IREFEL(E,2),1) - RNODAR(IREFEL(E,1),1)
      BB(2,1) = RNODAR(IREFEL(E,2),2) - RNODAR(IREFEL(E,3),2)
      BB(2,2) = 0.0
      BB(2,3) = RNODAR(IREFEL(E,3),2) - RNODAR(IREFEL(E,1),2)
      BB(2,4) = 0.0
      BB(2,5) = RNODAR(IREFEL(E,1),2) - RNODAR(IREFEL(E,2),2)
      BB(2,6) = 0.0
      AI = RNODAR(IREFEL(E,2),1)*RNODAR(IREFEL(E,3),2)
     1- RNODAR(IREFEL(E,3),1)*RNODAR(IREFEL(E,2),2)
      AJ = RNODAR(IREFEL(E,3),1)*RNODAR(IREFEL(E,1),2)
     1- RNODAR(IREFEL(E,1),1)*RNODAR(IREFEL(E,3),2)
      AM = RNODAR(IREFEL(E,1),1)*RNODAR(IREFEL(E,2),2)
     1- RNODAR(IREFEL(E,2),1)*RNODAR(IREFEL(E,1),2)
      ZBAR = (RNODAR(IREFEL(E,1),2)+RNODAR(IREFEL(E,2),2)+RNODAR(IREFEL(
     1E,3),2))/3.0
      RBAR = (RNODAR(IREFEL(E,1),1)+RNODAR(IREFEL(E,2),1)+RNODAR(IREFEL(
     1E,3),1))/3.0
      BB(3,1) = (AI/RBAR) + BB(2,1) + (ZBAR*BB(1,2))/RBAR
      BB(3,2) = 0.0
      BB(3,3) = (AJ/RBAR) + BB(2,3) + (ZBAR*BB(1,4))/RBAR
      BB(3,4) = 0.0
      BB(3,5) = (AM/RBAR) + BB(2,5) + (ZBAR*BB(1,6))/RBAR
      BB(3,6) = 0.0
```

```
      BB(4,1) = BB(1,2)
      BB(4,2) = -B(2,1)
      BB(4,3) = BB(1,4)
      BB(4,4) = BB(2,3)
      BB(4,5) = BB(1,6)
      BB(4,6) = BB(2,5)
      DO 1 I=1,4
      DO 1 J=1,6
    1 BB(I,J) = BB(I,J)/DELTA
      RETURN
      END


      SUBROUTINE FORMD(E)
      DOUBLE PRECISION BFORCE,ST
      COMMON BFORCE(2000),ST(24572),BNODAR(1000,2),IREFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,BB(4,6),D(4,4),PD(1500)
     2,MARK(3)
      INTEGER E
      A = (ELACON(E,1)*(1.0-ELACON(E,2)))/((1.0+ELACON(E,2))*(1.0-2.0*EL
     1ACON(E,2)))
      D(1,1) = A
      D(2,2) = A
      D(3,3) = A
      D(4,4) = ELACON(E,1)/(2.0*(1.0+ELACON(E,2)))
      D(1,2) = (A*ELACON(E,2))/(1.0-ELACON(E,2))
      D(2,1) = D(1,2)
      D(1,3) = D(1,2)
      D(3,1) = D(1,3)
      D(1,4) = 0.0
      D(4,1) = 0.0
      D(2,3) = D(1,2)
      D(3,2) = D(2,3)
      D(2,4) = 0.0
      D(4,2) = 0.0
      D(3,4) = 0.0
      D(4,3) = 0.0
      RETURN
      END
```

```
      SUBROUTINE CHATS(E)                                                  1
      DOUBLE PRECISION KEL,FEL,A,B,C,I1,I2,I3,I4,I5,LOGBI,LOGBJ,LOGBM,
     1AI,AJ,AM,BI,BJ,BM,CI,CJ,CM,AJI,AMJ,AIM,BJI,BMJ,BIM,RI,RJ,RM
      DOUBLE PRECISION BFORCE,ST
      COMMON BFORCE(3000),ST(24572),RNODAR(1000,2),IREFEL(1500,3),
     1FLACON(1500,2),NELEM,NPOIN,NBW,BB(4,6),D(4,4),BC(1500)
     2,FARR(3)
      INTEGER E                                                            2
      DIMENSION KEL(6,6),FEL(6),A(3),B(3),C(3)                             3
      CALL BBARR(E)                                                        5
      CALL DMATR(E)                                                        6
      DELTA = (RNODAR(IREFEL(E,3),2) * RNODAR(IREFEL(E,2),1))
     1        -(RNODAR(IREFEL(E,2),2) * RNODAR(IREFEL(E,3),1))
     2        -(RNODAR(IREFEL(E,1),1) * RNODAR(IREFEL(E,3),2))
     3        +(RNODAR(IREFEL(E,1),1) * RNODAR(IREFEL(E,2),2))
     4        +(RNODAR(IREFEL(E,1),2) * RNODAR(IREFEL(E,3),1))
     5        -(RNODAR(IREFEL(E,1),2) * RNODAR(IREFEL(E,2),1))
      PI=3.141415
      RBAR = (RNODAR(IREFEL(E,1),1)+RNODAR(IREFEL(E,2),1)+RNODAR(IREFEL(     9
     1E,3),1))/3.0                                                         10
      ZBAR = (RNODAR(IREFEL(E,1),2)+RNODAR(IREFEL(E,2),2)+RNODAR(IREFEL(
     1E,3),2))/3.0
      G = 981.0                                                            11
C*****  FORM AI,AJ,AM,BI,BJ,BM,CI,CJ,CM                                    12
      A(1) = RNODAR(IREFEL(E,2),1)*RNODAR(IREFEL(E,3),2)                  13
     1      -RNODAR(IREFEL(E,3),1)*RNODAR(IREFEL(E,2),2)
      B(1) = RNODAR(IREFEL(E,2),2)-RNODAR(IREFEL(E,3),2)
      C(1) = RNODAR(IREFEL(E,3),1)-RNODAR(IREFEL(E,2),1)
      A(2) = RNODAR(IREFEL(E,3),1)*RNODAR(IREFEL(E,1),2)                  17
     1      -RNODAR(IREFEL(E,1),1)*RNODAR(IREFEL(E,3),2)                  18
      B(2) = RNODAR(IREFEL(E,3),2)-RNODAR(IREFEL(E,1),2)
      C(2) = RNODAR(IREFEL(E,1),1)-RNODAR(IREFEL(E,3),1)
      A(3) = RNODAR(IREFEL(E,1),1)*RNODAR(IREFEL(E,2),2)                  21
     1      -RNODAR(IREFEL(E,2),1)*RNODAR(IREFEL(E,1),2)                  22
      B(3) = RNODAR(IREFEL(E,1),2)-RNODAR(IREFEL(E,2),2)
      C(3) = RNODAR(IREFEL(E,2),1)-RNODAR(IREFEL(E,1),1)
      DO 1 I=1,6                                                           25
      DO 1 J=1,6                                                           26
      KEL(I,J) = 0.0                                                       27
      DO 2 K=1,4                                                           28
      DO 2 L=1,4                                                           29
```

