The Lagrangian method for chiral symmetry

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THE LAGRANGIAN METHOD
FOR CHIRAL SYMMETRY

by

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PREFACE

The work presented in this thesis was carried out at the Department of Mathematics, University of Durham in the period from October 19 to June 19 under the supervision of Dr. D.B. Fairlie.

The author gratefully acknowledges his indebtedness to Dr. Fairlie for his continued guidance and encouragement as well as the introduction to the subject itself. He has also consented that the material in the papers written by him in corraboration with the present author may be used in this thesis. The author's thanks are also due to his colleagues in particular to Dr. M. Ahmed and Graham Ross for stimulating discussions.

Owing to the comparative complication of the mathematics involved, it was thought appropriate to include a fairly comprehensive description of the general framework of the subject. The quotations from the other authors are explicitly indicated in the text. Otherwise, the work is based essentially on two papers by Dr. Fairlie and the author and a paper by the author himself as well as some unpublished works carried out by the author.

Chapter I incorporates those works done by Dr. Fairlie and the author but also reviews the important works of other authors. No claim of originality is made on Chapter 2, which is necessary only to explain the basic idea of the subject. Chapter 3, Chapter 4 and most of Chapter 5 are claimed to be original.

CONTENTS

ABSTRACT

We describe the non linear realizations of chiral symmetry group and study some of its implications in elementary particle physics. In Chapter 1, the basic concepts of non linear realization techniques are introduced by the way of reviewing the special case of the chiral SU(2)xSU(2) group.

In Chapter 2 the general formalism for chiral SU(n)xSU(n) is developed. This part is wholly dependent on the work by Coleman, Wess and Zumino.

In Chapter 3 the method is generalized for local chiral invariance to describe the non-linear gauge fields.

The Chapter 4 illustrates the use of non linear realization techniques in conjunction with the phenomenological lagrangian. This chapter is introductory to the final Chapter, 5, in which we have attempted to use the phenomenological lagrangian with non linear realization of chiral SU(3)xSU(3) to calculate some low energy hadronic reactions. As an important addition, a description of broken chiral SU(3)xSU(3) is given. This follows the general scheme put forward by Gell-Mann, Oakes and Renner.

CHAPTER 1

Non-linear realization of chiral SU(2)xSU(2)

§1 Non-linear realization with phenomenological Lagrangian

The "non-linear realization" approach to chiral symmetry has received much attention recently. Weinberg[1] was the first to realise that the results of current algegra techniques which have been so successful in explaining several features of elementary particle physics can be reproduced very simply by considering the usual chiral (SU(2)) symmetry as a dynamical symmetry of a gauge-type rather than a conventional algebraic symmetry with a linear representation theory. By implementing this way of "realizing" chiral symmetry in a simple field theoretical model, not only the current algebra results may be obtained with much less labour but also we seem to get more insight into the physics we are trying to understand.

Weinberg's techniques have been quickly developed by a number of authors[2,3,4,5,6,7,8,9] and now there are seen to be fairly simple rules to construct a "chiral invariant" models for a wide range of physical situation.

This also includes a prescription of how to break the symmetry.

It is still very difficult to see if we can establish these techniques as being based on the orthodox field theory. Although there is considerable effort[10,11] to establish them as such by, for instance, studying the possible renormalizability of certain Lagrangian field theory connected to them, the complete success in this direction is not yet certain.

In this thesis, the discussion is confined strictly to the phenomenological side of these techniques, that is to say, the systematic construction of certain fairly simple dynamical model with a "Lagrangian", which, in turn, will be considered merely as a way to calculate physical S-matrix elements. This last statement means that I use this "Lagrangian" to calculate ordinary Feynman graphs with Wick's theorem but I only take a class of Feynman graphs which are obviously calculable (non divergent). These are the graphs without internal loops (so called tree graphs) and thus, strictly speaking, these techniques cannot be considered even as a perturbation approximation to quantum field theory at this stage.

Schwinger[3a] in his work of non-linear realization techniques has suggested the possibility of a new phenomenological theory" of elementary-particle physics which would give the physical basis for the disregard of various field theoretical difficulties in such a technique. Although I do not discuss the philosophy of Schwinger here his way of developing the non-linear realization techniques offers the convenient starting point.

§2 Schwinger's non-linear realization of chiral group

In this §, we follow and expand the analysis of chiral SU(2)xSU(2) symmetry of π-nucleon systems found in reference 3a[5,12].

The low energy pion-nucleon system can be described by the following phenomenological Lagrangian[2a].

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}\partial_\mu \underline{\pi}\, \partial^\mu \underline{\pi} - \frac{1}{2}\mu_\pi^2 \underline{\pi}^2 \\ & + \bar{\psi}(i\partial\!\!\!/ - m_N)\psi \\ & - \frac{f}{\mu_\pi}\bar{\psi}\gamma_\mu\gamma_5\underline{\tau}\psi\,\partial^\mu\underline{\pi} \\ & - \left(\frac{f_0}{\mu_\pi}\right)^2\bar{\psi}\gamma_\mu\underline{\tau}\psi\,(\underline{\pi}\wedge\partial^\mu\underline{\pi}) \end{aligned} \qquad (1.1)$$

with $f \sim$ 1.0 $f_0 \sim$ ... gives the correct value for s and p wave π-N scattering length (when calculated with tree graphs only).

Beside the invariance of $\mathcal{L}$ under usual isotopic spin rotations

$$\underline{\pi} \rightarrow \underline{\pi} - \delta\underline{\beta} \wedge \underline{\pi} \qquad (1.2a)$$

$$\psi \rightarrow \left(1 + i\frac{\underline{\tau}}{2}\cdot\delta\underline{\beta}\right)\psi \qquad (1.2b)$$

Schwinger observed that apart from the pion mass term $\mathcal{L}$ is also invariant under the gauge type transformation

$$\underline{\pi} \rightarrow \underline{\pi} + \delta\underline{\alpha}' \qquad (1.3a)$$

$$\psi \rightarrow \left\{ 1 + i\left(\frac{f_0}{\mu_\pi}\right)^2 \underline{\tau}\,(\underline{\pi} \wedge \delta\underline{\alpha}') \right\}\psi \qquad (1.3b)$$

where the real parameters $\delta\underline{\alpha}'$, $\delta\underline{\beta}$ are assumed to be infinitesimal.

Transformations of nucleon fields (1.2b and 1.3a) actually generate the chiral SU(2)xSU(2) group and we note that there the conventional γ_5 transformation of nucleon fields under the chiral group has been replaced by the addition of a single pion to nucleon states.

The transformation of pion fields , on the other hand, gives the semidirect product SU(2)xT_3. Now

Schwinger further observes that if we replace the "translation" (1.3a) by the non linear transformation

$$\underline{\pi} \rightarrow \underline{\pi} + \delta\underline{\alpha}' + \left(\frac{f_0}{\mu_\pi}\right)^2 \left(2\underline{\pi}(\underline{\pi}\cdot\delta\underline{\alpha}') - \pi^2\delta\underline{\alpha}'\right) \tag{1.4}$$

then both nucleon and pion transformations generate the group SU(2)xSU(2).

The generators of the transformation (1.3b) and (1.4) are the operators containing the differentiation with respect to pion fields II.

We may write (1.2a), (1.2b), (1.3b) and (1.4) in operator form as

$$\left.\begin{aligned} \underline{\pi} &\rightarrow \underline{\pi} + (i\delta\underline{\beta}\cdot\underline{I})\,\underline{\pi} \\ \psi &\rightarrow \psi + (i\delta\underline{\beta}\cdot\underline{I}')\,\psi \\ \underline{\pi} &\rightarrow \underline{\pi} + (i\delta\underline{\alpha}\cdot\underline{G})\,\underline{\pi} \\ \psi &\rightarrow \psi + (i\delta\underline{\alpha}\cdot\underline{G}')\,\psi \end{aligned}\right\} \tag{1.5}$$

with

$$I_i = -i\epsilon_{ij\ell}\,\pi_j \frac{\partial}{\partial \pi_\ell} \tag{1.6}$$

$$(I_i')_{\alpha\beta} = (\tau_i)_{\alpha\beta}/2 \tag{1.7}$$

$$G_i = -\frac{i}{2\lambda}\left\{\delta_{ij}(1-\lambda^2\pi^2) + 2\lambda^2\pi_i\pi_j\right\}\frac{\partial}{\partial\pi_j} \tag{1.8}$$

$$(G_i')_{\alpha\beta} = \lambda\left(\frac{\underline{\tau}}{2}\wedge\underline{\pi}\right)^i_{\alpha\beta} \tag{1.9}$$

where we write $\lambda_0 = f_0/\mu_\pi$ and also for the sake of simplicity, we have used the infinitesimal parameter of chiral transformations $\delta\underline{\alpha} = 2\lambda \cdot \delta\underline{\alpha}'$.

It is easy to show, by direct computation, that

$$[G_i, G_j] = i\epsilon_{ijk} I_k \tag{1.10a}$$

$$[I_i, G_j] = i\epsilon_{ijk} G_k \tag{1.10b}$$

in addition to the familiar iso-spin algebra

$$[I_i, I_j] = i\epsilon_{ijk} I_k \tag{1.10c}$$

The same chiral SU(2)xSU(2) algebra holds for G_i' and I_i'. (In showing this, we should remember that the nucleon transformations (1.2b) and (1.3b) are written in term of contragradient components of a vector).

The non-linear transformation (1.4) can be regarded as a simple generalization of the translation (1.3a) in the sense that it is a subgroup of conformal transformations in 3-dimensional euclidian space.

(1.2), (1.3b) and (1.4) with chiral SU(2)xSU(2) type algebra (1.11a,b,c) express the type of non-linear realization of chiral group first studied by Weinberg and Schwinger. Now it will be immediately observed that the lagrangian (1.1) is not strictly invariant under these transformations. But before trying to generalise (1.1) to "chiral invariant" form, it is more convenient to study some mathematical consequences of these non-linear transformations.

§3 The relation withcchiral 4-vector

The expression of generators of the chiral SU(2)xSU(2) group (1.6) and (1.8) can be used when the group is realized as a transformation group over the field of arbitrary polynomial or analytical function of $(\pi_i)_{i=1}^{3}$. Out of these polynomials, the construction of actual representation (linear) of chiral SU(2)xSU(2) group may be attempted. That we cannot get an arbitrary ineducible representation of chiral SU(2)xSU(2) group can be seen by computing the operator product $\underline{I}\ \underline{G}$ and $\underline{G}\ \underline{I}$.

I get from (1.6) and (1.9)

$$I_i G_j = \left(-i\epsilon_{ist}\pi_s\frac{\partial}{\partial\pi_t}\right)\left(-\frac{i}{2\lambda}\right)\left(\delta_{jp}(1-\lambda^2\pi^2)+2\lambda^2\pi_j\pi_p\right)\frac{\partial}{\partial\pi_p}$$

$$= \left(\frac{-1}{2\lambda}\right)\epsilon_{ist}\pi_s\left\{\left(\delta_{jp}(-2\lambda^2)\pi_t + 2\lambda^2(\delta_{jt}\pi_p + \delta_{tp}\pi_j)\right)\frac{\partial}{\partial\pi_p} + \left(\delta_{jp}(1-\lambda^2\pi^2)+2\lambda^2\pi_j\pi_p\right)\frac{\partial^2}{\partial\pi_t\partial\pi_p}\right\}$$

$\therefore$

$$\underline{I}\cdot\underline{G} = \left(\frac{-1}{2\lambda}\right)\epsilon_{ist}\pi_s\left\{(-2\lambda^2)\pi_t\frac{\partial}{\partial\pi_i} + (2\lambda^2)\pi_i\frac{\partial}{\partial\pi_t} + (1-\lambda^2\pi^2)\frac{\partial^2}{\partial\pi_t\partial\pi_i} + 2\lambda^2\pi_i\pi_p\frac{\partial^2}{\partial\pi_t\partial\pi_p}\right\}$$

$= 0$

i.e. $\underline{I} \cdot \underline{G} = 0$

Similarly I get

$$\underline{G} \cdot \underline{I} = 0$$

$\underline{I}\ \underline{G} = \underline{G}\ \underline{I} = 0$ implies $(\underline{I} + \underline{G})^2 = (\underline{I} - \underline{G})^2$ i.e. in constructing the representation of chiral SU(2)xSU(2) out of analytical function of π's, then any such representation contains the irreducible components of type ($n/2$, $n/2$) n, integer only. (I follow the usual notation for irreducible representations of chiral SU(2)xSU(2) by writing it as (J_1, J_2)).

This condition is also a sufficient one and any arbitrary representation of form (J_1, J_2) with $J_1 = J_2$ may be constructed. To see this, it is sufficient to explicitly construct the lowest $(\frac{1}{2}, \frac{1}{2})$ representation. This is the so-called 4-vector representation and must be constructed as a function of π's. $(\phi_\alpha)_{\alpha=1}^{4}$

$$\left.\begin{aligned} \phi_4 &= G(\pi^2) \\ \phi_i &= F(\pi^2)\,\pi_i \quad i=1,2,3 \end{aligned}\right\} \qquad (1.11)$$

which under the transformation of π's defined above (1.2a and 1.4) transform as chiral 4-vector. (Or vector in 4 dimensional euclidian space). The transformation of $\phi'_{\alpha\beta}$ under iso-spin rotation (1.2a) is obvious and the forms of G and F should be chosen so that under the chiral part of the transformation (1.4) ϕ_4 and ϕ_i transform like

$$\left.\begin{aligned} \delta\phi_4 &= \delta\underline{\alpha}\cdot\underline{\phi} \\ \delta\phi_i &= -\phi_4\cdot\delta\alpha_i \end{aligned}\right\} \qquad (1.12)$$

Using (1.4) $\delta\pi_i = \delta\alpha_j\left(\delta_{ji}(1-\lambda^2\pi^2)+2\lambda^2\pi_i\pi_j\right)\frac{1}{2\lambda}$ in (1.11) L.H.S. of (1.12) becomes

$$\delta\phi_4 = 2G'(1-\lambda^2\pi^2)\,\underline{\pi}\cdot\delta\underline{\alpha}\;/2\lambda$$

$$\delta\phi_i = 2F'(1+\lambda^2\pi^2)\,\underline{\pi}\cdot\delta\underline{\alpha}\;\pi_i/2\lambda$$
$$+\;F((1-\lambda^2\pi^2)\delta\alpha_i + 2\lambda^2\,\underline{\pi}\cdot\delta\underline{\alpha}\cdot\pi_i)/2\lambda$$

where $\quad G' = \dfrac{dG(\pi^2)}{d\pi^2} \qquad F' = \dfrac{dF(\pi^2)}{d\pi^2}$

So that (1.12) becomes

$$2G'(1+\lambda^2\pi^2)\,\underline{\pi}\cdot\delta\underline{\alpha}\;/2\lambda = -F\,\underline{\pi}\cdot\delta\underline{\alpha}$$

$$(2F'(1+\lambda^2\pi^2)+2\lambda^2F)\,\underline{\pi}\cdot\delta\underline{\alpha}\;\pi_i/2\lambda$$
$$+F(1-\lambda^2\pi^2)\,\delta\alpha_i/2\lambda \qquad = -G\,\delta\alpha_i$$

or

$$\left.\begin{aligned} F &= -2G'(1+\lambda^2\pi^2)/2\lambda \\ &= -\frac{1}{\lambda^2}F'\cdot(1+\lambda^2\pi^2) \\ G &= -\frac{1}{2\lambda}F\cdot(1-\lambda^2\pi^2) \end{aligned}\right\} \qquad (1.13)$$

we can easily integrate (1.13) and get the general solution which is regular at

$$G = -\frac{a}{2\lambda}\,\frac{1-\lambda^2\pi^2}{1+\lambda^2\pi^2}$$

$$F = a\,\frac{1}{1+\lambda^2\pi^2}, \qquad (1.14)$$

where a is an arbitrary constant.

It is easy to check that (1.14) actually satisfies the original condition (1.12).

The 4-vector components $(\phi_\alpha)_{\alpha=1}^{4}$ are not independent and satisfy a constraint

$$\phi_4^{\,2} + \boldsymbol{\phi}^2 = a^2/4\lambda^2$$

As for the arbitrary constant a, we choose it so that $\phi_i = \pi_i + O(\pi^2)$. Then a = 1 and the constraint equation is now

$$\phi_4^{\,2} + \boldsymbol{\phi}^2 = \frac{1}{4\lambda^2} \equiv \frac{\mu_\pi^{\,2}}{4f_0^{\,2}} \qquad (1.15)$$

§4 Gursey-Coleman-Zumino-Weiss parameterization of $\boldsymbol{\pi}$ fields; Linearization of nucleon fields

The simplicity of using a (linear) representation is that we can always formally integrate the differential expressions like (1.5) by using exponentials. In case of the non-linear transformation of $\boldsymbol{\pi}$ fields, we can derive anuseful parameterization of π'_α with the aid of $(\frac{1}{2},\frac{1}{2})$ representation constructed above.

The simplest way to express the linear transformation belonging to the 4-vector or $(\frac{1}{2},\frac{1}{2})$ representation of iso scalor - iso vector pair $(\phi_\alpha)_{\alpha=1}^{4}$ is to consider the 2x2 matrix

$$M(\phi) = -\phi_4 + i\,\underline{\phi}\cdot\underline{\tau} \tag{1.16}$$

where $(\tau_i)_{i=1}^{3}$ are Pauli matrices.

Then the chiral transformation generated by the infinitesimal forms (1.4) through (1.12) is(2,4)

$$M(\phi) \rightarrow M(\phi') = e^{i\frac{\underline{\tau}}{2}\cdot\underline{\alpha}}\, M(\phi)\, e^{i\frac{\underline{\tau}}{2}\cdot\underline{\alpha}}.$$

ϕ' being the transform of ϕ by an element of the chiral group with finite parameter $\underline{\alpha}$. Of course, with respect to iso-spin part of the group, we have the usual

$$M(\phi) \rightarrow M(\phi') = e^{i\frac{\underline{\tau}}{2}\cdot\underline{\beta}}\, M(\phi)\, e^{-i\frac{\underline{\tau}}{2}\cdot\underline{\beta}}$$

In terms of original π'_α. $M(\phi)$ is (from (1.14) with a=1).

$$\begin{aligned} M &= \frac{1}{2\lambda}\,\frac{1-\lambda^2\pi^2}{1+\lambda^2\pi^2} + i\,\underline{\pi}\cdot\underline{\tau}\,\frac{1}{1+\lambda^2\pi^2} \\ &= \frac{1}{2\lambda}\,\frac{1-\lambda^2\pi^2+2\lambda\underline{\pi}\cdot\underline{\tau}}{1+\lambda^2\pi^2} \\ &= \frac{1}{2\lambda}\,\frac{1+i\lambda\underline{\pi}\cdot\underline{\tau}}{1-i\lambda\underline{\pi}\cdot\underline{\tau}} \end{aligned} \tag{1.17}$$

i.e. $M(\phi) \equiv M(\pi) = \dfrac{1}{2\lambda}\,\dfrac{1+i\lambda\underline{\pi}\cdot\underline{\tau}}{1-i\lambda\underline{\pi}\cdot\underline{\tau}}$

and (1.17) can be taken as an integrated form of (1.4).[4]

$$\frac{1+i\lambda\boldsymbol{\pi}\cdot\boldsymbol{\tau}}{1-i\lambda\boldsymbol{\pi}\cdot\boldsymbol{\tau}} \rightarrow \frac{1+i\lambda\boldsymbol{\pi}'\cdot\boldsymbol{\tau}}{1-i\lambda\boldsymbol{\pi}'\cdot\boldsymbol{\tau}} = e^{i\frac{\boldsymbol{\tau}}{2}\cdot\boldsymbol{\alpha}}\,\frac{1+i\lambda\boldsymbol{\pi}\cdot\boldsymbol{\tau}}{1-i\lambda\boldsymbol{\pi}\cdot\boldsymbol{\tau}}\,e^{i\frac{\boldsymbol{\tau}}{2}\cdot\boldsymbol{\alpha}} \qquad (1.18)$$

These relations with linear representation automatically guarantee the consistency of the original non-linear transformation as a group operation.

When the integrated form of pion field transformations (1.19) are given, there is an element of the chiral group of special interest, i.e. we may look for a transformation which reduces π i's at given space time point x to zero. Putting the corresponding parameters of the chiral transformation $-\xi_i(x)\,;\, i=1,2,3$ i = 1,2,3, we have[8,9]

$$e^{-i\frac{\boldsymbol{\tau}}{2}\cdot\boldsymbol{\xi}(x)}\,\frac{1+i\lambda\boldsymbol{\pi}(x)\cdot\boldsymbol{\tau}}{1-i\lambda\boldsymbol{\pi}(x)\cdot\boldsymbol{\tau}}\,e^{-i\frac{\boldsymbol{\tau}}{2}\cdot\boldsymbol{\xi}(x)} = 1$$

or

$$e^{i\boldsymbol{\tau}\cdot\boldsymbol{\xi}(x)} = \frac{1+i\lambda\boldsymbol{\pi}(x)\cdot\boldsymbol{\tau}}{1-i\lambda\boldsymbol{\pi}(x)\cdot\boldsymbol{\tau}} \qquad (1.19)$$

(1.19) can be reduced by using the properties of Pauli matrices

$$\cos\sqrt{\xi^2} + i\,\underline{\tau}\cdot\underline{\xi}\,\frac{\sin\sqrt{\xi^2}}{\sqrt{\xi^2}} = \frac{1-\lambda^2\pi^2}{1+\lambda^2\pi^2} + 2i\lambda\,\underline{\tau}\cdot\underline{\pi}\,\frac{1}{1+\lambda^2\pi^2},$$

$\therefore$

$$\frac{1-\lambda^2\pi^2}{1+\lambda^2\pi^2} = \cos\sqrt{\xi^2}$$

$$\frac{2\lambda}{1+\lambda^2\pi^2} = \frac{\sin\sqrt{\xi^2}}{\sqrt{\xi^2}}$$

and (1.19) reduces to

$$\underline{\pi} = \frac{1}{\lambda}\,\underline{\xi}\,\frac{\tan(\sqrt{\xi^2}/2)}{\sqrt{\xi^2}} \tag{1.20}$$

also, the linear quantities can be expressed in term of new parameter

$$\left.\begin{aligned} \phi_4 &= \frac{1}{2\lambda}\,\frac{1-\lambda^2\pi^2}{1+\lambda^2\pi^2} = \frac{1}{2\lambda}\cos\sqrt{\xi^2} \\ \phi_i &= \frac{\pi_i}{1+\lambda^2\pi^2} = \xi_i\,\frac{\sin\sqrt{\xi^2}}{\sqrt{\xi^2}} \end{aligned}\right\} \tag{1.21}$$

The transformation formula (1.17) can be written as a transformation among corresponding parameters

$$e^{i\underline{\tau}\cdot\underline{\xi}} \longrightarrow e^{i\underline{\tau}\cdot\underline{\xi}'} = e^{i\underline{\tau}/2\cdot\underline{\alpha}}\, e^{i\underline{\tau}\cdot\underline{\xi}}\, e^{i\underline{\tau}/2\cdot\underline{\alpha}} \tag{1.22}$$

where $\underline{\xi}$ and $\underline{\xi}'$ are related to $\underline{\pi}$ and $\underline{\pi}'$ through (1.19).

(1.22) is remarkable in the following sense. The matrix equality (1.22) can be transformed as

$$\begin{aligned} &\left(e^{i\underline{\alpha}\cdot\tau/2}\, e^{i\xi\cdot\tau/2}\right)^{-1} e^{i\xi'\cdot\tau/2} \\ &= \left(e^{-i\underline{\alpha}\cdot\tau/2}\, e^{-i\xi\cdot\tau/2}\right)^{-1} e^{-i\xi'\cdot\tau/2} \qquad (1.23) \\ &\equiv U \end{aligned}$$

Since all the factors of (1.23) are unitary - unimodular, the whole matrix U is so too and can be written as an exponential form

$$U = e^{-i\tau\cdot\eta'/2}$$

introducing a set of real parameters η' depending on $\underline{\alpha}$ and ξ.

Thus I can write (1.23) as[8,9]

$$\left.\begin{aligned} e^{i\underline{\alpha}\cdot\tau/2}\, e^{i\xi\cdot\tau/2} &= e^{i\xi'\cdot\tau/2}\, e^{i\eta'\cdot\tau/2} \\ e^{-i\underline{\alpha}\cdot\tau/2}\, e^{-i\xi\cdot\tau/2} &= e^{-i\xi'\cdot\tau/2}\, e^{i\eta'\cdot\tau/2} \end{aligned}\right\} \qquad (1.24)$$

(1.24) actually gives the product of two successive chiral transformations as being decomposed into the product of a chiral transformation and an ordinary iso-spin transformation. Such a decomposition is essentially unique.

In the conventional treatment of chiral invariance the $(\frac{1}{2},0)$ $(0,\frac{1}{2})$ representation, assigned to the nucleon field is expressed by using γ_5 matrix[2]. (To avoid the introduction of a parity doublet). It is clear that (1.24) can be written as

$$e^{i\underline{\alpha}\cdot\underline{\tau}/2\cdot\gamma_5}e^{i\underline{\xi}\cdot\underline{\tau}/2\cdot\gamma_5} = e^{i\underline{\xi}'\underline{\tau}/2\cdot\gamma_5}e^{i\underline{\eta}'\underline{\tau}/2} \tag{1.25}$$

For an infinitesimal parameter $\delta\underline{\alpha}$, I shall compute explicitly the value of $\underline{\eta}'$ in term of $\underline{\xi}$ and $\delta\underline{\alpha}$. First eliminating $\underline{\xi}'$ from (1.24), and taking the term first order in $\delta\underline{\alpha}$ only, I get

$$\cos(\sqrt{\underline{\xi}^2}/2)(-i\underline{\eta}'\underline{\tau}) + \sin(\sqrt{\underline{\xi}^2}/2)(-\tfrac{1}{2}\delta\alpha_i\,\xi_j)[\tau_i,\tau_j] = 0$$

From this I get

$$\underline{\eta}' = \frac{\tan(\sqrt{\underline{\xi}^2}/2)}{\sqrt{\underline{\xi}^2}}\,\underline{\xi}\wedge\delta\underline{\alpha}\ \ (+O(\delta\alpha^2))$$

i.e. from (1.20).

$$\underline{\eta}' = \lambda\underline{\pi}\wedge\delta\underline{\alpha} \tag{1.26}$$

But (1.26) is precisely the parameter appearing in (1.3a) defining the infinitesimal non-linear transformation of nucleon field ψ under chiral group. Suppose that an iso-spinor ψ (which is also ordinary Dirac spinor) transforms under an infinitesimal element of the chiral group according to (1.3a).

$$\psi \to \{1 + i\underline{\tau}/2\cdot(\lambda\underline{\pi}\wedge\delta\underline{\alpha})\}\psi$$
$$= (1 + i\underline{\tau}/2\cdot\underline{\eta}')\psi$$

Let us define the new field Ψ with same spin and iso-spin by Weinberg ref.1,

$$\Psi = e^{i\xi\cdot\tau/2\cdot\gamma_5}\,\psi \qquad (1.27)$$

Then from (1.25), the transformation of Ψ under an infinitesimal element of chiral group will be

$$\begin{aligned}\Psi \rightarrow\ & e^{i\xi'\cdot\tau/2\cdot\gamma_5}\,\psi' \\ \simeq\ & e^{i\xi'\tau/2\,\gamma_5}\, e^{i\eta'\tau/2}\,\psi \\ =\ & e^{i\delta\alpha\cdot\tau/2\cdot\gamma_5}\, e^{i\xi\cdot\tau/2\cdot\gamma_5}\,\psi \\ =\ & e^{i\delta\alpha\cdot\tau/2\,\gamma_5}\,\Psi\end{aligned}$$

i.e.

$$\Psi \longrightarrow e^{i\delta\alpha\cdot\tau/2\,\gamma_5}\,\Psi$$

If we define η' for any order of α in the neighbourhood of $\alpha \sim 0$ by (1.25), we can consider

$$\psi \rightarrow e^{i\eta'\cdot\tau/2}\,\psi \qquad (1.28)$$

as the integrated form of (1.30). The consistency of (1.28) as a group operation is guaranteed through the relation (1.27) and matrix equality (1.25) by the linear transformation[2]

$$\Psi \rightarrow e^{i\alpha\cdot\tau/2\,\gamma_5}\,\Psi \qquad (1.29)$$

Thus I have shown that the non-linear transformation of nucleon fields can also be related to a linear representation of the chiral SU(2)×SU(2) group.

The transformation (1.27) was first used by Weinberg to obtain the non-linear realization starting from the conventional σ-model.

§3 Invariant lagrangian and covariant derivatives[6]

I now come back to the lagrangian (1.1). As have been remarked in §1, (1.1) is not really invariant under the non-linear transformation (1.3b) and (1.4). In particular, the derivatives like $\partial_\mu \pi$ or $\partial_\mu \psi$ naturally transform in rather complicated ways under the non-linear transformations and single iso-spin invariant coupling cannot produce an invariant lagrangian. To find the way to construct chiral invariant lagrangian in this non-linear realization scheme, and which gives the form like (1.1) as a relevent approximation, one can exploit the relation with linear representations of chiral group discussed above. It is easy to construct an invariant lagrangian in terms of the linear representation like Ψ or ϕ_α introduced in §2 onward. Thus a single chiral invariant generalization of the ordinary iso-spin invariant π-N lagrangian with

[illegible]

$$\mathcal{L}' = i\widehat{\bar{\Psi}}\not{\partial}\Psi - m\widehat{\bar{\Psi}}M(\phi)\Psi + \frac{1}{2}\partial_\mu\phi_\alpha\partial^\mu\phi_\alpha \qquad \text{[illegible]}$$

[illegible] ϕ [illegible]

[illegible] $\mathcal{L}'$ [illegible] λ [illegible]

[illegible] Ψ [illegible] ϕ_α [illegible] ψ [illegible] $\underline{\pi}$ [illegible]

$$\partial_\mu\phi_\alpha\partial^\mu\phi_\alpha = \frac{1}{4\lambda^2}\left(-\partial_\mu\frac{1-\lambda^2\pi^2}{1+\lambda^2\pi^2}\right)^2 + \left(\partial_\mu\frac{\pi_i}{1+\lambda^2\pi^2}\right)^2 = \frac{(\partial_\mu\pi)^2}{(1+\lambda^2\pi^2)^2}$$

$$\widehat{\bar{\Psi}}\not{\partial}\Psi = \bar{\psi}\not{\partial}\psi + \bar{\psi}\left(e^{i\tau\cdot\pi/2\,\gamma_5}\gamma_\mu\partial^\mu e^{-i\tau\cdot\pi/2\,\gamma_5}\right)\psi \qquad \text{[illegible]}$$

$$= \bar{\psi}\gamma_\mu\Big\{\partial^\mu + \frac{1}{2}\left(e^{-i\tau\cdot\pi/2}\partial^\mu e^{i\tau\cdot\pi/2} + e^{i\tau\cdot\pi/2}\partial^\mu e^{-i\tau\cdot\pi/2}\right) + \frac{1}{2}\gamma_5\left(e^{-i\tau\cdot\pi/2}\partial^\mu e^{i\tau\cdot\pi/2} - e^{i\tau\cdot\pi/2}\partial^\mu e^{-i\tau\cdot\pi/2}\right)\Big\}\psi$$

$\bar{\Psi}\gamma\Psi$

$$\bar{\Psi}\gamma_\mu\{\partial^\mu + \tfrac{1}{2}(e^{-i\xi\cdot\tau/2}\partial^\mu e^{i\xi\cdot\tau/2} + e^{i\xi\cdot\tau/2}\partial^\mu e^{-i\xi\cdot\tau/2})\}\psi$$

$$\bar{\Psi}\gamma_\mu\gamma_5\{\partial^\mu + \tfrac{1}{2}(e^{-i\xi\cdot\tau/2}\partial^\mu e^{i\xi\cdot\tau/2} - e^{+i\xi\cdot\tau/2}\partial^\mu e^{-i\xi\cdot\tau/2})\}\psi$$

ψ ... γ_5 ...

$$e^{-i\xi\cdot\tau/2}\partial_\mu e^{i\xi\cdot\tau/2} - e^{i\xi\cdot\tau/2}\partial_\mu e^{-i\xi\cdot\tau/2}$$

$$= e^{-i\xi\cdot\tau/2}(\partial_\mu e^{i\tau\cdot\xi})e^{-i\xi\cdot\tau/2}$$

$$= \left(\cos\sqrt{\xi^2}/2 - i\frac{\sin\sqrt{\xi^2}/2}{\sqrt{\xi^2}}\xi\cdot\tau\right)\partial_\mu\left(\frac{1+i\lambda\Pi\cdot\tau}{1-i\lambda\Pi\cdot\tau}\right)$$

$$\times\left(\cos\sqrt{\xi^2}/2 - i\frac{\sin\sqrt{\xi^2}/2}{\sqrt{\xi^2}}\xi\cdot\tau\right)$$

$$= \cos^2\sqrt{\xi^2}/2\,(1-i\lambda\Pi\cdot\tau)\partial_\mu\left(\frac{1+i\lambda\Pi\cdot\tau}{1-i\lambda\Pi\cdot\tau}\right)(1-i\lambda\Pi\cdot\tau)$$

$$= \frac{1}{1+\lambda^2\Pi^2}\,2i\lambda\partial_\mu\Pi\cdot\tau$$

$$i\lambda\bar{\Psi}\gamma_\mu\gamma_5\tau\psi\frac{\partial^\mu\Pi}{1+\lambda^2\Pi^2}$$

$$e^{-i\xi\cdot\tau/2}\partial^\mu e^{i\xi\cdot\tau/2} + e^{i\xi\cdot\tau/2}\partial^\mu e^{-i\xi\cdot\tau/2}$$

$$= e^{-i\xi\cdot\tau/2}\partial^\mu e^{i\xi\tau/2} - (\partial^\mu e^{i\xi\tau/2})e^{-i\xi\cdot\tau/2}.$$

$$= \left[\left(\cos\sqrt{\xi^2}/2 - i\,\boldsymbol{\xi}\cdot\boldsymbol{\tau}\,\frac{\sin\sqrt{\xi^2}/2}{\sqrt{\xi^2}} \right), \left(\partial_\mu \cos\sqrt{\xi^2}/2 \right.\right.$$

$$\left.\left. + i\,\boldsymbol{\xi}\cdot\boldsymbol{\tau}\,\partial_\mu \frac{\sin\sqrt{\xi^2}/2}{\sqrt{\xi^2}} + i\,\partial_\mu \boldsymbol{\xi}\cdot\boldsymbol{\tau}\,\frac{\sin\sqrt{\xi^2}/2}{\sqrt{\xi^2}} \right) \right]$$

$$= \left(\frac{\sin\sqrt{\xi^2}/2}{\sqrt{\xi^2}} \right)^2 \left[\boldsymbol{\xi}\cdot\boldsymbol{\tau} \, , \quad \partial_\mu \boldsymbol{\xi}\cdot\boldsymbol{\tau} \right]$$

$$= 2i\,(\boldsymbol{\xi}\wedge\partial_\mu\boldsymbol{\xi})\cdot\boldsymbol{\tau}\;\frac{\sin^2\sqrt{\xi^2}/2}{\sqrt{\xi^2}}$$

Using the relation $\lambda\boldsymbol{\pi} = \boldsymbol{\xi}\,\dfrac{\tan\sqrt{\xi^2}/2}{\sqrt{\xi^2}}$, the last expression reduces to

$$= 2i\lambda^2\,\frac{(\boldsymbol{\pi}\wedge\partial_\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}}{1+\lambda^2\pi^2}$$

(1.32) now can be written as

$$\bar{\psi}\gamma_\mu\left(\partial^\mu + i\lambda^2\,\frac{\boldsymbol{\pi}\wedge\partial^\mu\boldsymbol{\pi}}{1+\lambda^2\pi^2}\cdot\boldsymbol{\tau}\right)\psi \qquad (1.35)$$

Lastly the term

$$m\,\tilde{\bar{\Psi}}\,M(\phi)\,\Psi$$

obviously reduces to

$$m\,\bar{\psi}\cdot\psi \qquad (1.36)$$

Collecting (1.31), (1.34), (1.35) and (1.36), (1.30) becomes

$$\begin{aligned}
\mathcal{L}' = {} & i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi \\
& + \frac{1}{2}\,\frac{\partial_\mu\boldsymbol{\pi}\,\partial^\mu\boldsymbol{\pi}}{(1+\lambda^2\pi^2)^2} \\
& - \lambda\,\bar{\psi}\gamma_\mu\gamma_5\boldsymbol{\tau}\psi\;\frac{\partial^\mu\boldsymbol{\pi}}{1+\lambda^2\pi^2} \\
& - \lambda^2\,\bar{\psi}\gamma_\mu\boldsymbol{\tau}\psi\;\frac{\boldsymbol{\pi}\wedge\partial^\mu\boldsymbol{\pi}}{1+\lambda^2\pi^2}
\end{aligned} \qquad (1.37)$$

Now compare (1.37) with Weinberg's phenomenological Lagrangian (1.1). If we use the Lagrangian to calculate π-N interactions with only the tree diagrams, the relevant part of (1.37) is

$$\begin{aligned}\mathcal{L}' \sim\ & \tfrac{1}{2}\partial_\mu \underline{\pi}\, \partial^\mu \underline{\pi} \\ & + \bar{\psi}(i\not\partial - m)\psi \\ & - \lambda \bar{\psi}\gamma_\mu \gamma_5 \underline{\tau}\psi\, \partial^\mu \underline{\pi} \\ & - \lambda^2 \bar{\psi}\gamma_\mu \underline{\tau}\psi (\underline{\pi}\wedge\partial^\mu \underline{\pi})\end{aligned} \qquad (1.38)$$

Now it is an experimental fact that more or less correct π-π scattering lengths are obtained from (1.1) with

$$f \sim f_0$$

Thus, with the identification

$$\lambda = f_0/\mu_\pi$$

as in §1, (1.37) is approximately identical with (1.1). But in fact there is no need to approximate the actual ratio $f_0/f \sim 1.0$ by unity.

