A calculation of neutron-deuteron scattering using the su(3) basis

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A CALCULATION OF NEUTRON-DEUTERON SCATTERING USING THE Su(3) BASIS

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by

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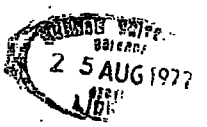


To My Mother

C O N T E N T S

ABSTRACT

The Faddeev equations for three particles are given a basis of representation according to the group Su(3). The usefulness of this representation is investigated by application to neutron-deuteron scattering.

CHAPTER ONE

INTRODUCTION

Investigation of the three-body problem has been found difficult partly because of the large number of degrees of freedom involved and partly because of difficulties with the formalism. Although the Faddeev equations put the problem into a clear framework[1,2], the three particles are still treated as identifiable and the large number of variables still remain. Even after the Omnes[3] angular momentum reduction three integrating variables still remain in the final equations. Osborn[4] succeeded in reducing Omnes' results for the Faddeev equations to two variables and actually solved the equations for the idealized case of three spinless bosons interacting through a Yukawa potential. However, the method is numerically complicated and it would be even more complicated and cumbersome when generalized to nucleons interacting through spin and isospin dependant potentials. However, it was realized that it is possible to reduce the problem to the solution of a single variable integral equation by assuming that the particles interact in pairs through non-local separable potentials[5,6,7]. When attempting to classify multiparticle states it is important to diagonalize these variables which are known to be constants of the motion from general invariance principles. Thus, plane wave states $|\underline{p}_1\underline{p}_2\underline{p}_3 \ldots.)$ which are eigenstates of the individual momenta $\underline{p}_i$ and the total momentum, $\underline{P} = \sum_i \underline{p}_i$, are useful in exploiting translational invariance. To benefit from rotational invariance as well, we need to abandon plane wave states in favour of eigenstates not only of total momentum but also of the total angular momentum. For scattering problems which do not involve rearrangements it is perhaps best to carry out a relatively simple partial wave analysis corresponding to the angular momentum in the CM system of the two incident bodies. Hetherington and Schick[5] have done this for k^- - D scattering.

However, for cases which go beyond the "effective two particle" situation then a more general three body partial wave analysis should be considered.

A discussion of how such states may be constructed for three particles has been given by Wick[8]. He forms states $|P\ J\ J_z, w\ j\ j_z)$ which are eigenstates of the total momentum and the angular momentum about the centre of mass. His states are also eigenstates of the total energy w and angular momentum j of particles 1 and 2 in their centre of mass frame. These states are partially localized and this is important when considering short range interactions. For example, states with moderate values of w and large j describe a situation in which particles 1 and 2 are almost always very far apart and therefore unlikely to interact. However, while the spatial correlation between particles 1 and 2 is well-defined, that between particles 1 and 3 or 2 and 3 has to be found using recoupling coefficients.

Three-body angular momentum states have also been constructed by Ahmadzadeh and Tjon[9] which were used by Chen, Ball and Wong[10] to study the (e,H) system. These states may be represented as $|p_i l M_l, q_i L\ M_L\rangle$ where p_i is the relative momentum co-ordinate for the two particles not specified by i and q_i is the momentum of particle i in the over-all CM system. The angular momenta l and L are the relative angular momentum of the two particles concerned and the angular momentum of the third particle i in the three-body CM system. Again, the criticism of partial localization can be directed at these states. It might be useful, therefore, to have a formalism which treats all three particles equivalently. That is, we should like to be able to construct states which have very simple symmetry properties.

The work of Smith[11] introduces a "Grand Angular Momentum Tensor", Λ^2, which plays a similar part for three particles as angular momentum does for two particles. The smaller is Λ^2, then the "closer" is the system. Thus, a partial wave analysis using Λ^2 might converge rapidly

if we are considering a system where every thing is happening "close in" such as a three-particle strong interaction system. The method of Smith leads to a single six-dimensional vector providing the co-ordinates needed to describe the three-particle CM System. Thus, it involves a single integration over the length of the six-dimensional vector and summations over the discrete quantum numbers arising from a consideration of the angular dependences in this six-dimensional space. It would be possible, therefore, to use a local potential and still be able to reduce the problem to one involving a single integration. This would be an advantage of the formalism if the summation over the discrete quantum numbers were to converge rapidly enough.

Indeed the main concern of this present work is to test quantitatively the reasonableness of this speculation for a realistic calculation. We have therefore turned our attention to the problem of neutron-deuteron (N-D) scattering. As will be seen, there is indeed some convergence, but it is not rapid enough to be a practical means of studying N-D scattering. Although certain simplications result in using these states, the calculation becomes rather cumbersome unless we can truncate the summation after only a few terms. Further discussion of this is left till later.

Chapter 2 provides a short account of the Faddeev formalism, while Chaper 3 provides a resume of the relevant work of Dragt[12] who extended the studies of Smith to the three-dimensional case. In section A of Chapter 4 the relevant coupled equations are reduced to one single variable integral equation using the three-particle states ("Dragt States") of reference 12, which have simple symmetry properties.

Section B of Chapter 4 gives the method of finding the three-body T matrix using the Su(3) basis.

Numerical results are presented in Chapter 5 and the conclusion in Chapter 6.

CHAPTER TWO

THE FADDEEV FORMALISM

Section A. The Non-Relativistic Three-body Problem.

The Lippmann-Schwinger (L-S) Equation for the T-matrix, which describes non-relativistic two-particle scattering, runs into trouble for the case of three particles[1,13]. The difficulties arise, in part, from the presence of disconnected diagrams corresponding to the possibility of a particle going straight through without interaction. The two-body L-S equation is:

$$\langle \underline{p}' | T(E) | \underline{p} \rangle = \langle \underline{p}' | V | \underline{p} \rangle + \int d\underline{p}'' \frac{1}{E - E_{p''}} \langle \underline{p}' | V | \underline{p}'' \rangle \langle \underline{p}'' | T(E) | \underline{p} \rangle \tag{2.1}$$

where $\underline{p}$ and $\underline{p}'$ are the relative momentum vectors for the incoming and outgoing states respectively and E is the total energy in the CM System.

For the three-particle case let us define channel α as that for which particle α is free while the other two particles form a bound state. Then the L-S equations for scattering from channel α to channel β are[14]

$$T^{+}_{\beta\alpha}(E) = V^{\beta} + V^{\beta} G^{+}_{\alpha}(E)\, T^{+}_{\alpha\alpha}(E)$$

$$= V^{\beta} + T^{+}_{\beta\alpha}(E)\, G^{+}_{\alpha}(E)\, V^{\alpha} \tag{2.2}$$

$$T^{-}_{\beta\alpha}(E) = V^{\alpha} + V^{\beta} G^{+}_{\beta}(E)\, T^{-}_{\beta\alpha}(E)$$

$$= V^{\alpha} + T^{+}_{\beta\alpha}(E)\, G^{+}_{\alpha}(E)\, V^{\alpha} \tag{2.3}$$

And $\alpha = 0,1,2,3$ where $\alpha = 0$ refers to the channel having all three particles free.

The quantities are defined for particles of masses M_α ($\alpha = 1,2,3$) as:

(a) $V_o = 0$ $\quad V_1$ = potential between particles 2 & 3

V_2 = potential between particles 1 & 3

V_3 = potential between particles 1 & 2

$V = V_1 + V_2 + V_3 \qquad V^\alpha = V - V_\alpha$.

(b) $G^+_\alpha(E)$ = Green's function in channel α

$$= \frac{1}{(E - H_\alpha \pm i\epsilon)} \quad , \epsilon \to 0+$$

with E = Energy in the three-particle CM System.

$$H_\alpha = H_o + V_\alpha$$

Equations (2.2) and (2.3) lead to different off energy-shell extensions of the T matrix and describe the same scattering in the limit as $\epsilon \to 0$. From now on we shall confine ourselves to the (+) -type of operator and omit the Superscript unless a distinction is important.

Equations of the type (2.3) can be solved by standard techniques if the kernel, $V^\beta G^+_\alpha(E)$, is Hilbert-Schmidt, (H-S).

That is, in operator form,

$$\text{trace } (K K^+) < \infty$$

where K is the kernel of the equation.

It fails to be H-S for three-body scattering because of the presence of delta-functions which arise in taking matrix elements of two-body potentials between three-body states. For example, the kernel $V^o G^+_o(E)$ leads to terms such as

$$\langle p'_1 p'_2 p'_3 \mid V_1 G^+_o(E) \mid p_1 p_2 p_3 \rangle$$

where p_1, p_2, p_3 are the momentum vectors for the three incoming particles and p'_1, p'_2, p'_3 likewise for the outgoing particles.

Particles 2 and 3 interact through V_1, but particle 1 is unaffected. That is, the matrix element gives

$$\delta(p_1-p_1') \quad \langle p_2'p_3' \mid V_1 \; G_o^+(E) \mid p_2p_3 \rangle$$

This may be represented diagramatically as in Figure (2.1)

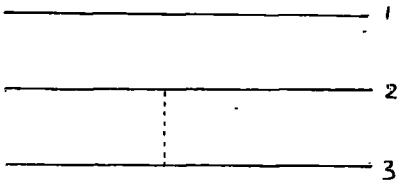


Figure 2.1

Therefore the kernel is not (H-S). There remains the possibility that it is compact, but in fact this is not the case[13]. The solution is then non-unique[24,25,26]. Faddeev overcame this by deducing coupled integral equations from the L-S equation which have connected kernels after one iteration[1]. Equations (2.2) may be written as

$$T_{\beta\alpha} = V^{\beta} + V^{\beta} G V^{\alpha} \qquad (2.4)$$

where $G^+ = \dfrac{1}{E-H+i\epsilon}$ is the full Greens function.

H = the complete Hamiltonian.

This leads to the break-down

$$T_{\beta\alpha} = \sum_{\substack{i\neq\beta \\ j\neq\alpha}} T^{ij} + V_{\alpha}\bar{\delta}_{\beta\alpha} \qquad (2.5)$$

$$\bar{\delta}_{\beta\alpha} = (1 - \delta_{\beta\alpha})$$

with

$$T^{ij} = t_i \delta_{ij} + \sum_{k=1}^{3} t_i G_o \bar{\delta}_{ik} T^{kj} \qquad (2.6)$$

and t_i = the two particle T matrix between the particles not denoted by i.

Equations (2.6) are the canonical Faddeev Equations and have connected kernels after one iteration. But even after an iteration the kernels are not H-S if we let $\epsilon \to 0$. However, after five iterations the kernels are

compact and it then follows that the solution is unique even on the real energy axis, except for a discrete set of real values of $E^{(13)}$.

Section B. Faddeev Equations for Neutron-Deuteron Scattering

It is our task to calculate the matrix element T_{ab} for elastic N-D scattering,

$$T_{ab} = \langle a | T | b \rangle \tag{2.7}$$

In particular we are dealing with a system of three identical particles, each with spin $\frac{1}{2}$ and isospin $\frac{1}{2}$. The states $|a\rangle$, $|b\rangle$ are made up of products of spatial wave functions, spinwave functions and isospin wave functions. Each spatial wave function is a product of a bound state wave function for the motion of the dueteron, and a plane wave for the motion of the third particle.

The state vector for a system of fermions must be anti-symmetric with respect to interchange of particles. Denote such a three particle state by $|123\rangle_{as}$, where the 1 includes space, spin and isospin information for particle 1 and likewise for particles 2 and 3.

In terms of states made up of direct products of single particle states, $|123\rangle_{as}$ may be written as [15]

$$|123\rangle_{as} = \frac{1}{\sqrt{6}} \left\{ \left(|123\rangle + |231\rangle + |312\rangle \right) - \left(|213\rangle + |132\rangle + |321\rangle \right) \right\} \tag{2.8}$$

defining an anti-symmetric 2-fermion state by

$$|12\rangle_{as} = \frac{1}{\sqrt{2}} \left\{ |12\rangle - |21\rangle \right\} \tag{2.9}$$

where $|12\rangle = |1\rangle \times |2\rangle$ then equation (2.8) may be re-written in the form

$$|123\rangle_{as} = \frac{1}{\sqrt{3}} \left\{ |3\rangle_{as} + |2\rangle_{as} + |1\rangle_{as} \right\} \tag{2.10}$$

with $|3\rangle_{as} = |12\rangle_{as} \times |3\rangle$ etc.

This is also true if $|12\rangle_{as}$ is any normalized anti-symmetric two-fermion state.

It is convenient here to introduce new momentum variables defined by

$$\underline{P} = \underline{p}_1 + \underline{p}_2 + \underline{p}_3 \tag{2.11}$$

$$\underline{q}_i = \underline{p}_j + \underline{p}_k \tag{2.12}$$

$$\underline{k}_i = \frac{(M_j \underline{p}_k - M_k \underline{p}_j)}{(M_j + M_k)} \tag{2.13}$$

where i,j,k is a cyclic permutation of 1,2,3 and

$\underline{P}$ = the total momentum

$\underline{q}_i$ = the momentum of particle i in the three particle CM system.

$\underline{k}_i$ = the relative momentum of particles j and k in the j k CM Subsystem.

M_i = mass of particle i.

It is then found that the Jacobian for the transformation from $(\underline{p}_1 \underline{p}_2 \underline{p}_3)$ to $(\underline{P}, \underline{q}_i, \underline{k}_i)$ is one.

That is

$$|\underline{p}_1\ \underline{p}_2\ \underline{p}_3\rangle = |\underline{P}, \underline{q}_i, \underline{k}_i\rangle \qquad i = 1,2,3$$

Including spin and isospin the state $|\beta\rangle_{as}$ can now be specified as

$$|\underline{P}, \underline{q}_\beta, D_\beta, I, I_z, I_\beta, S, S_z, S_\beta\rangle_{as} = \int d\underline{k}_\beta\ \phi_D(\underline{k}_\beta)\ |\underline{P}, \underline{q}_\beta, \underline{k}_\beta, I, I_z, I_\beta, S, S_z, S_\beta\rangle_{as} \tag{2.14}$$

Where we have now introduced the deuteron wave function

$$\phi_D(\underline{k}_\beta) = \langle \underline{k}_\beta \mid D_\beta \rangle \tag{2.15}$$

in its CM system, and the other quantities are

I = total isospin of the three particle system

I_z = Z component of I

I_β = total isospin of the β^{th} two-particle Subsystem

S = total spin of three particle system

S_z = Z component of S

S_β = total spin of the β^{th} two particle subsystem.

The deuteron isospin = 0, so the isospin wave function consists of a singlet function for the deuteron coupled with the isospin $\frac{1}{2}$ of the third nucleon.

As the deuteron spin = 1, we have two possible initial spin states, the quartet of total spin = $\frac{3}{2}$ and the doublet of total spin = $\frac{1}{2}$.

As we are considering only an S-wave non-local separable potential, we shall take the deuteron to be in a pure S-state. In this case the anti-symmetry is completely given by the isospin part of the deuteron wave function, using (2.10) then

$$T_{ab} = \frac{1}{3} \sum_{\beta,\alpha=1}^{3} {}_{as}\left\langle \underline{p}'q'_\beta D'_\beta I'I'_z I'_\beta S'S'_z S'_\beta \middle| T_{\beta\alpha} \middle| \underline{p}\, q_\alpha D_\alpha II_z I_\alpha SS_z S_\alpha \right\rangle_{as} \tag{2.16}$$

the values of I_α, in the above summation will be 0, 1. For the spin quartet scattering the S_α values must = 1. For an isospin independent interaction the functions

$$\left\langle I'I'_z I'_\beta \middle| I\, I_z\, I_\alpha \right\rangle = \delta_{II'}\, \delta_{I_z I'_z}\, U^{\beta\alpha}(I'_\beta, I_\alpha) \tag{2.17}$$

may be extracted. (See appendix 1).

Taking out a delta-function of total momentum conservation on both sides, together with Kronecker delta-functions of isospin and spin conservation we find,

$$\delta(\underline{p}-\underline{p}')\,\delta_{II'}\,\delta_{I_z I'_z}\,\delta_{SS'}\,\delta_{S_z S'_z}\,\mathcal{T}_{ab} =$$

$$\delta(\underline{p}-\underline{p}')\,\delta_{II'}\,\delta_{I_z I'_z}\,\delta_{SS}\,\delta_{S_z S'_z}\,\frac{1}{3}\left\{\sum_{\alpha=1}^{3}\hat{T}_{\alpha\alpha} - \frac{1}{2}\sum_{\alpha,\beta=1}^{3}\hat{T}_{\alpha\beta}\bar{\delta}_{\beta\alpha}\right\} \tag{2.18}$$

$$\text{where}\quad \hat{T}_{\beta\alpha} \equiv \left\langle q'_\beta D'_\beta S'_\beta \middle| T_{\beta\alpha} \middle| q_\alpha D_\alpha S_\alpha \right\rangle \tag{2.19}$$

It may be seen in appendix 1 that for the quartet interaction the spin dependence of the T matrix becomes trivial and we shall therefore ignore it in dealing with the equations.