```
    2 KEL(I,J) = KEL(I,J) + BB(K,I)*D(K,L)*BB(L,J)
    1 KEL(I,J) = KEL(I,J)*PI2*RBAR*ABS(DELTA)
C****** FORM AND ADD IN CORRECTION TERMS FOR KEL(E)
      RI = RNDDAR(IREFEL(E,1),1)
      RJ = RNDDAR(IREFEL(E,2),1)
      RM = RNDDAR(IREFEL(E,3),1)
      IF (RJ .EQ. RI) GO TO 5
      AJI=-A(3)/C(3)
      BJI=-B(3)/C(3)
      GO TO 6
    5 AJI = 0.
      BJI = 0.
    6 IF (RM .EQ. RJ) GO TO 7
      AMJ=-A(1)/C(1)
      BMJ=-B(1)/C(1)
      GO TO 8
    7 AMJ = 0.0
      BMJ = 0.0
    8 IF (RI .EQ. RM) GO TO 9
      AIM=-A(2)/C(2)
      BIM=-B(2)/C(2)
      GO TO 10
    9 AIM = 0.0
      BIM = 0.0
   10 CONTINUE
      IF (RI .EQ. 0.0) GO TO 11
      LOGRI = DLOG(RI)
      GO TO 12
   11 LOGRI = 0.0
   12 IF (RJ .EQ. 0.0) GO TO 13
      LOGRJ = DLOG(RJ)
      GO TO 14
   13 LOGRJ = 0.
   14 IF (RM .EQ. 0.) GO TO 15
      LOGRM = DLOG(RM)
      GO TO 16
   15 LOGRM = 0.
   16 CONTINUE.
C****** FORM I1,I2,I3,I4,I5
      I1 = ((AJI-AIM)*LOGRI +(AMJ-AJI)*LOGRJ + (AIM-AMJ)*LOGRM)/DELTA
      I2 = ( .5*((AIM**2-AMJ**2)*LOGRM + (AMJ**2-AJI**2)*LOGRJ + (AJI**2
```

```
     1-AIM**2)*LOGRI) + AIM*BIM*(RM-RI) + AMJ*BMJ*(RJ-RM) + AJI*BJI*(RI-
     2RJ) +0.25*(BIM**2*(RM**2-RI**2) + BMJ**2*(RJ**2-RM**2) + BJI**2*(R
     3I**2-RJ**2)))/DELTA
      I3 = (((AJI**3-AIM**3)*LOGRI + (AMJ**3-AJI**3)*LOGRJ + (AIM**3-AMJ
     1**3)*LOGRM)/3.0 + AIM**2*BIM*(RM-RI) + AMJ**2*BMJ*(RJ-RM) + AJI**2
     2*BJI*(RI-RJ) + 0.5*(AIM*BIM**2*(RM**2-RI**2) + AMJ*BMJ**2*(RJ**2-R
     3M**2) + AJI*BJI**2*(RI**2 - RJ**2)) + (BIM**3*(RM**3-RI**3) + BMJ*
     4*3*(RJ**3-RM**3) + BJI**3*(RI**3-RJ**3))/9)/DELTA
      I4 = (2.0/DELTA)*((AIM*(RM**3-RI**3)+AJI*(RI**3-RJ**3)+AMJ*(RJ**3-
     1RM**3))/3.0 +(BIM*(RM**4-RI**4)+BJI*(RI**4-RJ**4)+BMJ*(RJ**4-RM**4
     2))/4.0)
      I5 = (2.0/DELTA)*((AIM**2*(RM**2-RI**2)+AJI**2*(RI**2-RJ**2)+AMJ**
     12*(RJ**2-RM**2))/4.0  +(AIM*BIM*(RM**3-RI**3)+AJI*BJI*(RI**3-RJ**3
     2)+AMJ*BMJ*(RJ**3-RM**3))/3.0  + (BIM**2*(RM**4-RI**4)+BJI**2*(RI**
     34-RJ**4)+BMJ**2*(RJ**4-RM**4))/8.0)
      DO 17 L=1,3
      DO 17 M=1,3
      I =  2*L-1
      J =  2*M-1
   17 KEL(I,J) = KEL(I,J) + (PI/ABS(DELTA))*D(3,3)*(A(L)*A(M)*(2.0*I1-(I
     1. /RBAR)) +(A(M)*C(L)+ A(L)*C(M))*(2.0*I2-(ZBAR/RBAR)) + C(M)*C(L)
     2*(2.0*I3 -(ZBAR**2/RBAR)))
C*******  ADD KEL(E) INTO STIFFNESS MATRIX K AND STORE IN ST( ).
      DO 4 I=1,3
      KA = (IPEEL(E,I)-1)*2
      KB = (I-1)*2
      DO 4 J=1,3
      LA = (IPEEL(E,J)-1)*2
      LB = (J-1)*2
      DO 4 K=1,2
      KC = KA+K
      KD = KB+K
      DO 4 L=1,2
      LC = LA+L
      LD = LB+L
      IF (KC .GT. LC) GO TO 4
      MIN = (KC-1)*NBW + LC - KC + 1
      ST(MIN) = ST(MIN) + KEL(KD,LD)
    4 CONTINUE
C*******  FORM (F) MATRIX FOR ELEMENT E.
      DO 5 K=1,3
```

```
      J = 2*K
      I = J-1
      FEL(I) = 0.0
      FEL(J) = -ABS(DELTA)*PI*RO(E)*G*(A(K)*RBAR + B(K)*I4 + C(K)*I5)/
     1DELTA
    3 CONTINUE
C*******     ADD INTO  MATRIX F.
      DO 18 I=1,3
      DO 18 J=1,2
      K = 2*I-2+J
      M = 2*IPEFEL(E,I)+J-2
      RFORCE(M) = RFORCE(M) + FEL(K)
   18 CONTINUE
      RETURN
      END


      SUBROUTINE INDISP
      DOUBLE PRECISION RFORCE,ST
      COMMON RFORCE(2000),ST(24572),RNODAR(1000,2),IPEFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,BR(4,6),D(4,4),PO(1500)
     1,MARK(2)
C*******    PUT  ZERO DISPLACEMENT CONDITION IN FOR POINTS ON AXIS OF SYMMETRY
      DO 1 I=1,NPOIN
      IF (RNODAR(I,1) .EQ. 0.0) GO TO 2
      GO TO 1
    2 K = (2*I-2)*NBW+1
      N = 2*I-1
      ST(K) = ST(K)*1.0E12
      RFORCE(N) = 0.0
    1 CONTINUE
      K=NBW+1
      ST(K)=ST(K)*1.0E12
      RFORCE(2)=0.0
      RETURN
      END
```

