Some studies of the interactions of elementary particles

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T O

M A U R E E N

ABSTRACT

A model for the pion-production amplitude is developed in which it is possible to calculate the total and differential cross-sections for different assumed forms of the pion-pion $I = 0$, S-wave amplitude. The two final state pions in the process $\pi^- p \rightarrow \pi^+ \pi^- n$ are considered as an $I = 0$, scalar, spin-zero system σ with a continuous mass spectrum - the 'mass' being the di-pion invariant mass; and the model consists of assuming the peripheral interaction for all partial waves other than that corresponding to the S-wave σ n final state. For this 'lowest' partial wave, a phenomenological form is derived by assuming that of the three particles in the final state, only the two pions, in an $I = 0$ S-wave, provide an important final state interaction. If the further assumption is made that this final state interaction can be 'factored' from the rest of the amplitude, then an $I = 0$, S-wave pion-pion phase-shift with a negative scattering length and which turns up through zero is found to reproduce quite well the pion-production differential cross-section data. It is also shown from this model that almost any low-energy pion-pion interaction could be compatible with the low-energy total production cross-sections.

Corroboration for this type of phase-shift is sought in the pion-nucleon partial wave 'discrepancy' analysis. By increasing the parameterisation of the pion-pion amplitude in this analysis, such a 'turn-over' type of phase-shift is found as well as a very negative solution with a large negative scattering length, and no turn over, and the solutions previously found from this analysis with positive scattering lengths. The very negative solution is rejected as being incompatible with the pion production differential distributions calculated from our model.

The ABC effect is discussed in terms of the two enhancement factors usually assumed for this analysis. It is shown that for a phase-shift which passes through zero, these two factors are not equivalent and it is not clear which - if either - should be used.

The possibility of a CDD pole in the I = 0, S-wave pion-pion partial wave has recently been suggested. Both the model proposed for the pion-production amplitude, and the 'discrepancy' analysis are adapted to incorporate this possibility. It is found that at least three types of resonating phase-shifts - two similar to those found by Lovelace et al and one similar to that obtained by Wolf - could be compatible with the low-energy pion-production and pion-nucleon scattering data.

Finally, a survey is given of the other methods for obtaining the form of the low-energy pion-pion interaction. By discussing the possible sources of error inherent in these calculations, some fairly general conclusions are drawn and compared with the results of the above analyses.

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INTRODUCTION

I - INTRODUCTION

1. Strongly Interacting Particles

At the present time it is usually accepted that the forces between particles fall into four classes, gravitational, weak, electromagnetic and strong, depending on their strength. The most widely studied of these has been the electromagnetic force which, in the quantised theory, is propagated by the exchange of zero mass photons between two charged particles, the zero mass condition implying that the force has an infinite range.

However, it is clear that one cannot explain the properties of nuclear interactions in terms of these electromagnetic forces since the nuclear forces are known to operate only over a very short range and even between the charged proton and the neutral neutron. In order to account for this strong, short range force between the particles comprising the atomic nucleii, Yukawa[1] in 1935 introduced the hypothesis of heavy quanta, which to satisfy the experimental information require a mass about two hundred times heavier than the electron. If one assumes that the interactions between these heavy quanta π and a nucleon N is the result of virtual processes of the type $N \rightleftharpoons N' + \pi$ (i.e. assuming a Yukawa coupling) then angular momentum conservation implies that these quanta should have integer spin and hence they must obey Bose statistics. Also from a study of the two nucleon potential and from the empirical evidence that the nuclear forces are charge independent it may be concluded that a triplet of such particles should exist, one positively charged, one negatively charged and a neutral member[2].

In 1947 Powell and his fellow workers[3] using very sensitive photographic plates exposed to cosmic radiation at high altitudes, detected tracks of charged particles in the emulsion which induced nuclear disintergations or emitted secondary particles when they came to rest. These strongly interacting particles with zero spin and a mass of 139.59 Mev. may be identified as the heavy quanta postulated by Yukawa, and have now become known as pi-mesons or pions.

Since that time many more strongly interacting particles have been sought in experiments with cosmic rays, bubble chambers and particle accelerators, and there now exists good evidence for more than fifty such particles[4] which are identified by their mass and spin and various internal quantum numbers such as isospin and strangeness. These quantum numbers, with their corresponding conservation laws for strong interactions, are assigned to account for the observed production of only certain particles in any strong interaction process.

All of the strongly interacting particles may be classified into the two general species of baryons and mesons. The baryon class consists of the nucleons, that is the neutron (n) and proton (p), a hyperon singlet, doublet and triplet, carrying a quantum number of strangeness -1, -2, -1 respectively, as well as the nucleon and hyperon resonances. The meson class, until six years ago, consisted only of the pion triplet (π^+, π^-, π^0) and the 'strangeness' carrying kaon doublet with its anti-kaon counterpart. In the past few years, however, there has been a great increase in the 'discovery' of new mesons. Almost all of these are meson resonances, that is they decay via strong interactions into other mesons. Some of these resonances are well established, such as the ρ and the K* etc., but the existence of many is still open to question. One possible reason for this is that unlike the study of meson-baryon interactions, where data can be obtained by scattering mesons off nuclear targets in the large particle accelerators, no mesonic targets exist (mesons have a lifetime of about 10^{-8} seconds), so that no similar study can be performed for meson-meson interactions. Perhaps this will be possible when machines are built which can clash two mesonic beams together, but for the present, and for some time to come, meson-meson interactions must be interpreted either from the final state interactions of inelastic meson-baryon scattering experiments, meson decays, etc. or from a dispersion theory analysis of meson-baryon scattering.

However, before any detailed discussion can be given of the approximations and models used in the theoretical calculation of these processes, in particular those involving pions and nucleons, we must give a brief introduction to the S-matrix and the concept of the complex energy plane[5].

2. Introduction to the S-matrix

The earliest method of calculating scattering processes between strongly interacting particles was analogous to the procedure so successfully adopted in electromagnetism where, since the time of Maxwell, the electromagnetic force has been considered in terms of a field. This field is quantised by the use of a Lagrangian, the form of which is taken from classical physics. A solution of the equations thus formed is produced as a perturbation expansion in powers of the square of the electromagnetic coupling constant (i.e. the electric charge) which in rationalised units has a value

$$e^2 = 1/137.$$

There are inherent difficulties in this perturbation expansion. In particular there exists the problem of ultra-violet divergences, but, at least in principle, this can be overcome by renormalisation techniques and the smallness of the above constant means that the first few terms of the expansion, which in practice is all that can be calculated, give a very accurate result[6]. However, in such a field theory for strongly interacting particles, the square of the equivalent coupling constant is of the order of fifteen and therefore, except under very special conditions, one should not expect an analogous perturbation expansion to be a good approximation, even if the corresponding Lagrangian could be correctly surmised.

In the last decade a theory has been devised which attempts to calculate the transition amplitudes directly, without requiring a knowledge, or even the existence of a Lagrangian and its constituent fields. These transition amplitudes are the elements of the S-matrix.

For a scattering process in which the forces are of sufficiently short range the initial and final states can be assumed to consist of free particles which may be specified by the momentum of each particle together with the discrete quantum numbers of spin, isospin, etc. If one represents such an initial state by $|n\rangle$, where n denotes all the quantum numbers identifying the state, the superposition principle of quantum mechanics allows the final state to be written as $S|n\rangle$ where S is a linear operator.

The probability that a measurement on the final state gives a result corresponding to the initial state $|m\rangle$ is given by the square of the modulus of the matrix element,

$$\langle m|S|n\rangle.$$

Thus, assuming that the states $|m\rangle$ form a complete orthonormal set conservation of probability implies that

$$S^{+}S = SS^{+} = \mathbb{1}$$

where S^{+} denotes the adjacent of S; so that the operator S is unitary. Furthermore, if L is a proper Lorentz transformation such that

$$L|m\rangle = |m'\rangle$$

relativistic invariance requires that

$$|\langle m'|S|n'\rangle|^2 = |\langle m|S|n\rangle|^2$$

and the phase of the matrix element can be chosen so that

$$\langle m'|S|n'\rangle = \langle m|S|n\rangle$$

from which it follows that for spinless particles the matrix elements depend on the four momenta of the particles only through their invariant scalar products, and for particles with spin the matrix element is composed of a number of such invariant functions multiplied by certain vector or spinor terms. For example, the 'two-to-two' S-matrix element $\langle p_3 p_4|S|p_1 p_2\rangle$ which describes the scattering of two spinless particles into a final state of two spinless particles will be a function only of the invariants[7]

$$p_i^2 = -m_i^2 \; , \quad i = 1, 2, 3, 4 \qquad - (1.1)$$

where m_i and p_i are the mass and four momentum of the i^{th} particle,

$$\begin{aligned} s &= -(p_1 + p_2)^2 \\ t &= -(p_1 - p_3)^2 \\ u &= -(p_1 - p_4)^2 \end{aligned} \qquad - (1.2)$$

and even these are not all independent since the overall energy-momentum conservation condition

$$p_1 + p_2 = p_3 + p_4 \qquad - (1.3)$$

implies that

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \qquad - (1.4)$$

It is convenient to separate the S-matrix into two parts by subtracting off the term when the particles do not interact at all, in which case the transition probability will be unity of the particles in the initial and final states are identical and zero otherwise. Thus we can write

$$\langle p_3 p_4 | S | p_1 p_2 \rangle = \langle p_3 p_4 | \mathbb{1} | p_1 p_2 \rangle + + i(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \langle p_3 p_4 | \mathcal{T} | p_1 p_2 \rangle \qquad - (1.5)$$

where $\mathbb{1}$ is the identity operator and the δ-functions coming from translational invariance specify total energy-momentum conservation. The scattering cross-section is related to the scattering amplitude F(s,t,u) where

$$F \equiv \langle p_3 p_4 | \mathcal{T} | p_1 p_2 \rangle \qquad - (1.6)$$

by the equation

$$\sigma = \frac{1}{(8\pi)^2 qW} \int d\Omega \, |F|^2 \, p/W. \qquad - (1.7)$$

where $\vec{q}$ and $\vec{p}$ are the centre of mass momenta of a particle in the initial and final states respectively, W is the centre of mass energy and Ω is the solid angle in the final state. In deriving equation (1.7) a covariant normalisation of the states has been used, i.e.

$$\langle p' | p \rangle = (2\pi)^3 \, 2p^{(0)} \, \delta^3(\vec{p} - \vec{p}\,') \qquad - (1.8)$$

The unitary condition for the operator S and equation (1.5) produce the following relation for the amplitude F,

$$\langle p_3 p_4 | \mathcal{T} | p_1 p_2 \rangle - \langle p_1 p_2 | \mathcal{T} | p_3 p_4 \rangle^* = \frac{i}{(2\pi)^2} \int \frac{d\vec{k}_1 \, d\vec{k}_2}{W^2} \delta^4(p_1 + p_2 - k_1 - k_2) \times \qquad - (1.9)$$

$$\times \langle p_3 p_4 | \mathcal{T} | k_1 k_2 \rangle \langle p_1 p_2 | \mathcal{T} | k_1 k_2 \rangle^*$$

where the star denotes the complex conjugate. Above the energy threshold for inelastic scattering new terms must be added to the right hand scale of this unitarity relation, equation (1.9), since all intermediate states will occur which are allowed by energy conservation and quantum number selection rules. This implies a change in the left hand side of equation (1.9) and suggests that a scattering matrix element has a singularity at each energy corresponding to a threshold for a new allowed physical process. Thus, these thresholds are branch points of the amplitude F, with branch cuts conventionally drawn along the real axis in the complex energy squared plane, $s = W^2$. These branch cuts allow the amplitude to be single valued on a Riemann surface. By demanding that none of these cuts are crossed, a single sheet of this surface is defined which is called the physical sheet when the physical scattering amplitude is a boundary value on the real cut of the amplitude on this sheet.

The physical amplitude that gives $\langle p_3 p_4 | M | p_1 p_2 \rangle$ is defined as the limit onto the real axis of the complex s-plane from above,

$$F(\text{physical}) = \lim_{\varepsilon \to 0+} F(s+i\varepsilon, t, u) \qquad -(1.10)$$

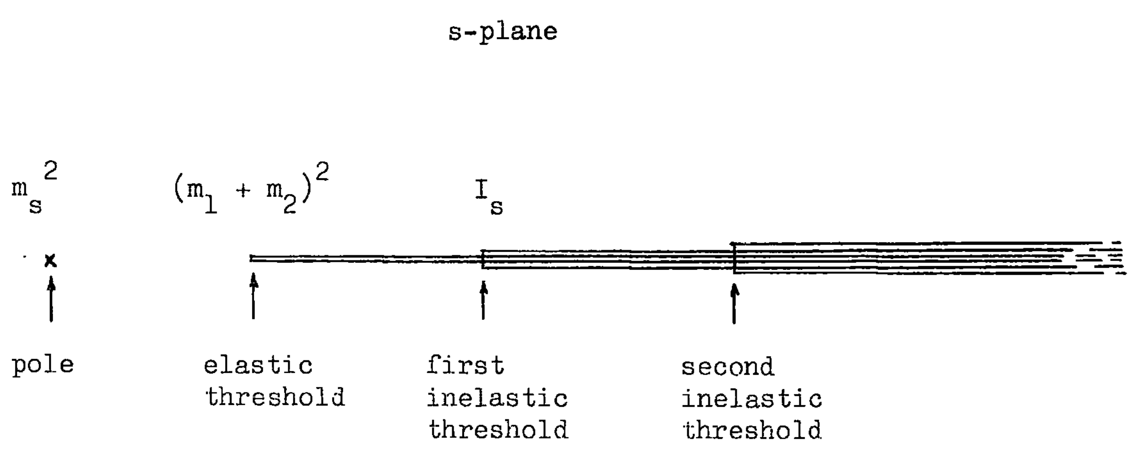
It is believed that this is related to the amplitude for $\langle p_1 p_2 | M | p_3 p_4 \rangle^*$ by analytic continuation, the latter being the limit of the same analytic functions onto the cut from below. This is the property of 'hermitian analyticity'. Thus the left hand side of equation (1.9) is the discontinuity of the amplitude across the branch cuts. If the symmetry condition

$$\langle a | S | b \rangle = \langle b | S | a \rangle$$

is true then this discontinuity is twice the imaginary part of the amplitude, but this condition holds generally only for 'two-to-two' particle scattering.