From equation (2.5)

$$\left\langle \beta \middle| T_{\beta\alpha} \middle| \alpha \right\rangle = \sum_{ij} \bar{\delta}_{i\beta}\,\bar{\delta}_{j\alpha}\left\langle \beta \middle| T^{ij} \middle| \alpha \right\rangle + \left\langle \beta \middle| V_\alpha \middle| \alpha \right\rangle \bar{\delta}_{\beta\alpha} \tag{2.20}$$

Equation (2.20) may be written in the form

$$\langle \beta | T_{\beta\alpha} | \alpha \rangle = \langle \beta | \gamma^{\beta\alpha} | \alpha \rangle + \langle \beta | V_\alpha | \alpha \rangle \bar{\delta}_{\beta\alpha} \qquad (2.21)$$

with

$$\langle \beta | \gamma^{\beta\alpha} | \alpha \rangle = \sum_{ij} \langle \beta | T^{ij} | \alpha \rangle \bar{\delta}_{i\beta} \bar{\delta}_{j\alpha} \qquad (2.22)$$

And, using (2.6) $\gamma^{\beta\alpha}$ must satisfy

$$\gamma^{\beta\alpha} = \sum_{ij} \bar{\delta}_{\beta i} \bar{\delta}_{\alpha j} t_i \delta_{ij} + \sum_{ij} \bar{\delta}_{\beta i} \bar{\delta}_{\alpha j} t_i G_o \sum_k \bar{\delta}_{ki} T^{kj}$$

which gives

$$\gamma^{\beta\alpha} = \sum_{ij} \bar{\delta}_{i\beta} \bar{\delta}_{j\alpha} t_i \delta_{ij} + \sum_i \bar{\delta}_{\beta i} t_i G_o \gamma^{i\alpha}$$

$$\therefore \quad = \sum_j \bar{\delta}_{\beta j} \bar{\delta}_{\alpha j} t_j + \sum_i \bar{\delta}_{\beta i} t_i G_o \gamma^{i\alpha} \qquad (2.23)$$

In order to be able to solve these equations, we need to introduce a complete set of states such that the number of integration variables is reduced. The following chapter establishes such a set in the Su(3) basis.

CHAPTER THREE

THREE PARTICLE STATES IN THE Su(3) BASIS

Section A. The Su(3) classification of three-particle states.

The purpose of this Chapter is to obtain states which have simple symmetry properties. Since we know that rotational invariance leads to conservation of angular momentum and its z component, then we would expect these quantities to emerge from the present analysis as two of the quantum numbers needed to label the states.

In momentum space the state of a three particle system can be characterized by the vectors $\underset{\sim}{p}_i$, Let us make an orthogonal transformation such that

$$(\underset{\sim}{p}_1\underset{\sim}{p}_2\underset{\sim}{p}_3) \rightarrow (\underset{\sim}{p}^{(1)}\ \underset{\sim}{p}^{(2)}\ \underset{\sim}{p}^{(3)})$$

with, for three equal-mass particles

$$\underset{\sim}{p}^{(1)} = \frac{1}{\sqrt{2}}(\underset{\sim}{p}_2 - \underset{\sim}{p}_1) \qquad (3.1a)$$

$$\underset{\sim}{p}^{(2)} = \frac{1}{\sqrt{6}}(2\underset{\sim}{p}_3 - \underset{\sim}{p}_1 - \underset{\sim}{p}_2) \qquad (3.1b)$$

$$\underset{\sim}{p}^{(3)} = \frac{1}{\sqrt{3}}(\underset{\sim}{p}_1 + \underset{\sim}{p}_2 + \underset{\sim}{p}_3) \qquad (3.1c)$$

These vectors span a 9-dimensional space, rotations in which are brought about by the orthogonal group O(9).

However, because of translational invariance, the total momentum of the three particles plays a straight-forward role.

That is, in the CM frame, $\underset{\sim}{p}^{(1)}$, $\underset{\sim}{p}^{(2)}$ are the only vectors required to specify the state. We therefore introduce a six-dimensional space spanned by vectors

$$\underset{\sim}{p} = (\underset{\sim}{p}^{(1)}, \underset{\sim}{p}^{(2)}) \qquad (3.2)$$

where the scalar product is defined in the usual way. In particular, we have the relation

$$p^2 = p^{(1)^2} + p^{(2)^2} = p_1^2 + p_2^2 + p_3^2 \qquad (3.2a)$$

Following Dragt, we now look for the linear transformations which leave invariant the form

$$2ME_o = p^2 = p_1^2 + p_2^2 + p_3^2 \qquad (3.3)$$

with M = mass of each particle

E_o = total kinetic energy in the CM System.

That is, we are looking for all orthogonal transformations in six-dimensional space, O(6). If the Lie algebra for O(6) is L_o, it will be characterized by the fifteen anti-symmetric 6 x 6 matrices.

$$R_{ij} = |i\rangle\langle j| - |j\rangle\langle i| \qquad i,j = 1 \ldots .6$$

where $|i\rangle$ denotes a six-dimensional column vector in a real vector space whose i^{th} component is unity whilst the others are zero. $\langle i|$ is the corresponding row vector.

The algebra L_o is given by the commutation rules

$$\left[R_{ij}, R_{mn}\right] = 0 \quad i \neq j \neq m \neq n \qquad (3.3a)$$

$$\left[R_{ij}, R_{jk}\right] = R_{jk} \qquad (3.3b)$$

with $$R_{ij} = -R_{ji} \qquad (3.3c)$$

Not all the elements of L_o treat all threeparticles equivalently. But we may obtain a sub algebra of L_o which does. Define L_1 to be the set of all elements F in L_o with the property

$$\left[G,F\right] = 0 \qquad (3.4)$$

where G effects a cyclic permutation on three objects e.g. 123 → 231. Then the effect of a cyclic permutation operation on the three-particle states we will construct is very simple. It is also found that the effect

of a transposition operation on the elements of L_1 is to produce a sign change at most.

The subalgebra L_1 has, therefore, very simple symmetry properties, and is in fact the algebra for U(3), having the elements

$$J_{ij} = R_{ij} + R_{i+3\ j+3} \qquad i,j \leqslant 3 \quad i \neq j$$
$$K_{ij} = R_{i,j+3} - R_{i+3,j} \qquad i,j \leqslant 3 \tag{3.5}$$

As $R_{ij} = - R_{ji}$ then it follows that

$$J_{ij} = - J_{ji} \qquad K_{ij} = - K_{ji} \tag{3.6}$$

Therefore, L_1 is nine-dimensional. If we extract from L_1 the linear Casimir operator of U(3).

$$S = \tfrac{1}{2} \sum_{i=1}^{3} K_{ii} \tag{3.7}$$

then the remaining eight elements form the Lie algebra, L_2, which is isomorphic to Su(3). For a quantum mechanical system, we need realizations of these algebras as Lie algebras of Hermitian operators acting in the Hilbert space of three particle states. Let $\underline{r} = (\underline{r}^{(1)}, \underline{r}^{(2)})$ be the six-dimensional co-ordinate vector corresponding to (3.2). That is, $\underline{p}$ and $\underline{r}$ are Hermitian and canonically conjugate, satisfying the commutation relation

$$\left[r_i, p_j\right] = i \hbar \delta_{ij}$$

The quantum analogue of the R_{ij} is then a set of operators, Λ_{ij}, with the following properties:

$$\left[\Lambda_{ij}, \underline{r}\right] = i R_{ij} \underline{r} \tag{3.8a}$$
$$\left[\Lambda_{ij}, \underline{p}\right] = i R_{ij} \underline{p} \tag{3.8b}$$

That is, in an exactly similar way as for orthogonal transformations in ordinary three dimensional space, we take

$$\Lambda_{ij} = r_i p_j - r_j p_i \tag{3.9}$$

By their definition in equation (3.9) the Λ_{ij} are subject to the identity

$$\Lambda_{ij}\Lambda_{kl} + \Lambda_{il}\Lambda_{jk} + \Lambda_{ik}\Lambda_{lj} = 0 \quad ; \; i\neq j\neq k\neq l \qquad (3.10)$$

The quadratic Casimir operator Λ^2 for O(6), which is also the square of the "Grand angular momentum tensor"[11] is

$$\Lambda^2 = \tfrac{1}{2} \sum_{i,j}^{6} (\Lambda_{ij})^2 \qquad (3.11)$$

The elements of L_1 in terms of the Λ_{ij} are:

$$\begin{aligned} J_{ij} &= \Lambda_{ij} + \Lambda_{i+3,j+3} && i,j \leq 3 \quad i \neq j \\ K_{ij} &= \Lambda_{i,j+3} - \Lambda_{i+3,j} && i,j \leq 3 \end{aligned} \qquad (3.12)$$

with $K_{ij} = K_{ji}$ and $J_{ij} = -J_{ji}$

Using (3.5), we obtain that

$$\Lambda^2 = \tfrac{1}{2} \sum_{ij} (J_{ij}^2 + K_{ij}^2) + \sum_{i,j} (\Lambda_{i,j+3}\Lambda_{i+3,j} - \Lambda_{ij}\Lambda_{i+3,j+3})$$

$$= \tfrac{1}{2} \sum_{i\,j} (J_{ij}^2 + K_{ij}^2) - S^2 \qquad (3.13)$$

$$S = \tfrac{1}{2} \sum_{i=1}^{3} K_{ii}$$

Λ^2 is the quadratic Casimir operator for L_1 and together with S their eigenvalues specify an irreducible representation (I.R.) of Su(3).

The Lie Algebra of L_1 is

$$\begin{aligned} [J_j, J_k] &= i\,\varepsilon_{jkl}\,J_l \\ [J_l, K_{kl}] &= i\,\varepsilon_{jkm}\,K_{ml} + i\,\varepsilon_{jlm}\,K_{km} \\ [K_{ij}, K_{mn}] &= i\,(\delta_{im}J_{jn} + \delta_{in}J_{jm} + \delta_{jm}J_{in} + \delta_{jn}J_{im}) \end{aligned} \qquad (3.14)$$

where $J_i = \tfrac{1}{2}\,\varepsilon_{ijk}\,J_{jk}$.

The J_i's form a vector given by

$$\underline{J} = \underline{r}_1 \times \underline{p}_1 + \underline{r}_2 \times \underline{p}_2 + \underline{r}_3 \times \underline{p}_3 - \underline{r}^{(3)} \times \underline{p}^{(3)} \qquad (3.15)$$

In the centre of momentum frame $\underline{p}^{(3)} = 0$, and $\underline{J}$ is then interpreted as the total angular momentum in that frame. The K_{ij}'s form a symmetric

tensor, the physical interpretation of which is not obvious.

Section B. The Construction of Su(3) Eigenstates.

In reference 12 Dragt has shown that there is a one to one correspondence between the irreducible representations of O(6) carried by three-particle states and those of Su(3). In order to construct representations for Su(3), we need a convenient system of co-ordinates. Since Su(3) treats the three particles equivalently, the co-ordinate system should do also. In the CM frame a system of three equal-mass particles with total kinetic energy E_o poses the constraint

$$\vec{p}_1+\vec{p}_2+\vec{p}_3 = 0 \qquad (3.16)$$

with

$$2mE_o = p_1^2 + p_2^2 + p_3^2 = p^{(1)2}+p^{(2)2} = p^2. \qquad (3.17).$$

We shall use the Dalitz-Fabri co-ordinates[18,19]. Consider the 'momentum triangle' whose vertices are the end points of the three momentum vectors directed from a common origin, the centre of momentum. The three Euler angles α,β,γ required to transform the triangle from a reference orientation to its actual orientation, may be conveniently taken to specify the orientation of this triangle in space. Then we need to parametrize the triangle itself.

Consider a second, equilateral, triangle of unit altitude. (See fig.(3.1)) This second triangle must not be confused with the first one, being constructed in a totally different space. For such a triangle the sum of the three distances from an interior point to each of the three sides is the same for each interior point and equal to 1.

Identify the three distances with the quantities $\frac{p_i^2}{p^2}$; then equation (3.17) is automatically satisfied. The polar co-ordinates ρ,ϕ may then be taken as the remaining two co-ordinates needed, and we have the relationships,

$$p_i^2 = \frac{1}{3} p^2 (1+\rho\zeta_i)$$

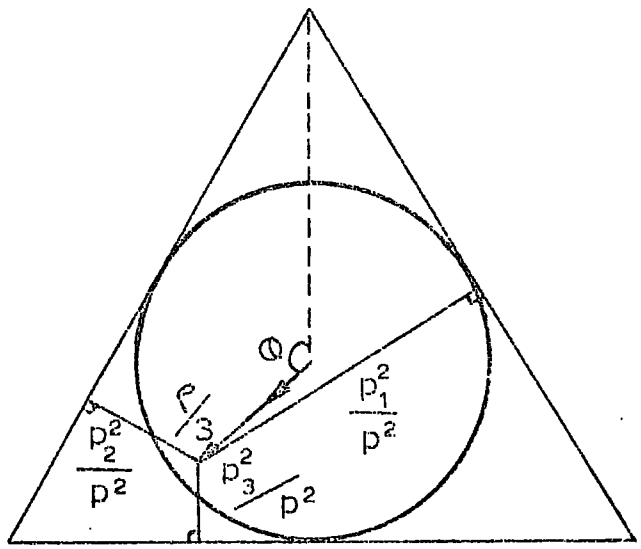


FIG (3·1)

with

$$\xi_1 = \cos\left(\phi - 2\frac{\pi}{3}\right) \tag{3.18a}$$

$$\xi_2 = \cos\left(\phi + 2\frac{\pi}{3}\right) \tag{3.18b}$$

$$\xi_3 = \cos\phi \tag{3.18c}$$

In order to be able also to satisfy equation (3.16) we find that the kinematically allowed region for our new co-ordinates is that given by the disc inscribed within the Daltiz triangle, i.e. such that

$$\rho^2 \leqslant 1$$

The choice of Dalitz-Fabri position co-ordinates can be made in complete analogy to the momentum case.

We need now to determine a complete set of three-particle states $|\, p^{(3)}, p^2, N \,\rangle$ which form a carrier space for irreducible representations of L_2. An index, N, is used to distinguish between states having the same energy and momentum, but different properties under L_2.

If $|\, \underline{r}^{(1)}\ \underline{r}^{(2)}\ \underline{r}^{(3)} \,\rangle$ denotes the eigenstate for the position operators $\underline{r}^{(1)}, \underline{r}^{(2)}, \underline{r}^{(3)}$ then the state $|\, \underline{p}^{(3)}\ p^2\ N\rangle$ is specified by

$$\langle \underline{r}^{(1)}\ \underline{r}^{(2)}\ \underline{r}^{(3)} |\, \underline{p}^{(3)}\ p^2 N\rangle = \exp\left(i\, \underline{p}^{(3)} \cdot \underline{r}^{(3)}\right)\psi$$

where $\psi = \langle \underline{r}^{(1)}\ \underline{r}^{(2)}\ 0 \mid 0\ \ p^2\ N \rangle$

must satisfy the six-dimensional Schrödinger equation:

$$-\nabla_6^2 \psi = p^2 \psi \tag{3.19}$$

where (∇_6/i) is the differential operator analog of $\underline{p}$. p^2 is related to the operator Λ^2 by

$$p^2 = \frac{\Lambda^2}{r^2} + p_r^{\,2} - \frac{5i}{r}\, p_r$$

where $p_r = \frac{1}{r}\ \underline{r} \cdot \underline{p}$

ψ may then be written in the form $\psi = f(r)\, g\, (\rho\phi\alpha\beta\gamma)$.

with f and g satisfying the equations:

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{5}{r}\frac{\partial}{\partial r} = \frac{\lambda(\lambda+4)}{r^2} + p^2 \right\} f = 0. \tag{3.20}$$

$$\Lambda^2 \, g = \lambda(\lambda+4) g \tag{3.21}$$

($\rho\phi\alpha\beta\gamma$ here refer to position Dalitz-Fabri co-ordinates).

$\hat{\Lambda}^2$ is the differential operator analog of Λ^2.

