Parallel foliations

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PARALLEL FOLIATIONS

by

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ABSTRACT

The basic theory of foliations is introduced in Chapter 1. Various classes of affine connexions associated with a foliation are discussed, in particular those which give rise to the notion of parallel foliation and those which give a realisation of the 1-Jet holonomy group of C. Ehresmann.

In Chapter 2, locally affine foliations are defined as parallel foliations for which the induced structure on each leaf is flat. A local characterisation is given in terms of the existence of a special sub-atlas of co-ordinate charts. Some results are obtained about the global structure of such foliations when certain completeness assumptions are made.

Chapter 3 gives a description, in terms of grid manifolds, of the work of S. Kashiwabara on the reducibility of an affinely connected manifold.

The work of the first three chapters is then used in Chapter 4 to discuss the question of parallel foliations on pseudoriemannian manifolds. Some new examples are given. An elementary proof of the De Rham-Wu decomposition theorem and some theorems about null foliations determined by submersions are obtained.

Chapter 5 is concerned with the properties of pseudoriemannian manifolds which admit systems of parallel vector fields. The problem is discussed in terms of parallel foliations and some recently developed techniques in foliation theory are used to obtain some strong global structure theorems.

INTRODUCTION AND ACKNOWLEDGMENTS

Many of the ideas in this thesis have been inspired by the work of A. G. Walker and S. A. Robertson. The central objective has been to find some kind of extension of the De Rham-Wu decomposition theorem to cover the case of parallel, partially null foliations on pseudoriemannian manifolds. It is hoped that the machinery developed in Chapters 2, 3 and 4 will serve as a foundation for further work on this problem. The concept of 'locally affine foliation' may well be of independent interest, particularly for the study of the flows of vector fields and differential equations on manifolds.

Although the results of Chapter 4 fall short of the main objective, the special case of strictly parallel foliations considered in Chapter 5 has met with more success. Theorem 1.5.2 and most of the material in Chapters 2, 4 and 5 is original. Some of the results in Chapter 4 were obtained independently by S. A. Robertson and together with the main results of Chapter 5 are to appear in a joint paper [6].

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CHAPTER 1

FOLIATIONS

§1.1 Definitions

Let R^m be euclidean m-space with coordinates z^i. Define $B^m(d^i,c^i)$ to be the open subset of R^m consisting of those points whose coordinates satisfy $-\infty \leqslant d^i < z^i < c^i \leqslant +\infty$.

Let M be an m-manifold of class C^s, $0 \leqslant s \leqslant \infty,\omega$ (see [15]). Then a coordinate chart (U,x^i) on M is an open set $U \subset M$ and coordinate functions $x^i : U \to R^1$ $i = 1,\ldots,m$ satisfying

(1) If $\phi_U : U \to B^m(d^i,c^i)$ is defined by $\phi_U(p) = (x^1(p),\ldots, x^m(p))$ for $p \in U$ then ϕ_U is a homeomorphism.

(2) If (V,y^i) is another coordinate chart and $V \cap U \neq \phi$ then $\phi_V \circ \phi_U^{-1} : R^m \to R^m$ is of class C^s where defined.

A C^s-Atlas $\mathcal{A}$ on M is a maximal collection of such coordinate charts, where maximality is defined with respect to an ordering by inclusion.

Definition 1.1.1 If N is an n-manifold of class C^s then a map $f : M \to N$ is said to be

(a) of class C^r $r \geqslant s$ if for all $p \in M$, $\dfrac{\partial^k f^i}{\partial x^{j_1}\ldots\partial x^{j_k}}(p)$ exist and are continuous for $0 \leqslant k \leqslant r$ where f is represented by

$$(x^1,\ldots,x^m) \mapsto (f^1(x^1,\ldots,x^m), \ldots, f^n(x^1,\ldots,x^m))$$

with respect to coordinate charts at p and f(p). Conditions (1) and (2) above ensure that this definition does not depend on the particular coordinate charts chosen.

(b) a homeomorphism of class C^r $r \leqslant s$ if f is a homeomorphism for which both f and f^{-1} are of class C^r.

(c) a local homeomorphism of class C^r $r \leqslant s$ if for all $p \in M$ there is a neighbourhood U of p such that $f : U \to f(U)$ is a homeomorphism of class C^r.

Definition 1.1.2 The Standard Foliation of R^m of codimension p.

This is the basic building block required for defining foliations on manifolds.

If y^i, $i = 1,\dots,m$ are coordinates for R^m then the (m-p) dimensional planes given by $y^{m-p+1},\dots,y^m$ = constant determine a product decomposition $R^m = R^{m-p} \times R^p$. This is the standard foliation of R^m of codimension p. If the discrete topology is put on R^p and the usual one on R^{m-p}, then, by taking the product toplogy (see [14] page 90) one obtains the leaf topology $T_o(R^m)$ on R^m. The leaves are defined as the connected components in this topology.

Throughout what follows, unless otherwise stated, late Greek suffices λ,μ,θ will denote integer values in the range $1,2,\dots,m-p$, early Greek, α,β,γ in the range $m-p+1,\dots,m$, and Roman i,j,k,ℓ in the range $1,\dots,m$.

Definition 1.1.3 A homeomorphism $h : U \subset R^{m-p} \times R^p \to h(U) \subset R^{m-p} \times R^p$ of class C^r is said to be leaf preserving (L.P.) if

$$h(y^1,\dots,y^{m-p},\dots,y^m) = (h^1(y^i),\dots,h^{m-p}(y^i),\ h^{m-p+1}(y^\alpha),\dots,h^m(y^\alpha))$$

Definition 1.1.4 A Foliation $\mathcal{F}$ of codimension p and class C^r $(0 \leqslant r \leqslant s)$ on an m-manifold M of class C^s, is a collection of leaf charts $\mathcal{A} = \{(U_a,h_a) : a \in J\}$, maximal with respect to:

(i) $U_a \subset M$, $\bigcup_{a\in J} U_a = M$.

(ii) $h_a : U_a \to B^m(d_a^{\,i}, c_a^{\,i})$ a homeomorphism of class C^r.

(iii) if $U_a \cap U_b \neq \phi$ then $h_a \circ h_b^{-1} : h_b\,(U_a \cap U_b) \to h_a(U_a \cap U_b)$ is an L.P. homeomorphism of class C^r.

$\mathcal{A}$ will be called a *leaf atlas* for the foliation $\mathcal{F}$.

If (U_a, h_a) is a leaf chart then one may consider the coordinate functions $x_a^{\,i} : U_a \to R^1$ $i = 1,\ldots,m$ defined by

$$x_a^{\,i}(z) = y^i \circ h_a(z) \qquad \text{for } z \in U_a$$

$$\text{clearly} \qquad d_a^{\,i} < x_a^{\,i} < c_a^{\,i}$$

If $U_a \cap U_b \neq \phi$, the coordinates $x_a^{\,i}$ and $x_b^{\,i}$ are related on the overlap by equations of the form

$$\left.\begin{array}{l} x_b^{\,\lambda} = P^{\lambda}_{ba}(x_a^{\,i}) \\ \\ \\ x_b^{\,\alpha} = Q^{\alpha}_{ba}(x_a^{\,\beta}) \end{array}\right\} \begin{array}{l} \text{where } P,\ Q \text{ are of class } C^r \\ \text{and } \left(\dfrac{\partial P^{\lambda}_{ba}}{\partial x_a^{\,\mu}}\right),\ \left(\dfrac{\partial Q^{\alpha}_{ba}}{\partial x_a^{\,\beta}}\right) \\ \text{are non singular matrices} \end{array}$$

Conversely, given coordinate charts $(U_a, x_a^{\,i})$ with overlap equations of the above form, one may recover h_a by defining

$$h_a(z) = (x^1(z),\ldots,x^m(z)) \quad \text{for } z \in U_a$$

The alternative form $(U_a, x_a^{\,i})$ for a leaf chart will often be used in what follows. If $z \in U_a$, then the points of U_a with coordinates $x_a^{\,\alpha} = x_a^{\,\alpha}(z)$ are called the *plaque* of the chart through z.

In the general theory of differentiable manifolds it is well known (see [28]) that a C^1-atlas always contains a C^∞-sub atlas. However, little appears to be known about the corresponding question for foliations. André

Haefliger has proved in [8] that if a compact C^ω-manifold M admits a codimension one foliation with C^ω leaf atlas then the fundamental group of M, $\pi_1(M)$ is infinite. Thus, the codimension one foliation of S^3 the three dimensional sphere given by G. Reeb in [20], does not admit a C^ω structure.

Definition 1.1.5 The Leaf Toplogy.

The leaf topology $T_o(R^m)$ on R^m induces a topology $T_o(B^m)$ on B^m by the inclusion map. A leaf atlas $\mathcal{A} = \{(U_a,h_a) : a \in J\}$ can now be used to put the leaf topology on M.

Consider the collection $\left\{ h_a^{-1}(V) : a \in J,\ V = \text{open set of } T_o(B^m) \text{ contained in } h_a(U_a). \right\}$

This collection defines a base for a topology $T_o(M)$ on M (see Kelley [14] page 47) because

(i) $\bigcup h_a^{-1}(V) = M$.

(ii) If $z \in h_a^{-1}(V) \cap h_b^{-1}(\bar{V})$, then $h_a(z) \in V \cap h_a \circ h_b^{-1}(\bar{V})$.

But $h_a\, h_b^{-1}$ is an L.P. homeomorphism

$\therefore h_a \circ h_b^{-1}(\bar{V}) \in T_o(B^m)$

thus there is $W \subset V \cap h_a \circ h_b^{-1}(\bar{V})$, $W \neq \phi$, $W \in T_o(B^m)$ and $h_a(z) \in W$

$\therefore z \in h_a^{-1}(W)$ and $h_a^{-1}(W) \subset h_a^{-1}(V) \cap h_b^{-1}(\bar{V})$.

The leaves of $\mathcal{F}$ are defined as the connected components of M in the leaf topology $T_o(M)$ and are clearly (m-p) dimensional submanifolds of class C^r in the sense of [15].

Definition 1.1.6 A map $f : M \to N$ of class C^s between two C^s-manifolds M and N with foliations, is said to be foliation preserving if f is continuous with respect to the leaf topologies $T_o(M)$ and $T_o(N)$.

Definition 1.1.7 Induced Foliations.

Let N be a C^s-manifold with a C^r-foliation $\mathcal{F}$ of codimension p and leaf atlas $\mathcal{A} = \{(U_a, h_a) : a \in J\}$. Suppose $f : M \to N$ is a local homeomorphism of class C^s.

Let $p \in M$. Pick a neighbourhood $\bar{U}$ of p such that $f(\bar{U})$ is contained in some U_a and $f|\bar{U}$ is a C^s-homeomorphism. It is not difficult to prove that there is an open set $W \subset h_a \circ f(\bar{U})$ such that there is an L.P. homeomorphism of class C^r, $g : W \to B^{m-p} \times B^p$ and $f(p) \in h_a^{-1}(W)$.

Consider the pair $((f|\bar{U})^{-1} \circ h_a^{-1} \circ g^{-1}(W),\ g \circ h_a \circ f)$. It can be shown that the collection of such pairs for all points $p \in M$ satisfies conditions (i), (ii), (iii) of definition 1.1.4 with respect to M, and so will generate a maximal leaf atlas. This gives the induced foliation $f^{-1}\mathcal{F}$ on M. The leaf atlas will be denoted by $f^{-1}\mathcal{A}$. With respect to $f^{-1}\mathcal{F}$ and $\mathcal{F}$, f is foliation preserving.

§1.2 The Ehresmann Holonomy Group

Let X and Y be two topological spaces and $f : X \to Y$ a map defined on some open subset of X such that $f(x_0) = y_0$. Then a map $g : X \to Y$ is in the same germ as f at x_0 if

(i) $g(x_0) = y_0$.

(ii) There is an open set U containing x_0 such that $g|U = f|U$.

This clearly defines an equivalence relation on the set of such maps f. The equivalence class of f is called the germ of f at x_0 and will be denoted by $G(x_0, f)$.

Consider the set $\mathcal{S}^p$ of maps $f : R^p \to R^p$ each defined on a neighbourhood $U(f)$ of the origin such that $f : U(f) \to f(U(f))$ is a homeomorphism of class C^r leaving the origin fixed.

Denote by $\mathcal{G}_{r,p}$ the set of germs of such maps at the origin. This set gives a group under the following multiplication

$$G(o,f) \times G(o,g) = G(o,f_o\, g)$$

If R^p has coordinates y^i, and $f \in \mathcal{S}^p$ has the coordinate representation $f(y^1,\dots,y^p) = (f^1(y^i),\dots,f^p(y^i))$ on $U(f)$, then the partial derivatives of f are defined at $y^i = 0$, up to order r. Furthermore it is clear that if $g \in G(o,f)$ then the partial derivatives of g agree with those of f up to order r. Thus the derivative of a germ $G(o,f)$ is well defined at the origin.

Consider the subset F_q of $\mathcal{G}_{r,p}$ consisting of those germs whose derivatives at the origin, up to order q are the same as the identity. Thus if $f \in F_q$ then $\left(\frac{\partial f^i}{\partial y^j}\right)_o = \delta_j^{\,i}$, $\left(\frac{\partial^k f^i}{\partial y^{i_1}\dots\partial y^{i_k}}\right)_o = 0$ for $1 < k \leqslant q$

F_q does not depend on the choice of y^i for if $\bar{y}^i = \bar{y}^i(y^j)$ is another coordinate system with the same origin, then

$$\left(\frac{\partial \bar{f}^i}{\partial \bar{y}^k}\right)_o = \frac{\partial \bar{y}^i}{\partial y^j}(f(o)) \cdot \frac{\partial f^j}{\partial y^\ell}(o) \cdot \frac{\partial y^\ell}{\partial \bar{y}^k}(o) = \delta_k^{\,i} \quad \text{since } f(o) = 0$$

Similarly for higher derivatives.

F_q is a normal subgroup of $\mathcal{G}_{r,p}$ because if $G(o,f)$, $G(o,g) \in F_q$ then

$$\left(\frac{\partial (f_o g)^i}{\partial y^j}\right)_o = \frac{\partial f^i}{\partial y^k}(g(o)) \cdot \frac{\partial g^k}{\partial y^j}(o) = \delta_j^{\,i} \qquad \text{etc.}$$

$$\delta_j^{\,i} = \left(\frac{\partial (f_o f^{-1})^i}{\partial y^j}\right)_o = \frac{\partial f^i}{\partial y^\ell}(f^{-1}(o)) \cdot \frac{\partial (f^{-1})^\ell}{\partial y^j}(o)$$

$$\therefore \quad \frac{\partial (f^{-1})^\ell}{\partial y^j}(o) = \delta_j^{\,\ell} \qquad \text{etc.}$$

and if $G(o,h) \in \mathcal{G}_{r.p}$ then

$$\left(\frac{\partial (h_o f_o h^{-1})^i}{\partial y^j}\right)_o = \frac{\partial h^i}{\partial y^\ell}(f_o h^{-1}(o)) \cdot \frac{\partial f^\ell}{\partial y^k}(h^{-1}(o)) \cdot \frac{\partial (h^{-1})^k}{\partial y^j}(o) = \delta_j^{\,i} \quad \text{etc.}$$

Put $\mathcal{G}^q_{r,p} = \mathcal{G}_{r,p}/F_q$ and let $\psi_q : \mathcal{G}_{r,p} \to \mathcal{G}^q_{r,p}$ be the quotient homomorphism.

For each f, ψ_q G(o,f) is called the q-Jet of f.

$\mathcal{G}^q_{r,p}$ could have been obtained by factoring $\mathcal{G}_{r,p}$ by the equivalence relation : $G(o,f) \sim G(o,g)$ if f and g have the same derivatives up to order r at the origin.

Let M be an m-manifold of class C^s admitting a class C^r foliation $\mathcal{F}$ of codimension p with leaf atlas $\mathcal{A}$.

Let L be a leaf of $\mathcal{F}$ $z \in L$ a given point and (W,y^i) a given leaf chart such that $z \in W$ and $y^i(z) = 0$.

One may identify R^p with the transversal set

$T = \{(0,\dots,0,y^{m-p+1},\dots,y^m) : |y^i| < 1\}$.

Let $\sigma : [0,1] \to L$, $\sigma(0) = \sigma(1) = z$ be a loop in L at z.

Since $\sigma([0,1])$ is compact it admits a finite cover $\Sigma = \{(U_a,x_a^{\;i}) : a = 0,1,\dots,n-1\}$ of charts of $\mathcal{A}$ with the following properties:

(i) There is a subdivision Λ of $[0,1]$ namely $[0,t_1],\dots,[t_a,t_{a+1}],\dots [t_{n-1},1]$, $t_o = 0$, $t_n = 1$, such that $\sigma([t_a,t_{a+1}]) \subset U_a$ $\sigma([t_{n-1},1]) \subset U_a$.

(ii) $x_a^{\;\alpha}(\sigma(t)) = 0$ for $t \in [t_a,t_{a+1}]$, and $x_a^{\;\lambda}(\sigma(t_a)) = 0$.

To construct such a cover, take any finite cover and choose a subdivision Λ so that $\sigma([t_a,t_{a+1}])$ is contained in the interior of a chart. Now index the charts so that (i) is satisfied and modify the coordinates by suitable affine transformations so that (ii) is satisfied.

Condition (i) ensures that there is a $\delta > 0$ such that the overlap transformations $x^{\alpha}_{a+1} = Q^{\alpha}_{a+1,a}(x_a^{\;\beta})$ $\alpha,\beta = m-p+1,\dots,m$ are defined for $0 \leqslant |x_a^{\;\beta}| < \delta$ for each $a = 0,1,\dots,n-1$. Put

$$V_a(\delta) = \{(0,\ldots,0,x_a^{m-p+1},\ldots,x_a{}^m) \in U_a : |x_a{}^\alpha| < \delta\}.$$

Define $f_{a+1,a} : V_a(\delta) \to V_{a+1}(1)$ by

$$f_{a+1,a}(0,..,0,x_a^{m-p+1},..,x_a^\alpha,..,x_a^m)=(0,..,0,Q_{a+1,a}^{m-p+1}(x_a^\beta),..,Q_{a+1,a}^{\alpha}(x_a^\beta),..,Q_{a+1,a}^{m}(x_a^\beta))$$

Clearly $f_{a+1,a}$ is a C^r-homeomorphism into $V_{a+1}(1)$ which sends the origin to the origin.

It is easy to see that there is an ε_1, $0 < \varepsilon_1 \leqslant \delta$ such that $f_{a,a-1} : V_{a-1}(\varepsilon_1) \to V_a(\delta)$ is a C^r homeomorphism into $V_a(\delta)$.

By induction there is ε_b, $0 < \varepsilon_b \leqslant \delta$ such that

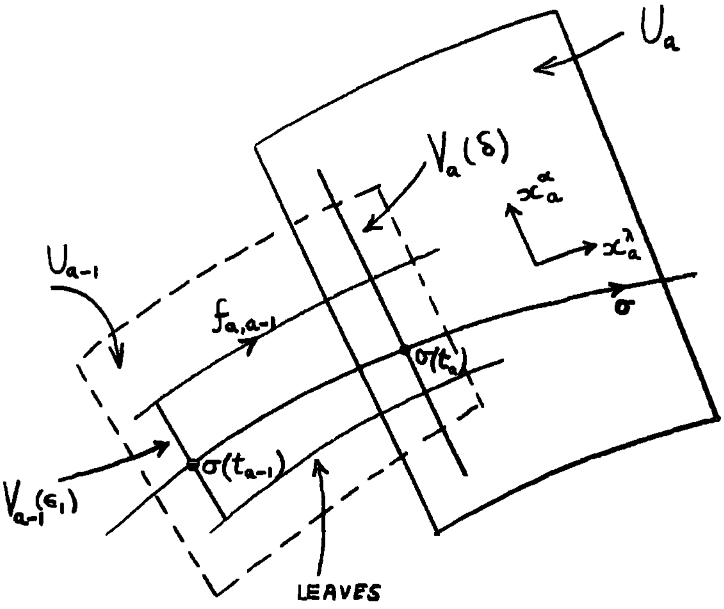
$$f_{a-b+1,a-b} : V_{a-b}\,(\varepsilon_b) \to V_{a-b+1}\,(\varepsilon_{b+1})$$

is a C^r-homeomorphism into $V_{a-b+1}\,(\varepsilon_{b+1})$ for all b, $0 \leqslant b \leqslant a$.

Thus there is $\varepsilon_a > 0$ such that

$$f_a = f_{a,a-1} \circ f_{a-1,a-2} \circ \cdots \circ f_{1,0} : V_0(\varepsilon_a) \to V_a(\delta)$$

is a C^r homeomorphism into $V_a(\delta)$.



One can define $(U_n, x_n^{\ i}) = (U_o, x_o^{\ i})$ and thus $f_n : V_o(\varepsilon_n) \to V_o(1)$ is a C^r homeomorphism into $V_o(1)$. Define a map $\Phi : \Omega\,(L, z) \to \mathcal{G}_{r,p}$ by $\Phi(\sigma) = G(0, \eta_{\Sigma} \circ f_n \circ \eta_{\Sigma}^{-1})$, where $\Omega\,(L, z)$ is the loop space of L at z and $\eta_{\Sigma} : V_o(\varepsilon) \to T$ is the C^r-homeomorphism sending the origin to the origin given by $(0,\dots,0,x_o^{m-p+1},\dots,x_o^{m}) \mapsto (0,\dots,0,y^{m-p+1}(x_o^{\ \alpha}),\dots,y^m(x_o^{\ \alpha}))$ defined for $\varepsilon > 0$. To show that Φ is well defined it will suffice to show that a different choice of subdivision Λ' and cover Σ' satisfying conditions (i) and (ii), give a C^r-homeomorphism $\eta_{\Sigma'} \circ f_{n'} \circ \eta_{\Sigma'}^{-1}$ with the same germ.

Let $\bar{\Lambda}$ be a subdivision of $[0,1]$ for which $\Lambda' \subset \bar{\Lambda}$ and $\Lambda \subset \bar{\Lambda}$. Suppose $\bar{\Lambda}$ has the form $[t_a, t_{a,1}], \dots, [t_{a,s}, t_{a,s+1}], \dots, [t_{a,k_a}, t_{a+1}]$ with $t_{a,0} = t_a$, $t_{a,k_a+1} = t_{a+1,0} = t_{a+1}$.

Then $\sigma([t_{a,s}, t_{a,s+1}])$ is contained in the interior of a chart of Σ and of a chart of Σ'.

A re-indexing of Σ gives a finite cover $\bar{\Sigma}$ of σ with leaf charts

$$(U_{a,s}, x_{a,s}^{\ i}), \text{ where } U_{a,s} = U_a$$
$$x_{a,s}^{\ \lambda} = x_a^{\ \lambda} - x_a^{\ \lambda}(\sigma(t_{a,s}))$$
$$x_{a,s}^{\ \alpha} = x_a^{\ \alpha} \qquad \text{for } 0 \leq s \leq k_a\ ., \ a = 0,\dots,n-1$$

Clearly $\sigma([t_{a,s}, t_{a,s+1}]) \subset U_{a,s}$ and

$$x_{a,s}^{\ \alpha}(\sigma(t)) = 0 \quad \text{for } t \in [t_{a,s}, t_{a,s+1}], \quad x_{a,s}^{\ \lambda}(\sigma(t_{a,s})) = 0$$

Put

$$V_{a,s}(\delta) = \{(0,\dots,0,x_{a,s}^{m-p+1},\dots,x_{a,s}^{\ m}) \in U_{a,s} : |x_{a,s}^{\ \alpha}| < \delta\}$$

Define

$$f_{a,s,s-1} : V_{a,s-1}(\delta) \to V_{a,s}(1)$$

by $(0,\dots,0,x^{m-p+1}_{a,s-1},\dots,x^{m}_{a,s-1}) \mapsto (0,\dots,0,x^{m-p+1}_{a,s},\dots,x^{m}_{a,s})$

for $0 \leqslant s \leqslant k_a-1$

For $s = k_a$ define $f_{a,k_a+1,k_a} : V_{a,k_a} \to V_{a,k_a+1} = V_{a+1,0} = V_{a+1}$

by $(0,\dots,0,x^{m-p+1}_{a,k_a},\dots,x^{m}_{a,k_a}) \mapsto (0,\dots,0,Q^{m-p+1}_{a+1,a}(x_a{}^{\alpha}),\dots,Q^{m}_{a+1,a}(x_a{}^{\alpha}))$

It is easy to see that on some neighbourhood of the origin in V_a

$$f_{a+1,a} = f_{a,k_a+1,k_a} \circ \cdots \circ f_{a,1,0}$$

and moreover $\eta_{\Sigma} = \eta_{\bar{\Sigma}}$.

Thus the germ obtained from $\bar{\Sigma}$ and $\bar{\Lambda}$ is the same as that from Σ and Λ.

Hence there is no loss of generality in assuming that the subdivisions Λ and Λ' are the same.

Let $\Sigma = \{(U_a,x_a{}^{i}) : a = 0,1,\dots,n-1\}$

$\Sigma' = \{(U'_a,x'_a{}^{i}) : a = 0,1,\dots,n-1\}$

For a given $b \in 1,\dots,n-1$ one may obtain a new cover Σ_b from Σ by replacing $(U_b,x_b{}^{i})$ by $(U'_b,x'_b{}^{i})$. Clearly, Σ_b satisfies conditions (i) and (ii) with respect to the subdivision Λ.

If on the overlap of U_b and $U_b{}'$, $x'_b{}^{i} = g_b{}^{i}(x_b{}^{j})$ then if $g_b : V_b(\varepsilon) \to V'_b(1)$ is defined by

$$(0,\dots,0,x_b^{m-p+1},\dots,x_b{}^{m}) \mapsto (0,\dots,0,g_b^{m-p+1}(x_b{}^{j}),\dots,g_b{}^{m}(x_b{}^{j}))$$

there is a sequence

$$\begin{tikzcd}
\cdots \arrow[r] & V_{b-1} \arrow[r, "f_{b,b-1}"] \arrow[dr, "\bar{f}_{b,b-1}"'] & V_b \arrow[r, "f_{b+1,b}"] \arrow[d, "g_b"] & V_{b+1} \arrow[r] & \cdots \\
& & V'_b \arrow[ur, "\bar{f}_{b+1,b}"'] & &
\end{tikzcd}$$

The nature of the coordinate transformations gives

$$\bar{f}_{b+1,b} \circ \bar{f}_{b,b-1} = (f_{b+1,b} \circ g^{-1}) \circ (g \circ f_{b,b-1})$$

$$= f_{b+1,b} \circ f_{b,b-1}$$

on a neighbourhood of the origin in V_{b-1}. Thus

$$\bar{f}_n = f_{n,n-1} \circ \cdots \circ \bar{f}_{b+1,b} \circ \bar{f}_{b,b-1} \circ \cdots \circ f_{1,0}$$

$$= f_{n,n-1} \circ \cdots \circ f_{b+1,b} \circ f_{b,b-1} \circ \cdots \circ f_{1,0}$$

$$= f_n \text{ on some neighbourhood of the origin in } V_0$$

furthermore

$$\eta_\Sigma = \eta_{\Sigma_b} \text{ and so } G(0,\eta_\Sigma \circ f_n \circ \eta_\Sigma^{-1}) = G(0,\eta_{\Sigma_b} \circ \bar{f}_n \circ \eta_{\Sigma_b}^{-1})$$

If $b = 0$ then $\eta_{\Sigma_b} = \eta_\Sigma \circ g_b^{-1}$ and $\bar{f}_n = g_b \circ f_n \circ g_b^{-1}$

$$\therefore \qquad G(0,\eta_{\Sigma_b} \circ \bar{f}_n \circ \eta_{\Sigma_b}^{-1}) = G(0,\eta_\Sigma \circ f_n \circ \eta_\Sigma^{-1})$$

By replacing each chart in turn it follows that

$$G(0,\eta_\Sigma \circ f_n \circ \eta_\Sigma^{-1}) = G(0,\eta_{\Sigma'} \circ f_n' \circ \eta_{\Sigma'}^{-1})$$

and so Φ is well defined.

Let σ' be another loop at z, homotopic to σ in L, relative to z. Then there is a continuous map $\xi : [0,1] \times [0,1] \to L$ satisfying

$$\xi(0,t) = \sigma(t),\ \xi(1,t) = \sigma'(t),\ \xi(s,0) = \xi(s,1) = z$$

Since $\xi([0,1]\times[0,1])$ is compact, it may be covered by a finite number of charts of $\mathcal{A}$. There is a subdivision of $[0,1]$ say $[0,u_1],\dots,[u_b,u_{b+1}],\dots,[u_{N-1},1]$ such that

$$\xi([u_b,u_{b+1}] \times [u_c,u_{c+1}])$$

is contained within one of these charts. Using this subdivision it is easy to obtain a sequence $\sigma = \sigma_1,\sigma_2,\dots,\sigma_i,\sigma_{i+1},\dots,\sigma_k=\sigma'$ of loops at z so that σ_i differs from σ_{i+1} only within a single plaque of a chart. It is straightforward to prove that $\Phi(\sigma_i) = \Phi(\sigma_{i+1})$ and thus by induction that $\Phi(\sigma) = \Phi(\sigma')$.

Also, if $\sigma_o\ \tau$ is the composition of two loops, it is clear that $\Phi(\sigma_o\ \tau) = \Phi(\sigma) \times \Phi(\tau)$ where × is the multiplication in $\mathcal{G}_{r,p}$.

Thus Φ determines a hom omorphism $\phi : \pi_1(L,z) \to \mathcal{G}_{r,p}$. If $H(L,z) = \phi(\pi_1(L,z))$ then it is not difficult to prove that a different choice of initial point z, or initial chart (W,y^i) will give an isomorphic group, where the isomorphism comes from conjugation by an element of $\mathcal{G}_{r,p}$.

Definition 1.2.1 The Ehresmann Holonomy Group H(L) of a leaf L is the group, determined up to isomorphism by H(L,z).

Definition 1.2.2 The Jet Group of order q, $J_q(L)$ of a leaf L is the group, determined up to isomorphism by $\psi_q(H(L,z))$. H(L) and $J_q(L)$ are isomorphic to factor groups of the fundamental group of L.

§1.3 Orientation of Foliations

Let M be an m-manifold of class C^s with a C^r, $r \geqslant 1$ foliation $\mathcal{F}$ and leaf atlas $\mathcal{A} = \{(U_a, x_a{}^i) : a \in J\}$.

If $U_a \cap U_b \neq \phi$, then $\det\left(\frac{\partial x_a{}^\alpha}{\partial x_b{}^\beta}\right)$ is defined on the overlap.

