Generalised symmetric spaces

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GENERALISED SYMMETRIC SPACES

by

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ABSTRACT

This thesis treats the symmetric spaces (originally studied by E. Cartan) and their various generalisations.

Chapter I presents the necessary fundamental definitions and results.

Chapter II describes the historical background to the subject: in Part A the relevant aspects of the theory of symmetric spaces are reviewed, the notion of k-symmetric space (due to A.J.Ledger) is introduced and various results (in particular those due to Gray for 3-symmetric spaces) are noted; in Part B the theory of Jordan algebras is summarised (as needed in this thesis) and the intimate relationship between Jordan algebras and symmetric spaces is discussed.

Chapter III contains largely original results on a class of manifolds, the symmetric spaces of order k (a generalisation of the symmetric spaces made in the spirit of the "algebraic" approach to symmetric spaces developed by O. Loos). A symmetric space of order k is a differentiable manifold M together with a smooth multiplication $\mu : M \times M \to M$ satisfying certain properties. The main result is that on such a manifold M an affine connexion ∇ may be defined in terms of the multiplication μ; M may then be shown to be a reductive homogeneous space and ∇ the (complete) canonical affine connexion of the second kind.

Chapter IV presents two original observations concerning the relationship between Jordan algebras and symmetric spaces (of order 2).

Chapter V contains a summary of results and various suggestions for further research. A bibliography follows Chapter V.

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TABLE OF CONTENTS

INTRODUCTION

Here I shall outline briefly how my interest in Jordan algebras, symmetric spaces and generalised symmetric spaces developed, how the contents of this thesis are arranged, and which of the results I claim as my original contributions to the subject.

At an early stage of my research work under his supervision, Professor T.J.Willmore suggested that a fruitful area of research might be the examination of Jordan algebras with a view to exploring their relationships with, or applications in, differential geometry. At that time only two references seemed readily available in the literature, namely Ottmar Loos's books on "Symmetric Spaces" (Loos [1]) and a paper by U. Hirzebruch [2] concerning Jordan algebras and compact Riemannian symmetric spaces of rank one. This relationship between Jordan algebras and symmetric spaces fascinated me and I began to study Loos's books and Hirzebruch's paper in some detail. In the course of these studies I conjectured and proved what is the main result (viz. Theorem IV.1) of Chapter IV of this thesis.

Aware of my growing interest in symmetric spaces, Professor Willmore mentioned to me the notion of k-symmetric space due to Dr. A.J.Ledger. Not long afterwards in the University of Leeds (at a Colloquium in honour of Professor Ruse on his retirement) Dr. A. Deicke delivered a lecture on k-symmetric spaces; during that lecture the question occurred to me as to whether the algebraic approach to symmetric spaces developed by Loos could be applied to the study of k-symmetric spaces. Some months later, after several stimulating conversations with Dr. Ledger, I became convinced that indeed they could be so applied, and I then began an earnest attempt to generalise the methods of Loos. Chapter III contains the results of that endeavour.

I now summarise, chapter by chapter, the contents of the thesis:

(i) Chapter I presents fundamental definitions and results which will be needed in the subsequent pages. Here the definitions of differentiable manifold, affine connexion, Lie group, Jordan algebra, etc. are presented, together with various known results relating these notions.

(ii) Chapter II describes the historical background to the new results of Chapters III and IV. In PART A the theory of symmetric spaces is reviewed in those aspects which are relevant to this thesis, the notion of k-symmetric space is introduced, and various results about s-regular k-symmetric spaces (especially for $k = 3$) are noted. In PART B the theory of Jordan algebras is summarised (as needed here), the intimate relationship between Jordan algebras and symmetric spaces is described and various details are discussed.

(iii) Chapter III contains largely original results on a class of manifolds - namely, the symmetric spaces of order k - which are a generalisation of the symmetric spaces, the generalisation being made in the spirit of the approach to symmetric spaces taken by Loos [1], [3]; a symmetric space of order k is a differentiable manifold M together with a smooth multiplication $\mu : M \times M \to M$ satisfying certain properties (cf. Definition III.1).

By carefully constructing an analogue of the affine connexion defined by Loos on a symmetric space (of order 2), I have defined in Definition III.6 an affine connexion on a symmetric space of order k for which all the "left multiplications" are affine maps (cf. Theorem III.1(a)). I then prove that with this connexion a symmetric space of order k is an s-regular k-symmetric space in the sense of Graham and Ledger [1]. The connexion of Definition III.6 is, moreover, complete; in fact, a symmetric space of order k may be represented as a reductive homogeneous space and the connexion of Definition III.6 is the canonical connexion of the second kind in the terminology of Nomizu [2].

The Definitions III.1 and III.6 and the proof of Theorem III.1(a) are modelled on the corresponding definitions and proof given by Loos [1] for the particular case k = 2; the necessary modification of his work to arbitrary integer k ⩾ 2 required some care and I consider this modification to be one of the major original contributions to the subject contained in this thesis. Theorem III.1(b), which asserts that the symmetric spaces of order k are exactly the s-regular affine k-symmetric spaces, follows fairly easily once Theorem III.1(a) has been established; this equivalence was conjectured during a conversation between Dr. Ledger and myself, and the desire to prove the validity of the conjecture motivated the work presented in Chapter III. The proof of the first part of Theorem III.1(c) which shows that the symmetric spaces of order k are homogeneous spaces of Lie groups follows the argument of Ledger and Obata [1], with the piece of their proof involving the exponential map replaced by a direct computation (using differentiability which Ledger and Obata had not assumed in their hypotheses). The proof of the part of Theorem III.1(c) which shows that the symmetric spaces of order k are reductive homogeneous spaces is a slightly more detailed but nevertheless direct adaptation for all k of the method of proof given by Gray [6] in the special case k = 3. I am completely responsible for the part of the proof of Theorem III.1(c) which shows that the connexion of Definition III.6 is in fact the canonical connexion of the second kind; the proof involves explicitly computing an expression for the connexion in terms of an appropriate Lie algebra.

Finally, Theorem III.2 gives a characterisation of the isotropy subgroup in the representation of a symmetric space of order k by the homogeneous space of Theorem III.1; the proof is a direct adaptation for all k of the method of proof given by Loos [1] in the special case k = 2 (cf. also Gray [6] in which the method of Loos was adapted to the case k = 3).

(iv) Chapter IV concerns the relationship between Jordan algebras and symmetric spaces (of order 2). One of the examples of symmetric spaces given by Loos [1] is the set I(A) of invertible elements in a Jordan algebra A. Essentially, I have examined the connexion on such a symmetric space and shown it to be related to the various isotopes of A (cf. Theorem IV.1). The statement and proof of Theorem IV.1 are original work carried out by me, but in a comprehensive paper on this subject Helwig [6] establishes the same result (by a less direct method of proof); it plays a significant role in his treatment. In Section 2 of Chapter IV I describe an observation that Jordan algebras are naturally associated to certain conformal transformations of Riemannian manifolds; this observation does not seem to appear in the literature.

(v) Chapter V contains a summary of results and some remarks on a programme for further research.

(vi) The Bibliography contains the relevant references required in establishing the new results of Chapters III and IV and contains also general references to other subjects discussed. In two areas it is intended to provide a reasonably complete survey of the existing literature: viz. firstly for the work on k-symmetric spaces and secondly the applications of Jordan algebras to differential geometry.

CHAPTER I

FUNDAMENTAL DEFINITIONS AND RESULTS

1. Differentiable Manifolds, Maps and Tensor Fields

This chapter contains basic definitions and fundamental results which will be used in the subsequent chapters; results here are quoted not in their full generality but simply in the form required for the work presented in this thesis. In the present chapter only a few proofs are given: viz. in those cases where there seems to be no standard reference in the literature. This first section introduces as briefly as possible some very basic notions and notational conventions; it concludes with a statement of the version of the inverse function theorem required in Chapter III.

A differentiable manifold (or simply, a manifold) is a Hausdorff topological space endowed with a differentiable (C^∞) structure of finite dimension. (For further details on this terminology and indeed on this whole section, refer to Chapter 1 in each of: Kobayashi and Nomizu [1], Helgason [1], Wolf [1]). A chart on a manifold of dimension n is a pair (ψ, U) of an (open) neighbourhood U in M (U being called a coordinate patch) and a homeomorphism $\psi : U \to \psi(U) \subset \mathbb{R}^n$ (ψ being called a chart map) where $\psi(U)$ is an open set in $\mathbb{R}^n$, the n-dimensional vector space over the real numbers $\mathbb{R}$ (with the usual topology on $\mathbb{R}^n$); a chart (ψ, U) defines "local coordinates $\{u^i\}_{i=1}^n$ on U" thus: relative to (ψ, U) and the standard basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$, a point $p \in U$ has coordinates $\{u^i(p)\}_{i=1}^n$ if $\psi(p) = u^i(p)\, e_i$ (the usual "summation convention" being used).

In the standard way (cf. the books by Helgason and Wolf referred to above), the following notions may be introduced: The class $F(M)$ of differentiable real-valued functions on M; the tangent space M_p at a point $p \in M$; the class $T^r_s(M)$ of differentiable tensor fields of type (r,s) on M, in particular the class $T^1_o(M) = T^1(M)$ of differentiable vector fields; a differentiable map from M into another manifold N; a diffeomorphism between two manifolds, a diffeomorphism of M onto itself being called a transformation of M; and the direct product manifold $M \times N$ of two manifolds M and N.

For each $i = 1,2,\ldots,n$, the "coordinate function" $u^i : U \to \mathbb{R}$ (associated to a chart (ψ, U)), where u^i maps $p \in U$ into $u^i(p)$, is an element of $F(U)$; i.e., u^i is a differentiable function on U. A differentiable map $\gamma : E \to M$ from an open interval E of $\mathbb{R}$ into M will be called a curve in M, and if $E = \mathbb{R}$, one will speak of "the curve $\gamma(t)$".

Analogously to the notion of differentiable (C^∞) structure just discussed, the notion of (real-) analytic (C^ω) or complex-analytic structure on a manifold may be formulated; likewise the related notions of analytic (resp. complex-analytic) maps, diffeomorphisms, functions, etc. may be introduced.

At a point $p \in M$ the differential $(d\Phi)_p$ of a differentiable map $\Phi : M \to N$ is the linear transformation $(d\Phi)_p : M_p \to N_{\Phi(p)}$ induced by Φ on the tangent space M_p; when it is obvious that $(d\Phi)_p$ is acting on a vector $X_p \in M_p$ the subscript "p" will be suppressed and the expression $d\Phi(X_p)$ will be written (instead of $(d\Phi)_p(X_p)$). If $\Phi : M \to N$ and $\eta : N \to P$ are two differentiable maps then their composition $\eta \circ \Phi : M \to P$ is differentiable; moreover for any $p \in M$: $(d(\eta \circ \Phi))_p = (d\eta)_{\Phi(p)} \circ (d\Phi)_p$. Given a transformation ψ of M and $X \in T^1(M)$, the vector field $d\psi(X) \in T^1(M)$ is defined by $(d\psi(X))_p := d\psi(X_{\psi^{-1}(p)})$ for $p \in M$.

Also, in terms of a system of local coordinates $\{u^i\}_{i=1}^n$ related to a chart (ψ, U) the (local) vector field $\frac{\partial}{\partial u^i} \in T^1(U)$ is introduced for $i = 1,2,\ldots,n$; at a point $p \in U$ the set $\left\{ \left(\frac{\partial}{\partial u^i} \right)_p \right\}_{i=1}^n$ forms a basis of the tangent space M_p. Indeed for a vector field $X \in T^1(M)$ the vector X_p for $p \in U$ may be represented as a differential operator on $F(M)$ thus:

$$X_p = X^i(p) \left(\frac{\partial}{\partial u^i} \right)_p \qquad \text{where the } X^i \text{ define}$$

differentiable functions on U; it is observed that $X^i(p) = X_p u^i$ (the real number obtained by the action of X_p on the element u^i of $F(U)$).

The juxtaposition, XY, of two vector fields $X, Y \in T^1(M)$ has the following significance: at a point u in a coordinate patch U (with local coordinates $\{u^i\}_{i=1}^n$), $X_u = X^i(u) \left(\frac{\partial}{\partial u^i} \right)_u$ and $Y_u = Y^i(u) \left(\frac{\partial}{\partial u^i} \right)_u$ and the differential operator XY is given at a point $p \in U$ by:

$$(XY)_p := X^i(p)\, Y^j(p) \left(\frac{\partial^2}{\partial u^i \partial u^j} \right)_p + X^i(p) \left(\frac{\partial Y^j}{\partial u^i} \right)_p \left(\frac{\partial}{\partial u^j} \right)_p .$$

$(XY)_p$ is also denoted X_pY (notice that with regard to X, $(XY)_p$ does in fact depend only on X_p); notice further that this juxtaposition of vector fields has a derivative nature in Y, i.e. considering the vector field fY defined by $(fY)_x := f(x)Y_x$ for $x \in M$ (where $f \in F(M)$ and $Y \in T^1(M)$) then for $X \in T^1(M)$ (or simply $X_p \in M_p$):

$$X_p(fY) = (X_pf)Y + f(X_pY).$$

In terms of this juxtaposition the Lie bracket of two vector fields may be defined; thus, given two vector fields $X,Y \in T^1(M)$ the vector field $[X,Y] \in T^1(M)$ is given at a point $x \in M$ by:

$$[X,Y]_x = X_x Y - Y_x X .$$

Finally in this section the following theorem is quoted:

Theorem I.1 Let M be a differentiable manifold and let $\Phi : M \to M$ be a differentiable map. If for a point $p \in M$ the linear transformation $(d\Phi)_p : M_p \to M_{\Phi(p)}$ is non-singular, then there exist neighbourhoods V_1 and V_2 of p and $\Phi(p)$ respectively such that $\Phi|_{V_1} : V_1 \to \Phi(V_1) = V_2$ is a diffeomorphism of V_1 onto V_2 .

A proof of this theorem is given in Wolf [1], Chapter 1.

2. Lie Groups and Homogeneous Spaces

Except for Theorems I.2 and I.5 the results of this section are proved in Chevalley [1], Helgason [1] or Hochschild [1]; Theorem I.2 is due to Ledger and Obata [1] and Theorem I.5 is proved here.

(i) A Lie group is an analytic manifold G with a group structure such that the maps $\lambda : G \times G \to G$ and $\upsilon : G \to G$ are analytic (where for $g,g' \in G : \lambda(g,g') = g.g' = gg'$ the group product of g and g', and $\upsilon(g) = g^{-1}$ the group inverse of g). e denotes the identity element of G. For $g \in G$ the two analytic diffeomorphisms $L_g : G \to G$ and $R_g : G \to G$ are defined by $L_g(g') := gg'$ and $R_g(g') := g'g$ for $g' \in G$.

The set $\underline{g}$ of "left-invariant vector fields on G" is defined as follows:

$$\underline{g} := \{X \in T^1(G) : X_{gg'} = dL_g(X_{g'}) \text{ for all } g,g' \in G\}$$

and the set $\underline{g}^R$ of "right-invariant vector fields on G" is defined by:

$$\underline{g}^R := \{X \in T^1(G) : X_{g'g} = dR_g(X_{g'}) \text{ for all } g,g' \in G\} .$$

An element of $\underline{g}$ or of $\underline{g}^R$ is in fact an analytic vector field. If $X \in T^1(G)$ then X^R will be understood to denote the (unique) element of $\underline{g}^R$ for which $(X^R)_e = X_e$. For $X,Y \in \underline{g}$ the Lie bracket $[X,Y]$ is also a vector field in $\underline{g}$, and this product of left-invariant vector fields defines on $\underline{g}$ the structure of an n-dimensional Lie algebra over $\mathbb{R}$ (where n = dimension of G); as a vector space $\underline{g}$ is isomorphic with G_e, a vector field $X \in \underline{g}$ being identified with $X_e \in G_e$, and the Lie bracket defines a Lie algebra on the n-dimensional vector space $\underline{g}$ because the two characteristic properties of a Lie algebra are satisfied: namely for all $X,Y,Z \in \underline{g}$:

$$[X,Y] = -[Y,X]$$

and $$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 .$$

For $X^R \in g^R$, $Y \in g$ the Lie bracket $[X^R,Y]$ vanishes at $e \in G$; this is the only result in the present section which does not seem to be explicitly proved in the literature: a proof is given at the end of this section (see Theorem I.5), because the result is needed in the proof of Theorem III.1.

For $X \in g$ the curve $\exp tX \subset G$ (defined for $-\infty < t < \infty$) is the integral curve of X passing through e at $t = 0$. Moreover, the curve $\exp tX$ gives an analytic homomorphism of the additive Lie group of the real numbers $\mathbb{R}$ into G, and the curve $\exp tX$ is called the one-parameter subgroup of G belonging to X; in particular, $(\exp t_0X) . (\exp t_1X) = \exp(t_0 + t_1)X$ for all $t_0,t_1 \in \mathbb{R}$. Also there exist a neighbourhood U_e of e in G and a neighbourhood V_0 of 0 in g such that the map $\exp : V_0 \to U_e$, defined by $\exp(X) := (\exp tX)_{t=1}$ for $X \in V_0$, is a diffeomorphism. These results have the straightforward consequence that given a positive integer k, there exists a neighbourhood U'_e of e in G such that if an element $g \in U'_e$ satisfies $g^k = e$, then $g = e$; (explicitly U'_e can be taken as $\exp(\frac{1}{k}(V_0))$ where $\frac{1}{k}(V_0) := \{X \in g : kX \in V_0\}$.)

The identity component G_0 of a Lie group G is the largest connected subset of G containing the identity e; G_0 is a closed (and open, also normal) subgroup of G and G_0 naturally inherits from G the structure of a Lie group (closed subgroups are discussed further in part (ii) below). Finally it is remarked that for a Lie group G, a continuous (group) automorphism $\tau : G \to G$ is necessarily an analytic diffeomorphism of G.

(ii) A Lie group H is a Lie subgroup of a Lie group G if (1) H is a subset of G, (2) H is a subgroup of G, (3) the inclusion $i : H \to G$ is analytic, and (4) the differential $(di)_h : H_h \to G_h$ is one-to-one for each $h \in H$. The Lie algebra $\underline{h}$ of H is then a subalgebra of the Lie algebra $\underline{g}$ of G; conversely, corresponding to a given subalgebra of $\underline{g}$ there is exactly one connected Lie subgroup of G whose Lie algebra is the given one. Given a Lie group G and a topologically closed subset H of G with H also a subgroup of G, then H admits a unique analytic manifold structure such that it becomes a Lie subgroup of G and such that the topology on H is exactly the topology induced on H as a subset of the underlying topological space of G; moreover in this case the Lie algebra $\underline{h}$ of H is given by:

$$\underline{h} = \{X \in \underline{g} : \exp tX \in H \text{ for all } -\infty < t < \infty\} .$$

(iii) A Lie transformation group G of a differentiable manifold M is a Lie group G whose elements are transformations of M, for which group multiplication is composition of transformations, and for which the map $\alpha : G \times M \to M$ is differentiable (where $\alpha(g,p) := g(p)$ for $p \in M$ and g a transformation of M belonging to G). Notice that if H is a Lie subgroup of such a Lie transformation group, then with $i : H \to G$ the corresponding inclusion map, the map $\alpha' : H \times M \to M$ (defined by $\alpha'(h,p) := h(p)$ for for $p \in M$, $h \in H$) may be written as $\alpha' = \alpha \circ (i \times id_M)$ whence, since α, i and id_M are differentiable maps, α' is a differentiable map and H is therefore also a Lie transformation group of M. Concerning Lie transformation groups, the following theorem (due to Ledger and Obata [1]) will be used in Chapter III :

<u>Theorem I.2</u>: Let G be a Lie transformation group of a connected differentiable manifold M. If G is locally transitive on M, i.e. if

for each point $p \in M$ there exists a neighbourhood U of p such that U is contained in the orbit $G(p) := \{z \in M : z = g(p) \text{ for some } g \in G\}$, then G is a transitive Lie transformation group of M. (G is transitive on M if for any two points $x,y \in M$, there exists an element $g \in G$ such such that $g(x) = y$).

The proof of this theorem is a straightforward topological argument.

The following theorem is valid:

Theorem I.3

(a) Let H be a closed Lie subgroup of a connected Lie group G_o. Let $\underline{g}$ and $\underline{h}$ denote the Lie algebras of G_o and H resp. and consider a vector subspace $\underline{m}$ of $\underline{g}$ chosen such that $\underline{g} = \underline{h} \oplus \underline{m}$ (direct sum).

Then the homogeneous space G_o/H of cosets of G_o modulo H admits a Hausdorff topological space structure uniquely determined by the requirement that $\pi : G_o \to G_o/H$ (defined by $\pi(g) = gH$) be an open and continuous map; G_o/H further admits an analytic manifold structure uniquely determined by the requirement that G_o be a Lie transformation group of G_o/H (for the natural action $g(g'H) = (gg')H$ for $g \in G_o$ and $g'H \in G_o/H$).

Then the projection $\pi : G_o \to G_o/H$ is analytic and moreover given a point $a \in G_o$ there exists a connected neighbourhood V_a of aH in the manifold G_o/H and an analytic "cross-section" $\psi_a : V_a \to G_o$ such that $\pi \circ \psi_a = id_{V_a}$. For $a = e$, V_e and ψ_e may be chosen to satisfy also the following two properties: $\psi_e(eH) = e$ and $d\psi_e((G_o/H)_{eH}) = \underline{m}$ ($\underline{m}$ being considered as a vector subspace of $(G_o)_e$ under the natural isomorphism between $\underline{g}$ and $(G_o)_e$).

(b) All the statements made in (a) remain valid if the connected Lie group G_o is replaced by any Lie group G.

This theorem follows from various results contained in Helgason [1], Chevally [1] and Hochschild [1]; the heart of the matter lies in the fact that for the decomposition $\underline{g} = \underline{h} \oplus \underline{m}$ there exist open balls $U_{\underline{h}}$ and $U_{\underline{m}}$ about 0 in $\underline{h}$ and $\underline{m}$ resp. such that the map $\phi : U_{\underline{h}} \times U_{\underline{m}} \to V \subset G_o$ (defined by $\phi(X,Y) = (\exp X).(\exp Y)$ for $X \in U_{\underline{m}}$, $Y \in U_{\underline{h}}$) is an analytic diffeomorphism of $U_{\underline{h}} \times U_{\underline{m}}$ onto an (open) neighbourhood V of e in G_o.

The following important theorem is also valid (cf. Helgason [1], Chapter II):

<u>Theorem I.4</u>: Let G be a transitive Lie transformation group of a connected differentiable manifold M; G_o denotes the identity component of G. Let p_o be a given point in M and define the ("isotropy") subgroup H of G_o by:

$$H := \{g \in G_o : g(p_o) = p_o\} .$$

Then H is a closed subgroup of G_o, G_o is a transitive Lie transformation group of M and the map $\eta : G_o/H \to M$ (defined by $\eta(gH) = g(p_o)$ for $gH \in G_o/H$) is a diffeomorphism (where G_o/H has the (unique) differentiable manifold structure compatible with its natural analytic manifold structure of Theorem I.3, H being endowed with the natural Lie group structure of a closed subgroup of G_o mentioned in part (ii)).

(iv) For a group G and an element $g \in G$ the automorphism $Ad_G(g) : G \to G$ is defined by $Ad_G(g)g' := gg'g^{-1}$ for $g' \in G$; if H is a subgroup of G then for $h \in H \subset G$: $Ad_H(h) = Ad_G(h)\big|_H$. If G is a Lie group then, since the identity e is a fixed point of the (analytic) map $Ad_G(g)$, the differential $(d(Ad_G(g)))_e$ is a linear transformation of G_e and the corresponding linear transformation of $\underline{g}$ (the Lie algebra of G, identified with G_e) will also be denoted $Ad_G(g)$. $Ad_G(g)$ is a Lie algebra automorphism of $\underline{g}$.

With this notation the following definition is made:

Definition I.1: Let G_0 be a connected Lie group and H a closed subgroup of G_0; let $\underline{g}$ and $\underline{h}$ denote the Lie algebras of G_0 and H resp. Then the homogeneous space G_0/H (with the analytic manifold structure of Theorem I.3) is said to be reductive if there exists a vector subspace $\underline{m}$ of $\underline{g}$ such that $\underline{g} = \underline{h} \oplus \underline{m}$ and $Ad_{G_0}(H)\underline{m} \subset \underline{m}$. (The latter inclusion means that for each $h \in H$ and $X \in \underline{m}$, $Ad_{G_0}(h)X \in \underline{m}$).

(v) In this part (v) the following theorem is established:

Theorem I.5: Let G be a Lie group and $X,Y \in G_e$; X^R, Y^L denote the related right-invariant, resp. left-invariant vector fields (uniquely determined by the requirement $(X^R)_e = X$, $(Y^L)_e = Y$).

Then $[X^R, Y^L]_e = 0$.

