The dynamical generation of symmetries

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THE DYNAMICAL GENERATION OF SYMMETRIES

by

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CONTENTS

PREFACE

The work performed in this thesis was performed at the University of Durham under the supervision of Dr. D.B. Fairlie. Except where indicated, the work is original.

The author wishes to thank Dr. D.B. Fairlie for his continued help, guidance and encouragement. He would also like to thank his colleagues, particularly Dr. P.J.S. Watson, for helpful conversations. The author's understanding of certain aspects of strong coupling theory has benefited greatly from conversations with Drs.Cremstrom and Nega, and from the latter's lectures at the 1968 Finnish summer school.

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INTRODUCTION

The fundamental bootstrap idea (1) is that it may be possible to find a small set of dynamical assumptions which, with the requirement that the nature be self consistent, imply that there is only one or a few possible worlds; this, or of one of these, being the one observed experimentally. As a workable dynamical scheme encompassing the whole field of physics, or even strong interactions, has yet to be found, it is necessary to seek an area of physics which is amenable to a bootstrap calculation.

The idea of bootstraps arose from the work of Chew and Mandelstam (2) on $\pi\pi$ scattering. They showed that the ρ resonance in $\pi\pi$ scattering could be produced qualititively by the exchange of a ρ in the other channels. Imposing the self consistency condition that the ρ has the same mass and couplings in each channel lead to the idea of a bootstrap, in which one considered a process involving a few particles and obtain consistency conditions of a few parameters the masses and couplings. The next advance was performed by Chew who showed using the N/D method that the N and N^* bootstrapped each other in π N scattering (3). The results for the couplings were in good agreement with experiment. With the discovery of su(3) symmetry (4), bootstraps were attempted using the baryon octet and decuplet (5). All these calculations were based on the Mandelstam representation (6) which says that the scattering amplitude is an analytic function of its variables apart from singularities at points corresponding to physical systems. However the bootstrap idea is not tied to any particular dynamical model and historically, as a new technique has emerged, so people have attempted to perform a bootstrap with it. This has been the case with the N/D method dispersion relations, superconvergence relations (7) and most recently finite energy sum rules (8).

In chapter one, we review the static model bootstrap calculations and discuss the relation ship between the N/D static model calculations and the consistency conditions imposed by saturating superconvergence relations with bound static and resonances. It has been observed (7) that the superconvergence relations obtained by considering the asympotic behaviour of an amplitude can be saturated quite well by the contributions from low-lying bound states and resonances, determined by experiment. Making this assumption gives relationship between the couplings which are often in agreement with the static model bootstrap results. We have investigated this situation in a more general model than that considered by Diu (9) in a recent paper. By considering the first moment sum rule along with the superconvergence relation, we find an elegant mathematical equivalence between the two methods which Diu did not observe due to the ad hoc nature of his calculation. We find that the bootstrap relation for the masses is related to the first moment sum rule and thus being less likely to be true, or to be saturated by isobars, provides a reason why the static model bootstrap calculations give bad or inconsistent results for the masses whilst giving good results for the couplings. The use of the moment sum rule also throws doubt on the validity of using a "universal cut-off".

In chapter two we present a review of strong coupling theory and discuss its relationship to bootstraps and superconvergence techniques. The strong coupling condition is known to give the static model bootstrap condition for a specific process (10). We see how the moment sum rule again appears as a condition on the masses, following the work of Cronstrom and Noga (11).

In chapter three, we investigate the boostrap model of Fulco and Wong (12), which attempts in a very ad hoc way to consider the effects of t- channel meson exchanges in meson-baryon scattering. We show that the model gives consistent results of all three processes involving the scattering of pseudoscalar mesons of the baryon octet and decuplet in the limit of su(3) symmetry. The couplings agree with those coming from the assumption of su(6) symmetry (13).

In chapter four, we consider the intermediate coupling theory of Kuriyan and Sudarshan (14) which is a generalisation of the strong coupling condition, writing the commutation of the meson source operators, not as zero, but as a linear combination of the generators of the symmetry group for the system. From this equation, the Fulco and Wong equation can be derived, identifiying the generators of the symmetry group with meson exchange terms. As the equations of the intermediate coupling group generate the algebra of su(6), it is clear why the model of Kuriyan and Sudarsham is obeyed by the octet and decuplet with couplings which agree with the assumption of su(6) as a symmetry group. Thus the self consistency of the Fulco and Wong model for the various processes is explained, as is the appearance of the results of su(6) and the consistency of Udgaonkar's su(6) bootstrap calculation (15). We also present the calculation of Gleeson and Musto (16) which derives the Fulco-Wong and Intermediate coupling equations from finite energy sum rules.

In chapter five, we discuss the use of sum rules and the mechanism by which the results of higher symmetries appear from the saturation of superconvergence relations (17) We show how the Fulco and Wong equation can be split up into sets of equations for each t- channel spin. Certain helicity amplitudes are shown to have the same decomposition into spin $\frac{1}{2}$ and 3/2 parts as t- channel spin amplitudes in the Fulco-Wong equation. Regge pole phenomenalogy gives Regge-pole terms in the finite energy sum rules which the same contribution to the Fulco-Wong equation as do the exchange meson terms assumed by Fulco and Wong. The finite energy sum rules give in a certain circumstances the su(6) results. In order to obtain these results it is necessary to assume mass degeneracy for the baryons. Putting in the experimental masses gives the su(6) breaking in a simple way. These results provide a possible explanation of why the results of higher symmetries appear, whilst these symmetries cannot be exact.

CHAPTER 1

Bootstraps and the Saturation of Sum Rules in the Static Model

1. Static Model.

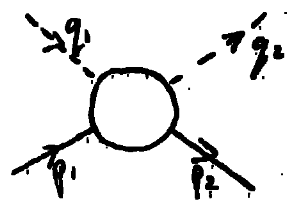
The first successful model of π-N scattering was developed by Chew and Low (21), using the static approximation. As much of this thesis is concerned with the static model, we begin by discussing its virtues, and its vices.

We will make use of the standard Mandelstam variables s,t,u which for the process $B + \pi \rightarrow B + \pi$ are:

$$s = -(p_1 + q_1)^2 = M^2 + m^2 + 2k_s^2 + 2\left[(k_s^2 + M^2)(k_s^2 + m^2)\right]^{\frac{1}{2}}$$

$$t = -(p_1 - p_2)^2 = -2k_s^2(1 - \cos\theta s)$$

$$u = -(p_1 - q_2)^2 = 2(M^2 + m^2) - s - t$$



p_1, p_2 are the 4 momenta of the baryons (mass M)

q_1, q_2 are the 4 - momenta of the mesons (mass m)

k_s and θs the centre of mass momentum and scattering angle.

We will also make use of the variable $\nu = \dfrac{s - u}{4M}$.

The static model consists of neglecting the nucleon recoil effects, and writing the energy of the system in the form $\sqrt{s} = M + w$ where $w^2 = m^2 + q^2$ and q^2 is the square of the 3-momentum of the meson. In this approximation $\sqrt{u} = M - w$, so that $\nu = 4Mw$ and $s - u$ crossing consists of putting $w \rightarrow -w$. If we denote this operation by a prime ('), $s = u'$, $u' = s$, $t = t'$. $t = t'$ implies that $q^2(1 - \cos\theta s) = q'^2(1 - \cos\theta' s)$

As $q' = - q$, this shows that $\cos\theta_s = \cos\theta_u$. Thus the partial wave expansions in the s and u channels are identical and under s - u crossing, the ℓ^{th} partial wave amplitude will cross into itself. This property does not hold when recoil effects are taken into account.

The error caused by neglecting the recoil effects is of order q^2/M^2 and hence the model is expected to work for $q^2 \ll M^2$. Unfortunately the nucleon resonances lie well outside this range and the success of the static model in describing them might be regarded as fortuitous. D.B. Fairlie (22) has pointed out that a possible reason for the success is contained in the work of Carruthers (23), who shows that in the "quasi-static limit" the crossing of partial wave amplitudes retains a simple form. Fairlie suggests that this simplicity allows to solutions of certain bootstrap requirements to be preserved beyond the static limit.

The introduction of particles with spin complicates the theory slightly, but again crossing is much simpler in the static model. Consider a spin 0 meson interacting in an ℓ-wave with a spin J particle. Consideration of the recoupling problem connected with s - u crossing reveals that the crossing matrix, connecting the various total angular momentum channels in the s-channel and u- channel ℓ- waves, is the same as for a spin ℓ particle scattering off a spin J particle with zero orbital angular momentum. Thus in the static model of pseudo-scalar mesons scattering off baryons, where the interaction is mainly p-wave, the angular momentum group assumes the role of an internal symmetry, with the meson belong to the spin 1 representation,

2. Partial Wave Dispersion Relations (24)

Consider a partial wave amplitude $a_\ell(s)$ which has the following properties:

(1) $a_\ell(s)$ can be analytically continued into the entire s- plane and is regular except for poles and cuts corresponding (à la Mandelstam (25)) to physical systems in the direct and crossed channel. (The direct channel will give a pole on (or near) the positive real axis for each bound state (or resonance) and the unitarity cut from threshold to $+\infty$. The crossed channels will give various cuts and poles depending on the kinematics. The (cut-structure for πN is given in Fig 1)

(ii) $a_\ell(s)$ is real analytic i.e. $a_\ell^*(s) = a_\ell(s^*)$

Using these properties we can apply Cauchy's Theorem and obtain: $a_\ell(s) = \frac{1}{2\pi i}\oint_C \frac{a(s')}{s'-s}\,ds'$ (1.1) where C is a contour enclosing all the cuts and poles and closed by sectors of a circle at infinity. By property (ii), the contribution from the circle at infinity vanishes. If (ii) doesn't hold, it is necessary to make subtractions in the dispersion relation, which action introduces further undetermined parameters into the problem.

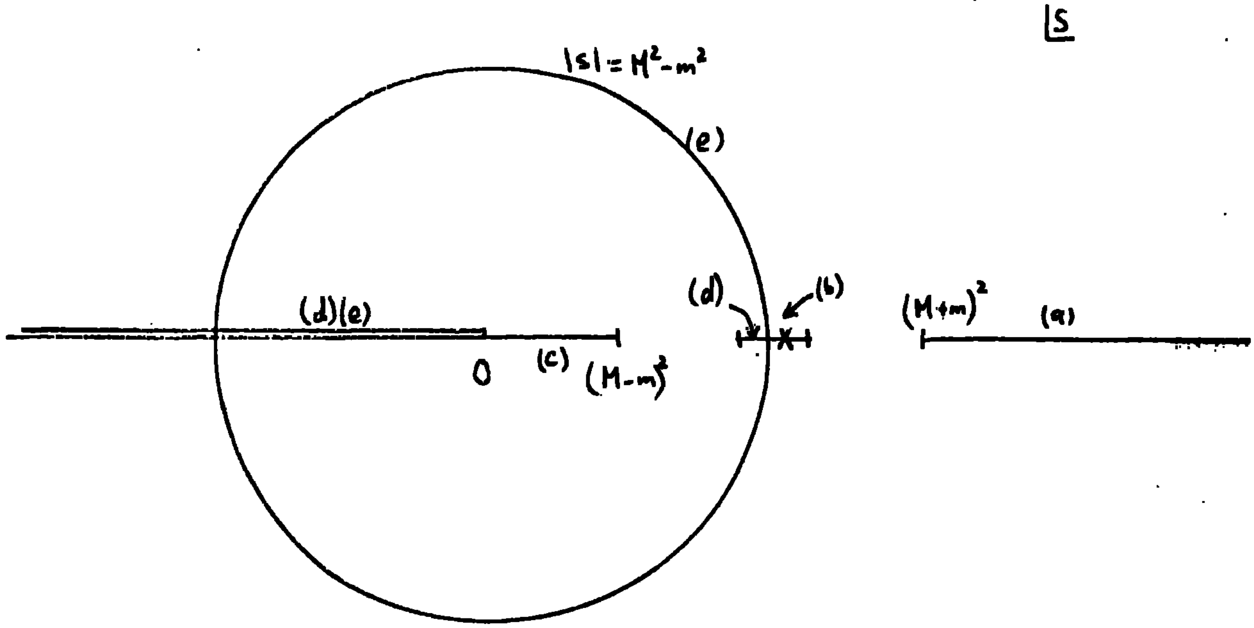


Fig 1. Cut-Structure for $\pi N \to \pi N$. (26)

(a) unitarity cut for direct channel.

(b) direct channel nucleon pole

(c) crossed unitarity cut for u - channel (also $\pi N \to \pi N$)

(d) cut from crossed N pole in u - channel.

(e) cut from t - channel process ($\pi\pi \to N\bar{N}$.)

In order to discuss πN scattering, it is necessary to know about the process $\pi\pi \to N\bar{N}$, for which there is little data. In order to say something about this channel, it is necessary to consider models in which the process is dominated by the ρ resonance. After considering such approximations, it seems likely that the effect of t- channel forces will be small, at least at low energies (27). We thus claim some justification for neglecting the circle cut and writing

$$a_\ell(s) = \frac{1}{\pi}\int_R \frac{\operatorname{Im} a_\ell(s')}{s'-s}\, ds' + \frac{1}{\pi}\int_L \frac{\operatorname{Im} a_\ell(s')}{s'-s}\, ds' \qquad (1.2)$$

This comes from integrating round the contour in Fig 2 and using the real-analyticity of $a_\ell(s)$ to write

$$\lim_{\epsilon \to 0+} \left\{ a(s + i\epsilon) - a(s - i\epsilon) \right\} = 2\, i \ \operatorname{Im} a(s)$$

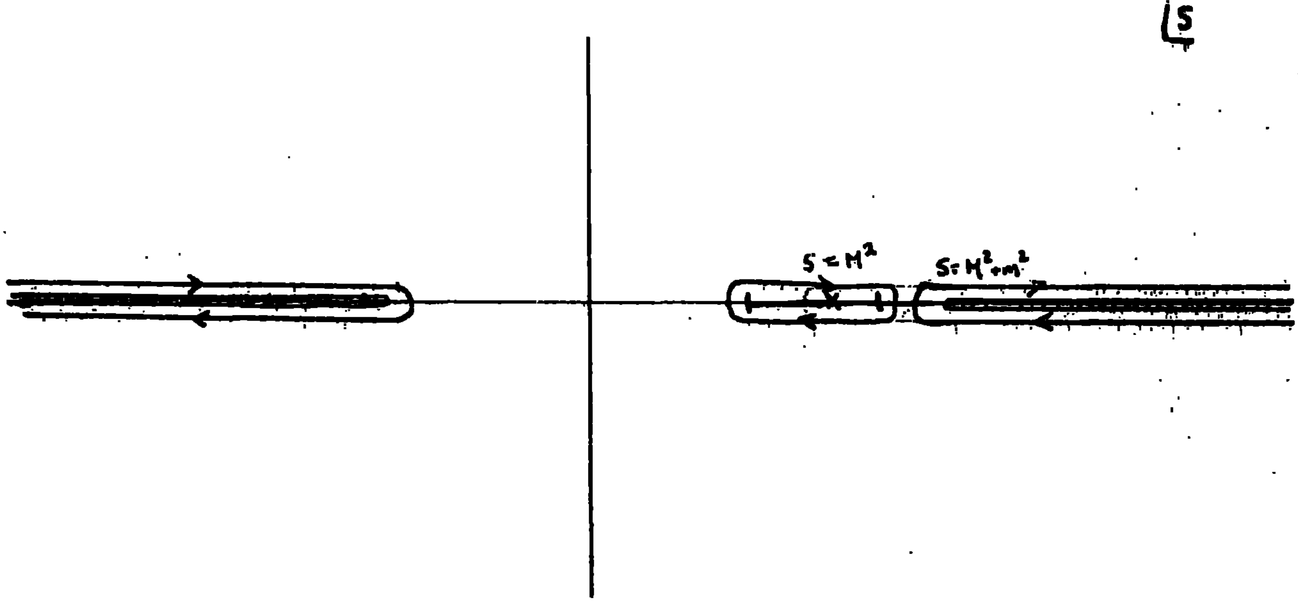


Fig. 2.

We remark here that if equations (1.2) hold for $\ell = 0, 1, 2, \ldots,$ one can combine them and obtain the $t = 0$ dispersion relation for the total amplitude $a(s,t)$:

$$a(s,0) = \frac{1}{\pi}\int_R \frac{\operatorname{Im} a(s',0)}{s'-s}\, ds' + \frac{1}{\pi}\int_L \frac{\operatorname{Im} a(s',0)}{s'-s}\, ds' \qquad (1.3)$$

Were $a(s,0)$ to be dominantly p- wave and were the integrals also dominated by p- wave contributions, then one would be justified in deriving equation (1.2) from equation (1.3) for the p- wave (i.e. $\ell = 1$). Is this likely to be true? The principal low lying reasonances are p- wave, and so, if $a(s,0)$

decreases rapidly as s increases, both the integrals and $a(s,c)$ may be dominated by the p- wave for small s. The rapid decrease of $a_\ell(s)$ with energy is a prime requirement for the bootstrap calculations, which follow to work, so if justified in these calculations, the above derivation of equation (1.2) for $\ell = 1$ may be considered as reliable as the earlier one. The method has the advantage that one may select amplitudes with good asymptotic behaviour, from experiment and Regge phenomenology, and also test p- wave dominance experimentally.

3. N/D Method (28)

One valuable property of equation (1.2) is that the unitarity equation relates $\mathrm{Im}\, a_\ell(s)$ to the amplitude on the right hand cut.

$$\mathrm{Im}\, a_\ell(s) = \rho_\ell(s) \, |a_\ell(s)|^2 \, R_\ell(s) \qquad 1.\ (1.4)$$

where $\rho_\ell(s) = q^{2\ell+1}/\sqrt{s}$. We note that $a_\ell(s)$ is related to the phase shift by: $\rho_\ell(s)\, a_\ell(s) = e^{i\delta_\ell}\sin\delta_\ell$. $R_\ell(s) = \dfrac{\sigma\,\mathrm{total}\,(s)}{\sigma\,\mathrm{elastic}\,(s)}$ is equal to 1 up to the inelastic threshold. $R_\ell(s) = 1$ is thus known as elastic unitarity. This is sometimes used as an approximation for the whole cut. This approximation will be good if $R_\ell(s)\, a_\ell(s)$ decreases rapidly with energy.

Given knowledge of $\mathrm{Im}\, a_\ell(s)$ on the left hand cut, one has then to solve a non-linear equation to find $a_\ell(s)$. Chew and Mandelstam converted this equation into a pair of coupled linear equations. This method has become known as the N/D method because of the conventional notation.

The fundamental tenet of the method is that one may write $a_\ell (s) = N_\ell (s) / D_\ell (s)$ where N_ℓ, D_ℓ are real analytic and $N_\ell (s)$ has a left hand cut only and $D_\ell (s)$ a right hand cut. The gap between the left and right hand cuts of a is greatly simplifies the procedure.

In mathematical form, the assumptions are:

$$\begin{aligned} \operatorname{Im} N_\ell (s) &= D_\ell (s) \operatorname{Im} a_\ell (s) \qquad S < S_L \qquad\qquad (1.5) \\ &= 0 \quad \text{otherwise} \\ \operatorname{Im} D_\ell (s) &= N_\ell \operatorname{Im} (1/a_\ell) \\ &= - N_\ell (s) \, \rho_\ell (s) \, R_\ell (s) \qquad S > S_R \qquad\qquad (1.6) \\ &= 0 \text{ otherwise.} \end{aligned}$$

It is further assumed that $N_\ell (s)$ can be chosen to go to zero at infinity so that one may write an unsubtracted dispersion relation for N.

Using equation (1.6): $$N_\ell (s) = \frac{1}{\pi} \int_{S_L}^{+\infty} \frac{D_\ell (s') \operatorname{Im} a_\ell (s')}{s' - s} \, ds' \qquad (1.7)$$

Before we can write a dispersion relation for D_ℓ, it is necessary to consider the C.D.D. (29) ambiguity. It is possible to insert abitrary poles into D_ℓ without changing the left hand cut. This corresponds to inserting in the partial wave, a particle not generated by the forces. There are two facets to the C.D.C. ambiguity which we shall refer to as the global and local problems. Firstly it may be that there exist "elementary particles" which are not generated by exchange forces. This is the negation of "pure" bootstrap philosophy. Even if there are no elementary particles and "global" C.D.D poles are not required, it may be necessary to insert them in a "local" calculation which concerns itself with a small sub-system. For example a process which is inelastic may require particles to be inserted as C.D.C poles whereas in a full multi-channel calculation they would be produced by the forces.

We assume that the system with which we are dealing is sufficiently elastic to allow us to neglect coupled systems and yet need no C.D.D poles. The normalisation of N_ℓ and D_ℓ is still undetermined so we normalise $D_\ell\,(so) = 1$. In an exact calculation, the solution of the equation would be independent of So. However an approximate solution may, and generally will, depend on So. We may now write the dispersion relation for D :

$$D_\ell\,(s) = 1 - \frac{(s - so)}{\pi} \int_{SR}^{+\infty} ds' \, \frac{\rho_\ell\,(s')\,R_\ell\,(s')\,N_\ell\,(s')}{(s' - s)(s' - so)} \qquad (1.8)$$

With knowledge of Im $a_\ell\,(s)$ on the left hand cut, equations (1.7) and (1.8) could be solved and $a_\ell\,(s)$ determined. In order to do practical calculations it is necessary to approximate the left hand cut in some way. One way, which is of particular value in the static model, is to replace the left hand cut by a sum of poles.