We have noted that

$$\bar{\psi}\gamma_\mu \gamma_5 \underline{\tau}\psi \frac{\partial^\mu \underline{\pi}}{1+\lambda^2\pi^2}$$

and

$$\bar{\psi}\gamma_\mu\left(\partial^\mu + i\lambda^2 \frac{\underline{\pi}\wedge\partial^\mu\underline{\pi}}{1+\lambda^2\pi^2}\cdot\underline{\tau}\right)\psi$$

are by themselves invariant under chiral transformations. In particular, this means that we can put an arbitrary constant coefficient to the interaction term $-\lambda \bar{\Psi}\gamma_\mu\gamma_5\tau\Psi \frac{\partial^\mu \pi}{1+\lambda^2\pi^2}$ in (1.37) without destroying the chiral invariance. Thus replacing this by

$$-f/f_0\, \lambda \bar{\Psi}\gamma_\mu\gamma_5\tau\Psi \frac{\partial^\mu \pi}{1+\lambda^2\pi^2}$$

(1.1) is recovered within the approximation of (1.36). Thus I get a chiral invariant generalization of (1.1)

$$\begin{aligned}\mathcal{L} &= i\bar{\Psi}\gamma\Psi + m\bar{\Psi}\Psi \\ &+ \frac{1}{2}\frac{\partial_\mu \pi\, \partial^\mu \pi}{(1+\lambda^2\pi^2)^2} \\ &- f/\mu_\pi\, \bar{\Psi}\gamma_\mu\gamma_5\tau\Psi \frac{\partial^\mu \pi}{1+\lambda^2\pi^2} \\ &- \left(\frac{f_0}{\mu_\pi}\right)^2 \bar{\Psi}\gamma_\mu\tau\Psi \frac{\pi\wedge\partial^\mu\pi}{1+\lambda^2\pi^2}\end{aligned} \qquad (1.39)$$

Only essential difference here is the absence of meson mass term and such a term cannot be accounted unless the symmetry breaking is introduced.

The invariance of terms in the Lagrangian discussed above also implies the special transformation property of each factors consisting these terms.

Thus from the invariance of

$$\bar{\Psi}\gamma_\mu\gamma_5\tau\Psi \frac{\partial^\mu \pi}{1+\lambda^2\pi^2}$$

and from the "iso-spin type" of non-linear transformation of ψ field (1.26), it can be seen that the quantity

$$\frac{\partial_\mu \underline{\pi}}{1+\lambda^2\pi^2}$$

transform like iso-spin-one object with same parameter η' (of 1.26) under chiral group. Similarly, the invariance of

$$\bar{\Psi}\gamma_\mu\left(\partial^\mu + i\lambda^2\,\frac{\underline{\pi}\wedge\partial^\mu\underline{\pi}}{1+\lambda^2\pi^2}\cdot\underline{\tau}\right)\psi$$

implies that

$$\left(\partial^\mu + i\lambda^2\,\frac{\underline{\pi}\wedge\partial^\mu\underline{\pi}}{1+\lambda^2\pi^2}\,\underline{\tau}\right)\psi$$

should transform exactly like ψ itself under chiral group. These transformation properties can be, of course, verified by explicit computation. I have in fact just introduced the covariant derivatives of Weinberg which transform simply by the "nucleon like" rule (1.26) (with corresponding iso-spin) and can be used instead of ordinary derivatives $\partial_\mu\underline{\pi}$ and $\partial_\mu\psi$. I shall write, after Weinberg,(7)

$$\nabla_\mu\underline{\pi} \equiv \frac{\partial_\mu\underline{\pi}}{1+\lambda^2\pi^2} \qquad (1.40)$$

$$\nabla_\mu\psi \equiv \left(\partial_\mu + \frac{i\lambda^2(\underline{\pi}\wedge\partial_\mu\underline{\pi})\cdot\underline{\tau}}{1+\lambda^2\pi^2}\right)\psi \qquad (1.41)$$

then (1.39) is written as

$$\begin{aligned}\mathcal{L} &= \bar{\Psi}(i\not\nabla - m)\psi \\ &+\tfrac{1}{2}\nabla_\mu\underline{\pi}\,\nabla^\mu\underline{\pi} \\ &-\frac{f}{\mu_\pi}\bar{\Psi}\gamma_\mu\gamma_5\underline{\tau}\psi\nabla^\mu\underline{\pi}\end{aligned} \qquad (1.42)$$

how general is the construction of invariant lagrangian in term of these covariant derivatives? To make the later generalization to chiral SU(3) more straightforward I will follow the argument of Coleman and Zumino[8] rather than original treatment of Weinberg, in answering this problem.

Suppose that $\mathcal{L}$ is an arbitrary chiral invariant lagrangian

$$\mathcal{L} = \mathcal{L}(\underline{\pi}(x), \partial_\mu \underline{\pi}(x), \psi(x), \partial_\mu \psi(x))$$

should be invariant under the non-linear chiral transformations (1.18), (1.14) and (1.28). In particular, the special transformation discussed for the introduction of the parameter $\underline{\xi}(x)$ (1.19) should keep $\mathcal{L}$ invariant since this can be formally considered as a chiral transformation with parameter $\underline{\alpha}$ being equal to $-\underline{\xi}(x)$ with arbitrary but definite space-time point x. If this transformation is denoted by g_x, as a member of chiral group we get from (1.19)

$$g_x \underline{\pi}(x) = 0 \qquad (1.43)$$

The replacement of $\underline{\alpha}$ by $-\underline{\xi}(x)$ in (1.24) gives $\underline{\eta}' = 0$

So, for the nucleon field

$$g_x \psi(x) = \psi(x) \qquad (1.44)$$

Thus at any given space-time point x, the invariant lagrangian $\mathcal{L}$ reduces as

$$\begin{aligned} &\mathcal{L}\left(\underline{\pi}(x), \partial_\mu \underline{\pi}(x), \psi(x), \partial_\mu \psi(x)\right) \\ &= \mathcal{L}\left(0, g_x \partial_\mu \underline{\pi}(x), \psi(x), g_x \partial_\mu \psi(x)\right) \\ &\equiv \mathcal{L}'\left(g_x \partial_\mu \underline{\pi}(x), \psi(x), g_x \partial_\mu \psi(x)\right) \end{aligned} \tag{1.45}$$

As for the quantities $g_x \partial_\mu$ ---- one must remember that the space-time point x must be considered as fixed so that the g_x is the chiral transformation with constant parameter (not the local chiral transformation). Thus

$$g_x \partial_\mu f(x) = \left[\frac{\partial}{\partial y_\mu} g_x f(x+y)\right]_{y=0} \tag{1.46}$$

Now

$$g_x \psi(x+y) = e^{i \underline{\eta}' \cdot \underline{\tau}/2} \psi(x+y)$$

$$e^{i \underline{\xi}' \cdot \underline{\tau}} = \frac{1 + i\lambda g_x \underline{\pi}(x+y)\cdot\underline{\tau}}{1 - i\lambda g_x \underline{\pi}(x+y)\cdot\underline{\tau}}$$

where the parameter $\underline{\xi}'$ and $\underline{\eta}'$ are given by the matrix equation (1.24)

$$e^{\mp i \underline{\xi}(x)\cdot\underline{\tau}/2} e^{\pm i \underline{\xi}(x+y)\cdot\underline{\tau}/2} = e^{\pm i \underline{\xi}'\cdot\underline{\tau}/2} e^{i \underline{\eta}'\cdot\underline{\tau}/2}.$$

To compute the first derivatives, it is enough to estimate the above relations upto first order in

(It is assumed that the "fields" $\xi(x)$ and $\psi(x)$ can be considered here as suitably smooth function of y).

Then I have

$$e^{\mp i\xi(x)\cdot\tau/2}\, e^{\pm i\xi(x+y)\cdot\tau/2} \simeq \left(1 + i\frac{\tau}{2}\cdot(\pm\xi' + \eta')\right)$$

and

$$g_x\,\psi(x+y) \simeq \psi(x+y) + i\eta'\cdot\tau\,\psi(x)$$

$$\lambda\tau g_x\,\pi(x+y) \simeq i\xi'\cdot\tau/2$$

So, from (1.46)

$$g_x\partial_\mu\psi = \partial_\mu\psi + i\partial_\mu\eta'\cdot\tau/2\,\psi$$

$$i\lambda\tau g_x\partial_\mu\pi = i\partial_\mu\xi'\cdot\tau/2.$$

also

$$e^{i\tau/2\cdot(\partial_\mu\eta' \pm \partial_\mu\xi')} = e^{\mp i\xi\cdot\tau/2}\,\partial_\mu\, e^{\pm i\xi\cdot\tau/2}.$$

$$\therefore\quad i\tau/2\,\partial_\mu\eta' = \frac{1}{2}\left(e^{-i\xi\cdot\tau/2}\partial_\mu e^{i\xi\cdot\tau/2} + e^{i\xi\cdot\tau/2}\partial_\mu e^{-i\xi\cdot\tau/2}\right)$$

$$i\tau/2\,\partial_\mu\xi' = \frac{1}{2}\left(e^{-i\xi\cdot\tau/2}\partial_\mu e^{i\xi\cdot\tau/2} - e^{i\xi\cdot\tau/2}\partial_\mu e^{-i\xi\cdot\tau/2}\right)$$

Thus

$$g_x\partial_\mu\psi = \left\{\partial_\mu + \frac{1}{2}\left(e^{-i\xi\cdot\tau/2}\partial_\mu e^{i\xi\cdot\tau/2} + e^{i\xi\cdot\tau/2}\partial_\mu e^{-i\xi\cdot\tau/2}\right)\right\}\psi$$

$$i\tau\cdot\lambda g_x\partial_\mu\pi = \frac{1}{2}\left(e^{-i\xi\cdot\tau/2}\partial_\mu e^{i\xi\cdot\tau/2} - e^{i\xi\cdot\tau/2}\partial_\mu e^{-i\xi\cdot\tau/2}\right)$$

But these expressions are just the ones appearing when I have defined covariant derivatives. Thus looking back at the transition from (1.32) and (1.33) to (1.35) and (1.34), it can be seen immediately[8]

$$g_\alpha \partial_\mu \psi = \left(\partial_\mu + \frac{i\lambda^2 \underline{\pi} \wedge \partial_\mu \underline{\pi}}{1+\lambda^2 \pi^2} \underline{\tau} \right) \psi$$

$$= \nabla_\mu \psi \tag{1.47}$$

$$g_\alpha \partial_\mu \underline{\pi} = \frac{\partial_\mu \underline{\pi}}{1+\lambda^2 \pi^2}$$

$$= \nabla_\mu \pi \tag{1.48}$$

From these discussions, I can conclude that an arbitrary chiral invariant lagrangian reduces to the form

$$\mathcal{L} = \mathcal{L}'(\nabla_\mu \underline{\pi}(x), \psi(x), \nabla_\mu \psi(x)) \tag{1.49}$$

On the other hand, I have just shown that these covariant derivatives transform similarly to an iso-spin type transformation (1.28) ($I=\frac{1}{2}$ for $\nabla_\mu \psi$ $I=1$ for $\nabla_\mu \underline{\pi}$). Therefore if $\mathcal{L}'$ in (1.49) is constructed in an iso-spin invariant way out of these arguments, then it is already invariant under the full chiral SU(2)xSU(2). So a general rule of constructing an invariant lagrangian is the following: Take any iso-spin invariant lagrangian which depends on π'_λ only through its derivatives. (The discussion above, in particular $g_\alpha \pi(x) = 0$ shows that there can be no explicit dependence on $\underline{\pi}(x)$. This excludes for example, $\underline{\pi}$ -mass term in an invariant lagrangian) and replace the derivatives of the fields by corresponding covariant derivatives.

This rule of course applies to the system with "pions" and any number of fields with arbitrary isospin.

§6 Currents[15]

a) Variational method

In the current algebra approach to chiral symmetry, it is the vector and axial vector currents rather than the lagrangian which are of central importance.

If the functional (1.39) is taken as a field theoretical lagrangian, the corresponding currents can be derived through Noether's theorem. (Gell-Mann Levy)[14]. Writing the infinitesimal parameters of local iso-spin and chiral transformation $\delta\underline{\beta}(x)$ and $\delta\underline{\alpha}(x)$ respectively, the usual expression of vector and axial vector currents are found.

$$\underline{V}^{\mu} = -\frac{\delta \mathcal{L}}{\delta \underline{\beta}_{,\mu}}\bigg|_{\underline{\beta}=0} \qquad (1.50)$$

$$\underline{A}^{\mu} = -\frac{\delta \mathcal{L}}{\delta \underline{\alpha}_{,\mu}}\bigg|_{\underline{\alpha}=0} \qquad (1.51)$$

where

$$\alpha_{,\mu} \equiv \partial_{\mu}\alpha \qquad \beta_{,\mu} \equiv \partial_{\mu}\beta$$

First, let us consider the system with π's only. The invariant term corresponding to this is (1.39) is

$$\mathcal{L}_{\pi} = \frac{1}{2} \frac{\partial_{\mu} \underline{\pi}\, \partial^{\mu} \underline{\pi}}{(1+\lambda^2 \pi^2)^2} = \frac{1}{2} \nabla_{\mu} \underline{\pi}\, \nabla^{\mu} \underline{\pi} \tag{1.52}$$

Then the variation of $\partial_{\mu} \underline{\pi}$ due to the space-time derivative of infinitesimal iso-spin and chiral transformation parameters are from (1.2a) and (1.4)

$$\delta_{is} \partial_{\mu} \underline{\pi} = -\partial_{\mu} \delta \underline{\beta} \wedge \underline{\pi} + \text{term proportional to } \delta \underline{\beta} \tag{1.53}$$

$$\delta_{ch} \partial_{\mu} \underline{\pi} = \frac{1}{2\lambda} \left\{ \partial_{\mu} \delta \underline{\alpha} (1-\lambda^2 \pi^2) + 2\lambda^2 (\underline{\pi} \cdot \partial_{\mu} \delta \underline{\alpha}) \underline{\pi} \right\} \tag{1.54}$$

The suffixes "is" and "ch" refer to local iso-spin and chiral transformation respectively.

Thus

$$\delta_{is} \mathcal{L} = \frac{-1}{(1+\lambda^2 \pi^2)^2} \partial_{\mu} \delta \underline{\beta} \cdot (\underline{\pi} \wedge \partial_{\mu} \underline{\pi}) + \delta \underline{\beta} \text{ terms} \tag{1.55}$$

$$\delta_{ch} \mathcal{L} = \frac{1}{(1+\lambda^2 \pi^2)^2} \Big\{ (1-\lambda^2 \pi^2) \partial_{\mu} \underline{\pi} \cdot \partial^{\mu} \delta \underline{\alpha} + 2\lambda^2 (\underline{\pi} \cdot \partial_{\mu} \delta \underline{\alpha})(\underline{\pi} \cdot \partial_{\mu} \underline{\pi}) \Big\} \frac{1}{2\lambda} + \delta \underline{\alpha} \text{ term} \tag{1.56}$$

So now (1.50) and (1.51) give

$$\underline{V}_\mu = \frac{\underline{\pi} \wedge \partial_\mu \underline{\pi}}{(1+\lambda^2\pi^2)^2} \tag{1.57}$$

$$\underline{A}_\mu = -\frac{1}{(1+\lambda^2\pi^2)^2}\frac{1}{2\lambda}\left\{(1-\lambda^2\pi^2)\partial_\mu\underline{\pi} + 2\lambda^2(\underline{\pi}\cdot\partial_\mu\underline{\pi})\underline{\pi}\right\} \tag{1.58}$$

Compare this with the bilinear form

$$V_\mu{}^2 + A_\mu{}^2 .$$

From (1.56) and (1.57) we have

$$\begin{aligned}
&(V_\mu{}^2 + A_\mu{}^2) \times (1+\lambda^2\pi^2)^4 \\
&= (\underline{\pi}\wedge\partial_\mu\underline{\pi})^2 + ((1-\lambda^2\pi^2)\partial_\mu\underline{\pi} + 2\lambda^2(\underline{\pi}\cdot\partial_\mu\underline{\pi})\underline{\pi})^2 \\
&= (\partial_\mu\underline{\pi})^2 - (\underline{\pi}\cdot\partial_\mu\underline{\pi})^2 \\
&\quad + \frac{(1+\lambda^2\pi^2)^2}{4\lambda^2}(\partial_\mu\underline{\pi})^2 + (1-\lambda^2\pi^2)(\underline{\pi}\cdot\partial_\mu\underline{\pi})^2 + \lambda^2(\underline{\pi}\cdot\partial_\mu\underline{\pi})\underline{\pi}^2 \\
&= \frac{(1+\lambda^2\pi^2)^2}{4\lambda^2}(\partial_\mu\underline{\pi})^2
\end{aligned}$$

Thus
$$V_\mu{}^2 + A_\mu{}^2 = \frac{1}{4\lambda^2}\frac{(\partial_\mu\underline{\pi})^2}{(1+\lambda^2\pi^2)^2}$$

$$= \frac{1}{2\lambda^2}\mathcal{L}$$

or

$$\mathcal{L} = 2\lambda^2(V_\mu{}^2 + A_\mu{}^2) \tag{1.59}$$

This is of the form considered by Sugawara(15,16).

The nucleon contribution to the currents can be derived by using the rest of lagrangian (1.39).

With the same notation as above

$$\delta_\alpha \partial_\mu \psi = i\lambda \partial_\mu \delta\underline{\alpha} \wedge \underline{\pi} \cdot \underline{\tau}/2\, \psi + \delta\underline{\alpha} \text{ term}$$

(cf 1.28)

and thus
$$\delta_\alpha \nabla_\mu \psi = i\lambda\left(\underline{\pi}\wedge\partial_\mu\delta\underline{\alpha}\cdot\underline{\tau}/2 + \frac{1-\lambda^2\pi^2}{1+\lambda^2\pi^2}\,\underline{\pi}\wedge\partial_\mu\delta\underline{\alpha}\cdot\underline{\tau}/2\right)\psi$$
$$+\ \delta\underline{\alpha} \text{ term}$$
$$= \left(i\lambda\,\underline{\pi}\wedge\partial_\mu\delta\underline{\alpha}\cdot\underline{\tau}/2\,\frac{2}{1+\lambda^2\pi^2}\right)\psi + \delta\underline{\alpha} \text{ term}$$
$$\therefore\quad \delta(i\bar{\psi}\not{\nabla}\psi) = \frac{-1}{1+\lambda^2\pi^2}\,\bar{\psi}\gamma_\mu\,\underline{\tau}\wedge\underline{\pi}\,\psi\,\partial_\mu\delta\underline{\alpha}$$
$$+\ \delta\underline{\alpha} \text{ term}$$

also
$$\delta_\alpha \bar{\psi}\gamma_\mu\gamma_5\underline{\tau}\psi\cdot\nabla^\mu\underline{\pi}$$
$$= \frac{1}{2\lambda}\,\frac{1}{1+\lambda^2\pi^2}\left\{\partial_\mu\delta\underline{\alpha}(1-\lambda^2\pi^2) + 2\lambda^2(\underline{\pi}\cdot\partial_\mu\delta\underline{\alpha})\underline{\pi}\right\}\bar{\psi}\gamma_\mu\gamma_5\underline{\tau}\psi$$
$$+\ \delta\underline{\alpha} \text{ term}$$

Thus the nucleon part of axial current
$$A_\mu' = -\left.\frac{\delta\mathcal{L}^{N\pi}}{\delta\alpha_\mu}\right|_{\alpha=0}$$
$$= \frac{1}{1+\lambda^2\pi^2}\,\bar{\psi}\gamma_\mu(\underline{\tau}\wedge\underline{\pi})\psi$$
$$+\frac{f}{f_0}\left\{\frac{1-\lambda^2\pi^2}{1+\lambda^2\pi^2}\,\bar{\psi}\gamma_\mu\gamma_5\underline{\tau}/2\,\psi + \frac{\lambda^2\underline{\pi}}{1+\lambda^2\pi^2}\,\bar{\psi}\gamma_\mu\gamma_5(\underline{\tau}\cdot\underline{\pi})\psi\right\}$$

(I-60)

For the iso-spin transformation, similar arguments lead to

$$\delta_{is}(i\bar{\Psi}\not{\partial}\Psi) = -\frac{1-\lambda^2\pi^2}{1+\lambda^2\pi^2}\bar{\Psi}(\tau/2\cdot\partial\delta\beta)\gamma^\mu\Psi - \frac{\lambda^2\pi}{1+\lambda^2\pi^2}\pi\cdot\partial\delta\beta\,\bar{\Psi}(\pi\cdot\tau)\Psi + \delta\beta\,6\tau\tau$$

$$\delta_{is}(\bar{\Psi}\gamma_\mu\gamma_5\tau\Psi\nabla^\mu\pi)$$

$$= -\bar{\Psi}\gamma_\mu\gamma_5\tau\Psi\,\partial^\mu\delta\beta\wedge\pi + \delta\beta\,6\tau\tau$$

and thus

$$V_\mu' = \frac{1-\lambda^2\pi^2}{1+\lambda^2\pi^2}\bar{\Psi}\tau/2\,\gamma_\mu\Psi + \frac{\lambda^2\pi}{1+\lambda^2\pi^2}\bar{\Psi}(\pi\cdot\tau)\Psi - \left(\frac{f}{f_0}\right)\frac{\lambda}{1+\lambda^2\pi^2}\bar{\Psi}\gamma_\mu\gamma_5(\pi\wedge\tau)\Psi \quad (1.61)$$

In the lowest order of π, (1.59) and (1.60) reduce to

$$A_\mu' \sim \frac{f}{f_0}\bar{\Psi}\gamma_\mu\gamma_5\tau/2\,\Psi \quad (1.62)$$

$$V_\mu' \sim \bar{\Psi}\gamma_\mu\tau/2\,\Psi \quad (1.63)$$

(1.61) gives the axial vector coupling constants to nucleon as

$$\frac{G_A}{G_V} = \frac{f}{f_0} \quad (1.64)$$

Thus G_A/G_V can be accounted by π-N contact interactions in (1.39) (or rather approximate (1.1)) which in fact dominate π-N scattering lengths. This is essentiallytthe Adler-Weissberger relation as have been noted by Weinberg[1]. Further, the pionic part of axial current (1.57) gives the axial vector coupling to single pion as

$$F_\pi/2 = -1/2\lambda = -1/2 \cdot \frac{\mu_\pi}{f}. \qquad (1.65)$$

$$\therefore \quad \frac{G_A}{G_V} = \frac{f}{f_0} = -\frac{F_\pi f}{\mu_\pi}$$

On the other hand f/μ_π is the coefficient of derivative type Yukawa coupling in (1.1). Resulting nucleon-pole term in π-N scattering amplitude (with tree-graph only) is used to define N-π coupling constant g as

$$\frac{g}{2m_N} = \frac{f}{\mu_\pi} \qquad (1.66)$$

and thus

$$\frac{G_A}{G_V} = -\frac{g F_\pi}{2m_N} \qquad (1.67)$$

(4, 7)

This is Goldberger-Treiman relation. ~~(Weinber Weis and Zumino~~.

It has been shown that pion Lagrangian (1.52) can be written in current-current form. (Sugawara type).

Fairlie[13,15] has shown that the same expression holds for the entire π-N lagrangian if certain simple terms are added to (1.39). The original derivation using directly the expression (1.56),(1.57), (1.59) and (1.60) involves lengthy arithmetic, so I shall leave the proof until the next chapter where the convenient expression for the current is obtained first. Here I shall state only the result. Using the expression of total currents

$$\begin{aligned} A_\mu^{tot} = {} & \bar\Psi\gamma_\mu \frac{1}{1+\lambda^2\pi^2}\,\underline{\tau}\wedge\underline{\pi}\,\psi \\ & + \frac{f}{f_0}\,\bar\Psi\gamma_\mu\gamma_5\left\{\frac{1-\lambda^2\pi^2}{1+\lambda^2\pi^2}\,\frac{\underline{\tau}}{2} + \frac{\lambda^2\underline{\pi}}{1+\lambda^2\underline{\pi}}\,\underline{\pi}\cdot\underline{\tau}\right\}\psi \\ & - \frac{i}{(1+\lambda^2\pi^2)^2}\,\frac{1}{2\lambda}\left\{(1-\lambda^2\pi^2)\partial_\mu\underline{\pi} + 2\lambda^2(\underline{\pi}\cdot\partial_\mu\underline{\pi})\,\underline{\pi}\right\} \end{aligned}$$

and

$$\begin{aligned} V_\mu^{tot} = {} & \bar\Psi\gamma_\mu\left\{\frac{1-\lambda^2\pi^2}{1+\lambda^2\pi^2}\,\frac{\underline{\tau}}{2} + \frac{\lambda^2\pi^2}{1+\lambda^2\pi^2}\,\underline{\pi}\cdot\underline{\tau}\right\}\psi \\ & - \frac{f}{f_0}\,\frac{\lambda}{1+\lambda^2\pi^2}\,\bar\Psi\gamma_\mu\gamma_5(\underline{\pi}\wedge\underline{\tau})\psi \\ & + \frac{(\underline{\pi}\wedge\partial_\mu\underline{\pi})}{(1+\lambda^2\pi^2)^2} \end{aligned}$$

it can be shown that

$$A_\mu^2 + V_\mu^2 = \frac{1}{2\lambda^2}\left(\mathcal{L} - i\bar\Psi(\not\partial - m)\psi + \left(\bar\psi\gamma_\mu\frac{\underline{\tau}}{2}\psi\right)^2 + \left(\frac{f}{f_0}\right)^2\left(\bar\Psi\gamma_\mu\gamma_5\frac{\underline{\tau}}{2}\psi\right)^2\right)$$

Since the addition of non-derivative term

$$2\lambda^2\left\{\left(\bar\Psi\gamma_\mu\,\underline{\tau}/2\,\psi\right)^2 + \left(\frac{f}{f_0}\right)^2\left(\bar\Psi\gamma_\mu\gamma_5\,\underline{\tau}/2\,\psi\right)^2\right\} \qquad (1.68)$$

to the lagrangian does not change the form of the currents, the new lagrangian

$$\mathcal{L}' = \mathcal{L} + 2\lambda^2\left\{ (\bar{\psi}\gamma_\mu \tau/2\, \psi)^2 + \left(\frac{f}{f_0}\right)^2 (\bar{\psi}\gamma_\mu\gamma_5 \tau/2\, \psi)^2 \right\} \quad (1.69)$$

is of Sugawara form

$$\mathcal{L}' = 2\lambda^2 (A_\mu^2 + V_\mu^2) + \text{chiral invariant kinematical term of nucleon.} \quad (1.70)$$

It has been proposed that the 4-point contact interaction of the type (1.69) can be useful in understanding high energy nucleon-nucleon interaction.[17] To relate this idea with our current-current form of lagrangian is attractive and even certain numerical success has been achieved[18].

According to the authors of (Ref. 17), differential cross section for pp elastic scattering for high energy, large momentum transfer becomes proportional to $G^4_{M_p}(t)$ where $G_{M_p}(t)$ is the proton magnetic form factor. They suggest the phenomenological form

$$\frac{d\sigma}{dt} = \left(\frac{d\sigma}{dt}\right)_{t=0} \left[a\, G^2_{M_p}(t) + \beta(t) \left(\frac{s}{s_0}\right)^{\alpha(t)-1} \right]^2$$

where a is constant and $\alpha(t)$ is Pomeranchuk trajectory. In their analysis, it was found

$$a \sim 0.85 \pm 0.15$$

Now suppose that we compute this amplitude quite naively according to the 4-fermion interaction (1.66) (Fairlie 16). It is found in Ref. 16 that this gives the corresponding differential cross-section

$$\frac{d\sigma}{dt} = \frac{(1+(f/f_0)^2)}{64\pi}\left(\frac{g}{m_\rho}\right)^4 G_{M_p}^4(-t)$$

In the region $s > -t > M_N^2$

$$\frac{d\sigma}{dt} \simeq \left(\frac{d\sigma}{dt}\right)_{t=0} a^2 G_{M_p}^4(-t)$$

where

$$a = \sqrt{2}\left(1+\left(\frac{f}{f_0}\right)^4\right)^{-1/2} \sim 0.8$$

On the other hand the amplitude itself from (1.66) is real while the physical amplitude has large imaginary part. The obvious difficulty here is that these techniques with phenomenological lagrangian cannot be used in the region where the restriction due to the unitarity is important. At this stage, there is no really convincing way of unitarise chiral lagrangian result.

(5) <u>Divergence equation, PCAC and symmetry breaking</u>

In the current algebra approach, the divergence equations of vector and axial vector currents are most important. It is through the partial conservation of axial currents (P.C.A.C.) which connects the divergence of axial currents with "interpolating fields" of pi-mesons that we can deduce physical prediction from the current commutation relation. It has even been shown[19,20] that these divergence equations in the presence of certain vector fields which act as a perterbative factor to the strongly interacting system (modified C.V.C and PCAC) are sufficient to reproduce most of the physical prediction from current algebra.

I have just introduced the vector and axial vector currents through the variational principle applied to the completely chiral invariant lagrangian (1.39). With respect such a variation of the field variables with infinitesimal parameter $\theta(x)$ (stands for both $\delta\alpha(x)$ and $\delta\beta(x)$), the Euler-Lagrange type equation holds (Gell-Mann, Levy Ref.14)

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \theta} = \frac{\delta \mathcal{L}}{\partial \theta} \qquad (1.71)$$

This comes from equations of motion for field variables and is quite independent of the invariance of the Lagrangian under the local transformations considered here.

From (1.71), it can be seen immediately that the currents defined by (1.50) and (1.51) satisfy the divergence equations

$$\partial_\mu \underline{V}^\mu = -\frac{\delta \mathcal{L}}{\delta \underline{\beta}}\Big|_{\beta=0} \tag{1.72}$$

$$\partial_\mu \underline{A}^\mu = -\frac{\delta \mathcal{L}}{\delta \underline{\alpha}}\Big|_{\alpha=0} \tag{1.73}$$

Thus for the invariant lagrangian (1.39), we have the conservation equation

$$\partial_\mu \underline{V}^\mu = 0 \tag{1.74}$$

$$\partial_\mu \underline{A}^\mu = 0 \tag{1.75}$$

To discuss the relation like PCAC, we should take account of symmetry breaking. I want to leave the more thorough discussion of chiral symmetry breaking till later when I should discuss chiral SU(3)xSU(3) symmetry where the symmetry breaking is essential. Here I merely follow Weinberg by asserting that the symmetry breaking should be introduced as a generalized form of pion mass terms which transform like 4^{th} component of chiral 4-vector - $(\frac{1}{2},\frac{1}{2})$ representation of chiral SU(2)xSU(2). (I have already mentioned that non-zero mass of pions necessiates the introduction of non-invariant pion-mass term into the lagrangian). From the discussion of §3, a single candidate for such pion mass term will be

$$\mathcal{L}'_{(\mu_\pi)} = C\phi_4 = \frac{C}{2\lambda}\,\frac{\lambda^2\pi^2-1}{\lambda^2\pi^2+1} \qquad (1.[illegible])$$

$$(C\,;\ \text{constant})$$

then, since the addition of constant does not change the physical content of Lagrangian, this is equivalent to

$$\mathcal{L}_{(\mu_\pi)} = \frac{C}{2}\,\frac{2\lambda\pi^2}{1+\lambda^2\pi^2} \quad \left(= \mathcal{L}'_{\mu_\pi} + \frac{C}{2\lambda} \right)$$

$\mathcal{L}_{(\mu_\pi)}$ in lowest order of π gives the pion mass term

$$\frac{1}{2}\,(2C\lambda)\,\pi^2$$

Thus I should make an identification

$$2C\lambda = -\mu_\pi^2$$

then

$$\mathcal{L}_{(\mu_\pi)} = -\frac{\mu_\pi^2}{2}\,\frac{\pi^2}{1+\lambda^2\pi^2} \qquad \text{(Weinberg Ref.7)} \quad (1.[illegible])$$

$$\sim -\frac{\mu_\pi^2}{2\lambda}\,\phi_4$$

Under the presence of this additional term, the axial vector will not be conserved

$$\partial_\mu \underline{A}^\mu = -\left.\frac{\delta \mathcal{L}_{(\mu_\pi)}}{\delta \underline{\alpha}}\right|_{\alpha=0} = \frac{\mu_\pi^2}{2\lambda}\left.\frac{\delta \phi_4}{\delta \underline{\alpha}}\right|_{\alpha=0}$$

From (1.1[illegible]) $\quad \delta\phi_4 = \delta\underline{\alpha}\cdot\underline{\Phi}$

$$\frac{\delta\phi_4}{\delta\underline{\alpha}} = \underline{\Phi}$$

Thus I obtain a form of PCAC

$$\partial_\mu \underline{A}^\mu = \frac{\mu_\pi^2}{2\lambda}\,\underline{\Phi}$$

By the identification of λ with pion decay constant introduced in the last paragraph this can be written as

$$\partial_\mu \underline{A}^\mu = \mu_\pi^2 F_\pi \underline{\phi} \tag{1.78}$$

The linear fields $\underline{\phi}$ are related to original $\underline{\pi}$ fields through (1.14) and (1.76) in term of

$$\partial_\mu \underline{A}^\mu = \mu_\pi^2 F_\pi \frac{\underline{\pi}}{1+\lambda^2\pi^2} \tag{1.79}$$

which is the generalization of conventional PCAC equation. Of course, if we want to stick to the ordinary form of exact PCAC. I can redefine physical pion fields by

$$\underline{\pi}' \equiv \underline{\phi} = \frac{\underline{\pi}}{1+\lambda^2\pi^2}$$

which makes (1.73) into

$$\partial_\mu \underline{A}^\mu = \mu_\pi^2 F_\pi \underline{\pi}' \tag{1.80}$$

With the addition of symmetry breaking term $\mathcal{L}(\mu_\pi)$, the lagrangian (1.59) becomes fully equivalent to the approximate form (1.1) so far as π -N interaction is concerned.

