Dispersion theoretic perturbation methods

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DISPERSION THEORETIC PERTURBATION METHODS

by

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ABSTRACT

The manuscript is organized as follows.
In Chapter 1 the Chew-Mandelstam equations are derived and there is a general discussion of the partial wave disperison relations and the CDD ambiguity.

The dispersion theoretic method of Dashen and Frautschi is presented in Chapter 2 both for single as well as multi channel case.PATON's investigation of the Dashen-Frautschi method is reviewed in Chapter 3.One of the criticisms concerned the poor convergence of the equations in the presence of short range forces, while the other dealt with the problem of including contributions coming from infra-red divergent terms in the input to the DF expressions.In order to handle the first difficulty a method of modified perturbed di~~s~~spersion relations is presented and applied to a model calculation in potential theory with good results.A modified Pagels-type procedure to solve the resulting equations for N and D functions is employed.This procedure is then applied to inve~~s~~tigate the modified perturbed dispersion relations technique in the presence of long range forces. All this is done in Chapter 4.The modified Pagels-type procedure is employed in Chapter 5 to generate Regge trajectories, the object being to see whether reasonable it is possible to Reggeize the direct channel while using unreggeized input in the crossed channels.It is shown that this is possible provided the cut-off is chosen suitably.

In Chapter 6 the problem of infra-red divergent contributions to the input in the Dashen-Frautschi method is again treated along the lines of a suggestion due to SQUIRES.The procedure is carried out within the context of potential theory where it is shown to give satisfactory results.The full details of the method are exposed in an Appendix to this Chapter.

In Chapter 7 a critical discussion of all previous attempts to calculate the neutron-proton mass difference is given.Chapter 8 is devoted to a detailed examination of the relation of Dashen-Frautschi perturbation theory to field theoretic self-energy calculations.It is found that Dashen's estimate of the contribution of γN intermediate state to the neutron-proton mass differnce is wrong by several orders of magnitude.This is one of many errors in Dashen's calculation of the neutron-proton mass differnce.

In Chapter 9 the neutron-proton mass difference is calculated with use of SQUIRE'S prescription for taking infra-red divergent contributions to the mass shift into account.In contrast with Dashen, who uses a simple form of expression for the D-function, and which is known to disagree with experimentally determined phase shifts , we construct the D-function from the phase shifts of Donnachie et. al (upto 2 Gev/c) and of Bransdon et al (upto 5 Gev/c).The resulting value for the mass differnce is opposite to the experimentally measured value, a result which Barton,and Shaw and Wong predicted on the basis of their criticisms of the Dashen calculation.It is likely that Dashen's unlikely result may be due to several factors , including 1) inadequate representation of the unperturbed strong interaction problem, the proper specification of which is demanded by the Dashen-Frautschi method; 2) Dashen's choice of the D-function which is shown to conflict with the correct D-function built from pion-nucleon phase shifts; 3) Dashen's neglect of all infra-red divergent contributions to the mass shift.

It is made clear that even with the above factors being put right there is the question of contributions coming from inelastic intermediate states. Nevertheless the ground has been prepared to attempt a multi-channel calculation of the neutron-proton mmass difference.

A computer programme to caluclate the phase shifts from the Schrödinger equation is attached.

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Acknowledgements

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NOTATION CONVENTION

Our space time metric is such that the fourth ℓ component of four vectors is imaginary. i.e., $P = (\vec{p}, ip_0)$. The inner product $p_1 \cdot p_2 = \vec{p}_1 \cdot \vec{p}_2 - p_{10}p_{20}$, for a free particle $p^2 = p \cdot p = -m^2$, m being the particle mass. $\hbar = c = 1$ units are used at some places in the text. e and $g_{\pi NN}$ are taken as rationalized, renormalized electronic charge with $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$ and $\frac{g^2_{\pi NN}}{4\pi} = 14.8$

Our γ matrices are hermitian, and $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}; \mu,\nu = 1,2,3,4.$

The Dirac equation for a free particle of momentum p is

$$(i\,\gamma \cdot p + m)\; u\,(p) = 0$$

Matrix $\underset{\sim}{A}$ is denoted by:

$$(\underset{\sim}{A}^T)_{ij} = (\underset{\sim}{A})_{ij}\,, \qquad (\underset{\sim}{A}^+)_{ij} = (\underset{\sim}{A}_{ji})^*$$

det $(\underset{\sim}{A})$ = (A) $\equiv$ determinant of A, and Tr$\underset{\sim}{A} = \sum_i A_{ii}$ = the trace of A.

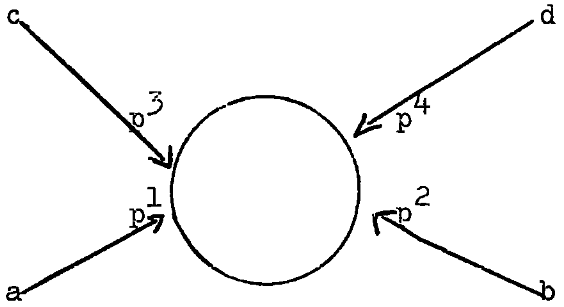
PART 1

CHAPTER ONE

In this part following the derivation of CHEW - MANDELSTAM EQUATIONS along the lines of CINI and FUBINI, partial wave dispersion relations are derived, and finally N/D equations are introduced and discussed.

§ 1. The Mandelstam Representation (1).

Let us consider a Feynman diagram with four external lines.



The reaction can be formally written as a + b + c + d ⟶ vacuum or more realistically in terms of possibly observable reactions

$$a + b \longrightarrow \bar{c} + \bar{d} \, ,$$
$$a + c \longrightarrow \bar{b} + \bar{d} \, , \qquad (A)$$
$$a + d \longrightarrow \bar{b} + \bar{c} \, ,$$

The conservation of energy and momentum implies.

$$p_1 + p_2 + p_3 + p_4 = 0 \, ,$$

and all four vectors are subject to the mass-shell conditions, i.e.

$$p_i^{\,2} + m_i^{\,2} = 0 \qquad (i = 1, 2, 3, 4).$$

We can form two independent scalar products out of four momenta.

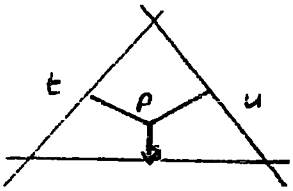
$$s_1 \text{ or } s = -(p_1 + p_2)^2 = -(p_3 + p_4)^2 \, ,$$
$$s_2 \text{ or } t = -(p_1 + p_3)^2 = -(p_2 + p_4)^2 \, , \qquad (B)$$
$$s_3 \text{ or } u = -(p_1 - p_4)^2 = -(p_2 - p_3)^2 \, ,$$

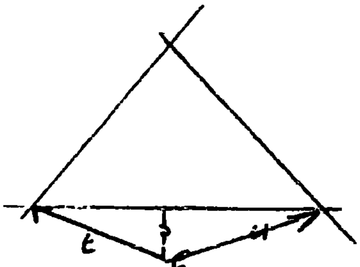
each of which represents the square of the total barycentric energy of a corresponding process given in (A). These three scalar products are not independent but satisfy a relation.

$$\Sigma s_i = s + t + u = \Sigma m_i^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 \quad .$$

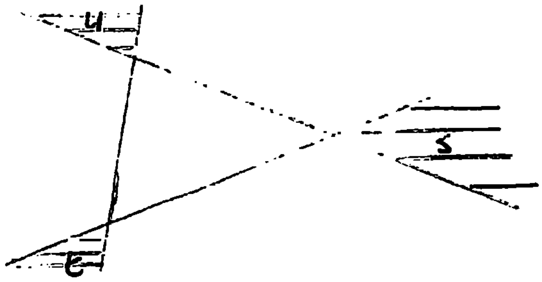
In order to represent a set of variables s, t, u we use the so-called Dalitz plot. For simplicity we shall use $\Sigma m_i^2 = M^2$.

Draw and equilateral triangle whose height is M^2. The sum of lengths of the perpendiculars to the sides from a point P is equal to M^2; $s_1 + s_2 + s_3 = M^2$.





When the point P is outside of the triangle we assign negative values to some of the variables so that the above equation is algebraically satisfied. When the variables s, t, and u are so chosen that one of the processes in (A) is physically realizable, we say that we are in the s, t, or u channel, respectively. The physical domains of these channels can be plotted on a two- dimensional graph introduced above (the Mandelstam plot).



Different processes in (A) correspond to different domains on this plot. For instance, for all $m_i = m$, the shaded domains above correspond to the three different processes mentioned above. In order to find the precise form of the physical domains one has to study the kinematics.

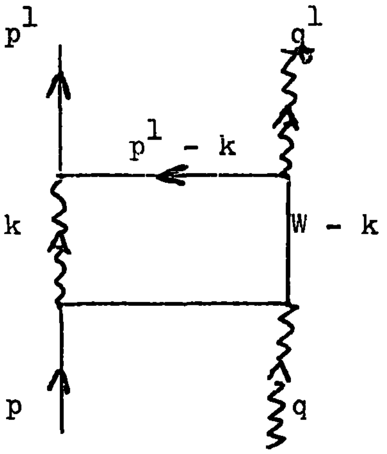
The invariant scattering amplitude $\mathfrak{J}$ becomes an invariant function of s, t, and u; we define a function F(s, t, u) which represents $\mathfrak{J}$ in the physical domains. Here we shall consider meson-nucleon scattering in the scalar model and shall identify

$$p1 = p \ , \ p_2 = q \ , \ p_3 = - p' \ , \ p_4 = -q' \ ,$$

then

$$s = -(p + q)^2 = -(p' + q')^2$$
$$t = --(p - q')^2 = -(q' - q)^2$$
$$u = -(p - q')^2 = -(p' - q)^2.$$

Let us study the structure of the contribution of a typical fourth order diagram .



$$W = p + q = p^1 + q^1$$
$$s = -W^2$$

The expression for F is

$$F = \frac{ig^4}{(2\pi)^4} \int \frac{d^4k}{\left[k^2 + \mu^2\right] \left[(W - k)^2 + M^2\right] \left[(p^1 - k)^2 + M^2\right] \left[(p - k)^2 + M^2\right]}$$

First, let us regard F as a function of s by fixing t and calculate the discontinuity of F across the branch cut starting from the Landau singularity caused by both intermediate particles being on the mass shell. By using Cutkosky's rule.

$$\Delta_s F = F(s + i\epsilon) - F(s - i\epsilon)$$

$$= \frac{ig^4}{(2\pi)^4} \int \frac{d^4k \; (2\pi i)^2 \delta_p(k^2 + \mu^2) \; \delta_p\left[(W - k)^2 + M^2\right]}{\left[(p^1 - k)^2 + M^2\right] \left[(p - k)^2 + M^2\right]}$$

$$= \frac{ig^4}{(2\pi)^4} \int \frac{d^4k d^4k^1 \; (2\pi i)^2 \delta_p(k^2 + \mu^2) \; \delta_p(k^{12} + M^2) \; \delta^4 \; (k + k^1 - W)}{\left[(p^1 - k)^2 + M^2\right] \left[(p - k)^2 + M^2\right]}$$

$$= \frac{-i}{(2\pi)^2} \int \frac{d^3k}{2k_o} \int \frac{d^3k^1}{2k_o^1} \; \delta^4 \; (k + k^1 - W) \; \frac{-g^2}{(p^1-k)^2 + M^2} \; \frac{-g^2}{(p - k)^2 + M^2}$$

$$= \frac{-i}{16\pi^2} \int \frac{d^3k}{k_o} \int \frac{d^3k^1}{k_o^1} \left[\delta \; (k + k^1 - W) \; F^*_{fn} \; F_{ni}\right]$$

where F_{ba} denotes the second order invariant scattering amplitude for a ⟶ b. In the s - channel $\Delta_s F = 2\,i\mathrm{Im}F$, so we get

$$\mathrm{Im}\,F_{fi} = -\frac{1}{3\;2\pi^2} \int \frac{d^3p_n}{(p_n)_o} \int \frac{d^3q_n}{(q_n)_o} \; \delta^4(Pn - Pi) \; F^*_{fn} \; F_{ni},$$

which is just the unitarity condition. From this example we see that Cutkosky's prescription is a generalization of the unitarity condition.

In order to show that the left - hand side is the absorptive part in the s - channel we should write

$$\Delta_s F = 2i\mathrm{Im}_s F.$$

Then we can write a dispersion relation for F in s as well as in t.

$$F(s,t) = \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \frac{ds^1}{s^1 - s - i\varepsilon} \, \mathrm{Im}_s F(s^1,t) = \frac{1}{\pi} \int_{(2M)^2}^{\infty} \frac{dt^1}{t^1 - t - i\varepsilon} \, \mathrm{Im}_t F(s,t^1)$$

The absorptive part can be computed again by using CUTKOSKY's rule. We now write the dispersion relation for F and $\Delta_s F$:

$$F(s,t) = \frac{1}{2\pi i} \int_{(M+\mu)^2}^{\infty} \frac{ds^1}{s^1 - s - i\varepsilon} \, \Delta_s F(s^1,t);$$

$$\Delta_s F(s^1,t) = \frac{1}{(2\pi i)} \int_{(2M)^2}^{\infty} \frac{dt^1}{t^1 - t - i\varepsilon} \, \Delta_t \Delta_s F(s^1,t^1)$$

and by combining them we get;

$$F(s,t) = \frac{1}{(2\pi i)^2} \int_{(M+\mu)^2}^{\infty} \frac{ds^1}{s^1 - s - i\varepsilon} \int_{(2M)^2}^{\infty} \frac{dt^1}{t^1 - t - i\varepsilon} \, \Delta_t \Delta_s F(s^1,t^1)$$

where

$$\Delta_t \Delta_s F(s,t) = \frac{ig^4}{(2\pi)^4} (2\pi i)^4 \int d^4k \, \delta_p(k^2+\mu^2) \, \delta_p\left[(W-k)^2+M^2\right] \delta_p\left[(p^1-k)^2+M^2\right] \delta_p\left[(p-k)^2 + M^2\right]$$

$$= ig^4 \int d^4k \, \delta_p(k^2+\mu^2) \, \delta\left[(W-k)^2+M^2\right] \delta_p\left[(p^1-k)^2+M^2\right] \delta_p\left[(p-k)^2 + M^2\right]$$

It is clear, however, that simultaneous discontinuity does not occur in the physical region. Therefore, this function survives only in the unphysical region. Let us denote the value of the integral by $\frac{1}{2}$ -D for later convenience; then

$$t \quad s \; F = \frac{ig^4}{2\sqrt{-D}} \equiv \frac{g^4}{2\sqrt{D}}$$

Hence we find

$$F(s,t,) = \frac{1}{\pi^2} \int \frac{ds^1}{s^1 - s} \int \frac{dt^1}{t^1 - t} \rho(s^1, t^1)$$

where

$$\rho(s,t) = \frac{-g^4}{8\sqrt{D}}$$

In defining the physical amplitude in the s channel we must take

$$\lim_{\varepsilon \to 0} F(s + i\varepsilon, t),$$

and a corresponding expression in the t or u channel.

The discontinuity integral can be done as follows:

$$\frac{1}{2\sqrt{-D}} = \int d^4k \; \delta_p(k^2+\mu^2)\,\delta_p\left[(W-k)^2+M^2\right]\delta_p\left[(p^1-k)^2+M^2\right]\delta_p\left[(p-k)^2+M^2\right]$$

$$= \int d^4k \; \delta_p(k^2+\mu^2) \; \delta_p\,(W^2+M^2-2kW-\mu^2)\;\delta_p(2p^1k+\mu^2)\;\delta_p(2pk+\mu^2)$$

We make a transformation of the variables of integration

$$k_1,\ k_2,\ k_3,\ k_0, \longrightarrow k^2,\ kW,\ p^1k,\ pk,$$

but this is a two to one correspondence, so we get;

$$\frac{1}{2\sqrt{-D}} = 2\int \left|\frac{\partial(k_1,\ k_2,\ k_3,\ k_0,)}{\partial(k^2,\ kW,\ kp^1,\ kp)}\right| \frac{1}{8} \; dk^2 \; d(2kW)\,d(2kp^1)\,d(2kp)$$

$$\times\; \delta_p(k^2+\mu^2)\; \delta_p\,(W^2+M^2-2kW-\mu^2)\;\; \delta_p(2kp^1+\mu^2)\; \delta_p(2kp+\mu^2)$$

$$= \frac{1}{4} \left|\frac{\partial(k^2,\ kW,\ kp^1,\ kp)}{\partial(k_1,\ k_2,\ k_3,\ k_0,)}\right|^{-1}$$

Hence

$$D = \begin{vmatrix} k^{2} & kW & kp^{1} & kp \\ kW & W^{2} & Wp^{1} & Wp \\ kp^{1} & Wp^{1} & p^{1}2 & p^{1}p \\ kp & Wp & p^{1}p & p^{2} \end{vmatrix}$$

The scalar products involving k should be replaced by those not depending on k putting the arguments of the four δ functions equal to zero. Then D is given explicitly in terms of external variables:

$$D = \begin{vmatrix} -\mu^{2} & \frac{1}{2}(M^{2} - s - \mu^{2}) & -\frac{1}{2}\mu^{2} & -\frac{1}{2}\mu^{2} \\ \frac{1}{2}(M^{2} - s - \mu^{2}\} & -s & \frac{1}{2}(\mu^{2} - s - M^{2}) & \frac{1}{2}(\mu^{2} - s - M^{2}) \\ -\frac{1}{2}\mu^{2} & \frac{1}{2}(\mu^{2} - s - M^{2}) & -M^{2} & -M^{2} + \frac{1}{2}t \\ -\frac{1}{2}\mu^{2} & \frac{1}{2}(\mu^{2} - s - M^{2}) & -M^{2} + \frac{1}{2}t & -M^{2} \end{vmatrix} =$$

$$\frac{t}{2} \begin{vmatrix} \mu^{2} & \frac{1}{2}(s + \mu^{2} - M^{2}) & \frac{1}{2}\mu^{2} \\ \frac{1}{2}(s + \mu^{2} - M^{2}) & s & \frac{1}{2}(s + M^{2} - \mu^{2}) \\ \mu^{2} & s + M^{2} - \mu^{2} & 2M^{2} - \frac{1}{2}t \end{vmatrix}$$

The double discontinuity function is different from zero in a domain where

$$D > 0, \quad s > (M+\mu)^2, \quad \text{and} \quad t > 4M^2,$$

as is clear from its derivation. If we put $M = \mu$ for simplicity, the boundary curve is described by

$$(s - 4\mu^2)(t - 4\mu^2) = 4\mu^2$$

and the domain for the discontinuity (support) is given by

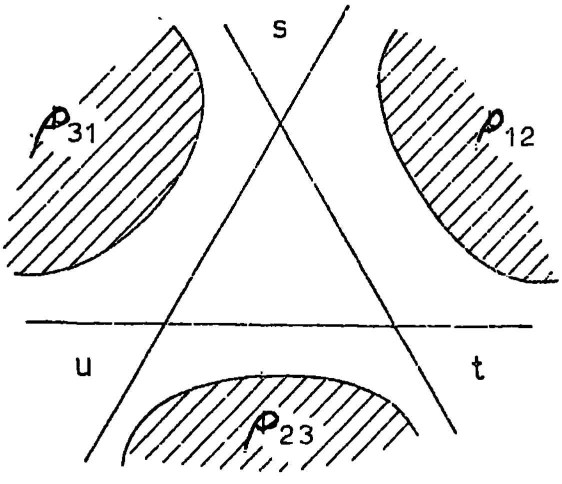
$$(s - 4\mu^2)(t - 4\mu^2) > 4\mu^4.$$

We can show through the fourth order by an explicit calculation that the most general form of F is given by

$$F(s,t,u) = \frac{1}{\pi}\int ds^1 \frac{\rho_1(s^1)}{s^1 - s} + \frac{1}{\pi}\int dt^1 \frac{\rho_2(t^1)}{t^1 - t} + \frac{1}{\pi}\int du^1 \frac{\rho_3(u^1)}{u^1 - u}$$

$$+ \frac{1}{\pi^2}\iint ds^1 dt^1 \frac{\rho_{12}(s^1,t^1)}{(s^1-s)(t^1-t)} + \frac{1}{\pi^2}\iint dt^1 du^1 \frac{\rho_{23}(t^1,u^1)}{(t^1-t)(u^1-u)}$$

$$+ \frac{1}{\pi^2}\iint du^1 ds^1 \frac{\rho_{31}(u^1,s^1)}{(u^1-u)(s^1-s)}$$



support of the double spectral functions.

This integral representation is the MANDELSTAM representation; it gives explicitly the analyticity properties of the amplitude F as function of two invariant variables.

As we have already mentioned, the physical amplitude in the s channel is given as the boundary value of the function F by

$$\lim_{\varepsilon \to 0} F(s + i\varepsilon,\ t,u).$$

Next, let us study the consequences of the crossing symmetry. The crossing transformation is

$$q \rightleftarrows -q^1 \quad \text{or} \quad p_2 \rightleftarrows p_4,$$

and in terms ~~of~~ the s, t, u, variables we get

$$s \rightleftarrows u, \qquad t \longrightarrow t.$$

This shows that F is symmetric in s and u, i.e.,

$$F(s,t,u) = F(u,t,s)$$

Finally we shall reproduce the dispersion relation for meson-nucleon scattering in the scalar model starting from the MANDELSTAM representation.

Assume that t is negative and fixed, then this domain includes both the s and u channels. We shall further split this domain in two according to whether $s > u$ or $u > s$. The absorptive part of the amplitude in the s channel is given by

$$\mathrm{Im}_s F(s,t,u) = \rho_1(s) + \frac{1}{\pi}\int dt^1 \frac{\rho_{12}(s,t^1)}{t^1 - t} + \frac{1}{\pi}\int du^1 \frac{\rho_{31}(u^1,s)}{u^1 - u}$$

for $s > u$,

and in the u channel by

$$\mathrm{Im}_u F(s,t,u) = \rho_3(u) + \frac{1}{\pi} \int dt^1 \frac{\rho_{23}(t^1,u)}{t^1 - t} + \frac{1}{\pi} \int ds^1 \frac{\rho_{31}(u^1,s)}{s^1 - s}$$

for $u > s$.

From these relations we get for $t < 0$ the expression

$$F(s,t,u) = \frac{1}{\pi} \int_{s^1 > u^1} ds^1 \frac{\mathrm{Im}_s F(s^1,t,u^1)}{s^1 - s - i\varepsilon} + \frac{1}{\pi} \int du^1 \frac{\mathrm{Im}_u F(s^1,t,u^1)}{u^1 - u - i\varepsilon}$$

provided that $\rho_2(t) = 0$. In carrying out s^1 and u^1 integrations it should be noticed that s^1 and u^1 are not independent since

$$s^1 + u^1 = \sum_i m_i^2 - t.$$

If we use crossing symmetry we find that the two dispersion integrals are related to one another through the transformation $s \rightleftarrows u$, so that

$$F(s,t,u,) = \frac{1}{\pi} \int_{s^1 > u^1} ds^1 \frac{1}{s^1 - s} \mathrm{Im}_s F(s^1,t,u^1) + (s \rightleftarrows u)$$

$$= \frac{1}{\pi} \int_{s^1 > u^1} ds^1 \left(\frac{1}{s^1 - s} + \frac{1}{s^1 - u} \right) \mathrm{Im}_s F(s^1,t,u^1)$$

§ 2. The Cini-Fubini Approximation

The analyticity properties of the scattering amplitudes as functions of two variables manifest themselves through the MANDELSTAM representation. When we combine the MANDELSTAM representation with unitarity in various channels we find a coupled set of non-linear integral equations in two variables. This is an extremely complicated mathamatical problem and we have to find some means to reduce the number of variables. The introduction of partial wave dispersion relations fits this purpose and the MANDELSTAM representations provides the appropriate basis for their derivation. In this section we shall discuss the problem à la Cini and Fubini(2).

Let us first consider meson-meson scattering and denote the meson mass by μ. The MANDELSTAM variables in this case satisfy

$$s + t + u = 4\mu^2 .$$

The s channel is characterized by

$$4\mu^2 < s < \infty , \qquad 4\mu^2 - s < t < 0.$$

If we write the four-momenta as

$$p_1 = (\vec{q}, w_q), \quad p_2 = (-\vec{q}, w_q), \quad p_3 = (-\vec{q}^{\,1}, -w_q), \quad p_4 = (\vec{q}^{\,1}, -w_q),$$

with

$$qq^1 = q^2 \cos\theta \equiv \hat{\nu} \cos\theta ,$$

$$w_q = q^2 + \mu^2 \equiv \sqrt{\nu + \mu^2} ,$$

then

$$s = 4(\gamma + \mu^2), \quad t = -2\gamma(1 - \cos\theta) , \quad u = -2\nu(1 + \cos\theta)$$

Similarly the domains

$$4\mu^2 < t < \infty , \qquad 4\mu^2 - t < u < 0,$$

and

$$4\mu^2 < u < \infty , \qquad 4\mu^2 - u < s < 0,$$

characterize the physical regions of the t and u channels, respectively. The MANDELSTAM representation can be written as

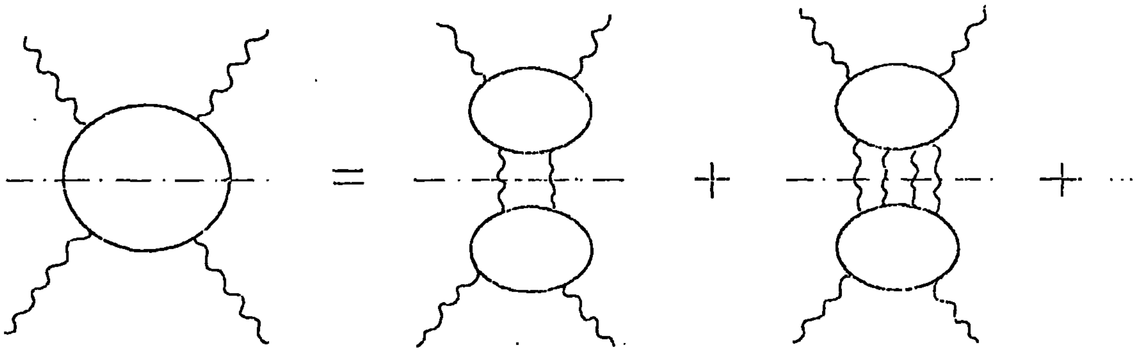
$$F(s,t,u) = \int_{4\mu^2}^{\infty} dx \int_{4\mu^2}^{\infty} dy \; A(x,y) \left[\frac{1}{(x-s)(y-t)} + \frac{1}{(x-t)(y-u)} + \frac{1}{(x-u)(y-s)} \right]$$

where A(x,y) is a real symmetric function corresponding to $\pi^{-2}\rho(x,y)$. The lower limit $4\mu^2$ is determined by the lowest possible mass in the intermediate state which can be reached by the two-meson system.

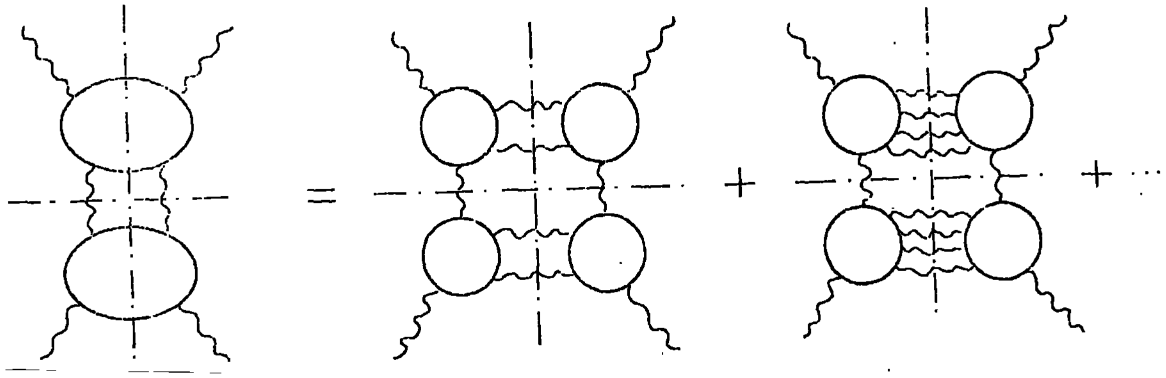
Now let us assume that the neutal meson under consideration is pseudo-scalar so that reactions of the type

odd number of mesons $\rightleftarrows$ even number of mesons

are forbidden. An important consequence of this assumption is that no two of the variables of integration reach the lower limit $4\mu^2$ at the same time. In order to see this let us insert a cut into a scattering diagram; then the various possible intermediate states involve 2,4,6,..... particles :

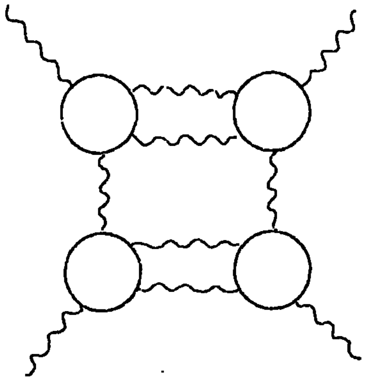


Of these diagrams only the first one can reach the lower limit $4\mu^2$ in the s channel; but if we cut this diagram again in the t or u channel, we find that the intermediate states now must have 4, 6, 8, 10, 12,.... particles because of the conservation of parity:



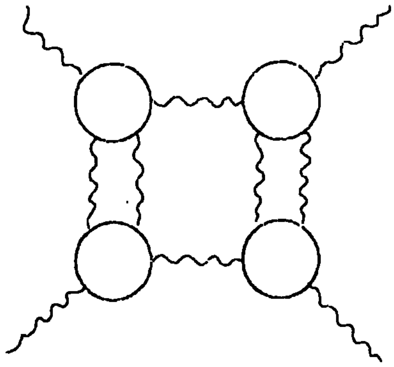
This shows that if the lower limit $4\mu^2$ is reached in one of the variables of integration, the lower limit for the other is $16\mu^2$.

If one takes the two alternative diagrams to compute the boundary curves for the support of the double spectral function one gets two intersecting curves.



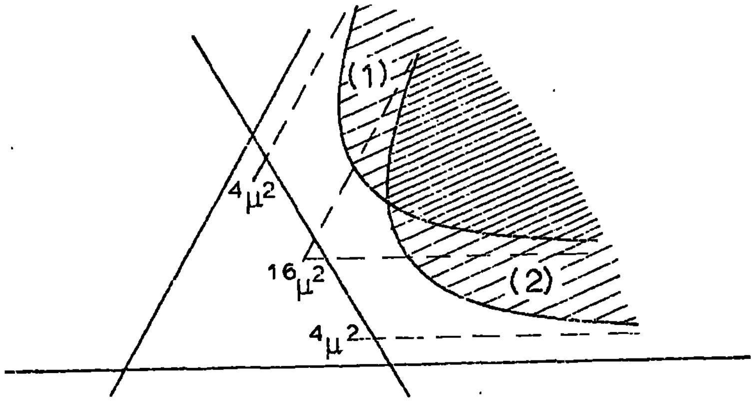
(1)

and



(2)

The boundary curves are



We can compute the boundary curves by the method studies in the preceding section:

$$A_1 \ (x,y) = 0, \quad \text{if} \quad y < \frac{16\mu^2 x}{x - 4\mu^2} ,$$

$$A_2 \ (x,y) = 0, \quad \text{if} \quad x < \frac{16\mu^2 y}{y - 4\mu^2} ,$$

Therefore we shall write each of the three integrals in the MANDELSTAM representation in the form

$$\int dx \int dy \frac{A(x,y)}{(x-s_i)(y-s_j)} = \frac{1}{2}\int_{4\mu^2}^{\infty} dx \int_{16\mu^2}^{\infty} dy \frac{A_1(x,y)}{(x-s_i)(y-s_j)}$$

$$+ \frac{1}{2}\int_{4\mu^2}^{\infty} dy \int_{16\mu^2}^{\infty} dx \frac{A_2(x,y)}{(x-s_i)(y-s_j)}$$

Where s_i, s_j, = s,t,u and $A_1(x,y) = A_2(x,y)$.

For the present purpose this representation is useful; the MANDELSTAM representation consists of three pairs of terms, each term having a cut in one variable starting at $4\mu^2$ and another cut in the other variable starting at $16\mu^2$. Now it is convenient to introduce the new variables

$$z_1 = t - u, \quad z_2 = u - s, \quad z_3 = s - t .$$

In the s channel we have

$$z_1 = 4\nu \cos \theta.$$

We recombine six terms in the MANDELSTAM representation as follows.

$$F(s,t,u) = \alpha(s,z_1) + \alpha(t,z_2) + \alpha(u,z_3) \quad ,$$

With

$$\alpha(s,z) = \int_{4\mu^2}^{\infty} \frac{dx}{x-s} \int_{16\mu^2}^{\infty} dy\, A_1(x,y) \quad \frac{1}{2y + s - 4\mu^2 + z} + \frac{1}{2y + s - 4\mu^2 - z}$$

As long as we deal with elastic scattering below the threshold energy for inelastic processes, the variables s, t and u are all smaller than $16\mu^2$, and the denominators in the integrals starting at $16\mu^2$ never vanish. Therefore, we introduce an expansion of the denominators to obtain an approximation valid in the elastic region.

$$\alpha(s,z) \simeq \int_{4\mu^2}^{\infty} \frac{dx}{(x-s)} \int_{16\mu^2}^{\infty} \frac{dy}{y} A_1(x,y) \left[1 + \frac{4\mu^2 - s}{2y} + \frac{(4\mu^2-s)^2}{4y^2} + \frac{z^2}{4y^2} + \dots \right]$$

First, let us keep only the first term in the expansion, then

$$\alpha(s,z) \simeq \int_{4\mu^2}^{\infty} \frac{dx}{x-s} \rho 0(x) ,$$

so

$$F(s,t,u) \simeq \int_{4\mu^2}^{\infty} \frac{dx}{x-s} \rho_0(x) + \int_{4\mu^2}^{\infty} \frac{dx}{x-t} \rho_0(x) + \int_{4\mu^2}^{\infty} \frac{dx}{x-u} \rho_0(x)$$

In order to determine the unknown function $\rho_0(x)$ we have ~~the~~ to use the unitarity condition in terms of partial waves, recalling the relation

$$F = - 8\pi W f(\theta) \quad ,$$

or for identical particles the modified relation

$$F = - 8\pi W \left[f(\theta) + f(\pi - \theta) \right] \quad .$$

Then we see that

$$h_l(\nu) \equiv \frac{1}{2} \int_{-1}^{1} d(\cos\theta) \; P_l \; (\cos\theta) \; F(\nu, \cos\theta)$$

$$= - 16\pi \sqrt{\frac{\nu - \mu^2}{v}} \; e^{i\delta_l} \sin\delta_l \left[1 - (-1)^l \right]$$

If we use the approximate one-dimensional representation only the first term has a non-vanishing absorptive part in the s channel, so we keep only the first term at low energies.

$$h_0(v) \simeq \int_{4\mu^2}^{\infty} dx \frac{\rho_0(x)}{x - 4\mu^2 - 4v - i\varepsilon}$$

Thus

$$\mathrm{Im}\ h_0\ (v) \simeq \pi \rho_0(x)$$

$$\mathrm{Im}\ h_\ell(v) \simeq 0, \text{ for } \ell > 0.$$

By introducing this approximation into the one-dimensional representation we get

$$F(\nu,\ \cos\theta) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\ h_0(v^1)}{\nu^1-\nu-i\epsilon}$$

$$+\frac{4}{\pi}\int_0^\infty dv^1 \mathrm{Im}\ h_0(\nu^1)\ \left(\frac{1}{4\nu^1+4\mu^2+2v(1-\mathrm{Cos}\ \theta)} + \frac{1}{4\nu^1+4\mu^2+2v(1-\mathrm{Cos}\theta)}\right)$$

This equation shows that only the s wave term has a non-vanishing absorptive part so that this approximation is valid only when $\sin\delta_\ell$ is smaller as compared with $\cos\delta_\ell$ for $\ell > 0$.