Proof: Observe firstly that for any $f_1 \in F(G)$:

$$Xf_1 = \frac{d}{dt} f_1(\exp tX)\Big|_{t=0}, \qquad (1)$$

For given $f \in F(G)$ take $f_1 = Y^L f$ in (1); therefore for $g \in G$:

$$f_1(g) = (Y^L)_g f$$

$$= \frac{d}{ds} f(g.(\exp sY))\Big|_{s=0} ,$$

and substituting this into (1) yields:

$$X(Y^L f) = \frac{d}{dt}\left(\frac{d}{ds} f((\exp tX).(\exp sY))\Big|_{s=0}\right)\Big|_{t=0} . \qquad (2)$$

In exactly similar fashion:

$$Y(X^R f) = \frac{d}{ds}\left(\frac{d}{dt} f((\exp tX).(\exp sY))\Big|_{t=0}\right)\Big|_{s=0} . \qquad (3)$$

The expressions in (2) and (3) differ only in the order of differentiation with respect to s and t; but $f \in F(G)$, exp is a differentiable map (on some neighbourhood of 0 in G_e), and group multiplication (denoted by "." above) is differentiable: explicitly denoting group multiplication by $\lambda : G \times G \to G$, one notes that the function $\phi := f \circ \lambda \circ (\exp \times \exp) \circ (\sigma \times \tau) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is differentiable at (0,0) (the differentiable maps $\sigma : \mathbb{R} \to G_e$ and $\tau : \mathbb{R} \to G_e$ are defined by $\sigma(s) := sX$ and $\tau(t) := tY$ for $s,t \in \mathbb{R}$); consequently

$$\frac{d^2\phi}{dsdt}\Big|_{(0,0)} = \frac{d^2\phi}{dtds}\Big|_{(0,0)} ,$$

i.e. from (2) and (3):

$$X(Y^L f) - Y(X^R f) = 0 \quad \text{for any} \quad f \in F(G);$$

i.e., recalling that $(X^R)_e = X, (Y^L)_e = Y$:

$$[X^R, Y^L]_e f = 0 \quad \text{for any } f \in F(G),$$

i.e. $$[X^R, Y^L]_e = 0.$$

This completes the proof of Theorem I.5.

3. Affine Connexions

Helgason [1] Chapter 1 and Kobayashi and Nomizu [1] volume I serve as general references for this section; the material on invariant connexions on reductive homogeneous spaces is taken from Nomizu [2].

(i) Definition I.2:

(a) An affine connexion ∇ on a manifold M is a map $\nabla : T^1(M) \times T^1(M) \rightarrow T^1(M)$ which maps $(X,Y) \in T^1(M) \times T^1(M)$ into $\nabla_X Y \in T^1(M)$ and satisfies the following four conditions for all X, X_1, X_2, Y, Y_1, $Y_2 \in T^1(M)$ and all $f \in F(M)$;

(1) $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

(2) $\nabla_{(X_1 + X_2)} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$

(3) $\nabla_{(fX)} Y = f(\nabla_X Y)$

(4) $\nabla_X(fY) = (Xf)Y + f(\nabla_X Y)$.

(b) An affine transformation Φ of a manifold M endowed with an affine connexion ∇ is a transformation $\Phi : M \rightarrow M$ such that $\nabla_{d\Phi(X)} d\Phi(Y) = d\Phi(\nabla_X Y)$ for all $X,Y \in T^1(M)$.

Henceforth in this section M denotes a given manifold endowed with a given affine connexion ∇. Defining now the differential operator $\Gamma(X,Y) := \nabla_X Y - XY$ for $X,Y \in T^1(M)$, it is seen that a transformation $\Phi : M \rightarrow M$ is an affine transformation of M iff $\Gamma(X,Y)_p(f \circ \Phi) = \Gamma(d\Phi(X), d\Phi(Y))_{\Phi(p)} f$ for all $X,Y \in T^1(M)$, all $p \in M$ and all $f \in F(M)$; this follows directly from the observation that for any transformation $\psi : M \rightarrow M$, $(XY)_p(f \circ \psi) = \{(d\psi(X))(d\psi(Y))\}_{\psi(p)} f$ for all

$X,Y \in T^1(M)$, all $p \in M$ and all $f \in F(M)$. Also given two affine connexions $\bar{\nabla}$ and ∇ on M it follows directly from Definition I.2 that a tensor field $D \in T^1_1(M)$ is defined by $D_p(X,Y) := (\bar{\nabla}_X Y - \nabla_X Y)_p$ for $X,Y \in T^1(M)$ and $p \in M$; D is called the difference tensor of $\bar{\nabla}$ and ∇.

The following result is quoted (cf. Loos [1] volume 1, Chapter 1):

Theorem I.6: Let Φ and Ψ be two affine transformations of a connected manifold M endowed with an affine connexion ∇. If for at least one point $p \in M$, $(d\Phi)_p = (d\Psi)_p$ as linear transformations from M_p onto $M_{\Phi(p)=\Psi(p)}$, then $\Phi = \Psi$ as transformations of M.

Consider a curve $\gamma : (a,b) \to M$ (with $a > b$); let $\dot{\gamma}(t)$ denote the tangent vector to this curve at the point $\gamma(t) \in M$. Then given $s \in (a,b)$ there exists a vector field $X^s \in T^1(M)$ such that $(X^s)_{\gamma(t)} = \dot{\gamma}(t)$ for t in some neighbourhood of s in (a,b); γ is called a geodesic (with respect to the affine connexion ∇) if $(\nabla_{X^s} X^s)_{\gamma(s)} = 0$ for each $s \in (a,b)$ (and each choice of X^s). The fundamental existence theorem for geodesics asserts that given $p \in M$ and $X_p \in M_p$ there exists an $\epsilon > 0$ and a unique geodesic $\gamma : (-\epsilon,\epsilon) \to M$ such that $\dot{\gamma}(0) = X_p$. If a geodesic γ may be extended (as a geodesic) to a curve $\gamma' : \mathbb{R} \to M$ whose restriction to each finite non-$\emptyset$ open interval of $\mathbb{R}$ is a geodesic (as just defined), then such a curve γ' is called a complete geodesic. If each geodesic on M admits of such an extension to a complete geodesic, then ∇ is called a complete affine connexion and M is said to be complete with respect to ∇. One notes also that affine transformations map geodesics into geodesics.

The vector field $\nabla_X Y$ is referred to as the covariant derivative of Y along X (with respect to ∇); the notion of parallel translating a vector field along a curve $\gamma : (a,b) \to M$ may be introduced (relative to ∇) in such a way that a vector field $Y \in T^1(M)$ is invariant under parallel translation along γ iff $(\nabla_{X^s} Y)_{\gamma(s)} = 0$ for all $s \in (a,b)$ (and for X^s as in the above paragraph). The notion of covariant derivation and parallel translation with respect to ∇ may be naturally extended to all tensor fields, in such a way that a tensor field $Q \in T^p_q(M)$ is invariant under any parallel translation iff $\nabla_X Q$ (the covariant derivative of Q along X) vanishes for all $X \in T^1(M)$. (For further details on the covariant differentiation and parallel translation of arbitrary tensor fields see Kobayashi and Nomizu [1]).

The set $A(M,\nabla)$ of all affine transformations of M (M being endowed with the affine connexion ∇) forms a group, with composition of transformations as the group multiplication; moreover $A(M,\nabla)$ admits the structure of a Lie group with which $A(M,\nabla)$ is a Lie transformation group of M (cf. Kobayashi and Nomizu [1], Chapter VI).

Related to a given affine connexion ∇ on a manifold M, the torsion tensor $T \in T^1_2(M)$ and the curvature tensor $R \in T^1_3(M)$ are defined by:

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$$

$$R(X,Y,Z) := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$$

for $X,Y,Z \in T^1(M)$.

(ii) In this part (ii) some of the results of Nomizu [2] concerning invariant affine connexions on reductive homogeneous spaces will be summarised. Consider, with the notation of Section 2, a reductive homogeneous space G_0/H (of a connected Lie group G_0) - reductive with respect to a given decomposition $\underline{g} = \underline{h} \oplus \underline{m}$ as in Definition I.1. $\psi_e : V \to G_o$ is the local cross-section (of Theorem I.3) defined on a neighbourhood V of eH in G_o/H such that $\psi_e(eH) = e$ and $d\psi_e((G_o/H)_{eH}) = \underline{m}$.

Suppose now that the manifold G_o/H is endowed with an affine connexion ∇ ; with the differentiable manifold structure naturally induced on V as an open subset of G_o/H, consider V endowed with the corresponding restriction ∇^+ of ∇ to vector fields in $T^1(V)$ as an affine manifold. For notational convenience define $\sigma : V \to \sigma(V) \subset G$ by $\sigma(p) := \psi_e(p)$ for $p \in V$: then σ is a diffeomorphism of the manifold V onto $\sigma(V)$; $\sigma(V)$ is in fact a submanifold of G, whence it follows that given $X \in T^1(G)$ and $p \in V$ a vector field $X^+ \in T^1(V)$ may be defined by:

$$(X^+)_p \quad := \quad d\pi(X_{\sigma(p)}).$$

With this notation established, the following important theorem (due to Nomizu [2]) is now quoted:

<u>Theorem I.7</u>: (The connexion ∇ on G_o/H is said to be G_o-invariant if G_o is a subgroup of $A(G_o/H, \nabla)$.)

(a) There exists a one-to-one correspondence between the set of all G_o-invariant affine connexions on the reductive homogeneous space G_o/H and the set of bilinear functions $\alpha : \underline{m} \times \underline{m} \to \underline{m}$ for which $\alpha(Ad_{G_o}(h)X, Ad_{G_o}(h)Y) = Ad_{G_o}(h) \circ \alpha(X,Y)$ for all $X,Y \in \underline{m}$ and all $h \in H$.

For a given G_o-invariant affine connexion ∇ on G_o/H the correspondence is explicitly given by:

$$\alpha(X,Y) \quad = \quad d\sigma(\nabla^+_{X^+} Y^+)_{eH} \; ,$$

where $d\sigma(\nabla^+_{X^+} Y^+)_{eH} = d\psi_e(\nabla^+_{X^+} Y^+)_{eH}$ is considered as an element of $\underline{m} \subset \underline{g}$ under the identification of $(G_o)_e$ with $\underline{g}$.

(Recall here that $\underline{m}$ is the particular subspace of $\underline{g}$ with respect to which G_o/H is reductive).

(b) The G_o-invariant affine connexion on G_o/H corresponding to the function $\alpha \equiv 0$ is called the canonical connexion of the second kind (on G_o/H). G_o/H endowed with this connexion is complete; in fact the geodesics through $eH \in G_o/H$ are exactly the projections (under π of section 2) of the one-parameter subgroups of G_o ; the images of such projections under the elements of G_o yield all geodesics.

(c) (For $X,Y \in \underline{g}$ let $[X,Y]_{\underline{m}}$ denote the $\underline{m}$-component of $[X,Y]$ with respect to the decomposition $\underline{g} = \underline{h} \oplus \underline{m}$). The G_o-invariant affine connexion on G_o/H corresponding to the function α given by

$$\alpha(X,Y) = \tfrac{1}{2}[X,Y]_{\underline{m}} \quad \text{for} \quad X,Y \in \underline{m}$$

is the unique torsion-free affine connexion on G_o/H having the same geodesics as the canonical connexion of the second kind; this torsion-free connexion is called the canonical connexion of the first kind (on G_o/H).

Nomizu showed also that the canonical connexion of the second kind is characterised by the fact that it is the unique G_o-invariant affine connexion on the reductive homogeneous space G_o/H such that, for any one-parameter subgroup $\exp tX$ in G_o, the parallel translation from $\pi(\exp t_oX)$ to $\pi(\exp t_1X)$ along the curve $\pi(\exp tX)$ in G_o/H is equivalent to the action of the differential of the transformation $(\exp(t_1 - t_o)X) \in G_o$.

4. Pseudo-Riemannian Manifolds

Wolf [1], Chapter 2, serves as a general reference for this section.

Definition I.3:

(a) A pseudo-Riemannian manifold is a differentiable manifold M endowed with a tensor field $g \in T_2^0(M)$ such that $g_p : M_p \times M_p \to \mathbb{R}$ is a non-degenerate symmetric bilinear form on M_p for each $p \in M$; g is then called a metric tensor on M.

(b) If the bilinear form g_p of (a) is positive-definite for each $p \in M$, then M endowed with g is called a Riemannian manifold; g is then called a Riemannian metric on M.

(c) An isometry of a pseudo-Riemannian manifold M is a transformation $\phi : M \to M$ such that for all $X, Y \in T^1(M)$ and all $p \in M$: $g_p(X_p, Y_p) = g_{\phi(p)}(d\phi(X_p), d\phi(Y_p))$.

(d) Two metric tensors $\bar{g}$ and g on a manifold M are said to be conformally equivalent if there exists a function $f \in F(M)$ such that $\bar{g} = fg$.

The fundamental theorem concerning pseudo-Riemannian manifolds is the following:

Theorem I.8: A pseudo-Riemannian manifold M with metric tensor g admits exactly one torsion-free affine connexion with respect to which the covariant derivative of the metric tensor vanishes. This connexion is called the Levi-Civita connexion of M (relative to the metric tensor g).

A useful fact is that isometries of a pseudo-Riemannian manifold M are affine transformations of M with respect to the Levi-Civita connexion. In general, on a pseudo-Riemannian manifold, affine concepts (such as affine transformation, completeness and parallel translation) always refer to the Levi-Civita connexion (unless otherwise indicated).

The following remark (from Section 3.5 of Klingenberg et al. [1]) will be used in Chapter IV: given two conformally related metric tensors $\bar{g}$ and g on a Riemannian manifold M (say $\bar{g} = fg$ with $f \in F(M)$), then the difference tensor D of their respective Levi-Civita connexions $\bar{\nabla}$ and ∇ satisfies for $X,Y \in T^1(M)$:

$$\begin{aligned} D(X,Y) &= \bar{\nabla}_X Y - \nabla_X Y \\ &= \tfrac{1}{2}\{(X\tilde{f})Y + (Y\tilde{f})X - g(X,Y)\vec{\nabla}\tilde{f}\} \end{aligned}$$

where $\tilde{f} = (\log f) \in F(M)$ (note that for each $x \in M$, $f(x) > 0$) and $\vec{\nabla}\tilde{f}$ denotes the usual gradient of $\tilde{f}$ (with respect to g) - namely, $\vec{\nabla}\tilde{f}$ is the unique element of $T^1(M)$ satisfying $g(Z, \vec{\nabla}\tilde{f}) = Z\tilde{f}$ for all $Z \in T^1(M)$.

5. A Miscellaneous Result.

Theorem I.9: Let M be a differentiable manifold and $\phi : M \to M$ a transformation of M of finite (positive integer) order k, i.e. $\phi^k = id_M$.

If moreover ϕ has a point $p \in M$ as an isolated fixed point, then the linear transformation $(d\phi)_p : M_p \to M_p$ does not have $+1$ as an eigenvalue.

Proof: Consider a coordinate chart (U', ψ') with $p \in U'$. Define $U := U' \cap \phi(U') \cap \ldots \cap \phi^{(k-1)}(U')$ and $\psi := \psi'|_{U}$. Then because $\phi^k = id_M$, $\phi(U) = U$; furthermore (U, ψ) is a coordinate chart with $p \in U$. U will now be considered as a manifold, with the manifold structure induced on it as an open (non-empty) subset of M.

Now $\psi(U) \subset \mathbb{R}^n$ ($n = \dim U = \dim M$) and so $\psi(U)$ admits the usual Kronecker metric δ (i.e. $\delta(e_i, e_j) = \delta_{ij}$ if $\{e_i\}_{i=1}^n$ is the standard basis on $\mathbb{R}^n$). Define the Riemannian metric g on U by:

$$g(X,Y) := \delta(d\psi(X), d\psi(Y)) \quad \text{for } X,Y \in T^1(U) \quad ,$$

Consider also the Riemannian metric $\bar{g}$ defined on U by:

$$\bar{g}(X,Y) := g(X,Y) + g(d\phi(X), d\phi(Y)) + \ldots$$

$$\ldots + g((d\phi)^{(k-1)}(X), (d\phi)^{(k-1)}(Y))$$

for $X,Y \in T^1(U)$.

Henceforth consider U as a Riemannian manifold with metric $\bar{g}$; then from the definition of $\bar{g}$ it follows that ϕ is an isometry of U.

Assume now that for $(d\Phi)_p$ some vector $X_p \in M_p$ has eigenvalue $+1$, i.e. $d\Phi(X_p) = X_p$. Let the curve $\gamma : (-\epsilon,\epsilon) \rightarrow M$ (for some sufficiently small $\epsilon > 0$) be the unique geodesic defined on $(-\epsilon,\epsilon)$ passing through p and having tangent vector X_p at $\gamma(0) = p$. Then the curve $\Phi \circ \gamma$ has tangent vector $d\Phi(X_p) = X_p$ at $\Phi \circ \gamma(0) = \Phi(p) = p$. But Φ is an isometry of U, hence an affine transformation of U whence $\Phi \circ \gamma : (-\epsilon,\epsilon) \rightarrow M$ is also a geodesic (through p with tangent vector X_p at $\Phi \circ \gamma(0) = p$); by the uniqueness of such a geodesic one concludes that $\Phi \circ \gamma(t) = \gamma(t)$ for all $t \in (-\epsilon,\epsilon)$. But $\gamma(0) = p$ and p is an isolated fixed point of Φ; hence $\gamma(t) = p$ for all $t \in (-\epsilon,\epsilon)$, whence $X_p = 0$. Therefore there exists no non-zero vector in M_p with eigenvalue $+1$ for $(d\Phi)_p$, i.e. $(d\Phi)_p$ does not have $+1$ as an eigenvalue.

This completes the proof of Theorem I.9.

6. Jordan Algebras

The books "Jordan-Algebren" by Braun and Koecher [1], "Structure and Representations of Jordan Algebras" by Jacobson [1] and "Symmetric Spaces" by Loos [1] serve as general references for this section; all the theorems quoted below are proved in at least one of these references; terminology and notation are basically those of Braun and Koecher [1].

In this section some basic definitions and theorems concerning Jordan algebras are concisely presented:

Definition I.4:

(a) A Jordan algebra is a commutative algebra A with unit element e, defined on a finite dimensional real vector space such that the algebraic multiplication (denoted by juxtaposition) satisfies:

$$a^2(ab) = a(a^2b) \quad \text{for } a,b \in A \qquad (4)$$

where $a^2 := aa$.

(b) The "left multiplication by $a \in A$" is the linear transformation $L(a) : A \to A$ defined by $L(a)b := ab$ for $b \in A$.

(c) The "quadratic representation of $a \in A$" is the linear transformation $P(a) : A \to A$ defined by $P(a) := 2L^2(a) - L(a^2)$.

Theorem I.10: Let A be a Jordan algebra.

For $a,b \in A$: $P(P(a)b) = P(a)P(b)P(a)$.

This identity is called the "fundamental formula".

Definition I.5: An element a in the Jordan algebra A is said to be invertible if det $P(a) \neq 0$; thus the set $I(A)$ of invertible elements is given by:

$$I(A) = \{a \in A : \det P(a) \neq 0\},$$

Theorem I.11: For a Jordan algebra A, let $a \in I(A)$. Then:

(a) Defining $a^{-1} := P^{-1}(a)a$, $a^{-1}a = aa^{-1} = e$;

(b) $P^{-1}(a) = P(a^{-1})$.

Definition I.6: Let A be a Jordan algebra.

(a) For $f \in A$ the f-mutation of A is the Jordan algebra A_f defined on the same real vector space underlying A but with the new multiplication "$\underset{f}{\perp}$" defined by:

$$a \underset{f}{\perp} b := (af)b + a(bf) - (ab)f.$$

(b) If $f \in I(A)$ the f-mutation is called the f-isotope of A.

Theorem I.12: Let A be a Jordan algebra. Then:

(a) If $f \in I(A)$ then f^{-1} is the unit in the f-isotope A_f.

(b) Defining, for $a,b \in A$, $P(a,b) := \frac{1}{2}\{P(a+b) - P(a) - P(b)\}$, then $P(a,b)$ is bilinear in a and b and for $f \in A$:

$$P(a,b)f = a \underset{f}{\perp} b.$$

(c) The quadratic representation P_f of the f-mutation A_f satisfies: $P_f(a) = P(a)P(f)$ for $a \in A_f$.

(d) For any $f, g \in A$: $A_{P(f)g} = (A_f)_g$;

i.e. for $a, b \in A$:

$$a \underset{P(f)g}{\perp} b = (a \underset{f}{\perp} g) \underset{f}{\perp} b + a \underset{f}{\perp} (b \underset{f}{\perp} g) - (a \underset{f}{\perp} b) \underset{f}{\perp} g \ .$$

It is remarked (with regard to $P(a,b)$ defined in (b) in this Theorem I.12) that for $t \in \mathbb{R}$ and $a, b \in A$:

$$P(a,tb) = tP(a,b);$$

by contrast $P(tb) = t^2 P(b)$.

Finally the following theorem (cf. Loos [1]) is quoted:

<u>Theorem I.13</u>: Let $\mathbb{R}^n$ be the real vector space underlying a Jordan algebra A. Then the set $I(A)$ of invertible elements has a natural differentiable manifold structure as an open subset of $\mathbb{R}^n$. Let $I_o(A)$ denote the component of $I(A)$ containing the algebraic unit e of A.

Endowed with the multiplication $\mu : I_o(A) \times I_o(A) \to I_o(A)$ defined by:

$$\mu(a,b) := P(a)b^{-1} \quad \text{for } a, b \in I_o(A),$$

$I_o(A)$ is a symmetric space (of order 2). (Cf. Definition III.1 with $k = 2$).

This symmetric space $I_o(A)$ will be called the "Jordan symmetric space of A".

CHAPTER II

HISTORICAL BACKGROUND

PART A: SYMMETRIC SPACES

1. Symmetric Spaces and Locally Symmetric Spaces

The notion of symmetric space evolved from the study of Riemannian manifolds on which the curvature tensor (of the Levi-Civita connexion) remains invariant under parallel translation along any curve in the manifold. The study of these manifolds may be said to have been initiated in 1926 in two notes by Harry Levy [1] and [2] in which he offered an erroneous proof that the only such manifolds were the Riemannian manifolds of constant sectional curvature and direct products of Riemannian manifolds of constant sectional curvature. Apparently Elie Cartan, however, had already studied the question in some detail and he was quick to point out (a little later in 1926) in Cartan [1] examples of Riemannian manifolds with curvature tensor invariant under parallel translation but with non-constant sectional curvature. In a series of papers Cartan [1] through [12] expounded more of the details of his work on these manifolds, exploring their geometric properties and moreover demonstrating their unexpected but intimate connection with the theory of Lie groups.

Consider now the following definitions:

Definition II.1:

(a) A (pseudo-)Riemannian locally symmetric space is a (pseudo-) Riemannian manifold on which the curvature tensor R (of the Levi-

Civita connexion ∇) is invariant under parallel translation along any curve in the manifold (i.e. on which $\nabla R = 0$).

(b) An affine locally symmetric space is an affine manifold with a torsion-free connexion ∇ whose curvature tensor R satisfies $\nabla R = 0$.

<u>Definition II.2</u>:

(a) A (pseudo-) Riemannian symmetric space is a (pseudo-) Riemannian manifold M which admits at each point $p \in M$ an isometry which is of order two and which has p as an isolated fixed point.

(b) An affine symmetric space is an affine manifold which admits at each point $p \in M$ an affine transformation which is of order two and which has p as an isolated fixed point.

In statements valid equally for a Riemannian, pseudo-Riemannian and affine (locally) symmetric space the adjective will be suppressed and the simple term "(locally) symmetric space" will be used.

As observed in Cartan [6] the locally symmetric spaces are characterised as follows (see also Whitehead [1]):

<u>Theorem II.1</u>:

(a) A (pseudo-) Riemannian manifold M is a (pseudo-) Riemannian locally symmetric space iff for each point $p \in M$ there exists a neighbourhood of p on which the geodesic symmetric about p is a local isometry.

(b) An affine manifold M with a torsion-free connexion is an affine locally symmetric space iff for each point $p \in M$ there exists a neighbourhood of p on which the geodesic symmetry about p is a local affine transformation.

Borel and Lichnerowicz [1] gave a proof of this theorem in 1952 for the Riemannian case; their proof was based on results of Ehresmann [1]. In the affine case the theorem follows from results of Ambrose [1] and Hicks [1] published in the late 1950's. (Further details may be found in Wolf [1], chapters 1 and 2).

The extension of the local isometries (affine transformations) of Theorem II.1 to global isometries (affine transformations) is not always possible; nevertheless the global extension is possible on a complete, simply-connected locally symmetric space - in particular therefore the simply-connected covering space of a complete locally symmetric space is a symmetric space. (Symmetric spaces themselves are always complete). Most significant for the present purpose, however, is the following theorem which shows that a complete locally symmetric space is in fact locally isometric (affinely diffeomorphic) to a symmetric space, thus justifying the use of the term "locally symmetric" :

<u>Theorem II.2</u>:

(a) Let M be a complete (pseudo-) Riemannian locally symmetric space and $p \in M$. Then there exist a neighbourhood V of p in M, a (pseudo-) Riemannian symmetric space N, and a neighbourhood U in N such that V is isometric to U.

(b) Let M be a complete affine locally symmetric space and $p \in M$. Then there exist a neighbourhood V of p in M, an affine symmetric space N, and a neighbourhood U in N such that V is affinely diffeomorphic to U.

The results of this theorem seem implicit in the work of Cartan and appear in the work of Whitehead [1] in 1932 when Whitehead treated locally symmetric spaces, giving explicit proofs for certain results stated by Cartan about symmetric spaces but proved by him only in the case of a Lie group manifold. A proof of the affine case of Theorem II.2 was given in 1954 by Nomizu [2].

Cartan ([1] and [2]) essentially posed the problem of the classification of all simply-connected Riemannian symmetric spaces. The solution to this problem, brilliantly worked out by Cartan, properly belongs to the Lie group aspect of the subject and will be discussed in the next section. Given the classification of simply-connected symmetric spaces, one has (by Theorem II.2) a local classification of the complete locally symmetric spaces; the global classification of these manifolds is a covering space problem: in this regard cf. Wolf [1] page 44 and Wolf [2]. Henceforth symmetric spaces will be the objects of central interest, but it is important to bear in mind that the geometrical motivation for their study arose from interest in the locally symmetric spaces.