Pole Approximations

The approximation is to set : $\text{Im}\, a_\ell\,(s) \underset{S < S_L}{=} \sum_i \gamma_i\,\delta(s - s_i)$,

$$(1.9)$$

Then from equation (1.7): $$N_\ell\,(s) = \frac{1}{\pi} \sum_i \frac{\gamma_i}{si - s} \, D_\ell\,(si) \qquad (1.10)$$

Substituting into equation (1.8) we find:

$$D_\ell\,(s) = 1 - \frac{(s - so)}{\pi} \sum_i \int_{SR}^{+\infty} \frac{ds' \;\; \rho_\ell\,(s')\,R_\ell\,(s')\,\gamma_i\,D\,(si)}{(si - s')(s' - s)(s' - so)} \qquad (1.11)$$

Putting $s = s_i$ in equation (1.1), one obtains a set of simultaneous equations for the $D_\ell(s_i)$ which can be solved, and inserting the solutions into equations (1.10), one may obtain $a_\ell(s)$ in the physical region.

The left hand cut comes from crossing the u- channel physical amplitude. In general, this crossing will be complicated and the approximation of the cut by poles will be of little significance physically. However in the static model, as previously remarked, s - u crossing merely consists of putting $w \to -w$. and u- channel poles do not spread out into cuts in the s- channel.

It is always possible to write a dispersion relation in ν instead of s. As $\nu = 4Mw$ in the static model, one can write the dispersion relations in w. This we do in what follows.

In the static models, a pole in a partial wave at $w = w_i$ will cross into a pole (with the same residue) at $w = -w_i$ in the same partial wave. Thus a set of resonances, with couplings γ_i and energies w_i in the u- channel process, will generate poles at $w = -w_i$, with residues γ_i, on the left hand cut of the s- channel process.

4. Static Model Bootstraps (30)

We consider a model of mesons scattering of baryons in an ℓ- wave. There is a symmetry group for the system and the invariant channels are labeled by Greek letters. We allow for some of these channels to contain particles and we label such channels, and the particles in them, by primed Greek letters. We rule out the possibility of there being, more than one particle in each invariant channel. The particle α' will have energy

w'_α and couple with strength $\gamma_{\alpha'}$ to the system.

The introduction of symmetries adds only a slight complication. Each invariant channel α in the s- channel will receive a contribution to its left hand cut from each invariant channel β' in the u- channel. In our model, we approximate Im $a_\alpha(s)$ (on the left hand cut) by:

$$\mathrm{Im}\ a_\alpha(w) = \sum_{\beta'} C_{\alpha\beta'}\ \delta(\omega+\omega_{\beta'})\,\gamma_{\beta'} \qquad (1.12)$$

where C is the s - u crossing matrix for our symmetry group. Then from equation (1.10): $N_\alpha(w) = \sum_{\beta'} C_{\alpha\beta'} \frac{\gamma_{\beta'}\ D_\alpha(-\omega_{\beta'})}{\omega+\omega_{\beta'}}$ (1.13)

Now from equation (1.11) we obtain:

$$D\ (s) = 1 - \frac{(w\quad -wo\quad)}{\pi}\sum_{\beta'}\int \frac{dw'\,\rho(w')\,R(w')\ \ C_{\alpha\beta'}\,'\,\gamma_{\beta'}\,'D_\alpha(-w_{\beta}\,')}{(w'-w)\,(w'-w_{o\alpha}\,)(w'+w_{\beta'}\,)} \qquad (1.4)$$

where we allow ourselves to choose a different subtraction point for each α if we so desire.

We are now faced with one of the central problems of the N/D method for if $\ell \geq 1$, the integral in equation (1.14) will diverge. In order to make it converge a cut-off function V(w) must be introduced into the integral, where V (w) has the property of being 1 up to large values of w and there after tend to zero in such a way as to make the integral converge. This ad hoc introduction of a cut-off is necessary because we are integrating over the range $(m, +\infty)$, whereas the model is only valid for small w. With the correct relativistic kinematic factors the integral will converge

converge, and is in fact tractable (31). Taking the static approximation at this point gives D linear for w<<M. Diu (32) argues that as the integral is initially divergent, with a proper cut-off the main contiubution to the integral will come from the high energy region. Thus the integral will be independent of ω and hence may be written as a constant, thus allowing us to consider D as linear. Experience of calculations in which D oscillates at high energy (33), makes this argument rather shaky and we prefer the former argument. Also it should be remarked that again experience in calculations shows that D tends to be linear in the resonance region (34). We write the linear D approximation in the form:

$$D_\alpha(\omega) = 1 - (\omega - \omega_{0\alpha})\lambda_\alpha K_\alpha \qquad (1.15)$$

where $$K_\alpha = \sum_{\beta'} C_{\alpha\beta'}\gamma_{\beta'} D_\alpha(-\omega_{\beta'}) \qquad (1.16)$$

and λ_α is related to the integral as described above.

If we believe Diu's argument:

$$\lambda_\alpha = \int_M^{+\infty} \frac{\rho(\omega')R(\omega')V_\alpha(\omega')\,d\omega'}{(\omega'-\omega)(\omega'-\omega_{0\alpha})(\omega'+\omega_{\beta'})} \qquad (1.17)$$

In any case we allow λ_α to depend on α. We discuss the possibility that the λ_αs are equal (the "universal cut-off assumption") later. We note that equations (1.15) and (1.16) are interdependent. Substituting equation (1.15) into equation (1.16), we obtain consistency conditons:

$$K_\alpha = \sum_{\beta'} C_{\alpha\beta'}\gamma_{\beta'}\left\{1 + (\omega_{\beta'}+\omega_{0\alpha})\lambda_\alpha K_\alpha\right\} \qquad (1.18)$$

which may be re-written as:

$$\left[1 - \sum_{\beta'} C_{\alpha\beta'}\gamma_{\beta'}(\omega_{\beta'}+\omega_{0\alpha})\lambda_\alpha\right]K_\alpha = \sum_{\beta'} C_{\alpha\beta'}\gamma_{\beta'} \qquad (1.19)$$

The bootstrap requirements are that each channel β' (into which we inserted a pole in the u- channel) should contain a direct channel pole corresponding to a particle with the same coupling and mass as the u- channel particle. Thus the condition is that each channel β' contain a pole at $\omega_{\beta'}$ with residue $\gamma_{\beta'}$. In mathematical terms:

$$\operatorname{Re} D_{\beta'}(\omega_{\beta'}) = 0 \tag{1.20}$$

and
$$-\gamma_{\beta'} = \frac{N_{\beta'}(\omega_{\beta'})}{\left[\operatorname{Re} D_{\beta'}\right]'_{\omega=\omega_{\beta'}}} \tag{1.21}$$

Using equation (1.15), equation (1.20) gives:

$$1 - (\omega_{\beta'} - \omega_{\beta'0})\lambda_{\beta'}K_{\beta'} = 0 \tag{1.22}$$

Using the consistency condition, equation (1.19), we obtain, after some simple algebra:

$$1/\lambda_{\beta'} - \sum_{\gamma'} C_{\beta'\gamma'}\gamma_{\gamma'}\omega_{\gamma'} = \sum_{\gamma'} C_{\beta'\gamma'}\gamma_{\gamma'}\omega_{\beta'} \tag{1.23}$$

Using equations (1.13) and (1.15), equation (1.21) gives:

$$-\gamma_{\beta'} = \sum_{\gamma'} C_{\beta'\gamma'} \frac{\gamma_{\gamma'} D_{\beta'}(-\omega_{\gamma'})}{(\omega_{\beta'} + \omega_{\gamma'})(-\lambda_{\beta'}K_{\beta'})}$$

Substituting for $D_{\beta'}(-\omega_{\gamma'})$ from equation (1.15) we obtain:

$$\gamma_{\beta'}\lambda_{\beta'}K_{\beta'} = \sum_{\gamma'} C_{\beta'\gamma'} \frac{\gamma_{\gamma'}\left\{1 + (\omega_{\gamma'} + \omega_{\beta'0})\lambda_{\beta'}K_{\beta'}\right\}}{\omega_{\beta'} + \omega_{\gamma'}}$$

Substituting for $\lambda_{\beta'}K_{\beta'}$ from equation (1.22) gives:

$$\gamma_{\beta'} = \sum_{\gamma'} \frac{C_{\beta'\gamma'}\gamma_{\gamma'}}{\omega_{\beta'} + \omega_{\gamma'}}\left\{1 + \frac{\omega_{\gamma'} + \omega_{\beta'0}}{\omega_{\beta'} - \omega_{\beta'0}}\right\} = \sum_{\gamma'} C_{\beta'\gamma'}\gamma_{\gamma'} \tag{1.24}$$

Using this result, equation (1.23) yields:

$$\gamma_{\beta'}\,\omega_{\beta'} + \sum_{\gamma'} C_{\beta'\gamma'}\,\gamma_{\gamma'}\,\omega_{\gamma'} = 1/\lambda_{\beta'} \qquad (1.25)$$

We have, as yet, imposed no condition on the channels into which we inserted no input pole. Were such a channel to have a pole in the direct channel, our bootstrap programme would be marred. We therefore wish to exclude this possibility.

From equations (1.15), the smaller $\lambda_\alpha K_\alpha$ the further away the pole will be in the α-channel. For no pole to occur in the low energy region, therefore, $\lambda_\alpha K_\alpha$ must be numerically small. If this is so, it may be hoped that second order terms will become important for larger energies and remove the pole from the α channel altogether. If $\lambda_\alpha K_\alpha < 0$ the pole would occur at unphysical values of w and correspond to a "ghost" state. Such states are physically not allowed. For the above reasons, it seems proper to impose the condition $\lambda_\alpha K_\alpha = 0$ or $\ll 1$ which will not allow the pole to occur at $\omega \ll M$. Reference (31) suggests that λ_α is of order 1/M, so this condition becomes $K_\alpha \ll 1$. This gives, using equation (1.18)

$$\sum_{\beta'} C_{\alpha\beta'}\,\gamma_{\beta'} = 0 \text{ or } \ll 1 \qquad (1.26)$$

Combining equations (1.24) and (1.26), we obtain the standard bootstrap equations (35): $\gamma_\alpha = \sum_\beta C_{\alpha\beta}\,\gamma_\beta$ (1.27)

(with the convention that $\gamma_\alpha = 0$ if there is no particle in that channel) or $\bar{\gamma}_\alpha = \sum_\beta C_{\alpha\beta}\gamma_\beta$, where $\bar{\gamma}_\alpha$ is small if $\gamma_\alpha = 0$

5. Superconvergence in the Static Model.

We have seen how, using the N/D equations and the pole approximations, a bootstrap calculation may be performed. As this method is based on the use of dispersion relations, it is interesting to see if superconvergences relations, another extension of dispersion relations, yield similar in-formation about the couplings and masses in the same pole approximation. From many calculations it is known that the saturation of superconvergence relations with single particle states gives relations between couplings. Also moment sum poles yield information about the particle masses.

The prima facie similarity between the two methods prompts one to look more closely to see if the methods are in fact eqdvalent in some way. Dim (32) looked at this problem and has shown how the similar results can be derived in a model with only two particles. The work in this section and the derviation of the canonical method of solving the N/D equations was undertaken in order to find the mathematical relation between the two methods. The in-sight provided by this work enables one to see a closer equivalence between the two methods than Dim found. Indeed conclusions can be drawn which throw light on the use of the bootstrap equations.

Let us first derive superconvergence relations from the dispersion relations. The model we use is the same as in the section of this chapter on Static Model Bootstraps.

An amplitude $a_\alpha(\omega)$ is said to superconverge (37) if $a_\alpha(\omega) \to \frac{1}{\omega^{1+\epsilon}}$, $\epsilon > 0$, as $\omega \to \infty$ in any direction

This condition is sufficient for $a_\alpha(\omega)$ to obey an unsubtracted dispersion relation:

$$a_\alpha(\omega) = \frac{1}{\pi}\int_m^\infty \frac{d\omega'}{\omega'-\omega}\, \mathrm{Im}\, a_\alpha(\omega) + \frac{1}{\pi}\int_m^\infty \frac{d\omega'}{\omega'+\omega} \sum_\beta C_{\alpha\beta}\, \mathrm{Im}\, a_\beta(\omega') \qquad (1.28)$$

If we expand a (w) in inverse powers of w, the superconvergence conditions tells us that the $1/\omega$ terms must vanish.

Thus:

$$-\int_m^\infty d\omega'\, \mathrm{Im}\, a_\alpha(\omega') + \int_m^\infty d\omega' \sum_\beta C_{\alpha\beta}\, \mathrm{Im}\, a_\beta(\omega') = 0 \qquad (1.29)$$

The assumption we make to derive relations between the couplings, is that the amplitudes $a_\alpha(\omega)$ are superconvergent and that the integrals in equation (1.29) can be saturated by the contributions of the single particles states $\{\beta'\}$. This latter assumption corresponds to putting $\mathrm{Im}\, a_\alpha(\omega) = \pi\gamma_\alpha\delta(\omega-\omega_\alpha)$ where again we use the convention $\gamma_\alpha = 0$ if there is no single particle state in the α-channel. The sum rules, equation (1.29), now become simple algebraic relations:

$$\gamma_\alpha = \sum_\beta C_{\alpha\beta}\gamma_\beta \qquad (1.30)$$

which are the bootstrap conditions, equation (1.27).

We next discuss the first moment sum rule. This may be derived from the dispersion relation in an analogous way to the superconvergence relation, on the assumption that $a_\alpha(\omega) \sim \frac{1}{\omega^{2+\epsilon}}, \epsilon > 0$ as $\omega \to \infty$ The relation is:

$$\int_m^\infty \omega' d\omega' \, \mathrm{Im}\, a_\alpha(\omega') + \int_m^\infty \omega' d\omega' \sum_{\beta'} C_{\alpha\beta} \, \mathrm{Im}\, a_\beta(\omega') = 0 \qquad (1.31)$$

If these relations hold, which is intrinsically less likely than the case of the superconvergence relations, it is still possible, even probable, that it will not be possible to saturate with the single particle states as the weighting factor ω' will enhance the contributions from higher energies. Because of this we allow for other contributions by writing

$$\int_m^\infty \omega' d\omega' \, \mathrm{Im}\, a_\alpha(\omega') = \omega_\alpha \gamma_\alpha + I_\alpha \qquad (1.32),$$ where

$\omega_\alpha \gamma_\alpha$ is the contribution to the integral from the pole term. With this, the first moment sum rule gives:

$$\omega_\alpha \gamma_\alpha + \sum_\beta C_{\alpha\beta}\, \omega_\beta \gamma_\beta + \left(I_\alpha + \sum_\beta C_{\alpha\beta} I_\beta \right) = 0 \quad (1.33)$$

If the relations do not hold, it may be possible to write a finite energy sum rule (38) which has the same form as equation (1.33) with I_α as a Regge pole term. (We deal with finite energy sum rules in chapter five) It is thus reasonable to assume an equation of the form of equation (1.33) holds, where I_α is an integral over the unitarity cut or a Regge pole term. In either case we can say little about the terms I_α without introducing assumptions, which would mean that our calculation would no longer be a "bootstrap". In chapter five, we introduce extra assumptions in an attempt to explain the existence of symmetries in a more re-liable model.

Conclusions

We are now able to discuss the connection between the bootstrap and superconvergence methods. We list several remarks to this end, below:

(i) The results of the standard static model bootstrap calculation are identical in almost all respects to those derived from taking single particle saturation of superconvergence relations written for the various amplitudes, and from a similar consideration of the first moment sum rules.

(ii) Using the stronger conditions in equation (1.26), both methods give the bootstrap consistency conditions for all channels. If the weaker the condition is used, the bootstrap method yields the conditions only for channels containing particles, whilst it says that the elements corresponding to particles with no particle should be small.

(iii) In the bootstrap calculation, the left hand cut is taken to contain only poles, whereas in the superconvergence calculation we allowed the moment sum rule to receive a contribution from the u- channel unitarity cut. If we are to be solving the same problem by each method we must neglect this cut contribution and take the same left hand cut for both calculations. Then, with the identification of I_α with $^{-1}/\lambda_\alpha$ equations (1.25) and (1.33) are the same for the channel which contains a particle. The first moment sum rule gives a mass relation for the case where the channel has no particle whereas the bootstrap does not.

We can see no reason for equating the λ_αs in any way and this casts doubt, via the above identification, on the assumption of a universal cut-off. As this assumption leads to inconsistencies in, for example, Diu's calculation (32) we are happy to discard it. As the mass relations all contain arbitrary parameters (I_α or λ_α) they are of little value and this situation puts the two methods on a par as far as masses are concerned.

(iv) It should be pointed out that the reasons that we differ from Diu in our conclusions are that Diu fails to look at the moment sum rule, puts no conditions on a bootstrap amplitude which should contain no pole, and assumes a universal cut-off. His ad hoc method of solving his two particle model obscures the simple mathematical relation between the two methods, which naturally leads to consideration of the moment sum rule. His use of a universal cut-off, against which useage we have argued, lends to the breakdown of his bootstrap equations in the πN case, where the internal and external nucleons are given equal masses, because there are insufficient parameters to satisfy the equations. Without the universal cut-off, one has no such problems.

6. Uses of the Bootstrap equations.

(a) $N - N^*$ bootstrap (3[illegible])

At low energies the πN scattering amplitude in largely p- wave and dominated by the existence of the nucleon and the N^*33 reasonance. Labelling states by their isospin and spin (I and J) we have the $N(\frac{1}{2},\frac{1}{2})$, $N^*(\frac{3}{2},\frac{3}{2})$ and the p- wave pion is effectively a (1,1) particle.

In this case the isospin and spin crossing matrixes are equal (39)

$$
\begin{array}{cc}
 & \begin{array}{cc} (\frac{1}{2}) & (\frac{3}{2}) \end{array} \\
C(su) \;\equiv & \begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}
\end{array}
\begin{array}{c} (\frac{1}{2}) \\ (\frac{3}{2}) \end{array}
$$

where the bracketed numbers beside the matrix indicate the channels.

If we assume that the N and N^* are the only single particle states which exist we have four equations:

$$
\begin{pmatrix} \gamma_{\frac{1}{2},\frac{1}{2}} \\ 0 \\ 0 \\ \gamma_{\frac{3}{2},\frac{3}{2}} \end{pmatrix} = \left(C^{I}_{su} \otimes C^{J}_{su} \right) \begin{pmatrix} \gamma_{\frac{1}{2}\frac{1}{2}} \\ 0 \\ 0 \\ \gamma_{\frac{3}{2},\frac{3}{2}} \end{pmatrix} \qquad (1.34)
$$

Due to what might be described as good fortune, these equations have a solution: $\gamma_{\frac{1}{2}\frac{1}{2}} = 2\ \gamma_{\frac{3}{2}\frac{3}{2}}$

If we identify the γ s with couplings as follows:

$\gamma_{\frac{1}{2}\frac{1}{2}} = g_{\pi NN}\ \ g_{\pi N^{*}N}$, $\quad \gamma_{\frac{3}{2}\frac{3}{2}} = g_{\pi NN^{*}}\ \ g_{\pi NN^{*}}$

we obtain: $g_{\pi NN} = 2\, g_{\pi NN^{*}}$, which is close to the experimental value.

The above solution is unique only because we put $\gamma_{\frac{1}{2}\frac{3}{2}} = \gamma_{\frac{3}{2}\frac{1}{2}} = 0$

It is however in some sense the simplest solution, requiring as it does a minimal number of particles. In general the bootstrap equation will not be exactly soluble with only the desired particles and it will be necessary to introduce other particles which one hopes will have small γ s thus corresponding to high lying resonances. Hwa and Patil (40) used this condition of using a minimal number of particles in an attempt to produce a meaningful bootstrap programme.

(b) <u>Baryon octet - decuplet bootstrap in SU(3)</u> (52)

After the success of the N - N* bootstrap, it was natural with the advent of unitary symmetries to attempt to extend this success to the SU(3) case of the pseudo-scalar meson octet scattering off the baryon octet, using, if possible, the baryon octet and decuplet as the internal states.