By taking the s wave projection of F we obtain an equation for $h_0(\nu)$:

$$h_0(\nu) = \frac{1}{\pi}\int_0^\infty dv^1\ \frac{\mathrm{Im}h_0(\nu^1)}{v^1-v-i} + \frac{2}{\pi}\int_{-1}^{+1} d(\cos\theta)\int_0^\infty d\nu^1\ \mathrm{Im}h_0(v^1)$$

$$\times\left[\frac{1}{4v^1+4\mu^2+2v(1-\mathrm{Cos}\ \theta)} + \frac{1}{4\nu^1+4\ {}^2+2\nu\ (1+\mathrm{Cos}\theta)}\right]$$

It is also possible to write this equation in the form

$$h_0(\nu) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\ h_0(v^1)}{v^1-v-i}\ dv^1 + \frac{1}{\pi}\int_{-\infty}^{-\mu^2} dv^1\ \frac{f(v^1)}{v^1-v}$$

with

$$f(\nu^{1}) = \frac{2}{\nu^{1}} \int_{0}^{-\nu^{1}-\mu^{2}} dv^{11}\mathrm{Im}\, h_{0}(\nu^{11})$$

The latter form shows that $h_0(\nu)$ has two cuts, one starting from 0 and continuing along the positive real axis and the other along the negative real axis.

The equation is satisfactory in that the unitarity condition for the s wave can be satisfied in all three channels. It is necessary, however, to introduce a subtraction in order to exclude the trivial solution $h_0 = 0$. Therefore we fix F(s,t,u) at the summetrical point $s = t = u = \frac{4}{3}\mu^2 \equiv s_0$:

$$F(s_0, s_0, s_0) = \lambda .$$

This defines a coupling constant for the effective interaction of the ϕ^4 type. Making the subtraction, we find that the one-dimensional representation is modified into

$$F(s,t,u) = \lambda + \sum_{i=1}^{3} (s_i - s_0) \int_{4\mu^2}^{\infty} dx \frac{\rho_0(x)}{(x - x_0)(x - s_i)}$$

with $s_1 = s$, $s_2 = t$, and $s_3 = u$. Or, if one keeps the second term $(4\mu^2 - s)/2y$ in the expansion introduced previously, one automatically gets a subtracted form:

$$\alpha(s,z) \simeq \int_{4\mu^2}^{\infty} \frac{dx}{x - s} \rho_0(x) + (s - 4\mu^2) \int_{4\mu^2}^{\infty} \frac{dx}{x - s} \rho_1(x)$$

$$= \alpha_0 + (s - s_0) \int_{4\mu^2}^{\infty} \frac{dx}{(x - s_0)(x - s)} \alpha(x)$$

with

$$\alpha(x) = \rho_0(x) + (x - 4\mu^2)\, \rho_1(x), \qquad \alpha_0 = \alpha(s_0, 0).$$

From the subtracted form of F we get

$$h_0(\nu) = \lambda + \frac{1}{\pi}(\nu + \tfrac{2}{3}\mu^2) \int_0^\infty \frac{d\nu^1 \ \mathrm{Im}\ h_0(v^1)}{(\nu^1 + \frac{2}{3}\mu^2)(\nu^1 - \nu - i\varepsilon)}$$

$$- \frac{1}{\pi} \int_{-1}^{+1} d(\mathrm{Cos}\ \theta)\ (2\nu(1 - \mathrm{Cos}\theta) + \tfrac{4}{3}\mu^2) \int_0^\infty \frac{d\nu^1 \ \mathrm{Im}\ h_0(v^1)}{(\nu^1 - \frac{2}{3}\mu^2)(4\mu^2 - 2\nu(1 - \mathrm{Cos}\theta))}$$

so the equation for $h_0(v)$ now reads

$$h_0(v) = a_0 + \frac{1}{\pi}(\nu + \tfrac{2}{3}\mu^2) \int_0^\infty \frac{dv^1 \ \mathrm{Im}\ h_0(v^1)}{(v^1 + \frac{2}{3}\mu^2)(v^1 - \nu - i\varepsilon)}$$

$$+ \frac{1}{\pi} \quad (\nu + \tfrac{2}{3}\mu^2) \int_{-\infty}^{-\mu^2} \frac{dv^1 \quad f(v^1)}{(v^1 + \frac{2}{3}\mu^2)(v^1 - v)}$$

with

$$a_0 = \lambda + \frac{2}{\pi} \int_0^\infty dv^1 \quad \mathrm{Im}\ h_0(v^1) \left[\frac{3}{\mu^2}\ \ln\ \left(\frac{v^1 + \mu^2}{v^1 + \frac{2}{3}\mu^2}\right) \quad - \quad \frac{1}{v^1 + \frac{2}{3}\mu^2} \right]$$

and $f(v^1)$ defined previously .

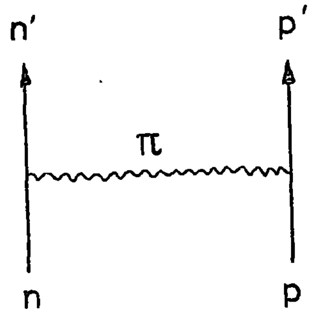
These equations were first derived by CHEW and MANDELSTAM(3); by solving them we can determine the scattering amplitude without recourse to the Feynman-Dyson theory. The advantage of the partial wave dispersion relations lies in the fact that the number of variables we have to deal with has been reduced to only one as compared with two in the original MANDELSTAM representation.

§ 3. The Partial Wave Dispersion Relations

In the preceding section we have discussed a dynamical formulation of the scattering problem based on the CINI-FUBINI approximation. In this section we shall now show that the partial wave dispersion relations are valid in general without making a particular approximation. We choose the problem of nucleon-nucleon scattering for the scalar model to illustrate this technique.

The choice of the MANDELSTAM variables is made as follows:

$$p_1 = p, \; p_2 = n, \; p_3 = -p^1, \; p_4 = -n^1 \; .$$



Then we get

$$s = -(p + n)^2, \quad t = -(p - p^1)^2, \quad u = -(p - n^1)^2$$

which correspond, respectively, to the channels

$$s: \; p + n \rightarrow p^1 + n^1, \quad t: \; p + \bar{p}^1 \rightarrow n^1 + \bar{n}, \quad u: \; p + \bar{m}^1 \rightarrow p^1 + \bar{n} \; .$$

In the present model only the t channel has a pole arising from the one (neutral) meson intermediate state. Therefore, in analogy with the analysis of the preceding section we can write the amplitude as

$$F(s,t,u) = \frac{g^2}{t + \mu^2} + \frac{1}{\pi^2} \int_{4M^2}^{\infty} ds^1 \int_{4\mu^2}^{\infty} dt^1 \, \frac{\rho_{12}(s^1,t^1)}{(s^1-s)(t^1-t)}$$

$$+ \frac{1}{\pi^2} \int_{4\mu^2}^{\infty} dt^1 \int_{4M^2}^{\infty} du^1 \, \frac{\rho_{23}(t^1,u^1)}{(t^1 - t)(u^1 - u)}$$

$$+ \frac{1}{\pi^2} \int_{4M^2}^{\infty} du^1 \int_{4M^2}^{\infty} ds^1 \, \frac{\rho_{31}(u^1, s^1)}{(u^1 - u)(s^1 - s)}$$

In addition we should add single-integral terms but we shall not write them down explicitly. Then in the centre-of-mass system of the s channel we introduce the relative momentum q and the scattering angle θ as in the preceding section, and recall the partial wave expansion of the amplitude

$$F = - 8\pi \sqrt{s}\ f(\theta)$$

$$f(\theta) = \frac{1}{q} \sum_{\ell} (2\ell + 1)\ e^{i\delta_\ell} \sin\delta_\ell\ P_\ell(\mathrm{Cos}\ \theta)$$

The partial wave amplitude h_ℓ is then defined, as in the preceding section by,

$$h_\ell = \frac{1}{2} \int_{-1}^{+1} d(\mathrm{Cos}\ \theta)\ P_\ell(\mathrm{Cos}\ \theta)\ F(q^2,\mathrm{Cos}\theta) = - 16\pi \sqrt{\frac{q^2 + M^2}{q^2}}\ e^{i\delta_\ell} \sin\delta_\ell$$

and the MANDELSTAM variables are

$$\begin{aligned} s &= 4(M^2 + q^2) &= 4(M^2 + \nu) \\ t &= -2q^2(1-\cos\theta) &= -2\nu(1-\cos\theta) \\ u &= -2q^2(1+\cos\theta) &= -2\nu(1+\cos\theta) \end{aligned}$$

Now we shall study the analytic structure of $F(q^2, \cos\theta)$ or $h_\ell(q^2)$.

There are four kinds of denominators in the MANDELSTAM representation.

(1). $s^1 - s = s^1 - 4M^2 - 4\nu$

s^1 runs from $4M^2$ to ∞, so that this denominator vanishes for

$$0 \leq \nu \leq \infty$$

giving rise to the right hand cut.

(2). $t^1 - t = t^1 - 2\nu(1 - \cos\theta)$ with $t^1 \geq 4\mu^2$

This denominator vanishes for

$$\nu^1 = - \frac{t'}{2(1 - \cos\theta)} \quad \text{or} \quad -v^1 \geq \frac{t^1}{4} \geq \frac{4\mu^2}{4} = \mu^2$$

or

$$-\infty \leq \nu^1 \leq -\mu^2 ,$$

which produces the left hand cut.

(3). $u^1 - u = u^1 + 2\nu(1 + \text{Cos}\,\theta)$ with $u^1 \geq 4M^2$.

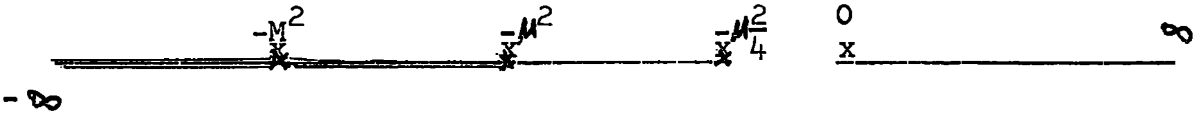
In this case we get a left hand cut beginning at $-M^2$.

(4). The pole term has the denominator

$$t - \mu^2 = -2\nu(1 - \text{Cos}\,\theta) - \mu^2$$

which generate a left hand cut beginning at $- \frac{\mu^2}{4}$.

The complete cut situation is illustrated below:



Hence Im h_ℓ vanishes for $-\frac{\mu^2}{4} < \nu < 0$ and we can write

$$h_\ell(\nu) = \frac{1}{\pi} \int_{-\infty}^{-\frac{\mu^2}{4}} d\nu^1 \, \frac{\text{Im}\, h_\ell(\nu^1)}{\nu^1 - \nu - i\epsilon} + \frac{1}{\pi} \int_{0}^{\infty} dv^1 \, \frac{\text{Im}\, h_\ell(\nu^1)}{\nu^1 - \nu - i\epsilon}$$

We can also give an explicit form of the pole contribution to the partial wave amplitude:

$$h_\ell(\nu)_{\text{Pole}} = \frac{1}{2} \int_{-1}^{+1} dx \; P_\ell(x) \; \frac{g^2}{-2v(1-x) - \mu^2} = \frac{-g^2}{2} \int_{-1}^{+1} dx \; \frac{P_\ell(x)}{2v(1-x) + \mu^2}$$

The right hand cut corresponds to the contributions from intermediate states in the s channel (N-N scattering) and the left hand cut results from those in the t and u channels (N - $\bar{N}$ scattering). The explicit form on the left hand cut contributions can be given only after the N - $\bar{N}$ scattering problem is solved. We shall simply assume here, however, that the result is known and shall write it as $f_\ell(v)$. Then

$$h_\ell(v) = \frac{1}{\pi}\int_0^\infty dv^1 \frac{\mathrm{Im}\, h_\ell(v^1)}{v^1 - v - i\varepsilon} + \frac{1}{\pi}\int_{-\infty}^{-\frac{\mu^2}{4}} dv^1 \frac{f_\ell(v^1)}{v^1 - v}$$

If $f_\ell(v)$ is known, this integral equation determines $h_\ell(v)$.

As a starting point we often replace the left hand cut by the pole contribution, then an approximate equation is

$$h_\ell(v) = \frac{1}{\pi}\int_0^\infty dv^1 \frac{\mathrm{Im}\, h_\ell(v)}{v^1 - v - i\varepsilon} + h_\ell(v)_{\mathrm{Pole}}$$

In the problem discussed in the preceding section this approximation cannot be used since there is no pole term in that process, but the unknown left hand cut can be expressed by the same function occurring on the right hand cut. In general

$$h_\ell(v)_{\mathrm{Pole}} = \frac{-g^2}{4}\int_{-\infty}^{\frac{-\mu^2}{4}} \frac{dv^1}{v^1 - v} \cdot \frac{1}{v^1} \cdot P_\ell(1 + \frac{\mu^2}{2v^1})$$

and in particular

$$h_0(v)_{\mathrm{Pole}} = \frac{-g^2}{4v}\,\ln\,(1 + \frac{4v}{\mu^2})$$

In order to determine $h_\ell(v)$ we have to take account of unitarity, the result of which can be seen from the expression for $h_\ell(v)$ in terms of the ℓ-th partial wave phase shift.

$$\mathrm{Im}\, h_\ell(\nu) = -\frac{1}{16\pi}\sqrt{\frac{\nu}{\nu + M^2}}\, \left| h_\ell(\nu) \right|^2, \text{ for } \nu > 0.$$

This form is obtained by neglecting the contributions from inelastic channels, and for that reason this relation is called the elastic unitarity. Upon inserting this into the dispersion integral we obtain a non linear integral equation for the partial wave amplitude $h_\ell(\nu)$.

§ 4. The N/D Method,

In order to simplify the unitarity condition we introduce the partial wave amplitude $F_\ell(v)$ by

$$F_\ell(v) = \frac{1}{16\pi} h_\ell(v) = \sqrt{\frac{v + M^2}{v}}\, e^{i\delta_\ell} \sin\delta_\ell$$

then the elastic unitarity assumes the form

$$\mathrm{Im}\, F_\ell(v) = \sqrt{\frac{v}{v + M^2}} \left| F_\ell(v) \right|^2, \quad \text{for } v > 0$$

The scattering equation then reads

$$F_\ell(v) = \frac{1}{\pi} \int_0^\infty dv^1 \frac{\mathrm{Im}\, F_\ell(v^1)}{v^1 - v - i\varepsilon} + F_\ell(v)_{\mathrm{Pole}}$$

In order to linearize the equations we introduce the N/D method devised by CHEW and MANDELSTAM. We write the amplitude as the quotient of two functions:

$$F_\ell(v) = \frac{N_\ell(v)}{D_\ell(v)}$$

where $N_\ell(v)$ has only a left hand cut and is real for $v > 0$, and $D_\ell(v)$ has only a right hand cut and is real for $v < 0$. The elastic unitarity can be written in the form

$$\mathrm{Im}\, (F_\ell(v))^{-1} = -\sqrt{\frac{v}{v + M^2}} \equiv -\rho(v), \quad \text{for } v > 0$$

For $v > 0$, this gives

$$\mathrm{Im}\, (F_\ell(v))^{-1} = \mathrm{Im}\, \frac{D_\ell(v)}{N_\ell(v)} = \frac{\mathrm{Im}\, D_\ell(v)}{N_\ell(v)} = -\rho(v)$$

or

$$\mathrm{Im}\, D_\ell(\nu) = -\rho(\nu)\, N_\ell(\nu)\,, \qquad \text{for } \nu > 0$$

For $\nu < 0$,

$$\mathrm{Im}\, F_\ell(\nu) = \frac{\mathrm{Im}\ N_\ell(\nu)}{D_\ell(\nu)}$$

or

$$\mathrm{Im} N_\ell(\nu) = D_\ell(\nu)\ \mathrm{Im} F_\ell(\nu) \simeq D_\ell(\nu) \mathrm{Im}\ F_\ell(\nu)_{\mathrm{Pole}}\ , \text{ for } \nu < 0$$

We define yet another function, which will in general be known, by

$$\mathrm{Im}\ F_\ell(\nu)\Big|_{\mathrm{Pole}} = -\frac{1}{16\pi}\ f_\ell(v) = \nu_\ell(\nu).$$

For the example considered in the preceding section

$$v_\ell(\nu) = \frac{1}{16\pi}\left(\frac{-\pi g^2}{4}\right)\frac{1}{\nu}\ P_\ell(1 + \frac{\mu^2}{2\nu}) = \frac{g^2}{64}.\ \frac{1}{\nu}\ .\ P_\ell(1 + \frac{\mu^2}{2\nu}),\ \text{for } \nu < -\frac{\mu^2}{4}$$

Then

$$\mathrm{Im}\ D_\ell(\nu) = -\rho(\nu)\ N_\ell(v), \qquad \text{for } \nu > 0$$

and

$$\mathrm{Im}\ N_\ell(v) = v_\ell(\nu)\ D_\ell(\nu), \qquad \text{for } \nu < 0$$

Let us normalize N and D by D (0) = 1, and write the once subtracted dispersion relation for $D_\ell(v)$:

$$D_\ell(\nu) = 1 + \frac{\nu}{\pi}\int_0^\infty dv^1\ \frac{\mathrm{Im}\ D_\ell(v^1)}{v^1(v^1-v-i\epsilon)} = 1 - \frac{\nu}{\pi}\int_0^\infty dv^1\ \frac{\rho(v^1)}{v^1}\ \frac{N_\ell(v^1)}{v^1-v-i\epsilon}$$

Then $N_\ell(0) = F_\ell(0)$, and the subtracted dispersion relation for

$N_\ell(\nu)$ is

$$N_\ell(\nu) = F_\ell(0) + \frac{\nu}{\pi} \int_{-\infty}^{0} d\nu^1 \; \frac{\mathrm{Im}\, N_\ell(\nu^1)}{\nu^1(\nu^1 - \nu - i\varepsilon)}$$

$$= F_\ell(0) + \frac{\nu}{\pi} \int_{-\infty}^{0} d\nu^1 \; \frac{v_\ell(\nu^1)\, D_\ell(\nu^1)}{\nu^1(\nu^1 - \nu - i\varepsilon)}$$

Together the dispersion relations for N and D form a coupled set of linear integral equations. We may also assume that $N_\ell(v)$ satisfies an unsubtracted dispersion relation.

$$N_\ell(\nu) = \frac{1}{\pi} \int_{-\infty}^{0} d\nu^1 \; \frac{v_\ell(\nu^1)\, D_\ell(\nu^1)}{\nu^1 - \nu - i\varepsilon}$$

With the help of this method we have succeeded in linearizing the original non-linear integral equation. The next step consists in transforming the singular equation into a non-singular equation.

Combining the dispersion relations we can eliminate $N_\ell(v)$.

$$D_\ell(v) = 1 + \frac{\nu}{\pi^2} \int_{-\infty}^{-\frac{\mu^2}{4}} d\nu^{11} \; \frac{1}{\nu - \nu^{11}} \left[\int_0^\infty d\nu^1 \frac{\rho(\nu^1)}{\nu^1} \left(\frac{1}{\nu^1 - \nu} - \frac{1}{\nu^1 - \nu^{11}} \right) \right]$$

$$\times \; v_\ell(\nu^{11})\; D_\ell(\nu^{11})$$

If we solve this equation for negative values of v, then $D_\ell(v)$ is real and there is no singularity since

$$D\,(\nu) = 1 + \frac{\nu}{\pi^2} \int_{-\infty}^{-\frac{\mu^2}{4}} d\nu^{11} \; \frac{K(\nu) - K(\nu^{11})}{\nu - \nu^{11}} \; v_\ell(\nu^{11})\; D_\ell(\nu^{11}),$$

with

$$K(v) = \int_0^\infty dv^1 \frac{\rho(v^1)}{v^1(v^1 - v)} = \int_0^\infty dv^1 \frac{1}{\sqrt{v^1(v^1 + M^2)}} \quad \frac{1}{v^1 - v}$$

Once $D_\ell(v)$ is known for negative values of v, one can compute $N_\ell(v)$ for all values of v and then $D_\ell(v)$ using dispersion relations. Let us put $v = -x$, $D_\ell(-x) = D(x)$, and $v_\ell(-x) = v(x)$, in order to discuss the integral equation for negative values of v. The integral equation for D is

$$D(x) = 1 - \frac{x}{\pi^2} \int_{\frac{\mu^2}{4}}^\infty dx^{11} \frac{K(-x)-K(-x^{11})}{x^{11} - x} \; v(x^{11}) \; D(x^{11}).$$

Defining the symmetric kernel

$$K(x,x^{11}) = \frac{K(-x)-K(-x^{11})}{x^{11} - x} = \int_0^\infty dv^1 \frac{1}{\sqrt{v^1(v^1 + M^2)}} \frac{1}{(v^1 + x)(v^1 + x^{11})}$$

we have

$$D(x) = 1 - \frac{1}{\pi^2} \int_{\frac{\mu^2}{4}}^\infty dx^{11}. \; x. \left[K(x,x^{11}) v(x^{11}) D(x^{11}) \right] .$$

In case v has a definite sign we can immediately transform this equation into the standard form. Take, for instance, the s wave amplitude for n-p scattering in the scalar model, then

$$v_0 \; (-x) = - \frac{g^2}{64} \; . \; \frac{1}{x}$$

Assume that $v(x^{11})$ is negative definite, and write

$$v(x) = - \left| \; v(x) \right| \quad ,$$

then the integral equation can be transformed into

$$\sqrt{\frac{|v(x)|}{x}}D(x) = \sqrt{\frac{|v(x)|}{x}} + \frac{1}{\pi^2}\int_{\frac{\mu_c^2}{4}}^{\infty} dx^{11}\sqrt{x|v(x)|}\,K(x,x^{11})\sqrt{x^{11}|v(x^{11})|}\;\sqrt{\frac{|v(x^{11})|}{x^{11}}}\,D(x^{11})$$

which is an integral equation of the FREDHOLM type.

To conclude, we have overcome three major difficulties step by step: First, we have reduced the number of variables from two to one by introducing the partial wave dispersion relations. Secondly, we have transformed the original non-linear integral equations into linear ones on the basis of the N/D method. Thirdly, we have reduced the linear but singular equations into the non-singular FREDHOLM type.

The FREDHOM equation is non-singular and is subject to various methods of solution. Thus the scattering problem in dispersion theory can be formulated in principle without recourse to the Feynman-Dyson theory.

§ 5. Further Discussion on the Scattering Equation

In the preceding section we have studied a general method of solving the scattering equation of the form

$$F_{\ell}(v) = \frac{1}{\pi}\int_{0}^{\infty} dv^{1} \frac{\mathrm{Im}F_{\ell}(v^{1})}{v^{1}-v-i\varepsilon} - \frac{1}{\pi}\int_{-\infty}^{-\frac{\mu^2}{4}} dv^{1} \frac{V_{\ell}(v^{1})}{v^{1}-v-i\varepsilon}$$

with

$$\mathrm{Im}\,F_{\ell}(v) = \rho(v)\left|F_{\ell}(v)\right|^{2} \quad , \text{ for } \quad v > 0$$

We have exploited the N/D method to linearize the equation and eliminate the singular kernal from the equation. Because of the non-linearity, however, it happens that the solution discussed in the preceding section is not unique, and occasionally it is not even the solution of the original equation.

Before discussing these points we shall study the relation between the D function and the phase shift. The function $D_{\ell}(v)$ satisfies a dispersion relation of the form

$$D_{\ell}(v) = 1 + \frac{v}{\pi}\int_{0}^{\infty} dv^{1} \frac{\mathrm{Im}\,D_{\ell}(v^{1})}{v^{1}(v^{1} - v - i\varepsilon)}$$

In order to evaluate Im D let us recall the relation

$$D_{\ell} = N_{\ell}/F_{\ell} \ ,$$

and also the fact that N_{ℓ} is real for $v > 0$. Thus we have

$$\frac{\mathrm{Im}\,D_{\ell}}{\mathrm{Re}\,D_{\ell}} = -\frac{\mathrm{Im}\,F_{\ell}}{\mathrm{Re}\,F_{\ell}} = -\tan\delta_{\ell} \ ,$$

or

$$\frac{D_\ell^*}{D_\ell} = \frac{F_\ell}{F_\ell^*} = e^{2i\delta_\ell} = s_\ell \ .$$

Combining the dispersion relation with

$$\text{Im } D_\ell(\nu) = - \tan\delta_\ell(\nu). \quad \text{Re } D_\ell(\nu).$$

we get the standard Muskhelishvili-Omnès equation for $D_\ell(v)$. The solution is

$$D\ (v) = \exp\left[-\frac{\nu}{\pi} \int_0^\infty \frac{d\nu^1 \ \delta_\ell(\nu^1)}{\nu^1(\nu^1 - \nu - i\epsilon)} \right]$$

We shall now discuss the problem of zeros and poles of the D - function(4). Let us consider a simple example

$$v_0(\nu) = -\pi \Gamma \delta(\nu + \nu_i) \qquad (\nu_i > 0)$$

The integral equation reduces to an algebraic equation, i.e.,

$$N_0(\nu) = \frac{1}{\pi} \int_{-\infty}^0 d\nu^1 \frac{v_0(\nu^1)\ D_0(\nu^1)}{\nu^1 - v - i\epsilon} = \frac{\Gamma}{\nu_i + \nu} \ D_0(-\nu_i).$$

Instead of normalizing D_0 by $D_0(0) = 1$ we may choose an alternative normalizing $D_0(-\nu_i) = 1$, then

$$N_0(\nu) = \frac{\Gamma}{\nu_i + \nu},$$

and

$$D_0(\nu) = 1 - \frac{\nu + \nu_i}{\pi} \int_0^\infty dv^1 \frac{\rho(v^1)}{\nu^1 + \nu_i} \cdot \frac{N_0(\nu^1)}{\nu^1 - \nu - i\epsilon}$$

$$= 1 - \frac{\Gamma}{\pi} (v - v_i) \int_0^\infty dv^1 \sqrt{\frac{v^1}{v^1 + M^2}} \frac{1}{(v^1 + v_i)^2 (v^1 - \nu - i\epsilon)}$$

$$\simeq 1 - \frac{\Gamma}{2M} \frac{\nu + \nu_i}{\sqrt{\nu_i} \left(\sqrt{\nu_i} + \sqrt{-\nu - i\epsilon} \right)^2}$$

where we have evaluated the dispersion integral in the non-relativistic approximation, i.e., $\nu \ll M^2$. This expression is certainly real for $\nu < 0$, but it develops an imaginary part for $\nu > 0$.

In the physical region $\nu > 0$, we find

$$\frac{\text{Re } D_0(v)}{N_0(v)} = \rho(v) \cot \delta_0 \simeq \frac{\sqrt{\nu}}{M} \cot \delta_0$$

$$= \left(\frac{\nu_i}{\Gamma} - \frac{\sqrt{v_i}}{2M} \right) + \nu \left(\frac{1}{\Gamma} + \frac{1}{2M\sqrt{\nu_i}} \right)$$

Comparing this formula with the standard non-relativistic effective range formula

$$q \cot \delta_0 = \frac{1}{a} + \frac{1}{2} r q^2 , \qquad (q^2 = \nu)$$

we see that

$$\frac{1}{a} = \frac{M}{\Gamma} \nu_i - \frac{1}{2} \sqrt{\nu_i}$$

$$\frac{1}{2}r = \frac{M}{\Gamma} + \frac{1}{2\sqrt{v_i}} = \frac{1}{\sqrt{v_i}} + \frac{1}{a\nu_i} .$$

If $\Gamma \gtrsim 2M\sqrt{\nu_i}$, we can find a solution of the equation

$$D_0(\nu) = 0,$$

that is,

$$-\nu = \alpha^2 = \nu_i \times \left(\frac{(\Gamma - 2M\sqrt{v_i})}{\Gamma + 2M\sqrt{v_i}} \right)^2$$

This determines the position of the bound state, since the zeros of $D_\ell(\nu)$ are the poles of $F_\ell(\nu)$ or $h_\ell(\nu)$, and we are forced to accept such states. When such is the case $\nu = -\alpha^2$ represents a pole, which is not present in the original dispersion relation.

There is another subject concerning the poles of $D_\ell(\nu)$. Assume that $D_\ell(\nu)$ has poles at ν_i $(i = 1,2,\ldots,n)$, then ν_i appears as zeros of the amplitude $F_\ell(v)$. The zeros are not singularities so that $F_\ell(v)$ can have poles without modifying the dispersion relation for $F_\ell(v)$. Therefore, the equation

$$\mathrm{Im}\, D_\ell(\nu) = -\rho(\nu)\, N_\ell(\nu), \quad (\nu > 0)$$

does not determine the dispersion relation for $D_\ell(\nu)$ uniquely, e.g., we may write it as

$$D_\ell(v) = 1 - \nu \left[\frac{1}{\pi} \int_0^\infty dv^1 \frac{\rho(\nu^1)}{\nu^1} \frac{N_\ell(\nu^1)}{\nu^1 - \nu - i\varepsilon} + \sum_{i=1}^{n} \frac{C_i}{\nu_i - \nu} + A \right]$$

The reality condition for $D_\ell(v)$, for $\nu < 0$, implies that all C_i, ν_i and A be real. This kind of non-uniqueness was first discussed by CASTILLEJO, DALITZ and DYSON$^{(5)}$, and these points are called CDD zeros. Whether or not the A term is present depends on the convergence of the unsubtracted dispersion relation for $N_{\ell}(v)$. The term A is associated with a CDD zero at $\nu = \infty$.(6)

One of the important conditions that has to be fulfilled is that $D_{\ell}(\nu)$ should not vanish between the branch cuts, otherwise this zero would show up as a pole in the amplitude $F_{\ell}(\nu)$ which originally does not have a pole in this domain, e.g., for the simple scalar model of Sec.3.

$$D_{\ell}(v) \neq 0, \quad \text{for } 0 > \nu > -\frac{\mu^2}{4}$$

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PART 2

CHAPTER TWO

In this chapter the equations of the DASHEN - FRAUTSCHI perturbation theory are derived, both for the single channel and the multichannel case.

Derivation of D.F. Equations

We consider a partial wave scattering amplitude $T_\ell(E)$ which can be written as (1):

$$T(E) = \frac{N(E)}{D(E)}, \qquad (1)$$

where N(E) is real for $E > 0$, and is in analytic function of E except for having the cuts of T(E) for $E < 0$,. D(E) is real for $E < 0$, goes to 1 as $E \rightarrow \infty$, and is analytic except for having the cut of T(E) for real $E > 0$ given by physical unitarity.

For $E > 0$, T(E) may be written

$$T(E) = \frac{e^{i\eta} \sin \eta}{\rho(E)} \qquad (2)$$

where n is a real phase shift and $\rho(E)$ is a phase space factor. D(E) has the phase e^{-in} for $E > 0$.(2)

Bound states of the potential appear as zeros of D(E), and thus poles of T(E) for $E < 0$. For such a bound state, we define

$$R = \lim_{E \rightarrow E_B} (E - E_B)T(E) = \frac{N(E_B)}{D'(E_B)} \qquad (3)$$

Now assume that a small perturbing potential is introduced. From Eq.(1) we can write to first order

$$(D^2\,\delta T^{(1)})(E) \;=\; D(E)\,\delta N^{(1)}(E) - N(E)\,\delta D^{(1)}(E) \qquad (4)$$

Evaluating Eq.(4) at $E = E_B$ and using the result

$$\delta E_B{}^{(1)} = -\frac{\delta D^{(1)}}{D^{(1)}(E_B)} \qquad (A)$$

we find

$$\delta E_B{}^{(1)} = \frac{(D^2\delta T^{(1)})(E_B)}{N(E_B)D^{1}(E_B)} = \frac{(D^2\delta T^{(1)}(E_B)}{R(D^{1}(E_B))^2} \qquad (5)$$

As is apparent from Eq.(4), the quantity $(D^2\delta T^{(1)})(E)$ is finite and, in general, non-zero at $E = E_B$. For $E > 0$, we have from Eq. (2),

$$\delta T^{(1)}(E) = \frac{e^{2in}}{\rho(E)}\,\delta\eta \qquad (6)$$

Since $D^2(E)$ has the phase $e^{-2i\eta}$ for $E > 0$, the quantity $(D^2\delta T^{(1)})(E)$ has no imaginary part for $E > 0$. If an <u>unsubtracted</u> dispersion relation is now written for $(D^2\ T(^1))(E)$ and evaluated at $E = E_B$, then we obtain from Eq.(6) exactly the equation of DASHEN and FRAUTSCHI$^{(2)}$ for the first order shift in energy of a bound state due to a perturbation:

$$\delta E_B{}^{(1)} = \frac{1}{R(D^{1}(E_B))^2}\;\frac{1}{\pi}\int_{-\infty}^{0}\frac{\mathrm{Im}(D^2\delta T^{(1)})(E^1)}{E^1 - E_B}\,dE^1 \qquad (7)$$

Now consider relativistic two particles scattering. We write the partial wave scattering amplitude, $T(s) = N(s)/D(s)$, where s is the square of the centre of mass energy. N(s) is an analytic function of s except for left hand cuts (LHC). D(s) is an analytic function of s except for the cuts given by physical unitarity for s real and above the threshold for two particle scattering. We refer to these as right hand cuts(RHC). Above the threshold for two particle scattering, we may write

$$T(s) \quad = \quad \frac{e^{2i\eta} - 1}{2i\rho(s)} \, , \qquad (8)$$

where $\rho(s)$ is a phase space factor and η is a phase shift, which becomes complex above the first inelastic threshold.

Bound states appear as zeros of D(s) and thus poles of T(s). Let us assume the existence of such a bound state at $s = s_B$. The residue at the bound state pole is then defined by

$$R \;=\; \lim_{s \to s_B} (s - s_B)T(s) \;=\; N(s_B)/D'(s_B) \qquad , \qquad (9)$$

where the prime now denotes the derivative with respect to s.

It is now simple to generalize what we did in the case of potential scattering, and to derive a first order expression for the change in position of the bound state when the relativistic partial wave amplitude is perturbed. Since the algebra used to derive Eq.(A) or (5) is independent of the assumption of potential scattering, the same equations are true for the relativistic case with s in place of E as an independent variable. It is more convenient for purposes of later calculation to have the change in the bound state energy expressed in terms of δT rather than δD. We therefore use Eq.(5), and find for the first order change in position of the bound state,

$$\delta s_B{}^{(1)} = \frac{(D^2\delta T^{(1)})(s_B)}{R(D'(s_B))^2} \tag{10}$$

In the following we will drop the superscripts on δs_B and δT, since we shall be concerned from here on only with first order quantities.