Cartan's extensive studies concerned Riemannian symmetric spaces; his papers in this subject have been a source of inspiration for a great deal of work on the geometry of these manifolds in particular and of homogeneous spaces in general. A comprehensive survey of the theory of Riemannian symmetric spaces was given by Helgason [1] in 1962. Treatments of affine symmetric spaces are to be found in Whitehead [1], Nomizu [2], Kobayashi and Nomizu [1], volume II, Loos [1], volume I, and also in Berger [1] where a classification is given.

2. The Lie Group Characterisation, the Geometry and the "Cartan" Classification of Symmetric Spaces.

(1) Lie Group Characterisation

Cartan [3] demonstrated that for a Riemannian symmetric space M, the group of isometries I(M) admits a Lie group structure (which in modern terminology is compatible with the compact open topology on I(M)) and that I(M) acts transitively as a Lie transformation group on M. Thus (cf. Theorem I.4) a Riemannian symmetric space is diffeomorphic to a homogeneous space of Lie groups; this result is of great importance in the study of these manifolds.

In the case of any Riemannian manifold work by van Dantzig and van der Waerden [1] of 1928 implies that the group of isometries admits the structure of a topological transformation group of the manifold. Extending these results Myers and Steenrod [1] showed that for any Riemannian manifold the group of isometries in fact admits a Lie group structure with which it is a Lie transformation group of the given manifold. Nomizu [1] proved the analogous result for the group of affine transformations of an affine manifold, and from this the result follows for the group of isometries of any pseudo-Riemannian manifold also. So, in particular, on a symmetric space the group of isometries (of affine transformations in the affine case) is always a Lie transformation group; that this group is transitive (on the symmetric space) follows for the pseudo-Riemannian and affine cases exactly as for the Riemannian case treated by Cartan. (An explicit proof is presented in Kobayashi and Nomizu [1]).

From the above results it follows (cf. Theorem I.4) that any symmetric space M may be diffeomorphically represented by a homogeneous space G_o/H where G_o can be taken as the identity component of the group of isometries (or affine transformations) of the given symmetric space M; the isotropy subgroup H in such a representation has the following characterisation (implicit in Cartan's work, but for an explicit proof see Loos [1]):

Theorem II.3:

With G_o and H as introduced immediately above, consider the involutive automorphism θ of G_o defined by $\theta(g) := s_{p_o} \circ g \circ s_{p_o}$ for $g \in G_o$ where p_o is the point in M for which H is the isotropy subgroup and s_{p_o} is the isometry (or affine transformation) associated to p_o (s_{p_o} having order two and having p_o as an isolated fixed point). Then the set $G_o^\theta := \{g \in G_o : \theta(g) = g\}$ is a closed Lie subgroup of G_o; $(G_o^\theta)_o$ denotes the identity component of G_o^θ. The subgroup H satisfies;

$$(G_o^\theta)_o \subset H \subset G_o^\theta \quad .$$

Conversely, given a connected Lie group B_o, a continuous automorphism τ of B_o of order two, and a closed Lie subgroup C of B_o satisfying $(B_o^\tau)_o \subset C \subset B_o^\tau$ (where the Lie group $B_o^\tau := \{b \in B_o : \tau(b) = b\}$ and $(B_o^\tau)_o$ denotes its identity component) then the homogeneous space B_o/C admits the structure of an affine symmetric space.

This characterisation of the isotropy subgroup is very important in the study of symmetric spaces; it lies at the heart of the classification theory (cf. part (iii) of the present section A.2).

(ii) The Geometry of Symmetric Spaces

Following Cartan's penetrating analysis in the Riemannian case, the geometry of symmetric spaces has been developed in considerable detail;

the key factor facilitating an elegant description of various geometrical ideas is the result that symmetric spaces are diffeomorphic to homogeneous spaces of Lie groups as discussed in the previous part (i). Indeed by presenting a geometrical question as an equivalent problem in Lie group theory, the solution usually becomes more readily tractable and, in many cases, can be reduced to simple algebraic considerations in the appropriate Lie algebras.

Emphasising the geometrical significance of Lie groups, Cartan [10] pointed out that geodesics on a Riemannian symmetric space are exactly the orbits of the one-parameter subgroups in the Lie group of isometries. Cartan [2] (with Schouten) studied invariant affine connexions on connected Lie groups, and Cartan [3] also showed how the (Levi-Civita) connexion on a Riemannian symmetric space M can be expressed in Lie algebra terms. In more detail, representing the manifold as G_o/H (as above) the connexion on M gives a G_o-invariant connexion on G_o/H; decomposing $\underline{g} = \underline{h} \oplus \underline{m}$ where $\underline{g}$ and $\underline{h}$ are the Lie algebras of G_o and H respectively and $\underline{m}$ is the (-1)-eigenspace of $(d\theta)_e$ (the differential at e of the automorphism θ of Theorem II.3), then $\underline{m}$ may be identified with M_{p_o} under the projection $\pi : G_o \to G_o/H$ where H is the isotropy subgroup of G_o at, say, the point $p_o \in M$. Thus X,Y,Z denote the vectors in $\underline{m}$ corresponding to $X,Y,Z \in M_{p_o}$; Cartan [3] showed that the curvature tensor R on M satisfies the following identity:

$$R(X,Y,Z) = -d\pi[[X,Y],Z]_e ,$$

In 1954 Nomizu [2] demonstrated that all the above considerations extend directly to the pseudo-Riemannian and affine symmetric spaces (the group of isometries being replaced by the group of affine transformations in the affine case). He introduced the notion of a "reductive" homogeneous space G_0/H as one for which the Lie algebra $\underline{g}$ of the connected Lie group G_0 admits a direct sum decomposition $\underline{g} = \underline{h} \oplus \underline{m}$ ($\underline{h}$ being the Lie algebra of H) such that $Ad_{G_0}(H)\underline{m} \subset \underline{m}$; symmetric spaces are a special class of reductive homogeneous spaces. Nomizu further showed that in the case of a G_0-invariant connexion on a reductive homogeneous space G_0/H the curvature tensor could again be expressed in terms of the appropriate Lie algebra; likewise for the torsion tensor. Moreover from his treatment of invariant connexions it becomes clear that on a symmetric space G_0/H the affine connexion ∇ (of Levi-Civita in the(pseudo-)Riemannian case and necessarily torsion-free also in the affine case) is distinguished in the sense that there is no other G_0-invariant torsion-free connexion on G_0/H with the same geodesics as ∇.

Another important contribution to the theory of symmetric spaces was Cartan's study of their totally geodesic submanifolds (cf. Cartan [3] and [4].) In Cartan [4] he introduced the important notion of the "rank" of a symmetric space - namely, the maximal dimension of flat, totally geodesic submanifolds. The significance of totally geodesic submanifolds of symmetric spaces may be appreciated by the insight they afford of the very nature of symmetric spaces: every symmetric space may be embedded as a totally geodesic submanifold of a certain Lie group (viz. the corresponding "group of displacements" defined in Cartan [4]; see also Loos [1]).

An almost complex structure on a manifold M is a tensor field $J \in T^1_1(M)$ such that $(J_p)^2 = -I_p$ for each $p \in M$. A Riemannian manifold M is called Hermitian if J preserves the metric tensor g of M; i.e. if $g(JX, JY) = g(X,Y)$ for all $X,Y \in T^1(M)$, and a Riemannian symmetric space with such a Hermitian structure is called a Hermitian symmetric space. The

Hermitian symmetric spaces were investigated by Cartan [12], who described their relationship with the bounded domains (i.e. bounded, open connected subsets of C^n for various n). Further studies of this relationship were made by Harish-Chandra [1] and Borel [1], [2].

At this point it seems natural to mention the following more general investigations of complex manifolds, and of almost complex structures (on "real" manifolds). The relationship between complex manifolds and almost complex structures was developed by Eckmann and Fröhlicher [1] and Newlander and Nirenberg [1]; a classification of homogeneous complex manifolds was given by Wang [1] in 1954, and the same results were obtained by a different method by Tits [1] in 1962.

An almost complex structure J on a homogeneous space G/H is called G-invariant if $dg \circ J_p = J_{g(p)} \circ dg_p$ for each $g \in G$, $p \in M$. Such G-invariant almost complex structures have been studied by Hermann [1] Passiencier [1], Gray [1] through [6], and Wolf and Gray [1].

(iii) The "Cartan" Classification of Symmetric Spaces

A very brief sketch of the classification of symmetric spaces will be given here; an excellent outline is to be found in Chapter XI of Kobayashi and Nomizu [1] and a complete treatment in Wolf [1] (see also Loos [1] and Helgason [1]).

The classification may be reduced to finding the simply-connected symmetric spaces and their "centres" (cf. Loos [1], chapter 4), and further reduced to finding those which are "irreducible" (cf. Helgason [1], chapter 8). The classification depends on the Lie group characterisation of symmetric spaces mentioned in part (i) above: viz. that a symmetric space is diffeomorphic to a homogeneous space of Lie groups G_0/H where the isotropy subgroup H satisfies the relation in Theorem II.3 in terms of a certain involutive automorphism of G_0. In fact the classification of irreducible simply-connected symmetric spaces may be in turn reduced to a study of involutive automorphisms of simple Lie algebras. Having a list of the involutive automorphisms of the simple Lie algebras, one finds the corresponding irreducible simply-connected symmetric spaces by careful geometric and Lie group theoretic arguments to establish the correct isotropy subgroups (knowing their Lie algebras); the centre of the isotropy group is determined from well-known results of Lie group theory and there follows the classification of the symmetric spaces covered by a given simply-connected one.

To determine which of the so classified affine symmetric spaces are in fact Riemannian or pseudo-Riemannian symmetric one applies nice arguments involving geometry and Lie group theory. For the Riemannian case the arguments were presented in 1927 by Cartan [4]; the pseudo-Riemannian case has been explicitly presented by Gray [6] in 1971. The general classification of the affine symmetric spaces was done by Berger [1] in 1957.

3. The "Algebraic" Definition of Symmetric Spaces

In the 1960's Ottmar Loos ([1] through [5]) developed a novel approach to the study of symmetric spaces, an approach which is algebraic in flavour and by which the study of symmetric spaces proceeds in a fashion elegantly analogous to the corresponding study of Lie groups. As is a Lie group, a (Loos) symmetric space is defined as a manifold admitting a smooth multiplication satisfying certain properties. The Lie triple system of a (Loos) symmetric space is introduced and plays a role analogous to that of a Lie algebra in Lie group theory; similarly the notion of a semi-simple and simple symmetric space is considered and the concepts "rank" and "centre" are defined. The centre $Z(M)$ of a (Loos) symmetric space M is an abelian Lie group acting freely on M; analogous to the case of the centre of a Lie group, the importance of the centre of a symmetric space is that the (Loos) symmetric spaces covered by M are exactly the quotients of M by discrete subgroups of $Z(M)$.

The full definition of a (Loos) symmetric space is now given:

Definition II.3: A (Loos) symmetric space is a differentiable manifold M endowed with a differentiable (C^∞) multiplication $\mu : M \times M \to M$ ($\mu(x,y)$ is also denoted by $x.y$) satisfying the following four properties for all $x,y,z \in M$:

(1) $x.x = x$

(2) $x.(x.y) = y$

(3) $x.(y.z) = (x.y).(x.z)$

and (4) there exists a neighbourhood V_x of x such that for $v \in V_x$, $x.v = v$ iff $v = x$.

A symmetric space (as introduced in Section 1) always admits the structure of a (Loos) symmetric space, the multiplication being determined simply by defining $\mu(x,y)$ to be the image of y under the (geodesic) symmetry at x. The following theorem, a fundamental result established by Loos ([1] and [3]) implies the converse, - namely that a (Loos) symmetric space always admits a symmetric space structure:

Theorem II.4: Let M be a (Loos) symmetric space with multiplication μ as in Definition II.3 and define for $x \in M$ the map $s_x : M \to M$ by $s_x(y) := \mu(x,y)$.

Then M admits an affine connexion, with respect to which the map s_x (for each $x \in M$) is an affine transformation of M.

From this theorem and the above remarks it is seen that the (Loos) symmetric spaces are exactly the affine symmetric spaces; i.e. the existence of an affine symmetric space structure on a given manifold M is equivalent to the existence of a smooth (i.e. C^∞) multiplication on M satisfying properties (1) through (4) of Definition II.3. This equivalence having been established, the study of symmetric spaces by Loos proceeds as outlined in the two previous sections of this chapter; Loos's development of the subject and his expositions of proofs are presented, however, with a more algebraic flavour. His demonstration that the theory of symmetric spaces so closely parallels that of Lie groups reminds one of Cartan's powerful insight into the structure of symmetric spaces, insight largely gained it would appear from exhaustive studies of group manifolds. Also one remarks that the Lie triple system, which appears significantly in Loos's algebraic treatment, was in fact originally introduced by Cartan [3] in connection with the totally geodesic submanifolds of symmetric spaces.

Finally it is pointed out that the applications of Jordan algebras to symmetric spaces (cf. Part B of this Chapter) are most naturally discussed in the framework of this algebraic approach to symmetric spaces developed by Loos.

4. Generalisations of Symmetric Spaces:

(i) k-symmetric Spaces

The Definition II.2 of a symmetric space M involves the existence at each point $p \in M$ of an involutive affine or (pseudo-)Riemannian symmetry, the symmetric spaces being locally characterised by the vanishing of the covariant derivative of the curvature tensor. The consideration of manifolds admitting at each point a symmetry of some integer order k different from two was initiated by A.J.Ledger in the 1960's; in 1967 Ledger [1] introduced "k-symmetric spaces" to the literature. For k = 3: a local characterisation of 3-symmetric spaces in terms of the curvature tensor was given in 1971 by Gray [6], who was led to the consideration of these manifolds from quite a different point of view: namely, an interest in manifolds admitting almost complex structures and in the existence of invariant almost complex structures on homogeneous spaces. Moreover, joint work by Wolf and Gray [1] essentially provides the classification for 3-symmetric spaces and indicates the procedure for classifying k-symmetric spaces for k > 3. Now a more detailed account of the theory of k-symmetric spaces will be presented.

In 1967 Ledger [1] introduced the following definition:

Definition II.4: A generalised Riemannian symmetric space is a connected Riemannian manifold M admitting at each point $p \in M$ an isometry s_p such that:

(a) p is an isolated fixed point of s_p

and (b) the tensor field S defined by $S_p := (ds_p)_p$ is differentiable (i.e., $S \in T^1_1(M)$).

Ledger [1] also introduced essentially the next definition:

Definition II.5: Let k be an integer ≥ 2.

A Riemannian k-symmetric space is a generalised Riemannian symmetric space (as in Definition II.4) for which the isometry s_p has order k (for each $p \in M$).

The important result presented by Ledger [1] was the proof that on a generalised Riemannian symmetric space the group of isometries is transitive; hence, of course, the manifold is diffeomorphic to a homogeneous space of Lie groups. An alternative proof of the transitivity of the group of isometries is due to F. Brickell (cf. Ledger and Obata [1]); this second proof makes no use of condition (b) of Definition II.4. Consequently in Ledger and Obata [1] the following definition and theorem are given:

Definition II.6: A Riemannian s-manifold is a connected Riemannian manifold M admitting at each point $p \in M$ an isometry s_p for which p is an isolated fixed point.

Theorem II.5: On a Riemannian s-manifold the group of isometries is transitive.

Ledger and Obata [1] and Graham and Ledger [1] made further studies of these manifolds; furthermore the notions of affine and pseudo-Riemannian s-manifolds were introduced:

Definition II.7: An affine (resp. pseudo-Riemannian) s-manifold is a connected affine (resp. pseudo-Riemannian) manifold M admitting at each point $p \in M$ an affine transformation (resp. isometry) s_p such that:

(a) for each $p \in M$, p is an isolated fixed point of s_p,

and (b) the tensor field S defined by $S_p := (ds_p)_p$ is differentiable (i.e., $S \in T^1_1(M)$).

The following analogue of Theorem II.5 is essentially proved in Ledger and Obata [1] (cf. Graham and Ledger [1]); (unlike the Riemannian case, the differentiability requirement of condition (b) seems necessary in the affine case):

Theorem II.6: On an affine s-manifold M the group G of affine transformations is transitive.

(From this result it follows that the group of isometries is transitive on a pseudo-Riemannian s-manifold).

The following definition (found essentially in Ledger and Obata [1]) is given for future reference:

Definition II.8: Let k be an integer ≥ 2.
An affine (resp. pseudo-Riemannian) k-symmetric space is an affine (resp. pseudo-Riemannian) s-manifold M such that for each $p \in M$ the symmetry s_p has order k.

Ledger and Obata further pointed out that for a compact connected non-abelian Lie group G and given integer $k > 2$, the Lie group G^k (the k-fold direct product of G with itself) admits a Riemannian metric with which G^k is a Riemannian s-manifold (and, in fact, a Riemannian k-symmetric space), but with this metric G^k is not a Riemannian symmetric space (indeed with the corresponding Levi-Civita connexion it is not even locally affine symmetric). Thus the notion of s-manifold is a non-trivial generalisation of the notion of symmetric space. Ledger and Obata also discuss the existence of almost complex structures on certain Riemannian k-symmetric spaces (for k odd).

One mentions here some studies of k-symmetric (homogeneous) spaces made by Field [1]; he considered related invariant affine connexions and decompositions of the associated Lie algebras.

In Graham and Ledger [1] the notion of an s-regular manifold was introduced:

<u>Definition II.9</u>: An s-regular Riemannian (resp. pseudo-Riemannian, affine) manifold is a Riemannian (resp. pseudo-Riemannian, affine) s-manifold M as in Definition II.6 (resp. II.7) such that:

(a) $s_p \circ s_x = s_{s_p(x)} \circ s_p$ for all $p, x \in M$,

and (b) the tensor field S defined by $S_p := (ds_p)_p$ is differentiable (i.e., $S \in T_1^1(M)$).

For an s-regular affine manifold Theorem II.6 has the following refinement (cf. Ledger and Obata [1] and Graham and Ledger [1]):

<u>Theorem II.6'</u> : Let M be an s-regular affine manifold, $A(M,\nabla)$ its group of affine transformations.

Define the following closed Lie subgroup of $A(M,\nabla)$:

$$G := \{g \in A(M,\nabla): s_{g(x)} = g \circ s_x \circ g^{-1} \text{ for all } x \in M\}.$$

Then G is transitive on M.

Hence denoting by H the isotropy subgroup of G at some point of M, M is diffeomorphic to the homogeneous space G/H.

Graham and Ledger further introduced in a natural way the notion of a locally s-regular manifold and showed that the relationship between s-regular and locally s-regular manifolds is analogous to the relationship between symmetric and locally symmetric spaces. They also obtained a characteristation of locally s-regular manifolds in terms of conditions on the curvature and torsion tensors and a tensor analogous to "S" in Definition II.9; this was accomplished by demonstrating (inter alia) that a locally s-regular manifold always admits a new affine connexion $\bar{\nabla}$ whose curvature tensor $\bar{R}$, torsion tensor $\bar{T}$ and difference tensor $\bar{D}$ (with respect to the original connexion) satisfy $\bar{\nabla}\bar{T} = 0$, $\bar{\nabla}\bar{R} = 0$ and $\bar{\nabla}\bar{D} = 0$.

Before discussing the link between s-regular manifolds and the studies of Gray, two conjectures are presented; the first is due to A. Deicke, the second to A.J.Ledger. These conjectures serve to emphasise the significance of the s-regular manifolds in this subject.

<u>Conjecture 1</u>: Every s-manifold is k-symmetric for some k.

<u>Conjecture 2</u>: Every k-symmetric space is an s-regular manifold.

(ii) A Local Characterisation and the Classification of s-regular pseudo-Riemannian 3-symmetric Spaces.

On a pseudo-Riemannian 3-symmetric space M a tensor field $J \in T^1_1(M)$ may be defined in terms of the tensor field S (of Definition II.7) by:

$$J_p := \frac{1}{\sqrt{3}}(I_p + 2S_p) \quad \text{for } p \in M.$$

J is in fact an almost complex structure on M (i.e. $(J_p)^2 = -I_p$ for all $p \in M$) - cf. Gray [6] or Ledger and Obata [1]; J is referred to as the canonical almost complex structure on M.

The following is essentially a definition due to Gray [6]:

Definition II.10: A pseudo-Riemannian holomorphically 3-symmetric space is an analytic pseudo-Riemannian manifold M endowed with a pseudo-Riemannian 3-symmetric space structure (as in Definition II.8) such that the multiplication $\mu : M \times M \to M$ is analytic and such that for each $p \in M$ the left multiplication s_p is "holomorphic" with respect to the canonical almost complex structure J on M: i.e.,

$$ds_p \circ J_x = J_{s_p(x)} \circ (ds_p)_x \quad \text{for all } p,x \in M.$$

It is observed that the manifolds of Definition II.10 are exactly the s-regular pseudo-Riemannian 3-symmetric spaces (of Definitions II.9 - II.8 with k = 3): the holomorphic condition on the symmetries is equivalent to the s-regular criterion (note the linear relationship between S and J); the various analyticity requirements of Definition II.10 were not specified in the definition of an s-regular pseudo-Riemannian 3-symmetric space, but such a manifold always admits an analytic structure for which they are in fact satisfied (in this regard cf. Graham and Ledger [1]).

One of the important results presented by Gray [6] is the following local characterisation of pseudo-Riemannian holomorphically 3-symmetric spaces in terms of their curvature tensors and canonical almost complex structures:

Theorem II.7: Let M be an analytic pseudo-Riemannian manifold with analytic almost complex structure J which is almost Hermitian (i.e. letting g denote the metric tensor on M, $g(JX,JY) = g(X,Y)$ for all $X,Y \in T^1(M)$).

Define the tensor field S by $S_p := \frac{\sqrt{3}}{2} J_p - \frac{1}{2} I_p$ for $p \in M$; in terms of the metric tensor g and associated Levi-Civita curvature tensor R on M define the tensor field $\tilde{R} \in T^0_4(M)$ by:

$$\tilde{R}(X,Y,Z,W) := g(Z,R(X,Y,W)) \quad \text{for } X,Y,Z,W \in T^1(M).$$

Then there exists a pseudo-Riemannian holomorphically 3-symmetric space N such that M is locally isometric to N and such that J corresponds to the canonical almost complex structure on N, if and only if the following three conditions are satisfied on M (for all $V,W\ X,Y,Z \in T^1(M)$):

(1) $\tilde{R}(W,X,Y,Z) = \tilde{R}(JW,JX,Y,Z) + \tilde{R}(JW,X,JY,Z) + \tilde{R}(JW,X,Y,JZ)$,

(2) $\nabla_V(\tilde{R})(W,X,Y,Z) + \nabla_V(\tilde{R})(JW,JX,JY,JZ) = 0$,

and (3) all covariant derivatives (of all orders) of J are invariant under S.

(Condition (3) is always satisfied on a nearly-Kaehler manifold; i.e. when $\nabla_X(J)X = 0$ for all $X \in T^1(M)$.)

Gray [6] furthermore shows that the pseudo-Riemannian holomorphically 3-symmetric spaces (equivalently s-regular pseudo-Riemannian 3-symmetric spaces) admit a Lie group characterisation in analogy with Theorem II.3 for symmetric spaces; indeed using methods similar to those of Ledger and Obata [1] and Loos [1], Gray established the following:

Theorem II.8:

(a) Let M be a pseudo-Riemannian holomorphically 3-symmetric space, G_o the largest connected group of holomorphic isometries of M and H the isotropy subgroup of G_o at a point $p_o \in M$. s_{p_o} denoting the holomorphic symmetry at p_o, define $\theta : G_o \to G_o$ by

$$\theta(g) := s_{p_o} \circ g \circ s_{p_o}^{-1} \quad \text{for} \quad g \in G_o .$$

Define the Lie group $G_o^\theta := \{g \in G_o : \theta(g) = g\}$ and let $(G_o^\theta)_o$ denote the identity component of G_o^θ.

Then: (1) θ is an analytic automorphism of G_o of order three,

(2) $(G_o^\theta)_o \subset H \subset G_o^\theta$,

and (3) M is diffeomorphic to G_o/H.

(b) Conversely, given a connected Lie group B_o, a continuous automorphism τ of B_o of order three, and a closed subgroup C of B_o satisfying $(B_o^\tau)_o \subset C \subset B_o^\tau$ (where the Lie group $B_o^\tau := \{b \in B_o : \tau(b) = b\}$ and $(B_o^\tau)_o$ denotes its identity component), then the homogeneous space B_o/C admits the structure of a pseudo-Riemannian holomorphically 3-symmetric space provided that, with a certain decomposition $\underline{b} = \underline{c} \oplus \underline{m}$ (see Gray [6] for details) with $\underline{b}$ and $\underline{c}$ the Lie algebras of B_o and C resp., the vector space $\underline{m}$ admit a non-degenerate symmetric bilinear form invariant under $Ad_{B_o}(C)$ and $(d\tau)_e$.

Moreover the following theorem of Gray [6] shows that the pseudo-Riemannian holomorphically 3-symmetric spaces are reductive homogeneous spaces, and neatly characterises those which are "naturally reductive": i.e. for which the Levi-Civita connexion is the canonical connexion of the first kind (cf. Theorem I.7(c)):

Theorem II.9: With the notation of the preceding theorem (statement (a)), the homogeneous space G_o/H diffeomorphic to M is a reductive homogeneous space.