Before we perform the calculation, we must do a little group theory:

In SU(3) : $8 \otimes 8 = 1 \oplus 8s \oplus 8a \oplus 10 \oplus \overline{10} \oplus 27.$

We have chosen linear combinations of the octet states which couple symmetrically and antisymmetrically to the $8 \otimes 8$. There are now eight channels for the process $8 \otimes 8 \rightarrow 8 \otimes 8$:

$1 \rightarrow 1$, $8s \rightarrow 8s$, $8s \rightarrow 8A$, $8A \rightarrow 8s$, $8A \rightarrow 8A$, $10 \rightarrow 10$, $\overline{10} \rightarrow \overline{10}$, $27 \rightarrow 27$

of which $8s \rightarrow 8A$ (8sA) and $8A \rightarrow 8s$(8As) are equal by time reversal.
The su(2) crossing matrix is the same as for the NN* case:

$$C_J = \begin{array}{cc} \begin{array}{cc} (\tfrac{1}{2}) & (\tfrac{3}{2}) \end{array} & \\ \begin{pmatrix} -\tfrac{1}{3} & \tfrac{4}{3} \\ \tfrac{2}{3} & \tfrac{1}{3} \end{pmatrix} & \begin{array}{c} (\tfrac{1}{2}) \\ (\tfrac{3}{2}) \end{array} \end{array}$$

The su(3) matrix is (63)

$$C^{su(3)} = \begin{array}{cc} \begin{array}{ccccccc} (27) & (10) & (\overline{10}) & (8ss) & (8sa) & (8aa) & 1 \end{array} & \\ \begin{pmatrix} 7/40 & 1/12 & 1/12 & 1/5 & 0 & \tfrac{1}{3} & \tfrac{1}{8} \\ 9/40 & \tfrac{1}{4} & \tfrac{1}{4} & 2/5 & 2/\sqrt{5} & 0 & -\tfrac{1}{8} \\ 9/40 & \tfrac{1}{4} & \tfrac{1}{4} & 2/5 & -2/\sqrt{5} & 0 & -\tfrac{1}{8} \\ 27/40 & \tfrac{1}{2} & \tfrac{1}{2} & -3/10 & 0 & -\tfrac{1}{2} & \tfrac{1}{8} \\ 0 & \sqrt{5}/4 & -\sqrt{5}/4 & 0 & 0 & 0 & 0 \\ 9/8 & 0 & 0 & -\tfrac{1}{2} & 0 & \tfrac{1}{2} & -\tfrac{1}{8} \\ 27/8 & -5/4 & -5/4 & 1 & 0 & 0 & \tfrac{1}{8} \end{pmatrix} & \begin{array}{c} (27) \\ (10) \\ (\overline{10}) \\ (8ss) \\ (8sa) \\ (8aa) \\ (1) \end{array} \end{array}$$

We seek a solution containing only the $(10, \tfrac{3}{2})$ and $(8, \tfrac{1}{2})$ channels.
To do this we attempt to solve the bootstrap condition for the sub-matrix (C') which contains only these channels and then see if this solution also yields a solution of the complete bootstrap equation. The sub-matrix in question is:

$$C' = \begin{array}{cc} \begin{array}{cccc} (10, \tfrac{3}{2}) & (8ss, \tfrac{1}{2}) & (8sa, \tfrac{1}{2}) & (8aa, \tfrac{1}{2}) \end{array} & \\ \begin{pmatrix} 1/12 & 4/15 & 4/3\sqrt{5} & 0 \\ \tfrac{2}{3} & 1/10 & 0 & 1/6 \\ \sqrt{5}/3 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & -1/6 \end{pmatrix} & \begin{array}{c} (10, \tfrac{3}{2}) \\ (8ss, \tfrac{1}{2}) \\ (8sa, \tfrac{1}{2}) \\ (8aa, \tfrac{1}{2}) \end{array} \end{array}$$

C' has an eigenvalue 0.85 which is near 1. However the couplings to the three octet channels are not independent, there being only two free parameters, an overall normalisation and the f/d ratio.

The best solution corresponds to $\gamma_{10}/\gamma_8 = 1.06$, $\alpha = 0.70$ where α is the f/d ratio. This gives

$$\gamma' = \begin{pmatrix} \gamma_{10} \\ \gamma_{ss} \\ \gamma_{sa} \\ \gamma_{aa} \end{pmatrix} \equiv \begin{pmatrix} \gamma_{10} \\ 20/9\,\alpha^2 \gamma_8 \\ 4\sqrt{5}/3\,\alpha(1-\alpha)\gamma_8 \\ 4(1-\alpha)^2\gamma_8 \end{pmatrix} = \begin{pmatrix} 1.06 \\ 1.09 \\ 0.626 \\ 0.0360 \end{pmatrix}$$

whilst

$$C'\gamma' = \begin{pmatrix} 0.752 \\ 0.876 \\ 0.790 \\ 0.122 \end{pmatrix}$$

It is open to argument whether the above represents a reasonable solution of the problem.

We now look at the complete bootstrap equation and put $\underline{\gamma}$ equal to γ' plus ten vanishing components. We already know four components of $C\gamma$ from the above. The remaining ones are:

$(27,\frac{3}{2})$	$(27,\frac{1}{2})$	$(10,\frac{1}{2})$	$(\overline{10},\frac{3}{2})$	$(\overline{10},\frac{1}{2})$	$(8ss,\frac{3}{2})$	$(8sa,\frac{3}{2})$	$(8aa,\frac{3}{2})$	$(1,\frac{3}{2})$	$(1,\frac{1}{2})$
0.255	0.005	0.021	0.006	0.395	−0.161	0.97	−0.243	0.045	−2.01

These elements are small or of the same order of magnitude as the error in solution of $C'\gamma' = \gamma'$, apart from the $(1,\frac{1}{2})$ element

(c) SU(6) (43)

In the SU(6) model of the baryons, the octet and decuplet are put in one representation of the group, the 56. In assuming this assignment we are discarding the idea that the baryons and the resonances bootstrap each other and assuming that both exist a priori.

The mesons are assigned to the 35 representations and one may ask whether the 56 can bootstrap itself in the meson-baryon scattering process. If this should prove correct and it is not the case for other multiplets such as the 20 or 70, it would provide a bootstrap argument for the existence of the 56 plet and not the other representations. The dynamical problems of SU(6) are avoided by letting the pseudo-scalar mesons act in a p- wave.

Balazs, Singh and Udgaonkar (48) carried out the above programme. Indeed the 20 plet is unlikely to bootstrap either singly or reciprocally However in 35 - 56 scattering, the diagonal 56 crossing-matrix element is very nearly one, which suggests that the 56 could bootstrap itself. This is also true for the 1134 in the same process, but a large negative 56 - 1134 crossing matrix elements suggests that the multiplets are unlikely to co-exist. If one believes in SU(6) as a symmetry group, the above may provide some reason for the 56 assignment of the baryons.

(d) Isobar chains (49, [illegible])

With the success of the N - N* bootstrap, people wondered whether there might not exists infinite chains of particles which could bootstrap each other in some way. The most interesting success in this field is the result of Abers, Bàlazs, and Hara (19), that in πN^* scattering the N^* and N^{**} $(I = J = 5/2)$ bootstrap each other, and so on. Thus the chain of nucleon isobars with $I = J$ bootstrap each other. As will appear in chapter II, this fact is no accident but derives from the existence of a non-invariance group for the system.

CHAPTER 2

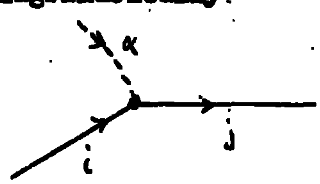
Strong Coupling Theory

Strong Coupling Theory, as developed by Cook, Goebel and Sakita (41,42) from the early work of Pauli et al. (43), sets out to describe, by means of the Chew-Low Equation, the scattering of pseudo-scalar mesons off baryons in a p- wave.

It is assumed that there exists an internal symmetry group (K) for the system, such that the mesons and isobars form representations of the group and such that the meson- baryon interaction is invariant under the group. As we will be working in the static model, $SU^J(2)$, the spin symmetry group, may be combined into the internal symmetry group K. Let us consider processes $Ni + \pi_\alpha \rightarrow Nj + \pi_\beta$, (45) with scattering amplitudes $(f_{\beta\alpha})^{ji}$ where i,j label isobar states and α, β, ... a set of mesons. $f_{\beta\alpha}$ and the A_α 's which we define later are operators in isobar space with the notation $(f_{\beta\alpha})^{ji} = \langle j | f_{\beta\alpha} | i \rangle$ and

$$(A_\alpha)^{ji} = \langle j | A_\alpha | i \rangle$$

The operation λA_α is defined as the Yukawa coupling for the absorption of the meson component α. Thus $\lambda (A_\alpha)^{ji}$ is the coupling corresponding to the isobar i absorbing the meson component and producing the isobar j. Diagramatically.



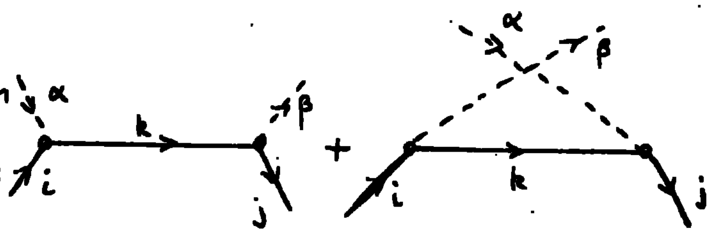
The Born terms for the process $N_i + \pi_\alpha \rightarrow N_j + \pi_\beta$ corresponds to possible isobar intermediate states in the process and in the $s \rightarrow u$ crossed process. Thus:

$$\left(f^{B}_{\beta\alpha}\right)^{ji} = -\lambda^2 \sum_k{}' \left[\frac{(A_\beta^\dagger)^{jk}(A_\alpha)^{ki}}{M_k - M_i - \omega} + \frac{(A_\alpha)^{jk}(A_\beta^\dagger)^{ki}}{M_k - M_j + \omega} \right] \tag{2.1}$$

where the sum over k is over all isobar states.

We can represent the Born term diagramatically as below:

$(f_{\beta\alpha})^{ji} \sim \sum_k$



The Chew - Low form for f, which satisfies analyticity, unitarity and crossing symmetry, is:

$$\begin{aligned}(f_{\beta\alpha})^{ji} = {} & (f^{B}_{\beta\alpha})^{ji} \\ & + \sum_{k,\gamma} \int_m^{+\infty} \rho(\omega')d\omega' \left[\frac{(f_{\gamma\beta}(\omega'))^{kj*}(f_{\gamma\alpha}(\omega'))^{ki}}{M_k - M_i + \omega' - \omega - i\epsilon} \right. \\ & \left. + \frac{(f_{\gamma\alpha}(\omega'))^{kj*}(f_{\gamma\beta}(\omega'))^{ki}}{M_k - M_j + \omega' - \omega} \right] \end{aligned}$$

+ (two or more meson intermediate states) (2.2) where Mi is the mass of the isobar i and $\rho(w)$ is a kinematic factor,

We have in the theory an undetermined parameter λ, which measures the overall strength of the meson couplings. Experience suggests that if λ is increased, the isobar masses tend to a common limit. We make this assumption and set:

$$M_i = M + \Delta_i/\lambda^2 \qquad (2.3)$$

where Δ_i remains finite as $\lambda^2 \to \infty$.

The limit $\lambda^2 \to +\infty$ is the strong coupling limit and the strong coupling model is derived on the assumption that the equations of the Chew-Low model are in some sense "analytic" in λ^2 in the limit $\lambda^2 \to \infty$. Unitarity requires the scattering amplitude to be finite in the physical region. By equation (2.2) the Born term is also constrained to be finite.

Using equation (2.3), the Born term can be expanded in terms of $1/\lambda^2$:

$$f^B_{\beta\alpha} = \frac{\lambda^2}{\omega}\left[A_\beta^\dagger, A_\alpha\right] - \frac{1}{\omega^2}\left[A_\beta^\dagger, \left[M, A_\alpha\right]\right] + O(1/\lambda^2) \qquad (2.4)$$

where m is the mass operation defined by $m|i\rangle = M_i|i\rangle$

Thus the finiteness of the Born term for all processes implies that

$$\left[A_\beta^\dagger, A_\alpha\right] = 0 . \qquad (2.5)$$

This equation, being true for all α, β, can be re-written in the standard form $[A_\alpha, A_\beta] = 0$. (2.6) This condition, derived from the dynamics of the problem, is sufficient to ensure that the algebra generated by the As and Js (the Js being the generators of the symmetry group K) closes. The problem of finding the isobars is thus reduced to the algebraic one of finding unitary irreducible representations of this algebra. The additional assumption required is that the mesons sources A_α transform like tensor operators of K. This gives an equation of the form:

$$[J_i, A_\alpha] = D_{i\alpha\beta} A_\beta \tag{2.7}$$

The generators of K obey an equation of the form:

$$[J_i, J_j] = C_{ijk} J_k. \tag{2.8}$$

where C_{ijk} and D_{ijk} are structure constants. Equations (2.6),(2.7) and (2.8) define the algebra of the strong coupling group G for the system. Inspection shows that G is the semidirect product of K with T, the translation or Abelian group generated by the A_α. $G = K \times T$. T is the translation group n n dimensions, where n is the dimension of the space spanned by the A_α. As G is non-compact its unitary irreducible representations are infinite dimensional.

Representations of the Strong Coupling Group

The methods used for deriving representation of the strong coupling group are mostly of a technical nature and physically unenlightening. The techniques of group contraction, used by C.G.S in their original paper, and method of induced representations are both standard group theory procedures. The methods derived by Fairlie (46) and also Udgoankar and Singh (47), are however of physical interest as they not only solve the problem in hand but also exhibit the close relationship between the strong coupling equations and the bootstrap consistency condition.

We therefore discuss these latter methods in some detail whilst contenting ourselves with a brief outline of the former.

Group Contraction

Given a strong coupling group G, the idea is to find a group H with the property that one may take linear combinations of the generators of H and by taking the coefficients to some limit obtain operators which obey the algebra of G. By seeing the effect of this limit on the parameters specifying a representation of H, one may find a corresponding representation of G. Naturally one tries to find a group H whose irreducible unitary representations are particularly simple and easy to find. Usually H will be chosen to be compact, thus enabling one to deal with finite representations. Of course, after contraction such representations will become infinite as G is be non-compact.

A group H, related to the strong coupling group G, as specified above, is referred to as an intermediate coupling group.

As an example of the use of this method consider the scattering of scalar mesons with isospin symmetry (42) This is the case where $K = SU^I(2)$ and $G = su(2) \times T_3$ H is chosen to be $su(2) \oplus su(2)$ with generators L^1_i and L^2_i which obey:

$$[L^r_i, L^r_j] = i \epsilon_{ijk} L^r_k \quad , r = 1,2$$

$$[L^1_i, L^2_j] = 0 \qquad i,j,k = 1,2,3$$

Put $I_i = L^1_i + L^2_i$, $A_i = \epsilon(L^1_i - L^2_i)$ In the limit $\epsilon \to 0$ keeping A_i finite, the I_i and A_i generate the algebra of $su(2) \times T_3$ The irreducible unitary representations of $su(2) \oplus su(2)$ are specified by (l_1, l_2) where $l_r(l_r + 1)$ is the value of the Casämir operator $(L^r)^2$ acting on the representation.

Putting $I^2 = t(t+1)$, t will assume the values $t = |l_1 - l_2|, \ldots, l_1 + l_2$, by the usual result for coupling time angular momenta. Thus for a useful representation of $su(2) \times T_3$ to emerge from our calculation we must keep $t\ t_0 = |l_1 - l_2|$ finite.

$$\underline{A} \cdot \underline{I} = \epsilon\{(L^1)^2 - (L^2)^2\} = \epsilon\,(l_1 - l_2)(l_1 + l_2 + 1).$$

Thus in order than A does not vanish we must choose $(l_1 + l_2)$ to become infinite. Thus we must contract $su(2) \oplus su(2)$ by making $l_1 + l_2 \to \infty$ whilst keeping $(l_1 - l_2)$ finite.

In fact t_o specifies the representation, and such a representation contains an infinite number of irredicible representations of the subgroup $su^I(2)$ with $I = t(t+1)$, $t=to, to+1,$

The use of this method is tedious for larger groups G. The details may be found in the literature.

At the end of their paper, C.G.S. remark that the connection between group contraction and taking the strong coupling limit might have physical significance. They suggest that for finite couplings, the precontracted intermediate coupling group might serve as a non-invariance group for the system. In chapter 4 , we discuss the theory of intermediate coupling built on this idea by Kuriyan and Sudarsham (14).

For completeness we list below various processes with the corresponding symmetry groups (K), strong coupling groups (G) and intermediate coupling groups (H)

K	G	H (compact)	H (non-compact)
$SU(2)$	$SU(2) \times T_3$	$SU(2) \otimes SU(2)$	$SL(2,C)$
$SU(2) \otimes SU(2)$	$[SU(2) \otimes SU(2)] \times T_9$	$SU(4)$	$SL(4,R)$
$SU(2) \otimes SU(3)$	$[SU(2) \otimes SU(3)] \times T_{24}$	$SU(6)$	$SL(6,R)$
$SU(4)$	$SU(4) \times T_{15}$	$SU(4) \otimes SU(4)$	$SL(4,C)$
$SU(6)$	$SU(6) \times T_{35}$	$SU(6) \otimes SU(6)$	$SL(6,C)$

Bootstrap consistency condition for strong coupling (10,47)

The diagram illustrating equation 2.1 shows how the strong coupling condition links the couplings of isobar intermediate states in the direct and crossed channels. It is thus not surprising to find that, using Clebsch-Gordon coefficients to project out specific invariants in the direct channel, the bootstrap consistency condition may be derived from equation (2.5). We use equation (2.5) rather than the more commonly used equation (2.6) because the analysis used for equation (2.5) is identical with that required later to deal with the total amplitude.

To illustrate the technique sketched out above, we take a simple case where K = su(2) and the mesons belong to the "spin 1" representations. Armed with this calculation, it is relatively easy job to construct the calculation for any group K.

To simplify the algebra, we take the meson operators A_α to form the spherical basis of the spin 1 representation. That is, α is the z- component of the meson concerned. (49,50). We label isobars by their spin and its z- component. We may use the Wigner-Eckart theorem for the symmetry group su(2) to write the matrix element of A between two isobar states as the product of an su(2) Clebsch - Gordon coefficient and a reduced matrix element, which is independent of the z- components. Thus:

$$\langle II_z \,|\, A_\alpha \,|\, JJ_z \rangle = C\begin{pmatrix} J & 1 & I \\ J_z & \alpha & I_z \end{pmatrix} \langle I \,\|\, A \,\|\, J \rangle \qquad (2.9)$$

for convenience we write $\langle I \| A \| J \rangle = g_I^J$

Inserting equation (2.5) between the states $\langle I I_z|$ and $|J J_z\rangle$ gives:

$$\sum_k C\begin{pmatrix} k & 1 & J \\ k_z & -\beta & J \end{pmatrix} g_J^k \quad C\begin{pmatrix} I & 1 & k \\ I & \alpha & k \end{pmatrix} g_k^I$$

$$= \sum_{k'} C\begin{pmatrix} k' & 1 & J \\ k'_z & \alpha & J \end{pmatrix} g_J^{k'} \quad C\begin{pmatrix} I & 1 & k' \\ I & -\beta & k'_z \end{pmatrix} \quad g_{k'}^I$$

(2.10)

using $A_\alpha^+ = (-1)^\alpha A_{-\alpha}$ and where $k_z = I_z + \alpha = J_z + \beta$, $k'_z = I_z - \beta = J_z - \alpha$

Equation (2.10) holds for all I, I_z, J, J_z and the summations are over all isobars K and K'. Charge conjugation invariance implies that:

$$\langle I I_z | A_\alpha | J J_z \rangle = (-1)^\alpha \langle J J_z | A_{-\alpha} | I I_z \rangle \qquad (2.11)$$

Using the Wigner-Eckart Theorem, this gives the vertex symmetry relation: $g_J^I = (-1)^{I-J} \left(\frac{2I+1}{2J+1}\right)^{\frac{1}{2}} g_I^J$ (2.12)

Using equation 2.11, equation (2.10) gives:

$$\sum_K g_k^I g_k^J \; C\begin{pmatrix} I & 1 & k \\ I & \alpha & k \end{pmatrix} C\begin{pmatrix} J & 1 & k \\ J & \beta & k \end{pmatrix} (-1)^\beta = \sum_{k'} g_{k'}^I g_{k'}^J \; C\begin{pmatrix} I & 1 & k' \\ I & -\beta & k' \end{pmatrix}$$

$$C\begin{pmatrix} J & 1 & k' \\ J & -\alpha & k' \end{pmatrix} \quad (-1)^\alpha$$

(2.13)

To project out the K channel we multiply by

$$C\begin{pmatrix} I & 1 & K \\ I_z & \alpha & k_z \end{pmatrix} \quad C\begin{pmatrix} J & 1 & K \\ J_z & \beta & k_z \end{pmatrix} \quad (-1)^{-\beta}$$

and sum over I_z and β keeping α fixed.