Demanding that f be regular at the origin then we find

$$\left\langle \underline{r}^{(1)}, \underline{r}^{(2)}, \underline{r}^{(3)} \,\middle|\, \underline{p}^{(3)}, p^2, N \right\rangle = g \exp(i\underline{p}^{(3)} \cdot \underline{r}^{(3)}) \frac{1}{r^2} J_{\lambda+2}(pr) \tag{3.22}$$

And the $J_{\lambda+2}$ is a Bessel function of the first kind.

$$\therefore \psi = g \frac{1}{r^2} J_{\lambda+2}(pr) \tag{3.23}$$

We can find g by looking at the 'harmonic oscillator' generalization of equation (3.19) i.e.

$$(-\nabla_6^2 + r^2)\psi_1 = 2E\psi_1 \tag{3.24}$$

As before we also write the solution as

$$\psi_1 = f_1 g_1 \tag{3.25}$$

with f_1 satisfying the equation

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{5}{r}\frac{\partial}{\partial r} - \frac{\lambda(\lambda+4)}{r^2} - r^2 + 2E \right\} f_1 = 0 \tag{3.26}$$

and g_1 satisfying the same equation as g. Therefore we can find g by determining the radial part of the solution of the harmonic oscillator problem.

Define vector creation operators $\underline{A}^+$ and $\underline{B}^+$ by the relations

$$\begin{aligned} \underline{A}^+ &= \frac{1}{\sqrt{2}} (\underline{r}^{(1)} - i\,\underline{p}^{(1)}) \\ \underline{B}^+ &= \frac{1}{\sqrt{2}} (\underline{r}^{(2)} - i\,\underline{p}^{(2)}) \end{aligned} \tag{3.27}$$

with their conjugate annihilation operators they obey:

$$\left[A_i, A_j^+\right] = \delta_{ij} \quad \text{etc.}$$

A simplification results if we pass to new variables $\underline{a}^+$ and $\underline{b}^+$ by the transformation

$$\begin{aligned} \underline{a}^+ &= \frac{1}{\sqrt{2}} (\underline{B}^+ + i\underline{A}^+) \\ \underline{b}^+ &= \frac{1}{\sqrt{2}} (\underline{B}^+ - i\underline{A}^+) \end{aligned} \tag{3.28}$$

The operators $\underline{J}$ and K together with the Hamiltonian H, now take the form

$$H = (\underline{a}^{+}.\underline{a} + \underline{b}^{+}.\underline{b}) \tag{3.29}$$

$$\underline{J} = i\underline{a}x\underline{a}^{+} + i\underline{b}x\underline{b}^{+} \tag{3.30}$$

$$K_{jk} = b_j^{+} b_k + b_k^{+} b_j - (a_j^{+} a_k + a_k^{+} a_j) \tag{3.31}$$

We need eigenstates of H which simultaneously transform irreducibly under L_2. To this end the following operators which commute with $\underline{J}$ and K are constructed:

$$N_a^{op} = \underline{a}^{+}.\underline{a} \tag{3.32a}$$

$$N_b^{op} = \underline{b}^{+}.\underline{b} \tag{3.32b}$$

$$\Delta_{+} = \underline{a}^{+}.\underline{b}^{+} \tag{3.32c}$$

$$\Delta_{-} = \underline{a}.\underline{b}. \tag{3.32d}$$

N_a^{op} and N_b^{op} are number operators measuring the number of excitations of types a and b. Since N_a^{op} and N_b^{op} commute with H and the elements of L_1 we may require that our states be simultaneous eigenstates of N_a^{op} and N_b^{op} with eigenvalues N_a and N_b respectively. As for the $\Delta_{\pm}$ it may be shown that if we have a representation of L_2 labelled by N_a, N_b than applying $\Delta_{\pm}$ produces another one labelled by $N_a \pm 1$, $N_b \pm 1$. Further since $\Delta_{\pm}$ commutes with L_2 this leaves us in the same representation. The only exception is the case for which Δ_{-} annihilates all the vectors in a representation. We will therefore impose the condition that all our states be annihilated by Δ_{-}, because we want the N_a, N_b to uniquely label a representation. Further, Δ_{-} lowers the energy of a "harmonic oscillator" state. Since it also commutes with Λ^2, the radial part of Ψ_1 must be that solution to equation (3.26) which has the lowest energy compatible with a given value of λ.

Therefore it can be shown that:

$$f_1 = r^{\lambda} e^{-\frac{1}{2}r^2} \tag{3.33}$$

Using equations (3.13), (3.30) and (3.31) we find that

$$\Lambda^2 = (N_a^{op} + N_b^{op})(N_a^{op} + N_b^{op} + 4) - 4\Delta_{+}\Delta_{-} \tag{3.34}$$

As Δ_{-} has zero eigenvalue for our states then Λ^2 has eigenvalues of the form

$\lambda(\lambda+4)$ with

$$\lambda = N_a + N_b. \tag{3.35}$$

Also

$$S = \tfrac{1}{2} \sum_i K_{ii} = N_b - N_a \tag{3.36}$$

Therefore S has eigenvalues $N_b - N_a$. Consequently the numbers N_a and N_b for a representation can be obtained completely in terms of operators constructed out of elements of L_1.

Now, each I.R of Su(3) is characterized by the two Cartan[20] indices λ_1, λ_2. The states we have constructed are eigenstates of Λ^2 and S, and since both these operators commute with all the elements of L_2, the numbers N_a and N_b must be related to λ_1 and λ_2. N_a and N_b are, in fact, identical to the Cartan indices. Once one has a given representation it is necessary to have further indices in order to specify the different vectors within a representation. The situation is analogous to the rotation group where one uses the angular momentum j to specify the representation and the additional index j_z to specify a state within the representation.

For Su(3) we need three additional indices. At this stage we have to consider the ways in which Su(3) contains Su(2) as a subalgebra. It is contained in two algebraically distinct ways, with elements, say $X_\pm$, X_o and and $Y_\pm$, Y_o. For the X set we find that there is a fourth element, X_4 of Su(3) which commutes with all the elements of the X-type of Su(2). However, for the Y set there is no fourth commuting element. Further, X_o has integral and half-integral eigenvalues, while Y_o has only integral eigenvalues. The Y-type is that used for the classification of three-particle states. In this case one gets with relative ease only the two labels provided by the eigenvalues of Y^2 and Y_o. For the three-body problem these operators represent the total angular momentum in the CM system and its Z component.

The third index is much harder to obtain. The question is, in a representation characterized by λ_1 and λ_2, how many states have the same J and J_z? Racah[16] has tackled the problem and his results are summarized in Table 1.

Table 1. Multiplicity within Su(3)

J.	λ_1, λ_2	Multiplicity
even	both even. one or both odd.	$\frac{1}{2}$ J +1 $\frac{1}{2}$J.
odd	both even one or both odd	$\frac{1}{2}$(J-1) $\frac{1}{2}$(J+1)

To produce a third index, say ω, it is necessary to construct a cubic operator, Ω, which is sufficient to break all degeneracies. However, from table 1, we see that a third label is only necessary for the states with $J \geqslant 2$. Further, we see that the degeneracy is basically determined by J.

If j is the total angular momentum in the CM system and m is its Z component then a state may be represented by

$$|N_a N_b\ j\ m\rangle$$

For $j < 2$, $N_a N_b$, j, m, will be sufficient to label our states and these will satisfy:

$$N_{a,b}^{op} |N_a N_b jm\rangle = N_{a,b} |N_a N_b\ jm\rangle \qquad (3.37a)$$

$$J^2 |N_a N_b jm\rangle = j(j+1) |N_a N_b jm\rangle \qquad (3.37b)$$

$$J_z |N_a N_b jm\rangle = m |N_a N_b jm\rangle \qquad (3.37c)$$

Also we have the annihilation condition:

$$\Delta_- |N_a N_b jM\rangle = 0 \qquad (3.38)$$

Equations (3.37b,c) are equivalent to the conditions

$$J_+ |N_a N_b jj\rangle = 0$$

$$J_z |N_a N_b jj\rangle = j |N_a N_b jj\rangle \qquad (3.39)$$

Where J_+ is the raising operator as known in the theory of angular momentum. Remaining states may be found by using the lowering operator J_-, where

$$J_\pm = J_x \pm i\ J_y.$$

The construction of explicit solutions to equations (3.37) is dealt with in Appendix 2. We now set out to find the explicit angular wave-functions.

The harmonic oscillator states are of the general form

$$\left|N_a N_b jm\right\rangle = P\ (\underline{a}^+, \underline{b}^+)\ \left|\ 0\right\rangle \qquad (3.40)$$

where P is a polynomial in $\underline{a}^+$ and $\underline{b}^+$. From equations (3.25) and (3.33) the angular functions, g, may be written

$$g = r^{-\lambda} e^{\frac{1}{2}r^2} \left\langle \underline{r}^{(1)}\ \underline{r}^{(2)} \left|\ P\ (\underline{a}^+, \underline{b}^+)\ \right|\ 0 \right\rangle \qquad (3.41)$$

If we specify each g function with a label n, then it may be shown that

$$g_n = A_n\ r^{-\lambda}\ P\ (\underline{r}^{(2)} + i\underline{r}^{(1)}\ ,\ \underline{r}^{(2)} - i\underline{r}^{(1)}) \qquad (3.42)$$

with A_n = a constant

g_n = g function for $(N_a N_b jm)$.

In the momentum representation then we have a similar expression for the corresponding g_n. Therefore, replacing the variables by their momentum counterparts,

$$g_n = A_n\ p^{-\lambda} P(\underline{p}^{(2)} + i\underline{p}^{(1)}\ ,\ \underline{p}^{(2)} - i\underline{p}^{(1)}) \qquad (3.43)$$

Each g function now needs to be converted into Dalitz-Fabri variables. The details of this for the first few states are contained in Appendix 3.

Finally, we give the momentum counterpart of the equation (3.22). The exponential and radial wave functions are replaced by momentum and energy delta functions. From equations (3.8)and (3.9), the operators Λ_{ij} treat momentum and position operators on equal footing. Therefore the angular function, g, remain unchanged. One simply has to replace the positron Euler angles and position Dalitz-Fabri co-ordinates by their momentum counterparts. One then obtains,

$$\left\langle \underline{p}^{(1)'}\ \underline{p}^{(2)'}\ \underline{p}^{(3)'} \left|\ \underline{p}^{(3)}\ p^2\ N_a N_b jm \right\rangle = C_1 \delta(\underline{p}^{(3)'} - \underline{p}^{(3)})\ \delta\left(\frac{p^2 - p'^2}{p^2}\right) \cdot$$

$$\cdot\ g(N_a N_b jm; \rho\phi\alpha\beta\gamma) \qquad (3.44)$$

with C_1 a constant.

Here, the states $\left| p^{(3)}\ p^2\ N_a N_b jm \right\rangle$ are defined for a particular channel. However, channels are related by cyclic permutations, brought about by the operator C,

$$C \left| N_a N_b jm \right\rangle = \exp\left(i \frac{2\pi}{3} (N_b - N_a)\right) \left| N_a N_b jm \right\rangle \tag{3.45}$$

That is, there is only a phase factor difference between corresponding states in the different channels. We also note that the operator P_3 which transposes particles 1 and 2 has the effect

$$P_3 \left| N_a N_b jm \right\rangle = \left| N_b N_a jm \right\rangle \tag{3.46}$$

More details are given in Appendix 4.

CHAPTER FOUR

SOLVING THE INTEGRAL EQUATIONS.

Section A Final Integral Equation Using Su(3) States

We are now in a position to conduct a decomposition of the elements of equation (2.18) using the Su(3) states $|p^2\ N_a N_b jm\rangle_\beta \equiv |p^2\ N_\beta\rangle$ corresponding to the channel β. The closure relation for these states is (see appendix 5)

$$\sum_{N_\beta} \int |p^2\ N_\beta\rangle \langle p^2\ N_\beta|\ dp^2 \quad = 1 \qquad (4.1)$$

It is easily seen that $\langle p^2\ N_\beta\ |\beta\rangle$ is invariant for $\beta = 1,2$ or 3 so that equation (2.18) can be cast into the form

$$\mathcal{T}_{ab} = \tfrac{1}{3}\int dp^{2"}\ dp^{2'''} \sum_{N_\alpha" N_\beta'''} \langle \beta=3 | p^{2"} N_3" \rangle \left\{ \sum_\alpha^3 \Pi^{\alpha\alpha} - \tfrac{1}{2} \sum_{\alpha,\beta}^3 \Pi^{\alpha\beta} \delta_{\alpha\beta} \right\} \langle p^{2'''} N_3''' | \beta=3 \rangle$$

$$- \frac{1}{6} \sum_{\beta\alpha} \delta_{\alpha\beta} \langle \beta | V_\alpha | \alpha \rangle \qquad (4.2)$$

where $$\Pi^{\beta\alpha} = \langle p^{2"} N_\beta" |\ \gamma^{\beta\alpha}\ | p^{2'''} N_\alpha''' \rangle \qquad (4.3)$$

Because of the simple symmetry properties of these Su(3) states, we are able to reduce the problem to the solution of a single integral equation except for later terms in the expansion in $N_a N_b$ when angular momentum then allows two or more contributions to couple together. In otherwords, we can now go to integral equations which do not involve the index α

Details of the procedure are given in Appendix 4.

Section B Finding the T matrix using the Su(3) Basis.

From equation (2.23) we see that we need to determine transformation expressions such as

$$F_\alpha \equiv \langle q'_\alpha D_\alpha |\ p^{2"}\ N_a" N_b" j" m" \rangle \qquad \text{in basis } \alpha.$$

Because of the identity of the particles then

$$F_1=F_2=F_3=F.$$

Let us then do the calculation for F_3. Introducing a complete set of states $|\underline{p}^{(1)}\ \underline{p}^{(2)}\rangle$ for the three-particle CM. system, we find

$$F = \int \langle q_3' D_3 | \underline{p}^{(1)''} \underline{p}^{(2)''} \rangle \langle \underline{p}^{(1)''} \underline{p}^{(2)''} | p''^2 N_3'' \rangle \, d\underline{p}^{(1)''} d\underline{p}^{(2)''} \tag{4.4}$$

From equations (2.12), (2.13) and (3.1) we obtain

$$\underline{k}_3 = \frac{\underline{p}^{(1)}}{\sqrt{2}} \qquad \underline{q}_3 = -\sqrt{\frac{2}{3}}\ \underline{p}^{(2)} \tag{4.5}$$

and

$$\langle \underline{p}^{(1)} \underline{p}^{(2)} | = \frac{1}{c^3} \langle \underline{q}_3 \underline{k}_3 | \tag{4.6}$$

with $c = 3^{\frac{1}{4}}$

Therefore

$$F = d\underline{p}^{(1)''} d\underline{p}^{(2)''} \frac{1}{c^3} \int \delta(q_3'-q_3'') \phi_D^*(\underline{k}_3'') \delta(p''^2-p''^2) \frac{1}{p^2} g(N_a'' N_b'' j'' m''; \rho'' \phi'' \alpha'' \beta'' \gamma'') \tag{4.7}$$

Before we can put this expression completely into Dalitz-Fabri co-ordinates we need to specify the body-fixed set of axes that we are using for the momentum triangle as introduced in Section 3B.

Consider a body-fixed set of axes (x,y,z) such that the positive z axis is along the vector $\underline{q}_3$ and the momentum triangle is in the x plane. Initially, the x,y,z axes coincide with the space-fixed set $\underline{x},\underline{y},\underline{z}$. The orientation of the final momentum triangle, that is for which $\underline{q}_3'$ lies along the z axis, may be specified by the Euler angles:

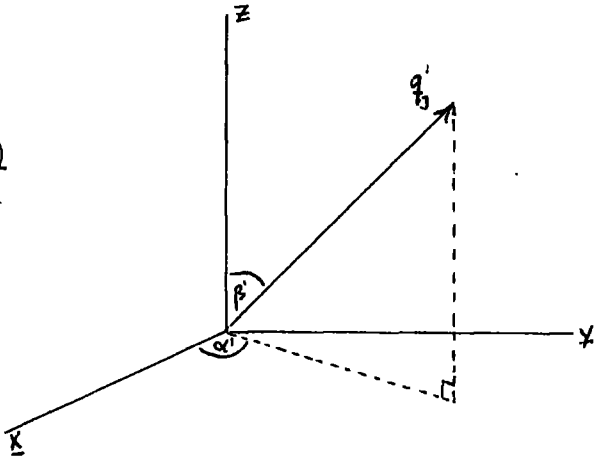
First, a rotation of angle γ' about z $\qquad \gamma' \leq 2\pi$

Second, a rotation of angle β' about y $\qquad \beta' \leq \pi$

Third, a rotation of angle α' about z $\qquad \alpha' \leq 2\pi$

(see Fig (4.1)).