G_o/H is naturally reductive iff the canonical almost complex structure J of M is nearly-Kaehler.

Finally Gray [6] presents a classification of the pseudo-Riemannian holomorphically 3-symmetric spaces. This classification depends on the joint work of Wolf and Gray [1] which treated the problem of finding for a simple Lie group all (continuous) automorphisms of a given order k; in the case k = 3 a complete list of the possible (continuous) automorphisms is given. Moreover for k = 3 Wolf and Gray examined the corresponding homogeneous spaces G_o/H (where as above $(G_o^\theta)_o \subset H \subset G_o^\theta$ for θ an automorphism of order three of a connected Lie group G_o); they also determined which such spaces admit G_o-invariant (pseudo-)Riemannian metrics and which admit various types of G_o-invariant almost complex structures.

Gray [6] discusses the decomposition of pseudo-Riemannian holomorphically 3-symmetric spaces into "primitive" ones using results of Wu [1] which extend the de Rham decomposition of Riemannian manifolds to the pseudo-Riemannian case. From these results and further observations Gray shows that the classification of pseudo-Riemannian holomorphically 3-symmetric spaces may be reduced to (1) consideration of homogeneous spaces G_o/H where $(G_o^\theta)_o \subset H \subset G_o^\theta$ for some (continuous) automorphism θ of order 3 of a connected simple Lie group G_o and (2) the question of which such spaces admit G_o-invariant pseudo-Riemannian metrics. Consequently the results of Wolf and Gray [1]enable Gray [6] to give a complete list of the "primitive" pseudo-Riemannian holomorphically 3-symmetric spaces (and to decide which are in fact Riemannian).

(iii) Other Generalisations of Symmetric Spaces

Apart from the study of general reductive homogeneous spaces by Nomizu [2] and those particular ones mentioned in the preceding parts (i) and (ii) of this section 4, other specific generalisations of the notion of symmetric space have been made.

(a) Firstly mention is made of the "reflexion spaces" studied by Loos [3], [4]; these are defined exactly as symmetric spaces except that the symmetry at a point p is required simply to leave p fixed, not necessarily to have p as an isolated fixed point. These reflexion spaces were shown to be fibre bundles over symmetric spaces.

(b) In a similar fashion, Robertson [1] has considered foliated Riemannian manifolds which admit for each leaf λ a leaf-preserving isometry s_λ leaving λ pointwise fixed and such that at each point $x \in \lambda$, $(ds_\lambda)_x$ does not have +1 as an eigenvalue. Robertson established that for a compact manifold such a foliation is in fact a fibration over a Riemannian s-manifold (cf. Definition II.6); s-manifolds themselves are considered, in this context, as the special case of point leaves.

(c) As pointed out by Helwig [6] the canonical affine connexion on certain symmetric spaces gives rise to a naturally associated Jordan algebra (cf. Theorem IV.1 of this thesis). More generally on any reductive homogeneous space the maps "$\alpha : \underline{m} \times \underline{m} \to \underline{m}$" introduced by Nomizu [2] (cf. Theorem I.7) also define algebras naturally related to affine connexions. A. Sagle [1], [2] made further studies along these lines, investigating the types of non-associative algebras related to connexions on reductive homogeneous spaces; he also considered the relationship between Lie triple systems and totally geodesic submanifolds of these homogeneous spaces (thus extending to a more general situation the results obtained by Cartan [4] in the case of symmetric spaces).

(d) As noted in section 1 it is a restriction on the curvature tensor (namely $\nabla R = 0$) which essentially characterises the symmetric spaces. Ambrose and Singer [1] and Singer [1] have presented a characterisation of the curvature on Riemannian homogeneous spaces (i.e. complete, connected Riemannian manifolds admitting a transitive group of isometries), and they have posed the general problem of classifying Riemannian homogeneous spaces. In this spirit one may consider the symmetric spaces (of Cartan) and the holomorphically 3-symmetric spaces (treated by Gray) to be two families in such a classification. As yet no other families have been studied so thoroughly but the programme suggested by Ambrose and Singer provides a broad geometrical framework for further generalisations of symmetric spaces.

PART B: JORDAN ALGEBRAS

1. General Development of Jordan Algebras

Jordan algebras originally arose from investigations of possible generalisations of the formalism of quantum mechanics; in the papers of P. Jordan [1], [2] and [3] appearing during 1932-33 these algebras were introduced into the literature. In 1934 a paper by Jordan, von Neumann and Wigner [1] together with a paper by A.A.Albert [1] completely resolved the classification of a family of Jordan algebras called formal-real (cf. the next section concerning this classification). This early work formed a firm foundation on which further studies have been based.

A.A.Albert and N. Jacobson have made important studies of Jordan algebras from a purely algebraic point of view; a German school led by M. Koecher has developed a theory of Jordan algebras having a strong flavour of geometry and analysis. Two excellent books treat these two approaches: namely "Structure and Representations of Jordan Algebras" by N.Jacobson [1] and "Jordan-Algebren" by H. Braun and M. Koecher [1]; in these books extensive bibliographies are to be found for work on Jordan algebras carried out prior to the mid-1960's. (The book "An Introduction to Non-associative Algebras" by R.D.Schafer [1] is mentioned as a relevant general reference).

H. Freudenthal (e.g. [1]), T.A.Springer and J. Tits have made contributions to the aspect of the subject concerning relations between the exceptional Lie algebras and Jordan algebras. Also, Koecher has been responsible for the development of calculus on Jordan algebras and of the related connections with complex analysis, in particular automorphic function theory. (For references to the work mentioned in this paragraph, cf. the bibliographies of Jacobson [1] and Braun and Koecher [1].)

The more geometrical approach to algebras over the real number field is important in the context of this thesis, but it should be pointed out that recently (during the last ten years) the study of Jordan algebras has been based on a very general formulation. Jacobson [5] and McCrimmon [1] have developed a "quadratic theory" based essentially on properties of the quadratic representation (cf. Definition I.4(c)), in particular the fundamental formula in a Jordan algebra (cf. Theorem I.10); in this approach, Jordan algebras over fields of arbitrary characteristic (including characteristic 2) may be given a unified treatment.

It is the recent advances in the more geometrical theory which are of interest here; these advances have been inspired by work of Koecher and concern basically the intimate relationship between Jordan algebras and differential geometry (in particular symmetric spaces) and the attendant insight into certain aspects of complex analysis. A lot of Koecher's results seem to be unpublished or relatively inaccessible (e.g. the 1962 "Lecture Notes" of Koecher [1]) - cf., however, Koecher [2]. Some of Koecher's results, summarised in Loos [1] chapter VIII, explore the connection between formal-real Jordan algebras and certain Hermitian symmetric spaces. Further studies made in this area by U. Hirzebruch, O. Loos and K.-H. Helwig during the 1960's will be discussed in section 4.

A school of Roumanian mathematicians has also studied independently connections between Jordan algebras and differential geometry. A recent publication by Popovici, Jordanescu and Turtoi [1] gives a detailed survey (in Roumanian) of their results. As in the case of the geometrical applications to be discussed in Section 4, one application is the definition of affine connexions (on differentiable manifolds) naturally related to various Jordan algebras; this approach developed from work of

G. Vranceanu [1] and [2]. Certain Wagner spaces are studied from the view-point of Jordan algebras by Turtoi [1]; further references on his work and that of other mathematicians in this Roumanian school are to be found in the book by Popovici et al. [1].

In closing this section, one mentions that K.-H. Helwig [1] presented in 1967 (as his habilitations-schrift) certain results on semi-simple Jordan algebras, elegantly demonstrating the analogy of their theory with the corresponding one for semi-simple Lie algebras: in particular, he showed that every semi-simple complex Jordan algebra is the complexification of a formal-real Jordan algebra; (thus a formal-real Jordan algebra is the analogue of a compact Lie algebra).

2. Examples and Classifications of Jordan Algebras

A detailed presentation of the various classification theorems will not be given here; an excellent treatment is to be found in Braun and Koecher [1] chapter X - cf. also Jacobson [1] chapter V.

In the basic paper by Jordan, von Neumann and Wigner [1] a complete classification of the formal-real Jordan algebras was given, a "formal-real" Jordan algebra A being one in which $a^2 + b^2 = 0$ $(a,b \in A)$ implies $a = b = 0$. These authors deemed such algebras significant from physical arguments related to the quantum mechanical questions which interested them; one remarks that it is these same algebras which in recent years have been shown to have a central connection with the Riemannian symmetric spaces.

More generally, as for Lie algebras, the semi-simple Jordan algebras have been classified: a notion of semi-simple algebra is introduced (in terms of the non-degeneracy of a certain bilinear form) and these algebras are then shown to be direct sums of ideals (the irreducible ones being the simple algebras). The simple Jordan algebras (over the reals) are then classified up to isotopy (due to Kalisch [1], and Jacobson and Jacobson [1] - cf. Braun and Koecher [1] chapter X, section 4); over the field of complex numbers, the simple complex Jordan algebras may be classified directly up to isomorphism (due to Albert [2], [3] and [4] and Jacobson [5] - cf. Braun and Koecher [1] chapter X, section 3.3). Further details of these classifications are also found in Chapter V of Jacobson [1]. (It is pointed out that a formal-real Jordan algebra is always semi-simple).

Although further details of the classifications will be omitted, a brief description of the types of algebras which occur will be given; (reference to Braun and Koecher [1], especially chapter VI, provides a full account of the examples given here).

Given any associative algebra $\tilde{A}$ (over the reals) with multiplication denoted by "o", a Jordan algebra $\tilde{A}^+$ may be defined on the vector space underlying $\tilde{A}$ by means of the multiplication (denoted by juxtaposition) given by

$$ab \;:=\; \frac{a\circ b + b\circ a}{2} \;.$$

A Jordan algebra A is said to be "special" if it is isomorphic to some subalgebra of such an $\tilde{A}^+$. The quadratic representation in such an algebra is given by:

$$P(a)b \;=\; a\circ b\circ a,$$

and the multiplication "$\perp_f$" in its f-mutation is given by

$$a \perp_f b \;=\; \frac{a\circ f\circ b + b\circ f\circ a}{2} \;.$$

Several different "types" of Jordan algebras will now be defined:

<u>Type (i)</u>: Taking $\tilde{A}$ above to be the algebra M_r of real $r\times r$ matrices with the usual (associative) matrix multiplication, $\tilde{A}^+$ is then denoted M_r^+. The set of symmetric matrices in M_r forms a subalgebra of M_r^+, denoted $H_r(\mathbb{R})$. $H_r(\mathbb{R})$ is the Jordan algebra of Type (i).

<u>Types (ii) and (iii)</u>: Type (ii): $H_r(\mathbb{C})$ and Type (iii): $H_r(\mathbb{H})$ are defined exactly analogously to Type (i) on the set of hermitean, resp. quaternionic symmetric $r\times r$ matrices over the complex numbers $\mathbb{C}$, resp.quaternions $\mathbb{H}$.

<u>Type (iv)</u>: Consider a real vector space V (of finite dimension) together with a symmetric real-valued bilinear form υ on V and an element $e \in V$ such that $\upsilon(e,e) = 1$; then V endowed with the multiplication:

$$ab \;=\; \upsilon(a,e)b + \upsilon(b,e)a - \upsilon(a,b)e$$

is a Jordan algebra (denoted $[V, \upsilon, e]$) with unit e. An element $a \in [V, \upsilon, e]$ is invertible iff $\upsilon(a,a) \neq 0$ and in that case $a^{-1} = \frac{a}{\upsilon(a,a)}$. For $\upsilon(f,f) \neq 0$ the f-isotope of $[V, \upsilon, e]$ is $[V, \upsilon_f, f^{-1}]$ where $\upsilon_f(a,b) = \upsilon(f,f)\, \upsilon(a,b)$.

Type (v): The Jordan algebras of Types (i) through (iv) are all, in fact, special Jordan algebras (over the reals). There is one other essentially different type which is not special: it is the exceptional Jordan algebra $H_3(\text{Cay})$ whose definition is similar to that of $H_3(\mathbb{R}, \mathbb{C}, \text{or } \mathbb{H})$. One thus considers 3×3 matrices over the Cayley numbers; the Cayley numbers admit an involution $x \to \bar{x}$ (the analogue of complex or quaternion conjugation) and the set of 3×3 Cayley number matrices (x_{ij}) such that $x_{ij} = \overline{x_{ji}}$ form the algebra $H_3(\text{Cay})$ with the multiplication

$$ab := \frac{a \circ b + b \circ a}{2}.$$

$H_3(\text{Cay})$ may be considered as a (27-dimensional) Jordan algebra over the reals; it is an exceptional algebra in the sense that it is (up to isotopy) the only simple Jordan algebra over the reals which is not special. This algebra was discovered by Jordan, von Neumann and Wigner [1]; that it is exceptional was established by Albert [1] shortly after its discovery.

3. The Automorphism Group of a Jordan Algebra; the Structure Group and Isotopes.

A number of Lie algebras (and corresponding Lie groups) may be associated with a given Jordan algebra (see, for example, chapters VI and VIII of Jacobson [1]). Two particular Lie groups play an extremely important role in the theory of Jordan algebras, and these will be discussed now.

Consider the set Hom(A,A) of all linear transformations of the vector space underlying a given Jordan algebra A, and consider the Lie group GL(A) of all non-singular linear transformations in Hom(A,A). The automorphism group Aut A of A is defined by:

$$\mathrm{Aut}\ A := \{W \in GL(A) : W(ab) = (Wa)(Wb) \quad \text{for all } a,b \in A\} ;$$

as closed subgroup of GL(A), Aut A is a Lie subgroup of GL(A).

One can define a larger subgroup of GL(A) leaving the algebraic structure of A invariant in a weaker sense than Aut A: namely, the structure group $\Gamma(A)$ of A defined by:

$$\Gamma(A) := \{W \in GL(A) : W(ab) = Wa \underset{(f_W)^{-1}}{\perp} Wb \quad (\text{with } f_W := We \in I(A)) \quad \text{for all } a,b \in A\} ;$$

$\Gamma(A)$ is a closed Lie subgroup of GL(A). The elements of $\Gamma(A)$ are "automorphisms up to isotopy": i.e., $W \in \Gamma(A)$ is an isomorphism of A onto some isotope of A; the subgroup Aut A of $\Gamma(A)$ is neatly characterised thus:

$$\mathrm{Aut}\ A = \{W \in \Gamma(A) : We = e\} .$$

One remarks here that (essentially as a consequence of the fundamental formula) $P(a) \in \Gamma(A)$ for $a \in I(A)$; consequently $P(a) \in \mathrm{Aut}\ A$

iff $P(a)e = e$, i.e. iff $a^2 = e$. The group $\Gamma_1(A) \subset \Gamma(A)$ generated by all the $P(a)$ with $a \in I(A)$ is called the inner structure group of A; $\Gamma_1(A) \cap \mathrm{Aut}\, A$ is the group of inner automorphisms. Finally the special structure group $S\Gamma(A)$ is the subgroup of $\Gamma(A)$ of elements with determinant $+1$.

It may be shown (cf. Braun and Koecher [1], chapter IX) that the Lie algebra of Aut A is the derivation algebra Der A of A defined by:

$$\mathrm{Der}\, A := \{W \in \mathrm{Hom}(A,A): W(ab) = (Wa)b + a(Wb) \quad \text{for all } a,b \in A\} ;$$

moreover the Lie algebra of $\Gamma(A)$ is the direct sum Der A $\oplus$ L(A) where L(A) denotes the set of left multiplications L(a) for $a \in A$. (The Lie algebra multiplication in Der A and L(A) is the usual Lie bracket $[S,T] = S \circ T - T \circ S$ defined for any $S,T \in \mathrm{Hom}(A,A)$.) The Lie algebras Der A and Der A $\oplus$ L(A) were treated by Jacobson [6] in 1950 and Meyberg [3] in 1966; one very interesting result in this area is the following realisation of two of the exceptional Lie algebras (and hence of the corresponding exceptional Lie groups). For the exceptional Jordan algebra $H_3(\mathrm{Cay})$, the Lie algebra Der $(H_3(\mathrm{Cay}))$ of Aut $(H_3(\mathrm{Cay}))$ is the exceptional Lie algebra f_4 ; the Lie algebra of $S\Gamma(H_3(\mathrm{Cay}))$ is the exceptional Lie algebra e_6. (One adds that the derivation algebra of the Cayley numbers themselves is the exceptional Lie algebra g_2.)

An important property of the structure group is the following result (cf. Braun and Koecher [1] Satz XI. 2.4):

<u>Theorem II.10</u>: (Recall Theorem I.13 which asserts that the set I(A) of invertible elements in a Jordan algebra A is a topological space - indeed a differentiable manifold; $I_0(A)$ denotes the topological component of I(A) which contains the algebraic unit e).

For a Jordan algebra A, the identity component $\Gamma_o(A)$ of the structure group $\Gamma(A)$ acts transitively on $I_o(A)$.

Mention is made of some work by Meyberg [2] on the relationship between the structure group and isotopes: namely, he proved that a central-simple Jordan algebra of given dimension is determined up to isotopy by its structure group. (The centre Z(A) of a Jordan algebra is defined by:

$$Z(A) := \{z \in A: L(a) \circ L(z) = L(z) \circ L(a) \text{ for all } a \in A\} ;$$

a simple Jordan algebra is called central-simple if $Z(A) = \mathbb{R}e$. Simple formal-real Jordan algebras are always central-simple.)

Further work on isotopes and mutations has been carried out by Helwig [1] and [4] - cf. also Braun and Koecher [1], chapter V.

4. Jordan Algebras and Symmetric Spaces

(i) Max Koecher discovered the relationship between formal-real Jordan algebras and Hermitian symmetric spaces; details of his results are given in Koecher [1]. A paper by Koranyi and Wolf [1]provides a characterisation of those Hermitian symmetric spaces realisable by Koecher's construction in terms of Jordan algebras. Related studies of certain Siegel domains were made by C. Hertneck [1] and U. Hirzebruch [1]; cf. also Resnikoff [1]. Hirzebruch [3] also studied the description of the two exceptional (Hermitian) bounded symmetric domains in terms of Jordan algebras.

A few results in this area will now be quoted. Consider a formal-real Jordan algebra A: the complexification of A is defined by $A^C := A + \sqrt{-1}A$; the map "exp" is defined on A (and on A^C) by $\exp a := e + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \dots$. $I_0(A)$ denotes, as in Theorem I.13, the topological component of $I(A)$ containing the algebraic unit e. Then $\exp\sqrt{-1}A$ is a compact Riemannian symmetric space whose non-compact dual is simply $\exp A = I_0(A)$ (with metrics naturally defined on these manifolds in terms of the Jordan algebra structure of A.) Moreover in this case the "half-space" $A + \sqrt{-1}I_0(A)$ admits a Riemannian metric with which it is a non-compact Hermitian symmetric space; the work of Koranyi and Wolf [1] shows that, in terms of complex analysis, $\exp\sqrt{-1}A$ may be interpreted as the Bergmann-Shilov boundary of the bounded symmetric domain corresponding to the half-space $A + \sqrt{-1}I_0(A)$.

The results just mentioned are a sophisticated generalisation of the following exceedingly special case: $A = \mathbb{R}$. $\exp\sqrt{-1}\mathbb{R}$ is the unit circle whose non-compact dual is exactly $\exp\mathbb{R} = I_0(\mathbb{R}) = \{x \in \mathbb{R} : x > 0\}$; the half-space $\mathbb{R} + \sqrt{-1}I_0(\mathbb{R})$ is just the usual "upper half" of the complex plane. Via stereographic projection this upper half-plane is realised as a hemisphere (without equator), which may be mapped onto the interior of the unit disc in the complex plane; the Bergmann-Shilov boundary of this (disc) domain is exactly the unit circle, i.e. $\exp\sqrt{-1}\,\mathbb{R}$.

(ii) In 1965 U.Hirzebruch [2] presented results on the correspondence between formal-real Jordan algebras and compact Riemannian symmetric spaces of rank one. For a formal-real Jordan algebra A consider the set $I_1(A)$ of "primitive idempotents" in A defined by:

$$I_1(A) := \{c \in A: \ (1)\ c^2 = c \text{ and } (2) \text{ if } c = c_1 + c_2 \text{ with } c_1c_2 = 0,\ c_1^2 = c_1 \text{ and } c_2^2 = c_2,\ \text{then } c_1 = 0 \text{ or } c_2 = 0\}.$$

Hirzebruch showed that for a simple formal-real Jordan algebra the set $I_1(A)$ is a connected differentiable manifold admitting a Riemannian metric defined in terms of the "trace form" λ on A (for $a,b \in A$, $\lambda(a,b) := \mathrm{Tr}L(ab)$); on a formal-real Jordan algebra λ is positive definite. Since, moreover, λ is Aut A-invariant, it may be shown that automorphisms of A are isometries of the Riemannian manifold $I_1(A)$; furthermore, $I_1(A)$ is a Riemannian symmetric space with multiplication μ (in the sense of Loos) given by:

$$\mu(c,d) := P(e - 2c)d,$$

i.e. the symmetry at $c \in I_1(A)$ is the automorphism $P(e - 2c)$. Hirzebruch showed that $I_1(A)$ is in fact a compact Riemannian symmetric space of rank one.

Conversely, by explicit construction he demonstrated that each compact Riemannian symmetric space of rank one could be represented as $I_1(A)$ as above for some simple formal-real Jordan algebra A. Hirzebruch makes some remarks about the geodesics of these particular symmetric spaces, using the Jordan algebraic representation to explicitly study a given geodesic.

To give one example of the correspondence just discussed, consider for $r \geq 3$ the simple formal-real Jordan algebra $H_r(\mathbb{R})$ defined on the set of symmetric $r \times r$ real matrices (cf. "Type (i)" in Section 2); by standard results on Jordan algebras the corresponding compact Riemannian symmetric space of rank one $I_1(H_r(\mathbb{R}))$ is the (n-1)-dimensional real projective space $P_{n-1}(\mathbb{R})$.

(iii) An elegant and comprehensive investigation of the interplay between Jordan algebras and symmetric spaces is given in a paper by Karl-Heinz Helwig [6], who inter alia constructs in a unified way via Jordan algebras all the Riemannian symmetric spaces treated by the authors mentioned in parts (i) and (ii) above (with the exception only of the two exceptional bounded symmetric domains considered by Hirzebruch [3]). Indeed Helwig realises (in a manner to be explained in more detail below) all the classical Riemannian symmetric spaces of non-compact type, all compact spaces of rank one, all the Grassmannians, the compact duals of the Siegel domains of types I, II and III, and the (compact) unitary, orthogonal and symplectic groups.

The method of Helwig's construction is as follows: given a Jordan algebra A together with an involutive automorphism J of A (i.e. $J \in \mathrm{Aut}\, A$ and $J^2 = \mathrm{id}_A$), one defines the following subset of the set I(A) of invertible elements in A:

$$I(A,J) \quad := \quad \{a \in I(A): \ a^{-1} = Ja\} \ .$$

It may be shown that I(A,J) is equivalently defined by the zeros of a certain set of polynomials (naturally related to the Jordan algebra structure of A), whence I(A,J) is shown to be a differentiable manifold - indeed an algebraic variety in the real vector space underlying A. The topological component of I(A,J) containing the unit e of A is denoted $I_o(A,J)$. In terms of the quadratic representation (cf. Definition I.4(c)) a smooth multiplication μ is defined on $I_o(A,J)$ by $\mu(a,b) := P(a)b^{-1}$, and $I_o(A,J)$ endowed with μ is in fact a symmetric space in the sense of Loos (cf. Definition II.3; also Theorem I.13).

Consider now an element $W \in \Gamma(A)$, the structure group of A (cf. Section 3); then, defining $w := (We)^{-1}$, W is an isomorphism of A with its isotope A_w; also $J_w := JP(w)$ is an involutive automorphism of A_w. Now the following natural Lie subgroup of $\Gamma(A)$ is introduced:

$$\Gamma(A,J) \;:=\; \{W \in \Gamma(A):\; W \circ J = J_w \circ W\}\;;$$

$\Gamma_o(A,J)$ denotes the identity component of $\Gamma(A,J)$.

One recalls that Der A $\oplus$ L(A) is the Lie algebra of $\Gamma(A)$ (cf. Section 3); its subalgebra which is the Lie algebra of $\Gamma_o(A,J)$ turns out to be the following:

$$\mathrm{Der}(A,J) \oplus L(A_-)$$

where $\qquad \mathrm{Der}(A,J) \;:=\; \{D \in \mathrm{Der}\,A:\; DJ = JD\}$

and $\qquad L(A_-) \;:=\; \{L(a) \in L(A):\; Ja = -a\}$.

Helwig establishes that $\Gamma_o(A,J)$ is a transitive Lie transformation group of $I_o(A,J)$, the isotropy subgroup at e being $K := \mathrm{Aut}\,A \cap \Gamma_o(A,J)$, and $I_o(A,J)$ is diffeomorphic to the homogeneous space of Lie groups $\Gamma_o(A,J)/K$. This representation plays a useful role in studying the symmetric space $I_o(A,J)$.