As in potential theory, we wish to write a dispersion relation for $(D^2\delta T)(s_B)$. One has to assume that: (1) $D(s_B) = 0$, (2) $D(s) \longrightarrow$ const. as $s \to \infty$, and (3) $D(s)$ is an analytic function of s except for having the right hand cut of $T(s)$. As $\rho(s)$ will be chosen so that $T(s) \to 0$ as $s \to \infty$ at least as fast as s^{-1}, $\delta T(s)$ will also $\to 0$ as $s \to \infty$ at least as fast as s^{-1}. We may then write an unsubtracted dispersion relation for the quantity $(D^2\delta T)(s_B)$ in Eq.(10). We then have

$$\delta s_B = \frac{1}{R(D'(s_B))^2} \quad \frac{1}{\pi}\int_{\text{cuts}} \frac{\mathrm{Im}(D^2\delta T)(s')}{s' - s_B}\, ds' \qquad 9(11)$$

On the left hand cut where $\mathrm{Im}D(s) = 0$, we have

$$\mathrm{Im}D^2\delta T = D^2\mathrm{Im}\delta T \tag{12}$$

On the right hand cut, we assume elastic unitarity before the perturbation is introduced,

$$\mathrm{Im}T(s) = \rho(s)\,\left|T(s)\right|^2 \tag{13}$$

where $\rho(s)$ is a phase space factor which depends on our choice of amplitude (see Eq.(8)). When the perturbation is introduced, $T \to T + \delta T$ and, since the masses of external particles may change, $\rho \to \rho + \delta\rho$, so that

$$\mathrm{Im}(T + \delta T) = (\rho + \delta\rho)\left|T + \delta T\right|^2 + \sum_i \rho_i \left|\delta T_i\right|^2 \tag{14}$$

where the second term on the right hand side of equation (14) is the contribution to the absorptive part of the partial wave amplitude coming from new inelastic states, i, and ρ_i is the corresponding phase space factor. For example, in pion-nucleon scattering with electromagnetism considered as a perturbation, a possible inelastic state is the photon-nucleon state, in which case δT_i is a pion photoproduction partial wave amplitude(3).

Combining Eq.(13) and (14), we have to first order

$$\text{Im}\,\delta T = \delta\rho \,\text{Re}\,(T\,\delta T) + \delta\rho\,|T|^2 + \sum_i \rho_i\,|\delta T_i|^2 \quad , \qquad (15)$$

or, on rearranging terms

$$\text{Im}\,\delta T = \frac{\delta\rho\,(\text{Re}\,T)(\text{Re}\,\delta T) + \delta\rho\,|T|^2 + \sum_i \rho_i |\delta T_i|^2}{1 - 2\rho\,\text{Im}T} \; . \qquad (16)$$

Then, since we have assumed elastic unitarity, $T = e^{i\eta}\sin\eta/\rho$ and $D = |D|\,e^{-i\eta}$ where n is real. From Eq.(16) we then find

$$\text{Im}(D^2\delta T) = |D|^2(\delta\rho\,|T|^2 + \sum_i \rho_i\,|\delta T_i|^2) = N^2\delta\rho + |D|^2 \sum_i \rho_i\,|\delta T_i|^2 \quad (17)$$

Although we shall not make use of it in this work, we note in passing that by the same methods one can easily derive an equation for the first order change in the residue, R, of the pole in T(s) as $s = s_B$:

$$\delta R = \frac{d}{ds}\left[\frac{(s - s_B)^2}{(D(s))^2}\;\frac{1}{\pi}\int_{\text{cuts}} \frac{\text{Im}(D^2\delta T)(s^1)}{s^1 - s}\, ds^1\right]_{s = s_B} \qquad (18)$$

This equation, and multichannel generalization of it(4), have been used in calculations involving the perturbation of strong interactions by the weak and electromagnetic interactions(5).

As an example of the use of Eq.(11) and its agreement with an independent calculation of δs_B, let us consider the application of the method of DASHEN and FRAUTSCHI to one channel elastic two particle scattering with a left hand cut given by a single pole. We take the imaginary part of the amplitude on the left cut to be

$$\mathrm{Im}T(s) = \pi g\,\delta(s - s_1) \quad , \tag{19}$$

from which we compute using the usual N/D equations,

$$N(s) = \frac{1}{\pi} \int_{LHC} \frac{(\mathrm{Im}T(s^1))\,D(s^1)}{s^1 - s}\, ds^1 = \frac{gD(s_1)}{s - s_1} \tag{20}$$

and

$$D(s) = 1 - \frac{1}{\pi}\int \frac{\rho(s^1)N(s^1)}{s^1 - s}\, ds^1 = 1 - \frac{1}{\pi}\int \frac{\rho(s^1)\; gD(s_1)}{(s^1 - s)(s^1 - s_1)}\, ds^1 \tag{21}$$

Eq.(20) can then be solved for $D(s_1)$ explicitly:

$$D(s_1) = \frac{1}{1 + \frac{g}{\pi}\int_{RHC} \frac{\rho(s^1)}{(s^1 - s_1)^2} ds^1} \tag{22}$$

Let us assume that there is a bound state at s_B due to the vanishing of $D(s_B)$:

$$0 = D(s_B) = 1 - \frac{gD(s_1)}{\pi} \int_{RHC} \frac{\rho(s^1)}{(s^1 - s_B)^2 (s^1 - s_1)} ds^1 \quad .(23)$$

Direct calculation from Eq.(21) then also gives

$$\frac{dD}{ds}(s_B) = -\frac{1}{\pi}\int_{RHC} \frac{\rho(s^1)\;\; gD(s_1)}{(s^1 - s_B)^2 (s^1 - s_1)} ds^1 \tag{24}$$

The residue at the pole at $s = s_B$ is then given by

$$R = \frac{gD\,(s_1)}{(s_B - s_1)\,\frac{dD}{ds}\,(s_B)} \tag{25}$$

Now let us consider the effect of making small perturbations in g, s_1 and $\rho(s)$ on the position of the bound state pole. We shall compute δs_B to first order directly from Eq.(23) and compare it with δs_B computed by the DASHEN - FRAUTSCHI formula,$^{(11)}$.

A. Vary g: $g \rightarrow g + \delta g$

From (23), using

$$\frac{d}{dg} \left(\frac{1}{D(s_1)}\right) = \frac{1}{g} \left(\frac{1}{D(s_1)} - 1\right), \tag{26}$$

and (24), we have

$$0 = \frac{\delta g}{g} \left(\frac{1}{D(s_1)} - 1\right) - \frac{\delta g}{gD(s_1)} + \frac{\delta s_B}{D(s_1)} \frac{dD(s_B)}{ds} , \tag{27}$$

or

$$\delta s_B = \frac{\delta g}{g} \frac{D(s_1)}{\frac{dD}{ds}(s_B)} . \tag{28}$$

On the other hand, putting $\mathrm{Im}\,\delta T(s) = -\pi \delta g \delta(s - s_1)$ in the DASHEN-FRAUTSCHI formula,$^{(11)}$, gives immediately

$$\delta s_B = \frac{1}{R \frac{dD}{ds} (s_B)^2} \frac{-\delta g (D(s_1))^2}{s_1 - s_B} = \frac{\delta g}{g} \frac{D(s_1)}{\frac{dD}{ds} (s_B)} . \tag{29}$$

B. Vary s_1: $s_1 \rightarrow s_1 + \delta s_1$

Direct computation from (23) gives

$$\delta s_B = \frac{D^2(s_1)\delta s_1 + (s_B - s_1) \frac{dD}{ds} (s_1) \delta s_1}{(s_B - s_1) \frac{dD}{ds}(s_B) D(s_1)} , \tag{30}$$

with

$$\frac{dD}{ds_1}(s_1) = -2(D(s_1))^2 \; \frac{g}{\pi} \int_{RHC} \frac{\rho(s^1)}{(s^1 - s_1)^3} ds^1 \qquad (31)$$

To use the DASHEN - FRAUTSCHI equation we need $\delta T(s)$, which we compute directly from

$$T(s) = \frac{N(s)}{D(s)} = \frac{\dfrac{gD(s_1)}{s - s_1}}{1 - \dfrac{1}{\pi} \displaystyle\int_{RHC} \frac{\rho(s^1) \; gD(s_1)}{(s^1-s)(s^1 - s_1)} ds^1}, \qquad (32)$$

we find

$$\delta T(s) = \frac{g}{(s - s_1)^2 (D(s))^2} \left[(s - s_1) \; \frac{dD}{ds_1}(s_1) + (D(s_1))^2 \right] \delta s_1 \qquad (33)$$

Putting Eq.(33) in Eq.(11) and carrying out the integral over the left hand cut as a contour integral around the pole at $s = s_1$ gives exactly Eq.(30)).

C. <u>Vary $\rho(s)$: $\rho(s) \rightarrow \rho(s) + \delta\rho(s)$</u>

Again, starting from the bound state Eq.(23), we find

$$\delta s_B = \frac{(s_B - s_1) \; D(s_1) g}{\frac{dD}{ds}(s_B)} \; \frac{1}{\pi} \int_{RHC} \frac{\delta\rho(s^1)}{(s^1 - s_1)^2 (s^1 - s_B)} ds^1 \qquad (34)$$

Recalling that on the right hand cut we have $\mathrm{Im}(D^2 \delta T) = N^2 \delta\rho$, the DASHEN - FRAUTSCHI expression (11) becomes

$$\delta s_B = \frac{1}{R \, \frac{dD}{ds}(s_B)^2} \; \frac{1}{\pi} \int_{RHC} \frac{N^2(s^1) \delta\rho(s^1)}{s^1 - s_B} ds^1 \qquad (35)$$

On using $N(s) = \frac{gD(s_1)}{s - s_1}$, Eq.(35) becomes

$$\delta s_B = \frac{g^2\ (D(s_1))^2}{R\ \frac{dD}{ds}(s_B)^2} \quad \frac{1}{\pi} \int_{RHC} \frac{\delta\rho(s^1)}{(s^1-s_1)^2(s^1-s_B)} ds^1 \tag{36}$$

$$=) \quad \frac{gD(s_1)(s_B - s_1)}{\frac{dD}{ds}(s_B)} \quad \frac{1}{\pi} \int_{RHC} \frac{\delta\rho\ (s^1)}{(s^1 - s_1)^2(s^1-s_B)} ds^1,$$

which is identical to Eq.(34)...

The results for δs_B computed directly from Eq.(23) thus agree in every case with those computed using the DASHEN - FRAUTSCHI formula, Eq.(11).

In the electromagnetic mass differences problem which we will consider in this thesis, it will be assumed that the strongly interacting particles of the unperturbed problem appear as bound state poles in two particle scattering amplitudes. In general we must consider a n - channel unperturbed scattering amplitude, T(s), where T(s) is a nxn symmetric partial wave scattering matrix which has a bound state pole. Along the two particle unitarity cut, we have in place of Eq.(13),

$$\text{Im}\ \underset{\sim}{T}(s) = \underset{\sim}{T}(s)\ \underset{\sim}{\rho}(s)\ \underset{\sim}{T}(s)^{\dagger}, \tag{37}$$

where $\rho(s)$ is a diagonal matrix containing phase space factors which are functions of the total centre of mass energy squared, s. We assume that the unperturbed amplitude has been written in the form

$$T(s) = \underset{\sim}{N}(s)\ \underset{\sim}{D}^{-1}(s), \tag{38}$$

where N(s) is a n x n matrix whose elements are analytic in s except for left hand cuts, and $\underset{\sim}{D}(s)$ ia a n x n matrix whose elements are analytic in s except for right hand cuts present in the partial wave amplitude, $\underset{\sim}{T}(s)$.

With assumption on $\underset{\sim}{D}(s)$ and $\delta\underset{\sim}{T}(s)$ similar to those given for D(s) and T(s) for the one channel case, the analogue of Eq.(11) for the multichannel is$^{(6)}$

$$R\delta s_B = \Delta^T \frac{1}{\pi} \int_{\text{CUTS}} \frac{\mathrm{Im}(\underset{\sim}{D}^T(s^1)\ \delta\underset{\sim}{T}(s^1)\ \underset{\sim}{D}(s^1))}{s^1 - s_B}\ ds^1\ \underset{\sim}{\Delta}\ , \tag{39}$$

where

$$\underset{\sim}{\Delta} = \lim_{s \to s_B} (s - s_B)\ \underset{\sim}{D}^{-1}(s), \tag{40}$$

and

$$\underset{\sim}{R} = \lim_{s \to s_B} (s - s_B)\ \underset{\sim}{T}(s) = \underset{\sim}{N}(s_B)\ \underset{\sim}{\Delta} \tag{41}$$

Multiplying both sides of Eq.(39) by $\underset{\sim}{R}$ and taking the trace of both sides of the resulting expression, we find

$$\delta s_B = \frac{\mathrm{Tr}\left[\underset{\sim}{R}\,\underset{\sim}{\Delta}^T \frac{1}{\pi}\int_{\text{CUTS}} \frac{\mathrm{Im}(\underset{\sim}{D}^T(s^1)\ \delta\underset{\sim}{T}(s^1)\ \underset{\sim}{D}(s^1))}{s^1 - s_B}\ \underset{\sim}{\Delta}\right]}{\mathrm{Tr}\left[\underset{\sim}{R}\,\underset{\sim}{R}\right]} \tag{42}$$

On the left hand cut we have

$$\mathrm{Im}(\underset{\sim}{D}^T \delta\underset{\sim}{T}\underset{\sim}{D}) = \underset{\sim}{D}^T \mathrm{Im}\,\delta\underset{\sim}{T}\ \underset{\sim}{D} \tag{43}$$

which generalizes Eq.(12). On the right hand cut, the generalization of Eq.(17) is

$$\mathrm{Im}(\underset{\sim}{D}^T \delta\underset{\sim}{T}\underset{\sim}{D}) = \underset{\sim}{N}^T \delta\underset{\sim}{\rho}\ \underset{\sim}{N} + \underset{\sim}{D}^{+}\ (\delta\underset{\sim}{T}_I^{\dagger}\ \underset{\sim}{\rho}_I\ \delta\underset{\sim}{T}_I)\underset{\sim}{D}\ , \tag{44}$$

where $\delta\underset{\sim}{T}_I$ is an m x n matrix if there are m new inelastic states, with $\underset{\sim}{\rho}_I$ an m x m diagonal matrix of phase space factors for the inelastic states.

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1a. As there is no ambiguity, we drop the partial wave amplitude subscripts for the remainder of this chapter.

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3. Note that for the perturbation of masses by electromagnetism, "first order" in the perturbation means terms of order e^2. The change in the elastic scattering amplitude, δT, is of order e^2, so the $(\delta T)^2$ in Eq. (14) will be dropped. δT_i can however be first order in e, e.g., a photoproduction amplitude, and so $(T_i)^2$ must be kept to order e^2.

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CHAPTER THREE

In the present chapter PATON's fundamental investigation of the DF-method is summarized. Then DF prescription for handling the infrared divergent contributions to the input is presented, and we end with a critical discussion of PATON's conclusions on this topic in the light of our findings.

i) REVIEW OF PATON's(1) INVESTIGATION OF DF-METHOD

The discussion of chapter 2 showed that if one were able to evaluate the DF expression for the mass shift exactly, the result must be the same as that obtained from the usual first order perturbation theory. In practice however one will inevitably be forced to make rather drastic approximation for the left-hand cut of $\delta T(s)$. This will usually consist of keeping one or two "nearby" singularities. Therefore, it is important to get some idea of how converges to the exact (first order) answer as one includes more and more distant contributions to the left-hand cut of $\delta T(s)$. PATON has carried out such an investigation. We shall summarize PATON's result, after which we describe our attempt at removing the deficiencies of the DF formalism exposed by PATON's study.

As the unperturbed problem PATON considers s-wave scattering in an exponential well, a problem for which one can solve the Schrödinger equation and obtain the N and D functions exactly. This problem is then perturbed in a number of ways.

One mainly wants to consider potential theory analogues of the case where the strong interactions are pertubed by electromagnetic forces. Then one type of perturbation, corresponding to photon exchange "driving terms" is of long range. A reasonable potential theory analogue of the photon exchange force might correspond to a Coulomb potential regularized at the origin $(1 - e^{-Kr}) /r$.

In the relativistic theory a second type of perturbation emerges as a result of the changes in the masses and coupling constants of the particles whose exchange produces the binding force. In contrast to the photon exchange perturbations, this second type of perturbation is of short range (generally of about the same range as the binding forces). For example, if we represent the binding potential as a simple Yukawa potential, then a change in the coupling constant will give a perturbing potential which is again a Yukawa potential of the same range, while a change in mass will be described by a perturbing potential of exponential form.

Both the Yukawa and Coulomb type of perturbations can be obtained as a superposition of exponential perturbations, since

$$\int_{K_1}^{K_2} e^{-Kr}\, dK = \frac{e^{-K_1 r} - e^{-K_2 r}}{r}$$

PATON therefore studies the exponential perturbations first and then sums them to get the others.

The Schrödinger equations for s - wave scattering in an exponential form reads

$$\frac{d^2 \varphi\ (rk)}{dr^2} + k^2\ \varphi(k,r) = a e^{-\mu r}\varphi(r,k)$$

Where $\frac{a}{2M}$ is the strength of the potential, the range and M the reduced mass, $K = \sqrt{s}$. The solutions of this equation are well-known/2/ and in particualr the Jost function is given by

$$f(k,r,) = e^{-i(\frac{k}{\mu}) \ln (-\frac{a}{\mu^2})} \Gamma (1 + \frac{2ik}{\mu}) \; J_{\frac{2ik}{\mu}} (2 \sqrt{-\frac{a}{\mu^2}} \, e^{-\frac{\mu r}{2}})$$

from which one can find N and D functions from the definitions

$$D(s) = f(- k,o), \; N(s) = \frac{1}{2ik} \left[f(-k,o) - f(k,o) \right]$$

Thus defined, D has a right hand cut arising from the square root branch point of k(s) at $s = o$. The singularities of N(s) in this case consist of an infinite set of poles at $s = -\frac{n\mu^2}{4}$, $n = 1, 2....$

For negative values of a, the potential is attractive and bound states can exist for suitable values of a. These occur for s_B such that

$$D(s_B) = 0 \quad \text{or} \quad J_{\frac{-2ik_B}{\mu}} (2(-\frac{a}{\mu^2})^{\frac{1}{2}}) = 0$$

The corresponding bound state wave functions are

$$\varphi(r) = N_B \; J_{\frac{-2ik_B}{\mu}} (2 \sqrt{\frac{-a}{\mu^2}} \; e^{\frac{-r}{2}})$$

where N_B is a normalizing factor.

The standard first order perturbation theory formula for δs_B is now written down:

$$\delta s_B = \frac{\int_0^\infty J_x^2 (be^{-\frac{r}{2}}) \; \delta V(r) \; dr}{\int_0^\infty J_x^2 (be^{-\frac{r}{2}}) \; dr} \qquad (1)$$

where $b = 2 \sqrt{\frac{-a}{\mu^2}}$ $\quad x = -2 \, ik_{B/\mu}$

This is to be compared with the result of DF - expression for δs_B, Eq. (3). PATON first considers perturbations of the form $ae^{-\mu r} \rightarrow a\, e^{-\mu r} + \delta a\, e^{-kr}$ (2) but δa is assumed small compared to a.

Calculating the perturbed Jost function $f(k,r) + \delta f(k,r)$ corresponding to the potential in Eq. (2) one can show that, to first order in δa the singularities of $\delta T(s)$ in the left hand plane will consist of poles at $s_n = -\frac{1}{4}(n\mu + K)^2$ n = 1, 2, 3... so that the total contribution of the left - hand singularities to $\delta T(s)$ is given by the infinite sum of poles:

$$\sum_{n=0}^{\infty} \frac{C'_{nl}}{\left[4s + (n\mu + K)^2\right]}$$

The coefficients C'_{nl} which are proportional to $\left(-\frac{a}{\mu^2}\right)^n \delta a$ have been evaluated by PATON. one then evaluates the DF expression for the mass shift

$$\delta s_B = \frac{1}{N(s_B)\, D^1(s_B)} \frac{1}{2\pi i} \int_C \frac{D^2(s^1)\ \delta T(s^1)}{s^1 - s_B}\, ds^1 \qquad (3)$$

where the contour C is around the left-hand cuts of $\delta T(s)$. In the present case the contour integral can be explicitly evaluated as the infinite sum of residue of $D^2 \delta T(s)$ at the poles of $\delta T(s)$. One finds

$$\delta s_B = \frac{1}{N(s_B)\, D^1(s_B)} \frac{1}{4} \sum_{n=0}^{\infty} \frac{D^2\left[= \frac{(n\mu - K)^2}{4}\right] C'_{nl}}{-\frac{(n\mu + K)^2}{4} - s_B} \qquad (4)$$

It can be written out more explicitly if C'_{nl} are put in as given by PATON. Eq. (14) explicitly exhibits the sum of contributions to δs_B coming from more and more distant parts of the left-hand cut of $D^2 \delta T(s)$. If all the terms in Eq. (..4) were kept then this expression for δs_B would be identical to that given by the standard formula, Eq (3). This can be verified in detail.

The question here, however, is how well the exact first order answer is approximated when only the first few terms in the infinite sum in Eq. (4) are kept. PATON investigated this point numerically. His conclusions can be summarized as follows:

i) The numerical results indicate that the rapidity of convergence of the DF - method, i.e. the number of terms in the sum (Eq B), needed to achieve a given accuracy compared to the exact solution, depends rather strongly on the binding energy;

ii) In the case that the binding energy is small compared to the energy as measured by the inverse range of the binding forces, both the long and short range types of perturbing potentials yield values for δs_B which are accurate to within $\sim$10% for the second Born approximation to δT. In the relativistic theory this approximation corresponds to taking into account one- and two-particle exchange. For short range perturbations the convergence of the DF - method gets worse more quickly with increasing binding energy. As can be seen from the table in the next chapter for $s_B = \frac{\mu^2}{4}$ and for the exponential perturbing potential even in the second Born approximation δs_B is underestimated by a factor or two. For $s_B = \mu^2$ even the sign is wrong in the second Born approximation for the perturbing potential $\delta a\, e^{-\mu r}$.

It will be recalled that DASHEN used the DF - method, in its relativistic variant to compute the neutron-proton mass difference.

If, for the moment, we were to forget about the infrared divergence problem, then we can say this for the DF - method. The restriction to "small binding" which is one aspect of the limitations of the DF - method and pointed out by PATON, is not disastrous for the calculation of the n - p mass difference. On the contrary, it would appear to be well satisfied for the case of a nucleon considered as pion - nucleon bound state though it would be violated in any Fermi-Yang-type compound model of the pion.

ii) DF - METHOD(S) FOR TREATING I.R.D.CONTRIBUTIONS/1,2/.

To begin with we shall recapitulate DF - prescription(s) for the I.R.D. problem. We remind that the problem is not just that of removing I.R.D. but in addition making sure that such a removal does not lead to a reduction in the convergence rate of the dispersion integrals for δs_B and δR. Indeed it is to be clearly understood that the DF - method is comparable to first order perturbation theory only if it converges rapidly enough to the exact result.

Since Coulomb - forces are of infinite range the only way to incorporate them in the DF - formalism is to work with an amplitude which has explicitly a fictitious photon mass in its formalism.

According to DF the necessary modification of their method, in the presence of long-range forces, is suggested by a study of a perturbing potential of the form

$$\frac{1}{t - \lambda^2} \left[\frac{m^2}{t - m^2} \right]^2 \qquad \text{where } t = - 2q^2(1 - \cos \theta).$$

Here λ is the fictitious photon mass which is ultimately supposed to be put equal to zero. DF now remark that such a potential is characteristic of 1 - photon exchange potential between two strongly interacting particles possessing rapidly converging form factors.

The co-ordinate space representation of this potential is

$$\delta V(r) = \delta b \left\{ \frac{e^{-\lambda r}}{r} - \frac{e^{=mr}}{r} + \frac{\lambda^2 - m^2}{2m} e^{-mr} \right\} \qquad (5)$$

This is a Coulomb potential regularized at the origin. If one were to evaluate the dispersion integrals for $D^2 \delta T$ using such a potential one would obtain a logarithmically divergent answer.

This is just the result of the long-range character of the Coulomb interaction, owing to which the phase shift acquires a divergent part proportional to $\ln(2qr)$ or $\ln\left(\frac{2q}{\lambda}\right)$, namely:

$$\delta_\ell = \arg\left(\ell + 1 + i\frac{\delta b}{q}\right) - \frac{\delta b}{2q}\ln\left(\frac{2q}{\lambda}\right)$$

We note that the divergent part of the phase shift δ_ℓ is independent of ℓ and does not depend on angle. It will therefore appear only as a phase factor $\exp\left[-\left(i\frac{\delta b}{q}\right)\ln\left(\frac{2q}{\lambda}\right)\right]$ which multiplies the entire S-matrix.

DF propose to deal with the infrared divergence as follows. First, remove from the S-matrix the infrared divergent factor

$$\exp\left[-i\left(\frac{\delta b}{2q}\right)\frac{g(s)}{\lambda^2}\right]$$

Where g(s) is an as yet unspecified function. Corresponding to this S-matrix one introduces a partial wave amplitude

$$\delta\hat{T}(s) = \frac{\exp\left\{2i(\eta + \delta\hat{\eta})\right\} - 1}{2iq}$$

$$= \frac{\exp\left\{2i\left(\eta + \delta\eta + \frac{\delta b}{4q}\ln\frac{g(s)}{\lambda^2}\right\} - 1}{2iq}$$

where η is the change in the strong-interaction phase shift caused by the electromagnetic interactions, and $\delta\hat{\eta} = \delta\eta - \delta\eta$born. It is clear that $\delta\hat{T}(s)$ is related to $\delta T(s)$ by

$$\delta\hat{T}(s) = \delta T(s) - \frac{\delta b}{4q^{1}} \ln\left(\frac{g(s)}{\lambda^{2}}\right) \left(\frac{e^{2i\eta}}{q}\right)$$

The amplitude $\delta\hat{T}(s)$ has the property that it is well behaved as $\lambda \rightarrow 0$ for any g(s). Consequently one may make use of this freedom to choose g(s) so as to minimize the sensitivity of the dispersion integrals to distant singularities.

The best choice for this purpose, according to DF, is to choose g(s) so that

$$\delta\hat{T}(s) = \delta\hat{\eta} \frac{e^{2i\eta}}{q} = (\delta T - \delta\eta \text{ Born}) \frac{e^{2i\eta}}{q} \qquad (6)$$

with $$\delta\eta\text{Born} = -\frac{1}{q} \int_{0}^{\infty} \sin^{2}qr \, \delta V(r) \, dr$$

Now since $\delta\eta$ Born contains the same $\ln\lambda$ dependence as $\delta\eta$, the infrared phase shift is thereby removed from $\delta\hat{T}(s)$.

Therefore δs_{B} and δR can be calculated in a way that is free of infrared divergences if one uses $\delta\hat{T}(s)$ in place of $\delta T(s)$. This is DF prescription No. 1, (DF - 1) (say).

Concerning the rate of convergence of dispersion integrals for $\delta\hat{T}(s)$, DF have this to say: DF remark that in potential theory any phase shift tends to have its Born approximation at high energies. Consequently, $\delta\hat{\eta} = \delta\eta - \delta\eta_{Born}$ will tend to zero more rapidly at high energies than either $\delta\eta$ or $\delta\eta_{Born}$, separately. This means that the dispersion intergral for $\delta\hat{T}(s)$ will almost inevitably be more rapidly convergent than that of $\delta T(s)$ and the influence of the distant singularities correspondingly less. DF investigate this point for the potential in Eq.(5). In this particular case it is possible to show that

$\delta\hat{\eta} \propto \frac{1}{q^2}$ as $s \to \infty$ while $\delta\eta_{Born} \propto \frac{1}{q}$

For this particular model in which V is short range, and δV is cut off at small distances, DF find that $\delta\hat{\eta}$ falls off like $\frac{1}{q^2}$ for large q no matter what the asymptotic behaviour of the strong interaction phase shift may be. DF therefore conclude that the dispersion integrals for $\delta\hat{T}(s)$ should be less sensitive to distant singularities than is usually the case.

In relativistic problems we are indeed forced to introduce redefined amplitudes free of infrared divergences in problems involving charged particles. In this case there are infrared divergences associated with inner bremstrahlung (a photon connects an initial with a final charged line) as well as Coulomb divergences similar to those encountered in potential theory. DF again recommend dealing with the redefined amplitude, free of $\ln\lambda$ dependence, by introducing $\delta\hat{T}(s)$ through the definition:

$$\delta\hat{T}(s) = \delta T(s) - \delta\eta_{Born} \frac{e^{2i\eta}}{\rho} \qquad (6)$$

Here, as in potential theory case, the freedom to choose the coefficient g(s) of $\ln\lambda$ is to be employed so as to minimize the sensitivity of the dispersion integrals to distant singularities.

It is not at all clear to what extent one may expect the potential theory arguments showing the more rapid rate of convergence of the dispersion integrals for $\delta\hat{T}(s)$ to carry over to the relativistic case.

Firstly, it is not implied that $\delta\eta \longrightarrow \delta\eta_{Born}$ at high energies more rapidly than the unperturbed phase shift $\eta \longrightarrow \eta_{Born}$.

Indeed, in strong interaction physics there is not the least evidence that $\eta \rightarrow \eta_{Born}$ at high energies. Indeed the contrary is most probably true.

Consequently one cannot use this argument to conclude anything about the convergence of the dispersion integrals, nor can one even say with certainty that g(s) chosen as to yield Eq.(6) necessarily represents the optimal choice from the point of view of convergence.

The second argument given for the rapid convergence of the dispersion integrals depended on the short range of the strong potential and the fact that the perturbing potential was cut off at small distances. These are properties which one can perhaps imagine as holding true for strong interactions as well. However one lacks any general demonstration in the relativisitic case these properties actually guarantee a rapidly convergent behaviour for the phase shift δ_η although it looks plausible.

The most serious objection, of course, arises from attempt to treat infrared divergence in relativistic case in the same fashion as in potential theory. It is clear that in practice working with Eq. (2) is easy to speak of but might prove extremely laborious to carry out in practice.

Having made the suggestion contained in Eq(2) DF now find a way to avoid the unpleasant task of actually computing Eq(2). Instead the task is eliminated in favour of a "simpler way to subtract". The prescription is simply to drop the terms containing $\ln \left(\frac{\lambda^2}{g(s_B)}\right)$ since "its coefficient should have vanish anyway". In addition it is claimed that the new procedure "would give the same results as an exact calculation and can be shown to give nearly the same result in approximate calculations".

Here $g(s_B)$ is an arbitrary function which according to DF has to be so chosen that it maximized the convergence of the dispersion integrals. The choice of $g(s_B)$ is made in the way described in the next paragraph. This is DF prescription No. 2. (DF - 2) (say).

PATON has examined both these suggestions in potential theory models. PATON found that if the modified amplitude $\delta T(s)$ is used in the dispersion integral than the I.R.D. associated with $\lambda \rightarrow 0$ can be avoided. In, addition, it improves the convergence of the dispersion integrals. We remark that DASHEN did not use the prescription contained in Eq. (6) in the calculation of neutron-proton mass difference. Instead he used the "subtraction" procedure. At least in potential theory models, PATON found that any prescription for removing I.R.D. which involves dropping in each order a term proportional to ln $\frac{\lambda}{K}$, where K is some constant, although it helps avoid I.R.D. destroys the nice property of making the dispersion integral converge more rapidly. The prescription for determining K is as follows.

One is to express the phase shift $\delta\eta_{Born}$ (Eq. 5) coming from the perturbing potential acting alone in the form

$$\frac{\delta\eta_B(s)}{K} = f(s) \ln \frac{\lambda^2}{g(s)} + O(\lambda) \qquad (7)$$

The K mentioned in the expression $\ln\left(\frac{\lambda}{K}\right)$ is now to be so chosen that $K = \sqrt{g(s_B)}$

Thus BARTON is entirely right in stating that actually DF suggest two separate prescriptions for dealing with I.R.D. BARTON's investigation showed that "contrary to the claim made by DF their second method (DF - 2) is not equivalent to their first (DF - 1); and that unlike the first it can easily give the wrong sign". This then is indeed the reason why DASHEN's calculation of the neutron-proton mass difference, using DF - 2 in its relativistic version, yielded the "physically impossible answer".

Thus we are left with the choice of either using DF - 1 to deal with I.R.D. or invent some other procedure. Any attempt to use DF - 1 (eq 6), in a relativistic problem, would entail the calculation of the Born phase shift $\delta\eta$ from the electromagnetic correction to the "generalized potential" defined by CHEW and FRAUTSCHI /3/. This choice would, hopefully lead to, though not guarantee, the best possible convergence of the dispersion integrals for δs_B and δR. One must always be aware of the limitations of an approach which attempted to simulate relativistic dynamics by a purely formal potential theoretic studies.

In a relativistic case one would have to calculate and take account of many terms contributing to $\delta\eta_{Born}$. It is not altogether surprising that DF swiftly abandon DF - 1. The fact is that no one has used DF - 1 in a relativistic problem. It remains an open problem to which we hope to return in due course.

Our approach is more transparent in that it honestly admits the existence of problems connected with the existence of $\lambda \rightarrow 0$ limit. Two separate methods were explored. In the first method, a potential theory model is studied along the lines of PATON's work discussed elsewhere, the photon mass λ is treated as a parameter. In the second approach, we arrange cancellations between different contributions to δT through the introduction of a function which serves the role of simulating contributions from distant left-hand cut contributions. One has sought to so choose the cut-off function as to minimize the dependence of mass shift integral on the choice of this function. However the results show a rather wide variation in the values of this function if it is to serve its purpose and thereby point to the need for inclusion of other channels in a realistic calculation.

Both suggestions have been studied in the potential theoretic context. The latter approach is then employed to calculate the neutron proton mass difference, with the further inclusion of a D- function constructed directly from the experimentally determined πN, $J = \frac{1}{2}$, $\ell = 1$, phase shifts, at least up to 2 Gev/c and beyond up to 5 Gev/c, using respectively the phase shift data from DONNACHIE et. al. /4/ and ROYCHOUDHURY et. al. /5/. The proton-neutron mass difference turns out to be of the opposite sign to its experimentally measured value/6/. Clearly the problem is impossible to treat as a one-channel calculation. Addition of CDD pole or poles, together with inclusion of other channels is clearly desirable. We hope to tackle this problem in due course.

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CHAPTER FOUR

1. INTRODUCTION

Let us recall that PATON's investigation in a potential model showed that whereas in the weak binding limit the DF - method is satisfactory, as the binding becomes stronger (approaching the realistic case of strong binding) the DF method gives very inaccurate results, even when the second and third order Born terms are included. The model considered was a particle in an exponential potential: When the interaction became almost strong enough to produce a second s - wave bound state, then even a combination of first and second Born terms proved to give the wrong signs.

The purpose of the present chapter is to propose a rather different method of treating perturbed dispersion relations and test it in similar circumstances to those in PATON's calculation. We work entirely in the framework of the usual N/D method but, in contrast to the DF - method, assume a perturbation in both N and D functions, caused by perturbations $\delta\rho$ in the kinematic factor and δB in the driving term. We derive an integral equation for δN and δD, and calculate the mass shift δs_B from these. Our method lacks the elegance of the DF - method, but has the considerable advantage of giving correct answers in the potential model considered.

In § 2 we describe our method with a description of the matrix inversion method used to solve the N/D integral equations, which is shown to be equivalent to (and more convenient) than the Pagels method of solving N/D equations/ /.

In § 3 we give the results of the model calculation, and in § 4 the problem of infrared divergence is discussed within the context of results of § 3. and some numerical work on the problem is reported.

The rest of the chapter contains an application of the modified Pagels procedure to generate the Nucleon as well as Ω^- trajectories using N/D equations.