Besides these branch points, with their corresponding branch cuts of the amplitude F in the complex s-plane, it can sometimes happen that except for energy conservation a one particle state of mass m_s, say, could be reached from the initial two particle state, in which case the amplitude F(s,t,u) has a pole singularity at an unphysical value of the variable $s = m_s^2$ below the first threshold. These singularities are represented diagrammatically in figure (1.1).

FIGURE 1.1



In this discussion of the S-matrix various general properties have been assumed which can be enumerated as follows:

i forces are of short range

ii the superposition principle of quantum mechanics

iii probability conservation

iv relativistic invariance

v transition amplitudes are the values of analytic functions on real boundaries.

The fifth of these is often stated as the condition of causality and the existence of macroscopic time, but although it is usually believed that this implies v. it is difficult to prove rigorously. In the following discussion it is hoped to illustrate the physical consequence of analytic continuation in the variables s, t and u together with assumption v. in the context of two-to-two particle amplitudes for spinless, equal mass particles.

If F(s,t,u) is the amplitude for the physical scattering process

$$A_1 + A_2 \rightarrow A_3 + A_4 \qquad - (1.11)$$

the energies $p_i^{(o)}$ and the momenta $\vec{p}_i$ of the four particles must be real. In the equal mass case this implies

$$s \geqslant 4m^2, \; t \leqslant 0, \; u \leqslant 0. \qquad - (1.12)$$

If s, t, and by condition (1.4), u are considered as complex variables then by analytic continuation to the region

$$t \geqslant 4m^2, \; s \leqslant 0, \; u \leqslant 0 \qquad - (1.13)$$

property v. implies that the resultant function F, evaluated in a suitable limit onto this region is the physical scattering amplitude for the process

$$A_1 + \bar{A}_3 \rightarrow \bar{A}_2 + A_4 \qquad - (1.14)$$

where the bar denotes the anti-particle. Similarly by analytic continuation to the region

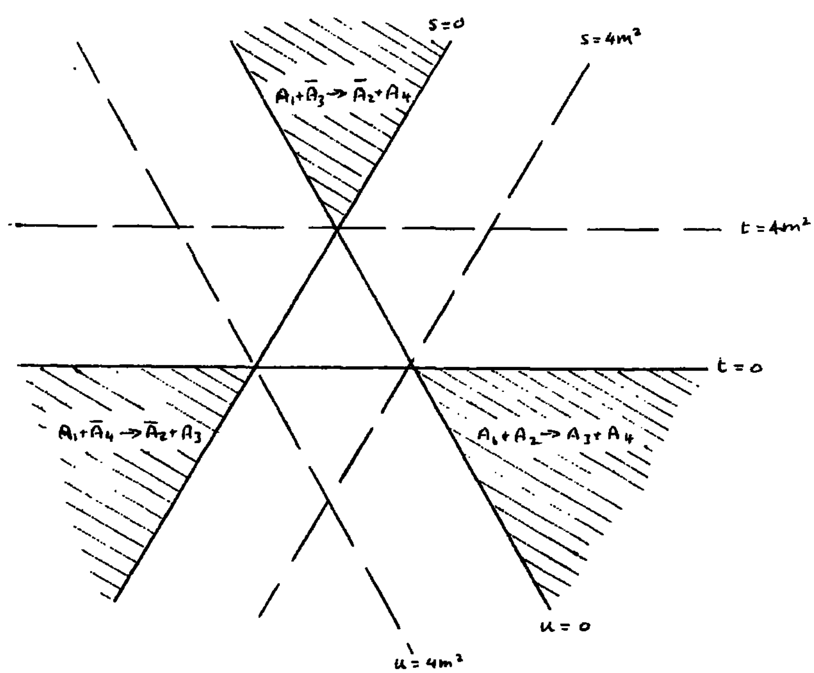
$$u \geq 4m^2, \quad s \leq 0, \quad t \leq 0 \qquad - (1.15)$$

the function F, again evaluated in a suitable limit, now gives the physical scattering amplitude for the process

$$A_1 + \bar{A}_4 \rightarrow \bar{A}_2 + A_3 \qquad - (1.16)$$

These 'crossing relations' thus state that the same analytic function can be used to describe three different physical processes by a suitable choice of s, t and u. In figure (1.2) are sketched the physical regions for these three processes in which s, t or u denote the square of the centre of mass energy, i.e. the so called s, t and u channels.

FIGURE 1.2



It should be noted that by using the unitarity equation (1.9) for processes (1.14) and (1.16) and the crossing relations it is possible to deduce further singularities of the amplitude $F(s,t,u)$. For example in the equal mass case there will be branch points at

$$t = 4m^2, (I_t)^2, \ldots \qquad \text{- (1.17)}$$
$$u = 4m^2, (I_u)^2, \ldots$$

where I_t and I_u are the first inelastic thresholds for processes (1.14) and (1.16) respectively; and possibly poles at $t = m_t^2$, $u = m_u^2$.

For a fixed value of u (at $u = u_o$, say,) the branch points will appear in the s-plane at

$$s = -u_o, 4m^2 - u_o - (I_t)^2, \ldots \qquad \text{- (1.18)}$$

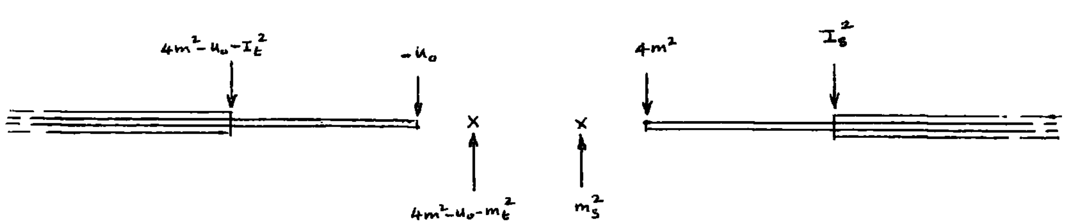
and the pole at

$$s = 4m^2 - u_o - m_t^2. \qquad \text{- (1.18)}$$

These singularities arising from the s and t 'channels' for a fixed real value of u are sketched in figure (1.3).

FIGURE 1.3

s-plane



It will be assumed, for the lack of any evidence to the contrary, that the S-matrix is an analytic function with only these singularities that are demanded by unitarity. This is the postulate of 'maximum analyticity'.

In the next section dispersion relations will be introduced which, with the unitarity conditions, define a set of dynamical equations for the physical amplitudes.

3. Dispersion Relations, Partial Wave Amplitudes and the N/D Equations

If figure (1.3) represents all the singularities of $F(s,t,u_0)$ on the physical sheet then by an application of Cauchy's theorem one obtains an expression of $F(s,t,u_0)$ in the form

$$F(s,t,u_0) = \frac{1}{2\pi i}\int_C ds' \frac{F(s',t',u_0)}{s'-s} \qquad - (1.20)$$

where C is the contour shown in figure (1.4)

If $F(s',t',u_0) \to 0$ as $|s'| \to \infty$ - (1.21)

then by allowing the radius of the contour to tend to infinity, equation (1.20) becomes

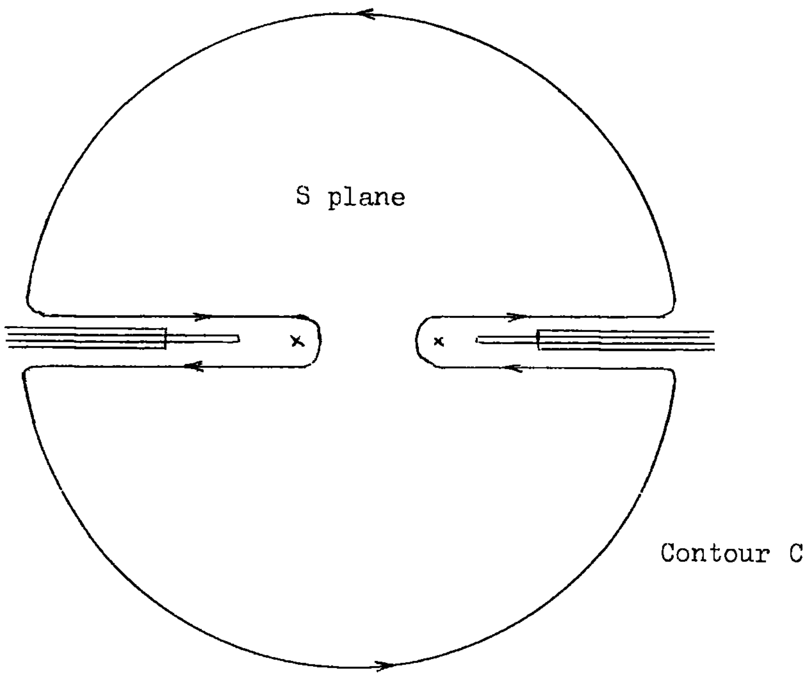
$$F(s,t,u_0) = \frac{g_s^2}{s-m_s^2} + \frac{g_t^2}{s+u_0-4m^2+m_t^2} + \frac{1}{2\pi i}\int_{4m^2}^{\infty} ds' \frac{F_s(s't'u_0)}{s'-s} + \frac{1}{2\pi i}\int_{-u_0}^{-\infty} ds' \frac{F_t(s't'u_0)}{s'-s} \qquad - (1.22)$$

which can be rewritten in the neater form

$$F(s,t,u_0) = \frac{g_s^2}{s-m_s^2} - \frac{g_t^2}{t-m_t^2} + \frac{1}{2\pi i}\int_{4m^2}^{\infty} ds' \frac{F_s(s',t',u_0)}{s'-s} + \frac{1}{2\pi i}\int_{4m^2}^{\infty} dt' \frac{F_t(s't'u_0)}{t'-t} \qquad - (1.23)$$

where F_s and F_t are the discontinuities of F across the right and left hand cuts respectively, and g_s and g_t are constants. This is usually known as a "dispersion relation" since a similar form was first used in the theory of dispersion of light in optics. Similar expressions can be written keeping t or s fixed instead of u.

FIGURE 1.4



Note that the discontinuity F_s is exactly the expression given by the unitarity equation (1.9). Thus, for two particle elastic scattering, ignoring inelastic processes

$$F_s(s,t,u) = 2i\,\mathrm{Im}\,F(s,t,u) = \frac{i}{(2\pi)^2}\sqrt{\frac{s-4m^2}{64\,s}}\int d\Omega\, F(s,t',u')\,F^*(s,t'',u'') \qquad - (1.24)$$

which with a similar expression for F_t in equation (1.23) gives an inhomogeneous integral equation for F(s,t,u). However, in physical applications one cannot ignore inelastic processes in the unitarity relation and to solve the equations rigorously one should write down equations analogous to (1.23) for all inelastic processes and solve the infinite set of coupled equations simultaneously. Obviously this is impossible, and so one must make approximations, either by ignoring inelastic processes, or by otherwise assuming the form of F_s and F_t above the inelastic thresholds, and performing an iteration procedure for F(s,t,u).

Since approximations must be made in the solution of these dispersion relations it is often preferable to use dispersion relations for the partial wave amplitudes which are defined as follows. In the centre of mass frame of reference, for equal mass particles one can write

$$\begin{aligned} s &= 4(q^2 + m^2) \\ t &= -2q^2(1 - \cos\theta) \\ u &= -2q^2(1 + \cos\theta) \end{aligned} \qquad - (1.25)$$

where $\vec{q}$ and θ are the momentum and scattering angle respectively. The partial wave amplitude $f_\ell(s)$ is defined by the relation

$$f_\ell(s) = \tfrac{1}{2}\int_{-1}^{1} d(\cos\theta)\; F(s,\cos\theta)\, P_\ell(\cos\theta) \qquad - (1.26)$$

where $P_\ell(\cos\theta)$ are Legendre polynomials and $F(s,\cos\theta)$ has been written for F(s,t,u). The function $F(s,\cos\theta)$ can be written as

$$F(s,\cos\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, f_\ell(s)\, P_\ell(\cos\theta) \qquad - (1.27)$$

where the series converges for physical s and complex $\cos\theta$, but only inside a certain ellipse.

In practice, since the partial wave amplitudes have the threshold behaviour

$$f_\ell(s) \sim q^{2\ell} \qquad \text{for small } q,$$

the partial wave expansion has the advantage that generally a few terms of the series will be sufficient to approximate the whole of the amplitude $F(s,\cos\theta)$ at low energies, although it must be noted that the truncated series, as a rule, does not have the same analytic continuation as the whole amplitude. A further advantage is that for each partial wave, the unitarity condition takes on a particularly simple form. For instance for the scattering of two spinless particles

$$\mathrm{Im}\, f_\ell(s) = \frac{1}{16\pi}\sqrt{\frac{s-4m^2}{s}}\,|f_\ell(s)|^2 R_\ell(s), \quad s \geq 4m^2 \qquad - (1.28)$$

where $R_\ell(s)$ is the ratio of total to elastic partial wave cross sections. Note that the amplitude $f_\ell(s)$ can be written as

$$f_\ell(s) = 16\pi\, \frac{e^{2i\delta_\ell(s)} - 1}{2i}\sqrt{\frac{s}{s-4m^2}} \qquad - (1.29)$$

where $\delta_\ell(s)$ is called the partial wave phase shift which from (1.28) is real below the inelastic threshold.