Being a symmetric space in the sense of Loos, $I_o(A,J)$ admits a canonical affine connexion (cf. Section A.3, Theorem II.4). Helwig proceeds to show how this connexion ∇, its curvature and geodesics may be expressed in terms of isotopes of A; in particular, his interpretation of ∇ is exactly that presented in Theorem IV.1 of this thesis (cf. remarks on Chapter IV in the INTRODUCTION). One remarks that for $a \in I_o(A,J)$ the (geodesic) symmetry s_a is given by:

$$s_a(b) = P(a)b^{-1} = P(a)Jb = JP(a)^{-1}b = J_{a^{-1}}(b).$$

In general, of course, $I_o(A,J)$ is not a Riemannian symmetric space; but further restrictions may be placed on the algebra A such that $I_o(A,J)$ does admit a Riemannian metric with which it is Riemannian symmetric. The Riemannian metrics constructed by Helwig are defined in terms of the trace form λ - namely, the symmetric bilinear form defined on A by $\lambda(a,b) := \mathrm{Tr}\, L(ab)$; the analogous form on the f-isotope A_f is denoted λ_f: in fact $\lambda_f(a,b) = \lambda(a,P(f)b)$. A Jordan algebra is semi-simple iff λ is non-degenerate; in particular, for a formal-real Jordan algebra λ is positive definite. Therefore, for example, given a formal-real Jordan algebra A, a Riemannian metric g on $I_o(A,J)$ may be defined as follows: for a point $q \in I_o(A,J) \subset I_o(A)$, the tangent space to $I_o(A)$ at q is identified naturally with $A_{q^{-1}}$ and the tangent space to $I_o(A,J)$ at q is identified with the (-1)-eigenspace of $A_{q^{-1}}$ under the involutive linear transformation $J_{q^{-1}} = JP(q^{-1})$; then

$$g_q(X,Y) := \lambda_{q^{-1}}(X,Y) \quad \text{for } X,Y \in (I_o(A,J))_q$$

defines the Riemannian metric g on $I_o(A,J)$. $I_o(A,J)$ endowed with g is then a Riemannian symmetric space: for each $a \in I_o(A,J)$ the (geodesic)symmetry $J_{a^{-1}}$ is an isometry with respect to g (this follows from standard properties of λ: cf. for example Chapter V of Braun and Koecher [1]). Actually a construction similar to the above may be applied to yield a Riemannian symmetric space structure on $I_0(A,J)$ for a wider class of algebras than the formal-real ones; the details are given in Helwig [6].

Helwig [6] studies extensively the Riemannian symmetric spaces of the form $I_0(A,J)$. He presents algebraic conditions sufficient that $I_0(A,J)$ be an irreducible symmetric space and conditions sufficient that it be simply-connected; he also gives a (Jordan) algebraic characterisation of those Riemannian symmetric spaces $I_0(A,J)$ which are Hermitian symmetric. From its definition $I_0(A,J)$ is shown to be a submanifold of a sphere; Helwig considers the question of when $I_0(A,J)$ is minimally embedded in this sphere (in this regard cf. the occurrence of Jordan algebras in minimal immersion problems considered by Kuiper[1]). Finally Helwig shows by explicit construction that all the Riemannian symmetric spaces mentioned in the first paragraph of this part (iii) can indeed be realised in the form $I_0(A,J)$ for some Jordan algebra A and some involutive automorphism J of A.

CHAPTER III

SYMMETRIC SPACES OF ORDER k

1. Basic Definition and Statement of Results

Generalisations of the notion of symmetric space have been made by several mathematicians, as discussed in Section A.4 of the previous chapter. It will be noted, however, that generalisations of the algebraic treatment developed for symmetric spaces by Loos (cf. Section II A.3) have not been examined; the present chapter develops this hitherto unconsidered topic.

Throughout this chapter k denotes a fixed integer ≥ 2. The following basic definition is introduced:

Definition III.1: (k is an integer ≥ 2).

A symmetric space of order k is a connected differentiable (C^∞) manifold M endowed with a smooth (C^∞) multiplication $\mu : M \times M \to M$ satisfying (with the definition of $s_p : M \to M$ by $s_p(q) := \mu(p,q)$ for $p,q \in M$) the following four properties for all $x,y \in M$:

(1) $s_x(x) = x$

(2) $s_x^k(y) = y$ (where s_x^k denotes the k-fold composition of s_x)

(3) $s_x \circ s_y = s_{s_x(y)} \circ s_x$

(4) x is an isolated fixed point of s_x.

Definition III.1 encompasses as the special case k = 2 the (connected) symmetric spaces as treated by Loos [1], [3] (cf. Definition II.3); indeed, all the definitions and results of this chapter reduce in the case $k = 2$ either to simple observations on symmetric spaces or to the corresponding important definitions and results appearing in the works of Loos. The preceding comment having been made, it will not be repeated throughout the subsequent pages.

Before stating the results to be proved in this chapter, some remarks and definitions are in order; throughout the whole chapter, M denotes a symmetric space of order k with multiplication μ, as in Definition III.1. Firstly, the definition of the "left multiplication by p", s_p, introduced in Definition III.1, will be restated together with the corresponding definition of the "right multiplication by p":

Definition III.2: With the notation of Definition III.1 for a symmetric space of order k:

(a) For $p \in M$ the "left multiplication by p" is the map $s_p : M \to M$ defined by $s_p(q) := \mu(p,q)$ for $q \in M$.

(b) For $p \in M$ the "right multiplication by p" is the map $r_p : M \to M$ defined by $r_p(q) := \mu(q,p)$ for $q \in M$.

It follows from property (2) of Definition III.1 that for each $x \in M$ the left multiplication s_x is an invertible mapping; its inverse s_x^{-1} is given by $s_x^{(k-1)}$, the (k-1)-fold composition of s_x. By the C^∞ nature of the multiplication μ, s_x (for each $x \in M$) is a (C^∞)differentiable map; so, therefore, is $s_x^{(k-1)}$ and hence s_x is a transformation (i.e., C^∞ diffeomorphism) of M. A consequence of the preceding remarks is that for a given $x \in M$ the differential $(ds_x)_p : M_p \to M_{s_x(p)}$ is defined at each point

$p \in M$, and the following definition is made:

Definition III.3: The (C^∞) tensor field $S \in T_1^1(M)$ is defined by

$S_x := (ds_x)_x$ for $x \in M$.

By property (1) of Definition III.1 (namely, $s_x(x) = x$), $(ds_x)_x$ is indeed a linear transformation of M_x into M_x ; moreover, $(ds_x)_x$ is C^∞ in x because the multiplication is C^∞. Thus S is well-defined as an element of $T_1^1(M)$.

Notice further that S_x is non-singular for each $x \in M$; in fact, property (2) of Definition III.1 implies that $(S_x)^k = I_x$, the identity linear transformation of M_x, and hence $(S_x)^{(k-1)}$ is the inverse $(S_x)^{-1}$ of S_x. From property (4) of Definition III.1 and Theorem I.9 it follows, moreover, that (for each $x \in M$) S_x has no eigenvalue +1; hence $(I_x - S_x) : M_x \to M_x$ is a non-singular linear transformation and the following definition is made:

Definition III.4: Let $x \in M$ and $Z \in M_x$.

Define $\tilde{Z} := (Z)^\sim := (I_x - S_x)^{-1}(Z)$.

Analogously to the case of the left multiplications, it follows from the C^∞ nature of the multiplication μ that (for each $x \in M$) the right multiplication r_x is a ~~transformation of M,~~ differentiable map, and hence the differential $(dr_x)_q : M_q \to M_{r_x(q)}$ is well-defined for each $q \in M$.

The next definition introduces, for any given vector in any given tangent space of M, a related vector field on the (whole) manifold:

Definition III.5: Let $Z \in M_x$. Define the vector field $Z^* \in T^1(M)$ by

$$(Z^*)_y \;:=\; \{dr_{s_x^{-1}(y)}\}\,(Z) \quad \text{for} \quad y \in M.$$

Z^* is indeed a well-defined C^∞ vector field: firstly $(Z^*)_y$ as defined is an element of M_y because

$$\begin{aligned} r_{s_x^{-1}(y)}(x) &= \mu(x, s_x^{-1}(y)) \\ &= s_x(s_x^{-1}(y)) \\ &= y, \end{aligned}$$

and secondly $(Z^*)_y$ is C^∞ in y because the multiplication μ is C^∞ .

In order to demonstrate the validity of the next definition, now it will be shown, as a lemma, that given $Z \in M_x$ the evaluation of the vector field $(\tilde{Z})^*$ at x yields exactly Z itself:

Lemma: Let $x \in M$ and $Z \in M_x$. Then $((\tilde{Z})^*)_x = Z$.

Proof: First it is shown that for any $W \in M_x$:

$$S_x(W) + dr_x(W) \;=\; W.$$

Let n denote the dimension of M and choose local coordinates $\{u^i\}_{i=1}^n$ defined on a coordinate patch U containing x (cf. Section I.1) and define $W^j := Wu^j$ for $j = 1,2,\dots,n$; then $W = W^i \left(\frac{\partial}{\partial u^i}\right)_x$. Now for any $f \in F(M)$:

$$\begin{aligned} \{S_x(W) + dr_x(W)\}f &= \{ds_x(W) + dr_x(W)\}f \\ &= W(f \circ s_x) + W(f \circ r_x) \\ &= W^i \left(\frac{\partial}{\partial u^i}\right)_x (f \circ s_x + f \circ r_x) \end{aligned}$$

$$= W^i\left\{\left(\frac{\partial}{\partial u^i}\right)_x (u^j \circ s_x + u^j \circ r_x)\right\}\left(\frac{\partial f}{\partial u^j}\right)_x \qquad (1)$$

But recall that $\mu(y,y) = y$ for all $y \in M$ by property (1) of Definition III.1; that is, defining the C^∞ map $\Delta : M \times M \to M$ by $\Delta(y) := (y,y)$ for $y \in M$,

$$\mu \circ \Delta = id_M .$$

For $y \in U$, $\mu \circ \Delta(y) = y \in U$, so that $\mu \circ \Delta|_U$ maps U onto U and, the coordinate functions u^j (for $j = 1,2,\ldots,n$) being defined on U, one obtains:

$$u^j \circ \mu \circ \Delta|_U = u^j \circ id_M|_U$$

$$= u^j \circ id_U ;$$

hence $\left(\frac{\partial}{\partial u^i}\right)_x (u^j \circ \mu \circ \Delta) = \delta^j_i$

(where $\delta^j_i = 0$ if $i \neq j$ and $\delta^i_i = 1$ for $i = 1,2,\ldots,n$).

It now follows from the differentiability of u^j, μ and Δ that:

$$\left(\frac{\partial}{\partial u^i}\right)_x (u^j \circ s_x + u^j \circ r_x) = \delta^j_i ; \qquad (2)$$

in more detail, observing that $u^j \circ \mu \circ \Delta(y) = u^j \circ \mu(y,y)$ for $y \in M$ and recalling that $\mu(x,y) = s_x(y)$ and $\mu(y,x) = r_x(y)$, the term $\left(\frac{\partial}{\partial u^i}\right)_x (u^j \circ s_x)$ in (2) arises from differentiating the expression $u^j \circ \mu(y,y)$ with respect to the "second y" leaving the "first y" fixed at the value x and the term $\left(\frac{\partial}{\partial u^i}\right)_x (u^j \circ r_x)$ arises from differentiating the same expression with respect to the "first y" leaving the "second y" fixed at the value x; that the sum of these two partial derivatives gives the value of $\left(\frac{\partial}{\partial u^i}\right)_x (u^j \circ \mu \circ \Delta)$ follows from the theory of differentiable functions of several variables (cf., for example, Burkill [1]).

Using identity (2) in (1) it is immediate that:

$$\{s_x(W) + dr_x(W)\}f = W^i \delta^j_i \left(\frac{\partial f}{\partial u^j}\right)_x$$

$$= W^j \left(\frac{\partial f}{\partial u^j}\right)_x$$

$$= Wf \ .$$

But this last result holds for each $f \in F(M)$ and so indeed:

$$s_x(W) + dr_x(W) = W \quad \text{for all} \quad W \in M_x \ . \tag{3}$$

This result having been established, the proof of the lemma proceeds as follows: from (3) it is clear that:

$$(I_x - s_x)(W) = dr_x(W) \quad \text{for all} \quad W \in M_x \ . \tag{4}$$

Recalling now that $(I_x - s_x)$ is a non-singular linear transformation of M_x and that $Z \in M_x$, put $W = (I_x - s_x)^{-1}(Z)$ in (4) to obtain:

$$Z = dr_x\{(I_x - s_x)^{-1}(Z)\}$$

$$= dr_x(\tilde{Z}) \qquad \text{by Definition III.4}$$

$$= ((\tilde{Z})^*)_x \qquad \text{by Definition III.5 and the fact that } s_x^{-1}(x) = x.$$

This completes the proof of the lemma.

In the following definition juxtaposition of vector fields has the (conventional) meaning as described in Section I.1. The notation introduced in this definition is suggestive of an affine connexion: this is justified in Theorem III.1.

Definition III.6: Let $X,Y \in T^1(M)$.

Define $\nabla_X Y \in T^1(M)$ by:

$$(\nabla_X Y)_x := X_x Y - Y_x(((X_x)^{\sim})^*) \quad \text{for} \quad x \in M.$$

Using the above lemma, it is now shown that $\nabla_X Y$ is well-defined as an element of $T^1(M)$. Firstly it is shown that for $X,Y \in T^1(M)$ and $x \in M$, $(\nabla_X Y)_x \in M_x$. As in the proof of the lemma choose local coordinates $\{u^i\}_{i=1}^n$ on a coordinate patch U containing x, and define for i = 1,2,...,n the functions $X^i : U \to \mathbb{R}$, $Y^i : U \to \mathbb{R}$, $A^i : U \to \mathbb{R}$ by $X^i(u) := X_u u^i$, $Y^i(u) := Y_u u^i$ and $A^i(u) := (((X_x)^{\sim})^*)_u u^i$ for $u \in U$; then:

$$X_u = X^i(u)\left(\frac{\partial}{\partial u^i}\right)_u$$

$$Y_u = Y^i(u)\left(\frac{\partial}{\partial u^i}\right)_u$$

$$(((X_x)^{\sim})^*)_u = A^i(u)\left(\frac{\partial}{\partial u^i}\right)_u \quad \text{for} \quad u \in U.$$

Because X,Y and $((X_x)^{\sim})^*$ are elements of $T^1(M)$, the functions X^i, Y^i, A^i are elements of $F(U)$ for i = 1,2,...,n.

Observe (cf. Section I.1):

$$X_x Y = X^i(x)Y^j(x)\left(\frac{\partial^2}{\partial u^i \partial u^j}\right)_x + X^i(x)\left(\frac{\partial Y^j}{\partial u^i}\right)_x\left(\frac{\partial}{\partial u^j}\right)_x$$

$$= Y^i(x)X^j(x)\left(\frac{\partial^2}{\partial u^i \partial u^j}\right)_x + X^i(x)\left(\frac{\partial Y^j}{\partial u^i}\right)_x\left(\frac{\partial}{\partial u^j}\right)_x,$$

because the "dummy" indices "i" and "j" in the first term can be interchanged without altering the value of the first term itself; this is the case because, considered as differential operators on F(U),

$$\left(\frac{\partial^2}{\partial u^i \partial u^j}\right)_x = \left(\frac{\partial^2}{\partial u^j \partial u^i}\right)_x .$$

Also:

$$Y_x((X_x)^{\sim})^* = Y^i(x)A^j(x)\left(\frac{\partial^2}{\partial u^i \partial u^j}\right)_x + Y^i(x)\left(\frac{\partial A^j}{\partial u^i}\right)_x\left(\frac{\partial}{\partial u^j}\right)_x$$

$$= Y^i(x)X^j(x)\left(\frac{\partial^2}{\partial u^i \partial u^j}\right)_x + Y^i(x)\left(\frac{\partial A^j}{\partial u^i}\right)_x\left(\frac{\partial}{\partial u^j}\right)_x$$

because $A^j(x) = (((X_x)^{\sim})^*)_x\, u^j$

$= X_x\, u^j$ by the lemma

$= X^j(x)$.

Hence as defined in Definition III.6 $(\nabla_X Y)_x$ is given by:

$$(\nabla_X Y)_x = X_x Y - Y_x((X_x)^{\sim})^*$$

$$= Y^i(x)X^j(x)\left(\frac{\partial^2}{\partial u^i \partial u^j}\right)_x + X^i(x)\left(\frac{\partial Y^j}{\partial u^i}\right)_x\left(\frac{\partial}{\partial u^j}\right)_x$$

$$- Y^i(x)X^j(x)\left(\frac{\partial^2}{\partial u^i \partial u^j}\right)_x - Y^i(x)\left(\frac{\partial A^j}{\partial u^i}\right)_x\left(\frac{\partial}{\partial u^j}\right)_x$$

$$= \left\{X^i(x)\left(\frac{\partial Y^j}{\partial u^i}\right)_x - Y^i(x)\left(\frac{\partial A^j}{\partial u^i}\right)_x\right\}\left(\frac{\partial}{\partial u^j}\right)_x .$$

Thus $(\nabla_X Y)_x$ is seen to be an element of M_x .

Moreover, as pointed out above, the functions X^i, Y^i and A^i are in $F(U)$ for $i = 1,2,\ldots,n$, and therefore from the expression just derived for $(\nabla_X Y)_x$ it is seen that $(\nabla_X Y)_x$ depends on x in a C^∞ fashion. Thus $\nabla_X Y$ is well-defined by Definition III.6 as a C^∞ vector field on M - i.e. as an element of $T^1(M)$.

The basic definitions and notation for this chapter have been introduced in the preceding few pages; the results to be established are now stated in the form of two theorems, the proofs of which occur in the subsequent sections of this chapter.

Theorem III.1: Let M be a symmetric space of order k. Then:

(a) M admits an affine connexion ∇ defined (by Definition III.6) in terms of the multiplication μ on M; moreover for each $p \in M$ the left multiplication s_p is an affine transformation of M with respect to ∇ .

(b) M endowed with the affine connexion ∇ is an s-regular affine k-symmetric space; conversely, every s-regular affine k-symmetric space admits the structure of a symmetric space of order k.

(c) M endowed with the affine connexion ∇ is affinely diffeomorphic to a reductive homogeneous space of Lie groups G_o/H endowed with the canonical connexion of the second kind; hence, in particular, ∇ is a complete affine connexion on M.

In the proof of part (c) of Theorem III.1 the group G_o is explicitly taken to be the identity component of the Lie group of those ∇-affine transformations of M whose differentials commute with the tensor field S (of Definition III.3); this group G_o acts transitively on M (cf. the proof of Theorem III.1(c)) and H denotes the isotropy subgroup of G_o at some point $p_o \in M$. Consider now the automorphism θ of G_o defined by $\theta(g) := s_{p_o} \circ g \circ s_{p_o}^{-1}$ for $g \in G_o$; θ is an automorphism of order k. $G_o^\theta := \{g \in G_o : \theta(g) = g\}$, called the "fixed point set of θ", is a closed Lie subgroup of G_o; let $(G_o^\theta)_o$ denote the identity component of G_o^θ. Then in the representation of the symmetric space of order k as the homogeneous space G_o/H, the isotropy subgroup H is characterised by the second theorem thus:

Theorem III.2:

(a) Let M be a symmetric space of order k. Then with the notation introduced above:

$$(G_o^\theta)_o \subset H \subset G_o^\theta \ .$$

(b) Conversely, given a connected Lie group B_o, a continuous automorphism τ of B_o of order k, and a closed Lie subgroup C of B_o satisfying

$$(B_o^\tau)_o \subset C \subset B_o^\tau$$

(where the Lie group $B_o^\tau := \{b \in B_o : \tau(b) = b\}$ and $(B_o^\tau)_o$ denotes the identity component of B_o^τ), then the homogeneous space B_o/C admits the structure of a symmetric space of order k.

2. The Affine Connexion ∇ Admitted by M.

(Proof of Statement (a) of Theorem III.1).

(i) To prove the first part of Statement (a) of Theorem III.1 it must be shown that Definition III.6 defines an affine connexion on M. Since $\nabla_X Y \in T^1(M)$ for $X, Y \in T^1(M)$, it remains to show that conditions (1) through (4) of Definition I.2 are satisfied; these conditions on ∇ are successively checked and shown to be satisfied by showing that for all $X, X_1, X_2, Y, Y_1, Y_2 \in T^1(M)$ and all $f \in F(M)$ the appropriate vector fields coincide at each $x \in M$:

Condition (1):

$$\begin{aligned}\{\nabla_X(Y_1 + Y_2)\}_x &= X_x(Y_1 + Y_2) - (Y_1 + Y_2)_x((X_x)^{\sim})^* \\ &= X_x Y_1 + X_x Y_2 - (Y_1)_x((X_x)^{\sim})^* - (Y_2)_x((X_x)^{\sim})^* \\ &= \{\nabla_X Y_1\}_x + \{\nabla_X Y_2\}_x .\end{aligned}$$

Condition (2): Using the fact that the operations signified by "~" and "*" (of Definitions III.4 and III.5) are $\mathbb{R}$-linear one obtains:

$$\begin{aligned}\{\nabla_{(X_1+X_2)} Y\}_x &= (X_1 + X_2)_x Y - Y_x(((X_1 + X_2)_x)^{\sim})^* \\ &= (X_1)_x Y + (X_2)_x Y - Y_x(((X_1)_x)^{\sim} + ((X_2)_x)^{\sim})^* \\ &= (X_1)_x Y + (X_2)_x Y - Y_x(((X_1)_x)^{\sim})^* - Y_x(((X_2)_x)^{\sim})^* \\ &= \{\nabla_{X_1} Y\}_x + \{\nabla_{X_2} Y\}_x .\end{aligned}$$

Condition (3): Again using $\mathbb{R}$-linearity of the "~" and "*" operations:

$$\{\nabla_{fX}Y\}_x = (fX)_xY - Y_x(((fX)_x)^{\sim})^*$$

$$= \{f(x)X_x\}Y - Y_x\{f(x)(X_x)^{\sim}\}^*$$

$$= f(x)\{X_xY\} - Y_x\{f(x)((X_x)^{\sim})^*\}$$

$$= f(x)\{X_xY\} - f(x)\{Y_x((X_x)^{\sim})^*\}$$

$$= f(x)\{\nabla_XY\}_x .$$

Condition (4): The derivative nature of juxtaposition of vector fields (cf. Section I.1) implies the required derivative nature of ∇_XY in Y, as follows:

$$\{\nabla_X(fY)\}_x = X_x(fY) - (fY)_x((X_x)^{\sim})^*$$

$$= (X_xf)Y_x + f(x)\{X_xY\} - \{f(x)Y_x\}((X_x)^{\sim})^*$$

$$= (Xf)_xY_x + f(x)\{X_xY\} - f(x)\{Y_x((X_x)^{\sim})^*\}$$

$$= \{(Xf)Y\}_x + f(x)\{\nabla_XY\}_x .$$

This completes the verification that conditions (1) through (4) are satisfied, so completing the demonstration that Definition III.6 defines an affine connexion ∇ on M.

(ii) To finish the proof of Statement (a) of Theorem III.1 it must be shown that for given $p \in M$, s_p is an affine transformation of M with respect to ∇.

Given two vector fields $X,Y \in T^1(M)$ the differential operator $\Gamma(X,Y)$ (acting on elements of $F(M)$) is defined in terms of the affine connexion ∇

by $\Gamma(X,Y) := \nabla_X Y - XY$; as remarked after Definition I.2 a transformation $\phi : M \to M$ is an affine transformation of M with respect to ∇ if and only if

$$\{\Gamma(X,Y)\}_x f \circ \phi \;=\; \{\Gamma(d\phi(X),\, d\phi(Y))\}_{\phi(x)} f \qquad (5)$$

for all $X,Y \in T^1(M)$, all $f \in F(M)$ and all $x \in M$. Explicitly in the present case:

$$\{\Gamma(X,Y)\}_x \;=\; -Y_x((X_x)^{\sim})^* ,$$

and now the identity (5) will be verified for $\phi = s_p$.

The left-hand side of (5) is given as follows for $\phi = s_p$:

$$\begin{aligned}\{\Gamma(X,Y)\}_x f \circ s_p &= \{-Y_x((X_x)^{\sim})^*\}(f \circ s_p)\\ &= -Y_x\{((X_x)^{\sim})^*(f \circ s_p)\} .\end{aligned}$$

The function enclosed in the braces will now be examined; its value at a point $y \in M$ is given as follows:

$$\begin{aligned}(((X_x)^{\sim})^*)_y(f \circ s_p) &= (X_x)^{\sim}(f \circ s_p \circ r_{s_x^{-1}(y)})\\ &= (X_x)^{\sim}(f \circ r_{s_q^{-1} \circ s_p(y)} \circ s_p) \quad \text{(where } q := s_p(x))^{\dagger}\\ &= ds_p((X_x)^{\sim})(f \circ r_{s_q^{-1} \circ s_p(y)}).\end{aligned}$$

---

† The Definition III.2(b) of right multiplication and repeated application of property (3) of Definition III.1 show that for all $z \in M$:

$$\begin{aligned}s_p \circ r_{s_x^{-1}(y)}(z) &= s_p \circ s_z \circ s_x^{-1}(y)\\ &= s_{s_p(z)} \circ s_p \circ s_x^{-1}(y)\\ &= s_{s_p(z)} \circ s_p \circ s_x^{(k-1)}(y)\\ &= s_{s_p(z)} \circ s_q^{(k-1)} \circ s_p(y) \quad \text{putting } q := s_p(x)\\ &= s_{s_p(z)} \circ s_q^{-1} \circ s_p(y)\\ &= r_{s_q^{-1} \circ s_p(y)} \circ s_p(z) ;\end{aligned}$$

that is, $s_p \circ r_{s_x^{-1}(y)} \;=\; r_{s_q^{-1} \circ s_p(y)} \circ s_p$.