Definition 1.3.1 $\mathcal{F}$ is said to be transversally orientable if there is a cover of M by charts of $\mathcal{A}$, such that on the overlap of two charts $(U_a, x_a{}^i)$ and $(U_b, x_b{}^i)$, $\det\left(\frac{\partial x_a{}^\alpha}{\partial x_b{}^\beta}\right)$ is positive

L E M M A 1.3.1. Let M be an m-manifold of class C^s, with a C^r, $r \geqslant 1$ foliation $\mathcal{F}$. Then there is a two fold cover $\tilde{M}$ of M such that, if $\pi : \tilde{M} \to M$ is the projection map, the induced foliation $\pi^{-1}\mathcal{F}$ on $\tilde{M}$ is transversally orientable.

Proof see Haefliger [8].

§1.4 Integrable and Involutive Distributions

From now on, only manifolds and geometric structures which are smooth (that is, of class C^∞), will be considered.

Definition 1.4.1 A q-dimensional distribution D on an m-manifold M is a q-dimensional, smooth sub-bundle of the tangent bundle TM (see [15]).

If M(x) denotes the tangent space of M at $x \in M$, then the fibre D(x) of D at x will be a q-dimensional subspace of M(x). Furthermore the local triviality of D implies that for each $x \in M$ there is a neighbourhood U of x and smooth vector fields $X_1, \ldots, X_q$ defined on U, such that D(x) is spanned by $X_1(x), \ldots, X_q(x)$ for each $x \in U$.

A vector field X defined on a set $V \subset M$, will be said to lie in D if $X(x) \in D(x)$ for each $x \in V$.

Let $\mathcal{F}$ be a smooth foliation of M, of codimension p, with leaf atlas $\mathcal{A}$. Let (U,x^i) be a chart of $\mathcal{A}$. Consider the smooth vector fields $\frac{\partial}{\partial x^\lambda}$, $\lambda = 1,\dots,m-p$.

For each point $z \in U$, $\frac{\partial}{\partial x^\lambda}(z)$, $\lambda = 1,\dots,m-p$ span an (m-p) dimensional subspace of $M(z)$. If $(\bar{U},\bar{x}^i)$ were another leaf chart with $z \in \bar{U}$, then $\frac{\partial}{\partial \bar{x}^\lambda}(z) = \frac{\partial x^\mu}{\partial \bar{x}^\lambda} \cdot \frac{\partial}{\partial x^\mu}(z)$ since $\frac{\partial x^\alpha}{\partial \bar{x}^\lambda} = 0$.

Hence this subspace does not depend on any particular leaf chart. Thus one obtains a smooth (m-p) dimensional distribution, the *tangent distribution* to $\mathcal{F}$.

Definition 1.4.2 A distribution D is *integrable* if it is tangent to a foliation.

Definition 1.4.3 A distribution D is *involutive* if given two smooth vector fields, with common domain, lying in D, then the Lie bracket $[X,Y]$ lies in D.

The classical Frobenius Theorem can be used to prove the following results (see Hicks [9] page 128).

LEMMA 1.4.1. *A distribution is integrable if and only if it is involutive.*

LEMMA 1.4.2. *Let M be a smooth m-manifold, and let $X_1,\dots,X_m$ be a set of independent smooth vector fields on a neighbourhood U of $z \in M$. Then there is a coordinate chart (V,x^i) with $V \subset U$ such that $X_i = \partial/\partial x^i$ on V for all i if and only if $[X_i,X_j] = 0$ for all i and j.*

§1.5 Connexions Associated with a Foliation

The material in this section stems directly from the work of A. G.

Walker [31], [32], [33] and shows how a foliated structure on a manifold gives rise to certain special classes of connexions. One such class, the Jet Connexions, of which the D-connexions of Walker, form a subclass, can be used to define another foliation holonomy group. The main result of this section says that this holonomy group is always isomorphic to the 1-Jet group.

Again, only smooth structures will be considered, and in addition all manifolds will be assumed to admit a positive definite riemannian metric.

Definition 1.5.1 A distribution D is said to be parallel with respect to an affine connexion Γ, if the action of parallel transport preserves D. That is, if X(x) ε D(x), then parallel transport of X(x) along any piecewise differentiable path from x to y yields a vector in D(y). (This vector will depend on the path in general).

Definition 1.5.2 A foliation $\mathcal{F}$ is said to be parallel with respect to an affine connexion Γ if its tangent distribution is parallel. The following result was proved by T. J. Willmore [39], and A. G. Walker [31].

LEMMA 1.5.1. A distribution is integrable if and only if it is parallel with respect to a torsion free affine connexion.

Proof Let D' be the distribution on the smooth manifold M.

By considering the orthogonal complement of D'(x) for each x ε M, with respect to the metric, one obtains a smooth complementary distribution D" such that M(x) = D'(x) ⊕ D"(x).

Associated with the structure (D',D") there are two smooth projector tensor fields of type (1,1) defined by

$$a'(X)\ (x) = X'(x) \qquad\qquad \text{for each } x \in M$$
$$a''(X)\ (x) = X''(x)$$

where X is a vector field on M and X'(x) is the component of X(x) lying in D'(x) and X''(x) is the component in D''(x). Clearly

$$a'a' = a',\ a''a'' = a'',\ a'a'' = a''a' = 0,\ a' + a'' = I \qquad (1)$$

where I is the identity tensor of type (1,1).

Take any smooth atlas of coordinate charts on M, and if (U,x^i) is one such chart, denote the basis vector fields $\frac{\partial}{\partial x^i}$ by e_i.

Lemma 1.4.1 implies that D' is integrable if and only if

$$a''\left[a'^i_j\, X^j\, e_i,\ a'^k_h\, Y^h\, e_k\right] = 0 \qquad \text{for all vector fields X, Y}$$

expanding

$$a''^s_k\{a'^i_j\, X^j\, Y^h\, a'^k_{h\cdot i} - a'^i_h\, Y^h\, X^j\, a'^k_{j\cdot i} + a'^i_j\, a'^k_h\, X^j\, Y^h_{\cdot i} - a'^i_h\, a'^k_j\, Y^h\, X^j_{\cdot i}\} = 0$$

where a dot denotes partial differentiation.

Using (1), one obtains

$$X^j\, Y^h\, a''^s_k(a'^i_j\, a'^k_{h\cdot i} - a'^i_h\, a'^k_{j\cdot i}) = 0 \qquad (2)$$

and

$$0 = (a''^s_k\, a'^k_h)_{\cdot i} = -a'^s_{k\cdot i}\, a'^k_h + a''^s_k\, a'^k_{h\cdot i} \qquad (3)$$

Substituting in (2) and noting that X, Y were arbitrary, one deduces that D' is integrable if and only if

$$a'^i_j\, a'^k_h\, (a'^s_{k\cdot i} - a'^s_{i\cdot k}) = 0$$

which will be written as

$$a'^{i}_{j}\, a'^{k}_{h}\, a'^{s}_{[k \cdot i]} = 0 \qquad (4)$$

Suppose a vector field X is parallel along a differentiable curve $\sigma : [0,1] \to M$, with respect to a connexion L. Then if a bar denotes covariant differentiation

$$X^{i}_{|j}\, \frac{d\sigma^{j}}{dt} = 0 \quad \text{along } \sigma$$

Thus

$$(a''^{i}_{j}\, X^{j})_{|k}\, \frac{d\sigma^{k}}{dt} = a''^{i}_{j|k}\, X^{j}\, \frac{d\sigma^{k}}{dt}$$

For parallelism, this expression must vanish if $X^{j}(\sigma(0)) = a''^{j}_{s}\, Y^{s}$ for some Y^{s}.

Hence a necessary and sufficient condition for D' to be parallel is

$$a''^{i}_{j|k}\, a'^{j}_{s} = 0$$

But since $0 \equiv (a''^{i}_{j}\, a'^{j}_{s})_{|k}$, this condition is equivalent to

$$a''^{i}_{j}\, a'^{j}_{s|k} = 0 \qquad (5)$$

The idea now is to find a connexion L for which (5) is satisfied, and which is torsion free if (4) is satisfied. Let Γ be any torsion free connexion on M (for instance the metric connexion).

Then L must have coefficients of the form $L^{i}_{jk} = \Gamma^{i}_{jk} + T^{i}_{jk}$ where T is a tensor field of type (1,2). If a comma denotes covariant differentiation with respect to Γ, then

$$a'^{p}_{j|k} = a'^{p}_{j,k} + a'^{q}_{j}\, T^{p}_{qk} - a'^{p}_{q}\, T^{q}_{jk} \qquad (6)$$

From (5), it follows that D' is parallel with respect to L if T satisfies

$$a''^{i}_{p}\, a'^{q}_{j}\, T^{p}_{qk} = -\, a''^{i}_{p}\, a'^{p}_{j,k} = -\, a'^{i}_{p,k}\, a'^{p}_{j} \tag{7}$$

But

$$0 \equiv a'^{i}_{p}\, (a'^{q}_{j}\, a''^{p}_{q})_{,k} = a'^{i}_{p}\, a'^{q}_{j}\, a''^{p}_{q,k} = -\, a'^{i}_{p}\, a'^{q}_{j}\, a'^{p}_{q,k} \tag{8}$$

thus one solution of equations (7) is

$$T^{i}_{jk} = -\, a''^{i}_{s}\, a'^{s}_{j,k}$$

The general solution of (7) is thus

$$T^{i}_{jk} = -\, a''^{i}_{s}\, a'^{s}_{j,k} + V^{i}_{jk} \tag{9}$$

where V is any symmetric tensor satisfying

$$a''^{i}_{p}\, a'^{q}_{j}\, V^{p}_{qk} = 0 \tag{10}$$

Now, V has to be chosen so that T is symmetric when (4) is satisfied.

It is straightforward to show that (4) is equivalent to

$$a'^{i}_{p,q}\, a'^{p}_{j}\, a'^{q}_{k} - a'^{i}_{p,q}\, a'^{p}_{k}\, a'^{q}_{j} = 0 \tag{11}$$

Thus for T to be symmetric

$$V^{i}_{jk} - V^{i}_{kj} = a''^{i}_{p}\, (a'^{p}_{j,k} - a'^{p}_{k,j}) \tag{12}$$

Using (8) and (11), a solution of (10) and (12) is

$$V^{i}_{jk} = -\, a''^{i}_{p}\, a'^{p}_{k,j} + a'^{i}_{p,q}\, a'^{p}_{k}\, a'^{q}_{j}$$

Thus the connexion L defined by the coefficients

$$L^i_{jk} = \Gamma^i_{jk} - a'^i_{p,j}\, a'^p_k - a'^i_{p,k}\, a'^p_j + a'^i_{p,q}\, a'^p_k\, a'^q_j$$

is a torsion free connexion for which D' is parallel.

Conversely, given a torsion free connexion L, with covariant derivative ∇, then

$$0 = \nabla_{a'(X)}\, a'(Y) - \nabla_{a'(Y)}\, a'(X) - [a'(X), a'(Y)]$$

for all vector fields X and Y.

It is not difficult to prove that condition (5) is equivalent to $a''(\nabla_X\, a'(Y)) = 0$ for all vector fields X and Y.

Thus $a''[a'(X), a'(Y)] = 0$ and so D' is involutive and hence integrable. Q.E.D.

This result implies that foliations can be characterised by distributions which are parallel with respect to torsion free connexions. The class of torsion free connexions which make the tangent distribution of a foliation $\mathcal{F}$ on M parallel will be denoted by $C(M, \mathcal{F})$.

Let $\mathcal{A}$ be a leaf atlas for $\mathcal{F}$ and (U, x^i) a chart of $\mathcal{A}$.

Then it can be shown that a', a" have components

$$\left.\begin{array}{l} a'^\lambda_\mu = \delta^\lambda_\mu,\ a'^\alpha_i = 0,\ a'^\lambda_\alpha = b^\lambda_\alpha \\ a''^i_\lambda = 0,\ a''^\alpha_\beta = \delta^\alpha_\beta,\ a''^\lambda_\alpha = -b^\lambda_\alpha \end{array} \qquad \text{for some } b^\lambda_\alpha \right\} \qquad (1)$$

Let $L \varepsilon C(M, \mathcal{F})$, and suppose a bar denotes covariant differentiation with respect to L. Parallelism implies

$$a''^i_j\, a'^j_{s|k} = 0$$

expanding

$$a''^i_j\, (a'^j_{s\cdot k} + L^j_{pk}\, a'^p_s - L^p_{sk}\, a'^j_p) = 0$$

From (1) this reduces to $a''^{i}_{\alpha}\, a'^{\mu}_{s}\, L^{\alpha}_{\mu k} = 0$ which is equivalent to

$$L^{\alpha}_{\mu k} = 0 \tag{2}$$

Let $(\bar{U},\bar{x}^{i})$ be another chart of $\mathcal{A}$ such that $U \cap \bar{U} \neq \phi$. Then, on the overlap

$$\begin{aligned}\bar{L}^{\lambda}_{\mu\theta} &= \frac{\partial \bar{x}^{\lambda}}{\partial x^{i}}\, \frac{\partial x^{j}}{\partial \bar{x}^{\mu}} \cdot \frac{\partial x^{k}}{\partial \bar{x}^{\theta}} \cdot L^{i}_{jk} + \frac{\partial^{2} x^{i}}{\partial \bar{x}^{\mu}\, \partial \bar{x}^{\theta}} \cdot \frac{\partial \bar{x}^{\lambda}}{\partial x^{i}} \\ &= \frac{\partial \bar{x}^{\lambda}}{\partial x^{\rho}} \cdot \frac{\partial x^{\sigma}}{\partial \bar{x}^{\mu}} \cdot \frac{\partial x^{\tau}}{\partial \bar{x}^{\theta}}\, L^{\rho}_{\sigma\tau} + \frac{\partial^{2} x^{\tau}}{\partial \bar{x}^{\mu}\, \partial \bar{x}^{\theta}} \cdot \frac{\partial \bar{x}^{\lambda}}{\partial x^{\tau}}\end{aligned}$$

from $\dfrac{\partial x^{\alpha}}{\partial \bar{x}^{\lambda}} = 0$ and (2).

Thus L induces a torsion free connexion on each leaf of $\mathcal{F}$ and each leaf is a totally geodesic submanifold.

This raises two interesting questions:

(A) What can be deduced about the global properties of $\mathcal{F}$ if $C(M,\mathcal{F})$ contains a complete connexion?

(B) What can be deduced about local or global properties of $\mathcal{F}$ if $C(M,\mathcal{F})$ contains a connexion for which the induced connexion on each leaf has special properties,

e.g. flat, locally symmetric, constant curvature, etc.

In Chapter 2, a partial answer to (B) is given when the induced connexion is flat. In Chapters 3 and 4, question (A) is discussed for the case of a complete riemannian and pseudo-riemannian metric connexion. However, the general question appears very difficult and must remain for future consideration.

Another class of connexions associated with a foliation is now defined.

Let (U,x^i) be a chart of $\mathcal{A}$. A basis for D' at each point of U is $e_\lambda = \partial/_{\partial x^\lambda}$ $\lambda = 1,\dots,m-p$, and a basis for D" is $E_\alpha = a''(e_\alpha) = e_\alpha - b^\mu_\alpha e_\mu$, $\alpha = m-p+1,\dots,m$.

<u>Definition 1.5.3</u> A <u>Jet-Connexion</u> Γ on a foliated manifold is a torsion free affine connexion for which the covariant derivative ∇ satisfies $\nabla_{e_\lambda} E_\beta = 0$ in each leaf chart.

This condition does not depend on the particular leaf chart used, for if $(\bar{U},\bar{x}^i)$ is another chart, then on the overlap

$$\bar{e}_\lambda = \frac{\partial}{\partial \bar{x}^\lambda} = \frac{\partial x^\mu}{\partial \bar{x}^\lambda} \cdot \frac{\partial}{\partial x^\mu} \qquad \text{since } \frac{\partial x^\alpha}{\partial \bar{x}^\lambda} = 0$$

$$\bar{E}_\beta = \frac{\partial \bar{x}^i}{\partial x^j} \cdot \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \cdot a''^j_\alpha \frac{\partial x^s}{\partial \bar{x}^i} e_s \qquad \text{since } a''^j_\lambda = 0$$

$$= \frac{\partial x^\alpha}{\partial \bar{x}^\beta} a''^s_\alpha e_s = \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \cdot E_\alpha$$

Thus

$$\nabla_{\bar{e}_\lambda} \bar{E}_\beta = \nabla_{\frac{\partial x^\mu}{\partial \bar{x}^\lambda} e_\mu} \frac{\partial x^\alpha}{\partial \bar{x}^\beta} E_\alpha = \frac{\partial^2 x^\alpha}{\partial \bar{x}^\lambda \partial \bar{x}^\beta} E_\alpha + \frac{\partial x^\mu}{\partial \bar{x}^\lambda} \cdot \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \nabla_{e_\mu} E_\alpha$$

$$= 0 + 0 = 0$$

To show that the class of Jet Connexions is non-empty, it will be proved that the D-connexions of A. G. Walker [33] are contained in the class.

Following Walker, some special parallelism conditions for a connexion are now defined.

(1) D' is said to be <u>parallel relative to D"</u> if parallel transport of vectors in D' along differentiable paths whose tangent fields lie in D", yields vectors in D'.

(2) Similarly, one may define D" parallel relative to D'.

(3) D' is said to be path parallel if geodesics with initial vectors in D' have their whole tangent field in D'.

(4) Similarly, D" path parallel.

Definition 1.5.4 A D-connexion is a torsion free connexion satisfying conditions (1) → (4).

If L is a given torsion free connexion then it can be shown [33] that the most general D-connexion is given by

$$D^i_{jk} = L^i_{jk} + 2a^i_{(jk)}(L) - a'^p_j\, a'^q_k\, a^i_{(pq)}(L)$$

$$- a''^p_j\, a''^q_k\, a^i_{(pq)}(L) + C^s_{pq}\,(a'^i_s\, a'^p_j\, a'^q_k + a''^i_s\, a''^p_j\, a''^q_k)$$

where $a^i_{(jk)} = a^i_{jk} + a^i_{kj}$ and $a^i_{jk}(L) = a'^i_s\, a'^s_{j|k} + a''^i_s\, a''^s_{j|k}$ and C^i_{jk} is any symmetric tensor field of type (1,2).

In fact, conditions (1), (2) are sufficient to give a Jet connexion. To prove this it will be necessary to obtain equivalent algebraic conditions.

It is not difficult to prove that the following conditions are respectively equivalent to (1) → (4), where a semi colon denotes covariant differentiation with respect to Γ.

(1)' $a''^i_{p;q}\, a'^p_j\, a''^q_k = 0$

(2)' $a'^i_{p;q}\, a''^p_j\, a'^q_k = 0$

(3)' $a''^i_{(p;q)}\, a'^p_j\, a'^q_k = 0$

(4)' $a'^i_{(p;q)}\, a''^p_j\, a''^q_k = 0$

In a leaf chart (1)' reduces to $\Gamma^s_{pq}\, a''^j_s\, a'^p_i\, a''^q_k = 0$

i.e. $\Gamma^{\alpha}_{\lambda q}\, a''^{j}_{\alpha}\, a'^{\lambda}_{i}\, a''^{q}_{\beta} = 0$

which is equivalent to $\Gamma^{\alpha}_{\lambda\beta} - b^{\mu}_{\beta}\, \Gamma^{\alpha}_{\lambda\mu} = 0$ (*)

(2)' reduces to $(a'^{j}_{p\cdot\lambda} - \Gamma^{s}_{p\lambda}\, a'^{j}_{s})\, a''^{p}_{\alpha}\, a'^{\lambda}_{k} = 0$

i.e. $a'^{\lambda}_{k}\, a''^{p}_{\alpha}\, (a'^{\theta}_{p\cdot\lambda} - \Gamma^{s}_{p\lambda}\, a'^{\theta}_{s}) = 0$

i.e. $a'^{\lambda}_{k}\, (b^{\theta}_{\alpha\cdot\lambda} - b^{\theta}_{\gamma}\, \Gamma^{\gamma}_{\alpha\lambda} - \Gamma^{\theta}_{\alpha\lambda} + b^{\tau}_{\alpha}\, b^{\theta}_{\gamma}\, \Gamma^{\gamma}_{\tau\lambda} + b^{\tau}_{\alpha}\, \Gamma^{\theta}_{\tau\lambda}) = 0$

which is equivalent to

$$b^{\theta}_{\alpha\cdot\lambda} + b^{\tau}_{\alpha}\, \Gamma^{\theta}_{\tau\lambda} - \Gamma^{\theta}_{\alpha\lambda} + b^{\theta}_{\gamma}\, (b^{\tau}_{\alpha}\, \Gamma^{\gamma}_{\tau\lambda} - \Gamma^{\gamma}_{\alpha\lambda}) = 0 \qquad (**)$$

If ∇ is the covariant derivative of Γ

$$\nabla_{e_\lambda} E_\beta = \nabla_{e_\lambda}(e_\beta - b^{\mu}_{\beta}\, e_\mu) = \Gamma^{i}_{\lambda\beta}\, e_i - e_\lambda\, (b^{\mu}_{\beta})\, e_\mu - b^{\mu}_{\beta}\, \Gamma^{i}_{\lambda\mu}\, e_i$$

$$= (\Gamma^{\alpha}_{\lambda\beta} - b^{\mu}_{\beta}\, \Gamma^{\alpha}_{\lambda\mu})\, e_\alpha - (b^{\theta}_{\beta\cdot\lambda} + b^{\tau}_{\beta}\, \Gamma^{\theta}_{\lambda\tau} - \Gamma^{\theta}_{\lambda\beta})\, e_\theta$$

$= 0 + 0$ if (*) and (**) are satisfied.

Thus (1) and (2) are sufficient for Γ to be a Jet-connexion. It follows that every D-connexion is a Jet-connexion. The most general Jet-connexion will be of the form $\Gamma^{i}_{jk} = D^{i}_{jk} + a'^{p}_{j}\, a''^{q}_{k}\, V^{i}_{pq}$ where V is any symmetric tensor of type (1,2) and D is a D-connexion.

Let x_o be a given point in a leaf L of $\mathcal{F}$ and $\sigma : [0,1] \to L$ a differentiable path in L such that $\sigma(0) = x_o$.

If (U,x^i) is a leaf chart containing x_o and Γ is a jet connexion with covariant derivative ∇, then

$(\nabla_{e_\lambda} Y^{\beta} E_\beta)\, \frac{d\sigma^\lambda}{dt} = 0$, yields a solution $Y^{\beta}(\sigma(t)) = Y^{\beta}(x_o)$

and so parallel transport of vectors in $D''(x_o)$ is independent of path in L locally. Furthermore, the translation does not depend on the particular choice of Jet connexion. It follows that if σ, τ are homotopic, piecewise differentiable loops at x_o, then parallel transport of a given vector in $D''(x_o)$ around σ yields the same result as that around τ.

Thus by transporting a given basis for $D''(x_o)$ one obtains a homomorphism $w : \pi_1(L,x_o) \to GL(p;R)$ the general linear group of order p if $\mathcal{F}$ is of codimension p.

Let $W(L,x_o) = w(\pi_1(L,x_o))$. A different choice of base point, or basis yields an isomorphic group. Similarly, for a different complementary distribution $\bar{D}''$ say, the vector bundle isomorphism defined by $Y^\alpha\ a''(e_\alpha)\ (x) \mapsto Y^\alpha\ \bar{a}''(e_\alpha)\ (x)$ for $x \in M$ shows that one again obtains an isomorphic group.

Definition 1.5.5 The Walker Holonomy Group W(L) of a leaf L is the group determined up to isomorphism by $W(L,x_o)$.

THEOREM 1.5.2 Let $\mathcal{F}$ be a smooth foliation of codimension p on a smooth m-manifold M. Then for each leaf of $\mathcal{F}$, the Walker Holonomy group and the 1-Jet group are isomorphic.

Proof

Let L be a leaf of $\mathcal{F}$ and $x_o \in L$ a given point.

Recall that $\psi_1 : \mathcal{G}_{\infty,p} \to \mathcal{G}^1_{\infty,p}$ was essentially obtained by taking equivalence classes of first derivatives at the origin. Thus, since the matrix of partial derivatives of a local homeomorphism of class C^∞ (that is, a local diffeomorphism) is non-singular at the origin, and moreover the derivative of a composition is the matrix product of the derivatives it follows that one may identify $\mathcal{G}^{1}_{\infty,p}$ with the general linear group $GL(p;R)$.

Thus $\psi_1 \circ \phi : \pi_1(L,x_0) \to GL(p;R)$ yields the 1-Jet group $J_1(L,x_0)$.

Take D' to be the tangent distribution to $\mathcal{F}$ and D'' a complementary distribution. Let Γ be a Jet-connexion.

If $\sigma : [0,1] \to L$ is a piecewise differentiable loop at $x_0 \,\varepsilon\, L$. Then there is a cover $\Sigma = \{(U_a, x^i_a) : a = 0,1,\ldots,n-1\}$ of σ by leaf charts and a subdivision Λ of $[0,1]$ satisfying conditions (i) and (ii) of §1.2.

Change coordinates in each chart (U_a, x^i_a) by the rule

$$y^\lambda_a = x^\lambda_a - b^\lambda_\alpha(0)\, x^\alpha$$

$$y^\alpha_a = x^\alpha_a$$

Then $\{(U_a, y^i_a)\}$ is easily shown to be a collection of leaf charts and furthermore

$$\frac{\partial}{\partial y^\alpha_a} = \frac{\partial x^i_a}{\partial y^\alpha_a} \cdot \frac{\partial}{\partial x^i_a} = \frac{\partial}{\partial x^\alpha_a} - b^\lambda_\alpha(0)\, \frac{\partial}{\partial x^\lambda_a}$$

Thus

$$\frac{\partial}{\partial y^\alpha_a} \Big| \sigma([t_a, t_{a+1}]) = E_\alpha \big| \, \sigma([t_a, t_{a+1}])$$

Hence if $Y^\beta \frac{\partial}{\partial y^\beta_a}$ is a vector in $D''(\sigma(t_a))$ then parallel transport along σ from $\sigma(t_a)$ to $\sigma(t_{a+1})$, with respect to Γ will yield the vector

$$Y^\beta \frac{\partial y^\alpha_{a+1}}{\partial y^\beta_a} \cdot \frac{\partial}{\partial y^\alpha_{a+1}} \text{ at } \sigma(t_{a+1}).$$

This is clearly the same as the action of $(f_{a+1,a})_*$ and so parallel transport around σ from x_0 to x_0 will yield the same result as the action of

$$(f_{n,n-1})_* \circ (f_{n-1,n-2})_* \circ \cdots \circ (f_{1,0})_* = (f_n)_*$$

which is the derivative of $\Phi(\dot{\sigma})$.

Thus $w([\sigma])$ and $\psi_1 \circ \phi([\sigma])$ give precisely the same linear maps of R^p to R^p (where $[\sigma]$ is the homotopy class of σ) and hence the groups $W(L,x_o)$ and $J_1(L,x_o)$ are isomorphic

Q.E.D.

The following example shows that the Walker Holonomy group and the Ehresmann Holonomy group are distinct in general.

EXAMPLE 1.5.1

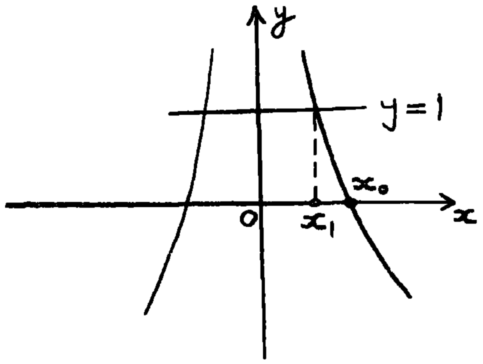
Take R^2 with coordinates (x,y). Consider the smooth vector field $X = -x \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.

X generates a one dimensional distribution, and since $[X,X] = 0$ this distribution is tangent to a one-dimensional smooth foliation $\mathcal{F}$. A simple calculation shows that the leaves of $\mathcal{F}$ consist of the y-axis and the curves $y = -\log|x| + c$ where c is an arbitrary constant.

The integers Z act on R^2 by $n(x,y) = (x,y+n)$, and X is clearly invariant under this action.

By taking the quotient structure, one obtains a smooth vector field on the cylinder $R \times S^1$. Let $\bar{\mathcal{F}}$ be the smooth foliation determined by this vector field.

The leaves of $\bar{\mathcal{F}}$ are homeomorphic to R except the image of the y-axis which is homeomorphic to S^1. Call this leaf L.



From the picture it is clear that H(L) is generated by the germ of the map f which sends x_o to x_1.

But $x_1 = \frac{x_o}{1+x_o}$

Thus $H(L) = < G(0,f) : f(x) = \frac{x}{1+x} >$

and is infinite cyclic.

However $J_1(L)$ is generated by $\frac{df}{dx}(0) = 1$, and this is the identity.

i.e. $H(L) \cong \mathbb{Z}$, $W(L) \cong \{1\}$

CHAPTER 2

Locally Affine Foliations

Throughout this chapter only smooth manifolds and maps will be considered.

§2.1 Locally Affine Manifolds

Definition 2.1.1 A locally affine (L.A.) manifold is a pair (M,Γ) where M is a smooth manifold carrying a smooth affine connexion Γ whose curvature and torsion tensors vanish identically.

Such (M,Γ) can be characterised by the existence of an atlas of affine coordinate charts. That is, an atlas in which the coordinate transformations have constant jacobian.

LEMMA 2.1.1 [1] Let (M,Γ) be an L.A. m-manifold then there is an affine atlas on M. Conversely, if M admits an affine atlas $\mathcal{A}$ then there is a uniquely determined connexion $\Gamma(\mathcal{A})$ for which $(M,\Gamma(\mathcal{A}))$ is an L.A. manifold and $\Gamma^i_{jk} \equiv 0$ in each chart.

Proof

Let (U,x^i) be a coordinate chart on M. From the classical Frobenius theorem (see Hicks [9] page 126) the equations

$$\frac{\partial X^i}{\partial x^j} + \Gamma^i_{kj} X^k = 0 \qquad (1)$$

one completely integrable if $R^{\,i}_{jkh}$, the components of the curvature tensor vanish on U.

Thus it follows that parallel transport of vectors in U between two points is independent of the choice of path between the points.

Let $p \in U$, $X_1(p),\dots,X_m(p)$ be m-independent vectors at p and $X_1,\dots,X_m$ the corresponding vector fields on U obtained by parallel translation. Let

$$X_j = X^i_j \frac{\partial}{\partial x^i}, \quad \text{then} \quad [X_i,X_j] = \left(X^h_i \frac{\partial X^s_j}{\partial x^h} - X^h_j \frac{\partial X^s_i}{\partial x^h}\right) \frac{\partial}{\partial x^s}$$

Substituting from (1), one obtains

$$[X_i,X_j] = \Gamma^s_{hk}\left(X^h_j X^k_i - X^h_i X^k_j\right) \frac{\partial}{\partial x^s} = 0$$

Hence, by lemma 1.4.2 there is a coordinate chart (V,y^i) such that $p \in V$ and $V \subset U$ and $X_i = \frac{\partial}{\partial y^i}$ on V.