For the left-hand side of equation (2.13), we can sum over with k_z fixed and then sum over k_z which is equivalent to summing over I_z From this we obtain from the left hand side.

$$\sum_k g_k^I g_k^J \sum_{I_z} C\begin{pmatrix} I & 1 & k \\ I_z & \alpha & k_z \end{pmatrix} C\begin{pmatrix} I & 1 & K \\ I_z & \alpha & k_z \end{pmatrix} \sum_\beta C\begin{pmatrix} J & 1 & k \\ J_z & \beta & k_z \end{pmatrix} C\begin{pmatrix} J & 1 & K \\ J_z & \beta & k_z \end{pmatrix}$$

$$= \sum_k g_k^I g_k^J \sum_{I_z} C\begin{pmatrix} I & 1 & k \\ I_z & \alpha & k_z \end{pmatrix} \quad C\begin{pmatrix} I & 1 & K \\ I_z & \alpha & k_z \end{pmatrix} \quad \delta_{Kk}^{kk}$$

$$= g_K^I g_K^J \quad \left(\frac{2K+1}{3}\right) \sum_{I_z} C\begin{pmatrix} I & K & 1 \\ I_z & -k_z & -\alpha \end{pmatrix} \quad C\begin{pmatrix} I & K & 1 \\ I_z & -k_z & -\alpha \end{pmatrix}$$

$$= g_K^I \; g_K^J \left(\frac{2k+1}{3}\right)$$

where we have used the relation

$$\sum_{i_z\,,\; k_z \text{ fixed}} C\begin{pmatrix} i & j & k \\ i_z & j_z & k_z \end{pmatrix} \quad C\begin{pmatrix} i & j & k' \\ i_z & j_z & k_z \end{pmatrix} \; = \; \delta\, kk'$$

and well known symmetry relations for C-G coefficients.

Under the same operation, the right hand side of equation (2.13) gives:

$$\sum_{k'} g_{k'}^{I}\, g_{k'}^{J} \sum_{I_z,\beta} C\begin{pmatrix} I & 1 & k' \\ I_z-\beta & & k'_z \end{pmatrix} C\begin{pmatrix} J & 1 & k' \\ J_z-\alpha & & k'_z \end{pmatrix} C\begin{pmatrix} J & 1 & K \\ J_z & \beta & k_z \end{pmatrix} C\begin{pmatrix} I & 1 & K \\ I_z & \alpha & k_z \end{pmatrix} (-1)^{\alpha-\beta}$$

Using the symmetry relations for C-G coefficients, the sum can be re-written as:

$$\sum_{k'} g_{k'}^{I}\, g_{k'}^{J} \sum_{I_z,\beta} C\begin{pmatrix} I & 1 & k' \\ I_z & -\beta & k'_z \end{pmatrix} C\begin{pmatrix} k' & J & 1 \\ k'_z & -J_z & -\alpha \end{pmatrix} C\begin{pmatrix} 1 & J & K \\ \beta & J_z & k_z \end{pmatrix} C\begin{pmatrix} I & K & 1 \\ I_z & -k_z & -\alpha \end{pmatrix}$$

$$\sqrt{\frac{(2k'+1)\,(2K+1)}{3}} \qquad (-1)^{2J}$$

The sum over C-G coefficients is simply related to a $6J$ symbol and we may write the term as:

$$\sum_{k'} g_{k'}^{I}\, g_{k'}^{J}\ \frac{(2k'+1)(2K+1)}{3}\ (-1)^{2J} \begin{Bmatrix} I & 1 & k' \\ J & 1 & K \end{Bmatrix}$$

$$= \sum_{k'} C_{Kk'} \left(\frac{2K+1}{3}\right) g_{k'}^{I}\ \ g_{k'}^{J}$$

where $C_{Kk'} = (-1)^{2J}(2k'+1) \begin{Bmatrix} I & 1 & k' \\ J & 1 & K \end{Bmatrix}$ is by definition the

s-u crossing matrix.

Thus we have the equation:

$$\left(\frac{2k+1}{3}\right) g_k^I g_k^J = \left(\frac{2k+1}{3}\right) \sum_{k'} C_{Kk'} g_{k'}^I g_{k'}^J \tag{2.14}$$

and hence obtain the bootstrap consistency condition

$$\Gamma_k = C_{Kk'} \Gamma_{k'} \tag{2.15}$$

where $g_k^I g_k^J = \Gamma_k$ and the summation over k' is assumed.

As remarked earlier, the bootstrap conditions may be derived from the strong coupling condition for any symmetry group K. Also the bootstrap conditions hold for any process $N_i + \pi \rightarrow N_j + \pi$ where N_i, N_j are isobars in the representation of the strong coupling group. Consideration of equation (2.1) quickly shows why this should be the case. We see that the Born term has direct channel poles and crossed channel poles. Moreover we notice that the strong coupling condition is also the condition that the Born term superconverges. Thus the residues at the poles for a particular process must obey the bootstrap conditions, as was shown in chapter 1, in the section on superconvergence. We will return to the topic of superconvergence and strong coupling theory later.

Uses of Bootstrap Condition

The fact that the bootstrap consistency condition holds for all processes within the isobar chain can be used to derive the meson isobar couplings, once the composition of the isobar chain (ie the representation of the strong coupling group) is known. One merely writes down the bootstrap equations for all processes (or as many as necessary) and puts in the isobars as intermediate states. The equations so derived are sufficient to determine the couplings. Indeed one could show that the fact that all the bootstrap equations hold with intermediate states from the isobar chain, is sufficient to guarantee the veracity of the strong coupling equation, acting between isobars within the chain.

The procedure outlined above is particularly suitable for use in a case where the isobars chain has a particularly simple structure, such as the $su^I(2) \otimes su^J(2)$ chain with $I = J$. In this case one need only consider two processes to cover all possible processes. Clearly the more complicated and the larger, the isobar chain is: the harder this method becomes.

The $I = J$ nucleon iso-bar chain is particularly suited to the above technique because we need only consider two processes:

(i) (I,I) + pion $\rightarrow$ (I,I) + pion and

(ii) (I,I) = pion $\rightarrow$ $(I+1,I+1)$ + pion

Consideration of the process (I,I) + pion $\rightarrow$ $(I-1,I-1)$ + pion gives no extra information, as it is the time reversed process to $(I-1, I-1)$ + pion $\rightarrow$ (I,I) + pion which is a process of type (ii)

D.B. Fairlie (46) has an elegent solution to the bootstrap problem for this case which makes use of the orthogonality properties of the crossing matrixes. The bootstrap equation for the invariant amplitude $(I,J) = (a,b)$ for the process (i,j) + pion $\rightarrow$ (i',j') + pion is:

$$(\delta_{aa'}\, \delta_{bb'} - C_{aa'}\, C_{bb'}) \quad g^{ij}_{ab} \quad g^{i'j'}_{ab} = 0$$

where the Cs are the appropriate isospin and spin crossing matrices. The properties of Racah coefficients allow us to write $C_{aa'} = O_{aa'} \left(\frac{2a'+1}{2a+1}\right)^{\frac{1}{2}}$ and $O_{aa'}$ orthogonal and symmetric, for both of crossing matrices. We can use this fact to write our equation as:

$$(\delta_{aa'}\, \delta_{bb'} - O_{aa'}\, O_{bb'})\, G^{ij}_{a'b'} \quad G^{i'j'}_{a'b'} = 0$$

where $G^{ij}_{ab} = \left\{(2a+1)(2b+1)\right\}^{\frac{1}{2}} g^{ij}_{ab}$

if we now restrict ourselves to isobars with $I = J$. The above equations gives: $(\delta_{aa'} - O^2_{aa'})\, G^{ii}_{a'a'}\, G^{i'i'}_{a'a'} = 0$

It is easy to see that if G^{ii}_{aa} is independent of a, the equation is satisfied. With this condition

$$\sum_{a'} O^2_{aa'} \quad G^{ii}_{a'a'} \quad G^{i'i'}_{a'a'} = \sum_{a'} O_{aa'}\, O_{a'a}\quad G^2\,(i) = G^2\,(i)$$

as O is symmetrical and orthogonal. Also $\delta_{aa'}\, G^{ii}_{a'a'}\, G^{i'i'}_{aa} = G^2\,(i)$

For completeness we give the isospin crossing matrix (Csu) for the process $1 + 1 \rightarrow 1 + 1$. (51)

$$\begin{array}{lcccc} & i-1 & i & i+1 & \\ C\,su = & \left(\begin{array}{c} \frac{1}{2i+1} \\ -\frac{(2i-1)}{i(2i+1)} \\ \\ \frac{2i-1}{2i+1} \end{array}\right. & \begin{array}{c} -1/i \\ \frac{i^2+i-1}{i(i+1)} \\ \\ \frac{1}{i+1} \end{array} & \left.\begin{array}{c} \frac{2i+3}{2i+1} \\ \frac{2i+3}{(i+1)(2i+1)} \\ \\ \frac{1}{(i+1)(2i+1)} \end{array}\right) & \begin{array}{c} i-1 \\ i \\ \\ i-1 \end{array} \end{array}$$

The crossing matrix for the process $i + 1 \rightarrow (i+1) + 1$ is: (51)

$$\begin{array}{lccc} & i & i+1 & \\ Csu = & \left(\begin{array}{c} \frac{-1}{(i+1)} \\ \left[\frac{i(i+2)(2i+1)}{2i+3}\right]^{\frac{1}{2}} \frac{1}{i+1} \end{array}\right. & \left.\begin{array}{c} \left[\frac{i(i+2)(2i+3)}{2i+1}\right]^{\frac{1}{2}} \frac{1}{i+1} \\ \frac{1}{i+1} \end{array}\right) & \begin{array}{c} i \\ i+1 \end{array} \end{array}$$

Inserting our solution $g^{ii}_{aa} = \frac{G(i)}{2a+1}$ gives, using vertex symmetry, $G(i)$ to be a constant (ie independent of i) Thus the couplings are: $g^{ii}_{ii} = 1$ (with a particular normalisation), and $g^{ii}_{i+1,i+1} = \left(\frac{2i+1}{2i+3}\right)^{\frac{1}{2}}$ and $g^{ii}_{i-1,i-1} = \left(\frac{2i+1}{2i-1}\right)^{\frac{1}{2}}$ from the vertex symmetry relation.

The above method seems not to be applicable beyond this simple example, so we now turn to another method of obtaining the couplings for the I = J isobar chain, which method was discovered by Fairlie,[46] and also by Udgaonkor and Singh (47) This calculation exemplifies the widely applicable technique of expanding Γ in terms of the eigen vectors with eigen value +1, which are the even columns of Cst (see appendix 1).

Briefly the method is as follows. Write Γ = Cst Γ'' where Γ'' multiplies only the even columns of Cst, as Γ obeys $(1 - C)\Gamma = 0$ and hence is even under s- u crossing. Γ'' has zero elements corresponding to the odd columns. Γ also has zeros corresponding to channels which do not contain a member of the isobar chain. The contraints the zeros of Γ put on Γ'' can be read off from Γ = Cst Γ''. If Γ'' is determined , one can now read off the values of the products of couplings which make up Γ . If Γ'' is undetermined, but contains only a few arbitary parameters, the equation may be inverted to find the constraints which the theory imposes on the elements of Γ . A little thought reveals that the above method of solving the bootstrap equations is, in general superior to the direct method. If the crossing matrix is of order m x m, the expansion Γ = Cst Γ'' immediately gives in terms of roughly m/2 parameters, corresponding to the number of even eigenvectors of Csu. In most problems Γ has half or more of its elements zero, so that Γ'' is determined or contains only a few parameters. The effort involved thus compares very favourably with that required to solve the m linear simultaneous equations of the direct method. As an example of this technique we follow the calculation of Fairlie, again for the isobar chains $I = J$ and $|I - J| = \frac{1}{2}$.

The isospin crossing matrix for $i + 1 \rightarrow i + 1$ has one odd eigenvector $(i + 1,\ 1,\ -i)$ and the even eigenvectors span a two dimensional space. For convenience we take $(0,\ 1/2i+1,\ 1/2i+3)$ and $(\frac{1}{2i-1},\ \frac{-(i+1)}{2i+1},\ 0)$ as our basis. Thus the crossing matrix for the process (i,j) + pion $\rightarrow$ (i,j) + pions has five even and four odd eigenvectors. If we consider the scattering of pions off isobars with $I = J$ or, for definiteness, $I = J + \frac{1}{2}$, the intermediate states, $(I,\ J-1)$, $(I+1, J)$, $(I+1,\ J-1)$ and $(I-1,\ J+1)$ cannot exist. Putting the corresponding elements of   to zero constrains Γ to be a particular linear combination of the five even eigenvectors which gives:

$$(I - 1,\ J - 1\ |A|\ I\ J)^2 = I\,\frac{(J + 1)(2I + 1)}{2I - 1}\,a$$

$$(I\ J\ |A|\ I\ J)^2 = \frac{(J + I)(I - J)\left[\lambda(I + J) + 1\right]}{(2I - 1)(2J + 1)}\,a$$

$$(I\ J\ |A|\ I\ J)^2 = \frac{J(I+1)}{(2I-1)(2J+1)}\left\{J(2J+3) + (I+1)(2I-1) + 2\right\}a$$

$$(I\ J + 1\ |A|\ I\ J)^2 = \frac{I(I - J)\left\{2(I+J) + 3\right\}}{(2I + 1)(2J + 3)}\,a$$

$$(I + 1\ J+1\ |A|\ I\ J)^2 = I\,\frac{(J + 1)(2J + 1)}{2J + 3}\,a$$

where a is some normalising function.

Vertex symmetry for the first and fifth relations gives the dependence of a on I and J as $a = \frac{1}{I(J+1)}$, and we put $a = \frac{1}{I(J+1)}$ as we are free to choose the overall normalisation.

Consider now the process (I,J) + pion $\rightarrow (I+1, J+1)$ + pion. Using the crossing matrix gives, one can derive, in the same way, couplings which are consistent with the above with respect to the factor a, providing $(I-J)\left[2(I-J) - 1\right] = 0$.

Thus the above equations give the solution for the $I = J$ chain. They are also consistent for the processes

$(J + \frac{1}{2}, J)$ + pion $\rightarrow (J + \frac{1}{2}, J)$ + pion and

$(J + \frac{1}{2}, J)$ + pion $\rightarrow (J + 3/2, J + 1)$ + pion

within the isobar chain $(I - J) = \frac{1}{2}$. There is also a third independent process not included in the previous case:

$(J + \frac{1}{2}, J)$+ pion $\rightarrow (J + \frac{1}{2}, J + 1)$ + pion. Inspection shows that the above solution is consistent for the process. As the theory is unchanged by ~~a~~ interchanging I and J, the above gives a consistent solution for all processes for the chain $|I - J| = \frac{1}{2}$

The recent work of Noga (52) has provided fresh insight into finding the representations of the strong coupling group. Using the identities relating 3-J, 6-J and 9-J symbols, Noga is able to find solutions of the problem for the case $K = su^I(2) \otimes su^J(2)$. Representations of the strong coupling group are classified by the value of $p = \max |I - J|$. For the scattering of pions off baryon isobars, Noga obtains the solution:

$$g^{ij}_{IJ} = (-1)^{i+j+p} \Big((2i+1)(2j+1)\Big)^{\frac{1}{2}} \begin{Bmatrix} 1 & i & I \\ p & J & j \end{Bmatrix} A_0(p)$$

A process involving strange mesons will change the p values of the isobars Noga considers such in-elastic processes of the type $\pi + B\,(i',j',p') \rightarrow K + B\,(i\ j\ p)$. Equating the s and u channels gives a solution for the couplings of strange mesons, since one vertex in each diagram is already known, being between isobars with the same p values. The solution is:

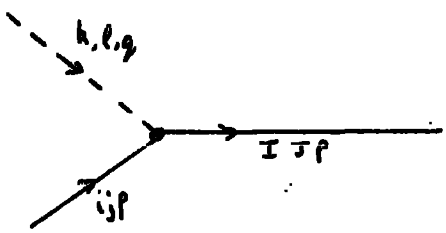
$$G^{i\ i\ p}_{I\ J\ p} = \Big((2i+1)\,(2j+1)\Big)^{\frac{1}{2}} \begin{Bmatrix} \frac{1}{2} & i & I \\ 1 & j & J \\ q & p & P \end{Bmatrix} G(p,q)$$

where $q = (I + J - i - j)$.

The symmetry properties of the 9-J symbol lead to invariance under a further su(2) group depending on the variable p. Postulating this invariance to exist for the scattering of ℓ - wave mesons, Noga, again equating s and u channels (this time for a process involving arbitrary wave mesons) obtains the coupling

$$G^{\begin{smallmatrix} k & \ell & q \\ i & j & p \\ I & J & P \end{smallmatrix}} = \begin{Bmatrix} K & i & I \\ \ell & j & J \\ q & p & P \end{Bmatrix} \Big((2i+1)(2j+1)(2p+1)\Big)^{\frac{1}{2}} a\,(k,\ell,q)$$

for the vertex



where the meson has I (J,P) spin K(ℓ,q).

By means of this sequence of boatstraps, Noga is able to obatin the solution for the scattering of mesons in an arbitrary wave which Sakita (53) obtained by group theoretical means. It should be mentioned that Bishari and Schwinmer (57) obtain the same solution for p-wave mesons using a similar method to Noga's. They do not however appear to see the significance of their result.

Mass formulae

Strong coupling theory also enables one to make statements about the masses of the isobars. Using the unitarity equation, constraints are imposed on the isobars masses sufficient to determine them, apart from a small number of arbitrary constants which may be fixed by setting the masses of the lowest isobars equal to their experimental values. The masses of the higher isobars are now fixed and appear in remarkable agreement with the experimental data, as remarked by Lovelace (54) at the Heidelberg conference.

The first step is to find a solution of the Chew-Low equation with the correct pole term as given by the strong coupling condition. The pole term is: $-\frac{D_{\beta\alpha}}{\omega^2}$

where $D_{\beta\alpha} = \left[A_\beta^+, \left[M, A_\alpha\right]\right]$

For s-waves, the solutions is: $f_{\beta\alpha}(w) = \dfrac{\Lambda_{\beta\alpha}}{2m(-m-ik)}$

where m is the meson mass. The unitarity equation is:

$$2m \; D_{\beta\alpha} = \sum_{\gamma} D_{\beta\gamma} \quad D_{\gamma\alpha} \tag{2.16}$$

For this to be valid in the physical region $w > m$, the mass differences must be small compared to the meson mass m. This means that the couplings must be large.

For p- waves, a cut-off is necessary. In this case the solution is: $f_{\beta\alpha}(w) = \dfrac{D_{\beta\alpha}}{-w^2 - iRk^3}$, where R is the cut-off radius. This gives the unitarity equation:

$$2\,RD_{\beta\alpha} = \sum_{\gamma} D_{\beta\gamma} \quad D_{\gamma\alpha} \tag{2.17}$$

The condition for this solution to be valid for $w > R^{-1}$ is that the mass differences be small compared to R^{-1}. This implies that the couplings must be large compared to the cut-off radius R.

The above deviations of the unitarity equations (2.16), (2.17) do not seem very satisfactory. Perhaps a better argument is that used by Sakita (53) who shows that a formal solution of the Chew-Low equation may be obtained in the strong coupling limit, with no extraneous singularities except at infinity, provided.

$$D_{\beta\alpha} = \sum_{\gamma} D_{\beta\gamma} \quad D_{\gamma\alpha} \quad (K) \tag{2.18}$$

where K is some kinematic factor which will be related to the cut-off.

Whether one takes equation (2.16), (2.17) or (2.18), one may divided D by 2m, 2R or K to obtain the same form for the unitarity equation:

$$\Lambda_{\beta\alpha} = \sum_{\gamma} \Lambda_{\beta\gamma} \Lambda_{\gamma\alpha} \tag{2.19}$$

where for s- waves $2m\,\Lambda_{\beta\alpha} = D_{\beta\alpha}$, for p- waves $2R\,\Lambda_{\beta\alpha} = D_{\beta\alpha}$

We take as our fundamental relations equation (2.19) and

$$\Lambda_{\beta\alpha} = \left[A_\beta^+, \left[\eta, A\alpha\right]\right] \qquad (2.20),$$ where

η = M/m for s- waves and η = M/R for p- waves.

Let us consider the case where the symmetry group K is $su^I(2)$ and the process $I + \pi \rightarrow J + \pi$.

The decomposition of the matrix elements of Λ in terms of invariant channels can easily be effected, using the techniques used earlier to obtain the bootstrap equation.

$$\begin{aligned}
&\langle JJ_z | \Lambda_{\beta\alpha} | II_z \rangle \\
&= \sum_K (\eta_K - \eta_I) \langle JJ_z | A_\beta^+ | KK_z \rangle \langle KK_z | A\alpha | II_z \rangle \\
&\quad + \sum_{K'} (\eta_{K'} - \eta_J) \langle JJ_z | A\alpha | K'K'_z \rangle \langle K'K'_z | A_\beta^+ | II_z \rangle \\
&= \sum_K (\eta_K - \eta_I)\, g_K^I\, g_K^J\, C\begin{pmatrix} I & 1 & K \\ I_z & \alpha & K_z \end{pmatrix} C\begin{pmatrix} J & 1 & K \\ J_z & \beta & K_z \end{pmatrix} (-1)^\beta \\
&\quad + \sum_{K'} (\eta_{K'} - \eta_J)\, g_{K'}^I\, {}^{J}_{K'}\, C\begin{pmatrix} I & 1 & K' \\ I_z-\beta & K'_z \end{pmatrix} C\begin{pmatrix} J & 1 & K' \\ J_z-\alpha & K'_z \end{pmatrix} (-1)^\alpha
\end{aligned}$$

(2.22)

Putting $R(JI;K) = (\eta_K - \eta_I)\, g_K^I\, g_K^J + \sum_{K'} C_{KK'}\, (\eta_{K'} - \eta_J)\, g_{K'}^I g_{K'}^J$ (2.23)

one obtains $\langle JJ_z | \Lambda_{\beta\alpha} | II_z \rangle = \sum_K C\begin{pmatrix} I & 1 & K \\ I_z & \alpha & K_z \end{pmatrix} C\begin{pmatrix} J & 1 & K \\ J_z & \beta & K_z \end{pmatrix} (-1)^\beta\, R(JI;K)$
(2.34)

R(JI;K) is the projection of Λ into the K invariant channel.