Thus defining $\alpha \in F(M)$ by

$$\alpha(y) \;:=\; ds_p((X_x)^{\sim})(f \circ r_{s_q^{-1} \circ s_p(y)}) \tag{6}$$

for $y \in M$, the left-hand side of (5) with $\Phi = s_p$ takes the form:

$$\{\Gamma(X,Y)\}_x \, f \circ s_p \;=\; -Y_x \alpha . \tag{7}$$

Computing now the right-hand side of (5) with $\Phi = s_p$:

$$\begin{aligned}\{\Gamma(ds_p(X), ds_p(Y))\}_{s_p(x)} f &= -ds_p(Y_x)\{((ds_p(X_x))^{\sim})^*\} f \\ &= -ds_p(Y_x)\{(((ds_p(X_x))^{\sim})^*) f\} \\ &= -Y_x\{\{(((ds_p(X_x))^{\sim})^*) f\} \circ s_p\} .\end{aligned}$$

The function enclosed in the outer braces will now be examined: its value at a point $y \in M$ is given as follows:

$$\begin{aligned}\{(((ds_p(X_x))^{\sim})^*) f\} \circ s_p(y) &= (((ds_p(X_x))^{\sim})^*)_{s_p(y)} f \\ &= dr_{s_q^{-1} \circ s_p(y)}((ds_p(X_x))^{\sim}) f \quad \text{by Definition III.5 of "}*\text{" and recalling } q = s_p(x), \\ &= ((ds_p(X_x))^{\sim})(f \circ r_{s_q^{-1} \circ s_p(y)}).\end{aligned}$$

Thus, defining $\beta \in F(M)$ by

$$\beta(y) \;:=\; ((ds_p(X_x))^{\sim})(f \circ r_{s_q^{-1} \circ s_p(y)}) \quad \text{for } y \in M, \tag{8}$$

the right-hand side of (5) with $\Phi = s_p$ takes the form:

$$\{\Gamma(ds_p(X), ds_p(Y)\}_{s_p(x)} f \;=\; -Y_x \beta \quad . \tag{9}$$

Now s_p is an affine transformation provided that the left- and right-hand sides of (5) are equal (for $\Phi = s_p$); from (7) and (9) it follows that this is indeed the case provided that α and β denote the same function in F(M). From the definitions of α and β in (6) and (8), this is seen to be the case if it can be shown that

$$ds_p((X_x)^{\sim}) = (ds_p(X_x))^{\sim} \quad ;$$

a proof of this identity is now given.

More explicitly (recalling the Definition III.4 of "$\sim$") what must be shown is that:

$$ds_p\{(I_x - S_x)^{-1}(X_x)\} = (I_{s_p(x)} - S_{s_p(x)})^{-1} \circ ds_p(X_x).$$

In fact the following (stronger) identity will be established:

$$ds_p \circ (I_x - S_x)^{-1} = (I_{s_p(x)} - S_{s_p(x)})^{-1} \circ (ds_p)_x \, .$$

Because $(I_z - S_z)$ is a non-singular linear transformation of M_z (for each $z \in M$), its inverse $(I_z - S_z)^{-1}$ can be represented as a linear combination of the identity linear transformation I_z and the first (n-1) positive powers of $(I_z - S_z)$; this follows from the Cayley-Hamilton Theorem (cf. Birkhoff and MacLane [1]). Hence $(I_z - S_z)^{-1}$ can be represented as a linear combination of I_z and the first (n-1) positive powers of S_z, thus:

$$(I_z - S_z)^{-1} = \alpha_0(z)I_z + \alpha_1(z)S_z + \ldots + \alpha_{n-1}(z)S_z^{(n-1)}, \qquad (10)$$

where the coefficients $\alpha_j(z)$ depend upon z; it will now be shown, however, that for $z = x$ and $z = s_p(x)$, S_x and $S_{s_p(x)}$ each satisfy (10) with $\alpha_j(x) = \alpha_j(s_p(x))$ for $j = 0,1,2,\ldots,(n-1)$.

Because $(ds_p)_x : M_x \to M_{s_p(x)}$ is non-singular (notice that indeed $((ds_p)_x)^{-1} = (ds_p^{(k-1)})_{s_p(x)})$, therefore a basis $\{F_i\}_{i=1}^n$ of M_x is mapped under $(ds_p)_x$ into a basis $\{ds_p(F_i)\}_{i=1}^{n}$ of $M_{s_p(x)}$. Furthermore notice that property (3) of Definition III.1 implies that $ds_p \circ S_x = S_{s_p(x)} \circ (ds_p)_x$,

i.e. $$ds_p \circ S_x \circ (ds_p)_x^{-1} = S_{s_p(x)}. \tag{11}$$

From (11) it follows that if the linear transformations S_p and $S_{s_p(x)}$ be expressed in matrix form relative to the bases $\{F_i\}_{i=1}^n$ and $\{ds_p(F_i)\}_{i=1}^n$ respectively, then they are in fact represented by the same matrix. It is manifest that if S_p satisfies (10) with $z = x$ with coefficients $\alpha_j(x)$ then so does its matrix, but moreover this shows that the matrix representing $S_{s_p(x)}$ and hence $S_{s_p(x)}$ itself satisfies (10) with $z = s_p(x)$ with coefficients $\alpha_j(s_p(x)) = \alpha_j(x)$. Thus, setting $\alpha_j(x) = \alpha_j(s_p(x)) = a_j$, $j = 0,1,2,\ldots,(n-1)$,

$$(I_x - S_x)^{-1} = a_o I_x + a_1 S_x + \ldots + a_{n-1}(S_x)^{(n-1)} \tag{12}$$

$$(I_{s_p(x)} - S_{s_p(x)})^{-1} = a_o I_{s_p(x)} + a_1 S_{s_p(x)} + \ldots + a_{n-1}(S_{s_p(x)})^{(n-1)}. \tag{13}$$

The following computation is now made:

$$ds_p \circ (I_x - S_x)^{-1} = ds_p \circ (a_o I_x + a_1 S_x + \ldots + a_{n-1}(S_x)^{(n-1)}) \quad \text{by (12)}$$

$$= (a_o I_{s_p(x)} + a_1 S_{s_p(x)} + \ldots + a_{n-1}(S_{s_p(x)})^{(n-1)}) \circ (ds_p)_x$$

by repeated application of (11),

$$= (I_{s_p(x)} - S_{s_p(x)})^{-1} \circ (ds_p)_x \quad \text{by (13).}$$

Thus the desired identity has been established, completing the proof that s_p is an affine transformation of M with respect to the affine connexion ∇ .

This completes the proof of Statement (a) of Theorem III.1.

3. The Relationship with s-regular Affine k-symmetric Spaces

(Proof of Statement (b) of Theorem III.1)

(i) First it will be shown that M endowed with the affine connexion ∇ (defined in Definition III.6 and examined in the previous section) is an s-regular affine k-symmetric space.

By the results of the preceding section (summarised in Statement (a) of Theorem III.1), the connected manifold M admits at each point $p \in M$ an affine transformation, namely s_p, with the following two properties:

(1) s_p has order k (by property (2) of Definition III.1)

(2) s_p has p as an isolated fixed point (by property (4) of Definition III.1).

Moreover, the tensor field S defined by $S_p := (ds_p)_p$ for $p \in M$ is indeed a differentiable (C^∞)tensor field (cf. Definition III.3 and the remarks immediately following that definition.) Hence M endowed with the affine connexion ∇ is an affine k-symmetric space (cf. Definitions II.8 and II.7).

Furthermore condition (3) of Definition III.1 is exactly the characteristic property of s-regular manifolds (cf. Definition II.9), and so M endowed with ∇ is indeed an s-regular affine k-symmetric space.

(ii) Conversely, consider a given s-regular affine k-symmetric space $\tilde{M}$.

As follows from Theorem II.6', $\tilde{M}$ is diffeomorphic to a homogeneous space $\tilde{G}/\tilde{H}$ of Lie groups, where the isotropy subgroup $\tilde{H}$ is a closed Lie subgroup of $\tilde{G}$ and $\tilde{G} := \{g \in A(\tilde{M},\tilde{\nabla}) : s_{g(x)} = g \circ s_x \circ g^{-1}$ for all $x \in \tilde{M}\}$,

$A(\bar{M},\bar{\nabla})$ being the group of affine transformations of $\bar{M}$. Identifying $\bar{M}$ and $\bar{G}/\bar{H}$, $\bar{G}$ being a Lie transformation group of $\bar{G}/\bar{H}$ (cf. Sections I.2 and I.3(i)), one observes that the symmetry $\bar{s}_{a\bar{H}}$ at a point $a\bar{H} \in \bar{G}/\bar{H}$ acts on $\bar{G}/\bar{H}$ as follows:

$$
\begin{aligned}
\bar{s}_{a\bar{H}}(b\bar{H}) &= \bar{s}_{a(e\bar{H})}(b\bar{H}) && \text{where } id_{\bar{M}} = \text{identity } e \text{ of } \bar{G}, \\
&= a \circ \bar{s}_{e\bar{H}} \circ a^{-1}(b\bar{H}) && \text{because } a \in \bar{G}, \\
&= a \circ \bar{s}_{e\bar{H}} \circ a^{-1} \circ b(e\bar{H}) \\
&= (a\bar{s}_{e\bar{H}}\, a^{-1}b)\bar{H} && \text{for } b\bar{H} \in \bar{G}/\bar{H}\,.
\end{aligned}
$$

Define $\bar{\mu} : (\bar{G}/\bar{H}) \times (\bar{G}/\bar{H}) \to \bar{G}/\bar{H}$ by:

$$
\begin{aligned}
\bar{\mu}(a\bar{H},b\bar{H}) &:= \bar{s}_{a\bar{H}}\,(b\bar{H}) \\
&= (a\bar{s}_{e\bar{H}}\, a^{-1}b)\bar{H} \quad \text{for } a\bar{H},b\bar{H} \in \bar{G}/\bar{H}; \qquad (14)
\end{aligned}
$$

it will now be proved that $\bar{M}$ endowed with this multiplication $\bar{\mu}$ is a symmetric space of order k.

Properties (1), (2) and (4) of Definition III.1 are immediately verified by a glance at Definition II.8 of an affine k-symmetric space; $\bar{M}$ being s-regular, property (3) of Definition III.1 is satisfied because it is exactly condition (a) of the s-regular criterion in Definition II.9. The manifold $\bar{M}$ is connected (cf. Definition II.8 of an affine k-symmetric space and Definition II.7); so to complete the proof that $\bar{M}$ endowed with $\bar{\mu}$ is a symmetric space of order k it remains to show that the multiplication $\bar{\mu}$ is a differentiable map.

To show that $\bar{\mu}$ is differentiable at a point $(a\bar{H},b\bar{H}) \in (\bar{G}/\bar{H}) \times (\bar{G}/\bar{H})$ consider local C^∞ cross-sections $\psi_a : V_a \to \bar{G}$ and $\psi_b : V_b \to \bar{G}$ where

V_a, V_b are neighbourhoods in $\bar{G}/\bar{H}$ of $a\bar{H}$, $b\bar{H}$ respectively and ψ_a, ψ_b are such that $\bar{\pi} \circ \psi_a = id_{V_a}$, $\bar{\pi} \circ \psi_b = id_{V_b}$ where $\bar{\pi} : \bar{G} \to \bar{G}/\bar{H}$ is the C^∞ map defined by $\bar{\pi}(g) := g\bar{H}$ for $g \in \bar{G}$; such cross-sections exist (cf. Theorem I.3(b)). It follows from (14) that on $V_a \times V_b$ (a neighbourhood of $(a\bar{H}, b\bar{H})$ in $(\bar{G}/\bar{H}) \times (\bar{G}/\bar{H})$), $\bar{\mu}$ is given thus:

$$\bar{\mu}|_{V_a \times V_b} = \bar{\pi} \circ \bar{\upsilon} \circ (\psi_a \times \psi_b) \tag{15}$$

where $\bar{\upsilon} : \bar{G} \times \bar{G} \to \bar{G}$ is defined by $\bar{\upsilon}(c,d) := c\bar{s}_{e\bar{H}}\, c^{-1}d$ for $c,d \in \bar{G}$. Observing that $\bar{\upsilon}$ is differentiable and recalling (as noted above) that $\bar{\pi}$ and ψ_a, ψ_b are differentiable (C^∞), it follows from (15) (and from the fact that the composition and direct products of differentiable maps are differentiable) that $\bar{\mu}|_{V_a \times V_b}$ is differentiable, and hence that $\bar{\mu}$ is differentiable at $(a\bar{H}, b\bar{H})$. Therefore $\bar{\mu}$ is a differentiable ($C^\infty \equiv$ "smooth") multiplication, and the proof that $\bar{M}$ endowed with $\bar{\mu}$ is a symmetric space of order k is completed.

This finishes the proof of Statement (b) of Theorem III.1.

4. Symmetric Spaces of Order k as Reductive Homogeneous Spaces

(Proof of Statement (c) of Theorem III.1)

(i) It is first shown that M is diffeomorphic to a homogeneous space of Lie groups.

$A(M,\nabla)$ denotes the group of all affine transformations of M endowed with the affine connexion ∇ discussed in Section 2; $A(M,\nabla)$ has the Lie group structure (mentioned in Section I.3) with which it is a Lie transformation group of M. Consider now the subgroup G of $A(M,\nabla)$ defined by:

$$G \ := \ \{g \in A(M,\nabla) : (dg)_p \circ S_p = S_{g(p)} \circ (dg)_p \ \text{ for all } p \in M\} ;$$

since (cf. Theorem I.6) two affine transformations of a connected manifold coincide if their differentials have the same action on at least one tangent space, it follows (recalling that $S_x := (ds_x)_x$) that G may also be described thus:

$$G \ = \ \{g \in A(M,\nabla) : g \circ s_p = s_{g(p)} \circ g \ \text{ for all } p \in M\} .$$

From this description of G, condition (3) of Definition III.1 and the result of Theorem III.1(a) proved in Section 2, it is clear that for each $x \in M$ the left multiplication s_x is a member of G. Moreover it follows, because $A(M,\nabla)$ is a Lie transformation group of M, that G is a closed subgroup of $A(M,\nabla)$; by remarks in the last paragraph of Section I.2(ii) and the first paragraph of Section I.2(iii), G therefore admits a natural Lie group structure with which, as a Lie subgroup of $A(M,\nabla)$, it is a Lie transformation group of M: from now on G will be considered a Lie group with that structure.

It will now be shown that G is locally transitive on M; i.e. that for given $p \in M$ there exists a neighbourhood (of p) which is contained in $G(p)$, the G-orbit of p. Given a point $p \in M$ consider the right multiplication r_p; as shown in the proof of the Lemma in Section 1 (cf. identity (4)): $(dr_p)_p = I_p - S_p$, a non-singular linear transformation of M_p (cf. the remark preceding Definition III.4). Hence, by the inverse function theorem (cf. Theorem I.1) there exist neighbourhoods V_1 and V_2 of p such that $r_p|_{V_1} : V_1 \to V_2$ is a diffeomorphism.

But observe that:

$$\begin{aligned} V_2 &= r_p(V_1) \\ &= \{r_p(x) : x \in V_1\} \\ &= \{s_x(p) : x \in V_1\} && \text{because } r_p(x) = \mu(x,p) = s_x(p) \\ &\subset G(p) && \text{because } s_x \in G \text{ for each } x \in V_1 \subset M. \end{aligned}$$

So V_2 is a neighbourhood of p and $V_2 \subset G(p)$; i.e., G is locally transitive on M.

Now Theorem I.2 implies that therefore G is transitive on M. Moreover Theorem I.4 implies that in fact G_o, the identity component of G, is a transitive Lie transformation group of M. Selecting a point $p_o \in M$, define the isotropy subgroup H of G_o at p_o by :

$$H := \{h \in G_o : h(p_o) = p_o\} ;$$

because G_o is a Lie transformation group of M it follows that H is a closed subgroup of G_o and H will be considered as a Lie subgroup of G_o (with the natural Lie group structure mentioned in Section I.2(ii) for a closed subgroup of a Lie group). Then Theorem I.4 implies that M is diffeomorphic to the homogeneous space G_o/H (endowed with the differentiable structure described in Theorem I.3).

From now on M and G_o/H will be identified under this diffeomorphism, which is explicitly given (cf. Theorem I.4) as the map $\eta : G_o/H \to M$ defined by:

$$\eta(gH) = g(p_o) \quad \text{for} \quad gH \in G_o/H \, ;$$

$$\eta^{-1}(x) = g'H \quad \text{for} \quad x \in M \quad \text{where } g' \text{ is an element of } G \text{ such that } g'(p_o) = x.$$

With M and G_o/H so identified, the various structures on M are transferred to G_o/H; in particular, G_o/H is thus endowed with the structure of a symmetric space of order k (given already on M). The various maps, tensor fields, the affine connexion, etc. thus induced on G_o/H by the corresponding objects given on M will be denoted by the same symbols; for example, if $\eta(gH) = g(p_o) = p$ for $gH \in G_o/H$, $p \in M$, then

$\eta^{-1} \circ s_p \circ \eta$ will be simply denoted by s_{gH} ,

$d\eta^{-1} \circ S_p \circ d\eta$ will be simply denoted by S_{gH} ,

and $d\eta^{-1}\{\nabla_{d\eta(X)} d\eta(Y)\}$ for $X, Y \in T^1(G_o/H)$ will be simply denoted by $\nabla_X Y$.

(ii) It will be shown in this part (ii) that the homogeneous space G_o/H introduced in part (i) is in fact a reductive homogeneous space (cf. Definition I.1).

Consider the Lie algebras $\underline{g}$ and $\underline{h}$ of G_o and H respectively; since H is a closed Lie subgroup of G_o, it follows that $\underline{h}$ is a subalgebra of $\underline{g}$ and that in fact:

$$\underline{h} = \{X \in \underline{g} : \exp tX \in H \ \text{ for all } -\infty < t < \infty\} . \qquad (17)$$

Recalling that p_o is the point in $M \equiv G_o/H$ for which H is the isotropy subgroup of G_o and that s_{p_o} is the left multiplication by p_o, one defines the following group automorphism of G_o:

<u>Definition III.7</u>: With G_o and s_{p_o} as above, define $\theta : G_o \to G_o$ by:

$$\theta(g) = s_{p_o} \circ g \circ s_{p_o}^{-1} \qquad \text{for } g \in G_o .$$

Clearly $\theta = Ad_G(s_{p_o})|_{G_o}$ (cf. Section I.2(iv)); because $Ad_G(s_{p_o})$ is a diffeomorphic group automorphism of G (and maps the group identity e into e itself and consequently maps the identity component G_o onto G_o itself), therefore θ is a diffeomorphic group automorphism of G_o. θ is of order k, because s_{p_o} is of order k. Notice also that θ leaves the subgroup H pointwise fixed; for if $h \in H$, then:

$$h \circ s_{p_o} = s_{h(p_o)} \circ h \quad \text{because } h \in G_o \subset G,$$

$$= s_{p_o} \circ h \quad \text{because } h \in H \Rightarrow h(p_o) = p_o ,$$

whence $$s_{p_o} \circ h \circ s_{p_o}^{-1} = h;$$

i.e. $$\theta(h) = h.$$

Before proceeding further notice that $\underline{h}$ may be characterised in terms of θ as follows:

$$\underline{h} = \{X \in \underline{g} : d\theta(X_e) = X_e\} \tag{18}$$

For if $X \in \underline{h}$ then exp tX, the integral curve of X (defined for all $-\infty < t < \infty$), lies completely in H by (17); therefore, by the remark in the preceding paragraph, exp tX remains pointwise fixed under θ, and consequently $d\theta(X_e) = X_e$. Conversely if for $X \in \underline{g}$, $d\theta(X_e) = X_e$, consider the integral curve exp tX in G_o; the action of s_{p_o} on the curve $\{\exp tX\}(p_o)$ in $M \equiv G_o/H$ is given by:

$$\begin{aligned} s_{p_o} \circ \{\exp tX\}(p_o) &= s_{p_o} \circ \{\exp tX\} \circ s_{p_o}^{-1}(p_o) \quad \text{because } s_{p_o}^{-1}(p_o) = p_o, \\ &= \{\theta(\exp tX)\}(p_o) \\ &= \{\exp t\, d\theta(X)\}(p_o) \\ &= \{\exp tX\}(p_o) \qquad \text{because } d\theta(X) = X, \end{aligned}$$

where the last two equalities are obtained by noting that because θ is an automorphism of G_o, $d\theta(X)$ is an element of $\underline{g}$, and recalling that such a left-invariant vector field is determined by its value at e (and in this case $d\theta(X_e) = X_e$ by hypothesis, whence $d\theta(X) = X$). So the above equality shows that the curve $\{\exp tX\}(p_o)$ is pointwise fixed under s_{p_o}, whence by the fact that p_o is an isolated fixed point of s_{p_o} (cf. property (4) of Definition III.1) it follows that $\{\exp tX\}(p_o) = p_o$ for $-\infty < t < \infty$ and consequently $\{\exp tX\} \in H$ for $-\infty < t < \infty$; therefore, by (17), $X \in \underline{h}$. This completes the verification of (18).

Denote by $\underline{g}^C$ and $\underline{h}^C$ the complexifications of $\underline{g}$ and $\underline{h}$ respectively. $(d\theta)_e$, the differential of θ at $e \in G_o$, may be considered as a linear transformation of $\underline{g}$ by the standard identification of $\underline{g}$ with $(G_o)_e$; interpreting $(d\theta)_e$ in that way, denote its C-linear extension to $\underline{g}^C$ by Θ: thus for a typical element $(X + \sqrt{-1}\,Y) \in \underline{g}^C$ (where $X, Y \in \underline{g}$), $\Theta(X + \sqrt{-1}\,Y) = (d\theta)_e(X) + \sqrt{-1}\,(d\theta)_e(Y)$. Observe that the results of the preceding paragraph imply that:

$$\underline{h}^C = \{Z \in \underline{g}^C : \Theta(Z) = Z\}$$

Because θ, and therefore $(d\theta)_e$, is of order k, it follows that $\theta^k = I$, the identity linear transformation on $\underline{g}^C$. Note that $\theta^k - I = (\theta - I)(\theta - \alpha_2 I) \ldots (\theta - \alpha_k I)$ where $\alpha_2, \alpha_3, \ldots, \alpha_k$ are the distinct k^{th} roots of unity different from $+1$; it is then clear that:

$$(\theta - I)(\theta - \alpha_2 I) \ldots (\theta - \alpha_k I) = 0;$$

therefore (cf. Birkhoff and MacLane [1]) $\underline{g}^C$ is the direct sum of the eigenspaces corresponding to the eignevalues $+1$, $\alpha_2, \alpha_3, \ldots, \alpha_k$ of θ. From the remark in the previous paragraph, $\underline{h}^C$ is exactly the (+1)-eigenspace; the following notation is introduced for the other eigenspaces:

Definition III.8: With $\underline{g}^C$, θ and α_ℓ $(\ell = 2,3,\ldots,k)$ as above, define the vector subspaces $\underline{m}_\ell$ of $\underline{g}^C$ by:

$$\underline{m}_\ell := \{Z \in \underline{g}^C : \theta(Z) = \alpha_\ell Z\} \quad \text{for} \quad \ell = 2,3,\ldots,k.$$

Then the θ-eignespace decomposition of $\underline{g}^C$ described above is expressed by the identity:

$$\underline{g}^C = \underline{h}^C \oplus \underline{m}_2 \oplus \ldots \oplus \underline{m}_k .$$

Hence, since $\underline{h} = \underline{h}^C \cap \underline{g}$ and $\underline{g} = \underline{g}^C \cap \underline{g}$:

$$\underline{g} = \underline{h} \oplus \{(\underline{m}_2 \oplus \ldots \oplus \underline{m}_k) \cap \underline{g}\} ,$$

and the following definition is introduced:

Definition III.9: With the above notation define the vector subspace $\underline{m}$ of $\underline{g}$ by:

$$\underline{m} := (\underline{m}_2 \oplus \ldots \oplus \underline{m}_k) \cap \underline{g} .$$

Then $\underline{g} = \underline{h} \oplus \underline{m}$; it will be shown now that $Ad_{G_o}(H)\underline{m} \subset \underline{m}$, thus establishing that G_o/H is a reductive homogeneous space (cf. Definition I.1).

Consider a vector $X \in \underline{m}$; it is (uniquely) expressible as:

$$X = X_2 + \dots + X_k$$

where $X_\ell \in \underline{m}_\ell \cap \underline{g}$ for $\ell = 2,3,\dots,k$. Now for a fixed value of ℓ and for a given $h \in H$ the action of $Ad_{G_o}(h)$ on X_ℓ will be examined: $\exp tX_\ell$ is a curve through e (at t = 0) with $(X_\ell)_e$ as tangent vector at t = 0.

Now $Ad_{G_o}(h)(\exp tX_\ell) = h(\exp tX_\ell)h^{-1}$;

therefore $\theta\{Ad_{G_o}(h)(\exp tX_\ell)\} = \theta(h)\theta(\exp tX_\ell)\theta(h)^{-1}$ because θ is an automorphism of G_o,

$$= Ad_{G_o}(\theta(h))\{\theta(\exp tX_\ell)\}$$

$$= Ad_{G_o}(\theta(h))\{\exp t.d\theta(X_\ell)\}$$

$$= Ad_{G_o}(h)\{\exp t.d\theta(X_\ell)\} \quad \text{because } \theta(h) = h.$$

Consequently:

$$(d\theta)_e \circ Ad_{G_o}(h)\{(X_\ell)_e\} = Ad_{G_o}(h) \circ (d\theta)_e\{(X_\ell)_e\}$$

whence $\theta\{Ad_{G_o}(h)X_\ell\} = Ad_{G_o}(h)\{\theta(X_\ell)\}$

$$= Ad_{G_o}(h)\{\alpha_\ell X_\ell\} \quad \text{because } X_\ell \in \underline{m}_\ell,$$

$$= \alpha_\ell\{Ad_{G_o}(h)X_\ell\};$$

whence $Ad_{G_o}(h)X_\ell \in \underline{m}_\ell$.