If $\bar{X}^k_i$ and $\bar{\Gamma}^k_{ij}$ are the respective components of X and Γ in (V,y^i) then, $\bar{X}^k_i = \delta^k_i$ and from (1),

$$\frac{\partial}{\partial y^i}(\delta^j_k) + \bar{\Gamma}^j_{is}\,\delta^s_k = 0$$

$$\therefore \quad \bar{\Gamma}^j_{ik} = 0 \quad \text{on } V \qquad (2)$$

Since one may find such a chart (V,y^i) about every point of M it follows that there is a cover S of M by coordinate charts in which the connexion coefficients of Γ vanish. From the transformation law for those coefficients it is clear that the coordinate transformation between two overlapping charts (V,y^i) and $(\bar{V},\bar{y}^i)$ must satisfy

$$\frac{\partial^2 \bar{y}^i}{\partial y^j\,\partial y^k} = 0$$

The required affine atlas will be the collection of coordinate charts containing S and maximal with respect to (2).

Conversely, if M admits an affine atlas $\mathcal{A}$ then one may define a connexion Γ on M by putting $\Gamma^i_{jk} \equiv 0$ in each chart. Q.E.D.

COROLLARY <u>If (M,Γ) is an L.A. manifold, and M is compact and connected then $\pi_1(M)$ is infinite</u>.

<u>Proof</u>

Assume $\pi_1(M) = \{1\}$, then from the proof of the theorem there are m-independent vector fields $X_1,\ldots,X_m$ defined over all of M satisfying $[X_i,X_j] = 0$.

Thus there is an affine atlas $\mathcal{A} = \{(U_a,x_a^i) : a \in J\}$ for which $\frac{\partial x_a^i}{\partial x_b^j} = \delta_j^i$ on the overlap of U_a and U_b.

Consider the 1-form ω, defined by $\omega = dx_a^m$ in U_a. Then ω is clearly defined globally and is non-vanishing.

If d is the exterior derivative (see [15]) then $d\omega = 0$ and ω is closed.

Now, any smooth real valued function f, on a compact manifold has at least two critical points (i.e. where $df = 0$), namely at the maximum and minimum values. Thus $\omega \neq df$ for any f, and so ω represents a non-trivial element in the first de Rham cohomology group (see [7]). It follows that the first singular homology group with integer coefficients is non trivial [27]. But this is just $\pi_1(M)$ made abelian and so $\pi_1(M) \neq \{1\}$ a contradiction.

If $\pi_1(M)$ were assumed finite, then the simply connected cover $\tilde{M}$ of M would be compact. The locally affine structure lifts in a natural way to $\tilde{M}$ and so $\pi_1(\tilde{M}) \neq \{1\}$ a contradiction. Thus $\pi_1(M)$ is infinite. Q.E.D.

So far only local properties of the connexion have been used. However, with an assumption of completeness, very strong global results may be obtained.

<u>Definition 2.1.2</u> A <u>connexion preserving</u> map f between two manifolds M and

M' with affine connexions Γ and Γ' is a smooth map satisfying

$$f_* \nabla_X Y = \nabla'_{f_* X} f_* Y$$

where ∇ and ∇' are the respective covariant derivatives and X, Y are any two vector fields on M.

The following result is due to Hicks [10].

LEMMA 2.1.2 Let M, M' be m-dimensional connected manifolds each carrying affine connexions. Let M' be complete and let $f : M' \to M$ be a connexion preserving local diffeomorphism of M' into M. Then f is a covering map.

Proof

To show that f is onto it will suffice to show that $f(M')$ is both open (which is trivial since f is a local diffeomorphism) and closed. Let $p \in \overline{f(M')}$ (the closure of $f(M')$). Though M is not assumed complete, the existence of a simple convex neighbourhood at p (see Whitehead [42]), ensures that the map $\exp_p$ is defined and non-singular in a neighbourhood U of $o \in M_p$, Such that if $\bar{p} \in U$ then $t\bar{p} \in U$ for all t $0 \leqslant t \leqslant 1$. Let $V = \exp_p U$ be the corresponding neighbourhood of p. Since p is a limit point of $f(M')$, there is a $p_1 \in V \cap f(M')$. Let $\bar{p} = (\exp_p | U)^{-1} (p_1)$. Then $\sigma : [0,1] \to M$ defined by $\sigma(t) = \exp_p t\bar{p}$ is a geodesic from p to p_1 with initial vector $T_\sigma(0) = \bar{p}$. Let $\alpha(t) = \sigma(1-t)$, then α is a geodesic from p_1 to p. Choose any $p' \in M'$ such that $f(p') = p_1$. Let $q = f_*^{-1} T_\alpha(0) \in M'_{p'}$. Define $\gamma : [0,1] \to M'$ by $\gamma(t) = \exp_{p'} tq$. Then γ is a geodesic in M', and hence $f_o \gamma$ is a geodesic in M since f is connexion preserving. Moreover, $f_o \gamma(0) = \alpha(0) = p_1$.

Also, $T_{f_o\gamma}(0) = f_*(q) = T_\alpha(0)$, which implies $f_o\gamma = \alpha$. Hence $f_o\ \gamma(1) = p$ and so f is onto. This argument also shows that M is complete.

To show that f evenly covers any $p \varepsilon M$, let U,V be as before. Then it can be shown that f evenly covers V, that is to say, $f^{-1}\ V$ consists of a union of disjoint sets each diffeomorphic by f to V. Let $p' \varepsilon M'$ such that $f(p') = p$. Since f is a local diffeomorphism, f_*^{-1} maps M_p isomorphically onto $M'_{p'}$.

Define $\phi : V \to M'$ by $\phi = \exp_{p'}\ {}_o\ f_*^{-1}\ {}_o\ (\exp_p|U)^{-1}$ and let $\phi(V) = V'$. Clearly ϕ is smooth, since it is a composition of smooth maps. $f_o\phi$ = identity map on V because ϕ lifts geodesics in V that emanate from p into geodesics in V' that emanate from p'; moreover, since f is connexion preserving, f projects these geodesics back into geodesics that have the same tangent vectors at p and hence for such geodesics σ, $f_o\phi_o\sigma = \sigma$.

Similarly $\phi_o\ (f|V')$ = identity map on V'. Thus f is a diffeomorphism of V' onto V and it follows trivially that V' is the connected component of p' in $f^{-1}(V)$. Q.E.D.

This result will be used several times in what follows.

Definition 2.1.3 An L.A. manifold (M,Γ) is complete if Γ is a complete connexion.

THEOREM 2.1.1. Let (M,Γ) be a complete L.A. m-manifold. Then for each $p \varepsilon M$, $\exp_p : M_p \to M$ is a covering map.

Proof

One can make M_p into a complete L.A. manifold as follows.

Pick a basis $e_1,\ldots,e_m$ for M_p. This defines a global coordinate chart (M_p,λ^i) for M_p, where if $X \varepsilon M_p$ and $X = \lambda^i\ e_i$ then X has coordinates

$\lambda^1, \ldots, \lambda^m$.

The coefficients $L^i_{jk} \equiv 0$ define a connexion L on M_p. Geodesics are just affine lines and the connexion is complete.

By Lemma 2.1.1 there is an atlas $\mathcal{A} = \{(U_a, x^i_a) : a \in J\}$ of affine charts on M such that the connexion coefficients of Γ have the form $\Gamma^i_{jk} \equiv 0$ in each chart.

Now, let $X \in M_p$ and $\sigma : [0,1] \to M$ be the geodesic at p with initial vector X. i.e. $\sigma(0) = p$, $T_\sigma(0) = X$.

Let $(U_1, x_1^i), \ldots, (U_a, x^i_a), \ldots, (U_n, x^i_n)$ be a cover of σ by charts of $\mathcal{A}$ for which there is a subdivision $[0,t_1], \ldots, [t_a, t_{a+1}], \ldots [t_{n-1}, 1]$ of $[0,1]$ satisfying $\sigma([t_a, t_{a+1}]) \subset U_a$. There is no loss of generality in assuming that $\frac{\partial}{\partial x_1^i}(p) = e_i$, $i = 1, \ldots, m$.

It follows by induction that in the chart (U_a, x^i_a), σ has coordinates of the form

$$\sigma^i_a(t) = A^i_j(a)\, X^j\, t + B^i(a), \quad t \in [t_a, t_{a+1}]$$

where $(A^i_j(a))$ is a constant non singular mxm matrix and $B^i(a)$, $i = 1, \ldots, m$ are m-constants.

Thus $\sigma^i_n(1) = A^i_j(n)\, X^j + B^i(n)$. These are the coordinates of $\exp_p X$ in the chart (U_n, x^i_n). Thus $\exp_p$ has the form $X^j \mapsto A^j_i(n)\, X^i + B^j(n)$, which has jacobian $(A^j_i(s))$ and so is non-singular, showing that $\exp_p$ is a local diffeomorphism. But the connexion L on M_p is also preserved (i.e. $L^i_{jk} \equiv 0 \to \Gamma^i_{jk} \equiv 0$). Hence, by Lemma 2.1.2 $\exp_p : M_p \to M$ is a covering map. Q.E.D.

COROLLARY. <u>Let (M,Γ) be a complete, L.A. m-manifold. Then with respect to the coordinate chart (M_p, λ^i) on M_p, the group of covering trans-</u>

formations of the covering map $\exp_p$, is a subgroup of the group of affine transformations $A(m; R)$ of R^m (see [15]).

Proof

Let $f : M_p \to M_p$ be a covering transformation (see [27]). Then by definition f is a homeomorphism and

$$\exp_p \circ f = \exp_p$$

Now, since $\exp_p$ is a connexion preserving local diffeomorphism it follows that f is a diffeomorphism and preserves the connexion L.

Thus if ∇ is the covariant derivative of L then

$$f_*\{(\nabla_{\partial/\partial\lambda^i}\, {}^{\partial/}\partial\lambda^j)\,(\lambda^k)\} = (\nabla_{f_*\,\partial/\partial\lambda^i}\, f_*\, {}^{\partial/}\partial\lambda^j)\,(f^k(\lambda^h))$$

Thus

$$L^s_{ij}\,\frac{\partial f^k}{\partial\lambda^s} = \frac{\partial f^s}{\partial\lambda^i}\left(\frac{\partial^2 f^k}{\partial\lambda^s\,\partial\lambda^j} + \frac{\partial f^h}{\partial\lambda^j}\,L^k_{hs}\right)$$

which gives $$\frac{\partial^2 f^k}{\partial\lambda^s\,\partial\lambda^j} = 0$$

and hence f is an affine transformation. Q.E.D.

Thus a complete, L.A. m-manifold can be considered as the quotient space of R^m by a subgroup of the affine group. This result was first proved by Auslander and Marcus in [1] using a different method.

§2.2 Locally Affine Foliations

Let M be a smooth m-manifold with a smooth r-dimensional (i.e. co-dimension m-r) foliation $\mathfrak{F}$.

Definition 2.2.1 A locally affine (L.A.) foliation on M is a triple

$(M,\mathcal{F},\Gamma)$ where Γ is a connexion in $C(M,\mathcal{F})$ which induces a locally affine structure on each leaf of $\mathcal{F}$. (See §1.5).

Several results, which generalise those in the previous section can be proved about such foliations.

Throughout what follows late greek suffices λ,μ,σ, etc. will denote integral values in the range $1,\dots,r$ early greek α,β,γ, etc. in the range $r+1,\dots,m$ and roman i,j,k, etc. in the whole range $1,\dots,m$.

The following result is due to A. G. Walker [35] and is quoted in a form suitable for use in the proof of the next theorem.

L E M M A 2.2.1. Let X_λ, $\lambda = 1,\dots, r$ be independent smooth vector fields defined on a neighbourhood U of a point $p \in M$, satisfying

$$[X_\lambda, X_\mu] = \Phi^\sigma_{\lambda\mu} X_\sigma$$

for some smooth $\Phi^\sigma_{\lambda\mu}$, and let ϕ_λ be r-smooth real valued functions defined on U. Then the system of equations

$$X_\lambda f = \phi_\lambda$$

for f admit a smooth solution on a neighbourhood $V \subset U$ of p if and only if

$$X_\lambda \phi_\mu - X_\mu \phi_\lambda = \Phi^\sigma_{\lambda\mu} \phi_\sigma$$

The next theorem is a direct generalisation of Lemma 2.1.1 and gives a local characterisation of an L.A. foliation.

THEOREM 2.2.1. Let $(M,\mathcal{F},\Gamma)$ be an L.A. foliation. Then there is an affine leaf atlas $\mathcal{A} = \{(U_a, x^i_a) : a \in J\}$ of charts for the foliation $\mathcal{F}$, such that in the overlap of two charts (U_a, x^i_a) and (U_b, x^i_b) the coordinates are related by equations of the form:

$$x_b^\lambda = A_\mu^\lambda(x_a^\beta)\ .\ x^\mu + B^\lambda(x_a^\beta)$$

$$x_b^\alpha = C^\alpha(x_a^\beta)$$

Furthermore the connexion coefficients Γ^i_{jk} satisfy $\Gamma^\lambda_{\mu\sigma} \equiv 0$ in each chart. Conversely, if M is paracompact and admits an affine leaf atlas $\mathcal{A}$ of the above form then there is a $\Gamma \in C(M,\mathcal{F})$ such that $(M,\mathcal{F},\Gamma)$ is an L.A. foliation and $\Gamma^\lambda_{\mu\sigma} \equiv 0$ in each chart of $\mathcal{A}$.

Proof

Let D be the tangent distribution to $\mathcal{F}$ and (U,x^i) a leaf chart from a smooth atlas. Let $p \in U$.

Consider the transversal neighbourhood at p consisting of those points of U whose coordinates satisfy $x^\lambda = 0$, $\lambda = 1,\dots,r$. W.

The smooth vector fields $\frac{\partial}{\partial x^\lambda}$, $\lambda = 1,\dots,r$ give a basis for D at each point of U, in particular along W.

The system of equations

$$\frac{\partial X^\mu_\lambda}{\partial x^\tau} + \Gamma^\mu_{\tau\sigma}\ (x^\theta, x^\alpha(w))\ X^\sigma_\lambda = 0 \qquad w \in W \tag{1}$$

with boundary condition $X^\mu_\lambda(0,x^\alpha(w)) = \delta^\mu_\lambda$, admits a unique solution $X^\mu_\lambda(x^\theta,x^\alpha(w))$ for each w because $R_{\mu\sigma\tau}^{\ \lambda} = 0$. Standard arguments (see for example [2]) show that the solution varies smoothly with x^i, if w is regarded as a parameter.

Thus one obtains smooth vector fields $X_\lambda = X^\mu_\lambda(x^i)\,\frac{\partial}{\partial x^\mu}$ $\lambda = 1,\dots,r$ on U. Furthermore

$$[X_\lambda, X_\mu] = \left(X^\tau_\lambda \frac{\partial X^\sigma_\mu}{\partial x^\tau} - X^\tau_\mu \frac{\partial X^\sigma_\lambda}{\partial x^\tau} \right) \frac{\partial}{\partial x^\sigma}$$

$$= X^\tau_\mu X^\theta_\lambda [\Gamma^\sigma_{\tau\theta} - \Gamma^\sigma_{\theta\tau}] \quad \text{from (1)}$$

$$= 0 \quad \text{since } \Gamma \text{ is torsion free.}$$

Thus by Lemma 2.2.1 there is a neighbourhood $V \subset U$ of p for which there are r smooth functions f^λ, $\lambda = 1,\ldots,r$ satisfying

$$X_\lambda f^\mu = \delta^\mu_\lambda \tag{2}$$

These functions are independent, for consider a functional relation $F(f^\lambda) = 0$. Then

$$0 \equiv X_\lambda F = \sum_{\mu=1,\ldots,r} \frac{\partial F}{\partial f^\mu} X_\lambda f^\mu = \frac{\partial F}{\partial f^\lambda} \quad \text{by (2)}$$

which is a contradiction.

Consider the transformation of coordinates defined by

$$y^\lambda = f^\lambda (x^i)$$

$$y^\alpha = x^\alpha$$

By suitably restricting the coordinate ranges one may obtain an open set $V' \subset V$ such that $p \in V'$, and (V', y^i) is a leaf chart.

From (2)

$$X^\theta_\lambda \frac{\partial y^\mu}{\partial x^\theta} = \delta^\mu_\lambda \quad \text{thus} \quad X^\theta_\lambda = \frac{\partial x^\theta}{\partial y^\lambda}$$

differentiating

$$0 = \frac{\partial X^\theta_\lambda}{\partial x^\tau} \cdot \frac{\partial y^\mu}{\partial x^\theta} + X^\theta_\lambda \frac{\partial^2 y^\mu}{\partial x^\tau \partial x^\theta}$$

$$= \left[-\Gamma^{\theta}_{\tau\sigma} \, X^{\sigma}_{\lambda} \frac{\partial y^{\mu}}{\partial x^{\theta}} + X^{\theta}_{\lambda} \frac{\partial^2 y^{\mu}}{\partial x^{\tau} \, \partial x^{\theta}} \right]$$

$$= \left[-\Gamma^{\theta}_{\tau\sigma} \frac{\partial x^{\sigma}}{\partial y^{\lambda}} \frac{\partial y^{\mu}}{\partial x^{\theta}} + \frac{\partial x^{\theta}}{\partial y^{\lambda}} \cdot \frac{\partial^2 y^{\mu}}{\partial x^{\tau} \, \partial x^{\theta}} \right]$$

which implies that $\bar{\Gamma}^{\mu}_{\lambda\sigma} = 0,$ (3)

where $\bar{\Gamma}^{i}_{jk}$ are the connexion coefficients of Γ with respect to the chart (V', y^{i}). Since p was arbitrary it is clear that one may cover M with such charts and thus generate a maximal atlas with the property (3). The affine nature of the coordinate transformations follows immediately from the transformation rule for the $\Gamma^{\mu}_{\lambda\sigma}$.

Conversely, let $\mathcal{A}$ be an affine leaf atlas. The assumption of paracompactness guarantees the existence of a postive definite riemannian metric on M (see [15]). Such a metric may be used to define a complementary distribution $\bar{D}$ to D.

The structure $(D,\bar{D})$ determines two smooth projector tensors a, $\bar{a}$ see §1.5.

Let $L \in C(M,\mathcal{F})$, and let (U,x^{i}) be a chart of $\mathcal{A}$. In this chart the components of a satisfy $a^{\lambda}_{\mu} = \delta^{\lambda}_{\mu}$, $a^{\alpha}_{i} = 0$. Define m^3 functions Γ^{i}_{jk} in each chart by

(i) $\Gamma^{\lambda}_{\mu\sigma} = 0$

(ii) $\Gamma^{\gamma}_{ij} = L^{\gamma}_{ij}$ (note that $L \in C(M,\mathcal{F})$ implies $L^{\alpha}_{i\lambda} = 0$).

(iii) $\Gamma^{\sigma}_{\alpha\lambda} = L^{\sigma}_{\alpha\lambda} - a^{\tau}_{\alpha} \, a^{\rho}_{\lambda} \, L^{\sigma}_{\tau\rho}$, $(= L^{\sigma}_{\lambda\alpha} - a^{\rho}_{\alpha} \, L^{\sigma}_{\lambda\rho})$.

(iv) $\Gamma^{\sigma}_{\alpha\beta} = L^{\sigma}_{\alpha\beta} - a^{\lambda}_{\alpha} \, a^{\mu}_{\beta} \, L^{\sigma}_{\lambda\mu}$.

To verify that these functions define a connexion on M, let (V,y^{i}) be another chart of $\mathcal{A}$ such that $U \cap V \neq \phi$ and let $\bar{L}^{i}_{jk}$ and $\bar{a}^{i}_{j}$ be the components of L and a in this chart. Then (i)' $$\frac{\partial y^\lambda}{\partial x^i} \cdot \frac{\partial x^j}{\partial y^\mu} \cdot \frac{\partial x^k}{\partial y^\sigma} \cdot \Gamma^i_{jk} + \frac{\partial^2 x^i}{\partial y^\mu \, \partial y^\sigma} \cdot \frac{\partial y^\lambda}{\partial x^i}$$

$$= \frac{\partial y^\lambda}{\partial x^\tau} \cdot \frac{\partial x^\theta}{\partial y^\mu} \cdot \frac{\partial x^\nu}{\partial y^\sigma} \cdot \Gamma^\tau_{\theta\nu} \quad \text{since} \quad \frac{\partial^2 x^\theta}{\partial y^\mu \, \partial y^\sigma} = 0 \, , \quad \frac{\partial x^\alpha}{\partial y^\mu} = 0$$

$$\text{and} \quad \Gamma^\beta_{\theta\nu} = L^\beta_{\theta\nu} = 0$$

$$= 0$$

$$\therefore \; \bar{\Gamma}^\lambda_{\mu\sigma} = 0,$$

(ii)' $$\bar{\Gamma}^\gamma_{ij} = \bar{L}^\gamma_{ij} = \frac{\partial y^\gamma}{\partial x^\alpha} \cdot \frac{\partial x^k}{\partial y^i} \cdot \frac{\partial x^h}{\partial y^j} \cdot L^\alpha_{kh} + \frac{\partial^2 x^\alpha}{\partial y^i \, \partial y^j} \cdot \frac{\partial y^\gamma}{\partial x^\alpha}$$

$$\text{since} \; \frac{\partial y^\gamma}{\partial x^\mu} = 0$$

But $L^\alpha_{kh} = \Gamma^\alpha_{kh}$

(iii)' $$\bar{\Gamma}^\sigma_{\alpha\lambda} = \bar{L}^\sigma_{\alpha\lambda} - \bar{a}^\tau_\alpha \, \bar{L}^\sigma_{\tau\lambda}$$

$$= \frac{\partial y^\sigma}{\partial x^\tau} \cdot \frac{\partial x^i}{\partial y^\alpha} \cdot \frac{\partial x^\mu}{\partial y^\lambda} L^\tau_{i\mu} + \frac{\partial^2 x^i}{\partial y^\alpha \, \partial y^\lambda} \cdot \frac{\partial y^\sigma}{\partial x^i}$$

$$- \left[\frac{\partial y^\tau}{\partial x^\nu} \cdot \frac{\partial x^\beta}{\partial y^\alpha} a^\nu_\beta + \frac{\partial y^\tau}{\partial x^\nu} \cdot \frac{\partial x^\nu}{\partial y^\alpha}\right]\left[\frac{\partial y^\sigma}{\partial x^\mu} \cdot \frac{\partial x^\theta}{\partial y^\tau} \cdot \frac{\partial x^\rho}{\partial y^\lambda} L^\mu_{\theta\rho}\right]$$

$$= \frac{\partial y^\sigma}{\partial x^\mu} \cdot \frac{\partial x^i}{\partial y^\alpha} \cdot \frac{\partial x^\tau}{\partial y^\lambda} \cdot L^\mu_{i\tau} - \frac{\partial y^\sigma}{\partial x^\mu} \, \frac{\partial x^\nu}{\partial y^\alpha} \cdot \frac{\partial x^\tau}{\partial y^\lambda} \cdot L^\mu_{\nu\tau}$$

$$- \frac{\partial y^\sigma}{\partial x^\mu} \cdot \frac{\partial x^\beta}{\partial y^\alpha} \cdot \frac{\partial x^\tau}{\partial y^\lambda} \cdot a^\nu_\beta \, L^\mu_{\nu\tau} + \frac{\partial^2 x^i}{\partial y^\alpha \, \partial y^\lambda} \cdot \frac{\partial y^\sigma}{\partial x^i}$$

$$= \frac{\partial y^\sigma}{\partial x^\mu} \cdot \frac{\partial x^\beta}{\partial y^\alpha} \cdot \frac{\partial x^\tau}{\partial y^\lambda} \left[L^\mu_{\beta\tau} - a^\nu_\beta \, L^\mu_{\nu\tau}\right] + \frac{\partial^2 x^i}{\partial y^\alpha \, \partial y^\lambda} \cdot \frac{\partial y^\sigma}{\partial x^i}$$

But from (iii) the bracketed term is $\Gamma^\mu_{\beta\tau}$

(iv)' $\bar{\Gamma}^{\sigma}_{\alpha\beta} = \bar{L}^{\sigma}_{\alpha\beta} - \bar{a}^{\lambda}_{\alpha}\, \bar{a}^{\mu}_{\beta}\, \bar{L}^{\sigma}_{\lambda\mu}$

$$= \frac{\partial y^{\sigma}}{\partial x^{i}} \cdot \frac{\partial x^{j}}{\partial y^{\alpha}} \cdot \frac{\partial x^{k}}{\partial y^{\beta}} \cdot L^{i}_{jk} + \frac{\partial^{2} x^{i}}{\partial y^{\alpha}\, \partial y^{\beta}} \cdot \frac{\partial y^{\sigma}}{\partial x^{i}}$$

$$- \left[\frac{\partial y^{\lambda}}{\partial x^{\phi}}\, \frac{\partial x^{\varepsilon}}{\partial y^{\alpha}}\, a^{\phi}_{\varepsilon} + \frac{\partial y^{\lambda}}{\partial x^{\phi}}\, \frac{\partial x^{\phi}}{\partial y^{\alpha}}\right] \left[\frac{\partial y^{\mu}}{\partial x^{\theta}} \cdot \frac{\partial x^{\gamma}}{\partial y^{\beta}}\, a^{\theta}_{\gamma} + \frac{\partial y^{\mu}}{\partial x^{\eta}} \cdot \frac{\partial x^{\eta}}{\partial y^{\beta}}\right] \left[\frac{\partial y^{\sigma}}{\partial x^{\nu}} \cdot \frac{\partial x^{\tau}}{\partial y^{\lambda}}\, \frac{\partial x^{\rho}}{\partial y^{\mu}}\, L^{\nu}_{\tau\rho}\right]$$

$$= \frac{\partial y^{\sigma}}{\partial x^{i}} \cdot \frac{\partial x^{j}}{\partial y^{\alpha}} \cdot \frac{\partial x^{k}}{\partial y^{\beta}} \cdot L^{i}_{jk} - \frac{\partial y^{\sigma}}{\partial x^{\nu}} \cdot \frac{\partial x^{\tau}}{\partial y^{\alpha}} \cdot \frac{\partial x^{\theta}}{\partial y^{\beta}}\, L^{\nu}_{\theta\tau} + \frac{\partial^{2} x^{i}}{\partial y^{\alpha}\, \partial y^{\beta}} \cdot \frac{\partial y^{\sigma}}{\partial x^{i}}$$

$$- a^{\tau}_{\varepsilon}\, a^{\theta}_{\gamma}\, L^{\nu}_{\tau\theta}\, \frac{\partial x^{\varepsilon}}{\partial y^{\alpha}} \cdot \frac{\partial y^{\sigma}}{\partial x^{\nu}}\, \frac{\partial x^{\gamma}}{\partial y^{\beta}}$$

$$- a^{\tau}_{\varepsilon}\, L^{\nu}_{\tau\theta}\, \frac{\partial x^{\varepsilon}}{\partial y^{\alpha}} \cdot \frac{\partial y^{\sigma}}{\partial x^{\nu}}\, \frac{\partial x^{\theta}}{\partial y^{\beta}}$$

$$- a^{\tau}_{\varepsilon}\, L^{\nu}_{\theta\tau}\, \frac{\partial x^{\theta}}{\partial y^{\alpha}} \cdot \frac{\partial y^{\sigma}}{\partial x^{\nu}} \cdot \frac{\partial x^{\varepsilon}}{\partial y^{\beta}}$$

$$= \frac{\partial y^{\sigma}}{\partial x^{\nu}} \cdot \frac{\partial x^{\varepsilon}}{\partial y^{\alpha}} \cdot \frac{\partial x^{\gamma}}{\partial y^{\beta}} \left[L^{\nu}_{\varepsilon\gamma} - a^{\tau}_{\varepsilon}\, a^{\theta}_{\gamma}\, L^{\nu}_{\tau\theta}\right] + \frac{\partial^{2} x^{i}}{\partial y^{\alpha}\, \partial y^{\beta}} \cdot \frac{\partial y^{\sigma}}{\partial x^{i}}$$

$$+ \frac{\partial y^{\sigma}}{\partial x^{\nu}} \cdot \frac{\partial x^{\theta}}{\partial y^{\alpha}} \cdot \frac{\partial x^{\varepsilon}}{\partial y^{\beta}} \left[L^{\nu}_{\theta\varepsilon} - a^{\tau}_{\varepsilon}\, L^{\nu}_{\theta\tau}\right]$$

$$+ \frac{\partial y^{\sigma}}{\partial x^{\nu}} \cdot \frac{\partial x^{\theta}}{\partial y^{\beta}} \cdot \frac{\partial x^{\varepsilon}}{\partial y^{\alpha}} \left[L^{\nu}_{\varepsilon\theta} - a^{\tau}_{\varepsilon}\, L^{\nu}_{\tau\theta}\right]$$

$$+ \frac{\partial y^{\sigma}}{\partial x^{\delta}}\, \frac{\partial x^{\gamma}}{\partial y^{\alpha}}\, \frac{\partial x^{\varepsilon}}{\partial y^{\beta}} \cdot L^{\delta}_{\gamma\varepsilon}$$

$$= \frac{\partial y^{\sigma}}{\partial x^{i}} \cdot \frac{\partial x^{j}}{\partial y^{\alpha}} \cdot \frac{\partial x^{k}}{\partial y^{\beta}} \cdot \Gamma^{i}_{jk} + \frac{\partial^{2} x^{i}}{\partial y^{\alpha}\, \partial y^{\beta}} \cdot \frac{\partial y^{\sigma}}{\partial x^{i}} \text{ as required.}$$

Thus the Γ^{i}_{jk} define a torsion free affine connexion Γ on M. The condition $\Gamma^{\alpha}_{i\lambda} = 0$ implied by (ii) ensures that $\Gamma \in C(M,\mathfrak{F})$. Furthermore since $\Gamma^{\lambda}_{\mu\sigma} = 0$, it follows that $R^{\lambda}_{\mu\sigma\tau} = 0$ and so Γ induces a L.A. structure on each leaf.

Thus $(M,\mathfrak{F},\Gamma)$ is an L.A. foliation. Q.E.D.

This result leads to the following conjecture.

CONJECTURE 2.2.1 *Let $(M,\mathcal{F})$ be a smooth foliation of a paracompact manifold M, in which each leaf is diffeomorphic to an L.A. manifold (where each leaf has the differentiable structure induced from a leaf atlas). Then there is a connexion Γ on M such that $(M,\mathcal{F},\Gamma)$ is an L.A. foliation.*

The following result shows that the conjecture is indeed true for the case of one dimensional foliations.