Using the bootstrap condition, one may re-write equation (2.23) to give:

$$R(JI;K) = \left(\mathcal{M}_{K} - \frac{\mathcal{M}_I - \mathcal{M}_J}{2}\right) \epsilon_K^I \epsilon_K^J + \sum_{K'} C_{KK'} \left(\mathcal{M}_{K'} + \frac{\mathcal{M}_I - \mathcal{M}_J}{2}\right) \epsilon_{K'}^I \epsilon_{K'}^J \qquad (2.25)$$

Note that R(JI;K) is the contribution the pole terms would give to the first moment sum rule for the process.

In terms of the Rs, the unitarity equation (2.19) becomes:

$$R(LJ;K) = \sum_{J'} R(LJ';K)\; R(J'J;K) \qquad (2.26)$$

where the summation is over J'=K-1,K,K+1.

These two facts are the essential ingredients in the calculation of Cronstrom and Noga, which we will discuss later.

Having obtained the unitarity equation in various forms, it is possible to put limitations on the form of $\mathcal{M}$. Firstly as $\mathcal{M}$ is an invariant of the symmetry group K, must be a function of the Casimir operators of K. The early strong coupling papers (41,42) assumed that $\mathcal{M}$ was a linear combination of the second order Casimir operators of K. Goebel (45) gives an argument for this:

Expand $\mathcal{M} = a + b_i J_i + C_{ij} J_i J_j + d_{ijk} J_i J_j J_k$

$$(2.27)$$

where J_i are the generators of K and a, b,.... are invariants of K. Then substituting gives:

$$\Lambda_{\alpha\beta} = - C_{ij} (J_i)_{\alpha\rho} (J_j)_{\beta\sigma} A_\rho A_\sigma - d_{ijk} (J_i)_{\alpha\rho} (J_j)_{\beta\sigma} A_\rho A_\sigma J_k + \ldots \qquad (2.28)$$

where $[A_\alpha, J_i] = i(J_i)_{\alpha\rho} A_\rho$.

Goebel argues that as a representation of the strong coupling group contains isobars with arbitrary high values of the Casimir operators, the matrix elements of Λ can be made arbitrarily large by taking them between sufficiently high isobars. This will contradict the unitarity condition, equation (2.19) which, being non-linear, limits the size of Λ. To prevent this happening, it is necessary for all terms beyond the C_{ij} term to vanish, thus giving the required form for η. Were the matrix elements of the As to decrease for higher isobars, it would be possible to retain some additional terms in the expansion (2.27). The present author can see no a priori reason why this decrease should not occur. However in the calculations performed, it does not occur and Goebels argument holds.

Rangwala (55) gives a method by which a difference equation may be found for η. As Λ is idempotent, it must have eigenvalues 0 or 1. Thus the trace of Λ is k I (I is the identity operater in isobar space) where k is an integer between zero and N, the dimension of the regular representation of K, which contains the mesons, i.e $\sum_{\alpha} \Lambda_{\alpha\alpha} = k\,I$ (2.29)
Putting $\eta = f/2A^2$ where $A^2 = \sum_{\alpha} A^+_{\alpha} A_{\alpha}$ and f is a function of the Casimir operators of K, one obtains:

$$\sum_{\alpha} \frac{A_{\alpha}}{\sqrt{A^2}} f \frac{A^+_{\alpha}}{\sqrt{A^2}} - f = k\,I \qquad (2.30)$$

This yields a linear difference equation for f. The solutions of this equation are not necessary solutions of equation (2.19) as taking the trace introduces spurious solutions. Substituting back, Rangwala obtains the usual mass formulas for the cases where $K = su^I(2)$ and $K = su^I(2) \otimes su^J(2)$.

We reproduce here Goebel's mass formulae for the case $K = su^J(2) \otimes su(3)$ (48). It can be seen that fixing the mass formulae by two experimental masses, the other masses are approximately correct, and that the ordering of the isobars with respect to their masses is the same for both experimental and theorectical masses. As Lovelace remarked (54), this success cannot be repeated by, for example, the quark model:

SU(3)	J^P	$\frac{M - M_{8,1/2}}{M_{10,3/2} - M_{8,1/2}}$	Predicted Mass	Observed Mass (πN)	Other members observed.
8	$1/2^+$	0	[1152]	N (938)	ALL
10	$3/2^+$	1	[1383]	Δ (1236)	ALL
$\overline{10}$	$1/2^+$	8/3	1768	N(1466)	Z_0(1865)?
27	$3/2^+$	26/9	1794	N (1863) Δ (1688)	Z_1(1900)
35	$5/2^+$	32/9	1974	Not Coupled	$N\pi\pi$ (1650)?
27	$1/2^+$	40/9	2176	N (1751)? Δ(1934)	

Strong Coupling and Superconvergence

Pandé (58) showed how saturating superconvergence relations lead to solving the I = J isobars series for $K = su^{I}(2) \otimes su^{J}(2)$. Knowing the similarity between bootstrap and superconvergence methods, and the relationship between the strong coupling equation and the bootstrap equation, this result comes as no surprise. However, strong coupling theory embraces more than the equation for the couplings and so it is worth while looking in greater detail at the connection between strong coupling and the saturation of superconvergence relations. Consider a process N_1 + meson $\rightarrow N_2$+ meson where N_1, N_2 form representations of K. Consider the invariant channel k.

$$f_k(w) = \lambda^2 (I - \frac{Ckk'}{w}) g_{k'}^{N1} g_{k'}^{N2} - \frac{1}{w2}\left\{g_k^{N1} \; g_k^{N2} \; w_k + Ckk' gk'^{N1} gk'^{N2} w_{k'}\right.$$

$$+ 0(\frac{1}{w}2) + \int_m^{+\infty} \mathrm{Im}\left[f_k(w) \frac{- Ckk' \; f_{k'}(w)}{w-w'}\right] dw' \qquad (2.31)$$

where the pole term contains only bound state terms.

If $f_k(w)$ superconverges, we obtain:

$$0 = \lambda^2 \left[I - Ckk'\right] g_{k'}^{N1} g_{k'}^{N2} + \int_m^{+\infty} \mathrm{Im}\left[fk(w) - Ckk' \; fk'(w)\right] dw' \quad (2.32)$$

If the integral can be saturated by resonances we obtain $\lambda^2 \left[I - Ckk'\right] g_{k'}^{N1} g_{k'}^{N2} = 0$ where the assumed summation is over both bound states and resonances. We thus obtain the static bootstrap, and hence the strong coupling condition.

If we further assume that the 1st moment of $f_k(w)$ superconverges, we obtain

$$-R(N_1, N_2; k) + \int_m^{+\infty} w' \left\{ \operatorname{Im} f_k(w') - C_{kk'} \operatorname{Im} f_{k'}(w) \right\} dw' = 0 \quad (2.33)$$

It is not possible to saturate this integral with resonances, it must have contributions from the two particle unitarity cut. If we assume only these contributions we are in a position to obtain information from the equation. In the s- wave case, putting $f_k(w)$ = resonance terms + $\frac{D_k}{w^{2+\epsilon}}$, $\epsilon > 0$, one obtains

$$-R(N_1, N_2; k) + \frac{D_k}{2m^{1+\epsilon}} = 0$$

In the limit $\epsilon \to 0$, one obtains the usual conditions $2m\, D_k = R\,(N_1, N_2; k)$ and the unitarity condition.

In the p- wave case, one runs into trouble because of the kinematic factors and it is not possible to find a simple solution. For f_k to superconverge in this case, it must also obey superconvergence relations for its first moment and a condition on its second moment. These do not in general hold for strong coupling theory solutions. However Cronström and Noga (59) found a situation in which one may apply these techniques.

Consider $K = su^I(2) \otimes su^J(2)$ and the $I = J$ isobar chain. The channel $(I+1, I-1)$ for the process $(I,I) + \pi \to (I,I) + \pi$ contains no isobar and obeys exact elastic unitarity. This means that $f_k(w) = 0$ is a possibility on the unitarity cut. Then using the moment sum rule one obtains $C_{I+1,k}\, C_{I-1,k}\, g_k^I\, g_k^I w_k = 0$ where the Cs are the su(2) crossing matrices for the process. This yields a difference equation for the masses which leads to the same solution as Rangwala. Cronström and Noga, use an N/D method and argue on the order of magnitudes of the mass differences of the different terms. The mechanism is essentially that given above.

The point at which the strong coupling condition can be of use in the case of sum rules is in allowing one to neglect all terms in the unitarity integral apart from the terms required to unitarise the Born term. This can be used to justify the simplified form of the unitarity condition.

CHAPTER 3

Bootstrap Model of Fulco and Wong

The work in this section is concerned with a bootstrap model, including all three channels (s,u and t), which was developed by Fulco and Wong (60) and modified and extended by Pat●l (61). The motivation for this work was a desire to obtain higher symmetries from a dynamical model, as happens with current algebra, and avoid the problems associated with the use of su(6) as a symmetry group. As the main concern of the authors seems to have been to indicate a rough model, the dynamical assumptions are rather tenuous (67)

Fulco and Wong obtain their bootstrap equation by representing the effect of crossed channel processes by effective poles in the partial wave amplitudes using the static model. Following the work in chapter one, if there is a particle in an invariant channel of the direct channel, one may write down the condition for it to exist, with mass m_α and coupling with strength γ_α to the system:

$$\lim_{s \to m_\alpha^2} \left\{ \frac{\gamma_\alpha}{s - m_\alpha^2} \right\} = \lim_{s \to m_\alpha^2} \left\{ \frac{1}{D(s)} \left(\frac{C_{\alpha\beta}^{su} \gamma_\beta^u D(-s_\beta^u)}{s + s_\beta^u} + \frac{C_{\alpha\rho}^{st} \gamma_\rho^t D(-s_\rho^t)}{s + s_\rho^t} \right) \right\} \tag{3.1}$$

where C^{su}, C^{st} are the crossing matrices for some internal symmetry group, $\frac{\gamma_\beta^u}{s+s_\beta^u}$ is the pole representing the effect of the invariant channel β in the u-channel and $\frac{\gamma_\rho^t}{s+s_\rho^t}$ represents the effect of the invariant channel ρ in the t-channel. Making the linear D approximation one obtains:

$$\gamma^{s}_{\alpha} = C^{su}_{\alpha\beta}\,\gamma^{u}_{\beta} + C^{st}_{\alpha\beta}\,\gamma^{t}_{\beta} \qquad (3.2)$$

The argument used to give meaning to this equation is that the u and t- channel poles correspond to the exchange of single particle states and the γs are the products of the couplings of these particles to the system. From chapter one, we believe that this may be a reasonable assumption for the u-channel poles, which corresponds to the heavy particles in the static approximation. It seems doubtful that this is also true for the t- channel poles, as in reality the t- channel gives a complicated cut structure. Even if the use of effective poles is permissible for the t- chanel contributions, it may not be possible to attach significance to the residues. For the u channel, the higher mass exchange poles are further from the physical region which may allow the neglect of higher masses exchanges without too much error. This is not true of the t- channel which contributes within a small finite region, and we have no guarantee that the lowest mass exchanges dominate to the extent that the effective poles have residues given by the couplings of the lowest mass exchanges.

With these assumptions we obtain, equating the s and u channels: $\Gamma - C_{su}\Gamma = C_{st}\Gamma'$ (3.3)
where $\Gamma_{\alpha} = \gamma^{s}_{\alpha} = \gamma^{u}_{\alpha}$ and $\Gamma'_{\beta} = \gamma^{t}_{\beta}$ This is the bootstrap equation of Fulco and Wong. We note that $\Gamma' = 0$ gives

the conventional bootstrap equation.

Patil, in his analysis, imposes less constnants on $\Gamma^{\gamma'}$ which still allows one to derive useful relations between the meson-baryon couplings in Γ^{γ}. From dispersion relations he obtains:

$$\gamma^{S}_{\alpha} = D_{\alpha\beta}\, C^{su}_{\alpha\beta}\, \gamma^{u}_{\beta} + A_{\alpha\rho}\, C^{st}_{\alpha\rho}\, \gamma^{t}_{\rho}$$

where the Ds and As are dynamical factors. He argues that in the static model it is reasonable to approximate the u- channel contribution by the crossed physical poles of the particles in that channel. Hence he obtains

$$\Gamma_{\alpha} - C_{su\,\alpha\beta}\, \Gamma_{\beta} = A_{\alpha\rho}\, C^{st}_{\alpha\rho}\, \gamma^{t}_{\rho} \qquad (3.4)$$

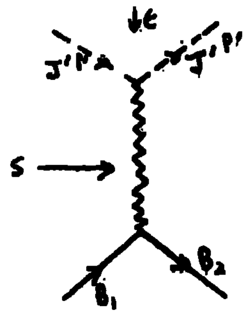
So far we have not mentioned spin. If we deal with particles with spin, it is again reasonable in the static limit to use static spin crossing to give the contribution of the u-channel poles. The concept of spin is not well defined in the static model of p- wave scattering and again there is trouble with the t- channel exchanges. Allowing the dynamical factors in A to commute with the internal symmetry group gives:

$$\Gamma - C^{I}_{su} \otimes C^{S}_{su}\, \Gamma = (C^{I}_{st} \otimes A)\, \Gamma' \qquad (3.5)$$

where I(J) refers to the internal (spin) symmetry group. From equation (3.5) it is possible to put constants on A from consideration of the spin and parities of the particles exchanged.

Spin and Statistics for meson exchanges

We forget for a moment the static model and simply consider the diagram for meson exchange in meson-baryon scattering and its symmetry under s- u crossing which is defined as the interchange of the external mesons.



The external meson have spin and parity $J'^{P'}$ and the internal one J^P. For what follows the baryon vertex is of no importance.

Let the external mesons be in an ℓ-wave in the t- channel. Then conservation of spin implies that

$$J \subset J' \otimes J' \otimes \ell \qquad (3.6)$$

Parity conservation at the vertex implies that $P = (-1)^{\ell} P'P' = (-1)^{\ell}$ (3.7)

Bose statistics implies that the three meson vertex is symmetrical under the interchange of the external mesons. Thus $(-1)^{\ell} \eta_I = 1$, (3.8) where η_I is $+1(-1)$ according as the vertex is symmetrical (anti-) under the internal symmetry crossing. Equations (3.7) and (3.8) imply that $\eta_I = P$. Knowledge of the internal symmetry representation to which the exchange belongs usually determines η_I (This is not the case where the external mesons are su(3) octets, as $8 \otimes 8$ contains octets which couple both symmetrically and antisymmetrically), and hence determines P. Equations (3.6) and (3.7) constrain ℓ and there may not be a value of ℓ to satisfy both. Equation (3.7) only determines whether ℓ is odd or even and it is easy to see that as long as $J' \neq 0$, there will exist both odd and even values to ℓ to satisfy equation (3.6). If $J' = 0$ $J = \ell$ and hence only exchanges of natural parity, with $P = (-1)^J$, are allowed.

Returning to equation (3.5), we see that the left hand side changes sign when operated on by $Csu^I \otimes Csu^J$. Hence the right hand side must be an odd eigenvector of $Csu^I \otimes Csu^J$. An exchange of positive (negative) parity will have $\eta_I = +1(-1)$. Expanding A in terms of the columns of Cst^J shows that exchange with $\eta_I = +1(-1)$ can only multiply odd (even) columns of Cst^J. When the spins are low, this result may be sufficient to determine A, apart from an arbitrary normalisation multiplying each individual exchange term of Γ''. We now indicate the differences between the approaches of Fulco and Wong, and Patil. The former consider axial vector mesons scattering off baryons. This enables them to obtain a consistent solution to their equations with the exchange of a vector octet and singlet and of an axial vector octet which may be identified with the external mesons. This identification, as we shall see, is consistent and leads to a model with a smaller number of mesons than required by Patil. He considers the p- wave scattering of pseudoscalar mesons, which is a physically observed process, whereas axial vector mesons have not been identified physically. The p.s octet cannot be exchanged, having unnatural parity and the external ps-mesons are spin 0. Patil introduces a tensor (2^+) octet which, having natural parity, can be exchanged and will contribute in the same way as an A.V octet exchange.

In the direct channel p- wave p.s. meson scattering is mathematically identical with A-V meson scattering. In the t- channel, Fulco and Wong's ($8,1^+$), ($8,1^-$), ($1,1^+$) exchanges contribute to the same elements of Γ' as the ($8,1^-$), ($1,1^-$) and ($8,2^+$) exchanges of Patil. Thus the two calculations are mathematically equivalent, except in so far as we are required to identify the internal and external ($8,1^+$) mesons in the F.W calculation. This constraint impose, the condition that the f/d ratios of the coupling to the baryon octet must be equal As both f and d contributions to Γ' are from ($8,1^+$) terms, the effect of A should be just an overall normalisation factor for the two terms, so dividing them should give the same as the ratio of the direct channel f and d contibutions.

In the case when C_{st}^{J} has only one odd and even column, and the contribution of a specific exchange is fixed, Fulco and Wong put the normalising factor derived from A, to unity. This gives significance to the elements of Γ' as products of couplings. The results for the meson-baryon couplings (63) are consistent for all processes involving the ($8,\frac{1}{2}$) and ($10,3/2$) iso-bars. As we shall see in chapter 4, this reflects the use of su(6) as a non-invariance group.

After this discussion, a brief remark on how to solve the mathematical problem. We have shown that the equation can be reduced to: $\Gamma^7 - \mathrm{Csu}\ \Gamma^7 = \mathrm{Cst}\ \Gamma^{7'}$ (3.3), where $C = C^J \otimes C^I$, and $\Gamma^{7'}$ is related, as indicated previously, to the t- channel exchanges, which fact imposes constraints, in the form of zero elements, on $\Gamma^{7'}$. Γ also has zero elements corresponding to iso-bars not in the chain considered.

The solution of equation (3.3) has most easily been effected by writing $\Gamma^7 = \mathrm{Cst}\ \Gamma^{7''}$ (3.9). Then operating with Csu, we see that the non zero elements of $\Gamma^{7'}$ equal twice the corresponding elements of $\Gamma^{7''}$. The constraints on Γ^7 and $\Gamma^{7'}$ can be imposed on Γ^7 and $\Gamma^{7''}$ in equation (3.9). The elements of $\Gamma^{7''}$ corresponding to even columns of Cst can be considered as free parameters with no physical significance. Using equation (3.9) and its inverse, the effects of the constants of $\Gamma^{7'}$ on Γ^7 and vice versa, may be found.

<u>1. $(8,\frac{1}{2}) + (8,1) \rightarrow (8,\frac{1}{2}) + (8,1)$</u>

$$8 \otimes 8 = 1 \oplus 8s \oplus 8A \oplus 10 \oplus \overline{10} \oplus 27.$$

As in previously we have chosen linear conbinations of the octets which couple symmetrically and anti-symmetrically. The non-zero elements of Γ^7 are Γ^7 ss, Γ^7sA, Γ^7As, Γ^7AA corresponding to the $(8,\frac{1}{2})$ state and $\Gamma^7 10$ corresponding to the $(10,3/2)$ state. The expansion of the Γ's as pairs of couplings shows that $\Gamma^{AS} = \Gamma^{SA}$ and Γ^{SS} $\Gamma^{AA} = \Gamma^{2}_{AS}$. We may thus write $\Gamma_{SS} = \gamma^{2}\Gamma$, $\Gamma_{SA} = \Gamma_{AS} = \gamma\Gamma_{1}$, $\Gamma_{SA} = \Gamma$

Let us now consider Γ'. In the s channel it does not matter which vertex we name first, as Γ_{AS} and Γ_{SA} are equal. In the t-channel we must pay attention to this point, as one vertex is a three meson vertex and the other a baryon-meson vertex. We refer to the three meson vertex by the first index. As we have defined $s \to u$ crossing as exchanging the mesons, this first index determines the symmetry of the channel under $s \to u$ crossing. Thus the 8_{AS}, 8_{AA} columns of Cst are odd and the 8_{SS}, 8_{SA} columns are even under $s \to u$ su(3) crossing.

With $P = -1$, the vector octet contributes to $(8_{AS}, 0)$ and $(8_{AA}, 0)$: the ratio of these terms giving the f/d ratio of the vector coupling to the $B\bar{B}$ system. The axial octet contributes to $(8_{SS}, 1)$ and $(8_{SA}, 1)$ (again with the f/d ambiguity) and the axial singlet to $(1,1)$.

The spin crossing metrices are: (28)

$$C^{S}_{st} = \begin{array}{cc} 0 & 1 \end{array} \begin{array}{c} \\ \end{array}
\begin{array}{c} \left(\begin{array}{cc} \sqrt{4/6} & 1 \\ \sqrt{4/6} & -1/2 \end{array}\right) \begin{array}{c} 1/2 \\ 3/2 \end{array} \end{array}
\qquad
C^{S}_{ts} = \begin{array}{cc} 1/2 & 3/2 \end{array}
\left(\begin{array}{cc} \sqrt{4/3} & 2\sqrt{4/3} \\ 2/3 & -2/3 \end{array}\right) \begin{array}{c} 0 \\ 1 \end{array}$$

The odd column is the spin 1 column.