Of course $Ad_{G_o}(h)X_\ell \in \underline{g}$, so that:

$$Ad_{G_o}(h)X_\ell \in \underline{m}_\ell \cap \underline{g} \quad \text{for each } \ell = 2,3,\dots,k.$$

It follows that:

$$\mathrm{Ad}_{G_o}(h)X = \sum_{\ell=2}^{k} \mathrm{Ad}_{G_o}(h)X_\ell \in \underline{m} \quad \text{for each} \quad h \in H \text{ and } X \in \underline{m};$$

i.e. $\mathrm{Ad}_{G_o}(H)\underline{m} \subset \underline{m}$.

Thus a subspace $\underline{m}$ of $\underline{g}$ has been defined such that $\underline{g} = \underline{h} \oplus \underline{m}$ and $\mathrm{Ad}_{G_o}(H)\underline{m} \subset \underline{m}$; therefore G_o/H is a reductive homogeneous space.

<u>Remark:</u>

Notice firstly that the restriction $\theta|_{\underline{g}}$ of θ to $\underline{g}$ is given by:

$$\theta|_{\underline{g}} = (d\theta)_e = \mathrm{Ad}_G(s_{p_o}) : \underline{g} \to \underline{g} ;$$

notice also that the restriction $\theta|_{\underline{m}}$ of θ to $\underline{m} \subset \underline{g}$ preserves $\underline{m}$ ($\underline{m}$ consisting of sums of eigenvectors of θ lying in $\underline{g}$). Consequently $\mathrm{Ad}_G(s_{p_o})$ preserves $\underline{m}$;

i.e. $\mathrm{Ad}_G(s_{p_o})\underline{m} \subset \underline{m}$.

(Of course in the case when $s_{p_o} \in H \subset G_o$, this latter result follows directly from the $\mathrm{Ad}_{G_o}(H)$-invariance of $\underline{m}$ established just above).

(iii) In this part (iii) the proof of Statement (c) of Theorem III.1 will be completed.

Under the diffeomorphism $\eta : G_o/H \to M$ the affine connexion ∇ defined on M induces an affine connexion (also denoted ∇) on G_o/H (cf. remarks at the end of part (i) of the present section); G_o/H and M are then affinely diffeomorphic. It will be shown here that this connexion on G_o/H is in fact the canonical connexion of the second kind on

G_o/H, a reductive homogeneous space with the $Ad_{G_o}(H)$-invariant decomposition $\underline{g} = \underline{h} \oplus \underline{m}$ of the Lie algebra $\underline{g}$ of G_o as introduced in part (ii) above.

In order to examine the connexion ∇ on G_o/H it is necessary to obtain explicit expressions for the left and right multiplications and the tensor field $\underset{\sim}{S}$ on G_o/H related to its structure as a symmetric space of order k. For notation refer to the concluding remarks in part (i) above. Firstly the left multiplication s_{aH} by a point $aH \in G_o/H$ maps a point $bH \in G_o$ as follows:

$$s_{aH}(bH) = \eta^{-1} \circ s_{a(p_o)} \circ \eta(bH)$$

$$= \eta^{-1} \circ a \circ s_{p_o} \circ a^{-1} \circ \eta(bH) \quad \text{because } a \in G_o \subset G \Longrightarrow s_{a(p_o)} = a \circ s_{p_o} \circ a^{-1},$$

$$= \eta^{-1} \circ a \circ s_{p_o} \circ a^{-1} \circ b(p_o) \quad \text{because } \eta(bH) = b(p_o),$$

$$= \eta^{-1} \circ a \circ s_{p_o} \circ a^{-1} \circ b \circ s_{p_o}^{-1}(p_o) \quad \text{because } s_{p_o}^{-1}(p_o) = p_o,$$

$$= (as_{p_o}\, a^{-1}bs_{p_o}^{-1})H \tag{19}$$

because $\eta^{-1} \circ g(p_o) = gH$ for $g \in G_o$; with regard to the last line notice that $(as_{p_o}a^{-1}bs_{p_o}^{-1})$ is indeed an element of G_o even if $s_{p_o}(\in G)$ is not an element of G_o: for the map $Ad_G(s_{p_o}) : G \to G$ is differentiable and consequently maps the connected component G_o into itself (as pointed out after Definition III.7), whence $Ad_G(s_{p_o})(a^{-1}b) = s_{p_o}a^{-1}b\, s_{p_o}^{-1}$ is in G_o (a and b being in G_o) and so therefore is $(a\, s_{p_o}a^{-1}b\, s_{p_o}^{-1})$ in G_o.

In particular, therefore, for a = e, b = y in (19):

$$s_{eH}(yH) = (s_{p_o} y s_{p_o}^{-1})H \quad \text{for} \quad yH \in G_o/H \ ; \qquad (20)$$

consequently:

$$s_{eH}^2(yH) = (s_{p_o}^2 y s_{p_o}^{-2})H,$$

and indeed by induction:

$$s_{eH}^m(yH) = (s_{p_o}^m y s_{p_o}^{-m})H \quad \text{for any integer} \quad m \geq 0.$$

Hence, recalling that $s_{p_o}^{(k-1)} = s_{p_o}^{-1}$, it follows that:

$$s_{eH}^{-1}(yH) = (s_{p_o}^{-1} y s_{p_o})H \quad \text{for} \quad yH \in G_o/H \ . \qquad (21)$$

Also from (19) it is seen that:

$$r_{bH}(aH) = s_{aH}(bH)$$

$$= (as_{p_o} a^{-1} bs_{p_o}^{-1})H \quad \text{for } aH, bH \in G_o/H \ . \qquad (22)$$

Before examining the tensor field S on G_o/H some notation to describe the relationship between G_o and G_o/H is needed; reference to Section I.2 will further elucidate the remarks about to be made. By definition of the differentiable structure on G_o/H, the map $\pi : G_o \to G_o/H$ (defined by $\pi(g) := gH$ for $g \in G_o$) is differentiable; moreover it has a differentiable local inverse: in particular there exists a neighbourhood V of eH in G_o/H and a differentiable map $\sigma : V \to G_o$ such that $\sigma(eH) = e$, $d\sigma((G_o/H)_{eH}) = \underline{m}$, and $\pi \circ \sigma = id_V$ ($\underline{m}$ is the vector space defined in Definition III.9). Under the linear transformation $(d\sigma)_{eH}$ and its inverse $((d\pi)_e)|_{\underline{m}}$, the vector spaces $(G_o/H)_{eH}$ and $\underline{m}$ are identified.

$S_{eH} = (ds_{eH})_{eH}$ will now be computed. For a vector $Z \in (G_o/H)_{eH}$, consider in G_o/H the curve $\pi(\exp t.d\sigma(Z)) = \{\exp t.d\sigma(Z)\}H$ which has tangent vector Z at t = 0 (because $(d\pi \circ d\sigma)_{eH} = I_{eH}$); the image of this curve under the left multiplication s_{eH} is given by (20) thus:

$$s_{eH}(\pi(\exp t.d\sigma(Z))) = \{s_{p_o}(\exp t.d\sigma(Z))s_{p_o}^{-1}\}H$$
$$= \pi\{s_{p_o}(\exp t.d\sigma(Z))s_{p_o}^{-1}\} ,$$

whence:

$$(ds_{eH})_{eH}(Z) = d\pi \circ Ad_G(s_{p_o}) \circ d\sigma(Z) \quad \text{for each } Z \in (G_o/H)_{eH};$$

therefore:

$$S_{eH} = d\pi \circ Ad_G(s_{p_o}) \circ (d\sigma)_{eH} . \tag{23}$$

Recalling that $(d\sigma)_{eH} : (G_o/H)_{eH} \to \underline{m}$ has inverse $((d\pi)_e)|_{\underline{m}}$ and recalling also the remark at the end of part (ii) asserting $Ad_G(s_{p_o})\underline{m} \subset \underline{m}$, it follows that:

$$d\sigma \circ d\pi \circ Ad_G(s_{p_o})|_{\underline{m}} = Ad_G(s_{p_o})|_{\underline{m}} ,$$

whence together with (23) the following useful identity is obtained:

$$(d\sigma)_{eH} \circ S_{eH} = Ad_G(s_{p_o}) \circ (d\sigma)_{eH} . \tag{24}$$

Finally before coming to an examination of the connexion on G_o/H, the operation "*" of Definition III.5 must be translated explicitly as an operation on tangent vectors to (G_o/H). Let $Z \in (G_o/H)_{eH}$. Now $\gamma(t) := \pi(\exp t.d\sigma(Z))$ is a curve in G_o/H which has tangent vector Z at t = 0 (because $(d\pi \circ d\sigma)_{eH} = I_{eH}$). Also, by Definition III.5 the vector field Z* on G_o/H is given as follows:

$$(Z^*)_{yH} = \{dr_{s_{eH}^{-1}(yH)}\}(Z) \quad \text{for } yH \in G_o/H ;$$

consequently $(Z^*)_{yH}$ is the tangent vector at t = 0 to the following curve, defined for t in some neighbourhood $(-\epsilon,\epsilon)$ of 0 in $\mathbb{R}$ $(\epsilon > 0)$:

$$r_{s_{eH}^{-1}(yH)} \circ \gamma(t) = r_{s_{eH}^{-1}(yH)}(\sigma(\gamma(t))H) \qquad \text{(because } \sigma(\gamma(t))H = \gamma(t) \text{ for } t \in (-\epsilon,\epsilon)\text{)},$$

$$= r_{(s_{p_0}^{-1} y\, s_{p_0})H}(\sigma(\gamma(t))H) \quad \text{by (21)},$$

$$= \{(\sigma \circ \gamma(t))s_{p_0}(\sigma \circ \gamma(t))^{-1}s_{p_0}^{-1}y\}H \quad \text{by (22)},$$

$$= \pi\{(\sigma \circ \gamma(t))s_{p_0}(\sigma \circ \gamma(t))^{-1}s_{p_0}^{-1}y\} \quad \text{by definition of } \pi;$$

evaluating the tangent vector to this curve at $t = 0$ one obtains (with the differentiable map $R_y : G_0 \to G_0$ defined by $R_y(g) := gy$ for $g \in G_0$) :

$$\begin{aligned}
(Z^*)_{yH} &= \{dr_{s_{eH}^{-1}(yH)}\}(Z) \\
&= d\pi \circ dR_y\{d\sigma(Z) - Ad_G(s_{p_0}) \circ d\sigma(Z)\} \\
&= d\pi \circ dR_y \circ \{I_e - Ad_G(s_{p_0})\} \circ d\sigma(Z) \\
&= d\pi \circ dR_y \circ d\sigma \circ \{I_{eH} - S_{eH}\}(Z),
\end{aligned} \tag{25}$$

by (24).

Now for $Z_1 \in (G_0/H)_{eH}$, $\tilde{Z}_1 = \{I_{eH} - S_{eH}\}^{-1}(Z_1)$ by Definition III.4 and from (25) with $Z = \tilde{Z}_1$ one obtains for $yH \in G_0/H$:

$$\begin{aligned}
((\tilde{Z}_1)^*)_{yH} &= d\pi \circ dR_y \circ d\sigma(Z_1) \\
&= d\pi(((d\sigma(Z_1))^R)_y),
\end{aligned} \tag{26}$$

where $(d\sigma(Z_1))^R$ is the right-invariant vector field on G_0 which has the same value as $d\sigma(Z_1)$ at the identity $e \in G_0$.

By means of Definition III.6 and (26) the expression for ∇ on G_o/H is now derived: for vector fields $X',Y' \in T^1(G_o/H)$, the vector field $\nabla_{X'}Y' \in T^1(G_o/H)$ is given by:

$$\left.\begin{aligned} &(\nabla_{X'}Y')_{xH} = X'_{xH}(Y') - Y'_{xH}((X'_{xH})^{\sim})^{*} \quad \text{for } xH \in G_o/H \\ &\text{where in the particular case } xH = eH \text{ (as needed below):} \\ &((X'_{eH})^{\sim})^{*} = d\pi((d\sigma(X'_{eH}))^{R}) \quad \text{by (26).} \end{aligned}\right\} \quad (27)$$

To show that this connexion on the reductive homogeneous space G_o/H is the canonical connexion of the second kind, one proceeds as follows, utilising the results and notation described in Section I.3(ii): the neighbourhood V of eH in G_o/H will be considered as a manifold with the affine connexion ∇^+, the restriction of ∇ to vector fields in $T^1(V)$ (the neighbourhood V has already been mentioned in the paragraph following (22) above). To show that ∇ is the canonical connexion of the second kind it suffices (cf. Theorem I.7) to prove that for any two vector fields $X,Y \in \underline{m} \subset \underline{g} \subset T^1(G_o)$ the following identity holds:

$$(\nabla^+_{X^+} Y^+)_{eH} = 0, \quad \text{where } X^+,Y^+ \in T^1(V) \quad \text{are defined}$$

in terms of X and Y as in Section I.3(ii), explicitly:

$$\left.\begin{aligned} (X^+)_p &= d\pi(X_{\sigma(p)}) \\ (Y^+)_p &= d\pi(Y_{\sigma(p)}) \end{aligned}\right\} \text{ for } p \in V,$$

where $\sigma : V \to \sigma(V) \subset G_o$ is the diffeomorphism of V onto the submanifold $\sigma(V)$ in G_o defined by the local cross-section $\psi_e : V \to G_o$ (for $p \in V$, $\sigma(p) := \psi_e(p)$). (For details refer again to Section I.3(ii)).

Consider the action of $(\nabla_{X^+} Y^+)_{eH}$ on a function $f \in F(V)$; by (27):

$$(\nabla_{X^+} Y^+)_{eH} f = (X^+)_{eH}(Y^+ f) - (Y^+)_{eH}((((X^+)_{eH})^{\sim})^* f) . \qquad (28)$$

Noting that the value of the function $(Y^+ f)$ at $p \in V$ is given by:

$$\begin{aligned}(Y^+ f)(p) &= (Y^+)_p f \\ &= d\pi(Y_{\sigma(p)})f \\ &= Y_{\sigma(p)}(f \circ \pi) \\ &= (Y(f \circ \pi)) \circ \sigma(p),\end{aligned}$$

the first term in (28) is computed thus:

$$\begin{aligned}(X^+)_{eH}(Y^+ f) &= d\pi(X_{\sigma(eH)})\{(Y(f \circ \pi)) \circ \sigma\} \\ &= d\pi(X_e)\{(Y(f \circ \pi)) \circ \sigma\} \\ &= d\sigma \circ d\pi(X_e)\{Y(f \circ \pi)\} \\ &= X_e(Y(f \circ \pi)) \qquad \text{because for } X \in \underline{m},\ d\sigma \circ d\pi(X_e) = X_e\ ; \\ &= (X_e Y)(f \circ \pi). \qquad (29)\end{aligned}$$

Noting that the value of the function $(((X^+)_{eH})^{\sim})^* f)$ at $p \in V$ is given by:

$$\begin{aligned}((((X^+)_{eH})^{\sim})^* f)(p) &= ((((X^+)_{eH})^{\sim})^*)_p f \\ &= (((d\pi(X_{\sigma(eH)}))^{\sim})^*)_p f \\ &= (((d\pi(X_e))^{\sim})^*)_p f \\ &= d\pi(((d\sigma(d\pi(X_e)))^R)_{\sigma(p)})f \qquad \text{by (26) and noting that } p = \sigma(p)H,\end{aligned}$$

$$= d\pi(((X_e)^R)_{\sigma(p)})f \qquad \text{because for } X \in \underline{m}\ \ d\sigma \circ d\pi(X_e) = X_e\ ,$$

$$= (X^R)_{\sigma(p)}(f \circ \pi) \qquad \text{recalling the definition of } X^R \text{ for } X \in \underline{m} \subset \underline{g} \text{ (cf. Section I.2(i))},$$

$$= (X^R(f \circ \pi)) \circ \sigma(p),$$

the second term of (28) is computed thus:

$$(Y^+)_{eH}((((X^+)_{eH})^{\sim})^*)f) = d\pi(Y_{\sigma(eH)})\{(X^R(f \circ \pi)) \circ \sigma\}$$

$$= d\pi(Y_e)\{(X^R(f \circ \pi)) \circ \sigma\}$$

$$= d\sigma \circ d\pi(Y_e)\{X^R(f \circ \pi)\}$$

$$= Y_e\{X^R(f \circ \pi)\} \qquad \text{because for } Y \in \underline{m},\ \ d\sigma \circ d\pi(Y_e) = Y_e\ ,$$

$$= (Y_e\ X^R)(f \circ \pi). \tag{30}$$

Combining (28), (29) and (30) one obtains:

$$(\nabla^+_{X^+}\ Y^+)_{eH}f = (X_eY)(f \circ \pi) - (Y_e\ X^R)(f \circ \pi)$$

$$= (X_eY - Y_eX^R)(f \circ \pi)$$

$$= [X^R,Y]_e(f \circ \pi) \qquad \text{for all } f \in F(V) \tag{31}$$

because $(X^R)_e = X_e$ and $(Y)_e = Y_e$.

But (cf. Theorem I.5) the Lie bracket of the left-invariant vector field Y and the right-invariant vector field X^R necessarily vanishes at e; therefore from (31):

$$(\nabla^+_{X^+} Y^+)_{eH} f = 0 \quad \text{for all} \quad f \in F(V),$$

$$\text{i.e.} \qquad (\nabla^+_{X^+} Y^+)_{eH} = 0 \quad (\text{for all} \quad X, Y \in \underline{m}).$$

Consequently the connexion ∇ is indeed the canonical connexion of the second kind on G_o/H.

From Theorem I.7 it follows that ∇ is a complete affine connexion on G_o/H; the corresponding affine connexion ∇ on M is therefore also complete (M and G_o/H being affinely diffeomorphic).

This completes the proof of Theorem III.1.

5. A Characterisation of the Isotropy Subgroup H.

(i) Proof of Statement (a) of Theorem III.2

Let M be a symmetric space of order k and its associated structures and notation be as in the preceding sections of this chapter.

Consider the automorphism θ of Definition III.7 and define the subgroup G_o^θ of G_o by:

$$G_o^\theta \;:=\; \{g \in G_o : \theta(g) = g\}.$$

Because θ is a (differentiable) transformation of G_o (and hence in particular continuous), it follows that G_o^θ is a topologically closed subset of G_o ; G_o^θ will therefore be considered as a Lie group, endowed with the natural Lie group structure mentioned in Section I.2(ii) for a closed subgroup of a Lie group. Let $(G_o^\theta)_o$ denote the identity component of G_o^θ. Observe that the Lie algebra $\underline{g}^\theta$ of $(G_o^\theta)_o$ (and of G_o^θ) is given by:

$$\begin{aligned}\underline{g}^\theta &= \{X \in \underline{g} : \exp tX \in G_o^\theta \quad \text{for } -\infty < t < \infty\} \\ &= \{X \in \underline{g} : \theta(\exp tX) = \exp tX \quad \text{for } -\infty < t < \infty\} \\ &= \{X \in \underline{g} : d\theta(X_e) = X_e\}.\end{aligned}$$

Recall that H denotes the isotropy subgroup of G_o at the point $p_o \in M$; in Section 4(ii) it was shown that the Lie algebra $\underline{h}$ of H is given by:

$$\underline{h} = \{X \in \underline{g} : d\theta(X_e) = X_e\},$$

whence $\underline{h} = \underline{g}^\theta$. Consequently (cf. Section I.2(ii)) the identity component H_o of H coincides with the identity component $(G_o^\theta)_o$ of G_o^θ. Therefore:

$$(G_o^\theta)_o \subset H.$$

But, as remarked following Definition III.7 (in Section 4(ii)), $\theta(h) = h$ for $h \in H$, i.e. $H \subset G_0^\theta$; therefore:

$$(G_0^\theta)_0 \subset H \subset G_0^\theta .$$

This completes the proof of Statement (a) of Theorem III.2.

(ii) Proof of Statement (b) of Theorem III.2

Consider a connected Lie group B_0 with a continuous group automorphism $\tau : B_0 \to B_0$ of finite order k; recalling the remark at the end of Section I.2(i), observe that τ is a diffeomorphism (in fact an analytic diffeomorphism) of B_0. Consider also a closed subgroup C of B_0 satisfying $(B_0^\tau)_0 \subset C \subset B_0^\tau$ where $B_0^\tau := \{b \in B_0 : \tau(b) = b\}$ - (B_0^τ is a Lie group as a closed subgroup of B_0: cf. the remarks in part (i) just above concerning G_0^θ) - and where $(B_0^\tau)_0$ denotes the identity component of B_0^τ. Considering the homogeneous space B_0/C as a differentiable manifold with the natural differentiable structure as described in Section I.2(iii), observe that the manifold B_0/C is connected because B_0 is connected and the projection $\tilde{\pi} : B_0 \to B_0/C$ (defined by $\tilde{\pi}(b) = bC$ for $b \in B_0$) is a differentiable (hence continuous) map.

Define on the connected manifold B_0/C the multiplication $\tilde{\mu} : (B_0/C) \times (B_0/C) \to B_0/C$ by:

$$\tilde{\mu}(aC,bC) := \{a\tau(a)^{-1}\tau(b)\}C \quad \text{for} \quad aC,bC \in B_0/C \; ; \qquad (32)$$

$\tilde{\mu}$ is well-defined because τ is an automorphism and $\tau(c) = c$ for $c \in C$: for if $(ac')C = aC$ and $(bc'')C = bC$ for $c',c'' \in C$, then

$$\tilde{\mu}((ac')C,(bc'')C) = \{(ac')\tau(ac')^{-1}\tau(bc'')\}C$$

$$= \{ac'\tau(c')^{-1}\tau(a)^{-1}\tau(b)\tau(c'')\}C$$

$$= \{ac'(c')^{-1}\tau(a)^{-1}\tau(b)c''\}C$$

$$= \{a\tau(a)^{-1}\tau(b)\}C$$

$$= \tilde{\mu}(aC,bC).$$

To show that $\tilde{\mu}$ is a differentiable map first consider the map $\tilde{\upsilon} : B_O \times B_O \to B_O$ defined by $\tilde{\upsilon}(a,b) := a\tau(a)^{-1}\tau(b)$ for $a,b \in B_O$ and the projection $\tilde{\pi} : B_O \to B_O/C$ as defined above; $\tilde{\upsilon}$ is a differentiable map because τ, group multiplication in B_O and the operation of taking the inverse in B_O are each differentiable maps, and $\tilde{\pi}$ is differentiable by the definition of the differentiable structure on B_O/C (cf. Section I.2(iii)). Now the proof that $\tilde{\mu}$ is differentiable is exactly the same as the proof in Section 3(ii) with the substitutions $\bar{\mu} \to \tilde{\mu}$, $\bar{\upsilon} \to \tilde{\upsilon}$, $\bar{G}_O \to B_O$, $\bar{H} \to C$, $\bar{\pi} \to \tilde{\pi}$.

It will now be verified that $\tilde{\mu}$ satisfies the properties (1) to (4) of Definition III.1. The map $\tilde{s}_{aC}$ for $aC \in B_O/C$ is defined by:

$$\tilde{s}_{aC}(bC) := \tilde{\mu}(aC,bC)$$

$$= \{a\tau(a)^{-1}\tau(b)\}C \quad \text{for } bC \in B_O/C.$$

<u>Property (1)</u>:

$$\tilde{s}_{aC}(aC) = \{a\tau(a)^{-1}\tau(a)\}C$$

$$= aC \qquad \text{for } aC \in B_O/C.$$

<u>Property (2)</u>:

$$\tilde{s}^2_{aC}(bC) = \tilde{s}_{aC}(\tilde{s}_{aC}(bC))$$

$$= \tilde{s}_{aC}(\{a\tau(a)^{-1}\tau(b)\}C)$$

$$= \{a\tau(a)^{-1}\tau(a\tau(a)^{-1}\tau(b))\}C$$

$$= \{a\tau(a)^{-1}\tau(a)(\tau^2(a))^{-1}\tau^2(b)\}C$$

$$= \{a(\tau^2(a))^{-1}\tau^2(b)\}C \qquad \text{for } aC, bC \in B_o/C;$$

and by induction:

$$\tilde{s}^k_{aC}(bC) = \{a(\tau^k(a))^{-1}\tau^k(b)\}C$$

$$= \{aa^{-1}b\}\, C \qquad \text{because } \tau \text{ has order } k,$$

$$= bC \qquad \text{for } aC, bC \in B_o/C.$$

Property (3):

$$\tilde{s}_{aC} \circ \tilde{s}_{bC}(dC) = \tilde{s}_{aC}(\{b\tau(b)^{-1}\tau(d)\}C)$$

$$= \{a\tau(a)^{-1}\tau(b\tau(b)^{-1}\tau(d))\}C$$

$$= \{a\tau(a)^{-1}\tau(b)\tau(\tau(b)^{-1})\tau(\tau(d))\}\, C$$

$$= \{\{a\tau(a)^{-1}\tau(b)\}\{\tau(\tau(b)^{-1})\}\{\tau(\tau(a))\tau(a)^{-1}\tau(a)\tau(\tau(a))^{-1}\} \times \{\tau^2(d)\}\}C$$

$$= \{\{a\tau(a)^{-1}\tau(b)\}\tau\{a\tau(a)^{-1}\tau(b)\}^{-1}\tau\{a\tau(a)^{-1}\tau(d)\}\}C$$

$$= \tilde{s}_{\{a\tau(a)^{-1}\tau(b)\}C}(\{a\tau(a)^{-1}\tau(d)\}C)$$

$$= \tilde{s}_{\tilde{s}_{aC}(bC)} \circ \tilde{s}_{aC}(dC) \qquad \text{for } aC, bC, dC \in B_o/C\,;$$

i.e. $$\tilde{s}_{aC} \circ \tilde{s}_{bC} = \tilde{s}_{\tilde{s}_{aC}(bC)} \circ \tilde{s}_{aC} \qquad \text{for } aC, bC \in B_o/C\,.$$

Property (4):

To show that aC is an isolated fixed point of $\tilde{s}_{aC}$ for each $aC \in B_0/C$ it suffices to prove the result for $\tilde{s}_{eC}$: for, considering B_0 as a Lie transformation group of B_0/C and observing that:

$$\begin{aligned} \tilde{s}_{aC}(bC) &= \{a\tau(a)^{-1}\tau(b)\}C \\ &= \{a\tau(a^{-1}b)\}C \\ &= a \circ \tilde{s}_{eC}(\{a^{-1}b\}C) \\ &= a \circ \tilde{s}_{eC} \circ a^{-1}(bC) \qquad \text{for } aC, bC \in B_0/C, \end{aligned}$$

it follows that if a neighbourhood U_0 of eC contains no fixed points under $\tilde{s}_{eC}$ except eC itself, then the neighbourhood $a(U_0)$ of aC contains no fixed points of $\tilde{s}_{aC}$ except aC itself.