THEOREM 2.2.2. *Let M be a smooth paracompact m-manifold admitting a smooth foliation $\mathcal{F}$ of dimension r, with leaf atlas $\mathcal{A} = \{(U_a,x^i_a) : a \in A\}$. Then there is a sub-atlas $\mathcal{A}' \subset \mathcal{A}$ for which the partial r × r jacobian determinants $J_{ab} = \det\left(\frac{\partial x^\lambda_a}{\partial x^\mu_b}\right)$ are ±1 (+1 if J_{ij} is always positive).*

Proof

The assumption of paracompactness guarantees the existence of a positive definite metric g. Let D' be the tangent distribution to $\mathcal{F}$ and D" the orthogonal distribution. As before (D',D") determines smooth projectors a' and a". In the chart (U_a,x^i_a), D' is spanned by $\frac{\partial}{\partial x^\lambda_a}$ $\lambda = 1,\ldots,r$, D" by $\left(\frac{\partial}{\partial x^\alpha_a} - a'^\lambda_\alpha \frac{\partial}{\partial x^\lambda_a}\right)$ $\alpha = r+1,\ldots,m$.

For the cotangent bundle there are the corresponding dual bases $\omega^\lambda_a = dx^\lambda_a + a'^\lambda_\beta\, dx^\beta_a$, $\lambda = 1,\ldots,r$ and dx^α_a $\alpha = r+1,\ldots,m$ respectively.

Then it is easy to show that g has a line element of the form

$$ds^2 = g_{a\lambda\mu}\, \omega^\mu_a \omega^\lambda_a + g_{a\alpha\beta}\, dx^\alpha_a\, dx^\beta_a$$

where

$$g_{b\lambda\mu} = \frac{\partial x_a^\sigma}{\partial x_b^\lambda} \cdot \frac{\partial x_a^\tau}{\partial x_b^\mu} \cdot g_{a\sigma\tau} \tag{1}$$

on the overlap of U_a and U_b.

Moreover: $\det(g_{a\lambda\mu}) \neq 0$.

From (1) $|\det(g_{b\lambda\mu})| = J_{ab}^2 \; |\det(g_{a\lambda\mu})|$

Writing $J_a = \sqrt{|\det(g_{a\lambda\mu})|}$ it is clear that

$$J_{ab} = \pm \frac{J_b}{J_a} \tag{2}$$

Now re-choose coordinates as follows

$$y_a^1 = \int_0^{x_a^1} J_a(t, x_a^2, \ldots x_a^m)\, dt$$

$$y_a^2 = x_a^2$$

$$\vdots$$

$$y_a^m = x_a^m$$

Then: $$\det\left(\frac{\partial y_a^\lambda}{\partial y_b^\mu}\right) = \det\left(\frac{\partial y_a^\lambda}{\partial x_a^\nu} \cdot \frac{\partial x_a^\nu}{\partial x_b^\theta} \cdot \frac{\partial x_b^\theta}{\partial y_b^\mu}\right)$$

$$= J_a \cdot J_{ab} \cdot \frac{1}{J_b} = \pm 1$$

Since J_a is always positive, one must obtain +1 if J_{ab} is always positive.

By suitably restricting the coordinate ranges one can obtain an open set $V_a \subset U_a$ such that (V_a, y_a^i) is a leaf chart.

Such charts will generate the required leaf atlas. Q.E.D.

COROLLARY. <u>Any 1-dimensional foliation $\mathcal{F}$ on a paracompact manifold M admits an L.A. structure $(M, \mathcal{F}, \Gamma)$.</u>

Proof

From the theorem there is a leaf atlas $\mathcal{A}$ on M for which

$$\det\left(\frac{\partial x_a^1}{\partial x_b^1}\right) = \pm 1 = \frac{\partial x_a^1}{\partial x_b^1} .$$

Thus $\mathcal{A}$ is an affine leaf atlas, and so by theorem 2.2.1 there is a connexion Γ on M for which $(M,\mathcal{F},\Gamma)$ is an L.A. foliation. Q.E.D.

EXAMPLE 2.2.1 Affine Bundles

Let $\mathcal{B} = (E,\pi,B,F,G)$ be a fibre bundle in the sense of Steenrod [29], with total space E, projection π, base space B, fibre F and structure group G, with the following properties:

(i) E,B,F are smooth manifolds.

(ii) π is a smooth map.

(iii) There is a connexion L on F such that (F,L) is a complete locally affine manifold.

(iv) G is the lie transformation group of connexion preserving diffeomorphisms of (F,L) (see Nomizu [18]).

(v) There is an atlas of coordinate charts $\mathcal{A}(B) = \{(V_a,y_a^i) : a \in J\}$ on B and diffeomorphisms $\phi_a : V_a \times F \to \pi^{-1}(V_a)$ satisfying

(a) $\pi_o\phi_a(y,x) = y$ for all $(y,x) \in V_a \times F$

(b) if $\phi_{a,y} : F \to \pi^{-1}(y)$ is defined by

$$\phi_{a,y}(x) = \phi_a(y,x)$$

Then the diffeomorphism $\phi_{b,y}^{-1} \circ \phi_{a,y} : F \to F$ defined on $V_a \cap V_b$, coincides with the operation of an element of G.

(c) For each $a,b \in J$, the map $g_{ba} : V_a \cap V_b \to G$ defined by $g_{ba}(y) = \phi_{b,y}^{-1} \phi_{a,y}$ is smooth.

Such a bundle $\mathcal{B}$ will be called an affine bundle. Condition (v) shows that E admits a smooth foliation $\mathcal{F}$, the leaves of which are all diffeomorphic to F. Moreover the connexion L induces a connexion L(y) on each leaf $\pi^{-1}(y)$ via the maps ϕ_a which does not depend on the particular choice of ϕ_a.

If $\mathcal{A}(F)$ is an affine atlas on F for L, then the maps ϕ_a together with $\mathcal{A}(B)$ and $\mathcal{A}(F)$ give a leaf atlas for $\mathcal{F}$ such that the induced atlas on each leaf $\pi^{-1}(y)$ is an affine atlas for the connexion L(y). Hence by theorem 2.2.1 there is a connexion Γ on E which induces L(y) on each leaf $\pi^{-1}(y)$ and such that $(E, \mathcal{F}, \Gamma)$ is an L.A. foliation. Clearly Γ induces a complete L.A. structure on each leaf.

This motivates the following definition.

Definition 2.2.2 An L.A. foliation $(M, \mathcal{F}, \Gamma)$ is leaf-wise complete if Γ induces a complete connexion on each leaf. (Of course, if Γ is complete then $(M, \mathcal{F}, \Gamma)$ is necessarily leafwise complete).

It might be hoped that a leaf-wise complete L.A. foliation always admits an affine bundle structure. However, the next result shows that this is certainly not true in general, even for simply connected manifolds. Thus, theorem 2.1.1 does not generalize in this direction.

THEOREM 2.2.3. Any 1-dimensional foliation $\mathcal{F}$ on a compact manifold M admits a leaf-wise complete, L.A. structure $(M, \mathcal{F}, \Gamma)$.

Proof

By theorem 2.2.2 there is an atlas $\mathcal{A} = \{(U_a, x_a^i) : a \in J\}$ on M such that the leaves of $\mathcal{F}$ are given locally by

$$x_a^2, \ldots, x_a^m = \text{constant}$$

If $U_a \cap U_b \neq \phi$ then $\dfrac{\partial x_a^1}{\partial x_b^1} = \pm 1$.

By theorem 2.2.1 there is a connexion Γ on M such that $\Gamma_{11}^{1} = 0$ in each chart of $\mathcal{A}$ and such that $(M,\mathcal{F},\Gamma)$ is an L.A. foliation.

Clearly, if $(U_a,x_a^i) \in \mathcal{A}$ then $(U_b,x_b^i) \in \mathcal{A}$ where $U_a = U_b$, $x_a^i = -x_b^i$. $i = 1,\ldots,m$.

Consider the set $\tilde{M} = \left\{ (p,X(p)) : X(p) \in M_p,\ X(p) = \dfrac{\partial}{\partial x_a^1}(p) \text{ for some } a \in J \right\}$

Consider the subsets of $\tilde{M}$, $\tilde{U}_a = \left\{ (p, \dfrac{\partial}{\partial x_a^1}(p)) : p \in U_a \right\}$.

It is straight forward to check that these give a base for a topology on $\tilde{M}$ such that $\pi : \tilde{M} \to M$, defined by $\pi(p,X(p)) = p$, is a 2-fold covering map.

Let $S = \{(\tilde{U}_a,x_a^i) : a \in J\}$. This is a smooth coordinate cover for $\tilde{M}$ and generates a smooth atlas for which π is smooth.

Moreover, if $\tilde{U}_a \cap \tilde{U}_b \neq \phi$ then it is easy to see that

$$\frac{\partial x_a^1}{\partial x_b^1} = +1 \tag{1}$$

Hence S generates a smooth leaf atlas $\tilde{\mathcal{A}}$ (satisfying (1)) for a foliation $\tilde{\mathcal{F}}$ on $\tilde{M}$. Obviously, $\tilde{\mathcal{F}} = \pi^{-1}\mathcal{F}$.

The induced connexion $\tilde{\Gamma}$ on $\tilde{M}$ clearly satisfies $\tilde{\Gamma}_{11}^{1} = 0$ in each chart of $\tilde{\mathcal{A}}$, and makes $(\tilde{M},\tilde{\mathcal{F}},\tilde{\Gamma})$ an L.A. foliation. Also, $(\tilde{M},\tilde{\mathcal{F}},\tilde{\Gamma})$ is leaf-wise complete if and only if $(M,\mathcal{F},\Gamma)$ is leafwise complete. Since $\tilde{M}$ is a 2-fold cover of M it is compact.

Because of (1), the vector field $X = \dfrac{\partial}{\partial x_a^1}$ on $\tilde{U}_a$, is defined globally on $\tilde{M}$.

Since $\tilde{M}$ is compact, X is a complete vector field (see [15]), that is to say, there is a smooth map $\sigma : \tilde{M} \times R \to \tilde{M}$, such that $\sigma(\tilde{p},0) = \tilde{p}$ and

$T_{\sigma(\tilde{p},\ )}(0) = X(\tilde{p})$.

But, in a chart $(\tilde{U},x^i)$ for which $\tilde{p} \in \tilde{U}$, one has

$$\sigma^1(\tilde{p},t) = x^1(\tilde{p}) + t$$

$$\sigma^\alpha(\tilde{p},t) = \sigma^\alpha(\tilde{p},0) \qquad \alpha = 2,\dots,m.$$

But these are the geodesic equations for $\tilde{\Gamma}$ along the leaves. Thus geodesics in the leaves can be extended for all values of the parameter and so $(\tilde{M},\tilde{\mathfrak{F}},\tilde{\Gamma})$ is a leafwise complete. Q.E.D.

COROLLARY. <u>There is 1-dimensional leafwise complete L.A. foliation on the 3-sphere S^3 which does not admit an affine bundle structure</u>.

<u>Proof</u>

It is well known that there is a complementary vector field X to the 2-dimensional Reeb foliation of S^3(see [20]) which has some integral curves homeomorphic to R and at least one which is homeomorphic to S^1. Thus the foliation determined by X cannot admit a bundle structure of any type.

Q.E.D.

EXAMPLE 2.2.2

Although one may always find a leafwise complete structure on a compact M in this way it is not true that any given L.A. structure is necessarily complete. For instance, consider the Christofel connexion Γ, on the plane R^2 defined by the Riemann metric $ds^2 = dx^2 + e^y\, dy^2$. Here $\overset{1}{\Gamma}_{11} = \overset{1}{\Gamma}_{12} = \overset{1}{\Gamma}_{21} = \overset{1}{\Gamma}_{22} = \overset{2}{\Gamma}_{11} = \overset{2}{\Gamma}_{12} = \overset{2}{\Gamma}_{21} = 0$ and $\overset{2}{\Gamma}_{22} = \frac{1}{2}$. A short calculation shows that curvature and torsion tensors are zero. The metric is not complete since geodesics do not have infinite length. On the torus $(x,y)(\text{mod } 1)$, the connexion Γ can be projected since the coefficients Γ^i_{jk}

are periodic.

This defines a non-complete L.A. structure on the torus T^2. By taking the affine product with itself (see example 3.1.1) one obtains a trivial foliation of T^4 by the T^2 factors with a non-leafwise complete L.A. structure.

The next result throws some light on the behaviour of these foliations in the large.

THEOREM 2.2.4. *Let $(M, \mathcal{F}, \Gamma)$ be a leafwise complete, r-dimensional L.A. foliation with an atlas $\mathcal{A} = \{(U_a, x_a^i) : a \in J\}$ of affine leaf charts. If $V_a = \{p \in U_a : x_a^\lambda(p) = 0, \lambda = 1, \dots, r\}$. Then for each $a \in J$ there is a local diffeomorphism*

$$\underline{\xi_a : V_a \times R^r \to M}$$

such that

(a) *There is a neighbourhood W_a of $0 \in R^r$ for which*

$$\underline{\xi_a : V_a \times W_a \to U_a \text{ is a diffeomorphism and}}$$

$$\underline{\xi_a : V_a \times 0 \to V_a \text{ is the inclusion map.}}$$

(b) *For each $v \in V_a$, $\xi_a \mid v \times R^r \to [\text{leaf through } v]$ is a covering map.*

(c) *If $\pi_a : U_a \to V_a$ is the obvious projection, and if $U_a \cap U_b$ is non-null and connected, then there is a diffeomorphism.*

$$\underline{\eta_{ba} : \pi_a(U_a \cap U_b) \times R^r \to \pi_b(U_a \cap U_b) \times R^r}$$

such that: $\underline{\xi_b \circ \eta_{ba} = \xi_a}$

$$\underline{\eta_{aa} = \text{identity}}$$

$$\underline{\eta_{ab} = \eta_{ba}^{-1}}$$

$$\underline{\eta_{cb} \circ \eta_{ba} = \eta_{ca} \quad \text{if } U_c \cap U_b \cap U_a \neq \phi.}$$

Two lemmas are required:

LEMMA 2.2.2. Let N be a smooth manifold with a complete connexion then the map Exp : T(N) → N, defined by $(p,X(p)) \mapsto \exp_p X_p$ for each $p \in M$ is smooth.

Proof

The theory of ordinary differential equations (see [2] page 22) can be used to show that this is true locally, in the sense that there is a neighbourhood U of p and a neighbourhood W of the zero section in $T(N)|U$ such that Exp : $W \to U$ is smooth.

Let $\sigma : [0,1] \to N$ be a geodesic starting at p with initial vector $T_\sigma(0) = X(p)$. Then by definition $\text{Exp}\,(p,X(p)) = \sigma(1)$.

If $[0,t_1],\ldots,[t_k,1]$ is a subdivision of $[0,1]$

$$\text{then} \quad \sigma(1) = \text{Exp}\,(\sigma(t_k),\ (1-t_k)\ T_\sigma(t_k))$$

$$= \text{Exp}\,(\text{Exp}(\sigma(t_{k-1}),\ (t_k-t_{k-1})\ T_\sigma(t_{k-1})),\ (1-t_k)\ T_\sigma(t_k))$$

etc.

Thus, by choosing the subdivision in a suitable way it is clear that there is a neighbourhood of (p,X(p)) in T(N) such the Exp can be expressed as a composition of smooth maps, and hence is smooth. Q.E.D.

LEMMA 2.2.3. Let (N,Γ) be an L.A. n-manifold and (U,x^i) an affine chart on N. Let $e_1,\ldots,e_n$ be a basis for N_p $p \in U$, and $f_1,\ldots,f_n$ a basis for N_q, $q \in U$. If $e_1(z),\ldots,e_n(z)$ and $f_1(z),\ldots,f_n(z)$ are the corresponding bases at $z \in U$ obtained by parallel translation along paths in U, and if $e_i(z) = A_i^j\, f_j(z)$ then A_i^j does not depend on z.

Proof

if $e_i = e_i^j \, \partial/\partial x^j \, (p)$ and $f_i = f_i^j \, \partial/\partial x^j \, (q)$

then $e_i(z) = e_i^j \, \partial/\partial x^j \, (z)$ and $f_i(z) = f_i^j \, \partial/\partial x^j \, (z)$

clearly $A_i^s f_s^k = e_i^k$, and since (f_s^k) is invertible, the result follows

Q.E.D.

Let D be the tangent distribution to $\mathcal{F}$. Consider the map $\xi_a : V_a \times R^r \to M$ defined by $\xi_a(v,(X^1,\dots,X^r)) = \exp_v X^\lambda \frac{\partial}{\partial x_a^\lambda}(v)$. Clearly $\xi_a = \mathrm{Exp}\,|(D|V_a)$ and hence is smooth by lemma 2.2.2. To show that ξ_a is a local diffeomorphism let $\sigma(v) : [0,1] \to M$ be the geodesic starting at v with initial vector $X^\lambda \frac{\partial}{\partial x_a^\lambda}(v)$. Since the leaves are totally geodesic submanifolds, σ will lie entirely in the leaf through v.

There is a subdivision $[0,t_1],\dots,[t_b,t_{b+1}],\dots,[t_k,1]$ of $[0,1]$, and a cover $\{(U_b,x_b^i) : b = 0,1,\dots,k\}$ of $\sigma(v)([0,1])$ by charts of $\mathcal{A}$ such that $\sigma(v)([t_b,t_{b+1}]) \subset U_b$.

It is not difficult to show that in the chart U_k, $\sigma(v)$ has coordinates of the form

$$\sigma_k^\lambda(v)\,(t) = P_\mu^\lambda(v)\, X^\mu t + Q^\lambda(v)$$

$$\sigma_k^\alpha(v)\,(t) = F^\alpha(v)$$

where $(P_\mu^\lambda(v))$ and $\left(\frac{\partial F^\alpha}{\partial x_0^\beta}(v(x_0^\gamma))\right)$ are non-singular matrices. But $(\xi_a(v,(X^1,\dots,X^r)))^i = \sigma_k^i(1)$ with respect to (U_k,x_k^i) and hence $(\xi_a)_*$ is non-singular, i.e. ξ_a is a local diffeomorphism. If the X^λ are sufficiently small then $\sigma(v)\,(1)$ will lie in U_a with, coordinates

$$\sigma^\lambda(v)\,(1) = X^\lambda + \sigma^\lambda(v)\,(0)$$

$$\sigma^\alpha(v)\,(1) = \sigma^\alpha(v)\,(0)$$

then (a) follows with

$W_a = \{(X^1,\dots,X^r) \in R^r : d_a^\lambda - \sigma^\lambda(v)\ (0) < X^\lambda < c_a^\lambda + \sigma^\lambda(v)\ (0)\}$ (see definition 1.1.4).

By Theorem 2.1.1, $\exp_p |D(p) : D(p) \to$ [leaf through v] is a covering map, and so (b) follows.

Put $Q_a = V_a \times W_a \subset V_a \times R^r$, then $\xi_a(Q_a) = U_a$.

Suppose now that $U_a \cap U_b \neq \phi$ and is connected.

If $z \in Q_a$ and $\xi_a(z) \in U_a \cap U_b$ define

$$\eta_{ba}(z) = (\xi_b|Q_b)^{-1} \circ \xi_a(z) \tag{I}$$

Put $N_{ab} = (\xi_a|Q_a)^{-1}\ (U_a \cap U_b)$ then clearly $\eta_{ba} : N_{ab} \to N_{ba}$ is a diffeomorphism.

The idea now is to extend linearly along the R^r fibres using the L.A. structure on the leaves.

For convenience, let e_λ, $\lambda = 1,\dots,r$ be a basis for R^r so that $(X^1,\dots,X^r)$ may be represented as $X^\lambda\ e_\lambda$.

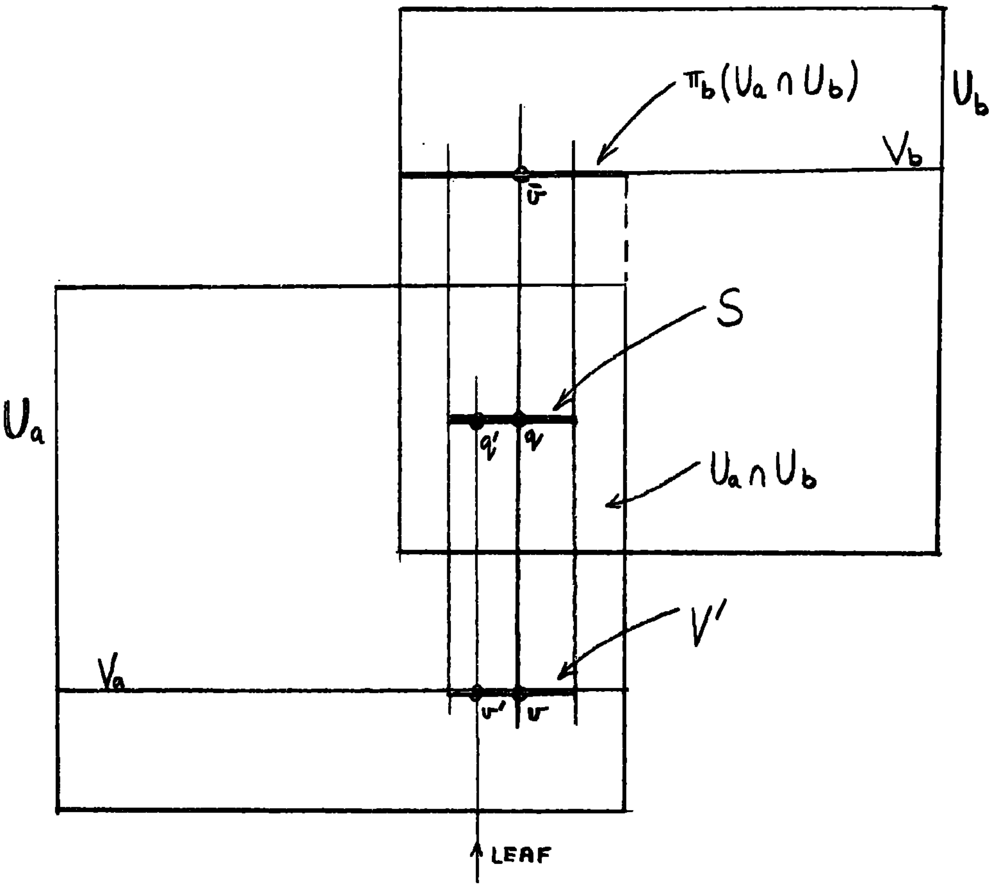
Suppose $z = (v, X_o^\lambda\ e_\lambda) \in N_{ab}$ and $\eta_{ba}\ (v, X_o^\lambda\ e_\lambda) = (\bar{v}, Y_o^\lambda\ e_\lambda)$, $\bar{v} \in V_b$. Let $q = \xi_a(z)$. Then $\frac{\partial}{\partial x_a^\lambda}(q)$, $\lambda = 1,\dots,r$ is the basis for $D(q)$ obtained from $\frac{\partial}{\partial x_a^\lambda}(v)$ by parallel translation within the leaf through v and $\bar{v}$ (in the plaque of U_a through q). This is because (U_a, x_a^i) is an affine leaf chart. Put

$$\frac{\partial}{\partial x_a^\lambda}(q) = A_\lambda^\mu \frac{\partial}{\partial x_b^\mu}(q) \tag{1}$$

(where (A_λ^μ) will be a non-singular $r \times r$ matrix).

By Lemma 2.2.3 A_λ^μ will not depend on the choice of $z \in Q_a$, provided q lies in the leaf through v and $\bar{v}$ (see picture) and hence is a function of v only. Now since N_{ab} is open, there is a neighbourhood V' of v in V_a such that the transverse neighbourhood S at q lies in $\xi_a N_{ab}$, where

$$S = \left\{ \begin{array}{l} q' \;\varepsilon\, \xi_a N_{ab} \;:\; x^\lambda_a(q') = x^\lambda_a(q) \;\; \lambda = 1,\ldots,r, \;\; x^\alpha_a(q') = x^\alpha_a(v') \;\; \alpha = r+1,\ldots,m \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad v' \;\varepsilon\; V' \end{array} \right\}$$



The equation $\frac{\partial}{\partial x^\lambda_a}(q') = A^\mu_\lambda(v') \,.\, \frac{\partial}{\partial x^\mu_b}(q')$ shows that $A^\mu_\lambda : V^1 \to R$ is smooth, and hence it is smooth on $\pi_a(U_a \cap U_b)$.

The domain of η_{ba} can be extended from N_{ab} to $\pi_a(U_a \cap U_b) \times R^r$ as follows. If

$$(v, X^\lambda_o\, e_\lambda) \;\varepsilon\; N_{ab}, \text{ and if } \eta_{ba}(v, X^\lambda_o\, e_\lambda) = (\bar{v}, Y^\lambda_o\, e_\lambda) \tag{2}$$

put

$$\eta_{ba}(v, Z^\lambda\, e_\lambda) = (\bar{v}, (Y^\lambda_o + A^\lambda_\mu(v)\{Z^\mu - X^\mu_o\})\, e_\lambda) \tag{II}$$

This does not depend on the choice of X_o because if

$$\eta_{ba}\,(v, X_1^\lambda\, e_\lambda) = (\bar{v}, Y_1^\lambda\, e_\lambda)$$

then

$$\exp_v\, X_1^\lambda \frac{\partial}{\partial x_a^\lambda}\,(v) = \exp_{\bar{v}}\, Y_1^\lambda \frac{\partial}{\partial x_b^\lambda}\,(\bar{v}) = q \quad \text{say.}$$

But

$$\exp_v (X_1^\lambda + Z^\lambda) \frac{\partial}{\partial x_a^\lambda}\,(v) = \exp_q\, Z^\lambda \frac{\partial}{\partial x_a^\lambda}\,(q) \quad \left(\begin{array}{l}\text{Since } v,q \text{ are in } U_a \text{ and } N_{ab}\\ \text{is connected.}\end{array}\right)$$

$$= \exp_q\, Z^\lambda\, A_\lambda^\mu(v) \frac{\partial}{\partial x_b^\mu}\,(q)$$

$$= \exp_{\bar{v}} (Y_1^\lambda + Z^\mu\, A_\mu^\lambda(v)) \frac{\partial}{\partial x_b^\lambda}\,(\bar{v})$$

$$\therefore\ \exp_v\, Z^\lambda \frac{\partial}{\partial x_a^\lambda}\,(v) = \exp_{\bar{v}} (Y_1^\lambda + A_\mu^\lambda(v)\{Z^\mu - X_1^\mu\}) \frac{\partial}{\partial x_b^\lambda}\,(\bar{v})$$

thus $\quad \eta_{ba}(v, Z^\lambda\, e_\lambda) = (\bar{v}, (Y_1^\lambda + A_\mu^\lambda(v)\{Z^\mu - X_1^\mu\})\, e_\lambda)$

and so η_{ba} is well defined.

By a similar argument one can show that definition (II) does in fact agree with definition (I) on N_{ab}.

η_{ba} is smooth because A_μ^λ is smooth, and is a diffeomorphism because the correspondence $v \to \bar{v}$ is a diffeomorphism and the correspondence $Z^\lambda \to Y_o^\lambda + A_\mu^\lambda(v)\ \{Z^\mu - X_o^\mu\}$ is a diffeomorphism. $R^r \to R^r$.

It is straight forward to show that $\xi_a = \xi_b \circ \eta_{ba}$.

Obviously η_{aa} = identity.

Suppose $U_a \cap U_b \cap U_c \neq \phi$ for $z \in Q_a$

$$\eta_{cb} \circ \eta_{ba} = \{(\xi|Q_c)^{-1} \circ \xi_b\} \circ \{(\xi|Q_b)^{-1} \circ \xi_a\}$$
$$= (\xi|Q_c)^{-1} \circ \xi_a = \eta_{ca}$$

By linearity it follows that

$$\eta_{cb} \circ \eta_{ba} = \eta_{ca} : \pi_a(U_a \cap U_b \cap U_c) \times R^r \to \pi_c(U_a \cap U_b \cap U_c) \times R^r$$

One can deduce immediately that $\eta_{ba}^{-1} = \eta_{ab}$. Q.E.D.

The maps ξ_a, η_{ba} closely resemble the structure one would expect from an affine bundle. However, since ξ_a is only a local diffeomorphism one cannot hope to obtain a bundle in general. It is hoped that this result may be used to study the existence or non-existence of codimension one, leafwise complete L.A. foliations on compact simply connected manifolds.

In Chapter 4 it will be seen that the foliation determined by a parallel field of null planes on a pseudo-riemannian manifold has an L.A. structure.

CHAPTER 3

Generalised Grid Manifolds

The work of this chapter has been inspired largely by the work of S. A. Robertson [22], S. Kashiwabara [13] and H. Wu [41]. The main structure theorem of §3.2 is due to Kashiwabara, although the proof given is more direct than the original.

§3.1 Equivalent Definitions

In [22], S. A. Robertson defined a grid as a set of complementary foliations, parallel with respect to a riemannian structure.

For our purposes it will suffice to consider only pairs of such foliations as all the results generalise easily to the more general situation. In Chapter 4 it will be seen that the following generalised definition of grid manifold reduces to Robertson's definition when the connexion is the Christoffel connexion of a riemannian metric.

Definition 3.1.1 A grid manifold $\mathcal{M} = (M, D_1, D_2, \Gamma)$ is a smooth m-manifold M, a pair of smooth complementary distributions D_1 and D_2 of dimensions $r > 0$ and $m-r > 0$ respectively, and a torsion free affine connexion Γ on M satisfying

(i) D_1 and D_2 are parallel.

(ii) If a_1 and a_2 are the projector tensors associated with the pair (D_1, D_2), and if R is the curvature tensor of Γ then $R(a_1X, a_2Y)Z = 0$ for all smooth vector fields X,Y,Z on M.

Condition (i) implies that D_1 and D_2 are integrable (by Lemma 1.5.1) and thus generate smooth foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ say, of dimensions r and (m-r) respectively.

This is essentially a local definition and hence it is not surprising that a grid manifold can be characterized by a special atlas of leaf charts in which the connexion coefficients Γ^i_{jk} have a special form.

As in previous chapters, late Greek suffices λ,μ,σ,τ etc. will denote integer values in the range $1,2,\ldots,r$, early Greek $\alpha,\beta,\gamma,\delta$ etc. in the range $r+1,\ldots,m$ and Roman i,j,k in the range $1,2,\ldots,m$.

THEOREM 3.1.1. Let $\mathcal{M} = (M,D_1,D_2,\Gamma)$ be a grid manifold. Then there is an atlas $\mathcal{A} = \{(U_a,z^i_a) : a \in J\}$ of coordinate charts on M such that on the overlap of two charts the coordinates z^i_a and z^i_b are related by equations of the form

$$\left.\begin{aligned} z^\lambda_b &= A^\lambda_{ba}\,(z^\mu_a) \\ z^\alpha_b &= B^\alpha_{ba}\,(z^\beta_a) \end{aligned}\right\} \qquad \text{(I)}$$

In the chart (U_a,x^i_a) the connexion coefficients satisfy

$$\Gamma^\lambda_{i\alpha} = \Gamma^\alpha_{\lambda i} = 0, \quad \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial z^\lambda} = \frac{\partial\Gamma^\lambda_{\mu\sigma}}{\partial z^\alpha} = 0$$

and D_1, D_2 are respectively spanned by $\dfrac{\partial}{\partial z^\lambda_a}$ $\lambda = 1,\ldots,r$ and $\dfrac{\partial}{\partial z^\alpha_a}$ $\alpha = r+1,\ldots,m$.