If this amplitude is decomposed into the ratio of two functions N_ℓ and D_ℓ, i.e.

$$f_\ell(s) = N_\ell(s)/D_\ell(s) \qquad - (1.30)$$

where $N_\ell(s)$ has only the left hand cut of the function and $D_\ell(s)$ has only the right hand cut, then

$$\begin{aligned} \text{Im } N_\ell(s) &= \text{Im} f_\ell(s) D_\ell(s) \quad \text{for } s < s_L \\ &= 0 \qquad\qquad\qquad \text{for } s \geq s_L \end{aligned} \qquad - (1.31)$$

if s_L is the start of the left hand cut; and

$$\begin{aligned} \text{Im } D_\ell(s) &= -N_\ell(s) \frac{1}{16\pi} \sqrt{\frac{s-4m^2}{s}}\, R_\ell(s) \qquad \text{for } s \geq 4m^2 \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{for } s < 4m^2 \end{aligned} \qquad - (1.32)$$

If N_ℓ is assumed to vanish at infinity then by an application of Cauchy's theorem one obtains

$$N_\ell(s) = 1/\pi \int_{-\infty}^{s_L} ds' \frac{\text{Im } N_\ell(s')}{s'-s} = 1/\pi \int_{-\infty}^{s_L} ds' \frac{\text{Im} f_\ell(s') D_\ell(s')}{s'-s} \qquad - (1.33)$$

and by normalising D_ℓ arbitrarily to unity at $s = s_0$, Cauchy's theorem for the function $(D_\ell - 1)/(s - s_0)$ leads to the result

$$D_\ell(s) = 1 - \frac{s-s_0}{\pi} \int_{4m^2}^{\infty} ds' \frac{R_\ell(s')}{16\pi} \sqrt{\frac{s'-4m^2}{s'}} \frac{N_\ell(s')}{(s'-s_0)(s'-s)} \qquad - (1.34)$$

Thus the phase shift $\delta_\ell(s)$ could be calculated from equations (1.29), (1.30), (1.33) and (1.34) given the discontinuity of the amplitude across its left hand cut, which corresponds to the interaction potential in a non-relativistic scattering problem, and the inelasticity factor $R_\ell(s)$.

A fuller discussion of the 'N over D' equations will be given in the Appendix, particularly for the case when the functions do not have the asymptotic properties assumed above. However, it is worth mentioning here that in dispersion relation theory generally if the amplitudes F do not tend to zero at infinity, functions of the form $(F(s) - F(s_o))/(s - s_o)$ which do have the correct asymptotic limits must be used. Thus, before the equations can be solved completely, the value of the function must be given at certain arbitrary points s_o, the subtraction points.

So far only spinless particles have been considered in this brief introduction to the S-matrix and dispersion relations. However when one considers the scattering of particles with spin, certain complications are introduced. For instance, in the case of pion-nucleon scattering the transition amplitude is written in terms of two invariant functions A and B such that

$$F(s,t,u) = \bar{u}(p_4)\left[A(s,t,u) - \tfrac{1}{2}i\,\gamma_\mu\,(p_1^\mu + p_3^\mu)B(s,t,u)\right]u(p_2) \qquad - (1.35)$$

where $u(p_2)$ and $u(p_4)$ are four-spinors representing the initial and final state nucleons with spin $\frac{1}{2}$ and four-momenta p_2 and p_4, and γ_μ are the well known gamma matrices. If the amplitudes for π^+p and π^-p elastic scattering are denoted by the subscripts + and - respectively, then crossing symmetry implies

$$\langle p_3 p_4 | \mathcal{T}_+ | p_1 p_2 \rangle = \langle -p_3, p_4 | \mathcal{T}_- | -p_1, p_2 \rangle \qquad - (1.36)$$

and it is often helpful to define new invariant functions

$$A^{+} = \tfrac{1}{2}(A_{-} + A_{+}); \quad A^{-} = \tfrac{1}{2}(A_{-} - A_{+}) \qquad - (1.37)$$

and similarly B^{+} and B^{-} for which it may be shown from (1.36) that A^{+}, B^{+} are symmetric under crossing and A^{-}, B^{-} are antisymmetric.

It is also often convenient to work with amplitudes defined in terms of eigenstates of isotopic spin. In the case of pion-nucleon scattering the πN system has values of isospin $I = \frac{1}{2}, 3/2$. With the help of (1.37) it may be shown that these pion-nucleon isospin amplitudes can be expressed in terms of the above crossing-symmetric and antisymmetric amplitudes as

$$A^{\frac{1}{2}} = A^{+} + 2A^{-}; \quad A^{3/2} = A^{+} - A^{-} \qquad - (1.38)$$

with identical expressions for $B^{\frac{1}{2}}$, $B^{3/2}$. For the pion-pion system, since the pion has unit isospin, there are three pion-pion isospin amplitudes corresponding to $I = 0,1,2$. The relationship between these and the amplitudes for the scattering of pions in definite charge states is discussed in detail in Appendix 1.

In the next section we shall describe how the ideas and techniques sketched here are utilized to derive information on the form of the pion-pion interaction from the available experimental data. In particular, we shall be concerned with the $I = 0$ amplitude T^{0} - which seems to be dominant at low energies - and since we shall be restricting our discussion to such low energies, the pion-pion interaction should be principally in the S-wave.

4. Some Studies of the Pion-Pion Interaction

A knowledge of the pion-pion interaction is of basic importance if one is ever to understand fully the interaction of elementary particles. Since there is no direct way at present of performing a pion-pion scattering experiment, information on the pion-pion scattering amplitude must be inferred from the studies of various scattering processes on which experimental data is available. A measure of the size of the low-energy pion-pion interaction in a state of angular momentum ℓ and isospin I may be given by the scattering length a_ℓ^I which is defined as

$$\mu . a_\ell^I = \lim_{k \to 0} \frac{A_\ell^I(k^2)}{16\pi k^{2\ell}} \qquad - (1.39)$$

where k is the magnitude of the centre of mass three momentum and A_ℓ^I is the pion-pion partial wave amplitude. In interpreting experimental data the low energy phase-shift is frequently parameterised in terms of this scattering length as

$$\frac{k}{\sqrt{k^2+\mu^2}} \cot \delta_\ell^I = \frac{\mu^{2\ell}}{\mu a_\ell^I . k^{2\ell}} \qquad - (1.40)$$

This is the so-called scattering length approximation. For the S-wave amplitude ($\ell = 0$) a two parameter form - the effective range approximation - is also often used in which

$$\frac{k}{\sqrt{k^2+\mu^2}} \cot \delta_0^I = \left(\frac{1}{\mu a_0^I} + \tfrac{1}{2} r_0^I \frac{k^2}{\mu^2} \right) \qquad - (1.41)$$

where r_0^I is the effective range. The Chew-Mandelstam[8] effective range formula is another useful parametric form

$$\frac{k}{\sqrt{k^2+\mu^2}} \cot \delta_0^I = \frac{1}{\mu a_0^I} + \frac{2}{\pi} \frac{k}{\sqrt{k^2+\mu^2}} \log\left(k+\sqrt{k^2+\mu^2}\right) \qquad - (1.42)$$

which can be derived from the N over D relations with the left hand singularities replaced by a pole at $-\infty$.

Much hard work has been done in attempting to calculate the size of these S-wave scattering lengths and many methods of varying degrees of accuracy have been proposed for analysing the available data. Unfortunately, the results of these calculations are by no means consistent and one must examine the approximations inherent in these various methods before the form of the pion-pion interaction can be firmly established. One of these methods analyses the low-energy S- and P-wave pion-nucleon data by using a partial wave dispersion relation in which the pion-pion interaction appears through the crossed channel process $\pi\pi \rightarrow N\bar{N}$ [9]. This dispersion relation can be written as

$$\mathrm{Re}\, f_{\ell\pm}^T(s) = B_{\ell\pm}(s) + \frac{P}{\pi} \int_{(m+\mu)^2}^{\infty} ds' \frac{\mathrm{Im}\, f_{\ell\pm}^T(s')}{s'-s} + \frac{1}{\pi} \int_0^{(m-\mu)^2} ds' \frac{\mathrm{Im}\, f_{\ell\pm}^T(s')}{s'-s} + D_{\ell\pm}^T(s) \qquad - (1.43)$$

where $f_{\ell\pm}^T(s)$ is the πN partial wave amplitude of total angular momentum, $J = \ell \pm \frac{1}{2}$ and ℓ is the orbital angular momentum. The term $B_{\ell\pm}(s)$ coming from the direct and crossed channel Born pole terms, together with the integrals over the right hand unitarity cut $(m+\mu)^2 \leqslant s < \infty$ and the cut $0 \leqslant s \leqslant (m-\mu)^2$ coming from the crossed terms, may be evaluated in terms of the reasonably well known pion-nucleon data. In this manner, the value of $\mathcal{D}_{\ell\pm}{}^{T}(s)$, the so called discrepancy, may be determined. However $\mathcal{D}_{\ell\pm}{}^{T}(s)$ is the contribution to the dispersion relation from the left hand cut $-\infty < s \leqslant 0$ and the circle $|s| = m^2 - \mu^2$ induced from the partial wave decomposition. These contributions arise from the crossed process $\pi\pi \to N\bar{N}$ which, from unitarity, has a phase equal to the pion-pion phase-shift, at least between the two- and the four-pion thresholds. Taking the '+' crossing-symmetric πN charge combination given by (1.37) it is possible to estimate $\mathcal{D}_{0+}{}^{+}(s)$ in terms of a low energy I = 0, S-wave pion-pion interaction. By parameterising this pion-pion contribution and fitting the discrepancy to the values given by the rest of the dispersion integral, it is possible to derive information on the form of the low-energy pion-pion amplitude. It was found by Hamilton et al[10] and Spearman[11] that the best fit to the data with $\mathcal{D}_{0+}{}^{+}$ described in terms of a simple parameterisation of the I = 0, S-wave pion-pion amplitude gave a value for $a_0{}^0$ of $1.3 \pm 0.4\ \mu^{-1}$.

A similar value for a_0^0 has been suggested by Abashian, Booth and Crowe[12] from a study of the ^{3}He momentum spectrum in the proton-deuteron scattering process

$$p + d \rightarrow {}^3He + 2\pi$$

A prominent bump appears in this spectrum which corresponds to a di-pion invariant mass of around 300 Mev. This is the so-called ABC effect and seems difficult to explain unless a strong $I = 0$, S-wave interaction is assumed. With an enhancement factor given by the square of the $I = 0$, S-wave amplitude multiplying the three-particle (^{3}He + 2π) phase-space it has been found that this bump may be reproduced with a value for the scattering length $|a_0^0|$ of $2 \pm 1\ \mu^{-1}$.

Another reaction which has received much attention in the search for the form of the pion-pion interaction is the pion-production process

$$\pi + N \rightarrow \pi + \pi + N$$

The usual model for investigating this process has been the peripheral model which assumes that the single pion exchange pole singularity is the dominant term of the pion-production scattering amplitude at least in certain regions of the momenta. With this model, one can obtain the pion-production cross-section σ_x from the relation[13]

$$d\sigma_{x} = 8/9 \; f^2/\pi\mu^2 \; \frac{\Delta^2}{(\Delta^2+\mu^2)^2} \sqrt{\frac{k^2}{k^2+\mu^2}} \; \frac{m}{\sqrt{p_N^2+m^2}} \; \frac{d^3 p_N}{(16\pi)^2 q_L} \int \left| 2A^0 + A^2 + 9A^1 \cos\theta_\pi \right|^2 d\cos\theta_\pi \qquad - (1.44)$$

where p_N, q_L are the magnitudes of the laboratory three momenta of the recoil nucleon and the ingoing pion and $f^2 = \frac{1}{4\pi}\left(\frac{G_r \mu}{2m}\right)^2$ where G_r is the rationalised renormalised pseudoscalar coupling constant. The square of the momentum transfer Δ^2 is equal to $2m(\sqrt{m^2 + p_N^2} - m)$ and θ_π , k are the pion-pion centre of mass scattering angle and magnitude of the relative three momentum respectively. Ceolin and Stroffolini[14] used equation (1.44) with the pion-pion isospin amplitudes A^I parameterised by Chew-Mandelstam effective range formulae (1.42) to calculate the low-energy total pion-production cross-sections. They found that the values of a_0^0 suggested from the π N partial-wave discrepancy analysis and the study of the ABC effect were too large to fit the experimental cross-sections and suggested that agreement could only be reached for $a_0^0 \prec 1$. It is also probably fair to say that even with a small scattering length, equation (1.44) fails to reproduce very closely the energy dependence of the low-energy total cross-sections. The reason for this may well be that the assumption of a dominant single pion exchange diagram is not justified over the whole region of the momenta considered, and that there are significant contributions from other more distant singularities corresponding to forces of shorter range. However the centrifugal barrier tends to shield states of high angular momentum from these shorter range forces and it should be reasonable to suppose that for such states the peripheral interaction is, in fact, dominant.

In Chapter II we present a phenomenological analysis of the process $\pi^- p \rightarrow \pi^+ \pi^- n$ near the production threshold which utilizes the above argument for states of higher angular momentum. The two final state pions are treated as a single spin zero system, $\tilde{\sigma}$, with $I = 0$ but with a continuous mass spectrum, and the process $\pi^- p \rightarrow \tilde{\sigma}\, n$ is decomposed into partial waves. It is supposed that, because of the centrifugal barrier damping of more distant singularities, all the partial waves other than that corresponding to the $\tilde{\sigma} n$ S-wave final state may be taken to have the values given by the single-pion exchange graph. For this other partial wave with the $\tilde{\sigma} n$ S_1-wave final state (and therefore by parity conservation the πN-P_{11} initial state) a phenomenological form is proposed by assuming that of the three final state particles only the two pions have an appreciable final state interaction. Thus the only right hand cut of this amplitude considered is associated with the two-pion interaction (assumed to be in an $I = 0$, S-wave) and can be removed by multiplying by an appropriate factor - the D function arising in an N/D solution for the $I = 0$, S-wave pion-pion amplitude. If one further assumes that this final state interaction can be factored from the rest of the amplitude so that the amplitude's dependence on the square of the di-pion centre of mass energy σ^2 comes only from the final state interaction then the effect of the left hand singularities can be approximated by a constant parameter C. By combining all these partial waves it is easily seen from the model that by varying this parameter C, practically any type of pion-pion interaction with a large or small, positive or negative, scattering length can be compatible with the $\pi^- p \rightarrow \pi^+ \pi^- n$ total cross-sections.