§7 The relation with the current algebra. Canonical field theory. (21, 22).

From the discussion in the preceding paragraph, the affinity of non-linear realization techniques to the current algebra approach is clear. But to show that this sort of lagrangian can be actually used as a field theoretical model to the current algebra, certain comlication should be met. The apparent difficulty here is that we should now take the operator nature of fields variables like π i's seriously and owing to the non commutativity of boson fields and their canonical momenta the standard argument of Gell-Mann and Levy[14] might not apply unless a careful consideration of ordering of field operators are taken. For the particular lagrangian (1.52), it can be seen easily that if we write it as

$$\mathcal{L} = \frac{1}{2} \frac{1}{(1+\lambda^2\pi^2)^2} (\partial_\mu \underline{\pi})^2 \qquad (1.81)$$

rather than

$$\nabla_\mu \underline{\pi} \; \nabla^\mu \underline{\pi}$$

and define the canonical momenta of i in the usual way

$$P_i = \frac{\delta \mathcal{L}}{\delta \frac{\partial \pi_i}{\partial t}} \qquad (1.82)$$

also the currents with respect to the infinitesimal transformation

$$\underline{\pi} \to \underline{\pi} + \delta\underline{\pi}(\theta)$$

as

$$j_\mu = -\left.\frac{\delta \mathcal{L}}{\delta \partial_\mu \theta}\right|_{\theta=0} \tag{1.83}$$

then the ordinary operator form of field transformation

$$\delta\,\underline{\pi}(\theta) = -i[Q\cdot\theta,\ \underline{\pi}\,] \tag{1.84}$$

with

$$Q = \int j_0(x)\,d^3\underline{x} \tag{1.85}$$

is obtained.

In fact, it has been shown by Isham(21) that (1.81) in which now the position of the derivative $\partial_\mu \underline{\pi}$ is important (the pion field is no longer a c number) is still invariant under the transformation of π_i's (1.4). His argument is very general and applies to wider class of meson-lagrangian. After this, we may derive the ordinary current algebra commutation relation among the vector and axial vector currents. The form of Schwinger term is exactly specified. Now at this point a rather interesting problem arises. Since we have now the currents in operator form we may consider the spectral representation of them. In particular, we may try to derive the spectral function sum rule considered by Weinberg[23].

It can be shown that if we use the commutation relation with Schwinger terms derived from the pionic lagrangian (1.81), then

$$\int \frac{\rho_V'(a)}{a}\, da + \int \frac{\rho_A'(a)}{a}\, da = 0. \qquad (1.86)$$

where ρ_V' and ρ_A' are the vector part of the spectral function of vector and axial vector currents respectively. They are, of course supposed to be positive definite and (1.86) implies $\rho_V = \rho_A = 0$ This is the contradiction first proposed by Jackiw[24] but should be considered as the indication that the self-consistent model requires at least vector and axial vector fields as an independent degree of freedom. This is connected to the problem of gauge fields in non-linear realization techniques which I shall discuss in the next chapter.

In the presence of such gauge fields, we can construct a model of the field algebra type and then the derivation of current commutation relation can be made very simple. The problem of quantizing the non linear lagrangian discussed in this paragraph is very clearly treated in the paper by Barnes and Isham[22].

CHAPTER 2

General theory of non-linear realization of chiral group

§1 The method of Coleman-Wess-Zumino and Isham

In this chapter we give the general formalism of the non-linear realization theory.

The main idea of the construction given here is first applied by Weinberg to the use of the chiral SU(2)xSU(2) group. The generalization to the case of KxK where K is any compact, simply connected lie group has been done by Coleman, Wess and Zumino[8], although the most physically important ideas are already to be found in the earlier paper by Cronin[2]. The mathematics employed by these authors reduces to the powerful techniques developed by Mackey[25]. The mathematical aspects of non-linear realization techniques have been fully investigated by Isham[9]. Salam and Strathdee[26] have treated a similar problem in less abstract level but in an intuitively appealing manner. It is surprising that this "theory of induced representation" by Wigner and Mackey is found to be relevant in such wide range of problems in quantum physics.

In my presentation of the material, I naturally emphasize the various relations and formulae which are necessary for actually constructing chiral SU(3) symmetric dynamical model, and skip over the most of the mathematics needed for considering the problem in its full generality.

Although most of the isolated formulae presented here are derived by the author independently the general formalism closely follows that of Coleman, Wess and Zumino[(6)] in the way presented by Coleman and Zumino at Erice Summer School, Sicily 1968. I was also influenced by an attractive presentation put forward by Salam and Strathdee[(26)].

Chiral group K_n = SU(n)xSU(N) can be regarded as a lie group associated with the lie algebra with 2N elements $(A_i, V_i)_{i=1}^{N}$ and with the commutation relation among them

$$[V_i, V_j] = -C_{ijk} V_k \qquad (2.1a)$$

$$[A_i, V_j] = -C_{ijk} A_k \qquad (2.1b)$$

$$[A_i, A_j] = -C_{ijk} V_k \qquad (2.1c)$$

where is the structure constant of SU( ) and can be taken as real, totally antisymmetric. The commutation relations given here are just the well known Gell-Mann relation fundamental in Current Algebra and the letter A, V have obvious implication with respect to space-reflection operation. (Note that commutation relation above differs by the factor i from the usual one).

An arbitrary element of chiral $SU(n) \times SU(n)$ group is characterized by 2N real co-ordinates $(\alpha_1' \cdots \alpha_N' \; \beta_1' \cdots \beta_N')$ and in a neighbourhood of identity element corresponding to $(\alpha_i' = 0 \quad \beta_i' = 0)$, they can be expressed as the exponential

$$g = e^{\Sigma \alpha_i' A_i + \Sigma \beta_i' V_i} \tag{2.2}$$

For our purpose it is enough to think of a group element as realized as an element of the group of continuous transformations in a certain manifold and thus and in the above expression should be interpreted as the operator operating in this same manifold.

The crucial point for the construction of Coleman, Wess and Zumino[8] is that the any element g of can be uniquely decomposed as

$$g = e^{\Sigma \alpha_i A_i} \; e^{\Sigma \beta_i V_i} \tag{2.3}$$

that is to say the product of the element of "diagonal subgroup SU(n) (I shall frequently denote it by letter H; $H \subset K_n$) and the elements which are characterized by $\beta_\varepsilon = 0$ in our way of representation. (I may call it the "chiral part" of K_n). Using (2.3), we consider the decomposition of particular elements

$$g\, e^{\xi A} = e^{\xi' A} e^{\eta' V} \tag{2.4}$$

where g is an arbitrary element of K_n and ξ are real numbers.

If the decomposition (2.3) is unique, we can consider (2.4) as defining ξ' and η' as the function of group element g as well as the quantity ,
and we can write

$$\xi' = \xi'(\xi, g) \tag{2.5}$$

$$\eta' = \eta'(\xi, g) \tag{2.6}$$

On the other hand, we can consider (2.5) as the definition of the operation of the element g of the group K_n realized as the transformation of real number field (ξ) by putting

$$g\, \xi = \xi' \tag{2.7}$$

Of course, we need a consistency condition for group operation

$$g_1(g_2\xi) = (g_1 g_2)\,\xi \qquad (2.8)$$

for any $g_1, g_2 \in K_n$. This can be proven trivially (Coleman-Wess-Zumino Ref.6).

$(\because)$ $g_2\xi = \xi'$ is defined by the help of (2.4) as

$$g_2 e^{\xi A} = e^{\xi' A} e^{\eta' V} \qquad (a)$$

Similarly, $g_1(g_2\xi) = \xi''$ is given by

$$g_1 e^{\xi' A} = e^{\xi'' A} e^{\eta'' V} \qquad (b)$$

On the other hand, by the associative law of group elements

$$(g_1 g_2) e^{\xi A} = g_1(g_2 e^{\xi A})$$
$$\overset{\text{by } a}{=} g_1 e^{\xi' A} e^{\eta' V} = (g_1 e^{\xi' A})\, e^{\eta' V}$$
$$\overset{\text{by } b}{=} e^{\xi'' A} e^{\eta'' V} e^{\eta' V} = e^{\xi'' A}(e^{\eta'' V} e^{\eta' V})$$

Since the diagonal elements ($e^{\eta V}$) form a subgroup (H) we have

$$e^{\eta'' V} e^{\eta' V} = e^{\eta''' V} \in H$$

and we get

$$(g_1 g_2) e^{\xi A} = e^{\xi'' A} e^{\eta''' V}$$

Assuming the uniqueness of the decomposition (2.4), this should be identical with the equation defining

$$(g_1 g_2)\xi = \xi'''$$

as

$$g_1 g_2 e^{\xi A} = e^{\xi''' A} e^{\eta'''' V}$$

i.e

$$\xi''' = \xi'' \quad , \quad \eta''' = \eta''''$$

That is to say

$$g_1(g_2 \xi) = (g_1 g_2)\xi$$

(QED)

When (2.4) and (2.7) are considered as defining the group operation of L_n in the field of quantity $\xi = (\xi_1 \cdots \xi_N)$, the analogy with Wigner's construction becomes apparent. Salam et al. (2.4) and (2.7) gives, in particular,

$$e^{\xi A}(0) = \xi \qquad (2.9)$$

and

$$h(0) = (0) \quad \text{for any } h \in H \, . \qquad (2.10)$$

It is also obvious that the second equation holds only for the element of H, and that H can be characterized as the group of elements which leave (0) (in ξ-field) invariant.

Assuming now the operation of K in field (2.7) together with (2.9) and (2.10) is known beforehand, we can write an arbitrary element of g of h as

$$g = e^{\xi' A}(e^{-\xi' A} g e^{\xi A}) e^{-\xi A}$$

where $\xi' = g\,\xi$

Then, by (2.9), the element $e^{-\xi' A} g e^{\xi A}$. leaves (0) in the ξ -field invariant, and thus belongs to H. Writing $e^{\eta' V} = e^{-\xi' A} g e^{\xi A}$, we have

$$g = e^{\xi' A} e^{\eta' V} e^{-\xi A} \qquad (2.11)$$

This is just recovering (2.4), but presented in this way, it is ana analogy of the construction of the Wigner rotation in studying the inhomogeneous Lorentz group[27].

In the case of chiral SU(2)xSU(2) = K_2, the relation (2.4) has been already introduced in (1) by an explicit construction (1.24). In the context of the more general treatment of the present chapter the importance of the parameter ξ is apparent.

The apparent resemblence should not let us think that the mathematics in both cases of chiral SU(n) and the Poincare group are identical in the simple way hinted here. In the case of the latter, we are given the four dimensional momentum space, as the homogeneous space of the group operation from beginning. The invariant subgroup consisting of the translations gives a label to the representations we are looking for. This is "mass". It is after determining this characteristic of the representation that we start constructing the Wigner Rotation. In the case of chiral SU(n)xSU(n), the lack of "momentum" in the proper sense makes the construction of the Wigner rotation in (2.11) more like an attractive way of presenting the decomposition theorem (2.4), and its power for studying further mathematical structures seems to be limited.

Yet, it may be possible to make this analogy more profound and useful by annexing to K 's structure something like a translation group. Then we will have to define the "orbit" in such a space which reduces to the manihold of $\xi = (\xi_1 \cdots \xi_N)$ introduced above. In case of K_2, we have already seen how to "embed" the space of "pi-mesons" or (ξ_1, ξ_2, ξ_3) in the space of 4-dimensional representation $(\frac{1}{2},\frac{1}{2})$ of K_2 (Chapter 1). The orbit condition in this case is, using the notation of Chapter 1.

$$\Phi_4^2 + \sum_{i=1}^{3} \Phi_i^2 = \frac{1}{4\lambda^2} = \left(\frac{\mu_\pi}{2f_0}\right)^2 \quad (\text{const}).$$

It should be noted that for higher , we will need the space of very high dimension. For chiral SU(_)xSU(3), for instance, 1_ dimensional momentum space will be a choice[28].

Returning to the equation (2.11), the obvious next step is to construct an "auxiliary" representation in an analogy with the construction of generalized spinor in the case of the relativistic free particle system[27]. First of all, I must introduce "the particle multiplet" which is just the irreducible representation of H.

We denote it as the set of "field operator" ψ_α, on which the action of $h \in H$ is defined as

$$(h\psi)_\alpha = \mathcal{D}_{\alpha\beta}(h)\psi_\beta \quad , \quad h \in H \qquad (2.12)$$

$\mathcal{D}$ is the matrix belonging to an irreducible representation of H transforming as h.

To define the operation of whole group K_n on ψ_α we consider ψ_α and an element of ξ field together as

$$\Phi_\alpha(\xi) = (\psi_\alpha , \xi_i) \qquad (2.13)$$

Then, remembering the definition of the "Wigner Rotation" (2.4), we define the transformation of the quantity $\Phi_\alpha(\xi)$ as

$$(g\Phi)_\alpha(\xi) = (\mathcal{D}_{\alpha\beta}(e^{\eta' V})\psi_\beta \, , \, \xi') \qquad (2.14)$$

where η' and ξ' are defined by (2.4). It is easy to see the transformation law

$$(g\psi)_\alpha = \mathcal{D}_{\alpha\beta}(e^{\eta' V})\psi_\beta = (e^{i\eta'\hat{t}})_{\alpha\beta}\psi_\beta \qquad (2.15)$$

is consistent as a group operation. The matrices $(\hat{t}_i)_{i=1}^{N}$ are the generator matrices corresponding to the irreducible representation of SU(n) belonging to ψ_α

The relations (2.7),(2.15) and (2.4)

$$g\xi = \xi'$$

$$(g\psi)_\alpha = (e^{i\eta'\hat{t}})_{\alpha\beta}\psi_\beta$$

$$\text{with} \quad g\, e^{\xi A} = e^{\xi' A}\, e^{\eta' V}$$

are fundamental for non-linear realization techniques for chiral SU(n) symmetry.

These relations are, of course, the result of the unique decomposition (2.4) and in analogy with the Wigner decomposition, boost our "auxiliary representation".

Now I write (2.15) as

$$\psi_\alpha \xrightarrow{g} \mathcal{D}_{\alpha\beta}(e^{-\xi' A} g\, e^{\xi A})\psi_\beta \qquad (2.16)$$

Take the arbitrary representation of k which when restricted to diagonal subgroup SU(n) contains this particular representation of SU(n) spanned by ψ'_β. Then, denoting the matrices associated with this representation of K_n in a suitable co-ordinate system as $D_{rq}(g)$, I can factorise the matrix element appearing in (2.16) as

$$\mathcal{D}_{\alpha\beta}(e^{-\xi' A}\, g\, e^{\xi A})$$
$$= \sum_{r,s} D_{\alpha r}(e^{-\xi' A})\, D_{rs}(g)\, D_{s\beta}(e^{\xi A}) \qquad (2.17)$$

where r, s correspond to a complete set of base vectors in the vector space carrying the representation D, ~~which~~ while α, β corresponds to the identification of the subset of these bases to the vectors carrying the representation $\mathcal{D}$ of SU(n) as assumed above.

Let us define the new quantity Ψ_r by

$$\Psi_r = D_{r\alpha}(e^{\xi A})\,\psi_\alpha \qquad (2.18)$$

Then, under the action of $g \in K_n$, I have from (2.7), (2.15) and (2.4)

$$\Psi_r \xrightarrow{g} D_{r\alpha}(e^{g\xi\cdot A})\,(g\psi)_\alpha$$
$$= D_{r\alpha}(e^{g\xi\cdot A})\,\mathcal{D}_{\alpha\beta}(e^{-\xi' A}\, g\, e^{\xi A})\,\psi_\beta$$

From (2.17), the last expression is equal to

$$D_{\lambda\alpha}(e^{\eta\xi\cdot A})\, D_{\alpha\lambda'}(e^{-\eta\xi\cdot A})\, D_{\lambda' s'}(g)\, D_{s'\beta}(e^{\xi A})\, \psi_\beta$$
$$= D_{\lambda s}(g)\, D_{s\beta}(e^{\xi A})\, \psi_\beta$$
$$= D_{\lambda s}(g)\, \Psi_s$$

Thus Ψ_λ transforms as the given linear representation of K_m

$$g\,\Psi_\lambda = D_{\lambda s}(g)\,\Psi_s \qquad (2.19)$$

This shows that we can obtain out of an arbitrary linear representation of K provided it does contain the representation of SU(m) spanned by ψ_α's

The converse is also true, we can ask(8) what sort of representation of K can be constructed as the function

$$U_\lambda = \sum_\alpha F_{\lambda\alpha}(\xi)\,\psi_\alpha \qquad (2.20)$$

where ξ and ψ_α obey the transformation rule (2.7) and (2.15) with (2.4).

We should transform according to some given representation of K

$$U_\lambda \xrightarrow{g} D_{\lambda s}(g)\, U_s \qquad (2.21)$$

Then it can be shown that the necessary and sufficient condition for (2.21) being realised by suitably choosing F(ξ) in (2.20) is that the representation $\mathcal{D}$ of SU(m) associated with ψ_α's when restricted to the subgroup $H = SU(n)$. The following proof is due to Coleman, Wess and Zumino(8).

As the result of (2.7), (2.15), (2.4) and (2.20), the transformation law (2.21) can be written as

$$\sum_{\alpha,\beta} F_{r\alpha}(\xi') \mathcal{D}_{\alpha\beta}(e^{\eta' V})\psi_\beta$$
$$= \sum_{s,\beta} D_{rs}(g) F_{s\beta}(\xi)\psi_\beta$$

choosing the element $h \in H$, and taking the particular value of ξ; $\xi \equiv 0$ we get

$$F_{r\alpha}(0)\mathcal{D}_{\alpha\beta}(h)$$
$$= D_{rs}(h) F_{s\beta}(0) \qquad (2.22)$$

Since $\sum_\alpha F_{r\alpha}(\xi)\psi_\alpha$ is supposed to transform linearly under K_n, and also K_n acts transitively on the ξ -field, $F_{r\alpha}(0) \equiv 0$ would mean

$$\sum F_{r\alpha}(\xi)\psi_\alpha \equiv 0$$

So I should look for the solution of (2.22) in which $F_{r\alpha}(0)$ are not identically zero. (Non-trivial solution). Then, since $\mathcal{D}$ is irreducible, the Schur's lemma tells us that $D_{rs}(h)$, which is now the direct sum of the irreducible representations of H, should contain at least one represent tion of H which is equivalent to $\mathcal{D}$.

If $D(h)$ contains only one such representation of h, then the construction (2.20) is essentially unique. The expression (2.18) is an exact analogue of the "auxiliary fields" of Weinberg and Matthews-Feldman for free relativistic particle. The coefficients $D_{\mu\alpha}(e^{\xi A})$ is taking the role of generalized spinor. (It is formally of same form - boost matrix elements). The construction of linear representation of K together with the theorem of Coleman, Wess and Zumino described above is important when we consider the problem of symmetry breaking.

§2 Explicit expressions

The three equations (2.7), (2.15) and (2.4) given in(§1) is fundamental to all the results of the present chapter. First I must show that we can determine the explicit form of the group operation on the field of ξ'^{α} and ψ'^{α}. I shall do it only for the infinitesimal element of K_n by these equations.

For this purpose, I regard the group K_n realized as a linear representation in a certain vector space. Then we can consider the A's and V's as anti-hermitian matrices. Writing them (as matrices) $i\mathcal{A}$ and $i\mathcal{V}$, the equation (2.4) reduces to the matrix formula

$$g\, e^{i\xi\mathcal{A}} = e^{i\xi'\mathcal{A}}\, e^{i\eta'\mathcal{V}} \tag{2.23}$$

To solve (2.23) in general, we may appeal to the formula due to Baker-Canbdell-Haussdorff. But we require the solution in all order of ξ but only in the first order in the parameters of a group element, and for this, we can find the explicit solution in an elementary way.

(1) Chiral part

I consider the case

$$g = e^{i\alpha\cdot\mathcal{A}}$$

where is considered as infinitesimal, (2.23) becomes

$$e^{i\alpha\mathcal{A}}\, e^{i\xi\mathcal{A}} = e^{i\xi'\mathcal{A}}\, e^{i\eta'\mathcal{V}} \tag{2.24}$$

I look for an analytical solution only and I may consider

$$\eta' \sim O(\alpha)$$

$$\delta\xi = \xi' - \xi \sim O(\alpha)$$

Thus, I may write the above formula as

$$1 + i\alpha\mathcal{A} = e^{i(\xi+\delta\xi)\cdot\mathcal{A}}\, e^{-i\xi\mathcal{A}} + i\, e^{i\xi\mathcal{A}}\, \eta'\cdot\mathcal{V}\, e^{-i\xi'\mathcal{A}} + O(\alpha^2)$$

Using the well known formula in elementary matrix calculations, we have, up to the first order in α ,

$$i\alpha\mathcal{A} = \sum_{N=1}^{\infty} \frac{1}{N!} \underbrace{[i\xi\mathcal{A} \cdots [i\xi\mathcal{A}}_{N-1}, i\delta\xi\mathcal{A}]\cdots]$$

$$+ \sum_{N=0}^{\infty} \frac{1}{N!} \underbrace{[i\xi\mathcal{A} \cdots [i\xi\mathcal{A}}_{N}, i\eta'\mathcal{V}]\cdots]$$

By using the commutation relation of matrices and from (1.1a) (1.1c) the N^{ple} commutators in R.H.S can be transformed, and we have

$$i\alpha\mathcal{A} = i\delta\xi \frac{E_1(-iF\xi)+E_1(iF\xi)}{2}\mathcal{A}$$

$$+ i\delta\xi \frac{E_1(-iF\xi)-E_1(iF\xi)}{2}\mathcal{V}$$

$$+ i\eta' \frac{e^{-iF\xi}+e^{iF\xi}}{2}\mathcal{V}$$

$$+ i\eta' \frac{e^{-iF\xi}-e^{iF\xi}}{2}\mathcal{A}$$

where NxN matrices F is defined by

$$(F^i)_{jk} = -i\, C_{ijk}$$

and the function $E_1(z)$ is defined by the series

$$E_1(z) = \sum_{N=1}^{\infty} \frac{z^{N-1}}{N!} = \frac{e^z - 1}{z}$$

Choosing the representation in which the matrices $\mathcal{A}$ and $\mathcal{V}$ are independent, we get the following equalities

$$\left.\begin{aligned} \alpha &= \delta\xi\,\frac{E_1(iF\xi)+E_1(iF\xi)}{2} + \eta'\,\frac{e^{-iF\xi}-e^{iF\xi}}{2} \\ 0 &= \delta\xi\,\frac{E_1(-iF\xi)-E_1(iF\xi)}{2} + \eta'\,\frac{e^{-iF\xi}+e^{iF\xi}}{2} \end{aligned}\right\} \quad (2.25)$$

From these, we get

$$\delta\xi = \alpha\,\frac{e^{iF\xi}+e^{-iF\xi}}{E_1(iF\xi)+E_1(-iF\xi)} \quad (2.26)$$

and

$$\eta' = \alpha\,\frac{1-e^{-iF\xi}}{1+e^{-iF\xi}} \quad (2.27)$$

(2) diagonal subgroup

Consider now $g = e^{i\beta\cdot\mathcal{V}} \in H$

The relation

$$e^{i\beta\mathcal{V}}\,e^{i\xi\mathcal{A}} = e^{i\xi'\mathcal{A}}\,e^{i\eta'\mathcal{V}} \quad (2.28)$$

gives instead of (2.25),

$$\left.\begin{aligned} 0 &= \delta\xi\,\frac{E_1(-iF\xi)+E_1(iF\xi)}{2} + \eta'\,\frac{e^{-iF\xi}-e^{iF\xi}}{2} \\ \beta &= \delta\xi\,\frac{E_1(-iF\xi)-E_1(iF\xi)}{2} + \eta'\,\frac{e^{-iF\xi}+e^{iF\xi}}{2} \end{aligned}\right\} \quad (2.29)$$

From (2.29), it is easy to see that up to the first order of β.

$$\eta' = \beta \quad (2.30)$$

and

$$\delta\xi = \beta\cdot(iF\xi)$$

The last equality can be written as

$$\delta\xi = (i\beta F)\cdot\xi \tag{2.31}$$

(2.30) and (2.31) express the fact that under the action of subgroup H. ξ field as well as ψ field transforms as linear representation (regular representation). In case of K_2 and K_3, this means that non linear meson fields behave as triplet and octet with respect to the SU(2) and SU(3) symmetry.

The result (2.26), (2.27), (2.30) and (2.31) is of course independent of particular representation used to calculate $\delta\xi$ and η' since, apart from the independence of matrices $\mathcal{A}$ and $\mathcal{V}$, we have needed only the commutation relation between $\mathcal{A}'_\rho$ and $\mathcal{V}'_\rho$

§3 Covariant derivatives

I have derived the transformation induced by chiral group K_n on the field of quantities ξ_i and ψ_α I am, of course, going to use these ξ'_ρ and ψ'_ρ as dynamical variables. More specifically, they are considered as the local "field variable" depending on space-time x_μ out of which I want to construct the lagrangian functional. But in studying their transformations, we more or less consider these quantities as C-number function of x_μ. Also we make an assumption that the expression $e^{\xi(x)A}$ for arbitrary x is meaningful as an element of chiral part of group .

To construct the lagrangian model with these quantities as field variables, we need at least the first order differential coefficients $\partial_\mu \xi(x)$ and $\partial_\mu \psi(x)$ together with original ξ'_A and ψ'_A.

Now from the non-linear transformation of ξ'_A and ψ'_A under the action of an element of K_n given by (2.7), (2.15) and (2.4) of §1, it is clear that these derivatives do not transform in a simple way under the group K_n.

To get the quantities which generalise these derivatives but have simple transformation under the group, we use the techniques analogous to the construction of covariant derivatives in general relativity theory.

Following Salam and Sthrathdee[26], I start from the following quantity

$$\Delta_\mu \psi_\alpha(x) = D_{\alpha\eta}(e^{-\xi(x)A})\, \partial_\mu D_{\eta\beta}(e^{\xi(x)A})\, \psi_\beta(x) \qquad (2.32)$$

where the matrix $\mathcal{D}$ is, as in §1, of any representation of K_n which contains the representation of H spanned by ψ_α. It is clear, in the light of construction given in §1, $\Delta_r \psi_\alpha$ transforms according to (2.15) i.e. in the same way as ψ_α itself.

$$\Delta_r \psi_\alpha \xrightarrow{g} \mathcal{D}_{\alpha\beta}(e^{\eta' V}) \Delta_r \psi_\beta \qquad (2.33)$$

(2.32) can be written as

$$\Delta_r \psi_\alpha = \partial_r \psi_\alpha + \{ D_{\alpha\lambda}(e^{-\xi \cdot A}) \partial_r D_{\lambda\beta}(e^{\xi A}) \} \psi_\beta$$

Now

$$D(e^{-\xi(x)\cdot A}) \frac{\partial}{\partial x^r} D(e^{\xi(x)\cdot A})$$

$$= \lim_{y \to 0} \frac{\partial}{\partial y^r} D(e^{-\xi(x)A}) D(e^{\xi(x+y)\cdot A}) \Big|_{y=0}$$

Using the fundamental formula (2.4), we can write

$$e^{-\xi(x)A} e^{\xi(x+y)A} = e^{\xi^{(0)} A} \; e^{\eta^{(0)} V}$$

with

$$e^{-\xi(x)\cdot A} \xi(x+y) = \xi^{(0)}$$

$$e^{-\xi(x)A} \psi_\alpha(x+y) = \mathcal{D}_{\alpha\beta}(e^{\eta^{(0)} V}) \psi_\beta(x).$$

I may assume $\xi^{(0)}, \eta^{(0)} \sim O(y)$, so that $D(e^{-\xi(x)\cdot A})$ $\times \partial_r D(e^{\xi(x)\cdot A})$ can be now written as

$$= i A \frac{\partial}{\partial y^r} \xi^{(0)} \Big|_{y=0} + i V \frac{\partial}{\partial y^r} \eta^{(0)} \Big|_{y=0}$$

Thus I get

$$\Delta_r \psi_\alpha = \partial_r \psi_\alpha + i \mathcal{V}_{\alpha\beta} \cdot \frac{\partial}{\partial y^r} \eta^{(0)} \Big|_{y=0} \psi_\beta$$

$$+ i \mathcal{A}_{\alpha\beta} \cdot \frac{\partial}{\partial y^r} \xi^{(0)} \Big|_{y=0} \psi_\beta$$

$$= \frac{\partial}{\partial y^r} \left(e^{-\xi(x)\cdot A} \psi_\alpha (x+y) \right) \Big|_{y=0}$$

$$+ i \frac{\partial}{\partial y^r} \left(e^{-\xi(x)A} \xi(x+y) \right) \Big|_{y=0} \cdot \mathcal{A}_{\alpha\beta} \psi_\beta(x) \quad (2.34)$$

where $\mathcal{V}$ and $\mathcal{A}$ have the same meaning as in §2 for a given representation of K_n, $\mathcal{D}$.

We consider the transformation of the first term of R.H.S. of (2.34) which contains $\partial_r \psi$. Under K_n

$$\frac{\partial}{\partial y^r} \{ e^{-\xi(x)A} \psi \}_\alpha (x+y) \xrightarrow{g} \frac{\partial}{\partial y^r} \left(e^{-g\xi(x)\cdot A} g \psi \right)_\alpha (x+y)$$

$$= \frac{\partial}{\partial y^r} \{ (e^{\eta'(x)V} e^{-\xi(x)A} g^{-1}) g \psi \}_\alpha (x+y) \qquad \text{from (1.4)}$$

$$= \frac{\partial}{\partial y^r} \{ e^{\eta'(x)V} e^{-\xi(x)A} \psi \}_\alpha (x+y)$$

$$= \frac{\partial}{\partial y^r} \mathcal{D}_{\alpha\beta} (e^{\eta'(x)V}) (e^{-\xi(x)A} \psi)_\beta (x+y)$$

$$= \mathcal{D}_{\alpha\beta} (e^{\eta'(x)V}) \frac{\partial}{\partial y^r} \{ (e^{-\xi(x)A} \psi)_\beta (x+y) \}$$

(2.35)

Thus the quantity $\frac{\partial}{\partial y^r}(e^{-\xi(x)A}\psi)_\alpha(x+y)$ transforms in the same way as ψ_α under K_n. On the other hand, clearly this quantity is independent of the choice of particular "auxiliary representation" D of K_n

So I shall put

$$\begin{aligned}\nabla_r\psi_\alpha(x) &= \frac{\partial}{\partial y^r}\left(e^{-\xi(x)A}\psi\right)_\alpha(x+y)\Big|_{y=0} \\ &= \partial_r\psi_\alpha(x) + i\,V_{\alpha\beta}\left(\frac{\partial}{\partial y^r}\eta^{(0)}\right)_{y=0}\psi_\beta(x) \qquad (2.36)\\ &= \partial_r\psi_\alpha + i\,\hat{t}_{\alpha\beta}\cdot\left(\frac{\partial}{\partial y^r}\eta^{(0)}\right)_{y=0}\psi_\beta(x)\end{aligned}$$

where $\hat{t}'_\rho$ as the same matrices appearing in (2.15) and call it the covariant derivatives of ψ_α.

Similarly

$$\begin{aligned}&\frac{\partial}{\partial y^r}\,e^{-g\xi(x)\cdot A}\,g\,\xi(x+y)\\ &= \frac{\partial}{\partial y^r}\left(e^{\eta' V}e^{-A\xi(x)}g^{-1}\right)g\,\xi(x+y)\\ &= \frac{\partial}{\partial y^r}\,e^{\eta' V}e^{-A\cdot\xi(x)}\,\xi(x+y)\end{aligned}$$

$$\begin{aligned}&\therefore\ \frac{\partial}{\partial y^r}\left(e^{-g\xi(x)\cdot A}\,g\,\xi(x+y)\right)_i\\ &= \left(e^{-iF\cdot\eta'}\right)_{ij}\frac{\partial}{\partial y^r}\left(e^{-A\xi(x)}\xi(x+y)\right)_j\end{aligned}$$

Thus if I put

$$\nabla_r\xi(x) = \frac{\partial}{\partial y^r}\,e^{-\xi(x)A}\,\xi(x+y)\Big|_{y=0} \qquad (2.37)$$

then $\nabla_r\xi(x)$ also transform covariantly according to the representation of H to which ξ belongs.

$$\nabla_\mu \xi_i(x) \xrightarrow{g} (e^{iF\cdot\eta'})_{ij} \nabla_\mu \xi_j(x) \qquad (2.38)$$

These quantities $\nabla_\mu \psi$, $\nabla_\mu \xi$ are just the generalization of covariant derivatives derived in Chapter 1 for K_2 and they coincide with the result of Chapter 1 for K_2.

