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THE LAGRANGIAN METHOD
FOR CHIRAL SYMMETRY

by

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A thesis presented for the degree of Doctor of Philosophy of the University of Durham.

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The work presented in this thesis was carried out at the Department of Mathematics, University of Durham in the period from October 19 to June 19 under the supervision of Dr. D.B. Fairlie.

The author gratefully acknowledges his indebtedness to Dr. Fairlie for his continued guidance and encouragement as well as the introduction to the subject itself. He has also consented that the material in the papers written by him in collaboration with the present author may be used in this thesis. The author's thanks are also due to his colleagues in particular to Dr. M. Ahmed and Graham Ross for stimulating discussions.

Owing to the comparative complexity of the mathematics involved, it was thought appropriate to include a fairly comprehensive description of the general framework of the subject. The quotations from the other authors are explicitly indicated in the text. Otherwise, the work is based essentially on two papers by Dr. Fairlie and the author and a paper by the author himself as well as some unpublished works carried out by the author.

Chapter I incorporates those works done by Dr. Fairlie and the author but also reviews the important
works of other authors. No claim of originality is made on Chapter 2, which is necessary only to explain the basic idea of the subject. Chapter 3, Chapter 4 and most of Chapter 5 are claimed to be original.
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ABSTRACT

We describe the non-linear realizations of chiral symmetry group and study some of its implications in elementary particle physics. In Chapter 1, the basic concepts of non-linear realization techniques are introduced by the way of reviewing the special case of the chiral SU(2)xSU(2) group.

In Chapter 2 the general formalism for chiral SU(n)xSU(n) is developed. This part is wholly dependent on the work by Coleman, Wess and Zumino.

In Chapter 3 the method is generalized for local chiral invariance to describe the non-linear gauge fields.

The Chapter 4 illustrates the use of non-linear realization techniques in conjunction with the phenomenological lagrangian. This chapter is introductory to the final Chapter, 5, in which we have attempted to use the phenomenological lagrangian with non-linear realization of chiral SU(3)xSU(3) to calculate some low energy hadronic reactions. As an important addition, a description of broken chiral SU(3)xSU(3) is given. This follows the general scheme put forward by Gell-Mann, Cakes and Renner.
CHAPTER 1

Non-linear realization of chiral SU(2)xSU(2)

§1 Non-linear realization with phenomenological Lagrangian

The "non-linear realization" approach to chiral symmetry has received much attention recently. Weinberg(1) was the first to realise that the results of current algebra techniques which have been so successful in explaining several features of elementary particle physics can be reproduced very simply by considering the usual chiral (SU(2)) symmetry as a dynamical symmetry of a gauge-type rather than a conventional algebraic symmetry with a linear representation theory. By implementing this way of "realizing" chiral symmetry in a simple field theoretical model, not only the current algebra results may be obtained with much less labour but also we seem to get more insight into the physics we are trying to understand.

Weinberg's techniques have been quickly developed by a number of authors (2,3,4,5,6,7,8,9) and now there are seen to be fairly simple rules to construct a "chiral invariant" models for a wide range of physical situation.
This also includes a prescription of how to break the symmetry.

It is still very difficult to see if we can establish these techniques as being based on the orthodox field theory. Although there is considerable effort (10, 11) to establish them as such by, for instance, studying the possible renormalizability of certain Lagrangian field theories connected to them, the complete success in this direction is not yet certain.

In this thesis, the discussion is confined strictly to the phenomenological side of these techniques, that is to say, the systematic construction of certain fairly simple dynamical model with a "Lagrangian", which, in turn, will be considered merely as a way to calculate physical $S$-matrix elements. This last statement means that I use this "Lagrangian" to calculate ordinary Feynman graphs with Wick's theorem but I only take a class of Feynman graphs which are obviously calculable (non divergent). These are the graphs without internal loops (so called tree graphs) and thus, strictly speaking, these techniques cannot be considered even as a perturbation approximation to quantum field theory at this stage.
Schwinger in his work of non-linear realization techniques has suggested the possibility of a new phenomenological theory of elementary-particle physics which would give the physical basis for the disregard of various field theoretical difficulties in such a technique. Although I do not discuss the philosophy of Schwinger here his way of developing the non-linear realization techniques offers the convenient starting point.

§2 Schwinger's non-linear realization of chiral group

In this §, we follow and expand the analysis of chiral $SU(2):SU(2)$ symmetry of $\pi$-nucleon systems found in reference $3a(2,12)$.

The low energy $\pi$-nucleon system can be described by the following phenomenological Lagrangian ($2a$).

$$\mathcal{L} = \frac{1}{2} \partial_\mu \pi \partial^\mu \pi - \frac{1}{2} \mu^2 \pi^2$$
$$+ \bar{\Psi} (\gamma \cdot \mathbf{r} - m \Psi) \Psi$$
$$- \frac{f}{\mu_n} \bar{\Psi} \gamma_5 \sigma \cdot \mathbf{r} \Psi \sigma^\dagger \pi$$
$$- \left( \frac{f_0}{\mu_n} \bar{\Psi} \gamma_5 \sigma \cdot \mathbf{r} \Psi \sigma^\dagger \mathbf{r} \right) (1.1)$$
with $f$ and $g$ give the correct value for $\pi$ and $\rho$ wave $\pi^-\pi^-$ scattering length (when calculated with true radii only).

Ledesma observed that apart from the $\rho\pi\rho\pi$ term, $L$ is also invariant under the gauge type transformation

$$\pi \to \pi + \delta \rho'$$

$$\psi \to \{ 1 + i \left( f_{\rho\pi} \pi \delta \rho' \right) \} \psi$$

(1.3a)

(1.3b)

where the real parameters $\delta \rho'$, $\delta \rho$ are assumed to be infinitesimal.

Transformations of nucleon fields (1.2a and 1.2b) actually generate the chiral $SU(2) \times SU(2)$ group and we note that there the conventional $\pi$ transformation of nucleon field under the chiral $\pi$-vec. has been replaced by the addition of a single $\pi$-vec. to nucleon states.

The transformation of $\rho\pi\rho\pi$ fields, on the other hand, gives the semidirect product $SU(2):T_{\pi}$. Now
Schwinger further observes that if we replace the "translation" (1.3a) by the non linear transformation

\[ \pi \rightarrow \pi + \delta \phi + \left( \frac{f_0}{\mu_n} \right) \left( 2\pi (\pi \cdot \delta \phi') - \pi^2 \delta \phi' \right) \]  \hspace{1cm} (1.4)

then both nucleon and pion transformations generate the group SU(2) \times SU(2).

The generators of the transformation (1.3b) and (1.4) are the operators containing the differentiation with respect to pion fields II.

We may write (1.2a), (1.2b), (1.3b) and (1.4) in operator form as

\[ \begin{align*}
\pi & \rightarrow \pi + (i \sigma_\beta \cdot \pi) \pi \\
\psi & \rightarrow \psi + (i \sigma_\beta \cdot \pi') \psi \\
\pi' & \rightarrow \pi' + (i \sigma_\beta \cdot \pi) \pi' \\
\psi' & \rightarrow \psi' + (i \sigma_\beta \cdot \pi') \psi'
\end{align*} \]  \hspace{1cm} (1.5)

with

\[ I_\beta = -i \epsilon_{\beta j k} \pi_j \frac{\partial}{\partial \pi_k} \]  \hspace{1cm} (1.6)

\[ (I_\beta)'_\mu = (T_\beta)_\mu / \lambda \]  \hspace{1cm} (1.7)

\[ \tilde{\sigma}_\beta = -\frac{i}{2\lambda} \left\{ \delta_{ij} (1-\lambda^2 \tilde{\pi}') + 2\lambda^2 \tilde{\pi} \tilde{\pi}_j \right\} \frac{\partial}{\partial \tilde{\pi}_j} \]  \hspace{1cm} (1.8)

\[ (\tilde{T}_\beta)'_\mu = \lambda \left( \frac{i}{2} \tilde{\pi} \tilde{\pi} \right)_\mu \]  \hspace{1cm} (1.9)
where we write \( \lambda = \frac{f_0}{\sqrt{\mu_n^2}} \) and also for the sake of
simplicity, we have used the infinitesimal parameter
of chiral transformations \( \delta \theta = 2 \lambda \delta \lambda' \).

It is easy to show, by direct computation, that

\[
[ G_i, G_j ] = i e : j e I_k \quad (1.10a)
\]

\[
[ I_i, G_j ] = i e : j e G_k \quad (1.10b)
\]

in addition to the familiar iso-spin algebra

\[
[ I_i, I_j ] = i e : j e I_k \quad (1.10c)
\]

The same chiral \( \text{SU}(2) \times \text{SU}(2) \) algebra holds for \( G_i' \) and
\( I_i' \). (In showing this, we should remember that the
nucleon transformations (1.2b) and (1.3b) are written
in term of contravariant components of a vector).

The non-linear transformation (1.4) can be regarded
as a simple generalization of the translation (1.3a)
in the sense that it is a subgroup of conformal
transformations in 3-dimensional euclidian space.

(1.2), (1.3b) and (1.4) with chiral \( \text{SU}(2) \times \text{SU}(2) \)
type algebra (1.11a,b,c) express the type of non-linear
realization of chiral group first studied by Weinberg
and Schwinger. Now it will be immediately observed
that the lagrangian (1.1) is not strictly invariant
under these transformations. But before trying to generalize (1.1) to "chiral invariant" form, it is more convenient to study some mathematical consequences of these non-linear transformations.

§5 The relation with chiral 4-vector

The expression of generators of the chiral SU(2):SU(2) group (1.6) and (1.8) can be used when the group is realized as a transformation group over the field of arbitrary polynomial or analytical function of \( \pi_i \). Out of these polynomials, the construction of actual representation (linear) of chiral SU(2)xSU(2) group may be attempted. That we cannot get an arbitrary inedducible representation of chiral SU(2)xSU(2) group can be seen by computing the operator product \( I \cdot G \) and \( G \cdot I \).

I derive from (1.6) and (1.9)

\[
I \cdot G_j = C \left( -i C \pi_i \frac{2}{\frac{2}{\lambda} } \right) \left( \frac{2}{\lambda} \right) \left( \delta_{i j} \left( 1 - \lambda^2 \pi^2 \right) + 2 \lambda \pi_i \pi_j \frac{\partial}{\partial \pi_i \pi_j} \right) \frac{\partial}{\partial \pi_j}
\]

\[
\frac{2^2}{\partial \pi_i \partial \pi_j}
\]

\[
= \left( \frac{-1}{\lambda} \right) C \pi_i \pi_j \left\{ \left( \delta_{i j} \left( -2 \lambda^2 \pi_i \pi_j + 2 \lambda \pi_i \pi_j + \delta_{i j} \pi_i \pi_j \right) \right) \frac{\partial}{\partial \pi_j} + \left( \delta_{i j} \left( -1 + \lambda^2 \pi^2 \right) - 2 \lambda \pi_i \pi_j \right) \frac{\partial}{\partial \pi_i \pi_j} \right\}
\]

\[
I \cdot G = \left( \frac{-1}{\lambda} \right) C \pi_i \pi_j \left\{ \left( -2 \lambda^2 \pi_i \pi_j + 2 \lambda \pi_i \pi_j + \delta_{i j} \pi_i \pi_j \right) \frac{\partial}{\partial \pi_j} + \left( -1 + \lambda^2 \pi^2 \right) \frac{\partial}{\partial \pi_i \pi_j} \right\}
\]

\[
+ 2 \lambda \pi_i \pi_j \frac{\partial^2}{\partial \pi_i \partial \pi_j}
\]
Similarly I get

\[ \mathbb{I} \cdot \mathbb{G} = 0 \]

\[ \mathbb{G} \cdot \mathbb{I} = 0 \]

\[ \mathbb{I} \cdot \mathbb{G} = 0 \] implies \((\mathbb{I} + \mathbb{G})^2 = (\mathbb{I} - \mathbb{G})^2\) i.e. in constructing the representation of chiral \(SU(2) \times SU(2)\) out of analytical function of \(\mathbb{\Pi}'\), then any such representation contains the irreducible components of type \((n/2, n/2)\, n,\text{integer only}\). (I follow the usual notation for irreducible representations of chiral \(SU(2) \times SU(2)\) by writing it as \((J_1, J_2)\)).

This condition is also a sufficient one and any arbitrary representation of form \((J_1, J_2)\) with \(J_1 = J_2\) may be constructed. To see this, it is sufficient to explicitly construct the lowest \((\frac{1}{2}, \frac{3}{2})\) representation. This is the so-called 4-vector representation and must be constructed as a function of \(\mathbb{\Pi}'\). \(\Phi^\alpha_{\xi}\) (1.11)

\[
\begin{align*}
\Phi_4 &= G \left( \mathbb{\Pi}' \right) \\
\Phi_\xi &= F \left( \mathbb{\Pi}' \right) \mathbb{\Pi}_\xi := 1, 2, 3
\end{align*}
\]

which under the transformation of \(\mathbb{\Pi}'\) defined above (1.2a and 1.4) transform as chiral 4-vector. (Or
vector in 4 dimensional euclidian space). The transformation of \( \phi' \) under iso-spin rotation (1.2a) is obvious and the forms of \( G \) and \( F \) should be chosen so that under the chiral part of the transformation (1.4) \( \phi_4 \) and \( \phi_i \) transform like

\[
\begin{align*}
\delta \phi_4 &= \delta \alpha \cdot \phi \\
\delta \phi_i &= -\phi_4 \cdot \delta \alpha_i
\end{align*}
\]  

(1.12)

Using (1.4) \( \delta \pi_i = 8 \delta_j (\delta_{ji} (1-\lambda^2 \pi^2) + 2 \lambda^2 \pi_j \pi_i) \frac{1}{2\lambda} \) in (1.11) L.H.S. of (1.12) becomes

\[
\begin{align*}
\delta \phi_4 &= 2G'(1-\lambda^2 \pi^2) \pi \cdot \delta \alpha / 2\lambda \\
\delta \phi_i &= 2F'(1+\lambda^2 \pi^2) \pi \cdot \delta \pi_i / 2\lambda \\
&\quad + F((1-\lambda^2 \pi^2) \delta \alpha_i + 2\lambda^2 \pi_i \cdot \delta \pi_i) / 2\lambda
\end{align*}
\]

where \( G' = \frac{dG(\pi^2)}{d \pi^2} \quad F' = \frac{dF(\pi^2)}{d \pi^2} \)

so that (1.12) becomes

\[
\begin{align*}
2G'(1+\lambda^2 \pi^2) \pi \cdot \delta \alpha / 2\lambda &= -F \pi \cdot \delta \pi_i \\
(2F'(1+\lambda^2 \pi^2) + 2\lambda^2 F) \pi \cdot \delta \pi_i / 2\lambda \\
&\quad + F((1-\lambda^2 \pi^2) \delta \alpha_i / 2\lambda \\
&= -G \delta \alpha_i
\end{align*}
\]

or

\[
\begin{align*}
F &= -2G'(1+\lambda^2 \pi^2) / 2\lambda \\
&= -\frac{1}{\lambda^2} F'(1+\lambda^2 \pi^2) \\
G &= -\frac{1}{2\lambda} F \cdot (1-\lambda^2 \pi^2)
\end{align*}
\]  

(1.13)
We can easily integrate (1.15) and get the general solution which is regular at

$$G = -\frac{a}{2\lambda} \left( \frac{1 - \lambda^2 \pi^2}{1 + \lambda^2 \pi^2} \right)$$

$$F = a \left( \frac{1}{1 + \lambda^2 \pi^2} \right)$$

where $a$ is an arbitrary constant.

It is easy to check that (1.15) actually satisfies the original condition (1.14).

The 4-vector components $(\phi_i)_{x=1}$ are not independent and satisfy a constraint

$$\phi_i^2 + \phi_{x}^2 = a \frac{1}{4\lambda}$$

As for the arbitrary constant $a$, we choose it so that

$$\phi_i = \pi_i + O(\pi^2)$$

Then $a = 1$ and the constraint equation is now

$$\phi_i^2 + \phi_{x}^2 = \frac{1}{4\lambda} = \frac{f_{\pi}^2}{4f_{0}^2}$$

4. Gell-Mann-Weinberg parameterization of $\pi$ fields; linearization of nucleon fields

The simplicity of using a (linear) representation is that we can always normally integrate the differential expressions like (1.5) by using exponentials. In case of the non-linear transformation of $\pi$ fields, we can
derive an useful parameterization of \( \pi' \) with the aid of \((\pi_a, \pi_{\bar{a}})\) representation constructed above.

The simplest way to express the linear transformation belonging to the 4-vector or \((\pi_a, \pi_{\bar{a}})\) representation of iso scalar - iso vector pair \((\Phi_{\pi_{\bar{a}}}, \Phi_{\pi_a})\) is to consider the 2x2 matrix

\[
M(\phi) = -\Phi_{\bar{a}} + i \Phi_a T
\]

where \((T_a)_{\pi_{\bar{a}}}^{\pi_a}\) are Pauli matrices.

Then the chiral transformation generated by the infinitesimal forms (1.4) through (1.12) is

\[
M(\phi) \rightarrow M(\phi') = e^{i(T_a_{\pi_{\bar{a}}}^{\pi_a})} M(\phi) e^{-i(T_a_{\pi_{\bar{a}}}^{\pi_a})}
\]

\(\phi'\) being the transform of \(\phi\) by an element of the chiral group with finite parameter \(d\). Of course, with respect to iso-spin part of the group, we have the usual

\[
M(\phi) \rightarrow M(\phi') = e^{i(T_a_{\pi_{\bar{a}}}^{\pi_a})} M(\phi) e^{-i(T_a_{\pi_{\bar{a}}}^{\pi_a})}
\]

In terms of original \(\pi_{\bar{a}}\), \(M(\phi)\) is (from (1.14) with \(a=1\)).

\[
M = \frac{1}{2\lambda} \left[ \frac{1 - \lambda^2 \pi^2}{1 + \lambda \pi \pi^*} + \frac{i \pi \cdot T}{1 + \lambda^2 \pi^2} \right]
\]

\[
= \frac{1}{2\lambda} \left[ \frac{1 - \lambda \pi^2 + 2\lambda \pi \cdot T}{1 + \lambda \pi \pi^*} \right]
\]

\[
= \frac{1}{2\lambda} \left[ \frac{1 + i \lambda \pi \cdot T}{1 - i \lambda \pi \cdot T} \right]
\]

i.e. \(M(\phi) = M(\pi) = \frac{1}{2\lambda} \frac{1 + i \lambda \pi \cdot T}{1 - i \lambda \pi \cdot T}\)
and (1.17) can be taken as an integrated form of (1.4).

\[
\frac{1 + i \lambda \pi \cdot T}{1 - i \lambda \pi \cdot T} \rightarrow \frac{1 + i \lambda \pi' \cdot T}{1 - i \lambda \pi' \cdot T} = e^{\frac{i}{2} \frac{\pi \cdot T}{\lambda}} e^{\frac{i}{2} \frac{\pi' \cdot T}{\lambda}} (1.18)
\]

These relations with linear representation automatically guarantee the consistency of the original non-linear transformation as a group operation.

When the integrated form of pion field transformations (1.18) are given, there is an element of the chiral group of special interest, i.e. we may look for a transformation which reduces \( \pi \)'s at given space time point \( \pi \) to zero. Putting the corresponding parameters of the chiral transformation \( \zeta \) \( i = 1, 2, 3 \)

\( i = 1, 2, 3 \), we have \( 6, 9 \)

\[
e^{-\frac{i}{2} \frac{\pi (1) \cdot T}{\lambda}} \frac{1 + i \lambda \pi (1) \cdot T}{1 - i \lambda \pi (1) \cdot T} e^{\frac{i}{2} \frac{\pi (1) \cdot T}{\lambda}} = 1
\]

or

\[
e^{\frac{i}{2} \frac{\pi (2) \cdot T}{\lambda}} \frac{1 + i \lambda \pi (2) \cdot T}{1 - i \lambda \pi (2) \cdot T} = (1.19)
\]

(1.19) can be reduced by using the properties of Pauli matrices
and (1.19) reduces to

\[
\lambda = \frac{1}{\lambda} \lambda \frac{\tan(\sqrt{\beta^2/2})}{\sqrt{\beta^2}}
\]

(1.20)
also, the linear quantities can be expressed in terms of new parameter

\[
\phi_\ell = \frac{1}{2\lambda} \frac{1-\lambda^2 \beta^4}{1+\lambda^2 \beta^4} = \frac{1}{2\lambda} \cosh \frac{\sqrt{\beta^2}}{}
\]

(1.21)
\[
\phi_c = \frac{\pi_c}{1+\lambda^2 \beta^4} = \frac{\pi_c}{1+\lambda^2 \beta^4} \frac{\cosh \sqrt{\beta^2}}{}
\]
The transformation formula (1.17) can be written as a transformation among corresponding parameters

\[
\mathbf{e}^{i\beta} \rightarrow \mathbf{e}^{i\beta'} = \mathbf{e}^{i\pi_c} \mathbf{e}^{i\beta/b} \mathbf{e}^{i\beta/\alpha}
\]
(1.22)
where \( \pi \) and \( \pi' \) are related to \( \pi_c \) and \( \pi'_c \) through (1.19).
\[ (1.2), \text{ is remarkable in the following sense.} \]

The matrix equality (1.2) can be transformed to
\[
\begin{align*}
(e^{\frac{i\pi}{2} I} e^{i\frac{3\pi}{2} I^2})^{-1} e^{i\frac{3\pi}{4} I^2} \\
= (e^{i\frac{3\pi}{2} I^2} e^{\frac{i\pi}{2} I})^{-1} e^{-i\frac{3\pi}{4} I^2} \quad (1.2')
\end{align*}
\]

Since all the factors of (1.2') are unitary - unipolar, the whole matrix \( U \) is also too and can be written as an exponential form
\[
U = e^{-i\pi \gamma' l^2}.
\]

Introducing a set of real parameters \( \gamma' \) ending on \( \gamma \) and \( \gamma \).

Thus I can write (1.2') as (1.5)
\[
\begin{bmatrix}
    e^{\frac{i\pi}{2} I} e^{i\frac{3\pi}{2} I^2} \\
    e^{-\frac{i\pi}{2} I} e^{-i\frac{3\pi}{2} I^2}
\end{bmatrix}
\]

(1.24) actually gives the product of two successive chiral transformations to being decomposed into the product of a chiral transformation and an ordinary iso-spin transformation. Such a decomposition is essentially unique.

In the conventional treatment of chiral invariance, the \( (1,0) \) \((0,1)\) representation, analogous to the nucleon field is expressed by using \( Y_5\) matrices. \((\cdot)\). (To avoid the introduction of a parity doublet). It is clear
that \(1.24\) can be written as
\[
e^{i \phi T/2} e^{i \phi T/2} = e^{i \phi T/2} e^{i \phi T/2}
\] (1.25)

For an infinitesimal parameter \(\delta \phi\), I shall compute explicitly the value of \(\phi'\) in terms of \(\phi\) and \(\delta \phi\).

First eliminating \(\hat{\phi}\) from \(1.24\), and taking the terms first order in \(\delta \phi\) only, I get
\[
\cos(\sqrt{2} \phi/2) \hat{\phi}' + \sin(\sqrt{2} \phi/2) \left( -\frac{1}{2} \delta \phi \hat{\Phi} \right) [T_i, T_j] = 0
\]

From this I get
\[
\phi' = \frac{(\cos(\sqrt{2} \phi/2))}{\sqrt{2}} \hat{\Phi} \delta \phi \left( + O(\delta \phi^2) \right)
\]

i.e. from \(1.20\).
\[
\phi' = \lambda \pi \Gamma \delta \phi
\] (1.26)

But \(1.26\) is precisely the parameter appearing in \(1.3a\) defining the infinitesimal non-linear transformation of nucleon field \(\psi\) under chiral group.

Suppose that an iso-spinor \(\Psi\) (which is also ordinary Dirac spinor) transforms under an infinitesimal element of the chiral group according to \(1.3a\).
\[
\Psi \rightarrow \left\{ 1 + i \frac{\gamma}{2} \cdot (\lambda \pi \Gamma \delta \phi) \right\} \Psi
\]
\[
= (1 + i \frac{\gamma}{2} \cdot \phi') \Psi
\]
Let us define the net field $\Psi$ with spin and iso-
i-metrical character as
\[ \Psi = e^{i \frac{\theta}{2} \cdot \vec{r}} \\Psi \quad (1.2') \]

Then from (1.2'), the transformation of the net field $\Psi$ under an
infinitesimal element of chiral group will be
\[
\Psi \rightarrow e^{i \frac{\theta}{2} \cdot \vec{r}} \Psi' = e^{i \frac{\theta}{2} \cdot \vec{r}} \Psi e^{i \frac{\theta}{2} \cdot \vec{r}} \Psi = e^{i \frac{\theta}{2} \cdot \vec{r}} \Psi e^{i \frac{\theta}{2} \cdot \vec{r}} \Psi
\]
i.e.
\[ \Psi \rightarrow e^{i \frac{\theta}{2} \cdot \vec{r}} \Psi \]

If we define $\Psi'$ for any order of $\Psi$ in the neighbourhood of $x = 0$, i.e., (1.2), we can consider
\[ \Psi \rightarrow e^{i \frac{\theta}{2} \cdot \vec{r}} \Psi \quad (1.2'a) \]
at the integration form of (1.3'). The consistency
of (1.2') is a new operation is guaranteed through
the relation (1.2') and matrix equality (1.5') by
the linear transformation (2)
\[ \Psi \rightarrow e^{i \frac{\theta}{2} \cdot \vec{r}} \Psi \quad (1.2'b) \]

I have shown that the non-linear transformation
of nucleon fields can also be related to a linear representation of the chiral \( SU(2) \times U(1) \) group.

The transformation (1.27) was first used by Weinberg to obtain the non-linear realization starting from the conventional \( \Theta \)-vacuum.

\[ g^2 \text{ Invariant Lagrangian and covariant derivatives (5)} \]

I now come back to the Lagrangian (1.1). As have been noted in \( g^1 \), (1.1) is not really invariant under the non-linear transformation (1.36) and (1.4).

In particular, the derivatives like \( \partial \psi \) or \( \gamma \psi \) naturally transform in rather complicated ways under the non-linear transformations and single iso-spin invariant coupling cannot produce an invariant Lagrangian. To find the way to construct chiral invariant Lagrangians in this non-linear realization scheme, and which gives the form like (1.1) as a relevant approximation, one can exploit the relation with linear representations of chiral group discussed above. It is easy to construct an invariant Lagrangian in terms of the linear representation like \( \Phi \) or \( \phi \) introduced in \( g^2 \) onward. Thus a single chiral invariant generalization of the ordinary iso-spin invariant \( \Pi N \text{Lagrangian with} \)
\[ L = i \hbar \frac{\partial}{\partial t} \Psi - \nabla \cdot \vec{M}(\phi) \vec{D} + \frac{i}{2} \phi \Psi \tilde{\Psi} \phi \]

\[ = \tilde{\Psi} \frac{\partial}{\partial t} \tilde{\phi} \phi \tilde{\phi} = \frac{1}{\phi \chi_0} \left( - \frac{1}{2} \frac{1 - \lambda^2 \frac{\pi^2}{\phi^2}}{1 + \lambda^2 \pi^2} \right)^2 + \left( \phi \frac{\pi z}{1 + \lambda^2 \pi^2} \right)^2 \]

\[ = \left( \frac{\phi \chi_0}{1 + \lambda^2 \pi^2} \right)^2 \]

\[ \overline{\Psi} \Psi = \overline{\Psi} \Psi + \Psi \left( e^{\frac{i \phi t}{2}} \gamma_5 \phi \gamma_5 e^{-\frac{i \phi t}{2}} \right) \overline{\Psi} \]

\[ = \overline{\Psi} \left\{ \gamma^0 + \frac{1}{2} \left( e^{-\frac{i \phi t}{2}} \gamma_5 \phi \gamma_5 + e^{\frac{i \phi t}{2}} \gamma_5 \phi \gamma_5 e^{\frac{i \phi t}{2}} \right) \right\} \psi \]

\[ + \frac{1}{2} \gamma_5 \left( e^{-\frac{i \phi t}{2}} \gamma_5 \phi \gamma_5 - e^{\frac{i \phi t}{2}} \gamma_5 \phi \gamma_5 e^{\frac{i \phi t}{2}} \right) \]
\[
\begin{align*}
&\psi(\theta_1, \theta_2, \theta_3, \theta_4) \\
&\quad = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_4 \\
&\quad \times \left\{ (\cos \theta_1 + e^{-i\lambda \pi/2} \theta_1 e^{i\theta_1/2} + e^{i\theta_1/2} \theta_1 e^{-i\lambda \pi/2}) \psi \right\} \\
&\quad \times \left\{ (\cos \theta_2 + e^{-i\lambda \pi/2} \theta_2 e^{i\theta_2/2} + e^{i\theta_2/2} \theta_2 e^{-i\lambda \pi/2}) \psi \right\} \\
&\quad \times \left\{ (\cos \theta_3 + e^{-i\lambda \pi/2} \theta_3 e^{i\theta_3/2} + e^{i\theta_3/2} \theta_3 e^{-i\lambda \pi/2}) \psi \right\} \\
&\quad \times \left\{ (\cos \theta_4 + e^{-i\lambda \pi/2} \theta_4 e^{i\theta_4/2} + e^{i\theta_4/2} \theta_4 e^{-i\lambda \pi/2}) \psi \right\} \\
&\quad = \frac{1}{1 + \lambda^2 \pi^2} \quad \text{(after integrating out the angular variables.)}
\end{align*}
\]
Collecting (1.34), (1.35), and (1.36) and the last expression reduces to

\[ \text{last expression reduces to} \]

\[ (1.2) \text{ now can be written as} \]

\[ = \sqrt{\frac{m}{m_\phi}} \]

\[ \text{collecting} \quad (1.34), \quad (1.35), \quad (1.36) \]
According to (1.2), it is necessary to include the term in (1.1) which is due to the corrections to the tree diagrams. The relevant part of (1.3) is

$$L' = \frac{1}{2} \psi \overline{\psi} \pi + \frac{\psi}{(\pi - m) \psi} - \lambda \overline{\psi} \gamma_{\nu} \psi \pi \overline{\psi} + \lambda \overline{\psi} \Gamma \pi (\pi \Gamma \pi)$$

Let it be an external fact that the core of the correct $\pi - \pi$ scattering length is given by its (1.1) with

$$f = f_0$$

Thus, with the identification

$$\lambda = f_0 / \rho_\pi$$

as in (1.4), (1.3) is approximately identical with (1.1). But in fact there is no need to approximate the real ratio $f_0 / f_0 \sim 2.5 \ | \ | \ $ units.

We have noted that

$$\overline{\psi} \gamma_\nu \gamma_\mu \psi \overline{\psi} \frac{\pi \Gamma \pi}{1 + \lambda^2 \pi^2}$$

and

$$\overline{\psi} \gamma_\nu (m \pi + i \lambda^2 \pi \lambda \psi \pi . \overline{\psi} \pi) \psi$$
are by themselves invariant under chiral transformations.

In particular, this can be seen if an arbitrary

covariant coefficient to the interaction term.

\[-\lambda \bar{\phi} \phi \frac{\partial^2 \Pi}{1 + \lambda^2 \Pi^2}\]
in (1.37) without destroying the

covariant invariance. Thus replacing this by

\[-f \rho \lambda \bar{\phi} \phi \gamma \gamma \frac{\partial^2 \Pi}{1 + \lambda^2 \Pi^2}\]

(1.1) is recovered within the approximation of (1.37).

Thus I get: chiral invariant generalization of (1.1)

\[\mathcal{L} = i \bar{\phi} \gamma^\mu \frac{\partial}{\partial x^\mu} \phi + m \bar{\phi} \phi\]

\[+ \frac{1}{2} \bar{\phi} \frac{\partial \Pi}{1 + \lambda^2 \Pi^2} \left( \frac{\partial^2 \Pi}{1 + \lambda^2 \Pi^2} \right)^2\]

\[-f \rho \lambda \bar{\phi} \phi \gamma \gamma \frac{\partial^2 \Pi}{1 + \lambda^2 \Pi^2}\]

\[-(e_0 \rho^\mu) \bar{\phi} \phi \gamma \gamma \frac{\pi \lambda \partial \Pi}{1 + \lambda^2 \Pi^2}\]

Only essential difference here is the absence of tensor

mass term and such a term cannot be accounted unless

the symmetry breaking is introduced.

The invariance of terms in the Lagrangian discussed

above also implies the special transformation property

of even factors consisting these terms.