2. INTEGRAL EQUATIONS FOR δN, δD, AND δS_B.

The unperturbed N- and D- functions are given by

$$N(S) = B(S) + \frac{1}{\pi}\int_R \frac{SB(S) - S^1B(S^1)}{S^1(S - S^1)}\rho(S^1)N(S^1)\,dS^1 \quad , \qquad (2.1)$$

$$D(S) = 1 - \frac{S}{\pi}\int_R \frac{\rho(S^1)N(S^1)}{S^1(S^1 - S)}dS^1 \quad , \qquad (2.2)$$

where the symbols have their usual meaning. If we apply a perturbation $B \rightarrow B + \delta B$, $\rho \rightarrow \rho + \delta\rho$

$$N(S) + \delta N(S) = B(S) + \delta B(S) + \frac{1}{\pi}\int_R \frac{S(B(S) + \delta B(S)) - S^1(B(S^1) + \delta B(S^1))}{S^1(S - S^1)}$$

$$\times\ (\rho(S^1) + \delta\rho(S^1))(N(S^1) + \delta N(S^1))dS^1, \qquad (2.3)$$

$$D(S) + \delta D(S) = 1 - \frac{S}{\pi}\int_R \frac{(\rho(S^1) + \delta\rho(S^1))(N(S^1) + \delta N(S^1))}{S^1(S^1 - S)}\,dS^1 \qquad (2.4)$$

Subtracting Eq.(2.1) from Eqs. (2.3) and (2.2) from Eq.(2.4) and, neglecting terms of order $\delta B \;.\; \delta N$ we find:

$$\delta N(S) = \delta B(S) + \frac{1}{\pi}\int_R \frac{(SB(S) - S^1B(S^1)) + (S\delta B(S) - S^1\delta B(S^1))}{S^1(S - S^1)}(\rho(S^1)$$

$$+ \delta\rho(S^1))\ \delta N(S^1)dS^1 + \frac{1}{\pi}\int_R \frac{S\delta B(S) - S^1\delta B(S^1)}{S^1\ (S - S^1)}(\rho(S^1) + \delta\rho(S^1))N(S^1)dS^1 \qquad (2.5)$$

$$D(S) = -\frac{S}{\pi}\int_R \frac{\rho(S^1)\delta N(S^1) + \delta\rho(S^1)N(S^1)}{S^1\ (S^1 - S)}dS^1\ , \qquad (2.6)$$

and finally the mass shift is given by

$$\delta S_B = \frac{\delta D}{dD/dS)}\Big|_{S = S_B}$$

These equations are much more complicated than the comparable DF equation.

$$\delta S_B \doteq \frac{1}{R\left[D^{1}(S_B)\right]^2}\int_L \frac{D(S^1)^2\cdot \mathrm{Im}\,\delta A(S^1)}{S^1 - S_B}\ \frac{dS^1}{\pi}$$

but they have three advantages: firstly our integrals are over the right hand cut, whereas the DF case the integrals are over the left (which is generally more complicated), secondly our method explicitly unitarises the perturbed amplitude, which we feel is important for the case of electromagnetic perturbations, and thirdly higher order perturbations may be simply included since Eqs. (2.5) and (2.6) are in principle exact.

We solve the N/D equations by a modified form of Gaussian quadrature. The N- equation is written in the approximate form

$$N(S) = B(S) + \sum_{i}^{n} \frac{a_i B(a_i) - SB(S)}{S(a_i - S)} c_i N(a_i), \qquad (2.7)$$

where we have included the effect of the kinematic factor in the Gaussian weights: a specific example of how to choose the weights and positions is given in section 3 below. This is equivalent to a form used by most workers in the field, and is usually solved by putting $S = a_j$ and solving for $N(a_i)$ by matrix inversion. Eq.(2.7) is clearly equivalent to the form suggested by Pagels/4/, with the advantage that the c_i and a_i can be easily calculated for any order, instead of being empirically fitted (which is incidentally, numerically a very unstable procedure for more than two points).

3. APPROXIMATION METHOD AND RESULTS

In the non-relativistic case, the integral in Eq.(2.1) may be written

$$I(S) = \frac{1}{\pi} \int_0^\infty \sqrt{\frac{1}{S^1}}\, F(S^1, S)\, dS^1.$$

substituting $S^1 = (1 + x)/(1 - x)$ we can convert it to an integral from -1 to 1, which may be evaluated by Tchebycheff quadrature, yielding, after some algebra, a form like Eq.(2.7) with

$$a_i = \frac{1 + x_i}{1 - x_i}, \qquad \text{where} \qquad x_i = \cos\left(\frac{(2i - 1)\pi}{n}\right)$$

$$c_i = \frac{2}{n(1 - x_i)} \qquad (3.1)$$

We specialize to the case where $\delta\rho = 0$: i.e. the only change comes from the Born term. This leads to the related equations

$$D(S) = 1 + i\sqrt{S}\,N(S) + \sum_{j=1}^{n} \left(\frac{N(a_j) - N(S)}{a_j - S}\right) c_j \qquad (3.2)$$

$$\delta N(S) = \delta B(S) + \sum_{j=1}^{n} \left(\frac{SB(S) - a_j B(a_j)}{S - a_j}\right) c_j \delta N(a_j) + \sum_{j=1}^{n} c_j \left(\frac{S\delta B(S) - a_j \delta B(a_j)}{S - a_j}\right) N(a_j), \qquad (3.3)$$

$$\delta N(S) = \delta B(S) + \sum_{j=1}^{n} \left(\frac{SB(S) - a_j B(a_j)}{S - a_j}\right) c_j \delta N(a_j), \qquad (3.4)$$

$$D(S_0) = - \sum_{j=1}^{n} \frac{c_i^1 \delta N(a_i)}{a_i - S_0}. \qquad (3.5)$$

The model we treat initially is an exponential well, for which the L.H. cut degenerates to a series of poles and the solutions are well known. We summarize the results below, using PATON's notation:

If

$$V(r) = ae^{-\mu r},$$

then

$$D(S) = \exp\left(\frac{ik}{\mu}\right) \ln\left(\frac{-a}{\mu^2}\right) \Gamma(-2ik - 1)\, J_{-2ik/\mu}\left(2\sqrt{\frac{-a}{\mu^2}}\right), \qquad k = \sqrt{S} \quad (3.6)$$

This has zeros(corresponding to bound states) when

$$J_{-2ik/\mu}\left(2\,\frac{-a}{\mu^2}\right) = 0 \qquad (3.7)$$

with the wave function

$$\phi(r) \quad = \quad J_{-2ik_B/\mu}(2\frac{\sqrt{-a}}{\mu^2}\exp 9(-\frac{r}{2})\ ) \qquad (3.8)$$

Hence the lowest order, the mass shift with a perturbing potential $\delta V(r)$

$$\delta S_B \quad = \quad \frac{\int_0^\infty J_x^2(be^{-\frac{1}{2}r})\ \delta V(r)\,dr}{\int_0^\infty J_x^2(be^{-\frac{1}{2}r})\,dr}, \qquad (3.9)$$

where $b = 2\sqrt{-a/\mu^2}$ and $x = -2ik_B$.

To check the basic numerical method, we compared the solution derived from Eq.(3.2) etc. with the exact solution for a two pole input.

Writing $B = g_1/(S + m_1)$, $\delta B = g_2/(S + m_2)$, the error in δN from Eq.(3.3) is about 0.3% when $n = 5$, and falls slowly as n increases.

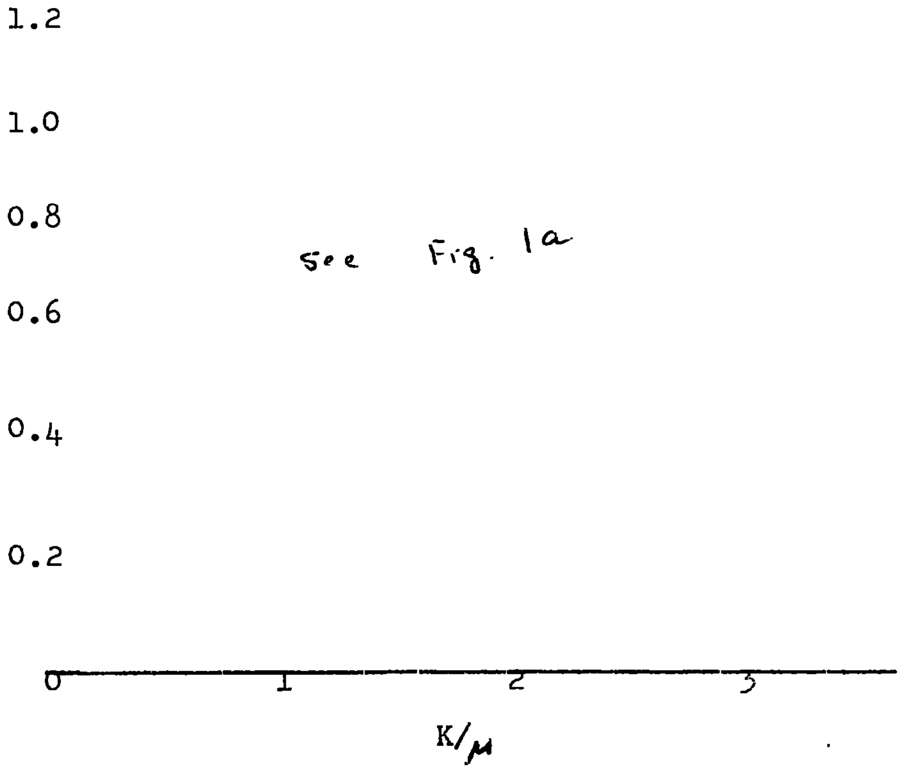
Turning to the case of an exponential potential, it is known that

$$B(+ S) \quad = \quad 2\pi \sum_{r=1}^{\infty} \frac{(-1)^r a^r}{r\ (r-1)!(4S + (\mu r)^2)} \qquad (3.10)$$

is the so called "Born term" which in this case exactly describes the interaction. A similar expression is used to give $\delta B(S)$ from $\delta V = \delta a e^{-Kr}$. A further check on the accuracy of the method is given by the erro in the unperturbed bound state energies: as similar accuracy to that above was found.

For a range of values of the (dimensionless) parameter K/μ the mass shifts $\delta S_B/ a$ were computed using eqs.(3.3) and (3.4).

In Eq.(314) we



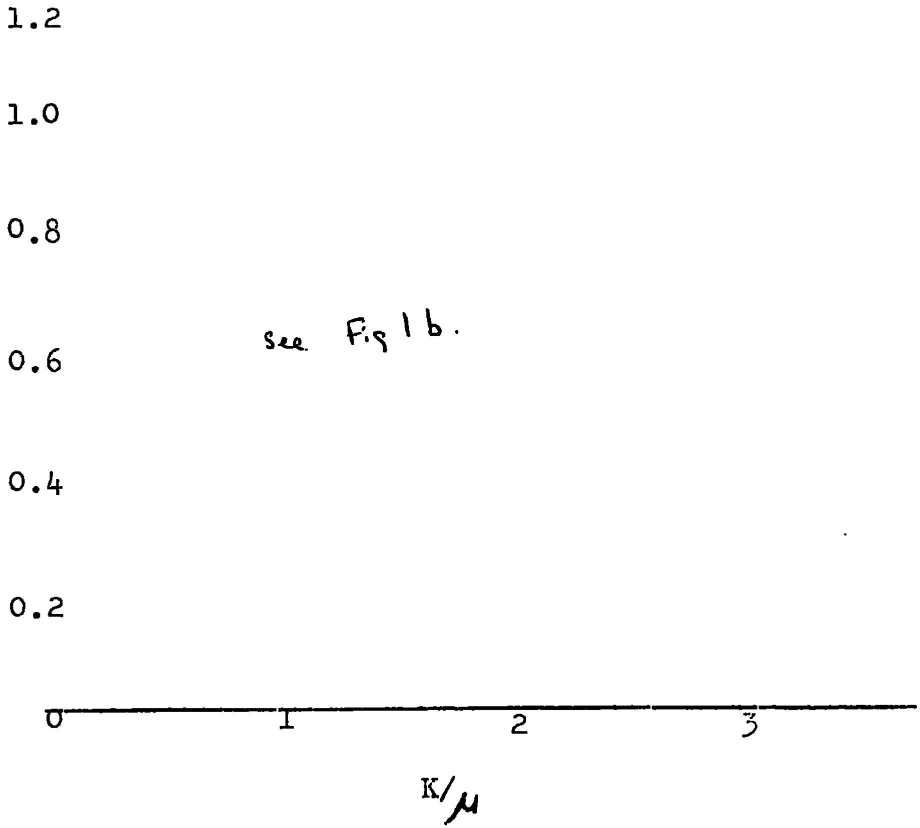


Fig. 1 Comparison of the mass shift δS_B for the perturbation δa exp (-Kr) using Eq.(3.4)(curveII) with the standard result (Curve I):

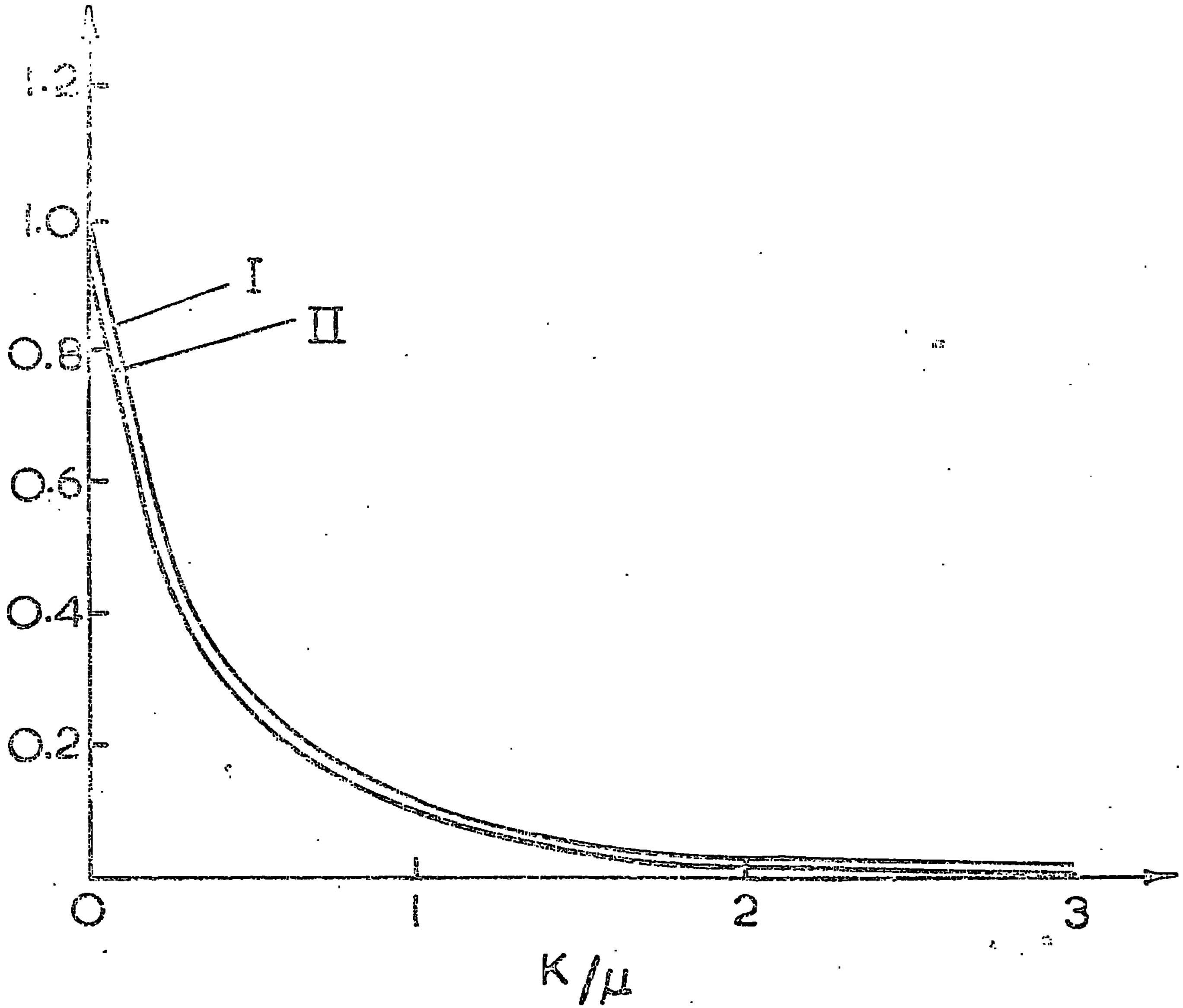


Fig.1a.

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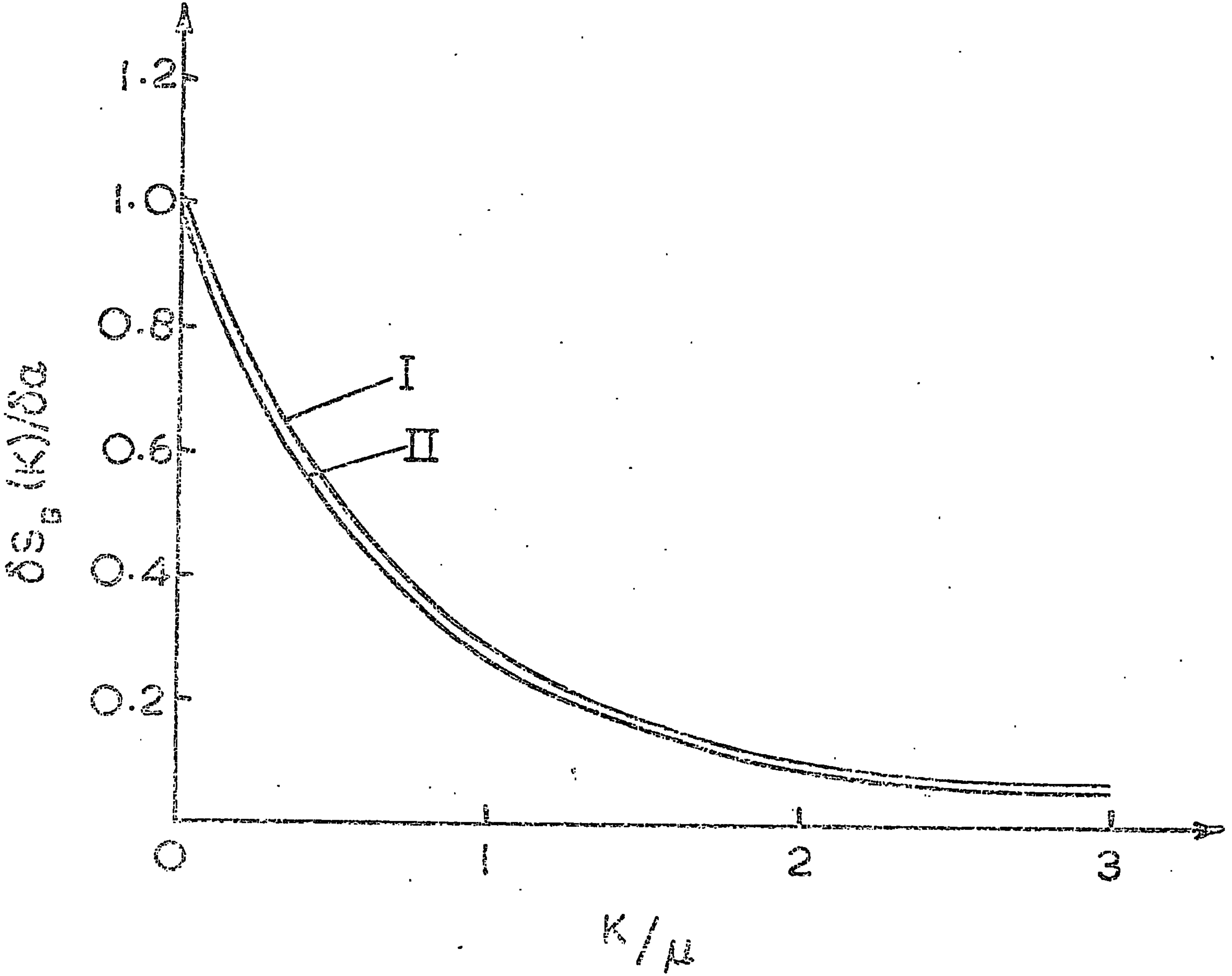

Fig. 1b.

256 Fig 1b

(a) binding energy $S_B = \mu^2$; (b) binding energy $S_B = \mu^2/16$.

made a further approximation in ignoring the third term in Eq.(3.3): in other words we made a determinantal approximation in line with the spirit of perturbation theory. Our results from Eqs.(3.3) and (3.4) are compared with some of Paton's in Table 1,, and it can be seen that they are verymmuch better : of course,, this is not s surprising, because we have employed an infinite series of terms to represent the input, whereas Paton only uses the first three Born terms. As the ratio K/μ decreases, it is necessary to increase the summation from $n = 6$ for $K/\mu = 0.1$ to n 11 for $K/\mu = 0.001$. The results are perfectly stable up to $n = 20$.

TABLE 1

S_B	μ^2	$\frac{1}{16}\mu^2$
DF estimates, third Born	34	94
Eq.(3.3)	99	99.3
Eq.(3.4)	97.7	98

Comparison of DF estimates of δS_B for the perturbation δa exp (-Kr), following Paton/3/, with estimates using eq.(3.3) and (3.4), respectively, with the input eq.(3.10). The numbers are percentages of the standard result.

For a perturbing potential of Yukawa form

$$\delta V(r) = \alpha M \pi \frac{e^{-Kr}}{r} \qquad (M = 100),$$

we are forced to consider only the first Born term. In the limit $K \longrightarrow 0$, this goes over to a Coulomb potential, which is of course our basic interest. In this limit our method fails; however we hope that for K small but finite we may obtain not unsatisfactory results. In this case

$$\delta B_0(S) \quad = \quad \frac{\alpha M \pi}{2S} Q_0 \left(1 + \frac{K^2}{2S}\right) \tag{3.11}$$

where the suffix emphasises that this is only the lowest order Born term: higher orders would improve the accuracy, but we cannot obtain them in closed form. As can be seen from Table 2, and figs. 2a, 2b results are satisfactory for K 0.03: we note that the first order potential theory result does not change very greatly between $K = 0.3$ and $K = 0.01$. Again numerical consistency was achieved for $n = 6$ for $K > 0.5$ to $n = 12$ for $K < 0.03$.

TABLE 2

S_B	μ^2	$\frac{1}{16}\mu^2$
DF estimates, third Born	58	94
Eq.(3.3)	98.9	99.1.
Eq.(3.4)	97	97.3

Comparison of DF estimates of δS_B for the perturbation δa exp $(-Kr)/r$, following Paton/3/, with estimates using Eqs.(3.3) and (3.4), respectively, with the input Eq.(3.11)($K > 0.03$)

6

5

4

Fig 2a
attached

3

2

1

0 1 2 3

K/μ

2.6

2.0

Fig 2b
attached

1.0

0 1 2 3

K/μ

Fig 2

Comparison of the mass shifts δS_B for the perturbation δa exp (-Kr)/r using Eq. (3.4)(Curve II) with the standard result (Curve I): (a) binding energy $S_B = \mu^2$; (b) binding energy $S_B = \mu^2/16$.

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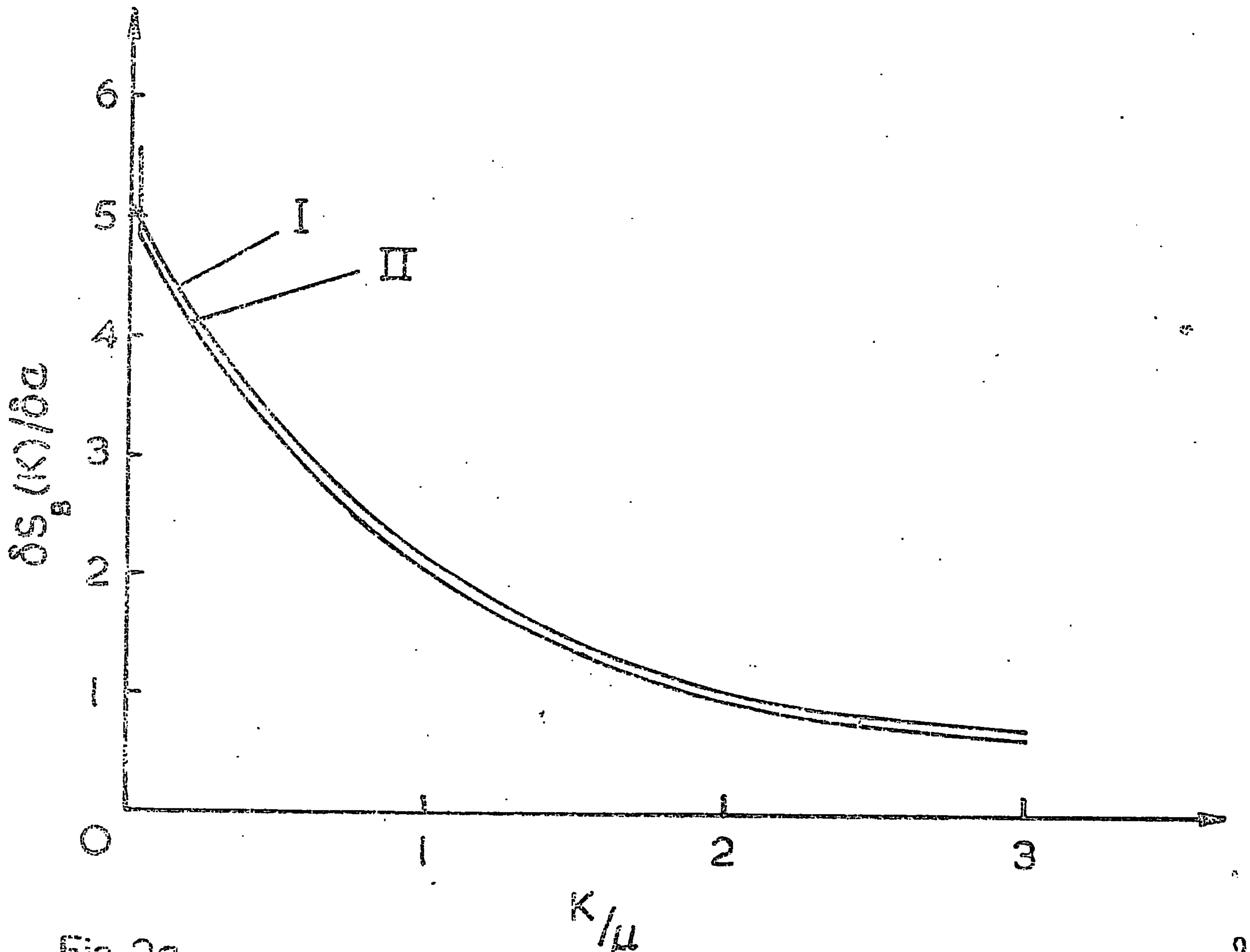

Fig. 2a.

256 Fig 2a

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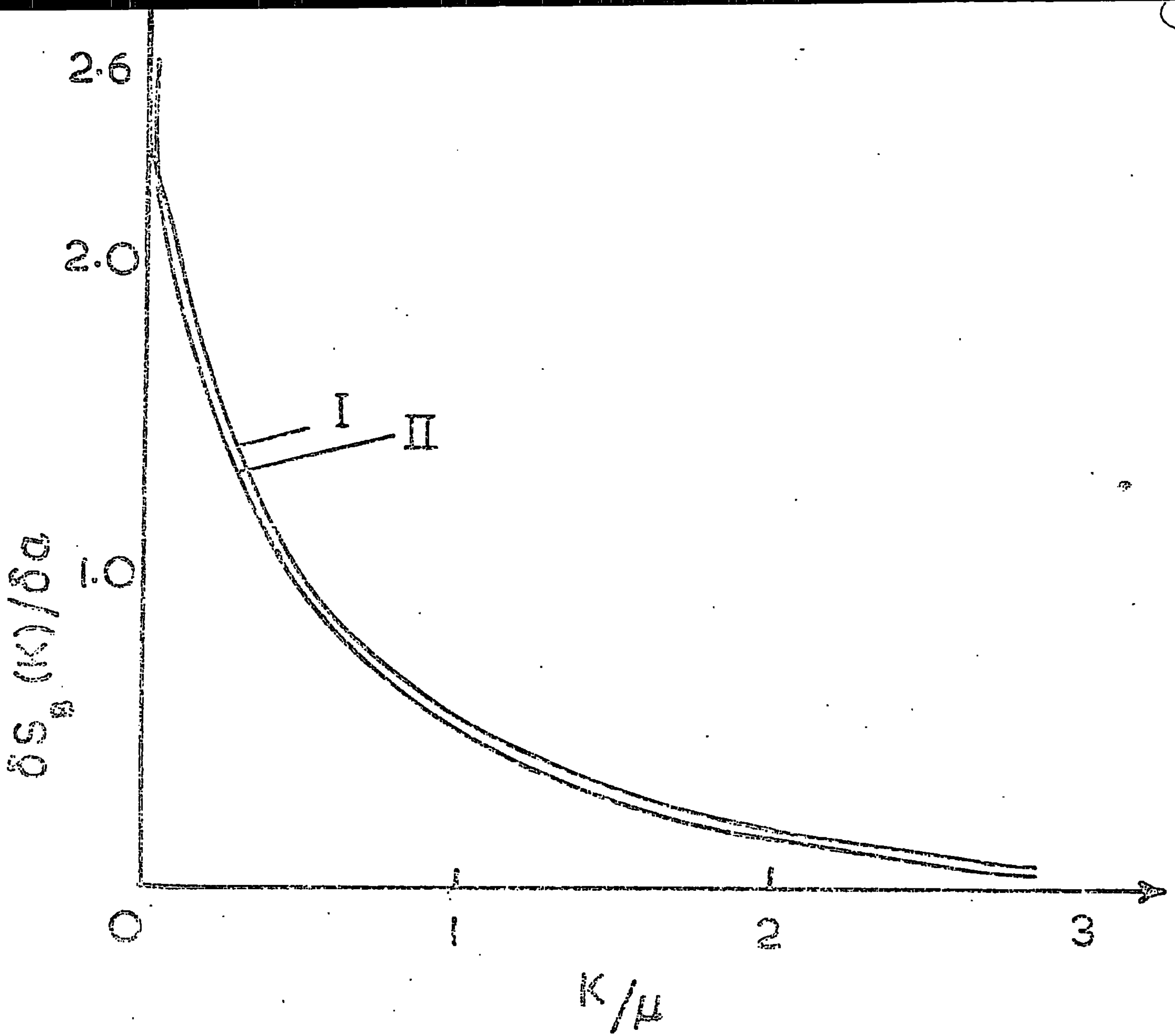

Fig. 2b.

256 Fig 2b

4. INFRA-RED DIVERGENCE

The infra-red divergence problem in the DF model has been the cause of considerable concern. We propose the simplest conceivable prescription: that it should be ignored. In other words the photon should be given the smallest finite mass consistent with numerical stability. This has a number of embarassing problems: in particular a realistic photon has spin and giving it mass introduces a helicity zero component. However, the results of sect. 3 suggests that in the scalar case the approximation is not bad. In this section we investigate the consequences of this assumption further.

Halpern and Rix(HR) (5) have obtained, by an elegant and exact method, a solution of the one-photon exchange N/D equations. As one would expect, the D-function develops an infinity of zeros to cancel the pathological behaviour of the input near threshold: this represents the infinite number of bound states which occur in the model. The central point here is the enforcement of unitary on the solution, which forces a finite solution despite a Born term which is infinite everywhere.

It must be emphasised that the Coulomb scattering problem is genuinely divergent in the following sense: the D-function really does contain an infinitude of zeros and the S-matrix has an essential singularity at threshold. To obtain these features in a dispersion relation approach it is clearly necessary to start with a singular input (see, ref (5), Eq.(3)}:

$$\mathrm{Im}\ B_{\ell}^{(1)}\ (S) = \frac{\alpha M \pi}{2S} P_{\ell}(1 + \frac{\lambda^2}{2S}) \quad 0 \quad (-S - \frac{\lambda^2}{4}) \qquad (4.1)$$

It must be admitted that the solution is not totally satisfactory, as the one photon exchange term does not reproduce the Coulomb force in its entirety.

The DF method is akin to the determinantal approximation in the HR equations. The method here proposed at least has the advantage of being demonstrably finite, but apparently suffers from two serious flaws: first we cannot hope to reproduce an infinitude of electromagnetically bound states by our rather crude approximation, and secondly the integral equation for N does not exist in the limit $\lambda \rightarrow 0$. We write

$$D(S) = 1 - \left\{ \int_{-\infty}^{-a} + \int_{-a}^{0} \left[\frac{\mathrm{Im}B(S^1)D(S^1)}{\sqrt{-S^1} + \sqrt{S}} \, dS^1 \right] \right\} \qquad (4.2)$$

where a is a small positive quantity. In the calculation of the neutron-proton mass difference, one is interested only in the behaviour of D(S) (in fact $\delta D(S)$) near the mass of the bound state, which lies wellbelow threshold; The lowest electromagnetically bound state lies about 1 MeV below threshold, while the proton (presumed to be a πN bound state) lies 137 MeV below threshold. Hence, although the second part of the integral in Eq.(4.2) has a somewhat peculiar behaviour as ImB blows up and D oscillates more violently near $S = 0$, we may hope that the net effect on D(S) with S large and negative may be negligible. The prescriptions of leaving the photon mass finite, or including a cutoff are essentially equivalent.

To check this idea, we compare the HR solution in the limit of large negative S

$$D(S) \xrightarrow[S \rightarrow -\infty]{} 1 - \frac{\alpha M}{\sqrt{-S}} \ln \left(\frac{\sqrt{(-S)}}{\alpha M} \right) + 1 - \ln 2\pi \qquad (4.3)$$

with our massive photon exchange solution for $S = -100$. As can be seen from table 3, the results are not unreasonable. As $\lambda \rightarrow 0$

instability is setting in, but for a value of λ = 0.50 or larger the approximate calculation agrees to within about 3 - 5%. (Note that the figures are rather worse than they appear, as we ought to be comparing $D_{HR}(S) -1$ with $D_{app}(S) - 1$). This is reasonable: although a photon mass of 0.50 sounds large, it is still a very long-range perturbation compared with the mass of the bound state.

TABLE 3

Eq.(4.3)	Eq.(4.1)	λ
0.9974	0.9406	0.01
	0.9903	0.50
	0.9933	1.00
	0.9971	1.50
	0.9976	1.75
	0.9986	2.50
	0.9991	3.50
	0.9993	4.00

Comparison of HR D-function from Eq.(4.3) with the approximate D-function using Eq.(4.1) as input as $\lambda \rightarrow 0$. (S = - 100, M = $\frac{1}{2}$).

An alternative method of handling the infra-red divergence problem has been proposed by Squires, Poston and ~~one of~~ the present author~~s~~. See chapter 6.

To conclude, we have proposed and investigated a method for evaluating perturbed dispersion relations. The method is very satisfactory for short range forces, and gives reasonable answers for long range (i.e. Coulombic)interactions. We intend to investigate the model further in a more realistic strong-interaction model.

~~We would like to thank Professor E.J. Squires, Dr. J. Paton and J.R.Poston for much advice and many comments~~.

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CHAPTER FIVE

1. APPLICATION OF MODIFIED PAGELS METHOD TO OBTAIN NUCLEON AND Ω TRAJECTORIES.

In this chapter the Pagels method in the modified form given in the last chapter is used to solve the N/D equations in an attempt to generate

i) the nucleon trajectory in the πN system from N, N^* and ρ exchanges in the crossed channels,

and

ii) the $\Omega^{(-)}$ trajectory in the $\bar{K}\ \Xi$, system with Λ and $\bar{\Sigma}$ exchange in the u-channel as input.