I can evaluate $\partial_\mu^y \xi^{(0)}$ and $\partial_\mu^y \eta^{(0)}$ explicitly as following. Consider

$$i\mathcal{A}\cdot\partial_\mu^y \xi^{(0)} + i\mathcal{V}\cdot\partial_\mu^y \eta^{(0)}$$
$$= D(e^{-\xi(x)\cdot A})\,\partial_\mu\, D(e^{\xi(x)\cdot A})$$
$$R\cdot H\cdot S = \partial_\mu^y D(e^{-\xi(x)\cdot A})\, D(e^{\xi(x+y)\cdot A})\Big|_{y=0}$$
$$= \partial_\mu^y \left\{ e^{-i\xi(x)\cdot\mathcal{A}}\; e^{i(\xi(x)+\partial_\mu\xi\cdot y^\mu)\cdot\mathcal{A}} \right\}\Big|_{y=0}$$
$$= \sum_{N=1}^{\infty} \frac{1}{N!} \underbrace{[-i\mathcal{A}\cdot\xi \cdots [-i\mathcal{A}\cdot\xi}_{N-1},\, i\mathcal{A}\cdot\partial_\mu\xi]\cdots]$$
$$= \frac{i}{2}\partial_\mu\xi\left\{E_1(iF\xi) + E_1(-iF\xi)\right\}\cdot\mathcal{A}$$
$$+ \frac{i}{2}\partial_\mu\xi\left\{E_1(iF\xi) - E_1(-iF\xi)\right\}\cdot\mathcal{V}$$

Thus, comparing the coefficients of $\mathcal{A}$ and $\mathcal{V}$, we get

$$\partial_\mu^y \xi^{(0)}\Big|_{y=0} = \partial_\mu\xi\,\frac{E_1(iF\xi) + E_1(-iF\xi)}{2} \qquad (2.39)$$

$$\partial_\mu^y \eta^{(0)}\Big|_{y=0} = \partial_\mu\xi\,\frac{E_1(iF\xi) - E_1(-iF\xi)}{2} \qquad (2.40)$$

Thus we obtain the following expression for the covariant derivatives

$$\nabla_\mu \psi_\alpha(x) = \partial_\mu \psi_\alpha(x) + i\, \hat{t}^{j}_{\alpha\beta}\, \partial_\mu \xi_i(x) \left(\frac{E_1(iF\xi) - E_1(-iF\xi)}{2}\right)_{ij} \qquad (2.41)$$

$$\nabla_\mu \xi_i(x) = \partial_\mu \xi_j(x) \left(\frac{E_1(iF\xi) + E_1(-iF\xi)}{2}\right)_{ji} \qquad (2.42)$$

The expression of covariant derivatives given in (2.36) and (2.37) can be readily generalized to the higher order derivatives like

$$\nabla_{\mu_1\cdots\mu_n} \psi_\alpha(x) = \frac{\partial^n}{\partial y^{\mu_1}\cdots\partial y^{\mu_n}} \left(e^{-\xi(x)\cdot A}\psi\right)_\alpha(x+y)\Big|_{y=0} \qquad (2.43)$$

$$\nabla_{\mu_1\cdots\mu_n} \xi_i(x) = \frac{\partial^n}{\partial y^{\mu_1}\cdots\partial y^{\mu_n}}\, e^{-\xi(x)\cdot A}\, \xi_i(x+y)\Big|_{y=0} \qquad (2.44)$$

To evaluate these forms explicitly, we need the higher order expansion of the matrix like

$$e^{-i\xi\cdot\mathcal{A}}\; e^{i(\xi+\delta\xi)\cdot\mathcal{A}}$$

used in deriving (2.41) and (2.42). The following formula proposed by Feynman[29] is useful

$$e^{-A} e^{A+B} = T\left(\exp \int_0^1 dt\, e^{-At} B\, e^{At}\right)$$

where A, B are arbitrary $n \times n$ matrices and T(....) means ordered product with respect to the parameter t. Using this up to second order in B, I can derive, for instance.

$$\nabla_{\mu\nu}\psi_\alpha = (\partial_\mu\partial_\nu + i\beta_\mu\cdot\hat{t}\,\partial_\nu + i\beta_\nu\cdot\hat{t}\,\partial_\mu + i\beta_{\mu\nu}\cdot\hat{t})_{\alpha\beta}\psi_\beta$$

where

$$\beta_\mu = \frac{1}{2}\partial_\mu \xi\,(E_1(iF\xi) - E_1(-iF\xi))$$

$$\beta_{\mu\nu} = \frac{1}{2}\partial_\mu\partial_\nu\xi\,(E_1(iF\xi) - E_1(-iF\xi)) \qquad (2.45)$$

$$\nabla_{\mu\nu}\xi = \frac{1}{2}\partial_\mu\partial_\nu\xi\,(E_1(iF\xi) + E_1(-iF\xi)) + \frac{1}{2}(\alpha_\mu\cdot(iF)\cdot\beta_\nu + \alpha_\nu\cdot(iF)\cdot\beta_\mu)$$

where

$$\alpha_\mu \equiv \nabla_\mu\xi = \frac{1}{2}\partial_\mu\xi\,(E_1(iF\xi) + E_1(-iF\xi)) \qquad (2.46)$$

Since we have not found yet how to use these higher derivatives of fields, we do not consider them any further.

§4 The case of chiral SU(2)xSU(2) (L_2)

The fundamental relation (2.4) in case of chiral SU(2)xSU(2) can be written in terms of $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation as

$$e^{\pm i\underline{\tau}\cdot\underline{\alpha}/2}\, e^{\pm i\underline{\tau}\cdot\underline{\xi}/2} = e^{\pm i\underline{\tau}\cdot\underline{\xi}'/2}\, e^{i\underline{\tau}\cdot\underline{\eta}'/2}$$

for chiral part of the group. This is just the (1.24) of Chapter 1. From this, it is clear that the quantities

introduced here as parameterizing the elements of chiral part of the group ($\sim G/H$) exactly corresponds to the ξ'_α introduced in Chapter 1 as parameterizing π-fields. In Chapter 1, ξ'_α are expressed as certain non-linear transform of π's. Since, by the equivalence principle, it is permissible to redefine physical pion fields with any non-linear transformation (without derivatives), we may use ξ'_α instead of π'_α as "pion-fields" in the field theoretical calculation of S-matrix elements. In the present chapter, we have treated the general case of H = SU(n)xSU(n). For n=3, we get "octets" $(\xi_i)_{i=1}^{8}$ transforming as the (8) dimensional representation of SU(3). In what follows I shall use these ξ'_α as the physical octet pseudo scalor meson fields (which contain pions) and will not seek the "simpler" form corresponding to of Chapter 1. The problem of finding the parallel treatment for $K_n; n \geq 3$ with Weinberg's and Schwinger's method for K_2 has been studied by Mcfarlane and Weisz[30]. But the general results seem to be very complicated.

It is easy to see that the general non-linear transformation formulae (2.26) and (2.27) or the expression of covariant derivatives in this chapter reduces to the familiar expression of Weinberg given in Chapter 1 in the case of SU(2)xSU(2).

For instance, consider (2.27). In case of chiral SU(2)xSU(2), this reduces to

$$\eta' = \alpha \frac{1 - e^{i\underline{J}\cdot\underline{\xi}}}{1 + e^{i\underline{J}\cdot\underline{\xi}}}$$

where $\underline{J} = (J^1, J^2, J^3)$ is the generator of I=1 representation of SU(2) group and

$$(J^i)_{jk} = -i\epsilon_{ijk} \qquad i, j, k = 1, 2, 3$$

The simplicity with SU(2) is that $\underline{J}\cdot\underline{\xi}$ satisfies the characteristic equation

$$(\underline{J}\cdot\underline{\xi})^3 = \underline{J}\cdot\underline{\xi}.$$

Consequently, the odd function of $\underline{J}\cdot\underline{\xi}$, $\Omega(\underline{J}\underline{\xi}) = \frac{e^{i\underline{J}\cdot\underline{\xi}} - 1}{e^{i\underline{J}\cdot\underline{\xi}} + 1}$ can be simplified as

$$\Omega(\underline{J}\cdot\underline{\xi}) = \frac{\Omega(\sqrt{\xi^2})}{\sqrt{\xi^2}}(\underline{J}\cdot\underline{\xi})$$

From this,

$$\eta' = \alpha(i\underline{J}\cdot\underline{\xi})\frac{1}{\sqrt{\xi^2}}\;\frac{e^{i\sqrt{\xi^2}}-1}{e^{i\sqrt{\xi^2}}+1}$$

$$= \alpha(i\underline{J}\cdot\underline{\xi})\;\frac{\tan(\sqrt{\xi^2}/2)}{\sqrt{\xi^2}}$$

This reduces to the familiar form of the " field" is defined as

$$\lambda\underline{\pi} = \frac{\tan(\sqrt{\xi^2}/2)}{\sqrt{\xi^2}}\;\underline{\xi}$$

But this is just the parameterization introduced in Chapter 1. In term of J matrices,

$$\lambda\underline{\pi}\cdot\underline{J} = \frac{e^{i\underline{J}\cdot\underline{\xi}}-1}{e^{i\underline{J}\cdot\underline{\xi}}+1}$$

This corresponds to I=$\frac{1}{2}$ form given in Chapter 1.

$$\lambda\underline{\pi}\cdot\underline{T} = \frac{e^{i\underline{T}\cdot\underline{\xi}}-1}{e^{i\underline{T}\cdot\underline{\xi}}+1}$$

They are, of course, all equivalent.

The correct transformation of $\pi'A$ given in Chapter 1 are guaranteed because of the similarity of (2.4) to (1.24) in case of chiral SU(2)xSU(2).

CHAPTER 3

Local chiral symmetry and gauge fields(31)

§1 The relevance of the gauge fields

This chapter is the continuation of the Chapter 2. I start with considering the problems which arise when the transformations of K_n over the field quantities are made space time dependent. This leads to the introduction of gauge fields in the ordinary way. On the other hand, for the non-linear realization of K_n , this is not the only way we can introduce gauge fields. The transformation of "linear" fields ψ_a , given by (2.15) looks very much like the ordinary SU(n) transformation with space-time dependent parameter $\eta'(x)$. This already suggests the introduction of SU(n) gauge fields. In his paper on the non-linear realization of chiral SU(2)xSU(2)[7] Weinberg has introduced vector gauge fields ("ρ mesons") in this way and constructed a lagrangian which is completely invariant under the chiral transformations with constant parameters. It is rather nice to consider the vector gauge fields as arising from chiral symmetry instead of conventional local SU(n) symmetry. In the case of the latter, the symmetry is always broken unless the mass term of the gauge fields is absent. Weinberg's construction does not require the axial vector mesons as the gauge fields. The role of the axial vector fields in a chiral symmetry scheme is not absolutely clear even forgetting the experimental uncertainty about their existance. In the application, such as a famous calculation of electromagnetic mass difference of pi-mesons, they are given an essential role[32]. But this is always tied to the soft meson approximation and to me it is not clear if the axial vector exchange diagram could not be interpreted as certain limit of pseudo-scalor meson exchange diagram[33].

From the point of view of the consistency of chiral lagrangian method as a field theory, we have already seen in Chapter 1 that we need at least the vector and axial vector gauge fields in addition to essential P-S meson fields to avoid the contradiction of zero spectral function.

§2 The local chiral transformation

I have determined the transformation of the field quantities both "linear" and "non-linear" (as well as their covariant derivatives) in Chapter 2. Now I want to consider what happens when the parameters of a group element depend on the co-ordinate (x_μ). By attaching different elements of the group to each space-time point, only the derivatives of field quantities will be affected, and thus I should determine the transformation of covariant derivatives $\nabla_\mu \xi$ and $\nabla_\mu \psi$ under the local chiral transformations.

Consider the first order variation of the field quantities ($\xi(x)$ or $\psi(x)$) under the action of an element of local chiral group where now the (infinitesimal) parameters of group are made dependent of the space time co-ordinates. I may write

$$\delta f(x) = \sum_i \gamma_i(x) X_i(x)$$

where $f(x)$ stands for $\xi(x)$ or $\psi(x)$ fields and γ_i stands for the group parameters.

Then the variation of derivatives are

$$\delta \partial_\mu f(x) = \sum_i \partial_\mu \gamma_i(x) X_i(x) + \sum_i \gamma_i(x) Y_i(x) + O(\gamma^2)$$

where $Y_i = \partial_\mu X_i$

Writing like this, I should assume that the space-time dependence of group parameters are smooth enough

so that we may consider $\partial_\mu \gamma_i(x)$ as being "small".
I call the term proportional to the derivatives of the parameters which arises from space-time dependence of the transformation the local term; $\delta \partial_\mu f \big|_{loc}$.
and the term independent of the derivatives of the parameters i.e. $\sum \gamma_i Y_i$, the symmetry term $\delta \partial_\mu f \big|_{sym}$
since this part has the same form as the variation under the transformation without space-time dependence.

Thus, I write in general

$$\left.\begin{aligned} \delta_g \nabla_\mu \xi &= \delta_g \nabla_\mu \xi \big|_{sym} + \delta_g \nabla_\mu \xi \big|_{loc} \\ \delta_g \nabla_\mu \psi &= \delta_g \nabla_\mu \psi \big|_{sym} + \delta_g \nabla_\mu \psi \big|_{loc} \end{aligned}\right\} \qquad (3.1)$$

δ_g signifies that the infinitesimal variation $\delta \ldots$ is due to the operation of group element g. In particular, I am going to consider $\delta_H \ldots$ and $\delta_{ch} \ldots$ due to the infinitesimal elements $e^{\beta V}$ and $e^{\alpha A}$ respectively (with the notation of Chapter 2). $\delta_g \ldots \big|_{sym}$ is, of course, given by (2.35) and (2.38), and I have to calculate only $\delta_g \ldots \big|_{loc}$ which depends on $\partial_\mu \alpha(x)$ and $\partial_\mu \beta(x)$

(a) $\nabla_\mu \xi(x)$

From (2.31) and (2.26),

$$\delta_H \xi = \beta \cdot (iF \cdot \xi)$$

$$\delta_\alpha \xi = \frac{e^{iF\xi} + e^{-iF\xi}}{E_1(iF\xi) + E_1(-iF\xi)}$$

So it can be seen that

$$\delta_H(\partial_\mu \xi)\big|_{loc} = \partial_\mu \beta \cdot (iF\xi) \qquad (3.2)$$

$$\delta_\alpha(\partial_\mu \xi)\big|_{loc} = \partial_\mu \alpha \frac{e^{iF\xi} + e^{-iF\xi}}{E_1(iF\xi) + E_1(-iF\xi)} \qquad (3.3)$$

$\partial_\mu \xi$ is, of course, ordinary derivatives. Putting (3.2) and (3.3) in the expression of $\nabla_\mu \xi$ given in (2.42), I get

$$\delta_H \nabla_\mu \xi\big|_{loc} = \frac{1}{2}\partial_\mu \beta \,(e^{iF\xi} - e^{-iF\xi}) \qquad (3.4)$$

$$\delta_\alpha \nabla_\mu \xi\big|_{loc} = \frac{1}{2}\partial_\mu \alpha (e^{iF\xi} + e^{-iF\xi}) \qquad (3.5)$$

(b) $\nabla_\mu \psi$

Let us first examine the transformation of appearing in (2.36). I shall call this quantity after Coleman and Zumino, since this is a rather

important quantity together with $\nabla_\mu \xi$.
By substituting (3.2) and (3.3) into the expression (2.40) of β_μ, I will get

$$\delta_H \beta_\mu \big|_{g\alpha} = \frac{1}{2} \partial_\mu \beta \left(e^{iF\xi} + e^{-iF\xi} - 2 \right) \qquad (3.6)$$

$$\delta_\alpha \beta_\mu \big|_{g\alpha} = \frac{1}{2} \partial_\mu \alpha \, \frac{E_1(iF\xi) - E_1(-iF\xi)}{E_1(iF\xi) + E_1(-iF\xi)} \left(e^{iF\xi} + e^{-iF\xi} \right) \qquad (3.7)$$

From the expression of covariant derivative (2.36), together with (2.15), I have

$$\delta_g \nabla_\mu \psi_\alpha \big|_{g\alpha} = \left(i \partial_\mu \eta' \big|_{g\alpha} + i\, \delta\beta_\mu \big|_{g\alpha} \right) \cdot \hat{t}_{\alpha\beta} \psi_\beta \qquad (3.8)$$

Substituting (3.6) or (3.7) as well as the expressions of η' (2.30) for δ_H or (2.27) for δ_α. I get

$$\delta_H \nabla_\mu \psi \big|_{g\alpha} = \frac{i \partial_\mu \beta}{2} \left(e^{iF\xi} + e^{-iF\xi} \right) \cdot \hat{t}\, \psi \qquad (3.9)$$

and

$$\delta_\alpha \nabla_\mu \psi \big|_{g\alpha} = \frac{i \partial_\mu \alpha}{2} \left(e^{iF\xi} - e^{-iF\xi} \right) \cdot \hat{t}\, \psi \qquad (3.10)$$

(3.6), (3.7), (3.9) and (3.10) give the required transformation law for the covariant derivatives under the local chiral group.

§3 The gauge fields

The simple transformation law for the covariant derivatives can be recovered only by introducing a set of gauge fields.

Following Wess and Zumino[4] in the case of chiral SU(2)xSU(2), I start by considering the gauge fields for the ordinary linear representation formalism of group K_n . The construction of such gauge fields is well known since the work of Yang and Mills[34], Gell-Mann and Glashow[35]. In the present case, we have set of vector and axial vector fields $(v_\mu^i, a_\mu^i)_{i=1}^N$ with Yang-Mills type transformation under the operation of the infinitesimal element

$$g = e^{\beta V} \begin{cases} \delta_H v_\mu = (iF\cdot\beta)\, v_\mu + \frac{1}{g}\partial_\mu \beta \\ \delta_H a_\mu = (iF\cdot\beta)\, a_\mu \end{cases} \qquad (3.11)$$

$$g = e^{\alpha A} \begin{cases} \delta_{ch} v_\mu = (iF\cdot\alpha)\, a_\mu \\ \delta_{ch} a_\mu = (iF\cdot\alpha)\, v_\mu + \frac{1}{g}\partial_\mu \alpha \end{cases} \qquad (3.12)$$

Except for the derivative term the above transformations constitute the reducible representation $(N,1)\oplus(1,N)$ of K_n.

To make them useful in non-linear scheme, I have to convert them to the quantities transforming under the rule like (2.15) with non-linear parameter η'. For this, I will try to use the relation between non-linear realization and linear representation of K_n discussed in Chapter 2.

I is convenient now to first construct the irreducible components as

$$\Psi_\mu(L) = v_\mu + a_\mu \quad (3.13)$$

$$\Psi_\mu(R) = v_\mu - a_\mu \quad (3.14)$$

Apart from the derivative term, and transform under the infinitesimal elements of as

$$\begin{cases} \delta_H \Psi(L)|_{sym} = (iF\cdot\beta)\Psi(L) \\ \delta_\alpha \Psi(L)|_{sym} = (iF\cdot\alpha)\Psi(L) \end{cases} \quad (3.15)$$

$$\begin{cases} \delta_H \Psi(R)|_{sym} = (iF\cdot\beta)\Psi(R) \\ \delta_\alpha \Psi(R)|_{sym} = (-iF\alpha)\Psi(R) \end{cases} \quad (3.16)$$

From these, it can be seen that the "boost" matrix $D(e^{\xi A})$ discussed in Chapter 2 should be taken as $e^{\pm iF\xi}$ for $\Psi(L)$ and $\Psi(R)$ respectively.

Thus by "antiboosting" the linear representation $\Psi(L)$ and $\Psi(R)$ according to the discussion of Chapter 2 §1 (cf e.g. (2.18) to get a non-linearly transforming object, I am lead to define the following fields
$\chi_r^i(L)$ and $\chi_r^i(R)$

$$\chi_r(L) = e^{-iF\xi}(\sigma_r + a_r) \qquad (3.17)$$

$$\chi_r(R) = e^{iF\xi}(\sigma_r - a_r) \qquad (3.18)$$

The symmetry part of the transformation of these quantities under $g \in K_n$ is clearly of the type (2.15)

$$\delta_g \chi_r(L)\big|_{sym} = (iF\cdot\eta')\chi_r(L) \qquad (3.19)$$

$$\delta_g \chi_r(R)\big|_{sym} = (iF\cdot\eta')\chi_r(R) \qquad (3.20)$$

I have already anticipated the usual parity assignments with the fields σ_r and a_r. So the quantities $\chi_r(L)$ and $\chi_r(R)$ have no definite parity. The fields with definite parity are defined as

$$\chi_r^{\pm} = \frac{\chi_r(L) \pm \chi_r(R)}{2} \qquad (3.21)$$

In terms of original σ_μ and a_μ, I can write

$$\chi_\mu^+ = \sigma_\mu \frac{e^{iF\xi}+e^{-iF\xi}}{2} + a_\mu \frac{e^{iF\xi}-e^{-iF\xi}}{2} \quad (5.22)$$

$$\chi_\mu^- = \sigma_\mu \frac{e^{iF\xi}-e^{-iF\xi}}{2} + a_\mu \frac{e^{iF\xi}+e^{-iF\xi}}{2} \quad (5.23)$$

under the action of the group K_N, symmetry part of transformation of $\chi_\mu^\pm$ are given by (5.19) and (5.20) i.e.

$$\delta_g \chi_\mu^\pm \Big|_{sym} = (iF\cdot\eta')\,\chi_\mu^\pm \qquad (5.24)$$

On the other hand, the "local" part of the transformation can be obtained by substituting (5.11) and (5.12) into (5.22) and (5.23). Thus

$$\delta_H \chi_\mu^\pm \Big|_{loc} = \frac{1}{g}\partial_\mu\beta \; \frac{e^{iF\xi}\pm e^{-iF\xi}}{2} \qquad (5.25)$$

$$\delta_A \chi_\mu^\pm \Big|_{loc} = \frac{1}{g}\partial_\mu\alpha \; \frac{e^{iF\xi}\mp e^{-iF\xi}}{2} \qquad (5.26)$$

But the forms of (5.25) and (5.26) are exactly the additional factors appearing in the transformation of covariant derivatives under local chiral SU(N) (5.6), (5.7), (5.8) and (5.10). It is now trivial to construct the covariant quantities under the local chiral transformation. Thus if I define

$$D_\mu \xi = \nabla_\mu \xi - g\chi_\mu^- \qquad (3.27)$$

$$D_\mu \psi = \nabla_\mu \psi - ig(\chi_\mu^+ \cdot \hat{t})\psi \qquad (3.28)$$

then I will have

$$\delta_g (D_\mu \xi)|_{loc} = 0$$
$$\delta_g (D_\mu \psi)|_{loc} = 0$$

and

$$\delta_g (D_\mu \xi(x)) = iF\cdot\eta'(x)(D_\mu \xi(x)) \qquad (3.29)$$

$$\delta_g (D_\mu \psi(x)) = i\hat{t}\cdot\eta'(x)(D_\mu \psi(x)) \qquad (3.30)$$

where $\eta'(x) \equiv \eta'(g, \xi(x))$

The substance of the foregoing discussion is already fully realised by Wess and Zumino for K_2. But it is due to the simpler and more general realization discussed in Chapter 2 that this elementary derivation could be readily applied to general .

§3 Covariant ~~angles~~ curls

To construct the chiral invariant dynamical models with a local lagrangian functional some further covariant quantities besides a_μ, v_μ or $\chi^\pm_\mu$ are needed.

(a) Covariant curls of vector and axial vector fields.

In constructing the lagrangian, the kinematical term of vector or axial vector fields should be modified so that it is invariant under the chiral transformations, and for that I need usual covariant curl of Yang-Mill fields. Since this kinematical term is not troubled by the presence of non-linear quantities, I may apply directly the techniques of Yang and Mills(34,35) to linear fields (v_μ, a_μ). The irreducible part of (3.11) and (3.12) can be written as

$$\delta \Phi_\mu = i F\cdot\gamma \; \Phi_\mu + \frac{1}{g} \partial_\mu \gamma \tag{3.31}$$

with some group parameters ($\gamma_1 \cdots \gamma_N$). Φ_μ are the same combination of a_μ and v_μ which appeared in (3.13) and (3.14). The correspond covariant curls are

$$\begin{aligned} \tilde{G}^i_{\mu\nu} &\equiv [\,\Phi_\mu , \Phi_\nu\,]^i \\ &\equiv \partial_\mu \Phi^i_\nu - \partial_\nu \Phi^i_\mu + g\, \Phi^j_\mu (i F^i_{jk}) \Phi^k_\nu \qquad i, j, k = 1 \cdots N \end{aligned} \tag{3.32}$$

Writing (3.32) explicitly in term of a_μ and v_μ , and again taking the parity eigen-vectors, I get the following covariant curls.

$$G^{+}_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + g\{v_\mu(iF)v_\nu + a_\mu(iF)a_\nu\} \quad (3.33)$$

$$G^{-}_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + g\{v_\mu(iF)a_\nu + a_\mu(iF)v_\nu\} \quad (3.34)$$

$G^{\pm}_{\mu\nu}$ are the covariant generalization of the $\partial_\mu v_\nu - \partial_\nu v_\mu$ and $\partial_\mu a_\nu - \partial_\nu a_\mu$ needed to construct free lagrangian of vector and axial vector fields.

(b) Covariant curls with non-linear transformations.

The type of linearly transforming curls discussed above is convenient when we need not consider more complicated coupling of gauge fields and non-linearly transforming fields. But when for instance, we want to discuss the "magnetic" coupling of gauge fields to non-linear type "Baryon" fields, we need the covariant curls with similar non-linear transformations.

If the general techniques used in introducing above is applied directly, the quantities like

$$\chi^{\pm}_{\mu\nu} = G^{+}_{\mu\nu}\,\frac{e^{iF\xi} \pm e^{-iF\xi}}{2} + G^{-}_{\mu\nu}\,\frac{e^{iF\xi} \mp e^{-iF\xi}}{2}$$

are obtained, and they do transform like (2.15) as required. However simpler construction is possible without further introducing non-linear factors like

Let us consider the quantities

$$\phi_\mu^{\,+} = \chi_\mu^{\,+} - \frac{1}{g}\beta_\mu \qquad (3.35)$$

$$\phi_\mu^{\,-} = \chi_\mu^{\,-} - \frac{1}{g}\nabla_\mu \xi \qquad (3.36)$$

From the transformations of χ_μ^{+} given by (3.25) and (3.26) and those of β_μ given by (3.6) and (3.7) under local chiral group, it can be seen that the transformation of ϕ_μ^{+} under the action of the group is

$$\delta_g \phi_\mu^{+}(x) = (iF\cdot\eta')\phi_\mu^{+}(x) + \frac{1}{g}\partial_\mu \eta' \qquad (3.37)$$

As for ϕ_μ^{-}, it is just the covariant derivative $D_\mu \xi$ for local chiral group defined in (3.27). Thus

$$\delta_g \phi_\mu^{-}(x) = (iF\cdot\eta')\phi_\mu^{-}(x) \qquad (3.38)$$

If we apply Yang-Mills techniques to (3.37) then we will immediately get the covariant curl of the vector fields and this is enough for giving, for instance, the invariant magnetic moment coupling of vector fields. Nevertheless, it is suggestive to treat vector and axial vector fields in a more symmetric looking way[4].

If I define the quantities

$$\phi_\mu(L) = \phi_\mu^{+} + \phi_\mu^{-}$$
$$\phi_\mu(R) = \phi_\mu^{+} - \phi_\mu^{-}$$

then the transformation of the fields derived from (2.37) and (2.38) are written in the form of (3.31) as

$$\delta_g \Phi_\mu = (iF\cdot\eta')\Phi_\mu + \frac{1}{g}\partial_\mu \eta' \qquad (3.39)$$

where Φ_μ here stands for $\phi_\mu(L)$ or $\phi_\mu(R)$ and $\eta' \equiv \eta'(g, \xi(\eta))$ with (2.4) and (2.15). From these, it can be seen that the forms $[\phi_\mu(L), \phi_\nu(L)]$ and $[\phi_\mu(R), \phi_\nu(R)]$ using the notation of (3.32) are the required non-linear covariant curls.

Again taking the linear combination with definite parity, I obtain the following covariant curls

$$G'^{+}_{\mu\nu} = \partial_\mu \phi^+_\nu - \partial_\nu \phi^+_\mu + g\{\phi^+_\mu (iF)\phi^+_\nu + \phi^-_\mu (iF)\phi^-_\nu\} \qquad (3.40)$$

$$G'^{-}_{\mu\nu} = \partial_\mu \phi^-_\nu - \partial_\nu \phi^-_\mu + g\{\phi^+_\mu (iF)\phi^-_\nu + \phi^-_\mu (iF)\phi^+_\nu\} \qquad (3.41)$$

$G'^{\pm}_{\mu\nu}$ of course transform with the transformation law (2.15) under the local chiral SU(n)xSU(n)

$$\delta_g G'^{\pm}_{\mu\nu} = (iF\cdot\eta')G'^{\pm}_{\mu\nu} \qquad (3.42)$$

The transformation law like (3.37) corresponds to the Weinberg's point of view for the gauge fields in a non-linear realization(7). (3.37) is valid independent of it the transformation is co-ordinate dependent or not. If the vector fields with (3.37) are used to merely replace non-linear β_μ factor in the expression of $\nabla_\mu \psi$ to keep the covariance under constant chiral transformation, this is just the Weinberg's definition of vector gauge fields.

§4 The relation between linear and non-linear form of gauge fields.

In §3, I have derived two different types of covariant curls. It can be seen that the "non-linear" type (3.40) and (3.41) can be obtained from conventional the linear type (3.33) and (3.34) by replacing v_μ , a_μ with ϕ_μ^+, ϕ_ν^- defined in (3.35) and (3.36).

Now I shall show that this relation between $\phi_\mu^\pm$ and v_μ , a_μ expressed in (3.35) and (3.36) can be interpreted as a local chiral transformation. For this, I consider the infinitesimal gauge transformation (3.31) which is obeyed by irreducible components $v_\mu \pm a_\mu$

$$\delta \Psi_\mu = (iF\cdot\delta\gamma)\Psi_\mu + \frac{1}{g}\partial_\mu \delta\gamma \qquad (3.43)$$

Let us try to solve this equation for finite value of γ by integrating it. This can be done by considering the group operation along the one parameter subgroup $\{\gamma t\}_{0\le t\le 1}$ (t is independent of co-ordinate). Then (3.43) will be converted to the ordinary differential equation

$$\frac{d}{dt}\Psi_\mu = (i\gamma.F)\,\Psi_\mu(t) + \frac{1}{g}\partial_\mu\gamma \qquad (3.44)$$

with $\Psi_\mu(0) = \Psi_\mu$

We should find $g(\gamma)\Psi_\mu$ as $\Psi_\mu(1)$

The matrix $\gamma\cdot F$ can be diagonalized by some unitary matrix as

$$U(\gamma\cdot F)U^{-1} = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix}$$

Putting

$$U\Psi_\mu(t) = \phi_\mu(t)$$
$$U\gamma = \Gamma$$

I have

$$\frac{d\phi_\mu^i}{dt} = i\lambda_i\phi_\mu^i + \frac{1}{g}\partial_\mu\Gamma^i \qquad (3.45)$$

(No summation involved)

The solution of the last differential equation is

$$\phi_\mu^i(1) = \phi_\mu^i(0) + \frac{1}{g}\partial_\mu\Gamma^i \quad \text{for } \lambda_i = 0$$

$$\phi_\mu^i(1) = \phi_\mu^i(0) + (e^{i\lambda_i}-1)\left(\phi_\mu^i(0) + \frac{1}{i\lambda_i g}\partial_\mu\Gamma^i\right) \quad \text{for } \lambda_i \ne 0$$

But the latter is regular at $\lambda_i = 0$ and reduces to the former if λ_i goes to zero. Thus the solution of (3.45) is

$$\phi_\mu^i(1) = \phi_\mu^i(0) + (e^{i\lambda_i} - 1)\left(\phi_\mu^i(0) + \frac{1}{i\lambda_i g}\partial_\mu T^i\right)$$

or

$$\phi_\mu^i(1) = e^{i\lambda_i}\phi_\mu^i(0) + \frac{e^{i\lambda_i} - 1}{i\lambda_i}\,\frac{1}{g}\partial_\mu T^i \qquad (3.46)$$

Transforming back to Φ_μ by the matrix U^{-1}, I get the solution of original (3.44) as

$$\Phi_\mu^i(1) = e^{iF\cdot\gamma}\Phi_\mu + \frac{1}{g}E_1(iF\cdot\gamma)\partial_\mu\gamma \qquad (3.47)$$

where the matrix function $E_1(z)$ has appeared already in Chapter 2 and defined by the series expansion of $(e^z - 1)/z$

Thus the finite gauge transformation generated by the infinitesimal form (3.43) is

$$\Phi_\mu \xrightarrow{g(\gamma)} e^{iF\cdot\gamma}\Phi_\mu + \frac{1}{g}E_1(iF\cdot\gamma)\partial_\mu\gamma \qquad (3.48)$$

Applying this result to the irreducible components (3.13) and (3.14) i.e. $\Phi_\mu(L) = V_\mu + a_\mu$ and $\Phi_\mu(R) = V_\mu - a_\mu$ with the element of the chiral part of the group; $g = e^{-A\cdot\xi(x)}$, I get

$$\Phi_\mu(L) \xrightarrow{e^{-A\xi(x)}} e^{-iF\cdot\xi}\Phi_\mu(L) + \frac{1}{g}E_1(-iF\cdot\xi)(-\partial_\mu\xi) \qquad (3.49)$$

$$\Psi_\mu(R) \xrightarrow{e^{-A\cdot\xi(x)}} e^{iF\cdot\xi}\Psi_\mu(R) + \frac{1}{g} E_1(iF\xi)(\partial_\mu\xi) \quad (3.50)$$

(cf; 3.15 and 3.16)

In terms of the parity eigen vectors v_μ and a_μ these can be written as

$$v_\mu \xrightarrow{e^{-A\cdot\xi(x)}} \frac{e^{-iF\xi}+e^{iF\xi}}{2} v_\mu + \frac{e^{-iF\xi}-e^{iF\xi}}{2} a_\mu$$
$$- \frac{1}{g}\{E_1(-iF\xi) - E_1(iF\xi)\}\partial_\mu\xi$$

$$a_\mu \xrightarrow{e^{-A\cdot\xi(x)}} \frac{e^{-iF\xi}-e^{iF\xi}}{2} v_\mu + \frac{e^{-iF\xi}+e^{iF\xi}}{2} a_\mu$$
$$- \frac{1}{g}\{E_1(-iF\xi) + E_1(iF\cdot\xi)\}\partial_\mu\xi$$

Comparing the above results with the expression derived in Chapter 2 §3 and §3 of the present chapter ((3.22) and (3.23)), these are seen to be equivalent to

$$v_\mu \xrightarrow{e^{-A\cdot\xi(x)}} \chi_\mu^{+} - \frac{1}{g}\beta_\mu \quad (3.51)$$

$$a_\mu \xrightarrow{e^{-A\cdot\xi(x)}} \chi_\mu^{-} - \frac{1}{g}\nabla_\mu\xi \quad (3.52)$$

Thus, from (3.35) and (3.36)

$$\begin{pmatrix} v_\mu \\ a_\mu \end{pmatrix} \xrightarrow{e^{-A\cdot\xi(x)}} \begin{pmatrix} \phi_\mu^{+} \\ \phi_\mu^{-} \end{pmatrix} \quad (3.53)$$

The special chiral transformation $e^{-A\cdot\xi(x)}$ is just the inverse-boost operator utilized so much in Chapter 2. But here I am taking it as the local chiral transformation and at each space-time point x_μ I take corresponding parameter $\xi_i(x)$

Thus, in particular, beside the usual

$$e^{-A\cdot\xi(x)}\,\xi(x) \equiv 0$$

I can also write

$$e^{-A\cdot\xi(x)}\,\partial_{\mu_1\cdots\mu_N}\,\xi(x) \equiv 0\,.$$

The results on covariance under local chiral transformation in §2 of the present chapter could be derived in somewhat simpler way if we utilize the relation (3.53). For this, I shall consider a simple invariant coupling of fields as I have done in Chapter 1.