Figure (4.1)



One of the delta-functions in equation (4.7) may now be written

$$\delta(q'_3 - q''_3) = \frac{2}{q'_3}\,\delta(\cos\beta' - \cos\beta'')\,\delta(q'^2_3 - q''^2_3)\,\delta(\alpha' - \alpha'') \qquad (4.8)$$

We use an S-wave non-local separable potential of the Yamaguchi type[17] with strength parameter λ ,

$$V_\alpha(\underline{k}_\alpha ; \underline{k}'_\alpha) = \lambda\, v_\alpha(k_\alpha)\, v_\alpha(k'_\alpha) \qquad (4.9)$$

$$\text{with} \quad v_\alpha(k_\alpha) = \frac{1}{(k_\alpha^2 + \beta_n^2)} \qquad (4.10)$$

The deuteron wave function for this potential is then

$$\phi_D(k) = \frac{A}{(\alpha_D^2 + k^2)\,(\beta_n^2 + k^2)} \qquad (4.11)$$

where $A^2 = \dfrac{\alpha_D \beta_n (\alpha_D + \beta_n)^3}{\pi^2}$ and $B = +\dfrac{\alpha_D^2}{2\mu}$

Here 2μ is the mass of the nucleon and B the deuteron binding energy.

From equation (4.5) we see that

$$p''^2 = 2\,k''^2_3 + \frac{3}{2}\,q''^2_3 \qquad (4.12)$$

$$= 2\,k''^2_3 + \frac{3}{2}\cdot p''^2_3\cdot(1 + \rho''\cos\phi'')$$

$$\therefore \quad k''^2_3 = p''^2_4\,(1 - \rho''\cos\phi'') \qquad (4.13)$$

This enables us to put equation (4.7) entirely into Dalitz-Fabri co-ordinates, with the volume element

$$dp^{(1)}dp^{(2)}dp^{(3)} = \frac{1}{8}\, dp^{(3)}\, p^5\, dp\, dR\, \rho d\rho\, d\phi \qquad (4.14)$$

where $dR = -d\alpha\, d(\cos\beta)\, d\gamma$

The final evaluation of F for each g function is shown in Appendix 6.

The next task is to find the inhomogeneous term (the two particle T operator, t_i, between 'Dragt' states). It is easily verified that the two particle T-matrix at energy E in the three-particle CM space defined by vectors $q_\alpha, \underline{k}_\alpha$ is related to the two-particle T-matrix in the two-particle CM space defined by $\underline{k}_\alpha$ according to

$$\left\langle q'_\alpha\, \underline{k}'_\alpha \,\middle|\, t_\alpha(E) \,\middle|\, q_\alpha \underline{k}_\alpha \right\rangle = \delta(q_\alpha - q'_\alpha) \left\langle \underline{k}'_\alpha \,\middle|\, t_\alpha(e_\alpha) \,\middle|\, \underline{k}_\alpha \right\rangle \qquad (4.15)$$

where $e_\alpha = E - \dfrac{q_\alpha^2}{2\mu^\alpha}$ $\qquad \mu^\alpha = \dfrac{M_\alpha (M_\beta + M_\gamma)}{M}$ $\qquad M = M_\alpha + M_\beta + M_\gamma$

and $\dfrac{q_\alpha^2}{2\mu^\alpha}$ is the energy of the particle α in the three particle CM system.

For the separable potential of type (4.9) then one obtains [14]

$$\left\langle \underline{k}'_\alpha \,\middle|\, t_\alpha(e_\alpha) \,\middle|\, \underline{k}_\alpha \right\rangle = v_\alpha(k'_\alpha)\, \tau_\alpha(e_\alpha)\, v_\alpha(k_\alpha) \qquad (4.16)$$

where $$\tau_\alpha(e_\alpha) = \frac{\lambda}{1 - \lambda \int \dfrac{4\pi k_\alpha^2 v_\alpha^2(k_\alpha)\, dk_\alpha}{\left(e_\alpha - \dfrac{k_\alpha^2}{\mu_\alpha} + i\eta\right)}} \qquad (4.17)$$

and $\mu_\alpha = \dfrac{M_\beta M_\gamma}{M_\beta + M_\gamma}$ i.e. reduced mass of particles $\beta + \gamma$ in the two particle subsystem.

We may now write

$$\left\langle p''^2\, N''_\alpha \,\middle|\, t_\alpha \,\middle|\, p'''^2\, N'''_\alpha \right\rangle = \int \left\langle p''^2\, N''_\alpha \,\middle|\, \hat{q}_\alpha\, \hat{\underline{k}}_\alpha \right\rangle \left\langle \hat{q}_\alpha\, \hat{\underline{k}}_\alpha \,\middle|\, t_\alpha \,\middle|\, \hat{q}'_\alpha\, \hat{\underline{k}}'_\alpha \right\rangle \left\langle \hat{q}'_\alpha\, \hat{\underline{k}}'_\alpha \,\middle|\, p'''^2\, N'''_\alpha \right\rangle \cdot d\hat{q}_\alpha d\hat{\underline{k}}_\alpha\, d\hat{q}'_\alpha d\hat{\underline{k}}'_\alpha \qquad (4.18)$$

and from (4.14) and (4.5)

$$dq_\alpha \, d\underline{k}_\alpha = \frac{1}{8}\,\frac{1}{c^6}\, p^5\, dp\, dR\, \rho\, d\rho\, d\phi$$

Using (3.44) this produces

$$\left\langle p''^2 N''_\alpha \,|\, t_\alpha \,|\, p'''^2 N'''_\alpha \right\rangle = \frac{c_1^2}{64\, c^{12}} \int \hat{p}^5 d\hat{p}\; d\hat{\alpha}_\alpha\; d(\cos\hat{\beta}_\alpha)\, d\hat{\gamma}_\alpha\; \hat{p}'^5\, d\hat{p}'\; d\hat{\alpha}'_\alpha\; d(\cos\hat{\beta}'_\alpha)\, d\hat{\gamma}'_\alpha$$

$$\hat{\rho}\, d\hat{\rho}\; d\hat{\phi}\; \hat{\rho}'\, d\hat{\rho}'\; d\hat{\phi}'\; \delta\left(\frac{p''^2 - \hat{p}^2}{p''^2}\right) g''_n\, g'''^*_n\, \frac{2}{\hat{q}_\alpha}\, \delta(\hat{q}_\alpha^2 - \hat{q}_\alpha'^2)\, \delta(\hat{\alpha}_\alpha - \hat{\alpha}'_\alpha)\, \delta(\cos\hat{\beta}_\alpha - \cos\hat{\beta}'_\alpha)$$

$$\nu_\alpha(\hat{k}_\alpha)\, \zeta_\alpha(\hat{e}_\alpha)\, \nu_\alpha(\hat{k}'_\alpha)\, \delta(\hat{p}'^2 - p'''^2)\, \frac{1}{p'''^2} \qquad (4.19)$$

The calculation is facilitated by changing from the circular variables (ρ,ϕ) to the Cartesian variables (x,y) as defined in Figure (4.2) for the Dalitz-Fabri triangle. Then, integrating over $\hat{\alpha}_\alpha$ and $\cos\hat{\beta}_\alpha$ equation (4.19) becomes

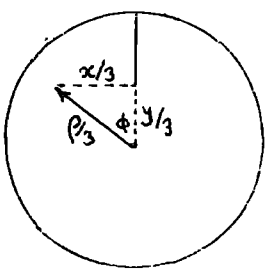


Figure (4.2)

$$\left\langle p''^2 N''_\alpha \,|\, t_\alpha \,|\, p'''^2 N'''_\alpha \right\rangle = -\frac{G_1^2}{64\, c^{12}} \int \hat{p}^5\, d\hat{p}\; d\hat{\gamma}_\alpha\; \hat{p}'^5\, d\hat{p}'\, d\hat{\alpha}'_\alpha\; d(\cos\hat{\beta}'_\alpha)\; d\hat{\gamma}'_\alpha$$

$$d\hat{x}\; d\hat{y}\; d\hat{x}'\, d\hat{y}'.\quad \frac{1}{p''^2}\, \delta(p''^2 - \hat{p}^2)\, g''_n g'''^*_n\, \frac{2}{\hat{q}_\alpha}\, \delta(\hat{q}_\alpha^2 - \hat{q}_\alpha'^2)\, \nu_\alpha(\hat{k}_\alpha)\, \zeta_\alpha(\hat{e}_\alpha)\, \nu_\alpha(\hat{k}'_\alpha)$$

$$\delta(\hat{p}'^2 - p'''^2)\, \frac{1}{p'''^2} \qquad (4.20)$$

This equation is invariant with respect to cyclic permutations of particles 1,2,3 and so we may take $\alpha = 3$.

We have, therefore, $q_3^2 = \frac{1}{3}p^2\ (1 + y)$

and $k_3^2 = \frac{1}{4}p^2\ (1 - y)$

Caution is needed when integrating over the x and y variables in equation (4.20). The function $\delta(\hat{q}_\alpha^2 - \hat{q}_\alpha'^2)$ becomes

$$\delta\left(\frac{p''^2}{3}\ (1+\hat{y}) - \frac{p'''^2}{3}\ (1+\hat{y}')\right) \qquad (4.21)$$

which imposes restrictions on the limits of the $\hat{y}$ & $\hat{y}'$ integrations. Two cases need consideration, (a) $p'' > p'''$, (b) $p''' < p'''$.

(a)

The limits of the $\hat{y}'$ integration are +1 and -1. Then the limits of the $\hat{y}$ integration are

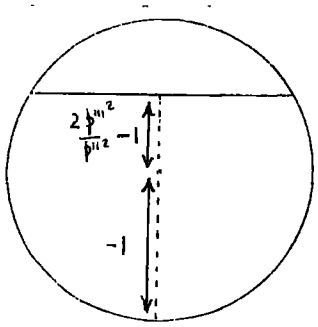
$$-1 \leqslant \hat{y} \leqslant 2\ \frac{p'''^2}{p''^2} - 1. \qquad \text{See figure 4.3}$$

(b)

The limits of the $\hat{y}$ integration are +1 and -1. Then the limits of the $\hat{y}'$ integration are

$$-1 \leqslant \hat{y}' \leqslant 2\ \frac{p''^2}{p'''^2} - 1.$$

Figure (4.3)



Range of $\hat{y}$ integration. (Likewise for (b) but with p''^2 & p'''^2 interchanged).

(a)

The limits of the x varibles are given by

$$-\sqrt{1 - y^2} \leqslant x \leqslant \sqrt{1 - y^2}$$

Therefore, we may easily carry out the integrations over $\hat{p}$, $\hat{p}'$, $\hat{x}$, $\hat{x}'$, and the integration over $\hat{y}'$ removes the δ function (4.21). Equation (4.20) becomes

$$\left\langle p^{2\,"} N" \left| t_3 \right| p^{2\,'''} N^{2\,'''} \right\rangle = -\frac{C_1^2}{64\,c^{12}} \frac{p^{"4}}{2} \frac{p^{'''4}}{2} \int d\hat{\gamma}_3\, d\hat{\alpha}_3'\, d(\cos\hat{\beta}_3')\, d\hat{\gamma}_3'\, d\hat{y}.$$

$$\frac{3}{p^{"2}}\; 2\sqrt{1-\hat{y}^2}\; 2.\sqrt{1-\hat{y}'^2}\; \frac{1}{p^{"2}}\; \frac{1}{p^{'''2}}\; g_n^{"}\, g_n^{'''*}\; \frac{2}{\hat{q}_3}\; v_3(\hat{k}_3)\mathcal{C}_3(\hat{e}_3)\, v_3(\hat{k}_3')$$

with $\hat{y}'^2$ now $= \frac{1}{p^{'''2}} (p^{"2} + p^{"2}\,\hat{y} - p^{'''2})$

$$\hat{q}_3^2 = \frac{p^{"2}}{3}(1+\hat{y}) \quad , \quad \hat{k}_3^2 = \frac{p^{"2}}{4}(1-\hat{y}) \, , \quad \hat{k}_3'^2 = \frac{p^{'''2}}{4}(1-\hat{y}')$$

To go further, we need to introduce explicit expressions for the g_n, v_3 and $\mathcal{C}_3$.

One finds

$$\mathcal{C}_3 = \frac{H_3}{(\hat{y} - \hat{y}_p)}.$$

where

$$H_3 = \frac{4}{p^{"2}} \frac{\chi(i\sqrt{2\mu_3 e_3} - \alpha)(\sqrt{2\mu_3 e_3} + i\beta_n)^2}{(\sqrt{2\mu_3 e_3} - i\beta_n)^2 (\alpha_n + 2\beta_n - i\sqrt{2\mu_3 e_3})}$$

and $$\hat{y}_p = \frac{4\alpha_n^2}{p^{"2}} + \frac{8\mu_3 E}{p^{"2}} - 1.$$

$\mathcal{C}_3$ exhibits a pole in the $\hat{y}$ range, the position of which depends upon $p^{"2}$. The pole may be physically interpreted as the deuteron bound state as one can see from Figure (4.4).

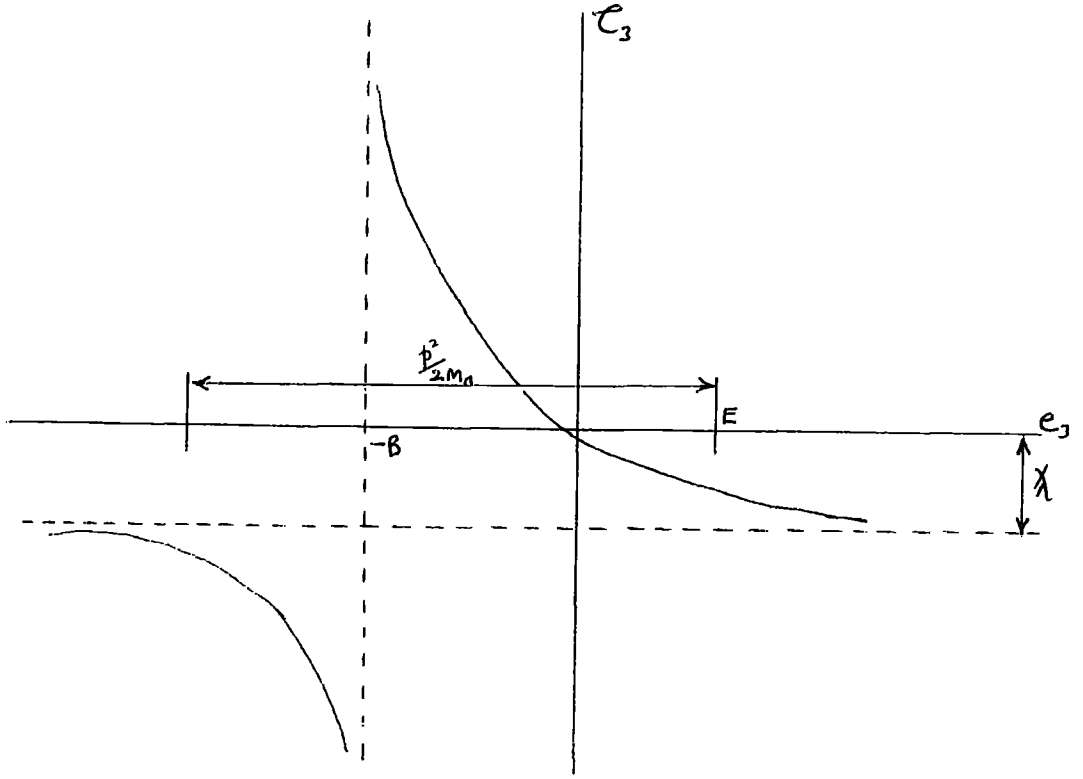


Figure (4.4) Graph of Real part of $\mathcal{C}_3$ for varying e_3.

(Using $e_3 = E - \frac{p^2}{4M_n}(1+\hat{y})$ with M_n = mass of nucleon, then the range of the $\hat{y}$ integration is given by the horizontal double headed arrow for p" = p"' . - B is the position of the deuteron pole).

The integration over the pole was carried out using the Cauchy relation.

$$\lim_{\eta \to 0} \frac{1}{(z-z_o \pm i\eta)} = P \frac{1}{(z-z_o)} \mp i\pi\delta(z-z_o) \quad , \; z_o = \text{constant.}$$

which gives

$$\lim_{\eta \to 0} \int \frac{f(z)}{(z-z_o \pm i\eta)} dz = P \int \frac{f(z)}{(z-z_o)} dz \mp \int i\pi\delta(z-z_o) f(z) dz.$$

Where P indicates a principal value integration. The integration over $\hat{y}$ could then be carried out numerically. The above relation was also used to "extract" the Green's function pole in the kernel of the integral equation. Using a matrix inversion sub-program, then the "Dragt T matrix" (i.e. between $|p^2 N\rangle$ states) could be found and finally $\mathcal{J}_{ab}$. Further details of the numerical calculation are given in appendix 7.