Thus from the invariance of

\[\bar{\phi} \gamma \gamma \frac{\partial^2 \Pi}{1 + \lambda^2 \Pi^2}\]
and are the "isospin type" or non-linear transformation of \( \psi \) fields (1.26), it can be seen that the quantity
\[
\frac{\partial \psi}{1 + \lambda^2 \pi^2}
\]
transforms like iso-spin-one object with a parameter \( \gamma \) (of 1.26) under chiral group. Similarly, the invariance of
\[
\varphi \gamma (\partial \gamma + i \lambda^2 \frac{\pi \lambda \gamma \partial \pi}{1 + \lambda^2 \pi^2} \varphi)
\]
implies that
\[
(\partial \gamma + i \lambda^2 \frac{\pi \lambda \gamma \partial \pi}{1 + \lambda^2 \pi^2} \varphi)
\]
should transform exactly like \( \psi \) itself under chiral group. These transformation properties can be, of course, verified by explicit computation. I have in fact just introduced the covariant derivatives of \( \text{Li} \) with which these rules, by the "nucleon like" rule (1.20) (with corresponding iso-spin) am can be used instead of ordinary derivatives \( \partial \pi \) and \( \partial \varphi \).

I shall write, after \( \text{Li} \), (7)
\[
\nabla \pi = \frac{\partial \pi}{1 + \lambda^2 \pi^2}
\]
(1.40)
\[
\nabla \varphi = \left( \partial \gamma + i \lambda^2 \frac{\pi \lambda \gamma \partial \pi}{1 + \lambda^2 \pi^2} \varphi \right)
\]
(1.41)
then (1.35) is written as
\[
\mathcal{L} = \varphi (i \gamma \varphi - m) \varphi
+ \frac{1}{2} \nabla \pi \nabla \varphi
+ \frac{1}{\rho} \left[ \gamma \delta \delta \varphi \gamma \partial \pi \varphi \right]
\]
(1.42)
how general is the construction of invariant lagrangian in terms of these covariant derivatives? To make the later generalization to chiral $SU(3)$ more straightforward, I will follow the argument of Coleman and Zumino (8) rather than original treatment of Weinberg, in answering this problem.

Suppose that $L$ is an arbitrary chiral invariant lagrangian

$$L = L \left( \pi(x), \partial \pi(x), \psi(x), \partial \psi(x) \right)$$

should be invariant under the non-linear chiral transformations (1.18), (1.14) and (1.28). In particular, the special transformation discussed for the introduction of the parameter $\xi(1)(1.19)$ should keep $L$ invariant since this can be formally considered as a chiral transformation with parameter $\xi$ being equal to $-\xi(1)$ with arbitrary but definite space-time point $x$. If this transformation is denoted by $\mathcal{G}_1$, as a member of chiral group we get from (1.19)

$$\mathcal{G}_1 \pi(1) = 0 \quad (1.43)$$

The replacement of $\xi$ by $-\xi(1)$ in (1.24) gives $\psi' = 0$.

So, for the nucleon field

$$\mathcal{G}_1 \psi(1) = \psi(1) \quad (1.44)$$
Thus at any given space-time point $x$, the invariant lagrangian $\mathcal{L}$ reduces as

$$\mathcal{L}(\pi(x), \varphi \pi(x), \psi(x), \varphi \psi(x))$$

$$= \mathcal{L}(0, g_2 \varphi \pi(x), \psi(x), g_2 \varphi \psi(x))$$

$$= \mathcal{L}'(g_2 \varphi \pi(x), \psi(x), g_2 \varphi \psi(x))$$

(1.45)

As for the quantities $g_2$ ... one must remember that the space-time point $x$ must be considered as fixed so that the $g_2$ is the chiral transformation with constant parameter (not the local chiral transformation). Thus

$$g_2 \partial f(x) = \left[ \frac{2}{\delta y \gamma} g_2 f(x+y) \right]_{y=0}$$

(1.46)

Now

$$g_2 \psi(x+y) = e^{i \gamma_{2y} \pi/2} \psi(x+y)$$

$$e^{i \gamma_{2y} \pi/2} = \frac{1 + i \lambda y \pi/2 (x+y) \pi}{1 - i \lambda y \pi/2 (x+y) \pi}$$

where the parameter $\gamma$ and $\lambda$ are given by the matrix equation (1.24)

$$e^{\gamma \pi/2} \pi/2 = \frac{\psi(x+y) \pi/2}{\psi(x+y) \pi/2}$$

To compute the first derivatives, it is enough to estimate the above relations up to first order in
(It is assumed that the "fields" $\xi(\tau)$ and $\psi(\tau)$ can be considered here as suitably smooth function of $\tau$).

Then I have

$$e^{\pm i \frac{\pi}{2} \cdot \frac{\tau}{2}} e^{\pm i \frac{\pi}{2} \cdot \frac{\tau}{2}} (1 + i \frac{\pi}{2} \cdot (\phi' + \phi'))$$

and

$$\frac{\partial}{\partial \tau} \psi(\tau + \phi') = \psi(\tau + \phi') + i \phi' \frac{\partial}{\partial \tau} \psi(\tau' + \phi')$$

So, from (1.46)

$$\frac{\partial}{\partial \tau} \psi = \frac{\partial}{\partial \tau} \psi + i \frac{\partial}{\partial \tau} \psi$$

also

$$e^{\pm i \frac{\pi}{2} \cdot \frac{\tau}{2}} e^{\pm i \frac{\pi}{2} \cdot \frac{\tau}{2}} = e^{\pm i \frac{\pi}{2} \cdot \frac{\tau}{2}} \cdot e^{\pm i \frac{\pi}{2} \cdot \frac{\tau}{2}}.$$

Thus

$$\frac{\partial}{\partial \tau} \psi = \frac{1}{2} (e^{-i \frac{\pi}{2} \cdot \frac{\tau}{2}} e^{i \frac{\pi}{2} \cdot \frac{\tau}{2}} + e^{i \frac{\pi}{2} \cdot \frac{\tau}{2}} e^{-i \frac{\pi}{2} \cdot \frac{\tau}{2}})$$

$$\frac{\partial}{\partial \tau} \psi = \frac{1}{2} (e^{i \frac{\pi}{2} \cdot \frac{\tau}{2}} e^{-i \frac{\pi}{2} \cdot \frac{\tau}{2}} e^{i \frac{\pi}{2} \cdot \frac{\tau}{2}} - e^{-i \frac{\pi}{2} \cdot \frac{\tau}{2}} e^{-i \frac{\pi}{2} \cdot \frac{\tau}{2}})$$

But these expressions are just the ones appearing when I have defined covariant derivatives. Thus looking back at the transition from (1.32) and (1.33) to (1.35) and (1.34), it can be seen immediately (8)
\[ q \cdot \partial_r \psi = \left( \partial_r + \frac{2r \pi \lambda^2 \eta \pi}{1 + \lambda^2 \eta^2} \right) \psi = \nabla_r \psi \quad (1.47) \]
\[ q \times \partial_r \pi = \frac{\partial_r \pi}{1 + \lambda^2 \pi^2} = \nabla_r \pi \quad (1.48) \]

From these discussions, I can conclude that an arbitrary chiral invariant lagrangian reduces to the form

\[ \mathcal{L} = \mathcal{L}'(\nabla \pi, \psi, \nabla \psi) \quad (1.49) \]

On the other hand, I have just shown that these covariant derivatives transform similarly to an iso-spin type transformation (1.28) \((I=-\frac{1}{2} \text{ for } \nabla_r \psi, \text{ for } I=1 \text{ for } \nabla_r \pi)\). Therefore if \(\mathcal{L}'\) in (1.49) is constructed in an iso-spin invariant way out of these arguments, then it is already invariant under the full chiral SU(2) x SU(2). So a general rule of constructing an invariant lagrangian is the following: Take any iso-spin invariant lagrangian which depends on the only through its derivatives. (The discussion above, in particular \(\partial \pi (I) = 0\) shows that there can be no explicit dependence on \(\pi (I)\). This excludes for example, \(-\) mass term in an invariant
la_{\text{Lagrangian}} and replace the derivatives of the fields by corresponding covariant derivatives.

Thus rule of course applies to the system with "pions" and any number of fields with arbitrary isospin.

\textbf{Currents} (1)

a) \textbf{Variational Method}

In the current algebra approach to chiral symmetry, it is the vector and axial vector currents rather than the la_{\text{Lagrangian}} which are of central importance.

If the functional \((1.39)\) is taken as a field theoretical la_{\text{Lagrangian}}, the corresponding currents can be derived through Noether's theorem. (Gell-hann Levy) (14). Writing the infinitesimal parameters of local iso-spin and chiral transformation \(s_{\theta}^{(i)}\) and \(s_{\tau}^{(j)}\) respectively, the usual expansion of vector and axial vector currents are found.

\[
\mathcal{V}_\mu = -\frac{\mathcal{L}}{\delta \Phi} \bigg|_{\Phi = 0} \tag{1.50}
\]

\[
\mathcal{A}_\mu = -\frac{\mathcal{L}}{\delta \alpha} \bigg|_{\alpha = 0} \tag{1.51}
\]

where

\[\alpha \cdot \tau = \Theta \rho \alpha, \quad \beta \cdot \tau = \Theta \rho \beta\]
First, let us consider the system with $\pi'$s only. The invariant term corresponding to this is (1.39) is

\[ \mathcal{L}_{\pi} = \frac{1}{2} \frac{\Theta_{\pi} \Theta^\dagger \pi}{(1 + \lambda^2 \pi^2)^2} \]

Then the variation of $\Theta_{\pi}$ due to the space-time derivative of infinitesimal iso-spin and chiral transformation parameters are from (1.2a) and (1.4)

\[ \delta \Theta_{\pi} \rightarrow \gamma \delta \Theta_{\pi} + \text{terms proportional to } \delta \Theta_{\pi} \]  

\[ \delta \Theta_{\pi} = \frac{1}{2\lambda} \left\{ \Theta_{\pi} \delta \Theta_{\pi} (1 - \lambda^2 \pi^2) + \lambda^2 (\pi', \pi') \right\} \]

The suffixes "is" and "ch" refer to local iso-spin and chiral transformation respectively.

Thus

\[ \delta : \mathcal{L} \rightarrow \frac{1}{(1 + \lambda^2 \pi^2)^2} \Theta_{\pi} \delta \Theta_{\pi} (1 + \lambda^2 \pi^2) + \delta \pi \leftrightarrow \]
So now (1.50) and (1.51) give

\[
\frac{V_f}{\lambda} = \frac{\pi \Lambda \partial_f \frac{\pi}{(1 + \lambda^2 \pi^2)^2}}{(1 + \lambda^2 \pi^2)^2}
\]

\[
A_f = -\lambda \left\{ \frac{1}{(1 + \lambda^2 \pi^2)^2} \cdot \frac{1}{2 \lambda} \left\{ (1 - \lambda^2 \pi^2) \partial_f \cdot \pi + 2 \lambda \partial_f \cdot \partial_f \right\} \right\}
\]

(1.58)

Compare this with the bilinear form:

\[V_f^2 + A_f^2 \]

From (1.56) and (1.57) we have

\[
(V_f^2 + A_f^2) \times (1 + \lambda^2 \pi^2)^4
\]

\[
= (\pi \Lambda \partial \cdot \pi + (\lambda^2 \pi) \partial_f \pi + 2 \lambda \partial_f \cdot \partial_f \pi)^2
\]

\[
= (\partial_f \cdot \pi)^2 - (\pi \cdot \partial_f \cdot \pi)^2
\]

\[
+ \frac{(1 + \lambda^2 \pi^2)}{4 \lambda^4} (\partial_f \cdot \pi)^2 + (\lambda^2 \pi^2) \pi \cdot \pi + \lambda \tilde{f} \cdot \partial_f \pi \pi^2
\]

\[
= \frac{(1 + \lambda^2 \pi^2)^2}{4 \lambda^4} (\partial_f \cdot \pi)^2
\]

Thus

\[V_f^2 + A_f^2 = \frac{1}{4 \lambda^4} \left( \frac{(\partial_f \cdot \pi)^2}{(1 + \lambda^2 \pi^2)^2} \right)
\]

\[= \frac{1}{2 \lambda^2} L
\]

or

\[L = 2 \lambda^2 (V_f^2 + A_f^2)
\]

(1.59)
This is of the form considered by Sugawara (15, 16).

The nucleon contribution to the currents can be derived by using the rest of lagrangian (1.39).

With the same notation as above

$$\delta \Pi \Psi = \partial (\lambda \Psi \bar{\Psi}) \Psi + \delta \Pi \bar{\Psi}$$

(cf. 1.28)

and thus

$$\delta \bar{\Psi} \bar{\Psi} \Psi = \partial (\lambda \Psi \bar{\Psi}) \Psi + \delta \Pi \bar{\Psi} = (i \lambda \pi \Psi \bar{\Psi} \partial \pi / 2 \frac{1}{1 + \lambda^2 \pi^2}) \Psi + \delta \Pi \bar{\Psi}$$

$$\delta (i \bar{\Psi} \bar{\Psi}) = \frac{1}{1 + \lambda^2 \pi^2} \bar{\Psi} \Psi \bar{\Psi} \bar{\Psi} \partial \pi \Psi + \delta \Pi \bar{\Psi}$$

also

$$\delta \bar{\Psi} \bar{\Psi} \bar{\Psi} \Psi = \frac{1}{2 \lambda} \frac{1}{1 + \lambda^2 \pi^2} \left\{ \Psi \bar{\Psi} \bar{\Psi} \Psi (1 - \lambda^2 \pi^2) + 2 \lambda \pi \Psi \bar{\Psi} \bar{\Psi} \Psi \right\} \bar{\Psi} \bar{\Psi} \Psi + \delta \Pi \bar{\Psi}$$

Thus the nucleon part of axial current

$$\partial \bar{\Psi} \bar{\Psi} \Psi = \frac{\delta \Pi}{\delta \bar{\Psi} \bar{\Psi} \Psi} \bigg|_{\lambda = 0}$$

$$= \frac{1}{1 + \lambda^2 \pi^2} \bar{\Psi} \Psi \bar{\Psi} \bar{\Psi} \Psi + \frac{\lambda^2 \pi^2}{1 + \lambda^2 \pi^2} \bar{\Psi} \bar{\Psi} \Psi \bar{\Psi} \bar{\Psi} \Psi \bar{\Psi} \bar{\Psi} \Psi \bar{\Psi} \bar{\Psi} \Psi$$

(1.60)
For the iso-spin transformation, similar arguments lead to

\[
\delta s_i(c \psi \bar{c} \psi) = - \frac{1 - \lambda^2 \pi^2}{1 + \lambda^2 \pi^2} \phi(T^a c, c) \xi^a + \phi
\]

\[
\delta s_i(c \psi + I c \pi) = - \frac{1 - \lambda^2 \pi^2}{1 + \lambda^2 \pi^2} \phi(T^a c, c) \xi^a + \phi
\]

and thus

\[
\psi' = \frac{1 - \lambda^2 \pi^2}{1 + \lambda^2 \pi^2} \phi(T^a c, c) \xi^a + \phi
\]

\[
+ \frac{\lambda^2 \pi^2}{1 + \lambda^2 \pi^2} \phi(T^a c, c) \xi^a + \phi
\]

\[
- \left( \frac{\lambda^2 \pi^2}{1 + \lambda^2 \pi^2} \right) \phi(T^a c, c) \xi^a + \phi
\]

In the lowest order of \( \pi \), (1.59) and (1.60) reduce to

\[
A \psi' = \frac{f}{f_0} \phi \xi^a \sigma \sigma \pi c \xi^a + \phi
\]

\[
\psi' \sim \phi \xi^a \sigma \sigma \pi c \xi^a + \phi
\]

(1.61) gives the axial vector coupling constants to nucleon as

\[
\frac{G_A}{G_V} = \frac{f}{f_0}
\]

Thus \( G_A/G_V \) can be accounted by \( \pi^- \pi^- \) contact interactions in (1.39) (or rather approximate (1.1)) which in fact dominate \( \pi^- \pi^- \) scattering lengths. This is
essentially the Adler-Weissberger relation as have been noted by Weinberg \(^1\). Further, the pionic part of axial current \((1.57)\) gives the axial vector coupling to single pion as

\[
\frac{F_\pi}{f} = -\frac{1}{2} \lambda = -\frac{1}{2} \frac{\mu_\pi}{f},
\]

\[
\therefore \quad \frac{G_A}{G_{1\nu}} = \frac{f}{f_0} = -\frac{F_\pi}{\mu_\pi}
\]

On the other hand \(f/\mu_\pi\) is the coefficient of derivative type Yukawa coupling in \((1.1)\). Resulting nucleon-pole term in \(\pi\) - \(\pi\) scattering amplitude (with tree-graph only) is used to define \(N - \pi\) coupling constant \(\alpha\) as

\[
\frac{\alpha}{2m_N} = \frac{f}{\mu_\pi}
\]

and thus

\[
\frac{G_A}{G_{1\nu}} = -\frac{\alpha}{2m_N} \frac{F_\pi}{\mu_\pi}
\]

\((4, 7)\)

This is Goldberger-Treiman relation. \((\text{Weinber-Weis and } \text{Zumino})\)

It has been shown that pion Lagrangian \((1.52)\) can be written in current-current form. \((\text{Sugawara type})\).
Fairlie (15, 15) has shown that the same expression holds for the entire $\mathcal{N}$-a lagrangian if certain simple terms are added to (1.39). The original derivation using directly the expression (1.56), (1.57), (1.59) and (1.60) involves lengthy arithmetic, so I shall leave the proof until the next chapter where the convenient expression for the current is obtained first. Here I shall state only the result. Using the expression of total currents

$$ A_\tau = \Phi \delta_\tau \left\{ \frac{1}{1+\lambda^2 \pi^2} - \Phi \lambda \pi \right\} \psi $$

$$ + \int_0^1 \Phi \delta_\tau \left\{ \frac{1}{1+\lambda^2 \pi^2} - \frac{\lambda^2 \pi^2}{2} + \frac{\lambda^2 \pi}{1+\lambda^2 \pi} \right\} \psi $$

$$ - \frac{1}{1+\lambda^2 \pi^2} \frac{1}{2\lambda} \left\{ (1-\lambda^2 \pi^2) \Phi \lambda \pi + 2 \lambda^2 (\Phi \lambda \pi) \pi \right\} \psi $$

and

$$ \Psi_{\psi} = \Phi \delta_\tau \left\{ \frac{1}{1+\lambda^2 \pi^2} - \frac{\lambda^2 \pi}{2} + \frac{\lambda^2 \pi}{1+\lambda^2 \pi} \right\} \psi $$

$$ - \int_0^1 \frac{\lambda^2 \pi}{1+\lambda^2 \pi^2} \Phi \delta_\tau (\Phi \lambda \pi) \psi $$

$$ + \left( \frac{\lambda^2 \pi}{1+\lambda^2 \pi^2} \right)^2 \frac{1}{2\lambda} \left( \frac{\lambda^2 \pi}{1+\lambda^2 \pi^2} \right)^2 \psi $$

it can be shown that

$$ A_\tau + \Psi_\psi = \frac{1}{\lambda^2} \left( L - i \Phi \delta_\tau \psi + (\Psi_\psi \psi) + (\phi \delta_\tau \psi) \right) \psi $$

$$ + \left( \frac{\lambda^2 \pi}{1+\lambda^2 \pi^2} \right)^2 \left( \frac{\lambda^2 \pi}{1+\lambda^2 \pi^2} \right)^2 \psi $$

Since the addition of non-derivative term

$$ 2 \lambda \left\{ (\Psi_\psi \psi \psi \psi) + (\phi \psi) \right\} \left( \Phi \delta_\tau (\Phi \lambda \pi \psi) \right)^2 $$

(1.68)
to the Lagrangian does not change the form of the currents, the new Lagrangian

\[ \mathcal{L}' = \mathcal{L} + 2\lambda \left\{ (\bar{\psi} \gamma_5 \psi)^2 + \left( \frac{\mu}{g_s^2} \right)^2 (\bar{\psi} \gamma_5 \not{\mathbf{U}} \psi)^2 \right\} \]  

(1.66)

is of Sugawara form:

\[ \mathcal{L}' = 2\lambda (A_\mu^2 + V_\mu^2) + \text{chiral invariant fermionic term of nucleon}. \]  

(1.76)

It has been resorted that the 4-point contact interaction of the type (1.67) can be useful in understanding high-energy nucleon-nucleon interaction (17) to relate this idea with the current-current form of Lagrangian in attractive and even certain numerical success has been achieved (16).

According to the authors of (Ref. 17), differential cross section for elastic scattering for high-energy, large momentum transfer becomes proportional to \( G_M^2(t) \), where \( G_M(t) \) is the proton magnetic form factor. They suggest the following general form:

\[ \frac{d\sigma}{d t} = \left( \frac{d\sigma}{d t} \right)_{t = 0} \left[ a G_M^2(t) + \beta(t) \left( \frac{s}{s_0} \right)^{s(t) - 1} \right]^2, \]

where \( s \) is constant and \( s(t) \) is low-energy behavior. In their analysis, it was found

\[ q \sim 0.85 \pm 0.15 \]
Suppose that we compute the amplitude quite naively, according to the S-matrix interaction (1.6) (small $\alpha$). It is found in fact that this gives the corresponding differential cross-section

$$\frac{d\sigma}{dt} = \left(1 + \left(\frac{f}{f_0}\right)^2\right) \left(\frac{g}{\pi\rho}\right)^2 G_{MP}^4(-t)$$

In the region $s > -t > M_N^2$

$$\frac{d\sigma}{dt} \sim \left(\frac{d\sigma}{dt}\right)_{t=0} a^2 G_{MP}^4(-t)$$

where

$$a = \sqrt{2} \left(1 + \left(\frac{f}{f_0}\right)^2\right)^{-1/2} \sim 0.8$$

On the other hand, the amplitude itself from (1.6) is real while the naively calculated amplitude has a large imaginary part. The obvious difficulty here is that these techniques with phenomenological input cannot be used in the region where the restriction due to the unitarity is important. At this stage, there is no really convincing way of unifying chiral la. ran. i.e. result.

(i) Divergence equation, PCAC and symmetry breaking.

In the current algebra approach, the divergence equations of vector and axial vector currents are most important. It is through the partial conservation of axial currents (PCAC) which connect the
divergence of axial currents with "interpolating fields" of \( \pi \)-mesons that we can deduce physical prediction from the current commutation relation. It has even been shown \((19, 20)\) that these divergence equations in the presence of certain vector fields which act as a perturbative factor to the strongly interacting system (modified C.V.C and PCAC) are sufficient to reproduce most of the physical prediction from current algebra.

I have just introduced the vector and axial vector currents through the variational principle applied to the completely chiral invariant lagrangian \((1.35)\). With respect such a variation of the field variables with infinitesimal parameter \( \theta (1) \) (stands for both \( \delta \mathcal{A}(1) \) and \( \delta \mathcal{B}(1) \)), the Euler-Lagrange type equation holds (Gell-Mann, Levy Ref.14)

\[
\mathbf{i} \frac{\delta \mathcal{L}}{\delta \theta} = \delta \mathcal{L} \tag{1.71}
\]

This comes from equations of motion for field variables and is quite independent of the invariance of the Lagrangian under the local transformations considered here.

From \((1.71)\), it can be seen immediately that the currents defined by \((1.50)\) and \((1.51)\) satisfy the divergence equations
Thus for the invariant lagrangian (1.35), we have the conservation equation

$$\partial_t V^r = 0$$  \hspace{1cm} (1.74)  

$$\partial_t A^r = 0$$  \hspace{1cm} (1.75)

To discuss the relation like FCAC, we should take account of symmetry breaking. I want to leave the more thorough discussion of chiral symmetry breaking till later when I should discuss chiral SU(3)xSU(3) symmetry where the symmetry breaking is essential. Here I merely follow Weinberg by asserting that the symmetry breaking should be introduced as a generalized form of ion mass terms which transform like 4th component of chiral 4-vector - (1/2, 1/2) representation of chiral SU(3)xSU(2). (I have already mentioned that non-zero mass of ions necessitates the introduction of non-invariant ion-mass term into the lagrangian).

From the discussion of this, a single candidate for such ion mass term will be
\[ L'_{\mathbf{\Phi}} = (C \Phi) - \frac{C}{2\lambda} \frac{2\pi^2 - 1}{1 + \lambda^2 \pi^2} \]

\[ \text{constant} \]

This is the action or \( C \) which can be written as:

\[ L'_{\mathbf{\Phi}} = \frac{C}{2\lambda} \frac{2\pi^2 - 1}{1 + \lambda^2 \pi^2} \]

\[ = L'_{\mathbf{\Phi}^*} + \frac{C}{2\lambda} \]

\[ L'_{\mathbf{\Phi}^*} \text{ hence must be given the positive sign} \]

\[ \frac{1}{2} (2C \lambda) \pi^2 \]

or in terms of a fundamental field:

\[ 2C \lambda = -\mu^2 \]

and

\[ L'_{\mathbf{\Phi}^*} = -\frac{\mu^2}{2} \frac{\pi^2}{1 + \lambda^2 \pi^2} \]

\[ \sim - \frac{\mu^2}{2\lambda} \Phi \]

Under the presence of the additional term, the axial vector will not be conserved.

\[ \gamma \mathbf{A} = - \frac{\delta L}{\delta \Phi} \bigg|_{\Phi = 0} = \frac{\mu^2}{2\lambda} \frac{\delta \Phi}{\delta \lambda} \bigg|_{\Phi = 0} \]

\[ \text{Eq. (1.1.1)} \]

\[ \delta \Phi = \delta x \cdot \Phi \]

\[ \frac{\delta \Phi}{\delta \alpha} = \Phi \]

with \( x \) extrinsic or of t.a.

\[ \gamma \mathbf{A} = \frac{\mu^2}{2\lambda} \Phi \]
By the identification of $\lambda$ with ion decay constant introduced in the last section, this can be written as
\[ \partial_{\mu} A^\mu = \mu^2 F_{\pi} \Phi \] (1.78)
The linear fields $\Phi$ are related to original $\pi$ fields through (1.14) and (1.76) in terms of
\[ \partial_{\mu} A^\mu = \mu^2 F_{\pi} \frac{\pi}{1 + \lambda^2 \pi^2} \] (1.79)
which is the generalization of conventional FCAC equation.
Of course, if we want to stick to the ordinary form of exact FCAC, I can redefine physical pion fields by
\[ \pi' = \frac{\pi}{1 + \lambda^2 \pi^2} \]
which makes (1.73) into
\[ \partial_{\mu} A^\mu = \mu^2 F_{\pi} \pi' \] (1.50)

With the addition of symmetry breaking term $L_{\mu^2}$, the lagrangian (1.59) becomes fully equivalent to the approximate form (1.1) so far as $\pi - \pi$ interaction is concerned.
§7 The relation with the current algebra. Canonical field theory. (21, 22).

From the discussion in the preceding paragraph, the affinity of non-linear realization techniques to the current algebra approach is clear. But to show that this sort of lagrangian can be actually used as a field theoretical model to the current algebra, certain complication should be met. The apparent difficulty here is that we should now take the operator nature of fields variables like $\pi_i$ seriously and owing to the non-commutativity of boson fields and their canonical momenta the standard argument of Gell-Mann and Levy (14) might not apply unless a careful consideration of ordering of field operators are taken. For the particular lagrangian (1.52), it can be seen easily that if we write it as

$$\mathcal{L} = \frac{1}{\lambda} \left(1 + \lambda^2 \pi_j \pi_i \right)^2$$

(1.81)

rather than

$$\nabla_{\tau} \pi \nabla^{\tau} \pi$$

and define the canonical momenta of $i$ in the usual way

$$\mathcal{P}_i = \frac{\delta \mathcal{L}}{\delta \frac{\partial}{\partial t} i}$$

(1.82)
also the currents with respect to the infinitesimal transformation
\[ \pi \rightarrow \pi + \delta \pi (\theta) \]
as
\[ j_\mu = -\frac{\partial L}{\partial (\partial_\mu \theta)} \bigg|_{\theta=0} \] (1.83)
then the ordinary operator form of field transformation
\[ \delta \pi (\theta) = -\left[ \mathcal{L}, \theta \right] \pi \] (1.84)
with
\[ \mathcal{L} = \int j_\mu (x) d^4x \] (1.85)
is obtained.

In fact, it has been shown by Isham\(^{(21)}\) that (1.81) in which now the position of the derivative \( \partial \pi \) is important (the pion field is no longer a \( \mathcal{C} \) number) is still invariant under the transformation of \( \pi \)'s (1.4). His argument is very general and applies to wider class of meson-lagrangian. After this, we may derive the ordinary current algebra commutation relation among the vector and axial vector currents. The form of Schwinger term is exactly specified.

Now at this point a rather interesting problem arises. Since we have now the currents in operator form we may consider the spectral representation of them. In particular, we may try to derive the spectral function
It can be shown that if we use the commutation relation with Schwinger terms derived from the pionic lagrangian (1.81), then

\[ \int \frac{\rho_V'(a)}{a} d a + \int \frac{\rho_A'(a)}{a} d a = 0. \]  

(1.86)

where \( \rho_V' \) and \( \rho_A' \) are the vector part of the spectral function of vector and axial vector currents respectively. They are, of course supposed to be positive definite and (1.86) implies \( \rho_V = \rho_A = 0 \). This is the contradiction first proposed by Jackiw\(^{(24)}\) but should be considered as the indication that the self-consistent model requires at least vector and axial vector fields as an independent degree of freedom. This is connected to the problem of gauge fields in non-linear realization techniques which I shall discuss in the next chapter.

In the presence of such gauge fields, we can construct a model of the field algebra type and then the derivation of current commutation relation can be made very simple. The problem of quantizing the non-linear lagrangian discussed in this paragraph is very clearly treated in the paper by Barnes and Isham\(^{(22)}\).
CHAPTER 2

General theory of non-linear realization of chiral group

§1 The method of Coleman-Wess-Zumino and Isham

In this chapter we give the general formalism of the non-linear realization theory.

The main idea of the construction given here is first applied by Weinberg to the use of the chiral SU(2)×SU(2) group. The generalization to the case of K×K where K is any compact, simply connected lie group has been done by Coleman, Wess and Zumino (5), although the most physically important ideas are already to be found in the earlier paper by Cronin (2). The mathematics employed by these authors reduces to the powerful techniques developed by Mackey (25). The mathematical aspects of non-linear realization techniques have been fully investigated by Isham (9). Salam and Strathdee (26) have treated a similar problem in less abstract level but in an intuitively appealing manner. It is surprising that this "theory of induced representation" by Wigner and Mackey is found to be relevant in such wide range of problems in quantum physics.
In my presentation of the material, I naturally emphasize the various relations and formulae which are necessary for actually constructing chiral SU(3) symmetric dynamical model, and skip over the most of the mathematics needed for considering the problem in its full generality.

Although most of the isolated formulae presented here are derived by the author independently the general formalism closely follows that of Coleman, Wess and Zumino\(^{(6)}\) in the way presented by Coleman and Zumino at Erice Summer School, Sicily 1968. I was also influenced by an attractive presentation put forward by Salam and Strathdee\(^{(26)}\).

Chiral group \(\mathbf{H} = \text{SU}(3) \times \text{SU}(3)\) can be regarded as a Lie group associated with the Lie algebra with elements \((A_i, V_i)\) and with the commutation relation among them

\[
\begin{align*}
\left[ V_i, V_j \right] &= -C_{ijk} V_k \\
\left[ A_i, V_j \right] &= -C_{ijk} A_k \\
\left[ A_i, A_j \right] &= -C_{ijk} V_k
\end{align*}
\]  

(2.1a)  

(2.1b)  

(2.1c)

where \(C_{ijk}\) is the structure constant of \(\text{SU}(3)\) and can be taken as real, totally antisymmetric. The
commutation relations given here are just the well known Gell-Mann relation fundamental in Current Algebra and the letter A, V have obvious implication with respect to space-reflection operation. (Note that commutation relation above differs by the factor i from the usual one).

An arbitrary element of chiral SU(n)xSU(n) group is characterized by 2N real co-ordinates \((\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N)\) and in a neighbourhood of identity element corresponding to \((\alpha_i = 0, \beta_i = 0)\), they can be expressed as the exponential

\[
\phi = e^{2\xi A_i + \sum \beta_i V_i}
\]

For our purpose it is enough to think of a group element as realized as an element of the group of continuous transformations in a certain manifold and thus and in the above expression should be interpreted as the operator operating in this same manifold.

The crucial point for the construction of Coleman, Wess and Zumino\((8)\) is that the any element \(g\) of can be uniquely decomposed as

\[
g = e^{2\xi A_i} e^{\sum \beta_i V_i}
\]

(2.3)
that is to say the product of the element of "diagonal subgroup \( SU(n) \) (I shall frequently denote it by \( K \); \( H \subset K \)) and the elements which are characterized by

\[ \beta_\zeta = 0 \]

in our way of representation. (I may call it the "chiral part" of \( K \)). Using (2.3), we consider the decomposition of particular elements

\[ g e^{3A} = e^{3'A} e^{\eta'} \quad (2.4) \]

where \( g \) is an arbitrary element of \( K \), and \( \zeta \) are real numbers.