```
      SUBROUTINE CHOLS
        DOUBLE PRECISION A,B,SUM
      DOUBLE PRECISION RFORCE,ST
     [illegible] RFORCE([illegible]),ST(24572),SNODAS([illegible],[illegible]),ITERAL([illegible],2),
     [illegible]([illegible],2),N[illegible],NPTIN,NBW,PR(4,[illegible]),[illegible](4,4),[illegible](15[illegible])
     2,MARK(3)
      DIMENSION [illegible]([illegible])
      DIMENSION MAR([illegible])
      EQUIVALENCE (RFORCE,U)
      NA = [illegible]
C ********     FORM V AND STORE IN ST
      IF (ST(1) .GE. [illegible]. ) GO TO 1
    2 MAR(1) = 1
      ST(1) = -ST(1)
      ST(1) = DSQRT(ST(1))
    [illegible] DO [illegible] I = 2,NBW
      ST(I) = -ST(I)/ST(1)
    [illegible] CONTINUE
      GO TO [illegible]
    1 MAR(1) = 0
      ST(1) = DSQRT(ST(1))
    [illegible] DO 7 I = 2,NBW
      ST(I) = ST(I)/ST(1)
    7 CONTINUE
C ********   I = ROW NO.           J - COLUMN NO.
      DO [illegible] I = 2,NA
      [illegible]
      [illegible] = I-1
    [illegible] DO [illegible] [illegible] = 1,[illegible]
      [illegible] = [illegible]+[illegible]-1
      IF (I .GT. [illegible]) GO TO [illegible]
    [illegible] IF ([illegible]AR([illegible]) .EQ. [illegible]) GO TO [illegible]
      [illegible] = SUM + ST([illegible]([illegible],I))**2
      GO TO [illegible]
      [illegible] = SUM - ST([illegible]([illegible],I))**2
    [illegible] CONTINUE
      [illegible] = ST([illegible](I,I)) + SUM
   [illegible] IF (A .GE. [illegible]) GO TO 1[illegible]
   [illegible] [illegible](I) = 1
      A = -A
      GO TO 14
```

```
   12 MAR(I) = 0
   14 ST(IN(I,I)) = DSQRT(A)
      NC = I+1
      NR = I+NRW-1
   15 DO 21  J = NC,NR
      SUM = 0.0
   16 DO 20  M = 1,ND
      NF = M+NRW-1
      IF (J .GT. NF) GO TO 20
   17 IF (MAR(M) .EQ. 0) GO TO 19
   18 SUM = SUM + ST(IN(M,I))*ST(IN(M,J))
      GO TO 20
   19 SUM = SUM - ST(IN(M,I))*ST(IN(M,J))
   20 CONTINUE
      ST(IN(I,J)) = (ST(IN(I,J))+SUM)/ST(IN(I,I))
      IF (MAR(I) .EQ. 1) ST(IN(I,J)) = -ST(IN(I,J))
   21 CONTINUE
   22 CONTINUE
C *******     FORM Y AND STORE IN U
   23 IF (MAR(1) .EQ. 0) GO TO 24
   25 U(1) = -U(1)/ST(1)
      GO TO 26
   24 U(1) = +U(1)/ST(1)
   26 DO 32 I = 2,NA
      ND = I-1
      SUM = 0.0
   27 DO 31 M = 1,ND
      NF = M+NRW-1
      IF (I .GT. NF) GO TO 31
   28 IF (MAR(M) .EQ. 0) GO TO 29
   30 SUM = SUM + ST(IN(M,I))*U(M)
      GO TO 31
   29 SUM = SUM - ST(IN(M,I))*U(M)
   31 CONTINUE
      U(I) = (U(I)+SUM)/ST(IN(I,I))
      IF (MAR(I) .EQ. 1) U(I) = -U(I)
   32 CONTINUE
C *******        FORM X AND STORE IN U
      U(NA) = U(NA)/ST(IN(NA,NA))
   33 DO 36  J=2,NA
      I=NA+1-J
```

```
      NC = I+1
      SUM = 0.0
   34 DO 35 M = NC,NA
      NE = I+NBW-1
      IF (M .GT. NE) GO TO 35
      SUM = SUM + ST(IN(I,M))*U(M)
   35 CONTINUE
      U(I) =(U(I)-SUM)/ST(IN(I,I))
   36 CONTINUE
C *******   PRINT NODAL DISPLACEMENTS
      WRITE (6,38)
   38 FORMAT(1H1,///,41X,11H NODAL PNT.,15X,19H DISPLACEMENT (M) ,/57X,
     112H X,17X,2H Y)
      DO 42  I = 1,NPOIN
      N = 2*I-1
      M = 2*I
C******* CHANGE UNITS OF DISPLACEMENTS
      DUMMYA = U(N)*1.0E-2
      DUMMYB = U(M)*1.0E-2
      WRITE(6,39) I,DUMMYA,DUMMYB
   39 FORMAT (43X,I4,15X,F10.3,4X,F10.3)
   42 CONTINUE
      RETURN
      END


      FUNCTION IN(I,J)
C ******  TO CALCULATE  THE ARRAY NO. FOR THE (IJ)TH ELEMENT OF A
C ******  SYMMETRIC MATRIX OF SEMI-BAND WIDTH NBW
      DOUBLE PRECISION RFORCE,ST
      COMMON RFORCE(2000),ST(24672),COORD(1000,2),IREFEL(1500,3),
     1ICLACDN(1500,2),NELEM,NPOIN,NBW,DB(4,4),D(4,4),R(1500)
     2,MARK(3)
      IN = (I-1)*NBW + J - I + 1
      RETURN
      END
```