For the su(3) crossing Cts = Cst and: (63)

$$
\begin{array}{r|cccccccc|}
 & 1 & 8ss & 8aa & 8as & 8aa & 10 & \overline{10} & 27 \\
\hline
Cst = 1 & \frac{1}{8} & 0 & 0 & 0 & 1 & 5/4 & 5/4 & 27/8 \\
8ss & \frac{1}{8} & -3/10 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 27/40 \\
8sa & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\sqrt{5}/4 & \sqrt{5}/4 & 0 \\
8as & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \sqrt{5}/4 & -\sqrt{5}/4 & 0 \\
8aa & \frac{1}{8} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & -9/8 \\
10 & \frac{1}{8} & -2/5 & -1/\sqrt{5} & +1/\sqrt{5} & 0 & \frac{1}{4} & \frac{1}{4} & -9/40 \\
\overline{10} & \frac{1}{8} & -2/5 & +1/\sqrt{5} & -1/\sqrt{5} & 0 & \frac{1}{4} & \frac{1}{4} & -9/40 \\
27 & \frac{1}{8} & 1/5 & 0 & 0 & -\frac{1}{3} & -1/12 & -1/12 & 7/40
\end{array}
$$

The odd columns are: 8as, 8aa, 10, $\overline{10}$

We now have to solve $\Gamma = (Cst^{J} \oplus Cst^{I})\,\Gamma''$

where $\Gamma = (0, \gamma^2\Gamma, \gamma\Gamma, \gamma\Gamma, \Gamma, 0,0,0; 0,0,0,0,0,\Gamma_{10}, 0,0)$

and

$$\Gamma'' = (x_1, x_2, x_3, \tfrac{1}{2}d_V, \tfrac{1}{2}f_V, 0, 0, x_4, \tfrac{1}{2}A, \tfrac{1}{2}d_A, \tfrac{1}{2}f_A, x_5, x_6, x_7, x_8, 0)$$

where $x_1, \ldots\ldots x_8$ are free parameters corresponding to even columns and the zeros of Γ'' correspond to these being no exchanges of 10, $\overline{10}$ and 27 plets of mesons. The way to solve this problem is to invert the equation and find the zero of Γ'' in terms of Γ. The 10 and $\overline{10}$ equations are the same for our choice of Γ and we have $\gamma_{10} = \gamma^2\, 4/5\, \Gamma$ and $\gamma = \sqrt{5/4}$.

Thus $\gamma_{10} = \Gamma$ and $\gamma = \sqrt{5/4}$

This gives $fA = +\sqrt{5}/a\,\Gamma$, $dA = 5/6\,\Gamma$, $A_1 = 4/3\,\Gamma$, d_V $f_V = \sqrt{3/4}$

$dA/fA = \sqrt{5/4}$ which shows that the coupling of the A-V mesons to the baryons has the same f/d ratio for the internal and external mesons. This corresponds to an f/d ratio of 3/2 which is the su(6) value. Since $d_V = 0$, the vector octet couples antisymmetrically to the baryons which is

If we refer to the mesons as A, V and 1, we have:

$g^2_{A8\beta a} = g^2_{A8,10}$, $g^2_{A\beta\beta s} = 5/4\ g^2_{A8,10}$. These give $g^2_{A\beta\beta} = 9/4\ g^2_{A8,10}$. Identifying the elements of Γ' with t-channel couplings gives:

$$g_{8\bar{8}1}\ g_{AA1} = 4/3\ g^2_{A\beta\beta a}$$
$$g_{8\bar{8}aV}\ g_{AAV} = = \sqrt{27/8}\ g^2_{A\beta\beta a}$$
$$g_{8\bar{8}sA}\ g_{AAA} = 5/6\ g^2_{A\beta\beta a}$$
$$g_{8\bar{8}aA}\ g_{AAA} = \sqrt{5/9}\ g^2_{A\beta\beta a}$$

These results are consistent with su(6). Fulco and Wong claim that this is also true for their results for the process A + B8 → A + B10 (ie decuplet production). Patil finds the results for baryon couplings for PS + B10 → PS. + B10 are also consistent with su(6) We have performed both these calculations in Fulco and Wong's model and have found results consistent with su(6) for both meson-baryon and meson-meson couplings.

<u>2. (10,3/2) + (8,1) → (10,3/2) + (8,1)</u>

$$8 \otimes 10 = 8 \oplus 10 \oplus 27 \oplus 35.$$
$$8 \otimes 8 = 1 \oplus 8s \oplus 8A \oplus 10 \oplus \overline{10} \oplus 27$$
$$10 \otimes \overline{10} = 1 \oplus 8 \oplus 27 \oplus 64$$

Thus the t- channel invariants are: 1, 8, 8A and 27 where again the index of the octets refers to the 3- meson coupling. The su(3) crossing matrices are (63)

$$
C_{st} = \begin{array}{c} \\ 8 \\ 10 \\ 27 \\ 35 \end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 8s & 8A & 27 \end{array} \\
\begin{pmatrix}
\sqrt{\frac{5}{20}} & \sqrt{2}/5 & \sqrt{10}/5 & 9\frac{\sqrt{7}}{20} \\
\frac{\sqrt{5}}{20} & 3\frac{\sqrt{2}}{10} & \frac{\sqrt{10}}{10} & -9\frac{\sqrt{7}}{20} \\
\frac{\sqrt{5}}{20} & -3\frac{\sqrt{2}}{10} & \frac{\sqrt{10}}{30} & -\frac{\sqrt{7}}{20} \\
\frac{\sqrt{5}}{20} & \frac{\sqrt{2}}{10} & -\frac{\sqrt{10}}{10} & 9\frac{\sqrt{7}}{140}
\end{pmatrix}
\end{array}
$$

$$
C_{ts} = \begin{array}{c} \\ 1 \\ 8sA \\ 8A \\ 27 \end{array}
\begin{array}{c}
\begin{array}{cccc} 8 & 10 & 27 & 35 \end{array} \\
\begin{pmatrix}
2/\sqrt{5} & \sqrt{5}/2 & 27/4\sqrt{5} & 35/4\sqrt{5} \\
\sqrt{2}/5 & 3\sqrt{2}/8 & -\frac{81}{80}\sqrt{2} & \frac{7}{16\sqrt{2}} \\
\sqrt{10}/5 & \sqrt{10}/8 & 9\frac{\sqrt{10}}{80} & -7/16\sqrt{10} \\
2\frac{\sqrt{7}}{15} & -\sqrt{7}/6 & -\sqrt{7}/20 & \frac{\sqrt{7}}{12}
\end{pmatrix}
\end{array}
$$

The only odd column of Cst is 8A.

The su(2) crossing matrices are (19):

$$
C_{st} = \begin{array}{c} \\ \frac{1}{2} \\ 3/2 \\ \frac{5}{2} \end{array}
\begin{array}{c}
\begin{array}{ccc} 0 & 1 & 2 \end{array} \\
\begin{pmatrix}
-\sqrt{3}/6 & -\frac{\sqrt{10}}{4} & -5\frac{\sqrt{6}}{12} \\
-\sqrt{3}/6 & -\frac{\sqrt{10}}{10} & \sqrt{6}/3 \\
-\sqrt{3}/6 & 3\frac{\sqrt{10}}{20} & -\frac{\sqrt{6}}{12}
\end{pmatrix}
\end{array}
$$

$$
Cts = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{c}
\begin{array}{ccc} \tfrac{1}{2} & 3/2 & 5/2 \end{array} \\
\begin{pmatrix}
-\sqrt{\frac{2}{3}} & -2\sqrt{\frac{2}{3}} & -\sqrt{5} \\
-\frac{\sqrt{10}}{6} & -2\frac{\sqrt{10}}{15} & 3\frac{\sqrt{10}}{10} \\
-\frac{\sqrt{6}}{6} & 4\frac{\sqrt{6}}{15} & -\frac{\sqrt{6}}{10}
\end{pmatrix}
\end{array}
$$

The odd columns of Cst is the spin 1 column.

Thus the odd t- channel invariants are (8A,0), (8A,2),(1,1)(8s,1) and (27,1). The axial vector mesons may contribute to the (1,1) and (8s,1) elements. The vector octet may contribute to the (8A,0) and (8A,2) elements. The (27,1) element is zero as we allow no 27- plet exchange. This is the only constrant on Γ''. Hence it is not surprising that we can solve for Γ, where Γ has contributions for the $(8,\frac{1}{2})$ and $(10,3/2)$ channels only.

$$\Gamma' 27 = 0 \Rightarrow \Gamma 8 = \Gamma 10$$

Then: $\Gamma' 8A,0 = -3\frac{\sqrt{30}}{10}$, $\Gamma' 1,1 = -4\sqrt{2}/3$, $\Gamma' 8s1 = -\sqrt{5}/3$ and $\Gamma' 8A,2 = 0$. It is interesting to note that if we had allowed for a spin 5/2 octet to contribute to Γ (this octet being the Regge recurrence of the spin $\frac{1}{2}$ octet), the vanishing of the $\Gamma' 8A,2$ would imply that its contribution vanished. Increasing the number of baryons in the theory seems in this case to imply a need for further mesons and vice versa.

The resulting implications for the couplings are:

$$g_{10,\overline{10}V}\; g_{AAV} = -3\sqrt{30}/10 \quad g^2_{A10,10}$$

$$g_{10,\overline{10}1}\; g_{AA1} = -4\sqrt{2}/3 \quad g^2_{A10,10}$$

$$g_{10,\overline{10}A}\; g_{AAA} = -\sqrt{5}/3 \quad g^2_{A10,10}$$

$$g^2_{A10,8} = g^2_{A10,10}$$

which are again consistent with su(6)

<u>3. $(8,\tfrac{1}{2}) + (8,1) \rightarrow (10,3/2) + (8,1)$</u>

For this process all channels contain two 8s, 10, 27. The su(3) crossing matrices are (64)

$$C_{st} = \begin{array}{c} \\ 8s \\ 8A \\ 10 \\ 27 \end{array} \begin{array}{c} \begin{array}{cccc} 8_s & 8a & 10 & 27 \end{array} \\ \begin{pmatrix} 2/5 & \sqrt{1/5} & \sqrt{2}/4 & 27/20 \\ -\sqrt{1/5} & 0 & -\sqrt{10}/4 & 9/4\,\sqrt{1/5} \\ \sqrt{2}/5 & \sqrt{2}/5 & -\tfrac{1}{2} & -\tfrac{9}{10}\sqrt{\tfrac{1}{2}} \\ -2/5 & \dfrac{2}{3\sqrt{5}} & -\tfrac{1}{3}\sqrt{\tfrac{1}{2}} & -1/10 \end{pmatrix} \end{array}$$

$$C_{ts} = \begin{array}{c} \\ 8s \\ 8a \\ 10 \\ 27 \end{array} \begin{array}{c} \begin{array}{cccc} 8s & 8a & 10 & 27 \end{array} \\ \begin{pmatrix} 2/5 & -1/\sqrt{5} & \sqrt{2}/4 & -27/20 \\ \sqrt{1/5} & 0 & \sqrt{10}/4 & 9/4\;\; 1/\sqrt{5} \\ \sqrt{2}/5 & -\sqrt{2}/5 & -\tfrac{1}{2} & \tfrac{9}{10}\sqrt{\tfrac{1}{2}} \\ 2/5 & \dfrac{2}{3\sqrt{5}} & -1/3\sqrt{\tfrac{1}{2}} & \dfrac{-1}{10} \end{pmatrix} \end{array}$$

Note that as all three channels contain the same invariants, the above matrices differ only by phase factors. The odd columns of Cst are 8^a, 10.

The spin crossing matrices are (39)

$$C_{st} = \begin{array}{c} \\ \tfrac{1}{2} \\ 3/2 \end{array}\begin{array}{c}\begin{array}{cc} 1 & 2 \end{array}\\ \begin{pmatrix} \tfrac{1}{2} & 5\sqrt{3}/6 \\ \sqrt{10}/4 & -\frac{\sqrt{30}}{12} \end{pmatrix}\end{array} \qquad C_{ts} = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c}\begin{array}{cc} \tfrac{1}{2} & 3/2 \end{array}\\ \begin{pmatrix} \tfrac{1}{3} & \sqrt{10}/3 \\ \sqrt{3}/3 & -\frac{\sqrt{30}}{15} \end{pmatrix}\end{array}$$

The spin 1 column of Cst is odd.

Thus the odd t- channel invariants are (8s,2), (27,1), (8a,2),(10,2). As we have no such multiplets we can put the 27 and 10 contributions to Γ', to zero. Inserting only the $(8,\frac{1}{2})$ and (10,3/2) elements into Γ,we obtain:

$$\Gamma_{8s} = \sqrt{5}/4 \quad \Gamma_{8A} \text{ and } \quad \Gamma_{8A} = \Gamma_{10}$$

This gives: $\Gamma'_{8s} = \sqrt{5}/6$ and $\Gamma'_{8a} = 0$

This gives for couplings:

The f/d ratio for the baryons octet - A V coupling is again 3/2

$$g_{A,8,8a} \quad g_{A,10,8a} = g_{A,8,10} \quad g_{A,8,10}$$

$$g_{A,\overline{10},A} \quad g_{A,A,A} = \sqrt{5}/3 \quad g_{A,8,10} \quad g_{A,10,10}$$

Again these are consistent with su(6). The ratio of the meson-baryon and meson-meson couplings is given using su(6) notation as $g_{MBB}/g_{MM} = 8\sqrt{3}/15$. This ties in with the results of Udgaonkar (65), for his su(6) bootstrap calculation.

We note, as do Fulco and Wong, that if one considers the scattering of the axial vector singlet off the baryons one obtains results consistent with the previous results. In fact, the scattering off the (8.2) gives, $g_{8,\overline{8},1}g_{1,1,1} = 4/3\ g^2_{1,8,8}$ and scattering off the (10,4) gives, $g_{10,\overline{10},1}\ g_{1,1,1} = -4\sqrt{10}/15\ g^2_{1,10,10}$.

The previous work would still be valid if the vector meson were instead scalar, as both particles have natural parity. In this case one could scatter the scalar octet off the baryons and obtain:

$$g 8 \bar{8}_a S \; gsss = g^2 s,8,8a$$

$$g 10 \overline{10} s \; gsss = \sqrt{10}/4 \; g^2 M,10,10$$

The results obtained above for the scattering of the axial vector singlet and the suggested scalar octet are a simple consequence of su(3) and $su^J(2)$ symmetry. The axial vector singlet belongs to the regular representation of $su^J(2)$ and the scalar octet to the regular representation of su(3). The regular representation transforms like the generators of the group and the above results reflect the commutation relations of the generators of $su^J(2)$ and su(3).

<u>su(6) Model of Udgaonkar</u>. (65)

Udgaonkar takes the Fulco and Wong bootstrap equation and applies it to meson-baryon and meson-meson scattering in su(6).

For scattering of the meson <u>35</u> plet off the baryons <u>56</u> plet, the crossing matrices are (66)

$$
C_{su} = \begin{array}{c}
\begin{array}{cccc} (70) & (1134) & (56) & (700) \end{array} \\
\begin{pmatrix}
\frac{1}{4} & -27/20 & -2/5 & 5/2 \\
-1/12 & 17/20 & -2/45 & 5/18 \\
-\frac{1}{2} & -9/10 & 11/15 & 5/3 \\
\frac{1}{4} & 9/20 & 2/15 & 1/6
\end{pmatrix}
\end{array}
\begin{array}{c} \\ (70) \\ (1134) \\ (56) \\ (700) \end{array}
$$

and

$$
C_{st} = \begin{array}{c} \\ \\ \\ \\ \end{array}
\begin{array}{cccc} (405) & (35s) & (35a) & 1 \end{array}
$$

$$
\begin{array}{c}
\\
\mathrm{Cst}=
\end{array}
\left(\begin{array}{cccc}
\frac{135}{28}\sqrt{1/10} & \sqrt{6}/8 & \sqrt{3}/4 & \frac{1}{14\sqrt{10}} \\
-5/28\sqrt{1/10} & -\sqrt{6}/24 & \sqrt{3}/36 & \frac{1}{14\sqrt{10}} \\
-\frac{45}{7}\sqrt{\frac{1}{10}} & \sqrt{6}/6 & \sqrt{3}/6 & \frac{1}{14\sqrt{10}} \\
-9/28\sqrt{\frac{1}{10}} & \sqrt{6}/24 & -\sqrt{3}/12 & \frac{1}{14\sqrt{10}}
\end{array}\right)
\begin{array}{c} (70) \\ (1134) \\ (56) \\ (700) \end{array}
$$

The simplest solution of Γ - Csu Γ = Cst Γ' is having only the 56 plet in Γ and the 35 plet in Γ' This gives gMBB/gMMM = $\frac{15}{8\sqrt{3}}$ which is the result of Fulco and Wong. This agreement of the two calculations will be explained in chapter 4.

We note in passing that Udgaonkar applies the F-W equation to meson-meson scattering, where the static model cannot be used to justify it.

All three channels are equal and $\Gamma = \Gamma'$. The equation has a solution with containing only 35 plet. As the 35 plet forms the regular representation of su(6), this results is similar to those for axial vector singlet and scalar octet scattering in the previous section. It follows from the commutation relations of su(6).

CHAPTER 4

Intermediate Coupling Theory

Kuriyan and Sudarshan (14) point out that the work of Cook, Goebel and Sakita (42) on the strong coupling model contains an error. There is an implicit assumption that the meson source operator is given by λA_α where A_α contains no further dependance on λ. The strong coupling condition, then implies that $[A_\alpha, A_\beta]=0$. Without the assumption that A_α is independent of λ one cannot extrapolate this equation to finite values of λ. Expanding A_α in terms of $1/\lambda^2$, the strong coupling condition implies that the 'constant terms' commute, but says nothing about the higher order terms. Thus, if $A_\alpha = A_\alpha^{(0)} + 1/\lambda^2 A_\alpha^{(1)} + 1/\lambda^4 A_\alpha^{(2)} + \ldots$ ...,(4.1) $[A_\alpha, A_\beta]=0$ (4.2). Unfortunately this weaker condition does not lead to the identification of a non-invariance group.

In order to obtain a non-invariance group for the system, it is necessary to identify the As with the non-invariant generators of such a group. The choice made follows the suggestion of Cook, Goebel and Sakita , and identifies the As with the non-invariance generators of the intermediate coupling groups.

Charge Symmetric Pseudo scalar Meson Theory

In the case of $K = su^I(2) \otimes su^J(2)$, the dynamical postulate is:

$$[A_{i\alpha}, A_{j\beta}] = i\theta \{ \delta_{\alpha\beta}\, \epsilon_{ijk}\, I_k + \delta_{ij}\, \epsilon_{\alpha\beta\gamma}\, J_\gamma \} \quad (4.3)$$

Putting $\theta = 0$ gives the strong coupling condition, $\theta > 0$ the compact intermediate coupling group su(4) and $\theta < 0$ the non-compact SL (4,R). Thus Kuriyan and Sudarshan's model contains strong coupling theory as a particular case. The strong coupling solution may be derived from the su(4) or SL(4,R) solutions by the usual process of group contraction which corresponds to putting θ to zero.

The use of su(4) as a non-invariance group differs in several ways from its use as a symmetry group. In order to satisfy the dynamical postulate, the isobars must form a representation of su(4). This is not true of the mesons, which are nine in number, and not fifteen, as in conventional su(4). Also there is no requirement that the meson-baryon states form a representation of su(4). In conventional su(4), there is no su(4) invariant $B\bar{B}M$ vertex for p- wave pions, and hence such processes as $N^* \to \pi N$ are forbidden. We have no such problem. In conventional su(4), the mesons belong to the regular representation and transform like the generators of the group. The isobars in the intermediate coupling model form representations of su(4). Acting within the isobars, our mesons transform as the non-invariant generators, as do a subset of the mesons in conventional su(4). Thus the couplings for these mesons must be the same. For this reason using su(4) as a non-invariance group gives results consistent with orthodox su(4) symmetry. The use of su(6) as a non-invariance group differs from its use as a symmetry group in an exactly equivalent way. After this digression we return to the calculation in hand.

Firstly we note that the solution of the equations for SL(4,R) and the strong coupling group may be obtained from those for su(4) using a method discovered by Kuriyan, Mukunda and Sudarshan(68) which depends on using Weyl's trick and introducing 'i' s into the commutation relations and using analytic continuation. We therefore derive the solution for the su(4) case, which is perhaps physically more interesting and present, without proof, the corresponding results for the other groups. We need only consider the case $\theta = 1$, as this differs from the other cases where $\theta > 0$ by an arbitrary factor which represents the overall strength of the meson couplings.