It must be noted that the total cross-sections for the production process $\pi N \rightarrow \pi\pi N$ are not the best means of investigating the pion-pion interaction since they involve an integration over the variables defining the di-pion system. A better method is to consider the differential pion-production cross-sections, for which in recent years some fairly detailed measurements have been performed[15, 16]. An interesting feature of the $\pi^- p \rightarrow \pi^+ \pi^- n$ and $\pi^- p \rightarrow \pi^0 \pi^0 n$ cross-sections is a peaking in the neutron energy distributions corresponding to the highest available value of the di-pion invariant mass. This effect is particularly marked for incident pion kinetic energies between 350 and 450 Mev and cannot be reproduced either by a statistical (phase-space) distribution or by a peripheral model calculation. Also the apparent absence of any such peaking for the process $\pi^- p \rightarrow \pi^- \pi^0 p$ suggests that the effect is due to the presence of a strong I = 0 interaction.

A moderately successful attempt to derive these differential distributions has been made by Goebel and Schnitzer[17] who proposed a static model comprising a single pion exchange diagram and a rescattering diagram where the scattering is principally due to the N*(3,3) resonance. Also Olsson and Yodh[18] have tried to fit production cross-sections from an isobaric model ($\pi N \rightarrow \pi N^* \rightarrow \pi\pi N$), where the N* is the '3,3' isobar or the $I = \frac{1}{2}$, $J = \frac{1}{2}$ S-wave π N interaction, which includes interference terms and the angular momentum dependence of the isobar decay. The model can account for the experimental distributions in both the reactions $\pi^+ p \rightarrow \pi^+ \pi^o p$, $\pi^+ p \rightarrow \pi^+ \pi^+ n$ but without a pion-pion interaction its inability to fit the data on $\pi^- p \rightarrow \pi^+ \pi^- n$ again is suggestive of the presence of a strong $\pi^+ \pi^-$ interaction in this reaction.

The phenomenological model which we propose above for this process $\pi^- p \rightarrow \pi^+ \pi^- n$ can also be used to study the peaking in the differential cross-sections. By suitably parameterising the pion-pion I = 0, S-wave amplitude and adjusting the parameters so that the calculated values for the distributions give an optimum fit to the experimental data, it is possible to deduce information on the I = 0, S-wave pion-pion phase-shift δ_o^o. In Chapter III it is shown that this procedure indicates that this phase-shift is initially negative (with a negative scattering length a_o^o) but soon turns over, passing through zero and becomes positive.

We cannot be sure that our assumptions for the phenomenological form of the lowest $\pi^- p \to \widetilde{\sigma} n$ partial wave are totally justified, but evidence for a negative scattering length a_0^0 from various analyses of the forward pion-pion dispersion relation[19,20] leads us to investigate more fully other methods for studying the pion-pion interaction - in particular the pion-nucleon partial wave discrepancy analysis. We find that by increasing the parameterisation of the pion-pion amplitude in this analysis, besides the solutions previously found with positive scattering lengths, a 'turn-over' phase-shift (similar in its gross features to that obtained above) is also produced as well as a very negative solution with a large negative scattering length. This very negative solution is rejected as being incompatible with the pion-production differential distributions calculated from our model, since this implies a 'bump' in the neutron energy spectrum which is not observed experimentally. This is likely to occur irrespective of the detailed parameterisation of the lowest $\pi^- p \to \widetilde{\sigma} n$ partial wave. We also discuss the two enhancement factors that are usually assumed for the analysis of the ABC effect and we note that if the phase-shift passes through zero these two factors are not equivalent and it is not clear which - if indeed either - should be used.

We end Chapter III by considering the possibility of a CDD pole in the I = 0, S-wave pion-pion partial wave amplitude. Such a possibility has been suggested recently by several authors from different theoretical standpoints. If the phase-shift $\delta_0^{\,0}$ is indeed of the turn-over type it is tempting to believe, by analogy with the situation of the P_{11} πN partial wave phase-shift, that this is indicative of a CDD pole. Certainly such a possibility must be considered and we discuss in some length how the inclusion of a CDD pole would affect the model we have proposed for the pion-production amplitude, the pion-nucleon partial wave discrepancy analysis and the calculation of the ABC effect. Having incorporated the necessary modifications, we show that the two resonating types of phase-shift $\delta_0^{\,0}$ obtained by Lovelace et al [21] from a backward pion-nucleon dispersion relation analysis, as well as the ε^0-resonance solution of Wolf [22], can all be compatible with the available data on pion-production, low energy pion-nucleon and proton-deuteron scattering.

Finally in Chapter IV we survey the knowledge of the low-energy pion-pion interaction which has been obtained from past studies. These include analyses of the three pion decay modes of the K and η mesons and the $K_{\ell 4}$ decays $K \rightarrow \pi\pi e \nu$, as well as pion-pion dispersion relation calculations, current algebra predictions and the $K_1^{\,0}$-$K_2^{\,0}$ mass difference interpreted in terms of the two-pion decay mode of the K_1^o-meson. Although the various results of these calculations are often contradictory, by investigating the assumptions and approximations used in these studies we deduce some fairly general conclusions which we compare with the results obtained from our analyses.

CHAPTER TWO

PION PRODUCTION IN A MODIFIED PERIPHERAL MODEL

II - PION PRODUCTION IN A MODIFIED PERIPHERAL MODEL

1. The Peripheral Model

A knowledge of the low energy pion-pion phase shifts, in particular their scattering lengths, is of basic importance in the interpretation of many phenomena involving pions and nucleons. Unfortunately this important problem of the pion-pion interaction at low energy is still some way from being satisfactorily resolved. For instance, on the one hand there is evidence of a large I = 0, S-wave pion-pion scattering length (a_o^o) from the ABC anomoly in the ^{3}He spectrum of proton-deuteron reactions[12] and the pion-nucleon dispersion relation 'discrepancy' analysis by Hamilton et al[10] and Spearman.[11] On the other hand there is the calculation by Ceolin and Stroffolini[14] of the low energy total cross sections for the process $\pi^- p \rightarrow \pi^+ \pi^- n$ using a peripheral model, which seems to exclude any value of a_o^o measured in natural units ($\hbar = \mu = c = 1$) that is greater than one. In order to try to reconcile these results, let us first consider in detail the description of inelastic processes by peripheral diagrams.

An interaction is said to be peripheral when it is propagated by the least massive system which can be exchanged between the colliding particles. In many cases this least massive system will be a single particle, and 'peripheral' is often used to describe any single particle exchange interaction. Exchanged systems correspond to the 'left hand' singularities of the scattering amplitude, and are the means by which forces can be transmitted. From the Uncertainty Principle it follows that the range of the force is $\sim E^{-1}$, where E is the total energy needed to produce the exchanged system, so that peripheral diagrams with low energy exchanged systems correspond to long range forces.

It should be noted that the long range forces should alone be sufficient to determine the scattering for particles in states of high orbital angular momentum since the centrifugal barrier shields these states from the unknown short range interactions. This is because particles with a high relative angular momentum 'see' each other only at a distance and consequently are little affected by forces which act only over a short distance.

The analytic scattering amplitude is determine through the Cauchy relations by pole and branch cut singularities. The residues of the poles and the discontinuities across the branch cuts are proportional to products of S-matrix elements (or their analytic continuations). These products may often be seen to be bounded, for example by the Unitarity conditions, so that the reciprocal dependence on distance which favours nearby singularities will not be overwhelmed by an increasing strength of singularity with distance.

In two-to-two particle scattering the one particle exchange diagram corresponds to the first term in a perturbation expansion which, as stated in the introduction, is of doubtful value as an approximation to the whole amplitude for strongly interacting processes. Nevertheless it can be of use in practical applications if care is taken to use it in approximating the amplitude only in certain regions of the variables where this interaction is known to dominate. For example, if instead of the complex energy plane we consider the complex Cos θ plane where θ is the centre of mass scattering angle then the physical region corresponds to the real line from -1 to +1. For the case of equal mass scattering the exchange of a single particle of mass m_1 corresponds to a pole at $t = m_1^2$ in the complex energy plane or at $\cos\theta = 1 + m_1^2/2q^2$ in the complex Cos θ plane. If m_1 is small this pole is seen to lie near the real line $-1 \leq \cos\theta \leq 1$. Also if the next exchanged system has a mass m_2 which is much greater than m_1 then the singularity in the Cos θ plane due to this exchanged system will either be a pole or a branch cut at $1 + \frac{m_2^2}{2q^2}$, depending on whether the exchanged system consists of one or more particles, which will be much further away from the physical region than $1 + m_1^2/2q^2$. Under these circumstances it should be reasonable to approximate the scattering amplitude in the physical region near $\cos\theta = 1$ by this single pole. Note also that as the magnitude of the three momentum of each particle in the centre of mass, q, increases the pole at $1 + m_1^2/2q^2$ moves nearer to $\cos\theta = 1$ (as, of course, do the other singularities) so enhancing its effect on the physical scattering amplitude near there.

The description of inelastic processes by peripheral diagrams was first introduced by Chew and Low[23] who proposed a method of extrapolating from the physical to the unphysical region in order to gain information on cross-sections which cannot be measured directly under Laboratory conditions. Their method employed this fact that a diagram with a one particle exchange contributes a pole to the amplitude of a physical process in the physical variable corresponding to the squared four momentum of the exchanged particle. As in the case of two-to-two particle scattering this pole is situated at an unphysical value of the variable to which it is related, but again, the exchange of the lightest allowed particle will result in a pole which is nearest to the physical region. In principle, at least, this pole can be reached by extrapolating from the physical region, and its residue, which will be proportional to the amplitude for processes involving the exchanged particle, can be determined. In this way it is possible to derive information about scattering processes which cannot be reproduced experimentally.

If the pole singularity is the one which is nearest to the physical region and if the exchanged particle is so light that the distance from the pole to the physical region is small then, as discussed above, it may be expected that in the physical region near the pole the peripheral interaction is dominant. This is particularly true when the exchanged particle is a pion, the lightest of the known strongly interacting particles.

Note that the nearby physical region corresponds to a quasi-real exchanged particle, that is to a small momentum transfer in the vertex where this particle is emitted. Thus, again if the exchanged particle is light, it may be reasonable to calculate the functions entering the vertices of a peripheral diagram as if the intermediate particle was in fact real. This is the 'pole approximation' of Ferrari and Selleri[13]. In S-matrix language the pole contribution to the amplitude has a residue given by a product of S-matrix elements which have one of the particles off its mass shell, that is with a squared mass equal to the squared momentum transfer $t < 0$. If the amplitude is considered in the region where t is close to zero and the physical mass, m, of the exchanged particle is small, the 'pole approximation' is equivalent to saying that there is negligible difference between these S-matrix elements and those analytically continued to m^2.

However these arguments for the dominance of the peripheral diagram are only qualitative because terms such as 'small momentum transfer' and 'physical region near the pole' cannot be precisely defined. It may be that in the calculation of total cross-sections for processes such as $\pi^- p \rightarrow \pi^+ \pi^- n$ using a pole model the dominance of the peripheral interaction is not so uniformly pronounced over the physical region considered as was supposed. In calculating these cross-sections one needs to integrate the square of the transition amplitude over a range of values for the square of the momentum transfer, and this range may include a region which is not dominated by the peripheral interaction.

Nevertheless for states of high orbital angular momentum it may be seen that the scattering amplitude is well approximated by processes involving low momentum transfer since the centrifugal barrier shields these states from the unknown short range forces. As an example, consider the scattering amplitude for the scattering of spinless, equal mass particles to be determined from the left hand singularities by a series of poles, i.e.

$$F(s,t) = \sum_i \left(\frac{R_i(s)}{t_i - t} \right)$$

Thus the partial waves are given by the equation

$$f_\ell(s) = \sum_i R_i(s) \, \tfrac{1}{2} \int_{-1}^{1} d(\cos\theta) \, P_\ell(\cos\theta) \frac{1}{t_i + 2q^2(1-\cos\theta)}$$

$$= \sum_i R_i(s) \frac{1}{2q^2} \, Q_\ell\left(\frac{t_i}{2q^2} + 1\right)$$

where Q_ℓ are the Legendre functions of the second kind. By inspection we see that

i. for small q^2 the partial waves behave like $(q^2)^\ell$

ii. for large t_i they behave like $(t_i)^{-\ell-1}$

From ii. it is clear that for large ℓ one need only consider the singularities near the physical region, i.e. for small t_i. Hence we should expect that collisions between particles in states of high angular momentum (large ℓ) would be reasonably estimated by the peripheral diagram - this being the nearest singularity to the physical region.

For this reason, in the next two sections, the peripheral contribution to the process $\pi^- p \rightarrow \pi^+ \pi^- n$ will be calculated explicitly for each orbital angular momentum state, and we shall assume that the contribution to the total cross-section for each value of the orbital angular momentum in the final state, except the lowest, is given by this one pion exchange graph alone. This is because the pion-exchange diagram has a pole which is near the physical threshold for the final state. For this 'lowest partial wave' corresponding to the S-wave final 'σ N' state, and by parity conservation to the $P_{2J=1}$ πN initial state, a phenomenological form will be discussed in Section 4. In terms of this model we will show that the total pion-production cross-section data can in fact be reconciled with a large pion-pion I = 0, S-wave scattering length.