Conversely given a torsion free connexion Γ on M and an atlas $\mathcal{A}$ with the above properties then there are smooth distributions D_1 and D_2 for which (M,D_1,D_2,Γ) is a grid manifold.

Proof

By Lemma 1.5.1 there is an atlas $\mathcal{A}_1$ of leaf charts for $\mathcal{F}_1$ $\{(U_a,x^i_a)\}$ so that D_1 is spanned by $\dfrac{\partial}{\partial x^\lambda_a}$ $\lambda = 1,\ldots,r$ on U_a.

Similarly there is an atlas $\mathcal{A}_2 = \{(V_b,y^i_b)\}$ so that D_2 is spanned by $\dfrac{\partial}{\partial y^\alpha_b}$

$\alpha = r+1,\dots,m$.

Let $p \in M$ and U a neighbourhood of p on which coordinates x^i and y^i are defined, and for which each plaque of $\mathcal{F}_1$ intersects each plaque of $\mathcal{F}_2$ exactly once. There is no loss of generality in assuming that the charts have a common origin O.

For each $q \in U$, denote by $P(q)$, $Q(q)$ the plaques of $\mathcal{F}_1$ and $\mathcal{F}_2$ through q in U.

Define new coordinates z^i on U by

$$z^\lambda(q) = x^\lambda(Q(q) \cap P(O))$$

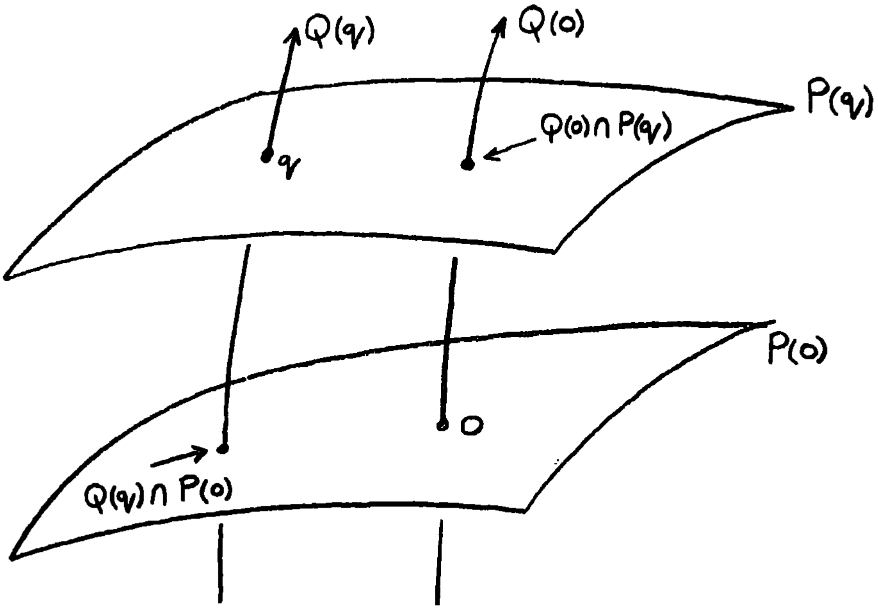
$$z^\alpha(q) = y^\alpha(Q(O) \cap P(q))$$

It is not difficult to prove that this defines a coordinate chart (U,z^i). Moreover, on U, D_1 is spanned by

$$\frac{\partial}{\partial z^\lambda}\ ,\ \lambda = 1,\dots,r \text{ and } D_2 \text{ by } \frac{\partial}{\partial z^\alpha}\ ,\ \alpha = r+1,\dots,m \qquad (1)$$

This procedure can be carried out for each $p \in M$ to obtain a cover $S = \{(W_c,z_c^i) : c \in J\}$ by such charts.

It follows immediately from (1) that in the overlap of two charts (W,z^i) and $(\bar{W},\bar{z}^i)$ of S the coordinates z^i and $\bar{z}^i$ are related by equations of the form (I).



D_1 is spanned by $\frac{\partial}{\partial z^\lambda}$ $\lambda = 1,\ldots,r$ and D_2 by $\frac{\partial}{\partial z^\alpha}$ $\alpha = r+1,\ldots,m$. With respect to z^i the projector tensors a_1 and a_2 have components

$$a_1{}^{\lambda}_{\mu} = \delta^{\lambda}_{\mu},\ a_2{}^{\alpha}_{\beta} = \delta^{\alpha}_{\beta},\ a_1{}^{\alpha}_{i} = 0,\ a_2{}^{\lambda}_{i} = 0$$

The parallelism of D_1 and D_2 implies that $\Gamma^{\alpha}_{i\lambda} = \Gamma^{\lambda}_{i\alpha} = 0$. Thus the curvature condition (ii) is equivalent to $R^{i}_{j\alpha\lambda} = 0$. Thus

$$0 = R^{\alpha}_{\beta\gamma\lambda} = \frac{\partial\ \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\lambda}} - \frac{\partial\ \Gamma^{\alpha}_{\beta\lambda}}{\partial x^{\gamma}} + \Gamma^{\alpha}_{i\lambda}\ \Gamma^{i}_{\beta\gamma} - \Gamma^{\alpha}_{i\gamma}\ \Gamma^{i}_{\beta\lambda}$$

$$= \frac{\partial\ \Gamma^{\alpha}_{\beta\gamma}}{\partial z^{\lambda}}$$

Similarly, one can deduce that $\frac{\partial\ \Gamma^{\lambda}_{\mu\theta}}{\partial z^{\alpha}} = 0$.

Thus the cover S will generate the required atlas.

Conversely given such an atlas, and torsion free connexion Γ the required smooth distributions D_1, D_2 are defined locally by $\frac{\partial}{\partial z^\lambda}$ $\lambda = 1,\ldots,r$, and $\frac{\partial}{\partial z^\alpha}$ $\alpha = r+1,\ldots,m$ respectively and the overlap equations (I) ensure that they are defined globally and are parallel.

Also, the atlas structure implies $R^{\alpha}_{\beta\lambda\gamma} = R^{\mu}_{\sigma\lambda\gamma} = 0$ and $R^{\mu}_{\beta\lambda\gamma} = R^{\beta}_{\mu\lambda\gamma} = 0$ and thus $R^{i}_{j\alpha\lambda} = 0$.

Hence (M,D_1,D_2,Γ) is a grid manifold. Q.E.D.

EXAMPLE 3.1.1 The Affine product (see [13]).

Let M,N be smooth manifolds of dimensions m and n carrying torsion free affine connexions Γ and L.

Let (U,x^λ) $\lambda = 1,\ldots,m$ be a coordinate chart on M and (V,y^α) $\alpha = 1,\ldots,n$ a chart on N.

If $E = M \times N$ is the smooth product then $(U\times V,\ (x^\lambda,y^\alpha))$ will be a coordinate chart on E. Such charts generate the product atlas on E.

Define $(m+n)^3$ functions P^i_{jk} on this chart by

$$P^\alpha_{\lambda\beta} = P^\mu_{\lambda\beta} = P^\alpha_{\lambda\mu} = P^\lambda_{\alpha\beta} = 0$$

$$P^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma},\ P^\lambda_{\mu\theta} = \Gamma^\lambda_{\mu\theta}$$

It is not difficult to show that P^i_{jk} give the connexion coefficients of a torsion free affine connexion P on E (defined globally by the product atlas).

Since $\dfrac{\partial P^\alpha_{\beta\gamma}}{\partial x^\lambda} = \dfrac{\partial P^\lambda_{\mu\theta}}{\partial y^\alpha} = 0$ it follows from Theorem 3.1.1 that P gives rise to a grid structure on E where the parallel distributions S_1 and S_2 are given by the product structure.

Let $\rho_1 : E \to M$, $\rho_2 : E \to N$ be the projection maps. It is clear that ρ_1 and ρ_2 are connexion preserving (see Definition 2.1.2).

Let $\sigma,\tau : [0,1] \to M,N$ be respectively geodesics on M and N, then $(\sigma,\tau) : [0,1] \to E$ will be a geodesic on E. Conversely if $h : [0,1] \to E$ is a geodesic on E then one can write $h = (\rho_1 h, \rho_2 h)$. Thus if Γ and L are complete then P will be complete.

This grid manifold will be denoted by $(M\times N, S_1, S_2, \Gamma\times L)$.

EXAMPLE 3.1.2. Take R^3 with coordinates (x,y,z). Let Γ be the complete, flat Christoffel connexion of the standard metric $ds^2 = dx^2 + dy^2 + dz^2$. The distributions D_1 and D_2 determined by the vector fields $(\partial/\partial x, \partial/\partial y)$ and $(\partial/\partial z)$ respectively are parallel. Since the curvature of Γ vanishes it is clear that (R^3, D_1, D_2, Γ) is a grid manifold.

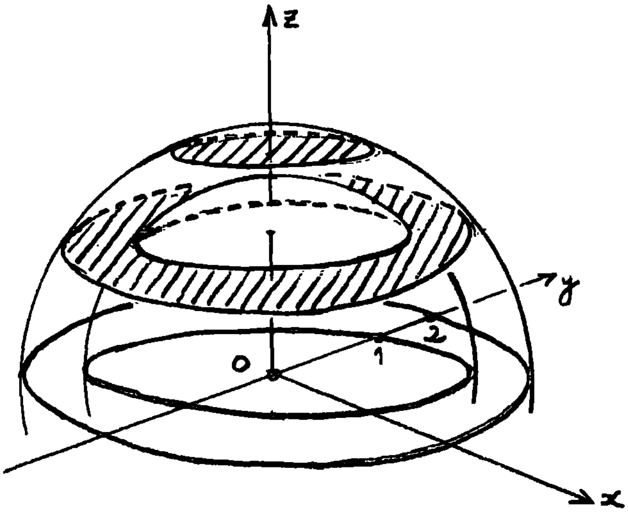
Consider the smooth embedding $f : S^2 \times (0,1) \to R^3$ defined as follows: Let $g : S^2 \to R^3$ be the standard embedding of the 2-sphere with radius 1 relative to the above metric

if $p \in S^2$ and $t \in (0,1)$, define $f(p,t)$ to be the point distant $(t+1)$

from the origin along the line joining the origin and g(p).

This embedding gives rise to a natural grid manifold structure on $S^2 \times (0,1)$ induced from that on R^3.

The foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on R^3 are given by the planes z = const and the lines x = const, y = const. The foliation on $S^2 \times (0,1)$ induced by $\mathcal{F}_1$ has leaves homeomorphic to R^2 and to $S^1 \times R$ and so the structure cannot arise from a product.



It should be noted that although the connexion Γ on R^3 is complete, the connexion induced on $S^2 \times (0,1)$ is not.

This example shows that, even in the case of simply connected manifolds, little can be deduced about the global structure of a grid without some extra conditions. In the next section it will be shown that if the connexion Γ is complete then $\mathcal{M} = (M, D_1, D_2, \Gamma)$ is covered by an affine product.

§3.2 Complete Grid Manifolds

Definition 3.2.1 A grid manifold $\mathcal{M} = (M, D_1, D_2, \Gamma)$ is complete if Γ is complete.

Definition 3.2.2 A grid morphism $f : \mathcal{M} \to \bar{\mathcal{M}}$ between two grid manifolds $\mathcal{M} = (M, D_1, D_2, \Gamma)$, $\bar{\mathcal{M}} = (\bar{M}, \bar{D}_1, \bar{D}_2, \bar{\Gamma})$ is a smooth connexion preserving map $f : M \to \bar{M}$ such that $D_1 = f^* \bar{D}_1$ and $D_2 = f^* \bar{D}_2$ as bundles (i.e. f preserves the foliations $\mathcal{F}_1$, $\bar{\mathcal{F}}_1$ and $\mathcal{F}_2$, $\bar{\mathcal{F}}_2$).

If in addition f is a diffeomorphism then f is a grid isomorphism.

THEOREM 3.2.1 [13]. Let $\mathcal{M} = (M, D_1, D_2, \Gamma)$ be a complete grid manifold for which M is connected and simply connected. Then $\mathcal{M}$ is grid isomorphic to an affine product.

Proof

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the foliations determined by the parallel distributions D_1 and D_2. Let $p \in M$ and suppose L_1 and L_2 are the leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ through p.

Let $\mathcal{A} = \{(U_a, x_a^i) : a \in J\}$ be the specially related atlas of leaf charts given by theorem 3.1.1. This atlas induces a smooth structure on L_1 and L_2 as submanifolds (via the leaf topology, see Definition 1.1.5) Γ induces connexions Γ_1 and Γ_2 on L_1 and L_2. Since L_1 and L_2 are totally geodesic (see §1.5) Γ_1 and Γ_2 will be complete.

Let $\tilde{L}_1$ and $\tilde{L}_2$ be the simply connected covers with smooth covering maps π_1 and π_2, and let $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ be the lifted (complete) connexions.

Consider $\tilde{\mathcal{M}} = (\tilde{L}_1 \times \tilde{L}_2, S_1, S_2, \tilde{\Gamma}_1 \times \tilde{\Gamma}_2)$ the affine product grid manifold. The idea now is to construct a grid morphism $f : \tilde{\mathcal{M}} \to \mathcal{M}$ for which $f : \tilde{L}_1 \times \tilde{L}_2 \to M$ is a covering map.

Let $\sigma, \tau : [0,1] \to$ be broken geodesics emanating from p and lying in L_1 and L_2 respectively.

Take a subdivision $[0 = t_0, t_1], \ldots, [t_a, t_{a+1}], \ldots, [t_{n-1}, t_n = 1]$ of $[0,1]$ for which $\sigma \mid [t_a, t_{a+1}]$ is a geodesic.

If σ passes through the chart (U, x^i) of $\mathcal{A}$ then the differential equations for σ reduce to

$$\left.\begin{aligned} \frac{d^2\sigma^\lambda}{dt^2} + \Gamma^\lambda_{\mu\theta}(\sigma^\tau)\,\frac{d\sigma^\mu}{dt}\cdot\frac{d\sigma^\theta}{dt} = 0 \\ \sigma^\alpha = \text{constant} \end{aligned}\right\} \tag{1}$$

Let X_a be the tangent vector at $\sigma(t_a)$ such that

$$\sigma(t) = \exp_{\sigma(t_a)}(t - t_a)\, X_a \quad \text{for } t \in [t_a, t_{a+1}]$$

(note that $X_a \in D_1\,(\sigma(t_a))$).

A broken geodesic $\bar{\sigma}$ corresponding to σ but emanating from $\tau(1)$ and lying in the leaf $\tilde{L}_1$ of $\mathcal{F}_1$ through $\tau(1)$, is now defined inductively.

Parallel translate X_0 along τ from $\tau(0)$ to $\tau(1)$. Denote by $Y_0(s)$ the vector so obtained at $\tau(s)$.

Locally, $Y_0(s)$ satisfies $\dfrac{d\,Y_0^i(s)}{ds} = 0$ (2)

since τ lies in a leaf of $\mathcal{F}_2$.

Define

$$\tau_1(s) = \exp_{\tau(s)} Y_0(s)$$

$$\bar{\sigma}(t) = \exp_{\tau(1)} t\, Y_0(1) \quad \text{for } t \in [0, t_1]$$

by virtue of equations (1) and (2) it is clear that τ_1 lies within a leaf of $\mathcal{F}_2$. Assume $\bar{\sigma} \mid [0, t_a]$ is defined and that $\tau_a : [0,1] \to M$ joining $\sigma(t_a)$ to $\bar{\sigma}(t_a)$, lying in a leaf of $\mathcal{F}_2$, is defined.

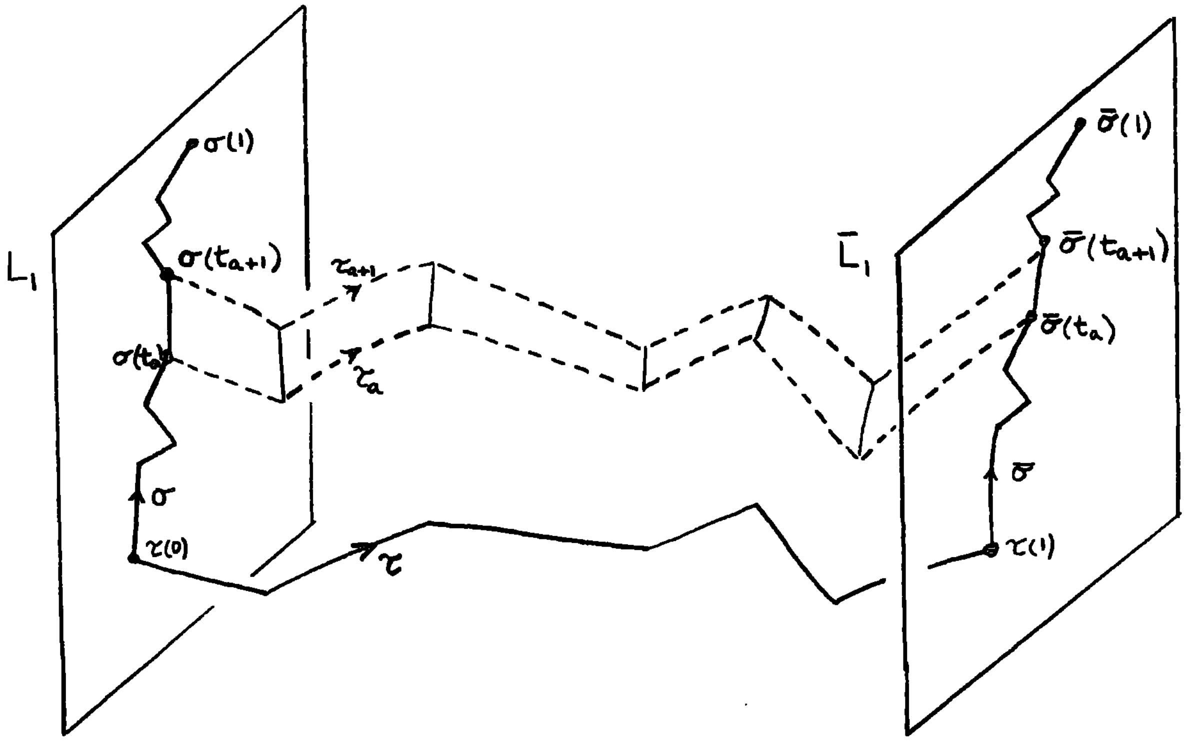
Denote by $Y_a(s)$ the vector at $\tau_a(s)$ obtained by translating X_a along τ_a. It satisfies (2) locally.

Define

$$\tau_{a+1}(s) = \exp_{\tau_a(s)} Y_a(s)$$

$$\bar{\sigma}(t) = \exp_{\tau_a(1)}(t-t_a).Y_a(1) \quad \text{for } t \in [t_a, t_{a+1}]$$

Again by virtue of equations (1), (2), τ_{a+1} will lie entirely in a leaf of $\mathcal{F}_2$. $\bar{\sigma} \mid [0, t_{a+1}]$ is clearly a broken geodesic. Thus by induction $\bar{\sigma}$ is defined on $[0,1]$.



Put $\bar{\sigma}(1) = F(\sigma(1), \tau(1))$.

F has the following property : If σ' and τ' are broken geodesics at p, lying in L_1 and L_2 respectively with $\sigma'(1) = \sigma(1)$, $\tau'(1) = \tau(1)$, and σ homotopic to σ', τ homotopic τ' relative to their respective end points, then $F(\sigma'(1), \tau'(1)) = F(\sigma(1), \tau(1))$.

Since equations (1), (2) do not depend on coordinates x^α $\alpha = r+1, \ldots, m$ it is clear that if σ' differs from σ only within a single chart of $\mathcal{A}$, then the property holds.

In general, if $H : [0,1] \times [0,1] \to L_1$ is a continuous map satisfying :

$$H(0,t) = \sigma(t)$$
$$H(1,t) = \sigma'(t)$$
$$H(s,0) = p$$
$$H(s,1) = \sigma(1) = \sigma'(1)$$

then, one may subdivide $[0,1]$ as $[0,u_1],\ldots,[u_{b,b+1}],\ldots,[u_N,1]$ so that $H([u_b,u_{b+1}] \times [u_c,u_{c+1}])$ is contained within a simple convex neighbourhood U of Γ_1 in L_1 with $U \subset U_a$, $(U_a,x_a^i) \in \mathcal{A}$. One can now use this subdivision to obtain a sequence $\sigma = \sigma_1,\sigma_2,\ldots,\sigma_i,\ldots\ \sigma_\ell = \sigma'$ of broken geodesics satisfying $\sigma_i(1) = \sigma(1) = \sigma'(1)$ for $i = 1,\ldots,\ell$ and such that σ_i differs from σ_{i+1} only within one chart of $\mathcal{A}$.

It follows by induction that $F(\sigma'(1),\tau(1)) = F(\sigma(1),\tau(1))$. But since the homotopy argument did not depend on τ

$$F(\sigma'(1),\tau'(1)) = F(\sigma(1),\tau'(1))$$

One may use similar arguments to show that $F(\sigma(1),\tau'(1)) = F(\sigma(1),\tau(1))$. Hence

$$F(\sigma'(1),\tau'(1)) = F(\sigma(1),\tau(1))$$

Now fix σ,τ. Let $(U_0,x_0^i),\ldots,(U_a,x_a^i),\ldots\ (U_d,x_d^i)$ be a cover of $\bar{\sigma}([0,1])$ by charts of $\mathcal{A}$ satisfying

(i) There exists a subdivision $[0,v_1],\ldots,[v_a,v_{a+1}],\ldots,[v_{d-1},1]$ of $[0,1]$ for which $\bar{\sigma}([v_a,v_{a+1}]) \subset U_a$.

(ii) For all $a = 1,\ldots,d$ $x_a^\alpha\ (\bar{\sigma}(t)) = 0$ if $t \in [v_a,v_{a+1}]$ $\alpha = r+1,\ldots,m$.

The piecewise smoothness and compactness of $\bar{\sigma}([0,1])$ guarantees the existence of such a cover.

Let P be the plaque of L_2 in U_1 through $\tau(1)$. Let $\bar{L}_2$ be the leaf of $\mathcal{F}_2$ which passes through $\bar{\sigma}(1)$, and $\bar{P}$ be the plaque of $\bar{L}_2$ through $\bar{\sigma}(1)$ in U_d. It may be assumed without loss of generality that P is given by $x_1^\lambda = 0 \quad \lambda = 1,\dots,r$ and $\bar{P}$ by $x_d^\lambda = 0,\ \lambda = 1,\dots,r$. Suppose

$$x^\alpha_{a+1} = B^\alpha_{a+1,a}\,(x^\beta_a) \tag{3}$$

on $\sigma([v_a,v_{a+1}]) \cap U_a \cap U_{a+1}$.

Then the map $(0,\dots,0,x_a^{r+1},\dots,x_a^m) \mapsto (0,\dots,0,B^{r+1}_{a+1,a}(x^\beta_a),\dots,B^m_{a+1,a}(x^\beta_a))$ defines a diffeomorphism of a neighbourhood of $q(x^i_a=0) \in U_a$ in the plaque given by $x^\lambda_a = 0,\ \lambda = 1,\dots,r$, onto a neighbourhood of $q(x^i_{a+1}=0) \in U_{a+1}$ in the plaque given by $x^\lambda_{a+1} = 0,\ \lambda = 1,\dots,r$.

Furthermore by (ii) $q(x^i_a=0) \mapsto q(x^i_{a+1}=0)$.

Then by an inductive argument one obtains a diffeomorphism ξ of a neighbourhood W of $\tau(1)$ in P onto a neighbourhood $\bar{W}$ of $\bar{\sigma}(1)$ in $\bar{P}$.

Let us suppose that with respect to the charts (U_1,x_1^i) and (U_d,x_d^i) that ξ has the form

$$(0,\dots,x_1^{r+1},\dots,x_1^m) \mapsto (0,\dots,0,\xi^{r+1}(x_1^\beta),\dots,\xi^m(x_1^\beta)).$$

By virtue of equations (1) it is clear that if $\tau' : [0,2] \to L_2$ is a broken geodesic satisfying $\tau' \mid [0,1] = \tau \mid [0,1]$ and $\tau'([1,2]) \subset W$ then $F(\sigma(1),\tau'(2)) = \xi(\tau'(2))$.

Since τ_n lies entirely in $\bar{L}_2$, one may do an exactly similar analysis to obtain a diffeomorphism η of a neighbourhood V of $\sigma(1)$ in L_1 onto a neighbourhood $\bar{V}$ of $\bar{\sigma}(1)$ in $\bar{L}_1$.

If (U_o,x_o^i) is a chart of $\mathcal{A}$ with $\sigma(1) \in U_o$ and $x^i_o(\sigma(1)) = 0 \quad i = 1,\dots,m$, then with respect to (U_o,x^i_o) and (u_d,x^i_d), η will have the form

$$(x_0^1,\dots,x_0^r,0,\dots,0) \mapsto (\eta^1(x_0^\mu),\dots,\eta^r(x_0^\mu),0,\dots,0)$$

Furthermore, if $\sigma' : [0,2] \to L_1$ is a broken geodesic satisfying $\sigma' \mid [0,1] = \sigma \mid [0,1]$ and $\sigma'([1,2]) \subset V$.

Then $F(\sigma'(2),\tau'(2)) = (\eta^1(\sigma'(2)),\dots,\eta^r(\sigma'(2)),\xi^{r+1}(\tau'(2)),\dots,\xi^m(\tau'(2))$ with respect to (U_d,x_d^i).

Thus for fixed σ,τ there is a diffeomorphism

$$g : V \times W \to U_d \subset M \text{ defined by}$$

$$g(a,b) = (\eta^1(a),\dots,\eta^r(a),\xi^{r+1}(b),\dots,\xi^m(b))$$

If $V \times W$ has the product connexion defined by Γ_1 and Γ_2 then equations (3), together with the corresponding equations used to define η, show that g is connexion preserving and foliation preserving.

Let $\rho_1 : \tilde{L}_1 \times \tilde{L}_2 \to \tilde{L}_1$, $\rho_2 : \tilde{L}_1 \times \tilde{L}_2 \to \tilde{L}_2$ be the projections.

Choose $p_1 \in \tilde{L}_1$ and $p_2 \in \tilde{L}_2$ such that $\pi_1 \circ \rho_1(p_1) = p = \pi_2 \circ \rho_2(p_2)$.

Let $\bar{p} = (p_1,p_2) \in \tilde{L}_1 \times \tilde{L}_2$.

Take any point $q \in \tilde{L}_1 \times \tilde{L}_2$ and let $h : [0,1] \to \tilde{L}_1 \times \tilde{L}_2$ be a broken geodesic from $\bar{p}$ to q (which always exists because any path from $\bar{p}$ to q can be covered by a finite number of simple convex neighbourhoods).

Define $f : \tilde{L}_1 \times \tilde{L}_2 \to M$ by

$$f(q) = F(\pi_1 \circ \rho_1 \circ h(1),\pi_2 \circ \rho_2 \circ h(1))$$

The various properties of $\pi_1,\pi_2,\rho_1,\rho_2,F$ and g show that this does not depend on the choice of h and is a smooth connexion preserving, foliation preserving, local diffeomorphism.

Thus by Lemma 2.1.2 f is a covering map.

Since M is simply connected it follows that f is a diffeomorphism and thus f : $\tilde{\mathcal{M}} \to \mathcal{M}$ is a grid isomorphism. Q.E.D.

This theorem shows that simple connectivity plus completeness is sufficient for a global product decomposition. Example 3.1.2 showed that the assumption of completeness cannot in general be dropped.

Thus the general problem of classifying complete grid manifolds reduces to an algebraic one, namely the classification of certain groups of covering transformations.

Let G be a properly discontinuous group of diffeomorphisms (see Spanier [27] page 87) of a smooth m-manifold M. Then one may take the quotient space M/G. M/G inherits a smooth hausdorf manifold structure from the quotient map $\rho : M \to M/G$. Furthermore, with respect to these structures ρ is a *regular* covering map (see [11] page 92) and if M is simply connected then $\pi_1(M/G) \cong G$.

If M has some geometric structure which is invariant under the action of G then there is a corresponding structure induced on M/G. Thus if $\mathcal{M} = (M, D_1, D_2, \Gamma)$ is a grid manifold and G is also a group of grid automorphisms then there is a grid structure on M/G for which the quotient map ρ induces a grid morphism. This grid manifold will be denoted by $\mathcal{M}/G$. This leads to the following result.

THEOREM 3.2.2. *Let $\mathcal{M} = (M, D_1, D_2, \Gamma)$ be a complete grid manifold. Then there is an affine product $\mathcal{N} = (M_1 \times M_2, S_1, S_2, \Gamma_1 \times \Gamma_2)$ and a properly discontinuous group G of grid automorphisms of $\mathcal{N}$ such that $\mathcal{M}$ is grid isomorphic to $\mathcal{N}/G$. Futhermore $\pi_1(M) \cong G$.*

Proof

Let $\tilde{M}$ be the simply connected cover of M. Then the grid structure on M lifts to one on $\tilde{M}$ in such a way that the covering transformations act as a properly discontinuous group G of grid automorphisms. But $G \cong \pi_1(M)$ and so the result follows by theorem 3.2.1. Q.E.D.

It is possible that a global product decomposition might result even if M is not simply connected.

Suppose that G decomposes as the direct product of two normal subgroups G_1 and G_2 such that for all $(x,y) \in M_1 \times M_2$

$$g \in G \Rightarrow g(x,y) = g_1 g_2(x,y) \quad g_1 \in G_1,\ g_2 \in G_2$$

$$\text{where } g_1(x,y) = (g_1'(x),y)$$

$$\text{and } \quad g_2(x,y) = (x,g_2'(y))$$

where g_i' $i = 1,2$ is a connexion preserving diffeomorphism of (M_i,Γ_i).
If $G_i' = < g_i' : g \in G>$ then G_i' will be a properly discontinuous group of diffeomorphisms of M_i.
It is not difficult to show that ${}^{M_1}/G_1' \times {}^{M_2}/G_2'$ admits an affine product structure $\mathcal{N}'$ induced from $\mathcal{N}$, and $\mathcal{M}$ is grid isomorphic to $\mathcal{M}'$.

Conversely if $\mathcal{M}$ is grid isomorphic to a product then G will factor in this way.

This motivates the definition (due to A. G. Walker [34]) of the multiplicity p(z) of the point $z \in M$ as the number of intersections of the leaf of $\mathcal{F}_1$ through z with the leaf of $\mathcal{F}_2$ through z. p(z) is obviously closely related to the action of G.