Take the "linear" type "baryon" field ψ_α described in Chapter 2, §1. This is associated with some irreducible representation $\mathcal{D}$ of SU($\mathcal{N}$). Further, let us assume, for the sake of simplicity of notation, that it is also an ordinary Dirac spinor as in Chapter 1. Then, as has been shown in Chapter 2, §1 (also Chapter 1, §3), the new fields

$$\Psi_\alpha = \left(e^{i\hat{t}\cdot\xi\,\gamma_5}\right)_{\alpha\beta}\psi_\beta \tag{3.54}$$

transform as the representation $(D,0)\oplus(0,D)$ of chiral $SU(n)\times SU(n)$. For instance

$$\Psi \xrightarrow{e^{A\cdot\alpha}} e^{(i\hat{t}\cdot\alpha)\gamma_5}\,\Psi$$

Then the following coupling

$$\mathcal{L}_{inv} = \bar{\Psi}\gamma^\mu(\partial_\mu - ig\hat{t}(v_\mu + \gamma_5 a_\mu))\Psi \qquad (3.55)$$

is clearly invariant under local chiral transformation. (This is just the ordinary covariant derivative for the representation of chiral $SU(n)\times SU(n)$). Thus, transforming the fields involved in (3.55) by the local chiral transformation $e^{-A\cdot\xi(x)}$ with

$$\Psi \longrightarrow e^{(-i\hat{t}\cdot\xi)\cdot\gamma_5}\,\Psi = \psi$$

$$\partial\Psi \longrightarrow \frac{\partial}{\partial x_\mu}\left\{e^{(-i\hat{t}\xi(x))\gamma_5}\,\Psi(x)\right\} = \partial_\mu\psi$$

$$\begin{pmatrix} v_\mu \\ a_\mu \end{pmatrix} \longrightarrow \begin{pmatrix} \phi_\mu^+ \\ \phi_\mu^- \end{pmatrix}$$

I get

$$\mathcal{L}_{inv} = \bar{\psi}\gamma^\mu(\partial_\mu - ig\hat{t}(\phi_\mu^+ + \gamma_5\phi_\mu^-))\psi \qquad (3.56)$$

From the invariance of (3.56), I can conclude as in §4 of Chapter 1, that

$$(\partial_\mu - ig\hat{t}\cdot\phi_\mu^+)\psi$$

and

$$\gamma_5 (\hat{t} \cdot \phi_\mu^-) \psi$$

are covariant, and from the covariance of the second form, the covariance of ϕ_μ^- itself can be concluded. This is just showing the covariance of $\mathcal{D}_\mu \psi$ and $\mathcal{D}_\mu \xi$ given in (3.78) and (3.27) of §3.

CHAPTER 4

Chiral invariant lagrangians

§1 The lagrangian and the currents

In this chapter, I would like to apply the results of the preceding chapters to construct a few examples of chiral invariant lagrangian models. Although these models are chosen with the application to the actual physical problems in mind, in the present chapter I shall discuss mainly the formula structure of these lagrangians, and leave the more practical problems to the next chapter where the detailed discussion of how to break symmetry will be given.

(a) The lagrangian and the currents without the gauge fields

The simplest example of a chiral invariant lagrangian is one with only the multiplet of non-linear fields ($\xi_1 \cdots \xi_N$) which is, in the physical case of chiral SU(3), identified with the octet of pseudoscalar mesons (π, K, η). Consider the chiral invariant lagrangian density

$$\mathcal{L}_\xi = \frac{a}{2} \sum_{i=1}^{N} (\nabla_\mu \xi)^2 \qquad (4.1)$$

where $\nabla_\mu \xi$ is the covariant derivative discussed in Chapter 2, and a is a constant. (We use the metric

convention $g_{00} = 1$, $g_{ii} = -1$ for i=1,2,3 and $x \cdot y = x_\mu y^\mu$
$= x_0 y_0 - \sum_{i=1}^{3} x_i y_i$).

(4.1) is the generalization of the pion lagrangian $\mathcal{L}_\pi$ (1.52) discussed in Chapter 1, §5.

The vector and the axial vector currents are defined by Noether's theorem with respect to the infinitesimal local chiral transformations

$$g = e^{\delta\beta(x)\cdot V} \sim 1 + \delta\beta(x)\cdot V$$

and

$$g = e^{\delta\alpha(x)\cdot A} \sim 1 + \delta\alpha(x)\cdot A$$

As in Chapter 1, §5, I have

$$V_\mu^{\,i} = -\frac{\delta\mathcal{L}_\xi}{\delta\partial^\mu\beta_i} \qquad \text{vector currents}$$

$$A_\mu^{\,i} = -\frac{\delta\mathcal{L}_\xi}{\delta\partial^\mu\alpha_i} \qquad \text{axial vector currents}$$

By using the transformation laws for given in (3.4) and (3.5), I get immediately

$$V_\mu = -a \cdot \frac{e^{iF\xi} - e^{-iF\xi}}{2} \nabla_\mu \xi \qquad (4.2)$$

$$A_\mu = -a \, \frac{e^{iF\xi} + e^{-iF\xi}}{2} \nabla_\mu \xi \qquad (4.3)$$

Then, comparing them with the original lagrangian (4.1), I find immediately

$$\mathcal{L}_\xi = \frac{1}{2a} \sum_{i=1}^{N} \left(V_\mu^{\,i} V^{\mu i} + A_\mu^{\,i} A^{\mu i} \right) \qquad (4.4)$$

This is the extention of the result found in Chapter 1 §5 (1.59) to chiral SU(n)xSU(n).

Let us consider now what will happen when the arbitrary SU(n) multiplet (ψ_α) transforming by (2.15) is included. I take the simplest model lagrangian with essentially Yukawa type coupling of non linear "meson" fields to this multiplet ("baryons")

$$\mathcal{L}_{\xi\psi} = \frac{a}{2}(\nabla_r \xi)^2 + \bar{\psi}(i\gamma_r \nabla^r - M)\psi + G' \bar{\psi} \gamma_r \gamma_5 \hat{t}' \psi \nabla^r \xi \qquad (4.5)$$

where are essentially C-G coefficients. For instance, in the case of the chiral SU(3), if we consider the coupling of octet p-s mesons to octet $\frac{1}{2}^+$ baryons with BBM Yukawa coupling, t has the well known form

$$\hat{t}'^i = (1-\alpha)F^i + \alpha D^i \qquad i = 1, 2, \cdots 8$$

The spin of the -fields is not essential although I have taken it for the ordinary spin $\frac{1}{2}$ dirac spinor to avoid unnecessary complications. To define the currents, the transformation laws for $\nabla_r \psi$ given in (3.9) and (3.10) are needed. Taking the variation of (4.5), I find

$$\begin{aligned} V_r &= -\frac{\delta \mathcal{L}}{\delta \partial_r \beta} \\ &= -\Big\{ a\, \frac{e^{iF\xi} - e^{-iF\xi}}{2} \nabla_r \xi - \frac{e^{iF\xi} + e^{-iF\xi}}{2} \bar{\psi} \gamma_r \hat{t} \psi \\ &\quad + G' \frac{e^{iF\xi} - e^{-iF\xi}}{2} \bar{\psi} \gamma_r \gamma_5 \hat{t}' \psi \Big\} \end{aligned} \qquad (4.6)$$

$$A_\mu = -\frac{\delta \mathcal{L}}{\delta \partial_\mu \alpha}$$
$$= \left\{ a\,\frac{e^{iF\xi}+e^{-iF\xi}}{2} \nabla_\mu \xi - \frac{e^{iF\xi}-e^{-iF\xi}}{2} \bar{\psi}\gamma_\mu \hat{t} \psi \right. \tag{4.7}$$
$$\left. + G' \frac{e^{iF\xi}+e^{-iF\xi}}{2} \bar{\psi}\gamma_\mu \gamma_5 \hat{t}' \psi \right\}.$$

From (4.6) and (4.7), I get the following expressions for the bilinear forms of the currents.

$$\sum_{i=1}^{N} V_\mu^i V^{i\mu}$$
$$= -a^2 \nabla_\mu \xi \left(\frac{e^{iF\xi}-e^{-iF\xi}}{2}\right)^2 \nabla^\mu \xi$$
$$+ N_\mu \left(\frac{e^{iF\xi}+e^{-iF\xi}}{2}\right)^2 N^\mu$$
$$- G'^2 N_\mu^5 \left(\frac{e^{iF\xi}-e^{-iF\xi}}{2}\right)^2 N^{5\mu}$$
$$+ 2a \nabla_\mu \xi \,\frac{e^{iF\xi}-e^{-iF\xi}}{2}\,\frac{e^{iF\xi}+e^{-iF\xi}}{2} N^\mu$$
$$- 2G' N_\mu \,\frac{e^{iF\xi}+e^{-iF\xi}}{2}\,\frac{e^{iF\xi}-e^{-iF\xi}}{2} N^{5\mu}$$
$$- 2aG' N_\mu^5 \left(\frac{e^{iF\xi}-e^{-iF\xi}}{2}\right)^2 \nabla^\mu \xi$$

(4.8)

$$\sum_{i=1}^{N} A_\mu^i A^{i\mu}$$
$$= a^2 \nabla_\mu \xi \left(\frac{e^{iF\xi}+e^{-iF\xi}}{2}\right)^2 \nabla^\mu \xi$$
$$- N_\mu \left(\frac{e^{iF\xi}-e^{-iF\xi}}{2}\right)^2 N^\mu$$
$$+ G'^2 N_\mu^5 \left(\frac{e^{iF\xi}+e^{-iF\xi}}{2}\right)^2 N^{5\mu}$$
$$- 2a \nabla_\mu \xi \,\frac{e^{iF\xi}+e^{-iF\xi}}{2}\,\frac{e^{iF\xi}-e^{-iF\xi}}{2} N^\mu$$

$$+ 2G'N_r \frac{e^{iF\xi} - e^{-iF\xi}}{2} \frac{e^{iF\xi} + e^{-iF\xi}}{2} N^{sr}$$

$$+ 2aG'N_r^s \left(\frac{e^{iF\xi} + e^{-iF\xi}}{2}\right)^2 \nabla^r \xi \tag{4.9}$$

where I write

$$N_r^i = \bar{\psi} \gamma_r \hat{t}^i \psi$$

$$N_r^{is} = \bar{\psi} \gamma_r \gamma_5 \hat{t}'^i \psi$$

Thus, I get

$$\sum_{i=1}^{iv} (V_r^i V^{ir} + A_r^i A^{ir})$$

$$= a^2 \nabla_r \xi \nabla^r \xi + 2aG' N_r^s \nabla^r \xi$$

$$+ N_r N^r + G'^2 N_r^s N^{sr}$$

or

$$\frac{1}{2a} \{ V_r^i V^{ir} + A_r^i A^{ir} - (N_r^i N^{ri} + G'^2 N_r^{is} N^{sir}) \}$$

$$= \frac{a}{2} \nabla_r \xi \nabla^r \xi + G' N_r^s \nabla^r \xi$$

Comparing this with (4.5), I get

$$\mathcal{L}_{\xi,\psi} = \bar{\psi}(i\gamma_r \nabla^r - M)\psi$$

$$+ \frac{1}{2a}(V_r^i V^{ri} + A_r^i A^{ri}) \tag{4.10}$$

$$- \frac{1}{2a}(N_r^i N^{ri} + G'^2 N_r^{si} N^{sir})$$

This is the result stated in §5, Chapter 1, for pion-nucleon lagrangian. Suppose we add the additional 4-point contact term[13]

$$+ \frac{1}{2a}(N_r N^r + G'^2 N_r^s N^{sr})$$

to the original lagrangian of (4.5). Then the modified lagrangian

$$\mathcal{L}'_{\xi,\psi} = \bar{\psi}(i\gamma_\mu \nabla^\mu - M)\psi + \frac{1}{2a}(V_\mu V^\mu + A_\mu A^\mu) \qquad (4.11)$$

is of "Sugawara form"(16) except for the chiral invariant kinematical term of ψ fields.

(b) The introduction of gauge fields(4)

As it can be seen from the results of Chapter 3, the chiral invariant lagrangian can be made invariant under the local chiral transformation by replacing the covariant derivatives $\nabla_\mu\xi$ and $\nabla_\mu\psi$ by $D_\mu\xi$ and $D_\mu\psi$ of (3.27) and (3.28).

$$\begin{aligned} &\mathcal{L}(\nabla_\mu\xi, \psi, \nabla_\mu\psi) \\ \rightarrow &\mathcal{L}(D_\mu\xi, \psi, D_\mu\psi) \\ &D_\mu\xi = \nabla_\mu\xi - g\chi_\mu^- \\ &D_\mu\psi = \nabla_\mu\psi - ig\hat{t}\cdot\chi_\mu^+\psi \end{aligned} \qquad (4.12)$$

χ_μ^+, χ_μ^- are defined according to (3.22) and (3.23) in terms of gauge fields a_μ, v_μ and non-linear "meson" fields ξ. Besides the replacement (4.12), we need to introduce a kinematical term of dynamical variables v_μ and a_μ which must be itself invariant.

Starting from the chiral invariant lagrangian $\mathcal{L}_{\xi,\psi}$ (4.5), the replacement (4.12) induces the following change

$$\mathcal{L}_{\xi,\psi}(\nabla_\mu \xi, \psi, \nabla_\mu \psi)$$
$$\rightarrow \mathcal{L}_{\xi,\psi}(\nabla_\mu \xi, \psi, \nabla_\mu \psi) + \Delta_1 + \Delta_2. \qquad (4.13)$$

where

$$\Delta_1 = a(-g\nabla_\mu \xi \cdot \chi^{-\mu} + \frac{g^2}{2}\chi_\mu^- \chi^{-\mu}) - gG' N_\mu^5 \chi^{-\mu}$$

$$\Delta_2 = g N_\mu \chi^{+\mu}$$

Using (3.22) and (3.23), I can write it as

$$\Delta_1 = \frac{ag^2}{2}\chi_\mu^- \chi^{-\mu} - ag\left\{ v_\mu \frac{e^{iF\xi} - e^{-iF\xi}}{2} + a_\mu \frac{e^{iF\xi} + e^{-iF\xi}}{2} \right\} \nabla^\mu \xi$$
$$- gG' \left\{ v_\mu \frac{e^{iF\xi} - e^{-iF\xi}}{2} + a_\mu \frac{e^{iF\xi} + e^{-iF\xi}}{2} \right\} N^{5\mu}$$

$$\Delta_2 = g\left\{ v_\mu \frac{e^{iF\xi} + e^{-iF\xi}}{2} + a_\mu \frac{e^{iF\xi} - e^{-iF\xi}}{2} \right\} N^\mu$$

And thus

$$\Delta_1 + \Delta_2 = \frac{ag^2}{2}\chi_\mu^- \chi^{-\mu}$$
$$+ g v^\mu \left\{ -a \frac{e^{iF\xi} - e^{-iF\xi}}{2} \nabla_\mu \xi - G' \frac{e^{iF\xi} + e^{-iF\xi}}{2} N_\mu^5 \right.$$
$$\left. + \frac{e^{iF\xi} + e^{-iF\xi}}{2} N_\mu \right\}$$
$$+ g a^\mu \left\{ -a \frac{e^{iF\xi} + e^{-iF\xi}}{2} \nabla_\mu \xi - G' \frac{e^{iF\xi} + e^{-iF\xi}}{2} N_\mu^5 \right.$$
$$\left. + \frac{e^{iF\xi} + e^{-iF\xi}}{2} N_\mu \right\}$$

Comparing this with the expressions (4.6) and (4.7) for the currents, it can be seen that

$$\Delta_1 + \Delta_2 = \frac{ag^2}{2}\chi_\mu^- \chi^{-\mu} + g(v^\mu V_\mu + a^\mu A_\mu) \qquad (4.14)$$

where A_μ and V_μ are the same functional of the fields ξ and ψ as defined in (4.6) and (4.7).

The invariant kinematical term can be constructed out of covariant curls discussed in §3, Chapter 3 and can be written as

$$\mathcal{L}'_{kin} = -\frac{1}{4}\left(G^{+}_{\mu\nu}G^{+\mu\nu} + G^{-}_{\mu\nu}G^{-\mu\nu}\right) \qquad (4.15)$$

If the lagrangian is constructed out of (4.13), (4.14) and (4.15), it is completely invariant under the local chiral transformation. But then the field variables v_μ and a_μ can represent particles with zero mass only and we cannot consider them as, for instance, the phenomenological description of known heavy vector or axial vector mesons. To give masses to these fields, I should break the invariance under the local chiral transformation. The mass term which is still invariant under constant chiral transformation is

$$\mathcal{L}'_{mass} = \frac{m_0^2}{2}\sum_{i=1}^{N}\left(a_\mu^i a^{i\mu} + v_\mu^i v^{i\mu}\right) \qquad (4.16)$$

This is the unique expression since the chiral invariant bilinear forms are $(a_\mu \pm v_\mu)^2$ in terms of irreducible quantities and we must eliminate the parity non-conserving product term $a_\mu \cdot v^\mu$.

The addition of (4.16) violates the invariance under the local chiral transformations, but on the other hand, it has an attractive feature when the currents are considered. Taking the usual variation of the lagrangian

$$\mathcal{L}_{tot} = \mathcal{L}_{3,4} + \Delta_1 + \Delta_2 + \mathcal{L}'_{bi} + \mathcal{L}''_{m}$$

I get immediately

$$V_\mu' = -\frac{\delta \mathcal{L}_{tot}}{\delta \partial^\nu \beta} = -\frac{m_0^2}{g} v_\mu \qquad (4.17)$$

$$A_\mu' = -\frac{\delta \mathcal{L}_{tot}}{\delta \partial^\nu \alpha} = -\frac{m_0^2}{g} a_\mu \qquad (4.18)$$

This is just the field current identity of Lee and Zumino(36,37). It is shown(38) by Lee, Weinberg and Zumino that if a_μ and v_μ are considered as canonical variables and ordinarly canonical commutationsrelations from $\mathcal{L}_{tot}$ are assumed, then the chiral SU(n)xSU(n) current algebra type commutation relations between the currents A_μ' and V_μ' can be derived and that the Schwinger terms appearing there are finite and c-numbers. It should be noted that from the derivation of (4.17) and (4.18) the quantities like masses, coupling constants and field themselves are unrenormalized. But it has been shown(36) also that if the lagrangian $\mathcal{L}_{tot}$ is renormalizable with finite propagation then the quantities appearing in the R.L..s of (4.17) and (4.18) can be replaced by the corresponding renormalized quantities.

$\mathcal{L}_{\omega}$ is still invariant under the chiral transformation and the corresponding currents are conserved

$$\partial_\mu V^{\mu\,\prime} = \partial_\mu v^\mu = 0 \tag{4.19}$$

$$\partial_\mu A^{\mu\,\prime} = \partial_\mu a^\mu = 0 \tag{4.20}$$

This means that if $\mathcal{L}_{\omega}$ can be considered as defining a quantized field theory then the spectral functions of the field operators $V^{\mu\,\prime}$ (or v^μ) and $A^{\mu\,\prime}$ (or a^μ) contain the vector (spin 1) parts only except the mass loss excitation corresponding to the ξ fields (Goldstone boson).

Finally, collecting (4.10), (4.14), (4.15) and (4.16), the form of $\mathcal{L}_{\omega}$ becomes

$$\begin{aligned}\mathcal{L}_{\omega} = {} & \bar{\psi}(i\partial\!\!\!/ - M)\psi - \frac{1}{4}\left((G^+_{\mu\nu})^2 + (G^-_{\mu\nu})^2 \right) \\ & - \frac{1}{2a}\left(N_\mu N^\mu + G'^2 N_\mu{}^5 N^{5\mu} \right) \\ & + \frac{1}{2a}(V_\mu^2 + A_\mu^2) + g(v_\mu V^\mu + a_\mu A^\mu) + \frac{m^2}{2}(a_\mu^2 + v_\mu^2) \\ & + \frac{a g^2}{2}(\chi_\mu^-)^2 \end{aligned} \tag{4.21}$$

(c) The further invariant couplings[4]

We can add a few other invariant couplings of physical interest to (4.21). These can contribute to the anomalous magnetic moments of particles.

(1) Magnetic coupling to ψ field

$$\mathcal{L}_1 = \mu \bar{\psi} \sigma_{\mu\nu} \hat{t}'' \psi G'^{+\mu\nu}_{\mu\nu} \qquad (4.22)$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ and $\hat{t}''$ like $\hat{t}'$ of (4.5) represents general SU( ) coupling between , and $(G'^{+i}_{\mu\nu})_{i=1}^{N}$. $G'^{+}_{\mu\nu}$ are "non-linear" covariant curls defined in (3.40). μ is a constant.

(2) Trilinear coupling of the gauge fields

$$\mathcal{L}_2 = kg\, t'''^{i}_{jk} \phi^{-j}_{\mu} \phi^{-k}_{\nu} G'^{+i\mu\nu} \qquad (4.23)$$

where t''' again represents the SU(N) coupling between ϕ^{-}_{μ}, ϕ^{-}_{ν} and $G'^{+i}_{\mu\nu}$. k is a constant.

§2 The phenomenological lagrangian

To use (4.21) (with possible (4.22) and (4.23)), as a phenomenological lagrangian, we can impose some further restrictions. The argument used by Wess and Zumino[4] for the chiral SU(2)xSU(2) invariant model can be applied to (4.21) without alteration. First, it should be noted that the non-linear lagrangian (4.21) does contain the term like $a_\mu D^\mu \xi$, which modifies the propagator by the direct transition between a_μ fields and ξ fields. To eliminate such a term, the a_μ field is assumed to be a mixture of an "axial vector field" $\hat{a}_\mu$ and pseudo-scalar field ξ as

$$a_\mu = \hat{a}_\mu + c\,\nabla_\mu \xi$$

Instead of taking this straightforward decomposition, I follow Wess and Zumino to write it

$$a_\mu = \hat{a}_\mu + c\left(\nabla_\mu \xi + g\,\frac{e^{i\Gamma\xi} - e^{-i\Gamma\xi}}{2}\,v_\mu\right) \quad (4.22)$$

The additional term containing v_μ field looks a little arbitrary. One way of justifying this form is the simplicity with which the electro-magnetic interaction of ξ field can be introduced. (c.f. §4). Then, looking at the term $\frac{a}{2}(\nabla_\mu\xi)^2 + \frac{m^2}{2}(a_\mu{}^2 + v_\mu{}^2)$ the coefficients of $\frac{1}{2}\nabla_\mu\xi\,\nabla^\mu\xi$, $\frac{1}{2}\hat{a}_\mu\hat{a}^\mu$, $\frac{1}{2}v_\mu v^\mu$ and $\nabla_\mu\xi\cdot\hat{a}^\mu$ in (4.21) are $c^2m^2+a(1-gc)^2$, m^2+ag^2, m^2 and $m^2c-ag(1-gc)$ respectively. (According to the argument about (4.17) and (4.18), I now assume m and g as renormalized quantities and use the letter m instead of m). The condition of the absence of the term $\nabla_\mu\xi\cdot\hat{a}_\mu$ in the phenomenological lagrangian now imposes

$$m^2 c = ag(1 - gc)$$

or

$$c = \frac{ag}{m^2 + ag^2}$$

The coefficients of v_μ^2 and a_μ^2 can be considered as the mass squares of vector and axial vector fields. So putting $\bar{m}^2 = m^2 + a g^2$ (mass square of axial vector field) the above condition can be written as

$$C = \frac{1}{g} \frac{\bar{m}^2 - m^2}{\bar{m}^2} \tag{4.25}$$

I have stated the $(\xi_i)_{i=1}^N$ are SU(n) generalization of pion fields of Chapter 1. But the coefficient of $(\nabla_\mu \xi)^2$ tells us that it is not properly normalized as the phenomenological description of the particles. Thus defining the "physical" fields by $\phi = \frac{F}{2}\xi$, the condition for the correct normalization is

$$\left(\frac{2}{F}\right)^2 (c^2 m^2 + a(1-gc)^2) = 1.$$

In terms of introduced above,

$$\left(\frac{2}{F}\right)^2 \frac{\bar{m}^2 - m^2}{g^2} \frac{m^2}{\bar{m}^2} = 1$$

or

$$F = 2 \frac{m}{\bar{m}} \sqrt{\frac{\bar{m}^2 - m^2}{g^2}} \tag{4.26}$$

It is impossible to get further restrictions by the chiral invariance alone. One of the well known results suggested from the calculation in the current algebra is the ratio $\bar{m}/m$ and the relation between g and F. One way of recovering these results is to introduce an additional physical assumption of "vector meson dominance". This is of course, the ρ-dominance model of Sakurai in the case of chiral SU(2)xSU(2) but probably can be extended to chiral SU(3)xSU(3) too. (4.21) does contain ξ-ψ contact term due to $\bar{\psi}\gamma^{\mu}\rho_{\mu}\psi$. If "$\xi$-$\psi$ scattering amplitude" is required to be always mediated by vector fields v_μ, then the coefficient of a possible ξ-ψ 4-point contact term should be put equal to zero. Since

$$D_\mu \psi = \{\partial_\mu + i\hat{t}(\rho_\mu - g\chi_\mu^{+})\}\psi$$

$$\simeq \{\partial_\mu + i\hat{t}(\partial_\mu \xi \frac{iF\xi}{2} - g(v_\mu + a_\mu(iF\xi)))\}\psi$$

and

$$a_\mu \to \hat{a}_\mu + c\,\partial_\mu \xi$$

the coefficient in question is $(\frac{1}{2} - gc)$

Thus I may conclude

$$c = \frac{1}{2g} \qquad (4.27)$$

Then, from (4.25),

$$\frac{m^2}{\bar{m}^2} = \frac{1}{2} \qquad (4.28)$$

which is the Weinberg's relation[2[illegible]]. Also from (4.26)

$$F = \frac{\sqrt{2}\,m}{g}. \qquad (4.30)$$

which is the Kawarabayashi-Suzuki relation[39].

Finally, I note that with the relations obtained above I get

$$a g^2 = \bar{m}^2 - m^2 = m^2$$

$$\therefore a = \frac{m^2}{g^2}.$$

The lagrangian (4.21) reduces to rather simple form

$$\begin{aligned} \mathcal{L}_{tot} &= \bar{\psi}(i\not{\partial} - M)\psi - \tfrac{1}{4}\left((G^+_{\mu\nu})^2 + (G^-_{\mu\nu})^2\right) \\ &\quad - \frac{g^2}{2m^2}\left(N_\mu^2 + G'^2 N_\mu^{5\,2}\right) \\ &\quad + \frac{g^2}{2m^2}\left(\left(V_\mu + \frac{m^2}{g}v_\mu\right)^2 + \left(A_\mu + \frac{m^2}{g}a_\mu\right)^2\right) + \frac{m^2}{2}(\chi_\mu^-)^2 \end{aligned} \qquad (4.21)$$

The model discussed above due to Wess and Zumino does not represent unique chiral invariant lagrangian with gauge fields. Weinberg has emphasized the likeness of chiral SU(2)xSU(2) invariance in non-linear realization technique to the ordinary local SU(2) invariant coupling. According to this idea, instead of (4.21), we have

$$\begin{aligned} \mathcal{L}' &= \frac{a'}{2}(\nabla_\mu \xi)^2 + \frac{m^2}{2}(\chi_\mu^+)^2 - \frac{1}{2}\Gamma'^2_{\mu\nu} \\ &\quad + \bar{\psi}(i\gamma_\mu(\partial^\mu - ig\hat{t}\,\Gamma'^\mu) + M)\psi \\ &\quad + G'\bar{\psi}\gamma_\mu\gamma_5\hat{t}'\psi\,\nabla^\mu\xi \end{aligned} \qquad (4.31)$$

where
$$\chi_\mu^{+\prime} = v_\mu' + \frac{1}{g}\beta_\mu$$
$$v_{\mu\nu}' = \partial_\mu v_\nu' - \partial_\nu v_\mu' + ig\, v_\mu'(iF)v_\nu'$$

The new gauge field v_μ' transform under the chiral SU(n)xSU(n) according to (3.37)

$$\delta v_\mu' = (iF\cdot\eta')v_\mu' + \frac{1}{g}\partial_\mu\eta' \qquad (4.32)$$

In this model, where there is no need for an a_μ field, the identification of various arbitrary constants can be done somewhat more simply. In particular (4.30) can be obtained as the result of the universal coupling of vector gauge fields. Note that (4.32) gives according to the expressions given in Chapter 2.

$$\delta v_\mu'\big|_{\mathrm{ta}} = \frac{1}{g}\partial_\mu\eta'$$
$$= \frac{1}{g}\partial_\mu\left\{\delta\alpha\,\xi\,\frac{E_1(iF\xi)+E_1(-iF\xi)}{e^{iF\xi}+e^{-iF\xi}}\,\frac{e^{iF\xi}-1}{e^{iF\xi}+1}\right\}$$
$$\simeq \frac{1}{g}\partial_\mu\left(\delta\xi\cdot\frac{iF\xi}{2}\right)$$

In term of "physical" fields $\phi = \frac{F}{2}\xi$ with Kawarabayashi-Suzuki relation, the last expression reduces to

$$\frac{g}{m^2}\partial_\mu(\delta\phi\cdot iF\phi)$$
$$= \frac{g}{m^2}\partial_\mu(C_{ijk}\,\phi_j\,\delta\phi_k)$$

which is equivalent to the transformation used by Schwinger[3].

§3 The equivalence relations

We can use a phenomenological lagrangian like (4.21) according to the idea discussed in §1 Chapter 1. But then it should be remembered that the definition of "physical" fields is not unique. If, for instance, $\phi_i = \frac{F}{2}\xi_i$ is used as second quantized P-S meson fields to compute the Feynman graph with given lagrangian, any transformation

$$\chi_i = f_i(\phi) \tag{4.33}$$

with $f_i(0)=0$ can be used with the same (transformed) lagrangian. This has the analogy in formal field theory with the non uniqueness of interpolating field operator. Coleman and Zumino[6] give a general proof that the "canonical transformation" like (4.33) leaves not only the exact on mass shell S-matrix elements invariant, but it leaves the value of the sum of Feynman graphs with a fixed number of internal loops invariant. In particular, the value of amplitudes obtained in the tree approximation (Chapter 1) is not affected by such a transformation. Instead of quoting their proof. I would like to give some examples of field transformations.

(a) Weinberg-Kawarabayashi form of gauge fields.

In Chapter 3, I have introduced the functions of gauge fields $\phi_\mu^{\pm}(v_\mu, a_\mu, \xi)$. (3.35 and 3.36). They have rather simple transformations under the local chiral group. (3.37) and (3.38). I have shown that $\phi_\mu^{\pm}$ can be regarded as chiral transform of the original linear gauge fields v_μ and a_μ. (§4, Chapter 3). Kawarabayashi[40] has suggested to use this transformed $\phi_\mu^{\pm}$ as the vector and axial vector meson fields instead of v_μ and a_μ of Weiss and Zumino. Thus, write

$$v_\mu' \equiv \phi_\mu^{+} \equiv \chi_\mu^{+} - \frac{1}{g}\beta_\mu$$
$$a_\mu' \equiv \phi_\mu^{-} \equiv \chi_\mu^{-} - \frac{1}{g}\nabla_\mu \xi$$

where

$$\chi_\mu^{\pm} = v_\mu \frac{e^{iF\xi} \pm e^{-iF\xi}}{2} + a_\mu \frac{e^{iF\xi} \mp e^{-iF\xi}}{2}$$

$$\left.\begin{array}{c}\nabla_\mu \xi \\ \beta_\mu\end{array}\right\} = \frac{1}{2}\partial_\mu \xi \, \frac{E_1(iF\xi) \pm E_1(-iF\xi)}{2}$$

Since the transformation $\begin{pmatrix} v_\mu \\ a_\mu \end{pmatrix} \longrightarrow \begin{pmatrix} v_\mu' \\ a_\mu' \end{pmatrix}$ is an element of local chiral transformation $e^{-A\xi(2)}$, the part of the gauge fields Lagrangian which is invariant under local chiral transformation will be left unchanged

$$\mathcal{L}_{inv}(a_\mu, v_\mu) \longrightarrow \mathcal{L}_{inv}(a_\mu', v_\mu')$$

The local chiral invariance of the Lagrangian (4.21) is broken through the mass term

$$\mathcal{L}''_{mass} = \frac{m^2}{2} \sum (a_\mu a^\mu + v_\mu v^\mu)$$

But according to the definition of the functions above, I have

$$\sum_{i=1}^{N} (\chi_\mu^{+i} \chi^{+i\mu} + \chi_\mu^{-i} \chi^{-i\mu})$$
$$= \sum_{i=1}^{N} (a_\mu^i a^{i\mu} + v_\mu^i v^{i\mu})$$

Thus the mass term $\mathcal{L}''_{mass}$ can be written in terms of new variables as

$$\mathcal{L}''_{mass} = \frac{m^2}{2} \left\{ (v_\mu' + \frac{1}{g} \beta_\mu)^2 + (a_\mu' + \frac{1}{g} \nabla_\mu \xi)^2 \right\}.$$

Note also the invariant term $\frac{a}{2}(\mathcal{D}_\mu \xi)^2$ is transformed into $\frac{ag^2}{2}(a_\mu')^2$ and in general, the expression containing ξ or $\nabla_\mu \xi$ can be replaced by v_μ' and a_μ' alone. Thus if the mass m is put to zero, ξ field will disappear, which is of course obvious because of

$$e^{A \cdot \xi(x)} \partial_{\mu_1} \cdots \partial_{\mu_n} \xi(x) = 0 \qquad \text{(Chapter 3)}$$

This disappearance of ξ's corresponds to the decoupling of Goldstone boson suggested by Kibble[41].

The identification of arbitrary constants can be carried out as in the last section. But it should be noted that the transformation $(v_\mu, a_\mu) \to (v_\mu', a_\mu')$ is not a proper canonical transformation since this contains the derivatives of ξ fields. Thus, even with the theorem of Coleman and Zumino, there is no reason why the new scheme (of Kawarabayashi) should be compatible with the old one (of Wess and Zumino).