2. S-Matrix Elements

If p and q represent the four momenta of the ingoing nucleon and meson respectively and p', q_1', q_2' the outgoing nucleon and meson then from Möller's formula[24] the S-matrix elements for the process $\pi N \to \pi\pi N$ may be written in terms of F_{fi}^{L}, the Lorentz invariant scattering amplitude as

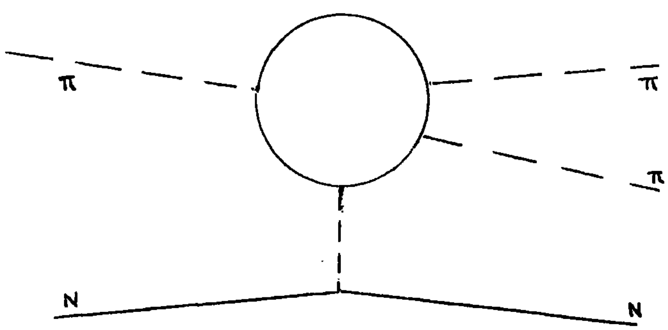
$$\langle p' q_1' q_2' | S | p q \rangle = i (2\pi)^4 \delta^4(p+q-p'-q_1'-q_2') \frac{F_{fi}^{L}}{\sqrt{32 E E' w w_1' w_2'}} \qquad - (2.1)$$

where E, E', w, w_1', w_2' are the energies of the five particles, i.e.

$$E = (p^2 + m^2)^{1/2}$$
$$w = (q^2 + \mu^2)^{1/2}, \text{ etc.}$$

Considering only the single pion exchange diagram shown in figure (2.1) F_{fi}^{L} will contain a factor F_{π}^{L} the invariant pion-pion amplitude normalised between states $|q_1', q_2'\rangle$ in the overall centre of mass frame. However, since F_{π}^{L} is an invariant it may be calculated in any Lorentz frame of reference and for convenience it will be evaluated in the pion-pion centre of mass system.

FIGURE 2.1



The inelastic cross-sections are given by the relation

$$\sigma_x = \int \frac{(2\pi)^4}{v} \frac{|F_{fi}|^2}{32 E E' \omega \omega_1' \omega_2'} \delta^4 (p + q - p' - q_1' - q_2') \frac{d^3\vec{p}'}{(2\pi)^3} \frac{d^3\vec{q}_1'}{(2\pi)^3} \frac{d^3\vec{q}_2'}{(2\pi)^3} \qquad - (2.2)$$

where $\quad v = Wp / E\omega$

To facilitate the 'partial wave analysis' the two pion system will be regarded as a single system $\tilde{\sigma}$ with a definite value of spin and a 'mass' σ, where σ is the total energy of the two pions in their own centre of mass system. By this means the contribution to these cross-sections for each value of the orbital angular momentum of the π N initial state or $\tilde{\sigma}$ N final state may be determined. To do this, however, we need to evaluate the Jacobian J which expresses the three particle element of phase space given in (2.2) in terms of the two particle phase space factor for the $\tilde{\sigma}$ N final state such that

$$\frac{d^3\vec{p}'}{(2\pi)^3} \frac{d^3\vec{q}_1'}{(2\pi)^3} \frac{d^3\vec{q}_2'}{(2\pi)^3} = \frac{J}{(2\pi)^2} d\sigma^2 \frac{d\Omega_\pi}{2\pi} \frac{d^3\vec{p}'}{(2\pi)^3} \cdot \frac{d^3\vec{q}'}{(2\pi)^3} \qquad - (2.3)$$

where $\quad q' = q_1' + q_2'; \qquad - (2.4)$

and $d\Omega_\pi = d(\cos\theta_\pi) d\phi_\pi$ where θ_π , ϕ_π are the polar angles of the relative three-momentum of the two pions measured in their own centre of mass system.

From equation (2.4) it follows that

$$d^3\vec{q}_1' \cdot d^3\vec{q}_2' = d^3\vec{q}_1' \cdot d^3\vec{q}' \qquad - (2.5)$$

where $\vec{q}_1{}'$, $\vec{q}_2{}'$, $\vec{q}\,'$ are measured in any frame of reference. Since $d^3\vec{q}_1{}'/w_1'$ and $d^3\vec{q}_2/w_2'$ are invariants their product is also an invariant and as such may also be evaluated in any reference frame; e.g.

$$\frac{d^3\vec{q}_1{}'}{\omega_1'}\cdot\frac{d^3\vec{q}_2{}'}{\omega_2'} = \{d^3\vec{q}_1{}'\cdot d^3\vec{q}_2{}'\}_{cm\pi}\Big/(\sigma/2)^2 = \{d^3\vec{q}_1{}'\cdot d^3\vec{q}\,'\}_{cm\pi} \qquad - (2.6)$$

where $cm\pi$ refers to the di-pion centre of mass frame in which $w_1' = w_2' = \sigma/2$. In this frame of reference

$$|\vec{q}_1{}'|^2 = |\vec{q}_2{}'|^2 = \sigma^2/4 - \mu^2 = k^2 \ (\text{say})$$

so that $(d^3\vec{q}_1{}')_{cm\pi} = k^2 dk d\Omega_\pi$

and $$\frac{d^3\vec{q}_1{}'}{\omega_1'}\cdot\frac{d^3\vec{q}_2{}'}{\omega_2'} = (k^2 dk\, d\Omega_\pi)\cdot(d^3\vec{q}\,')_{cm\pi}\Big/(\sigma/2)^2 \qquad - (2.7)$$

Momentum four-vectors are defined as $p_\mu = (\vec{p}\,,\ p_4 = ip_0)$ so that $P.P = -m^2 = \vec{P}.\vec{P} - P_0P_0$. Hence in the di-pion centre of mass frame where $\vec{q}\,' = 0$ and $q_0' = \sigma$ we have the relation

$$(dq_0')_{cm\pi} = \frac{2k\,dk}{\sqrt{k^2+\mu^2}} = 4k\,dk/\sigma$$

so that

$$\frac{d^3\vec{q}_1{}'}{\omega_1'}\cdot\frac{d^3\vec{q}_2{}'}{\omega_2'} = (k/\sigma\ d\Omega_\pi)(d^4q')_{cm\pi} \qquad - (2.8)$$

$$= (k/\sigma\ d\Omega_\pi)(d^4q')_{\text{any frame}}$$

since d^4q' is co-variant.

In the overall centre of mass system it follows that

$$d^3\vec{q}_1{}'\cdot d^3\vec{q}_2{}' = \omega_1'\omega_2'\, k/\sigma\ d\Omega_\pi\ d^3\vec{q}\,'\ d\sigma^2\ \frac{\partial(q_0',\vec{q}\,')}{\partial(\sigma^2,\vec{q}\,')} \qquad - (2.9)$$

where w_1' and w_2' are the energies of the final state pions measured in this system and

$$\frac{\partial(q_0',\vec{q}\,')}{\partial(\sigma^2,\vec{q}\,')} = \frac{\partial q_0'}{\partial\sigma^2} = \frac{1}{2\omega'} \qquad - (2.10)$$

where w' is the total energy of the two pions considered as a single system of 'mass' σ ,

i.e. $w' = w_1' + w_2' = (\vec{q}'^2 + \sigma^2)^{\frac{1}{2}}$.

Therefore

$$\frac{d^3\vec{p}'}{(2\pi)^3}\cdot\frac{d^3\vec{q}_1'}{(2\pi)^3}\cdot\frac{d^3\vec{q}_2'}{(2\pi)^3} = \frac{\sqrt{\sigma^2-4\mu^2}}{8\sigma}\cdot\frac{d\sigma^2}{(2\pi)^2}\cdot\frac{d\Omega_\pi}{(2\pi)}\cdot\frac{d^3\vec{p}'}{2\pi}\;\frac{d^3\vec{q}'}{2\pi}\left(\frac{4\omega_1'\omega_2'}{2\omega'}\right) \qquad - (2.11)$$

and by multiplying the S-matrix element in (2.1) by $\left(\frac{4w_1' \cdot w_2'}{2w'}\right)^{\frac{1}{2}}$ this may be written as a pseudo two-to-two matrix element

$$\langle p' q' (\sigma^2, \Omega_\pi) | S | p q \rangle = i(2\pi)^4 \delta^4(p+q-p'-q') \frac{F_{fi}^L}{\sqrt{16EE'\omega\omega'}} \qquad - (2.12)$$

in which case using equation (2.11) the cross-section (2.2) may be written as

$$\sigma_x = \int \frac{(2\pi)^4}{V} \frac{|F_{fi}^L|^2}{16EE'\omega\omega'} \delta^4(p+q-p'-q') \frac{\sqrt{\sigma^2-4\mu^2}}{8\sigma} \frac{d\sigma^2}{(2\pi)^3} d\Omega_\pi \left(\frac{d^3\vec{p}'}{(2\pi)^3}\cdot\frac{d^3\vec{q}'}{(2\pi)^3}\right) \qquad - (2.13)$$

where the last bracket is the usual two particle element of phase space.

Before discussing the partial wave analysis of the matrix elements for the process $\pi N \to \sigma N$ it is necessary to approximate the pion-pion scattering amplitude in the following way. We are interested in deriving an expression for the inelastic cross-sections at low values of T_L , the incident pion kinetic energy. As shown in Appendix II the maximum value of the total pion-pion centre of mass energy σ_{max} is an increasing function of T_L . Hence for values of T_L up to 400 Mev the two-pion centre of mass energy σ has a range of values from the two pion threshold 2μ ($\approx$ 279 Mev) to about 417 Mev. In this low energy range it should be reasonable to approximate the pion-pion amplitude by the S-wave amplitude alone. In this approximation the S-matrix element (2.12) is independent of the angles θ_π and ϕ_π so that the integration over Ω_π in (2.13) just gives the numerical factor 4π ; and the two particle system $\tilde{\sigma}$ can be considered for the purpose of partial wave analysis as a scalar, spin zero system of mass σ . It must be stressed that we are not presupposing the existence of the so-called 'sigma resonance' in pion-pion S-wave scattering. We are merely considering the two pion final state as a single system with spin zero and a mass which ranges from 2μ up to σ_{max}.

In the process $\pi^- p \to \pi^+ \pi^- n$ the pion-pion scattering amplitude contained in (2.12) is the combination of isospin amplitudes

$$\tfrac{1}{3}\, T_0 \;+\; \tfrac{1}{2}\, T_1 \;+\; 1/_6\, T_2$$

where T_0, T_1, T_2 are the amplitudes for scattering between states of isospin I = 0,1,2 respectively. As discussed in the Appendix, because of the Pauli Principle, T_1 is a sum only of the odd partial waves ℓ = 1,3, etc. so that this term drops out in the above approximation. Also there is some reason to believe that the I = 2 pion-pion interaction is less important than the I = 0 interaction at low energies. For instance the total cross-section for the inelastic process $\pi^+ p \rightarrow \pi^+ \pi^+ n$ may be reproduced quite well in the medium energy range by a peripheral diagram which, for this process, is a function of the T_2 amplitude alone. The experimental cross-sections may be fitted with the I = 2 interaction characterised by an S-wave scattering length with values $|a_0^2| \leq 0.6$. This is in agreement with most calculations for this amplitude at low energies. To simplify the low energy analysis therefore, we shall assume the I = 2 interaction is negligible in the following discussion.

These approximations imply that the invariant pion-pion amplitude F_π^L in (2.12) will be proportional to the I = 0, S-wave partial wave, and hence the πN and $\tilde{\sigma} N$ states must have a total isotopic spin of $I = \frac{1}{2}$ only.

3. Partial Wave Analysis

In order to calculate the peripheral contribution to each partial wave for the two-to-two matrix element defined in (2.12) and the contributions of these partial waves to the cross-sections given by (2.13) we shall use the helicity formalism of Jacob and Wick[25].