THEOREM 3.2.3. <u>Let $\mathcal{M} = (M,D_1,D_2,\Gamma)$ be a complete grid manifold. Then $\mathcal{M}$ is grid isomorphic to an affine product if and only if $p(z) = 1$ for all $z \in M$.</u>

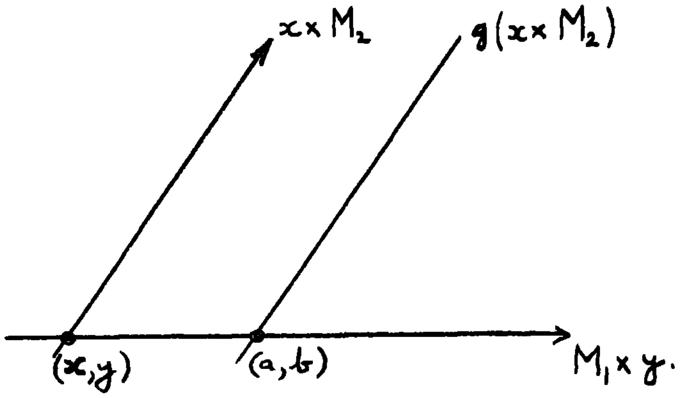
Proof necessity is obvious.

Let $\mathcal{N} = (M_1 \times M_2, S_1, S_2, \Gamma_1 \times \Gamma_2)$ be the covering affine product of theorem 3.2.2 and G the covering group of grid automorphisms.

Let $(x,y) \in M_1 \times M_2$ and $g \in G$.

Since G preserves the product structure one may write $g(x,y) = (A(x),B(y))$ where A,B are connexion preserving diffeomorphisms of M_1 and M_2 respectively.

If $\pi : M_1 \times M_2 \to M$ is the covering map then $p(\pi(x,y)) = 1$ tells us that if $(a,b) = g(x \times M_2) \cap (M_1 \times y)$ then there exists $g_1 \in G$ such that $g_1(x,y) = (a,b)$ ((a,b) will always exist because g preserves the product).



now, $(a,b) = (A(x),y)$

$$\therefore \quad g_1(x,y) = (A(x),y)$$

Put $G_1 = < g_1 : g \in G >$ then it is easy to show that G_1 is a normal subgroup of G.

Similarly, one may construct a normal subgroup G_2 such that

$$g = g_1 \circ g_2 = g_2 \circ g_1, \; g_1 \in G_1, \; g_2 \in G_2$$

The representation is obviously unique. Thus $G \cong G_1 \times G_2$ and so $\mathcal{M}$ is an affine product. Q.E.D.

EXAMPLE 3.2.1. A complete grid structure is now constructed on the torus T^2 which has infinite multiplicity and hence is not isomorphic to an affine product.

Take R^2 with coordinates (x,y). Let Γ be the Christoffel connexion of the standard complete metric $ds^2 = dx^2 + dy^2$. Clearly $\Gamma^i_{jk} \equiv 0$ and so Γ is invariant under the usual action of $Z \times Z$. Consider the distributions D_1 and D_2 spanned by the smooth vector fields $\frac{\partial}{\partial x} + \sqrt{2}\,\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x}$. These are parallel, and invariant under $Z \times Z$ and so give rise to a complete grid structure on $T^2 = R^2 / Z \times Z$.

Let L_1 and L_2 be the leaves on R^2 through $(0,0)$. L_1 is the line $y = \sqrt{2}\,x$ and L_2 is the line $y = 0$. If $(m,n) \in Z \times Z$ then $(m,n)(L_2)$ is the line $y = n$. This intersects L_1 at the point with coordinates $(n, {}^n/\sqrt{2})$. Since $\sqrt{2}$ is irrational, ${}^n/\sqrt{2}$ is never an integer and so there is no $g \in Z \times Z$ such that

$$(n, {}^n/\sqrt{2}) = g(0,0)$$

Furthermore it is easy to show that if $n \neq n'$ then there is no $g \in Z \times Z$ such that

$$(n, n'/\sqrt{2}) = g(n, n/\sqrt{2})$$

Thus if $\pi : R^2 \to T^2$ is the projection then $\pi(L_1)$ and $\pi(L_2)$ must intersect infinitely many times.

CHAPTER 4

Parallel Foliations on Pseudoriemannian Manifolds

§4.1 Pseudoriemannian Metrics

Let M be a smooth m-manifold. A riemannian metric g on M is a smooth symmetric tensor field of type (0,2) which is positive definite as a bilinear form on the tangent space at each point of M.
If the positive definite condition is relaxed to non-degeneracy then one obtains:

Definition 4.1.1. A pseudoriemannian metric g on M is a smooth symmetric tensor field of type (0,2) which is non-degenerate as a bilinear form on the tangent space at each point of M.

A pseudoriemannian manifold will be denoted by the pair (M,g).
Let $x \varepsilon M$, then the signature of g at x is the pair (k,m-k) where k is the number of negative eigenvalues of the bilinear form.
A simple continuity argument shows that the signature of g is constant over a neighbourhood of x and hence is constant on M if M is connected.
It is well known that a paracompact manifold always admits a riemannian metric. The situation in the pseudoriemannian case is more complicated.
The following result is proved in [29].

LEMMA 4.1.1. A compact smooth m-manifold admits a pseudoriemannian metric of signature (k,m-k) if and only if it admits a smooth k-dimensional distribution.

Hence the 2-sphere S^2 admits a riemannian but no pseudoriemannian structure of signature (1,1).

A pseudoriemannian manifold is said to be complete if the Christoffel connexion is complete.

Subspaces at a point.

Let $x \in M$ and suppose E_1 and E_2 are two vector subspaces of M_x. $E_1 \cap E_2$ will denote the intersection subspace and $E_1 + E_2$ the sum.

Then $\dim(E_1+E_2) = \dim(E_1) + \dim(E_2) - \dim(E_1 \cap E_2)$.

E_2 is said to be orthogonal to E_1 if $g(X,Y) = 0$ for every $X \in E_1$, $Y \in E_2$.

The conjugate subspace $E_\perp$ to a subspace E of M_x is defined as the collection of vectors which are orthogonal to every vector of E.

It can be shown that $\dim(E_\perp) = m - \dim(E)$. Clearly, if E_1 is orthogonal to E_2 then E_2 is orthogonal to E_1 and so $(E_\perp)_\perp = E$.

The null part $E_\cap$ of E is $E_\cap E_\perp$ and consists of vectors X for which $g(X,X) = 0$. If $E_\cap = \{0\}$ then E is said to be non-null. If $\dim(E_\cap) > 0$ then E is said to be partially null.

The subspace $E + E_\perp$ will be denoted by E_+. It is not difficult to prove that $E_+ = (E_\cap)_\perp$ and hence $\dim(E_+) = m - \dim(E_\cap)$.

Since $E_\perp$ contains $E_\cap$, it follows that $(m - \dim(E)) \geqslant \dim(E_\cap) \leqslant \dim(E)$.

Hence $\dim(E_\cap) \leqslant \frac{1}{2}m$.

Parallel Foliations.

Let Γ be the torsion free Christoffel connexion of g and suppose that $\mathcal{F}$ is a parallel foliation on M of dimension r in the sense of definition 1.5.2.

Denote the tangent distribution to $\mathcal{F}$ by $T\mathcal{F}$.

By taking the conjugate subspace at each point one obtains a conjugate distribution $(T\mathcal{F})_\perp$ say. $(T\mathcal{F})_\perp$ is a parallel distribution because parallel translation preserves orthogonality. The corresponding parallel foliation is denoted by $\mathcal{F}_\perp$. Parallel translation also preserves the null part of $T\mathcal{F}$ at each point and hence one can define $\mathcal{F}_\cap$ to be the foliation with tangent distribution $(T\mathcal{F}) \cap (T\mathcal{F}_\perp)$.

$\mathcal{F}$ will be called a parallel foliation of type (r,s) if $\dim(\mathcal{F}) = r + s$ and $\dim(\mathcal{F}_\cap) = r$. This implies that $\dim(\mathcal{F}_\perp) = m - r - s$ and $\dim(\mathcal{F}_+) = m - r$.

Definition 4.1.2. A parallel foliation of type (r,s) is said to be

non-null	if $r = 0$
partially null	if $r > 0$ and $s \geqslant 0$
null	if $s = 0$.

Clearly, $\mathcal{F}$ is non-null if and only if $\mathcal{F}_\perp$ is non null.

§4.2. Parallel Non-null Foliations

In this section an alternative proof of the De-Rham, Wu decomposition theorem [3], [41] is given, using theorem 3.2.1. The proof is simpler than that given by Wu in [41].

THEOREM 4.2.1. Let $\mathcal{F}$ be a parallel non-null foliation on a pseudoriemannian manifold (M,g). Then $\mathcal{M} = (M, T\mathcal{F}, T\mathcal{F}_\perp, \Gamma)$ is a grid manifold (see §3.1). Furthermore, each leaf of $\mathcal{F}$ has an induced pseudoriemannian structure.

Proof.

For convenience put $D = T\mathcal{F}$.

Since $D \cap D_\perp = \{0\}$ it is clear that D and $D_\perp$ are complementary.

Suppose $\dim(D) = r$. From the proof of theorem 3.1.1 there is a leaf atlas $\mathcal{A} = \{(U_a, x_a^i) : a \in J\}$ such that on U_a, D is spanned by $\dfrac{\partial}{\partial x_a^\lambda}$ $\lambda = 1,\ldots,r$ and $D_\perp$ is spanned by $\dfrac{\partial}{\partial x_a^\alpha}$ $\alpha = r+1,\ldots,m$. The orthogonality of D and $D_\perp$ implies that

$$g_{\lambda\alpha} = g_{\alpha\lambda} = g^{\alpha\lambda} = g^{\lambda\alpha} = 0 \tag{1}$$

in each chart. The parallelism of D and $D_\perp$ implies that

$$\Gamma^{\alpha}_{\lambda i} = \Gamma^{\lambda}_{\alpha i} = 0 \tag{2}$$

If a comma denotes partial derivative then from (1) and (2)

$$0 = \tfrac{1}{2}\, g^{\alpha\beta}\left[g_{\beta\lambda,i} + g_{\beta i,\lambda} - g_{\lambda i,\beta}\right]$$

$$\therefore \quad g_{\beta\alpha,\lambda} = g_{\lambda\mu,\beta} = 0 \tag{3}$$

Also

$$\Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2}\, g^{\alpha\varepsilon}\left[g_{\varepsilon\beta,\gamma} + g_{\varepsilon\gamma,\beta} - g_{\beta\gamma,\varepsilon}\right]$$

and thus from (3)

$$\Gamma^{\alpha}_{\beta\gamma,\lambda} = 0, \text{ similarly } \Gamma^{\lambda}_{\mu\theta,\alpha} = 0 .$$

Hence by theorem 3.1.1 $\mathcal{M} = (M, D, D_\perp, \Gamma)$ is a grid manifold.
The components $g_{\alpha\beta}$ $\alpha, \beta = 1, \ldots, r$ induce the required pseudoriemannian structure on the leaves of $\mathcal{F}$. Q.E.D.

In particular, if g is riemannian then any parallel foliation is non-null and thus the grid manifold definition due to S. A. Robertson [22] is a special case of definition 3.1.1.

EXAMPLE 4.2.1. <u>The Pseudoriemannian Product</u>.

Let (M,g) and (N,h) be pseudoriemannian manifolds of dimensions m and n. Consider the smooth product $P = M \times N$. Let (U,x^λ) and (V,y^α) $\lambda = 1,\dots,m$, $\alpha = 1,\dots,n$ be coordinate charts on M and N respectively. Then $(U\times V,(x^\lambda,y^\alpha))$ gives a chart of the product atlas on P.

Define $(m+n)^2$ functions k_{ij} in each such chart by

$$k_{\lambda\mu} = g_{\lambda\mu},\ k_{\alpha\beta} = h_{\alpha\beta},\ k_{\lambda\alpha} = k_{\alpha\lambda} = 0$$

These define a pseudoriemannian metric on P. Moreover the distribution D determined by the field $\frac{\partial}{\partial x^\lambda}$ $\lambda = 1,\dots,m$ in each product chart, is parallel and non-null.

By similar arguments to example 3.1.1 the structure is complete if and only if both (M,g) and (N,h) are complete.

LEMMA 4.2.1. (Wolf [40]). Let $f : M \to M'$ be a map of connected pseudoriemannian manifolds. Then the following are equivalent.

(i) f is an isometry.

(ii) f is connexion preserving and $f_* : M_x \to M'_{f(x)}$ is a linear isometry for every $x \in M$.

(iii) f is connexion preserving and there exists $x \in M$ for which $f_* : M_x \to M'_{f(x)}$ is a linear isometry.

Proof

(i) implies (ii) implies (iii) is trivial.

Assume (iii), given $z \in M$ choose a smooth path σ in M from x to z and let $\sigma' = f_o \sigma$. If T and T' denote parallel translation along σ and σ', then because f is connexion preserving $(f_*)_z : M_z \to M'_{f(z)}$ is given by $(f_*)_z = T'_o (f_*)_x {}_o T^{-1}$. But T and T' are linear isometries thus $(f_*)_z$ is a linear isometry. Thus (iii) → (i). Q.E.D.

THEOREM 4.2.2. Let (M,g) be a connected, simply connected, complete pseudoriemannian m-manifold which admits a parallel non-null foliation $\mathcal{F}$ of dimension r. Then there is a foliation preserving isometry from (M,g) onto the pseudoriemannian product of an r-manifold and an (m-r) manifold.

Proof

Let $x \in M$ and suppose M_1 is the leaf of $\mathcal{F}$ through x and M_2 is the leaf of $\mathcal{F}_\perp$ through x. By theorem 4.2.1 g induces metrics g_1 and g_2 say, on M_1 and M_2. The Christoffel connexions Γ_1 and Γ_2 determined by g_1 and g_2 are clearly the connexions induced on M_1 and M_2 by Γ.
Denote the pseudoriemannian product by $(M_1 \times M_2, g_1 \times g_2)$.
Obviously the Christoffel connexion of $g_1 \times g_2$ is $\Gamma_1 \times \Gamma_2$ (the affine product connexion). By theorem 3.2.1 $\mathcal{M} = (M, T\mathcal{F}, T\mathcal{F}_\perp, \Gamma)$ is grid isomorphic to $(M_1 \times M_2, S_1, S_2, \Gamma_1 \times \Gamma_2)$.
Hence there is a connexion preserving diffeomorphism.

$$f : (M_1 \times M_2, \Gamma_1 \times \Gamma_2) \to (M, \Gamma)$$

But f is clearly an isometry at the point $(x,x) \in M_1 \times M_2$ with respect to the metrics $g_1 \times g_2$ and g. Hence by Lemma 4.2.1 f is a global isometry.

Q.E.D.

COROLLARY Let (M,g) be a connected, complete pseudoriemannian m-manifold which admits a parallel non-null foliation. Then there is a pseudoriemannian product $(\tilde{M},\tilde{g})$ and a properly discontinuous group G of isometries of $(\tilde{M},\tilde{g})$ such that (M,g) is isometric to $(\tilde{M},\tilde{g})/G$. Furthermore $\pi_1(M) \cong G$.

Proof immediate.

The theorem determines completely the global structure of a parallel non-null foliation on a complete simply connected, pseudoriemannian manifold.

§4.3. Parallel Partially-null Foliations

Whereas the global structure for the non-null case is well understood, the situation for parallel partially-null foliations is far more complicated. The reason for this seems to be the loss of a local product structure. However, it will be seen that the null part of a parallel foliation is in fact a locally affine foliation in the sense of definition 2.2.1. This property is used to deduce several global results.

The next result is due to A. G. Walker [35], [36] and gives a local characterisation of the structure.

LEMMA 4.3.1. Let $\mathcal{F}$ be a parallel foliation of type (r,s) on a connected pseudoriemannian m-manifold (M,g). Then there is an atlas $\mathcal{A}$ of coordinate charts on M such that in each chart (U,x^i) the metric has the canonical form.

$$(g_{ij}) = \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & A & 0 & F \\ 0 & 0 & B & G \\ I & F' & G' & C \end{bmatrix}$$

where the non-zero submatrices satisfy the following conditions.

(i) I is the unit r × r matrix and A,B are non-singular and symmetric of of orders s × s and (n-2r-s) × (n-2r-s) respectively. C is symmetric of order r × r. F and G are of order s × r and (n-2r-s) × r respectively with transposes F' and G'.

(ii) A and F (and thus F') are independent of the coordinates $x^1, \dots, x^r$ and $x^{r+s+1}, \dots, x^{m-r}$; and B and G (and thus G') are independent of $x^1, \dots, x^r$ and $x^{r+1}, \dots, x^{r+s}$.

Furthermore the tangent distributions to $\mathcal{F}$, $\mathcal{F}_\cap$, $\mathcal{F}_\perp$, $\mathcal{F}_+$ are spanned respectively by $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{r+s}}\right)$, $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^r}\right)$, $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^{r+s+1}}, \dots, \frac{\partial}{\partial x^{m-r}}\right)$, $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{m-r}}\right)$.

Conversely, given such an atlas with a canonical form, then the above distributions are parallel.

Definition 4.3.1. An atlas $\mathcal{A}$ of the above form will be called a Walker atlas.

THEOREM 4.3.1. Let $\mathcal{F}$ be a parallel foliation on a pseudoriemannian m-manifold (M,g), then (M, $\mathcal{F}_\cap$, Γ) is an L.A. foliation (where Γ is the Christoffel connexion of g).

Proof

Let $\mathcal{A}$ be a Walker Atlas and (U, x^i), $(\bar{U}, \bar{x}^i)$ be overlapping charts. Then on the overlap $\bar{g}_{ij} = \frac{\partial x^p}{\partial \bar{x}^i} \cdot \frac{\partial x^q}{\partial \bar{x}^j} g_{pq}$.

If $\rho, \lambda, \mu, \theta, \tau \in (1, \dots, r)$; $\rho', \lambda', \mu', \theta', \tau' \in (m-r+1, \dots, m)$ then from the lemma

$$\bar{g}_{\lambda\mu'} = \frac{\partial x^p}{\partial \bar{x}^\lambda} \cdot \frac{\partial x^q}{\partial \bar{x}^{\mu'}} g_{pq} = \frac{\partial x^\theta}{\partial \bar{x}^\lambda} \cdot \frac{\partial x^{\tau'}}{\partial \bar{x}^{\mu'}} g_{\theta\tau'} \tag{1}$$

Differentiating

$$0 = \frac{\partial^2 x^\theta}{\partial \bar{x}^\rho \, \partial \bar{x}^\lambda} \cdot \frac{\partial x^{\tau'}}{\partial \bar{x}^{\mu'}} g_{\theta\tau'}$$

thus

$$\frac{\partial^2 x^\theta}{\partial \bar{x}^\rho \, \partial \bar{x}^\lambda} = 0$$

Furthermore $\Gamma^\mu_{\theta\lambda} = \frac{1}{2} g^{\mu i}(g_{i\theta,\lambda} + g_{i\lambda,\theta} - g_{\theta\lambda,i}) = 0$.

Thus by theorem 2.2.1 $(M, \mathcal{F}_\Lambda, \Gamma)$ is an L.A. foliation. Q.E.D.

In [37], E. M. Patterson and A. G. Walker exhibit a pseudoriemannian structure on the cotangent bundle of an affinely connected manifold. This structure makes the foliation determined by the vector space fibres, null and parallel.

A similar structure is now put on the sub-bundle of the cotangent bundle which is canonically determined by a foliation.

Definition 4.3.2. The Co-normal Bundle.

Let $\mathcal{F}$ be an arbitrary codimension p foliation on a manifold M and $\mathcal{A} = \{(U_a, x^i_a) : a \in J\}$ a leaf atlas for $\mathcal{F}$. The leaves are determined locally by x^α_a = const., $\alpha = m-p+1, \ldots, m$.

Consider the 1-forms $\omega^\alpha_a = dx^\alpha_a$. These span a p-dimensional subspace of the cotangent space of M at each point of U_a. Moreover, since $dx^\alpha_b = \left(\frac{\partial x^a_b}{\partial x^\beta_a}\right) . dx^\beta_a$ it is clear that this subspace does not depend on the particular choice of chart. Thus one obtains a smooth p-dimensional sub-bundle of the cotangent bundle T*M.

This is the co-normal bundle of $\mathcal{F}$ and is denoted by $\nu^* \mathcal{F}$.

THEOREM 4.3.2. Let $\mathcal{F}$ be a smooth codimension p foliation on a paracompact m-manifold M. Then there is a pseudoriemannian structure on the conormal bundle $\nu^* \mathcal{F}$ which makes the foliation by the vector space fibres, parallel and null.

Proof

Let $\mathcal{A}$ be a leaf atlas for $\mathcal{F}$ and $\Gamma \in C(M,\mathcal{F})$ (which is non-empty by theorem 1.5.1).

If $(U,x^i) \in \mathcal{A}$, then $\omega^\alpha = dx^\alpha$ $\alpha = m-p+1,\ldots,m$ span $\nu^*\mathcal{F}|U$.
Thus one may take coordinate functions (ξ_α,x^i) on $\nu^*\mathcal{F}|U$ where $v \in \nu^*\mathcal{F}(x)$ has coordinates $(\xi_1,\ldots,\xi_p,x^1(x),\ldots,x^m(x))$ and $v = \xi_\alpha\,\omega^\alpha(x)$.
Put $W = \nu^*\mathcal{F}|U$, then $(W,(\xi_\alpha,x^i))$ is a coordinate chart on $\nu^*\mathcal{F}$ (since W is diffeomorphic to $U\times R^p$). Such a chart will be called a bundle chart.
Let h be a positive definite metric on M. This can be used to determine a complementary distribution to $T\mathcal{F}$, and projector tensors a and I - a. With respect to (U,x^i), a has components $a^\lambda_\mu = \delta^\lambda_\mu$, $a^\alpha_i = 0$, a^λ_α and h has the form $ds^2 = h_{\lambda\mu}\,\omega^\lambda\,\omega^\mu + h_{\alpha\beta}\,dx^\alpha\,dx^\beta$ where $\omega^\lambda = dx^\lambda + a^\lambda_\alpha\,dx^\alpha$.
Define

$$\left.\begin{aligned} A &= (A_{\lambda\mu}) = (h_{\lambda\mu}) \\ H &= (H_{\lambda\alpha}) = (a^\mu_\alpha\, h_{\mu\lambda}) \\ B &= (B_{\alpha\beta}) = (-2\,\Gamma^\gamma_{\alpha\beta}\,\xi_\gamma + a^\mu_\alpha\, a^\lambda_\beta\, h_{\mu\lambda}) \end{aligned}\right\} \qquad (1)$$

Consider the following $(m+p)\times(m+p)$ matrix defined on $W \subset \nu^*\mathcal{F}$

$$(g_{rs}) = \begin{bmatrix} 0 & 0 & I \\ 0 & A & H \\ I & H' & B \end{bmatrix}$$

$r,s \in (1,\ldots,p,1,\ldots,m)$
where I is the unit
$p\times p$ matrix.

For (g_{rs}) to define a pseudoriemannian metric tensor globally on $\nu^*\mathcal{F}$ it is not difficult to show that the following conditions must be satisfied on on the overlap of two bundle charts $(W\,(\xi_\alpha,x^i))$, $(\bar{W},(\bar{\xi}_\alpha,\bar{x}^i))$. (For convenience α' will denote components with respect to ξ_α).

$$\text{(i)}\ \bar{g}_{\alpha\beta'} = g_{\alpha\beta'}$$

$$\text{(ii)}\ \bar{A}_{\lambda\mu} = \frac{\partial x^{\theta}}{\partial \bar{x}^{\lambda}} \cdot \frac{\partial x^{\tau}}{\partial \bar{x}^{\mu}} \cdot A_{\theta\tau} \quad \left[\text{note that } \frac{\partial \xi_{\alpha}}{\partial \bar{x}^{\lambda}} = 0\right]$$

$$\text{(iii)}\ \bar{H}_{\lambda\alpha} = \frac{\partial x^{\mu}}{\partial \bar{x}^{\lambda}} \left(\frac{\partial x^{\theta}}{\partial \bar{x}^{\alpha}} \cdot A_{\mu\theta} + \frac{\partial x^{\beta}}{\partial \bar{x}^{\alpha}} \cdot H_{\mu\beta}\right)$$

$$\begin{aligned}\text{(iv)}\ \bar{B}_{\alpha\beta} = \sum_{\varepsilon} &\left(\frac{\partial \xi_{\varepsilon}}{\partial \bar{x}^{\alpha}} \cdot \frac{\partial x^{\gamma}}{\partial \bar{x}^{\beta}} + \frac{\partial x^{\gamma}}{\partial \bar{x}^{\alpha}} \cdot \frac{\partial \xi_{\varepsilon}}{\partial \bar{x}^{\beta}}\right) g_{\varepsilon'\gamma} + \frac{\partial x^{\lambda}}{\partial \bar{x}^{\alpha}} \cdot \frac{\partial x^{\mu}}{\partial \bar{x}^{\beta}} \cdot A_{\lambda\mu} \\ &+ \left(\frac{\partial x^{\lambda}}{\partial \bar{x}^{\alpha}} \cdot \frac{\partial x^{\gamma}}{\partial \bar{x}^{\beta}} + \frac{\partial x^{\gamma}}{\partial \bar{x}^{\alpha}} \ \frac{\partial x^{\lambda}}{\partial \bar{x}^{\beta}}\right) H_{\lambda\gamma} + \frac{\partial x^{\gamma}}{\partial \bar{x}^{\alpha}} \cdot \frac{\partial x^{\varepsilon}}{\partial \bar{x}^{\beta}} \ B_{\gamma\varepsilon}\end{aligned}$$

These conditions are now verified in turn.

$$\text{(i) clearly } \xi_{\beta} = \bar{\xi}_{\alpha} \frac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}} \tag{2}$$

$$\text{Thus } \sum_{\gamma} \frac{\partial x^{\varepsilon}}{\partial \bar{x}^{\alpha}} \cdot \frac{\partial \xi_{\gamma}}{\partial \bar{\xi}_{\beta}} \cdot g_{\varepsilon\gamma'} = \frac{\partial x^{\gamma}}{\partial \bar{x}^{\alpha}} \cdot \frac{\partial \bar{x}^{\varepsilon}}{\partial x^{\gamma}} \cdot g_{\varepsilon\beta'} = g_{\alpha\beta'}$$

$$\text{(ii)}\ \bar{A}_{\lambda\mu} = \bar{h}_{\lambda\mu} = \frac{\partial x^{\theta}}{\partial \bar{x}^{\lambda}} \cdot \frac{\partial x^{\tau}}{\partial \bar{x}^{\mu}} \cdot h_{\theta\tau} = \frac{\partial x^{\theta}}{\partial \bar{x}^{\lambda}} \cdot \frac{\partial x^{\tau}}{\partial \bar{x}^{\mu}} \cdot A_{\theta\tau}$$

$$\begin{aligned}\text{(iii)}\ \bar{H}_{\lambda\alpha} = \bar{a}^{\mu}_{\alpha}\ \bar{h}_{\mu\lambda} &= \left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\theta}} \cdot \frac{\partial x^{\varepsilon}}{\partial \bar{x}^{\alpha}} \cdot a^{\theta}_{\varepsilon} + \frac{\partial \bar{x}^{\mu}}{\partial x^{\theta}} \cdot \frac{\partial x^{\theta}}{\partial \bar{x}^{\alpha}}\right) \left(\frac{\partial x^{\tau}}{\partial \bar{x}^{\lambda}} \cdot \frac{\partial x^{\rho}}{\partial \bar{x}^{\mu}} \cdot A_{\tau\rho}\right) \\ &= \frac{\partial x^{\tau}}{\partial \bar{x}^{\lambda}} \cdot \frac{\partial x^{\theta}}{\partial \bar{x}^{\alpha}} \cdot A_{\tau\theta} + \frac{\partial x^{\varepsilon}}{\partial \bar{x}^{\alpha}} \ \frac{\partial x^{\tau}}{\partial \bar{x}^{\lambda}} \cdot H_{\tau\varepsilon}\end{aligned}$$

$$\begin{aligned}\text{(iv)}\ \bar{B}_{\alpha\beta} &= -2\ \bar{\Gamma}^{\gamma}_{\alpha\beta}\ \bar{\xi}_{\gamma} + \bar{a}^{\mu}_{\alpha}\ \bar{a}^{\lambda}_{\beta}\ \bar{h}_{\mu\lambda} \\ &= -2\ \xi_{\delta} \left(\Gamma^{\delta}_{\varepsilon\gamma} \frac{\partial x^{\varepsilon}}{\partial \bar{x}^{\alpha}} \cdot \frac{\partial x^{\gamma}}{\partial \bar{x}^{\beta}} + \frac{\partial^{2}\ x^{\delta}}{\partial \bar{x}^{\alpha}\ \partial \bar{x}^{\beta}}\right) + \end{aligned}$$

$$+ \left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\theta}} \cdot \frac{\partial x^{\varepsilon}}{\partial \bar{x}^{\alpha}}\ a^{\theta}_{\varepsilon} + \frac{\partial \bar{x}^{\mu}}{\partial x^{\theta}} \cdot \frac{\partial x^{\theta}}{\partial \bar{x}^{\alpha}}\right) \left(\frac{\partial \bar{x}^{\lambda}}{\partial x^{\phi}} \ \frac{\partial x^{\delta}}{\partial \bar{x}^{\beta}}\ a^{\phi}_{\delta} + \frac{\partial \bar{x}^{\lambda}}{\partial x^{\phi}} \ \frac{\partial x^{\phi}}{\partial \bar{x}^{\beta}}\right) \frac{\partial x^{\sigma}}{\partial \bar{x}^{\mu}} \cdot \frac{\partial x^{\tau}}{\partial \bar{x}^{\lambda}} \cdot h_{\sigma\tau}$$

(noting that $\Gamma^{\alpha}_{\lambda i} = 0$ because $\Gamma \in C(M, \mathfrak{F})$).

$$\begin{aligned}&= \frac{\partial x^{\varepsilon}}{\partial \bar{x}^{\alpha}} \cdot \frac{\partial x^{\gamma}}{\partial \bar{x}^{\beta}} \left(a^{\sigma}_{\varepsilon}\ a^{\tau}_{\gamma}\ h_{\sigma\tau} - 2\ \xi_{\delta}\ \Gamma^{\delta}_{\varepsilon\gamma}\right) \\ &\quad + \left(\frac{\partial x^{\tau}}{\partial \bar{x}^{\beta}} \ \frac{\partial x^{\varepsilon}}{\partial \bar{x}^{\alpha}} + \frac{\partial x^{\tau}}{\partial \bar{x}^{\alpha}} \ \frac{\partial x^{\varepsilon}}{\partial \bar{x}^{\beta}}\right) a^{\sigma}_{\varepsilon}\ h_{\sigma\tau} + \frac{\partial x^{\sigma}}{\partial \bar{x}^{\alpha}} \cdot \frac{\partial x^{\tau}}{\partial \bar{x}^{\beta}}\ h_{\sigma\tau} \\ &\quad - 2\ \xi_{\delta} \frac{\partial^{2}\ x^{\delta}}{\partial \bar{x}^{\alpha}\ \partial \bar{x}^{\beta}}\end{aligned} \tag{3}$$

now, from (2)

$$\frac{\partial \xi_\varepsilon}{\partial \bar{x}^\alpha} = \sum_\beta \xi_\beta \, g_{\delta\beta'} \frac{\partial x^\gamma}{\partial \bar{x}^\alpha} \cdot \frac{\partial^2 \, \bar{x}^\delta}{\partial x^\gamma \, \partial x^\varepsilon}$$

thus

$$\sum_\varepsilon \frac{\partial \xi_\varepsilon}{\partial \bar{x}^\alpha} \cdot \frac{\partial x^\gamma}{\partial \bar{x}^\beta} \cdot g_{\varepsilon'\gamma} = \frac{\partial x^\varepsilon}{\partial \bar{x}^\beta} \cdot \frac{\partial x^\gamma}{\partial \bar{x}^\alpha} \cdot \frac{\partial^2 \, \bar{x}^\delta}{\partial x^\gamma \, \partial x^\varepsilon} \cdot \bar{\xi}_\delta$$

and by differentiating $\delta^\gamma_\beta = \dfrac{\partial x^\delta}{\partial \bar{x}^\beta} \cdot \dfrac{\partial \bar{x}^\gamma}{\partial x^\delta}$ one obtains

$$\xi_\varepsilon \cdot \frac{\partial^2 \, x^\varepsilon}{\partial \bar{x}^\alpha \, \partial \bar{x}^\beta} = - \frac{\partial x^\gamma}{\partial \bar{x}^\beta} \cdot \frac{\partial x^\varepsilon}{\partial \bar{x}^\alpha} \cdot \frac{\partial^2 \, \bar{x}^\delta}{\partial x^\varepsilon \, \partial x^\gamma} \cdot \bar{\xi}_\delta$$

Hence

$$- 2 \, \xi_\delta \frac{\partial^2 \, x^\delta}{\partial \bar{x}^\alpha \, \partial \bar{x}^\beta} = \sum_\varepsilon \left(\frac{\partial \xi_\varepsilon}{\partial \bar{x}^\alpha} \cdot \frac{\partial x^\gamma}{\partial \bar{x}^\beta} + \frac{\partial x^\gamma}{\partial \bar{x}^\alpha} \cdot \frac{\partial \xi_\varepsilon}{\partial \bar{x}^\beta} \right) g_{\varepsilon'\gamma}$$

Substituting in (3) one obtains the required form for $\bar{B}_{\alpha\beta}$.