```
      SUBROUTINE ESTRES(E)
      DOUBLE PRECISION RR,ZZ,RZ
      DOUBLE PRECISION RFORCE,ST
      COMMON RFORCE(3000),ST(24572),RNODAR(1500,2),IREFEL(1500,3),
     1SLACIN(1500,2),NELEM,NPOIN,NBW,BB(4,6),D(4,4),FR(1500)
     2,MARK(3)
      INTEGER E
      DIMENSION UE(6),RA(3)
      REAL NN
      IF (E .EQ. 1)K=1
      CALL FCOND(E)
      CALL FBBAR(E)
C******  EXTRACT DISP. MATRIX FOR ELEMENT E FROM DISP. MATRIX U          7
      DO 1 I=1,3
      DO 2 J=1,2
      M = 2*I-2+J
      L = 2*IREFEL(E,I)-2+J
       UE(M) = RFORCE(L)
    2 CONTINUE
    1 CONTINUE
      RR=0.
      ZZ=0.0
      TT=0.0
      RZ=0.0
      DO 3 I=1,4
      DO 4 J=1,6
      RR = RR + D(1,I)*BB(I,J)*UE(J)
      ZZ = ZZ + D(2,I)*BB(I,J)*UE(J)
      TT = TT + D(3,I)*BB(I,J)*UE(J)
      RZ = RZ + D(4,I)*BB(I,J)*UE(J)
    4 CONTINUE
    3 CONTINUE
C*****    CALC. PRINCIPAL STRESSES AND AXES
    6 IF ((RR-ZZ) .EQ. 0.0) GO TO 7
      ALPHA = (2.*RZ)/(RR-ZZ)
      IF (ALPHA .GE. 0.0) GO TO 10
      ALPHA = 0.5*(ATAN(ALPHA)*57.29+180.0)
      GO TO 8
   10 ALPHA = 0.5*ATAN(ALPHA)*57.29
      GO TO 8
    7 ALPHA = 45.0
```

```
    8 NN = (RR*(COS(ALPHA/57.29)**2))+(ZZ*(SIN(ALPHA/57.29)**2))
     1+(RZ*SIN(2.0*ALPHA/57.29))
      SS = RR+ZZ-NN
C*******     PRINT STRESSES
      A=AMAX1(SS,NN,TT)
      C=AMIN1(SS,NN,TT)
      A = 0.5*(A-C)
      WRITE(6,9)E,ALPHA,NN,SS,TT,A
    9 FORMAT( 4X,I3,12X,F7.2,15X,F10.4,13X,F10.4,13X,F10.4,13X,F10.4)
      IF (MARK(3) .EQ. 0)GO TO 13
      IF (K .EQ. 1)GO TO 12
      K=E-1
      WRITE(7,11)K,(IREFEL(K,J),J=1,3),RA(1),RA(2),RA(3),E,
     1(IREFEL(E,J),J=1,3),ALPHA,NN,SS
   11 FORMAT(2(4I4,F5.2,2E9.3))
      K=1
      GO TO 13
   12 RA(1)=ALPHA
      RA(2)=NN
      RA(3)=SS
      K=K+1
   13 CONTINUE
      ST(E)=A
      RETURN
      END


      SUBROUTINE ERROR(I)
      DOUBLE PRECISION RFORCE,ST
      COMMON RFORCE(2000),ST(24572),RNODAR(1000,2),IREFEL(1500,3),
     1ILOCON(1500,2),NELEM,NPOIN,NEW,BB(4,4),D(4,4),RG(1500)
     2,MARK(3)
      GO TO (1,2,3),I
    1 WRITE (6,101)
  101 FORMAT(/111HNO.OF NODAL PNTS. TOO LARGE. MUST ENLARGE STORAGE ALLO
     1CATED IN COMMON STATEMENTS AND ERROR CHECK IN MAIN PROG.)
      GO TO 201
    2 WRITE (6,102)
  102 FORMAT(/108HNO. OF ELEMENTS TOO LARGE. MUST ENLARGE STORAGE ALLOCA
     1TED IN COMMON STATEMENTS AND ERROR CHECK IN MAIN PROG.)
```

```
      GO TO 201
    2 WRITE (6,100)NBW
  100 FORMAT(/106HSTORAGE SPACE ALLOCATED FOR STIFFNESS MATRIX NOT ENOUG
     1H AND THE STORAGE ALLOCATED TO ST IN COMMON MUST BE INCREASED TO A
     2T LEAST 2XNPOINXNBW WHERE NBW EQUALS      ,I4)
  201 RETURN
      END


      SUBROUTINE ISOS
      DOUBLE PRECISION ST,REFORCE
      COMMON REFORCE(2000),ST(24572),RNODAR(1000,2),IREFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,BB(4,6),D(4,4),RC(1500)
     2,MARK(3)
      COMMON/3/ YSUB,RDSUB,IBASER(200),NBASER
      PI=3.141518
      A = ABS((RNODAR(IBASER(1),1)-RNODAR(IBASER(2),1))/2.0)*0.01
      R=ABS((RNODAR(IBASER(1),1)+RNODAR(IBASER(2),1)))
      A=A*PI*R
      J = 2*IBASER(1)
      ST(IN(J,J)) = ST(IN(J,J)) + A*RDSUB
      K = NBASER - 1
      DO 30 I = 2,K
      J = I+1
      L = I-1
      M=2*IBASER(I)
      A =(ABS(RNODAR(IBASER(J),1)-RNODAR(IBASER(I),1))+ABS(RNODAR(IBASER
     1(L),1)-RNODAR(IBASER(I),1)))*0.5*0.01
      R=ABS(RNODAR(IBASER(J),1)+RNODAR(IBASER(L),1))
      A=A*PI*R
      ST(IN(M,M)) = ST(IN(M,M)) + A*RDSUB
   30 CONTINUE
      A = ABS(RNODAR(IBASER(NBASER),1)-RNODAR(IBASER(K),1))*0.5*0.01
      R=ABS(RNODAR(IBASER(NBASER),1)+RNODAR(IBASER(K),1))
      A=A*PI*R
      J = 2*IBASER(NBASER)
      ST(IN(J,J)) = ST(IN(J,J)) + A*RDSUB
      RETURN
      END
```

```
      SUBROUTINE VIS
      DOUBLE PRECISION ST,FFORCE
      COMMON FFORCE(2000),ST(24572),RNODAR(1000,2),IREFEL(1500,3),
     1ELACON(1500,2),NELEM,NPOIN,NBW,FB(4,4),D(4,4),RC(1500)
     2,RANK(2)
      COMMON/A/ YSUB,RQSUB,IBASEP(200),NBASEP
      DIMENSION AS(23),COL(100),AT(90)
      INTEGER E
      READ(5,102)BLANK,(AS(I),I=1,22),NR,NC
  102 FORMAT(23A1,I2,2X,I2)
      SCLSTR = 1.0E-9
      DO 111 I=1,100
  111 COL(I) = BLANK
      XM = 0.0
      YM = 0.0
      DO 105 I=1,NPOIN
      IF (RNODAR(I,1) .GT. XM) XM=RNODAR(I,1)
  105 IF (RNODAR(I,2) .LT. YM) YM=RNODAR(I,2)
      SPV = -YM/NR
      SPH = XM/NC
      WRITE(6,101)
  101 FORMAT(1H1,10X,29HSTRESS DIFFERENCES*1.0E-9.//)
      YONE=0.0
      YTWO=0.0
      DO 103 NROW=1,NR
      XONE=0.0
      XTWO=0.0
C*******      SET UP ROW
      YONE=YTWO
      YTWO=YTWO-SPV
      DO 104 NCOL=1,NC
      XONE=XTWO
      XTWO=XTWO+SPH
      DO 104 E=1,NELEM
      XBAR = (RNODAR(IREFEL(E,1),1)+RNODAR(IREFEL(E,2),1)+
     1RNODAR(IREFEL(E,3),1))/3.
      YBAR = (RNODAR(IREFEL(E,1),2)+RNODAR(IREFEL(E,2),2)+
     1RNODAR(IREFEL(E,3),2))/3.0
      IF (YBAR .LT. YTWO)GO TO 104
      IF (XBAR .GE. YONE)GO TO 104
      IF (XBAR .GT. XTWO)GO TO 104
```