If the decomposition (2.3) is unique, we can consider (2.4) as defining \( \zeta' \) and \( \eta' \) as the function of group element \( g \) as well as the quantity, and we can write

\[ \zeta' = \zeta'(\zeta, g) \quad (2.5) \]

\[ \eta' = \eta'(\zeta, g) \quad (2.6) \]

On the other hand, we can consider (2.5) as the definition of the operation of the element \( g \) of the group \( K \) realized as the transformation of real number field \( (\xi) \) by putting

\[ g \xi = \xi' \quad (2.7) \]
Of course, we need a consistency condition for group operation
\[ g_1 (g_2 \mathcal{G}) = (g_1 g_2) \mathcal{G} \]  
for any \( g_1, g_2 \in K_n \). This can be proven trivially (Coleman-Weiss-Zumino Ref.6).

\[ \therefore g_2 \mathcal{G} = \mathcal{G} \]  
is defined by the help of (2.4) as
\[ g_2 e^\mathcal{E}A = e^{\mathcal{E}A} e^{\mathcal{G}'} \]  
(a)

Similarly, \( g_1 (g_2 \mathcal{G}) = \mathcal{G}'' \) is defined by
\[ g_1 e^{\mathcal{E}A} = e^{\mathcal{E}''A} e^{\mathcal{G}''} \]  
(b)

On the other hand, by the associative law of group elements
\[ (g_1 g_2) e^{\mathcal{E}A} = g_1 (g_2 e^{\mathcal{E}A}) \]
\[ = g_1 e^{\mathcal{E}A} e^{\mathcal{G}'} = (g_1 e^{\mathcal{E}A}) e^{\mathcal{G}'} \]
\[ = e^{\mathcal{E}''A} e^{\mathcal{G}''} = e^{\mathcal{E}''A} (e^{\mathcal{G}''} e^{\mathcal{G}''}) \]

Since the diagonal elements (\( e^{\mathcal{E}''} \)) form a subgroup \( H \) we have
\[ e^{\mathcal{G}''} e^{\mathcal{G}''} = e^{\mathcal{G}''} e^{\mathcal{G}''} \in H \]
and we set
\[ (g_1 g_2) e^{\mathcal{E}A} = e^{\mathcal{E}''A} e^{\mathcal{G}''} \]
Assuming the uniqueness of the decomposition (2.4), this should be identical with the equation defining
\[(g_1 g_2) \xi = \xi''\]
as
\[g_1 g_2 e^{g^A} = e^{g''^A} e^{g''^V}\]
\[
\xi'' = \xi'', \quad \eta'' = \eta''
\]
That is to say
\[g_1 (g_2 \xi) = (g_1 g_2) \xi\] (CEL)

When (2.4) and (2.7) are considered as defining the operation of \(\xi\) in the field of quantity \(\xi = (\xi_1 \cdots \xi_N)\), the analogy with Wigner's construction becomes apparent. Salam et al. (2.4) and (2.7) give, in particular,
\[e^{g^A} (0) = \xi\] (2.9)
and
\[h(0) = (0) \text{ for any } h \in H.\] (2.10)

It is also obvious that the second equation holds only for the element of \(H\), and that \(h\) can be characterized as the _group_ of elements \(h(0)\) (in \(\xi\)-field) invariant.
Assuming no. the operation of \( K \) in field (2.7) together with (2.9) and (2.10) is known beforehand, we can write an arbitrary element of \( \mathfrak{g} \) of \( K \) as

\[ q = e^{\mathfrak{s}'A} (e^{-\mathfrak{s}'A} g e^{\mathfrak{s}A}) e^{-\mathfrak{s}A} \]

where \( \mathfrak{s}' = q \mathfrak{o} q^{-1} \).

Then, by (2.9), the element \( e^{-\mathfrak{s}'A} g e^{\mathfrak{s}A} \) leaves (0) in the \( \mathfrak{s}' \)-field invariant, and thus belongs to \( K \). Writing \( e^{\mathfrak{y}V} = e^{-\mathfrak{s}'A} g e^{\mathfrak{s}A} \), we have

\[ q = e^{\mathfrak{s}A} e^{\mathfrak{y}V} e^{-\mathfrak{s}A} \quad (2.11) \]

This is just recovering (2.4), but presented in this way, it is a analogy of the construction of the Wigner rotation in studying the inhomogeneous Lorentz group (27).

In the case of chiral \( SU(2) \times SU(2) = \mathbb{R}_2 \), the relation (2.4) has been already introduced in (1) by an explicit construction (1.24). In the context of the more general treatment of the present chapter the importance of the parameter \( \mathfrak{s} \) is apparent.

The apparent resemblance should not let us think that the mathematics in both cases of chiral \( SU(N) \) and the Poincare group are identical in the simple way hinted here. In the case of the latter, we are
given the four dimensional momentum space, as the homogeneous space of the group operation from beginning. The invariant subgroup consisting of the translations gives a label to the representations we are looking for. This is "mass". It is after determining this characteristic of the representation that we start constructing the Wigner Rotation. In the case of chiral SU(\(\mathcal{N}\))xSU(\(\mathcal{M}\)), the lack of "momentum" in the proper sense makes the construction of the Wigner rotation in (2.11) more like an attractive way of presenting the decomposition theorem (2.4), and its power for studying further mathematical structures seems to be limited.

Yet, it may be possible to make this analogy more profound and useful by annexing to \(K\)'s structure something like a translation group. Then we will have to define the "orbit" in such a space which reduces to the manifold of \(S = (\mathfrak{g}_1 \cdots \mathfrak{g}_n)\) introduced above.

In case of \(K_\zeta\), we have already seen how to "embed" the space of "pi-mesons" or \((\mathfrak{g}_1 , \mathfrak{g}_2 , \mathfrak{g}_3)\) in the space of 4-dimensional representation \((\zeta , \zeta)\) of \(K_\zeta\) (Chapter 1). The orbit condition in this case is, using the notation of Chapter 1,

\[
\Phi_4^2 + \sum_{z=1}^{3} \Phi_z^2 = \frac{1}{4\Lambda^2} = \left(\frac{m_\zeta}{2f_\zeta}\right)^2 \quad (cond).
\]
It should be noted that for higher \( n \), we will need the space of very high dimension. For chiral SU(\( n \))xSU(3), for instance, 16 dimensional momentum space will be a choice \( (2.8) \).

Returning to the equation \( (2.11) \), the obvious next step is to construct an "auxiliary" representation in an analogy with the construction of generalized spinor in the case of the relativistic free particle system \( (2.7) \). First of all, I must introduce "the particle multiplet" which is just the irreducible representation of \( H \).

We denote it as the set of "field operator" \( \Psi_a \), on which the action of \( h \in H \) is defined as

\[
(h \Psi)_a = \mathbb{D}_{h \rightarrow h} \Psi_\beta , \quad h \in H \quad (2.12)
\]

\( \mathbb{D} \) is the matrix belonging to an irreducible representation of \( H \) transforming as \( h \).

To define the operation of whole group \( K_n \) on \( \Psi_a \) we consider \( \Psi_a \) and an element of \( \Xi \) field together as

\[
\overline{\Psi}_a(\Xi) = (\Psi_a , \Xi) \quad (2.15)
\]

Then, remembering the definition of the "Wigner Rotation" \( (2.4) \), we define the transformation of the quantity \( \Phi_a(\Xi) \) as
\[(g \psi)_\alpha (\xi) = (D_\beta (e^{\xi_\nu}) \psi_\beta, \xi') \quad (2.14)\]

where \(\xi'\) and \(\xi'\) are defined by (2.4). It is easy to see the transformation law

\[(g \psi)_\alpha = D_\beta (e^{\xi_\nu}) \psi_\beta = (e^{i\xi_\nu})_{\alpha \beta} \psi_\beta \quad (2.15)\]

is consistent as a group operation. The matrices \((\hat{t}_i)_{\xi z}^\nu\) are the generator matrices corresponding to the irreducible representation of \(SU(n)\) belonging to \(\psi_\alpha\).

The relations (2.7), (2.15) and (2.4)

\[g \xi = \xi' \]
\[g \psi_\alpha = (e^{i\xi_\nu})_{\alpha \beta} \psi_\beta \]

with \(g e^{g A} = e^{g A} e^{g A}\)

are fundamental for non-linear realization techniques for chiral \(SU(n)\) symmetry.

These relations are, of course, the result of the unique decomposition (2.4) and in analogy with the Wigner decomposition, boost our "auxiliary representation".

Now I write (2.15) as

\[\psi_\alpha \rightarrow D_\beta (e^{g A} e^{g A}) \psi_\beta \quad (2.16)\]

Take the arbitrary representation of \(k\) which when
restricted to diagonal subgroup $SU(n)$ contains
this particular representation of $SU(n)$ spanned by
$\psi'$. Then, denoting the matrices associated with
this representation of $K_a$ in a suitable co-ordinate
system as $D_{n\beta}(g)$, I can factorise the matrix element
appearing in (2.16) as
\[
D_{\alpha\beta}(e^{-\frac{i}{2}A} g e^{\frac{i}{2}A})
= \sum_{\alpha, \beta} D_{n\alpha}(e^{-\frac{i}{2}A}) D_{s\beta}(g) D_{s\beta}(e^{\frac{i}{2}A})
\] (2.17)
where $n, s$ correspond to a complete set of base
vectors in the vector space carrying the representation
$D$, while $\alpha, \beta$ corresponds to the identification of
the subset of these bases to the vectors carrying the
representation $G$ of $SU(n)$ as assumed above.

Let us define the new quantity $\Omega_n$ by
\[
\Omega_n = D_{n\alpha}(e^{i\theta A}) \psi_{\alpha}
\] (2.18)
Then, under the action of $g \in K_a$, I have from (2.7),
(2.15) and (2.4)
\[
\Omega_n \rightarrow D_{n\alpha}(e^{i\theta A}) (g \psi_{\alpha})
= D_{n\alpha}(e^{i\theta A}) D_{s\alpha}(e^{-\frac{i}{2}A} g e^{\frac{i}{2}A}) \psi_{\beta}
\]
From (2.17), the last expression is equal to
\[ D_{n_0}(e^\mathbf{33}^A) D_{n_1}(e^{-\mathbf{33}^A}) D_{n_2}(g) D_{s_3}(e^{3/4}) \Psi_{\beta} = D_{n_3}(g) D_{s_3}(e^{3/4}) \Psi_{\beta} - D_{n_3}(g) \mathbf{\Psi}_3 \]

Thus \( \mathbf{\Psi}_n \) transforms as the given linear representation of \( K \)

\[ \Phi \Psi_n = D_{n_3}(g) \Phi \Psi_3 \quad (2.19) \]

This shows that we can obtain out of an arbitrary linear representation of \( K \) provided it does contain the representation of \( \text{SU}(\mathfrak{m}) \) spanned by \( \Psi_\alpha'^{\mathfrak{m}} \).

The converse is also true, we can ask what sort of representation of \( K \) can be constructed as the function

\[ \mathbf{U}_n = \sum_{\alpha} F_{n\alpha}(g) \Psi_\alpha \quad (2.20) \]

where \( g \) and \( \Psi_\alpha \) obey the transformation rule (2.7) and (2.15) with (2.4).

We should transform according to some given representation of \( K \)

\[ \mathbf{U}_n \stackrel{g}{\longrightarrow} D_{n_3}(g) \mathbf{U}_3 \quad (2.21) \]

Then it can be shown that the necessary and sufficient condition for (2.21) being realised by suitably choosing \( F(g) \) in (2.20) is that the representation \( \mathfrak{d} \) of \( \text{SU}(\mathfrak{n}) \) associated with \( \Psi_\alpha'^{\mathfrak{m}} \) when restricted to the
subgroup $H = SU(n)$. The following proof is due to Coleman, Wess and Zumino\(^{(5)}\).

As the result of (2.7), (2.15), (2.4) and (2.20), the transformation law (2.21) can be written as

$$\sum_{s_{ij}} F_{\alpha \beta} (\xi_{ij}) D_{\alpha \beta} (e^{\alpha^{(j)\beta}}) \psi_{\beta}$$

$$= \sum_{s_{ij}} D_{s_{ij}} (\xi_{ij}) F_{\alpha \beta} (\xi_{ij}) \psi_{\beta}$$

choosing the element $h \in H$, and taking the particular value of $\xi_{ij}; \xi_{ij} = 0$ we get

$$F_{\alpha \beta}(0) D_{\alpha \beta}(h)$$

$$= D_{s_{ij}}(h) F_{\alpha \beta}(0) \quad (2.22)$$

Since $\sum_{\alpha} F_{\alpha \beta}(\xi_{ij}) \psi_{\alpha}$ is supposed to transform linearly under $K_n$, and also $K_n$ acts transitively on the $\xi_{ij}$-field, $F_{\alpha \beta}(\xi_{ij}) \equiv 0$ would mean

$$\sum_{\alpha} F_{\alpha \beta}(\xi_{ij}) \psi_{\alpha} \equiv 0$$

So I should look for the solution of (2.22) in which $F_{\alpha \beta}(0)$ are not identically zero. (Non-trivial solution). The $\xi_{ij}$, since $\xi_{ij}$ is irreducible, the Schur's lemma tells us that $D_{s_{ij}}(h)$, which is now the direct sum of the irreducible representations of $H$, should contain at least one representation of $H$ which is equivalent to $\xi_{ij}$. 
If $D(h)$ contains only one such representation of $h$, then the construction (2.20) is essentially unique. The expression (2.18) is an exact analogue of the "auxiliary fields" of Weinberg and Mattheus-Feldman for free relativistic particle. The coefficients $D_{\mu}(e^{\gamma})$ is taking the role of generalized spinor.

(It is formally of same form - boost matrix elements).

The construction of linear representation of $h$ together with the theorem of Coleman, Weas and Zumino described above is important when we consider the problem of symmetry breaking.

§2 Explicit expressions

The three equations (2.7), (2.15) and (2.4) given in (§1) is fundamental to all the results of the present chapter. First I must show that we can determine the explicit form of the group operation on the field of $\mathcal{F}'$ and $\psi'$. I shall do it only for the infinitesimal element of $K_n$ by these equations.

For this purpose, I regard the group $K_n$ realized as a linear representation in a certain vector space. Then we can consider the A's and V's as anti-hermitian matrices. Writing them (as matrices) $iA$ and $iV$,
the equation (2.4) reduces to the matrix formula
\[ e^{i\beta} = e^{i\gamma} e^{-i\gamma} \] (2.23)

To solve (2.23) in general, we may appeal to the formula due to Ecker-Canbell-Haussdorff. But we require the solution in all order of ε but only in the first order in the parameters of a group element, and for this, we can find the explicit solution in an elementary way.

(1) Chiral part

I consider the case
\[ q = e^{iA} \]
where is considered as infinitesimal, (2.23) becomes
\[ e^{iA} e^{-i\gamma} = e^{i\gamma} e^{-i\gamma} \] (2.24)

I look for an analytical solution only and I may consider
\[ \eta' \sim O(\epsilon) \]
\[ \delta \approx \frac{\eta' - \eta}{\epsilon} \sim O(\epsilon) \]

Thus, I may write the above formula as
\[
1 + iA = e^{i(\varphi + \delta)} e^{-i\varphi} + i e^{i\varphi} \eta' + O(\epsilon^2)
\]
Using the well known formula in elementary matrix calculations, we have, up to the first order in \( \alpha \),

\[
\hat{\text{id}} \mathcal{A} = \sum_{N=1}^{\infty} \frac{1}{N!} \left[ i^{N} \mathcal{A} \right] \quad \text{for } N \geq 1
\]

\[
+ \sum_{N=1}^{\infty} \frac{1}{N!} \left[ i^{N} \mathcal{A} \right] \quad \text{for } N \leq 0
\]

By using the commutation relation of matrices and from (1.1a) (1.1c) the \( n \)-ple commutators in R.H.S can be transformed, and we have

\[
\hat{\text{id}} \mathcal{A} = \frac{i \delta_{3} \left( E_{\mathcal{F}} + i \mathcal{F} \right)}{2} \mathcal{A}
\]

\[
+ \frac{i \delta_{3} \left( E_{\mathcal{F}} - i \mathcal{F} \right)}{2} \mathcal{A}
\]

\[
+ \frac{\eta' e^{-i \mathcal{F}} + e^{i \mathcal{F}}}{2} \mathcal{A}
\]

\[
+ \frac{\eta' e^{-i \mathcal{F}} - e^{i \mathcal{F}}}{2} \mathcal{A}
\]

where \( N \times N \) matrices \( \mathcal{F} \) is defined by

\[
(F^i)_{j,k} = -i C_{j}^{i} \delta_{j,k}
\]

and the function \( E_{1}(z) \) is defined by the series

\[
E_{1}(z) = \sum_{N=1}^{\infty} \frac{z^{N-1}}{N!} = \frac{e^{z} - 1}{z}
\]
Choosing the representation in which the matrices $\mathbf{A}$ and $\mathbf{V}$ are independent, we get the following equalities:

\[
\begin{align*}
\alpha &= \delta \left( \frac{E_i + E_j}{2} \right) + \gamma' \left( e^{iF} - e^{-iF} \right) \\
\beta &= \delta \left( \frac{E_i - E_j}{2} \right) + \gamma' \left( e^{iF} + e^{-iF} \right)
\end{align*}
\]

From these, we let

\[
\delta = \alpha \frac{e^{iF} + e^{-iF}}{E_i + E_j + E_i - E_j}
\]

and

\[
\gamma' = \alpha \frac{1 - e^{-iF}}{1 + e^{-iF}}
\]

($\alpha$) diagonal subgroup.

Consider now

\[ \mathbf{g} = e^{i\beta} \mathbf{v} \in \mathbf{H} \]

The relation

\[ e^{i\beta} \mathbf{v} e^{i\mathbf{F}} = e^{i\mathbf{F}} e^{i\beta} \mathbf{v} \]

gives instead of (2.25),

\[
\begin{align*}
0 &= \delta \left( \frac{E_i - E_j}{2} \right) + \gamma' \left( e^{iF} - e^{-iF} \right) \\
\beta &= \delta \left( \frac{E_i - E_j}{2} \right) + \gamma' \left( e^{iF} + e^{-iF} \right)
\end{align*}
\]

From (2.29), it is easy to see that up to the first order of $\beta$,

\[
\gamma' = \beta
\]
and
\[ \delta \beta = \beta \cdot (iF \cdot \beta) \]

The last equality can be written as
\[ \delta \beta = (i \beta F) \cdot \beta \quad (2.31) \]

(2.30) and (2.31) expresses the fact that under the action of subgroup \( H \), \( \beta \) field as well as \( \psi \) field transforms as linear representation (regular representation). In case of \( K_1 \) and \( K_3 \), this means that non-linear resonon fields behave as triplet and octet with respect to the \( SU(2) \) and \( SU(3) \) symmetry.

The result (2.26), (2.27), (2.30) and (2.31) is of course independent of particular representation used to calculate \( \delta \beta \) and \( \psi' \), since, apart from the independence of matrices \( A \) and \( V \), we have needed only the commutation relation between \( A' \) and \( V' \).

§3 Covariant derivatives

I have derived the transformation induced by chiral group \( K \) on the field of quantities \( \beta \) and \( \psi \). I am, of course, going to use these \( \beta' \) and \( \psi' \) as dynamical variables. More specifically, they are considered as the local "field variable" depending on
space-time $\mathcal{L}$ out of which I want to construct the lagrangian functional. But in studying their transformations, we more or less consider these quantities as $\mathcal{G}$-number function of $\mathcal{L}$. Also we take an assumption that the expression $e^{\theta_{\mu}^A}$ for arbitrary $2$ is meaningful as an element of chiral part of group $\mathcal{G}$.

To construct the lagrangian model with these quantities as field variables, we need at least the first order differential coefficients $\partial\cdot \varphi(1)$ and $\partial\cdot \varphi(1)$ together with original $\varphi_{\mu}$ and $\varphi'_{\mu}$.

Now from the non-linear transformation of $\varphi_{\mu}$ and $\varphi'_{\mu}$ under the action of an element of $\mathcal{G}$ given by (2.7), (2.15) and (2.4) of §1, it is clear that these derivatives do not transform in a simple way under the group $\mathcal{G}$.

To get the quantities which generalize these derivatives but have simple transformation under the group, we use the techniques analogous to the construction of covariant derivatives in general relativity theory.

Following Salam and Strathdee (26), I start from the following quantity

$$\Delta_{\nu} \varphi(1) = D_{\alpha \nu} e^{\varphi_{\mu} A} \partial_{\alpha} \varphi_{\mu} e^{\varphi_{\mu} A} \varphi_{\mu}(1)$$

(2.32)
where the matrix $\mathbf{D}$ is, as in §1, of any representation of $\mathbf{K}$, which contains the representation of $\mathbf{H}$ spanned by $\Psi_{\alpha}$. It is clear, in the light of construction given in §1, $\Delta_{\gamma} \Psi_{\alpha}$ transforms according to (2.15) i.e. in the same way as $\Psi_{\alpha}$ itself.

$$\Delta_{\gamma} \Psi_{\alpha} \rightarrow \mathbf{D}_{\alpha\beta}(e^{\gamma y}) \Delta_{\gamma} \Psi_{\beta}$$  \hspace{1cm} (2.33)

(2.32) can be written as

$$\Delta_{\gamma} \Psi_{\alpha} = \xi_{\alpha} \Psi_{\alpha} + \{ \mathbf{D}_{\alpha \lambda}(e^{\gamma y}) \xi_{\beta} \mathbf{D}_{\beta \lambda}(e^{\gamma y}) \} \Psi_{\beta}$$

Now

$$\mathbf{D}(e^{-\frac{3(2y)}{2}}) \frac{\partial}{\partial y_{0}} \mathbf{D}(e^{\frac{3(2y)}{2}})$$

$$= \lim_{y \rightarrow 0} \frac{\partial}{\partial y_{0}} \mathbf{D}(e^{-\frac{3(2y)}{2}}) \mathbf{D}(e^{\frac{3(2y)}{2}}) \bigg|_{y=0}$$

Using the fundamental formula (2.4), we can write

$$e^{-\frac{3(2y)}{2}} e^{\frac{3(2y)}{2}} = e^{\frac{3(i0)}{2}} e^{\frac{3(0)}{2}}$$

with

$$e^{-\frac{3(2y)}{2}} \xi_{(2+y)} = \xi_{(0)}$$

$$e^{-\frac{3(2y)}{2}} \Psi_{\alpha}(2+y) = \mathbf{D}_{\alpha \beta}(e^{\gamma(0) y}) \Psi_{\beta}(0)$$

I may assume $\xi_{(0)}$, $\eta_{(0)} \sim O(y)$, so that $\mathbf{D}(e^{-\frac{3(2y)}{2}})$ x $\frac{\partial}{\partial y_{0}} \mathbf{D}(e^{\frac{3(2y)}{2}})$ can be now written as

$$= i \mathbf{A} \frac{\partial}{\partial y_{0}} \xi_{(0)} \bigg|_{y=0} + i \mathbf{V} \frac{\partial}{\partial y_{0}} \eta_{(0)} \bigg|_{y=0}$$
Thus I get

\[
\Delta_x \psi_\alpha = \partial_x \psi_\alpha + i \gamma_\mu \partial_{\gamma \mu} \gamma^{(1)} \psi_\alpha \bigg|_{y=0} \psi_\beta \\
+ i A_\mu \partial_{\gamma \mu} \gamma^{(1)} \psi_\alpha \bigg|_{y=0} \psi_\beta \\
= \frac{\partial}{\partial y} \left( e^{-\frac{1}{2}(1)} \psi_\alpha (x+y) \right) \bigg|_{y=0} \\
+ i \frac{\partial}{\partial y} \left( e^{-\frac{1}{2}(1)} \psi_\alpha (x+y) \right) \cdot A_\mu \psi_\beta (x) \tag{2.34}
\]

where \( \gamma \) and \( A \) have the same meaning as in \( \S 2 \) for a given representation of \( \kappa_n, D \).

We consider the transformation of the first term of R.H.S. of (2.34) which contains \( \partial_x \psi \). Under \( K_n \)

\[
\frac{\partial}{\partial y} \left( e^{-\frac{1}{2}(1)} \psi_\alpha (x+y) \right) \rightarrow \frac{\partial}{\partial y} \left( e^{-\frac{1}{2}(1)} \gamma \psi_\alpha (x+y) \right)
\]

from (1.4)

\[
= \frac{\partial}{\partial y} \left\{ e^{\gamma(1)} e^{-\frac{1}{2}(1)} \psi_\alpha \right\}_\alpha (x+y)
\]

\[
= \frac{\partial}{\partial y} \left( \gamma \psi_\alpha \right) \tag{2.35}
\]
Thus the quantity \( \frac{\partial}{\partial y} \left( e^{g_{ij}A} \psi \right) \) transforms in the same way as \( \psi \) under \( K \). On the other hand, clearly this quantity is independent of the choice of particular "auxiliary representation" \( D \) of \( K \).

So I shall put
\[
\nabla \psi_n (y) = \frac{\partial}{\partial y} \left( e^{-g_{ij}A} \psi \right) \bigg|_{y=0} = \partial \psi_n (y) + i \gamma \psi_n \left( \frac{\partial}{\partial y} \eta^{(w)} \right) \bigg|_{y=0} \psi_n (y) = \partial \psi_n + i \hat{\gamma} \psi_n \left( \frac{\partial}{\partial y} \eta^{(w)} \right) \bigg|_{y=0} \psi_n (y)
\]
where \( \hat{\gamma} \) as the same matrices appearing in (2.15) and call it the covariant derivatives of \( \psi_n \).

Similarly
\[
\frac{\partial}{\partial y} \left( e^{-g_{ij}A} \right) \bigg|_{y=0} = \frac{\partial}{\partial y} \left( e^{\gamma_{ij}^w} e^{-A_{ij}^w} \right) \bigg|_{y=0} = \frac{\partial}{\partial y} \left( e^{\gamma_{ij}^w} e^{-A_{ij}^w} \right) \bigg|_{y=0} \]

Thus if I put
\[
\nabla \eta^w \bigg|_{y=0} = \frac{\partial}{\partial y} \left( e^{-g_{ij}A} \right) \bigg|_{y=0} = \frac{\partial}{\partial y} \left( e^{-g_{ij}A} \right) \bigg|_{y=0}
\]
then \( \nabla \eta^w \) also transform covariantly according to the
representation of \( H \) to which \( \mathfrak{g} \) belongs.

\[
\mathfrak{g}_j^j(x) \to (e^{iF^y_j y})^q \mathfrak{g}_j^j(x) \tag{2.38}
\]

These quantities \( \mathfrak{g}_j^y \), \( \mathfrak{g}_y^y \) are just the generalization of covariant derivatives derived in Chapter 1 for \( K_2 \) and they coincide with the result of Chapter 1 for \( K_2 \).

I can evaluate \( \mathfrak{g}_j^y \mathfrak{g}_y^y \) and \( \mathfrak{g}_j^y \mathfrak{g}_y^{(w)} \) explicitly as following. Consider

\[
\mathfrak{g}_j^y \mathfrak{g}_y^{(w)} = D(e^{-\frac{x}{(1+y)^4}}) \mathfrak{g}_j^y D(e^{\frac{y}{(1+y)^4}})
\]

Thus, comparing the coefficients of \( A \) and \( \mathcal{V} \), we get

\[
\mathfrak{g}_j^y \mathfrak{g}_y^{(w)} \mid_{y=0} = \mathfrak{g}_j^y \frac{E_1(iF^y_j) + E_1(-iF^y_j)}{2} \tag{2.39}
\]

\[
\mathfrak{g}_j^y \mathfrak{g}_y^{(w)} \mid_{y=0} = \mathfrak{g}_j^y \frac{E_1(iF^y_j) - E_1(-iF^y_j)}{2} \tag{2.40}
\]
Thus we obtain the following expression for the covariant derivatives

\[
\nabla \Psi_{\mu}(\tau) = \nabla \Phi_{\mu}(\tau) + i e^{\frac{e^{iF_{\mu}} - e^{-iF_{\mu}}}{2}} \left( \frac{e^{iF_{\mu}} + e^{-iF_{\mu}}}{2} \right)_{\mu} (2.41) \\
\n\nabla \Sigma_{\mu}(\tau) = \nabla \Xi_{\mu}(\tau) \left( \frac{e^{iF_{\mu}} + e^{-iF_{\mu}}}{2} \right)_{\mu} (2.42)
\]

The expression of covariant derivatives given in (2.36) and (2.37) can be readily generalized to the higher order derivatives like

\[
\nabla_{\mu_1 \ldots \mu_n} \Psi_{\mu}(\tau) = \frac{\partial^n}{\partial y_{\mu_1} \ldots \partial y_{\mu_n}} \left( e^{-\frac{\partial^2 y_{\mu_1} \Psi_{\mu}(\tau)}{e^{y_{\mu_1} \Psi_{\mu}(\tau)}}} \right)_{\tau = \infty} (2.43) \\
\n\nabla_{\mu_1 \ldots \mu_n} \Sigma_{\mu}(\tau) = \frac{\partial^n}{\partial y_{\mu_1} \ldots \partial y_{\mu_n}} \left( e^{-\frac{\partial^2 y_{\mu_1} \Sigma_{\mu}(\tau)}{e^{y_{\mu_1} \Sigma_{\mu}(\tau)}}} \right)_{\tau = \infty} (2.44)
\]

To evaluate these forms explicitly, we need the higher order expansion of the matrix like

\[e^{-\frac{\partial^2 y_{\mu_1} \Phi_{\mu}(\tau)}{e^{y_{\mu_1} \Phi_{\mu}(\tau)}}} \Phi_{\mu}(\tau)\]

used in deriving (2.41) and (2.42). The following formula proposed by Feynman(29) is useful
\[ e^{-A} e^{A+B} = T \left( \exp \int_0^1 dt \ e^{-At} B \ e^{At} \right) \]

where \( A, B \) are arbitrary \( M \times M \) matrices and \( T(\ldots\ldots) \) means ordered product with respect to the parameter \( t \). Using this \( \psi \) to second order in \( B \), I can derive, for instance.

\[
\partial_t \psi_\mu = (\partial_\mu + i A^\mu \cdot \hat{A} + i \beta^\mu \cdot \hat{\beta} + i \beta^\mu \cdot \hat{t}) \psi_\mu
\]

where

\[
\beta_\mu = \frac{1}{2} \partial_\mu \left( E_1 (iF^\mu) - E_1 (-iF^\mu) \right)
\]

\[
\beta^\mu = \frac{1}{2} \partial^\mu \left( E_1 (iF^\mu) - E_1 (-iF^\mu) \right)
\]

\[
\partial_t \frac{\partial}{\partial \phi} = \frac{1}{2} \partial_\mu \left( E_1 (iF^\mu) + E_1 (-iF^\mu) \right)
\]

+ \frac{1}{2} \left( \partial^\mu (iF^\cdot \beta) + \partial^\mu (iF^\cdot \beta^\mu) \right)

where

\[
\partial^\mu = \partial_{\phi} \frac{\partial}{\partial \phi} = \frac{1}{2} \partial_{\phi} \left( E_1 (iF^\mu) + E_1 (-iF^\mu) \right)
\]

(2.46)

Since we have not found yet how to use these higher derivatives of fields, we do not consider them any further.
§4 The case of chiral $SU(2) \times SU(2)$ (i.e.)

The fundamental relation (2.4) in case of chiral $SU(2) \times SU(2)$ can be written in terms of $(\frac{\pi}{2}, 0) \Theta (0, \frac{\pi}{2})$ representation as

$$e^{\pm i\frac{\pi}{2}} e^{\pm i\frac{\pi}{2}} = e^{\pm i\frac{3\pi}{2}} e^{\pm i\frac{3\pi}{2}}$$

for chiral part of the group. This is just the $(1, 24)$ of Chapter 1. From this, it is clear that the quantities introduced here as parameterizing the elements of chiral part of the group $(\sim \frac{\pi}{2})$ exactly corresponds to the $\mathcal{A}'$ introduced in Chapter 1 as parameterizing $\pi$-fields. In Chapter 1, $\mathcal{A}'$ are expressed as certain non-linear transform of $\pi$'s. Since, by the equivalence principle, it is possible to redefine physical pion fields with any non-linear transformation (without derivatives), we may use $\mathcal{A}'$ instead of $\pi'$ as "pion-fields" in the field theoretical calculation of $S$-matrix elements. In the present chapter, we have treated the general case of $\mathcal{H} = SU(3) \times SU(3)$. For $\mathcal{H} = 3$, we get "octets" $(\mathcal{A})^8_{\mathcal{H}}$ transforming as the $(3)$ dimensional representation of $SU(3)$. In what follows I shall use these $\mathcal{A}'$ as the physical octet pseudo scalar meson fields (which contain pions) and
will not see the "similar" form corresponding to
of Chapter 1. The problem of finding the parallel
treatment for $K_{3}$ with Weinberg's and Schwinger's
method for $K_{2}$ has been studied by Lefarlane and Weiss (3c).
Furthermore, the general results seem to be very complicated.

It is easy to see that the general non-linear

transformation formulae (2.26) and (2.27) or the
expression of covariant derivatives in this chapter
reduce to the familiar expression of Weinberg given
in Chapter 1 in the case of $SU(2) \times SU(2)$.