Thus (g_{rs}) defines a pseudo-riemannian metric on $\nu^* \mathcal{F}$. By Lemma 4.3.1 it makes the foliation by the vector space fibres, parallel and null. Q.E.D.

This theorem enables one to construct on simply connected manifolds, parallel foliations which do not admit a global product structure.

For instance the cotangent bundle of S^2, T^*S^2 is a simply connected 4-manifold with such a metric. However T^*S^2 is not homeomorphic to $S^2 \times R^2$.

Also, by considering the co-normal bundle of the 2-dimensional Reeb foliation of S^3 (see [20]) one can use the theorem to obtain a metric on $R \times S^3$ which makes the 1-dimensional foliation by the R factors, parallel and null. The conjugate 3-dimensional foliation does not even admit a fibred structure, because the Ehresmann holonomy group of at least one leaf is non-trivial.

Definition 4.3.3. A pseudoriemannian co-normal bundle (E,g) is a co-normal bundle E and a pseudoriemannian metric g which makes the foliation by the vector space fibres, parallel and null.

It might be hoped that, just as the pseudoriemannian product was the 'canonical' example for parallel non-null foliations, so the co-normal bundle might be the 'canonical' example for parallel null foliations. However, little appears to be known on the subject. In the next section some special cases are discussed.

The next result is due to S. A. Robertson.

THEOREM 4.3.3. Let $\mathfrak{F}$ be a parallel foliation of type (r,s) on a connected, pseudoriemannian m-manifold M. Then there is a natural vector bundle isomorphism $f : T\mathfrak{F}_\wedge \to \nu^* \mathfrak{F}_+$.

Proof

Let $\mathfrak{A} = \{(U_a, x_a^i) : a \in J\}$ be a Walker atlas for $\mathfrak{F}$. Then $\nu^* \mathfrak{F}_+$ is spanned on U_a by the 1-forms $dx_a^{m-r+1}, \ldots, dx_a^m$ and $T\mathfrak{F}_\cap$ is spanned by the vector fields $\frac{\partial}{\partial x_a^1}, \ldots, \frac{\partial}{\partial x_a^r}$.

Define f on $T\mathfrak{F}_\cap | U_a$ by $f\left(X^\lambda \frac{\partial}{\partial x_a^\lambda}\right) = X^\lambda g_{\lambda\alpha} dx_a^\alpha$ where α runs from (m-r+1) to m and λ from 1 to r.

From Lemma 4.3.1 $(g_{\lambda\alpha})$ is the unit r × r matrix.

If $U_a \cap U_b \neq \phi$ then

$$f\left(X^\lambda \frac{\partial}{\partial x_b^\lambda}\right) = f\left(X^\lambda \frac{\partial x_a^\mu}{\partial x_b^\lambda} \cdot \frac{\partial}{\partial x_a^\mu}\right) = X^\lambda \frac{\partial x_a^\mu}{\partial x_b^\lambda} \cdot g_{\mu\beta}\, dx_a^\beta$$

$$= X^\lambda \frac{\partial x_a^\mu}{\partial x_b^\lambda} \cdot \frac{\partial x_a^\beta}{\partial x_b^\alpha} \cdot g_{\mu\beta}\, dx_b^\alpha = X^\lambda g_{\lambda\alpha}\, dx_b^\alpha$$

and thus f does not depend on the particular chart used and so is defined globally. It is easy to see that f is a vector bundle isomorphism. Q.E.D.

COROLLARY. *Let $\mathfrak{F}$ be a parallel null foliation of dimension m on a paracompact connected pseudoriemannian 2m-manifold M. Then*

(i) *$TM \cong T\mathfrak{F} \oplus T\mathfrak{F}$ (Whitney sum).*

(ii) *M admits an almost complex structure.*

(iii) *The Stiefel Whitney classes of M are given by*

$W_{2i+1}(M) = 0, \; W_{2i}(M) = (W_i(T\mathfrak{F}))^2.$

Proof

Since $\mathfrak{F} = \mathfrak{F}_{\cap}$ it follows that $T\mathfrak{F} \cong \nu^* \mathfrak{F}$.

But $TM \cong T\mathfrak{F} \oplus \nu\mathfrak{F}$ where $\nu\mathfrak{F}$ is any normal bundle (determined by some positive definite metric).

Also $\nu\mathfrak{F} \cong \nu^*\mathfrak{F}$ and hence $TM \cong T\mathfrak{F} \oplus \nu^*\mathfrak{F} \cong T\mathfrak{F} \oplus T\mathfrak{F}$.

The almost complex structure J is defined by

$$J(a,b) = (-b,a)$$

(iii) follows directly from the product formula $W(\lambda \oplus \mu) = W(\lambda).W(\mu)$. (See [16]). Q.E.D.

As a consequence of theorem 4.3.2 and this corollary it follows that the cotangent bundle (and hence the tangent bundle) of a paracompact manifold admits an almost complex structure.

§4.4 Submersions

Definition 4.4.1. A submersion $f : M \to N$ between two smooth manifolds is a smooth surjective map such that f_* is surjective on each tangent space. N will be called the base of the submersion.

In this section some global results will be obtained about parallel foliations by assuming that there is a submersion f for which $T\mathcal{F}_\wedge(x) = \text{kernal } (f_*)(x)$ i.e. the inverse image of a point of N is a union of leaves of $\mathcal{F}_\wedge$.

In the corollary to theorem 2.1.1 it was proved that complete L.A. manifolds could be considered as the quotient space of R^m by a group of transformations contained in the affine group A(m;R).

Definition 4.4.2. A euclidean cylinder is a complete L.A. manifold for which the group of covering transformations is a group of translations.

THEOREM 4.4.1. Let $\mathcal{F}$ be a parallel foliation of type (r,0), given by a submersion, on a complete, connected, pseudoriemannian m-manifold (M,g). Then each leaf of $\mathcal{F}$ with the induced connexion is affinely equivalent to a euclidean cylinder.

Proof

Let $f : M \to N$ be the submersion and $\mathcal{A}$ a Walker atlas for $\mathcal{F}$. Let L be a leaf of $\mathcal{F}$ and $w = f(L) \in N$. Take any point $p \in L$ and let $(U,x^i) \in \mathcal{A}$ such that $p \in U$.

Put $V = \{p' \in U : x^\lambda(p') = x^\lambda(p), \lambda = 1,\ldots,r\}$. Because kernal $(f_*)(p) = T\mathcal{F}(p)$ it follows that there is a neighbourhood $U' \subset U$ of p such that

$$f : V' \to f(V') \text{ is a diffeomorphism.}$$

Let $W = f(V')$. Let E = union of leaves of $\mathcal{F}$ through V'. Then $f(E) = W$ $\therefore$ E is open. Now because each leaf of $\mathcal{F}_{\perp}$ is a union of leaves of $\mathcal{F}_{\wedge}$ it follows that $f|E$ induces a foliation $\mathcal{G}$ say on W where $T\mathcal{G}(f(x)) = f_*(T\mathcal{F}_{\perp}(x))$ for $x \in E$.

Define coordinates y^i $i = 1,\dots,m-r$ on W by

$$y^1(z) = x^{r+1}((f|V')^{-1}(z))$$
$$\cdot$$
$$\cdot$$
$$y^{m-r}(z) = x^m((f|V')^{-1}(z))$$

It is clear that (W,y^i) is a leaf chart for $\mathcal{G}$ (leaves are $y^{m-2r},\dots,y^{m-r}$=constant).

Now, let q be any other point of L and $(U_o,x_o^i) \in \mathcal{A}$ such that $q \in U_o$. Then there is $U_o' \subset U_o$ such that

$$f : V_o' \to f(V_o') \subset W \text{ and is a diffeomorphism.}$$

Change to new coordinates $\bar{x}_o^{r+1},\dots,\bar{x}_o^m$ by the rule

$$\bar{x}_o^{r+1}(q') = y^1(f(q'))$$
$$\cdot$$
$$\cdot$$
$$\bar{x}_o^m(q') = y^{m-r}(f(q')),\ q' \in U_o'$$

It follows directly from Walker's original construction (see [35] Theorem 1) that one may find coordinates $\bar{x}_o^1,\dots,\bar{x}_o^r$ (defined in terms of $x_o^1,\dots,x_o^r$ and $\bar{x}_o^{r+1},\dots,\bar{x}_o^m$) and a neighbourhood $U_o'' \subset U_o'$ of q such that $(U_o'',\bar{x}_o^i) \in \mathcal{A}$.

Let $S = \{(U_a, x_a^i) : a \in J\}$ be a cover of L by such charts. From the construction it is clear that if $U_a \cap U_b \neq \phi$ then

$$\frac{\partial x_a^\alpha}{\partial x_b^\beta} = \delta_\beta^\alpha , \quad \alpha,\beta = r+1,\ldots,m$$

Thus from equation (1) of theorem 4.3.1 it follows that $\frac{\partial x_a^\lambda}{\partial x_b^\mu} = \delta_\mu^\lambda$ $\lambda,\mu = 1,\ldots,r$ and so S gives rise to a cover of L by affine charts in which the coordinate transformations are translations. It is now easy to show that the group of covering transformations of L with respect to the covering map, $\exp_p : T\mathcal{F}(p) \to L$, is a group of translations of R^r. Q.E.D.

Definition 4.4.3. A submersion $f : M \to N$ is injective if $f^{-1}(y)$ is connected for all $y \in N$.

THEOREM 4.4.2. Let $\mathcal{F}$ be a parallel foliation of type (r,0) given by an injective submersion, on a complete, connected, paracompact pseudoriemannian m-manifold (M,g). If each leaf of $\mathcal{F}$ is simply connected then (M,g) is isometric to a pseudoriemannian co-normal bundle.

Proof

Let $f : M \to N$ be the injective submersion. Because each leaf of $\mathcal{F}_\perp$ consists of a union of leaves of $\mathcal{F}$ it follows that $\mathcal{F}_\perp$ induces a foliation $\mathcal{G}_\perp$ on N given by $T\mathcal{G}_\perp(f(x)) = f_*(T\mathcal{F}_\perp(x))$. (The injectivity ensures that images of the leaves of $\mathcal{F}_\perp$ do not have self intersections) see picture.(p88). Let $\mathcal{A}$ be a Walker atlas. Let $(U,x^i) \in \mathcal{A}$ such that if $V = \{q \in U : x^\lambda(a) = 0 \quad \lambda = 1,\ldots,r\}$ then $f : V \to f(V)$ is a diffeomorphism. By theorems 4.3.1 and 2.2.4 there is a local diffeomorphism

$\xi : V \times R^r \to M$ which is leaf preserving and is a covering space of each leaf. Thus, since each leaf is simply connected

$\xi : v \times R^r \to$ (leaf through v) is a diffeomorphism.

Also, if $\xi(v,X) = \xi(\bar{v},\bar{X})$ then $f(v) = f(\bar{v})$ which gives $v = \bar{v}$ and so $X = \bar{X}$. Hence ξ is 1-1 and so is a diffeomorphism.

Consider $\psi : f(V) \times R^r \to M$ defined by $\psi(f(v),X) = \xi(v,X)$.

ψ is clearly a diffeomorphism. The collection of all such ψ together with the transition maps $\eta_{b\beta}$ of theorem 2.2.4 show that M admits an affine bundle structure with projection f, fibre R^r, base N and structure group A(r,R) (see example 2.2.1).

Now it is well known (see [15] and [29]) that any smooth fibre bundle with fibre R^r over a paracompact base manifold, admits a smooth cross section. It follows that the affine bundle structure can be reduced to a vector bundle structure, with structure group GL(r;R). (The general linear group).

It is not difficult to show that there is a cover of M by coordinate charts of the form $(\psi(W \times R^r),x^i)$ where $x^1,\ldots,x^r$ span the fibres, $(W,x^{r+1},\ldots,x^m)$ is a leaf chart on N for the foliation $\mathcal{G}_\perp$, and on the overlap of $(\psi(W \times R^r),x^i)$ and $(\psi(\bar{W} \times R^r),\bar{x}^i)$; $\bar{x}^\lambda = \dfrac{\partial \bar{x}^\lambda}{\partial x^\mu} x^\mu$ $\lambda,\mu = 1,\ldots,r$.

Thus $T\mathcal{F}, T\mathcal{F}_\perp$ are spanned by $\dfrac{\partial}{\partial x^1},\ldots,\dfrac{\partial}{\partial x^r}$ and $\dfrac{\partial}{\partial x^1},\ldots,\dfrac{\partial}{\partial x^{m-r}}$ respectively.

Also $g_{\lambda i} = 0$ $\lambda = 1,\ldots,r$, $i = r+1,\ldots,m-r$.

Consider now the map $h : M \to \nu^* \mathcal{G}_\perp$ defined by

$$h(x^1,\ldots,x^r,x^{r+1},\ldots,x^m) = (g_{\alpha\lambda} x^\lambda dx^\alpha, x^{r+1},\ldots,x^m)$$

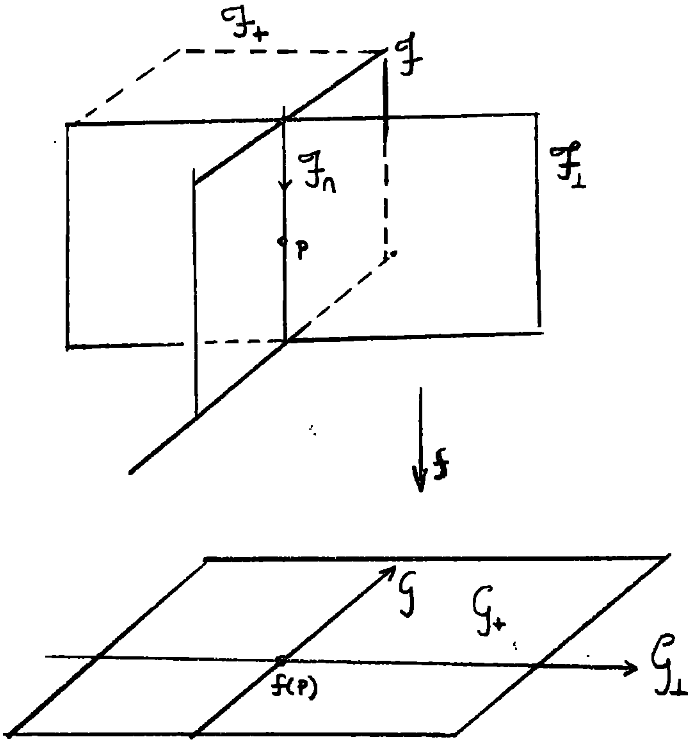
where $\alpha = m-r+1,\ldots,m$ and $\lambda = 1,\ldots,r$.

If $(\psi(\bar{W} \times R^r),\bar{x}^i)$ is an overlapping chart then

$$\bar{g}_{\alpha\lambda}\ \bar{x}^{\lambda}\ d\bar{x}^{\alpha} = \frac{\partial x^{\beta}}{\partial \bar{x}^{\alpha}} \cdot \frac{\partial x^{\mu}}{\partial \bar{x}^{\lambda}} \cdot g_{\beta\mu} \cdot \frac{\partial \bar{x}^{\lambda}}{\partial x^{\tau}} \cdot x^{\tau}\ \frac{\partial \bar{x}^{\alpha}}{\partial x^{\gamma}} \cdot dx^{\gamma}$$

$$= g_{\beta\mu}\ x^{\mu}\ dx^{\beta}$$

This shows that h does not depend on any particular chart and so is indeed defined globally. It is easy to show that h is a diffeomorphism. The required metric on $\nu^* \mathcal{G}_\perp$ is given by $(h^{-1})^*g$. Q.E.D.



This theorem shows that if the null part $\mathcal{F}_\cap$ of a parallel, partially null foliation $\mathcal{F}$ of type (r,s) is given by an injective submersion then it can be considered as a co-normal bundle $\nu^* \mathcal{G}_+$ where $\mathcal{G}_+$ is the foliation on the base induced from $\mathcal{F}_+$. By looking at the canonical form for the metric given in Lemma 4.3.1 it can be shown that each leaf L of $\mathcal{G}_+$ admits a complete pseudoriemannian structure for which the foliations induced on L by $\mathcal{G}$ and $\mathcal{G}_\perp$ are parallel, non-null and complementary. Thus, by theorem 4.2.2 L is covered by the product of an (m-2r-s) manifold and an s-manifold.

It would seem reasonable to conjecture that the submersion assumption is unnecessary if M is simply connected.

CONJECTURE 4.4.1. Let $\mathcal{F}$ be a parallel foliation of type (r,0) on a complete, connected, paracompact, simply connected, pseudoriemannian m-manifold (M,g). If each leaf of $\mathcal{F}$ is simply connected then (M,g) is isometric to a, pseudoriemannian co-normal bundle.

§4.5 Parallel Fields of Lines

In [23], S. A. Robertson proved that a compact, connected, complete, 3-dimensional pseudoriemannian manifold which admits a parallel 1-dimensional foliation (i.e. a parallel field of lines) has infinite fundamental group. His proof for the null case used a deep theorem of Novikov [19]. In [5], it was claimed that the result generalised to n-dimensional manifolds. Unfortunately, there is a gap in the proof of the null case and it only works for a strictly parallel field of lines (see Chapter 5). However, the proof of the non-null case is valid, and in fact does not make use of completeness. In this section a much stronger result is obtained by using a theorem of Reeb [20].

LEMMA 4.5.1. Let ω be a closed non-vanishing smooth 1-form on a smooth m-manifold, then the smooth co-dimension-1 distribution D defined by $\omega | D = 0$ is integrable.

Proof

If $X,Y \in D$ then in the chart (U,x^i)

$$[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right) \frac{\partial}{\partial x^j}$$

$$\therefore \quad \omega[X,Y] = X^i \frac{\partial Y^j}{\partial x^i}\,\omega_j - Y^j \frac{\partial X^i}{\partial x^j}\,\omega_i$$

$$= X^i \frac{\partial}{\partial x^i}(\omega_j Y^j) - Y^j \frac{\partial}{\partial x^j}(\omega_i X^i) + X^i Y^j \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right)$$

$$= 0 \quad \text{since } \omega(X) = \omega(Y) = 0 \text{ and } d\omega = 0.$$

$\therefore$ $[X,Y] \in D$ and so D is involutive and hence integrable by Lemma 1.4.1. Q.E.D.

LEMMA 4.5.2. (Reeb [20]). *Let M be a compact riemannian manifold and ω a closed non-vanishing 1-form satisfying $||\omega|| = 1$. Let $\mathcal{F}$ be the foliation of M defined by $\omega = 0$ (see lemma 4.5.1). Then the leaves of $\mathcal{F}$ are homeomorphic and if L is a typical leaf, there is a covering map $f : R \times L \to M$ which preserves the foliation and for which $f|t\times L$ is a homeomorphism for each t.*

THEOREM 4.5.1. *Let $\mathcal{F}$ be a parallel foliation of type (0,1) on a compact connected, pseudoriemannian m-manifold (M,g). Then M is covered by $R \times V$ for some (m-1) manifold V.*

Proof

By lemma 4.3.1 there is a Walker Atlas $\mathcal{A}$ of charts for which the metric g has the canonical form.

$$(g_{ij}) = \begin{bmatrix} a & 0 \\ 0 & B \end{bmatrix}$$

where B is a non-singular, symmetric (m-1) × (m-1) matrix function of the coordinates $x^2,\dots,x^m$.

and a is a non-vanishing function of x^1 only.

It follows that on the overlap of two charts (U,x^i), $(\bar{U},\bar{x}^i)$ the jacobian matrix has the form

$$\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) = \begin{bmatrix} r & 0 \\ 0 & S \end{bmatrix}$$

where $\bar{a} = r^2 a$.

Change coordinates by the rule $y^1 = \int_0^{x^1} |a(u)|^{-\frac{1}{2}}\, du$, $y^i = x^i \quad i \geqslant 2$.

Then

$$\left(\frac{\partial \bar{y}^i}{\partial y^j}\right) = \begin{bmatrix} \pm 1 & 0 \\ 0 & S \end{bmatrix}$$

By using similar arguments to theorem 2.2.3 it is possible to show that there is a 2-fold covering map $\phi_1 : \tilde{M} \to M$ such that $\tilde{M}$ admits a cover by coordinate charts (W,u^i) with jacobian matrices of the form

$$\left(\frac{\partial \bar{u}^i}{\partial u^j}\right) = \begin{bmatrix} 1 & 0 \\ 0 & S' \end{bmatrix} \tag{1}$$

There is a globally defined closed non-vanishing 1-form ω on $\tilde{M}$ given by $\omega = du^1$ in each chart. Let h be any positive definite metric on $\tilde{M}$. Then $\frac{h}{||\omega||^2}$ is a riemannian metric for which ω has unit norm. Let $\mathcal{G}$ be the (m-1) dimensional foliation on $\tilde{M}$ determined by $\omega = 0$. By lemma 4.5.2 the leaves of $\mathcal{G}$ are homeomorphic, and if V is a typical leaf there is a covering map $\phi_2 : R \times V \to \tilde{M}$ thus $f = \phi_1 \circ \phi_2 : R \times V \to M$ is a covering map. Q.E.D.

COROLLARY. <u>If (M,g) is a compact, pseudoriemannian manifold which admits a parallel field of non-null lines then $\pi_1(M)$ is infinite</u>.

See also [5].

It should be noted that if (M,g) is complete then the theorem follows from theorem 4.2.2.

CHAPTER 5

Parallel Framings on Pseudoriemannian Manifolds

§5.1 Related Atlases

Let M be a smooth, connected, pseudoriemannian m-manifold. An orthogonal k-frame at $x \in M$ is an ordered set $\lambda = (\lambda_1,\ldots,\lambda_k)$ of mutually orthogonal, linearly independent tangent vectors to M at x. The set of all orthogonal k-frames at $x \in M$ forms a Stiefel manifold S_x^k (see [29]) which is the fibre over x of the Stiefel bundle S^kM. A smooth section σ of S^kM is called a k-framing of M and determines an ordered set $(\sigma_1,\ldots,\sigma_k)$ of smooth, linearly independent, mutually orthogonal vector-fields σ_i. The section σ also determines a sub-bundle Σ of TM generated by $\sigma_1,\ldots,\sigma_k$.

Definition 5.1.1. The framing is said to be parallel of type (r,k-r) if and only if:

(1) For all $i = 1,\ldots,k$, σ_i is a parallel vector field.
(2) $\sigma_1,\ldots,\sigma_r$ are null.
(3) $\sigma_{r+1},\ldots,\sigma_k$ are non-null and unit (i.e. $g(\sigma,\sigma)=\pm 1$).
(4) $\sigma_1,\ldots,\sigma_r$ generate $\Sigma \cap \Sigma_\perp$.

There is no loss of generality in assuming condition (4), because if some linear combination of $\sigma_{r+1},\ldots,\sigma_k$ say X were null then since parallel translation preserves nullity the system could be reduced to one of type (r+1,k-r-1).

If σ is parallel, then Σ is a strictly parallel field of k-planes of nullity r, in the terminology of [35]. The results of Chapter 4 can now be strengthened considerably for such parallel fields.

As before the foliations determined by $\Sigma,\Sigma_\perp,\Sigma\ \ \Sigma_\perp$ and $\Sigma+\Sigma_\perp$ will be denoted by $\mathcal{F}, \mathcal{F}_\perp, \mathcal{F}_+, \mathcal{F}_\cap$ respectively.

Suppose that σ is a parallel k-framing of M of type $(r,k-r)$. Then by a result of Eisenhart [4] (see also Walker [36]), in the notation of §4.3 there is a Walker atlas $\mathcal{A}$ on M such that in each chart (with coordinates $(x,y,z,t) \in R^r \times R^s \times R^u \times R^r$, $u=m-k-r$) the matrix of the metric tensor has the form

$$(g_{ij}) = \begin{bmatrix} 0 & 0 & 0 & I_r \\ 0 & A & 0 & 0 \\ 0 & 0 & B(z,t) & G(z,t) \\ I_r & 0 & G'(z,t) & C(z,t) \end{bmatrix}$$

where I_r is the unit $r \times r$ matrix and A,B,C are symmetric matrices of order $s \times s$, $u \times u$ and $r \times r$ respectively, where $r + s = k$ and $u + r = m - k$. Also, A and B are invertible and A is a constant diagonal matrix with entries of the form ± 1.

If $x = (x^1,\dots,x^r)$ etc. then $\sigma_1 = \frac{\partial}{\partial x^1}, \dots, \sigma_r = \frac{\partial}{\partial x^r}$ $\sigma_{r+1} = \frac{\partial}{\partial y^1}, \dots, \sigma_k = \frac{\partial}{\partial y^s}$. It follows that the coordinates (x_*,y_*,z_*,t_*) and (x,y,z,t) on the overlap of two charts are related by equations of the form

$$\begin{aligned} x_* &= x + \alpha(z,t) \\ y_* &= y + \beta \\ z_* &= Z(z,t) \\ t_* &= t + \gamma \end{aligned}$$

where $\beta \in R^s$ and $\gamma \in R^r$ are constants and Z,α are smooth functions of the coordinates z,t.

The existence of $\mathcal{A}$ leads to the following result.

THEOREM 5.1.1. Let (M,g) be a connected, pseudoriemannian m-manifold with a parallel k-framing of type $(r,k-r)$, then:

(i) $TM \cong c^{k+r} \oplus \xi$ for some sub bundle ξ of TM (where c is the trivial line bundle.

(ii) If M is compact then the leaves of $\mathcal{F}$ and $\mathcal{F}_\Lambda$ are affinely equivalent in the induced structure to euclidean cylinders and there is a k-dimensional subspace in $H^1(M;R)$. Furthermore M is a bundle over T^k (the k-torus).

Proof

(i) follows from theorem 4.3.3.

(ii) The atlas $\mathcal{A}$ induces a locally euclidean structure on the leaves of $\mathcal{F}$ and $\mathcal{F}_\Lambda$. Since M is compact the integral curves of $\sigma_1,\ldots,\sigma_k$ are complete and so this induced structure is complete. Hence the leaves are affinely equivalent to euclidean cylinders (see theorem 4.4.1). $dt = (dt_1,\ldots,dt_r)$ determines globally, r-independent closed non-vanishing 1-forms and thus gives an r-dimensional linear subspace in $H^1(M;R)$ (see corollary to lemma 2.1.1). It follows also that M is a bundle over T^k by theorem 1 of Tischler [30] (see lemma 5.3.2). Q.E.D.

§5.2 Parallel Framings of Maximum Nullity

The extreme case of parallel framings of type $(r,0)$ on manifolds of dimension $m = 2r$ or $(2r+1)$ is now considered. The metric of M has signature (r,r) if m is even and $(r+1,r)$ or $(r,r+1)$ if m is odd. If m is even it follows immediately from theorem 5.1.1 that M is parallelizable (and hence orientable).

THEOREM 5.2.1. Let (M,g) be a complete, connected, pseudoriemannian $2r$ or

(2r+1)-manifold with a parallel framing of type (r,0). Then, for all *$x \in M$, $\exp_x : M_x \to M$ is a covering map.*

Proof. *Case (1). M is even dimensional*.

There is a Walker Atlas $\mathcal{A}$ on M such that in each chart the metric tensor has the form

$$(g_{ij}) = \begin{bmatrix} 0 & I_r \\ I_r & C(t) \end{bmatrix}$$

Each chart has coordinates $(x,t) \in R^r \times R^r$ and on the overlap of two charts the coordinates (x_*,t_*), (x,t) are related by equations of the form

$$\left.\begin{aligned} x_* &= x + \alpha(t) \\ t_* &= t + \gamma \qquad \text{where } \gamma \in R^r \text{ is constant} \end{aligned}\right. \tag{1}$$

For ease of notation x will be denoted by x^λ $\lambda = 1,\ldots,r$ and t by $t^1 = x^{r+1},\ldots,t^r = x^{2r}$. Late Greek suffices $\lambda,\mu,\tau,\ldots$ will denote integers in $(1,\ldots,r)$ and early Greek $\alpha,\beta,\gamma,\ldots$ integers in $(r+1,\ldots,2r)$. Roman suffices i,j,k will denote integers in $(1,\ldots,2r)$. The coefficients of the Levi-Civita connexion satisfy

$$\Gamma^\alpha_{jk} = \Gamma^i_{\mu k} = 0,\ \Gamma^\lambda_{\beta\gamma} = \tfrac{1}{2}\, g^{\lambda\alpha}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha})$$

The equations for a geodesic $\theta : [0,1] \to M$ reduce to

$$\left.\begin{aligned} \frac{d^2\theta^\lambda}{du^2} + \Gamma^\lambda_{\alpha\beta}(\theta^\alpha(u))\, X^\alpha X^\beta &= 0 \\ \frac{d^2\theta^\alpha}{du^2} &= 0 \end{aligned}\right\} \tag{2}$$

where $\frac{d\theta^\lambda}{du}(0) = X^\lambda$ and $\frac{d\theta^\alpha}{du}(0) = X^\alpha$.