```
      IF (YBAR .LE. YONE)GO TO 104
C*******        PUT STRESS DIFFERENCE IN
      S = ST(F)*SCLSTD
      IF (S .LT. 0.75)GO TO 107
      R=0.75
      DO 103 I=1,19
      IF (S .LT. R)GO TO 109
  103 R=R+0.5
      IF (S .LT. 10.0)GO TO 110
      COL(NCOL)=AS(1)
      GO TO 105
  110 COL(NCOL)=AS(2)
      GO TO 105
  107 COL(NCOL) = AS(21)
      GO TO 105
  109 I=22-I
      COL(NCOL)=AS(I)
      GO TO 105
  104 CONTINUE
      COL(NCOL)=BLANK
  105 CONTINUE
C*******          PRINT THE ROW THAT HAS JUST BEEN SET UP
  106 FORMAT(15X,100A1)
      WRITE(6,106)COL
      WRITE(6,112)
  112 FORMAT(//10X,5HA----,14X,12HS   .GE. 10.0)
  113 FORMAT(10X,A1,7H----   ,F4.2,13H .LE.  S  .LT. ,F4.2)
      DO 114 I=2,22
      IF (I .EQ. 2) GO TO 115
      IF (I .EQ. 22) GO TO 116
      F1=F1-0.5
      F2=F1+0.5
      WRITE(6,113)AS(I),F1,F2
      GO TO 114
  115 F1=9.75
      F2=10.
      WRITE(6,117)AS(2),F1,F2
  117 FORMAT(10X,A1,7H----   ,F4.2,15H .LE.  S  .LT. ,F4.1)
      GO TO 114
  116 F1=0.0
      F2=0.75
```

```
      WRITE(6,[illegible])XS(27),F1,F2
 14   CONTINUE
      DO 1 I=1,N[illegible],6
      K=I+7
      [illegible]=[illegible]
      DO 2 J=[illegible],N
      L = J-I+1
      S=[illegible](J)*SCLSTO
      IF(S .LT. [illegible])GO TO 7
      [illegible] = [illegible]
      [illegible] = [illegible],1[illegible]
      IF(S .LT. [illegible])GO TO [illegible]
 [illegible]  [illegible] = [illegible]+[illegible]
      IF(S .LT.[illegible]. )GO TO [illegible]
      AT(L) =AS(1)
      GO TO 5
 [illegible]  AT(L)=AS(2)
      GO TO 5
 7    AT(L)=[illegible]
      GO TO 5
 [illegible]  K= [illegible]
      AT(L)=AS(K)
 5    CONTINUE
 [illegible]  CONTINUE
        [illegible](7,[illegible])AT
 1    CONTINUE
 [illegible]  [illegible]
      [illegible]
      END
```

3. NOMOFV

Purpose

To produce the normal modes and natural frequencies of vibration of a 2-D model with 3 noded elements subjected to displacements in the plane of the model only.

Data Input

As for FINEL.

Theory

See Chapter 8.

```
      SUBROUTINE DATA
      [illegible] (  ,2),[illegible](  ,3),[illegible](   ,2),[illegible] ),
     1ISDISP(  ,2),NPOIN,NELEM
C     PRINT HEADING FOR NODAL POINT DATA
      WRITE (6,1) NPOIN,NELEM
    1 FORMAT (///[illegible]X,19H NO. OF NODAL PNTS.,[illegible]X,15H NO. OF ELEMENTS//39X,
     1I4,[illegible]X,I4)
      [illegible] (  ,2)
    3 FORMAT (////4X,12H REF. NO. OF,18X,8H CO-ORDS,22X,14H SPECIFIED [illegible]
     1[illegible],7[illegible]X,[illegible] NODAL PNT,15X,2H X,12X,2H Y,[illegible]X,2H X,12X,2H Y//)
C     [illegible] NODAL POINT DATA
      DO 2 I = 1,NPOIN
      READ(5,4)NREF,(CNODAR(NREF,J),J=1,2),ISDISP(NREF,1),ISDISP(NREF,2)
    4 FORMAT(I10,2F1[illegible],I2,8X,I2)
    2 CONTINUE
C     PRINT NODAL POINT DATA
      DO 1[illegible] I=1,NPOIN
      [illegible] (I,2) = CNODAR(I,2)/1.5
      [illegible]
      [illegible]
      [illegible]
C[illegible] IS Y DISP. SPECIFIED.
      IF (ISDISP(I,1) .EQ. 0) GO TO 13
C[illegible] Y DISP SPECIFIED.  IS Y DISP. SPECIFIED.
      IF (ISDISP(I,2) .EQ. 0) GO TO 1[illegible]
C[illegible] BOTH X AND Y DISP. SPECIFIED.
      [illegible](I,J),J=1,2),7,7
      [illegible]
C[illegible] X DISP. ONLY SPECIFIED.
      [illegible](CNODAR(I,J),J=1,2),7
      [illegible]
C[illegible] X DISP. NOT SPECIFIED.  IS Y DISP. SPECIFIED
      [illegible] (I,2) .EQ. [illegible]) GO TO 15
C[illegible] Y DISP. ONLY SPECIFIED.
   1[illegible] WRITE([illegible])I,(CNODAR(I,J),J=1,2),7
      [illegible]
C[illegible] [illegible] SPECIFIED
   15 WRITE(6,[illegible])I,(CNODAR(I,J),J=1,2)
   [illegible] FORMAT (7X,I4,1[illegible]X,2(F[illegible].3,[illegible]X),2([illegible]),1[illegible]X,[illegible],3X,[illegible].3)
   [illegible] FORMAT (7X,I[illegible],1[illegible]X,2(F[illegible].3,[illegible]X),[illegible]X,F[illegible],[illegible]X,[illegible].3,[illegible]X,[illegible])
   [illegible] FORMAT (7X,I4,15X,2(F[illegible].3,[illegible]X),[illegible]X,[illegible],1[illegible]X,E1[illegible].[illegible],[illegible]X,E1[illegible].3)
```