For instance, consider (2.27). In case of chiral

$SU(2) \times SU(2)$, this reduces to

$$
\eta' = \lambda \frac{1 - e^{i \mathcal{J} \cdot \mathcal{K}}}{1 + e^{i \mathcal{J} \cdot \mathcal{K}}}
$$

where $\mathcal{J} = (J^1, J^2, J^3)$ is the generator of $I=1$
representation of $SU(2)$ group and

$$(\mathcal{J}^\mathcal{I})_{jk} = -i \epsilon_{ijk}$$

The simplicity with $SU(2)$ is that $\mathcal{J} \cdot \mathcal{K}$ satisfies the
characteristic equation

$$(\mathcal{J} \cdot \mathcal{K})^{2} = \mathcal{J} \cdot \mathcal{K}.$$

Consequently, the odd function of $\mathcal{J} \cdot \mathcal{K}$, $\Omega(\mathcal{J} \cdot \mathcal{K}) = \frac{e^{i \mathcal{J} \cdot \mathcal{K}} - 1}{e^{i \mathcal{J} \cdot \mathcal{K}} + 1}$ can be simplified as

$$
\Omega(\mathcal{J} \cdot \mathcal{K}) = \frac{\Omega(\sqrt{\mathcal{J} \cdot \mathcal{K}})}{\sqrt{\mathcal{J} \cdot \mathcal{K}}}.
$$
From this,
\[ \mathcal{L}' = \alpha (i \vec{J} \cdot \vec{\mathcal{L}}) \frac{e^{\frac{e^{iJ}}{\sqrt{3}}} - 1}{e^{\frac{e^{iJ}}{\sqrt{3}}} + 1} \]
\[ = \alpha (i \vec{J} \cdot \vec{\mathcal{L}}) \frac{\tan(\sqrt{3}/2)}{\sqrt{3}/2} \]
This reduces to the familiar form of the "field"
is defined as
\[ \lambda \pi = \frac{\tan(\sqrt{3}/2)}{\sqrt{3}/2} \]
But this is just the parameterization introduced in
Chapter 1. In terms of J matrices,
\[ \lambda \pi \cdot \vec{J} = \frac{e^{iJ} - 1}{e^{iJ} + 1} \]
This corresponds to \( l = \frac{1}{2} \) form given in Chapter 1.
\[ \lambda \pi \cdot \vec{T} = \frac{e^{iT} - 1}{e^{iT} + 1} \]
They are, of course, all equivalent.

The correct transformation of \( \pi ' A \) given in
Chapter 1 are guaranteed because of the similarity of
(2.4) to (1.24) in case of chiral SU(2)xSU(2).
CHAPTER 3

Local chiral symmetry and gauge field

§1 The relevance of the gauge fields

This chapter is the continuation of the Chapter 2. I start with considering the problems which arise when the transformations of \( K_u \) over the field quantities are made space time dependent. This leads to the introduction of gauge fields in the ordinary way. On the other hand, for the non-linear realization of \( K_u \), this is not the only way we can introduce gauge fields. The transformation of "linear" fields \( \psi_a \), given by (2.15) looks very much like the ordinary SU(\( N \)) transformation with space-time dependent parameter \( \gamma'(t) \). This already suggests the introduction of SU(\( N \)) gauge fields. In his paper on the non-linear realization of chiral SU(\( N \))xSU(2) Weinberg has introduced vector gauge fields ("rho mesons") in this way and constructed a lagrangian which is completely invariant under the chiral transformations with constant parameters. It is rather nice to consider the vector gauge fields as arising from chiral symmetry instead of conventional local SU(\( N \)) symmetry. In the case of
the latter, the symmetry is always broken unless the mass term of the gauge fields is absent. Weinberg's construction does not require the axial vector mesons as the gauge fields. The role of the axial vector fields in a chiral symmetry scheme is not absolutely clear even forgetting the experimental uncertainty about their existence. In the application, such as a famous calculation of electromagnetic mass difference of pi-mesons, they are given an essential role\(^\text{(32)}\). But this is always tied to the soft meson approximation and to me it is not clear if the axial vector exchange diagram could not be interpreted as certain limit of pseudo-scalar meson exchange diagram\(^\text{(33)}\).

From the point of view of the consistency of chiral lagrangian method as a field theory, we have already seen in Chapter 1 that we need at least the vector and axial vector gauge fields in addition to essential \(P-S\) meson fields to avoid the contradiction of zero spectral function.

§2 The local chiral transformation

I have determined the transformation of the field quantities both "linear" and "non-linear" (as
well as their covariant derivatives) in Chapter 2.

Now I want to consider what happens when the parameters of a group element depend on the co-
ordinate \( x^\mu \). By attaching different elements of the group to each space-time point, only the
derivatives of field quantities will be affected, and thus I should determine the transformation of
covariant derivatives \( \nabla^\mu \xi \) and \( \nabla^\mu \psi \) under the local chiral transformations.

Consider the first order variation of the
field quantities \( (\xi(x) \) or \( \psi(x) ) \) under the action
of an element of local chiral group where now the
(infinitesimal) parameters of group are made dependent
of the space time co-ordinates. I may write
\[
\delta f(x) = \sum \gamma_i(x) \chi_i(x)
\]
where \( f(x) \) stands for \( \xi(x) \) or \( \psi(x) \) fields and \( \gamma_i \)
stands for the group parameters.

Then the variation of derivatives are
\[
\delta \partial^\mu f(x) = \sum \partial^\mu \gamma_i(x) \chi_i(x) + \sum \chi_i(x) \gamma_i(x)\\
+ O(\delta^2)
\]
where \( \gamma_i = \partial^\mu \chi_i \)
Writing like this, I should assume that the space-
time dependence of group parameters are smooth enough
so that we may consider \( \mathcal{B}_r \psi_i (x) \) as being "small". I call the term proportional to the derivatives of the parameters which arises from space-time dependence of the transformation the local term; \( \mathcal{S} \Theta \psi \rho \alpha \). and the term independent of the derivatives of the parameters i.e. \( \Sigma \Psi \xi \psi \), the symmetry term \( \mathcal{S} \Theta \psi \rho \gamma \), since this part has the same form as the variation under the transformation without space-time dependence.

Thus, I write in general

\[
\begin{align*}
\delta \xi \psi \rho \xi &= \left[ \delta \xi \psi \xi \rho \xi + \delta \xi \psi \xi \rho \xi \right]_{\text{sym}} \\
\delta \xi \psi \xi &= \delta \xi \psi \xi + \delta \xi \psi \xi \text{sym} \\
\delta \xi \psi \rho &= \delta \xi \psi \rho + \delta \xi \psi \rho \text{sym}
\end{align*}
\]

(3.1)

\( \delta \) signifies that the infinitesimal variation is due to the operation of group element \( g \). In particular, I am going to consider \( \mathcal{S}_{\mathcal{H}} \) and \( \mathcal{S}_{\mathcal{A}} \) due to the infinitesimal elements \( e^{2v} \) and \( e^{2A} \) respectively (with the notation of Chapter 2). \( \delta \gamma \) is, of course, given by (2.35) and (2.38), and I have to calculate only \( \delta \psi \rho \xi \) which depends on \( \mathcal{S} \xi \rho \gamma \) and \( \mathcal{S} \xi \rho \gamma \) (a)

\( \delta \xi \rho \xi \gamma (x) \)

From (2.31) and (2.26),
\[ \delta_H \xi = \beta \cdot (iF \cdot \xi) \]

\[ \delta \alpha \xi = \frac{e^{iF \xi} + e^{-iF \xi}}{E_1(iF \xi) + E_1(-iF \xi)} \]

So it can be seen that

\[ \delta_H (\partial \xi) \big|_{\partial \alpha} = \partial \beta \cdot (iF \xi) \quad (3.2) \]

\[ \delta \alpha (\partial \xi) \big|_{\partial \alpha} = \partial \alpha \frac{e^{iF \xi} + e^{-iF \xi}}{E_1(iF \xi) + E_1(-iF \xi)} \quad (3.3) \]

\( \xi \) is, of course, ordinary derivatives. Putting (3.2) and (3.3) in the expression of \( \nabla \xi \) given in (2.42), I get

\[ \delta_H \nabla \xi \big|_{\partial \alpha} = \frac{1}{2} \partial \beta (e^{iF \xi} - e^{-iF \xi}) \quad (3.4) \]

\[ \delta \alpha \nabla \xi \big|_{\partial \alpha} = \frac{1}{2} \partial \alpha (e^{iF \xi} + e^{-iF \xi}) \quad (3.5) \]

(b) \( \nabla \psi \)

Let us first examine the transformation of appearing in (2.36). I shall call this quantity after Coleman and Zumino, since this is a rather
important quantity together with $\nabla \xi$.

By substituting (3.2) and (3.3) into the expression (2.40) of $\beta_\mu$, I will get

$$\delta_H (\beta_{\mu})_{\lambda \alpha} = \frac{1}{2} \partial_\mu \beta (e^{iF^a} + e^{-iF^a} - 2)$$  \hspace{1cm} (3.6)

$$\delta_A (\beta_{\mu})_{\lambda \alpha} = \frac{1}{2} \partial_\mu \alpha \left( \frac{E_i(F^a) - E_i(-iF^a)}{E_i(iF^a) + E_i(-iF^a)} \right) (e^{iF^a} + e^{-iF^a})$$  \hspace{1cm} (3.7)

From the expression of covariant derivative (2.36), together with (2.15), I have

$$\delta_g \nabla_{\alpha} \psi_{\lambda | \kappa} = \left( i \partial_{\alpha} \xi_{\lambda | \kappa} + i \delta (\beta_{\mu})_{\lambda \alpha} \right) \cdot \hat{\nabla}_{\alpha} \psi_{\lambda}$$ \hspace{1cm} (3.8)

Substituting (3.6) or (3.7) as well as the expressions of $\gamma$ (2.30) for $\delta_H$ or (2.27) for $\delta_A$. I get

$$\delta_H \nabla_{\alpha} \psi_{\lambda | \kappa} = i \frac{\partial_{\alpha} \beta}{2} (e^{iF^a} + e^{-iF^a}) \cdot \hat{\nabla}_{\alpha} \psi$$ \hspace{1cm} (3.9)

and

$$\delta_A \nabla_{\alpha} \psi_{\lambda | \kappa} = i \frac{\partial_{\alpha} \alpha}{2} (e^{iF^a} - e^{-iF^a}) \cdot \hat{\nabla}_{\alpha} \psi$$ \hspace{1cm} (3.10)

(3.6), (3.7), (3.9) and (3.10) give the required transformation law for the covariant derivatives under the local chiral group.
§3 The gauge fields

The simple transformation law for the covariant derivatives can be recovered only by introducing a set of gauge fields.

Following Mess and Zumino\(^{(4)}\) in the case of chiral SU(2) x SU(2), I start by considering the gauge fields for the ordinary linear representation formalism of group \(K_n\). The construction of such gauge fields is well known since the work of Yang and Mills\(^{(34)}\), Gell-Mann and Glashow\(^{(35)}\). In the present case, we have set of vector and axial vector fields \((U^i_f, A^i_f)\), with Yang-Mills type transformation under the operation of the infinitesimal element

\[
q^j = e^{\rho \nu} \left\{ \begin{array}{l}
S_{\mu} U^i_f = (i \mathbf{F} \cdot \mathbf{\beta}) U^i_f + \frac{1}{2} \partial_\mu \mathbf{\beta} \\
S_{\mu} A^i_f = (i \mathbf{F} \cdot \mathbf{\beta}) A^i_f
\end{array} \right. \tag{3.11}
\]

\[
q^j = e^{a A} \left\{ \begin{array}{l}
S_{\mu} U^i_f = (i \mathbf{F} \cdot \mathbf{a}) U^i_f \\
S_{\mu} A^i_f = (i \mathbf{F} \cdot \mathbf{a}) A^i_f + \frac{1}{2} \partial_\mu \mathbf{a}
\end{array} \right. \tag{3.12}
\]

Except for the derivative term the above transformations constitute the reducible representation \((N, 1) \oplus (1, N)\) of \(K_n\).
To make them useful in non-linear scheme, I have to convert them to the quantities transforming under the rule like (2.15) with non-linear parameter \( \gamma' \).

For this, I will try to use the relation between non-linear realization and linear representation of \( \kappa_n \) discussed in Chapter 2.

I is convenient now to first construct the irreducible components as

\[
\Psi_{\mu}(L) = \Omega_{\mu} + \alpha_{\mu}
\]

(3.15)

\[
\Psi_{\mu}(R) = \Omega_{\mu} - \alpha_{\mu}
\]

(3.14)

Apart from the derivative term, and transform under the infinitesimal elements of as

\[
\begin{align*}
S_{H, V} \left[ \Psi(L) \right]_{\mu \gamma \nu} &= \left( i \mathbf{F} \cdot \beta \right) \Psi(L) \\
S_{\Delta, V} \left[ \Psi(L) \right]_{\mu \gamma \nu} &= \left( i \mathbf{F} \cdot \omega \right) \Psi(L) \\
S_{H, \Psi} \left[ \Psi(R) \right]_{\mu \gamma \nu} &= \left( i \mathbf{F} \cdot \beta \right) \Psi(R) \\
S_{\Delta, \Psi} \left[ \Psi(R) \right]_{\mu \gamma \nu} &= \left( -i \mathbf{F} \cdot \omega \right) \Psi(R)
\end{align*}
\]

(3.15)

(3.16)

From these, it can be seen that the "boost" matrix \( D(\Gamma^A) \) discussed in Chapter 2 should be taken as \( e^{\pm i \mathbf{F} \cdot \mathbf{s}} \) for \( \Psi(L) \) and \( \Psi(R) \) respectively.
Thus by "antiboosting" the linear representation $\Phi^+(L)$ and $\Phi^-(R)$ according to the discussion of Chapter 2 §1 (cf e.g. (2.18) to get a non-linearly transforming object, I am lead to define the following fields

$\chi^+_f(L)$ and $\chi^-_f(R)$

$$\chi^+_f(L) = e^{-iF^+} (U_f + a_f)$$  \hspace{1cm} (3.17)

$$\chi^-_f(R) = e^{iF^-} (U_f - a_f)$$  \hspace{1cm} (3.16)

The symmetry part of the transformation of these quantities under $g \in K$ is clearly of the type (2.15)

$$\delta_g \chi^+_f(L)_{\text{sym}} = (iF \cdot g') \chi^+_f(L)$$  \hspace{1cm} (3.19)

$$\delta_g \chi^-_f(R)_{\text{sym}} = (iF \cdot g') \chi^-_f(R)$$  \hspace{1cm} (3.20)

I have already anticipated the usual parity assignments with the fields $U_f$ and $a_f$. So the quantities $\chi^+_f(L)$ and $\chi^-_f(R)$ have no definite parity. The fields with definite parity are defined as

$$\chi^+_f \pm = \frac{\chi^+_f(L) \pm \chi^-_f(R)}{2}$$  \hspace{1cm} (5.21)
In terms of original $\chi^\pm$, i.e.,

$$
\chi^\pm = \left( \frac{e^{iF^g} + e^{-iF^g}}{2} \right) + \alpha \left( \frac{e^{iF^g} - e^{-iF^g}}{2} \right)
$$

under the action of the group $K_{\alpha}$, e.g. everyone of transformation of $\chi^\pm$ are given by (2.15) and (2.16)

i.e.

$$
\delta g \chi^\pm \big|_{\text{sym}} = (iF\cdot\gamma') \chi^\pm
$$

On the other hand, the "local" part of the transformation can be obtained by substituting (2.11) and (2.12) into (2.22) and (2.23). Then

$$
\delta H \chi^\pm |_{\alpha} = \frac{1}{4} \partial \alpha \left( \frac{e^{iF^g} \pm e^{-iF^g}}{2} \right)
$$

$$
\delta A \chi^\pm |_{\alpha} = \frac{1}{4} \partial \alpha \left( \frac{e^{iF^g} \pm e^{-iF^g}}{2} \right)
$$

but the form of (2.22) and (2.23) are exactly the additional factor appearing in the transformation of covariant derivatives under local chiral $SU(N)$ (2.6), (2.7), (2.8) and (2.10). It is non-trivial to construct the covariant quantities under the local chiral transformation. Thus if $I$ define
\[
\mathcal{D} \mathcal{S} = \nabla \mathcal{S} - g \mathcal{X}^- \\
\mathcal{D} \mathcal{Y} = \nabla \mathcal{Y} - i g (X^+ \cdot \mathcal{Y}) \mathcal{Y}
\]

(3.27) \hspace{1cm} (3.28)

then I will have

\[
\delta g (\mathcal{D} \mathcal{S})_{\mathcal{Y}^a} = 0 \\
\delta g (\mathcal{D} \mathcal{Y})_{\mathcal{Y}^a} = 0
\]

and

\[
\delta g (\mathcal{D} \mathcal{S} (\mathcal{Y})) = i \mathcal{F} \gamma'(\mathcal{Y}) (\mathcal{D} \mathcal{S} (\mathcal{Y})) \\
\delta g (\mathcal{D} \mathcal{Y} (\mathcal{Y})) = i \mathcal{F} \gamma'(\mathcal{Y}) (\mathcal{D} \mathcal{Y} (\mathcal{Y}))
\]

(3.29) \hspace{1cm} (3.30)

where

\[
\gamma'(\mathcal{Y}) = \gamma'(\mathcal{Y}, \mathcal{S} (\mathcal{Y}))
\]

The substance of the foregoing discussion is already fully realised by Wess and Zumino for $K_a$. But it is due to the simpler and more general realization discussed in Chapter 2 that this elementary derivation could be readily applied to general.

\section{Covariant

To construct the chiral invariant dynamical models with a local lagrangian functional some further
covariant quantities besides \( q_r \), \( q_f \) or \( \chi^r \) are needed.

(a) Covariant curls of vector and axial vector fields.

In constructing the lagrangian, the kinematical term of vector or axial vector fields should be modified so that it is invariant under the chiral transformations, and for that I need usual covariant curl of Yang-Mills fields. Since this kinematical term is not troubled by the presence of non-linear quantities, I may apply directly the techniques of Yang and Mills\(^{(34,35)}\) to linear fields \( (q_f, q_f) \). The irreducible part of (3.11) and (3.12) can be written as

\[
\delta \Phi_f = i E \cdot \gamma \Phi_f + \frac{1}{4} \partial \Phi_f \gamma \quad (3.31)
\]

with some group parameters \( \gamma_1 \cdots \gamma_N \). \( \Phi_f \) are the same combination of \( q_f \) and \( q_f \) which appeared in (3.13) and (3.14). The corresponding covariant curls are

\[
\tilde{\Phi}_f = [ \Phi_f, \Phi_f ]^i \quad (3.32)
\]

Writing (3.32) explicitly in term of \( q_f \) and \( q_f \), and again taking the parity eigen-vectors, I get the following covariant curls.
(b) Covariant curls with non-linear transformations.

The type of linearly transforming curls discussed above is convenient when we need not consider more complicated coupling of gauge fields and non-linearly transforming fields. But when for instance, we want to discuss the "magnetic" coupling of gauge fields to non-linear type "Daryon" fields, we need the covariant curls with similar non-linear transformations.

If the general techniques used in introducing above is applied directly, the quantities like

\[ \chi_{\mu \nu}^\pm = G_{\mu \nu} \frac{e^{if\Phi} \pm e^{-if\Phi}}{2} + G_{\mu \nu} \frac{e^{if\Phi} \mp e^{-if\Phi}}{2} \]

are obtained, and they do transform like (2.15) as required. However simpler construction is possible without further introducing non-linear factors like

\[ G_{\mu \nu} = \partial_\mu \phi - \partial_\nu \phi + g \{ \nu \gamma (iF) \nu_\gamma + q \gamma (iF) a_\gamma \} \]  

\[ G^\pm_{\mu \nu} = \partial_\mu \phi - \partial_\nu \phi + g \{ \nu \gamma (iF) \nu_\gamma + q \gamma (iF) a_\gamma \} \]
Let us consider the quantities

\[ \phi^+ = \chi^+-\frac{i}{\varrho} \beta \]  
\[ \phi^- = \chi^-+\frac{i}{\varrho} \beta \]  

From the transformations of \( \chi^+ \) given by (3.25) and (3.26) and those of \( \beta \) given by (3.6) and (3.7) under local chiral group, it can be seen that the transformation of \( \phi^+ \) under the action of the group is

\[ \delta g \phi^+(\gamma) = (\overline{\gamma} \cdot \beta' \gamma') \phi^+(\gamma) + \frac{i}{\varrho} \gamma \frac{\partial}{\partial \gamma} \]  

As for \( \phi^- \), it is just the covariant derivative \( \overline{\gamma} \) for local chiral group defined in (3.27). Thus

\[ \delta g \phi^-(\gamma) = (\overline{\gamma} \cdot \beta' \gamma') \phi^-(\gamma) \]  

If we apply Yang-Mills techniques to (3.37) then we will immediately get the covariant curl of the vector fields and this is enough for giving, for instance, the invariant magnetic moment coupling of vector fields. Nevertheless, it is suggestive to treat vector and axial vector fields in a more symmetric looking way.  

If I define the quantities

\[ \phi^+ (L) = \phi^+ + \phi^- \]  
\[ \phi^- (R) = \phi^+ - \phi^- \]
then the transformation of the fields derived from (2.37) and (2.38) are written in the form of (3.31) as

\[ S_{\phi} \hat{\Phi}_\mu = (iF_\gamma') \Phi_\mu + \frac{i}{2} \partial_\mu \hat{\Phi} \]

(3.39)

where \( \hat{\Phi} \) here stands for \( \Phi_4(L) \) or \( \Phi_4(R) \) and \( \gamma' \equiv \gamma'(\alpha, \beta) \) with (2.4) and (2.15). From these, it can be seen that the forms \([\Phi_4(L), \Phi_5(L)]\) and \([\Phi_4(R), \Phi_5(R)]\) using the notation of (3.32) are the required non-linear covariant curls.

Again taking the linear combination with definite parity, I obtain the following covariant curls

\[ G^{+}_{\mu \nu} = \partial_\mu \Phi_+ - \partial_\nu \Phi_+ + \frac{i}{2} \{ \Phi^+(iF)_\mu \Phi^+ + \Phi^-(iF)_\mu \Phi^- \} \]

(3.40)

\[ G^{-}_{\mu \nu} = \partial_\mu \Phi_- - \partial_\nu \Phi_- + \frac{i}{2} \{ \Phi^+(iF)_\mu \Phi^- + \Phi^-(iF)_\mu \Phi^+ \} \]

(3.41)

\( G^\pm_{\mu \nu} \) of course transform with the transformation law (2.15) under the local chiral \( SU(\mathcal{N}) \times SU(\mathcal{N}) \)

\[ S_{\phi} G^{\pm}_{\mu \nu} = (iF_\gamma') G^{\pm}_{\mu \nu} \]

(3.42)

The transformation law like (3.37) corresponds to the Weinberg's point of view for the gauge fields.
in a non-linear realization. (5.37) is valid independent of it the transformation is co-ordinate dependent or not. If the vector fields with (5.37) are used to merely replace non-linear $\beta_\mu$ factor in the expression of $\nabla_\mu \psi$ to keep the covariance under constant chiral transformation, this is just the Weinberg's definition of vector gauge fields.

§4 The relation between linear and non-linear forms of gauge fields.

In §3, I have derived two different types of covariant curls. It can be seen that the 'non-linear' type (3.40) and (3.41) can be obtained from conventional the linear type (3.33) and (3.34) by replacing $\phi^+, \phi^-$ with $\phi^+, \phi^-$ defined in (3.35) and (3.36).

Now I shall show that this relation between $\phi^+$ and $\phi^-$ expressed in (3.35) and (3.36) can be interpreted as a local chiral transformation. For this, I consider the infinitesimal gauge transformation (5.31) which is obeyed by irreducible components $\phi^+ \pm \phi^-$

$$S \phi_\mu = (i F^\mu_\nu) \phi^\nu_\nu + \frac{1}{g} \partial_\mu S \phi$$

(3.43)

Let us try to solve this equation for finite value of $\phi$ by integrating it. This can be done by considering
the group operation along the one parameter subgroup
\{ e^{t} \}_{t \in \mathbb{R}} \ (t \text{ is independent of co-ordinate}). Then
(3.43) will be converted to the ordinary differential equation

\[ \frac{d}{dt} \Psi_f(t) = (i \mathbf{r} \cdot \mathbf{F}) \Psi_f(t) + \frac{i}{\xi} \partial_f \mathbf{r} \]

with \[ \Psi_f(0) = \Psi_f \]

we should find \[ \mathbf{g}(\mathbf{r}) \mathbf{F} \] as \[ \Psi_f(1) \]

The matrix \( \mathbf{F} \) can be diagonalized by some unitary matrix as

\[ \mathbf{U}(\mathbf{r}, \mathbf{F}) \mathbf{U}^{-1} = \begin{pmatrix} 
\lambda_1 & & \\
& \ddots & \\
& & \lambda_n 
\end{pmatrix} \]

Putting

\[ \mathbf{U} \Psi_f(t) = \Phi_f(t) \]
\[ \mathbf{U} \mathbf{r} = \mathbf{r} \]

I have

\[ \frac{d \Phi_f^i}{dt} = i \lambda_i \Phi_f^i + \frac{i}{\xi} \partial_f r^i \]

(No summation involved)

The solution of the last differential equation is

\[ \Phi_f^i(1) = \Phi_f^i(0) + \frac{i}{\xi} \partial_f r^i \]
\[ \text{for } \lambda_i = 0 \]

\[ \Phi_f^i(1) = \Phi_f^i(0) + (e^{i \lambda_i}) (\Phi_f^i(0) + \frac{i}{i \lambda_i} \partial_f r^i) \]
\[ \text{for } \lambda_i \neq 0 \]
But the latter is regular at $\lambda_i = 0$ and reduces to the former if $\lambda_i$ goes to zero. Thus the solution of (3.45) is

$$\phi_i^o(t) = \phi_i^o(0) + (e^{i\lambda i} - 1) (\phi_i^o(0) + \frac{1}{i\lambda_i} \partial_7 \Gamma^i)$$

or

$$\phi_i^o(t) = e^{i\lambda_i} \phi_i^o(0) + e^{i\lambda_i} - 1 \frac{1}{i\lambda_i} \partial_7 \Gamma^i \quad (3.46)$$

Transforming back to $U_7$, the matrix $U_7$, I get the solution of original (3.44) as

$$\phi_i^o(t) = e^{iF_7} \phi_i^o + \frac{1}{\bar{g}} E_i (iF_7) \partial_7 \phi_i \quad (3.47)$$

where the matrix function $E_i(z)$ has appeared already in Chapter 2 and defined by the series expansion of $(e^{\bar{z}} - 1)/\bar{z}$

Thus the finite gauge transformation generated by the infinitesimal form (3.43) is

$$U_7 \xrightarrow{\partial_7} e^{iF_7} U_7 + \frac{1}{\bar{g}} E_i (iF_7) \partial_7 \phi_i \quad (3.48)$$

Applying this result to the irreducible components (3.13) and (3.14) i.e. $\phi_7^o(L) = U_7 + \phi_7$ and $\phi_7^o(P) = U_7 - \phi_7$

with the element of the chiral part of the group;

$$g = e^{-A \bar{z}^o}$$

I get

$$\phi_7^o(L) \xrightarrow{e^{-A \bar{z}^o}} e^{-iF_7} \phi_7^o(L) + \frac{1}{\bar{g}} E_i (-iF_7)(\bar{z}) \quad (3.49)$$
\[ \Psi_f(R) e^{A_{\text{fin}}} \rightarrow e^{i \vec{F}_f} \Psi_f(R) + \frac{1}{\hbar} E_l(i \vec{F}_f) \tilde{\sigma}_f \]  \hspace{1cm} (3.50)

(\text{cf}; \hspace{0.2cm} 3.15 \hspace{0.2cm} \text{and} \hspace{0.2cm} 3.16)

In terms of the parity eigen vectors \( \psi_f \) and \( \phi_f \), these can be written as

\[ \psi_f \rightarrow \frac{e^{-i \vec{F}_f} + e^{i \vec{F}_f}}{2} \psi_f + \frac{e^{-i \vec{F}_f} - e^{i \vec{F}_f}}{2} \phi_f \]

\[ - \frac{1}{\hbar} \left\{ E_l(-i \vec{F}_f) - E_l(i \vec{F}_f) \right\} \tilde{\sigma}_f \]

\[ \phi_f \rightarrow \frac{e^{-i \vec{F}_f} - e^{i \vec{F}_f}}{2} \psi_f + \frac{e^{-i \vec{F}_f} + e^{i \vec{F}_f}}{2} \phi_f \]

\[ - \frac{1}{\hbar} \left\{ E_l(-i \vec{F}_f) + E_l(i \vec{F}_f) \right\} \tilde{\sigma}_f \]

Comparing the above results with the expression derived in Chapter 2 §3 and §5 of the present chapter ((3.22) and (3.23)), these are seen to be equivalent to

\[ \psi_f \rightarrow \chi_f^+ - \frac{1}{\hbar} \beta f \]  \hspace{1cm} (3.51)

\[ \phi_f \rightarrow \chi_f^- - \frac{1}{\hbar} \psi f \]  \hspace{1cm} (3.52)

Thus, from (3.35) and (3.36)

\[ \begin{pmatrix} \psi_f \\ \phi_f \end{pmatrix} \rightarrow \begin{pmatrix} \phi_f^+ \\ \phi_f^- \end{pmatrix} \]  \hspace{1cm} (3.53)
The special chiral transformation \( e^{-A \cdot \frac{S}{2}(a)} \) is just the inverse-boost operator utilized so much in Chapter 2. But here I am taking it as the local chiral transformation and at each space-time point \( \chi \) I take corresponding parameter \( \frac{S}{2}(a) \).

Thus, in particular, beside the usual
\[
e^{-A \cdot \frac{S}{2}(a)} \mathcal{F}(\chi) = 0
\]
I can also write
\[
e^{-A \cdot \frac{S}{2}(a)} \partial_\mu \ldots \partial_\nu \mathcal{F}(\chi) = 0.
\]

The results on covariance under local chiral transformation in \( \mathbb{R}^2 \) of the present chapter could be derived in somewhat simpler way if we utilize the relation (3.53). For this, I shall consider a simple invariant coupling of fields as I have done in Chapter 1.

Take the "linear" type "baryon" field \( \psi_\alpha \) described in Chapter 2, \( \S 1 \). This is associated with some irreducible representation \( D \) of \( SU(n) \). Further, let us assume, for the sake of simplicity of notation, that it is also an ordinary Dirac spinor as in Chapter 1. Then, as has been shown in Chapter 2, \( \S 1 \) (also Chapter 1, \( \S 3 \)), the new fields
\[
\psi_\alpha = \left( e^{i \frac{A}{2} \cdot \mathbf{S}} \right)_\alpha^\beta \psi_\beta
\]
transform as the representation \((\mathbb{U}, 0) \oplus (0, D)\) of chiral \(SU(\mathbb{U}) \times SU(\mathbb{U})\). For instance
\[
\psi \rightarrow e^{A_{\mathbb{U}}} \psi = \psi
\]

Then the following coupling
\[
\hat{L}^{\text{int}} = \frac{g}{2} \eta^f \left( \xi_f - i g \Gamma (\mathbb{U} + \xi_f \phi_f) \right) \psi
\]  

(3.55)
is clearly invariant under local chiral transformation.
(This is just the ordinary covariant derivative for the representation of chiral \(SU(\mathbb{U}) \times SU(\mathbb{U})\)). Thus,
transforming the fields involved in (3.55) by the local chiral transformation \(e^{-A_{\mathbb{U}}(\mathbb{U})}\),
\[
\psi \rightarrow e^{-(i \mathcal{A}(\mathbb{U})) \cdot \xi_f} \psi = \psi
\]
\[
\frac{\partial \psi}{\partial \chi_f} \rightarrow \frac{\partial}{\partial \chi_f} \left\{ e^{(i \mathcal{A}(\mathbb{U})) \cdot \xi_f} \psi(\chi_f) \right\} = \frac{\partial}{\partial \chi_f} \psi
\]

\[
\left( \begin{array}{c}
\psi_f \\
\phi_f
\end{array} \right) \rightarrow 
\left( \begin{array}{c}
\phi_f^+
\\
\phi_f^-
\end{array} \right)
\]

I get
\[
\hat{L}^{\text{int}} = \frac{g}{2} \eta^f \left( \xi_f - i g \Gamma (\phi_f^+ + \xi_f \phi_f^-) \right) \psi
\]  

(3.56)
from the invariance of (3.56), I can conclude as in §4 of Chapter 1, that
\[
(\xi_f - i g \Gamma \phi_f^+) \psi
\]
and

$$\delta_s (F \cdot \phi^-) \psi$$

are covariant, and from the covariance of the second form, the covariance of $\phi^-$ itself can be concluded. This is just showing the covariance of $\delta_s \psi$ and $\phi_s \psi$ given in (3.78) and (3.27) of §3.
CHAPTER 4

Chiral invariant lagrangians

§1 The lagrangian and the currents

In this chapter, I would like to apply the results of the preceding chapters to construct a few examples of chiral invariant lagrangian models. Although these models are chosen with the application to the actual physical problems in mind, in the present chapter I shall discuss mainly the formula structure of these lagrangians, and leave the more practical problems to the next chapter where the detailed discussion of how to break symmetry will be given.