CHAPTER FIVE

NUMERICAL RESULTS

Our results fall into two parts. That is, calculations for

(a) The quartet scattering length, a_4

and (b) The quartet contributions to elastic neutron-deuteron differential cross-sections.

(a) The quartet scattering length

Of the lowest λ terms, those containing zero angular momentum contributions are $\lambda = 0, \lambda = 2$ and $\lambda = 4$. Therefore, a_4 was initially calculated using only $\lambda = 0$, then including $\lambda = 2$. The results are summarized in table 2, where they are also compared with those of Phillips [7], Aaron, Amado and Yam [21] and Zakharyev, Pustovalov and Efros [22]. The latter reference adopts a method similar to our own developed by Simonov [23], but uses a square-well potential. We are able to compare with the results of Amado et al. for which the renormalization constant, Z, is zero as this is the separable potential limit of the model of reference 21. Phillips also introduces a three-body force, V_4, but we are able to compare with his result for $V_4=0$. We see that our result for a_4 is quite good and in keeping with the other theoretical results.

Table 2

	Quartet Scattering length, a_4. (Fermis).
Experimental	6.38 ± 0.06 2.6 ± 0.2
Phillips ($V_4=0$)	6.28
Aaron, Amado & Yam (z=0)	6.32
Z., P. and Efros.	6.71
This work ($\lambda=0$)	5.05
This work ($\lambda=0$) + ($\lambda=2$)	6.59

(b) The quartet differential cross-sections

The quartet contribution to elastic n-D scattering has been found at two values of the laboratory energy (E_L). As Aaron, Amado and Yam are the only ones to present the quartet contribution to the differential cross-section explicitly, we naturally make the comparison with their work. Therefore, the two values of E_L chosen are 2.45 MeV: and 14.1 MeV, which are the maximum and minimum values studied by Aaron, Amado and Yam. The differential cross-section is related to the T matrix by

$$\frac{d\sigma}{d\Omega} = \left(\frac{2\pi}{\hbar}\right)^4 \mu_{nD} \left| T_{ab} \right|^2 \qquad (5.1)$$

where μ_{nD} is the reduced mass of the neutron-deuteron system. The effects of adding contributions corresponding to successive values of λ are shown in figs. (5.1a) and (5.2a). The quartet part of $\frac{d\sigma}{d\Omega}$, say $\left(\frac{d\sigma}{d\Omega}\right)_Q$, including terms less than $\lambda = 3$, are compared with the quartet contributions given in reference 21.

In order to see more clearly the nature of the convergence we are getting, we haveplotted also $\left(\frac{d\sigma}{d\Omega}\right)_Q$ against the maximum λ term taken into account, at each of the scattering angles $\theta = 0, \pi$. These results are given in figs. (5.1b) and (5.2b) where the points have been joined in order to indicate trends.

Evidently there seems to be some convergence, but not as much as we had hoped for or were led to believe might occur from the scattering length result. But we should remember that because of the awkward (a) multiplicity we have not included the J = 2 states, (2A.14) and (2A.15) of Appendix 2, in the $\lambda = 2$ contribution considered. However, because of the disappearance of the J = 2 contribution at scattering angles given by $\cos\theta = \pm \frac{1}{\sqrt{3}}$, then at these angles the $\lambda = 2$ contribution we have found must be the total $\lambda = 2$ term. We have therefore plotted

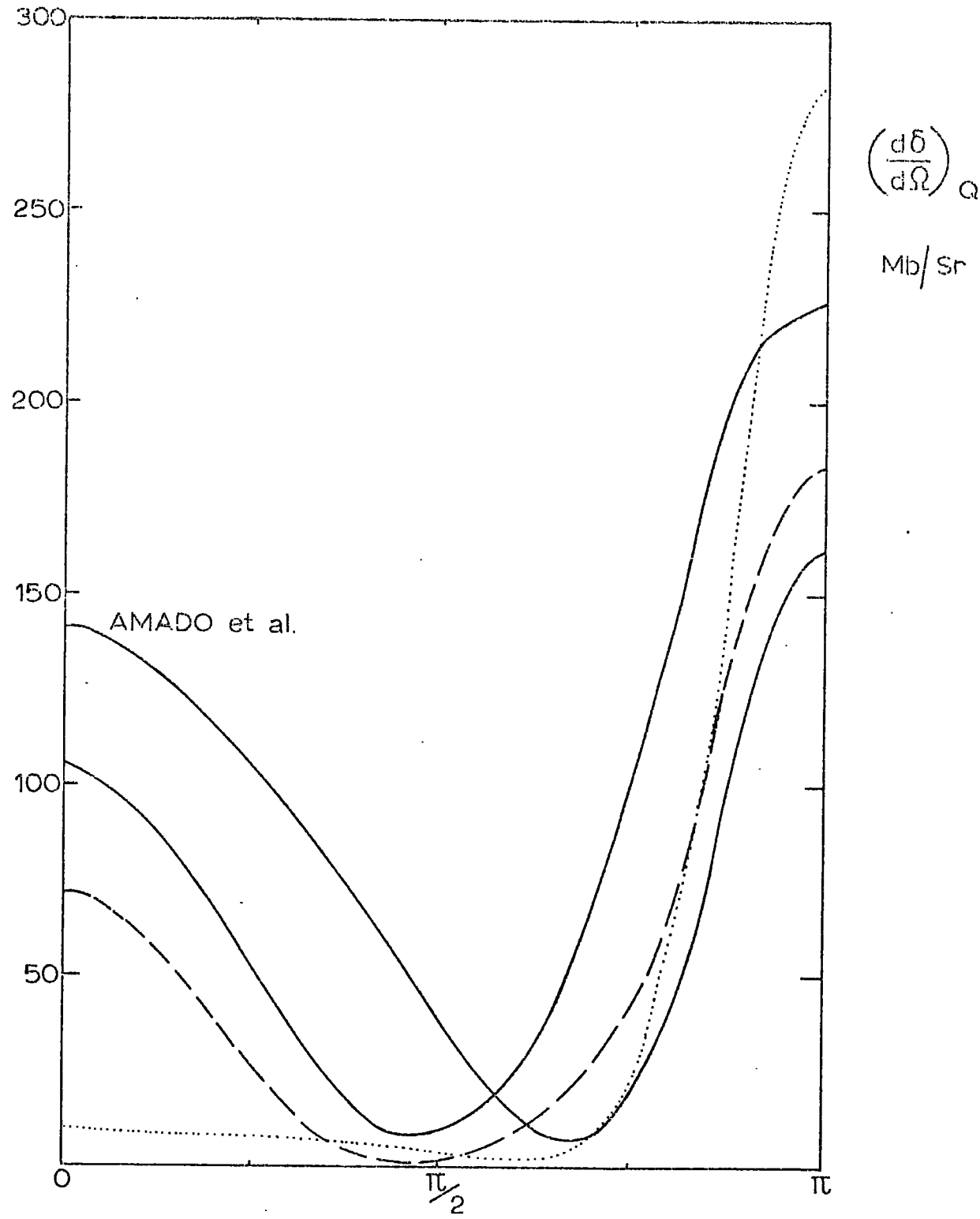
$(\frac{d\sigma}{d\Omega})_Q$ against the maximum λ term taken into account for these two angles. The graphs are presented in figs. (5.1c) and (5.2c).

By projecting out the J=0 and J=1 parts of the potential term, we are able to compare more directly with the results of Amado et al. by looking at the T-matrix contributions they give. The results are compared in table 3.

TABLE 3. $T = t\ P_J(\cos\theta)$.

	J = 0		J = 1		E_L
	10^7xReal (t)	10^7xImag (t)	10^7xReal (t)	10^7xImag (t)	(MeV)
This work	-1.16	-2.41	1.90	-5.51	14.1
Amado et al.	-1.04	-2.39	-3.66	-2.12	
This work	3.20	-5.92	3.95	24.2	2.45
Amado et al.	2.88	-6.12	-7.82	-3.26	

We see that the comparison with the results of reference 21 is fairly good for J=0, as expected, but surprisingly poor for J = 1. Indeed there seems to be no sign of convergence at all for J = 1.



—— $\equiv (\lambda=0)+(\lambda=1)+(\lambda=2)$

– – – $\equiv (\lambda=0)+(\lambda=1)$

····· $\equiv (\lambda=0)$

FIG(5·1a) $E_L = 14{\cdot}1$ mev. (excluding states (2A·14) and (2A·15) and including the full potential contribution.)

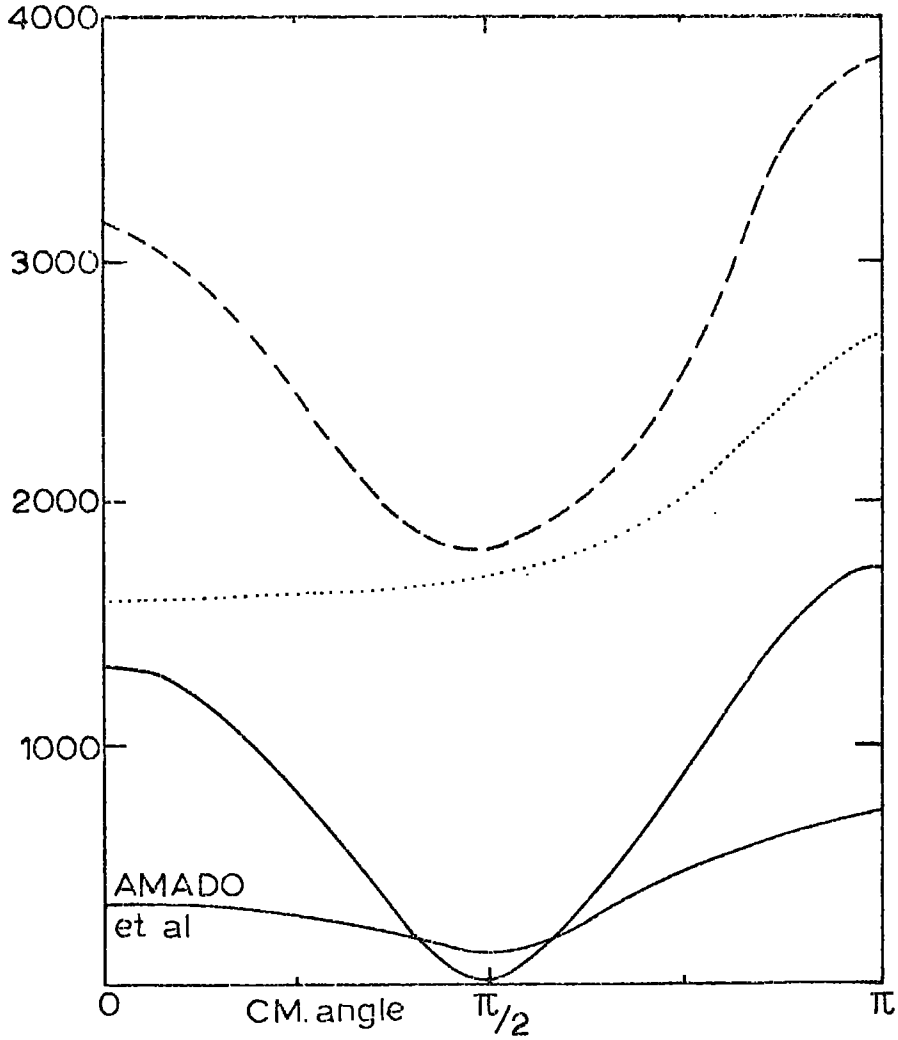
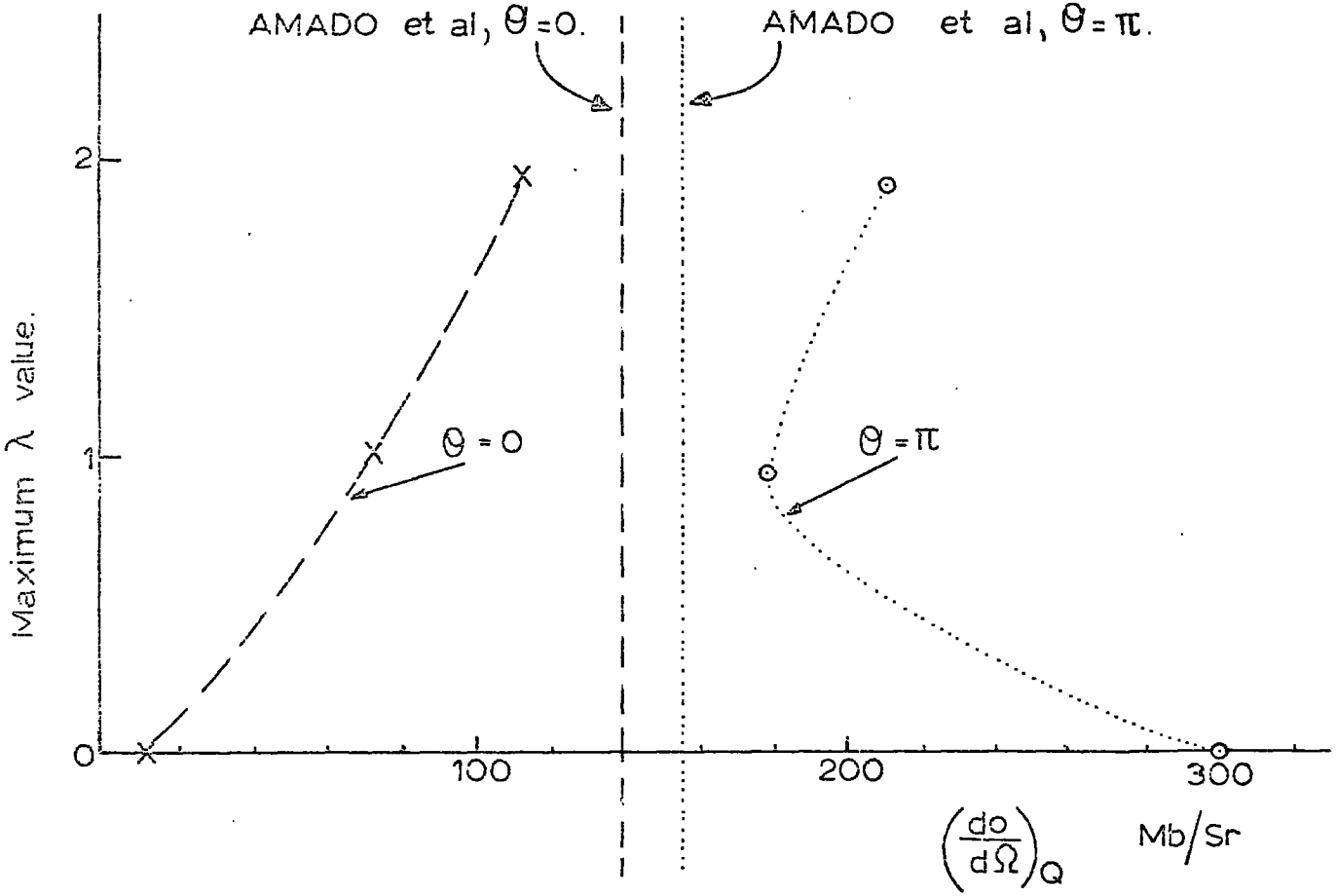


FIG (5·2a) E_L = 2·45 mev. (excluding states (2A·14) and (2A·15) and including the full potential contribution.