It will now be shown that such a neighbourhood U_0 does indeed exist. Observe first of all (cf. Section I.2(i)) that there exists a neighbourhood U_2 of e in B_0 such that:

if an element $d \in U_2$ satisfies $d^k = e$, then $d = e$.

Also there exists a neighbourhood U_1 of e in B_0 such that:

$$b^{-1}\tau(b) \in U_2 \quad \text{for all } b \in U_1 ;$$

the existence of such a neighbourhood U_1 follows from the differentiability (and hence continuity) of τ, group multiplication in B_0 and the operation of taking the inverse in B_0.

Also because (cf. Theorem I.3) there exists a differentiable cross-section σ_B from a connected neighbourhood V'_e of eC in B_0/C into a (connected) neighbourhood of e in B_0, it follows that there exists in B_0/C a connected neighbourhood $U_0 \subset V'_e$ of $p_0 = eC$ such that there corresponds to each point $p \in U_0$ a unique point $\sigma_B(p) \in \sigma_B(U_0) \subset U_1$ for which $p = \{\sigma_B(p)\}C$.

Consider now a given point $p \in U_0$ and suppose that:

$$\tilde{s}_{p_0}(p) = p\ ;$$

that is, defining $b := \sigma_B(p)$ (whence $p = bC$):

$$\tilde{s}_{eC}(bC) = bC,$$

i.e. $\{\tau(b)\}C = bC$ (for (32) implies that

$$\tilde{s}_{eC}(bC) := \mu(eC, bC) = \{\tau(b)\}C)$$

therefore $b^{-1}\tau(b) = c$ for some $c \in C$, where in fact $c \in U_2$

because by the above remarks $p \in U_0 \Longrightarrow b = \sigma_B(p) \in U_1$

$$\Longrightarrow b^{-1}\tau(b) \in U_2\ .$$

But because $c \in C \Longrightarrow \tau(c) = c$ therefore:

$$c = \tau(c) = \tau^2(c) = \ldots = \tau^{(k-1)}(c)\ ;$$

consequently:

$$\begin{aligned} c^k &= c\tau(c)\tau^2(c)\ \ldots\ \tau^{(k-1)}(c) \\ &= b^{-1}\tau(b)\tau(b^{-1}\tau(b))\tau^2(b^{-1}\tau(b))\ \ldots\ \tau^{(k-1)}(b^{-1}\tau(b)) \\ &= b^{-1}\tau(b)\tau(b)^{-1}\tau^2(b)\tau^2(b)^{-1}\tau^3(b)\ \ldots\ \tau^{(k-1)}(b)^{-1}\tau^k(b) \\ &= b^{-1}\tau^k(b) \\ &= b^{-1}b \\ &= e, \end{aligned}$$

(because τ is an automorphism (of B_C) of order k).

Thus the point $c \in U_2$ satisfies $c^k = e$, whence by the definition of U_2 one concludes that $c = e$; i.e., $\tau(b) = b$, therefore $b \in B_o^\tau$. Thus $b \in B_o^\tau \cap \sigma_B(U_o)$; but because U_o is connected $\sigma_B(U_o)$ is connected, and because B_o^τ (as a closed Lie subgroup of B_o) has the subspace topology, so also is $B_o^\tau \cap \sigma_B(U_o)$ connected in B_o^τ - whence $b \in (B_o^\tau)_o$, the identity component of B_o^τ. Because $(B_o^\tau)_o \subset C$, therefore $b \in C$ which implies that $p = bC = eC = p_o$. Consequently it has been shown that for the neighbourhood U_o of p_o :

if an element $p \in U_o$ satisfies $\tilde{s}_{p_o}(p) = p$, then $p = p_o$;

that is, $p_o = eC$ is an isolated fixed point of the map $\tilde{s}_{p_o} = \tilde{s}_{eC}$.

This completes the proof that, endowed with the multiplication $\tilde{\mu}$ of (32), B_o/C is a symmetric space of order k.

This completes the proof of Theorem III.2.

CHAPTER IV

SOME OBSERVATIONS CONCERNING JORDAN ALGEBRAS AND DIFFERENTIAL GEOMETRY

1. The Canonical Connexion on a "Jordan Symmetric Space"

The notation introduced in Section I.6 will be assumed here: in particular, A denotes a Jordan algebra with unit e defined on the real vector space $\mathbb{R}^n$, I(A) denotes the set of invertible elements of A, P(a) denotes the quadratic representation of the element $a \in A$, and A_f denotes the f-mutation of A with respect to an element $f \in A$ (A_f is called the f-isotope if $f \in I(A)$). By Theorem I.13 I(A) is naturally a differentiable manifold; moreover, the topological component $I_0(A)$ of I(A) containing the unit e may be endowed with the multiplication μ defined by $\mu(p,q) := P(p)q^{-1}$ for $p,q \in I_0(A)$, and $I_0(A)$ so endowed is a symmetric space (of order 2) - called the Jordan symmetric space of A.

$I_0(A)$ admits two affine connexions. Firstly, as an open subset of $\mathbb{R}^n$ (cf. Theorem I.13), $I_0(A)$ admits the affine connexion $\bar{\nabla}$ induced by the standard flat connexion on $\mathbb{R}^n$. In terms of the standard basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$ the vector field $\left(\frac{\partial}{\partial u^i}\right) \in T^1(\mathbb{R}^n)$ is defined for $i = 1,2,\ldots,n$: as a differential operator on $F(\mathbb{R}^n)$, $\left(\frac{\partial}{\partial u^i}\right)_x \in (\mathbb{R}^n)_x$ is simply the partial derivative in the direction e_i, evaluated at the point $x \in \mathbb{R}^n$; the restriction of $\frac{\partial}{\partial u^i}$ to $I_0(A)$ gives a vector field in $T^1(I_0(A))$, denoted E_i for $i = 1,2,\ldots,n$. Now $\bar{\nabla}$ on $I_0(A)$ is given by:

$$(\bar{\nabla}_X Y)_p = X^i(p)(E_i\, Y^j)_p (E_j)_p \quad \text{for } X,Y \in T^1(I_o(A)) \text{ and } p \in I_o(A),$$

$$= X^i(p)\left(\frac{\partial Y^j}{\partial u^i}\right)_p \left(\frac{\partial}{\partial u^j}\right)_p , \qquad (1)$$

where $X^i(q) := X_q u^i$ and $Y^i(q) := Y_q u^i$ for $q \in I_o(A)$ and $i = 1,2,\ldots,n$.

Secondly, as a symmetric space (of order 2), $I_o(A)$ admits the (canonical) affine connexion ∇ defined in Definition III.6 (cf. Theorem III.1). Because $I_o(A)$ is a symmetric space of order $k = 2$, the tensor field S of Definition III.3 is given by $S_p = -I_p$ at each $p \in I_o(A)$; for since in this case $(S_p)^2 = I_p$ and S_p has no eigenvalue (+1) (cf. remarks preceding Definition III.4), S_p is diagonalizable over the reals, all its eigenvalues being (-1) (cf. Birkhoff and MacLane[1]), whence $S_p = -I_p$ as asserted. Hence from Definition III.6 it follows that ∇ is given by:

$$(\nabla_X Y)_p = X_p Y - \tfrac{1}{2} Y_p (X_p)^* \qquad (2)$$

(where " * " denotes the operation of Definition III.5).

Consider now the difference tensor $D \in T^1_2(I_o(A))$ defined by:

$$D(X,Y) := \bar{\nabla}_X Y - \nabla_X Y \quad \text{for } X,Y \in T^1(I_o(A)) ; \qquad (3)$$

the following theorem provides an interpretation of D (and hence of ∇) in terms of the isotopes of the algebra A:

<u>Theorem IV.1</u>: Let $\beta_x : \mathbb{R}^n \to (\mathbb{R}^n)_x$ be the natural identification of $\mathbb{R}^n$ with the tangent space $(\mathbb{R}^n)_x$ at $x \in \mathbb{R}^n$; β_x is a linear isomorphism with inverse denoted by $\beta_x^{-1} : (\mathbb{R}^n)_x \to \mathbb{R}^n$.

Then the tensor field D defined above admits the following description:

given a point $a \in I_o(A)$, then

$$D_a(X_a,Y_a) = \beta_a(\beta_a^{-1}(X_a) \underset{a^{-1}}{\perp} \beta_a^{-1}(Y_a)) \quad \text{for } X_a,Y_a \in (I_o(A))_a ,$$

where $\underset{a^{-1}}{\perp}$ denotes the multiplication in the a^{-1}-isotope $A_{a^{-1}}$ of the Jordan algebra A.

<u>Proof:</u> Because $D \in T_2^1(I_o(A))$, D_a is completely determined by its action on a basis of the tangent space $(I_o(A))_a$: considering the vector fields E_i defined above $\{(E_i)_a\}_{i=1}^n$ is such a basis of $(I_o(A))_a$; $D_a((E_i)_a,(E_j)_a)$ will now be computed for $1 \leqslant i,j \leqslant n$.

Notice firstly that for the connexion $\bar{\nabla}$:

$$(\bar{\nabla}_{E_i} E_j)_a = 0, \tag{4}$$

because $(E_i)_a E_j = 0$: cf. (1) and the definition of E_i, E_j ; thus:

$$\begin{aligned} D_a((E_i)_a,(E_j)_a) &= (D(E_i,E_j))_a \\ &= -(\nabla_{E_i} E_j)_a && \text{by (3) and (4),} \\ &= -(E_i)_a E_j + \tfrac{1}{2}(E_j)_a((E_i)_a)^* && \text{by (2),} \\ &= \tfrac{1}{2}(E_j)_a((E_i)_a)^* && \text{again because } (E_i)_a E_j = 0, \\ &= \tfrac{1}{2}\left(\frac{\partial\phi^k}{\partial u^j}\right)_a (E_k)_a , && (5) \end{aligned}$$

where $\phi^k \in F(I_o(A))$ for $k = 1,2,\ldots,n$ is the k^{th} component of the vector field $((E_i)_a)^*$: namely $\phi^k = ((E_i)_a)^* u^k$, u^k being the k^{th} coordinate function, i.e. $\phi^k(y) = \left(\frac{\partial}{\partial u^i}\right)_a (u^k \circ r_{\mu(a,y)})$ for $y \in I_o(A)$;

this last step follows from Definition III.5 (of "*"), the remark that because $I_o(A)$ is a symmetric space: $s_a^{-1}(y) = s_a(y) = \mu(a,y)$, and the fact that $(E_i)_a = \left(\frac{\partial}{\partial u^i}\right)_a$ as differential operators on $F(I_o(A))$.

Using the definition of partial derivative and other basic analysis on $\mathbb{R}^n$ one obtains:

$$\phi^k(y) = \left(\frac{\partial}{\partial u^i}\right)_a u^k \circ r_{\mu(a,y)}$$

$$= \lim_{t\to 0} \frac{1}{t}\{u^k \circ r_{\mu(a,y)}(a + te_i) - u^k \circ r_{\mu(a,y)}(a)\}$$

$$= \lim_{t\to 0} \frac{1}{t}\{u^k \circ \mu(a + te_i, \mu(a,y)) - u^k \circ \mu(a, \mu(a,y))\}$$

by Definition III.2(b) for $r_{\mu(a,y)}$,

$$= \lim_{t\to 0} \frac{1}{t}\{u^k \circ P(a + te_i) - u^k \circ P(a)\}(\mu(a,y))^{-1}$$

by definition of μ on $I_o(A)$,

$$= u^k \circ \lim_{t\to 0} \frac{1}{t}\{2P(a,te_i) + P(te_i)\}(\mu(a,y))^{-1}$$

cf. Theorem I.12(b),

$$= u^k \circ \lim_{t\to 0} \frac{1}{t}\{2t(P(a,e_i) + t^2P(e_i)\}(\mu(a,y))^{-1}$$

by the remark following Theorem I.12,

$$= 2u^k \circ P(a,e_i)(\mu(a,y))^{-1} . \qquad (6)$$

Recall that $\mu(a,y) = P(a)y^{-1}$,

whence

$$(\mu(a,y))^{-1} = P^{-1}(P(a)y^{-1}) \circ P(a)y^{-1} \qquad \text{by Theorem I.11(a)}$$

$$= P^{-1}(a)P^{-1}(y^{-1})P^{-1}(a) \circ P(a)y^{-1}$$

$$\text{cf. Theorem I.10}$$

$$= P(a^{-1})P(y)y^{-1} \qquad \text{by Theorem I.11(b)}$$

$$= P(a^{-1})P(y)P^{-1}(y)y \qquad \text{by Theorem I.11(a)}$$

$$= P(a^{-1})y. \tag{7}$$

Substituting (7) into (6) one obtains:

$$\phi^k(y) = 2u^k \circ P(a,e_i)P(a^{-1})y \quad \text{for} \quad y \in I_o(A). \tag{8}$$

Now the partial derivative $\left(\frac{\partial \phi^k}{\partial u^j}\right)_a$ may be evaluated:

$$\left(\frac{\partial \phi^k}{\partial u^j}\right)_a = \lim_{t \to 0} \frac{1}{t}\{\phi^k(a + te_j) - \phi^k(a)\}$$

$$= 2u^k \circ P(a,e_i) \circ P(a^{-1}) \circ \lim_{t \to 0} \frac{1}{t}\{(a + te_j) - a\}$$

$$\text{by using (8),}$$

$$= 2u^k \circ P(a,e_i) \circ P(a^{-1})e_j \; . \tag{9}$$

Therefore, substituting (9) into (5) one obtains:

$$D_a((E_i)_a,(E_j)_a) = (u^k \circ P(a,e_i) \circ P(a^{-1})e_j).(E_k)_a$$

$$= \beta_a\{u^k \circ P(a,e_i) \circ P(a^{-1})e_j)e_k\} \quad \text{by definition of } \beta_a \, ,$$

$$= \beta_a\{P(a,e_i) \circ P(a^{-1})e_j\} \; . \tag{10}$$

Now $P(a,e_i) \circ P(a^{-1})e_j = a \underset{P(a^{-1})e_j}{\perp} e_i$ where $\underset{P(a^{-1})e_j}{\perp}$ denotes the algebraic multiplication in the $P(a^{-1})e_j$-mutation of the Jordan algebra A (cf. Definition I.6(a) and Theorem I.12(b)). But by Theorem I.12(d), $A_{P(a^{-1})e_j} = (A_{a^{-1}})_{e_j}$ = the e_j-mutation of the a^{-1}-isotope of A; so denoting the algebraic multiplication in $A_{a^{-1}}$ by @ (for convenience, in place of $\underset{a^{-1}}{\perp}$):

$$P(a,e_i) \circ P(a^{-1})e_j = (a \text{ @ } e_j) \text{ @ } e_i + a \text{ @ } (e_i \text{ @ } e_j) - (a \text{ @ } e_i) \text{ @ } e_j$$

cf. Theorem I.12(d),

$$= e_j \text{ @ } e_i + e_i \text{ @ } e_j - e_i \text{ @ } e_j ,$$

because a is the unit in $A_{a^{-1}}$. Because $A_{a^{-1}}$ is a commutative algebra, therefore $e_j \text{ @ } e_i = e_i \text{ @ } e_j$ and consequently:

$$P(a,e_i) \circ P(a^{-1})e_j = e_i \text{ @ } e_j . \qquad (11)$$

Substituting (11) into (10) one obtains:

$$D_a((E_i)_a,(E_j)_a) = \beta_a\{e_i \text{ @ } e_j\}$$

$$= \beta_a\{\beta_a^{-1}((E_i)_a) \text{ @ } \beta_a^{-1}((E_j)_a))\},$$

because $\beta_a^{-1}((E_i)_a) = e_i$ and $\beta_a^{-1}((E_j)_a) = e_j$.

Because D_a is bilinear (over the reals) and β_a and β_a^{-1} are $\mathbb{R}$-linear, it follows that:

$$D_a(X_a,Y_a) = \beta_a\{\beta_a^{-1}(X_a) \text{ @ } \beta_a^{-1}(Y_a)\}$$

i.e. $D_a(X_a,Y_a) = \beta_a\{\beta_a^{-1}(X_a) \underset{a^{-1}}{\perp} \beta_a^{-1}(Y_a)\}$ for all $X_a,Y_a \in (I_o(A))_a$.

This completes the proof of Theorem IV.1.

2. Conformal Transformations and Jordan Algebras of Type (iv)

In this section it will be shown that Jordan algebras of type (iv) arise naturally in differential geometry when one considers conformal transformations.

Recall the remark at the end of Section I.4: given two conformally related Riemannian metric tensors $\bar{g}$ and g on a manifold M (say $\bar{g} = fg$ with $f \in F(M)$), then the difference tensor D of their respective Levi-Civita connexions $\bar{\nabla}$ and ∇ satisfies:

$$D(X,Y) = \tfrac{1}{2}\{(X\tilde{f})Y + (Y\tilde{f})X - g(X,Y)\vec{\nabla}\tilde{f}\}$$

where $\tilde{f} = (\log f) \in F(M)$ (note that for each $x \in M$, $f(x) > 0$) and $\vec{\nabla}\tilde{f}$ denotes the usual gradient of $\tilde{f}$ (with respect to g) - namely $\vec{\nabla}\tilde{f}$ is the unique element of $T^1(M)$ satisfying $g(Z,\vec{\nabla}\tilde{f}) = Z\tilde{f}$ for all $Z \in T^1(M)$. Thus:

$$D(X,Y) = \tfrac{1}{2}\{g(X,\vec{\nabla}\tilde{f})Y + g(Y,\vec{\nabla}\tilde{f})X - g(X,Y)\vec{\nabla}\tilde{f}\} . \qquad (12)$$

Consider now a point $p \in M$ for which $(\vec{\nabla}\tilde{f})_p \neq 0$ (such points exist provided f is not constant on some connected component of M); then in terms of the symmetric positive-definite bilinear form $g_p : M_p \times M_p \to M_p$ define:

$$\|(\vec{\nabla}\tilde{f})_p\| := \{g_p((\vec{\nabla}\tilde{f})_p, (\vec{\nabla}\tilde{f})_p)\}^{\frac{1}{2}} \qquad (13)$$

and

$$E_p := \frac{(\vec{\nabla}\tilde{f})_p}{\|(\vec{\nabla}\tilde{f})_p\|} . \qquad (14)$$

Then:

$$2\,\frac{D_p(X_p, Y_p)}{\|(\vec{\nabla}\tilde{f})_p\|} = g_p(X_p, E_p)Y_p + g_p(Y_p, E_p)X_p - g_p(X_p, Y_p)E_p . \qquad (15)$$

By glancing at the definition of Jordan alebras of type (iv) in Section II.B.2, it is seen that (15) defines on M_p a Jordan algebra of type (iv): namely $[M_p, g_p, E_p]$ where E_p given by (14) is the algebraic unit.

Thus, given two conformally related metric tensors on M as described above ($\bar{g} = fg$), then at each point $p \in M$ for which $(\vec{\nabla}\tilde{f})_p \neq 0$ the Jordan algebra $[M_p, g_p, E_p]$ with unit $E_p = \frac{(\vec{\nabla}\tilde{f})_p}{\|(\vec{\nabla}\tilde{f})_p\|}$ is defined by (15) in terms of the difference tensor D of the two Levi-Civita connexions for the two conformally related metric tensors.

CHAPTER V

CONCLUSION AND FINAL REMARKS

In this chapter I shall briefly summarise the results obtained and suggest those directions in which further research should be pursued in this subject.

1. Brief Summary of Results

The main result of Chapter III on symmetric spaces of order k (namely, Theorem III.1) establishes that basically the "algebraic" approach which Loos applied to symmetric spaces may be successfully applied to the s-regular k-symmetric spaces. In view of Conjectures 1 and 2 of Section II.A.4(i) it is very interesting to notice that the s-regular condition arises naturally in this approach. It was also proved in Chapter III (Theorem III.2) that in a homogeneous space representation G_o/H of a symmetric space of order k the isotropy subgroup H may be characterised in terms of an automorphism of G_o of order k; this result (as do all the results of Chapter III) generalises the analogous result for symmetric spaces.

The main result of Chapter IV (namely, Theorem IV.1) presents an interpretation of the canonical connexion on a "Jordan symmetric space" $I_o(A)$ in terms of the isotopes of A. The observation in Section 2 of Chapter IV indicates how Jordan algebras are associated with certain conformal transformations of Riemannian manifolds.

2. Programme for Further Research

(i) Although the relationship between Jordan algebras and symmetric spaces has been developed extensively the following two studies remain to be pursued.

Firstly, the work that the Roumanian school of differential geometers has done concerning applications of Jordan algebras to differential geometry seems to be little known and its connection with the other recent studies discussed in Section II.B.4 has not been examined. Investigation of the extent to which the various results overlap or complement one another would be a useful and probably fruitful endeavour.

Secondly, it should be determined whether the observation in Section IV.2 (which associates Jordan algebras of the type $[X, \upsilon, e]$ with certain conformal transformations) gives rise to any significant information about conformal transformations.

(ii) In Chapter III the fundamental aspects of an "algebraic" approach to symmetric spaces of order k have been established; it remains, however, to pursue further studies in this context:

(a) For example, it should be investigated whether a meaningful "centre" of a symmetric space can be defined, generalising the notion of the centre of a Lie group or symmetric space (of order 2) - cf. Section II.A.3. There also remain the Conjectures 1 and 2 of Section II.A.4(i). These questions would entail examination of the homogeneous space representation of a symmetric space of order k, particular attention being paid to the structure of the isotropy subgroup.

(b) A fascinating problem is the search for a construction of symmetric spaces of order k (for $k > 2$) in terms of Jordan, or possibly other non-associative algebras, in analogy of course with such a construction for symmetric spaces as outlined in Section II.B.4(iii); in this regard the work of Sagle [1], [2] is of interest. At first sight it might appear that the methods of Helwig [6] would naturally generalise for symmetric spaces of order $k > 2$, but this does not seem to be the case; a heuristic approach to the problem probably lies in developing the "algebraic" approach to symmetric spaces of order k. The justification for such a suggestion is the following: Loos's "algebraic" approach to symmetric spaces reveals an intriguing fact; namely, the embedding $Q : M \to G$ of a symmetric space into its group of displacements (cf. Section II.A.2(ii)) satisfies $Q(Q(x)y) = Q(x)Q(y)Q(x)$ which is formally the "fundamental formula", satisfied by the quadratic representation of a Jordan algebra (cf. Theorem I.10) and on which the whole theory of Jordan algebras can essentially be based (cf. Section II.B.1, fourth paragraph). (That Q satisfies the above identity is proved in Loos [1], chapter 2). Consequently I suggest that further study of the "algebraic" approach to symmetric spaces of order k might well provide insight into the type of algebra appropriate to the study of these spaces.

A second means of tackling the question of which algebras are relevant is suggested by Theorem IV.1. Thus a study of difference tensors between the canonical connexion on a symmetric space of order k (as defined in Definition III.6) and the connexion induced when the space is embedded into various Euclidean spaces may well provide a clue as to the type of algebras sought.

(c) I point out that a comparison of Table V in Gray [6] and the classification by Boothby [1] of certain compact homogeneous complex contact manifolds reveals that the manifolds in Boothby's classification are Riemannian symmetric spaces of order 3. Further exploration of this phenomenon (for example, a classification-free proof of this very observation) would shed further light on the geometry of symmetric spaces of order 3 (and of homogeneous complex contact manifolds); also the existence of weaker structures (e.g., almost contact) should be investigated on the other symmetric spaces of order 3. Moreover analogous inquiries can be made for symmetric spaces of order k in general.

The realisation of symmetric spaces of order k as fibre bundles should be examined; in the case $k = 2$, for instance, every symmetric space is a vector bundle over a compact symmetric space (cf. Loos [1], chapter 4). Furthermore along these lines one can examine generalisations of the notion of reflexion spaces (cf. Section II.A.4(iii)) - obtained by relaxing the condition in Definition III.1 that x be <u>isolated</u> as a fixed point of the left multiplication s_x .

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