Let us consider the coupling of ξ's to v_μ. First the relevant part of Wess-Zumino lagrangian is

$$\frac{m^2}{2} a_\mu^2 + \frac{a}{2} (\partial_\mu \xi)^2$$

$$\Rightarrow \frac{m^2}{2} c^2 (\partial_\mu \xi - g v_\mu \cdot i F \xi)^2$$

$$+ \frac{a}{2} \left((1-gc)\, \partial_\mu \xi - g(1-gc) v_\mu \cdot i F \xi \right)^2$$

$$\Rightarrow g \left(m^2 c^2 + a(1-gc)^2 \right) \left(\frac{2}{F} \right)^2 \left\{ -\partial_\mu \phi \cdot (v_\mu \cdot i F \phi) \right\}$$

But because of the normalization condition discussed in the last section, the last expression reduces to

$$g \left\{ -\partial_\mu \phi (v_\mu \cdot i F \phi) \right\} \qquad (4.34)$$

To examine the analogous coupling term in the Kawarabayashi lagrangian, I should first repeat the analysis of the last section to relate various arbitrary constants. Thus, I introduce again the decomposition

$$a_\mu' = \hat{a}_\mu' + c' \nabla_\mu \xi \qquad (4.35)$$

The elimination of term like $a_\rho \xi \cdot \hat{a}_\rho{}'$ gives

$$C' = -\frac{m^2}{\overline{m}'^2}\frac{1}{g} \qquad (4.36)$$

where $\overline{m}'$ is the mass of $\hat{a}_\rho{}'$ field.

The normalization of ξ-fields is completely analogous to the Wess-Zumino lagrangian and normalization constant $\phi = \frac{F'}{2}\xi$ satisfies (4.36) i.e.

$$\left(\frac{F'}{2}\right)^2 = \frac{1}{g^2}\frac{m^2}{\overline{m}'^2}\left(\overline{m}'^2 - m^2\right) \qquad (4.37)$$

Now the trilinear coupling of the form $\partial_\rho \phi (v_\rho \cdot iF\phi)$ in Kawarabayashi lagrangian comes solely from the term $\frac{m^2}{2}(v_\rho + \frac{1}{g}\beta_\rho)^2$. (We may modify the decomposition (4.35) to include vector field term like

$$a_\rho' \sim \hat{a}_\rho' + C'(\partial_\rho \xi + x\, v_\rho \cdot (iF\xi))$$

But the contribution coming from such an addition in the case of the Kawarabayashi form does cancel.)

Thus the corresponding coupling is

$$\frac{1}{2}\frac{m^2}{g}\left(\frac{2}{F'}\right)^2\left\{-\partial\phi(v_\rho \cdot iF\phi)\right\} \qquad (4.38)$$

(4.34) and (4.37) represent the "2 ϕ decay process of v_ρ" in both Wess-Zumino and Kawarabayashi lagrangian. Thus if I now require the compatability of two lagrangians in spite of the improper transformations connecting them, I should conclude

$$\frac{1}{2}\frac{m^2}{g}\left(\frac{2}{F'}\right)^2 = g.$$

Using (4.37), this reduces to

$$\frac{\overline{m}'^2}{\overline{m}'^2 - m^2} = 2$$

or

$$\overline{m}'^2/m^2 = 2.$$

Of course, I will also require the equivalence of $\hat{a}_r'$ to $\hat{a}_l$ and put $\overline{m}' = \overline{m}$. So I have

$$\overline{m}^2/m^2 = 2 \tag{4.39}$$

and

$$F = F' = \frac{\sqrt{2}\, m}{g}. \tag{4.40}$$

These are just Weinberg and Kawarabayashi-Suzuki relations, which are, in case of Wess-Zumino model alone, derived with the assumption of vector dominance. This result is derived by Kawarabayashi although I have presented it in a little different way. Also, the essence of the argument is contained already in the original paper by Weinberg[7]. The model of Kawarabayashi can be regarded as the generalization of Weinberg's model discussed in the last section. Unlike Weinberg's it contains an a_r field but it does not satisfy a field-current identity like Wess and Zumino's.

(b) Cronin form of meson-nucleon lagrangian

The lagrangians without gauge field (4.5) contain as the special case the pion-nucleon lagrangian (1.42). In Chapter 1, I have shown that (1.42) which is the chiral invariant generalization of the Schwinger's phenomenological lagrangian (1.1), can be approximately derived from the simple "linear field" lagrangian (1.30). The approximation here is that I put in (1.42) which is equivalent to $G'=1$ in (4.5). I keep this approximation in the discussion below for the sake of simplicity. Consider two forms of chiral (SU(2) invariant (apart from the meson mass term) lagrangians

$$\mathcal{L}_1 = i N \not\partial N - m\bar{N}N + \frac{\lambda^2}{2}(\nabla_\mu \xi)^2 - \frac{1}{2}\lambda^2\mu^2\xi^2 + \bar{N}\gamma_\mu\gamma_5\frac{\tau}{2}N\cdot\nabla^\mu\xi \qquad (4.41)$$

and

$$\mathcal{L}_2 = i\bar{N}'\not\partial N' - \bar{N}'M(\xi)N' + \frac{\lambda^2}{2}(\nabla_\mu\xi)^2 - \frac{1}{2}\lambda^2\mu^2\xi^2 \qquad (4.42)$$

where $$M(\xi) = m\, e^{-2i\frac{\tau}{2}\cdot\xi\,\gamma_5}$$

m and μ are nucleon and meson masses. The mesons are represented by the ξ fields discussed in Chapter 1 and Chapter 2 and the physical fields is just $\phi = \xi/\lambda$ rather than more complicated functions of ξ/λ as in Chapter 1.

It has been explained in Chapter 1[1] that (4.41) can be obtained from (4.42) by the transformation of the nucleon fields

$$N'_\alpha = (e^{A\xi} N)_\alpha = (e^{i\frac{\tau}{2}\cdot\xi\,\gamma_5})_{\alpha\beta} N_\beta \qquad (4.43)$$

Weinberg in (Ref.1) has already noted that (4.41) and (4.42) give the same meson-nucleon scattering amplitude for tree graphs. Of course, we can construct the equivalent form of (4.42) in chiral SU(3)xSU(3) scheme and it is this form rather than the "derivative coupling form" (4.5) which has been earlier proposed by Cronin[2] as the model of octet meson-baryon interaction.

Now, the transformation (4.43) is a form of the canonical transformation in the sense of Coleman and Zumino, and the invariance of on-mass-shell S-matrix element under the transformation N' → N $\xi \to \xi$ should be expected from their equivalence theorem.

To see the relevance of Coleman-Zumino theorem a little further, I consider the slightly more general transformation than (4.43)

$$N' \to N; \quad N' = e^{i\gamma\frac{\tau}{2}\cdot\xi\,\gamma_5} N \qquad (4.44)$$

where γ is an arbitrary real constant. (4.44) cannot be considered as the chiral transformation of the nucleon field like (4.43).

Let us examine the nucleon-nucleon and the meson-meson amplitudes in the tree approximation using the lagranagian (4.42) and its transform by (4.44) (which is equivalent to (4.41) if γ =1). By the transformation (4.44).

$$\mathcal{L}_2 \rightarrow \mathcal{L}(\gamma) \,;$$

$$i\bar{N}'\not{\partial}N' \rightarrow i\bar{N}\gamma_\mu\, e^{-i\gamma\frac{\tau}{2}\cdot\xi\gamma_5}\partial^\mu e^{i\gamma\frac{\tau}{2}\cdot\xi\gamma_5}N$$

$$= i\bar{N}\gamma_\mu\{\partial^\mu + i\tfrac{\tau}{2}\cdot(\beta_\mu(\gamma.\xi) + \gamma_5\alpha_\mu(\gamma.\xi))\}N$$

$$m\bar{N}'M(\xi)N' = m\bar{N}'e^{-2i\frac{\tau}{2}\cdot\xi\gamma_5}N'$$

$$\rightarrow m\bar{N}e^{-2i(1-\gamma)\frac{\tau}{2}\cdot\xi\gamma_5}N$$

where $\beta_\mu(\gamma.\xi) = \partial_\mu(\gamma.\xi)\dfrac{E_1(i\gamma\xi\cdot F) - E_1(-i\gamma\xi\cdot F)}{2} \simeq \gamma^2\xi\wedge\partial_\mu\xi$

$$\alpha_\mu(\gamma.\xi) = \partial_\mu(\gamma.\xi)\frac{E_1(i\gamma\xi\cdot F) + E_1(-i\gamma\xi\cdot F)}{2} \simeq \gamma\partial_\mu\xi$$

Thus the relevant part of $\mathcal{L}'(\gamma)$ for our purpose is

$$\begin{aligned}\mathcal{L}(\gamma) \simeq\; & i\bar{N}\not{\partial}N - \frac{\gamma^2}{4\lambda^2}\bar{N}\gamma^\mu\tau N(\phi\wedge\partial_\mu\phi) \\ & - \frac{\gamma}{2\lambda}\bar{N}\gamma^\mu(\tau\cdot\partial_\mu\phi)\gamma_5 N \\ & - m\bar{N}N + i\frac{(1-\gamma)m}{\lambda}\bar{N}(\tau\cdot\phi)\gamma_5 N \\ & + \frac{(1-\gamma)^2 m}{2\lambda^2}\bar{N}(\tau\cdot\phi)^2 N\end{aligned} \qquad (4.45)$$

The N-N scattering in the tree approximation is solely due to one pion exchange diagrams. The relevant coupling term in the above $\mathcal{L}(\gamma)$ is

$$\frac{i(1-\gamma)}{\lambda} m \bar{N}(\underline{\tau}\cdot\underline{\phi})\gamma_5 N - \frac{\gamma}{2\lambda}\bar{N}(\underline{\tau}\cdot\partial\underline{\phi})\gamma_\mu\gamma_5 N$$

But for calculating on-mass-shell N-N scattering in the pion exchange graph, the derivative coupling term $\bar{N}\gamma_\mu\gamma_5(\underline{\tau}\cdot\partial\underline{\phi})N$ is equivalent to $-2im\bar{N}\gamma_5(\underline{\tau}\cdot\underline{\phi})N$ thus making the second term of the above expression equivalent to

$$\frac{i\gamma}{\lambda}\bar{N}(\underline{\tau}\cdot\underline{\phi})\gamma_5 N$$

and the whole interaction is simply equivalent to

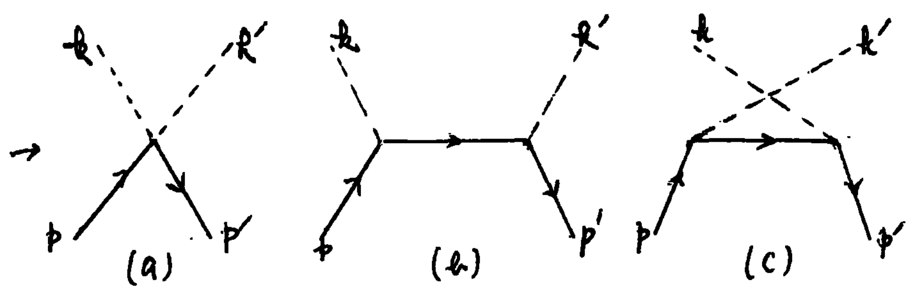
$$\frac{im}{\lambda}\bar{N}(\underline{\tau}\cdot\underline{\phi})\gamma_5 N$$

Thus the N-N amplitude in the tree approximation will not be affected by the transformation (4.44).

The nucleon-meson scattering is slightly more complicated. For the process

$$N_\alpha(p) + \pi_i(k) \to N_\beta(p') + \pi_j(k') \quad \left(\alpha, \beta, i, j \text{ refer to I-spin}\right)$$

I must calculate the diagrams



The contact diagram (a) includes both the second and the last term of (4.45). They each give the contribution to T-matrix element

$$\left.\begin{matrix} \dfrac{-i\gamma^2}{2\lambda^2} T^{l}_{\beta\alpha}\, \epsilon_{lji}\, \not{Q} \\ \dfrac{m(1-\gamma)^2}{\lambda^2}\, \delta_{ij}\delta_{\alpha\beta} \end{matrix}\right\} \qquad (4.46)$$

respectively. The exchange diagrams (b) and (c) give, on the other hand, the following contribution

$$\begin{aligned} &\delta_{ij}\delta_{\alpha\beta}\Big[\Big\{\frac{\gamma^2}{4\lambda^2}\Big(\frac{s+3m^2}{s-m^2}-\frac{u+3m^2}{u-m^2}\Big) \\ &\quad+\frac{m^2\gamma(1-\gamma)}{\lambda^2}\Big(\frac{-1}{s-m^2}+\frac{-1}{u-m^2}\Big) \\ &\quad+\frac{m^2(1-\gamma)^2}{\lambda^2}\Big(\frac{1}{s-m^2}+\frac{1}{u-m^2}\Big)\Big\}\not{Q} \\ &\quad+m\Big\{-\frac{\gamma^2}{\lambda^2}+\frac{\gamma(1-\gamma)}{\lambda^2}\Big\}\Big] \\ &+i\epsilon_{lji}T^{l}_{\alpha\beta}\Big[\frac{\gamma^2}{4\lambda^2}\Big(\frac{s+3m^2}{s-m^2}+\frac{u+3m^2}{u-m^2}\Big) \\ &\quad-\frac{m^2\gamma(1-\gamma)}{\lambda^2}\Big(\frac{1}{s-m^2}+\frac{1}{u-m^2}\Big) \\ &\quad+\frac{m^2(1-\gamma)^2}{\lambda^2}\Big(\frac{1}{s-m^2}+\frac{1}{u-m^2}\Big)\Big]\not{Q} \end{aligned} \qquad (4.47)$$

where $Q=(k+k')/2 \qquad s=(p+k)^2 \quad u=(p-k')^2$

The factors γ and $(1-\gamma)$ come respectively from the derivative and non derivative Yukawa couplings in (4.45). Putting (4.46) and (4.47) together, the amplitude is found to be

$$T_{N\pi\to N\pi} = \frac{1}{\lambda^2}\left\{3\gamma(\gamma-1)+1\right\}\Big[\delta_{ij}\delta_{\alpha\beta}\left\{-m^2+\not{\!\alpha}\left(\frac{m^2}{s-m^2}-\frac{m^2}{u-m^2}\right)\right\} + i\epsilon^{k}_{ji}T^{k}_{\alpha\beta}\,\not{\!\alpha}\left(\frac{m^2}{s-m^2}+\frac{m^2}{u-m^2}\right)\Big] \qquad (4.48)$$

The only change caused by the transformation of the nucleon field (4.44) is the overall constant factor $(3\gamma^2-3\gamma+1)$. On the other hand, the only way to identify the arbitrary constant λ is to compare the residue of the pole term of (4.48) with the known pion nucleon coupling constant, and this will fix the value of $\frac{1}{\lambda^2}(3\gamma^2-3\gamma+1)$ uniquely. Thus the phenomenological lagrangian $\mathcal{L}_\gamma$ (4.47) with the tree approximation gives a unique value for the meson-nucleon scattering amplitude too. In the case of meson-nucleon amplitude, we can consider an even more general transformation

$$N' = f\left(i\frac{\tau}{2}\cdot\xi\right)N$$

where $f(z) = 1+\gamma z + \frac{x\gamma^2 z^2}{2} + O(z^3)$ with arbitrary real x. This causes the additional interaction of a contact type like diagram (a) discussed above. These additional contributions do, however, cancel each other when the nucleons are on mass-shell.

§4 The weak and the electromagnetic interaction

(a) The field current identity in electro-magnetic interaction.

I would like to discuss the problem of introducing the electromagnetic interaction into the chiral invariant model like (4.21). In what follows, I naturally consider the chiral SU(3)xSU(3) scheme only.

My aim is to put in the additional electromagnetic interaction in such a way that both the ordinary gauge invariance and the field current identity may be satisfied. The latter scheme introduced by Kroll, Lee and Zumino(36,37,42,43) for the electromagnetic interaction has several attractive features and gives a theoretical basis for the assumption of vector meson dominance which is very successful in explaining various features of electromagnetic interaction of hadrons. The Maxwell's equation for such system is written as

$$\partial_\mu F^{\mu\nu} = \text{linear combination of vector meson fields} + \text{leptonic currents} \qquad (4.49)$$

Following the prescription by Kroll, Lee and Zumino for the iso-spin invariant system. I make the following replacement in the chiral SU(3) invariant lagrangian model discussed above

$$(v_\mu^i)_{i=1}^{8} \to (\hat{v}_\mu^i)_{i=1}^{8} \left\{ \begin{array}{l} \hat{v}_\mu^3 = v_\mu^3 + e/g\, A_\mu \\ \hat{v}_\mu^8 = v_\mu^8 + \sqrt{3}\, e/g\, A_\mu \\ \hat{v}_\mu^i = v_\mu^i \quad i \neq 3, 8 \end{array} \right. \tag{4.50}$$

But I leave the vector meson mass term intact. Because of this, I get immediately the right hand side of (4.49) as

$$J_\mu^{em.} \text{ (= hadronic electro magnetic current)}$$

$$= \frac{m^2}{g} \left(v_\mu^3 + \frac{1}{\sqrt{3}} v_\mu^8 \right) \tag{4.51}$$

This is the field current identity.

To see that this replacement (4.50) guarantees the gauge invariance, the following expressions for the covariant quantities discussed so far should be noted

$$\nabla_\mu \xi = \partial_\mu \xi \left(E_1(iF\xi) + E_1(-iF\xi) \right)/2.$$

$$\beta_\mu = \partial_\mu \xi \left(E_1(iF\xi) - E_1(-iF\xi) \right)/2$$

$$\chi_\mu^- = v_\mu \cdot iF\xi \left(E_1(iF\xi) + E_1(-iF\xi) \right)/2 + a_\mu \left(e^{iF\xi} + e^{-iF\xi} \right)/2,$$

$$\chi_\mu^+ = v_\mu \left\{ 1 + iF\xi \left(E_1(iF\xi) - E_1(-iF\xi) \right)/2 \right\} + a_\mu \left(e^{iF\xi} - e^{-iF\xi} \right)/2.$$

From these, I get

$$\mathcal{D}_\mu \xi = \nabla_\mu \xi - g\chi_\mu^- \\ = (\partial_\mu \xi - g v_\mu \cdot iF\xi)(E_1(iF\xi) + E_1(-iF\xi))/2 \\ - g a_\mu (e^{iF\xi} + e^{-iF\xi})/2 \quad (4.52)$$

also $$\beta_\mu - g\chi_\mu^+ = (\partial_\mu \xi - g v_\mu \cdot iF\xi)(E_1(iF\xi) - E_1(-iF\xi))/2 \\ - g v_\mu - g a_\mu (e^{iF\xi} - e^{-iF\xi})/2$$

and thus

$$\mathcal{D}_\mu \psi = \nabla_\mu \psi - ig\chi_\mu^+ \cdot \hat{t}\psi \\ = (\partial_\mu - ig\hat{t}\cdot v_\mu)\psi \\ + i\hat{t}(\partial_\mu \xi - g v_\mu \cdot iF\xi)(E_1(iF\xi) - E_1(-iF\xi))/2\,\psi \\ - ig\hat{t}\cdot a_\mu (e^{iF\xi} - e^{-iF\xi})/2\,\psi \quad (4.53)$$

Thus, the derivative $\partial_\mu \xi$ and $\partial_\mu \psi$ in the lagrangian model (4.21) always appears as the combinations $(\partial_\mu - ig v_\mu \cdot F)\xi$ and $(\partial_\mu - ig v_\mu \cdot t)\psi$ respectively. It should be remembered also that there is an extra term $\nabla_\mu \xi$ introduced into a_μ field. But in the case of Wess and Zumino[4] decomposition (4.24), $\nabla_\mu \xi$ term appears as only

$$\nabla_\mu \xi + g(e^{iF\xi} - e^{-iF\xi})/2 \cdot v_\mu \\ = (\partial_\mu \xi - g v_\mu \cdot iF\xi)(E_1(iF\xi) + E_1(-iF\xi))/2 \quad (4.54)$$

As the result of these particular combination of vector fields and ordinary derivatives of the fields, the gauge invariance of electromagnetic interaction generated by the replacement (4.50) is guaranteed. The replacement (4.50) is completely equivalent to the ordinary formalism of introducing the electro-magnetic interaction in which $\partial_\mu \psi$ and $\partial_\mu \xi$ should be replaced by $(\partial_\mu - ieA_\mu)\psi$ and $(\partial_\mu - ie'A_\mu)\xi$. The assignments of electromagnetic charge for the $(\xi_i)_{i=1}^{8}$ is, of course, that of pseudo scalor meson octet (π, K, η).

(4.54) justifies the Wess-Zumino way of introducing $\xi - \hat{a}_\mu$ mixing (4.24) from the point of view of simplicity.

(b) The modified divergence equation in the presence of "weak" perturbations.

The divergence equation (4.19) and (4.20) for v_μ and a_μ fields should be modified in the presence of the weak and electromagnetic interaction. In certain approach to the current,algebra, such modified divergence equations were given the important role. (Veltman, Nauenberg Refs. 19 and 20).

The modification of (4.19) and (4.20) (or as in Chapter 1, the PCAC form with meson mass term) with the electromagnetic interaction can be done by the method of Adler[(44)]. The following argument is more or less parallel to the Adler's discussion found in (Ref. 44).

First let us consider the vector field v_μ and the divergence equation (4.19). The electromagnetic interaction is introduced by the replacement (4.50). Starting from the lagrangian <u>without</u> the electromagnetic interaction

$$\mathcal{L}_0 = \mathcal{L}(v_\mu, \underline{\Psi}) + \frac{m^2}{2}\sum v_\mu^{\,i} v^{\mu i} \qquad (4.55)$$

where   represents all the fields other than   ,

I have for the lagrangian with e-m interaction

$$\begin{aligned}\mathcal{L}_{tot} &= \mathcal{L}_0 + \mathcal{L}_{em} \\ &= \mathcal{L}(\hat{v}_\mu, \underline{\Psi}) + \frac{m^2}{2}\sum v_\mu^{\,i} v^{\mu i}\end{aligned} \qquad (4.56)$$

Here, of course, v_μ, $\underline{\Psi}$ etc. in (4.56) are different from the ones in (4.55) since they obey the different equation of motion.

Consider the infinitesimal virtual displacement of the field variables

$$\left.\begin{aligned}\delta\underline{\Psi} &= i\hat{T}\cdot\beta\,\underline{\Psi} \\ \delta\hat{v}_\mu &= iF\cdot\beta\,\hat{v}_\mu + \frac{1}{g}\partial_\mu\beta \\ \delta A_\mu &= 0\end{aligned}\right\} \qquad (4.57)$$

where $\hat{T}$ is SU(3) generator matrix corresponding to the multiplet $\underline{\Psi}$. Then for this variation as long as it is compatible with the general constraints, Gell-Mann Levy type variational equations hold.

$$\partial_\mu \frac{\delta \mathcal{L}_{tot}}{\delta \partial_\mu \beta_i} = \frac{\delta \mathcal{L}_{tot}}{\delta \beta_i} \qquad (4.58)$$

Under the variation (4.57), the term $\mathcal{L}(\hat{V}_\mu, \underline{\Psi})$ is invariant and the mass term $\frac{m^2}{2} \sum V_\mu^i V^{i\mu}$ gives

$$\delta\left(\tfrac{1}{2} m^2 V_\mu^{\,2}\right)$$

$$= \delta\left(\tfrac{1}{2} m^2 (\hat{V}_\mu - \tfrac{e}{g} A_\mu^i)^2\right)$$

$$= \delta\left(\tfrac{1}{2} m^2 (\hat{V}_\mu^{\,2} - \tfrac{2e}{g} \hat{V}_\mu \cdot A^\mu + \tfrac{e^2}{g^2} A_\mu^{\,2})\right)$$

$$= - m^2 \frac{e}{g} \beta^i (i F^i_{jk} \hat{V}_\mu^k) A^{j\mu}$$

$$= - m^2 \frac{e}{g} \beta^i (i F^i_{jk}) V_\mu^k A^{j\mu}$$

where

$$A_\mu^i = 0 \qquad i \neq 3, 8$$
$$\quad\;\; A_\mu \qquad i = 3$$
$$\quad\;\; \sqrt{3}\, A_\mu \qquad i = 8$$

Thus

$$\frac{\delta \mathcal{L}_{tot}}{\delta \beta_i} = - \frac{m^2 e}{g}\, i F_{jk}^i\, V_\mu^k A^{j\mu}.$$

and

$$\frac{\delta \mathcal{L}_{tot}}{\delta \partial_\mu \beta_i} = - \frac{m^2}{g} V_\mu^\mu.$$

From (4.58), I get now

$$\partial_\mu v^{i\mu} = e(iF^i_{jk}\, v_\mu^k)\mathcal{A}^{j\mu} \qquad (4.59)$$

As for fields, consider the virtual displacement corresponding to an infinitesimal chiral transformation

$$\left.\begin{aligned} \delta a_\mu &= iF\cdot\alpha\, \hat{v}_\mu + \frac{1}{g}\partial_\mu \alpha \\ \delta \hat{v}_\mu &= iF\cdot\alpha\ a_\mu \\ \delta \mathcal{A}_\mu &= 0 \end{aligned}\right\} \qquad (4.60)$$

And instead of (4.56), I write

$$\mathcal{L}_{tot} = \mathcal{L}'(\hat{v}_\mu^i, \Psi) + \frac{m^2}{2}(v_\mu^2 + a_\mu^2) \qquad (4.61)$$

is again invariant under (4.60) and I get

$$\frac{\delta \mathcal{L}}{\delta \alpha_i} = -\frac{m^2 e}{g}\, iF^i_{jk}\, a_\mu^k \mathcal{A}^{j\mu}.$$

$$\frac{\delta \mathcal{L}}{\delta \partial_\mu \alpha_i} = -\frac{m^2}{g}\, a_\mu^{i\mu}$$

And from

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \alpha} = \frac{\delta \mathcal{L}}{\delta \alpha}.$$

I get

$$\partial_\mu a^{i\mu} = e\,(iF^i_{jk})\,a_\mu^k \mathcal{A}^{j\mu} \qquad (4.62)$$

(4.59) and (4.62) are required modified divergence equations. In the case of SU(2) with the only iso-vector part of e-m interaction considered, they reduces to the form considered by Veltman

$$\partial_\mu \underline{v}^\mu = e\, \underline{A}^\mu \wedge \underline{v}_\mu$$
$$\partial_\mu \underline{a}^\mu = e\, \underline{A}^\mu \wedge \underline{a}_\mu$$

Veltman[20] further considered the modification due to the perturbation of the weak interaction type. This divergence equations can be obtained by the replacements of the field variables of the following form

$$\left.\begin{aligned} a_\mu &\rightarrow a_\mu + \frac{G}{g} W_\mu^A \\ v_\mu &\rightarrow v_\mu + \frac{G}{g} W_\mu^V \end{aligned}\right\} \qquad (4.63)$$

where G is the weak interaction constant. W_μ^A and W_μ^V correspond to different parity parts of "weak boson" fields. Then for chiral SU(2), I can get the divergence equations used by Veltman[20].

$$\left.\begin{aligned} \partial_\mu \underline{a}^\mu &= G(\underline{W}_\mu^A \wedge \underline{v}^\mu + \underline{W}_\mu^V \wedge \underline{a}^\mu) \\ \partial_\mu \underline{v}^\mu &= G(\underline{W}_\mu^V \wedge \underline{v}^\mu + \underline{W}_\mu^A \wedge \underline{a}^\mu) \end{aligned}\right\} \qquad (4.64)$$

CHAPTER 5

The breaking of chiral SU(3)xSU(3) symmetry in the non-linear realization techniques and the application to the interaction of the hadrons.

§1 The introduction

In the previous three chapters the principles of the non-linear realization techniques for chiral SU(n)xSU(n) symmetry have been discussed. I would now like to present some applications of chiral SU(3)xSU(3) symmetry with the phenomenological lagrangian of the type discussed in Chapter 4.

There are already many examples of successful application of non-linear realization techniques for chiral SU(2)xSU(2) symmetry[(45)]. Because it is free of the laborious computations involved in the current algebra techniques, it has been found to be quite useful in understanding some aspects of elementary particle interaction even though in many cases it just reproduces current algebra results. Also, this technique found the appeal to some people because it emphasizes more strongly the symmetry or the group theoretical point of view for the "chiral dynamics".

The similar applications for the chiral SU(3)xSU(3) case are hindered by the fact that there is yet no definite prescription of how to take account of the symmetry breaking. Of course, this problem has its parallel in the current algebra approach. To estimate "σ-terms", for instance, one is always forced to make one or another of the plausible assumptions. In the phenomenological lagrangian method with chiral SU(3)xSU(3), the good agreements with experiments were achieved often by putting in the symmetry breaking "by hand", for instance by replacing some invariant mass term in the lagrangian by the physical masses(46). (The best example of an earlier application of chiral SU(3) symmetry with non-linear realization techniques is found in the paper by Cronin(2), where many of the ideas which have more conveniently formulated later are already present).

Recently, however, a theory of broken chiral SU(3) symmetry has been proposed by Gell-Mann, Oakes and Renner(47). Although many of their ideas can be found in the works of previous authors(48), they presented their method in such a way that it formulates the prescription to the given problems with seemingly much less ambiguity. In particular, the definite ratio of the strength of SU(3) singlet and octet component of symmetry breaking part of strong interactions is suggested as a kind of universal constant.

The original authors treat the problem in terms of currents and their commutation relations. But it is straightforward to construct a parallel theory in terms of a chiral lagrangian with non linear realization. In fact the simplicity of the Gell-Mann, Oakes and Renner scheme becomes most apparent in the latter approach. A very thorough study of the general structure of such a theory has been given by Macfarlane, Sundbery and Weisz. But their emphasis on the most general definition of the fields within the non-linear realization techniques seems to give their work a forbiddingly complicated appearance without reaching the essential simplicity expected from the group theory.

§2 The breaking of chiral symmetry in the non-linear realization.

It is well known within the framework of ordinary unitary symmetry that one can represent the symmetry breaking part of the interaction as the combination of simple representations of the SU(3) group. And the simplest and the most popular one is to consider the strong interaction hamiltonian as the combination of an SU(3) singlet (i.e. symmetry preserving) and the octet representation. Gell-Mann, Oakes and Renner generalize this idea to chiral SU(3) symmetry. They assume that the symmetry breaking can be considered as the simple (linear) representation of the chiral SU(3) group, K_3. They choose a single $(3.\bar{3})\oplus(\bar{3}.3)$ representation as it is the simplest one which satisfies various physical requirements. They express this idea in term of hamiltonian density responsible for the symmetry breaking.

$$\mathcal{H} = -u_0 - c u_8 \qquad (5.1)$$

where u_0 and u_8 are the SU(3) singlet and octet contained in $(3.\bar{3})\oplus(\bar{3}.3)$ representation. Of course, the chirality implies the different parities and u_0, u_8 should be considered as the scalors since the symmetry breaking interaction $\mathcal{H}$ still conserves the parity. The interaction of the form (5.1) has been used previously in the chiral lagrangian method(49). But Gell-Mann, Oakes and Renner further propose to put the constant c uniquely determined number independent of the particular physical process under the consideration. They determine the value of the number c from the consideration of simple matrix elements of the currents with the PCAC assumption, derived from the transformation of $(3.\bar{3})+(\bar{3}.3)$ representation under the chiral group K_3, and propose to use the same value for treating more complicated processes.

In expressing the idea of Gell-Mann, Oakes and Renner in terms of the chiral lagrangian scheme discussed in the previous chapters, the first step is, of course, to construct the quantities like u_0 or u_φ in term of non-linearly transforming quantities like ξ'_μ or ψ'_ρ. But the construction of (linear) representation of chiral group within the non-linear realization scheme has been fully discussed in Chapter 2. This is the problem solved by Coleman, Wess and Zumino(8).

Following the notation of Chapter 2, the representation D of the chiral group can be constructed out of non linear "fields" ψ_α as

$$U_\pi = D_{\pi\alpha}(e^{\xi A})\psi_\alpha \qquad (5.2)$$

provided that the representation $\mathcal{D}$ of the diagonal subgroup H (=SU(3) in the case considered here) spanned by ψ'_α is contained in D when restricted to H.

It should be noted that Gell-Mann, Oakes and Renner start from the hamiltonian formalism, and the non-linear realization scheme discussed so far is is conveniently expressed only in terms of the lagrangian. But they also demand that the symmetry breaking hamiltonian should be a Lorentz scalor. This requirement is satisfied by the interactions which do not involve the derivative of field variables. For such an interaction, the relation between the lagrangian and the hamiltonian is trivial.

For the sake of illustration, I shall give first the construction of the form like (5.1) using the non-linear fields $(\xi_a^i)_{a=1}^{8}$ only. The construction of the particular representation $(3,\bar{3})+(\bar{3},3)$ from ξ^i_a only is possible, according to the theorem of Coleman, Wess and Zumino discussed in Chapter 2, because this representation contains the scalor representation (singlet) when restricted to SU(3) diagonal subgroup. Choosing a suitable co-ordinate system, the eighteen components of $(3.\bar{3})+(\bar{3}.3)$ representation can be given, by (5.2), as

$$\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_8 \\ v_0 \\ v_1 \\ \vdots \\ v_8 \end{pmatrix} = e^{i a_r \cdot \xi} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (5.3)$$

[illegible] ψ [illegible] tiel [illegible] tion [illegible]trix

$$D(e^{\xi A}) = e^{iQ_r\cdot\xi}$$

[illegible]

$$Q_r\cdot\xi = i\begin{pmatrix} 0 & D'\xi \\ -D'\xi & 0 \end{pmatrix}$$

[illegible]

$$D'\xi = \left(\begin{array}{c|c} 0 & -\sqrt{2/3}\,\xi \\ \hline \sqrt{2/3}\,\xi & D\cdot\xi \end{array}\right) \qquad (5.4)$$

$(Q_r^{\alpha})^{\gamma}_{\alpha\beta}$ [illegible] tor [illegible] of [illegible]

[illegible] by

$$(D^i)_{jk} = d_{ijk}, \qquad i, j, k = 1, 2 \cdots 8$$

[illegible] get

$$u_\alpha(\xi) = \{\cos(D'\xi)\}_{\alpha 0} \qquad [illegible]$$

$$v_\alpha(\xi) = \{\sin(D'\xi)\}_{\alpha 0} \qquad [illegible]$$

where α [illegible] to [illegible].

[illegible]

$$\mathrm{Exp}\begin{pmatrix} 0 & -D'\xi \\ D'\xi & 0 \end{pmatrix}\begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \\ b \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \{a \cos D'\xi - b \sin D'\xi\}_{\alpha 0} \\ \cdots \\ \{a \sin D'\xi + b \cos D'\xi\}_{\alpha 0} \end{pmatrix}$$

On the other hand, ξ fields are going to be treated as pseudo scalor with respect to space reflection. Therefore if, for instance, the octet and singlet components $\{U_a\}$ are required to be scalor so that it can be used to construct the hamiltonian of the type (5.1), then the above generalization does not give essentially new construction over (5.5) and (5.6).

§3 The pseudo scalor meson lagrangian.