Consider as an example the nucleon isobars with $I = J = \lambda$ Inserting (4.3) between states with different values of λ, the right hand side vanishes, as each term may change I or J but not both. Thus the relations between coupling derived from this equation are the same as the relations derived in chapter 2. i.e

$$\langle \lambda \| A \| \lambda \rangle = r \text{ (constant)} \qquad (4.4)$$

Inserting the commutator between states with equal λ, we obtain

$$\langle \lambda + 1 \| A \| \lambda \rangle = \sqrt{\frac{2\lambda+1}{2\lambda+3}} \sqrt{r^2 - 16(\lambda+1)^2} \qquad (4.5).$$

Thus the representations are labelled by a non-negative even integer r: This allows λ to go in integer steps from 0 or $\frac{1}{2}$ to

$r/4 - 1$. Equation (4.5) means that a state with $\lambda > r/4-1$ cannot couple to the isobar chain and we have a finite representation. By appropriate choice of r one can include as many isobars in the chain as one wishes.

The corresponding results for SL(4,R) are:

$$\langle \lambda \| A \| \lambda \rangle = R \tag{4.6}$$

$$\langle \lambda+1 \| A \| \lambda \rangle = \sqrt{\frac{2\lambda+1}{2\lambda+3}} \sqrt{R^2+16(\lambda+1)^2} \tag{4.7}$$

Equations (4.4) and (4.5) give: $g^2{}_{\pi NN^*}/g^2{}_{\pi NN} = 2(1 - 36/r^2)$ which ratio gives an N^* width of 80 MeV when $r = 10$, so that the representation includes the N and N^* only. $r \to \infty$ gives the strong coupling values for the ratios of couplings. In the limit $r \to \infty$, $g^2{}_{\pi NN^*}/g^2{}_{\pi NN} = 2$ which gives an N^* width of 125 MeV which is very close to the experimental value of 120MeV.

For the group SL(4,R), $g^2{}_{\pi NN^*}/g^2{}_{\pi NN} = 2\,(1 + 36/R^2)$. Thus for this non-invariance group, the N^* width is always greater than 125MeV. Again as the number of isobars grows, the N^* width approaches 125MeV.

One interesting consequence of this theory is that $g_{\pi^0 N^{*+} N^{*+}}/g_{\pi^0 pp} = 1/5$ for su(4), SL(4,R) and the strong coupling group, independent of r and R. This may justify the models of the π-N-N^* system which neglect the $\pi N^* N^*$ coupling in comparison with $g_{\pi NN}$ and $g_{\pi NN^*}$

Unitary symmetric pseudoscalar theory

Consider the p- wave scattering of the octet of pseudoscalar mesons off baryon isobars which contain the usual baryon octet. The symmetry group $K = su(3) \otimes su^{J}(2)$.

The dynamical postulate for the compact intermediate coupling group su(6) is:

$$[\breve{A}_{i\alpha}, \breve{A}_{j\beta}] = i\theta \{ \delta_{ij} \epsilon_{\alpha\beta\gamma} J_\gamma + \delta_{\alpha\beta} f_{ijk} F_k + d_{ijk} \epsilon_{\alpha\beta\gamma} \breve{A}_{k\gamma} \qquad (4.8)$$

Where $\breve{A}_{i\alpha}$ is a definite multiple of $A_{i\alpha}$, chosen so that the commutation relation may be written in the usual form. This is necessary because of the linear term in $A_{i\alpha}$ in equation (4.8) For the same reason one cannot use the Weyl trick to obtain a non-compact intermediate coupling group.

We content ourselves with considering the 56 representation of su(6) which contains the $\frac{1}{2}^{+}$ octet and $3/2^{+}$ decuplet The couplings derived from this are the standard su(6) ones (62).

$$\langle 10 \| A \| 10 \rangle = \alpha$$

$$\langle 8 \| A \| 10 \rangle = -\alpha$$

$$\langle 8 \| A \| 8 \; s \rangle = \alpha/\sqrt{2}$$

$$\langle 8 \| A \| 8 \; a \rangle = \alpha\sqrt{2/5}$$

$$\langle 10 \| A \| 8 \rangle = -\alpha\sqrt{2/5} \qquad \text{(by vertex symmetry)}$$

Fulco-Wong equation from the dynamical postulate

Consider the case where K = su(2). The dynamical postulate can be written as:

$$\left[A_\alpha, A_{-\beta}\right] = \frac{-1}{\sqrt{3}} \; C\begin{pmatrix} 1 & 1 & 1 \\ \alpha & -\beta & \gamma \end{pmatrix} J_\gamma$$

Inserting these between isobars and using the projection operators as in chapter II, one obtains for the left hand side

$$g_K^I \, g_K^J - C^{su}_{Kk'} \; g_{k'}^I \, g_{k'}^J$$

The right hand side is again a sum of four C- G. coefficients. Performing this sum one obtains $C^{st}_{K1} \Gamma_1$ where $\Gamma_1 = g_I^J \gamma$ where γ is a constant. Thus one obtains the Fulco-Wong equation:

$$\Gamma - C_{us}\Gamma = C_{st}\Gamma' \quad \text{where} \quad \Gamma' \text{ is zero.}$$

apart from the isospin 1 element. This calculation may be performed for a general symmetry group and shows the equivalence of the Fulco and Wong and intermediate coupling methods for a specific process, where the terms on the right hand side of the dynamical postulate are identified with t- channel exchanges. As we shall see, the meson exchanges assumed by Fulco and Wong correspond to the non-invariant generators of the intermediate coupling groups used by Kuriyan and Sudarshan.

Fulco and Wong Re-visited

In the su(6) case the dynamical postulate which gives the intermediate coupling group su(6) leads to the Fulco and Wong Bootstrap condition, $\Gamma - C_{su}\Gamma = C_{st}\Gamma'$ where Γ contains the $\frac{1}{2}^+$ octet and $3/2^+$ decuplet as intermediate states and Γ' contains the following t-channel exchange contributions:

1) $\delta_{ij}\,\epsilon_{\alpha\beta\gamma}\,J_\gamma$ gives an su(3) singlet (from δ_{ij}) spin 1 exchange (as J_γ belongs to the spin 1 representation: $\epsilon_{\alpha\beta\gamma}$ represents the coupling of a spin 1 particle γ to two spin 1 particles α, β)

2) $\delta_{\alpha\beta}\,f_{ijk}\,F_k$ gives a scalar (from $\delta_{\alpha\beta}$), su(3) octet exchange (F_k belongs to the octet representation and f_{ijk} represents the antisymmetric coupling of an octet k to the octet i,j). Note that as the exchange meson transforms like F_k, its coupling to two baryon octets must be totally antisymmetric.

3) $d_{ijk}\,\epsilon_{\alpha\beta\gamma}\,\check{A}_{k\gamma}$ gives a spin 1 octet exchange with d coupling to the mesons, and coupling to two baryon octets with the same f/d ratio as the direct channel mesons.

Fulco and Wong made exactly these assumptions in their model, and hence arrived at the su(6) results found by Kuriyan and Sudarshan. This mathematical equivalence also explains the fact that the results for axial vector singlet scattering, and for the scattering of the scalar octet we proposed in chapter 3, are consistent with su(6)

The bootstrap equations for these process may be derived identifying the singlet with J and the scalar octet with F and using the equations for the generators of the symmetry group i.e

$$[J_\alpha, J_\beta] = i\,\epsilon_{\alpha\beta\gamma}\,J_\gamma$$

and

$$[F_i, F_j] = i\,f_{ijk}\,F_k$$

Were Fulco and Wong to have proposed the exchange of a scalar octet instead of a vector octet, one might argue that the two models were equivalent physically. However, what they have is a vector exchange which in their model for axial-vector meson scattering acts like a scalar particle. In another context, this particle will behave as a vector and the simple equivalence of the two models will not be so evident.

Meson-Baryon Scattering

In order to derive any relationship between scattering amplitudes, it is necessary to make a additional assumption. Consider the amplitude $T_{\alpha\beta}(\omega)$ for the process $M_\alpha + B \rightarrow \bar{M}_\beta + B'$ where α, β refer to a general symmetry group K. The amplitude has well defined transformation properties under K, but none for the intermediate coupling group.

The assumption made by Kariyan and Sudarshan that

$$T_{\alpha\beta}(\omega) - T_{\beta,\alpha}(\omega) = f(\omega)\,[A_\alpha, A_\beta] \qquad (4.14)$$

where f(w) is some function. This can be made plausible by the following arguments:

(i) $T_{\alpha,\beta}(w) - T_{\beta,\alpha}(w)$ and $[A_\alpha, A_\beta]$ transform in the same way under K and are both antisymmetric in α and β.

(ii) The Born Term for the Chew-Low amplitude is given by: $\frac{1}{w}[A_\alpha, A_\beta]$ to lowest order in $\frac{1}{w}$, which agrees with the assumption. However it should not be infered that the symmetry of the Born term is necessarily a symmetry of the amplitude.

Expanding T as the sum of non-spin-flip and spin-flip terms, facilitates the discovery of relations between amplitudes.

Put $T = f + \bar{\sigma}.\bar{n}\, g$ and define

$$X(iB;jB') = f(iB \quad jB') - f(jB \quad iB')$$

$$Y(iB;jB') = g(iB \quad jB') - g(jB \quad iB')$$

where i and j refer to the internal symmetry group only, as we have extracted the spin behaviour. We now consider the implication of the above for the su(4) and su(6) theories.

<u>(1) su(4) theory</u>

X,Y are both proportional to the matrix elements of the commutater of two non-invariant generators of su(4) been baryon states.

As X is a non spin-flip amplitude, it can receive no contribution from the term $\delta_{ij}\epsilon_{\alpha\beta\gamma}J_\gamma$ Thus:

$$X(iB;jB') = \langle B' | [A_{i\alpha}, A_{j\beta}] | B \rangle$$
$$= i \langle B' | \epsilon_{ijk} I_k | B \rangle \qquad (4.5)$$

Isospin implies that X(iB,jB') can be expressed as a linear conbination (given by C-G coefficients) of amplitudes for specific t- channel invariants. In the processes we are considering there is only one such invariant, the isospin 1 channel. This equation (4.15) gives us no information that could not be obtained from isospin symmetry.

The spin flip term Y can only have contributions from the term $\delta_{ij}\epsilon_{\alpha\beta\gamma}J_\gamma$ which, being an isospin singlet, gives zero contribution if B $\neq$ B'. This one may obtain relations of the form $g(p\pi^- \to n\pi^\circ) = g(p\pi^\circ \to n\pi^+)$ In terms of isospin $\frac{1}{2}$ and 3/2 amplitudes, $A\frac{1}{2} = A3/2$ which result is not well satisfied by experiment (69)

In the case of baryon resonance production, the commutator must vanish between the external baryons. This gives, for example:
$T(\pi^+ p \to \pi^+ N^{*+}) = T(\pi^- p \to \pi^- N^{*+})$ In terms of the isospin $\frac{1}{2}$ and 3/2 amplitudes, A = 10 A3, which compares well with the relation A = 3,34A3 obtained by Olsson from experimental data (70)

(ii) su(6) theory

Only the first term is a spin singlet and hence:

$$X(i\alpha B; j\beta B') = \langle B' | [\tilde{A}_{i\alpha}, \tilde{A}_{j\beta}] | B \rangle = i f_{ijk} \langle B' | F_k | B \rangle \quad (4.16)$$

If B = B' is the baryon octet, there are from su(3) invariance, four odd invariants in the t- channel. Equation (4.16) writes X in terms of only one and hence gives information additional to that derived from su(3). The results involve the Johnson-Trieman relation ... (71) for the non-spin-flip amplitude. Some of the realtions obtained are in agreement with experiment andsome not.

The relations obtained for baryon resonance production give the result A1 = 10A3, already derived in su(4). Also

$$T(K^- p \rightarrow K^0 \Xi^{*0}) - T(\bar{K}^0 p \rightarrow K^+ \Xi^{*0}) = 0.$$

As su(3) symmetry is broken, it is difficult to say whether the successes and failure of the above relations give any indication as to the validity of the theory as a whole.

Intermediate Coupling Theory and Finite Energy Sum Rules

Gleeson and Muste (72), use the theory of finite energy sum rules to provide a mechanism for deriving the non-invariance group. We shall discuss these sum rules fully in Chapter 5 but it is worth while considering this particular model as it relates to the intermediate coupling method as the saturation of superconvergence relations does to strong coupling theory.

Let $f^{(-)}$ be the forward amplitude for isospin 1 in the t-channel for a process $i + \pi \rightarrow j + \pi$ where i, j are nucleon isobars. This amplitude is dominated by the ρ Regge trajectory at high energies. One is lead (see chapter 5) to a finite energy sum rule of the form:

$$\int_0^{\infty} \text{Im } f^{(-)}(\nu)d\nu = \frac{b(0)}{\alpha_\rho(0)+1} \; K^{\alpha_\rho(0)+1} \qquad (5.17)$$

where $\alpha_\rho(t)$ is the ρ trajectory and b is a product of couplings. In terms of direct channel isospin indices, $f^{(-)}$ is antisymmetric and

and $f^{\alpha\beta} - f^{\beta\alpha} = i\,\epsilon_{\alpha\beta\gamma} < i \,|\, I_\gamma \,|\, j > \; f^{(-)}$ (4.8)

where α, β are meson isospin indices. Thus if equation (4.18), can be saturated isobar states, one obtains.

$$\sum_n{}' \left(g_{in}^{\alpha} \, g_{nj}^{\beta} - g_{in}^{\beta} \, g_{nj}^{\alpha} \right) = < i \,|\, [A_\alpha, A_\beta] \; j >$$

$$= i\,\epsilon_{\alpha\beta\gamma} < i \,|\, I_\gamma \,|\, j > \quad C \qquad (4.19)$$

where $C = b_{NN\rho\pi\pi}(o) \, \frac{K^{\alpha_\rho(o)+1}}{\alpha_\rho(o)+1}$ and $g_{in}^{\alpha} = < i \,|\, A_\alpha \,|\, n >$ If we assume that the ρ couples universally to the isospin current and that it is possible to take a fixed K for all processes, C is independent of the process and we obtain the purely algebraic expression $[A_\alpha, A_\beta] = i\,\epsilon_{\alpha\beta\gamma} I_\gamma$ between isobars, which is the usual dynamical postulate for $K = su^{I}(2)$. One can extend this to su(3) by assuming the existence of Regge poles corresponding to the various terms on the right hand side of the dynamical postulate equation. The same technique cannot be applied to an amplitude even under crossing, as this involves an anticommutator on the left hand side of equation (4.19), and its value depends on the representation, unlike the commutator.

If equation (4.13) holds only for a specific process, then it is possible to derive the Fulco-Wong equation for that process. In fact, the above is merely a sophisticated way of deriving Fulco and Wong's model using sum rules and Regge poles instead of loose arguments about meson exchanges. In chapter five, we look at the relationship between saturating sum rules and symmetries in a more realistic situation. We shall see that there are mechanisms to explain the appearance of higher symmetry results. It will not be possible, however, to elevate these mechanisms into what might be termed a model.

CHAPTER 5.

Superconvergence and Finite Energy Sum Rules

Regge Poles

Despite the various difficulties which exist in the theory, Regge poles have been remarkably successful in describing the high energy behaviour of scattering amplitudes. We consider first particles without spin, which will enable us to introduce the concept of signature with all essential details without getting entangled in a mass of spin idices.

The idea behind Regge theory is that the partial wave amplitude $a^J(s)$ for some scattering processes may be represented by a function $a(J,s)$, which equals $a^J(s)$ for physical values of J and is meromorphic (i.e only has poles) in the J- plane. (This is possible for potential scattering but probably not otherwise where there are probably moving cuts.) The attraction of this scheme is that as the position of these Regge poles $\alpha(s)$ varies, $\alpha(s)$ will sometime pass through or by an integer point Jo which will give a pole in $a^{Jo}(s)$ which will correspond to a particle of spin Jo. Thus Regge poles may link up particles with the same quantum numbers but different spins. For simplicity we consider the scattering of spin less equal mass in particles. We expand the scattering amplitude in a partial wave series in the t-channel.

$$a(s,t) = \sum_J (2J+1)\, a^J(t)\, P_J(zt) \qquad (5.1)$$

where $zt = \frac{s-u}{t-4m^2} = \frac{\nu}{qt^2}$ This series converges in an ellipse which includes the physical region. The series may be inverted to give.

$$a^J(t) = \tfrac{1}{2}\int_{-1}^{+1} dz_t P_J(z_t)\, a(s,t) \qquad (5.2)$$

As $P_J(zt)$ is not well behaved for large J, equation (2) will not serve to define our interpolating function. To get round this difficulty, we write a fixed t dispersion relation for a(s,t), which we assume to be free of kinematic singularities. (We will discuss this point when we come to consider particles with spin).

$$a(s,t) = \frac{1}{\pi}\int_{so}^{+\infty} \frac{ds'}{s'-s}\, a_s(s't) = \frac{1}{\pi}\int_{u_o}^{+\infty} \frac{du'}{u'-u}\, a_u(s',t) \qquad (5.3)$$

where $a_i(s,t)$ is the absorptive part of a(s,t) in the i channel.

Substituting equation (5.3) into equation (5.4):

$$a^J(t) = \frac{1}{\pi}\int_{x_o}^{+\infty} \frac{ds'}{2q_t^2}\, Q_J(zt(s'))\left\{a_s(s',t) + (-1)^J a_u(s',t)\right\} \qquad (5.4)$$

where Xo = min (So,Uo) and

$$Q_J(z) = \tfrac{1}{2}\int_{-1}^{+1} \frac{dx}{z-x}\, P_J(x)$$ is a Legendre function

of the second kind. For large J,

$$Q_J(z) \sim \sqrt{\pi/2J}\; \frac{e^{-(J+\frac{1}{2})\phi}}{(\sinh\phi)^{\frac{1}{2}}} \quad \text{where } \phi = \cosh^{-1} z.$$

Following Froissart and Gribov, we define

$$b^{\pm}(J,t) = \frac{1}{\pi} \int_{x_0}^{+\infty} \frac{ds'}{2q_t^2} \; Q_J(z_t) \left\{ a_s(s',t) \pm a_u(s',t) \right\}$$

This removes the unpleasant $(-1)^J$ factor and the functions are suitable for interpolating between into J. $b^{\pm}(J,t)$ are called even and odd signatured amplitudes. For even (odd) J the even (odd) signatured amplitude which gives the physical amplitude. It is these signatured amplitudes which are belived to contain the Regge poles.

Statistics demand that bosons occur in a symmetric state. Thus if the mesons coupling to a Regge pole are in an even (odd) wave, they must be in a symmetrical (anti-) state of the internal symmetry group. Thus even (odd) signatured Regge poles correspond to symmetrical (anti-) representations of the internal symmetry group.

The derivation of Regge poles from equation (5.4) appears in the standard texts on the subject (73) we shall not perform this task. A Regge pole, which has the form $Q_{-\alpha-1}(\nu)$, can be expanded as a series of Khuri poles of the form ν^{α}. For convenience we shall expand amplitudes in terms of Khuri poles.

Finite Energy Sum Rules (74)

Consider meson-baryon scattering and an amplitude corresponding to a specific t- channel invariant, which is antisymmetric. Then the amplitude will have odd signature as will the Regge Poles contributing to the asymptotic behaviour. We assume that sufficiently large energies, the amplitude is given by a sum of poles:

$$f(\nu) = \sum_i \frac{1 - e^{-i\pi\alpha_i(t)}}{\sin \pi\alpha_i(t)\Gamma(\alpha_i+1)} \nu^{\alpha_i(t)} = \sum_i R_i \qquad (5.5)$$

f being antisymmetric under $\nu \to -\nu$ will obey a dispersion relation

$$f(\nu) = \frac{2\nu}{\pi} \int_0^\infty \frac{\operatorname{Im} f(\nu')}{\nu'^2 - \nu^2} \, d\nu' \qquad (5.6)$$

If the leading trajectory has $\alpha < -1$, then $f(\nu)$ will obey the usual superconvergence relation:

$$\int_0^\infty \operatorname{Im} f(\nu) \, d\nu = 0 \qquad (5.7)$$

If a Regge term has $-1 < \alpha < 1$ it also satisfies the dispersion relation

$$R(\nu) = \frac{2\nu}{\pi} \int_0^\infty \frac{\beta}{\Gamma(\alpha+1)} \frac{\nu'^{\alpha}}{\nu'^2 - \nu^2} \, d\nu' \qquad (5.8)$$

Thus if the leading Regge trajectory has $\alpha_i < 1$, one may subtract off sufficient Regge poles from f to give a function which superconverges. $\left(f(\nu) - \sum_{\alpha_i > -1} R_i\right)$ obeys the

dispersion relation and it goes down faster than $1/\nu$ as $\nu \to \infty$. Thus

$$\int_0^\infty \left[\operatorname{Im} f(\nu) - \sum_{\alpha_i > -1} R_i\right] d\nu = 0 \qquad (5.9)$$

Then:

$$\int_0^\infty \left[\operatorname{Im} f - \sum_{\alpha_i > -1} R_i\right] d\nu = \beta \qquad (5.10)$$

where β is the residue of the pole at -1, if such exists. Assuming the Regge expansion is exact for $\nu > N$, we can split up the integral as:

$$\int_0^N \left[\operatorname{Im} f - \sum_{\alpha_i > -1} R_i\right] d\nu + \int_N^{+\infty} \left[\sum_{\alpha_i < -1} R_i\right] d\nu = \beta$$

Performing the integrals of the Regge terms explicitly:

$$\sum_{\alpha_i > -1} \frac{\beta_i N^{\alpha_i}}{\Gamma(\alpha_i+2)} + \sum_{\alpha_i < -1} \frac{\beta_i N^{\alpha_i}}{\Gamma(\alpha_i+2)} + \frac{\beta}{N} = \frac{1}{N}\int_0^N \mathrm{Im}\, f(\nu)\, d\nu$$

$$= \sum_{\text{all } i} \frac{\beta_i N^{\alpha_i}}{\Gamma(\alpha_i+2)}$$

Note that the final result treats all the Regge terms on an equal footing, despite the different ways in which the Regge terms with $\alpha > -1$ and $\alpha < -1$ entered the equations. The point $\alpha = -1$ no longer plays the special role it has in superconvergence relations. If all the $\alpha_i < -1$, one can let $N \to +\infty$ and obtain the usual superconvergence relation. It has not been necessary to assume that the amplitude has the Regge form below N.