(a) The lagrangian and the currents without the gauge fields

The simplest example of a chiral invariant lagrangian is one with only the multiplet of non-linear fields \( \vec{\phi}_1, \ldots, \vec{\phi}_N \) which is, in the physical case of chiral SU(3), identified with the octet of pseudoscalar mesons \( (\pi, K, \eta) \). Consider the chiral invariant lagrangian density

\[
\mathcal{L}_\phi = \frac{a}{2} \sum_{i=1}^{N} \left( \nabla_\mu \phi_i \right)^2
\]

where \( \nabla_\mu \phi_i \) is the covariant derivative discussed in Chapter 2, and \( a \) is a constant. (We use the metric
convention $\varepsilon_{00} = 1$, $\varepsilon_{ii} = -1$ for $i = 1, 2, 3$ and $\gamma \cdot \gamma = 2$. 

(4.1) is the generalization of the pion lagrangian $\mathcal{L}_\pi$ (1.52) discussed in Chapter 1, §5.

The vector and the axial vector currents are defined by Noether's theorem with respect to the infinitesimal local chiral transformations

$$ J^a = e^{\delta B^a} - 1 + gB^a \cdot V $$

and

$$ J^i = e^{\delta A^i} - 1 + gA^i \cdot A $$

As in Chapter 1, §5, I have

$$ V^i = - \frac{\delta \mathcal{L}_\pi}{\delta \partial^i \theta_i} \quad \text{vector currents} $$

$$ A^i = - \frac{\delta \mathcal{L}_\pi}{\delta \partial^i \theta_i} \quad \text{axial vector currents} $$

By using the transformation laws for given in (3.4) and (3.5), I get immediately

$$ V^i = - \alpha \frac{e^{iF^0} - e^{-iF^0}}{2} \nabla \bar{\psi} $$

(4.2)

$$ A^i = - \alpha \frac{e^{iF^i} + e^{-iF^i}}{2} \nabla \bar{\psi} $$

(4.3)

Then, comparing these with the original lagrangian (4.1), I find immediately

$$ \mathcal{L}_\pi = \frac{1}{2a} \sum_{i=1}^{N} (V^i \cdot \nabla \bar{\psi} + A^i \cdot A^i ) $$

(4.4)

This is the extension of the result found in Chapter 1 §5 (1.59) to chiral SU($n$) x SU($n$).
Let us consider now what will happen when the arbitrary SU(3) multiplet \((\psi_a)\) transforming by (3.15) is included. I take the simplest model lagrangian with essentially Yukawa type coupling of non linear "meson" fields to this multiplet ("baryons")

\[
L_{\Psi} = \frac{\alpha}{2} (\bar{\psi} \gamma^2 \psi)^2 + \nabla \cdot (\bar{\psi} \gamma^\mu \partial^\mu \psi - M) \psi + G' \bar{\psi} r \sigma^i \gamma^i \psi \frac{\partial}{\partial y} \bar{\psi}
\]  

(4.5)

where \(\sigma^i\) are essentially C-G coefficients. For instance, in the case of the chiral SU(3), if we consider the coupling of octet \(\pi\) mesons to octet \(\frac{3}{2}^+\) baryons with BBH: Yukawa coupling, it has the well known form

\[
\frac{\hat{v}^c}{(1-\alpha)} \mathcal{F}^c + \mathcal{D}^c \quad c = 1, 2, \ldots 8
\]

The spin of the \(\gamma\)-fields is not essential although I have taken it for the ordinary spin \(\frac{3}{2}\) Dirac spinor to avoid unnecessary complications. To define the currents, the transformation laws for \(\bar{\gamma}\psi\) given in (3.9) and (3.10) are needed. Taking the variation of (4.5), I find

\[
\delta \psi = \frac{\delta \phi}{\delta \sigma^\mu} \frac{\partial}{\partial y} \bar{\psi} \left\{ \alpha \frac{e^{i \theta^3} - e^{-i \theta^3}}{2} \nabla^\mu - \frac{e^{i \theta^3} + e^{-i \theta^3}}{2} \bar{\psi} \gamma^\mu \bar{\psi} 
+ G' \frac{e^{i \theta^3} - e^{-i \theta^3}}{2} \bar{\psi} r \sigma^i \gamma^i \psi \right\}
\]  

(4.6)
\[
A_{r} = -\frac{8e}{\delta \phi_{\alpha}} \\
= \left\{ \frac{a}{2} e^{i\phi_{\beta}} + e^{-i\phi_{\beta}} \nabla_{\beta} \phi - \frac{e^{i\phi_{\beta}} - e^{-i\phi_{\beta}}}{2} \nabla \phi \right\} \\
+ \frac{G'}{2} \left( e^{i\phi_{\beta}} + e^{-i\phi_{\beta}} \right) \nabla \phi \right\}.
\]

(4.7)

From (4.6) and (4.7), I get the following expressions for the bilinear forms of the currents:

\[
\sum_{r=1}^{N} \nabla_{r} N_{r} \left( \frac{e^{i\phi_{\beta}} - e^{-i\phi_{\beta}}}{2} \right)^{2} N_{r}^{\mu} \\
+ 2a \nabla_{\beta} \phi \left( \frac{e^{i\phi_{\beta}} + e^{-i\phi_{\beta}}}{2} \right) \nabla \phi \right\} \\
+ 2G' \left( \frac{e^{i\phi_{\beta}} + e^{-i\phi_{\beta}}}{2} \right) \nabla \phi \right\} \\
- 2a G' \left( \frac{e^{i\phi_{\beta}} - e^{-i\phi_{\beta}}}{2} \right)^{2} \nabla \phi \right\}.
\]

(4.8)

\[
\sum_{r=1}^{N} A_{r}^{c} A_{r}^{c} \\
= a^{2} \nabla_{\beta} \phi \left( \frac{e^{i\phi_{\beta}} + e^{-i\phi_{\beta}}}{2} \right)^{2} \nabla \phi \\
- N_{r} \left( \frac{e^{2i\phi_{\beta}} - e^{-i\phi_{\beta}}}{2} \right)^{2} N_{r}^{\mu} \\
+ G' \left( \frac{e^{i\phi_{\beta}} + e^{-i\phi_{\beta}}}{2} \right) \nabla \phi \right\} \\
- 2G' \left( \frac{e^{i\phi_{\beta}} - e^{-i\phi_{\beta}}}{2} \right)^{2} \nabla \phi \right\} \\
- 2a G' \left( \frac{e^{i\phi_{\beta}} + e^{-i\phi_{\beta}}}{2} \right) \nabla \phi \right\}.
\]
\[ + 2G'N_f \frac{e^{i\mathbf{F}_\mu} - e^{-i\mathbf{F}_\mu}}{2} \frac{e^{i\mathbf{F}_\nu} + e^{-i\mathbf{F}_\nu}}{2} N^{\mu\nu} \]
\[ + 2aG'N_f \left( \frac{e^{i\mathbf{F}_\mu} + e^{-i\mathbf{F}_\mu}}{2} \right)^2 \nabla^\mu \mathbf{F}_\mu \]

where I write
\[ N^{\mu}_f \mathcal{P} = \mathcal{P} \sigma_{\mu}^* \mathcal{P} \Psi_f \]
\[ N^{\mu}_i = \mathcal{P} \sigma_{\mu}^* \mathcal{P} \Psi_i \]

Thus, I get
\[ \sum_{\mu=1}^2 \left( \nabla_{\mu} \mathbf{F}_\mu + A_{\mu}^i A_i^\mu \right) \]
\[ = a^2 \nabla_{\mu} \mathbf{F}_\mu \nabla^\mu \mathbf{F}_\mu + 2aG'N_f \nabla^\mu \mathbf{F}_\mu \\
+ N_f N^\mu_i + G'N_i N^\mu_i N^{\mu}_f \]
or
\[ \frac{1}{2a} \left\{ \nabla_{\mu} \mathbf{F}_\mu + A_{\mu}^i A_i^\mu \right\} - \left( N_f N^\mu_i + G'N_i N^{\mu}_i N^{\mu}_f \right) \]
\[ = \frac{a}{2} \nabla_{\mu} \mathbf{F}_\mu \nabla^\mu \mathbf{F}_\mu + G'N_f \nabla^\mu \mathbf{F}_\mu \]

Comparing this with (4.5), I get
\[ \mathcal{L}_{\text{4-point}} = \mathcal{P} \left( i\sigma_{\mu}^* \nabla_{\mu} - M \right) \Psi_f \]
\[ + \frac{1}{2a} \left( \nabla_{\mu} \mathbf{F}_\mu + A_{\mu}^i A_i^\mu \right) \quad (4.10) \]
\[ - \frac{1}{2a} \left( N_f N^\mu_i + G'N_i N^{\mu}_i N^{\mu}_f \right) \]

This is the result stated in \$5$, Chapter 1, for pion-nucleon lagrangian. Suppose we add the additional 4-point contact term (13)
\[ + \frac{1}{2a} \left( N_f N^\mu + G'N_f N^{\mu}_i N^{\mu}_f \right) \]
to the original lagrangian of (4.5). Then the modified lagrangian
\[ L'_{\bar{\psi}, \psi} = \overline{\psi} (i \gamma^\mu D^\mu - M) \psi + \frac{1}{2a} (\gamma^\mu V^\mu + A^\mu A^\mu) \]  
\tag{4.11}

is of "Sugawara form"(16) except for the chiral invariant kinematical term of \( \psi \) fields.

(b) **The introduction of gauge fields**:

As it can be seen from the results of Chapter 3, the chiral invariant lagrangian can be made invariant under the local chiral transformation by replacing the covariant derivatives \( \gamma^\mu \) and \( \gamma^\mu \psi \) by \( \gamma^\mu \) and \( \gamma^\mu \psi \) of (3.27) and (3.28).

\[ L (\gamma^\mu, \psi, \gamma^\mu \psi) \]
\[ \rightarrow L (\gamma^\mu, \psi, \gamma^\mu \psi) \]
\[ \gamma^\mu \rightarrow \gamma^\mu + \gamma^\mu \gamma^\mu - \gamma^\mu \gamma^\mu \]
\[ \gamma^\mu \psi \rightarrow \gamma^\mu \psi - \gamma^\mu \gamma^\mu \psi \]

\( \gamma^+ \), \( \gamma^- \) are defined according to (3.22) and (3.23) in terms of gauge fields \( \gamma^\mu \), \( \gamma^\mu \) and non-linear "meson" fields \( \xi^\mu \). Besides the replacement (4.12), we need to introduce a kinematical term of dynamical variables \( \gamma^\mu \) and \( \gamma^\mu \) which must be itself invariant.

Starting from the chiral invariant lagrangian \( L_{\bar{\psi}, \psi} (4.5) \), the replacement (4.12) induces the following change.
\[ L_{j+4}(\gamma, \psi, \gamma, \psi) - L_{j+4}(\gamma, \psi, \gamma, \psi) + \Delta_1 + \Delta_2. \]  

(4.13)

where

\[ \Delta_1 = g (-g_{jF}^3 \cdot \gamma^r + \frac{g}{2} \gamma r - \gamma^r) - g G' \gamma^r \gamma^r \]

\[ \Delta_2 = g N_F^r \gamma^r \]

Using (3.22) and (3.23), I can write it as

\[ \Delta_1 = \frac{ag^2}{2} \gamma^r - \gamma^r - ag \left\{ \nu_T \frac{e^{-iF^3}}{2} + \nu_T \frac{e^{iF^3}}{2} \right\} \Delta F^3 \]

\[ - g G' \left\{ \nu_T \frac{e^{iF^3}}{2} - \nu_T \frac{e^{-iF^3}}{2} \right\} N_f^r \]

\[ \Delta_2 = g \left\{ \nu_T \frac{e^{iF^3} + e^{-iF^3}}{2} + \nu_T \frac{e^{iF^3} - e^{-iF^3}}{2} \right\} N_f^r \]

And thus

\[ \Delta_1 + \Delta_2 = \frac{ag^2}{2} \gamma^r - \gamma^r + \frac{g}{2} \left\{ \nu_T \frac{e^{iF^3} + e^{-iF^3}}{2} - \nu_T \frac{e^{iF^3} - e^{-iF^3}}{2} \right\} N_f^r \]

Comparing this with the expressions (4.6) and (4.7) for the currents, it can be seen that

\[ \Delta_1 + \Delta_2 = \frac{ag^2}{2} \gamma^r - \gamma^r + g (\nu_T N_f + a r A_f) \]

(4.14)
where \( A_\mu \) and \( V_\mu \) are the same functional of the fields \( \bar{\psi} \) and \( \psi \) as defined in (4.6) and (4.7).

The invariant kinematical term can be constructed out of covariant curls discussed in §3, Chapter 3 and can be written as

\[
\mathcal{L}^\text{kin} = -\frac{i}{2} \left( G^{\mu\nu}_+ G^{+\mu\nu}_+ + G^{\mu\nu}_- G^{-\mu\nu}_- \right) \tag{4.15}
\]

If the lagrangian is constructed out of (4.13), (4.14) and (4.15), it is completely invariant under the local chiral transformation. But then the field variables \( U_\mu \) and \( Q_\mu \) can represent particles with zero mass only and we cannot consider them as, for instance, the phenomenological description of known heavy vector or axial vector mesons. To give masses to these fields, I should break the invariance under the local chiral transformation. The mass term, which is still invariant under constant chiral transformation is

\[
\mathcal{L}_\text{mass} = \frac{m^2}{2} \sum_i (\bar{U}_i U_i + \bar{Q}_i Q_i) \tag{4.16}
\]

This is the unique expression since the chiral invariant bilinear forms are \( (A_\mu t U_\mu) \) in terms of irreducible quantities and we must eliminate the parity non-conserving product term \( A_\mu U^\mu \).
The audition of (4.16) violates the invariance under the local chiral transformations, but on the other hand, it has an attractive feature when the currents are considered. Taking the usual variation of the lagrangian

\[ L^{tt} = L_{5,4} + \Delta_1 + \Delta_2 + L_{\mathbf{e}i} + L_m \]

I get immediately

\[ \chi^t = -\frac{\delta L^{tt}}{\delta \phi^t} = -\frac{m_0^2}{g} \chi^t \] (4.17)

\[ \chi^t = -\frac{\delta L^{tt}}{\delta \phi^t} = -\frac{m_0^2}{g} \chi^t \] (4.18)

This is just the field current identity of Lee and Zumino.\(^{36,37}\) It is shown\(^{38}\) by Lee, Weinberg and Zumino that if \( q_f \) and \( \chi^t \) are considered as canonical variables and ordinarily canonical commutations relations from \( L^{tt} \) are assumed, then the chiral \( SU(\mathbb{N}) \times SU(\mathbb{N}) \) current algebra type commutation relations between the currents \( \chi^t \) and \( q_f \) can be derived and that the Schwinger terms appearing there are finite and c-numbers. It should be noted that from the derivation of (4.17) and (4.18) the quantities like masses, coupling constants and field themselves are unrenormalized. But it has been shown\(^{36}\) also that if the lagrangian \( L^{tt} \) is renormalizable with
finite propagation then the quantities appearing in the R.I.S. of (4.17) and (4.18) can be replaced by the corresponding renormalized quantities.

\[ L' \] is still invariant under the chiral transformation and the corresponding currents are conserved

\[ \partial \cdot \psi' = \partial \cdot \psi_\tau = 0 \quad (4.19) \]

\[ \partial \cdot A' = \partial \cdot a_\tau = 0 \quad (4.20) \]

This means that if \( L' \) can be considered as defining a quantized field theory then the spectral functions of the field operators \( \psi' \) (or \( \psi_\tau \)) and \( A' \) (or \( a_\tau \)) contain the vector (spin 1) parts only except the mass loss excitation corresponding to the \( \Theta \) fields (Goldstone boson).

Finally, collecting (4.10), (4.14), (4.15) and (4.16), the form of \( L' \) becomes

\[
L' = \bar{\psi} (i\gamma^\mu - m) \gamma^\mu \psi - \frac{1}{4} ( (\gamma^\mu \gamma^\nu + (\gamma^\mu \gamma^\nu))^2 ) \\
- \frac{1}{2a} ( N^r \gamma^r + N^r \gamma^r ) \\
+ \frac{1}{4a} ( \gamma^r A^r ) + g ( \gamma^r A^r + q_\gamma A^r ) + \frac{m^2}{2} ( \gamma^r A^r ) + m^2 \gamma^r A^r \\
+ \alpha \frac{q^2}{2} (\gamma^r \gamma^r )^2
\]
(c) **The further invariant couplings**

We can add a few other invariant couplings of physical interest to (4.21). These can contribute to the anomalous magnetic moments of particles.

1. **Magnetic coupling to \( \psi \) field**

\[
L_1 = \frac{1}{2} \psi \sigma_{\mu\nu} \psi G^{\mu\nu}_{\rho}
\]

(4.22)

where \( \sigma_{\mu\nu} = \frac{i}{2} [\gamma_{\mu}, \gamma_{\nu}] \) and \( \hat{\sigma}^\mu \) like \( \hat{F} \) of (4.5) represents general SU(2) coupling between \( \psi \) and \( (G^{\mu\nu})_{i=1}^N \). \( G^{\mu\nu} \) are "non-linear" covariant curls defined in (2.40). \( \rho \) is a constant.

2. **Trilinear coupling of the gauge fields**

\[
L_2 = kg \tilde{t}^{\mu} \phi^{-i} \phi^{-j} G^{\mu\nu}_{\rho}
\]

(4.23)

where \( \tilde{t}^{\mu} \) again represents the SU(2) coupling between \( \phi^{-i} \), \( \phi^{-j} \) and \( G^{\mu\nu}_{\rho} \). \( k \) is a constant.

§2 **The phenomenological lagrangian**

To use (4.21) (with possible (4.22) and (4.23)), as a phenomenological lagrangian, we can impose some further restrictions. The argument used by Wess and Zumino(4) for the chiral SU(2)xSU(2) invariant model can be applied to (4.21) without alteration. First, it should be noted that the non-linear lagrangian (4.21) does contain the term like \( q \cdot \mathcal{D} \bar{\psi} \), which modifies
the procedure by the direct transition between $q_\mu$ fields and $\pi$ fields. To eliminate such a term, the $a_\mu$ field is assumed to be a mixture of an "axial vector field" $\hat{a}_\mu$ and a "scalar field" $\pi$ as

$$a_\mu = \hat{a}_\mu + c \, \pi \, \pi$$

Instead of using this straightforward decomposition, I follow next the hint to write it

$$a_\mu = \hat{a}_\mu + c \left( \pi^2 + \frac{e_F q^2}{2} \frac{e^{-i\beta}}{e^{-i\beta}} \right) \tag{4.2.3}$$

The additional term containing $\pi$ field looks a little arbitrary. One way of justifying this form is the simplicity with which the electromagnetic interaction of $\pi$ field can be introduced. (cf. 6.4). Then, looking at the term $\frac{1}{2} \dot{\pi}^2 + \frac{m^2}{2} (a_\mu^2 + \pi^2)$ the coefficients of $\frac{1}{2} \dot{\pi}^2 \pi^2$, $\frac{1}{2} \dot{a}_\mu^2 \pi$, $\frac{1}{2} \pi^2 \pi$, and $\pi \cdot \pi \pi$ in (4.2.1) are $e^2 \pi^2 + \frac{m^2}{2}$, $m^2 + a q^2$, $m^2$ and $m^2 - a g^2 (1 - g^2)$ respectively. (According to the argument about (4.17) and (4.19), I now assume $m$ and $g$ as renormalized quantities and use the letter $m$ instead of $m$). The condition of the absence of the term $\nabla \cdot \pi \pi$ in the phenomenological lagrangian now imposes

$$m^2 c = a g (1 - g c)$$

or

$$c = \frac{a g}{m^2 + a g^2}$$
The coefficients of $\eta_f^2$ and $a_T^2$ can be considered as the mass squares of vector and axial vector fields. So putting $\overline{m}^2 = m^2 + q^2$ (mass square of axial vector field) the above condition can be written as

$$C = \frac{1}{q} \frac{\overline{m}^2 - m^2}{\overline{m}^2}$$

(4.25)

I have stated the \((\frac{\phi}{\sqrt{2}})_2\) are \(SU(N)\) generalization of pion fields of Chapter 1. But the coefficient of \((\frac{\phi}{\sqrt{2}})_2^2\) tells us that it is not properly normalized as the phenomenological description of the particles. Thus defining the "physical" fields by $\phi = \frac{F}{\sqrt{2}} \frac{\phi}{\sqrt{2}}$, the condition for the correct normalization is

$$\left(\frac{2}{F}\right)^2 (\overline{m}^2 - m^2) + a (1 - q^2) = 1.$$ 

In terms of introduced above,

$$\left(\frac{2}{F}\right)^2 \frac{\overline{m}^2 - m^2}{q^2} \frac{m^2}{\overline{m}^2} = 1$$

or

$$F = \frac{2}{a} \frac{m}{\overline{m}} \sqrt{\frac{\overline{m}^2 - m^2}{q^2}}$$

(4.26)

It is impossible to get further restrictions by the chiral invariance alone. One of the well known results suggested from the calculation in the current algebra is the ratio $\frac{\overline{m}}{m}$ and the relation between $g$ and $F$. One way of recovering these results is to introduce an additional physical assumption of "vector
noon dominance”. This is of course, the $\rho$-dominance model of Ishikawa in the case of chiral $SU(2)\times SU(\rho)$ but probably can be extended to chiral $SU(\rho):SU(\bar{\rho})$ too.

(4.21) does contain $\frac{3}{2}-\bar{\psi}$ contact term due to $\bar{\psi}\gamma^\mu\psi$. If "$\frac{3}{2}-\bar{\psi}$ scattering amplitude" is required to be always radiated by vector fields $\phi_f$, then the coefficient of a possible $\frac{3}{2}-\bar{\psi}$ 4-point contact term should be put equal to zero. Since

$$D\bar{\psi} = \{D_f + i\bar{\psi}(\not{\partial} - g \not{\gamma}_f)\}\bar{\psi}$$

$$\times \{D_f + i\bar{\psi}(\not{\partial} \not{\phi}_f + \frac{g \not{\gamma}_f}{2} - g (\not{\gamma}_f + \not{\gamma}_f(\not{\gamma}_f))\}\bar{\psi}$$

and

$$a_{\mu} \not{\partial} \bar{\psi} + c \bar{\psi} \not{\phi}$$

the coefficient in question is

$$\left(\frac{1}{2} - g \not{\gamma}\gamma\right)$$

Thus I may conclude

$$c = \frac{1}{2g}$$

Then, from (4.25),

$$\frac{m_1}{m_2} = \frac{1}{2}$$

(4.26)

which is the Weinberg’s relation$^{(28)}$. Also from (4.26)

$$F = \frac{\sqrt{2} m}{g}$$

(4.30)

which is the Kawarabayashi-Suzuki relation$^{(39)}$. 
Finally, I note that with the relations obtained above I get

\[ \alpha = \frac{m^2}{\sigma^2}. \]

The Lagrangian (4.21) reduces to rather simple form

\[ \mathcal{L}_{\text{lag}} = \overline{\psi} \left( \gamma^a \gamma^5 \psi - \frac{\lambda}{4} \left( \mathcal{G}_{\mu}^+ \right)^2 + \left( \mathcal{G}_{\mu}^- \right)^2 \right) - \frac{g^2}{2m^2} \left( \mathcal{V}_{\mu}^r \right)^2 + \frac{g^2}{2m^2} \left( \mathcal{V}_{\mu}^l \right)^2 + \frac{g^2}{2m^2} \left( \mathcal{A}_{\mu}^r \right)^2 + \frac{g^2}{2m^2} \left( \mathcal{A}_{\mu}^l \right)^2. \]  

The model discussed above due to Wess and Zumino does not represent unique chiral invariant Lagrangian with gauge fields. Weinberg has emphasized the likeness of chiral SU(2) x SU(2) invariance in non-linear realization technique to the ordinary local SU(2) invariant coupling. According to this idea, instead of (4.21), we have

\[ \mathcal{L} \cdot = \frac{g}{2} \left( \mathcal{D}_\mu \bar{\psi} \right)^2 + \frac{m^2}{2} (\chi^r)^2 - \frac{1}{2} \mathcal{V}_{\mu}^r \mathcal{V}_{\mu}^r + \overline{\psi} (\gamma^5 \mathcal{D}_\mu \chi^r + m) \psi + \frac{g}{2} \overline{\psi} \gamma^5 \gamma^r \slashed{\mathcal{D}} \psi \]  

(4.31)
where
\[ \chi_r' = \nu_r' + \frac{1}{g} \beta_r \]
\[ \nu_{r\nu}' = \chi_r' \nu' - \nu_r \nu_r' + i g \nu_r' (iF) \nu' \]
The new gauge field \( \nu_r' \) transform under the chiral \( SU(\mathfrak{g}) \times SU(\mathfrak{h}) \) according to (5.37)
\[ S \nu_r' = (iF' \gamma') \nu_r' + \frac{1}{g} \gamma_r' \gamma' \] (4.32)

In this model, where there is no need for an \( \chi_r' \) field, the identification of various arbitrary constants can be done somewhat more simply. In particular (4.30) can be obtained as the result of the universal coupling of vector gauge fields. Note that (4.32) gives according to the expressions given in Chapter 2.
\[ S \nu_r' \chi = \frac{1}{g} \gamma_r' \gamma' \]
\[ = \frac{1}{g} \gamma_r' \left\{ \frac{S \gamma}{2} \left( \frac{e^{iF} + e^{-iF}}{e^{iF} + e^{-iF}} \right) \right\} \]
\[ = \frac{1}{g} \gamma_r' \left( \frac{S \gamma}{2} \cdot \frac{iF}{2} \right) \]

In term of "physical" fields \( \phi = \frac{F}{2} \) with Kawarabayashi-Suzuki relation, the last expression reduces to
\[ \frac{g}{m} \gamma_r' \left( S \phi \cdot iF \phi \right) \]
\[ = \frac{g}{m} \gamma_r' \left( C \cdot J_{A} \phi_{q} \cdot \delta \phi_{q} \right) \]
which is equivalent to the transformation used by Schwinger (3).

§3 The equivalence relations

We can use a phenomenological lagrangian like (4.21) according to the idea discussed in §1 Chapter 1. But then it should be remembered that the definition of "physical" fields is not unique. If, for instance,

\[ \phi_i \rightarrow \frac{e}{2} \hat{\phi}_i \]

is used as second quantized P-S coton fields to compute the Feynman grath with given lagrangian, any transformation

\[ \chi_i = f_i (\phi) \]  

(4.33)

with \( f(\omega) = 0 \) can be used with the same (transformed) lagrangian. This has the analogy in formal field theory with the non uniqueness of interpolating field operator. Coleman and Zumino (\(^{1}\)) give a general proof that the "canonical transformation" like (4.33) leaves not only the exact on mass shell S-matrix elements invariant, but it leaves the value of the sum of Feynman graphs with a fixed number of internal loops invariant. In particular, the value of amplitudes obtained in the tree approximation (Chapter 1) is not affected by such a transformation. Instead of quoting their proof, I would like to give some examples of field transformations.
(a) \[ \text{Heitmann's field for the \( u_a \) field.} \]

In Chapter 5, I have introduced the functions of some fields: \( \phi^\pm(u_F, \phi_T, \phi^T) \). (\( \text{3.37 and 3.39.} \))

They have rather simple transformations under the local chiral group: (\( \text{1.37} \)) and (\( \text{2.3c} \)). I have shown that \( \phi_T \) can be recovered by a chiral transformation of the original linear gauge field: \( u_T \) and \( \phi_T \).

(\( \text{5.4, Chapter 5.} \)) [\( \text{Heitmann's (4.4)} \]) has suggested to use this transformation \( \phi_T^+ \) as the vector and axial vector meson fields instead of \( u_T \) and \( \phi_T \) of \( \text{2.3} \) and \( \text{4.4} \).

Thus, write

\[
\begin{align*}
\nu_T' &= \phi_T^+ = \chi_T^+ - \frac{1}{2} \Theta_T \\
\phi_T' &= \phi_T^- = \chi_T^- - \frac{1}{2} \Gamma_T^T
\end{align*}
\]

where

\[
\begin{align*}
\chi_T^\pm &= \nu_T \frac{e^{i\Phi^T} \pm e^{-i\Phi^T}}{2} + a_T \frac{e^{i\Phi^T} \mp e^{-i\Phi^T}}{2} \\
\beta_T^T &= \frac{1}{2} \left( \frac{E_i(i\Phi^T) \pm E_i(-i\Phi^T)}{2} \right)
\end{align*}
\]

Since the transformation \( (\mu_T^\text{'} \ \phi_T^\text{'}) \rightarrow (\mu_T \ \phi_T) \) is an element of local chiral transformation \( e^{-A^\text{M} \mu} \), the part of the \( \text{H} \) which is invariant under local chiral transformation will be left unchanged.

\[
\text{L}_\text{inv} (\phi_T, \nu_T) \rightarrow \text{L}_\text{inv} (\phi_T', \nu_T')
\]

The local chiral invariance of the \( \text{H} \), (\( \text{4.41} \))
in the last term.

\[ \mathcal{L}^{\text{mass}} = \frac{m^2}{2} \sum (\mathbf{q} \cdot \mathbf{a} + v_t \mathbf{u}^t) \]

But according to the definition of the functions above, I have

\[ \sum (\mathbf{q} \cdot \mathbf{a} + v_t \mathbf{u}^t) = \sum (\mathbf{q} \cdot \mathbf{a} + v_t \mathbf{u}^t) \]

Thus the mass term \( \mathcal{L}^{\text{mass}} \) can be written in terms of new variables as

\[ \mathcal{L}^{\text{mass}} = \frac{m^2}{2} \left\{ (v_t^r + \frac{1}{8} \mathbf{q}^r)^2 + (v_t^t + \frac{1}{8} \mathbf{q}^t)^2 \right\} \]

Note also the invariance term \( \frac{\mathbf{q}}{2} (\mathbf{q}^r)^2 \) is transformed into \( \frac{\mathbf{q}}{2} (\mathbf{q}^r)^2 \) and in general, the expression containing \( \mathbf{q} \) or \( v_t \) can be replaced by \( v_t^r \) and \( v_t^t \) alone. Thus if the mass \( m \) is put to zero, \( \mathcal{L} \) field will disappear, which is of course obvious because

\[ e^{-A \sum_{\mu=1}^{n} \partial_{\mu} \partial_{\mu} \mathcal{F}(x)} = 0 \quad \text{(Chapter 3)} \]

This disappearance of \( \mathcal{F} \)'s corresponds to the decoupling of solitonic bosons suggested by Hibble (41).

The identification of arbitrary constants can be carried out as in the last section. But it should be noted that the transformation \( (v_t, \mathbf{q}) \rightarrow (v_t^r, \mathbf{q}^r) \) is not a proper canonical transformation since this contains the derivatives of \( \mathcal{F} \) fields. Thus, even
with the theorem of Coleman and Zumino, there is no reason why the new scheme (of Kawarabayashi) should be compatible with the old one (of Wess and Zumino).

Let us consider the coupling of $\mathcal{G}$'s to $\mathcal{U}_T$. First the relevant part of Wess-Zumino lagrangian is

$$\frac{m^2}{2} a_T^2 + \frac{a}{2} \left( \mathcal{F}_1 \mathcal{F}_2 \right)^2$$

$$\Rightarrow \frac{m^2}{2} C^2 \left( \mathcal{F}_1 \mathcal{F}_2 - g \mathcal{U}_T \cdot \mathcal{F}_2 \right)^2$$

$$+ \frac{a}{2} \left( \left( 1 - g C \right) \mathcal{F}_1 \mathcal{F}_2 - g \left( 1 - g C \right) \mathcal{U}_T \cdot \mathcal{F}_2 \right)^2$$

$$\Rightarrow g \left( m^2 C^2 + a \left( 1 - g C \right)^2 \right) \left( \frac{2}{F} \right)^2 \left\{ - \partial_T \phi \cdot \left( \mathcal{U}_T \cdot \mathcal{F}_2 \right) \right\}$$

But because of the normalization condition discussed in the last section, the last expression reduces to

$$g \left\{ - \partial_T \phi \cdot \left( \mathcal{U}_T \cdot \mathcal{F}_2 \right) \right\} \quad (4.34)$$

To examine the analogous coupling term in the Kawarabayashi lagrangian, I should first repeat the analysis of the last section to relate various arbitrary constants. Thus, I introduce again the decomposition

$$a_T = \tilde{a}_T + C \mathcal{F}_1 \mathcal{F}_2 \quad (4.35)$$
The elimination of terms like \( \bar{q}_t \gamma^5 \lambda_t' \) gives
\[
C' = -\frac{m'^2}{\bar{m}'^2} \bar{\lambda}_t' \tag{4.36}
\]
where \( \bar{m}' \) is the mass of \( \lambda_t' \) field.