It is obvious that our calculation is not of any intrinsic interest since it is essentially a fixed-spin exchange calculation. In a realistic calculation one would have to use the strip approximation of CHEW and JONES/1/, with full Reggeization and with the input left hand cut derived from the leading trajectories in all three channels. In view of the ad-hoc nature of our calculation, the strip-width parameter, which in the full calculation marks the transition from direct channel resonance dominance to crossed channel Regge dominance, is here simulated by the cut off W_1. The continuation of the partial wave amplitude for complex ℓ values is defined through the Froissart - Gribov projection formula. The presence of the u channel leads to two different continuations corresponding to odd and even integral values of ℓ. The amplitudes for which the correct N/D separation can be made is obtained by writing the partial wave amplitude as

$$B_\ell (W) = kf_{\ell\pm}(W) \; / \rho_\ell (W) \qquad (0)$$

where

$$\rho_{\ell_{\pm}}(W) = (E \pm) \left(\frac{k}{W}\right)^{2\ell \pm 1}$$

with

and f_ℓ is the partial wave amplitude given by

$$f_\ell(W) = \frac{e^{i\delta_\ell} \sin\delta_\ell}{k}$$

Notation $\ell\pm$ coresponds to $j = \ell \pm \frac{1}{2}$, $s = W^2$ is the square of the centre of mass energy, and k^2 the centre of mass momentum

$$k^2 = \frac{\left[(W + M_i)^2 - \mu_i^2\right]\left[(W - M_i)^2 - \mu_i^2\right]}{4W^2}$$

The index i labels the relevent channel, πN or $\bar{K}\ \Xi$, respectively and $\mu = m_\pi$ or m_K respectively. For brevity we are treating

both πN and $\bar{K}\ \Xi$ systems on one footing, i.e. notations are interchangeable, except for $j = \ell \pm \frac{1}{2}$. The dispersion relation for the ℓ-the partial wave amplitude is given by

$$B_\ell(W) = B_\ell^V(W) + \frac{1}{\pi}\int_{-W_1}^{W_0} \frac{dW^1 \ \mathrm{Im}\, B_\ell(W^1)}{W^1 - W} + \frac{1}{\pi}\int_{W_0}^{W_1} \frac{\mathrm{Im}\, B_\ell(W^1)}{W^1 - W}\, dW^1$$

Here W_0 is the threshold energy in the relevant channel. B_ℓ^V is the so-called generalized potential obtained from crossed channel exchanges. As mentioned the object here is to simulate a strip-approximation calculation by fixed spin exchanges in the relevant channels with W_1, the cut off, simulating the role of strip width.

2. MODIFIED PAGELS TYPE APPROXIMATION/2/

We will not repeat the full description of Pagels method except to remark the following. In the strip approximation type calculation the equation for N is singular. The usual method separates the singular part of the kernel leading to a Wiener-Hopf type integral equation whose inhomogenous term satisfies a Fredholm equation. Here we solve the N/D equations by replacing the Pagels pole fitiing procedure by a Gaussian interpolation, and then solving the equations for N and D by matrix inversion.

On carrying out the procedure the N and D equations take the form

$$N_\ell(W) = B_\ell^V(W) - \sum_{i=1}^{n} C_i a_i N(a_i) \left[\frac{W B_\ell^V(W) - a_i B_\ell^V(a_i)}{W - a_i} - \frac{W_0 B_\ell^V(W)}{W_0 - a_i} \right] \qquad (1)$$

$$D_\ell(W) = 1 + W_0 F_\ell(W_0)\, N_\ell(W_0) - W F_\ell(W)\, N_\ell(W) + \sum_{i=1}^{n} C_i \left[\frac{W}{W - a_i} \left\{ W N_\ell(W) - a_i N_\ell(a_i) \right\} - \frac{W_0}{W_0 - a_i} \left\{ W_0 N_\ell(W_0) - a_i N_\ell(a_i) \right\} \right] \qquad (2)$$

The function $B_\ell^V(W)$ giving that part of the amplitude containing the cuts which lie outside the strip is defined by Eq.(a). The C_i and a_i are the parameters for pole-fitting of the function

$$F(x) = \frac{1}{\pi} \int_R \frac{\rho_\ell(W^1)\, dW^1}{W^2 (W^1 - X)} = \sum_{i=1}^{n} \frac{C_i}{X - a_i}, \qquad (A)$$

and are the same as in Chapter 4 . The right-hand cuts cover the strips $-W_1 < W < -W_0$ and $W_0 < W < W_1$, W_0 being the threshold energy in the relevent channel and W_1 the cut off. $D_\ell(W)$, outside the cuts is given by

$$D_{l}(W) = 1 - \sum_{i=1}^{n} C_i \, a_i \, N(a_i) \left[\frac{W}{W - a_i} - \frac{W_0}{W_0 - a_i} \right] \qquad (3)$$

By putting $W = a_i$ in Eq.(1) we get a set of simultaneous equations in $N(a_i)$ which are then solved by matrix inversion.

We now treat the Nucleon and Ω^- trajectories separately.

BALL and WONG/3/ in 1963 used PWDRs(partial wave dispersion relations) to obtain integral equations for the PWAs (partial Wave amplitudes) for πN scattering, using interaction terms arising from the exchange of a Nucleon, the N^* and ρ- meson. Adjusting the value of the cut off to produce N^* at correct energy, they found a bound state, the Nucleon in the p- wave, $I = J = \frac{1}{2}$ amplitude. The effect of varying the coupling constant was also treated.

In our case the crossed channel exchanges are obtained in the narrow resonance approximation. The relevant expressions were taken from FRAUTSCHI and WALECKA/4/ after correcting some misprints in their expressions

i) <u>U - CHANNEL: N AND N^*EXCHANGE</u>

Nucleon exchange in lth partial wave in $T = \frac{1}{2}$ channel is given by the expression:

$$g_{l-}^{N} = \frac{f^2}{4k^2} \left[-(E + M)(W - M) \, Q_{l}(y_1) + (E - M)(W + M) Q_{l-1}(y_1) \right]$$

where k^2 is defined earlier, $f^2 = 15$ the πN coupling constant,

$$y_1 = \frac{W^2 - M^2 - \mu^2}{2k^2} - 1,$$

M is the nucleon mass, and μ the pion mass. The subscript $l-$ means $j = l - \frac{1}{2}$ is the total angular momentum.

The contribution from N^* exchange is

$$g_{\ell}^{N^*} = \frac{M^2_{N^*}}{6k^4} \; \gamma^2_{33} \quad - (W+M)^2 - \mu^2 \left\{ \frac{3x_{N^*}(M_{N^*}+2M-W)}{(M^* + M)^2-\mu^2} + \frac{M_{N^*}-2M-W}{(M_{N^*} - M)^2-\mu^2} \right\}$$

$$\times \; Q_{\ell}(y_2) \Big] + \Big[(W-M)^2-\mu^2\Big] \left\{ \frac{3X_{N^*}(M_{N^*} + 2M + W)}{(M_{N^*} + M)^2 - \mu^2} + \frac{M_{N^*} - 2M - W}{(M_{N^*} - M)^2 - \mu^2} \right\}$$

$$\times \; Q_{\ell - 1}(y_2) \Big]\Big] \Big\}$$

where
$$y_2 = \frac{W^2 + M^2_{N^*} - 2M^2 - 2\mu^2}{2k^2} - 1 ;$$

and γ_{33} is a coupling parameter, determined by BALL and WONG from the experimental width of N^* .

ii) T - CHANNEL: ρ - EXCHANGE

$$g_{\ell}^{\rho} - = \frac{1}{16\pi W^2} \Big\{ - \Big[(W-M)^2 - \mu^2\Big] \Big[2\gamma (W-M) + \gamma_2 \, (4M(W-M) + 2W^2$$

$$- M_{\rho}^2 + 2M^2 - 2\mu^2) \quad \frac{Q_{\ell}^{(a_{\rho})}}{2k^2} \Big\} \quad - \Big\{ \Big[(W - M)^2 - \mu^2\Big] \times$$

$$(2\gamma_1 \, (W + M) + \gamma_2(4M(W + M) + 2s + M_{\rho}^2 - 2M^2 - 2\mu^2)) \quad \frac{Q_{\ell-1}^{(a_{\rho})}}{2k^2} \Big\}$$

where
$$a_{\rho} = 1 + \frac{M_{\rho}^2}{2k^2}$$

Here γ_1 and γ_2 are determined from the electromagnetic form factors of the nucleon and are taken from BALL and WONG's paper. They are given by

$$\frac{\gamma_1}{\gamma_2} \simeq \frac{M}{1.83}$$

iii) RESULTS

With

$$B_{\ell}^{V} = \frac{k}{\rho_{\ell}(W)} \left(g^{N}_{\ell-} + g^{N^{*}}_{\ell-} + g^{\rho}_{\ell-} \right)$$

and choosing the subtraction point at $W = W_0 = 0$, 32^{pole} terms we retained in the sum in Eq.(A).

The nucleon trajectory for 4 separate values of the cut off W_1 are illustrated in the figure 1 W_1 is in units of M_π^2

$\alpha_N(W)$	$W_1 = 10.0$	$W_1 = 10.5$	$W_1 = 10.8$	$W_1 = 11.$
.5	7.00	6.60	6.31	6.21
.6	7.50	7.18	6.81	6.77
.7	——	7.59	7.35	7.28
.8	——	——	——	7.68

$\alpha(W)$ is the position of the Regge pole for given W. It is seen that the nucleon pole comes closest to the physical mass, $6.60\, m_\pi^2$ for $W_1 = 10.5$.
"_____" denotes that the trajectory arises above threshold.

Ω^- - DYNAMICS

For this problem the most thorough work has been done by JOHNSON and KAHANE/5/. These authors generated Ω^- through a proper Reggerized calculation. We are only interested in seeing how well the truncated appraoch using Pagels-type solution method works. Following JOHNSON and KAHANE, we exclude Y^* (1385 MeV) exchange since this gives a repulsive contribution, whereas the exchange of a victor meson in the t channel appears to have no significant effect on the binding energy of Ω^-. Our only input consists of Σ and Λ

exchange in u channel. The kinematics is similar to πN system except for the presence of two isospin amplitudes, T = 0 and T = 1. We are interested only in T = 0 amplitude in the direct

$$\frac{f^2_{\Xi\Lambda K}}{4\pi} = 1.68$$ These renormalized coupling constants were taken from JOHNSON and KAHANE.

$$g^2_{\Xi\Sigma K}/4\pi = 14.0$$
$$f^2_{\Xi\Lambda K}/4\pi = 1.68$$

RESULTS

For various cut off values $\alpha_{\Omega^-}(w)$ was calculated. The results are tabulated below and illustrated in the Figure 2.

W_1 is in Kaon mass unit

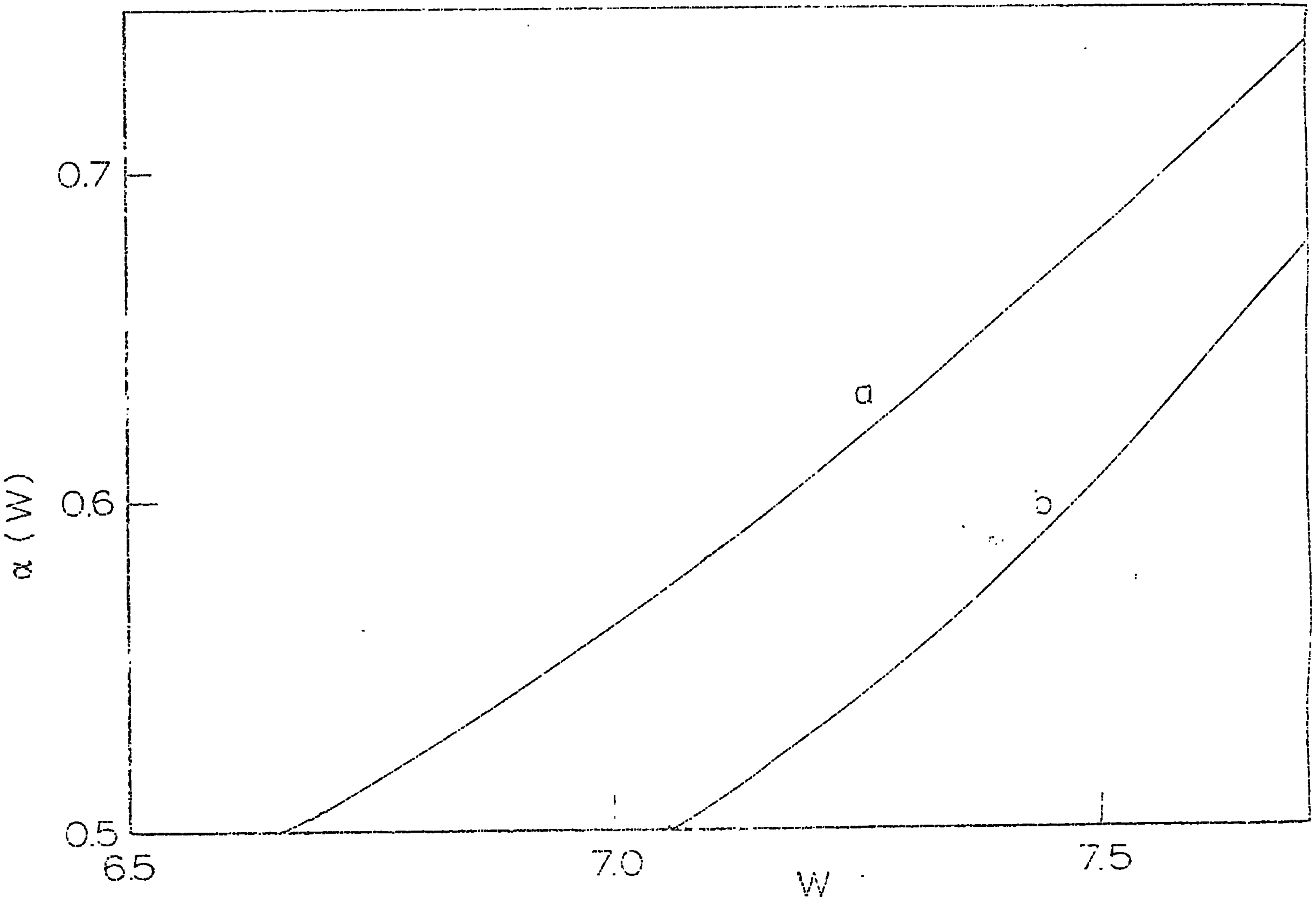
$\alpha_{\Omega}(W)$	$W_1 = 7.25$	$W_1 = 7.5$	$W_1 = 7.75$	$W_1 = 8.0$
1.2	2.92	2.78	2.64	2.50
1.3	3.02	3.09	2.98	2.85
1.4	3.45	3.35	3.25	3.15
1.5	3.64	3.56	3.48	3.40
1.6	——	——	3.67	3.60

$\alpha_{\Omega}(W)$ stands for the position of the Regge pole for given W. The Ω^- mass comes closest to physical mass for $W = 8.0, \alpha_{\Omega} = 3.3$

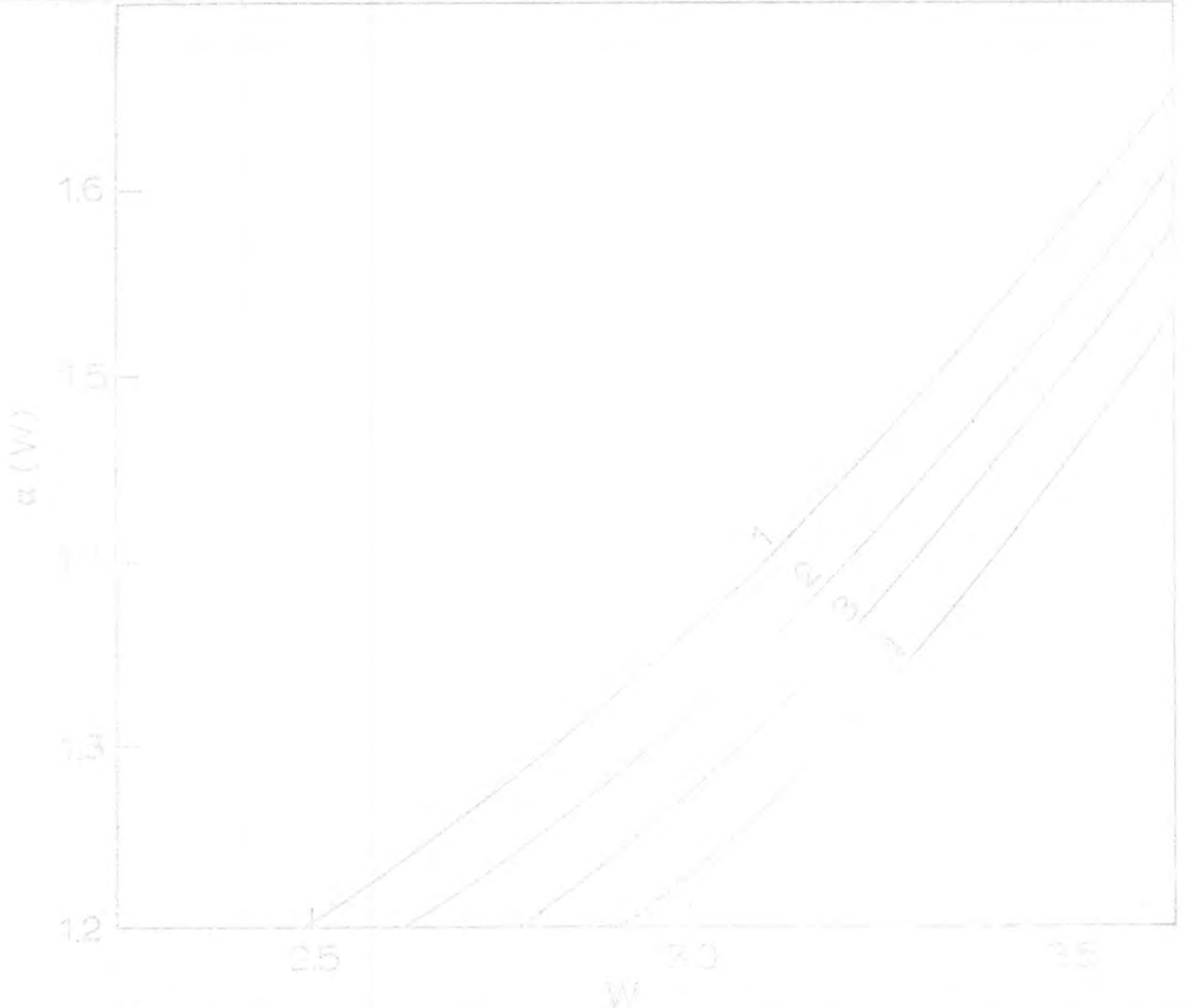
CONCLUSIONS

It is clear that the approximation method of replacing Pagels pole-fitting technique by straightforward Gaussian interpolation is easy to use and appears to generate reasonable trajectories. The results are off by about 10 - 15% from the "honest" calculations of BALL and WONG and JOHNSON and KAHANE, respectively. Our object was to see if it is feasible to reggerize the direct channels with unreggerized input in crossed channels.

The results are not too bad though of little significance for any deep insight into the dynamics.



NUCLEON TRAJECTORY $\alpha(W)$ for cut-off a) W_1=10.6, b) W_1=10.1



Ω^-- TRAJECTORY FOR CUT-OFF : 1) W_1=8.1, 2) W_1 =7.7;3) W_1=7.5; 4) W_1 =7.3

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CHAPTER SIX

1. Introduction

In 1964 DASHEN /2/ attempted to calculate the neutron-proton mass difference due to electromagnetic effects, using a perturbation technique developed for the N/D method by DASHEN and FRAUTSCHI /2/, and discussed in Chapter 2. The technique suffers from the defect that it diverges for infinite range (e.g. Coulomb) forces and so some method of introducing a cut-off had to be introduced. BARTON /3/ and PATON /4/ showed that this introduced such a large measure of uncertainty into the calculation as to make DASHEN's result meaningless.

It is the purpose of this note to discuss a particular method, due to SQUIRES /1/, of removing this divergence and to test it in a situation where exact results are available, namely potential scattering. We find that in the cases considered the method works well, and agree with the exact results for a wide range of binding energies.

The method consists in the inclusion of box-diagram contributions to the left-hand cut and arranging for the total infrared contributions to cancel with the help of a fudge factor, the exact details of which are worked out in Sec. 3 and the Appendix.

2. The DASHEN-FRAUTSCHI method

We consider the non-relativistic scattering of two spinless particles by a Yukawa potential $-g^2 \frac{e^{-\mu r}}{r}$. We suppose that there is an S-wave bound state with binding energy $-s_B$, and attempt to calculate the first-order change in s_B due to the perturbing Yukawa potential, $-e^2 e^{-\lambda r}/r$ which becomes a Coulomb potential when $\lambda \to 0$.

We use the N/D method and write the unpertubed s-wave scattering amplitude as N/D:

$$a(s) = \frac{N(s)}{D(s)}, \tag{1}$$

where

$$D(s_B) = 0 \tag{2}$$

the expression for the first order change in s_B, was derived earlier and is

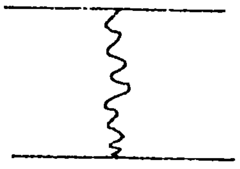
$$\Delta s_B = \frac{1}{R(D^1(s_B))^2} \int_L \frac{D(s^1)^2 \mathrm{Im}\Delta a(s^1)}{(s^1 - s_B)} \frac{ds^1}{\pi}, \tag{3}$$

where R is the residue of the pole in a(s) at $s = s_B$, and $\mathrm{Im}\Delta a(s^1)$ is the first order change in Im a due to the perturbation interaction; the integration is over the region of the real axis where $\mathrm{Im}\Delta a$ differs from zero.

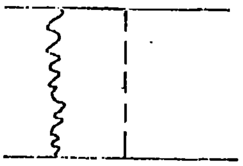
Now, Eq.(3) is an exact expression for the first order mass shift and as such must be finite for any value of λ, i.e. it is equal to the more usual expression

$$\Delta s_B = \int |\Psi|^2 \frac{\cancel{e} -e^2 e^{-\lambda r}}{r} dr, \tag{4}$$

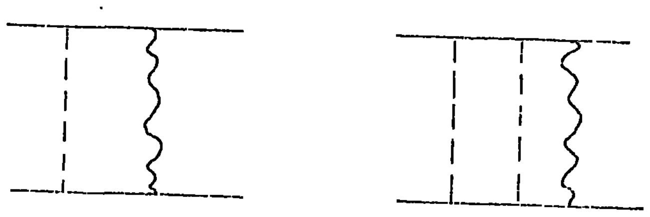
where Ψ is the bound state wave function. However, in applications of (3) it is usually necessary to make approximations; in particular, in general one will not have an exact solution for the unperturbed problems so D(s) will not be known exactly, and, more seriously, it is usually necessary to approximate $\mathrm{Im}\Delta a$. In fact $\mathrm{Im}\Delta a$ has contributions from diagrams of the type shown in fig.1., all of which are of lowest (first) order in the perturbing interaction .



(a)



(b)



(c) (d)

Fig 1. showing some of the diagrams which contribute to $\mathrm{Im}\Delta a(s)$. The dashed line represents the unperturbed interaction and the wavy line the perturbation. The previous calculations have just included (a); here we include also (b) and (c).

Fig. 1a gives a LH cut starting at $s = -\frac{1}{4}\lambda^2$, figs. 1b and 1c give a cut starting at $s = -\frac{1}{4}(\lambda+\mu)^2$, other cuts start at $-\frac{1}{4}(\lambda + 2\mu)^2$, etc. Since it has not been found possible in general to sum all diagrams of the form of fig. 1. it is necessary to make some approximation, and it is clearly necessary that such an approximation conserves the property that (3) is finite as $\lambda \rightarrow 0$. However, Dashen's original calculation(1) ignored all diagrams other than fig 1a which contribute $e^2\pi/2s$ to $\mathrm{Im}\Delta a(s)$ in the region $-\infty < s < -\frac{1}{4}\lambda^2$, and hence leads to a contribution to Δs_B given by

$$\Delta s_B(a) = \frac{-e^2}{R(D^1(s_B))^2} \frac{D^2(0)}{s_B} \log\lambda + O(1). \qquad (5)$$

Since, in general $D(0) \neq 0$, we see that this is infrared divergence ($\lambda \rightarrow 0$) and the approximation of taking only fig. 1a fails badly for long range forces.

We therefore try including in addition the contribution of the box diagrams, figs. 1b and 1c. Our hope that this might lead to a significant improvement is strngthened by the work of Lumming/5/ and of Collins and Johnson/6/, who found that the use of just single-particle exchange diagrams for left hand cuts is always a

bad approximation when there is an s-wave bound state, but that inclusion of the box diagrams gives a good result for a wide range of interactions. This hope is confirmed by the results we present below.

3. INCLUSION OF BOX DIAGRAMS

The method of calculating the contribution of the diagrams was first given by Mandelstam/7/ and we give the details elsewhere. It is worth noting however that the infrared divergence now arises already in $\mathrm{Im}\Delta a$ and not from the integration over $\mathrm{Im}\Delta a$ in Eq.(3). In fact the divergent part of $\mathrm{Im}\Delta a$ is given by

$$\mathrm{Im}\Delta a^{(b,c)} = \frac{-e^2g^2}{4s\sqrt{-s}}\pi\log\lambda + \theta(1), \qquad -\infty < s < -\frac{1}{4}\mu^2 . \qquad (6)$$

When we put this into (3) and integrate, the $\log\lambda$ term partially cancels with that given in Eq.(5). Of course, we cannot in general expect that there will be complete cancellation since we have still not kept all terms of the left hand cut. However, the coefficient of $\log\lambda$ will certainly be smaller, see below, so the result will not be so sensitive on the cut off.

Alternatively we suggest that one could use the knowledge that the coefficient of the $\log\lambda$ term should be zero to remove some of the other uncertainties. There are two possible approaches here:

i) Since our left hand cut (including figs. 1a, 1b, and 1c) is correct down to $s = -\mu^2$ (for $\lambda = 0$) we could multiply its contribution by a factor which is essentially unity for $-\mu^2 < s < 0$ but which permits some deviation for $s < -\mu^2$. This deviation, containing some free parameter, would account for the effect of higher order terms (in g^2) in $\mathrm{Im}\Delta a$. The free parameter could be determined by the requirement that the coefficient of $\log\lambda$ in s_B be zero, and with this value of the parameter we could evaluate the finite contribution uniquely. Here we use the factor

$$f(c,s) = \frac{(1 + c)\, e^{-s/\mu^2}}{c + e^{-s/\mu^2}}, \qquad (7)$$

where c is the free parameter chosen to make the coefficient of $\log\Lambda$ equal to zero.

ii) In practice, in the relativistic case, D(s) is not known exactly, and indeed a linear approximation,

$$D(s) = \text{const} \frac{s - s_B}{s - s_0}, \qquad (8)$$

has been used in some applications of the Dashen-Frautshhi method. With this form for D(s) we can regard s_0 as the free parameter to be determined by the requirement that the $\log \Lambda$ term vanishes. The most important aspect of the investigation is to obtain the D function.

4. RESULTS AND CONCLUSIONS

These are summarised in table 1, where we have used units such that $\mu = 1$. We see from this table that, with the exact D function, the inclusion of the box diagrams reduces the coefficient of the $\log \Lambda$ term by more than 50%. When we modify the left hand cut by the factor f(c,s) then the values of the mass shift agree to within 5% with the exact values. This agreement is remarkable when we note that the values of c required to cancel the $\log \Lambda$ term vary considerably with s_B. The use of the second method, involving the approximate D function is not so accurate but the qualitative agreement is good (particularly when we remember that even the sign of the result is in dispute in methods where only fig. 1a is included (see ref. (3,4)).

TABLE 1

g^2	$-s_B$	Coefficient of logλ	Corrected Coefficient (arbitrary units)	Method II		Method I		Exact
				s_0	Δs_B	c	Δs_B	Δs_B
2.34	0.09	11.10	4.70	0.45	0.50	0.4	0.75	0.73
2.76	0.25	4.00	1.75	1.10	0.76	1.30	1.04	1.01
3.00	0.39	2.56	1.16	1.8	0.92	3.00	1.19	1.17
3.30	0.56	1.78	0.82	3.0	1.09	4.50	1.34	1.33
3.82	1.0	1.00	0.47	10.4	1.42	6.22	1.69	1.64

We conclude therefore that this is a reasonable way to remove the infrared divergence from this type of calculation. Calculations using the method in the framework of a reasonable model of the nucleon are presented.

Computer Program to Solve the Schrödinger equation following BURGESS

```
[illegible]
      [illegible]
      [illegible]
      [illegible](1500),[illegible](1500),[illegible](1500),W(1500),Z(1500)
     [illegible]
        [illegible]
      [illegible]
      [illegible]
        [illegible]
      [illegible]
      [illegible]
      [illegible]
      AML=L*(L+1)
      R(1)=0.0
      Y(1)=0.0
      DO 1 I=1,1500
      R(I+1)=R(I)+H
    1 CONTINUE
      G=H**2
      [illegible]
      CA=G/12.
      CB=5.*G/6.
      DO 2 I=2,5
      R1=R(I)
      R2=R(I)**2
      R3=R(I)**3
      R4=R(I)**4
      Y(I)=(1.+A1*R1+A2*R2+A3*R3+A4*R4)*(R1**(L+1))
    2 CONTINUE
      DO 3 I=2,1501
      R1=R(I)
      R2=R(I)**2
      F(I)=AML/R2-S-C*DEXP(-R1)/R1
    3 CONTINUE
      DO 4 I=6,1501
      AJ=1.-CA*F(I-2)
      AK=1.-CA*F(I)
      AL=2.+CB*F(I-1)
      Y(I)=(AL*Y(I-1)-AJ*Y(I-2))/AK
    4 CONTINUE
      DO 5 I=1201,1501
      R1=R(I)
      R2=R(I)**2
      R3=R(I)**3
      R4=R(I)**4
      W1=S-AML/R2
      W2=W1**(-2)
      W3=(AML/R3)**2
      W4=W1**(-1)
      W5=-1.5*AML/R4
      W(I)=DSQRT(W1+1.25*W2*W3-W4*W5)
    5 CONTINUE
      AI=0.0
      BI=0.0
      DO 6 I=1202,1500,2
      AI=AI+W(I)
    6 CONTINUE
      DO 7 I=1203,1499,2
      BI=BI+W(I)
```

```
FORTRAN IV G LEVEL  1, MOD 1                    THT                        DATE = 69311

          [illegible]
             [illegible]2.*(RI+G(I)*G(I)))/3.
             [illegible]
           [illegible]*Y(I5C(I)
            [illegible]**2)
             [illegible]**2)
              [illegible]
          [illegible]
            [illegible]
         [illegible]
          DO 11 I=1001,1501,100
           S1=R(I)
           [illegible]
           IF (L) 9,9,10
      9     Z(I)=Z1+[illegible](S)
          GO TO 11
     10   Z1=DSQRT(S1*S2-AML)
          Z2=Z1**(-1)
          Z3=Z1**(-3)
          Z4=(AL+0.125)/DSQRT(AML)
          Z5=DARSIN(DSQRT((AML/S)/R))
          Z6=Z5*Z4
          Z(I)=Z1-1.5707963*L*Z6-0.125*Z2 - 5.*AML*Z3/24.
     11   CONTINUE
          DO 16 I=1201,1401,100
          SA=Y(I)*DSQRT(R(I))
          SB=Y(I+100)*DSQRT(R(I+100))
          SC=SA*DSIN(Z(I+100))-SB*DSIN(Z(I))
          SD=SB*DCOS(Z(I))-SA*DCOS(Z(I+100))
           SE=SC/SD
          THETA = DATAN(SE)
           IF (SD)12,13,13
     12   THETA=THETA+3.14159265
          IF(DABS(THETA)-3.14159265) 13,13,120
    120    IF (THETA)  121,121,122
    121        CI=+1
           GO TO 123
    122      CI=-1
    123             THETA=THETA+CI*3.14159265
     13   CONTINUE
     16   CONTINUE
          THETA=THETA-3.14159265
          RETURN
          END
```

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APPENDIX TO CHAPTER SIX.

In the present section we derive the input for the diagrams shown in Figs. 1a - 1c of the text. However, before doing that a few general remarks on the relationship between the MANDELSTAM representation.

1. MANDELSTAM REPRESENTATION IN POTENTIAL SCATTERING.

In 1958 MANDELSTAM suggested for a relativistic scattering amplitude a double dispersion relation.

While even now this conjecture has not been proved it started a new line of approach which proved fruitful. An immediate consequence was the rush to prove the conjecture in simpler models like the potential scattering.

In Khuri's form of dispersion relations the MR for Yukawian potential scattering reads

$$f(E,t) = f_0(t) + \sum_n \frac{g_n(t)}{E - En} + \frac{1}{\pi} \int_0^{\infty} \frac{Imf(E^1 t^1)}{E^1 - E} dE^1$$

$$V(x) = \int_M^{\infty} \frac{e^{-\mu x}}{x} d$$

Then

$$f_0(t) = - \int_M^{\infty} \frac{\sigma(\mu)}{\mu^2 + t} d\mu$$

It is known that if the potential is Yukawian f(E,t) is analytic in the t - plane with the cut $-\infty \leq t \leq - M^2$. Let E real $>$ 0 and let us define (Imf)(E,t) as the analytic continuation of Im f(E,t), that is,

$$(Imf)(E,t) = \frac{1}{2i} \left\{ f(E,t) - \left[f(E,E^*) \right]^* \right\}$$

Clearly Imf(E,t) is not an analytic function of E,t; however (Imf)(E,t) has at least the same analyticity domain as f(E,t). Furthermore, since $(\mathrm{Imf}_0)(E,t) = 0$ as a consequence of the analytic properties of the perturbation terms we have

$$(\mathrm{Imf})(E,t) = \frac{1}{\pi} \int_{4M^2}^{\infty} \frac{\rho(E,t^1)}{t^1 + t} \, dt^1$$

where $\rho(E,t)$ will, in general, be a distribution. The contribution to the integral $M^2 \leq t^1 < 4M^2$ vanishes because the first Born approximation is real. Putting Imf into the first equation, and neglecting bound state contributions and interchanging the order of integration we have

$$f(E,t) = -\int_{M}^{\infty} \frac{\sigma(\mu)}{\mu^2 - t} + \frac{1}{\pi^2} \int_{4M^2}^{\infty} dt^1 \int_{0}^{\infty} \frac{\rho(E^1,t^1) \, dE^1}{(E^1 - E)\,(t^1 - t)} \qquad (1)$$

If bound states are present, subtractions are needed in (1)

We owe to MANDELSTAM the idea of combining UNITARITY with the above representation in the relativistic context to obtain the ρ's, The unitarity property of the scattering amplitude is expressed as

$$I = 4\pi \sum_{\ell=0}^{\infty} (2\ell + 1) \left| a_\ell \right|^2 P_\ell(\cos \theta) = \frac{4\pi}{k} \mathrm{Im}\, f(\cos \theta^1)$$

with

$$f(t) = \sum_{\ell=0}^{\infty} (2\ell + 1)\, a_\ell P_\ell(\cos \theta) \text{ and}$$

$$a_\ell = \frac{S_\ell - 1}{2ik} \qquad \text{and} \qquad S_\ell = e^{2i\delta_\ell(k)}$$

In order to determine ρ in potential theory directly BLANKENBELLER/1/ et. al. now introduce the representation in Eq. (1) into the unitarity equation above

$$\frac{4\pi}{k} \text{ Im } f(E,t) = \int f(E,t_1)\, f^{*}(E,t_2)\, d\Omega_q$$

where E is the energy, t the momentum transfer according to

$$t_1 = (\vec{k}_i - \vec{q})^2 = 2k^2(1 - \text{Cos } \theta_1)$$

$$t_2 = (\vec{k}_f - \vec{q})^2 = 2k^2(1 - \text{Cos } \theta_2)$$

The integral in question is of type

$$I = \int \frac{d\Omega q}{(A - \vec{V}_f \cdot \vec{V}_q)(B - \vec{V}_i \cdot \vec{V}_q)} = 4\pi \int_{\varphi_0}^{\infty} \frac{1}{H(\varphi)} \quad \frac{d\varphi}{\varphi - \text{Cos } V} \qquad (2)$$

$$\varphi_o = AB + \sqrt{\left\{(A^2 - 1)(B^2 - 1)\right\}}$$

$$H(\varphi) = \sqrt{\left\{(\varphi - AB)^2 - (A^2 - 1)(B^2 - 1)\right\}}$$

Using Eq.(2) we find

$$\rho(E,t) = \int_M^{\infty} \sigma(\mu_1)\, d\mu_1 \int_M^{\infty} \sigma(\mu_2)\; K(E,t,\mu_1^2,\mu_2^2,)\; d\mu_2$$

$$- \frac{2}{\pi^2} \int_0^{\infty} dt_1 \quad P\int_0^{\infty} \frac{dE_1\, \rho(E_1,t_1)}{E_1 - E} \int_M^{\infty} \sigma(\mu)\; K(E,t;\; t_1,\mu^2)\; d\mu$$

$$+ \frac{1}{\pi^4} \int_0^{\infty} dt_1 \int_0^{\infty} dt_2 \int_0^{\infty} dE_1 \int_0^{\infty} dE_2 \; \frac{\rho(E_1,\, t_1)\;\; K(E,t;\; t_1,\, t_2)}{(E_1 - E - i\epsilon)\;(E_2 - E - i\epsilon)} \qquad (3)$$

with

$$K(E,t;t_1t_2) = \frac{\pi}{2}\; \frac{\Theta \left\{t - t_1 - t_2 - \frac{t_1 t_2}{2E}\left[\frac{\sqrt{t_1 t_2}}{2E}\right]\right\} \left[16E^2 + 4E(t_1 - t_2) + t_1 t_2\right]^{\frac{1}{2}}}{\left\{E\left[t - (\sqrt{t_1} + \sqrt{t_2})^2\right]\left[t - (\sqrt{t_1} - \sqrt{t_2})^2\right] - t\, t_1 t_2\right\}^{\frac{1}{2}}}$$

The solution of Eq.(3) is simple although it looks very complicated.