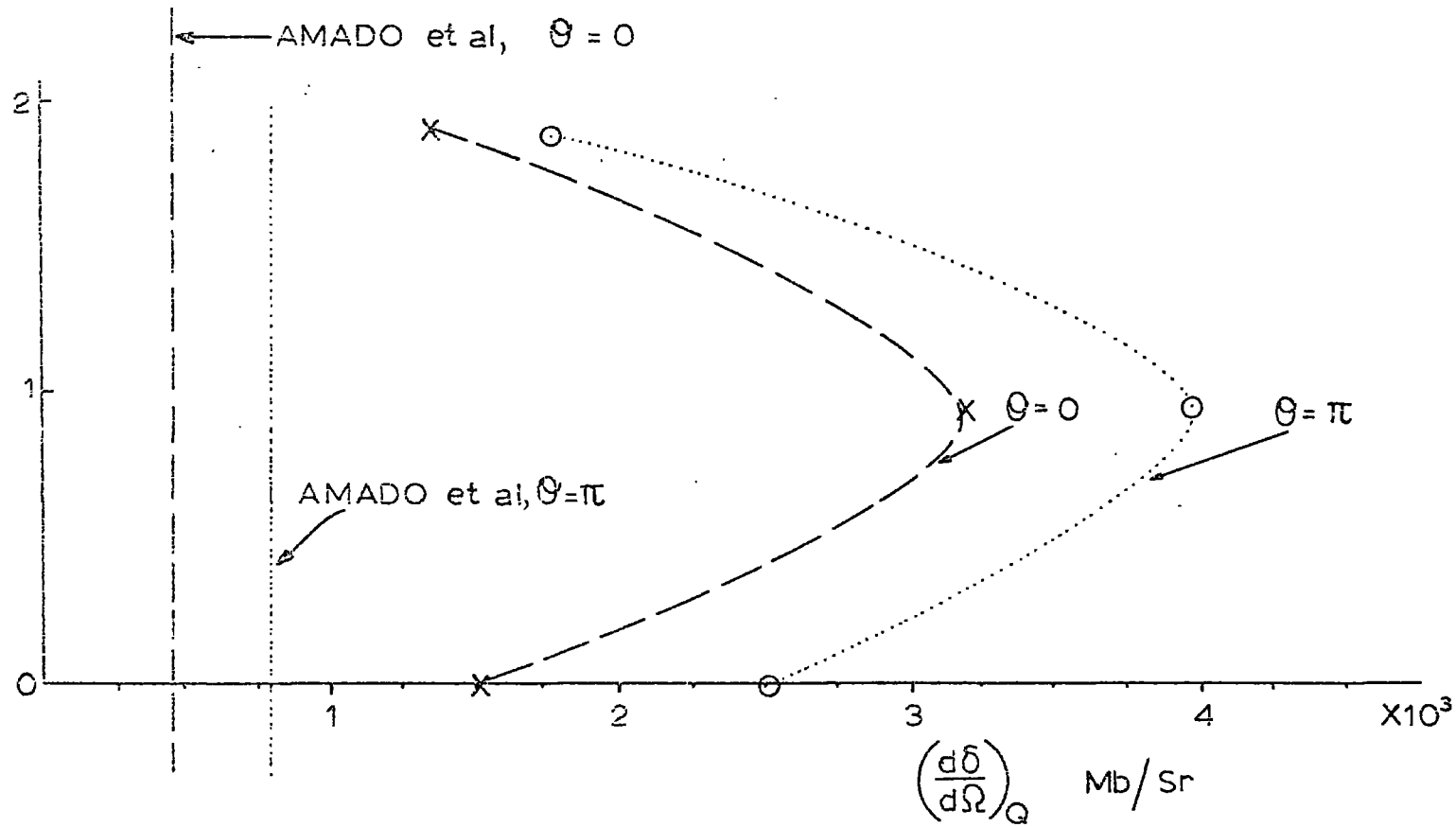
——— ≡ (λ = 0) + (λ = 1)

..... ≡ λ = 0

——— ≡ (λ = 0) + (λ = 1) + (λ = 2)



FIG(5·1b) $\theta = 0, \pi$ E_L = 14·1 Mev (excluding states (2A·14) and (2A·15), and including the full potential contribution.)



FIG(5·2b). $\vartheta = 0, \pi$ $E_L = 2{\cdot}45$ mev. (excluding states (2A·14) and (2A·15), and including full potential contribution)

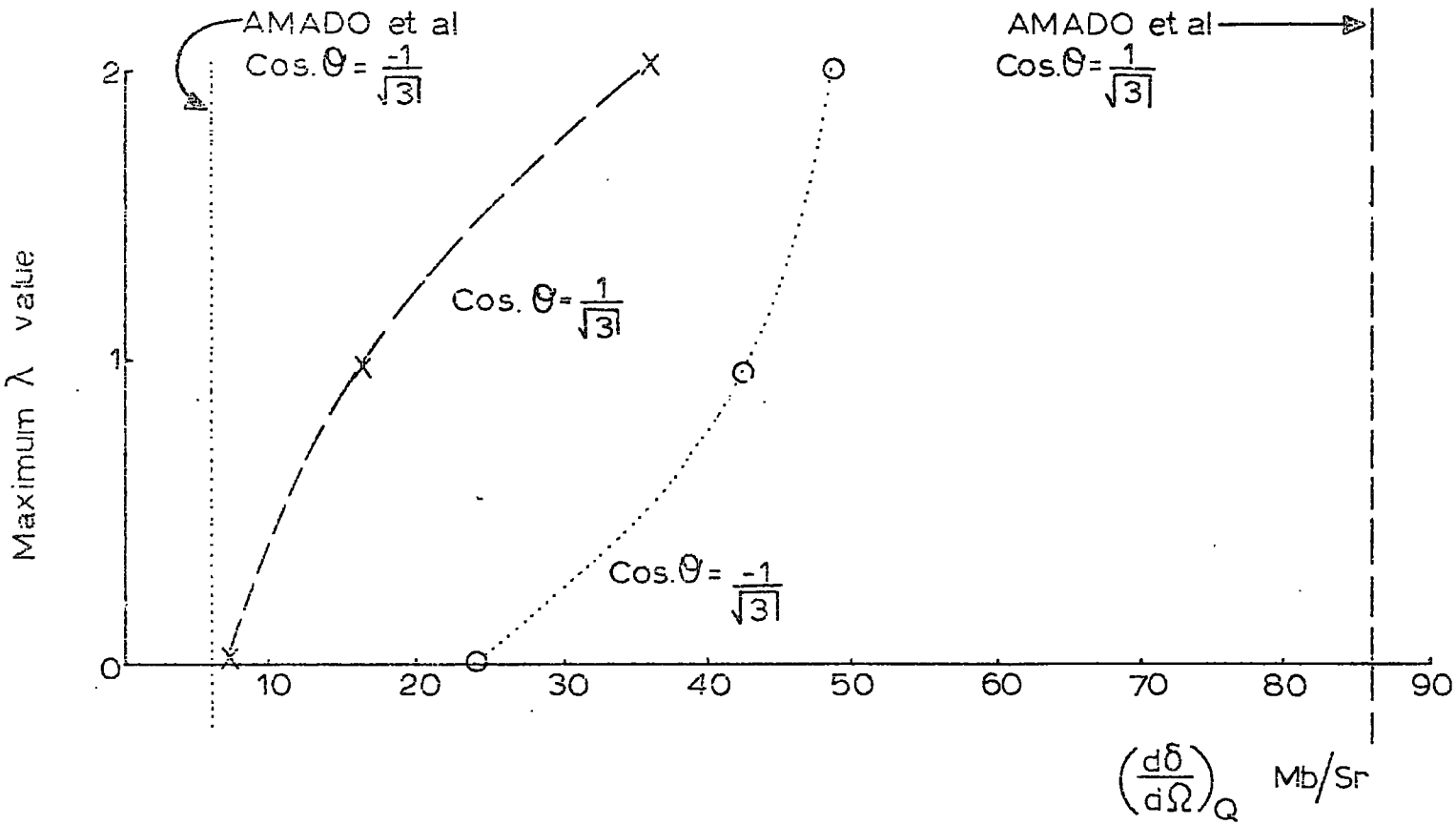
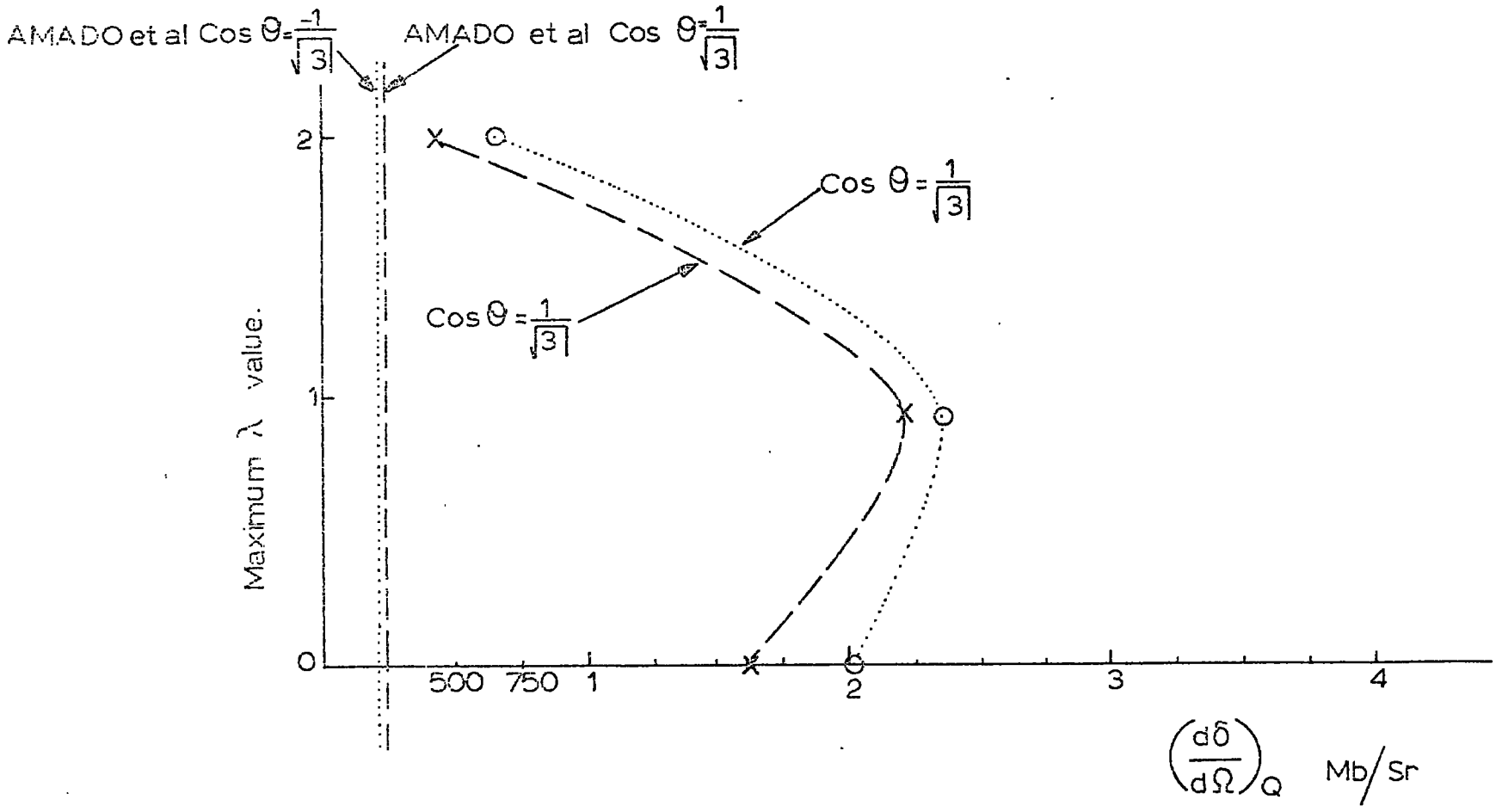


FIG. (5·1c) E_L = 14·1 mev (including full potential contribution)



FIG(5·2c) E_L= 2·45 mev (including full potential contribution.)

CHAPTER SIX

CONCLUSION

Our prime concern in this work has been to ascertain how useful Su(3) three-particle states really are in a practical calculation, that of the quartet contribution to neutron-deuteron scattering. It was hoped that contributions from successive λ terms would decrease rapidly enough to make this method of solution preferable, or at least a reasonable alternative to other methods for systems of short-range interaction, especially sinceusing the Su(3) formalism means that local potentials can be used instead of the usual non-local separable potentials. If the calculation had been successful we would then have had a useful method for studying this important problem of n-D scattering.

In fact, although a good value for the scattering length has been obtained, the results for the quartet contribution to the differential cross-section, $(\frac{d\sigma}{d\Omega})_Q$, are disappointing. At the higher energy there seemed to be some hope of there being reasonably good convergence in λ, although already a fair number, nine to be exact, of different Dragt states have been brought into play. When we look more closely at the T-matrices and compare these with the results of Amado et al, as in table 3 of chaper 5, we find that there is good correspondence for $J = 0$, but for $J = 1$ the comparison is very poor indeed. The apparent convergence in $(\frac{d\sigma}{d\Omega})_Q$ is now seen to be due to chance cancellations. Perhaps this is not really so surprising since an examination of the next higher Dragt states, given by $\lambda = 3$, shows that of the ten states, (not counting the awkward ω multiplicity of some of these states) there are four with $J = 1$. Furthermore, no attempt was made to include in the $\lambda = 2$ contribution the $J = 2$ terms of ω multiplicity 2. This was partly due to an uncertainty as to how to deal with these terms, at least within the formalism as given by Dragt, and partly because there seems to be no point in doing so, since there is no real convergence in sight for the

$J = 1$ contributions. It was originally hoped that because Su(3) states for small λ describe "close-in" particles then this might imply good convergence in λ. This also seemed to be implicit in previous calculations. From the results we have obtained, though, it would appear that this hope is without any foundation.

A further contributing factor to the poorness of the results is probably the "looseness" of the deuteron itself. It might have been interesting, although purely academic, to have repeated the calculation with a much greater deuteron binding energy. However, as there is nothing similar to the Aaron, Amado and Yam results to compare with, the calculation would also have to be done by one of the usual methods.

We note that although there is a simplicity in using the Su(3) states (because of the symmetry properties, equations (3.45) and (3.46)) it however gets quite involved because conservation of angular momentum allows coupling of two or more simultaneous integral equations. That is, states with different values of N_a and N_b are linked if they have the same values of J. In addition the ω multiplicity causes trouble. According to Lee[27], ω is in general irrational and it is difficult to express the eigenfunctions in closed form. However, he does construct states which, although they are not eigenstates of the operator Ω, are nevertheless a complete set for a three-particle system.

To summarize, looking at specific angular momentum contributions we have found that those with $J = 0$ for both the scattering length and the differential cross-section at both energies are well represented, but the $J = 1$ contribution to $(\frac{d\sigma}{d\Omega})_Q$ for $\lambda \gtrsim 3$ must still make an appreciable difference. This feature of $J = 0$ calculations seems to be born out by Lee's calculation of the triton binding energy and Dragt's discussion

of the decay process $K^+ \rightarrow \pi^+ + \pi^+ + \pi^-$ in reference 12. Thus it would seem that the Dragt states are really only useful when we are simply after the J=0 part of a three-body system in nuclear physics.

APPENDIX ONE

SPIN AND ISOSPIN U FUNCTIONS.

In this appendix we look at the spin functions, although isospin may be treated in a similar manner. Each of the nucleons has spin $\frac{1}{2}$, so that we expect one quartet group of spin states ($S = \frac{3}{2}$) and two distinct doublet groups of spin states ($S = \frac{1}{2}$). If we denote products of spin-functions as e.g. (+ - +) etc. where + indicates $+\frac{1}{2}$ and - indicates $-\frac{1}{2}$, then the quartet states may be expressed as[28]

$$(+\ +\ +)$$

$$\frac{1}{\sqrt{3}}\left\{(+\ +\ -) + (+\ -\ +) + (-\ +\ +)\right\}$$

$$\frac{1}{\sqrt{3}}\left\{(-\ -\ +) + (-\ +\ -) + (+\ -\ -)\right\}$$

$$(-\ -\ -)$$

One then finds that the U functions are 1 for $S = \frac{3}{2}$.

The doublet states may be denoted as

$$\frac{1}{\sqrt{6}}\left\{(+\ +\ -) + (+\ -\ +) - 2\ (-\ +\ +)\right\}$$

$$\frac{1}{\sqrt{6}}\left\{(-\ -\ +) + (-\ +\ -) - 2\ (+\ -\ -)\right\}$$

$$\frac{1}{\sqrt{2}}\left\{(+\ +\ -) - (+\ -\ +)\right\}$$

$$\frac{1}{\sqrt{2}}\left\{(-\ -\ +) - (-\ +\ -)\right\}$$

where the states have been chosen so that the first pair is symmetric with respect to interchange of particles 2 and 3 and the second pair are anti-symmetric with respect to interchange of particles 2 and 3.

It may be shown that the $U^{\beta\alpha}$ (I_β, I_α) functions for $S = \frac{1}{2}$ are,

$$U^{31} = (U^{13})^{+} = U^{12} = (U^{12})^{+} = U^{23} = (U^{32})^{+} =$$

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

where one goes from one channel to another by a cyclic permutation.