The initial and final states in the centre of mass frame can be labelled by the total energy E, the angular momentum J and its third component M, together with λ, the helicity of the nucleon. The parity of each state is given by

$$\mathcal{P}|E,J,M,\lambda\rangle = \eta_1\eta_2(-1)^{J-s}|E,J,M,-\lambda\rangle \qquad - (2.14)$$

where $\mathcal{P}$ is the parity operator, $\eta_1\ \eta_2$ are the intrinsic parities of the two particles in the state and $s = \frac{1}{2}$ is the spin of the nucleon. Also the S-matrix element for the process $\pi N \to \sigma N$ defined by such states can be written in terms of sub matrix elements of definite values of E and J by the relation

$$\langle E'J'M'\lambda_c|S|E\,J\,M\,\lambda_a\rangle = \delta(E-E')\,\delta_{JJ'}\,\delta_{MM'}\,\langle\lambda_c|S^J|\lambda_a\rangle. \qquad - (2.15)$$

Using the notation

$$S^J_{++} = \langle\lambda_c = +\tfrac{1}{2}|S^J|\lambda_a = +\tfrac{1}{2}\rangle, \quad \text{etc.,} \qquad - (2.16)$$

it will be seen that all four elements of this submatrix, S_{++}^{J}, S_{+-}^{J}, S_{-+}^{J}, S_{--}^{J} may be expressed in terms of two eigen values of S^{J}. Since the submatrix is unitary these eigen values are of the form $e^{2i\delta}$ where δ is a real phase, and we shall denote the two eigen-values by the phases $\delta_{\ell+}$, $\delta_{\ell-}$ depending on whether J = $\ell + \frac{1}{2}$ or $\ell - \frac{1}{2}$, where ℓ is the orbital quantum number. Neglecting the intrinsic parity factors, from equation (2.14) we see that the states

$$|JM, +\tfrac{1}{2}\rangle \pm |JM, -\tfrac{1}{2}\rangle \qquad - (2.17)$$

have parity $(-1)^{J \mp \frac{1}{2}}$ for the π N initial state. Also, since $\tilde{\sigma}$ is composed of two pseudo-scalar pions, $\tilde{\sigma}$ will have opposite intrinsic parity to that of the pion so that for the $\tilde{\sigma}$ N final states (2.17) will have parity $-(-1)^{J \mp \frac{1}{2}} = (-1)^{J \pm \frac{1}{2}}$. In terms of the states (2.17)

$$\{\langle JM, +\tfrac{1}{2}| - \langle JM, -\tfrac{1}{2}|\}\, S\, \{|JM, +\tfrac{1}{2}\rangle + |JM, -\tfrac{1}{2}\rangle\}$$

will have the phase $\delta_{\ell+}$ where ℓ = J - $\frac{1}{2}$ is the orbital angular momentum of the initial state, and

$$\{\langle JM, +\tfrac{1}{2}| + \langle JM, -\tfrac{1}{2}|\}\, S\, \{|JM, +\tfrac{1}{2}\rangle - |JM, -\tfrac{1}{2}\rangle\}$$

will have $\delta_{\ell'-}$ with ℓ' = J + $\frac{1}{2}$ = ℓ + 1. Assuming parity conservation we see that

$$\begin{aligned} S_{++}^{J} &= - S_{--}^{J} = \tfrac{1}{2}\left(e^{2i\delta_{\ell+}} + e^{2i\delta_{(\ell+1)-}}\right) \\ S_{+-}^{J} &= - S_{-+} = \tfrac{1}{2}\left(e^{2i\delta_{\ell+}} - e^{2i\delta_{(\ell+1)-}}\right) \end{aligned} \qquad - (2.18)$$

In order to relate the matrix element in (2.12) to these phase shifts $\delta_{\ell+}, \delta_{(\ell+1)-}$, ℓ = 0,1,2 ... we must expand it in terms of submatrices $S^J_{\lambda_a \lambda_c}$. This is done in two stages. We can write

$$\langle p' q' \lambda_c | S | p q \lambda_a \rangle = (2\pi)^6 \delta^4(p+q-p'-q')(vv')^{1/2}(\bar{p}\bar{p}')^{-1} \langle \theta \phi \lambda_c | S(W) | 0 0 \lambda_a \rangle \quad - (2.19)$$

where θ, ϕ are the scattering angles measured in the overall centre of mass frame of reference in which $\bar{p}, \bar{p}'; v, v'$ are the magnitudes of the relative three momenta and velocities of $p, q ; p', q'$ respectively. The S-matrix element on the right hand side of equation (2.19) can then be written in terms of the submatrix elements as

$$\langle \theta \phi \lambda_c | S(W) | 0 0 \lambda_a \rangle = \frac{1}{4\pi} \sum_J (2J+1) \langle \lambda_c | S^J | \lambda_a \rangle d^J_{\lambda_c \lambda_a}(\theta) \, e^{i(\lambda_a - \lambda_c)\phi} \quad - (2.20)$$

where the functions $d^J_{\lambda_a \lambda_c}$ are defined in reference 25. Therefore combining equations (2.12), (2.19) and (2.20) we have

$$\frac{i F^L_{\lambda_c \lambda_a} \sqrt{\bar{p}\bar{p}'}}{8\pi W} = \sum_J (J+\tfrac{1}{2}) \langle \lambda_c | S^J | \lambda_a \rangle d^J_{\lambda_a \lambda_c}(\theta) \, e^{i(\lambda_a - \lambda_c)\phi} \quad - (2.21)$$

From equation (2.13) and Appendix III we see that the spin averaged differential cross-sections are given by the relation

$$\frac{d^2\sigma_x}{d\sigma^2 d\Omega} = \sum \frac{\bar{p}'}{\bar{p}} \left| f_{\lambda_c \lambda_a}(\theta, \phi, \sigma^2) \right|^2 \frac{\sqrt{\sigma^2 - 4\mu^2}}{16\pi^2 \sigma} \quad - (2.22)$$

where the sum denotes an average over the helicities of the initial states and a summation over the final states; and

$$f_{\lambda_c \lambda_a} = - \frac{F^L_{\lambda_c \lambda_a}}{8\pi W} \quad - (2.23)$$

If we define the partial wave amplitudes $f_{\ell\pm}$ for the inelastic process $\pi N \rightarrow \tilde{\sigma} N$ in terms of the phase-shifts $\delta_{\ell\pm}$ as

$$f_{\ell\pm}(W^2, \sigma^2) = \frac{e^{2i\delta_{\ell\pm}}}{2i\bar{p}} \sqrt{\bar{p}/\bar{p}'} \qquad - (2.24)$$

then, by combining equations (2.18), (2.21), (2.23) and (2.24) and using the values

$$\begin{aligned} d^{J}_{1/2, 1/2} &= (\ell+1)^{-1} \cos\theta/2 \, (P'_{\ell+1} - P'_{\ell}) \\ d^{J}_{-1/2, 1/2} &= (\ell+1)^{-1} \sin\theta/2 \, (P'_{\ell+1} + P'_{\ell}) \end{aligned} \qquad - (2.25)$$

where $J = \ell + \frac{1}{2}$ and $P'_{\ell} = dP_{\ell}(\cos\theta)/d(\cos\theta)$ where P_{ℓ} are Legendre polynomials, the following relations may be obtained

$$\begin{aligned} f_{++} &= \sum_{\ell=0}^{\infty} (f_{\ell+} + f_{(\ell+1)-}) (P'_{\ell+1}(z) - P'_{\ell}(z)) \cos\theta/2 \\ f_{+-} &= \sum_{\ell=0}^{\infty} (f_{\ell+} - f_{(\ell+1)-}) (P'_{\ell+1}(z) + P'_{\ell}(z)) \sin\theta/2 \, e^{-i\phi} \end{aligned} \qquad - (2.26)$$

where $z = \cos\theta$. A more helpful way of writing (2.26) is

$$\begin{aligned} f_{++} &= (f_1 + f_2) \cos\theta/2 \\ f_{+-} &= (f_1 - f_2) \sin\theta/2 \, e^{-i\phi} \end{aligned} \qquad - (2.27)$$

where
$$f_1 = \sum_{\ell=0}^{\infty} f_{\ell+} P'_{\ell+1}(z) - \sum_{\ell=2}^{\infty} f_{\ell-} P'_{\ell-1}(z)$$
and
$$f_2 = \sum_{\ell=1}^{\infty} \left(f_{\ell-} P'_{\ell}(z) - f_{\ell+} P'_{\ell}(z) \right) \qquad - (2.28)$$

which, with the orthogonality property

$$\int_{-1}^{1} dz \, P'_{k}(z) \left(P_{\ell+1}(z) - P_{\ell-1}(z)\right) = \delta_{k\ell} \qquad - (2.29)$$

give the inverse relations

$$f_{\ell\pm} = \tfrac{1}{2} \int_{-1}^{1} dz \left(f_1 P_{\ell}(z) + f_2 P_{\ell\pm1}(z) \right) \qquad - (2.30)$$

Combining (2.22) and (2.28) and performing the spin averaging the differential cross-sections for the process $\pi N \rightarrow \sigma N$ may finally be expressed in terms of the partial waves as

$$\frac{d^2\sigma_\pi}{d\sigma^2 d\Omega} = \frac{\bar{p}'}{\bar{p}} \left\{ \left| \sum_{l=0}^{\infty} \left[(l+1) f_{(l+1)-}(w^2;\sigma^2) + l f_{(l-1)+}(w^2;\sigma^2) \right] P_l(z) \right|^2 + \left| \sum_{l=1}^{\infty} \left[f_{(l-1)+}(w^2;\sigma^2) - f_{(l+1)-}(w^2;\sigma^2) \right] \sin\theta \, P_l'(z) \right|^2 \right\} \frac{\sqrt{\frac{\sigma^2-4\mu^2}{\sigma^2}}}{16\pi^2} \qquad - (2.31)$$

and the total cross-sections as

$$\sigma_\pi = \int 4\pi \frac{\bar{p}'}{\bar{p}} \sum_{l=0}^{\infty} \left[(l+1) \left| f_{(l+1)-}(w^2;\sigma^2) \right|^2 + l \left| f_{(l-1)+}(w^2;\sigma^2) \right|^2 \right] \frac{d\sigma^2}{16\pi^2} \sqrt{\frac{\sigma^2-4\mu^2}{\sigma^2}} \qquad - (2.32)$$

We are now in a position to determine the peripheral contribution to each partial wave $f_{l\pm}$ for the process $\pi N \rightarrow \sigma N$. To do this let us return to the invariant amplitude $F^L_{\lambda_c \lambda_a}$. The general form of $F^L_{\lambda_c \lambda_a}$ will be a product of factors (including spinors) with the initial (final) state variables on the right (left). For the peripheral diagram sketched in figure (2.1) the internal pion line with momentum $\Delta = p - p'$ and mass μ has associated with it a propagator factor $\frac{i}{\Delta^2+\mu^2}$. The πNN vertex is represented by the term $-i\sqrt{4\pi}\, g\, \gamma_5\, K(\Delta^2)$ where K is the form factor normalised such that $K(-\Delta^2) = 1$, and g is the unrationalised coupling constant ($g^2 \approx 14.6$); and for the four pion vertex there is $A(\sigma^2, \cos\theta_\pi \; ; \Delta^2)$ which is the invariant amplitude for pion-pion scattering with the Δ^2- dependence indicating that one of the pions is off the mass shell. For the external nucleon lines we write the spinors $u_\lambda(\bar{p})$ on the right and $\bar{u}_{\lambda'}(\bar{p}')$ $(= u^+\gamma_0)$ on the left. Collecting these terms together we obtain for the peripheral diagram

$$F^{L}_{\lambda_c \lambda_a}(\text{peripheral}) = \mathcal{C}\,\bar{u}_{\lambda_c}\left\{ \gamma_5 \frac{\sqrt{4\pi}\, g\, K(\Delta^2)\, A(\sigma^2, \cos\theta_\pi ; \Delta^2)}{\Delta^2 + \mu^2} \right\} u_{\lambda_a} \qquad - (2.33)$$

where $\mathcal{C}$ is a product of Clebsch Gordan coefficients which determine the contribution for each isospin state.

In the pole approximation of Ferrari and Selleri discussed in the first section of this chapter we can consider that for small Δ^2 the functions entering the numerator of the right hand side of equation (2.33) have the values given by the on-the-mass-shell functions, i.e.

$$K(\Delta^2) = 1$$

$$A(\sigma^2, \cos\theta_\pi ; \Delta^2) = A(\sigma^2, \cos\theta_\pi)$$

where $A(\sigma^2, \cos\theta_\pi)$ is the on-the-mass-shell amplitude for pion-pion scattering. In performing the partial wave analysis it was also assumed that, for the process $\pi^- p \to \pi^+ \pi^- n$ at low energies, this amplitude A could be approximated by the I = o, s-wave amplitude $A_0^0(\sigma^2)$ defined in terms of the phase shift $\delta_0^0(\sigma^2)$ by the relation

$$A_0^0(\sigma^2) = 16\pi\, e^{i\delta_0^0} \sin\delta_0^0 \frac{\sigma}{\sqrt{\sigma^2 - 4\mu^2}} \qquad - (2.34)$$

in which case $C = 2\sqrt{2}/3$ and we obtain

$$F^{L}_{\lambda_c \lambda_a} = \frac{2\sqrt{2}}{3}\,\bar{u}_{\lambda_c}\left\{ \gamma_5 \frac{\sqrt{4\pi}\, g\, A_0^0(\sigma^2)}{\Delta^2 + \mu^2} \right\} u_{\lambda_a} \qquad - (2.35)$$

If we take the spinors

$$u_+(p) = \frac{\sqrt{E+m}}{\sqrt{2(n_3+1)}} \begin{bmatrix} \begin{pmatrix} n_3+1 \\ n_1+in_2 \end{pmatrix} \\ \frac{|\vec{p}|}{E+m} \begin{pmatrix} n_3+1 \\ n_1+in_2 \end{pmatrix} \end{bmatrix}, \qquad u_-(p) = \frac{\sqrt{E+m}}{\sqrt{2(n_3+1)}} \begin{bmatrix} \begin{pmatrix} -n_1+in_2 \\ n_3+1 \end{pmatrix} \\ \frac{|\vec{p}|}{E+m} \begin{pmatrix} -n_1+in_2 \\ n_3+1 \end{pmatrix} \end{bmatrix}$$

to represent the two helicity states of the nucleon, where $\vec{n} = \vec{p}/|\vec{p}|$, when from (2.23) and (2.27) we find

$$-\frac{F^L_{++}}{8\pi W} = (f_1+f_2)\cos\theta/2 = \frac{2/3\sqrt{8\pi}\, g\, A_0^0}{8\pi W\,(\Delta^2+\mu^2)}\left(\bar{p}'\sqrt{\frac{E+m}{E'+m}} - \bar{p}\sqrt{\frac{E'+m}{E+m}}\right)\cos\theta/2 \qquad - (2.36)$$

and

$$-\frac{F^L_{+-}}{8\pi W} = (f_1-f_2)\sin\theta/2\, e^{-i\phi} = \frac{2/3\sqrt{8\pi}\, g\, A_0^0}{8\pi W\,(\Delta^2+\mu^2)}\left(\bar{p}'\sqrt{\frac{E+m}{E'+m}} + \bar{p}\sqrt{\frac{E'+m}{E+m}}\right)\sin\theta/2\, e^{-i\phi}$$