It is easy to see that if the unsubtracted dispersion relation holds for $\nu^{2n} f(\nu)$ (n integrer) as long as the functions goes to zero as $\nu \to \infty$. Thus one may derive finite energy sum rules for the even moments of the amplitude. For negative n, an extra term appears corresponding to the pole at $\nu = 0$. The odd moments of f will not obey the dispersion relation. It is however possible to write sum rules for these amplitudes if there are no fixed poles at wrong signature points. However such poles may exist (75). The position for symmetric amplitudes is the opposite of that for the antisymmetric ones. The odd moment sum rules will hold given the correct asymptotic behaviour but the even ones will only hold in the absence of the fixed poles. The value of superconvergence relations and finite energy sum rules lies in the assumption that the integrals can be saturated by the contributions from bound states and resonances. We note that this assumption is less likely to be valid for the higher moments as the integrals become increasingly sensitive to the behaviour of f just below N. The saturation assumption is clearly only an approximation which may be valid for a particular sum rule. It will, for example, not be possible to fit the sum rules for different values of t with a finite number of resonances. We consider the t=0 sum rules which as the bound states and resonances lie around t=o, might be though to be the sum rules most likely to be saturated by pole terms. Secondly we consider the lowest moment sum rules (i.e the zero moment for odd and the first moment for even amplitudes), as off all the moments, these are most likely to allow saturation with low lying states.

Kinematic factors and spin

The introduction of particles with spin enables one to find more sum rules than in the spinless case. The additional amplitudes contain kinematic factors which may lead to these amplitudes having a better asymptotic behaviour than the total amplitude. If one is working in invariant amplitudes, the asympotitic behaviour of an amplitude can be read off from the expansion of the total amplitude in terms of Lorentz invariants. In the case of helicity amplitudes,

t

the factors arise because, in order to write a dispersion relation for an amplitude it must contain no kinematic singularities. In removing these singularities from the helicity amplitudies (76), factors are introduced which improve the asymptotic behaviour. We will deal with the helicity formalism as it enables one to show simply how higher symmetry results arise from sum rules.

The t- channel helicity amplitude $f^t_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s,t)$ for the process a + b → c + d has partial wave expansion:

$$f^t_{\lambda_c,\lambda_d;\lambda_a,\lambda_b}(s,t) = \sum (2J+1)\, d^J_{\lambda\mu}(\theta_t)\, F^J_{\lambda_c\lambda_d;\lambda_a\lambda_b} \quad ,$$

where $\lambda = \lambda_a - \lambda_b$, $\mu = \lambda_c - \lambda_d$ and θ_t is the t- channel scattering angle. Each $d^J_{\lambda\mu}(\theta_t)$ equals a factor $D_{\lambda\mu} = |\cos\theta_t/2|^{|\lambda+\mu|}$ $|\sin\theta_t/2|^{|\lambda-\mu|}$ times a Jacobi polynonical in θ_t. f has kinematic singularities which we must remove in order to write a dispersion relation: In the high energy region $D_{\lambda\mu} \sim s^{\Delta}$ where Δ = max $\{|\lambda|, |\mu|\}$ One can now show that $\bar{f}_{\lambda\mu} = f_{\lambda\mu}/D_{\lambda\mu} \sim s^{\alpha-\Delta}$ where $\alpha(t)$ is the leading Regge trajectory. Detailed proofs of these statements may be found in the literature (73,77).

Let us now consider pions scattering off baryons. The pions being spinless, have zero helicity so we may put $\lambda_b = \lambda_d = 0$. In the forward direction, there can be no helicity flip in the direct channel and so $f_{\lambda;\mu} = 0$ unless $\lambda = \mu$. Thus if we deal with helicity amplitudes at t = 0, we find only a subset of the possible sum rules: those exactly true at $t = 0$ (what Gilman and Harari (78) call "Class I" sum rules) Others ("Class 2") could be obtained by taking out certain factors which go to zero as $t \to 0$. This corresponds to taking the sum rules for small t and extrapolating to $t=0$. As Gilman and Harari pointed out (78), it is the class 1 sum rules which lead to the results of higher symmetries and it is these that we will consider. We thus lose nothing by taking helicity amplitudes at the point $t=0$. We also make the approximations of Gilman and Harari that the mesons have zero mass and that the baryons are mass degenerate. In this limit the crossing matrix is a constant and its functional dependence on the masses of the saturation isobars does not appear. It is only in this equal mass case that su(6) like results emerge from the sum rules. This is not unexpected as su(6) itself implies mass degeneracy for the octet and decuplet. If the physical masses of the particles are used, the results will of course differ somewhat from su(6). This is comparable with the breaking of exact su(6) due to mass differences.

In $\pi N \to \pi N$ and $\pi N \to \pi N^*$, ther is only one non-vanishing s- channel helicity amplitude: $a_{\frac{1}{2};\frac{1}{2}}$. Thus all t- channel amplitudes are equal apart from a multiplicative constant. In the case of $\pi N^* \to \pi N^*$, however, there are two non-vanishing amplitudes: $a_{\frac{1}{2};\frac{1}{2}}$ and $a_{\frac{3}{2};\frac{3}{2}}$. Thus the t- channel amplitudes must be linear combinations of these amplitudes. We tabulate the various amplitudes for the processes mentioned above. The figures in brackets next to the helicity amplitude give the invariant amplitude which has the same asymptatic behaviour. The entries indicate where superconvergence relations hold and the moment of the sum rule.

(1) $\pi N \rightarrow \pi N$

	$\Delta = 0$ $a_{\frac{1}{2}\frac{1}{2};00}$ (A)	$\Delta = 1$ $a_{\frac{1}{2}-\frac{1}{2};00}$ (B)
$-1 \leq \alpha < 0$		1
$\alpha < -1$	1	ν

$$a_{\frac{1}{2};\frac{1}{2}}(s,0) = \sum_J (2J+1)\, d^J_{\frac{1}{2}\frac{1}{2}}(0)\; a^J_{\frac{1}{2}\frac{1}{2}}(s)$$

Using the wellknown relations between helicity amplitudes and those between arbital angular momentum states (79), it is possible to expand $a_{\frac{1}{2};\frac{1}{2}}$ in terms of angular momentum transactions.

Performing this operation and retaining only the transitions between p- waves. $a_{\frac{1}{2};\frac{1}{2}} = 2/3\,(a^{\frac{1}{2}} + 2a^{3/2})$ where a^J is the amplitude for total angular momentum. We do this as we wish to saturate the sum rules with p- wave resonances only.

(11) $\pi N \rightarrow \pi N^*$

	$\Delta = 0$ $a_{\frac{1}{2}\frac{1}{2}}$ (A1)	$\Delta = 1$ $a_{\frac{1}{2};\frac{3}{2}}$ (A2) $a_{\frac{1}{2};-\frac{1}{2}}$ (B1)	$\Delta = 2$ $a_{\frac{1}{2};-\frac{3}{2}}$ (B2)
$\alpha > 0$			
$-1 \leq \alpha \leq 0$			1
$\alpha < -1$		1	ν

The non-vanishing amplitude $a_{\frac{1}{2};\frac{1}{2}} \sim (a_{1/2} - \sqrt{2/5}\;\; a_{3/2})$

(111) $\pi N^* \to \pi N^*$

	$\Delta = 0$ $a_{\frac{1}{2}\frac{1}{2}}$ (App) $a_{3/2\,3/2}$(Ag)	$\Delta = 1$ $a_{\frac{1}{2},\frac{3}{2}}$ (Bpp) $a_{\frac{1}{2},-\frac{1}{2}}$ (Bg)	$\Delta = 2$ $a_{-\frac{1}{2},\frac{3}{2}}$ (AQQ)	$\Delta = 3$ $a_{3/2,-3/2}$(BQQ)
$\alpha > 0$				1
$-1 \leq \alpha \leq 0$			1	ν
$\alpha < -1$		1	ν	$1, \nu^2$

$a_{3/2\,-\frac{1}{2}}$: oo $\backsim$ $(8/5\ a_{\frac{3}{2}\frac{3}{2}} - a_{\frac{1}{2}})$

Sum Rules and Symmetries

Consider the finite energy sum rules for all the odd t- channel invariants, $\int_0^{Ni} \mathrm{Im}\, f^{i}(\nu) d\nu = \rho^{i}(Ni)$ (5.13)
where $\rho^{i}(Ni)$ is the Regge term. Expand the t- channel amplitudes in terms of the direct channel ones. $f^{i}(\nu) = C_{ts}^{ij} F^{j}(\nu)$ (5.14)
If we can choose the same cut-off Ni for all the t- invariants, one can combine these equations.

$$\rho^{i}(N) = \int_0^N C_{ts}^{ij}\, \mathrm{Im}\, F^{j}(\nu)\, d\nu \qquad (5.15)$$

Using the results in Appendix 1, it can be seen that this equation is equivalent to

$$(1 - C_{us}) \Gamma = C_{st} \Gamma' \qquad (5.16)$$

where $\Gamma_j = \int_0^N \mathrm{Im}\, F^{J}(\nu)\, d\nu$ and $\Gamma'_i = \rho^{i}(N)$ for odd invariants and zero otherwise. Thus the Fulco-Wong bootstrap equation is obtained from the sum rules. If the integrals are saturated by terms from bound states and resonances, we have the Fulco and Wong model provided the terms in Γ' correspond to the appropriate meson exchanges. The crossing matrices in equation (5.16) are those for the internal symmetry group. The question now arises as to how spin symmetry comes into the theory.

Consider the Fulco and Wong equation for the symmetry group $su^J(2) \otimes K$, $K = su^I(2)$ or $su(3)$

$$(I - \overset{J}{Cus} \otimes \overset{K}{Cus})\Gamma = (\overset{J}{Cst} \otimes \overset{K}{Cst})\Gamma' \qquad (5.17)$$

This can be re-written as:

$$\overset{J}{Cts}\,[\,I - \overset{J}{Cus} \otimes \overset{K}{Cus}]\;\; \overset{J}{Cst}\;\; (\overset{J}{Cts}\;\Gamma\;) = \overset{J}{Cts}\;(\overset{J}{Cst} \otimes \overset{K}{Cst})\Gamma'$$

Now $$(Cus)^{ij}(Cst)^{jk} = \eta_k\,(Cst)^{ik}$$

where $\eta k = \pm 1$ depending on whether the k^{th} column of Cst corresponds to an even or odd invariant. Using this property:

$$(I - \eta k\,\overset{K}{Cus})\;\;\left[(Cts)^{k\ell}\Gamma_\ell\right] = \overset{K}{Cst}\;\Gamma'_k \qquad (5.18)$$

where $\Gamma = (\Gamma_1, \Gamma_2, \ldots\ )$ and $\Gamma' = (\Gamma'_1, \Gamma'_2, \ldots\ )$ are the decompositions of Γ and Γ' into representations of $su^J(2)$

For the case of meson-baryon scattering where

$$Cts = \begin{pmatrix} \sqrt{6}/3 & 2\sqrt{6}/3 \\ 2/3 & -2/3 \end{pmatrix}$$ we obtain the two equations

$$(I - Cus)\;\;(\Gamma_{\frac{1}{2}} + 2\,\Gamma_{3/2}) = 3/\sqrt{6}\;\;Cst\;\Gamma'_0 \qquad (5.19)$$

$$(I + Cus)\;\;(\Gamma_{\frac{1}{2}} - \Gamma_{3/2}) = 3/2\;\;Cst\,\Gamma'_1$$

where $\Gamma_{\frac{1}{2}}(3/2)$ are the spin $\frac{1}{2}(3/2)$ terms in Γ and $\Gamma'_0\,(\,1\,)$ are the spin O(1) contribution to Γ'.

For the case of πN scattering, the Γ in equation (5.16) indeed has the form $(\Gamma_{\frac{1}{2}} + 2\Gamma_{3/2})$. Thus for this case the equation derived from the sum rules is identical with one of those derived from the assumption of spin symmetry. Thus if the t- channel exchange terms are the same as in the Fulco and Wong model, we obtain from the sum rules the same solution as Fulco and Wong, which, as we have seen, is the same as that coming from the assumption of su(6) or su(4) symmetry. Further investigation reveals that the amplitude $a_{-\frac{1}{2},3/2}$ for $\pi N \rightarrow \pi N^*$ and $a_{-\frac{1}{2},3/2}$ for $\pi N^* \rightarrow \pi N^*$ correspond to those for t- channel spin 2. in their respective processes. Thus for these amplitudes the su(4) or su(6) results may be obtained again assuming the same t- channel terms. Gilman and Harari (78) show that all class one superconvergence relations for $\Delta = 2$ ampltidudes agree with the results derived from the algebra of charges. This agrees with our results, which show how for a small number of processes the results of higher symmetries come from sum rules.

Numerous people (80) have found the superconvergence relations which as we indicated above give su(6) results. However, as far as we know, no one has looked at all the sum rules for the different invariants at once. As we shall see, the results of this investigation are consistent with what is believe, at present, about Regge poles.

Sum Rules in su(3) for $\pi N \rightarrow \pi N$

From present knowledge about meson spectra, where no 10, $\overline{10}$ or 27 plet of mesons are known, it was assumed by Sakita and Wali (81) and by Babu, Gilman and Suzuki (82), that $\alpha(t) < 0$ for the leading 10, $\overline{10}$ and 27 plet trajectories. As the 27 is a symmetric invariant, this will lead to a superconvergence relation for $B^{27}(\nu)$ (Using the invariant amplitudes defined by $T = A + i\, Q{:}\gamma\, B$ for $\pi N \rightarrow \pi N$)

$$\int_{0}^{\infty} B^{27}(\nu)\, d\nu = 0$$

which is reasonably well satisfied, though it is impossible to test it exactly (81,82). There is no corresponding sum rules for the 10 and $\overline{10}$, because, being anti-symmetric invariants, they can only appear in sum rules for A or B which have the asymptotic behaviour

If one assumes, as Palmer does (83), that $\alpha_{10,\overline{10}} < -1$, it is possible to write superconvergence relations for the 10 and $\overline{10}$ amplitudes. In the forward direction A and νB are proportional so one has the relations:

$$\int_{0}^{\infty} A^{10}(\nu)\, d\nu = \int_{0}^{\infty} A^{\overline{10}}(\nu)\, d\nu = 0$$

Palmer saturates these three superconvergence relations with the octet and decuplet assuming degenerate mass. With mass degeneracy, the two Fulco-Wong equations, of which these relations are part, are for the same Γ. The result is that the couplings are those of su(6). We have seen how the 10 and $\overline{10}$ relations should agree with su(6), but the fact that the 27 relation gives the same needs to be explained. The 27 equation is part of the Fulco-Wong equation $\Gamma + \mathrm{Cus}\,\Gamma = \mathrm{Cst}\,\Gamma'$. The left hand side is related to the anti-commutator of the non-invariant generators of su(6). This contains no 27 part in the 56 representation of su(6). which accounts for Palmer's result.

We now consider the antisymmetric part of the amplitude corresponding to the 10, $\overline{10}$, 8aa, 8as t- channel amplitudes. Assuming we can choose a common cut-off which allows us to saturate with just the octet and decuplet, we have:

$$\Gamma - \mathrm{Cus}\,\Gamma = \mathrm{Cst}\,\Gamma'$$

Where Γ contains just the octet and decuplet terms and Γ' is zero apart from the 10, $\overline{10}$, 8aa, 8as term.

$$\Gamma'_i = \rho^i(N)$$

We know from our assumptions that in the limit $N \to \infty$, $\rho^{10} = \rho^{\overline{10}} = 0$. We have no guarantee that this is so when N is finite. However as saturation of superconvergence relations by reasonances seems successful we feel justified in assuming that we can chose the cut-off N to make $\rho^{10} = \rho^{\overline{10}} = 0$ a good approximation. With this saturation scheme, $\rho^{8as}=0$ which tells us that the ρ Regge pole which we associate with the octet exchange, couples anti-symmetrical to baryons.

We now look at the symmetric part of the amplitude. In the limit of degenerate mass saturation, the first moment sum rule for $a_{\frac{1}{2};\frac{1}{2}}$ is the same as the zero moment one, which for the symmetric invariants is only valid in the absence of fixed poles. Thus the results from this process are shakier than those for the odd invariants. However it is interesting to saturate the sum rules with the octet and decuplet. The results of inverting the process and finding the Regge terms from the resonances, is that the 27 plet contribution is zero as already stated. $\rho 1 = 19/4$, $\rho 8ss = -7/8$, $\rho 8sa = -\sqrt{5}$ ([illegible]). This corresponds to a large singlet contribution from a Regge pole which we identify with the Pomeranchon and a siseable one from a Regge pole which we identify with the A2. These results are quantitatively in agreement with present knowledge. Similar results are produced if one looks at the appropriate $\pi N \to \pi N^*$ and $\pi N^* \to \pi N^*$ amplitudes in the same way.

The above rough and ready calculations points to the way in which higher symmetry results can be produced from sum rules. Similar work could be performed for meson-meson scattering and

Fulco-Wong type solutions obtained in a place where the static model could not be used to justify the equations. Indeed work has been performed to justify the equations. Indeed work has been performed which uses the fact that all three channels are similar in meson-meson scattering to effect a new type of bootstrap (84, 8) More exact calculations on sum rules may well provide further insight into why higher symmetry results emerge from dynamical calculations.

Appendix 1 Crossing Matrices of Meson-baryon scattering

To obtain the properties of the crossing matrices for a process assuming a symmetry group K, we first define the operator F which is related to the s- channel amplitude by $f_\alpha^s(w) = \langle \alpha | F^s(w) | \alpha \rangle$ (A.1) where $|\alpha\rangle$ is a representation of K. F(w) is expanded in terms of t- channel invariants:

$$F^s(w) = \sum_T A_T(w)\, P_T \qquad (A.2)$$

where P_T is the operator which projects out the t- channel state T. Now combining A.1 and A.2 :

$$f^s_\alpha(w) = \sum_T A_T(w) \langle \alpha | P_T | \alpha \rangle \qquad (A.3)$$

By definition Cst, defined by $(Cst)_{\alpha T} = \langle \alpha | P_T | \alpha \rangle$, (A.4) is the s to t crossing matrix. Crossing from s to u consists of sending w to - w and exchanging the two mesons. Under this operation each P_T has well definded properties:

$P_T \rightarrow \eta_T P_T$ (A.5) where $\eta_T = \pm 1$ according as T is a symmetric or antisymmetric state of the mesons.

Thus: $F^u(w) = F^s(-w) = \sum_T A_T(-w)\, \eta_T\, P_T$ (A.6)

As the s and u channels are equal, one has

$$A_T(w) = \eta_T\, A_T(-w) \qquad (A.7)$$

Thus $f^u_\alpha(w) = \sum_T A_T(w)\, \eta_T \langle \alpha | P_T | \alpha \rangle$ (A.8)

Thus the u to t crossing matrix Cut is given by

$$(Cut)_{\alpha\tau} = \eta_\tau < \alpha | P_\tau | \alpha > \qquad (A.9)$$

From (A.4) and (A.9): $(Cut)_{\alpha\tau} = (Cst)_{\alpha\tau}\,\eta_\tau$ (A.10)

Expanding the u channel invariants in terms of t- channels invariants and then expanding these in terms of s channel invariants gives the expansion of u channel invariants in terms of s- channel invariants. Thus Cut Cts = Cus. Thus from A.10

$$(Cus)_{\alpha\beta} = (Cst)_{\alpha\tau}\,\eta_\tau\,(Cts)_{\tau\beta} \qquad (A.10)$$

From (A.10, we see immediately that $Cus^2 = 1$ (A.11) Moreover

$$(Cus)_{\alpha\beta}(Cst)_{\beta\tau} = (Cst)_{\alpha\mu}\,\eta_\mu\,(Cts)_{\mu\beta}\,(Cst)_{\beta\tau}$$
$$= \eta_\tau (Cst)_{\alpha\tau} \qquad (A.12)$$

Thus the column of Cst corresponding to the invariant τ is an eigen-vector of Cus with eigenvalue η_τ, where $\eta_\tau = \pm 1$ according as τ is a symmetric or antisymmetric respresentation.

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