The normalization of \( \bar{\lambda}_t' \)-fields is completely analogous to the Wess-Zumino lagrangian and normalization constant \( \phi = \frac{\bar{\lambda}_t'}{\bar{m}'} \) satisfies (4.36) i.e.
\[
\left( \frac{\bar{\lambda}_t'}{\bar{m}'} \right)^2 = \frac{1}{\bar{m}^2} \frac{m^2}{\bar{m}^2} (\bar{m}'^2 - m^2) \tag{4.37}
\]
Now the trilinear coupling of the form \( \bar{q}_t \phi (\gamma \cdot i \Gamma \phi) \) in Kawarabayashi lagrangian comes solely from the term
\[
\frac{m^2}{2} (\gamma_t + \frac{1}{\bar{m}'^2} I_t) \gamma_t. \] (We may modify the decomposition (4.35) to include vector field term like
\[
\bar{q}_t' \gamma_t' \bar{\lambda}_t' + C' (\gamma_t \bar{\lambda}_t' + \alpha \gamma_t \cdot (i F \phi)) \]
But the contribution coming from such an addition in the case of the Kawarabayashi form does cancel.)

Thus the corresponding coupling is
\[
\frac{i}{2} \frac{m^2}{\bar{m}^2} \left( \frac{2}{\bar{m}'^2} \right)^2 \left\{ -\bar{q}_t' \phi (\gamma_t \cdot i \Gamma \phi) \right\} \tag{4.38}
\]
(4.34) and (4.37) represent the "\( 2\phi \) decay process of \( \gamma_t \)" in both Wess-Zumino and Kawarabayashi lagrangian. Thus if I now require the compatibility of two lagrangians in spite of the improper transformations
connecting the, I should conclude
\[ \frac{1}{2} \frac{m^2}{g^2} \left( \frac{2}{F} \right)^2 = g. \]
Using (4.37), this reduces to

\[ \frac{m_1^2}{m_1^2 - m_2^2} = 2 \]
or

\[ \frac{m_1^2}{m_2^2} = 2. \]

Of course, I will also require the equivalence of \( \hat{\phi}' \) to \( \hat{\phi} \) and put \( m' = m \). So I have

\[ \frac{m_1^2}{m_2^2} = 2 \] (4.39)
and

\[ F = F' = \sqrt{\frac{m_1}{g}}. \] (4.40)

These are just Weinberg and Kawarabayashi-Suzuki relations, which are, in case of Wess-Zumino model alone, derived with the assumption of vector dominance. This result is derived by Kawarabayashi although I have presented it in a little different way. Also, the essence of the argument is contained already in the original paper by Weinberg (7). The model of Kawarabayashi can be regarded as a generalization of Weinberg's model discussed in the last section. Unlike Weinberg's it contains an \( q_f \) field but it does not satisfy a field-current identity like Wess and Zumino's.
(b) Cronin form of meson-nucleon Lagrangian

The Lagrangians without gauge field (4.5) contain as the special case the pion-nucleon Lagrangian (1.42). In Chapter 1, I have shown that (1.42) which is the chiral invariant generalization of the Schwinger's phenomenological Lagrangian (1.1), can be approximately derived from the simple "linear field" Lagrangian (1.30). The approximation here is that I put in (1.42) which is equivalent to $G'=1$ in (4.5). I keep this approximation in the discussion below for the sake of simplicity. Consider two forms of chiral (SU(2) invariant (apart from the meson mass term) Lagrangians

$$L_1 = i N \not\sigma N - m \bar{N} N$$

$$+ \frac{\lambda^2}{2} \left( \not\sigma \frac{G}{2} \right)^2 - \frac{i}{2} \lambda' \not\sigma \frac{G}{2}$$

$$+ \bar{N} \gamma_5 \sigma_{\frac{G}{2}} N \not\sigma \frac{G}{2}$$

and

$$L_2 = i \bar{N}' \not\sigma N' - \bar{N}' M \left( \frac{G}{2} \right) N'$$

$$+ \frac{\lambda^2}{2} \left( \not\sigma \frac{G}{2} \right)^2 - \frac{i}{2} \lambda' \not\sigma \frac{G}{2}$$

(4.42)

where

$$M \left( \frac{G}{2} \right) = M e^{-2i \frac{G}{2} \cdot \frac{G}{2} \not\sigma_{\frac{G}{2}}}$$

$m$ and $\mu$ are nucleon and meson masses. The mesons are represented by the $\frac{G}{2}$ fields discussed in Chapter 1
and Chapter 2 and the physical fields is just $\Phi = \frac{g}{\lambda}$ rather than more complicated functions of $\frac{g'}{\lambda}$ as in Chapter 1.

It has been explained in Chapter 1(1) that (4.41) can be obtained from (4.42) by the transformation of the nucleon fields

$$N'_\alpha = (e^{i \frac{3}{\sqrt{g}} S} N)_\alpha = (e^{i \frac{3}{2} \frac{g'}{g} T_f})_{\alpha \beta} N^\beta \quad (4.43)$$

Weinberg in (Ref.1) has already noted that (4.41) and (4.42) give the same meson-nucleon scattering amplitude for tree graphs. Of course, we can construct the equivalent form of (4.42) in chiral SU(3)xSU(3) scheme and it is this form rather than the "derivative coupling form" (4.5) which has been earlier proposed by Cronin(2) as the model of octet meson-baryon interaction.

Now, the transformation (4.43) is a form of the canonical transformation in the sense of Coleman and Zumino, and the invariance of on-mass-shell S-matrix element under the transformation $N' \rightarrow N$ $S \rightarrow S$ should be expected from their equivalence theorem.

To see the relevance of Coleman-Zumino theorem a little further, I consider the slightly more general transformation than (4.43)

$$N' \rightarrow N; \quad N' = e^{i \frac{3}{2} \frac{g'}{g} \frac{g}{g} T_f} N \quad (4.44)$$
where \( \theta \) is an arbitrary real constant. (4.44)
cannot be considered as the chiral transformation
of the nucleon field like (4.43).

Let us examine the nucleon-nucleon and the
meson-meson amplitudes in the tree approximation using
the lagrangian (4.42) and its transform by (4.44)
(which is equivalent to (4.41) if \( \gamma = 1 \)). By the
transformation (4.44).

\[
\mathcal{L}_2 \rightarrow \mathcal{L}(\phi) \ ;
\]

\[
i\bar{N} \gamma^\mu N' \rightarrow i\bar{N} \gamma^\mu e^{-i\frac{\theta}{2} \frac{\sigma}{3} \frac{\gamma}{3}} \gamma^\mu e^{i\frac{\theta}{2} \frac{\sigma}{3} \frac{\gamma}{3}} N
\]

\[
= i\bar{N} \gamma^\mu \{ \Theta^\mu + i \frac{\sigma}{3} \cdot (\beta(y, \phi) + i \gamma^\rho (y, \phi)) \} N
\]

\[
m \bar{N}' M(\phi) N' = m \bar{N} e^{2i \gamma^\rho \phi} \gamma^\rho N
\]

\[
\rightarrow m \bar{N} e^{2i(1-\gamma) \frac{\phi}{2}} \gamma^\rho N
\]

where

\[
\beta(y, \phi) = \beta(y, \phi) \left( \gamma^\rho \phi \right) - \gamma^\rho \phi + \gamma^\rho \phi = \gamma \phi \frac{\phi}{2}
\]

\[
\phi(y, \phi) = \phi(y, \phi) \left( \gamma^\rho \phi \right) + \gamma^\rho \phi + \gamma^\rho \phi = \gamma \phi \frac{\phi}{2}
\]

Thus the relevant part of \( \mathcal{L}(\phi) \) for our purpose is

\[
\mathcal{L}(\phi) \approx i\bar{N} \gamma^\mu N - \frac{\theta^2}{4\lambda^2} \bar{N} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\nu \gamma^\sigma \gamma^\mu N (\phi \cdot \phi) \phi.
\]

\[
- \frac{\gamma}{2\lambda} \bar{N} \gamma^\mu (\gamma \cdot \gamma) \phi \cdot \phi \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\nu \gamma^\sigma \gamma^\mu N
\]

\[
- m \bar{N} N + i(1-\gamma) \frac{\partial}{\partial \lambda} \bar{N} (\gamma \cdot \phi) \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\nu \gamma^\sigma \gamma^\mu N
\]

\[
+ \frac{(1-\gamma)^2 \frac{\partial}{\partial \lambda} \bar{N} (\gamma \cdot \phi)^2}{2\lambda^2} N
\]

(4.45)
The $N-N$ scattering in the tree approximation is solely due to one pion exchange diagrams. The relevant coupling term in the above $\mathcal{L}(\phi)$ is

$$\frac{c(1-y)}{\lambda} m \bar{N} (\tau \cdot \phi) \gamma_\tau N - \frac{y}{2\lambda} \bar{N} (\tau \cdot \phi) \gamma_\tau \tau N$$

But for calculating on-mass-shell $N-N$ scattering in the pion exchange graph, the derivative coupling term

$$\bar{N} \gamma_\tau \gamma_\sigma (\tau \cdot \phi) N$$

is equivalent to

$$-2im \bar{N} (\tau \cdot \phi) \gamma_\sigma N$$

thus making the second term of the above expression equivalent to

$$\frac{c}{\lambda} \bar{N} (\tau \cdot \phi) \gamma_\sigma N$$

and the whole interaction is simply equivalent to

$$\frac{im}{\lambda} \bar{N} (\tau \cdot \phi) \gamma_\sigma N$$

Thus the $N-N$ amplitude in the tree approximation will not be affected by the transformation (4.44).

The nucleon-meson scattering is slightly more complicated. For the process

$$N\alpha(p) + \pi^i(\not{q}) \rightarrow N\beta(p') + \pi_j(\not{q'})$$

I must calculate the diagrams

(a) \hspace{2cm} (b) \hspace{2cm} (c)
The contact diagram (a) includes both the second and the last term of (4.45). They each give the contribution to T-matrix element

\[
\frac{-i}{2\lambda^2} \gamma^d \rho_\alpha \bar{E}_{\delta j} i \delta_{\phi \lambda} \\
\frac{m(1-i)^s}{\lambda^2} \delta_{\phi \lambda}
\]

respectively. The exchange diagrams (b) and (c) give, on the other hand, the following contribution

\[
\delta_{ij} \delta_{\phi \lambda} \left[ \frac{1}{4\lambda^2} \left( \frac{s+3m^2}{s-m^2} - \frac{u+3m^2}{u-m^2} \right) + \frac{m^2(i-1)}{\lambda^2} \left( \frac{-1}{s-m^2} + \frac{1}{u-m^2} \right) + \frac{m^2(i-1)^2}{\lambda^2} \left( \frac{1}{s-m^2} + \frac{1}{u-m^2} \right) \right] \delta_{\phi \lambda} \\
+ m \left\{ -\frac{s^2}{\lambda^2} + \frac{s(1-s)}{\lambda^2} \right\} \\
+ \bar{E}_{\delta j} \gamma^d \rho_\alpha \left[ \frac{1}{4\lambda^2} \left( \frac{s+3m^2}{s-m^2} + \frac{u+3m^2}{u-m^2} \right) - \frac{m^2(i-1)}{\lambda^2} \left( \frac{1}{s-m^2} + \frac{1}{u-m^2} \right) + \frac{m^2(i-1)^2}{\lambda^2} \left( \frac{1}{s-m^2} + \frac{1}{u-m^2} \right) \right] \delta_{\phi \lambda}
\]

where \( \Theta = (k + k')/2 \), \( s = (p+k)^2 \), \( u = (p-k')^2 \)
The factors \( \gamma \) and (1-\( \delta \)) come respectively from the derivative and non derivative Yukawa couplings in (4.45). Putting (4.46) and (4.47) together, the amplitude is found to be

\[
T_{\pi \pi \rightarrow \pi \pi} = \frac{1}{\lambda^2} \left\{ 3 \gamma (\delta-1) + 1 \right\} \left[ \delta \gamma \delta \gamma \left\{ -m^2 + \lambda \left( \frac{m^2}{s-m^2} - \frac{m^2}{u-m^2} \right) \right\} \right.
\]

\[
+ \epsilon^{\ell} j^i \Gamma_{\gamma} \left( \frac{m^2}{s-m^2} + \frac{m^2}{u-m^2} \right) \right]\]

(4.48)

The only change caused by the transformation of the nucleon field (4.44) is the overall constant factor (3 \( \delta^2 - 3 \gamma + 1 \)). On the other hand, the only way to identify the arbitrary constant \( \lambda \) is to compare the residue of the pole term of (4.48) with the known pion nucleon coupling constant, and this will fix the value of \( \frac{1}{\lambda^2} (3 \gamma^2 - 3 \gamma + 1) \) uniquely. Thus the phenomenological lagrangian \( \mathcal{L} \) (4.47) with the tree approximation gives a unique value for the meson-nucleon scattering amplitude too. In the case of meson-nucleon amplitude, we can consider an even more general transformation

\[
N' = f \left( \frac{\pi}{2} \cdot \frac{\pi}{2} \right) N
\]

where \( f(z) = 1 + \delta z + \lambda \frac{z^2}{2} + \mathcal{O}(z^3) \) with arbitrary
real x. This causes the additional interaction of a contact type like diagram (a) discussed above. These additional contributions do, however, cancel each other when the nucleons are on mass-shell.

§4 The weak and the electromagnetic interaction

(a) The field current identity in electro-magnetic interaction.

I would like to discuss the problem of introducing the electromagnetic interaction into the chiral invariant model like (4.21). In what follows, I naturally consider the chiral SU(3)×SU(3) scheme only.

My aim is to put in the additional electromagnetic interaction in such a way that both the ordinary gauge invariance and the field current identity may be satisfied. The latter scheme introduced by Kroll, Lee and Zumino (36, 37, 42, 43) for the electromagnetic interaction has several attractive features and gives a theoretical basis for the assumption of vector meson dominance which is very successful in explaining various features of electromagnetic interaction of hadrons. The Maxwell's equation for such system is written as

\[ \partial_{\mu} F^{\mu
u} = \text{linear combination of vector meson fields + leptonic currents} \]

(4.49)
Following the prescription by Kroll, Lee and Zumino for the iso-spin invariant system. I make the following replacement in the chiral SU(3) invariant lagrangian model discussed above

\[
(U^i_f)_{s=i} \rightarrow (\tilde{U}^i_f)_{s=i} \quad \begin{cases} 
\tilde{U}^3_f = U^3_f + e/g \ A^\tau_f, \\
\tilde{U}^8_f = U^8_f + e/3g \ A^\tau_f, \\
\tilde{A}^{\tau i}_f = U^i_f \ i \neq 3,8 
\end{cases}
\]  

(4.50)

But I leave the vector meson mass term intact. Because of this, I get immediately the right hand side of (4.49) as

\[
\mathcal{J}_f = \frac{m^2}{g} \left( U^3_f + \frac{1}{\sqrt{3}} \ A^8_f \right) 
\]

(4.51)

This is the field current identity.

To see that this replacement (4.50) guarantees the gauge invariance, the following expressions for the covariant quantities discussed so far should be noted

\[
\chi^3_f = \gamma^3_f (E_i (i \tilde{F}^3_f) + E_i (-i \tilde{F}^3_f)) / 2, \\
\beta^3_f = \gamma^3_f (E_i (i \tilde{F}^3_f) - E_i (-i \tilde{F}^3_f)) / 2, \\
\chi^+_{f i} = \chi^3_f (E_i (i \tilde{F}^3_f) + E_i (-i \tilde{F}^3_f)) / 2 + q_f (e^{i \tilde{F}^3_f} + e^{-i \tilde{F}^3_f}) / 2, \\
\chi^-_{f i} = \chi^3_f (E_i (i \tilde{F}^3_f) - E_i (-i \tilde{F}^3_f)) / 2 + q_f (e^{i \tilde{F}^3_f} - e^{-i \tilde{F}^3_f}) / 2. 
\]
From these, I get

$$\varphi_{\mu} = \nabla_{\mu} \Phi - g \chi_{\mu}$$

$$= (\varphi_{\mu} - g \nabla_{\mu} \cdot \mathbf{i} \mathbf{F}) (E_{\mu}(\mathbf{i} \mathbf{F}) + E_{\mu}(\mathbf{i} \mathbf{F})) / 2$$  (4.52)

$$- g a_{\mu} (e^{i\mathbf{F}} + e^{-i\mathbf{F}}) / 2$$

also

$$\beta_{\mu} = g \chi_{\mu}^+ = (\varphi_{\mu} - g \nabla_{\mu} \cdot \mathbf{i} \mathbf{F}) (E_{\mu}(\mathbf{i} \mathbf{F}) - E_{\mu}(\mathbf{i} \mathbf{F})) / 2.$$

$$- g a_{\mu} (e^{i\mathbf{F}} - e^{-i\mathbf{F}}) / 2$$

and thus

$$\hat{\nabla} \Phi = \hat{\nabla} \Phi - ig \chi_{\mu}^+ \cdot \hat{F} \Psi$$

$$= (\hat{\nabla} - ig \hat{\nabla} \cdot \mathbf{U}) \Psi$$

$$+ i \mathbf{U} (\varphi_{\mu} - g \nabla_{\mu} \cdot \mathbf{i} \mathbf{F}) (E_{\mu}(\mathbf{i} \mathbf{F}) - E_{\mu}(\mathbf{i} \mathbf{F})) / 2 \Psi$$

$$- ig \hat{\nabla} \cdot a_{\mu} (e^{i\mathbf{F}} - e^{-i\mathbf{F}}) / 2 \Psi$$

Thus, the derivative $\varphi_{\mu}$ and $\beta_{\mu}$ in the lagrangian model (4.21) always appears as the combinations

$$\left( \varphi_{\mu} - g \nabla_{\mu} \cdot \mathbf{U} \right) \Phi$$

and

$$\left( \varphi_{\mu} - g \nabla_{\mu} \cdot \mathbf{U} \right) \Psi$$

respectively.

It should be remembered also that there is an extra term $\varphi_{\mu}$ introduced into $a_{\mu}$ field. But in the case of Wess and Zumino decomposition (4.24), $\varphi_{\mu}$ term appears as only

$$\nabla_{\mu} \Phi + g (e^{i\mathbf{F}} - e^{-i\mathbf{F}}) / 2 \cdot \mathbf{U}_{\mu}$$

$$+ (\varphi_{\mu} - g \nabla_{\mu} \cdot \mathbf{i} \mathbf{F}) (E_{\mu}(\mathbf{i} \mathbf{F}) + E_{\mu}(\mathbf{i} \mathbf{F})) / 2$$  (4.54)
As the result of these particular combination of vector fields and ordinary derivatives of the fields, the gauge invariance of electromagnetic interaction generated by the replacement (4.50) is guaranteed. The replacement (4.50) is completely equivalent to the ordinary formalism of introducing the electromagnetic interaction in which $\varphi$ and $\mathcal{A}^3$ should be replaced by $(\varphi - i\alpha \mathcal{A}_\mu)\varphi$ and $(\varphi - i\alpha \mathcal{A}_\mu)\mathcal{A}^3$. The assignments of electromagnetic charge for the $(\mathcal{F}_i)_\mu$ is, of course, that of pseudo scalar meson octet $(\pi, K, \eta)$.

(4.54) justifies the Wess-Zumino way of introducing $\mathcal{F}_\mu - \mathcal{A}^\mu$ mixing (4.24) from the point of view of simplicity.

(b) The modified divergence equation in the presence of "weak" perturbations.

The divergence equation (4.19) and (4.20) for $\varphi$ and $\mathcal{A}_\mu$ fields should be modified in the presence of the weak and electromagnetic interaction. In certain approach to the current algebra, such modified divergence equations were given the important role. (Veltman, Nauenberg, Refs. 19 and 20).

The modification of (4.19) and (4.20) (or as in Chapter 1, the FCAC form with meson mass term) with...
the electromagnetic interaction can be done by the method of Adler. The following argument is more or less parallel to the Adler's discussion found in (Ref. 44).

First let us consider the vector field \( \mathbf{A} \) and the divergence equation (4.19). The electromagnetic interaction is introduced by the replacement (4.50).

Starting from the Lagrangian without the electromagnetic interaction

\[
\mathcal{L}_0 = \mathcal{L}(\mathbf{A}, \mathbf{\phi}) + \frac{\hbar^2}{2} \sum \mathbf{A}_i \mathbf{A}^i \quad (4.55)
\]

where \( \mathbf{\phi} \) represents all the fields other than \( \mathbf{A} \).

I have for the Lagrangian with e-m interaction

\[
\mathcal{L}_\text{adm} = \mathcal{L}_0 + \mathcal{L}_\text{em} = \mathcal{L}(\mathbf{\phi}, \mathbf{\phi}_0) + \frac{\hbar^2}{2} \sum \mathbf{\phi}_i \mathbf{\phi}^i \quad (4.56)
\]

Here, of course, \( \mathbf{A}, \mathbf{\phi}_0 \) etc. in (4.56) are different from the ones in (4.55) since they obey the different equation of motion.

Consider the infinitesimal virtual displacement of the field variables

\[
\begin{align*}
\delta \mathbf{\phi} &= i \mathbf{\rmi} \cdot \beta \mathbf{\phi} \\
\delta \mathbf{A} &= i \mathbf{\rmi} \cdot \beta \mathbf{\phi} + \frac{1}{g} \mathbf{\phi}_0 \beta \\
\delta \mathbf{\phi}_0 &= 0
\end{align*}
\quad (4.57)
\]
where \( \vec{\gamma} \) is \( \text{SU}(3) \) generator matrix corresponding to the multiplet \( \vec{\psi} \). Then for this variation as long as it is compatible with the general constraints, Gell-Mann Levy type variational equations hold.

\[
\frac{\partial}{\partial \gamma_i} \frac{\delta L}{\delta \beta_i} = \frac{\delta L}{\delta \beta_i}
\]  

(4.58)

Under the variation (4.57), the term \( L(\vec{\psi}, \bar{\psi}) \) is invariant and the mass term \( \frac{m^2}{2} \sum \psi \bar{\psi} \) gives

\[
\delta \left( \frac{1}{2} m^2 \psi \bar{\psi} \right)
= \delta \left( \frac{1}{2} m^2 (\hat{\psi}_i - \frac{e}{g} \hat{A}_i \bar{\psi})^2 \right)
= \delta \left( \frac{1}{2} m^2 \left( \psi_i^2 - \frac{2e}{g} \hat{\psi}_i \hat{A}^t + \frac{e^2}{g^2} \hat{A}_i^2 \right) \right)
= -m^2 \frac{e}{g} \beta^c \left( \hat{F}^{\hat{c}}_{jk} \hat{\psi}_i \hat{A}^j + \frac{1}{g} \hat{A}_i \hat{A}_j \right)
= -m^2 \frac{e}{g} \beta^c \left( \hat{F}^{\hat{c}}_{jk} \psi_j \right) \hat{A}^i \hat{A}^j
\]

where

\[
\hat{A}^{\hat{c}}_\hat{i} = 0 \quad \hat{c} \neq 3, 8
\]

\[
\hat{A}^{3}_\hat{i} = \hat{8} \quad \hat{i} = 3
\]

\[
\sqrt{3} \hat{A}^{8}_\hat{i} = \hat{3} \quad \hat{i} = 8
\]

Thus

\[
\frac{\delta L}{\delta \beta_i} = -\frac{m^2 e}{g} \hat{F}^{\hat{c}}_{jk} \psi_j \hat{A}^i \hat{A}^j
\]

and

\[
\frac{\delta L}{\delta \gamma_i} = -\frac{m^2}{g} \psi_i \hat{A}^t
\]
From (4.58), I get now
\[
\Theta_f U^i_f = e^{(iF^i_f U^i_f)} A^i_f \tag{4.59}
\]

As for fields, consider the virtual displacement corresponding to an infinitesimal chiral transformation
\[
\begin{align*}
\delta A_f &= i F \cdot \delta \gamma_f + \frac{i}{g} \Theta_f \gamma_f \\
\delta \gamma_f &= i F \cdot A_f \\
\delta A_f &= 0
\end{align*} \tag{4.60}
\]

And instead of (4.56), I write
\[
\mathcal{L}_{bd} = \mathcal{L}'(\gamma_f; \Phi) + \frac{m^2}{4} (U_f^2 + A_f^2) \tag{4.61}
\]

is again invariant under (4.60) and I get
\[
\begin{align*}
\frac{\delta \mathcal{L}}{\delta \gamma^i_f} &= - \frac{m^2}{g} i F^i_f A^i_f A^{j^*} \\
\frac{\delta \mathcal{L}}{\delta A^i_f} &= - \frac{m^2}{g} A_i^i \\
\end{align*}
\]

And from
\[
\frac{\delta}{\delta \gamma^i_f} = \frac{\delta \mathcal{L}}{\delta \delta \gamma^i_f}
\]
I get
\[
\Theta_f A^i_f = e^{(iF^i_f A^i_f)} A^i_f A^{j^*} \tag{4.62}
\]

(4.59) and (4.62) are required modified divergence equations. In the case of SU(2) with the only iso-
vector part of e–m interaction considered, they reduces to the form considered by Veltman

\[ \partial_t U^r = e \mathcal{A}^r \wedge U^r \]

\[ \partial_t A^l = e \mathcal{A}^r \wedge A^r \]

Veltman\(^{(20)}\) further considered the modification due to the perturbation of the weak interaction type. This divergence equations can be obtained by the replacements of the field variables of the following form

\[
\begin{align*}
    a_r & \to a_r + \frac{G}{f} \mathcal{W}^A_r \\
    v_r & \to v_r + \frac{G}{f} \mathcal{W}^V_r
\end{align*}
\]

where \(G\) is the weak interaction constant. \(\mathcal{W}^A_r\) and \(\mathcal{W}^V_r\) correspond to different parity parts of "weak boson" fields. Then for chiral SU(2), I can get the divergence equations used by Veltman\(^{(20)}\).

\[
\begin{align*}
    \partial_t a^r &= G \left( \mathcal{W}^A_r \wedge U^r + \mathcal{W}^V_r \wedge a^r \right) \\
    \partial_t U^l &= G \left( \mathcal{W}^V_r \wedge U^l + \mathcal{W}^A_r \wedge a^l \right)
\end{align*}
\]
CHAPTER 5

The breaking of chiral $SU(3) \times SU(3)$ symmetry in the non-linear realization techniques and the application to the interaction of the hadrons.

§1 The introduction

In the previous three chapters the principles of the non-linear realization techniques for chiral $SU(n) \times SU(n)$ symmetry have been discussed. I would now like to present some applications of chiral $SU(3) \times SU(3)$ symmetry with the phenomenological Lagrangian of the type discussed in Chapter 4.

There are already many examples of successful application of non-linear realization techniques for chiral $SU(2) \times SU(2)$ symmetry\(^{45}\). Because it is free of the laborious computations involved in the current algebra techniques, it has been found to be quite useful in understanding some aspects of elementary particle interaction even though in many cases it just reproduces current algebra results. Also, this technique found the appeal to some people because it emphasizes more strongly the symmetry or the group theoretical point of view for the "chiral dynamics".

The similar applications for the chiral $SU(3) \times SU(3)$ case are hindered by the fact that there is yet no
definite prescription of how to take account of the symmetry breaking. Of course, this problem has its parallel in the current algebra approach. To estimate "$\sigma$-terms", for instance, one is always forced to make one or another of the plausible assumptions. In the phenomenological lagrangian method with chiral SU(3) x SU(3), the good agreements with experiments were achieved often by putting in the symmetry breaking "by hand", for instance by replacing some invariant mass term in the lagrangian by the physical masses\(^{(46)}\). (The best example of an earlier application of chiral SU(3) symmetry with non-linear realization techniques is found in the paper by Cronin\(^{(2)}\), where many of the ideas which have more conveniently formulated later are already present).

Recently, however, a theory of broken chiral SU(3) symmetry has been proposed by Gell-Mann, Oakes and Renner\(^{(47)}\). Although many of their ideas can be found in the works of previous authors\(^{(48)}\), they presented their method in such a way that it formulates the prescription to the given problems with seemingly much less ambiguity. In particular, the definite ratio of the strength of SU(3) singlet and octet
component of symmetry breaking part of strong interactions is suggested as a kind of universal constant.

The original authors treat the problem in terms of currents and their commutation relations. But it is straightforward to construct a parallel theory in terms of a chiral lagrangian with non linear realization. In fact the simplicity of the Gell-Mann, Oakes and Renner scheme becomes most apparent in the latter approach. A very thorough study of the general structure of such a theory has been given by Macfarlane, Sundbery and Weisz. But their emphasis on the most general definition of the fields within the non-linear realization techniques seems to give their work a forbiddingly complicated appearance without reaching the essential simplicity expected from the group theory.

§2 The breaking of chiral symmetry in the non-linear realization.

It is well known within the framework of ordinary unitary symmetry that one can represent the symmetry breaking part of the interaction as the combination of simple representations of the SU(3) group. And the simplest and the most popular one is to consider
the strong interaction hamiltonian as the combination of an SU(3) singlet (i.e. symmetry reserving) and the octet representation. Gell-Mann, Oakes and Renner generalize this idea to chiral SU(3) symmetry. They assume that the symmetry breaking can be considered as the simple (linear) representation of the chiral SU(3) group, \( K \). They choose a single \((5,\bar{3}) \oplus (\bar{5},3)\) representation as it is the simplest one which satisfies various physical requirements. They express this idea in terms of hamiltonian density responsible for the symmetry breaking.

\[
\mathcal{H} = - u_0 - c u_8
\]

where \( u_0 \) and \( u_8 \) are the SU(3) singlet and octet contained in \((5,\bar{3}) \oplus (\bar{5},3)\) representation. Of course, the chirality implies the different parities and \( u_0, u_8 \) should be considered as the scalors since the symmetry breaking interaction \( \mathcal{H} \) still conserves the parity. The interaction of the form (5.1) has been used previously in the chiral lagrangian method \((49)\). But Gell-Mann, Oakes and Renner further propose to put the constant \( c \) uniquely determined number independent of the particular physical process under the consideration. They determine the value of the number \( c \) from the consideration of single matrix elements of the currents.
with the PCAC assumption, derived from the transformation of $\pi^+(\pi^0)$ representation under the chiral group $K$, and propose to use the same value for treating more complicated processes.

In expressing the idea of Gell-Mann, Oakes and Renner in terms of the chiral lagrangian scheme discussed in the previous chapters, the first step is, of course, to construct the quantities like $\mathcal{U}_0$ or $\mathcal{U}_\pi$ in terms of non-linearly transforming quantities like $\mathcal{F}_\rho$ or $\mathcal{F}_\pi$. But the construction of (linear) representation of chiral group within the non-linear realization scheme has been fully discussed in Chapter 2. This is the problem solved by Coleman, Wess and Zumino.

Following the notation of Chapter 2, the representation $D$ of the chiral group can be constructed out of non-linear "fields" $\psi_a$ as

$$\mathcal{U}_n = D_{\alpha \alpha} (e^{\gamma A}) \psi_a$$

provided that the representation of the diagonal subgroup $H (=SU(3)$ in the case considered here) spanned by $\psi_a$ is contained in $D$ when restricted to $H$.

It should be noted that Gell-Mann, Oakes and Renner start from the hamiltonian formalism, and the non-linear realization scheme discussed so far is
is conveniently expressed only in terms of the lagrangian. But they also demand that the symmetry breaking hamiltonian should be a Lorentz scalar. This requirement is satisfied by the interactions which do not involve the derivative of field variables. For such an interaction, the relation between the lagrangian and the hamiltonian is trivial.

For the sake of illustration, I shall give first the construction of the form like (5.1) using the non-linear fields \((\frac{3}{2}, \frac{1}{2})^8\) only. The construction of the particular representation \((\frac{3}{2}, \frac{1}{2})^8\) only is possible, according to the theorem of Coleman, Wess and Zumino discussed in Chapter 2, because this representation contains the scalar representation (singlet) when restricted to SU(3) diagonal subgroup. Choosing a suitable co-ordinate system, the eighteen components of \((\frac{3}{2}, \frac{1}{2})^8\) representation can be given, by (5.2), as

\[
\begin{pmatrix}
0 \\
\ldots \\
0 \\
\ldots \\
0 \\
\ldots \\
u_{0,1} \\
u_{0,2} \\
u_{0,3} \\
\ldots \\
u_{0,6} \\
\ldots \\
u_{0,9} \\
\ldots \\
u_{0,12} \\
\ldots \\
u_{0,15} \\
\ldots \\
u_{0,18} \\
\end{pmatrix} = e^{i \alpha \cdot \mathbf{3}} \begin{pmatrix}
0 \\
\ldots \\
0 \\
\ldots \\
0
\end{pmatrix}
\] (5.3)
where $\psi$ is the wave function of the particle. The representation matrix

$$D(e^{i\alpha}) = e^{i\alpha D^3}$$

can be written as

$$Q \cdot \overline{\Omega} = i \begin{pmatrix} 0 & D^3 \\ -D^3 & 0 \end{pmatrix}$$

with

$$D^3 = \begin{pmatrix} 0 & -\sqrt{\frac{\overline{\Omega}}{\Omega}} \\ \sqrt{\frac{\overline{\Omega}}{\Omega}} & 0 \end{pmatrix}$$

as defined

$$(D^i)_{jk} = d_{ij}.$$

We may set

$$U \alpha(\overline{\Omega}) = \{ \cos(D^3) \}_{\lambda_{\sigma}}$$

and

$$V \alpha(\overline{\Omega}) = \{ \sin(D^3) \}_{\lambda_{\sigma}}$$

where $\lambda_{\sigma}$ come from to $\lambda$.