```
   34 FORMAT (7X,I4,15X,2(F9.3,5X),41X,F10.3,3X,F10.3)
   12 CONTINUE
C*********     CHANGE UNITS OF CO-ORDS
   50 DO 40 I=1,NPOIN
      DO 30  JY = 1,2
      RNODAR(I,JY) = RNODAR(I,JY)*1.0E5
   30 CONTINUE
   40 CONTINUE
C      READ AND STORE ELEMENT DATA
       WRITE (6,14)
   14 FORMAT (1H1,//9X,8H ELE NO.,16X,23H NODAL PNTS. OF ELEMENT,12X,19H
     1 ELASTIC CONSTANTS,15X,3H RO,/7X,11H OF ELEMENT,16X,4H (1),4X,4H (
     22),4X,4H (3),21X,7H E(CGS),7Y,2H V,15X,8H (GM/CC))
      DO 26 I=1,NELEM
      READ (5,27)IERFF   ,(IREFFL(IERFF   ,J),J=1,3),(ELACON(IERFF    ,K)
     1,K=1,2),RO(IERFF)
   27 FORMAT (4I1 ,F1 .0,2F10.0)
   26 CONTINUE
C      PRINT ELEMENT DATA
      DO 30 I=1,NELEM
      WRITE (6,10) I,(IREFFL(I,J),J=1,3),(ELACON(I,K),K=1,2),RO(I)
   10 FORMAT (11X,I4,19X,3(I4,4X),16X,F9.3,4X,F6.3,15X,F6.3)
   30 CONTINUE

      SUBROUTINE FORMD(N)
      COMMON RNODAR(100,2),IREFFL(250,3),ELACON(250,2),RO(250),
     1IRDIS(1 ,2),NPOIN,NELEM
      COMMON/B/ D(3,3),B(3,6)
C FORM D MATRIX FOR ELEMENT N
C  PLANE STRAIN
      A= ELACON(N,1)*(1.0 -ELACON(N,2))/((1.0 + ELACON(N,2))*(1.0 - 2*EL
     1ACON(N,2)))
      B= ELACON(N,2)/(1.0 - ELACON(N,2))
    2 D(1,1) = A
      D(1,2) = A*B
      D(1,3) = 0.
      D(2,1) = A*B
      D(2,2) = A
```

```
[illegible](2,3) = .
[illegible] = .
[illegible](4,2) = .
[illegible] = [illegible](1. - [illegible])/2.
[illegible]
[illegible]

[illegible]TINE [illegible] (N)
COMMON [illegible](1[illegible]),IPEREL(2[illegible],3),ELACON(200,2),R2(2[illegible]),
1IEDISP([illegible],2),N[illegible],NELEM
[illegible] (B) THE STAIN - DISP. MATRIX FOR ELEMENT N
COMMON/B/ P(3,3),S(3,6)
DIMENSION [illegible](3),C(3)
[illegible] = ([illegible] * [illegible](N,[illegible],1))
1        -([illegible](IPEREL(N,2),2) * [illegible](IPEREL(N,3),1))
2        -([illegible](N,1),1) * [illegible](N,3),2))
3        +([illegible](IPEREL(N,1),1) * [illegible](IPEREL(N,2),2))
4        +([illegible](N,3),1))
         -([illegible](N,1),2) * RNODAR(IPEREL(N,2),1))
[illegible](1)=(RNODAR(IPEREL(N,2),2)-RNODAR(IPEREL(N,3),2))/DELTA
[illegible]/DELTA
[illegible]/DELTA
[illegible](3)=([illegible](IPEREL(N,1),1)-RNODAR(IPEREL(N,3),1))/DELTA
[illegible]/DELTA
[illegible]=([illegible](IPEREL(N,2),1)-RNODAR(IPEREL(N,1),1))/DELTA
[illegible](1,1)=B(1)
[illegible](1,2)=[illegible]
[illegible](1,3)=B(2)
[illegible](1,4)=[illegible]
[illegible]
[illegible]
[illegible](2,1)=[illegible]
[illegible](2,2)=C(1)
[illegible](2,3)=[illegible]
[illegible](2,4)=C(2)
[illegible]
[illegible]=C(1)
[illegible]=C(1)
[illegible]=[illegible](1)
```

```
      B(3,3)=C(2)
      B(3,4)=B(2)
      B(3,5)=C(3)
      B(3,6)=B(3)
      RETURN
      END


      SUBROUTINE FRAME(N,NM,KK,X,XL,RMD,SKD,MD,IJ)
      COMMON /A/ RNDEAR(100,2),IREFEL(200,3),ELACON(200,2),PR(200),
     1ISDISP(100,2),NPOIN,NELEM
      DIMENSION KEL(6,6),RMEL(6,6),RM(N,N),RK(N,N),X(IJ,IJ),XL(IJ)
      DIMENSION SKD(MD),RMD(MD)
      COMMON/A/ NUNW(100),MBI
      COMMON /B/ C(3,3),B(3,6)
      INTEGER E
      PI = 3.14159265
      DO 11 I=1,N
      DO 11 J=1,N
      RM(I,J) = 0.0
   11 RK(I,J) = 0.0
      [illegible] KTL
      DO 101 E=1,NELEM
      DELTA = (RNDEAR(IREFEL(E,3),2) * RNDEAR(IREFEL(E,2),1))
     1       -(RNDEAR(IREFEL(E,2),2) * RNDEAR(IREFEL(E,3),1))
     2       -(RNDEAR(IREFEL(E,1),1) * RNDEAR(IREFEL(E,3),2))
     3       +(RNDEAR(IREFEL(E,1),1) * RNDEAR(IREFEL(E,2),2))
     4       +(RNDEAR(IREFEL(E,1),2) * RNDEAR(IREFEL(E,3),1))
     5       -(RNDEAR(IREFEL(E,1),2) * RNDEAR(IREFEL(E,2),1))
C********        DELTA = 2*AREA IN THIS ROUTINE
      DELTA = ABS(DELTA)
      Q=121.
C       FORM (B) MATRIX FOR ELEMENT (E)
      CALL FORMB(E)
C       FORM (C) MATRIX FOR ELEMENT (E)
      CALL FORMC(E)
C       FORM (K) MATRIX FOR ELEMENT (E)
      DO 1 I=1,6
      DO 1 J=1,6
      KEL(I,J) = 0.0
```