(a)

The chiral symmetric lagrangian of the pseudo scalor meson octet is given by

$$\mathcal{L} = \frac{a}{2}\left(\nabla_\Gamma \xi\right)^2 \tag{5.7}$$

This represents the mass-less particles interacting with each other. If, in addition to (5.7), I may take the symmetry breaking interaction of type (5.1) there is the possibility of giving them the finite masses as well as their physical mass splitting within

within the octet. Taking the iso-spin hypercharge conserving members of the scalor part of the multiplet (5.3), I will write Gell-Mann, Oakes, Renner type symmetry breaking interaction as

$$\mathcal{H} = -\mu_0^2 \lambda^2 (u_0 + c\, u_8) \qquad (5.8)$$

where u_i' are given by (5.5), μ_0 is an arbitrary constant and λ is to cancel the normalization constant of the physical P-S octet fields. I start by putting (as in Chapter 4)

$$\phi_i \text{ (physical P-S octet)} = \lambda_i \xi_i$$

Since the interaction (5.8) does not contain the derivatives, the kinematical term of the meson lagrangian is contained in the chiral invariant part (5.7). As given in Chapter 2,

$$\nabla_\mu \xi = \partial_\mu \xi \, \frac{\sin iF\xi}{iF\xi}$$
$$\simeq \partial_\mu \xi + O(\xi^3)$$

and

$$\frac{a}{2} (\nabla_\mu \xi)^2 \simeq \frac{a}{2} (\partial_\mu \xi)^2 = \frac{1}{2} \sum_{i=1}^{8} \frac{a}{\lambda_i^2} (\partial_\mu \phi_i)^2$$

Thus I get from the condition of correct kinematical term

$$\lambda_i^2 = a = \text{independent of } i \qquad (5.9)$$

This in fact implies that within our simple model of the symmetry breaking, we cannot account for the difference between the leptonic decay constants of P-S mesons within the octet.

For the purpose of the present discussion, it is sufficient to consider the first few powers of in $U_\alpha(\xi)$ and $V_\alpha(\xi)$ given by (5.5) and (5.6). Expanding (5.5) and (5.6), I get

$$U_\alpha(\xi) = \left(1 - (D'\xi)^2/2 + (D'\xi)^4/24\right)_{\alpha 0} + O(\xi^6)$$

$$V_\alpha(\xi) = \left(D\xi - (D'\xi)^3/6\right)_{\alpha 0} + O(\xi^5)$$

Which can be explicitly written in terms of

$$U_0 = 1 - \xi^2/3 + (\xi^2)^2/36 + O(\xi^6) \qquad (5.10)$$

$$U_i = \frac{1}{\sqrt{6}}\left(-1 + \xi^2/12\right) d_{ijk}\, \xi_j \xi_k + \frac{1}{18\sqrt{6}} d_{lmn}\, \xi_l \xi_m \xi_n \xi_i + O(\xi^6) \qquad i = 1, \cdots 8 \qquad (5.11)$$

$$V_0 = -\frac{1}{9} d_{ijk}\, \xi_i \xi_j \xi_k + O(\xi^5) \qquad (5.12)$$

$$V_i = \sqrt{\frac{2}{3}}\left(1 - \frac{\xi^2}{6}\right) \xi_i + O(\xi^5) \qquad i = 1, \cdots 8 \qquad (5.13)$$

Putting (5.10) and (5.11) into (5.8) and taking it only upto the quadratic term in , I get

$$\mathcal{H} \simeq \mu_0 \lambda^2 \left(\frac{\xi^2}{3} + \frac{c}{\sqrt{6}}\, d_{8ij}\, \xi_i \xi_j \right)$$

Choosing λ^2 to be equal to $\lambda_i^2 = a$, and introducing the physical P-S meson octet with the usual assignment for charge etc. this can be written as

$$\frac{\mu_0^2}{2}\Big\{\frac{\sqrt{2}}{3}(\sqrt{2}+c)(2\pi^+\pi^- + \pi_0^2) + \frac{\sqrt{2}}{3}(\sqrt{2}-\frac{c}{2})(2K^+K^- + 2\bar{K}^0K^0) + \frac{\sqrt{2}}{3}(\sqrt{2}-c)\,\eta^2\Big\} \qquad (5.14)$$

This is just the mass term we want. I can identify the masses of P-S meson octet as

$$\begin{aligned} \mu_\pi^2 &= \mu_0^2\,\frac{\sqrt{2}}{3}(\sqrt{2}+c) \\ \mu_K^2 &= \mu_0^2\,\frac{\sqrt{2}}{3}(\sqrt{2}-\frac{c}{2}) \\ \mu_\eta^2 &= \mu_0^2\,\frac{\sqrt{2}}{3}(\sqrt{2}-c) \end{aligned} \qquad (5.15)$$

If I fit the experimental value for $\bar{\mu}_\pi^2/\bar{\mu}_K^2$ (averaged over isospin multiplet), I get

$$c = -0.889\times\sqrt{2} = -1.26 \qquad (5.16)$$

and $\mu_0^2 = 0.96\,\bar{\mu}_K^2$

This gives $\mu_\eta^2 = 30.26\text{x}\ 10^6(\text{Mev})^2$ while experimentally $\mu_\eta^2 = 30.11\text{x}10^6(\text{Mev})^2$.

I should remark at this point that here and throughout the following, I entirely neglect the problem of mixing.

These results with (5.9) are the ones obtained in the paper by Gell-Mann, Oakes and Renner. (5.15) satisfies the Gell-Mann, Okubo mass formula for squared masses

$$\bar{\mu}_K^{\,2} = \frac{1}{4}\left(\bar{\mu}_\pi^{\,2} + 3\mu_\eta^{\,2}\right) \qquad (5.17)$$

Next I consider the problem of PCAC. When the chiral symmetry breaking part of interaction is given by (5.8), the axial currents given by the variational principle from the given lagrangian, as in Chapter 4,

$$A_\mu^{\,i} = -\frac{\delta \mathcal{L}}{\delta \partial^\mu \alpha_i}$$

satisfy the divergence conditions

$$\partial^\mu A_\mu^{\,i} = \frac{\delta H'}{\delta \alpha_i} = -[Q_i^5, \mathcal{H}']$$
$$= \mu_0^2\lambda^2\left(-\sqrt{\tfrac{2}{3}}\,\sigma_i - c\,d_{8ij}\,\sigma_j\right)$$
$$= \mu_0^2\lambda^2\left(\tfrac{2}{3}\,\xi_i + \sqrt{\tfrac{2}{3}}\,c\,d_{8ij}\,\xi_j\right)\left(1-\frac{\xi^2}{6}\right)$$

From (5.15), the right hand sides can be written in terms of the physical meson fields

$$\mu_0^2\lambda^2\left(\tfrac{2}{3}\,\xi_i + \sqrt{\tfrac{2}{3}}\,c\,d_{8ij}\,\xi_j\right)$$
$$= \begin{cases} \mu_\pi^2\lambda\,\phi_i & i = 1, 2, 3 \\ \mu_K^2\lambda\,\phi_i & i = 4, 5, 6, 7 \\ \mu_\eta^2\lambda\,\phi_i & i = 8 \end{cases}$$

Thus I have

$$\partial_\mu A^{i\mu} = \lambda\,\mu_i^2\,\phi_i\left(1-\frac{\Phi^2}{6\lambda^2}\right) \qquad (5.18)$$

Upto the linear term in P-S meson fields, (5.18) is just the expression of PCAC, and coefficients gives the "residue of one meson singularity".

Thus $\lambda - F_i$ gives the (uniform) leptonic decay constants of pseudo-scalor meson octet. We have within our approximation

$$F_\pi = F_K = F_\eta \qquad (5.19)$$

But (5.18) also has cubic correction term, this implies that PCAC cannot be assumed in calculating thee meson-meson scattering amplitude. Thus if the off-mass shell meson-meson amplitude is calculated using ϕ_i's as the physical meson fields it will not satisfy the Adler consistency condition.

Unlike the mass relation, the leptonic decay constants F_K and F_π (F_η cannot be measured because of fast radiative decay) are not too different and (5.19) $F_\pi \sim F_K$ can be considered as reasonable. Nevertheless, the experimental value $F_K/F_\pi \sim 1.26$ should be somehow accounted for. I may, for instance, incorporate into the lagrangian terms containing the derivatives of ξ fields like

$$\mathcal{L}'_{deriv} = -\frac{\lambda'^2}{2}(u_0' + c\,u_8') \qquad (5.20)$$

with $(3,\bar{3}) + (\bar{3},3)$ multiplet now constructed out of covariant derivatives

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = e^{i a_r \cdot \xi} \begin{pmatrix} 0 \\ d_{ijk} \nabla_r \xi_j \nabla^r \xi_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

It should be noted that an interaction like (5.20) cannot be considered as being within the scheme of Gell-Mann, Oakes, Renner. The Hamiltonian corresponding to (5.20) is not a Lorentz scalar.

Taking (5.20) together with the invariant term

$$\mathcal{L}_0 = \frac{\lambda^2}{2} (\nabla_r \xi)^2$$

I get the following modified relation for the leptonic decay constant.

$$\left.\begin{aligned} \lambda^2 - c\,\lambda'^2/\sqrt{3} &= F_\pi^2 \\ \lambda^2 + c\,\lambda'^2/2\sqrt{3} &= F_K^2 \\ \lambda^2 + c\,\lambda'^2/\sqrt{3} &= F_\eta^2 \end{aligned}\right\} \qquad (5.21)$$

The mass relations (5.15) should be also modified and these can be solved again by fitting the value of μ_K/μ_π as well as F_K/F_π. This modifies the value of c and brings it even closer to $-\sqrt{2}$.

$$c \sim -1.37 \qquad (5.22)$$

μ_η and F_η are calculated to give

$$\mu_\eta \sim 540 \text{ MeV}$$

$$F_\eta/F_\pi \sim 1.34$$

On the other hand, the ratio λ'^2/λ^2 is turned out to be rather large

$$c\lambda'^2/\lambda^2 \sim 0.49 \qquad (5.23)$$

The way of accounting for F_K/F_π described above is probably unsatisfactory, and it will be found later that for other calculations like $K\pi$ scattering length it is important to use the physical F_K value to get the reasonable agreement. Thus this simple minded scheme cannot be considered as satisfactory unless the F_K/F_π ratio is correctly described. It has been suggested that this and related problems can be treated in a satisfactory way when at least the vector and axial vector gauge fields are incorporated[50]. The trouble seems to be that it is not straightforward to construct Gell-Mann - Oakes - Renner type interaction to describe, for instance, bector meson mass splitting or the mixing between octet and singlet mesons.

(b) Meson-meson scattering.

Let us return to the non-derivative interaction (5.8) or the corresponding term in the lagrangian

$$\mathcal{L}' = \mu_0^2 \lambda^2 (u_0 + c\, u_8)$$

with (5.5) for u_i.

Interesting point about the non-linear realization scheme is that the introduction of mass term like (5.14) does automatically generate higher order terms in ξ fields required from symmetry. These terms in general contribute to the other physical processes. Thus the fourth order term of ξ fields ((5.10) and (5.11)) give rise to the 4-point contact interaction among mesons which modifies P-S meson scattering amplitudes. Computing $\mathcal{L}'$ above upto fourth order using (5.10) and (5.11), I get

$$\begin{aligned}\mathcal{L}' = & -\frac{1}{2}\left(\mu_\pi^2 \underline{\Pi}^2 + 2\mu_K^2 \bar{K}\cdot K + \mu_\eta^2 \eta^2\right) \\ & + \frac{1}{24\lambda^2}\left(\underline{\Pi}^2 + 2\bar{K}\cdot K + \eta^2\right)\left(\mu_\pi^2 \underline{\Pi}^2 + 2\mu_K^2 \bar{K}\cdot K + \mu_\eta^2 \eta^2\right) \\ & + \frac{\mu_0^2}{18\sqrt{6}\,\lambda^2}\, d_{ijk}\, \phi_i \phi_j \phi_k\, \eta\end{aligned} \qquad (5.24)$$

Term contributing to the scattering of $\pi\pi$, πK and KK [redacted] is

$$\mathcal{L}'^{\,scattering}_{K,\pi} = \frac{1}{24\lambda^2}\left(\underline{\Pi}^2 + 2\bar{K}\cdot K\right)\left(\mu_\pi^2 \underline{\Pi}^2 + 2\mu_K^2 \bar{K}\cdot K\right) \qquad (5.55)$$

The contribution to the scattering amplitudes from the chiral invariant term (5.7) has also been computed by Isham and Patani[51].

I get

$$\mathcal{L}^{0\ \text{scattering}}_{K,\pi} = \frac{1}{3\lambda^2}\Big[\tfrac{1}{2}\big\{(\partial_\mu \underline{\pi}\cdot\underline{\pi})^2 - (\partial_\mu \underline{\pi})^2\underline{\pi}^2\big\}$$

$$+\big\{\tfrac{1}{2}(\underline{\pi}\cdot\partial_\mu\underline{\pi})(\partial^\mu\bar{K}\cdot K + \partial^\mu K\cdot\bar{K})$$

$$+\tfrac{3}{2}i(\underline{\pi}\wedge\partial_\mu\underline{\pi})(\partial^\mu\bar{K}^T\underline{\tau}K - \bar{K}^T\underline{\tau}\partial^\mu K)$$

$$-\tfrac{1}{2}\underline{\pi}^2\partial_\mu\bar{K}\partial^\mu K - \tfrac{1}{2}(\partial_\mu\underline{\pi})^2\ \bar{K}\cdot K\big\} \qquad (5.26)$$

$$+\big\{(\partial_\mu\bar{K}\cdot K)^2 + (\partial_\mu K\cdot\bar{K})^2 - (\bar{K}\partial_\mu K)(K\partial^\mu\bar{K})$$

$$-(\bar{K}\cdot K)(\partial_\mu\bar{K}\partial^\mu K)\big\}\Big]$$

Then I get the contribution from (5.25) and (5.26) for each scattering process as follows

(A)

$$\mathcal{L}^{\text{scattering}}_{\pi\pi} = \frac{1}{6\lambda^2}\Big((\partial_\mu\underline{\pi}\cdot\underline{\pi})^2 - (\partial_\mu\underline{\pi})^2(\underline{\pi})^2 + \tfrac{1}{4}\mu_\pi^2\,\pi^4\Big) \qquad (5.27)$$

Computing the scattering amplitude for the process

$$\pi_a(k_a) + \pi_c(k_c) \longrightarrow \pi_b(k_b) + \pi_d(k_d)$$

we get

$$T^{\pi\pi} = \frac{1}{\lambda^2}\Big[(s-\mu_\pi^2)\delta_{ac}\delta_{bd} + (u-\mu_\pi^2)\delta_{ad}\delta_{bc}$$

$$+(t-\mu_\pi^2)\delta_{ab}\delta_{cd}\Big] - \frac{1}{3\lambda^2}\big(\Sigma k^2 - 4\mu_\pi^2\big)(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} + \delta_{ab}\delta_{cd}) \qquad (5.28)$$

where

$$s = (k_a + l_c)^2, \quad u = (k_a - k_d)^2 \quad t = (l_a - k_d)^2$$

(B) $K\pi$

$$\mathcal{L}^{\text{scattering}}_{K\pi} = \frac{1}{12\lambda^2}\Big\{ (\underline{\pi}\cdot\partial_\mu\underline{\pi})(\partial^\mu\bar{K}\cdot K + \partial^\mu K\cdot\bar{K})$$

$$- \pi^2\partial_\mu\bar{K}\partial^\mu K - (\partial_\mu\underline{\pi})^2\,\bar{K}\cdot K \tag{5.29}$$

$$+ 3i(\underline{\pi}\wedge\partial_\mu\underline{\pi})(\partial^\mu\bar{K}^T\underline{\sigma}K - \bar{K}^T\underline{\sigma}\partial^\mu K)$$

$$+ (\mu_\pi^2 + \mu_K^2)\,\underline{\pi}^2\,\bar{K}\cdot K \Big\}$$

The scattering amplitude for the process

$$\pi_a(k_a) + K_\alpha(k_\alpha) \longrightarrow \pi_b(k_b) + K_\beta(k_\beta)$$

is

$$T^{\pi,K} = \frac{1}{12\lambda^2}\Big\{ (3t - (\Sigma k_i^2 - 2\mu_\pi^2 - 2\mu_K^2)\delta_{ab}\delta_{\alpha\beta} \tag{5.30}$$

$$+ 3i(s-u)\,\epsilon_{bac}(\sigma_c)_{\beta\alpha} \Big\}$$

where $s = (k_a + k_\alpha)^2, \quad u = (k_a - k_\beta)^2 \quad t = (k_a - k_b)^2.$

(C) K, K

$$\mathcal{L}^{\text{scattering}}_{KK} = \frac{1}{6\lambda^2}\Big\{ (\partial_\mu\bar{K}\cdot K)^2 + (\partial_\mu K\cdot\bar{K})^2 - (\bar{K}\partial_\mu K)(K\partial^\mu\bar{K})$$

$$- (\bar{K}\cdot K)(\partial_\mu\bar{K}\partial^\mu K) + \mu_K^2\,\bar{K}\cdot K \Big\} \tag{5.31}$$

and for the process

$$K_\alpha(k_\alpha) + K_\gamma(k_\gamma) \longrightarrow K_\beta(k_\beta) + K_\delta(k_\delta)$$

I get

$$T^{KK} = \frac{1}{6\lambda^2}(-3s+\Sigma k^2+2\mu_K^2)(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}) \quad (5.32)$$

where $s=(k_\alpha+k_\gamma)^2$

It should be noted that the amplitudes (5.28), (5.30) and (5.32) do <u>not</u> satisfy the Adler consistency condition, in accordance with the discussion on the modified PCAC equation (5.18).

To recover such off-mass-shell condition, I can redefine the physical meson fields so that the PCAC relation holds upto higher order of meson fields. For instance I can put, from (5.18), as

$$\phi_i' \text{ (physical meson fields)} \equiv \phi_i(1-\phi^2/6\lambda^2)$$

Then (5.18) becomes

$$\partial^\mu A_\mu^i = \lambda\mu_i^2\phi_i' + O(\phi_i'^5)$$

The effect of such transformation (i.e. to use ϕ_i' instead of ϕ_i as second quantized meson fields in our "tree approximation" calculation) is to add to the T matrix elements the correction terms proportional to

$$\Sigma k^2 - \Sigma\mu^2$$

In this way, I modify the off-mass-shell value of the scattering amplitude so that it now satisfies the Adler condition

$$T_{\pi\pi} = \frac{1}{\lambda^2}\left[(s-\mu_\pi^2)\delta_{ac}\delta_{bd} + (u-\mu_\pi^2)\delta_{ad}\delta_{bc} + (t-\mu_\pi^2)\delta_{ab}\delta_{cd}\right] \qquad (5.33)$$

$$T_{\pi K} = \frac{1}{4\lambda^2}\left[(t+\Sigma k_i^2-2\mu_\pi^2-2\mu_K^2)\delta_{ac}\delta_{\alpha\beta} + i(s-u)\epsilon_{bac}(\sigma_c)_{\beta\alpha}\right] \qquad (5.34)$$

$$T_{KK} = \frac{1}{2\lambda^2}(-s+\Sigma k_i^2-2\mu_K^2) \qquad (5.35)$$

We may even have "exact PCAC" instead of (5.18) if it is put directly

$$\phi_i' = \lambda\sqrt{\frac{3}{2}}\,U_i \qquad i=1,\cdots,8$$

where $\{U_i\}_{i=1}^{8}$ is defined in (5.6)

$$\partial^\mu A_\mu^{\,i} = \lambda\mu_i^2\phi_i' \qquad (5.36)$$

(5.36) is the parallel of chiral SU(2) divergence equation (1.80).

At this point, I can compare the results of the present section with those of Chapter 1 (§5b). If I take from the expression of $U'_{\alpha\beta}$ in (5.5) the terms containing "pion fields" ξ_1, ξ_2, ξ_3 only, these can be explicitly summed up and I get

$$\mathcal{H}_{\pi}' = (-u_0 - c\,u_8)_{\pi}$$

$$= -\mu^2 \frac{2}{3}(\sqrt{2}+c)\cos\sqrt{\xi^2} + \text{constant term}$$

That is

$$\mathcal{L}_{\mu\pi}' = \mu_\pi^2 \lambda^2 \cos\sqrt{\xi^2} \qquad (5.37)$$

But this is just the expression (1.76) on chiral SU(2) and shows that the symmetry breaking term $\mathcal{L}_{\mu\pi}'$ is essentially the 4th components of 4-vector representation $(\frac{1}{2},\frac{1}{2})$ of chiral SU(2)xSU(2) group. The representation $(3,\overset{*}{3}) + (\overset{*}{3},3)$ of the chiral SU(3)xSU(3) contains the representation $(\frac{1}{2},\frac{1}{2})$ when restricted to the chiral SU(2)xSU(2), and the results of this section can be considered as the generalization of Weinberg's scheme explained in Chapter 1. §5

§5

The analysis of the meson-lagrangian presented in this section is essentially not new. It can be found in the paper by Cronin[2] where the identical form of the symmetry breaking term is used. Moreover, so far as mesons are concerned his point of view is more general. The results of this section (5.27)~(5.32) coincide with Cronin's if, using his notation, $a_3 = 4/3$ in his formulae. This is as it should be since $a_3=4/3$ in the expansion of mason matrix in Cronin's paper corresponds to the exponential meson matrix used in the present thesis. (Cronin considers the wider form of meson matrix instead of redefining the meson fields like I have done. For instance, PCAC results with Adler condition can be obtained in Cronin's formalism by putting the coefficients of 3rd power of meson fields in the meson matrix a_3 to zero. These two approaches should be equivalent as has been shown by Weinberg(7) for chiral SU(2) and studied by Macfarlane and Weisz(30) for general case).

(c) <u>The vector gauge fields</u>

I have derived the expression of meson-meson scattering amplitudes from non-linear lagrangian of the form

$$\mathcal{L} = \frac{\lambda^2}{2}(\nabla_\mu \xi)^2 + \text{mass term} \qquad (5.38)$$

From these amplitudes, the scattering length of mesons can be derived and the results agree with the current algebra(52). On the other hand, it is well known that the low energy meson-meson interaction can be accounted for rather well by the vector dominance model. Weinberg in (Ref. 7) shows that his chiral SU(2) invariant lagrangian with vector mesons gives the same result for low energy π-π scattering as the non linear lagrangian of form (5.38), except the difference of the order of μ_π^2/m_ρ^2.

This result can be formally extended to the case of chiral SU(3). From the point of view of physics, this may not be so useful since, for instance, $\mu_K^2/m_\rho^2 \sim 0.5$ is not too small.

For the sake of the simplicity, let us consider the SU(3) version of Weinberg's model with non linear type vector mesons (Chap. 4), rather than Wess and Zumino or Kawarabayashi model. Let us also disregard the symmetry breaking with respect to vector mesons. As it has been described in Chapter 4, the lagrangian in question is of the following form

$$\mathcal{L} = \frac{\lambda^2}{2}(\nabla_\mu \xi)^2 + \text{mass term} + \frac{m^2}{2}(\chi_\mu^+)^2 - \frac{1}{4}G_{\mu\nu}^2 \qquad (5.39)$$

where
$$\chi_\mu^+ \equiv v_\mu + \frac{1}{g}\beta_\mu$$
$$G_{\mu\nu} \equiv \partial_\mu v_\nu - \partial_\nu v_\mu + g\, v_\mu (iF) v_\nu$$

and β_μ is, as usual, equal to

$$\partial_\mu \xi \; \frac{E_1(iF\xi) - E_1(-iF\xi)}{2}$$

Here of course F's are ordinary F matrices of SU(3) and defined by

$$(F^i)_{jk} = -if_{ijk} \qquad i, j, k = 1, \cdots, 8$$

The mass terms in (5.38) and (5.39) are the same and the parts of the scattering amplitude coming from them are independent of the presence or the absence of the vector fields.

So far as the meson-meson scattering is concerned (5.39) differs from (5.38) by the presence of the term

$$\frac{m^2}{2g^2}\beta_\Gamma^{\;2} + \frac{m^2}{g} V_\Gamma \cdot \beta^\Gamma \tag{5.40}$$

The first term gives the contribution to the amplitude because of the 4-pt contact term

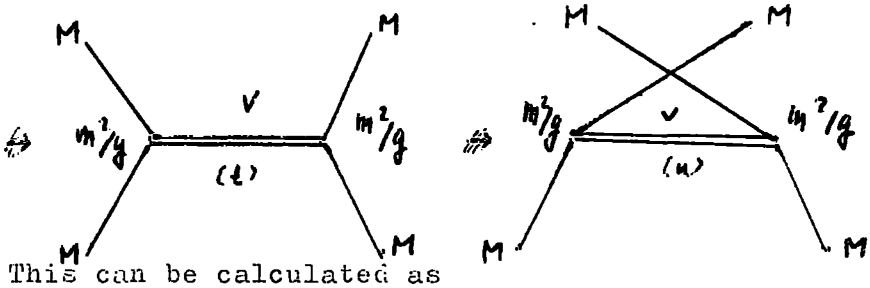
$$i\,\frac{m^2}{2g^2} \langle f | \int d^4x \, (\beta_\Gamma(x))^2 | i \rangle \tag{5.41}$$

where $|i\rangle$ and $|f\rangle$ represent the initial and the final two meson states.

On the other hand, the second term gives rise to VMM type interaction

$$\frac{m^2}{2g} f_{ijk} V_\Gamma^{\,i} \xi^j \partial^\Gamma \xi^k \tag{5.42}$$

The contribution to the scattering amplitude comes from the vector meson exchange diagrams



This can be calculated as

$$\left(-\frac{1}{2}\right)\frac{m^4}{g^2}\langle f|\int d^4x\int d^4y\, T(\beta_\mu(x)\beta^\mu(x),\beta_\nu(y)U^\nu(y))|i\rangle$$

$$= -\frac{1}{2}\frac{m^4}{g^2}\int d^4x\int d^4y\,\langle f|:\beta^\mu(x)\,\beta^\nu(y):|i\rangle$$
$$\times(-i)\int d^4K\, e^{-iK(x-y)}\delta_{ij}\left(g_{\mu\nu}-\frac{K_\mu K_\nu}{m^2}\right)\Big/(K^2-m^2)$$

On the other hand, from the consideration of corresponding diagrams K is seen to represent the momentum of the exchanged vector meson and the main contribution to the integral over K comes from K $\simeq$ P-S meson momenta. In the low energy region where $p_i^2 \simeq \mu_i^2$ for the external momenta. I may conclude $K_\mu K_\nu \sim \mu_i^2$ thus if $\mu_i^2 \ll m^2$, $K_\mu K_\nu/m^2$ in the vector meson propagator can be neglected compared with $g_{\mu\nu}$ term and the above expression reduces as

$$-\frac{1}{2}\frac{m^4}{g^2}\int d^4x\int d^4y\,\langle f|:\beta^\mu(x)\,\beta^\nu(y):|i\rangle$$
$$\times(-i)\frac{1}{-m^2}\,\delta_{ij}\,\delta^4(x-y)$$
$$= (-i)\frac{m^2}{2g^2}\int d^4x\,\langle f|:\beta_\mu^2(x):|i\rangle$$

which does cancel (5.41). Thus we will be left with the contribution coming from original $\frac{\lambda^2}{2}(\nabla_\mu \xi)^2$ term only. This result is independent of Kawarabayashi-Suzuki relation, and the vector meson exchange term disappears rather than dominates low energy scattering. On the other hand if the K-S relation is assumed, then (5.42) takes the usual form in the vector dominance model

$$g f_{ijk} \, v_\mu^i \phi^j \partial^\mu \phi^k$$

where ϕ's are the physical P-S meson fields.

In addition ρ_μ^2 and $(\nabla_\mu \xi)^2$ terms give the 4-pt contact term

$$-\frac{1}{24} \frac{m^2}{g^2} \partial_\mu \xi (iF\xi)^2 \partial^\mu \xi$$

This is still half as large as original contribution from $(\nabla_\mu \xi)^2$ term

$$+\frac{1}{12} \frac{m^2}{g^2} \partial_\mu \xi (iF\xi)^2 \partial^\mu \xi$$

§4 <u>Meson-Baryon interaction</u>

(a)

In this chapter, I would like to consider the interaction of P-S meson octet with known baryon octet. The chief interest here is again that the symmetry breaking term in lagrangian which primarilly accounts for the baryon mass splitting within its octet gives rise to the modification to law energy meson-baryon scattering amplitude.

Following Gell-Mann, Oakes and Renner I am going to use the interaction of the type (5.1) with the same value of c estimated in the preceding section by fitting pseudo-scalor meson masses.

This time $U'_{\alpha\Lambda}$ in (5.1) will be constructed, according to (5.2), with "linear fields" φ_α taken as some bilinear combination of physical octet baryon fields so that the interaction (5.1) gives rise first of all to the baryon mass terms. Since the baryon fields $(B_i)_{i=1}^{8}$ transform according to (2.15) with

$$B_i \xrightarrow{g} (e^{iF\eta'})_{ij} B_j \qquad (5.43)$$

I can get the following SU(3) covariant bilinear combination (with right parity).

(i) The octet $(X_i)_{i=1}^{8}$

$$-X_i = m_0 \alpha d_{ijk} \bar{B}_j B_k + m_0 \beta (-i) f_{ijk} \bar{B}_j B_k \qquad (5.44)$$

(ii) The singlet X_0

$$-X_0 = m_0 \gamma \sum_{i=1}^{8} \bar{B}_i B_i \qquad (5.45)$$

where α, β and γ are yet undetermined constants and m_0 represents the coefficient of chiral invariant mass term

$$\mathcal{H}^0_{mass} = m_0 \sum_{i=1}^{8} \bar{B}_i B_i \qquad (5.46)$$

Using the same notation as the last section ((5.3), (5.4)), I can write down the required chiral SU(3)xSU(3) multiplet as

$$\begin{pmatrix} u_0 \\ \vdots \\ u_8 \\ v_0 \\ \vdots \\ v_8 \end{pmatrix} = e^{i a_r \xi} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_8 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (5.47)$$

The symmetry breaking interaction hamiltonian $\mathcal{H} = -u_0 - c u_8$ with (5.47), (5.44) and (5.45) above together with the chiral invariant term (5.46) gives the baryon mass term

$$\mathcal{H}_{mass} = m_0(1+\gamma) \sum_{i=1}^{8} \bar{B}_i B_i$$
$$+ c m_0 \left\{ \alpha\, d_{8jk} \bar{B}_j B_k + \beta (-i) f_{8jk} \bar{B}_j B_k \right\}$$

If Bi's are identified in ordinary way with this can be written as

$$\mathcal{H}_{mass} = m_0 \left(1+\gamma+\frac{c\alpha}{\sqrt{3}}\right)\left(\bar{\Sigma}^+\Sigma^- + \bar{\Sigma}^-\Sigma^+ + \bar{\Sigma}^0\Sigma^0\right)$$

$$+ m_0\left(1+\gamma - \frac{c\alpha}{\sqrt{3}} + \frac{\sqrt{3}}{2} c\beta\right)(\bar{P}P + \bar{N}N)$$

$$+ m_0\left(1+\gamma - \frac{c\alpha}{\sqrt{3}} - \frac{\sqrt{3}}{2} c\beta\right)(\bar{\Xi}^-\Xi^- + \bar{\Xi}^0\Xi^0) \qquad (5.48)$$

$$+ m_0\left(1+\gamma - \frac{c\alpha}{\sqrt{3}}\right)\bar{\Lambda}\Lambda$$

Thus the masses of baryon octet is identified as

$$\left.\begin{aligned} m_\Sigma &= m_0\left(1+\gamma + \frac{c\alpha}{\sqrt{3}}\right) \\ m_N &= m_0\left(1+\gamma - \frac{c\alpha}{2\sqrt{3}} + \frac{\sqrt{3}}{2} c\beta\right) \\ m_\Xi &= m_0\left(1+\gamma - \frac{c\alpha}{2\sqrt{3}} - \frac{\sqrt{3}}{2} c\beta\right) \\ m_\Lambda &= m_0\left(1+\gamma - \frac{c\alpha}{\sqrt{3}}\right) \end{aligned}\right\} \qquad (5.49)$$

(5.49) satisfies Gell-Mann, Okubo mass formula for linear mass. The experimental $\bar{m}_\Sigma$ = 1193 MeV, $\bar{m}_N$ = 939 MeV, $\bar{m}_\Xi$ = 1318 MeV and m_Λ = 1115 MeV can be fitted within one percent. Thus I get the estimate of the parameters $m_0\alpha$, $m_0\beta$ and $m_0(1+\gamma)$ as

$$\left.\begin{aligned} m_0(1+\gamma) &= 1154 \ \text{Mev} \\ m_0 c\alpha \frac{1}{\sqrt{3}} &= 39 \ \text{Mev} \\ m_0 \frac{\sqrt{3}}{2} c\beta &= -190 \ \text{Mev} \end{aligned}\right\} \qquad (5.50)$$

Here again, the derivation of the baryon octet mass formula is equivalent to the elementary derivation of G-O formula under broken SU(3) with exclusion of 27plet from mass term[53].

(b) Meson-baryon scattering.

To get the estimate of γ in chiral symmetry breaking term (5.45), I must try to fit scattering data. Unfortunately, many of the known hadronic reactions are very inelastic and I cannot hope to get good agreement with experiment by essentially a perturbation approach. The way to tackle the problem related to the unitarity with the phenomenological lagrangian method is not yet fully developed. Thus I will have to confine myself mainly to the examination of elastic KN and π N reactions at threshold.

The contribution to the scattering amplitude can be obtained by computing the chiral symmetry breaking term in the lagrangian.

$$\mathcal{L}' = u_0 + c\,u_8$$

upto second order in ξ fields. From (5.47) this can be written as

$$\begin{aligned}\mathcal{L}'_{scattering} = &-\Big[\sqrt{\tfrac{1}{6}}\, d_{ijk}\,\xi_i\,\xi_j\,X_k \\ &+ c\left(\tfrac{1}{2}(D\cdot\xi)^2_{8j}\,X_j + \xi_8\,\frac{X\cdot\xi}{3}\right)\Big] \\ &-\left(\frac{\xi^2}{3} + \frac{c}{\sqrt{6}}\,d_{8jk}\,\xi_j\,\xi_k\right)X_0\end{aligned} \qquad (5.51)$$

Instead of writing down full RHS of (5.51) in term of physical meson and baryon fields. I shall extract terms contributing to KN and πN scattering

$$\mathcal{L}'_{N\pi} = \frac{m_0}{\lambda_\pi^2}(\sqrt{2}+c)\left(-\frac{\frac{\alpha}{3}-\beta}{4\sqrt{3}} + \frac{\gamma}{3\sqrt{2}}\right)\bar{N}\cdot N\,\pi^2 \qquad (5.52)$$

$$\mathcal{L}'_{NK} = \frac{m_0}{\lambda_K^2}\Big[\frac{i}{4\sqrt{3}}(\sqrt{2}-\frac{c}{2})(\alpha+\beta)(\bar{N}\sigma_i N)(\bar{K}\sigma_i K)$$

$$+(\sqrt{2}-\frac{c}{2})\left(\frac{\frac{\alpha}{3}-\beta}{4\sqrt{3}}+\frac{\sqrt{2}\gamma}{3}\right)(\bar{N}N)(\bar{K}K)\Big] \qquad (5.53)$$

Using the result of the previous section on mesons, (5.52) and (5.53) can also be written as

$$\mathcal{L}'_{N\pi} = \frac{m_0}{\lambda_\pi^2}\frac{\mu_\pi^2}{\mu_0^2}\left(-\frac{1}{4}\sqrt{\frac{3}{2}}\left(\frac{\alpha}{3}-\beta\right)+\frac{\gamma}{2}\right)\bar{N}N\pi^2$$

$$\mathcal{L}'_{NK} = \frac{m_0}{\lambda_K^2}\frac{\mu_K^2}{\mu_0^2}\Big[-\frac{1}{4}\sqrt{\frac{3}{2}}(\alpha+\beta)(\bar{N}\sigma_i N)(\bar{K}\sigma_i K)$$

$$+\left(\frac{1}{4}\sqrt{\frac{3}{2}}\left(\frac{\alpha}{3}-\beta\right)+\gamma\right)(\bar{N}N)(\bar{K}K)\Big]$$

Since $\mu_0 \sim \mu_\pi$ the influence of symmetry breaking interaction to π N (or any π B) reaction is expected to be small, and the results obtained from chiral symmetric lagrangian may give good approximation[54].