In the next section we give a general treatment of the quantum states as spinors (e.g.) and consider the additional conditions. The spinors in the next section are generally different, and the spinors come in the form of one with no indication of the spinors. Here comes a brief introduction of the theory.
On the other hand, $\xi$ fields are going to be treated as pseudo scalar with respect to space reflection. Therefore if, for instance, the octet and singlet components $\{U_a\}$ are required to be scalar so that it can be used to construct the hamiltonian of the type (5.1), then the above generalization does not give essentially new construction over (5.5) and (5.6).

§3 The pseudo scalar meson lagrangian.

(a)

The chiral symmetric lagrangian of the pseudo scalar meson octet is given by

$$\mathcal{L} = \frac{\alpha}{2} \left( \nabla^2 \xi \right)^2$$  \hspace{1cm} (5.7)

This represents the mass-less particles interacting with each other. If, in addition to (5.7), I may take the symmetry breaking interaction of type (5.1) there is the possibility of giving them the finite masses as well as their physical mass splitting within
within the octet. Taking the iso-spin hypercharge conserving members of the scalar part of the multiplet (5.5), I will write Gell-Mann, Cakes, Renner type symmetry breaking interaction as

\[ H = -\mu_0^2 \lambda^2 (u_0 + cu_8) \]  

(5.8)

where \( u_0 \) are given by (5.5), \( \mu_0 \) is an arbitrary constant and \( \lambda \) is to cancel the normalization constant of the physical P-S octet fields. I start by putting (as in Chapter 4)

\[ \phi_i (\text{physical P-S octet}) = \lambda \xi \xi_i \]

Since the interaction (5.8) does not contain the derivatives, the kinematical term of the meson lagrangian is contained in the chiral invariant part (5.7). As given in Chapter 2,

\[ \nabla_\mu \phi = \phi_\mu \frac{\sin \tilde{F}_\phi}{\tilde{F}_\phi} \]

\[ \sim \phi_\mu + O(\xi^3) \]

and

\[ \alpha \left( \nabla_\mu \phi \right)^2 = \frac{\alpha}{2} \left( \phi_\mu \right)^2 = \frac{1}{2} \xi^2 \frac{\alpha}{\lambda^2} (\xi \phi_i)^2 \]

Thus I get from the condition of correct kinematical term

\[ \lambda \xi^2 = \alpha = \text{independent of } i \]  

(5.9)
This in fact implies that within our simple model of the symmetry breaking, we cannot account for the difference between the leptonic decay constants of P-S mesons within the octet.

For the purpose of the present discussion, it is sufficient to consider the first few powers of in \( U_0(3) \) and \( U_a(3) \) given by (5.5) and (5.6).

Expanding (5.5) and (5.6), I get

\[
U_0(3) = \left( 1 - \left( D^2 \right)^{3/2} + \left( D^2 \right)^{9/2} \right) a_0 + O(3^6),
\]

\[
U_a(3) = \left( D^2 - \left( D^2 \right)^{3/2} \right) a_0 + O(3^5),
\]

which can be explicitly written in terms of

\[
U_0 = 1 - \frac{3^2}{3} + \left( \frac{3^2}{3} \right)^{3/2} + O(3^6) \tag{5.10}
\]

\[
U_a = \frac{1}{\sqrt{3}} \left( -1 + \frac{3^2}{3} \right) d_{i,j} \frac{3_i}{3_j} \frac{3_k}{3_k} + \frac{1}{18 \sqrt{6}} d_{i,j,k} \frac{3_i}{3_i} \frac{3_j}{3_j} \frac{3_k}{3_k} + O(3^6) \tag{5.11}
\]

\[
U \sigma = -\frac{1}{q} d_{i,j,k} \frac{3_i}{3_i} \frac{3_j}{3_j} \frac{3_k}{3_k} + O(3^5) \tag{5.12}
\]

\[
U_c = \sqrt{\frac{1}{3}} \left( 1 - \frac{3^2}{6} \right) \frac{3_i}{3_i} + O(3^5) \tag{5.13}
\]

Putting (5.10) and (5.11) into (5.8) and taking it only upto the quadratic term in \( \mathcal{A} \), I get

\[
\mathcal{H} \propto \frac{\mu^0}{\lambda^2} \left( \frac{3_i}{3_i} + \frac{3_j}{3_j} d_{i,j} \frac{3_i}{3_i} \right)
\]
Choosing $\lambda^2$ to be equal to $\lambda^2 = \alpha$, and introducing the physical $P-S$ meson octet with the usual assignment for charge etc., this can be written as

$$\frac{\mu_0^2}{2} \left\{ \frac{\sqrt{2}}{3} \left( \sqrt{2} + c \right) \left( 2 \pi^+ \pi^- + \pi^0 \right) \right. + \frac{\sqrt{2}}{3} \left( \sqrt{2} - \frac{c}{2} \right) \left( 2 K^+ K^- + 2 \bar{K}^0 K^0 \right) + \left. \frac{\sqrt{2}}{3} \left( \sqrt{2} - c \right) \eta^2 \right\}$$

(5.14)

This is just the mass term we want. I can identify the masses of $P-S$ meson octet as

$$\mu_{\pi}^2 = \mu_0^2 \frac{\sqrt{2}}{3} \left( \sqrt{2} + c \right)$$

$$\mu_K^2 = \mu_0^2 \frac{\sqrt{2}}{3} \left( \sqrt{2} - \frac{c}{2} \right)$$

$$\mu_{\eta}^2 = \mu_0^2 \frac{\sqrt{2}}{3} \left( \sqrt{2} - c \right)$$

(5.15)

If I fit the experimental value for $\mu_{\pi}^2 / \mu_K^2$ (averaged over isospin multiplet), I get

$$c = -0.889 \times \sqrt{2} = -1.26$$

(5.16)

and $\mu_0^2 = 0.96 \mu_K^2$

This gives $\mu_{\eta}^2 = 30.26 \times 10^5$ (MeV)$^2$ while experimentally $\mu_{\eta}^2 = 30.11 \times 10^5$ (MeV)$^2$.

I should remark at this point that here and throughout the following, I entirely neglect the problem of mixing.
These results with (5.9) are the ones obtained in the paper by Gell-Mann, Oakes and Renner. (5.15) satisfies the Gell-Mann, Okubo mass formula for squared masses

\[
\bar{m}_k^2 = \frac{1}{4} \left( \bar{m}_\pi^2 + 3 \bar{m}_\eta^2 \right) \tag{5.17}
\]

Next I consider the problem of PCAC. When the chiral symmetry breaking part of interaction is given by (5.8), the axial currents given by the variational principle from the given lagrangian, as in Chapter 4,

\[
A^\mu_\tau = - \frac{\delta L}{\delta \partial_\tau \phi_\tau}
\]
satisfy the divergence conditions

\[
\partial^\tau A^\mu_\tau = \frac{\delta H'}{\delta \phi_\tau} = - [Q, H']
\]

\[
= \mu_0^2 \lambda' \left( -\sqrt{2} \phi_2 - c \delta \varepsilon_{ij} \phi_j \right)
\]

\[
= \mu_0^2 \lambda' \left( \frac{\mu_0^2}{3} \phi_2 + \sqrt{2} c \delta \varepsilon_{ij} \phi_j (1 - \frac{\mu_0^2}{3}) \right)
\]

From (5.15), the right hand sides can be written in terms of the physical meson fields

\[
\mu_0^2 \lambda' \left( \frac{\mu_0^2}{3} \phi_2 + \sqrt{2} c \delta \varepsilon_{ij} \phi_j \right)
\]

\[
= \left\{ \begin{array}{ll}
\mu^2 \lambda \phi_i & \text{if } i = 1, 2, 3 \\
\mu^2 \lambda \phi_i & \text{if } i = 4, 5, 6, 7 \\
\mu^2 \lambda \phi_i & \text{if } i = 8
\end{array} \right.
\]

Thus I have

\[
\partial^\tau A^\mu_\tau = \mu^2 \phi_c (1 - \frac{\phi_c^2}{8 \lambda^2}) \tag{5.18}
\]
Up to the linear term in $\pi$ meson fields, (5.18) is just the expression of PCAC, and coefficients gives the "residue of one meson singularity".

Thus $\lambda - F_i$ gives the (uniform) leptonic decay constants of pseudo-scalar meson octet. We have within our approximation

$$ F_{\pi} = F_{\eta} = F_{\eta} $$

(5.19)

But (5.18) also has cubic correction term, this implies that PCAC cannot be assumed in calculating the meson-meson scattering amplitude. Thus if the off-mass shell meson-meson amplitude is calculated using $\phi_0$ as the physical meson fields it will not satisfy the Adler consistency condition.

Unlike the mass relation, the leptonic decay constants $F_{\pi}$ and $F_{\eta}$ ($F_{\eta}$ cannot be measured because of fast radiative decay) are not too different and (5.19) $F_{\pi} \approx F_K$ can be considered as reasonable. Nevertheless, the experimental value $F_{\pi}/F_{\eta} \sim 1.26$ should be somehow accounted for. I may, for instance, incorporate into the Lagrangian terms containing the derivatives of $\phi$ fields like

$$ \mathcal{L}_{\text{der}} = -\frac{\lambda^2}{2} (u \delta' + c u' \delta) $$

(5.20)
with \((\gamma, \bar{\gamma}) + (\sigma, \bar{\sigma})\) multilet not constructed out of covariant derivatives

\[
\begin{pmatrix}
v' \\
n' \\
\end{pmatrix} = e^{i\varphi}\begin{pmatrix} 0 \\
d_{i\ell} \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \theta_j} \end{pmatrix}
\]

It should be noted that an interaction like (5.20) cannot be considered as being within the scheme of cill-ion, i.e., lepton. The Hamiltonian corresponding to (5.20) is not Lorentz scalar.

Taking (5.20) to either with the invariant term

\[
L_0 = \frac{\lambda^2}{2} (\nabla_{\lambda})^2
\]

I get the following modified relation for the leptonic decay constant

\[
\begin{align*}
\lambda^2 - c \lambda' / \sqrt{3} &= F_{\pi}^2 \\
\lambda^2 + c \lambda' / \sqrt{3} &= F_{\pi}^2 \\
\lambda^2 + c \lambda' / \sqrt{3} &= F_{\eta}^2
\end{align*}
\]  

(5.21)

The first relations (5.15) should be also modified and these can be solved again as fitting the value of \(F_{\eta} / F_{\pi}\) as well as \(F_{\pi} / F_{\eta}\). This modifies the value of \(c\) and brings it even closer to \(-\sqrt{2}\).

\[
c \approx -1.37
\]  

(5.22)
\[ \mu_\eta \text{ and } F_\eta \text{ are calculated to give} \]
\[ \mu_\eta \sim 5.40 \text{ MeV} \]
\[ F_\eta / F_n \sim 1.34 \]

On the other hand, the ratio \( \lambda_1 / \lambda^2 \) is turned out to be rather large

\[ c \lambda_1 / \lambda^2 \sim 0.49 \quad \text{(5.23)} \]

The way of accounting for \( F_\eta / F_n \) described above is probably unsatisfactory, and it will be found later that for other calculations like in scattering length it is important to use the physical \( F_\eta \) value to get the reasonable agreement. Thus this simple minded scheme cannot be considered as satisfactory unless the \( F_\eta / F_n \) ratio is correctly described. It has been suggested that this and related problems can be treated in a satisfactory way where at least the vector and axial vector gauge fields are incorporated \( (50) \). The trouble seems to be that it is not straightforward to construct Gell-Mann - Oakes - Renner type interaction to describe, for instance, vector meson mass splitting or the mixing between octet and singlet mesons.

(b) Baryon-meson scattering.

Let us return to the non-derivative interaction \( (5.8) \) or the corresponding term in the lagrangian
with (5.5) for $\mathcal{L}_0$. 

Interesting point about the non-linear realization scheme is that the introduction of mass term like (5.14) does automatically generate higher order terms in $\mathcal{F}$ fields required from symmetry. These terms in general contribute to the other physical processes. Thus the fourth order term of $\mathcal{F}$ fields ((5.10) and (5.11)) give rise to the 4-point contact interaction among mesons which modifies $\pi\pi$ meson scattering amplitudes. Computing $\mathcal{L}'$ above up to fourth order using (5.10) and (5.11), I get

\[
\mathcal{L}' = -\frac{i}{2} \left( \mu_\pi^2 \pi^2 + 2\mu_k^2 \vec{K} \cdot \vec{K} + \mu_\eta^2 \eta^2 \right) \\
+ \frac{i}{24\lambda^2} \left( \pi^2 + 2 \vec{K} \cdot \vec{K} \right) \left( \mu_\pi^2 \pi^2 + 2\mu_k^2 \vec{K} \cdot \vec{K} + \mu_\eta^2 \eta^2 \right) \\
+ \frac{\mu_0^2}{18\lambda^2} \, d:j: \Phi: \Phi; \Phi \phi \eta \tag{5.24}
\]

Terms contributing to the scattering of $\pi\pi$, $\pi K$ and $K K$ is

\[
\mathcal{L}_{\text{scattering}} \propto \frac{1}{24\lambda^2} \left( \pi^2 + 2 \vec{K} \cdot \vec{K} \right) \left( \mu_\pi^2 \pi^2 + 2\mu_k^2 \vec{K} \cdot \vec{K} \right) \tag{5.55}
\]

The contribution to the scattering amplitudes from the chiral invariant term (5.7) has also been computed
Then I get the contribution from (5.25) and (5.26) for each scattering process as follows

(A)

\[
\int S^{\text{scattering}} = \frac{1}{2 \lambda^2} \left[ (\Omega \cdot \Pi)^2 - (\Omega^2 \Pi^2) \right] + \frac{i}{\lambda^2} \left\{ \frac{1}{2} (\Pi \cdot \Omega)(\vartheta f K \cdot K + \vartheta f K \cdot K) - \frac{3}{2} (\Omega \cdot \Pi)(\vartheta f K \cdot K - \vartheta f K \cdot K) - \frac{1}{2} \pi^2 \vartheta f K \vartheta f K - \frac{1}{2} (\vartheta f K^2) K \cdot K \right\} (5.26)
\]

Computing the scattering amplitude for the process

\[ P_{\text{a}}(k) + P_{\text{c}}(k) \rightarrow P_{\text{a}}(k') + P_{\text{d}}(k'') \]

we get

\[
T^{\pi_\pi} = \frac{i}{\lambda^2} \left[ (5 - t \eta^2) \delta_{ac} \delta_{ad} + (u - t \eta^2) \delta_{ad} \delta_{ac} + (t - \eta^2) \delta_{ac} \delta_{cd} \right] - \frac{i}{3 \lambda^2} (2 \eta^2 - \eta^4) (\delta_{ac} \delta_{ad} + \delta_{ad} \delta_{ac} + \delta_{ac} \delta_{cd}) (5.28)
\]
where

\[ S = (k_a + k_c)^2, \quad U = (k_a - k_d)^2, \quad t = (k_a - k_c)^2 \]

(B)

\[
\mathcal{L}_{\text{scattering}}^{\mathcal{K} \pi} = \frac{i}{12\lambda^2} \left\{ (\pi \cdot \pi \cdot \pi)(k_a \cdot k_a - k_c \cdot k_c) \ight. \\
- \pi^2 k_a \cdot k_b k_c - (\pi \cdot \pi \cdot \pi) k_a \cdot k_b k_c \\
+ 3i(\pi \cdot \pi \cdot \pi)(k_a \cdot k_c - k_a \cdot k_b) \\
\left. + \left( a^2 + \mu_k^2 \right) \pi^2 k_a \cdot k_b k_c \right\} (5.29)
\]

The scattering amplitude for the process

\[ \pi a(k_a) + k_c(k_d) \rightarrow \pi a(k_a) + k_d(k_d) \]

is

\[
T_{\pi a}^{k_c} = \frac{1}{12\lambda^2} \left\{ (3t + \left( 2k_a^2 - 2k_c^2 - 2\mu_k^2 \right) sa \cdot sa \right\} \\
+ 3i(s - u) e_b a_c (\sigma_e)_{ba} \right\} (5.30)
\]

where \( s = (k_a + k_d)^2, \quad U = (k_a - k_d)^2, \quad t = (k_a - k_c)^2. \)

(C) \( \mathcal{K}, \mathcal{K} \)

\[
\mathcal{L}_{\text{scattering}}^{\mathcal{K} \mathcal{K}} = \frac{i}{6\lambda^2} \left\{ (\pi \cdot \pi \cdot \pi)(k_a \cdot k_a - k_c \cdot k_c) \ight. \\
- (k_a \cdot k_a - k_c \cdot k_c) \\
\left. + \left( \pi \cdot \pi \cdot \pi \right) \pi^2 k_a \cdot k_b k_c \right\} (5.31)
\]

and for the process

\[ k_a(k_a) + k_c(k_c) \rightarrow k_d(k_d) + k_s(k_s) \]
I let

$$T^{\alpha \beta} = \frac{1}{\delta \lambda} (-3 \delta^{\alpha \beta} + \gamma \delta^{\alpha \beta}) (\delta^{\beta \alpha} \delta^{\gamma \delta} + \delta^{\gamma \delta} \delta^{\beta \alpha})$$  \hspace{1cm} (5.33)

where \( S = (k_{\lambda} + \delta \gamma)^2 \)

It should be noted that the amplitudes (5.28), (5.30) and (5.32) do not satisfy the Adler consistency condition, in accordance with the discussion on the modified PCAC equation (5.18).

To recover such off-mass-shell condition, I can redefine the physical meson fields so that the PCAC relation holds up to higher order of meson fields. For instance I can put, from (5.18), as

$$\phi_{\gamma} (\text{physical meson fields}) = \phi_{\gamma} (1 - \Phi^2 / \delta \lambda^2)$$

Then (5.18) becomes

$$\partial^\mu A_{\mu} = \gamma \phi_{\gamma} \phi'_{\gamma} + O(\phi'_{\gamma}^3)$$

The effect of such transformation (i.e. to use \( \phi'_{\gamma} \) instead of \( \phi_{\gamma} \) as second quantized meson fields in our "tree approximation" calculation) is to add to the \( T \) matrix elements the correction terms proportional to

$$\sum_k h_k - \sum p^2$$

In this way, I modify the off-mass-shell value of the scattering amplitude so that it now satisfies the Adler
\[
\begin{align*}
T_{\pi\pi} &= \frac{1}{\lambda^2} \left[ (s-\mu^2) \delta_{ac} \delta_{ad} + (u-\mu^2) \delta_{ad} \delta_{ac} ight. \\
&\quad \left. + (t-\mu^2) \delta_{ae} \delta_{cd} \right] \\
T_{\pi K} &= \frac{1}{4\lambda^2} \left[ (t+2q^2-2\mu^2-2p_k^2) \delta_{ae} \delta_{ef} \right. \\
&\quad \left. + i(s-u)e\epsilon\lambda_{ef} \right] \\
T_{KK} &= \frac{1}{2\lambda^2} (-s + \Sigma k^2 - 2p_k^2) 
\end{align*}
\]

We may even have "exact PCAC" instead of (5.18) if it is put directly
\[
\phi_i' = \lambda \sqrt{\frac{3}{2}} \epsilon_i \quad c = 1, \ldots, 8
\]
where \((\epsilon_i)_{a}^{b}\) is defined in (5.6)
\[
\partial^i A_{\mu}^i = 2\mu_i^2 \phi_i' \quad (5.36)
\]

(5.36) is the parallel of chiral SU(2) divergence equation (1.80).

At this point, I can compare the results of the present section with those of Chapter 1 (§5b). If I take from the expression of \(U_{\mu \rho}^i\) in (5.5) the terms containing "pion fields" \(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3\) only, these can
be explicitly summed up and I get
\[ H_\pi' = (-u_0 - c u_8) \tau \]
\[ = -m^2 \frac{2}{3} (\sqrt{2} + c) \cos \sqrt{3} + \text{constant} \bar{u} \bar{u} \]
That is
\[ L_{\pi'} = m' \lambda' \cos \sqrt{3} \]  
(5.37)
But this is just the expression (1.76) on chiral SU(2) and shows that the symmetry breaking term \( L_{\pi'} \) is essentially the 4th components of 4-vector representation \( (\frac{1}{2}, \frac{3}{2}) \) of chiral SU(2)xSU(2) group.
The representation \( (3,3) + (3,\bar{3}) \) of the chiral SU(2)xSU(3) contains the representation \( (\frac{3}{2}, \frac{3}{2}) \) when restricted to the chiral SU(2)xSU(2), and the results of this section can be considered as the generalization of Weinberg's scheme explained in Chapter 1. § 5

The analysis of the meson-lagrangian presented in this section is essentially not new. It can be found in the paper by Cronin(2) where the identical form of the symmetry breaking term is used. Moreover, so far as mesons are concerned his point of view is more general. The results of this section (5.27)~(5.32)
coincide with Cronin's if, using his notation, \( a_\beta = 4/3 \) in his formulae. This is as it should be since \( a_\beta = 4/3 \) in the expansion of meson matrix in Cronin's paper corresponds to the exponential meson matrix used in the present thesis. (Cronin considers the wider form of meson matrix instead of redefining the meson fields like I have done. For instance, PCAC results with Adler condition can be obtained in Cronin's formalism by putting the coefficients of 3rd power of meson fields in the meson matrix \( a_\beta \) to zero. These two approaches should be equivalent as has been shown by Weinberg\(^{(7)}\) for chiral SU(2) and studied by Macfarlane and Weisz\(^{(10)}\) for general case).

(c) The vector, axial fields

I have derived the expression of meson-meson scattering amplitudes from non-linear lagrangian of the form

\[
\mathcal{L} = \frac{a}{2} \left( p_1 \cdot p_2 \right)^2 + \text{mass terms} \quad (5.36)
\]

From these amplitudes, the scattering length of mesons can be derived and the results agree with the current algebra\(^{(52)}\). On the other hand, it is well known that the low energy meson-meson interaction can
be accounted for rather well by the vector dominance model. Weinberg in (Ref. 7) shows that his chiral SU(2) invariant lagrangian with vector mesons gives the same result for low energy π-π scattering as the non linear lagrangian of form (5.30), except the difference of the order of $\frac{\mu^2}{m^2}$.

This result can be formally extended to the case of chiral SU(5). From the point of view of physics, this may not be so useful since, for instance, $\frac{\mu^2}{m^2}$ ~ 0.3 is not too small.

For the sake of the simplicity, let us consider the SU(5) version of Weinberg's model with non linear type vector mesons (Chap. 4), rather than West and Lumino or Kawaiabayashi model. Let us also disregard the symmetry breaking with respect to vector mesons. As it has been described in Chapter 4, the lagrangian in question is of the following form

$$\mathcal{L} = \frac{\lambda^2}{2} (\nabla^2 \phi)^2 + \text{mass terms}$$

$$+ \frac{m^2}{2} (\chi^+)^2 - \frac{1}{4} G_{\mu\nu}^2$$

(5.38)

where

$$\chi^+ = \varphi + \frac{1}{8} \varphi^4$$

$$G_{\mu\nu} = \partial_{\mu} \varphi - \partial_{\nu} \varphi + g \varphi (iF_{\mu\nu}$$

and $\varphi$ is, as usual, equal to

$$\varphi \approx \frac{E_1(\mathcal{E}^2) - E_1(-\mathcal{E}^2)}{2}$$
Here of course F's are ordinary F matrices of SU(5) and defined by

\[
(F^i)_{jk} = - i f_{jk} \quad i, j, k = 1, \ldots, 8
\]

The mass terms in (5.38) and (5.39) are the same and the parts of the scattering amplitude coming from them are independent of the presence or the absence of the vector fields.

So far as the meson-meson scattering is concerned (5.39) differs from (5.38) by the presence of the term

\[
\frac{m^2}{2g^2} \beta^2 + \frac{m^2}{4} \nu \beta^I \beta^J 
\]

(5.40)

The first term gives the contribution to the amplitude because of the 4-pt contact term

\[
\frac{m^2}{2g^2} \langle i \mathbf{f} \int d^4x (\beta^I (x)) \mathbf{f} \rangle
\]

(5.41)

where \( \downarrow \mathbf{i} \rangle \) and \( \uparrow \mathbf{f} \rangle \) represent the initial and the final two meson states.

On the other hand, the second term gives rise to VMD type interaction

\[
\frac{m^2}{2g^2} \int d^4x \nu \beta I \mathbf{f} \Theta \mathbf{f} \mathbf{f} \Theta 
\]

(5.42)
The contribution to the scattering amplitude comes from the vector meson exchange diagrams.

\[
\begin{align*}
& \left( -\frac{1}{2} \right) \frac{m_v^2}{g^2} \left< \phi \int \! \! d^4x \int \! \! d^4y \, \mathcal{T} \left( \beta_1(\gamma) \beta_2(\gamma) \gamma^\dagger \right) \right> \\
& = -\frac{1}{2} \frac{m_v^2}{g^2} \left< \phi \int \! \! d^4x \int \! \! d^4y \, \mathcal{T} \left( \beta_1(\gamma) \beta_2(\gamma) \gamma^\dagger \right) \right> \\
& \times \left( -\frac{i}{\mu^2} \right) \int \! \! d^4\kappa \, e^{-i\kappa \cdot \mathbf{L}} \, \delta(\mathbf{p}_v - \mathbf{K}) \, \left( \mathbf{k}^2 - m_v^2 \right)
\end{align*}
\]

On the other hand, from the consideration of corresponding diagrams \( k \) is seen to represent the momentum of the exchanged vector meson and the main contribution to the integral over \( k \) comes from \( k \propto P-S \) meson momenta. In the low energy region where \( P \propto P' \) for the external momenta, I may conclude \( k \propto K_v \propto \mu \) thus if \( \mu^2 \ll m^2 \), \( K_v \propto \mu \) in the vector meson propagator can be neglected compared with \( \mathcal{G}_{\mu\nu} \) term and the above expression reduces as

\[
\begin{align*}
& \left( -\frac{1}{2} \right) \frac{m_v^2}{g^2} \left< \phi \int \! \! d^4x \int \! \! d^4y \, \mathcal{T} \left( \beta_1(\gamma) \beta_2(\gamma) \gamma^\dagger \right) \right> \\
& \times \left( -\frac{i}{\mu^2} \right) \int \! \! d^4x \, e^{-i\mathbf{p}_v \cdot \mathbf{x}} \, \delta(\mathbf{x} - \mathbf{y}) \\
& = \left( -\frac{1}{2} \right) \frac{m_v^2}{g^2} \left< \phi \int \! \! d^4x \, \mathcal{T} \left( \beta_1^\dagger(\gamma) \beta_2(\gamma) \gamma^\dagger \right) \right>
\end{align*}
\]
which does cancel (5.41). Thus we will be left with the contribution coming from original \( \frac{3}{2} (\Omega \Omega')^2 \) term only. This result is independent of Kawarabayashi-Suzuki relation, and the vector meson exchange term disappears rather than dominates low energy scattering. On the other hand if the \( \lambda - 5 \) relation is assumed, then (5.42) takes the usual form in the vector dominance model

\[
\mathcal{L} = f_{ijk} \phi_j^i \phi_k^i \phi^0 \phi^6
\]

where \( \phi_0 \) are the physical \( P-S \) meson fields.

In addition \( \beta f^2 \) and \( (\Omega \Omega')^2 \) terms live the 4-pt contact term

\[
-\frac{1}{24} \frac{m^2}{\alpha^2} \phi_0^3 (i F^3) \phi^0 \phi^6
\]

This is still half as large as original contribution from \( (\Omega \Omega')^2 \) term:

\[
+\frac{1}{12} \frac{m^2}{\alpha^2} \phi_0^3 (i F^3)^2 \phi^0 \phi^6
\]

§4 \textbf{Meson-Baryon interaction}

(a)

In this chapter, I would like to consider the interaction of \( P-S \) meson octet with known baryon octet. The chief interest here is again that the symmetry
breaking term in lagrangian which primarily accounts for the baryon mass splitting within its octet gives rise to the modification to low energy meson-baryon scattering amplitude.

Following Gell-Mann, Oakes and Renner I am going to use the interaction of the type (5.1) with the same value of c estimated in the preceding section by fitting pseudo-scalar meson masses.

This time $U_{\alpha\Lambda}$ in (5.1) will be constructed, according to (5.2), with "linear fields" $\phi_{\alpha}$ taken as some bilinear combination of physical octet baryon fields so that the interaction (5.1) gives rise first of all to the baryon mass terms. Since the baryon fields $(B_i)^{\gamma}_{\xi=1}$ transform according to (2.15) with

$$B_i \xrightarrow{g} (e^{-F_{\gamma}^0})_{ij} B_j \quad (5.43)$$

I can get the following SU(3) covariant bilinear combination (with right parity).

(i) The octet $(X_i)^{\Phi}_{\xi=1}$

$$-X_i = m_0 \delta_{ij} B_j B_k + m_0 \delta_{ij} (-\delta_{jk}) B_j B_k \quad (5.44)$$

(ii) The singlet $X_0$

$$-X_0 = m_0 \delta \sum_{i=1}^{8} \overline{B}_i B_i \quad (5.45)$$
where $\alpha$, $\beta$ and $\gamma$ are yet undetermined constants and $m_0$ represents the coefficient of chiral invariant mass term

$$\mathcal{H}^{\text{mass}}_0 = m_0 \sum_{i=1}^{9} \bar{B}_i B_i$$

(5.46)

Using the same notation as the last section ((5.3), (5.4)), I can write down the required chiral $SU(3) \times SU(3)$ multiplet as

$$\begin{pmatrix}
U_0 \\
\vdots \\
U_8 \\
U_0 \\
\vdots \\
U_8
\end{pmatrix} = e^{i\alpha \delta} \begin{pmatrix}
\chi_0 \\
\vdots \\
\chi_8 \\
\chi_0 \\
\vdots \\
\chi_8
\end{pmatrix}$$

(5.47)

The symmetry breaking interaction hamiltonian $\mathcal{H} = -U_0 - cU_8$ with (5.47), (5.44) and (5.45) above together with the chiral invariant term (5.46) gives the baryon mass term

$$\mathcal{H}^{\text{mass}}_0 = m_0 (1 + \delta) \sum_{i=1}^{9} \bar{B}_i B_i + c m_0 \left\{ \alpha d \delta \varepsilon^{ijk} \bar{B}_j B_k + \beta (-c) f_2 \varepsilon^{ijk} \bar{B}_j B_k \right\}$$

(5.48)

If $\beta$'s are identified in ordinary way with this can be written as

$$\mathcal{H}^{\text{mass}}_0 = m_0 (1 + \delta + \frac{c d}{\sqrt{3}}) \left( \bar{\Sigma}^+ \Sigma^- + \bar{\Sigma}^- \Sigma^+ + \bar{\Sigma}^0 \Sigma^0 \right)$$
Thus the masses of baryon octet is identified as

\[
\begin{align*}
\mathcal{m}_Z &= m_0 \left(1 + \gamma + \frac{cd}{\sqrt{3}} \right) \\
\mathcal{m}_N &= m_0 \left(1 + \gamma - \frac{cd}{\sqrt{3}} + \frac{\sqrt{3}}{2} \beta \right) \\
\mathcal{m}_\Xi &= m_0 \left(1 + \gamma - \frac{cd}{\sqrt{3}} - \frac{\sqrt{3}}{2} \beta \right) \\
\mathcal{m}_\Lambda &= m_0 \left(1 + \gamma - \frac{cd}{\sqrt{3}} \right)
\end{align*}
\]

(5.49) satisfies Gell-Mann-Okubo mass formula for linear mass. The experimental \( \mathcal{m}_Z = 1153 \) MeV, \( \mathcal{m}_N = 939 \) MeV, \( \mathcal{m}_\Xi = 1318 \) MeV and \( \mathcal{m}_\Lambda = 1115 \) MeV can be fitted within one percent. Thus I get the estimate of the parameters \( m_0 \alpha \), \( m_0 \beta \) and \( m_0 (1 + \gamma) \) as

\[
\begin{align*}
m_0 (1 + \gamma) &= 1154 \text{ MeV} \\
m_0 \alpha \frac{1}{\sqrt{3}} &= 39 \text{ MeV} \\
m_0 \frac{\sqrt{3}}{2} \beta &= -190 \text{ MeV}
\end{align*}
\]

(5.50)
Here again, the derivation of the baryon octet mass formula is equivalent to the elementary derivation of G-O formula under broken SU(3) with exclusion of \( \Xi \)plet from mass term (53).

(b) Meson-baryon scattering.

To get the estimate of \( \mathcal{G} \) in chiral symmetry breaking term (5.45), I must try to fit scattering data. Unfortunately, many of the known hadronic reactions are very inelastic and I cannot hope to get good agreement with experiment by essentially a perturbation approach. The way to tackle the problem related to the unitarity with the phenomenological lagrangian method is not yet fully developed. Thus I will have to confine myself mainly to the examination of elastic KN and \( \pi N \) reactions at threshold.