```
      IF (I .GT. J) GO TO 5
      [illegible] = 1,[illegible]
      [illegible] = 1,[illegible]
      KEL(I,J) = KEL(I,J)+(S(L,I)*D(L,M)*B(M,J))                       24
      [illegible]                                                       25
      [illegible](I,J) = (EELTA*[illegible](I,J))/2.                    26
      [illegible]
      [illegible](I,J) = [illegible](J,I)
    1 CONTINUE                                                          27
C[illegible] MATRIX FOR ELEMENT([illegible])
      [illegible] I=1,[illegible]
      [illegible] J=1,[illegible]
      IF (I .EQ. J)GO TO 13
      L=(I+J)/2 + (I+J)/2
      [illegible]=I+J
      [illegible] (I .[illegible]. M)GO TO 14
      [illegible](I,J) = .[illegible]
      [illegible] TO 13
   1[illegible] [illegible](I,J) = [illegible]LTA/24.
      GO TO 13
   [illegible] [illegible](I,J) = [illegible]LTA/12.[illegible]
   [illegible] [illegible]
      [illegible] STIFFNESS MATRIX (K) AND STORE IN ST( ).              35
      [illegible]=1,[illegible]                                         [illegible]
      [illegible] = ([illegible](F,I)-1)*2                              [illegible]
      [illegible] = (I-1)*2                                             38
      [illegible] I=1,[illegible]                                       [illegible]
      [illegible] = ([illegible](F,J)-1)*2                              4[illegible]
      [illegible] = (J-1)*2                                             4[illegible]
      [illegible]                                                       4[illegible]
      [illegible] = [illegible]+[illegible]                             4[illegible]
      [illegible] = [illegible]+[illegible]                             4[illegible]
      [illegible],2                                                     4[illegible]
      [illegible] = [illegible]+1                                       4[illegible]
      [illegible] = [illegible]+1                                       4[illegible]
      [illegible]([illegible],[illegible]) = [illegible]([illegible],[illegible])+[illegible]([illegible],[illegible])
      [illegible]([illegible],[illegible]) = [illegible]([illegible],[illegible])+[illegible]([illegible],[illegible])
    4 CONTINUE                                                          51
      [illegible]
C[illegible] COEFFICIENTS
      I=1
```

H. LOVE

Purpose.

To produce the normal modes and natural frequencies of vibration of a 2-D model with 3 noded elements subjected to displacement normal to the plane of the model.

Data Input

As for FINEL.

Theory See Chapter 8.

```
C      [illegible] PROP(1[illegible],2),IPREFL(2[illegible],3),FLAG[illegible](2[illegible],2),PQ([illegible]),
      [illegible]([illegible]),[illegible]PROT[illegible],NEL[illegible]
      [illegible] /[illegible]/ [illegible]([illegible]),[illegible]
      DIMENSION [illegible]([illegible]),PX([illegible]),X([illegible],1[illegible]),XL(100)
      [illegible] [illegible]([illegible]),[illegible]([illegible])
      [illegible] EQUIVALENCE ([illegible]),(PK,PKU)
      INTEGER [illegible]
      READ (5,4) NJOB
    4 FORMAT (I[illegible])
      DO [illegible] JN=1,NJOB
C     READ PRINT TITLE
      READ (5,1)
    1 FORMAT (1H1,'
     1                                          ')
      WRITE (6,1)
      READ(5,2) NPOIN,NELEM
    2 FORMAT(2I[illegible])
      IF ([illegible] .GT. [illegible]) GO TO [illegible]
      IF (NELEM .GT. [illegible]) GO TO 1[illegible]
      [illegible] DATA
C     [illegible] FORM STIFFNESS MATRIX K
      [illegible]
      [illegible],NPOIN
      IF ([illegible]) .EQ. [illegible] GO TO [illegible]
      [illegible]
      [illegible]
      [illegible]
      [illegible]
      [illegible]
      [illegible]
      [illegible]
      [illegible]
      [illegible]
      [illegible]
      [illegible]
      [illegible]
```

```
      SUBROUTINE RDATA
      COMMON RNODAR(1[illegible],2),IPREL([illegible],3),ELACON([illegible],2),RO([illegible]),
     1ISDISP([illegible]),NPOIN,NELEM
C     PRINT HEADING FOR NODAL POINT DATA
      WRITE (6,1) NPOIN,NELEM
    1 FORMAT (//[illegible]X,19H NO. OF NODAL PNTS.,2[illegible]X,14H NO. OF ELEMENTS//3[illegible]X,
     1I4,3[illegible]X,I4)
      WRITE (6,3)
    3 FORMAT (/////4X,12H REF. NO. OF,19X,8H CO-ORDS,22X,16H SPECIFIED DI
     1[illegible],/5X,10H NODAL PNT,15X,2H X,12X,2H Y,//)
      DO 2 I = 1,NPOIN
      READ(5,4)NREF,(RNODAR(NREF,J),J=1,2),ISDISP(NREF)
    4 FORMAT(I1[illegible],2F1[illegible].[illegible],I2)
    2 CONTINUE
      Z=[illegible]
C     PRINT NODAL POINT DATA
      DO 1[illegible] I=1,NPOIN
      RNODAR(I,2) = RNODAR(I,2)/[illegible]
      IF (ISDISP(I) .EQ. [illegible])GO TO 13
      WRITE(6,[illegible])I,(RNODAR(I,J),J=1,2),Z
   [illegible] FORMAT(7X,I[illegible],1[illegible]X,2(F[illegible].[illegible],5X),1[illegible]X,F[illegible].[illegible])
      GO TO 12
   13 WRITE(6,[illegible])I,(RNODAR(I,J),J=1,2)
   [illegible] FORMAT(7X,I[illegible],1[illegible]X,2(F[illegible].[illegible],5X))
   1[illegible] CONTINUE
C*********     CANGE UNITS OF CO-ORDS
      DO 4[illegible] I=1,NPOIN
      DO 2[illegible] JY=1,2
      RNODAR(I,JY) = RNODAR(I,JY)*1.[illegible]
   2[illegible] CONTINUE
    4 CONTINUE
C     READ AND STORE ELEMENT DATA
      WRITE (6,[illegible])
   [illegible] FORMAT (1H1,//[illegible]X,[illegible]H [illegible],[illegible]X,23H NODAL PNTS. OF ELEMENT,1[illegible]X,1[illegible]H
     1 ELASTIC CONSTANTS,1[illegible]X,[illegible]H RO,/7X,11H OF ELEMENT,15X,4H (1),4X,4H (
     2[illegible]),4X,4H ([illegible]),2[illegible]X,7H E(CGS),7X,[illegible]H V,1[illegible]X,[illegible]H (GM/CC))
      DO [illegible] I=1,NELEM
      READ (5,27)IPREF    ,(IPREL(IPREF    ,J),J=1,3),(ELACON(IPREF    ,K)
     1,K=1,2),RO(IPREF)
   27 FORMAT (4I[illegible] ,[illegible]F1[illegible].[illegible],2F1[illegible].[illegible])
   [illegible] CONTINUE
```

```
C     PRINT ELEMENT DATA
      DO 40 I=1,NELEM
      WRITE (6,100) I,(IRFEL(I,J),J=1,3),(ELACON(I,K),K=1,2),RO(I)
 100  FORMAT (11X,I4,10X,3(I4,4X),16X,E9.2,4X,F5.3,15X,F5.2)
 40   CONTINUE
      END