Notice that

$$T = t_1 + t_2 + \frac{t_1 t_2}{2E} + \frac{\sqrt{t_1 t_2}}{2E}\left[16E^2 + 4E(t_1 + t_2) + t_1 t_2\right]^{\frac{1}{2}}$$

is the largest root of the second degree equation

$$D \equiv \quad E\left[t - (\sqrt{t_1} + \sqrt{t_2})^2\right]\left[t - (\sqrt{t_1} - \sqrt{t_2})^2\right] - t,t_1 t_2 = 0.$$

If $t > T$ we have $D > 0$, and

$$\left[t - (\sqrt{t_1} + \sqrt{t_2})^2\right]\left[t - (\sqrt{t_1} - \sqrt{t_2})^2\right] > 0$$

Since $t > T > t_1 + t_2$ the second factor is positive and so is the first. Hence $K(E,t;\, t_1, t_2)$ vanishes unless

$$t > (\sqrt{t_1} + \sqrt{t_2})^2$$

The second Born approximation therefore vanishes unless $t > 4M^2$ and if $4M^2 < t < 9M^2$ it coincides with $\rho(E,t)$ because the other terms in Eq.(3) vanish. Take now $9M^2 < t < 16M^2$. In the right hand side of Eq. (3) we have either

$$\sqrt{t_1} + M < \sqrt{t_1} + \mu < \sqrt{t} < 4M \qquad \text{or} \quad \sqrt{t_1} + \sqrt{t_2} < 4M.$$

In the first case the integration over t_1 runs over the range $t_1 < 9M^2$ where $\rho(E,t_1)$ is known exactly, and in the second there is no contribution because $\rho(E,t_1) = \rho(E,t_2) = 0$ unless $\sqrt{t_1} > 2M$, $\sqrt{t_2} > 2M$ which contradicts $\sqrt{t_1} + \sqrt{t_2} < 4M$.

We can therefore compute $\rho(E,t)$ up to $t < 16M^2$. Proceeding in this way it is possible to show recursively that $\rho(E,t)$ in $n^2m^2 \leq t < (n + 1)^2 M^2$ can be computed by straight forward integration from the value of $\rho(E,t)$ in $(n-1)^2M^2 \leq t < n^2 M^2$. We can therefore compute $\rho(E,t)$ in a finite (though increasing with t) number of steps up to any value of t. At each point E, t, $\rho(E,t)$

will be exactly given by a polynomial in the coupling constant, the degree of the polynomial increasing with t.

The fact that we may compute $\rho(E,t)$ exactly in any point does not warrant that conclusion that we also know the scattering amplitude exactly at any point in a finite number of steps because in order to obtain $f(E,t)$ we need to know simultaneously for all t the value of $\rho(E,t)$, and this still takes an infinite number of iterations. However, it appears reasonable that a convenient approximation can be reached by pushing the number of iterations high enough since the higher iterations will contribute to points which are far away in the t - plane. Once $\rho(E,t)$ is known one may compute the left-hand cut needed in the N/D method. We remark that E is the Energy and this is relabelled $\sqrt{S}$ in next section. For relativistic purposes $S = W^2 = E^2$ is the notation.

2. BORN AND BOX DIAGRAM INPUT

We now proceed to derive the input.

The kinematics is that of scalar particles, equal masses in final state, unequal masses for the intermediate particles.

Since our real object is to treat n-p mass difference, which incidentally differs little from the potential theory calculation, the 1γ exchange graph contributes a pole in the t - channel, the box graph may be said to simulate a two particle exchange depending on which channel we look at, although for ~~the~~ potential theory ~~calculate~~ this is unimportant. The Born contribution, as is known, is a pole in the channel in question.

$B_0(s,t) = - \frac{e^2}{t - \lambda^2}$ represents the contribution from Fig. 1a of the text. Partial wave projection yields for

$$\mathrm{Im}\,\delta T = \frac{e^2\pi}{2S} \quad \text{for} \quad -\infty < s < -\frac{\lambda^2}{4}$$

2. i) BOX DIAGRAM

In our case since we have only a Yukawa potential, and for the box diagram the double spectral function is just $\rho(s, t) = K(s,t;\mu^2, \lambda^2)$ where λ is the fictitious photon mass; μ is the mass of the other intermediate particle. One simple integrates t to get the contribution to the amplitude δT in the dispersion integral for the mass shift.

Writing

$$G = \frac{\pi}{2} \int \frac{dt^1}{(t^1-t)\left\{ s\left[t^1-(\mu+\lambda)^2\right] \times \left[t^1-(\mu-\lambda)^2\right]-\left[t^1\mu^2\lambda^2\right]\right\}^{\frac{1}{2}}}$$

As stated above the integral is to be taken over region of positive real axis where the denominator D is real. D is real when

$$s\left[t^1-(\mu+\lambda)^2\right]\left[t^1-(\mu-\lambda)^2\right] - t^1\mu^2\lambda^2 > 0$$

i.e. when

$$\left\{t^1 - \left[\mu^2 + \lambda^2 + \frac{\mu^2\lambda^2}{2S}\right]\right\}^2 > \frac{\mu^2\lambda^2}{4S^2}\left[4S+\mu^2\right]\left[4S+\lambda^2\right]$$

since the roots of D are given by

$$t^1 = \mu^2 + \lambda^2 + \frac{\mu^2\lambda^2}{2S} \pm \frac{\mu\lambda}{S}\left[(4S+\mu^2)(4S+\lambda^2)\right]^{\frac{1}{2}}$$

$$= A \pm B \text{ (say)}$$

Hence

$$G = \frac{\pi}{2} \int_0^{\infty} \frac{dt^1}{t^1 - t} \qquad \frac{1}{\sqrt{s}} \qquad \frac{1}{[t^1 - (A + B)]^{\frac{1}{2}} [t^1 - (A - B)]^{\frac{1}{2}}}$$

Putting $t^1 - t = x$, and doing a little algebra we get

$$= \frac{-\pi}{2\sqrt{s}} \int_{A + B - t}^{\infty} \frac{dx}{\{(x - (A - t)^2 - B\}^{\frac{1}{2}}}$$

Finally, after some more algebra, we obtain

$$G(s,t) = + \frac{\pi}{2\sqrt{s}} \frac{1}{[(A - t)^2 - B^2]^{\frac{1}{2}}} \qquad \log \left\{ \frac{\sqrt{A - B - t} - \sqrt{A - B - t}}{\sqrt{A - B - t} - \sqrt{A - B - t}} \right\}$$

with

$$A = t_1 + \lambda^2 + \frac{t_1\lambda^2}{2S}$$

$$B = \frac{\sqrt{t_1}\lambda}{2S} \quad \{[4S + t_1][4S + \lambda^2]\}^{\frac{1}{2}}$$

where

$t_1 = \mu^2$.

The object is to obtain terms proportional to log λ as well as those not containing log λ. Terms with λ or powers of λ vanish when we take the limit $\lambda \rightarrow 0$.

one may write approximately

$$A = t_1 = \mu^2$$

$$B = \frac{\lambda\mu}{S} \quad [S(4S + \mu)]^{\frac{1}{2}} \quad - \quad \lambda F(S,\mu) \text{ (say)}$$

Remembering that λ is small,

Now simplifying

$\log\left[\sqrt{A+B-t}+\sqrt{A-B-t}\right]$ after some algebra

we get in the limit $\lambda \to 0$

$$= \log\left\{2\ (\mu^2 - t)\right\}$$

Similarly for the term

$$\log\left[\sqrt{A+B-t}-\sqrt{A-B-t}\right] \underset{\lambda\to 0}{\sim} \log(\mu^2-t) + \log\left[\frac{\lambda F(s,\mu)}{\mu^2-t}\right]$$

the term proportion to $\log\lambda$ is thus

$$\frac{\pi}{2\sqrt{s}}\ \frac{\log\lambda}{(t-\mu^2)}$$

without taking the limit $\lambda \to 0$

one can easily show that

$\log\left[\sqrt{A+B-t}+\sqrt{A-B-t}\right]$ can be reduced, after some

algebra to the form

$$\mathrm{Log}\left[(\mu^2-t)+\lambda^2+\frac{\Delta^2\mu^2}{2S}+\lambda\mu\left[4S+\mu^2\right]^{\frac{1}{2}}\left[4S+\lambda^2\right]^{\frac{1}{2}}\right]$$

$$+\log\left[1+\frac{\left(\lambda^2+\mu^2+\frac{\lambda^2\mu^2}{2S}-t\right)-\frac{\lambda\mu}{2S}\left[\quad\right]^{\frac{1}{2}}\left[\quad\right]^{\frac{1}{2}}}{\left[\quad\downarrow\quad\right]+\left[\quad\downarrow\quad\right]}\right]$$

Consider a term like

$\log\left[\mu^2-t+\lambda B(S,\mu^2,\lambda^2)+\lambda^2 D(S,t)\right.$ as $\lambda \to 0$

Away from $W+\lambda V = 0$ $\log W + V\lambda$ is continuous in λ. If

$$\mu^2 - t \neq 0 \quad \log\left[\mu^2 - t + \lambda B + \lambda^2 D\right] \longrightarrow \log(t_1 - t).$$

The next term in an expansion in powers of λ is

$$\log \quad (\mu^2 - t) \quad \frac{\lambda}{\mu^2 - t} \qquad B(S, \mu^2, 0)$$

Now let us look at

$$\log \left\{ 1 + \frac{\sqrt{[\quad] + [\qquad]}}{\sqrt{[\quad] + [\qquad]}} \right\}$$

It can be reduced to the form:

$$\sim \log \left\{ 1 + \frac{\sqrt{(\mu^2 - t)^2 + M\lambda^2 + N\lambda^4}}{(t_1 - t) + G\lambda + H\lambda^2} \right.$$

$$\cong \quad \log \left\{ 1 + \frac{\sqrt{A^2 + B^2}}{A + C} \right.$$ where B and C are functions of S, t and λ.

$$\Longrightarrow \log \left\{ 1 + \frac{\sqrt{A + B\lambda^2}}{A} \right\{ 1 - C\lambda + \ldots\ldots$$

$$= \quad \log \left\{ 1 + \quad \left(1 - \frac{1}{2}\,\frac{B}{A^2}\,\lambda^2 + \cdots\right)(1 - C\lambda) ---- \right\}$$

$\Longrightarrow$ log 2. The next term is $\dfrac{-C(s,t,0)}{2}\,\lambda \log 2$

Similar analysis applies to

$$\log \left[\sqrt{A + B - t} - \sqrt{A - B - t} \right]$$

$$= \log \left\{ (\mu^2 - t)^{\frac{1}{2}} \left[1 + \frac{1}{2}\,\frac{\mu\lambda}{2S}\,\frac{[4S + \mu^2 \quad 4S]^{\frac{1}{2}}}{t_1 - t} + \lambda^2 C(\lambda^2, s, \mu^2) \right. \right.$$

$$- \left(1 - \frac{1}{2}\,\frac{\mu\lambda}{2s}\,\frac{[(4s + \mu^2) \quad 4s]^{\frac{1}{2}}}{t_1 - t} + \lambda^2 D(\lambda^2, s, \mu^2) \right\}$$

$$= \log \left\{ \frac{1}{(\mu^2 - t)^{\frac{1}{2}}} \left[\frac{\Delta\mu}{S} \left[S(4S + \mu^2) \right]^{\frac{1}{2}} + \lambda^2 (C - D) \right] \right.$$

$$\equiv \log \frac{\mu\,(4S + \mu)^{\frac{1}{2}}\,\lambda}{(\mu^2 - t)^{\frac{1}{2}} \sqrt{S}}\, \lambda + \frac{\lambda^2}{(\mu^2 - t)^{\frac{1}{2}}} (C - D) \left. \right\}$$

$$\equiv \log \lambda + \log \left[\frac{\mu (4S + \mu)^{\frac{1}{2}}}{(\mu^2 - t)^{\frac{1}{2}} \sqrt{S}} \right] + \frac{(C - D)(0, S, \mu^2)\, \lambda \log \lambda}{\mu (4S + \mu^2)^{\frac{1}{2}}} + O(\lambda^2) \quad \ldots$$

Finally one has

$$G = \frac{\pi}{2\sqrt{S}} \; \frac{1}{((\mu^2 - t)^2 + S\lambda^2)^{\frac{1}{2}}} \log 2(t_1 - t) - \log \left(\frac{\mu (4S + \mu^2)^{\frac{1}{2}}}{(\mu^2 - t)\sqrt{S}} \right)$$

$$- \log \lambda + O(\lambda) \; \ldots\ldots$$

$$\equiv \frac{\pi}{2\sqrt{S}} \, \frac{1}{(\mu^2 - t)} \cdot \left[\log (2\mu^2 - t) - \frac{1}{2} \log \left[\frac{\mu^2 (4S + \mu^2)}{(\mu^2 - t)\, S} - \log \lambda \right] \right]$$

In the DF method we need the imaginary part of the box-graph contribution which we now calculate.

Thus, for the box-graph we have, up to order $e^2 g^2$

$$\frac{e^2}{\pi^2} \pi g^2 \quad G(s,t;\, \mu^2, \lambda^2) = \operatorname{Im} f(s,t)$$

As $\lambda \rightarrow 0$ the leading term was obtained to be

$$\frac{e^2}{\pi^2} \; \frac{\pi}{2\sqrt{S}} \; \pi g^2 \; \frac{1}{(t - \mu^2)} \log \lambda = \frac{e^2 g^2}{2\sqrt{S}} \; \frac{1}{t - \mu^2} \log \lambda$$

The real part is obtained via Cauchy integral

$$G \quad g(s,t) = \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Im} g(t,s^1)}{s^1 - s}\, ds^1$$

$$= \frac{e^2 g^2}{2(t - \mu^2)} \log \lambda \, \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{s^1}\,(s^1 - s)} = \frac{e^2 g^2}{2(t - {}^2)} \; \frac{1}{\sqrt{-s}} \log \lambda$$

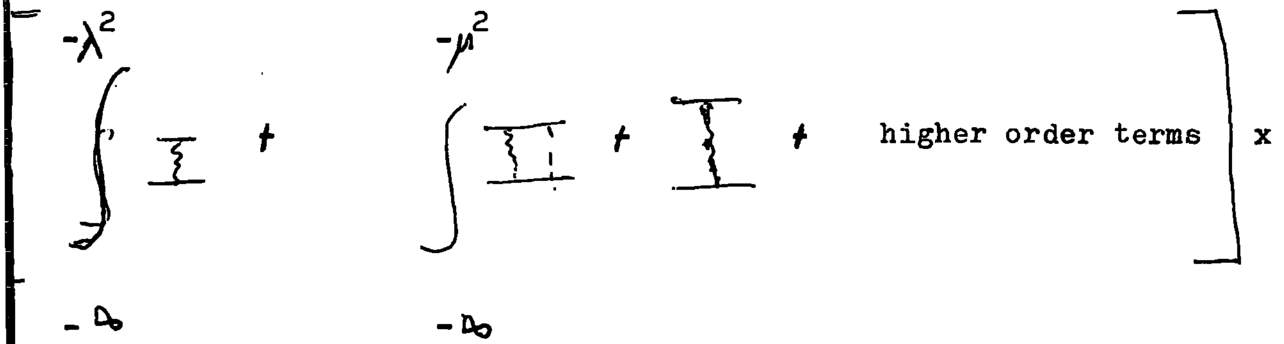
Partial wave projection gives

$$g_0(s,t) = \frac{e^2 g^2}{4s\sqrt{-s}} \log \lambda \; \log \left[\frac{\mu^2}{4\, s - \mu^2} \right]$$

The cut runs from $-\infty < s < -\frac{\mu^2}{4}$

We now recall our prescription for cancellation of I.R.D. contributions: the coefficient of log λ from first and higher orders is made to vanish with proper choice of the fudge factor.

It can be pictorially described as follows:



$$\times \left[\begin{matrix} \text{FUDGE factor} \\ \parallel \\ f(c,s) \end{matrix} \right] = 0.$$ Here c is the parameter.

For the present case we have infrared contributions coming from

$-\infty \leq s \leq \frac{\lambda^2}{4}$ $\qquad \frac{e^2 \pi}{2s}$ $\qquad$ 1γ exchange

$-\infty \leq s \leq \frac{-\mu^2}{4}$ $\qquad -\left[\frac{e^2 g^2 \pi}{4s\sqrt{-s}} \quad - \frac{e^2 \pi}{2s} \right]$ Box + 1γ exchange

Having determined f(C,S) from above prescription one simply computes the mass shift integral but now, with the inclusion of fudge factor, $f(c,s)$

$$\int_{-\infty}^{\frac{\mu^2}{4}} \frac{\text{Im}\,\delta T \quad D^2(s^1)\;ds^1}{s^1 - s} \quad f(c,s)$$

After some algebra it is easy to obtain the contribution to ImδT. For the finite contribution one has

$$-\frac{\pi e^2}{2s} - \frac{e^2 g^2}{4s}\;\frac{1}{2\sqrt{-s}}\;\log\left[\frac{(2\sqrt{-s}+\mu)\,\mu^2)}{-4s}\right]$$

$$-\frac{3g^2 e^2}{8s\sqrt{-s}}\;\log\;\left|4s + \mu^2\right|$$

The contribution proportional to logλ is given as

i)
$$\frac{-\;e^2 D^2(0)}{s_B}\quad\log\lambda$$

coming from the integration of 1γ exchange mass shift expression;

ii) There is the contribution from the box graph whose integral itself contains logλ contribution of the form

$$\frac{e^2 g^2}{4S\sqrt{-S}}\;\log\lambda$$

3. D - FUNCTION

Our D-function was chosen in Omnés form

$$D(s) = (s - s_B)\;\exp\left[-\frac{s - s_o}{\pi}\int_0^{\infty}\frac{ds^1\,\delta_o(s^1)}{(s^1 - s_o)(s^1 - s - i\epsilon)}\right]$$

with $s = k^2$, with square of momentum

This is the choice used also by SHAW and WONG/1/, and we use exactly the same form for later n-p mass difference calculation.

The phase shift $\delta_0(E^1)$ was obtained by solving the Schrödinger equation with a Yukawa potential input, the coupling constant g^2 being the parameter giving bound states for various values of g^2 ($\ell = 0$ for our case)

$$\frac{d^2 \mu_\ell(r)}{dr^2} + \left[k^2 + \frac{\ell(\ell - 1)}{r^2} + \frac{g^2 e^{-\mu r}}{r} \right] \mu_\ell(r) = 0$$

where $\frac{u(r)}{r}$ satisfies the boundary condition

$$u_r \sim r^{\ell - 1} \qquad r \to 0$$ and k is the momentum

Our phase shifts agreed correctly with those of LUMING /2/. Our computer program is able to obtain phase shifts for any ℓ, any energy. Following SHAW and WONG, and from Levinson's theorem we normalize the phase shifts by

$$\delta_0(s) \underset{s \to \infty}{\longrightarrow} -\pi$$

$$\delta_0(0) = 0$$

In actual fact $\delta_0(s) \underset{s \to \infty}{\longrightarrow} 0$ but the normalization of SHAW and WONG is the correct one for us, since we have no CDD poles in our calculation, and also inelastic channels are absent.

Concerning phase shifts and residue of the bound state pole we used the method of BURGESS/3/, which is perhaps the most sophisticated available for the determination of the wave function and the phase shifts. NUMEROV's method of solving differential equations is used throughout.

DF - results were compared with the first order perturbation theory results obtained from

$$\delta s_B = \frac{\int \psi^* \delta V \psi \, dV}{\int \psi^* \psi \, dV}$$

Here the ψ's was obtained from those tabulated by HULTHEN and LAURIKAINEN/4/. The numerical solution is accurate to ~~99%~~ 1% for a three parameter fit to the expression (i.e. n = 3)

$$\psi(r) = (1 - e^{-r}) \left\{ \exp\left[- (-s)^{\frac{1}{2}} r \right] \right\} x \sum_{V=0}^{n} h_V e^{-Vr} .$$

For full details of Burgess' method we refer to the original paper from which it is easy to understand the computer program.

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PART THREE

CHAPTER SEVEN

1. INTRODUCTION

Within the past few years there has been much theoretical interest in electromagnetic mass difference within baryon isospin multiplets. Part of this interest stems from the fact that theories of strong interaction symmetries can be used to relate the mass splittings in one isospin multiplet to those in other isospin multiplets(1). The subject of electromagnetic mass differences can thus be looked upon as forming a testing ground for conjectures about the strong interactions and strong interaction symmetries.

An interesting conjecture about the strong interactions is the hypothesis that all strongly interacting particles are composite(2). From this point of view, one looks at electromagnetic mass differences of particles in an isospin multiplet as arising from a difference in their binding energies due to the electromagnetic interaction.

Within the past years, an "S matrix perturbation theory" hass been developed by Dashen and Frautschi(3) and has been used by Dashen(3) to calculate the neutron proton mass difference. In Dashen's calculation, the nucleon is viewed as a composite particle appearing in the πN scattering amplitude(4). In the absence of electromagnetic interactions it is assumed that the proton and neutron have the same mass, M, and result in a pole in the $J^P = (\frac{1}{2})^+$, $I = \frac{1}{2}$, $I_z = +\frac{1}{2}$ and $I_z = -\frac{1}{2}$ πN scattering amplitudes respectively. The neutron proton mass difference is viewed as arising because of a difference of binding forces in the

$I_z = + \frac{1}{2}$ and $I_z = - \frac{1}{2}$ channels when the electromagnetic interaction is turned on. The proton neutron mass difference is then calculated from an expression of the form

$$M_p - M_n = \frac{1}{R\left(\frac{dD}{dW}(M)\right)^2} \frac{1}{\pi} \int_{\text{cuts}} \frac{\text{Im}\,(D^2\delta T)\,(W^1)}{W^1 - M} dW^1 \qquad 91- \quad (1)$$

where R is the residue at the nucleon pole, D(W) is the denominator function for the $J^P = (\frac{1}{2})^+$, $I = \frac{1}{2}$ partial wave scattering amplitude, and δT is the difference between the πN partial wave scattering amplitudes in the proton and neutron channels.

Historically, the first calcaulation of a mass difference between members of a baryon isospin multiplet was the calculation by Feynman and Speisman of the neutron-proton mass difference(5). Using the Dirac equation with a Pauli anomalous moment term to represent the nucleon, they calculated the contribution to the nucleon self-energy of the perturbation theory "bubble" diagram shown in Figure 1. Since they did not know the high energy behaviour of the propagators or vertex functions, they used cut off functions for the photon propagator and for the anomalous moment which could be regarded as charge and magnetic moment form factors. For cut off energies of the order of several nucleon masses Feynman and Speisman found that they could obtain the correct experimental mass difference of $M_p - M_n \simeq - 1.3$MeV.

A similar analysis of the neutron proton mass difference was made by Huang(6) who calculated the self-energy diagram shown in Figure 1 in perturbation theory without form factors, but with a momentum space cut off. He found that for a spin $\frac{1}{2}$ fermion of mass M, charge e, and Pauli anomalous moment in units of e/2M, the self-energy is

$$\delta M_{e.m} = \left\{ \frac{\alpha}{2\pi} \quad 3M \log\left[V + (1 + V^2)^{\frac{1}{2}} \right] + \frac{\alpha}{2\pi} \quad M\left[V^2 - V(1 + V^2)^{\frac{1}{2}} \right] \right\} \quad (2)$$

$$- \left\{ (\tfrac{5}{4}\mu^2 + 3\mu) \quad \frac{\alpha}{2\pi} \quad M \, (V(1 + V^2)^{\frac{1}{2}} \; - \log\left[V + (1 + V^2)^{\frac{1}{2}} \right]) \right\}$$

Where $V = k/M$ is the cut off momentum. The first term is the usual expression for the electromagnetic self energy of a Dirac particle in second order perturbation theory and diverges logarithmically with the cut off momentum. Taken alone this term is positive and would make the proton heavier than the neutron. However, the terms linear and quadratic in the anomalous moment diverges dquadratically with the cut off momentum and, using experimental values for the neutron and proton anomalous moments, tend to make the neutron heavier than the proton. So for a sufficiently high value of the cut off momentum the contribution from the anomalous moment terms will dominate that from the charge terms and one can obtain the experimental mass difference. In fact, for a value of the cut off momentum of $V = 1.12$ (corresponding to an energy of $(V + \sqrt{1 + V^2}) \; Mc^2 = 2.72 \; Mc^2)$ one can reproduce the observed proton neutron mass difference$^{(6)}$.

A different method of calculating the electromagnetic self-energy of strongly interacting particles was proposed in 1957 by Wick and Sörensen$^{(7)}$ and by Goldberger$^{(8)}$. To second order in e their expression for the nucleon electromagnetic self-energy can be written

$$\bar{U}(p)U(p)\delta M_{em} = -\frac{1}{2} \int d^4 x \, d^4 y \;\; D_f(y-x) \left\{ \langle p \left| T(j_\mu(y), j_\mu(x) \right| p \rangle \right.$$

$$\left. - \langle 0 \left| T \right| j_\mu(y), \; j_\mu(x) \right| \;\; 0 \rangle \Big\} \qquad (3)$$

Where $D_F(y-x)$ is the Feynman photon propagator, $T(j_\mu(y),j_\mu(x))$ is the time ordered product of the Heisenberg electromagnetic current operators, and $|p\rangle$ and $|0\rangle$ are the physical one nucleon and vacuum states respectively.

One might now consider inserting a sum $\sum_K |K\rangle\langle K|$ over a complete set of ingoing physical states between the Heisenberg current operators in Eq.(3) and then trying to evaluate (3) keeping only the lowest mass intermediate states. Sunakawa and Tanaka$^{(9)}$ have shown that keeping just the one nucleon and one nucleon plus nucleon antinucleon pair states leads directly to the perturbation theory expression of Feynman and Speisman with charge and moment form factors at the nucleon photon vertices. Using one parameter fits to the nucleon form factors obtained from electron scattering experiments, Sunakawa and Tanaka obtained for the neutron proton mass difference a number roughly half of the experimental magnitude, but of the wrong sign$^{(9)}$.

The expression given by Feynman and Speisman has since been recalculated several times by various other authors$^{(10)}$. If form factors are used which(1) agree with the low momentum transfer data for the nucleon form factors and (2) tend to zero as the momentum transfer goes to infinity, i.e., no hard core, then the calculations give results of the wrong sign for the neutron proton mass difference. To obtain agreement with experiment using the Feynman Speisman formula alone one must introduce a hard core and then a cut off momentum of several BeV/c so that the contribution from the anomalous moment terms dominates that from the charge terms(11). However, if important contributions to the Feynman Speisman expression for the slef energy come from the high energy region of integration, then one is beggin the question of whether other intermediate states make an important contribution to Eq.(3)

at such high energies. In fact, as pointed out by Wick(7) and also more recently by Cottingham(12) there is abinitio no reason to believe that other intermediate states, such as pion plus nucleon, are not important. These "inelastic" contributions to Eq.(3) can be related to quantities obtainable from inelastic electron nucleon scattering experiments(12), but as yet there is not enough data to draw any conclusions.

Coleman and Schnitzer(13) have taken an alternative viewpoint in calculating baryon electromagnetic mass differences. They calculate the contribution of Figure 1 of Chapter 8 to the self energies using form factors without hard cores and neglect contributions to Eq.(3) from higher mass states, but they assume the existence of "scalar meson tadpole diagrams" which add a constant to the unphysical photon nucleon scattering amplitude involved in Eq.(3), but do not contribute to the absopptive part of that amplitude(2). In their actual calculation the "tadpole" contribution to the baryon mass differences overshadows that from Figure 1. The resulting mass differences (often of opposite sign to the contribution from Figure 1) are in rather good agreement with experiment, whereas the contribution of Figure 1 alone is in uniformly poor agreement with experiment(13).

When one considers the previous methods of calculation of electromagnetic mass differences, a number of questions about Dashen's calculation arise: What is the relation of the S-matrix perturbation theory of Dashen and Frautschi to other perturbation methods? In particular, is the contribution to the self energy calculated by Feynman and Speisman contained in Dashen's calculation? Can the method of calculation of Dashen also explain other baryon electromagnetic mass differences?

In Chapter 8 we shall investigate the question of how older calculations of the baryon electromagnetic self energy are contained in a Dashen-Frautschi type calculation. In particular, we shall see that the perturbation theory result of Huang(6) is contained in a calculation to lowest order in the strong and electromagnetic interactions of the contribution of the photon nucleon inelastic state to the right hand cut of the dispersion relation of Dashen for the neutron proton mass difference. We then go on to consider the general contribution of the photon baryon inelastic state to a Dashen-Frautschi calculation of baryon electromagnetic mass differences. We find, that the net contribution of the photon baryon inelastic state to the dispersion integral of Dashen and Frautschi for the mass difference is the same as in a dispersion theoretic calculation of the "bubble" diagram using the full (strongly renormalized)photon baryon proper vertex function.

We conclude with a brief mention of the latest work on the so called Cottingham formula(12) for calculating mass differences among isospin multiplets. The work in question is by Harari and Elitsur(14). According to Cottingham, to the lowest order in the electromagnetic interactions and to all orders in the strong interaction, the electromagnetic self energy of a hadron can be expressed as an integral over the amplitude of forward Compton scattering of virtual photons on the same hadron (see fig.2.)

$$\Delta M_{e.m.} = \frac{i}{(2\pi)^2} \int_{-\infty}^{+\infty} \frac{d^4q}{q^2 + i\varepsilon} g_{\mu\nu} M_{\mu\nu} (q, v) \qquad (4)$$

where p and q are the hadron and photon momenta respectively; M is the hadron mass and $v = \frac{p.q}{M}$ is the phton energy in the lab. system.

Now Harari and Elitsur transform Eq.(4) into an expression involving integration over space like photon momenta only.

This is accomplished by rotating the integration contour in the complex ν - plane.

$\Delta M_{e.m.}$ is then expressed in terms of the absorptive parts of the Compton amplitudes and the subtraction functions entering into the calculations. The subtraction function can be expressed in terms of the contributions of the t channel Regge poles (and, possible fixed poles). The Regge pole contributions, in principle, can be calculated from the low energy inelastic data by the use of EESR. Harari and Elitsur then conclude that, *if the above procedure is valid,* "the electromagnetic mass difference can be expressed only in terms of low lying electron scattering data:" A calculation of the neutron proton mass difference was carried out by expressing the subtraction function for the $\Delta I = 1$ mass differences in terms of the A_2 residue function. The conclusion was that the contribution of the A_2 trajectory, as computed from FESR, cannot explain the observed n-p mass difference.

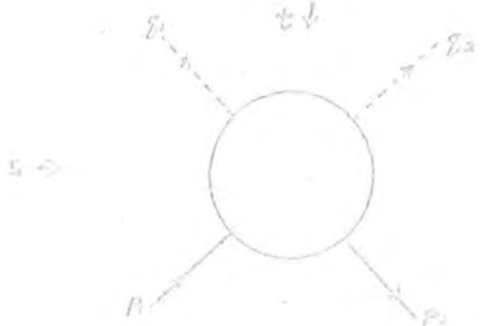
We reported on the above calculation in detail since this calculation with the many others cited earlier in the text all testify to the lack of success in calculating the observed n-p mass difference.

Only Dashen claims to have successfully solved this problem. In the present work we shall attempt to show that the DF method is, by itself, just as good as the Cottingham formula. The troubles arise only when one attempts to make use of them in practice to obtain answers to physically relevant problems. The principle difficulty in all the approaches thus far adopted is the same: lack of success in fully presenting the strong interaction part of the problem. The DF method assumes this to be given.

We shall see in the following how a direct application of the DF method yields the wrong answer for the n-p mass difference. Clearly the problem is a multichannel one.

CHAPTER SEVEN

Fig. 1. The bubble diagram for the baryon electromagnetic self-energy. The blobs represent form factors.



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CHAPTER EIGHT

RELATION OF DF METHOD TO PERTURBATION CALCULATIONS IN FIELD THEORY.

DASHEN'S NEGLECT OF INELASTIC INTERMEDIATE Nγ - STATE FOUND UNJUSTIFIABLE.

Here we investigate the connection between the old field theoretic self energy calculations and dispersion theoretic perturbation theory of Dashen and Frautschi. It emerges that the inelastic contribution from Nγ intermediate state is many times that taken by Dashen, thus invalidating his 'successful' calculation of neutron proton mass difference. Other aspect weakening Dashen's result, the choice of the D function is only touched on partially. A fuller discussion is given elsewhere.

We try to discover in what senge ,calculations of the baryon electromagnetic self energy which invole the "bubble" diagram [1.2.] are contained in a Dashen Frautschi type calculation. For this purpose, let us imagine temporarily a world with only neutral pseudoscalar mesons of mass m ("pions") coupled to charged spin $\frac{1}{2}$ baryons of mass M ("nucleons"). We assume the baryons are coupled to the electromagnetic field with a coupling constant e.