The contribution to the scattering amplitude can be obtained by computing the chiral symmetry breaking term in the lagrangian.

\[
\mathcal{L}' = u_0 + c U g
\]

upto second order in \( \mathcal{S} \) fields. From (5.47) this can be written as

\[
\mathcal{L}'_{\text{scattering}} = -\left[ \frac{1}{\sqrt{6}} d_{ij} \mathcal{S}_i \mathcal{S}_j X_k \right. \\
+ c \left( \frac{1}{2} (D \mathcal{S})^2_r X_j + \mathcal{S}_r X_0 \right) \] \\
- \left( \frac{g^2}{3} + \frac{c}{\sqrt{6}} d_{ij} \mathcal{S}_i \mathcal{S}_j X_k \right) X_0 \tag{5.51}
\]
Instead of writing down full RHS of (5.51) in terms of physical meson and baryon fields. I shall extract terms contributing to $\pi N$ and $\pi N$ scattering:

\[
L'_N = \frac{m_0}{\lambda} \left( \frac{3}{2} + \frac{\beta}{4} \right) \vec{N} \cdot \vec{N} \Pi^2
\]

\[
L'_{\pi N} = \frac{m_0}{\lambda} \left[ \frac{1}{\sqrt{3}} \left( \frac{3}{2} - \frac{\beta}{2} \right) \left( \frac{d}{2} - \beta \right) \left( \vec{N} \vec{\sigma} \cdot \vec{N} \right) \left( \vec{K} \vec{\sigma} \cdot \vec{K} \right) + \left( \frac{3}{2} \left( \frac{d}{2} - \beta \right) + \frac{\gamma}{2} \right) \left( \vec{N} \vec{N} \vec{K} \vec{K} \right) \right]
\]

Using the result of the previous section on mesons, (5.52) and (5.53) can also be written as

\[
L'_N = \frac{m_0}{\lambda} \frac{M_N^2}{M_\pi^2} \left( \frac{3}{2} - \frac{\beta}{2} \right) \left( \frac{d}{2} - \beta \right) \left( \vec{N} \vec{N} \Pi^2 \right)
\]

\[
L'_{\pi N} = \frac{m_0}{\lambda} \frac{M_N^2}{M_\pi^2} \left[ \frac{1}{\sqrt{3}} \left( \frac{3}{2} - \frac{\beta}{2} \right) \left( \frac{d}{2} - \beta \right) \left( \vec{N} \vec{\sigma} \cdot \vec{N} \right) \left( \vec{K} \vec{\sigma} \cdot \vec{K} \right) + \left( \frac{3}{2} \left( \frac{d}{2} - \beta \right) + \frac{\gamma}{2} \right) \left( \vec{N} \vec{N} \vec{K} \vec{K} \right) \right]
\]

Since $m_0 \sim M_\pi$, the influence of symmetry breaking interaction to $\pi N$ (or any $\pi B$) reaction is expected to be small, and the results obtained from chiral symmetric lagrangian may give good approximation (54).

In addition to (5.51), there is a chiral invariant meson-baryon interaction term. This, I take to be essentially the form given in (4.5), the relevant term is
The first term which is the form of 4-pt contact interaction comes from the covariant derivative of baryon fields $\nabla_i B_i$ in the kinematical term of baryon lagrangian. The second term comes from the chiral invariant form of Yukawa-Coupling and $G_A$ (written $G'$ in Chapter 4) is renormalized axial vector form factor for baryons. $\frac{1}{\Lambda}$ is ordinary d/f ratio. This term gives rise to the Goldberger-Treiman relation for chiral $SU(3)$. In general, the contribution of the derivative Yukawa coupling in (5.54) through Born term is small compared with the contribution from contact term. The latter, of course, corresponds to the current commutator term in ordinary current algebra calculation (5.2). This can be replaced by the vector meson exchange term according to the idea of vector dominance which unlike in the case of non-linear P-S mesons explained in the last chapter works in a straightforward way.

Extracting the relevant term for $\pi N$ and $K\bar{N}$ from (5.54) I get

$$\mathcal{L}_{\text{chiral}}^{\text{mixed}} = \frac{\zeta}{\Lambda^2} \left( \bar{N} \sigma_i \mathcal{J} \sigma_i N \right) \left( -2i \mathcal{J}_{ij} \bar{N} \mathcal{T}_j N \right) + \frac{G_A}{2} \bar{N} \sigma_i \mathcal{J} \sigma_i N$$  

(5.55)
Let us evaluate first the contact terms, i.e. the (5.53) and the first half of (5.56).

To compute the amplitude in term of its iso-spin components, it is convenient to write (5.53) and (5.56) in the following way.

\[
\mathcal{L}_{NW}^\prime = \frac{m_0}{\lambda_k^2} \left[ (\frac{1}{2} - \frac{e}{3}) \left( \frac{d}{3\lambda^2} + \frac{1}{3\lambda^2} \right) (\bar{N}\sigma_i K_i) \right] \text{ (5.57)}
\]

and

\[
\mathcal{L}_{NW} \text{, contact} = -\frac{e}{4\lambda_k^2} \left[ (\bar{N}\sigma_i K^c_i) \gamma_5 K^c_i \sigma_i N - (\bar{N}\sigma_i \gamma_5 K^c_i) \sigma_i N \right] \text{ (5.58)}
\]
where \( K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \) while \( K^c = (-K^+, K^0) \)

(5.58) shows that the chiral invariant Lagrangian gives the vanishing of the \( I=0 \) amplitude (except for the Born term).

The contribution to a wave isospin amplitude from (5.57) and (5.58) are

\[
\bar{f}_{I=0}^{KN} = (A + \mu_k B)_{\text{contact}}^{I=0} = \frac{2m_0}{\lambda k^2} \left( \sqrt{2} - \frac{c}{2} \right) \left( \frac{1}{2\sqrt{3}} \left( -\frac{2\phi}{3} - \beta \right) + \frac{f}{3\sqrt{2}} \right) \tag{5.55}
\]

\[
\bar{f}_{I=1}^{KN} = (A + \mu_k B)_{\text{contact}}^{I=1} = -\frac{\mu_k}{\lambda k^2} + \frac{2m_0}{\lambda k^2} \left( \sqrt{2} - \frac{c}{2} \right) \left( \frac{d}{6\sqrt{3}} + \frac{f}{3\sqrt{2}} \right) \tag{5.60}
\]

As for the contribution from Yukawa coupling, I must compute the diagram below from the latter half of (5.56)

I get the \( \pi \)-wave \( PK^+ \) and \( NK^+ \) amplitude at threshold as

\[
\bar{f}_{PK^+} = (A + \mu_k B)_{\text{contact}}^{PK^+}
\]
\[
\begin{align*}
&= \left( \frac{C A}{2 \lambda_F} \right)^2 \left[ \frac{(2d'-3)}{3} \frac{-M_k^2}{m_N + m_{\pi} - m_{k}} + (2d' - 1)^2 \frac{-M_k^2}{m_N + m_{\pi} - m_{k}} \right] \\
&= \left( \frac{C A}{2 \lambda_F} \right)^2 2 \frac{(2d' - 1)^2}{m_N + m_{\pi} - m_{k}} \frac{-M_k^2}{m_N + m_{\pi} - m_{k}} \\
&= \left( \frac{C A}{2 \lambda_F} \right)^2 2 \frac{(2d' - 1)^2}{m_N + m_{\pi} - m_{k}} \frac{-M_k^2}{m_N + m_{\pi} - m_{k}} \\
&= \left( \frac{C A}{2 \lambda_F} \right)^2 2 \frac{(2d' - 1)^2}{m_N + m_{\pi} - m_{k}} \frac{-M_k^2}{m_N + m_{\pi} - m_{k}}
\end{align*}
\]

Thus the expectation of this term is small compared with the threshold value and contact terms which is an order of \( M_k^2/\lambda_F^2 \).

There is an uncertainty in the experimental value for \( K^+ n \) threshold effect \( f_{I=0} \). If I try to fit the value of \( \alpha \) using the data \( q_{I=1} = q_{PK^+} \) given by Sichel et al., (56) quoted in (ref. 46),

\[ q_{PK^+}(EX) = -0.22 (\mu^+) \]

and use the value of \( \alpha \) derived in the last section

\[ \alpha = -1.4 \text{ with experiments} \]

\[ \lambda_k = 1.2 \times \lambda_N \]

\[ = 1.2 \times \lambda_N \]

I get the estimate of \( m_{\pi} \) from (2.46) as

\[ m_{\pi} \sim 200 \text{ Mev} \]
This gives the value of $\alpha_{\text{not}}$ rather smaller than the value quoted in same (Ref. 40), $\alpha_{\text{not}} \sim 0.97 \mu_\text{e}^{-1}$ (56). On the other hand if I consider $f_0 \sim 0$ for (5.59) as a good approximation as suggested by some data, I get the estimate of $M_0$ as

$$M_0 \sim 170 \text{ MeV}$$

This corresponds to $\alpha_{\text{not}} \sim 0.27 \mu_\text{e}^{-1}$. The last evaluation with $f_0 \sim 0$ is identical with the calculation by Von Nij enables and Kn (56) with current commutator techniques.

( $f_0 = 0$ for (5.59) actually gives $M_0 \sim 174$ MeV which is Von Nij enables et al.'s estimate).

As the matter of interest, I can formally compute the amplitude for the $\bar{K}K$ reaction from the lagrangian (5.51) and (5.54). Corresponding to (5.59) and (5.60), I get

$$f_{\text{I} = 0}^\text{Kn} = \frac{3M_0}{\lambda_\infty} + \frac{2\sqrt{2}}{\lambda_\infty^2} M_0 \bigg( \sqrt{1 - \frac{c}{2}} \bigg) \bigg\{ \frac{1}{4\sqrt{6}} \bigg( \frac{d}{3} + \beta \bigg) + \frac{\nu}{\beta} \bigg\}$$

$$f_{\text{I} = 1}^\text{Kn} = \frac{M_0}{2\lambda_\infty^2} - \frac{2\sqrt{2}}{\lambda_\infty^2} M_0 \bigg( \sqrt{1 - \frac{c}{2}} \bigg) \bigg\{ \frac{1}{4\sqrt{6}} \bigg( \frac{d}{3} + \beta \bigg) + \frac{\nu}{\beta} \bigg\}$$

The contribution of Born terms are again small. (They can be obtained from (5.61) and (5.62) by changing the
(5.66) gives a large (about the twice of $a_{\text{exp}}$) positive scattering length. Experimentally this is given by an enormous negative value. The contribution due to $s$-wave unitarity cut for $(\bar{K}N)_{I=0}$ is supposed to be particularly large $^{(56)}$. The situation is not so bad for $I=1$ case (5.67) but again experimentally the scattering length is negative.

It may be of interest to compare the calculation of the $KN$ amplitude presented here with the earlier work by Schechter, Ueda and Venturi $^{(46)}$. In the latter, the mass splitting of baryon octets are put "by hand" in the quadratic baryon mass term, and there is no 4-pt term directly arising from such symmetry breaking. However, they treat the chiral symmetry following the model by Cronin discussed at the end of Chapter 4.

Thus the Yukawa coupling appears in the non-derivative form and the contribution of Born terms is as large as the contribution of 4-pt contact term (also of non derivative form). As a result in their model the mass splitting of baryon octets can affect scattering amplitude at threshold through $\Sigma$ and $\Lambda$ poles in the Born terms. The good agreement with experiment has been obtained in this way.
Peccei calculated $\pi N$ scattering length using the phenomenological lagrangian which is equivalent to the chiral invariant meson-baryon lagrangian given here. Apart from the problem of the effect of the symmetry breaking which is always supposed to be small, his treatment is very thorough, and reasonable agreement with experiments is obtained for the s and p wave scattering length.

Let us now discuss the effect of the symmetry breaking $N\pi$ interaction (5.52). Using the values of parameters which have been determined already, and putting $m_0$ to 200 MeV, I get the following estimate for the s-wave iso-spin amplitudes

$$f_{I=3/2}^{\pi N} = -\frac{1}{\lambda_{\pi}} \left( \frac{m_\pi}{2} - 22 \text{ MeV} \right)$$

$$f_{I=1/2}^{\pi ' N'} = \frac{1}{\lambda_{\pi}} \left( m_\pi + 22 \text{ MeV} \right)$$

The part $\frac{22 \text{ MeV}}{\lambda_{\eta}}$ only comes from (5.52), and the rest comes from the chiral symmetric contact term in (5.55). The contribution of Born term is extremely small.
The symmetry breaking affects only the isospin even combination of the amplitude
\[ f^+ = \frac{1}{3} \left( 2 f^{3/2} + f^{1/2} \right) \]
and has no effect on the isospin odd
\[ f^- = \frac{1}{3} \left( f^{1/2} - f^{3/2} \right) \]
Chiral symmetric part gives \( f^+ = 0 \), and symmetry breaking part makes it to
\[ f^+/f^- \sim 0.3 \]
which gives the corresponding scattering length as
\[ a^+ \sim 0.3 (\mu_\pi^{-1}) \]
Experimentally (57), \( a^+ \) is smaller and some data is consistent with \( a^+ \neq 0 \). One way is to appeal to the effect of (53) resonance. According to the calculation in (Ref. 53), the \( N^* \) resonance contributes significantly as
\[ a^+ (N^*) \sim -0.5 (\mu_\pi^{-1}) \]
If this value is added to the result above, \( a^+ \) will be reduced to
\[ a^+ \sim -0.02 (\mu_\pi^{-1}) \]
which is not too far from the estimate by Woodcock and Samarayake (57)
\[ a^+ \sim -0.013 \pm 0.003 (\mu_\pi^{-1}) \]
This interpretation is not an unique one. Reclai rejects the use of a symmetry breaking term and proposes to modify the exchange term so as to get more reasonable asymptotic behaviour at high energy \(5\) and lets almost complete cancellation (with the original \(K\) contribution).

\[
\alpha^+ \sim 0.001 (\mu \sigma^2)
\]

The arguments for the "reasonable asymptotic behaviour of tree diagrams in general has been put forward by Weinberg\(5\) and several interesting consequences have been derived. But the above result just quoted certainly cannot be regarded as showing the relevance of such a scheme, since this will leave the symmetry breaking contribution unaccounted for.

**Other amplitudes**

\(\pi \pi\)

The \(I=0\) component of \(\Sigma \pi\) amplitude has no inelastic channels opening at threshold. The part of chiral symmetry breaking term \((5.51)\) which contributes to \(\pi \pi\) scattering is

\[
\mathcal{L}'_{\pi\pi} = \frac{1}{\lambda_\pi^2} \frac{C+\sqrt{2}}{C} \left( \frac{1}{6} \frac{m_0 \sigma d}{\sqrt{2}} + \frac{C}{\sqrt{2}} \right) \overline{\Sigma} \cdot \Sigma \pi^2 \tag{5.69}
\]
and the chiral symmetric part of the interaction

\[ L^{\text{chiral}}_{\pi\pi} = \frac{i}{2\lambda_{\pi}^2} \varepsilon_{abc} \sum \bar{j} [\gamma_{\mu} D_{\mu} G_{\text{em}} \Pi E \partial^\dagger \Pi^\dagger] \quad (5.70) \]

The second term which represents the effect of the symmetry breaking is extremely small and the amplitude is more or less "chiral invariant".

\[ f_{\pi\pi}^{\text{chiral}} \sim \frac{2}{\lambda_{\pi}^2} \quad (5.71) \]

The corresponding scattering length is

\[ a_{\pi\pi} (I=0) \sim 0.43 \left( \frac{\hbar^2}{m} \right)^{-1} \quad (5.72) \]

According to him, \( a_{\pi\pi} (I=0) \) is about 0.7(\( \hbar^2 \)) but the uncertainty is very large.

(\( \pi\Lambda \))

\( \pi\Lambda \) scattering term does not occur in the chiral invariant contact meson-baryon interaction (the first part of (5.54)). In the current algebra calculation with soft meson approximation, \( a_{\pi\Lambda} = 0 \).

The symmetry breaking interaction \( L^{(\text{scattering})} \) of (5.51) gives the contribution to \( \pi\Lambda \) scattering as

\[ L'_{\pi\pi} = \frac{1}{\lambda_{\pi}^2} \left( \frac{1}{2} + c \right) \left\{ \frac{m_0 \delta}{3} - \frac{m_0 \delta}{6 \sqrt{3}} \right\} \pi^2 \Lambda \quad (5.73) \]
The $s$-wave scattering amplitude at threshold is

$$f^{\pi\Lambda} = \frac{2}{\Lambda^2} (1^2 + c) \left( \frac{m_0}{3} - \frac{m_0 c}{3\sqrt{5}} \right) \quad (5.74)$$

The corresponding $s$-wave scattering length is

$$\alpha^{\pi\Lambda} \sim 0.004 \left( \mu \right)^{-1}$$

and still very small. On the other hand, I must still consider the Born term due to the coupling

$$\mathcal{L}^{\pi\Lambda\Sigma} = \frac{2d'}{\sqrt{3}} \frac{G_F}{2\Lambda^2} \alpha^{\pi\Pi} \left( \tilde{\Sigma} \tilde{\Sigma}^{\pi\Lambda} + \tilde{\Lambda} \tilde{\Lambda}^{\pi\Sigma} \right) \quad (5.75)$$

coming from the second half of $(5.54)$. The contribution to the $s$-wave $\pi\Lambda$ amplitude is

$$f^{\pi\Lambda} \sim \left( \frac{2d'}{\sqrt{3}} \frac{G_F}{2\Lambda^2} \right)^2 (m_\Sigma + m_\Lambda)^2 \left[ - \frac{1}{m_\Lambda + m_\Sigma - \mu} + \frac{m_\Sigma - m_\Lambda - m_\pi}{(m_\Sigma + m_\Lambda - m_\pi) X (m_\Lambda - m_\Sigma + m_\pi)} \right] \quad (5.76)$$

Numerically, for $\alpha' = 0.75$ this gives the scattering length

$$\alpha^{\pi\Lambda} \sim 1.5 \left( \mu^{-1} \right) \quad (5.77)$$
The breaking of the coupling constant

The construction like (5.47) is of course not the unique symmetry breaking interaction within Gell-Mann, Oakes, Renner scheme. The one advantage of the Lagrangian method is the ease with which various possibilities within a given symmetry scheme can be exploited, and we may try to study some other examples of broken chiral SU(3) than the one discussed above.

Let us consider the following multiplet

$$\begin{pmatrix}
    u_0 \\
    v_0 \\
    u_1 \\
    v_1 \\
    \vdots \\
  \end{pmatrix}
  = e^{i \theta_\phi} \begin{pmatrix}
    0 \\
    \cdot \\
    \cdot \\
    \cdot \\
  \end{pmatrix}$$

with

$$\gamma_c = \bar{\psi} \left( \alpha' D^c + \beta' F^c \right) B$$

(5.78)

The corresponding Gell-Mann, Oakes, Renner type Lagrangian is

$$L' = i \chi Q' \left[ \frac{\sqrt{2}}{2} \bar{\psi} \gamma_i Y + c (D \psi) Y_j Y_j \right] + O(\psi^6)$$

$$= i \chi Q' \left[ \frac{\sqrt{2} + c}{\sqrt{3}} \left( \bar{\psi} \gamma_i Y_i + \bar{\psi} \gamma_2 Y_2 + \bar{\psi} \gamma_3 Y_3 \right) \right.$$  

$$+ \frac{\sqrt{2} - c}{\sqrt{3}} \left( \bar{\psi} \gamma_4 Y_4 + \cdots + \bar{\psi} \gamma_8 Y_8 \right) \right]$$  

$$+ \frac{\sqrt{2} - c}{\sqrt{3}} \bar{\psi} \gamma_8 Y_8 \right] + O(\psi^6)$$  

(5.76)
The interaction like (5.7) gives rise to an additional Yukawa type coupling and breaks the symmetry of the Born term expressed by the chiral and SU(3) symmetric coupling in (5.54). Comparing the residue of Born terms with or without (5.7), and defining the meson baryon coupling constants $\tilde{g}_{\pi \bar{B}B}$ by generalized Goldberger-Treiman relation, I get

$$\left( \frac{\tilde{g}'_{\pi \bar{B}B}}{\tilde{g}^0_{\pi \bar{B}B}} \right)^2 = \left( \frac{m + m' + \frac{\sqrt{2} + C}{\sqrt{3}} \chi}{2 m_0} \right)^2$$

$$\left( \frac{\tilde{g}'_{K \bar{B}B}}{\tilde{g}^0_{K \bar{B}B}} \right)^2 = \left( \frac{m + m' + \frac{\sqrt{2} - C}{\sqrt{3}} \chi}{2 m_0} \right)^2$$

$$\left( \frac{\tilde{g}'_{\eta \bar{B}B}}{\tilde{g}^0_{\eta \bar{B}B}} \right)^2 = \left( \frac{m + m' + \frac{\sqrt{2} - C}{\sqrt{3}} \chi}{2 m_0} \right)^2$$

where $\tilde{g}'$ and $\tilde{g}^0$ represent the coupling constants with or without (5.7). Numerically, taking $\theta = -1.26$,

$$\frac{\sqrt{2} + C}{\sqrt{3}} \approx 0.01, \quad \frac{\sqrt{2} - C}{\sqrt{3}} \approx 0.9, \quad \frac{\sqrt{2} - C}{\sqrt{3}} \approx 1.5$$

As to be expected, the change of the $\pi \bar{B}B'$ coupling constant is small even if $\chi$ is order of 2000 keV.

Although the meson baryon coupling constant is suspected to differ considerably from SU(3), the interaction like (5.7) has no immediate application.

Lastly, it should be noted that the Yukawa type interaction (5.7) cannot be used to fit the KN
scattering data discussed above. The corresponding u-channel Born term gives negative contribution and the contribution of chiral invariant contact term is already negative and too large. So the modification by the last term the interaction with, in particular, chiral symmetry breaking term of strength $\rho_0$ described before is the only way to fit the data. This strengthens a little the argument for the quantity like $\rho_0$ being physically meaningful.
Discussion

In conclusion of the work presented here, I would like to add the following remarks.

First, the importance of the "equivalence relation" of the kind discussed in the end of Chapter 4 should be emphasized. Certain arbitrariness in choosing the physical fields operators can leave the underlying chiral symmetry and related group theoretical structure as the only physically meaningful concept, and it may be possible to reformulate the whole algorithm of "chiral lagrangian calculation" in group theoretical terms avoiding the redundancy which seems to accompany field theory. The derivation of the relation and Weinberg mass relation seems to suggest that such an approach may have interesting physical results.

In connection with this point about the non-uniqueness of the choice of physical field, the notion of F.C.A.C seems to be a little puzzling. In the current algebra approach, the divergence equation

$$\partial_t A^\mu_c = \star F \Phi_c$$

itself is the matter of defining the right hand side which, due to the right quantum numbers, has the singularity corresponding to single meson state. But
when we start to ask that such a pole term actually dominates the matrix element

\[ \langle \alpha | \Phi | A^\pm | \beta \rangle \]

over certain range of momentum transfer

\[ t = (p_\alpha - p_\beta)^2 \]

i.e.

\[ \langle \alpha | \Phi | A^\pm | \beta \rangle \approx \mu^2 F_\alpha^2 \sqrt{t} \]

then it is a meaningful assumption and can impose the restriction on the physics. It is in this form, we use "FCAC" or (pole dominance assumption) in current algebra. Thus, when we write the scattering amplitude including these mesons in L.E.E reduced form, the residue after the removal of meson pole factors may be assumed as "smooth"\(^{(52)}\). For the derivation of Adler's consistency relation, FCAC interpreted in this way is essential. Also, this smoothness assumption gives the certain prescription for obtaining physical amplitude from soft meson limit which can be determined from current algebra. Thus, FCAC or the pole dominance assumption is the most important assumption which enables current algebra scheme to make predictions. On the other hand, it is difficult to recognize the role of FCAC in the
chiral lagrangian scheme. The field theoretical divergence equation has its correspondence in the lagrangian scheme so that the first term of divergence of axial currents discussed in §5 is determined by the symmetry argument.

$$\partial A^\mu = i \mathbf{F} \phi + O(\phi^3)$$

this again cannot be considered as an additional assumption. Now in the theory in which we have a definite lagrangian, that is to say a dynamical equation of motion, the assertion about the particular form of the higher order term in the R.H.S above certainly gives a non trivial restriction. If we change the definition of the physical field to modify the PCAC equation this will change the result of calculation of the off-mass shell amplitude. At the same time, unlike the current algebra, these off-mass shell amplitude is not important. The lagrangian perturbation theory gives the on-mass shell amplitude directly. Even in the example of weak decay of K mesons(2), the correction term to the amplitudes due to the redefinition of the amplitude (i.e. PCAC or "non" PCAC) vanishes when all the external mesons are on mass shell. This is because even the weak or electromagnetic interaction is written in term of
definite symmetric quantities (i.e. vector or axial vector currents) and thus independent of the particular choice of "physical field". Our feeling is that PCAC is deeply connected with the orthodox field theoretical notions which underlines the current algebra and gives the formal expression for the off mass shell amplitude through L.S.Z techniques.

Lastly in practical calculations like scattering length, the limitation due to the problem of unitarity is strongly felt. Recently the possibility of using certain techniques of summing up the perturbation series to calculate the higher order rescattering processes in the Lagrangian approach has been suggested. It may be that a practicable and convincing prescription of "unitarizing" chiral results will emerge from the study of this technique. But for the moment, it is hard to do anything beyond the analysis of the rather formal structure of the theory.

It is also possible to go to an extreme "phenomenological" approach. For instance, we may use higher order covariant derivatives disregarding all the field theoretical difficulties. On the other hand, this means that we will effectively abandon hope of extending the restriction due to chiral symmetry beyond what can
be calculated by the tree approximation.
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59. The clear exposition on this point is given by M. Jacob's lecture "Algèbre des Courants" (Saclay preprint, also published by CERN).


Appendix

"Meson and Matrices"

To study the formal structure of chiral Lagrangian, it is sometimes more convenient to consider the "meson matrices"

\[
\mathcal{M} = e^{-2i\lambda_8}; \quad \lambda_8 = \sum_{d=1}^{9} \lambda_d \mathcal{F}_d \quad (A.1)
\]

instead of meson fields. Here \((\lambda_d)_{d=1}^{9}\) are the generators of (3) representation of SU(3) and equal to the half of Gell-Mann matrices.

As has been used by previous authors, I can write the chiral invariant Lagrangian in terms of \(\lambda_8\)'s and appropriate traces. Thus, for instance,

\[
(\lambda_8 \bar{\psi})^2 = -\frac{1}{2} \text{Tr} (e^{-2i\lambda_8} \bar{\psi} e^{i\lambda_8} - e^{i\lambda_8} \bar{\psi} e^{-i\lambda_8})^2 = \frac{1}{2} \text{Tr} (\bar{\psi} e^{-2i\lambda_3} \mathcal{D}_\pi e^{2i\lambda_3})
\]

and the chiral invariant meson Lagrangian can be written as

\[
\mathcal{L} = \frac{g}{4} \text{Tr} (\bar{\psi} M \mathcal{D} \mathcal{F} M^+) \quad (A.2)
\]

I would like to study some consequences of the expression like (A.2) to show the use of \(\lambda\) matrices.
First, let us evaluate the contribution of the \( n \)-point contact term to the \( n \)-particle amplitude like

\[
\begin{align*}
\text{Fig. 1}
\end{align*}
\]

\( a_1 \ldots a_n \) are SU(3) indices of the corresponding mesons

Taking the matrix element

\[
\langle 0 \mid T \mid M^+ M^+ \mid a_1 \ldots a_n \rangle
\]

the corresponding amplitude can be obtained immediately.

This is

\[
T = -\frac{i}{2} \sum_{s=0}^{n} \sum_{\text{perm}(a_1 \ldots a_n)} \begin{vmatrix}
2i\lambda_{a_1} & \ldots & 2i\lambda_{a_{n-s}} \\
-2i\lambda_{a_{n-s+1}} & \ldots & -2i\lambda_{a_n}
\end{vmatrix} \\
\times (q_{a_1} + \ldots + q_{a_{n-s}} - (q_{a_{n-s+1}} + \ldots + q_{a_n}) \times \frac{1}{(n-s)!}) \\
= i \frac{1}{2} \sum_{s=0}^{n} (-1)^s \sum_{\text{cyclic permutation}} T_n (\lambda_{a_1}, \ldots, \lambda_{a_n}) \\
\times \sum_{\text{cyclic}} (q_{a_1} + \ldots + q_{a_{n-s}})^2
\]

where \( q_{a_1}, \ldots, q_{a_n} \) are the momentum of meson \( a_1 \ldots a_n \).

Thus \( T \) has the form:

\[
T = \sum_{\text{perm}(a_1 \ldots a_n)} T_n (\lambda_{a_1}, \ldots, \lambda_{a_n}) \times (q_{a_1}, \ldots, q_{a_n}) \quad (A.3)
\]

where \( \sum_{(a_1 \ldots a_n)} \) means the sum over the class of cyclic permutations. \( \times \) is of course invariant under the
cyclic permutation of \((a_1, \ldots, a_n)\). More generally, the single vertex like (Fig. 2) can be stuck together to form an arbitrarily tree graph like

![Diagram](image)

(Fig. 2)

Each internal line is represented by the free propagator from the factor like

\[
<\Gamma(\lambda \bar{\sigma}_a, \lambda \bar{\sigma}_f)>
\]

In case the eight mesons corresponding to \((\bar{\sigma}_a)_a^p\) are all degenerate. (SU(3) symmetric), the resultant propagator with internal momentum \(q\) is proportional to

\[
\sum_{\alpha, \beta} \lambda_\alpha \lambda_\beta \frac{\delta_{\alpha \beta}}{\mu^2 - q^2} = \frac{1}{\mu^2 - q^2} \sum_{\alpha=1}^8 \lambda_\alpha \lambda_\alpha
\]

where if \(p^2 \neq 0\), the appropriate SU(3) invariant mass term like

\[
T_a (M + M^+) \]

should be a ded. Because of the form of SU(3) factor in the propagator above, sticking two vertices like
Give the overall factor

\[ \sum_{d=1}^{8} \text{Tr}(\lambda_{d1} \cdots \lambda_{dm} \lambda_{d}) \text{Tr}(\lambda_{d} \lambda_{d1} \cdots \lambda_{dm}) \]  

(A.4)

Now it is well known that sums like this can be transformed using the completeness relation of -matrices.

\[ \sum_{d=1}^{8} \text{Tr}(\lambda_{d1} \cdots \lambda_{dm} \lambda_{d}) \text{Tr}(\lambda_{d} \lambda_{d1} \cdots \lambda_{dm}) \]

\[ = 2 \text{Tr}(\lambda_{d1} \cdots \lambda_{dm} \lambda_{d1} \cdots \lambda_{dm}) \]  

(A.5)

\[ - \frac{2}{3} \text{Tr}(\lambda_{d1} \cdots \lambda_{dm}) \text{Tr}(\lambda_{d1} \cdots \lambda_{dm}) \]

The simple trace factor seems to be lost. On the other hand, suppose there is ninth meson with same mass \( \mu \) and which comes into the interacting system in such a way that the sum \( \sum_{d=1}^{8} \lambda_{d} \lambda_{d} \) in the propagator should be replaced by

\[ \sum_{d=0}^{8} \lambda_{d} \lambda_{d} \]

where

\[ \lambda_{0} = \sqrt{\frac{2}{3}} \]  

(A.6)
Then, instead of (A.6), the overall trace factor of two vertices stuck together is
\[
\sum_{\lambda_1} T_n(\lambda_1, \ldots, \lambda_n, \lambda_0) T_n(\lambda_0, \lambda_1, \ldots, \lambda_n)
\]
\[
+ T_n(\lambda_1, \ldots, \lambda_n, \lambda_0) T_n(\lambda_0, \lambda_1, \ldots, \lambda_n)
\]
\[
= 2 T_n(\lambda_1, \ldots, \lambda_n, \lambda_0, \ldots, \lambda_n)
\]

Thus the trace factor will be recovered. In this way, any number of vertices can be stuck together to give any tree graph while retaining the general form of (A.5). The ninth meson can be introduced by simply replacing matrices by "nonet matrices"

\[
M' = \exp\left(\sum_{\lambda=0}^{n} -2i\lambda_0 \frac{\hat{X}}{\lambda_0}\right)
\]

(A.7)

Then
\[
M' = e^{-2i\lambda_0 \frac{\hat{X}}{\lambda_0}} M
\]

and
\[
T_n \Theta_\gamma M' \Theta_\gamma M'^+ = T_n \Theta_\gamma e^{i\lambda_0 \frac{\hat{X}}{\lambda_0}} \Theta_\gamma e^{-2i\lambda_0 \frac{\hat{X}}{\lambda_0}} + T_n \Theta_\gamma M \Theta_\gamma M'^+
\]

(A.9)

\[
= T_n \left(2\lambda_0 \frac{\hat{X}}{\lambda_0}\right)^2 + 2 \Theta_\gamma M \Theta_\gamma M'^+
\]

If \( \Theta_\gamma \) is assumed to be chiral scalar, the resultant lagrangian
\[
L = \frac{q}{4} \Theta_\gamma M \Theta_\gamma M'^+ + \frac{q}{2} (\Theta_\gamma \frac{\hat{X}}{\lambda_0})^2
\]

(A.9)

is chiral invariant.