APPENDIX TWO

EXPLICIT SU(3) EIGENSTATES

In this appendix we give the forms of the eigenstates used. It is only necessary to have the states with $j = j_z$ as the remaining states can then be obtained by successive applications of J_-. Equations (3.37) may then be replaced by the conditions,

$$J_+ \left| N_a N_b jj \right\rangle = 0. \tag{2A.1}$$

$$J_z \left| N_a N_b jj \right\rangle = j \left| N_a N_b jj \right\rangle \tag{2A.2}$$

It is helpful to define creation and destruction operators in a spherical basis, according to

$$a_1^+ = -\frac{1}{\sqrt{2}} \left(a_x^+ + i\, a_y^+\right) \tag{2A.3}$$

$$a_{-1}^+ = \frac{1}{\sqrt{2}} \left(a_x^+ - i\, a_y^+\right) \tag{2A.4}$$

$$a_o^+ = a_z^+ \quad ; \quad a_m = (a_m^+)^+$$

Likewise for b. the commutation rules are,

$$\left[a_m, a_n^+\right] = \delta_{mn} \tag{2A.5}$$

The states of concern to us are:

(1) $\lambda = 0$

Then $N_a = N_b = 0$. This is the vacuum state,

$$\left| 0000 \right\rangle = \left| 0 \right\rangle \tag{2A.6}$$

That is, the state has zero angular momentum.

(2) $\lambda = 1$

Here we have the two possibilities,

$$\left| 1011 \right\rangle = a_1^+ \left| 0 \right\rangle \tag{2A.7}$$

$$\left| 0111 \right\rangle = b_1^+ \left| 0 \right\rangle \tag{2A.8}$$

Both states have angular momentum = 1.

(3) $\lambda = 2$

The possibilities are, for $N_a = N_b = 1$.

$$|1100\rangle = \underline{a}^+ . \underline{b}^+ |0\rangle \qquad (2A.9)$$

$$|1111\rangle = (\underline{a}^+ x \underline{b}^+)_1 |0\rangle \qquad (2A.10)$$

$$|1122\rangle = a_1^+ b_1^+ |0\rangle \qquad (2A.11)$$

That is, we have angular momenta 0,1,2.

However we must also conform to the rule that Δ- annihilates the states, and this is not the case for state (2A.9). This means, in fact, that states $|1100\rangle$ and $|0000\rangle$ differ only in the radial part.

For $\lambda = 2$ we may also have states,

$$|2000\rangle = \underline{a}^+ . \underline{a}^+ |0\rangle \qquad (2A.12)$$

$$|0200\rangle = \underline{b}^+ . \underline{b}^+ |0\rangle \qquad (2A.13)$$

$$|2022\rangle = a_1^+ a_1^+ |0\rangle \qquad (2A.14)$$

$$|0222\rangle = b_1^+ b_1^+ |0\rangle \qquad (2A.15)$$

which have angular momenta 0 and 2. States (2A.14) and (2A.15), having multiplicity two, have not been included in the present calculation.

We note that the maximum value of j occuring in any given representation is restricted by the inequality

$$j \leqslant \lambda$$

with $\lambda = N_a + N_b$.

APPENDIX THREE

g FUNCTIONS IN DALITZ-FABRI CO-ORDINATES.

Using equ. (3.43) we are able to determine the explicit forms of the g functions.

g_1. This corresponds to state (2A.6)

We find $g_1 = A_1 = \dfrac{1}{\sqrt{(2\pi)^3}}$

g_2. This corresponds to state (2A.7)

We find $g_2 = A_2 \dfrac{1}{p} (\underline{p}^{(2)} + i\underline{p}^{(1)})_{M'}$

$$= A_2 \frac{1}{p} N_{M'} \cdot (\underline{p}^{(2)} + i\underline{p}^{(1)})$$

where $N_{M'}$ denotes a spherical space-fixed triad defined by:-

$$N_1 = -\frac{1}{\sqrt{2}} (N_x + iN_y) \quad ; \quad N_o = N_z \quad ; \quad N_{-1} = \frac{1}{\sqrt{2}} (N_x - iN_y)$$

$$\text{and } N_M \cdot N_{M'} = (-1)^M \delta_{M,-M'}$$

We also define a spherical body-fixed triad by:-

$$E_1 = -\frac{1}{\sqrt{2}} (E_x + iE_y) \quad ; \quad E_o = E_z \quad ; \quad E_{-1} = \frac{1}{\sqrt{2}} (E_x - iE_y)$$

with E_x, E_y, E_z all unit vectors.

Let I be the unit dyad given by

$$I = \sum_M (-1)^M E_M E_{-M}$$

Then one finds

$$g_2 = A_2 \frac{1}{p} \sum_M (-1)^M N_{M'} \cdot E_{-M} E_M \cdot (\underline{p}^{(2)} + i\underline{p}^{(1)})$$

If we go from a space-fixed triad to a body-fixed triad by a rotation [29]

$$E_i = R N_i \qquad (i = 1,2,3)$$

then we find $g_2 = A_2 \frac{1}{p} \sum_M D^{1*}_{M',M}(\alpha,\beta,\gamma)\, E_M \cdot (\underline{p}^{(2)} + i\underline{p}^{(1)})$

where $D^{1*}_{M',M}(\alpha,\beta,\gamma) = (-1)^M N_{M'} \cdot E_{-M}$

Using a body-fixed set of axes such as is described in Chapter 4, we find

$$g_2 = A_2 \frac{1}{p} \left\{ \frac{i\, p^{(1)}}{\sqrt{2}} \left[\frac{1-\rho^2}{1-\rho^2\cos^2\phi}\right]^{\frac{1}{2}} \left[D^{1*}_{M',-1}(\alpha,\beta,\gamma) - D^{1*}_{M',1}(\alpha,\beta,\gamma) \right] + D^{1*}_{M',0}(\alpha,\beta,\gamma) \left[p^{(2)} - i p^{(1)} \frac{\rho \sin\phi}{\left[1-\rho^2\cos^2\phi\right]^{\frac{1}{2}}} \right] \right\}$$

with $(A_2)^2 = \frac{3}{8\pi^3}$

g_3. This corresponds to state (2A.8). By using a similar method to that which was used for finding g_2, one obtains

$$g_3 = A_3 \frac{1}{p} \left\{ \frac{i\, p^{(1)}}{\sqrt{2}} \left[\frac{1-\rho^2}{1-\rho^2\cos^2\phi}\right]^{\frac{1}{2}} \left[D^{1*}_{M',1}(\alpha,\beta,\gamma) - D^{1*}_{M',-1}(\alpha,\beta,\gamma) \right] + D^{1*}_{M',0}(\alpha,\beta,\gamma) \left[p^{(2)} + i\, p^{(1)} \frac{\rho \sin\phi}{\left[1-\rho^2\cos^2\phi\right]^{\frac{1}{2}}} \right] \right\}$$

And $A_3 = A_2$

g_4. This corresponds to state (2A.9) and is omitted

g_5. This corresponds to state (2A.10). We have

$$g_5 = A_5 \frac{1}{p^2} N_{M'} \cdot (\underline{p}^{(2)} + i\underline{p}^{(1)}) \times (\underline{p}^{(2)} - i\underline{p}^{(1)})$$

$$= A_5 \frac{1}{p^2} \sum_M (-1)^M N_{M'} \cdot E_{-M}\, E_M \cdot (\underline{p}^{(2)} + i\underline{p}^{(1)}) \times (\underline{p}^{(2)} - i\underline{p}^{(1)})$$

$$= A_5 \frac{1}{p^2} \sum_M D^{1*}_{M',M}(\alpha,\beta,\gamma)\, E_M \cdot (\underline{p}^{(2)} + i\underline{p}^{(1)}) \times (\underline{p}^{(2)} - i\underline{p}^{(1)})$$

leading to

$$g_5 = -A_5\, 2\sqrt{2}\; p^{(1)}\, p^{(2)} \left[\frac{1-\rho^2}{1-\rho^2\cos^2\phi}\right]^{\frac{1}{2}} \left(D^{1*}_{M',1}(\alpha,\beta,\gamma) + D^{1*}_{M',-1}(\alpha,\beta,\gamma)\right)$$

g_6. This corresponds to state (2A.12)

$$g_6 = A_6\, \frac{1}{p^2}\, (p^{(2)} + ip^{(1)}) \cdot (p^{(2)} + ip^{(1)})$$

giving

$$g_6 = \rho e^{i\phi}$$

and $$A_6^2 = \frac{1}{4\pi^3}$$

g_7. Corresponds to state (2A.13)

We obtain $g_7 = \rho e^{-i\phi}$

and $$A_6 = A_7$$

g_8. Corresponds to state (2A.11)

Although this state has J = 2, the ω multiplicity is 1.

$$g_8 = A_8\, \frac{1}{p^2} \left\{ \frac{i\, p^{(1)}}{\sqrt{2}} \left[\frac{1-\rho^2}{1-\rho^2\cos^2\phi}\right]^{\frac{1}{2}} \left[D^{1*}_{M',-1}(\alpha,\beta,\gamma) - D^{1*}_{M',1}(\alpha,\beta,\gamma)\right] + D^{1*}_{M',0}(\alpha,\beta,\gamma) \left[p^{(2)} - ip^{(1)} \left[\frac{\rho \sin\phi}{1-\rho^2\cos^2\phi}\right]^{\frac{1}{2}}\right]\right\} \cdot$$

$$\cdot \left\{ \frac{i\, p^{(1)}}{\sqrt{2}} \left[\frac{1-\rho^2}{1-\rho^2\cos^2\phi}\right]^{\frac{1}{2}} \left[D^{1*}_{M',-1}(\alpha,\beta,\gamma) - D^{1*}_{M',1}(\alpha,\beta,\gamma)\right] + D^{1*}_{M',0}(\alpha,\beta,\gamma) \left[p^{(2)} + ip^{(1)} \left[\frac{\rho \sin\phi}{1-\rho^2\cos^2\phi}\right]^{\frac{1}{2}}\right]\right\}$$

giving

$$g_8 = A_8\, \sqrt{6}\, \frac{1}{p^2} \left\{ 2\, p^{(2)\,2} + p^{(1)\,2} \left[\left[\frac{3\rho^2 \sin^2\phi}{1-\rho^2\cos^2\phi}\right] - 1\right]\right\} D^{2*}_{M',0}(\alpha,\beta,\gamma)$$

with $$A_8^2 = \frac{3}{2\pi^3}$$

APPENDIX FOUR

THE INTEGRAL EQUATION

The permutation group on three objects, S_3, has six elements which may be taken to be the transpositions, P_1, P_2, P_3, which interchange particle pairs, the cyclic permutations C and C^2 and the identity operation I. These elements may be represented by the group of 6 x 6 orthogonal matrices acting on the vector space spanned by the six-component vector $\underline{p}$. As C and P_3 generate all of S_3, we need only look at these. P_3 has the matrix representation

$$P_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

And C may be expressed as

$$C = \exp\left(\tfrac{2}{3}\pi S\right) = I \cos\frac{2\pi}{3} + S \sin\frac{2\pi}{3}$$

with $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

The unitary analogue for Cgives equ. (3.45) i.e.

$$C\left|N_a N_b jm\right\rangle = \exp\left[i\,\frac{2\pi}{3}\,(N_b - N_a)\right] \left|N_a N_b jm\right\rangle$$

Therefore a state which is symmetric with respect to interchange of the two non-β particles may be constructed.

$$\left|\beta(s)\right\rangle = \left|N_a N_b jm\right\rangle + P_\beta \left|N_a N_b jm\right\rangle$$

$$= \left|N_a N_b jm\right\rangle + \left|N_b N_a jm\right\rangle$$

We now set about casting the integral equations into a convenient form for solution.

From equ. (2.23) we find, symbolically,

$$\Gamma^{ll} = \sum_j \bar{\delta}_{lj} \left\langle l\left|t_j\right|l\right\rangle + \sum_i \bar{\delta}_{li} \left\langle l\left|t_i G_o\right|i\right\rangle \Gamma^{il} \qquad (4A.1)$$

Also,

$$\Gamma^{M\ell} = \sum_k \bar{\delta}_{Mk} \; \bar{\delta}_{\ell k} \; \langle M | t_k | \ell \rangle + \sum_q \bar{\delta}_{Mq} \langle M | t_q G_o | q \rangle \Gamma^{q\ell} \quad (4A.2)$$

with ℓ, M, n a cyclic permutation of 1,2,3. (For convenience the primes are being omitted).

A third equ. is

$$\Gamma^{\ell M} = \sum_{k'} \bar{\delta}_{\ell k'} \; \bar{\delta}_{Mk'} \langle \ell | t_{k'} | M \rangle + \sum_{q'} \bar{\delta}_{\ell q'} \langle \ell | t_{q'} G_o | q' \rangle \Gamma^{q'M} \quad (4A.3)$$

Using the property that $\Gamma^{\ell\ell}$, $\Gamma^{\ell M}$, $\Gamma^{M\ell}$ and $\langle \ell | t_M G_o | n \rangle$ are invariant with respect to cyclic permutations of ℓ, M, n, equation (4A.1) then becomes

$$\Gamma^{\ell\ell} = \sum_j \bar{\delta}_{\ell j} \langle \ell | t_j | \ell \rangle + \langle \ell | t_M G_o | M \rangle \Gamma^{M\ell} + \langle \ell | t_n G_o | n \rangle \Gamma^{\ell M} \quad (4A.4)$$

i.e. using $\Gamma^{n\ell} = \Gamma^{\ell M}$.

Also (4A.2) becomes

$$\Gamma^{M\ell} = \langle M | t_n | \ell \rangle + \langle \ell | t_M G_o | M \rangle \Gamma^{\ell M} + \langle \ell | t_n G_o | n \rangle \Gamma^{\ell\ell} \quad (4A.5)$$

i.e. using $\langle M | t_n G_o | n \rangle = \langle \ell | t_M G_o | M \rangle$;

$\Gamma^{n\ell} = \Gamma^{\ell M}$; $\langle M | t_\ell G_o | \ell \rangle = \langle \ell | t_n G_o | n \rangle$.

And (4A.3) is

$$\Gamma^{\ell M} = \langle \ell | t_n | M \rangle + \langle \ell | t_M G_o | M \rangle \Gamma^{\ell\ell} + \langle \ell | t_n G_o | n \rangle \Gamma^{M\ell} \quad (4A.6)$$

i.e. using $\Gamma^{\ell\ell} = \Gamma^{MM}$; $\Gamma^{nM} = \Gamma^{M\ell}$.

To proceed further we need to look more closely at the sum over N_i. The summation will include such terms as $(N_a N_b jm)$ and $(N_b N_a jm)$, for specific values.

If a state $|\ell\rangle$ in the equs. above corresponds to the combination $(N_a N_b jm)$, then a state $|\ell\rangle_I$ may be defined corresponding to $(N_b N_a jm)$.

Each of equations (4A.4), (4A.5), (4A.6) above will become four equations because of the possibilities:

$$\begin{aligned}
(N_a N_b \ldots.) &\longleftrightarrow (N_a N_b \ldots.) \\
(N_b N_a \ldots.) &\longleftrightarrow (N_b N_a \ldots.) \\
(N_a N_b \ldots.) &\longleftrightarrow (N_b N_a \ldots.) \\
(N_b N_a \ldots.) &\longleftrightarrow (N_a N_b \ldots.)
\end{aligned} \qquad (4A.7)$$

We are looking for a combination of the Γ's which will give us the same combination of Γ's within the homogeneous term of an integral equation,

Examine the combination

$$\Gamma^{\ell\ell} - \tfrac{1}{2}\left(\Gamma^{M\ell} + \Gamma^{\ell M}\right)$$

for each of the possibilities (4A.7) above:

(1) $(N_a N_b \ldots.) \longleftrightarrow (N_a N_b \ldots.)$.

From equations (4A.4), (4A.5) and (4A.6), the homogeneous term is

$$-\tfrac{1}{2}\left[\langle \ell | t_M G_o | M\rangle + \langle \ell | t_n G_o | n\rangle\right]\Gamma^{\ell\ell} + \left[\langle \ell | t_n G_o | n\rangle - \tfrac{1}{2}\langle \ell | t_M G_o | M\rangle\right]\Gamma^{\ell M} + \left[\langle \ell | t_M G_o | M\rangle - \tfrac{1}{2}\langle \ell | t_n G_o | n\rangle\right]\Gamma^{M\ell}$$

Introduce now the transformation factor U by defining

$$\langle \ell | t_M G_o | M\rangle = U''_{\ell M}\langle M | t_M G_o | M\rangle$$

$$\langle \ell | t_n G_o | n\rangle = U''_{\ell n}\langle n | t_n G_o | n\rangle$$

The homogeneous term therefore becomes

$$-\tfrac{1}{2}\left[U''_{\ell M} + U''_{\ell n}\right]\langle M | t_M G_o | M\rangle\Gamma^{\ell\ell} + \left[U''_{\ell n} - \tfrac{1}{2}U''_{\ell M}\right]\langle M | t_M G_o | M\rangle\Gamma^{\ell M} + \left[U''_{\ell M} - \tfrac{1}{2}U''_{\ell n}\right]\langle M | t_M G_o | M\rangle\Gamma^{M\ell} \qquad (4A.8)$$

Where we have used the condition $\langle M | t_M G_o | M\rangle = \langle n | t_n G_o | n\rangle$.