With an eye to using the Dashen Frautschi method, we consider pseudoscalr meson-baryon scattering (see Figure 2). Let $q_1 = (q_1, iw_1)$ and $p_1 = (p_1, iE_1) = (-q_1, iE_1)$ be the initial four momenta of the meson and baryon in the centre of mass system, and let $q_2 = (q_2, iw_2)$ and $p_2 = (p_2, iE_2) = (-q_2, iE_2)$ be the final meson and baryon four momenta respectively. Then

$$p_1 + q_1 = p_2 + q_2 \qquad (1)$$

by conservation of four momentum. We define the convential variables

$$s = -(p_1 + q_1)^2,$$

$$t = -(q_1 - q_2)^2, \qquad (2)$$

and $$u = -(p_1 - q_2)^2, \qquad \text{with } s + t + u = 2(M^2 + m^2)$$

If $W = E_1 + W_1 = E_2 + W_2 =$ the total centre of mass energy, $q = |\vec{q_1}| = |q_2|$ and $z = \hat{q}_1 \cdot \hat{q}_2$, then

$$s = W^2,$$

$$t = -2q^2(1 - z).$$

$$q^2 = \frac{[W^2 - (M + m)^2 \quad W^2 - (M - \mu)^2]}{4W^2} \qquad (3)$$

We define the usual invariants A(s,t,u) and B(s,t,u) of pseudoscalar meson-baryon scattering in terms of the S matrix element for scattering from an initial state i to a final state f by

$$S_{fi} = \delta_{fi} + (2\pi)^4 \, i \, \delta(p_2 + q_2 - p_1 - q_1)$$

$$\sqrt{\frac{M^2}{4W_1 W_2 E_1 E_2}} \; \bar{U}(p_2)\left[A - i\gamma . \frac{q_1 + q_2}{2} B\right] U(p_1) \qquad (4)$$

In the following we will be working with the partial wave amplitudes for meson-baryon scattering. As is usual in doing calculations involving such partial wave amplitudes we shall for convenience work in the complex W plane rather than the complex s plane. We refer the reader to the standard literature on the definitions and analyticity properties of the partial wave amplitudes[3,4]

To avoid difficulties with kinematic singularities we shall work with the $\ell = 1$, $J = \ell - \frac{1}{2}$ partial wave amplitude defined by [3,4]

$$T_{1-}(W) = \frac{16\pi \; W^2}{(W-M)^2-m^2} f_{1-}(W) \qquad (5)$$

where $f_{1-}(W)$ is the usual partial wave amplitude which satisfies elastic unitarity in the form

$$\mathrm{Im} f_{1-}(W) = q \left[f_{1-}(W) \right|^2, \qquad f_{\ell\pm}(W) = \frac{1}{16\pi W}(E-M)$$

$$\left[A_{\ell}^{(I)} + (W-m) \; B_{\ell}^{(I)} \right] - (E-m) \left[- A_{\ell\pm 1} - (W-m) \; B_{\ell\pm 1} \right] \qquad (6)$$

with $E = \dfrac{(W^2 + M^2 - M^2)}{2W}$

$T_{1}-(W)$ defined as above has a pole at $W = M$(the "nucleon pole") with residue equal to $-g^2$. We assume the pole is a bound state due to the vanishing of the D function at $W = M$.

In the W plane, the expression for the change in mass of the baryon due to electromagnetic interactions given by Dashen and Frautschi now takes the form[5,6].

$$\delta M = \frac{1}{(-g^2)((\frac{dD}{dW})(M))^2} \; \frac{1}{\pi} \int_{M}^{\infty} dW^1 \left(\frac{\mathrm{Im}(D^2 \delta T_1)(W^1)}{W^1 - M} - \frac{\mathrm{Im}(D^2 \delta T_1)(-W^1)}{W^1 + M} \right)$$

$$+ \frac{1}{\pi} \int_{LHC} dW^1 \quad \frac{\mathrm{Im}(D^2 \delta T_{1-}(W^1)}{W^1 - M}$$

Where $\delta T_1-(W)$ is the change in the meson-baryon scattering amplitude due to the presence of the electromagnetic interaction. Here the integral from M to ∞ receives contributions from diagrams containing s channel discontinuties with inelastic intermediate states such as the photon baryon state. We are not interested at the moment in contributions to δM due to external mass shifts which also contribute to this integral. From Eq. (17) we find that on the right hand cut we have simply

$$\mathrm{Im}\, D^2\, \delta T_{1-} = \left|D\right|^2 \sum_i \rho_i \left|\delta T_i\right|^2 \qquad (8)$$

where we recall from Eq. (14) that $\sum_i \rho_i \left|\delta T_i\right|^2$ is just the contribution to the absorptive part on the right hand cut of the partial wave amplitude due to new inelastic states. The sum in Eq.(8) is over new inelastic states, δT_i is a partial wave amplitude for the proxess: meson + baryon (inelastic states)$_i$, and ρ_i is a phase space factor for the i-th inelastic state. The integral over the left hand cut, which in the W plane includes cuts on the real axis from -M to +M, along the imaginary axis, and a circular cut about the origin[3,4], receives contributions from diagrams with t and u channel discontinuities.

Before proceeding, let us also review our assumptions on D from Pt.1, Chapter two: (1) $D(M) = 0$; (2) $D(W) \to$ const. as $W \to \infty$; (3) $D(W)$ has the right hand cut[5] of $T_{1-}(W)$ and is otherwise analytic in the W plane.

Now that we have taken care of the preliminary definitions and kinematics let us consider the contribution to $\mathrm{Im}(D^2 \delta T_{1-})$ of the photon-baryon inelastic intermediate state. We start by considering the amplitude $\delta T_{1-}(W)$ which comes from all Feynman diagrams which contain a photon- baryon intermediate state in the s channel, and which are second order in e and lowest order in g (also second order). These diagrams are shown in Figure 4a-d (remember that our "pions" are neutral and have no electromagnetic interactions).

Now recall that to obtain Eq.(7) we simply wrote an unsubtracted dispersion relation for the quantity $(D^2\, T_{1}-)(W)$. Since D^2 has a double zero at $W = M$, only contributions to $\delta T_{1}-$ which have a double pole at $W = M$, give a non-zero contribution to δM. However, an inspection of Figs. 4a-d leads to the conclusion that only Fig. 4a will give a contribution to $T_{1-}(W)$ which has a double pole at $W = M$, while Figs. 4b,c, and d give contributions to δT_{1-} which have either a single pole or no pole at all at $W = M$. Thus we see that only Fig.4a will give a non zero contribution to the electromagnetic mass shift when we evaluate the dispersion integral in Eq.(7) with $\delta T_{1}-(W)$ from Figs. 4a-d.

However, if we want to calculate the contribution of the imaginary parts of the scattering amplitudes corresponding to Figures 4a - d to the dispersion relation, we must be somewhat careful because the amplitude corresponding to Figure 4d has a t channel cut (from a baryon-antibaryon intermediate state which gives cuts along the imaginary axis in the W plane) which cannot be neglected. If one took only the contributions to the s channel photon-baryon cut from Figure 4d, then the result of evaluating the dispersion relation with the contributions from Figures 4b, c, and d would not be zero. It is only when all the singularities are taken together that these contributions cancel.

To actually compute the absorptive part of the scattering amplitude corresponding to Figures 4a - d is a straightforward but somewhat laborious exercise in the use of the Feynman and Cutkosky rules. We find for the absorptive part of the invariant amplitudes A and B due to the photon-baryon intermediate state in the s channel$^{(7)}$

$$\left[A\ (W^2,t,u)\right]_{\gamma B} = \frac{e^2 g^2 k}{8\pi W} \left\{ \frac{M(W^2 - M^2)(3W^2 - M^2)}{W^2(W^2 - M^2)^2}\ 4MJ_1 \right.$$

$$\left. + \left[\frac{M}{W^2}\ (W^2 + M^2)(M^2 - U)\ - 2M\ (t - 2m^2)\right] J_2 - 2\frac{W_1}{W}\ M(M^2 - U)\ J_3 \right\} \qquad (9)$$

and

$$\Big[B(W^2, t, u)\Big]_{\gamma B} = \frac{e^2g^2k}{8\pi W}\Bigg\{\frac{W^4 - 6M^2W^2 + M^4}{W^2(W^2 - M^2)^2} - \frac{4}{W^2 - M^2}$$

$$+ 4\,\frac{M^2 - m^2}{W^2 - M^2}\,J_1 - (U + M^2 - 2m^2)\,\frac{W^2 + M^2}{W^2}\,J_2 \tag{10}$$

where J_1, J_2 and J_3 are integrals defined by

$$J_1 = \int \frac{1}{2E_1k - 2\vec{k}.\vec{q}} \cdot \frac{d\Omega_{\hat{k}}}{4\pi}\Big) = \frac{1}{4kq}\log\frac{2E_1k + 2kq}{2E_1k - 2kq} \tag{11}$$

$$J_2 = \int_0^1 \frac{dy}{4E_1{}^2k^2 - 4k^2q^2 - k^2t(1-y^2)}$$

$$J_3 = 4k^3E_1 \int_0^1 \frac{(1 - y^2)\,dy}{(4E_1{}^2k^2-4k^2q^2-k^2t(1-y^2))(4E_1{}^2k^2-4k^2q^2+4k^2q^2(1-y^2))}$$

The quantities s,t,u,W_1,E_1, and $\vec{q}$ are all defined above in Eq.(2) and following. $\vec{k}$ is the centre of mass momentum of the photon (on the mass shell when we compute the absorptive part), and has the magnitude $k = \dfrac{W^2 - M^2}{2W}$.

The first terms on the right hand side of Eq.(9) and (10) come from Fig 4a and characteristically have a double pole at $W = M$. The second and third terms on the right hand side of Eq.(10) come from Figures 4b and 4c and have a single pole at $W = M$, as was expected.

As we have just seen, figure 4a will give the only non zero contribution to Eq.(7) for δM. Let us therefore first consider its contribution to the dispersion integral. Rewriting the first terms on the right hand side of Eq.(9) and (10), we have

$$(A)_{\gamma B,4a} = \cancel{(}\left(\frac{eg}{W^2-M^2}\right)^2 \quad \frac{M(W^2 - M^2)(3W^2 - M^2}{16\pi W^4}$$

$$(B)_{\gamma B,\ 4a} = \left[\frac{eg}{W^2 - M^2}\right]^2 \quad \frac{(W^2 - M^2)(W^4 - 6M^2W^2 + M^4)}{16\pi\ W^4}$$

(12)

We then compute, using the usual formalism for partial wave amplitudes for pion-nucleon scattering[3,4],

$$(\ T_{1-}(W))_{\gamma B,4a} = \left(\frac{eg}{W^2-M^2}\right)^2 \quad \frac{(W + M)^2\ (W^2 - M^2)(W^2 + M^2 - 4MW)}{16\pi W^3}$$

(13)

Note that as $W \rightarrow \infty$, $(\delta T_{-1-}(W))_{\gamma B} \rightarrow 0$, as $1/W$ so that the dispersion integral in Eq.(7) converges rapidly if, as assumed, $D(W) \rightarrow 1$ as $W \rightarrow \infty$.

In order to see directly that the contribution to δM from Fig ,4a is related to the older calculations of δM involving Fig 1, let us impose one more assumption on our imaginary world. We assume g is "small" and work only to lowest order in g for meson-baryon scattering. To second order in g, there are only two diagrams which contribute to meson baryon scattering (see Fig. 3). Also, as noted before, the diagrams in Figs 4a - d are the only diagrams which are second order in g and in e. To this order in g, $T_{1-}(W)$ has a left hand cut coming from the partial wave projection of Fig. 3b and a pole at $W = M$ coming from Fig. 3a, but no right hand cut. Writing $T_1-(W) = N/D$, we assign the left hand cut of $T_1-(W)$ to N and the pole of T to a zero of D. Therefore, to second order in g we take

$$D(W) = \left(\frac{dD(M)}{dW}\right) (W - M)) \tag{14}$$

The actual value of $\frac{dD}{dW}(M)$ is not of interest, since it will drop out of the calculation in the end. The D(W) given in Eq.(14) does not satisfy the condition that $D(W) \rightarrow 1$ as $W \rightarrow \infty$. We expect this behaviour only from the complete D(W) obtained by taking diagrams of all orders in g^2. Eq.(14) is to be regarded as simply the first term in an expression of D in powers of g^2.

Substituting Eq.(13) for $(\delta T_{1-}(W))_{\gamma B,4a}$ and Eq.(14) for D(W) in Eq.(7), we find

$$\delta M = \frac{-e^2}{16\pi^2}\left\{\int_M^\infty \frac{dW'}{W'^3}(W'+M)(W'^2+M^2-4MW')\right. \tag{15}$$

$$\left. -\int_M^\infty \frac{dW'}{W'^3}(W'-M)(W'^2+M^2+4MW')\right\}$$

or

$$\delta M = \frac{\alpha M}{2\pi}\int_M^\infty \frac{dW'}{W'^3}(3W'^2-M^2) \tag{16}$$

The two integrals in Eq.(15) are linearly divergent, but their sum diverges only logarimically. If we introduce a cut off energy W_{max}, we have

$$\delta M = \frac{\alpha M}{2\pi}\left(3\log\frac{W_{max}}{M} - \frac{1}{2} + \frac{M^2}{2W^2_{max}}\right) \tag{17}$$

This is exactly the perturbation theory result for the bubble diagram without form factors given by Weisskopf(8) and by Huang(9) if we write $\frac{W_{max}}{M} = \nu + \sqrt{\nu^2+1}$ where ν is a momentum cut-off.

Using Eq.(14) for D(W) to lowest order in g, let us also consider the contribution of the other terms in Eq.(9) and (10) to the dispersion integral in Eq.(7). First consider the second and third terms on the right hand side of Eq.(10), which come from figs. 4b and 4c and have a single pole at W = M. We find for their contribution to the absorptive part of the partial wave amplitude,

$$(\delta T_{1-}(W))_{\gamma B} = e^2 g^2 \frac{W + M}{4W^2} ((M^2 - m^2) J_1 - 1) \qquad (18)$$

The integral of the dispersion integral in Eq.(7) then receives a contribution

$$\frac{\mathrm{Im}(D^2 \delta T_{1-}(W))}{W - M} + \frac{\mathrm{Im}(D^2 \delta T_{1-})(-W)}{W + M} = \left(\frac{dD}{dW}(M)\right)^2 \frac{e^2 g^2}{4\pi W^2}$$

$$((M^2 - m^2)J_1 - 1) \left\{ \frac{(W - M)^2 (W + M)}{W - M} + \frac{(-W - M)^2 (- W + M)}{W + M} \right\}$$

$$= 0 \qquad (19)$$

The terms in Eq.(10) with a single pole at $W = M$ thus make no contribution to the dispersion integral. The remaining terms in Eq.(9) and Eq.(10) have no pole at $W = M$. Calculating their contribution to $(\delta T_{1-}(W))_{\gamma B}$ we find a complicated sum of products of Legendre polynomials of the second kind which gives a non zero contribution to the integral over the right hand cut. This is not unexpected, for it is only when the contribution of Fig.4d to the left hand cut is taken into account that we expect a cancellation resulting in zero net contribution to δM of the terms with no pole at $W = M$. We shall leave the direct verification of this cancellation to a future calcaulation.

Now that we have a better feeling for what is going on, let us remove some of the restrictions on our imaginary world. First of all, instead of neutral mesons we can consider isospin multiplet of pseudoscalar mesona (e.g. pions) coupled to an isospin multiplet of baryons (e.g. nucleons). In our lowest order calculation this gives rise to the additional diagrams with s channel photon baryon intermediate states shown in Figs. 4e - i. However, none of these new diagrams gives a contribution to $\delta T_{1-}(W)$ with a double pole at $W = M$, and therefore give no contribution to δM. Again note that

Fig. 4i has a t channel cut which must be included in the dispersion integral. The inclusion of meson and baryon isospin multiplets in the calculation also results in the multiplication of the residue at the "nucleon" pole of T_1-(W) by some isospin factor. It is however not difficult to verify that this isospin factor cancels out of the contribution of Fig. 4a to Eq.(7) and thus leaves Eq.(16) or (17) for M unchanged.

We could now also consider diagrams which are higher order in g. In a calculation to fourth order in g and second order in e, D(W) is no longer $(\frac{dD}{dW}(M))(W - M)$, but acquires a right hand cut. Also, in place of Figs. 4a - i we would have meson baryon scattering diagrams in which both the meson baryon and photon baryon vertices acquire mesonic corrections. Instead of doing such a calculation it is just as simple to consider the general contribution of the photon baryon intermediate state to the dispersion relation for δM to all orders in the strong interactions.

For definiteness let us consider pion nucleon scattering in the $\ell = 1$, $J = \ell - \frac{1}{2}$, $I = \frac{1}{2}$ partial wave, The partial wave amplitude T_1-(W) then has a pole at W = M with residue $-3g^2$ (the 3 is an isospin factor). We then wish to consider the contributions to Eq.(17) from all graphs with a photon-nucleon intermediate state in the s channel. δT_1-(W) will then be related to the "square" of a photoproduction amplitude (integrated over the photon nucleon intermediate state).

Such a photoproduction amplitude can in general be split into a sum of a one nucleon reducible part and a one nucleon irreducible part in a unique way[10,11]. The one nucleon reducible part has a pole at W = M and is equal to the Born contribution with all (strong interaction) radiative correction. The one nucleon irreducible part has no pole at W = M. Thus, if we let M_μ(W) be the partial wave photoproduction amplitude in the nucleon channel for photons of polarization μ, then we write [11]

$$M_{\mu}(W) = \sqrt{3}\ g\ K(W) . \frac{1}{W - M} . \Gamma_{\mu}(W) + M_{\mu}(W)_{\text{irred}} \tag{20}$$

Where K(W) is the form factor (improper vertex function) for the pion nucleon vertex with one nucleon off the mass shell[12], and $\Gamma_{\mu}(W)$ is the proper vertex function[13] for the photon nucleon vertex with one nucleon off the mass shell. $M_{\mu}(W)$ is defined to have no pole at $W = M$.

Furthermore, within the approximation of two particle unitarity, $K(W)/(W-M)$ is proportional to $1/D(W)$, since both have a cut from $W = (M + m)$ to ∞ with the same phase and both have a pole at $W = M$[11]. In fact we have

$$\frac{K(W)}{W - M} = \frac{\left(\frac{dD(M)}{dW}\right)}{D(W)} \tag{21}$$

if the residues at the pole are to agree $\left(K(M) = 1\right)$. Therefore

$$M_{\mu}(W) = \sqrt{3}\ g\ \frac{\frac{dD}{dW}(M)}{D(W)}\ \Gamma_{\mu}(W) + M_{\mu}(W)_{\text{irred}} \tag{22}$$

When "squared" and integrated over intermediate states we will get a contribution to $(D^2 \delta T_1-)(M)$ only from the "square" of the first term of (22) since only it has a double pole at $W = M$. Furthermore the first term of Eq.(22) leads to a $\delta T_1-(W)$ with only a right hand cut. Substituting the "Sqare" of Eq.(22) into Eq(7), we find the net contribution of the photon nucleon intermediate state to δM to be

$$\delta M = \frac{1}{\pi} \int_{M}^{\infty} dW^1 \left\{ \frac{\sum_{\mu} \rho_{\gamma} (W^1) \left| \Gamma_{\mu}(W) \right|^2}{W^1 - M} + \frac{\sum_{\mu} \rho_{\gamma N}(-W^1) \left| \Gamma_{\mu}(-W^1) \right|^2}{W^1 + M} \right. \tag{23}$$

where $\rho_{\gamma N}(W)$ is a phase space factor for the intermediate photon nucleon state. Factors from the pion nucleon scattering have thus cancelled out, leaving the contribution given in Eq.(23). Moreover, Eq.(23) is exactly what one would obtain if one had set for himself the problem of computing the contribution of the bubble diagram of Fig.1 to the nucleon self energy by means of dispersion theory, and had used the fully renormalized proper vertex function at the photon nucleon vertices.

The cancellation of factors from meson baryon scattering leaving Eq.(23) occurs in the case of multichannel scattering as well. As an interesting exercise, let us see briefly how this occurs.

We assume that a baryon, B, of mass M occurs as a bound state in n pseudoscalar meson baryon scattering channels. $\underset{\sim}{D}$, $\underset{\sim}{T}$, and $\delta\underset{\sim}{T}$ are now n x n matrices, and the generaliztion of eq(7) is

$$\delta M = \frac{\mathrm{Tr}\left[\underset{\sim}{R}\,\underset{\sim}{\Delta}^T\,\frac{1}{\pi}\int_{cuts} dW^1\,\frac{\mathrm{Im}(\underset{\sim}{D}^T\,\delta\underset{\sim}{T}\,\underset{\sim}{D})(W^1)}{W^1 - M}\,\underset{\sim}{\Delta}\right]}{\mathrm{Tr}\,(R\,R)} \qquad (24)$$

where

$$\underset{\sim}{\Delta} = \lim_{W \to M}\,(W - M)\,\underset{\sim}{D}^{-1}(W) \qquad (25)$$

and

$$\underset{\sim}{R} = \lim_{W \to M}\,(W - M)\,\underset{\sim}{T}\,(W) \qquad (26)$$

Since the residue matrix may be factored(14); $R_{ij} = r_i r_j (i,j = 1,..n)$, we may write $\underset{\sim}{R} = \underset{\sim}{r}^T\underset{\sim}{r}$ (27)

where $\underset{\sim}{r} = r_1 \ldots r_n$ is a 1 x n row matrix whose elements we take to be real. In place of Eq(8), we have on the right hand cut (see Eq.42)

$$\mathrm{Im}\,(\underset{\sim}{D}^T\,\delta\underset{\sim}{T}\,\underset{\sim}{D}) = \underset{\sim}{D}^T\,\Big(\sum_\mu \underset{\sim}{M}^T_\mu\,\rho_{\gamma B}\,\underset{\sim}{M}_\mu\Big)\,\underset{\sim}{D} \qquad (28)$$

where $\underset{\sim}{M}_\mu$ is a 1 x n matrix for the process: γ + B → meson + baryon.

As in the single channel case, we separate $\underset{\sim}{M}_\mu$ into one baryon reducible and one baryon irreducible parts:

$$\underset{\sim}{M}_\mu(W) = \Gamma_\mu(W)\,\frac{1}{W \to M}\,\underset{\sim}{K}(W) + M_\mu(W)_{irred} \qquad (29)$$

where the meson baryon form factor, K(W), is now a 1 x n matrix(16).

For the multichannel case the generalization of Eq.(21) is (15,16)

$$\frac{K(W)}{W-M}\;) = \underset{\sim}{r}\,\underset{\sim}{\Delta}^{-1}\,\underset{\sim}{D}^{-1}\,(W) \qquad (30)$$

Substituting Eq.(30) for K(W)/(W - M) in Eq.(29), we find

$$M_\mu(W) = \Gamma_\mu(W)\, \underset{\sim}{r}\, \underset{\sim}{\Delta}^{-1}\, \underset{\sim}{D}^{-1}(W) + \underset{\sim}{M}_\mu(W)_{\text{irred}} \qquad (31)$$

Since only the reducible part of $\underset{\sim}{M}_\mu(W)$ gives a non zero contribution to the dispersion integral, we have from Eq.(28) and (31) on dropping terms containing $M_\mu(W)$ irred

$$\begin{aligned} \text{Im}(\underset{\sim}{D}^T \delta \underset{\sim}{T}\underset{\sim}{D})_{\text{RHC}} &= \underset{\sim}{D}^T \sum_\mu (\underset{\sim}{r}\, \underset{\sim}{\Delta}^{-1}\, \underset{\sim}{D}^{-1}\Gamma_\mu)^{\dagger}\, \rho_{\gamma B}\, (\underset{\sim}{r}\, \underset{\sim}{\Delta}^{-1}\, \underset{\sim}{D}^{-1}\Gamma_\mu)\underset{\sim}{D} \\ &= (\underset{\sim}{\Delta}^{\dagger})^{-1}\, \underset{\sim}{r}^T \,(\sum_\mu \Gamma_\mu^{*} \rho_{\gamma B} \Gamma_\mu)\, \underset{\sim}{r}\, \underset{\sim}{\Delta}^{-1} \end{aligned} \qquad (32)$$

Since $\underset{\sim}{r}^T = \underset{\sim}{r}^{\dagger}$ and $\Delta^T = \Delta^{\dagger}$

Finally, Eq.(24) becomes

$$\delta M = \frac{\frac{1}{\pi}\int_{\text{RHC}} \frac{dW^1}{W' - M}\, \text{Tr}\,(\underset{\sim}{R}\, \underset{\sim}{\Delta}^T\, (\underset{\sim}{\Delta}^T)^{-1} \underset{\sim}{r}^T (\sum_\mu \Gamma_\mu^{*} \rho_{\gamma B} \Gamma_\mu)\, \underset{\sim}{r}\, \underset{\sim}{\Delta}^{-1}\, \underset{\sim}{\Delta})}{\text{Tr}\,(\underset{\sim}{R}\, \underset{\sim}{R})}$$

$$= \frac{\text{Tr}\,(\underset{\sim}{R}\, \underset{\sim}{r}^T \underset{\sim}{r})}{\text{Tr}\,(\underset{\sim}{R}\, \underset{\sim}{R})}\; \frac{1}{\pi}\int_{\text{RHC}} \frac{dW^1}{W^1 - M} \sum_\mu \Gamma_\mu^{*}(W^1)\, \rho_{\gamma B}(W^1)\, \Gamma_\mu(W^1) \qquad (33)$$

Using $r^T r = R$, we have

$$\delta M = \frac{1}{\pi}\int_{\text{RHC}} \frac{dW^1}{W^1 - M} \sum_\mu \rho_{\gamma B}(W^1)\, \left|\Gamma_\mu(W^1)\right|^2 \qquad (34)$$

which is the same as Eq.(23).

Now that we have generalized to the multichannel case our result Eq.(23), for the contribution of the photon baryon inelastic state to the dispersion integral for δM, let us note the following about this result:

1) Let us stress again that taking the contributions of figures 4a - i to just the right hand cut does not lead to Eq.(15). One must consider the left hand cut as well if the contribution of all but Fig 4a is to cancel. Similarly, one must take the left hand cut into account to obtain the more general result, Eq.(23) for the contribution of the photon baryon intermediate state to the dispersion relation for the mass shift of the baryon.

Note also that the diagrams in Figs. 4d,g,h,and i involve photons connecting initial and final external lines. We find that these "inner bremstrahlung diagrams" not only give a negligible contribution to the proton neutron mass difference as estimated by DASHEN$^{(17)}$ but in fact give zero contribution to the mass difference when considered together with the contributions from Figs. 4b, c, e and f and when both the right and left hand cuts are taken into account.

2) Numerically we find the contribution of the photon baryon intermediate state to the electromagnetic shift in the mass of the baryon is not negligible. For example, using Eq. (15) or the more general Eq. (23), and integrating over just the part of the photon nucleon cut within a pion mass of the nucleon pole, we find a contribution to the neutron proton mass difference several orders of magnitude greater than the 2 % effect on $M_p - M_n$ estimated by DASHEN$^{(17)}$. In fact it has a 12 % effect on $\delta M = M_p - M_n$. DASHEN simply ignored inelastic contributions. His calculation is thus completely unreliable. To take proper account of these contributions presents formidable problems.

3) Eq. (16) is not exactly equivalent to the calculation of WICK$^{(2)}$ or CINI et. al$^{(18)}$ whose equations involve the photon baryon improper vertex function with the photon of the mass shell. One expects the two expressions to be related, but their exact relationship is not clear. We hope to examine this and other questions about the role of inelastic states in a DASHEN-FRAUTSCHI calculation of electromagnetic mass differences in the course of future research.

Fig. 1. The bubble diagram for the baryon electromagnetic self-energy. The blobs represent form factors.

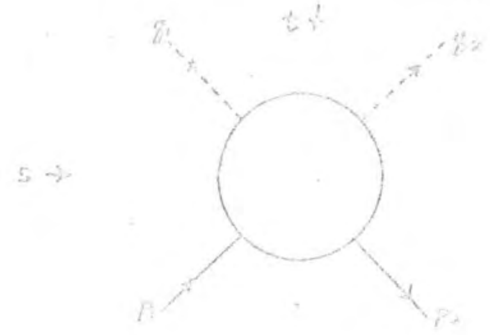
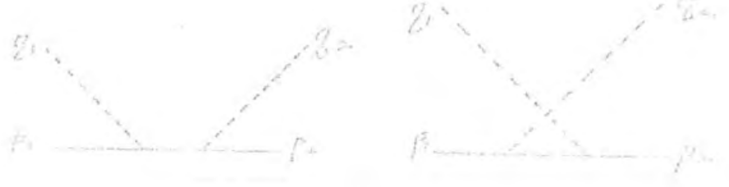


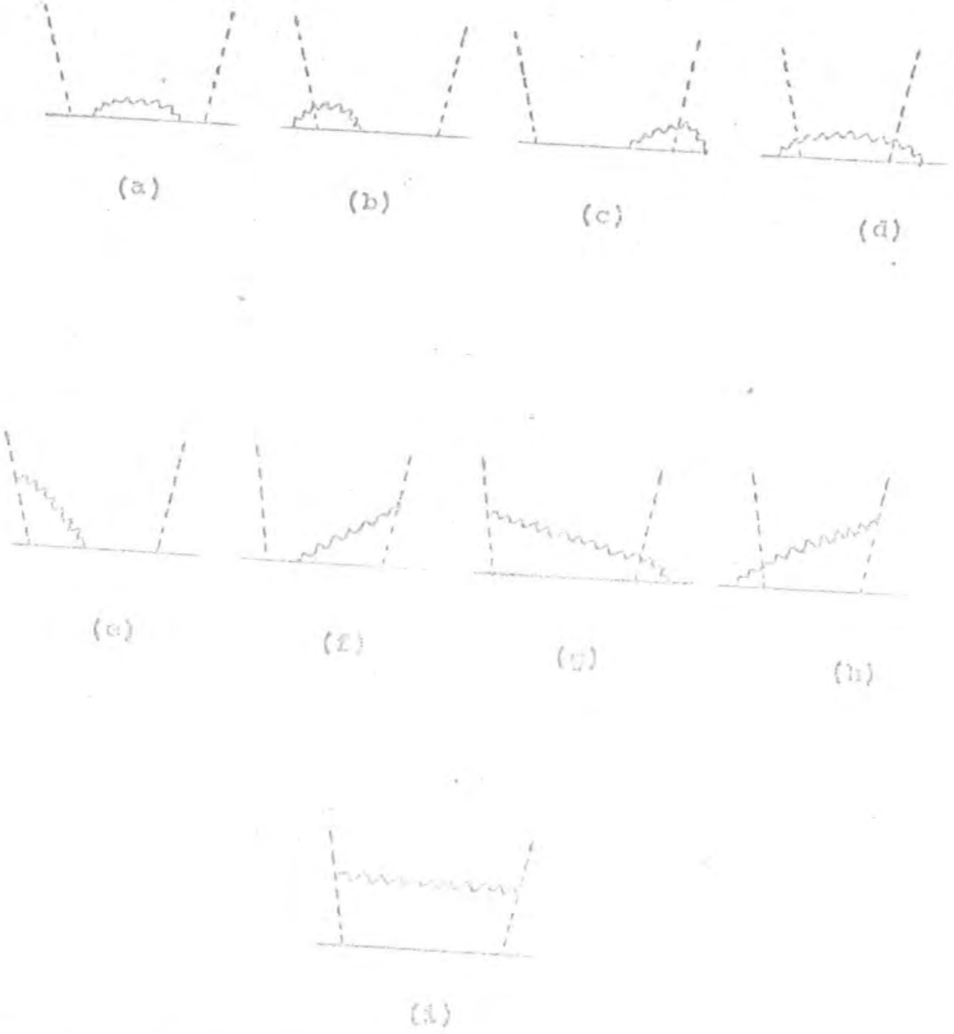
Fig. 2. Diagram for meson-baryon scattering. Solid lines represent baryons, dashed lines mesons.



(a)

(b)

Fig. 3. Lowest order diagrams for meson-baryon scattering.


(a) (b) (c) (d)

(e) (f) (g) (h)

(i)

Fig. 4. Lowest order diagrams for meson-baryon scattering with a photon-baryon intermediate state in the s channel. The wavy lines represent photons.

Fig. 5. Diagram for one photon exchange contribution to the driving force. The blobs represent electromagnetic form factors.

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5. The right hand cut in the s plane from $s = (M + m)^2$ to ∞ becomes two cuts in the W plane: one from $W = +(M + m)$ to ∞ and another from $W = -(M + m)$ to $-\infty$. By a change of variables we have written the integral over both cuts as going from $+(M + m)$ to ∞. See Ref. 3

6. Because of the photon baryon intermediate state, the integral over the right hand cut actually goes from M to ∞, even though the region from M to $M + m$ is below the threshold for meson baryon scattering.

7. Our "nucleons" have only a charge and no anomalous magnetic moments. In computing Eq.(9) and (10) we used simply γ_μ at the photon nucleon vertices.

8. V. Weisskopf, Phys. Rev. 56, 72 (1939)

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10. See the general discussion in M. Ida, Phys, Rev. 135, B499 (1964). For the case of pion nucleon scattering see Ref. 11.

11. M. Ida, Phys. Rev. 136 B1767 (1964)

12. We define K(W) as in Ref 11 by

$$\langle 0|f|N\pi\gamma\rangle \overset{\uparrow}{=} g\sqrt{\frac{M}{2q_0E_0}}\left(\frac{W - i\gamma(p + q)}{2W}K(W) + \frac{W + i\gamma(p + q)}{2W}K(-W)\right) i\gamma_5 \text{ X } U(p)$$

where M, p, E_0 and m, q, q_0 are the mass, momentum, energy of the nucleon and pion respectively. $W = (-(p + q)^2)^{\frac{1}{2}}$. K(M) is normalized to 1, and $K(W) = 1$ for all W corresponds to lowest order perturbation theory.

13. We are using $\Gamma_\mu(W)$ as a sumbolic shorthand for a sum of vertex functions. We define the form factors for the photon nucleon vertex with the nucleon off mass shell by

$$\langle 0 \left| f \right| N\gamma\rangle = \mathcal{E}_\mu \sqrt{\frac{M}{2k_0E_0}} \left\{ \frac{W - i\gamma(p + k)}{2W} F_1(W) + \frac{W + i\gamma(p + k)}{2W} F_1(-W) \right)$$

$$+ \left(\frac{W - i\gamma(p + k)}{2W} F_2(W) + \frac{W + i\gamma(p + k)}{2W} F_2(-W) \; i\sigma_{\mu v}k_v \right\} U(P)$$

where M, p, E_0 and 0, k, k_0 are the mass, momentum, energy of the nucleon and photon respectively. $W = (-(p + k)^2)^{\frac{1}{2}}$, and $\mathcal{E}_\mu$ is the photon polarization $F_1(M)$ and $F_2(M)$ are the nucleon charge and magnetic moment respectively. $F_1(W) = e$, $F_2(W) = 0$ correspond to lowest order perturbation theory. The proper vertex functions are defined in terms of the form factors by $F_i(W) = S_F^{-1} S_F^{1} \Gamma_i(W)$, $i = 1,2$, where S_F is the Born approximation for the nucleon propagator and $S_F^{\prime}$ is the fully renormalized propagator. $\Gamma_1(W) = e$, $\Gamma_2(W) = 0$, $K(W) = 1$, $D(W) = (\frac{dD}{dW}(M))(W - M)$ would reproduce the result for M given in Eq.(15).

14. R. Dashen and S. Frautschi, Phys. Rev 137, 1318 (1965).

15. We obtain the equality in Eq.(30) by again demanding that the residues at the pole at $W = M$ agree. Note: $\underset{\sim}{K}(M) = \underset{\sim}{r}$.