Hence $\quad f_1(\text{peripheral}) = g_1/(\Delta^2+\mu^2)$

$\qquad\quad f_2(\text{peripheral}) = g_2/(\Delta^2+\mu^2)$

where

$$g_1 = 2/3\sqrt{8\pi}\, g\, \bar{p}'\sqrt{\frac{E+m}{E'+m}}\cdot\frac{A_0^0}{8\pi W} \qquad - (2.37)$$

$$g_2 = -2/3\sqrt{8\pi}\, g\, \bar{p}\sqrt{\frac{E'+m}{E+m}}\cdot\frac{A_0^0}{8\pi W}$$

and it follows from equation (2.30) that the peripheral contribution to each partial wave is given by

$$f_{\ell\pm}(\text{peripheral}) = \frac{1}{2\bar{p}\bar{p}'}\left[g_1 Q_\ell(z_0) + g_2 Q_{\ell\pm1}(z_0)\right] \qquad - (2.38)$$

where $Q_\ell(z)$ are Legendre functions of the second kind, and

$$z_o = \frac{EE' - m^2 - \mu^2/2}{p p'} \qquad - (2.39)$$

As a check on our partial wave analysis we calculated the total cross-sections for $\pi^- p \rightarrow \pi^+ \pi^- n$ given by equation (2.32). This calculation requires a knowledge of the $\pi\pi$ I = 0, S-wave phase shifts over a continuous range of values of σ^2. We used an N/D decomposition for this amplitude since this form will be required in the next section; and assumed that the amplitude on the left hand cut to be approximated by a single delta function, i.e.

$$A_o^{\,o}(\sigma^2) = -i\, 16\, \pi^2\, \Gamma\, \delta(\sigma^2 - \sigma_p^{\,2}) \qquad - (2.40)$$

From equation (1.33), if $D(\sigma^2)$ is normalised to unity at $\sigma^2 = \sigma_p^{\,2}$ then

$$N(\sigma^2) = 16\pi\, \Gamma / (\sigma^2 - \sigma_p^{\,2}) \qquad - (2.41)$$

and from equation (1.34)

$$D(\sigma^2) = 1 - \frac{\Gamma}{\sigma_p^2 \pi} \left\{ \begin{array}{l} 1 + \alpha \left(\frac{2}{\sigma_p^2 - 4} + \frac{\sigma_p^2}{\sigma^2 - \sigma_p^2} \right) \log \left(\frac{1+\alpha}{1-\alpha} \right) \\ - \beta \left(\frac{\sigma_p^2}{\sigma^2 - \sigma_p^2} \right) \log \left(\frac{1+\beta}{1-\beta} \right) \end{array} \right\} \qquad - (2.42)$$

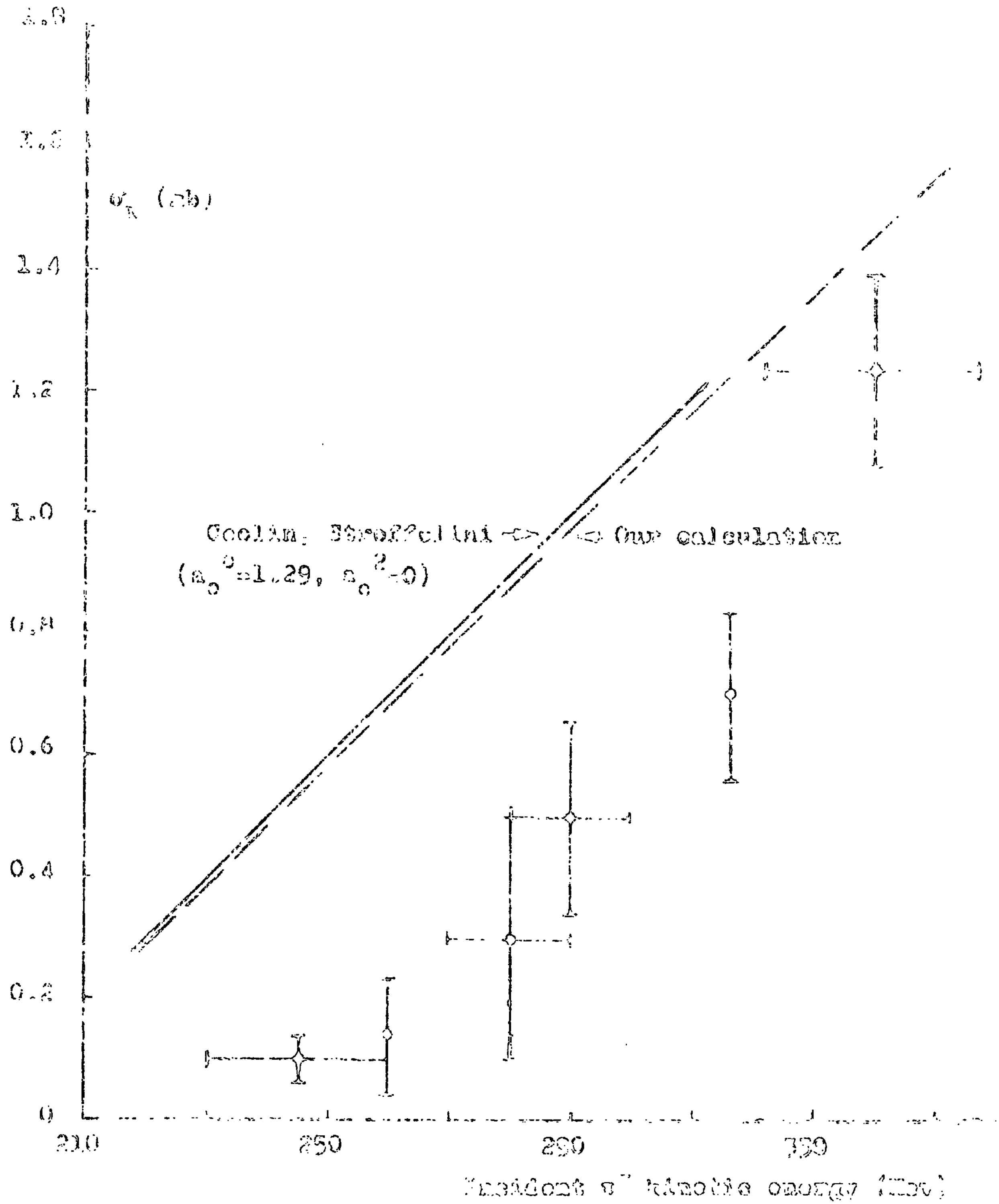
where $\alpha = \sqrt{\frac{\sigma_p^2 - 4}{\sigma_p^2}}$

$\beta = \sqrt{\frac{\sigma^2 - 4}{\sigma^2}}$

and μ, the pion mass, has been taken as unity.

Values for Γ and σ_p^2 of 168 and -396 respectively were taken which, corresponding to Spearman's solution (4) for his 'Discrepancies' analysis, give a scattering length for a_0^0 of 1.29 in natural units. It should be noted that the cross-sections given by equation (2.32), summing over <u>all</u> $\pi N \to \tilde{\pi} N$ partial waves, and using the N/D decomposition (2.41) and (2.42) should be in close agreement with the values obtained from the Ceolin and Stroffolini calculation, equation (1.44), using $a_0^0 = 1.29$, $a_0^2 = 0$. This is because both calculations use a similar peripheral approximation to the invariant amplitude $F^L_{\lambda_c\lambda_a}$ (see equation (2.33)). Also the Chew-Mandelstam effective range formula with which they approximate the pion-pion amplitudes are derived from these N and D equations by assuming the delta function singularity to be at $-\infty$ so that N is a constant. There should be very little difference, therefore, between this effective range formula and our one pole approximation with $\sigma_p^2 = -396$. By making a simple linear interpolation of the Ceolin and Stroffolini results obtained from $a_0^0 = 1, 1.5$ and $a_0^2 = \pm 0.54$ to the values $a_0^0 = 1.29$, $a_0^2 = 0$ we found that the cross-sections agreed to within five per cent. These results are represented in figure (2.2) together with the experimental observations for these cross-sections.[15,26]

FIGURE 2.2


0.6
0.4
0
290
Coolidge, Stroffolini → ← Our calculation
(a_0^0=1.29, a_0^2=0)

4. Parameterisation of the P_{11} Partial Wave

The extent of the disagreement of a peripheral calculation of the $\pi^- p \rightarrow \pi^+ \pi^- n$ total cross-sections using a large scattering length ($a_o^o > 1$) compared with the experimentally observed values is clearly demonstrated in figure (2.2). This disagreement may be due to the assumption of dominance of the peripheral interaction over the whole range of values for the squared momentum transfer (Δ^2) considered in determining these cross-sections. However, as already mentioned, the centrifugal barrier shields states of high angular momentum from the unknown short range forces, and so for these states the amplitudes should be well approximated by processes involving only long range interactions, that is by processes which give rise to the singularities close to the physical region. This physical region is bounded by the threshold value for the σ N final state, and the singularity nearest to this is the pole coming from the single pion exchange diagram. It follows, therefore, that in attempting to improve the reliability of the cross-section calculation one might still assume that this peripheral diagram should give reasonable estimates for the 'higher partial waves', defined in terms of the orbital angular momentum of the final state, although this may not be the case for the lower partial waves.

In practice we shall take 'higher' to mean all waves except that corresponding to the final state with the lowest value of orbital angular momentum, i.e. the σN S-state. By parity conservation this corresponds to the $P_{2J=1}$ initial πN state and since the two pions are considered to be in a pure I = 0 state this will be the $P_{2J=1,\ 2I=1}$ (i.e. the P_{11}) πN state. From table (2.1), in which the peripheral contributions to the $\pi^- p \to \pi^+ \pi^- n$ total cross-section are given for values of the angular momentum of the initial state, it is seen that the contribution for this P_{11} initial state (which in the notation of the last section is the f_{1-} $\pi N \to \sigma N$ partial wave) gives by far the largest effect.

TABLE 2.1

	$\int \lvert f_{\ell\pm}(W^2;\sigma^2) \rvert^2 \frac{\bar{p}'}{\bar{p}} \sqrt{\frac{\sigma^2-4\mu^2}{\sigma^2}} \frac{d\sigma^2}{4\pi}$			
$\ell\pm$ \ T_L	212 Mev	268 Mev	324 Mev	375 Mev
0+	0.008	0.047	0.098	0.135
1+	0.002	0.004	0.009	0.013
2+	-	-	0.001	0.003
1-	0.182	0.638	0.970	1.171
2-	0.007	0.035	0.069	0.095
3-	-	0.005	0.010	0.017
σ_x in mb	0.210	0.778	1.257	1.582

If we are not to assume that the peripheral interaction is dominant for the P_{11} amplitude we must derive some form for $f_{1-}(W^2, \sigma^2)$ which is suggested by a phenomenological study of this inelastic process. We shall do this in two stages. Firstly, we must notice that the expansion of the invariant scattering amplitude F^L into partial waves introduces certain kinematic singularities. These can be determined for each wave, and from Appendix III we see that $f_{\ell\pm}$ can be written in terms of $h_{\ell\pm}(W^2, \sigma^2)$, the partial wave amplitude for $\pi N \rightarrow \sigma N$ which is free from right hand kinematic singularities, in the form[27]

$$f_{\ell\pm} = h_{\ell\pm} \cdot (\bar{p}\bar{p}')^{\ell} \frac{(\bar{p}')^{\pm 1}}{W} \sqrt{\frac{E'+m}{E+m}} \qquad - (2.43)$$

Secondly, in order to derive a phenomenological form for $h_{\ell\pm}$ we must consider the dynamical singularities of this amplitude. We are concerned with the calculation of cross-sections for values of the incident pion kinetic energy up to 400 Mev, which is the threshold for N* production. Below this threshold it will be assumed that the only final state reaction between the three particles π, π, N is the interaction between the two pions so that the only right hand dynamical singularity of $h_{\ell\pm}$ will be the cut in σ^2 from $4\mu^2$ to infinity. By Unitarity the phase of the amplitude on this cut will just be the phase of the two-pion interaction δ_0^0 the elastic I = 0, S-wave pion-pion phase shift. We can make use of this in the following way. If we define the function $T(\sigma^2)$ such that

$$T(\sigma^2) = \exp\left\{ -\frac{(\sigma^2 - \sigma_s^2)}{\pi} \int_{4\mu^2}^{\infty} \frac{\delta_0^0(\sigma'^2)}{(\sigma'^2 - \sigma_s^2)(\sigma'^2 - \sigma^2)} d\sigma'^2 \right\} \qquad - (2.44)$$

and ignoring inelastic contributions to the pion-pion interaction above the first inelastic threshold $\sigma^2 \geqslant 16\mu^2$, then $T^{-1}(\sigma^2)$ has the same phase δ_o^o as $h_{\ell\pm}$ on the right hand cut, and therefore the product $h_{\ell\pm}.T$ has only the left hand singularities in W^2 and σ^2. Thus, for a fixed value of the square of the overall centre of mass energy $W^2 = W_o^2$ (say), the partial wave amplitude $h_{\ell\pm}(W_o^2, \sigma^2)$ may be written in the form

$$h_{\ell\pm}(W_o^2,\sigma^2).T(\sigma^2) = \int_{L.H.} \frac{T(\sigma'^2)\, \mathrm{Im}\, h_{\ell\pm}(W_o^2,\sigma'^2)}{\sigma'^2 - \sigma^2}\, d\sigma'^2 \qquad - (2.45)$$

If the integrand of this equation were known, the integral could be evaluated and would provide us with the functional form of $h_{\ell\pm}$. Unfortunately, this integrand is not known and so to proceed further we shall assume that the right hand side of equation (2.45) can be approximated by a constant C for each value of W_o^2; in which case we obtain

$$h_{\ell\pm}(W_o^2, \sigma^2) = C.\ T^{-1}(\sigma^2) \qquad - (2.46)$$

This approximation of the integral over the left hand cut by a constant is the same as that used in deriving the Chew-Mandelstam effective range formula from the one pole approximation of equations (2.41) and (2.42) by assuming that the discontinuity across the cut can be approximated by a delta function singularity at $\sigma^2 = -\infty$. Unlike the other approximations made in deriving this model, which were based on physical intuition, this assumption is made because of a lack of knowledge of $\mathrm{Im}\, h_{\ell\pm}(W_o^2, \sigma^2)$ for negative values of

σ^2, and is, therefore, a possible source of error. We shall return to this point again in the next chapter.