(2) $(N_a N_b \ldots) \longleftrightarrow (N_b N_a \ldots)$

Adopting a similar procedure as in (1), the homogeneous term is

$$-\tfrac{1}{2}\left[U''_{\ell m} + U''_{\ell n}\right]\langle M|t_M G_o|M\rangle\langle \ell|\gamma^{\ell\ell}|\ell\rangle_I + \left[U''_{\ell n} - \tfrac{1}{2}U''_{\ell m}\right]\langle M|t_M G_o|M\rangle\langle \ell|\gamma^{\ell M}|M\rangle_I$$

$$+ \left[U''_{\ell m} - \tfrac{1}{2}U''_{\ell n}\right]\langle M|t_M G_o|M\rangle\langle M|\gamma^{M\ell}|\ell\rangle_I . \quad (4A.9)$$

(3) $(N_b N_a \ldots) \longleftrightarrow (N_b N_a \ldots)$

Gives an inhomogeneous term

$$-\tfrac{1}{2}\left[U''^{*}_{\ell m} + U''^{*}_{\ell n}\right]\ {}_I\langle M|t_M G_o|M\rangle_I\ {}_I\langle \ell|\gamma^{\ell\ell}|\ell\rangle_I + \left[U''^{*}_{\ell n} - \tfrac{1}{2}U''^{*}_{\ell m}\right]\ {}_I\langle M|t_M G_o|M\rangle_I \cdot$$

$$\cdot\ {}_I\langle \ell|\gamma^{\ell m}|M\rangle_I$$

$$+ \left[U''^{*}_{\ell m} - \tfrac{1}{2}U''^{*}_{\ell n}\right]\ {}_I\langle M|t_M G_o|M\rangle_I\ {}_I\langle M|\gamma^{M\ell}|\ell\rangle_I \quad (4A.10)$$

(4) $(N_b N_a \ldots) \longleftrightarrow (N_a N_b \ldots)$

Produces

$$-\tfrac{1}{2}\left[U''^{*}_{\ell m} + U''^{*}_{\ell n}\right]\ {}_I\langle M|t_M G_o|M\rangle_I\langle \ell|\gamma^{\ell\ell}|\ell\rangle + \left[U''^{*}_{\ell n} - \tfrac{1}{2}U''^{*}_{\ell m}\right]\ {}_I\langle M|t_M G_o|M\rangle\ {}_I\langle \ell|\gamma^{\ell m}|M\rangle$$

$$+ \left[U''^{*}_{\ell m} - \tfrac{1}{2}U''^{*}_{\ell n}\right]\ {}_I\langle M|t_M G_o|M\rangle_I\ {}_I\langle M|\gamma^{M\ell}|\ell\rangle \quad (4A.11)$$

Using the conditions, $\mathrm{Re}(U_{M\ell}) = \mathrm{Re}\,(U_{n\ell})$

and $\langle M|t_M G_o|M\rangle = {}_I\langle M|t_M G_o|M\rangle_I$ then the

contributions (1), (2), (3); (4) to the total homogeneous term give

$$-2\mathrm{Re}\,(U''_{\ell m})\ \langle M|t_M G_o|M\rangle\left[\Gamma^{\ell\ell} - \tfrac{1}{2}\Gamma^{\ell m} - \tfrac{1}{2}\Gamma^{m\ell}\right]$$

$$-2\mathrm{Re}\,(U''_{\ell m})\ \langle M|t_M G_o|M\rangle\left[\langle \ell|\gamma^{\ell\ell}|\ell\rangle_I - \tfrac{1}{2}\langle \ell|\gamma^{\ell m}|M\rangle_I - \tfrac{1}{2}\langle M|\gamma^{m\ell}|\ell\rangle_I\right] \quad (4A.12)$$

with $\langle \ell|\gamma^{\ell\ell}|\ell\rangle_I = {}_I\langle \ell|\gamma^{\ell\ell}|\ell\rangle$ and similarly for the $\gamma^{\ell m}$ and $\gamma^{m\ell}$ cases.

We have, therefore, the expression

$$\Gamma^{\ell\ell} - \tfrac{1}{2}\Gamma^{\ell m} - \tfrac{1}{2}\Gamma^{m\ell} + \langle \ell | \gamma^{\ell\ell} | \ell \rangle_I - \tfrac{1}{2} \langle \ell | \gamma^{\ell m} | m \rangle_I - \tfrac{1}{2} \langle m | \gamma^{m\ell} | \ell \rangle_I \qquad (4A.13)$$

on both sides of the equation that we have constructed.

The inhomogeneous term is made up by adding the four contributions as has been done for the homogeneous term. This gives a total inhomogeneous term of

$$2 \langle M | t_M | M \rangle \left[\mathrm{Re} \left(U''_{\ell M} U^{*'''}_{\ell M} + U''_{\ell n} U^{'''*}_{\ell n} - \tfrac{1}{2} U''_{\ell n} U^{'''*}_{Mn} - \tfrac{1}{2} U''_{MN} U^{'''*}_{\ell n} \right) \right.$$
$$\left. + \left(U''_{\ell M} U'''_{\ell M} + U''_{\ell n} U'''_{\ell n} - \tfrac{1}{2} U''_{\ell n} U'''_{\ell n} - \tfrac{1}{2} U''_{\ell n} U''_{mn} - \tfrac{1}{2} U''_{Mn} U'''_{\ell n} \right) \right]$$

And taking

$$U_{\ell M} = U_{Mn} = U_{n\ell} = U^*_{\ell n} = U^*_{Mn} = U^*_{M\ell} \quad , \qquad \text{then}$$

the inhomogeneous term becomes

$$\langle M | t_M | M \rangle \ \mathrm{Re} \ \left(U''_{\ell M} U^{*'''}_{\ell M} + U''_{\ell n} U^{*'''}_{\ell n} + U''_{\ell M} U'''_{\ell M} + U''_{\ell n} U'''_{\ell n} \right)$$
$$= 2 \langle M | t_M | M \rangle \ \mathrm{Re} \ \left(U''_{\ell M} U^{*'''}_{\ell M} + U^{*''}_{\ell M} U'''_{\ell M} \right) \qquad (4A.14)$$

Denoting the expression (4A.13) by $\mathcal{J}$, we have a final iterative equation represented by

$$\mathcal{J} = E_N \langle M | t_M | M \rangle + D_N \langle M | t_M G_o | M \rangle \mathcal{J} \qquad (4A.15)$$

with

$$E_N = 2 \ \mathrm{Re} \left(U''_{\ell M} U^{*'''}_{\ell M} + U''_{\ell M} U'''_{\ell M} \right)$$

and $$D_N = - \ \mathrm{Re} \left(U''_{\ell M} \right)$$

We note that for terms with $N_a = N_b$, this equation will give four times the necessary contribution. Also $\mathcal{J}$ gives onethird of the contribution from the Γ functions. The transformation functions have the explicit expression

$$U''_{\ell M} = \exp \left[i \ \frac{2\pi}{3} \ (N''_b - N''_a) \right]$$

Equation (4A.15) may be expressed diagrammatically as in Fig. (4A).

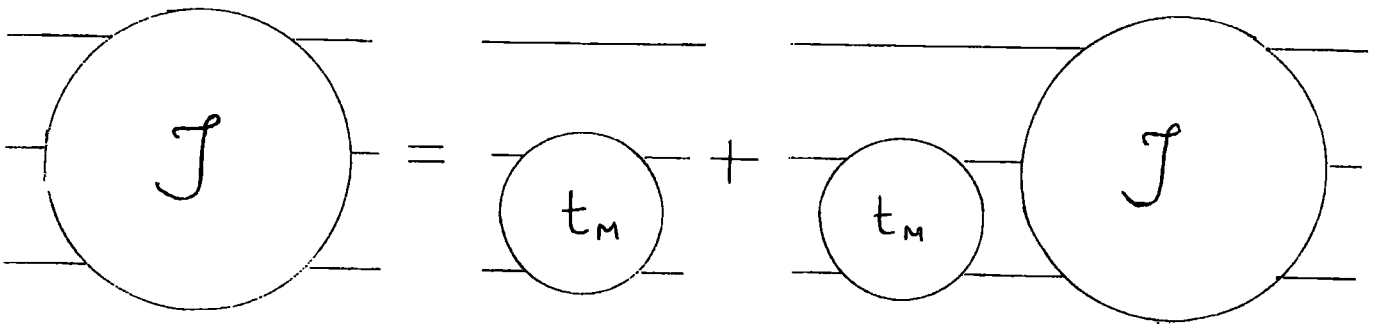


Figure (4A). Diagrammatical representation of equation (4A.15)

APPENDIX FIVE

NORMALIZATION OF THE STATES

Here we summarize the normalization conditions.

1.
$$\langle p_1' p_2' p_3' \mid p_1 p_2 p_3 \rangle = \delta(p_1'-p_1)\,\delta(p_2'-p_2)\,\delta(p_3'-p_3)$$

2. $$\langle p^{2\prime} N_a' N_b' j'm' \mid p^2 N_a N_b jm \rangle = \delta(p'^2-p^2)\,\delta_{N_a' N_a}\,\delta_{N_b' N_b}\,\delta_{j'j}\,\delta_{m'm}$$

This may be written more neatly as

$$\langle p^{2'} N' \mid p^2 N \rangle = \delta(p'^2 - p^2)\ \delta_{N'N}$$

which leads to the closure relation (4.1)

The constant C_1 of equation (3.44) is found to be 4.

APPENDIX SIX

EVALUATION OF F FOR EACH g FUNCTION

The F functions for each g are given in this Appendix. The starting point is equation (4.7). The integrations involved are made more tractable by changing from variables (ρ, ϕ) to (x,y) as indicated in Fig. (4.2).

We then find for the various g_n functions:

g_1.
$$F = F(1) = \frac{8c\, A_1 \left[\alpha_D\, \beta_n (\alpha_D+\beta_n)^3\right]^{\frac{1}{2}} \left[p''^2 - \frac{3}{2} q_3^2\right]^{\frac{1}{2}}}{(2\beta_n^2 + p''^2 - \frac{3}{2} q_3^2)\ (2\alpha_D^2 + p''^2 - \frac{3}{2} q_3^2)}$$

g_2.
$$F = F(2) = \frac{A_2}{A_1} \sqrt{\frac{3}{2}}\ \frac{q_3}{p''}\ F(1)\ D^{1*}_{M'',0}\ (\alpha', \beta', o)$$

Considering the F function corresponding to p''' and using

$$P_J(\cos\theta) = \sum_M D^{J*}_{M,0}\ (\phi_1, e_1)\ D^{J}_{M,0}\ (\phi_2, e_2)$$

where $P_J(\cos\theta)$ is a Legendre polynomial and $\theta = \theta_1 - \theta_2$, then a $\cos\theta$ factor emerges, where θ is the CM. scattering angle.

g_3. This produces the same F function as g_2.

g_4. Omitted

g_5. One finds for this case that F = 0.

g_6.
$$F = F(6) = \frac{A_6}{A_1} \left[\frac{3q_3^2}{p''^2} - 1\right] F(1)$$

g_7. This produces the same F function as g_6.

g_8.

$$F = F(8) = \frac{A_8}{A_1} \sqrt{\frac{3}{2}} \frac{q_3^2}{p''^2} F(1) D^{2*}_{M'',0} (\alpha', \beta', 0).$$

We find that a P_2 (cos θ) factor emerges, when also considering the F function for p''' .

APPENDIX SEVEN

NUMERICAL CALCULATION

In this appendix we give the computational techniques used in evaluating the T-matrices. Symbolically the "Dragt integral equation" may be represented by

$$T(p''^2\ p'''^2) \simeq t\ (p''^2\ p'''^2) + K\ (p''^2\ p_I^2)\ T\ (p_I^2\ p'''^2)$$

The integral equations were solved using a Simpson's Rule procedure. That is, they were approximated by matrix equations and matrix inversions carried out with the use of the IBM Subroutine MINV. In order that points could be effectively taken to infinity, a change of variable was made from p^2 to R such that

$$p^2 = W \frac{R}{(1-R)}$$

and R varies between 0 and 1. W is a constant determining the varying concentration of points throughout the range $p^2 = 0$ to ∞. The optimum value of W was found to be $2\ q_3^2$. The R range was divided into 40 equal intervals, which was found to represent with reasonable accuracy the meshes involved.

In order to be able to determine the matrix elements for the inhomogeneous and kernel parts then integrations over y variables have to be completed. For this a 12-point Gaussian quadrature integration subprogram was used which had a built-in convergence criterion. Doubling the number of Gaussian points made no difference to the matrix elements obtained to within 0.4 per cent for thereal part and 0.2 per cent for the imaginary part.

The technique therefore mainly involved 41 by 41 point meshes, each point of which is a complex number. This becomes then a 82 by 82 point real mesh.

Because the states (2A.6), (2A.12) and (2A.13) are coupled this involves the solution of three simultaneous integral equations. However, because of the symmetry properties, it is possible to effectively reduce this to the solution of two simultaneous integral equations. The solution then involves the inversion of a 164 by 164 point matrix.

One finds from Chapter 4, section B, that the lower limit for each of the integrations over p''^2 and p'''^2 is $\frac{3}{2}\ q_3^2$. The final integration was therefore taken over 23 points in the R variable.

In order to obtain an estimate of the errors involved in the values obtained for the T-matrix, then the number of points for the solution of a single integral equation was doubled. Specifically, the $\lambda = 0$ contribution was found with just one Dragt term in the kernel part both by the inversion of an 82 by 82 point mesh and then using a 164 by 164 point mesh.

The error at the higher energy in the real part was found to be about 4 per cent and the error in the imaginary part about 1 per cent. The greater error in the real part is perhaps to be expected as there is a good deal of cancellation involved in the calculation of this part. At the lower energy the corresponding values for real and imaginary parts were approximately 5.5 per cent and 1.5 per cent.

The potential contributions were determined using a 12-point Gaussian sub-program as was used to determine the elements of the meshes. Doubling the number of Gaussian points produced a difference of less than 0.3 per cent, at both energies. The partial wave projections of the potential parts were determined using Simpson's Rule, and doubling the number of points produced a difference of less than 1 per cent at both energies.

Error estimation for the two parts contributing to a_4 produced approximately 6 per cent for the T matrix part and 0.8 per cent for the potential part.

We might expect more realistic values for the errors to be not a lot greater than those figures quoted.

The computing was carried out on the I.B.M. 360/67 machine in the University of Newcastle upon Tyne.

APPENDIX EIGHT

THE INHOMOGENEOUS TERMS

The inhomogeneous terms, denoted by t, are given. They are determined from equation (4.20). In each case the final integration over $\hat{y}$ needs to be carried out.

t (i,j) indicates that g_i is taken on the left and g_j on the right.

(1)
$$t(1,1) = 8\pi^3 A_1^2\, p''\, (1-\hat{y})^{\frac{1}{2}}\, (1-Q^2)^{\frac{1}{2}}\, v_3(p'',\hat{y})\, \mathcal{C}_3(p'',\hat{y})\, v_3(p''',Q)$$

where $Q = \frac{1}{p''^2}\,(p''^2\hat{y} + p''^2 - p'''^2)$.

(2)
$$t(2,2) = \frac{1}{6}\,\frac{A_2^2}{A_1^2}\,(1+\hat{y})^{\frac{1}{2}}\,(1+Q)^{\frac{1}{2}}\, t(1,1)$$

(3)
$$t(1,6) = \frac{A_6}{A_1}\, Q\, t(1,1).$$

(4)
$$t(6,1) = \frac{A_6}{A_1}\, \hat{y}\, t(1,1)$$

(5)
$$t(6,6) = \frac{A_6^2}{A_1^2}\, \hat{y}\, Q\, t(1,1)$$

(6)
$$t(1,7) = t(1,6)$$

(7)
$$t(7,1) = t(6,1)$$

(8)
$$t(7,7) = t(6,6)$$

(9)
$$t(8,8) = \frac{1}{30}\,(1+\hat{y})\,(1+Q)\,\frac{A_8^2}{A_1^2}\, t(1,1).$$

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