Let $x \in M$ and $X_o \in M_x$. Let $\theta : [0,1] \to M$ be the geodesic determined by X_o such that $\theta(1) = \exp_x X_o$. Cover $\theta([0,1])$ with charts $(U_o, x_o^j), \ldots, (U_N, x_N^j)$ of $\mathcal{A}$ for which there is a subdivision $[0,u_1], \ldots, [u_i, u_{i+1}], \ldots [u_{N-1}, 1]$ of $[0,1]$ satisfying $\theta([u_i, u_{i+1}]) \subset U_i$. Suppose that X_o has components X_o^j with respect to (U_o, x_o^j).

It follows from (2) that in the chart (U_i, x_i^j), θ has components

$$\theta_i^\lambda(u) = X_o^\lambda(u-u_i) + \theta_i^\lambda(u_i) - X_o^\alpha X_o^\beta \int_{u_i}^{u} \int_{u_i}^{v} {}_i\Gamma_{\alpha\beta}^\lambda \, (\theta_i^\gamma(s)) \, ds \, dv$$

$$\theta_i^\alpha(u) = X_o^\alpha(u-u_i) + \theta_i^\alpha(u_i)$$

By using an inductive argument with (1), one can obtain

$$(3) \begin{cases} \theta_i^\alpha(u) = X_o^\alpha u + A_i^\alpha \\ \theta_i^\lambda(u) = X_o^\lambda u - X_o^\alpha X_o^\beta \left[\int_{u_i}^{u} \int_{u_i}^{v} {}_i\Gamma_{\alpha\beta}^\lambda (X_o^\gamma s + A_i^\gamma) ds \, dv + \sum_{j=0}^{i-1} \int_{u_j}^{u_{j+1}} \int_{u_j}^{v} {}_j\Gamma_{\alpha\beta}^\lambda (X_o^\gamma s + A_j^\gamma) ds \, dv \right] \\ \qquad + \sum_{j=0}^{i-1} K_j^\lambda {}_{j+1}(X_o^\gamma u_j + A_j^\gamma) \end{cases}$$

where A_i^α is constant and $K_j^\lambda {}_{j+1}$ is a smooth real valued function defined on the overlap of U_j and U_{j+1}. Thus, in the chart (U_N, x_N^j), one can represent $\exp_x X_o$ by

$$(\exp_x X_o)^\alpha = \theta_N^\alpha(1)$$

$$(\exp_x X_o)^\lambda = \theta_N^\lambda(1)$$

It is clear from (3) that the Jacobian of this map has the matrix form

$$\begin{bmatrix} I_r & Q(X_o^\alpha) \\ 0 & I_r \end{bmatrix} \quad \text{for some smooth } Q \qquad (4)$$

This matrix is non-singular and so $\exp_x$ is a local diffeomorphism. It follows that there is a neighbourhood W of X_o in M_x and an open set $U \subset U_N$ such that $\exp_x : W \to U$ is onto. It will now be shown that $\exp_x$ is onto $\bar{U}$ (the closure of U in U_N).

Let $z = (x^\lambda, x^\alpha)$ be a limit point of U in U_N and let $X(p) \varepsilon W \quad p = 1,2,3,\ldots,$ be a sequence such that $\exp_x X(p)$ converges to z.

Clearly $X^\alpha(p)$ converges to $x^\alpha - A^\alpha_N$. Using equations (3) is is not difficult to show that $\exp_x |\{X \varepsilon M_x : X^\alpha = x^\alpha - A^\alpha_N\} \to$ (leaf through z) is a covering map. It follows that $\lim\limits_{p \to \infty} (x^\lambda - X^\lambda(p))$ exists. Thus $X^i_1 = \lim\limits_{p \to \infty} X^i(p)$ exists.

Clearly, $z = \exp_x X_1$ and so $\exp_x$ is onto $\bar{U}$ and hence the whole of U_N. It is easily seen from (3) that there is a neighbourhood W' of X such that $\exp_x : W' \to U_N$ is 1-1. A straightforward induction shows that $\exp_x : M_x \to M$ is onto.

The connexion on M can now be pulled back to a connexion on M_x so that $\exp_x$ is connexion preserving.

Let $\sigma : [0,u_1) \to M_x$ be a geodesic and let $\tau = \exp_x \circ \sigma$ be the corresponding geodesic on M. Since M is complete, $\tau(u_1)$ is defined. One may pick a chart (U, x^i) around $\tau(u_1)$ so that $\exp_x$ has the form (3) for a neighbourhood W of $\sigma(u_2)$, $u_2 < u_1$, $\sigma([u_2,u_1)) \subset W$ and $\exp_x : W \to U$ is a diffeomorphism. Put $\sigma(u_1) = (\exp_x|W)^{-1} \tau(u_1)$. Thus σ is defined on the whole of R and hence M_x is complete.

By lemma 2.1.2 $\exp_x : M_x \to M$ is a covering map.

<u>Case (2). M is odd dimensional.</u>

There is a Walker atlas $\mathcal{A}$ on M (see Walker [35]) such that in each chart the metric tensor has the form

$$(g_{ij}) = \begin{bmatrix} 0 & 0 & I_r \\ 0 & \pm 1 & 0 \\ I_r & 0 & C(t) \end{bmatrix}$$

Each chart has coordinates $(x,z,t) \in R^r \times R \times R^r$, and on the overlap of two charts the coordinates (x_*,z_*,t_*) and (x,z,t) are related by equations of the form

$$\left.\begin{aligned} x_* &= x + \alpha_1(t)z + \alpha_2(t) \\ z_* &= \pm z + \beta(t) \\ t_* &= t + \gamma \end{aligned}\right\} \qquad (5)$$

where $\gamma \in R^r$ is constant.

The result now follows by an exactly analogous method to case (1). Q.E.D.

COROLLARY. <u>If M^m is simply connected and connected, with a parallel framing of maximum nullity then M^m is diffeomorphic to R^m.</u>

§5.3 Parallel Framings of Maximum Nullity on Compact Manifolds.

The results of the previous section can be strengthened considerably if M is assumed to be compact.

LEMMA 5.3.1. <u>If (M,g) is a compact, pseudoriemannian m-manifold with a parallel framing of maximum nullity of type (r,0), then (M,g) is complete.</u>

Proof

Only the case m = 2r is proved. The proof for the odd case is exactly analogous. It will again be convenient to work with closed charts. By the nature of equations (3) of theorem 5.2.1 it is clear that normal coordinate systems are compatible with the Walker Atlas $\mathcal{A}$. By using proposition 8.1 of Chapter III of [15] and the compactness of M it follows that there exists an $\varepsilon > 0$ and a Walker chart centred at each point of M whose coordinate ranges are greater than ε. That is to say, for each $p \varepsilon M$ there is $(U,x^i) \varepsilon \mathcal{A}$ such that $x^i(p) = 0$ and $|\max(x^i)-\min(x^i)| > \varepsilon$, $i = 1,\ldots,m$. Denote this collection of charts by S. It is clear that there is a $K > 0$ such that in every chart of S, $|\Gamma^\lambda_{\alpha\beta}| < K$.

Fix $p \varepsilon M$ and let $\sigma : [0,1) \to M$ be a geodesic emanating from p with initial vector $X \varepsilon M_p$ so that $\sigma(u) = \exp_p uX$ for $u \varepsilon [0,1)$.

Let (U_o,x^i_o) be a chart at p and X^i the components of X with respect to $\frac{\partial}{\partial x^i_o}(p)$ pick $u_1 \varepsilon [0,1)$ such that $|(1-u_1)X^i| < \frac{\varepsilon}{2}$ and $\left|\frac{(1-u_1)^2}{2} X^\alpha X^\beta K\right| < \frac{\varepsilon}{2}$ for $i = 1,\ldots,m$. $\alpha,\beta = r+1,\ldots,m$. Let $(U,x^i) \varepsilon S$ be a chart at $\sigma(u_1)$ then σ has coordinates

$$\sigma^\alpha(u) = X^\alpha(u-u_1)$$

$$\sigma^\lambda(u) = X^\lambda(u-u_1) - X^\alpha X^\beta \int_{u_1}^{u} \int_{u_1}^{v} \Gamma^\lambda_{\alpha\beta}(\sigma^\gamma(s))\, ds\, dv$$

The conditions on u_1 ensure that for $u \varepsilon [u_1,1]$, the right hand sides of both equations are within the respective coordinate ranges.

Thus $\sigma(1)$ is defined. It follows easily that σ is defined on the whole of R, and hence (M,g) is complete. Q.E.D.

LEMMA 5.3.2. (Tischler [30]). *Let M be a compact m-manifold which admits r-independent, closed non vanishing 1-forms $\omega^1,\ldots,\omega^r$. Then there is a bundle map $f : M \to T^r$ and if $\{\theta^\alpha : \theta^\alpha + 1 \sim \theta^\alpha, \alpha = 1,\ldots,r\}$ are standard coordinates on T^r then for any $\varepsilon > 0$ there exists a rational number q such that $||f^*(d\theta^\alpha) - q\,\omega^\alpha|| < \varepsilon$ (where the norm is induced from some riemannian metric on M).*

This result can be strengthened as follows.

THEOREM 5.3.1. Let M be a smooth, compact, connected m-manifold which has a foliation $\mathcal{F}$ of codimension r, determined by r independent closed non-vanishing 1-forms, $\omega^{m-r+1},\dots,\omega^{m}$. Then all the leaves of $\mathcal{F}$ are homeomorphic, and there is a bundle map $f : M \to T^r$ such that if F is the fibre and L is a typical leaf then $F \times R^r$ and $L \times R^r$ have the same universal cover and $\pi_1(F)$ is isomorphic to an extension of a subgroup of $\pi_1(L)$ by Z^n (where Z^n is the free abelian group on n generators). Furthermore, if L is simply connected then $\pi_1(M)$ is abelian.

Proof

Let $\mathcal{A} = \{(U,x^i)\}$ be a leaf atlas for $\mathcal{F}$ so that the leaves are given locally by x^α = constant, $\alpha = m-r+1,\dots,m$.

$d\omega^\alpha = 0$ implies that there are r smooth functions y^α defined on U such that $\omega^\alpha = dy^\alpha$.

There exists a leaf chart (U,z^i) such that $z^\alpha = y^\alpha$, $\alpha = m-r+1,\dots,m$.

Let h be a riemannian metric on M. Then, by defining an orthogonal complementary distribution to $T\mathcal{F}$ one can obtain projector tensors a and $\bar{a}$ in the usual way.

Suppose h has line element $ds^2 = h_{\lambda\mu}\,\omega^\lambda\omega^\mu + h_{\alpha\beta}\,dz^\alpha dz^\beta$ where $\omega^\lambda = dz^\lambda + a^\lambda_\alpha\,dz^\alpha$.

Define a new metric g by $ds^2 = h_{\lambda\mu}\,\omega^\lambda\omega^\mu + \sum_\alpha (dz^\alpha)^2$.

Now, because $\omega^\alpha = dz^\alpha$ it follows that g is defined globally and is bundle like in the sense of Reinhart [21].

If g has components g_{ij} with respect to the new chart (U,z^i) then the vector fields $X_\alpha = \omega^\alpha_i\,g^{ij}\,\partial/\partial z^j$ are defined globally and satisfy $\omega^\beta(X_\alpha) = \delta^\beta_\alpha$ and $g(X_\alpha,X_\alpha) = 1$. Let $X = \xi^\alpha X_\alpha$ be a non zero combination with ξ^α = constant. Then the one parameter group of diffeomorphisms $\psi : R \times M \to M$ associated with X (the flow of X) corresponds to a geodesic flow normal to the leaves. But $g(X(x),X(x))$ is constant as x varies over M and thus because g is bundle like (and complete since M is compact), $\psi(s,\ ) : M \to M$ sends leaves to leaves for each $s \in R$. It follows easily that all the leaves are homeomorphic.

Denote the r-tuple of 1-forms $(\omega^{m-r+1},\dots,\omega^{m})$ by $\underline{\omega}$. Thus $T\mathcal{F}$ is defined by $\underline{\omega}|T\mathcal{F} = 0$. Operations on $\underline{\omega}$ are carried out component-wise.

Let $a \in M$ and consider $H_a = \{[\sigma] : [\sigma] \in \pi_1(M,a)$, σ smooth loop, $\int_\sigma \underline{\omega} = 0\}$.

Clearly H_a is a normal subgroup of $\pi_1(M,a)$, and moreover it contains the commutator subgroup C.

Let $\tilde{M}$ be the connected covering space of M with respect to the group H_a (see Rosenburg [25]) then $\tilde{M}$ is a regular covering space of M with covering group $\pi_1(M,a)/H_a$. Denote the covering projection by p.

Defined on $\tilde{M}$ is the r-tuple $\underline{\omega}^* = p^*\,\underline{\omega} = (p^*\,\omega^{m-r+1},\dots,p^*\,\omega^{m})$.

$\underline{\omega}^*$ is never zero, and $d\underline{\omega^*} = 0$. Let $\mathcal{F}^*$ be the foliation determined by $\underline{\omega}^*|T\mathcal{F}^* = 0$.

Let σ be a closed curve in $\tilde{M}$ based at some point $\tilde{a}$ in $p^{-1}(a)$.

Now, $\int_\sigma \underline{\omega}^* = \int_{p_o\sigma} \underline{\omega}$ and because $[p_o\sigma]$ represents an element in H_a (from the construction of $\tilde{M}$) it follows that $\int_\sigma \underline{\omega}^* = 0$.

Thus the integral of $\underline{\omega}^*$ about any closed curve in $\tilde{M}$ is zero and so $\underline{\omega}^* = d\,\underline{\ell}$ where $\underline{\ell}$ is an r-tuple $(\ell^{m-r+1},\dots,\ell^{m})$ of smooth real valued functions on $\tilde{M}$. The level surfaces of $\underline{\ell}$ are precisely the leaves of $\mathcal{F}^*$.

The vector fields X_α lift to X^*_α on $\tilde{M}$ so that $\omega^{\alpha *}(X^*_\beta) = \delta^\alpha_\beta$.

Thus $X^*_\beta(\ell^\alpha) = \delta^\alpha_\beta$, and so if $\ell^\alpha = c^\alpha$, $\alpha = m-r+1,\dots,m$ is a leaf of $\mathcal{F}^*$ then the flow of ξX^*_β for a real number ξ takes this leaf to the leaf $\ell^{m-r+1} = c^{m-r+1},\dots,\ell^\beta = \xi + c^\beta,\dots,\ell^m = c^m$. It follows that if $\gamma \in R^r$ then $\underline{\ell} = \gamma$ is a leaf of $\mathcal{F}^*$.

Thus $\tilde{M}$ is diffeomorphic to $L_o \times R^r$ where L_o is a leaf of $\mathcal{F}^*$. For each

$\gamma \in R^r$, $L_o \times \gamma$ corresponds to a leaf of $\mathcal{F}^*$. It may be assumed without loss of generality that $p(L_o) = L$ the leaf of $\mathcal{F}$ through a.

Define a map $q : L \to L_o$ as follows. Let $b \in L$ and $\tau : [0,1] \to L$ a path from a to b. Lift τ to a path $\tilde{\tau}$ in L_o with initial point $\tilde{a}$. Put $q(b) = \tilde{\tau}(1)$. This map does not depend on τ because closed paths in L are represented in H_a and so lift to closed paths in L_o.

Thus $p : L_o \to L$ is a diffeomorphism and $\tilde{M}$ can be identified with $L \times R^r$. $p : L \times R^r \to M$ is a regular covering with covering group G, isomorphic to $\pi_1(M,a)/H_a$. Now because $p_\#$ is a monomorphism it follows that H_a may be identified with $i_\# \pi_1(L,a) \subset \pi_1(M,a)$ (where $i : L \to M$ is the inclusion map). Thus $G \cong \pi_1(M,a)/i_\# \pi_1(L,a)$ and is abelian because $C \subset H_a$. To show that G is free abelian a further lemma is required.

Definition 5.3.1. An *oriented closed transversal* to $\mathcal{F}$ is a smooth path $j : S^1 \to M$ such that $\omega^\alpha(j_*(\partial/\partial t))$ $\alpha = m-r+1,\dots,m$ are not all zero and all have constant sign for $t \in S^1$ ($= \{t \in R : t \sim t+1\}$).

LEMMA 5.3.2. *Under the hypotheses of the theorem an element of $\pi_1(M,a)$ can be represented by an oriented closed transversal if and only if it belongs to $\pi_1(M,a) - i_\# \pi_1(L,a)$.*

Proof

This result is a direct generalization of a theorem of Moussu [17] for the case r = 1.

If τ is a closed oriented transversal then obviously $\int_\tau \underline{\omega} \neq 0$ and thus $[\tau] \notin H_a = i_\# \pi_1(L,a)$.

Conversely, let σ be a loop at a such that $[\sigma] \in \pi_1(M,a) - i_\# \pi_1(L,a)$.

Let $\tilde{\sigma}$ be the lift of σ in $L \times R^r$ with initial point $\tilde{a} = (a \times 0)$ say. Put $\tilde{a}_1 = (a_1, \underline{t}_1) = \tilde{\sigma}(1)$, $\underline{t}_1 \in R^r$ and $L_1 = L \times \underline{t}_1$. Now, since $[\sigma] \notin i_{\#}\,\pi_1(L,a)$, $\underline{t}_1$ is non zero. The path $\tilde{\tau} : [0,1] \to L \times R^r$ defined by $\tilde{\tau}(u) = (a, u\,\underline{t}_1)$ is a transversal segment, oriented with respect to $\omega^{\alpha *}$ for $\alpha = m-r+1, \ldots, m$, with end point $\tilde{a}_2 = (a, \underline{t}_1) \in L_1$.

Let $\tilde{\theta}_1$ be a smooth path in L_1 which joins $\tilde{a}_2$ to $\tilde{a}_1$. Since $\pi_1(L \times R^r, \tilde{a}) = i_{\#}\,\pi_1(L_0, \tilde{a})$, the loop $\tilde{\sigma}^{-1} \circ \tilde{\theta}_1 \circ \tilde{\tau}$ is homotopic relative to $\tilde{a}$ to a loop $\tilde{\theta}_0$ in L_0. Let θ_1, τ, θ_0 a' be the projections under p of $\tilde{\theta}_1, \tilde{\tau}, \tilde{\theta}_0$, $(a, \underline{t}_1)$. Then it is clear that $[\sigma] = [\theta_1 \circ \tau \circ \theta_0^{-1}]$. By a suitable deformation along one of the flows, one may construct an oriented closed transversal τ' homotopic to $\theta_1 \circ \tau \circ \theta_0^{-1}$ and oriented with respect to each ω^{α} in the same sense as τ. Q.E.D.

COROLLARY. <u>$\pi_1(M,a)/i_{\#}\,\pi_1(L,a)$ is free abelian.</u>

<u>Proof</u>

It is abelian because $C \subset H_a$.

To show it has no torsion let $[\sigma] \in \pi_1(M,a)$ and let $[\bar{\sigma}] \neq 0$ be its coset in $\pi_1(M,a)/i_{\#}\,\pi_1(L,a)$. By the lemma $[\sigma]$ is representable by an oriented closed transversal τ say. τ^k is always an oriented closed transversal and thus $[\tau^k] = [\sigma]^k \notin i_{\#}\,\pi_1(L,a)$ i.e. $[\bar{\sigma}]^k \neq 0$. Q.E.D.

Hence G is free abelian, and moreover is finitely generated because M is compact (see [27]).

Now, by lemma 5.3.1 there is a bundle map $f : M \to T^r$. If F is the fibre, then F is compact and there is no loss of generality in assuming that F is connected - because if it were not then one could construct a k-fold cover (k=number components of F) of T^r (which is diffeomorphic to T^r) and a new bundle map onto this covering manifold, with connected fibre and the same properties.

Let $\bar{M}$ be the universal cover of M, then there is a regular covering $\phi : \bar{M} \to L \times R^r$ such that $p \circ \phi : \bar{M} \to M$ is the projection.

Let F(a) be the fibre of f through a and $\tilde{F}$ the component of $p^{-1}(F(a))$ through $\tilde{a}$. Then $p : \tilde{F} \to F(a)$ is a regular covering with covering group G', a subgroup of G. Clearly G' is free abelian and finitely generated, i.e. $G' \cong Z^n$ for some n.

Since $\pi_1(L\times R^r) \cong \pi_1(L)$ it follows that the covering group of ϕ is isomorphic to $\pi_1(L)$.

Let $\tilde{\tilde{F}}$ be a connected component of $\phi^{-1}(\tilde{F})$, then $\phi : \tilde{\tilde{F}} \to \tilde{F}$ is a regular covering with covering group isomorphic to a subgroup A of $\pi_1(L)$. Then, if $\tilde{\tilde{F}}$ is simply connected, $\pi_1(F)$ will be isomorphic to an extension of A by G'.

If T^r has coordinates $\{\theta^\alpha \in R : \theta^\alpha \sim \theta^\alpha+1,\ \alpha = m-r+1,\dots,m\}$, let $\xi : R^r \to T^r$ be the regular covering induced by the standard Z^r action. Then ξ induces a pull back bundle on R^r with fibre F. But since R^r is contractable this bundle is reducible to the trivial bundle and so there is a covering map $\eta : F \times R^r \to M$ such that for each $\underline{t} \in R^r$, $\eta | F \times \underline{t}$ is a diffeomorphism onto a fibre of f in M.

Let $\bar{F}$ be a simply connected cover of F, then there is a covering map $\phi' : \bar{F} \times R^r \to F \times R^r$. Clearly $\eta \circ \phi' : \bar{F} \times R^r \to M$ is a simply connected cover. Hence by the uniqueness of simply connected covers (see [27]) there is a homeomorphism $\lambda : \bar{F} \times R^r \to \bar{M}$ such that $(p \circ \phi) \circ \lambda = \phi'$. Thus $\lambda : \bar{F} \to \tilde{\tilde{F}}$ is a homeomorphism and so $\tilde{\tilde{F}}$ is simply connected. This proves the first part of the theorem.

The second part follows immediately from the fact that $C \subset i_{\#}\, \pi_1(L,a)$ and so if L is simply connected, C is trivial. Q.E.D.

The next theorem uses this result to show that there are very strong topological restrictions on compact manifolds with parallel framings of maximum nullity.

THEOREM 5.3.2. Let (M,g) be a compact, connected, pseudoriemannian m-manifold with a parallel framing of type $(r,0)$ and maximum nullity. Then the leaves of $\mathfrak{F}$ are all homeomorphic to $T^q \times R^{r-q}$ for some fixed $q \leqslant r$ and M is a bundle over T^r. The fibre F is a compact, connected $(m-r)$ manifold for which $F \times R^r$ is covered by R^m. If $m = 2r$ then $\pi_1(F)$ is isomorphic to an extension of Z^k by Z^h for some k and h with $k \leqslant q$. Furthermore, M is covered by R^m and if $m = 2r$ then $\pi_1(M)$ is isomorphic to an extension of Z^q by Z^s for some s.

Proof

By theorems 5.1.1 and 5.3.1 the leaves of $\mathfrak{F}$ are homeomorphic euclidean cylinders and hence are all homeomorphic to $T^q \times R^{r-q}$ for some q (see [15] page 210).

(i) $m = 2r$

There is a Walker Atlas $\mathcal{A}$ on M such that on the overlap of two charts (U,x^i) and (U_*,x^i_*) the coordinates are related by equations of the form

$$x^\lambda_* = x^\lambda + a^\lambda(x^\alpha) \qquad \lambda = 1,\dots,r$$

$$x^\alpha_* = x^\alpha + c^\alpha \quad \text{where } c^\alpha \text{ is constant, } \alpha = r+1,\dots,m=2r$$

Put $\omega^\alpha = dx^\alpha$ in each chart. This defines r independent closed non-vanishing 1-forms which determine $\mathfrak{F}$. The result now follows from theorem 5.3.1.

(ii) $m = 2r + 1$

There is a Walker Atlas with coordinates related by

$$x_*^\lambda = x^\lambda + a_1^\lambda(x^\alpha)\, x^{r+1} + a_2^\lambda(x^\alpha) \qquad \lambda = 1,\ldots,r$$

$$x_*^{r+1} = \pm\, x^{r+1} + b^{r+1}(x^\alpha)$$

$$x_*^\alpha = x^\alpha + c^\alpha \qquad \alpha = r+2,\ldots,m=2r+1,\ c^\alpha \text{ is constant.}$$

Then $\omega^\alpha = dx^\alpha$ define r-independent, closed non-vanishing 1 forms on M. Again the result follows by theorem 5.3.1.

That M is covered by R^m follows from lemma 5.3.1 and theorem 5.2.1 (in the case m=2r it also follows from the fact that M is covered by $T^q \times R^{r-q} \times R^{m-r}$).

Now, $\pi_1(T^q \times R^{r-q} \times R^{m-r}) \cong Z^q$ and so, if m = 2r then $\pi_1(M)$ is isomorphic to an extension of Z^q by Z^s where Z^s is isomorphic to the group G in the proof of theorem 5.3.1. Q.E.D.

COROLLARY 1. *Suppose m = 2r. If q = r then F is diffeomorphic to T^r and if q = 0 then M has the homotopy type of T^m and is homeomorphic to T^m if $m \neq 4$.*

Proof

It is easy to prove that the Ehresmann Holonomy group of each leaf of $\mathfrak{F}$ is trivial. Thus, by results of [21] and [30] it follows that $\mathfrak{F}$ can be given a bundle structure if the leaves are all compact (i.e. q = r). Thus if q = r then one may assume that F is diffeomorphic to T^r.

If q = 0 then theorem 5.3.1 shows that $\pi_1(M)$ is abelian. Let $h : R^m \to M$ be a covering map and K the group of covering transformations. A theorem of P. A. Smith [26] says that any homeomorphism ψ of R^m of finite order a prime has a fixed point. Thus if $\phi \in K$ were of finite order then some power of ϕ would have a fixed point, contradicting the fact that

K is properly discontinuous. Hence $\pi_1(M)$ is free abelian, and is finitely generated because M is compact i.e. $\pi_1(M) \cong Z^k$ for some k.
Also $\pi_i(M) = 0$ for $i > 1$ because M is covered by R^m.
Hence M has the homotopy type of T^k. Homology considerations (see [11]) and the compactness of M show that $k = m$.
A theorem of Rosenburg [24] says that M is irreducible if M is covered by R^m. Hence by results of C. T. C. Wall [38], M is homeomorphic to T^m if $m \neq 4$. Q.E.D.

COROLLARY 2. Let (M,g) be a compact, connected, pseudoriemannian 4-manifold with a parallel framing of type (2,0) then M is a T^2 bundle over T^2.

Proof

There are three cases, $q = 0,1,2$. Since $F \times R^2$ is covered by R^4 it follows that F is covered by R^2. If $q = 0$ then M is a homotopy T^4, $\pi_1(F)$ is free abelian and so F is diffeomorphic to T^2.
If $q = 2$ then the result follows immediately from corollary 1.
If $q = 1$ then $\pi_1(F)$ is at worst isomorphic to an extension of Z by Z^h for some h. Thus there is a regular Z^h cover $p : \tilde{F} \to F$ where $\pi_1(\tilde{F}) \cong Z$. It can be shown that $\tilde{F}$ is homeomorphic to $S^1 \times R$ as F is orientable (because $H_1(M;Z)$ has no torsion). Let $x \in S^1 \times 0 \subset S^1 \times R^r$ and $\sigma : S^1 \times 0 \to S^1 \times R$ be the inclusion map. Then $[\sigma]$ represents a generator of $\pi_1(\tilde{F},x)$. Let ψ be any orientation preserving homeomorphism of $S^1 \times R$. Then $\psi \circ \sigma$ is an embedded S^1 and hence if $x \in S^1 \times 0$, $y = \psi(x)$ and $\tau : [0,1] \to S^1 \times R^r$ is a path joining x to y then $[\tau^{-1} \circ (\psi \circ \sigma) \circ \tau] = [\sigma]$.
Thus $\sigma^{-1} \circ \tau^{-1} \circ (\psi \circ \sigma) \circ \tau$ is null homotopic. It follows easily that the commutator subgroup of $\pi_1(F)$ is trivial and hence F is diffeomorphic to T^2.

Q.E.D.

The following example shows that this bundle structure is non-trivial in general.

EXAMPLE 5.3.1. Take R^4 with coordinates (x,y,z,t) and pseudoriemannian metric $ds^2 = 2dx\,dz + 2dy\,dt$. With respect to this metric the vector fields $X_1 = {}^{\partial}/\partial x$ and $X_2 = {}^{\partial}/\partial y$ are mutually orthogonal, parallel and null. Consider the group G of transformations of R^4 generated by A,B,C,θ defined as follows:

$$A(x,y,z,t) = (x+1,y,z,t),$$
$$B(x,y,z,t) = (x,y+1,z,t),$$
$$C(x,y,z,t) = (x,y,z,t+1),$$
$$\theta(x,y,z,t) = (x+t,y-z,z+1,t).$$

It is not difficult to show that G is a properly discontinuous group of isometries leaving X_1 and X_2 invariant. Since θ does not commute with C, G is non-abelian.

Let $M = {}^{R^4}/G$. Then M admits a parallel framing of type (2,0). Furthermore, M is compact and the fields X_1, X_2, $z\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$, $\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$ on R^4 are invariant under G, showing that M is parallelizable.

The projection $\pi : R^4 \to R^2$ defined by $(x,y,z,t) \mapsto (z,t)$ is equivariant with respect to the action of G on R^4 and the usual action of Z^2 on R^2 and so

$$\pi : {}^{R^4}/G = M \to {}^{R^2}/Z^2 = T^2 \text{ is well defined}$$

It is not difficult to show that π gives a fibre bundle projection with fibre T^2 and structure group a subgroup of T^2.

M is not the trivial bundle because $\pi_1(M) \cong G \neq Z^4$.

It would be interesting to know whether, in higher dimensions, the fibre of theorem 5.3.2 is always a torus. A good problem would be to try and use the general technique of this example to find a bundle which admits a framing of maximum nullity but whose fibre has non-abelian fundamental group.

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