      SUBROUTINE FORMD(N)
      COMMON RADLAR(1[illegible],2),IREFEL(2[illegible],3),ELACON(2[illegible],2),RO(2[illegible]),
     1ISDISP([illegible]),NPOIN,NELEM
      COMMON/B2/ D(3,3),B(2,3)
C     D MATRIX FOR ELEMENT N
      U=ELACON(N,1)/(2.0*(1.0+ELACON(N,2)))
      D(1,1) = [illegible]
      D([illegible]) = [illegible]
      D([illegible],3) = [illegible]
      D(2,1) = [illegible]
      RETURN
      END


      SUBROUTINE FORMB(N)
      COMMON RADLAR([illegible]),IREFEL([illegible],3),ELACON([illegible],2),[illegible]([illegible]),
     1ISDISP([illegible]),NPOIN,NELEM
C     [illegible] (B) THE STRAIN - DISP. MATRIX FOR ELEMENT [illegible]
      COMMON/B2/ D(3,3),B(2,3)
      DIMENSION Q(3),C(3)
      DELTA = (RADLAR(IREFEL(N,2),1) * RADLAR(IREFEL(N,3),2))
     1       -(RADLAR(IREFEL(N,3),1) * RADLAR(IREFEL(N,2),2))
     2       -(RADLAR(IREFEL(N,1),1) * RADLAR(IREFEL(N,3),2))
     3       +(RADLAR(IREFEL(N,3),1) * RADLAR(IREFEL(N,1),2))
     4       +(RADLAR(IREFEL(N,1),1) * RADLAR(IREFEL(N,2),2))
     5       -(RADLAR(IREFEL(N,2),1) * RADLAR(IREFEL(N,1),2))
      Q(1)=(RADLAR(IREFEL(N,2),2)-RADLAR(IREFEL(N,3),2))/DELTA
      C(1)=(RADLAR(IREFEL(N,3),1)-RADLAR(IREFEL(N,2),1))/DELTA
      Q(2)=(RADLAR(IREFEL(N,3),2)-RADLAR(IREFEL(N,1),2))/DELTA
      C(2)=(RADLAR(IREFEL(N,1),1)-RADLAR(IREFEL(N,3),1))/DELTA
      Q(3)=(RADLAR(IREFEL(N,1),2)-RADLAR(IREFEL(N,2),2))/DELTA
```

```
C(I)=(RNODAR(IREFEL(N,2),I)-RNODAR(IREFEL(N,1),I))/DELTA
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]

[illegible] MATS(N,EM,EK,X,XI,2MD,EKD,ND,IJ)
[illegible] RNODAR([illegible]),IREFEL([illegible]),FLAGGN([illegible]),[illegible],
[illegible]NUMBER(100),NPOIN,NELEM
[illegible] KEL([illegible]),[illegible](N,K),[illegible](N,N),X(IJ,IJ),XI(IJ)
[illegible]
[illegible]
[illegible]
[illegible]
[illegible]
[illegible] I=1,N
[illegible] J=1,N
[illegible]
[illegible]
[illegible]
[illegible]
DELTA = [illegible]RNODAR(IREFEL(N,2),2) * [illegible]RNODAR(IREFEL(N,3),1))
      -( [illegible] ) * [illegible] ))
      -( [illegible] ) * [illegible] ))
      +( [illegible] ) * [illegible] ))
      +( [illegible] ) * [illegible] ))
      -( [illegible] ) * [illegible] ))
C****** DELTA = 2AREA IN THIS ROUTINE
DELTA = ABS(DELTA)
[illegible]
[illegible] AT [illegible]
[illegible]
[illegible] MATRIX [illegible]
[illegible]
[illegible] IN THE [illegible]
```

```
      DO 1 I=1,3
      DO 1 J=1,3
      KEL(I,J) = 0.0
      IF (I .GT. J) GO TO 5
      DO 2 L=1,2
      DO 2 M=1,2
      KEL(I,J) = KEL(I,J)+(B(L,I)*D(L,M)*B(M,J))                    24
    2 CONTINUE                                                       25
      KEL(I,J) = (DELTA*KEL(I,J))/2.0                                26
      GO TO 1
    5 KEL(I,J) = KEL(J,I)
    1 CONTINUE                                                       27
C*********FORM INERTIA MATRIX FOR ELEMENT(E)
      DO 12 I=1,3
      DO 12 J=1,3
      IF (I .EQ. J)GO TO 13
   14 RMEL(I,J) = RO(E)*DELTA/24.0
      GO TO 12
   13 RMEL(I,J) =RO(E)*DELTA/12.0
   12 CONTINUE
C     FORM STIFFNESS MATRIX (K) AND STORE IN ST( ).                  35
      DO 4 I=1,3
      DO 4 J=1,3
      KA = IREFEL(E,I)
      KB = IREFEL(E,J)
      RK(KA,KB) = RK(KA,KB)+KEL(I,J)
    4 RM(KA,KB) = RM(KA,KB)+RMEL(I,J)
  101 CONTINUE
      L=1
      DO 116 J=1,N
      DO 116 I=1,N
      CALL WAN(I,J,K)
      IF (K .EQ. 0)GO TO 116
      RMD(L) =RM(I,J)
      RKD(L) = RK(I,J)
      L=L+1
  116 CONTINUE
      M=N-MEL
      CALL NROOT(M,RMD,RKD,XL,X)
C*******   PRINT NORMAL FREQUENCIES AND NORMAL MODES(REL. DISP. AMPLITUDES)
      DO 50 I=1,M
```

```
      FN = 1.0/(2.0*PI*SQRT(XL(I)))
      WRITE(6,51)I
   51 FORMAT(11H1        MODE   ,I4)
      WRITE(6,52)FN
   52 FORMAT(///12H FREQUENCY   ,F10.3,17H  CYCLES PER SEC.)
      WRITE(6,53)
   53 FORMAT(/////20X,12H NODAL POINT,20X,15HDISP. AMPLITUDE,//)
      L=1
      Z=0.0
      DO 50 J=1,NPOIN
      CALL WAN(J,J,K)
      IF (K .EQ. 0)GO TO 150
      WRITE(6,54)J,X(L,I)
      L=L+1
      GO TO 151
  150 WRITE(6,54)J,Z
  151 CONTINUE
   50 CONTINUE
   54 FORMAT(23X,I4,28X,F10.3)
      RETURN
      END


      SUBROUTINE WAN(I,J,K)
      COMMON/A/ NUNW(100),MEL
      DO 1 L=1,MEL
      IF (I .EQ. NUNW(L))GO TO 2
      IF (J .EQ. NUNW(L))GO TO 2
    1 CONTINUE
      K=1
      GO TO 3
    2 K=0
    3 RETURN
      END
```

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