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CHAPTER NINE

Summary

In the present chapter DASHEN's calculation of the neutron-proton mass difference is critically examined . Various inadequacies of DASHEN's calculation are pointed out, and some are overcome in the present calculation In contrast with DASHEN's calculation here we build the D - function from experimental πN phase shifts for $J = \frac{1}{2}$, $I = \frac{1}{2}$ state. Furthermore the infrared divergent contributions to the mass-shift integral are explicitly taken into account by using the prescription of SQUIRES which was developed in Chapter 6. A cut off on the dispersion integrals at 2 and 5 GeV/c, respectively, corresponding to the available phase-shifts from the work of Donnachie, et. al (4), and Roychoudhury et. al (5), respectively, is employed. The resulting change in the answer was of the order 15 - 20 % showing thereby the importance of inelastic contributions. Our final value of $\delta M = M_p - M_n$ difference is in conflict with the experimental value of - 1.3 MeV. Our answer is of the order + 1.01 MeV.

i) INTRODUCTION

DF start with the Chew - Frautschi;$^{(8)}$ bootstrap view of the nucleon prior to the onset of electromagnetic perturbations. It may be recalled that in bootstrap - type calculations there arises a so - called generalized potential for πN scattering. The potential includes various exchanges, such as Nucleon, $N^{*}(1238)$ resonance, ρ etc. plus inelastic effects. It has been shown, to various degrees of confidence depending on one's point of view since doubters are a legion, that these exchanges provide sufficient attractive force in the $J = \frac{1}{2}^{+}$, $I = \frac{1}{2}$ channel to give rise to a bound state, to be identified with the nucleon. Realistically speaking the nucleon, according to the bootstrap point of view ought to be treated as a bound state with components ranging from $N\pi$, $N\pi\pi$ .... upwards. However in practice, DF assume it to be a pole in πN amplitude. The neutron and the proton have the same mass since electromagnetic effects have not yet been included.

Now electromegnetic perturbations are switched on. They alter the generized potential of Chew and Frautschi, or for that matter any mechanism which gave the nucleon as a bound state, be it iterated as in Chew - Frautschi approach or otherwise. The claim of DF - method is that the electromagnetic perturbations cause the potential to change in such a way that it discriminates between the proton and the neutron. These changes may be of various types.

i) One photon exchange between π and N, $\gamma\pi$, γN etc. acting as intermediate states, and in general the perturbations may be looked upon as being represented through all possible diagrams where the intermediate state explicitly contains a photon or photons.

ii) All diagrams representing πN scattering where the intermediate states are unchanged but the parameters characterising the exchange, viz. the mass and the coupling constant are changed.

For example in ρ exchange $V^{\rho} \sim \frac{g\, e^{-mr}}{r}$, one can have an electromagnetic shift in ρNN coupling constant via $\delta V \sim \delta g \, \frac{e^{-mr}}{r}$ and/or a shift in the mass via $\delta V \sim \frac{g \times \delta(e^{-mr})}{r}$.

iii) Changes in the mass of the constituents (here π and N). These are called the external mass shifts. These work through the mechanism of changing the range of the potential and by changing the phase space factor in the unitarity relation used for the purpose of obtaining the amplitude from the potential. For the realistic but complicated Nucleon exchange, for instance, the range is deduced from the location of singularities which depend on the external mass.

An ideal neutron - proton mass difference calculation would be a multi-channel calculation. DF do not claim to have done this but rather to have calculated the mass difference δM, which is written as a dispersion integral in $\delta T(s)D^2(s)$, where $\delta T(s)$ is the potential or the perturbed amplitude, and D is the denominator function of the N/D method and which is supposed to represent the strong interactions exactly before electromagnetism is switched on. After some algebra, done in earlier chapters one gets for the mass-shift, the expression,

$$\delta M = \frac{1}{R[D^2(M)]^2} \frac{1}{\pi}\left[\int_{\text{L.H. Cuts}} \frac{D^2(s^1) \; \text{Im}\, \delta T(s^1)}{s^1 - s_B} ds^1 + \int_R \frac{\text{Im}\, D^2(s^1) \; \delta T(s^1)}{s^1 - s_B} ds^1 \right. \tag{1}$$

or equivalently in the W - plane, for $\ell = 1$,

$$\delta M = \frac{1}{R\, D^2(M)^2}\left\{ \frac{1}{\pi} \int_M^{\infty} dw^1 \, \frac{\text{Im}(D^2 \delta T_1)(W^1)}{W^1 - M} + \frac{\text{Im}(D^2 \delta T_{1-})(-W^1)}{W^1 + M} \right.$$

$$+ \frac{1}{\pi} \int_{\text{L.H.C.}} dw^1 \, \frac{\text{Im}(D^2 \delta T_{1-}(W^1)}{W^1 - M} \tag{2}$$

Where $\delta T_{1-}(W)$ is the change in the πN scattering amplitude due to electromagnetism. The expression as well as its full significance were exposed earlier in an exhaustive manner, and will not be repeated.

As mentioned elsewhere in this work the DF - work has been criticised by BARTON, PATON, and SHAW and WONG(1), and others. The criticism of BARTON and PATON was directed against the DF - treatment of Coulomb-type perturbations, We suggested in a previous chapter a procedure for handling infra - red divergent contributions to $\delta T(W)$, the essential idea of which is the introduction of an energy dependent function to simulate the effect of distant singularities and which is uniquely determined through the prescription that all contributions from $\delta T(s)$, which contain infrared divergent contributions in the limit $\lambda \to 0$, where λ is the fictitious photon mass, must add up to zero. Of course, in an exact calculation up to all orders this would or rather should come out naturally, though this is as yet a hypothesis taken over bodily from the experience of Q.E.D. For our purposes the practical aspect is simple: it is to demand that the contribution proportional to log λ arising from the discontinuity of the left-hand cut contribution due to the one photon exchange in the t - channel of πN scattering be cancelled by the cut contribution arising from other diagram(s) of the same order. The terms proportional to log λ are to be multiplied by the cut-off energy dependent fudge factor $f(C,W)$ of Chapter Six. The demand that C be so chosen that the full contribution proportional to log λ in every order cancel, guarantees that high mass singularities have been,,at least simulated, although at the price of introducing a parameter in each channel. However, this presumably is unavoidable if one is going to do something with infra-red divergent contributions other than drop them altogether as DF suggest. The practical application of our prescription poses no problems. It is elementary. We worked out the procedure explicitly in the Appendix for a particular potential theory model.

There is, however, a more serious criticism of the DASHEN's calculation of the neutron - proton mass difference calculation. This hinges on the proper choice of the D - function. SHAW and WONG repeated DASHEN's calculation but with a physical D - function in P_{11} and P_{33} channel and found that the DASHEN result could not possibly be right since DASHEN represented the unperturbed strong interaction problem through a D - function for $I = \frac{1}{2}$, $J = \frac{1}{2}$ channel by

$$D_{11} = \frac{W - M}{(M - M^1)} \Bigg/ (W - M^1) \qquad \text{where } M^1 = \frac{7}{3} M$$

Dashen chose D_{11} in order to simulate BALÁZS's[(2)] D - function, which has the serious defect that is suppresses the N^* - contribution. DASHEN introduces a factor $C = 6$ to account for the detailed shape of the N^* resonance. Unfortunately such a D_{11} function corresponds to a P_{11} - partial wave with a negative definite phase-shift, in contradiction to experiment. In other words DASHEN misrepresented the unperturbed problem altogether. We would like to point out that in a later publication(3) DASHEN admits this error, though he puts the error in δM at around 20%. this, together with nearly 10% from inelastic $N\gamma$ contribution we obtained in Chapter Eight plus another 15 - 20% error in which DASHEN admits arises from neglected effects of other channels to $\delta T(s)$ but which he did not calculate, just about destroys the value of DASHEN's claim. In addition all I.R.D. divergent contributions were also ignored. Thus it is clear that DASHEN's calculation is of no~~t~~ use except to point out the difficulty of doing a realistic calculation of the neutron-proton mass difference. A realistic calculation would be far more difficult, and would certainly involve CDD - poles. Here we follow SHAW and WONG in taking D - function(~~s~~) from the work of Donnachie et. al$^{(4)}$ which correctly reproduces the experimental data up to 2 Gev/C. In addition we also used the phase-shifts due to Roychoudhury et.al. $^{(5)}$ to create the D - function(~~s~~) up to 5Gev/C. The difference in the two values for δM obtained with the phase shift of ref. 4 with those from ref. 5 was of the order 15 - 20%. In addition we used the prescription of Squires et, al. discussed in Chapter Five in the potential theory context, to take account of the I.R.D. contributions. The final answer gave the wrong sign as well as the wrong order of magnitude for $\delta M = M_p - M_N = + 1.01$, in comparison with the experimental value of N - 1.3 Mev.

ii) MASS - SHIFT CALCULATION - REQUIREMENTS

A realistic calculation of the neutron - proton mass difference even in a 2 channel framework would have to include the following contributions, although some of them would undoubtedly give negligible contribution.

	CONTRIBUTION	CHANNEL
1.	γ exchange	t
2.	external nucleon mass shift	s, t, u
3.	external pion mass shift	nil
4.	mass shift of the exchanged nucleon	s, u
5.	mass and coupling shift of N^* exchange	u
6.	γ N exchange	s, u
7.	$\gamma N \pi$	s, u
8.	$\pi \gamma$	t
9.	ρ, W, ϕ	t
10.	$N\gamma$; $N\pi\gamma$	inelastic contribution

DASHEN has claimed that No. 1 contribution is responsible for the whole of the mass difference δM. We shall do this explicitly to show that DASHEN's result, even with his poor D - function represented in either of the forms given by DASHEN, gives a result which is a sort of warning against treating the problem as simply a one - particle exchange problem.

iii) DETAILS OF $I\gamma$ EXCHANGE CONTRIBUTION

The S - matrix is given by

$$S_{fi} = \delta_{fi} + (2\pi)^4 \, i \, \delta(p_f + q_f - p_i - q_i) \; \frac{M_i M_f}{4 W_i W_f E_i E_f} \quad x$$

$$\bar{U}(p_f) \left[A - i\gamma \, \frac{q_1 + q_2}{2} \, B \right] U(p_i) \qquad (3)$$

Where A and B are functions of the usual invariants.

$$\begin{aligned} s &= -(p_i + q_i)^2 = W^2 \\ t &= -(q_i - q_f)^2 \\ u &= -(p_i - q_f)^2 \end{aligned}$$

The partial wave amplitudes are easily defined if one introduces $z = \hat{q}_i \cdot \hat{q}_f$ = cosine of the scattering angle in the centre of mass. They are given by

$$f_{\ell\pm} = \frac{1}{2} \, dz \qquad f_1 P_\ell(z) + f_2 P_{\ell \mp 1}(z)$$

Where ℓ is the orbital angular momentum of the partial wave, $P = (-1)^{\ell + 1}$ is the parity and the total angular momentum $J = \ell \pm \frac{1}{2}$.

We shall be working in the complex W - plane where $W = \sqrt{s}$ is the total centre of mass energy of the baryon and pion. The analytic properties of the partial wave amplitudes in the W - plane are thoroughly discussed in the literature and we shall not repeat them here.

To avoid trouble with kinematic singularities we work with partial wave amplitudes with $\ell = 1$, $J^P = (\frac{1}{2})^+$ and $(\frac{3}{2})^+$

$$T_{fi}^{J}(W) = \frac{2W}{\sqrt{(E_f - M_f)(E_i - M_i)}} \qquad f_J(W) \qquad (4)$$

with $q = \dfrac{\sqrt{((W + M)^2 - M^2)((W - M)^2 - M^2)}}{2W}$

Let us stress that the case J $=(\frac{3}{2})^{+}$ has nothing to do with our present one-channel calculation. It would be useful if N^{*} exchange were brought in, but this is a multichannel matter.

We now consider contributions to the driving terms from diagrams which involve intermediate states with photons. First let us examine the t channel singularities, where the one photon state is the intermediate state of lowest mass. For the sake of future reference we treat J $= \frac{1}{2}, \frac{3}{2}$ case simultaneously.

The exchange of a single photon gives a contribution to the left hand cut which is not present before the electromagnetic interaction is turned on, and which is different, in general, for states in the same isospin multiplet but with different values of I_z. Let us define $\delta T_{\gamma}^{(\frac{1}{2})}(W)$ and $\delta T^{(\frac{3}{2})}(W)$ as the $J^P = (\frac{1}{2})^{+}$ and $J^P = (\frac{3}{2})^{+}$ partial wave amplitudes for pi meson-baryon scattering with exchange of a single photon (see Chapter Eight, Figure 5). Also, let Q_{π} and Q_B be the pion and baryon charges in units of /e/, μ_B the baryon anomalous magnetic moment in units of /e/2M/, and λ a fictitious photon mass. Recalling the kinematics and definitions of the partial wave amplitudes in Chapter   it is an exercise in the use of the Feynman rules and the formalism for pseudoscalar meson-baryon scattering to show that

$$\overset{T}{\delta T}_{\gamma}^{(\frac{1}{2})}(W) = \alpha Q_{\pi} \left\{ \frac{(W+M)^2 - M^2}{(W-M)^2 - M^2} \; (W-M)\; I_1 Q_B + (W+M)\; I_2 Q_B \right\}$$

$$- \frac{1}{2} \frac{(W+M)^2 - M^2}{(W-M)^2 - M^2} I_3 \, \mu_B - \frac{1}{2} \left(\frac{(W+M)^2 - M^2}{q^2} - 1 \right) I_4 \, \mu_B \Bigg\} \qquad (5)$$

$$\delta T_{\gamma}^{\overset{(3)}{2}}(W) = \alpha Q_{\pi}\left\{\frac{(W+M)^2-M^2}{(W-M)^2-M^2}\right. (W-M) I_1 Q_B + (W+M)\, I_5 Q_B$$

$$\left. - \frac{1}{2}\left\{\frac{(W+M)^2-M^2}{(W \pm M)^2-M^2}\right\} I_3 Q_B + \frac{1}{2} I_6 \mu_B \mp \frac{(W+M)^2-M^2}{4q^2} I_7 \mu_B \right. \qquad (6)$$

where

$$I_1 = \int_{-1}^{+1} dz \, \frac{F_{\pi}(t)\; F_1(t)}{t-\lambda^2}$$

$$I_2 = \int_{-1}^{+1} dz \, \frac{F_{\pi}(t)\; F_1(t)}{t-\lambda^2}$$

$$I_3 = \int_{-1}^{+1} dz \, . \, z \, . \, F_{\pi}(t) . \; F_2(t)$$

$$I_4 = \int_{-1}^{+1} dz \, \frac{3z^2-1}{2} \, \frac{F_{\pi}(t)\; F_1(t)}{t-\lambda^2} \qquad (7)$$

$$I_5 = \int_{-1}^{+1} dz \, \frac{3z^2-1}{2} \, \frac{F_{\pi}(t) . \; F_1(t)}{t-\lambda^2}$$

$$I_6 = \int_{-1}^{+1} dz \, \frac{3z^2-1}{2} \, F_{\pi}(t) . \; F_2(t)$$

$$I_7 = \int_{-1}^{+1} dz \, (3z+1) \; F_{\pi}(t) \, . \, F_2(t)$$

and q is the centre of mass momentum of the meson or baryon,

$t = -2q^2(1 - z)$, and M is the external baryon mass. We have also written $F_\pi(t)$ as the pion electromagnetic form factor, and $F_1(t)$ as the baryon Dirac and Pauli electromagnetic form factors, normalized so that $F(0) = F_1(0) = F_2(0) = 1.$/3/ Terms proportional to λ^2 have been dropped in Eqs (5, 6) since λ will be set equal to zero at the end of the calculation.

In calculating the contribution of one photon exchange to the driving terms we shall be substituting the expressions in Eq. 5 for δT into the dispersion intergrals in Eq 2. We must then do integrals of the form (M^* is, of course, the nucleon mass).

$$\frac{1}{\pi} \int_{LHC} \frac{dW^1}{W^1 - M^*} \quad Im(D^2 \delta T_\gamma)(W^1) \qquad (7)$$

where M^* is the mass of the bound state (resonance), and D(W) is given by DASHEN a~~pproximation~~, expression

$$D(W) = D^1(M^*) \, (W - M^*) \, \frac{M^* - Wo}{W - Wo}$$

The integral in Eq. (7) is most easily done by contour methods.(+)
If we use DASHEN's linear approximation for D(W), i.e. $Wo = \infty$, with the specific representations of the form factors,

$$F_\pi(t) = \frac{M_p^2}{M_p^2 - t} \quad , \quad F_1(t) = \frac{M_1^{\ 2}}{M_1^{\ 2} - t}$$

and

$$F_2(t) = \frac{M_2^{\ 2}}{M_2^{\ 2} - t}$$

(+) For the purpose of explicitly exhibiting DASHEN's results in obtaining Eqs (8,9) we have neglected the contribution of the s-wave cuts, i.e. contributions from Ao, Bo, near the pole $W = -M$. If this is done the coefficient of Q_B in Eq(8) is in agreement with Eq(9) of Dashen's paper /3/.

then the results of doing the integral in Eq. (7) /3/ are

$$\frac{1}{D^1(M^*)^2} \cdot \frac{1}{\pi} \cdot \int_{L.H.C.} dW^1 \; \mathrm{Im}(D^2 \delta T_\gamma^{(\frac{1}{2})}(W^1) = 2\alpha Q_\pi \Big\{ (7M - 2M^*) \Big[\mathrm{Log}\, \frac{Mp}{\lambda}$$

$$- \frac{M_p{}^2}{M_p{}^2 - M_1{}^2} \quad \mathrm{Log}\, \frac{M_p}{M_1} \Big] \quad Q_B + \frac{19M}{4} \Big[\frac{M_p{}^2 M_2{}^2}{4M^2(M_p{}^2 - M_2{}^2} \quad \mathrm{Log}\, \frac{M_p{}^2}{M_2{}^2} \Big] \mu B \qquad (8)$$

$$\frac{1}{D^1(M^*)} \; \frac{1}{\pi} \int_{L.H.C.} dW^1 \; \frac{\mathrm{Im}\big[D^2 \delta T_\gamma^{(\frac{3}{2})} \big] (W^1)}{W^1 - M^*} = 2\alpha Q_\pi \Big\{ (7M - 2M^*) \Big[\mathrm{Log}\, \frac{M_p}{\lambda}$$

$$- \frac{M_p{}^2}{M_p{}^2 - M_1{}^2} \quad \mathrm{Log}\, \frac{M_p}{M_1} \Big] \quad Q_B - \frac{29}{4} M \Big[\frac{M_p{}^2 M_2{}^2}{4M^2(M_p{}^2 - M_2{}^2} \quad \mathrm{Log}\, \frac{M_p{}^2}{M_2{}^2} \Big] \mu B \Big\} \qquad (9)$$

As one expects in computing partial waves of coulomb scattering, our result, Eqs (8), (9), contains a characteristic infrared divergence, i.e., a term which diverges logarithmically as we let the photon mass $\lambda \to 0$. It should be noted that this infrared divergence only occurs in the coefficient of Q_B, but not of μ_B. Dashen and Frautschi have treated the problem of eliminating this spurious infrared divergence in their original paper, and have given a prescription for removing the infrared divergence which we shall follow here for showing the flaw in Dashen's calculation. For one photon exchange, their prescription boils down to computing the Born approximation to $\delta T_\gamma(W)$ (without form factors) for W near the bound state pole, and identifying the term of the form $\log \frac{g(W)}{\lambda}$ which one then subtracts from the expression for $\delta T_\gamma(W)$ computed above(with form factors), thus removing the infrared divergent part. For the case of interest here, this means subtracting out the term which diverges as $\mathrm{Log}\, \frac{2M}{e\lambda}$ as $\lambda \to 0$ (e = 2.718).

Carrying this out, we obtain for the integrals in Eqs. (8) (9):

$$\frac{1}{(D^1(M^*))^2}\,\frac{1}{\pi}\int_{L.H.C.} dW^1\,\frac{\mathrm{Im}\,(D^2\,\delta T_\gamma^{(\frac{1}{2})})(W^1)}{W^1 - M^*} = 2\alpha Q_\pi \Big\{ (7M - 2M^*) \Big[\mathrm{Log}\,\frac{eM_p}{2M} - \frac{M_p^2}{M_p^2 - M_1^2}\,\mathrm{Log}\,\frac{M_p}{M_1} \Big] Q_B + \frac{19}{4}M \Big[\frac{M_p^2 M_2^2}{4M^2(M_p^2 - M_2^2)}\,\mathrm{Log}\,\frac{M_p^2}{M_2^2} \Big] \mu B, \qquad (10)$$

$$\frac{1}{(D^1(M^*))^2}\,\frac{1}{\pi}\int_{L.H.C.} dW^1\,\frac{\mathrm{Im}(D^2\,\delta T_\gamma^{(\frac{3}{2})}\ (W)}{W^1 - M^*} = 2\alpha Q_\pi \Big\{ (7M - 2M^*) \Big[\mathrm{Log}\,\frac{eM_p}{M} - \frac{M_p^2}{M_p^2 - M_1^2}\,\mathrm{Log}\,\frac{M_p}{M_1} \Big] Q_B - \frac{29}{4}M \Big[\frac{M_p^2 M_2^2}{4M^2(M_p^2 - M_2^2)}\,\mathrm{Log}\,\frac{M_p^2}{M_2^2} \Big] \mu B \qquad (11)$$

For the pion form factor we shall use $M_p = 750$ MeV. For both $F_1(t)$ and $F_2(t)$ we shall use the results of one pole fits to the low momentum transfer behaviour of the nucleon form factors which give $M_1^2 = M_2^2 \simeq 20M^2$. We then have $(M^* = M)$

$$\left.\begin{aligned} &\Big[\mathrm{Log}\,\frac{eM_p}{2M} - \frac{M_p^2}{M_p^2 - M_1^2}\,\mathrm{Log}\,\frac{M_p}{M_1} \Big] = 1.4 \\ &\Big[\frac{M_p^2\,M_2^2}{4M^2\,(M_p^2 - M_2^2)}\,\mathrm{Log}\,\frac{M_p^2}{M_2^2} \Big] = .082 \end{aligned}\right\} \qquad (12)$$

If for $F_2(t)$ we had used a Pauli form factor

$$F_2(t) = \left(\frac{M_2^{\,2}}{M_2^{\,2} - t} \right)^2$$, i.e. a two pole fit,

which also fits the data, then the coefficient of μ_B in brackets would have been

$$\left[\frac{M_p^{\,2} M_2^{\,4}}{4M^2(M_p^{\,2} - M_2^{\,2})^2} \left(\frac{M_p^{\,2} - M_1^{\,2}}{M_2^{\,2}} - \mathrm{Log}\, \frac{M_p^{\,2}}{M_2^{\,2}} \right) \right] \qquad (13)$$

If we require that $F_2'(0) = \frac{1}{20M_\pi^2}$, as for the one pole form factors for $F_2(t)$, then

$$\left[\frac{M_p^{\,2} M_2^{\,4}}{4M^2(M_p^{\,2} - M_2^{\,2})^2} \left(\frac{M_p^{\,2} - M_2^{\,2}}{M_2^{\,2}} - \mathrm{Log}\, \frac{M_p^{\,2}}{M_2^{\,2}} \right) \right] = .062 \qquad (14)$$

If now in the first term of Eq. (10), we put $M^* = M$, i.e. the nucleon mass which occurs in the direct channel, realize that there is a kinematic factor $\rho(W)$ in DASHEN's definition of the perturbed amplitude δT, multiply with crossing factor, then we obtain, as the first term contribution to

$$\delta M, \quad - \frac{5}{9} \quad \frac{\alpha}{f^2} \quad \frac{\mu^2}{M} \left[\mathrm{Log}\, \frac{eM_p}{2M_\pi} - \frac{M_p^{\,2}}{M_p^{\,2} - M_1^{\,2}} \, \mathrm{Log} \left(\frac{M_p}{M_1} \right) \right] ;$$

have the residue at the nucleon pole is taken, as in DASHEN, equal to $\frac{3f^2}{M^2}$, where $f^2 = .08$ and M_π is the pion mass.

This is precisely Eq. (9) of DASHEN. The term in square bracket, gave) +1.4 Mev, a moment ago. When all is done one gets the magic number - 1.4 Mev.... H~~owever~~,

iv) CONCLUSIONS AND DETAILS OF OUR CALCULATION

Our first reaction would be one of great surprise since no one has suceeded, prior to DASHEN's work or afterwards in obtaining the observed mass difference $\delta M = M_p - M_n \simeq -1.29$ Mev. The errors were indicated all along by us

i) neglect of an infinitely divergent contribution from the infrared divergent terms;

ii) wrong choice of the D - function;

iii) the unpredictable and probably decisive role of inelastic effects.

The best place to find out DASHEN's omissions is to repeat the calculation with our D - function generated from the πN phase shifts in P_{11} - state. We carried that out, first without altering DASHEN's prescription of neglecting infrared divergent contributions. The answer came out to be +2.1 Mev.

It is clear that the trouble is clearly connected with

1) DASHEN's attempt to treat mass difference problem as a single channel problem, although never explicitly admitting it;

2) and secondly with using the D - function

$$D_{11} = \frac{(W - M)\ (M - M^1)}{(W - M^1)} \quad \text{with } M^1 = (\tfrac{7}{3})M \qquad (15)$$

in an attempt to simulate BALÁZ's D - function.

Now D_{11} given by Eq (15) has the feature that its slope continually decreases for $W < M$, which leads to the suppression of N^* exchange. Indeed the true phase shifts used in the definition of our D_{11} - function show that it has characteristic feature that δ_{11} starts off negative and small but quickly turns over and becomes large and positive going through $\frac{\pi}{2}$ at the pion laboratory kinetic energy $E_L \sim 600$ Mev, ("Roper resonance"). Then assuming that the

"Roper Resonance" as well as the nucleon bound state are predominantly due to forces in the πN-channel we may write, following SHAW and WONG

$$D_{11} = (W - M) \exp [- \frac{(W-M)}{\pi} \int_{M+m_\pi}^{\infty} \frac{\delta_{11}(W^1)\, dW^1}{(W^2 - W)(W^1 - M)} \tag{16}$$

with $\delta_{11}(\infty) = - \pi$

On the other hand if the "Roper Resonance" is supposed to be due mainly to inelastic channels than a pair of CDD zeros in the s-matrix for $J = \frac{1}{2}$, $I = \frac{1}{2}$ state defined by $s_{11} = \eta_{11} e^{2i\delta_{11}}$ (η_{11} is the inelastic factor), located at $W = W_R + iW_I$ appears on the physical sheet. This result is well known and is due to BANDER, COULTER and SHAW /6/. Then $\delta_{11}(\infty) \equiv 0$ and Eq (16) for D_{11} may be changed to

$$D_{11} = (W - M) \frac{(M - W_R)^2 + W_I^2}{(W - W_I)^2 + W_I^2} \times \mathrm{EXP}\left[- \frac{W-M}{\pi} \int_{M+m_\pi}^{\infty} \frac{\delta_{11}(W^1)\, dW^1}{(W^1 - W)(W^1 - M)} \right] \tag{17}$$

We note that Eq (16) and (17) both approach a constant as $W \to \infty$.

On the other hand, as mentioned already, the P_{11} phase shifts of BALÁZS is always negative, which is contrary to experiment. Thus, if DASHEN's calculation is done with correct phase shifts, it will indeed give exactly the opposite sign, as it is shown by our result.

In fact SHAW and WONG used the multichannel, Λ-matrix method of DASHEN et.al /3/ to compute n-p mass difference. Symbolically the 2 x 2 problem of N and $N^*(1238)$ splitting is written as

$$\delta_{n,p} = - \frac{1}{27} \left(5 + 8\beta_{13} \right) \delta_{n,p} - \left(\frac{40}{81} \right) \beta_{13} \delta_{-,++} + \Gamma$$

$$\delta_{-,++} = \frac{1}{9} \left(9 + 16\, \beta_{31} + \beta_{33} \right) \delta_{n,p} + \frac{1}{27} \beta_{33} \delta_{-,++} + \Gamma^*$$

Where the β's depend on D - function of the various channels, as well on coupling constants. It is clear that the treatment of N^* on the same footing as N could be the only way to get a reasonable answer. However, we have other doubts even on this program (see later). Even this procedure is not free from ambiguities as SHAW and WONG admit. The presence of CDD zeros in the s - matrix for P_{11} - state might imply that inelastic channels are important.

We used existing phase shift analysis up to 5 Gev/c to determine our D_{11}- function. Beyond 5Gev/c we put $\delta_{11} = 0$.

When using Eq (17) we used a CDD zero near the pole of the DF D_{11} function (we followed SHAW and WONG once again).

$W_R = 16$, $W_I = 2$ Here also the cut off was fixed at 5 Gev/c.

Since we did not do a multi-channel calculation like SHAW and Wong, who use a Chew - Low model as their static limit, and full DF - multichannel A - matrix formalism, our calculation clearly is not as good as SHAW and WONG. In addition we took account of our prescription for removing infrared divergent contributions by cancelling Iγ exchange in t channel -vs - correction to the nucleon exchange in the U - channel. The whole calculation was carried out exactly as in the potential theory case. Let us remind the reader that the only non zero contribution to δM came from fig 4a of Chapter Seven both in the s - channel and in the U - channel. For the s - channel, in Chapter Seven it was already shown that in field theory a cut off has to be introduced and that δM is then equal to

$$\delta M = \frac{\alpha}{2\pi}\left[3\log\frac{W_{max}}{M} - \frac{1}{2} + \frac{M^2}{2W^2_{\ max}}\right]$$

Now HUANG (7) has shown that if the quantity within the square bracket were so chosen, (here $W_{max} = \sqrt{\nu^2 + 1} + \nu$, where ν is a momentum cut off, equal to 1.12 Mc, M = nucleon mass), then, one obtains the observed mass difference. However, in view of our having obtained the wrong sign with just the left-hand cut - input, the overall answer is still of the wrong sign. Perhaps field theoretic and dispersion theoretic calculations are going to be plagued by the same trouble which has haunted high energy physics ever since 1929, divergences at high energies. We reluctantly agree with SHAW and WONG that δM is "sensitive to the details of the strong interactions"; not only is the magnitude uncertain but also the sign. It is clearly going to be necessary to have more information about the high energy behaviour of form factors, and above all better knowledge of the input. This is clearly a multichannel calculation for which our present work has given us a fairly good preparation, we hope. The role of inelastic contributions would still threaten any "would-be" optimistic calculator.

The numerical results are summarised in the attached table. For purpose of completeness the phase-shifts of Roychoudhury et. al are also attached in Appendix II.

We summarise the results

$\delta M = M_p - M_N \simeq -1.29$ Experimental number

OUR CALCULATIONS	OUTPUT δM WITH USE OF	
INPUT	D - Fn of Eq(16)	D - Fn of Eq(17)
	(CUT OFF 5 Gev/c)	
1γ in t - channel + $N\gamma$ u - channel (without infrared contributions) + $N\gamma$ in s-channel	+ 2.1Mev	+ 1.87 Mev
The same but with the cut off factor f(c,s) of chapter from $-(M+\mu)$ to $-\infty$	+ 1.08 Mev	+ 1.00 Mev
Value of C needed to just cancel ~~the~~ ~~two~~ infrared divergent terms was $C \simeq 3.73$		
INDIVIDUAL EFFECTS		
1γ - t channel	1.73 Mev	.76 Mev
{ $N\gamma$ - u channel~~s~~ " - s-channel } No I.R.D. contribution	.38	.24 Mev
SHAW and WONG		
N - and N^* - multichannel Reciprocal bootstrap with cut off $W_{max} = 15$, D - Fn same as in Eqs (16) and (17) but D_{33} also since calculation was multichannel.	+ 6.5 Mev	

RELATIVE COMPARISON OF OUR NUMERICAL D - FUNCTION - vs ÷ DASHEN's D - FUNCTIONS. (in pion mass unit).

DASHEN D - FUN. $D_{11} = (W - M)$	$D_{11} = \frac{(W-M)^2 (M - 15.5)}{(W - 15.5)}$	D_{11} - from Eq 16 via numerical phase shifts	ENERGY (W) in Mev
-1314	-638	-1224	-375
-1239	-620	-1167	-300
-1189	-607	-1130	-250
-1139	-594	-1092	-200
-1014	-558	- 996	- 75
- 939	-534	- 939	0
61	64	18	1000
			1500
1061	7320	5.4	2000
~ 2100			3000
		3.07	5000

Pion - Nucleon
P_{11} - Phase shifts

Column No. 1. is pion energy in Lab.

Column No. 2 is δ

Column No. 3 is η

14.00	-0.1000	1.0000
21.50	-0.3400	1.0000
24.70	-0.4000	1.0000
31.00	-0.5400	1.0000
37.00	-0.6000	1.0000
41.50	-0.7300	1.0000
58.00	-1.0700	1.0000
63.00	-1.2100	1.0000
98.00	-1.2100	1.0000
113.00	-1.1000	1.0000
120.00	-1.0400	1.0000
137.00	-0.8400	1.0000
142.00	-0.5300	1.0000
151.00	-0.2200	1.0000
157.00	0.1800	1.0000
170.00	0.5400	1.0000
176.00	0.8940	0.9991
187.00	1.4200	0.9981
194.00	1.7600	0.9979
206.00	2.3900	0.9974
210.00	3.1000	0.9976
217.00	3.5700	0.9977
220.00	3.7000	0.9977
233.00	4.4500	0.9977
247.00	7.0100	0.9951
270.00	10.2500	0.9674
311.00	17.7300	0.9257
335.00	22.9400	0.8636
370.00	32.1600	0.7722
410.00	42.7800	0.6120
457.00	57.6100	0.4700
490.00	[illegible]	0.4051
533.00	91.4800	0.3613
570.00	[illegible]	0.3595
591.00	107.0700	0.3641
630.00	[illegible]	[illegible]
655.00	[illegible]	[illegible]
692.00	[illegible]	0.4211
696.00	135.7400	0.4211
745.00	144.1000	0.4342
775.00	145.5400	0.4376

795.90	151.7800	0.4376
820.60	155.3400	0.4352
845.40	158.7900	0.4301
870.10	162.1100	0.4222
899.80	165.9600	0.4089
915.00	167.8700	0.4005
924.60	169.0600	0.3946
949.40	172.0600	0.3773
990.00	176.7500	0.3426
1048.60	182.8900	0.2790
1078.00	187.2400	0.2141
1148.00	179.9100	0.1401
1227.60	160.9100	0.0360
1311.80	129.5900	0.1536
1371.90	131.8900	0.2677
1467.60	138.3400	0.3873
1556.00	143.8100	0.5491
1625.90	147.2500	0.6074
1745.60	170.6200	0.5391
1875.30	144.7600	0.0496
1935.10	159.5000	0.6347
1960.40	25.1720	0.1352
2060.40	22.7027	0.1391
2160.40	27.6561	0.1487
2260.40	23.9170	0.1591
2360.39	22.3371	0.1863
2460.39	19.4510	0.2002
2560.39	17.5073	0.2104
2660.39	16.1682	0.2142
2760.39	15.3077	0.2097
2860.39	15.0623	0.1861
2960.39	15.4077	0.1752
3060.39	15.9334	0.1535
3160.39	15.4221	0.1420
3260.39	13.4494	0.1437
3360.39	10.0938	0.1525
3460.39	6.4789	0.1652
3560.39	5.5678	0.1781
3660.39	5.1441	0.1851
3760.39	3.4991	0.2001
3860.39	2.1841	0.2071
3960.39	1.0911	0.2167
4060.38	1.2[illegible]	0.2277
4160.38	0.5[illegible]	0.2[illegible]
[illegible]	[illegible]	[illegible]
4360.38	-0.[illegible]	0.2[illegible]
[illegible]	-[illegible]	[illegible]
4560.38	-1.1[illegible]	0.2[illegible]
[illegible]	-1.[illegible]	0.2[illegible]
4760.38	-1.8[illegible]	[illegible]
[illegible]	-2.[illegible]	0.2[illegible]
4960.38	-2.3[illegible]	0.2[illegible]
		0

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