In addition to (5.51), there is a chiral invariant meson-baryon interaction term. This, I take to be essentially the form given in (4.5), the relevant term is

$$\mathcal{L}^{chiral}_{scattering} = -\frac{i}{2}(-i)f_{ijk}\xi_j \partial_\mu \xi_k (-i) f_{\ell\ell m} \bar{B}_\ell \gamma^\mu B_m \quad (5.54)$$
$$+ G_A \bar{B}_j \gamma_\mu \gamma_5 (\alpha' d_{ijk} + (1-\alpha')(-i) f_{ijk}) B_k \partial^\mu \xi_i$$

The first term which is the form of 4-pt contact interaction comes from the covariant derivative of baryon fields $\nabla_\mu B_i$ in the kinematical term of baryon lagrangian. The second term comes from the chiral invariant form of Yukawa-Coupling and G_A (written G' in Chapter 4) is renormalized axial vector form factor for baryons. α' is ordinary d/f ratio. This term gives rise to the Goldberger-Treiman relation for chiral SU(3). In general, the contribution of the derivative Yukawa coupling in (5.54) through Born term is small compared with the contribution from contact term. The latter, of course, corresponds to the current commutator term in ordinary current algebra calculation[52]. This can be replaced by the vector meson exchange term according to the idea of vector dominence which unlike in the case of non-linear P-S mesons explained in the last chapter works in a straightforward way.

Extracting the relevant term for πN and KN from (5.54) I get

$$\mathcal{L}^{chiral}_{N\pi} = -\frac{i}{8\lambda_\pi^2}(\bar{N}\sigma_i \gamma^\mu N)(-2i\epsilon_{ijk}\partial_\mu \pi_j \pi_k)$$
$$+ \frac{G_A}{2\lambda_\pi}\partial_\mu \underline{\pi}\, \bar{N}\underline{\sigma}\gamma_5\gamma^\mu N \quad (5.55)$$

$$\mathcal{L}_{NK}^{chiral} = -\frac{i}{8\lambda_K^2}\Big[(\bar{N}\sigma_i\gamma^\mu N)(\bar{K}\sigma_i\partial_\mu K - \partial_\mu\bar{K}\sigma_i K)$$
$$+\ 3(\bar{N}\gamma^\mu N)(\bar{K}\partial_\mu K - \partial_\mu\bar{K}\cdot K)\Big]$$
$$+\frac{G_A}{2\lambda_K}\Big[\frac{2\alpha'-3}{\sqrt{3}}\Big\{\partial_\mu K_\alpha(\bar{N}^\alpha\gamma_5\gamma_\mu\Lambda) + h.c\Big\} \tag{5.56}$$
$$+(2\alpha'-1)\Big\{\partial_\mu\bar{K}^\beta\sigma^k_{\beta\alpha}(\bar{\Sigma}_k\gamma_5\gamma^\mu N_\alpha) + h.c\Big\}\Big]$$

<u>NK scattering length</u>

It is obvious that NK should be more advantageous for the purpose of estimating $m_0\gamma$, chiral symmetry breaking mass, since the corresponding term in πN is expected to be small.

Let us evaluate first the contact terms, i.e. the (5.53) and the first half of (5.56).
To compute the amplitude in term of its iso-spin components, it is convenient to write (5.53) and (5.56) in the following way.

$$\mathcal{L}'_{NK} = \frac{m_0}{\lambda_K^2}\Big[(\sqrt{2}-\frac{c}{2})\Big(\frac{\alpha}{6\sqrt{3}}+\frac{\gamma}{3\sqrt{2}}\Big)(\bar{N}\sigma_i\bar{K}^c)(K^c\sigma_i N)$$
$$+(\sqrt{2}-\frac{c}{2})\Big(\frac{1}{2\sqrt{3}}(-\frac{2\alpha}{3}-\beta)+\frac{\gamma}{3\sqrt{2}}\Big)(\bar{N}\bar{K}^c)(K^c N)\Big] \tag{5.57}$$

and

$$\mathcal{L}_{NK}^{chiral,\,contact} = -\frac{i}{4\lambda_K^2}\Big[\bar{N}\sigma_i\bar{K}^c\gamma^\mu\partial_\mu K^c\sigma_i N$$
$$-\bar{N}\sigma_i\gamma^\mu\partial_\mu\bar{K}^c K^c\sigma_i N\Big] \tag{5.58}$$

where $K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}$ while $K^c = (-K^+, K^0)$

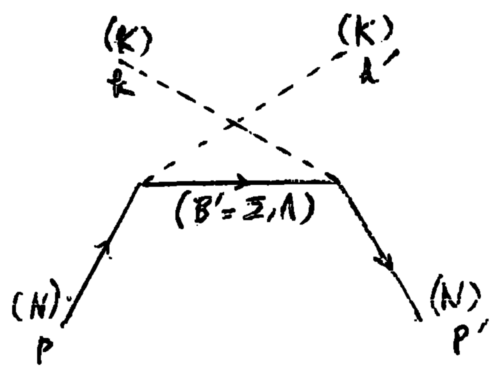
(5.58) shows that the chiral invariant lagrangian gives the vanishing of the I=0 amplitude (except for the Born term).

The contribution to s wave isospin amplitude from (5.57) and (5.58) are

$$f^{KN}_{I=0} = (A + \mu_K B)^{contact}_{I=0} = \frac{2m_0}{\lambda_K^2}\left(\sqrt{2} - \frac{c}{2}\right)\left(\frac{1}{2\sqrt{3}}\left(-\frac{2\alpha}{3} - \beta\right) + \frac{\gamma}{3\sqrt{2}}\right) \tag{5.59}$$

$$f^{KN}_{I=1} = (A + \mu_K B)^{contact}_{I=1} = -\frac{\mu_K}{\lambda_K^2} + \frac{2m_0}{\lambda_K^2}\left(\sqrt{2} - \frac{c}{2}\right)\left(\frac{\alpha}{6\sqrt{3}} + \frac{\gamma}{3\sqrt{2}}\right) \tag{5.60}$$

As for the contribution from Yukawa coupling, I must compute the diagram below from the latter half of (5.56)



I get the s-wave pK^+ and nK^+ amplitude at threshold as

$$f'_{pK^+} \equiv (A + \mu_K B)^{Exch.}_{pK^+}$$

$$= \left(\frac{G_{TA}}{2\lambda_K}\right)^2 \left[\frac{(2\alpha'-3)^2}{3} \frac{-\mu_K^2}{m_N + m_\Lambda - \mu_K} + (2\alpha'-1)^2 \frac{-\mu_K^2}{m_N + m_\Sigma - \mu_K} \right] \quad (5.21)$$

$$f'_{nK^+} \equiv (A + \mu_K B)^{exch}_{nK^+}$$
$$= \left(\frac{G_{TA}}{2\lambda_K}\right)^2 2\,(2\alpha'-1)^2 \frac{-\mu_K^2}{m_N + m_\Sigma - \mu_K} \quad (5.22)$$

Taking α' = [illegible] = 1.1 the estimation for the s-wave amplitudes is

$$f^{exch}_{I=0} \sim 0.06\ \mu_K/\lambda_K^2$$
$$f^{exch}_{I=1} \sim -0.1\ \mu_K/\lambda_K^2 \quad (5.23)$$

that the contribution of exchange term is small compared with the chiral invariant contact term which is order of μ_K/λ_K^2

There is an uncertainty in the experimental value for K^+N which affect $f_{I=0}$. If I try to fit the value of scattering length $a_{I=1} = a_{pK^+}$ given by Goldhaber et al.(55) quoted in (Ref. 46).

$$a_{pK^+}(Exp) = -0.22\ (\mu_\pi^{-1})$$

and the SU value of c derived in the last section as, c = −1.[illegible] with experimental λ_K = 1.2 λ_π = 1.2 F_π, I get the estimate of $m_0\gamma$ from (5.36) as

$$m_0\gamma \sim 200\ \text{MeV} \quad (5.24)$$

This gives the value of a_{nK^+} rather smaller than the value quoted in Game (Ref. 46), $a_{nK^+} \sim -0.97\mu_\pi^{-1}$ (56). On the other hand if I consider $f_0 \sim 0$ for (5.59) as a good approximation as suggested by some data, I get the estimate of $m_0\gamma$ as

$$m_0\gamma \sim 170 \text{ MeV}$$

This corresponds to $a_{pK^+} \sim -0.29\mu_\pi^{-1}$. The last evaluation with $f_0 \sim 0$ is identical with the calculation by Von Hippel and Kim (56) with current commutator techniques. ($f_0 \equiv 0$ for (5.59) actually gives $m_0\gamma$ = 174 MeV which is Von Hippel et al.'s estimate).

As the matter of interest, I can formally compute the amplitude for the $\bar{K}N$ reaction from the lagrangian (5.51) and (5.54). Corresponding to (5.59) and (5.60), I get

$$f^{\bar{K}N}_{I=0} = \frac{3\mu_K}{\lambda_K^2} + \frac{2\sqrt{2}}{\lambda_K^2} m_0\left(\sqrt{2}-\frac{c}{2}\right)\left\{\frac{1}{4\sqrt{6}}\left(\frac{5\alpha}{3}+\beta\right)+\frac{\gamma}{6}\right\} \qquad (5.66)$$

$$f^{\bar{K}N}_{I=1} = \frac{\mu_K}{2\lambda_K^2} - \frac{2\sqrt{2}}{\lambda_K^2} m_0\left(\sqrt{2}-\frac{c}{2}\right)\left\{\frac{-1}{4\sqrt{6}}\left(\frac{\alpha}{3}+\beta\right)+\frac{\gamma}{6}\right\} \qquad (5.67)$$

The contribution of Born terms are again small. (They can be obtained from (5.61) and (5.62) by changing the

denominator $m_N + m_B' - \mu_K$ by $m_N + m_B' + \mu_K$).
(5.66) gives a large (about the twice of a_{K^-p}) positive scattering length. Experimentally this is given by enormous negative value. The contribution due to s-wave unitarity cut for $(\bar{K}N)_{I=0}$ is supposed to be particularly large[56]. The situation is not so bad for I=1 case (5.67) but again experimentally the scattering length is negative.

It may be of interest to compare the calculation of the KN amplitude presented here with the earlier work by Schechter, Ueda and Venturi[46]. In the latter, the mass splitting of baryon octets are put "by hand" in the quadratic baryon mass term, and there is no 4-pt term directly arising from such symmetry breaking. However, they treat the chiral symmetry following the model by Cronin discussed at the end of Chapter 4. Thus the Yukawa coupling appears in the non-derivative form and the contribution of Born term is as large as the contribution of 4-pt contact term (also of non derivative form). As a result in their model the mass splitting of baryon octets can affect scattering amplitude at threshold through Σ and Λ poles in the Born terms. The good agreement with experiment has been obtained in this way.

N π scattering length

Peccei[53] calculated πN scattering length using the phenomenological lagrangian which is equivalent to the chiral invariant meson-baryon lagrangian given here. Apart from the problem of the effect of the symmetry breaking which is always supposed to be small, his treatment is very thorough, and reasonable agreement with experiments is obtained for the s and p wave scattering length.

Let us now discuss the effect of the symmetry breaking $N\pi$ interaction (5.52). Using the values of parameters which have been determined already, and putting $m_0\gamma$ to 200 MeV, I get the following estimate for the s-wave iso-spin amplitudes

$$f^{\pi N}_{I=3/2} = -\frac{1}{\lambda_\pi^2}\left(\frac{\mu_\pi}{2} - 22^{\text{Mev}}\right)$$
$$f^{\pi N}_{I=1/2} = \frac{1}{\lambda_\pi^2}\left(\mu_\pi + 22^{\text{Mev}}\right) \qquad (5.68)$$

The part $22^{\text{Mev}}/\lambda_\pi^2$ only comes from (5.52), and the rest comes from the chiral symmetric contact term in (5.55). The contribution of Born term is extremely small.

The symmetry breaking affects only the iso-spin even combination of the amplitude

$$f^{+} = \frac{1}{3}\left(2 f^{3/2} + f^{1/2}\right)$$

and has no effect on the iso-spin odd

$$f^{-} = \frac{1}{3}\left(f^{1/2} - f^{3/2}\right)$$

Chiral symmetric part gives $f^{+} \cong 0$ and symmetry breaking part makes it to

$$f^{+}/f^{-} \sim 0.3$$

which gives the corresponding scattering length as

$$a^{+} \sim 0.03\ (\mu_\pi^{-1})$$

Experimentally[57], a^{+} is smaller and some data is consistent with $a^{+} \cong 0$. One way is to appeal to the effect of (33) resonance. According to the calculation in (Ref. 53), the N^{*} resonance contribute significantly as

$$a^{+}(N^{*}) \sim -0.05\ (\mu_\pi^{-1})$$

If this value is added to the result above, a^{+} will be reduced to

$$a^{+} \sim -0.02 (\mu_\pi^{-1})$$

which is not too far from the estimate by Woodcock and Samarayake[57]

$$a^{+} \sim -0.013 \pm 0.003 (\mu_\pi^{-1})$$

This interpretation is not an unique one. Kecai rejects the use of a symmetry breaking term and proposes to modify the N^* exchange term so as to get more reasonable assymptotic behaviour at high energy[56] and gets almost complete cancellation (with the original N^* contribution).

$$a^+ \sim -0.001 \, (\mu_\pi^{-1})$$

The arguments for the "reasonable assymptotic behaviour of tree diagrams in general has been put forward by Weinberg[57] and several interesting consequences have been derived. But the above result just quoted certainly cannot be regarded as showing the relevance of such a scheme, since this will leave the symmetry breaking contribution unaccounted for.

Other amplitudes

$(\pi \Sigma)$

The I=O component of $\Sigma\pi$ amplitude has no inelastic channels opening at threshold. The part of chiral symmetry breaking term (5.51) which contributes to $\pi\Sigma$ scattering is

$$\mathcal{L}'_{2\pi} = \frac{1}{\lambda_\pi^2} \frac{c+\sqrt{2}}{c} \left(\frac{1}{6} \frac{m_0 c \alpha}{\sqrt{3}} + \frac{c}{3\sqrt{2}} m_0 \gamma \right) \bar{\Sigma} \cdot \Sigma \pi^2 \qquad (5.69)$$

and the chiral symmetric part of the interaction

$$\mathcal{L}^{chiral}_{\Sigma\pi} = \frac{i}{2\lambda_\pi^2}\,\epsilon_{ijk}\,\bar{\Sigma}_j\,\gamma_\mu\,\Sigma_k\,\epsilon_{ilm}\,\pi_l\,\partial^\mu\pi_m \qquad (5.70)$$

I=0 amplitude is

$$f^{\pi\Sigma}_{I=0} = \frac{1}{\lambda^2}\left[2\mu_\pi + \frac{c+\sqrt{2}}{c}\left(\frac{1}{6}\,\frac{m_0\alpha}{\sqrt{3}} + \frac{c\,m_0\gamma}{3\sqrt{2}}\right)\right] \qquad (5.71)$$

The second term which represents the effect of the symmetry breaking is extremely small and the amplitude is more or less "chiral invariant".

$$f^{\pi\Sigma}_{I=0} \sim \frac{2\mu_\pi}{\lambda_\pi^2}$$

The corresponding scattering length is

$$a_{\pi\Sigma}(I=0) \sim 0.43\ (\mu_\pi^{-1}) \qquad (5.72)$$

According to him, $a_{\pi\Sigma}(I=0)$ is about 0.7(μ_π^{-1}) but the uncertainty is very large.

$(\pi\Lambda)$

$\pi\Lambda$ scattering term does not occur in the chiral invariant contact meson-baryon interaction (the first part of (5.54)). In the current algebra calculation with soft meson approximation, $a_{\pi\Lambda} = 0$.

The symmetry breaking interaction $\mathcal{L}'$(scattering) of (5.51) gives the contribution to $\pi\Lambda$ scattering as

$$\mathcal{L}'_{\Lambda\pi} = \frac{1}{\lambda_\pi^2}(\sqrt{2}+c)\left\{\frac{m_0\gamma}{3} - \frac{m_0\alpha}{6\sqrt{3}}\right\}\pi^2\bar{\Lambda}\Lambda \qquad (5.73)$$

The s -wave scattering amplitude at threshold is

$$f^{\pi\Lambda} = \frac{2}{\lambda^2}(\sqrt{2}+c)\left(\frac{m_0\gamma}{3} - \frac{m_0\alpha}{b\sqrt{3}}\right) \qquad (5.74)$$

The corresponding s wave scattering length is

$$a_{\pi\Lambda} \sim 0.004\,(\mu_\pi^{-1})$$

and still very small. On the other hand, I must still consider the Born term due to the coupling

$$\mathcal{L}_{\pi\Lambda\Sigma} = \frac{2\alpha'}{\sqrt{3}}\frac{G_{\pi}}{2\lambda_\pi}\,\partial_\mu \underline{\pi}\,(\bar{\underline{\Sigma}}\,\gamma_5\gamma^\mu\Lambda + \bar{\Lambda}\gamma_5\gamma^\mu\underline{\Sigma}) \qquad (5.75)$$

coming from the second half of (5.54).

The contribution to the s-wave $\pi\Lambda$ amplitude is

$$f^{Born}_{\pi\Lambda} \simeq \left(\frac{2\alpha'}{\sqrt{3}}\frac{G_{\pi}}{2\lambda_\pi}\right)^2 (m_\Sigma + m_\Lambda)^2\left[-\frac{1}{m_\Lambda + m_\Sigma - \mu_\pi} + \frac{m_\Sigma - m_\Lambda + \mu_\pi}{(m_\Sigma + m_\Lambda + \mu_\pi)(m_\Lambda - m_\Sigma + \mu_\pi)}\right] \qquad (5.76)$$

Numerically, for $\alpha' = 0.75$ this gives the scattering length

$$a_{\pi\Lambda} \simeq 1.5\,(\mu_\pi^{-1}) \qquad (5.77)$$

The breaking of the coupling constant

The construction like (5.47) is of course not the unique symmetry breaking interaction within Gell-Mann, Oakes, Renner scheme. The one advantage of the lagrangian method is the ease with which various possibilities within a given symmetry scheme can be exploited, and we may try to study some other examples of broken chiral SU(3) than the one discussed above.

Let us consider the following multiplet

$$\begin{pmatrix} u_0 \\ \vdots \\ u_8 \\ v_0 \\ \vdots \\ v_8 \end{pmatrix} = e^{iQ_5\xi} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \delta \\ y_1 \\ \vdots \\ y_8 \end{pmatrix}$$

with

$$y_i = i\bar{B}\gamma_5(\alpha' D_i + \beta' F_i)B \tag{5.78}$$

The corresponding Gell-Mann, Oakes, Renner type lagrangian is

$$\begin{aligned} \mathcal{L}' &= i\chi G'\left(\sqrt{\tfrac{2}{3}}\,\xi\cdot y + c(D\cdot\xi)_{8j}\,y_j\right) + O(\xi^3) \\ &= i\chi G'\Big\{\frac{\sqrt{2}+c}{\sqrt{3}}(\xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3) \\ &\quad + \frac{\sqrt{2}-c/\sqrt{2}}{\sqrt{3}}(\xi_4 y_4 + \cdots + \xi_7 y_7) \\ &\quad + \frac{\sqrt{2}-c}{\sqrt{3}}\,\xi_8 y_8\Big\} + O(\xi^3) \end{aligned} \tag{5.79}$$

The interaction like (5.7) gives rise to an additional Yukawa type coupling and breaks the symmetry of the Born term expressed by the chiral and SU(3) symmetric coupling in (5.54). Comparing the residue of Born terms with or without (5.7), and defining the meson baryon coupling constants $g_{\pi BB'}$ by generalized Goldberger-Treiman relation, I get

$$\left(\frac{g'_{\pi BB'}}{g^0_{\pi BB'}}\right)^2 = \left(\frac{m + m' + \frac{\sqrt{2}+c}{\sqrt{3}}\chi}{2m_0}\right)^2$$

$$\left(\frac{g'_{KBB'}}{g^0_{KBB'}}\right)^2 = \left(\frac{m + m' + \frac{\sqrt{2}-c/2}{\sqrt{3}}\chi}{2m_0}\right)^2$$

$$\left(\frac{g'_{\eta BB'}}{g^0_{\eta BB'}}\right)^2 = \left(\frac{m + m' + \frac{\sqrt{2}-c}{\sqrt{3}}\chi}{2m_0}\right)^2$$

where g' and g^0 represent the coupling constants with or without (5.7). Numerically, taking $c = -1.26$,

$$(\sqrt{2}+c)/\sqrt{3} \sim 0.01 \qquad \left(\frac{\sqrt{2}-c/2}{\sqrt{3}}\right) \sim 0.9 \qquad \left(\frac{\sqrt{2}-c}{\sqrt{3}}\right) \sim 1.5$$

As to be expected, the change of the $\pi BB'$ coupling constant is small even if χ is order of 2000 MeV. Although the meson baryon coupling constant is suspected[59] to differ considerably from SU(3), the interaction like (5.7) has no immediate application.

Lastly, it should be noted that the Yukawa type interaction (5.7) cannot be used to fit the KN scattering data discussed above. The corresponding u-channel Born term gives negative contribution and the contribution of chiral invariant contact term is already negative and too large. So the modification by the mass term type interaction with, in particular, chiral symmetry breaking term of strength $m_0 \gamma$ described before is the only way to fit the data. This strengthens a little the argument for the quantity like $m_0 \gamma$ being physically meaningful.

Discussion

In conclusion of the work presented here, I would like to add the following remarks.

First, the importance of the "equivalence relation" of the kind discussed in the end of Chapter 4 should be emphasized. Certain arbitrariness in choosing the physical fields operators seems to leave the underlying chiral symmetry and related group theoretical structure as the only physically meaningful concept, and it may be possible to reformulate the whole algorithm of "chiral lagrangian calculation" in group theoretical terms avoiding the redundancy which seems to accompany field theory. The derivation of K-S relation and Weinberg mass relation seems to suggest that such an approach may have interesting physical results. In connection with this point about the non-uniqueness of the choice of physical field, the notion of P.C.A.C seems to be a little puzzling. In the current algebra approach, the divergence equation

$$\partial_\mu A^\mu_i = \mu^2 F \phi_i$$

itself is the matter of defining the right hand side which, due to the right quantum numbers, has the singularity corresponding to single meson state. But when we start to say that such a pole term actually dominates the matrix element

$$\langle \alpha | \partial_\mu A^\mu_i | \beta \rangle$$

over certain range of momentum transfer

$$t = (p_\alpha - p_\beta)^2$$

i.e.

$$\langle \alpha | \partial_\mu A^\mu_i | \beta \rangle \cong \mu^2 F^i_{\alpha\beta} / (\mu^2 - t)$$

then it is a meaningful assumption and can impose the restriction on the physics. It is in this form we use "PCAC" or (pole dominence assumption) in current algebra. Thus, when we write the scattering amplitude including these mesons in L.S.Z reduced form, the residue after the removal of meson pole factors may be assumed as "smooth"[52]. For the derivation of Adler's consistency relation, PCAC interpreted in this way is essential. Also, this smoothness assumption gives the certain prescription for obtaining physical amplitude from soft meson limit which can be determined from current algebra. Thus PCAC or the pole dominence assumption is the most important assumption which enables current algebra scheme to make predictions. On the other hand, it is difficult to recognize the role of PCAC in the chiral lagrangian scheme. The field theoretical divergence equation has its correspondence in the lagrangian scheme so that the first term of divergence of axial currents discussed in §5 is determined by the symmetry argument.

$$\partial_\mu A^\mu_i = \mu^2 F \phi_i + O(\phi^3)$$

this again cannot be considered as an additional assumption. Now in the theory in which we have a definite lagrangian, that is to say a dynamical equation of motion, the assertion about the particular form of the higher order term in the R.H.S above certainly gives a non trivial restriction. If we change the definition of the physical field to modify the PCAC equation this will change the result of calculation of the off-mass shell amplitude. At the same time, unlike the current algebra, these off-mass shell amplitude is not important. The lagrangian perturbation theory gives the on-mass shell amplitude directly. Even in the example of weak decay of K mesons(2), the correction term to the amplitudes due to the redefinition of the amplitude (i.e. PCAC or "non" PCAC) vanishes when all the external mesons are on mass shell. This is because even the weak or electromagnetic interaction is written in term of definite symmetric quantities (i.e. vector or axial vector currents) and thus independent of the particular choice of "physical field". Our feeling is that PCAC is deeply connected with the orthodox field theoretical notions which underlines the current algebra and gives the formal expression for the off mass shell amplitude through L.S.Z techniques.

Lastly in practical calculations like scattering length, the limitation due to the problem of unitarity is strongly felt. Recently the possibility of using certain techniques of summing up[60,61] the perturbation series to calculate the higher order rescattering process in the lagrangian approach has been suggested. It may be that a practicable and convincing prescribtion of "unitarizing" chiral results will emerge from the study of this technique. But for the moment, it is hard to do anything beyond the analysis of the rather formal structure of the theory.

It is also possible to go to an extreme "phenomenological" approach. For instance, we may use higher order covariant derivatives disregarding all the field theoretical difficulties. On the other hand, this means that we will effectively abandon hope of extending the restriction due to chiral symmetry beyond what can be calculated by the tree approximation.

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Appendix

"Meson and Matrices"

To study the formal structure of chiral lagrangian, it is sometimes more convenient[2] to consider the "meson matrices"

$$M = e^{-2i\lambda\cdot\xi} \quad ; \quad \lambda\cdot\xi = \sum_{\alpha=1}^{8} \lambda_\alpha \xi_\alpha \qquad \text{(A.1)}$$

instead of meson fields . Here $(\lambda_\alpha)_{\alpha=1}^{8}$ are the generators of (3) representation of SU(3) and equal to the half of Gell-Mann matrices.

As has been used by previous authors[2], I can write the chiral invariant lagrangian in term of M's and appropriate traces. Thus , for instance,

$$(\nabla_\mu \xi)^2 = -\frac{1}{2}\, Tr\, (e^{-i\lambda\xi}\partial_\mu e^{i\lambda\xi} - e^{i\lambda\xi}\partial_\mu e^{-i\lambda\xi})^2$$
$$= \frac{1}{2}\, Tr\, (\partial_\mu e^{-2i\lambda\xi}\, \partial^\mu e^{2i\lambda\xi})$$

and the chiral invariant meson lagrangian can be written as

$$\mathcal{L} = \frac{a}{4}\, Tr\, (\partial_\mu M \partial^\mu M^+) \qquad \text{(A.2)}$$

I would like to study some consequences of the expression like (A.2) to show the use of M matrices.

First, let us evaluate the contribution of the n-point contact term to the n-particle amplitude like

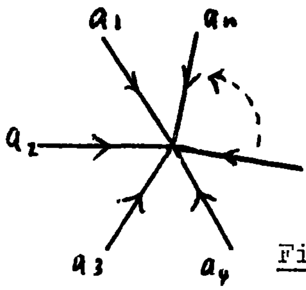


Fig. 1

$a_1 \ldots . a_n$ are SU(3) indeces of the corresponding mesons

Taking the matrix element

$$\langle 0 | \mathrm{Tr}\, \partial_\mu M \partial^\mu M^+ | a_1 \cdots a_n \rangle$$

the corresponding amplitude can be obtained immediately. This is

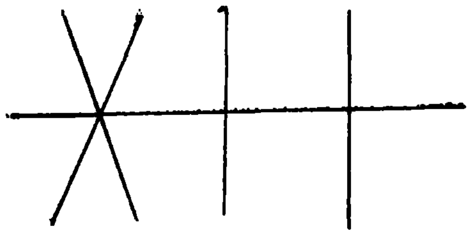
$$T = -\frac{1}{2} \sum_{s=0}^{n} \sum_{\mathrm{perm}(a_1 \cdots a_n)} \mathrm{Tr}\{(2i\lambda_{a_1} \cdots 2i\lambda_{a_{n-s}})$$
$$\times' (-2i\lambda_{a_{n-s+1}} \cdots\cdots -2i\lambda_{a_n})\}$$
$$\times (q_{a_1} + \cdots + q_{a_{n-s}}) \cdot (q_{a_{n-s+1}} + \cdots + q_{a_n}) \times \frac{1}{(n-s)!\, s!}$$
$$= \frac{i^n}{2} \sum_{s=0}^{n} (-1)^s \sum_{\substack{\text{class of} \\ \text{cyclic permutation}}} \mathrm{Tr}(\lambda_{a_1} \cdots\cdots \lambda_{a_n})$$
$$\times \sum_{\text{cyclic}} (q_{a_1} + \cdots\cdots + q_{a_{n-s}})^2$$

where $q_{a_1}, \cdots\cdots, q_{a_n}$ are the ingoing momentum of meson $a_1 \cdots a_n$.

Thus T has the form

$$T = \sum_{(a_1 \cdots a_n)} \mathrm{Tr}(\lambda_{a_1} \cdots \lambda_{a_n}) X(q_{a_1}, \cdots q_{a_n}) \quad \text{(A.3)}$$

where $\sum_{(a_1 \cdots a_n)}$ means the sum over the class of cyclic permutations. X is of course invariant under the cyclic permutation of $(a_1 \ldots\ldots a_n)$. More generally the single vertex like (Fig. 2) can be stuck together to form an arbitrarly tree graph like



(Fig. 2)

Each internal line is represented by the free propagator from the factor like

$$\langle T(\lambda_\alpha \xi_\alpha \, , \, \lambda_\beta \xi_\beta)\rangle_0$$

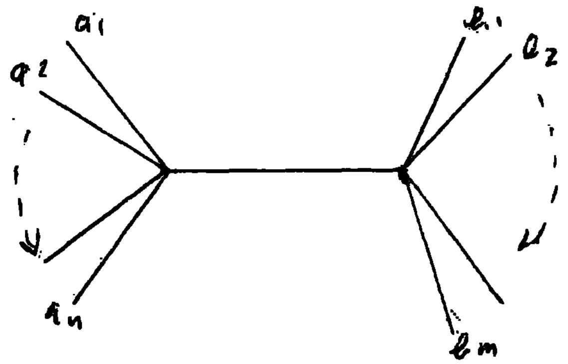
In case the eight mesons corresponding to $(\xi_\alpha)_{\alpha=1}^{8}$ are all degenerate. (SU(3) symmetric), the resultant propagator with internal momentum q is proportional to

$$\sum_{\alpha,\beta} \lambda_\alpha \cdot \lambda_\beta \frac{\delta_{\alpha\beta}}{\mu^2 - q^2} = \frac{1}{\mu^2 - q^2} \sum_{\alpha=1}^{8} \lambda_\alpha \lambda_\alpha$$

where if $\mu^2 \neq 0$, the appropriate SU(3) invariant mass term like

$$Tr\,(M + M^{+})$$

should be a ded. Because of the form of SU(3) factor in the propagator above, sticking two verteces like



give the overall factor

$$\sum_{\alpha=1}^{8} \mathrm{Tr}(\lambda_{a_1} \cdots \lambda_{a_n} \lambda_\alpha)\, \mathrm{Tr}(\lambda_\alpha \lambda_{b_1} \cdots \lambda_{b_m}) \tag{A.4}$$

Now it is well known that sums like this can be transformed using the completeness relation of -matrices.

$$\sum_{\alpha=1}^{8} \mathrm{Tr}(\lambda_{a_1} \cdots \lambda_{a_n} \lambda_\alpha)\, \mathrm{Tr}(\lambda_\alpha \lambda_{b_1} \cdots \lambda_{b_m})$$

$$= 2\, \mathrm{Tr}(\lambda_{a_1} \cdots \lambda_{a_n} \lambda_{b_1} \cdots \lambda_{b_m}) - \frac{2}{3} \mathrm{Tr}(\lambda_{a_1} \cdots \lambda_{a_n})\, \mathrm{Tr}(\lambda_{b_1} \cdots \lambda_{b_m}) \tag{A.5}$$

The simple trace factor seems to be lost. On the other hand, suppose there is ninth meson with same mass μ and which comes into the interacting system in such a way that the sum $\sum_{\alpha=1}^{8} \lambda_\alpha \lambda_\alpha$ in the propagator should be replaced by

$$\sum_{\alpha=0}^{8} \lambda_\alpha \lambda_\alpha$$

where

$$\lambda_0 = \sqrt{\frac{2}{3}}\, \mathbb{1} \tag{A.6}$$

Then, instead of (A.6), the overall trace factor of two verteces stuck together is

$$\sum_{\alpha=1}^{8} Tr(\lambda_{a_1} \cdots \lambda_{a_n} \lambda_\alpha) Tr(\lambda_\alpha \lambda_{e_1} \cdots \lambda_{e_m})$$

$$+ Tr(\lambda_{a_1} \cdots \lambda_{a_n} \lambda_0) Tr(\lambda_0 \lambda_{e_1} \cdots \lambda_{e_m})$$

$$= 2 Tr(\lambda_{a_1} \cdots \lambda_{a_n} \lambda_{e_1} \cdots \lambda_{e_m})$$

Thus the trace factor will be recovered. In this way, any number of verteces can be stuck together to give any tree graph while retaining the general form of (A.5). The ninth meson can be introduced by simply replacing M matrices by "nonet M matrices"

$$M' = Exp\left(\sum_{\alpha=0}^{8} - 2i \lambda_\alpha \xi_\alpha\right) \quad (A.7)$$

Then $M' = e^{-2i\lambda_0 \xi_0} M$

and

$$\begin{aligned} &Tr\, \partial_\mu M' \partial^\mu M'^\dagger \\ &= Tr\, \partial_\mu e^{2i\lambda_0\xi_0} \partial^\mu e^{-2i\lambda_0\xi_0} + Tr\, \partial_\mu M \partial^\mu M^\dagger \\ &= Tr\, (2i\lambda_0 \partial_\mu \xi_0)^2 + Tr(\partial_\mu M \partial^\mu M^\dagger) \end{aligned} \quad (A.8)$$

If is assumed to be chiral scalor, the resultant lagrangian

$$\mathcal{L} = \frac{a}{4} \partial_\mu M \partial^\mu M^\dagger + \frac{a}{2} (\partial_\mu \xi_0)^2 \quad (A.9)$$

is chiral invariant.