It should be noted that $T(\sigma^2)$ as defined in (2.44) is the Omnes-Mushkelishvilli form for the D-function arising in an N/D solution for the I = 0, S-wave pion-pion scattering amplitude. Hence, combining equations (2.43) and (2.46) the phenomenological form which we shall use for the partial wave amplitude corresponding to the P_{11} initial state is

$$f^{\rho}_{1-}(W^2,\sigma^2) = \frac{C}{D(\sigma^2)} \cdot \sqrt{\frac{E'+m}{E+m}} \cdot \frac{\bar{p}}{W} \qquad - (2.47)$$

where C is an arbitrary constant for each value of W^2. We can adjust this constant to obtain a good fit to the data for the $\pi^- p \rightarrow \pi^+ \pi^- n$ total cross-sections at each value of the incident pion kinetic energy and in table (2.2) we list these values of C for the one pole approximation to the pion-pion amplitude given in the last section.

<u>TABLE 2.2</u>

T_L in Mev	C^2
240	0.5 ± 0.5
268	1.9 ± 1.9
296	4.3 ± 3.0
324	9.7 ± 3.5

With the freedom of choosing the value of C at each energy the total cross-sections can be reproduced for practically any I = 0 S-wave pion-pion phase shift with a scattering length which is either large or small. Thus, this model which should be a better approximation to the amplitude for the process $\pi^- p \rightarrow \pi^+ \pi^- n$ than the unmodified pole approximation has the advantage that it can reconcile a large pion-pion scattering length a_0^0 with the data for the inelastic total cross-sections. On the other hand, by just fitting to the total cross-sections, the model is unable to give us any information on the shape of the I = 0, S-wave phase shift δ_0^0 since one integrates over the whole allowed range of σ^2. It is clearly seen, however, that this model is also applicable to the study of the differential cross-sections given by $\frac{d^2\sigma_x}{d\Omega\, d\sigma^2}$ which are defined at each value of Ω and σ^2. In the next chapter we shall discuss the values for the pion-pion phase shift obtained by fitting the experimental data for the differential cross-sections from this model, and then compare these results with the values obtained from some other theoretical calculations.

C H A P T E R   T H R E E

DIFFERENTIAL PRODUCTION CROSS-SECTIONS AND THE PION-PION INTERACTION

III - DIFFERENTIAL PRODUCTION CROSS-SECTIONS & THE PION-PION INTERACTION

1. Differential Pion-Production Cross-Sections

In the past few years some extensive experimental analyses have been performed for the single pion-production processes $\pi N \rightarrow \pi \pi N$ at low energies. For instance Barish et.al.[15] have measured the differential cross-sections for positive pions, protons and neutrons resulting from inelastic $\pi^- p$ collisions at 310 Mev to 454 Mev incident pion kinetic energy. The pion source was an internal target of the Berkeley 184 inch synchro-cyclotron and the pion beam was focused at a liquid hydrogen target. The energy distributions of the final state particle of interest were measured at a series of angles defined in the laboratory frame for $\pi^+\pi^-n$, $\pi^0\pi^0n$, $\pi^-\pi^0p$ final states. It was observed that the distributions of the final state nucleon show a strong preference for low centre-of-mass neutron energies in both $\pi^+\pi^-n$ and $\pi^0\pi^0n$ final states. This effect was not present in the observed proton distribution for the $\pi^-\pi^0p$ reaction which, if one assumes the pion-pion interaction is responsible for the enhancement of these differential cross-sections, corresponds to a dominant I = 0 pion-pion interaction. This is because the $(\pi^+\pi^-)$ state is a combination of all three isotopic spin states I = 0, 1 and 2, and $(\pi^0\pi^0)$ of the I = 0 and 2 states whilst $(\pi^-\pi^0)$ is a combination only of the I = 1 and 2 states.

The observed peaking of the neutron distribution at the lowest neutron centre of mass energy (i.e. the highest available values of the pion-pion energy) was also observed by Kirz et al[16] for the process $\pi^- p \rightarrow \pi^+ \pi^- n$ at incident pion kinetic energies between 350 and 450 Mev. This is a definite deviation from the behaviour expected on the basis of a statistical (phase-space) distribution or a peripheral model calculation. A further point of interest is the apparent absence of any observable effects of the very strong I = 0 low energy pion-pion interaction suggested by the results of Abashan, Booth and Crowe[12]. That is, there is no evidence of a 'bump' in the distributions corresponding to low values of the pion-pion total energy (σ) which is so marked in the ^3He distributions in proton-deuteron scattering. We shall show in this section that these distributions may be explained in terms of the model for the $\pi^- p \rightarrow \pi^+ \pi^- n$ transition amplitude which was proposed in the last chapter.

The model consisted of taking the peripheral contribution for partial waves corresponding to all values of the $\tilde{\sigma} N$ final state orbital angular momentum except the lowest - the f_{1-} partial wave amplitude - which is defined by the $S_1 - \tilde{\sigma} N$ final state and the $P_{1\frac{1}{2}} - \pi N$ initial state. For this partial wave a phenomenological form was proposed which, with the peripheral contributions to the other waves, was able to reproduce the experimental data for the $\pi^- p \rightarrow \pi^+ \pi^- n$ total cross-sections. This was possible because the lack of knowledge of the f_{1-} amplitude for negative values of σ^2 allowed us to introduce a parameter to approximate the left hand integral in equation (2.45). The parameter, we assumed, had the value necessary so that the partial wave sum (2.32) fitted the experimental data. If, as we believe, this model is a good approximation to the $\pi^- p \rightarrow \pi^+ \pi^- n$ transition amplitude at low energies and low momentum transfer, it should be possible to reproduce at least the main features of the neutron energy distributions found by Barish et al at various scattering angles. Also since these distributions are described in terms of variables which define definite values of σ^2, it may be possible to deduce the shape of the pion-pion I = 0, S-wave phase shift $\delta_0^0(\sigma^2)$ near threshold. This was not possible by fitting to the total cross-sections alone since in that case an integration over the whole allowed range of values for σ^2 was required, and the parameter C was used to 'normalise' the result to the experimental value.

The inelastic neutron energy distributions for the process $\pi^- p \rightarrow \pi^+ \pi^- n$ at various values of the scattering angle measured in the laboratory frame of reference are calculated from our model in the following way. If $d\Omega = d(\cos\theta)d\phi$ and $d\Omega_L = d(\cos\theta_L)d\phi_L$ where (θ, ϕ), (θ_L, ϕ_L) are the scattering angles measured in the overall centre of mass frame and the laboratory frame respectively, then in Appendix II it is shown that

$$dT_N \, d\Omega_L = d\sigma^2 \, d\Omega \, \frac{\bar{p}\,\bar{p}'}{2m\, p_N q_L} \qquad - (3.1)$$

where T_N is the kinetic energy of the outgoing neutron and p_N, q_L are the magnitudes of the three momenta of the ingoing pion and outgoing neutron all measured in the laboratory frame of reference. (As before $\bar{p},\bar{p}'$ are the magnitudes of the relative three momenta of the initial pion and proton, and final two pion system $\tilde{\sigma}$ and neutron states measured in the overall centre of mass system.) Therefore from (3.1) and (2.31) we have that

$$\frac{d^2\sigma_x}{dT_N d\Omega_L} = \frac{d^2\sigma_x}{d\sigma^2 d\Omega}\left(\frac{2m p_N q_L}{\bar{p}\bar{p}'}\right)$$

$$= \left\{ \begin{array}{l} \left|\sum_{\ell=0}^{\infty}\left[(\ell+1)f_{(\ell+1)-} + \ell f_{(\ell-1)+}\right] P_\ell(z)\right|^2 \\ + \left|\sum_{\ell=1}^{\infty}\left[f_{(\ell-1)+} - f_{(\ell+1)-}\right]\sin\theta\, P_\ell'(z)\right|^2 \end{array}\right\} \frac{\sqrt{\frac{\sigma^2-4\mu^2}{\sigma^2}}\, 2m p_N q_L}{16\pi^2\bar{p}^2} \qquad - (3.2)$$

where σ_x is the inelastic cross-section defined in terms of quantities measured in the laboratory frame of reference. For the $\pi^- p \rightarrow \tilde{\sigma} N$ partial wave amplitudes we shall use the form derived in the last section, i.e.

$$f_{\ell\pm} = g_1 Q_\ell(z_0) + g_2 Q_{\ell\pm1}(z_0) \qquad - (3.3)$$

for all $\ell+$ and all $\ell-$ except 1-,

where $$g_1 = \tfrac{2}{3}\sqrt{8\pi}\, g\, \bar{p}'\sqrt{\frac{E+m}{E'+m}}\cdot\frac{A_0^0}{8\pi W}$$

and $$g_2 = -\tfrac{2}{3}\sqrt{8\pi}\, g\, \bar{p}\sqrt{\frac{E'+m}{E+m}}\;\frac{A_0^0}{8\pi W} \qquad - (3.4)$$

and for f_{1-} we shall assume the form

$$f_{1-} = \frac{C}{D_0^0(\sigma^2)}\sqrt{\frac{E'+m}{E+m}}\cdot\frac{\bar{p}}{W} \qquad - (3.5)$$

where D_0^0 is the D-function arising in an N over D solution for the pion-pion I = 0, S-wave amplitude A_0^c. From these equations it is clear that each partial wave will have the phase δ_0^0 (the I = 0, S-wave pion-pion phase-shift) since this is the phase of A_0^0

in (3.4) and $1/D_0^{\,0}$ in (3.5). Also each partial wave other than f_{1-} will have a factor $\sin \delta_0^{\,0}$ since this is a factor of $A_0^{\,0}$.

We wish to study the differential cross-sections $d^2\sigma_x/dT_N\, d\Omega_L$ given by the model for various assumed forms of the pion-pion S-wave interaction. In order that the phase-shift has sufficient freedom we use a two pole solution of the N over D equations so that

$$N(\sigma^2) = 16\pi \left\{ \frac{\Gamma_1}{\sigma^2 - \sigma_1^2} + \frac{\Gamma_2}{\sigma^2 - \sigma_2^2} \right\} \quad \text{where } \sigma_1^2, \sigma_2^2 < 0 \qquad - (3.6)$$

Elastic unitarity for the pion-pion amplitude $A_0^{\,0}$ requires that

$$\text{Im.}\, A_0^0 = \frac{1}{16\pi}\sqrt{\frac{\sigma^2 - 4}{\sigma^2}}\ |A_0^0|^2 , \qquad \sigma^2 > 4 \qquad - (3.7)$$

which in terms of the N over D decomposition ($A_0^{\,0} = N/D$) implies that

$$\text{Im.}\, D = -\frac{N}{16\pi}\sqrt{\frac{\sigma^2 - 4}{\sigma^2}} , \qquad \sigma^2 > 4 \qquad - (3.8)$$

so that

$$\text{Re}\, D = 1 - \frac{\sigma^2 - \sigma_1^2}{\pi}\int_4^\infty \sqrt{\frac{\sigma'^2 - 4}{\sigma'^2}}\ \frac{N(\sigma'^2)/16\pi}{(\sigma'^2 - \sigma^2)(\sigma'^2 - \sigma_1^2)}\, d\sigma'^2 \qquad - (3.9)$$

and therefore

$$A_0^0(\sigma^2) = N/D = \frac{N}{\text{Re}\, D - \frac{iN}{16\pi}\sqrt{\frac{\sigma^2 - 4}{\sigma^2}}} = 16\pi\sqrt{\frac{\sigma^2}{\sigma^2 - 4}}\, e^{i\delta_0^0} \sin\delta_0^0 \qquad - (3.10)$$

By taking different values for σ_1^2, σ_2^2, Γ_1, Γ_2, various forms for the amplitude A_0^0 and hence for the phase-shift δ_0^0 may be produced. For each set of values σ_1^2, σ_2^2, Γ_1, Γ_2, we evaluate the partial wave amplitudes (3.3) and (3.5) and determine the value of the parameter C^2 by fitting the partial wave sum (2.32) to the value of the total cross-section given by Barish et al. A selection of these solutions is given in table (3.1) and the phase shifts generated by these pole positions and residues are given in figure (3.1).

Equation (3.2) then allows us to investigate the differential cross-sections $\frac{d^2\sigma_x}{dT_N\, d\Omega_L}$ given by our model for various assumed forms of the pion-pion I = 0, S-wave interaction. In figure (3.2) we show the neutron energy distributions at various angles for the process $\pi^- p \rightarrow \pi^+ \pi^- n$ at 374 Mev incident pion energy which are computed for the phase shifts shown in figure (3.1). It should be mentioned that, in fitting to the total cross-section, only the absolute value of C is determined, not the sign, so that in deriving the differential cross-sections both the values $\pm$ C must be considered. We show in figure (3.2) the distributions corresponding to the sign of C which gives the 'better' agreement with the experimental data.

TABLE 3.1

Solution Number	Pole Positions σ_1^2	σ_2^2	Residues Γ_1	Γ_2	Scattering length a_o^o	$\lvert C \rvert$ at 374 Mev	$\lvert C \rvert$ at 417 Mev
1	-36	-4	120	-36	-1.66	3.68	3.48
2	-36	-4	48	-12	-0.33	6.56	
3	-156	-36	520	-128	+0.32	1.08	
4	-396	-	168	0	+1.29	4.00	

From these results it may be seen that the various two-pole solutions for the pion-pion amplitude give widely differing shapes for the differential cross-sections. One of these, solution 1, is in good agreement with the experimental data. This solution can reproduce the preference the neutron distribution shows for low centre of mass momentum, although it should be noted that it does not reproduce the increase in the differential cross-sections near threshold which is shown by the experimental data at values of the laboratory scattering angles above 30^o. However as discussed by Barish et al this increase is perhaps not due to the process $\pi^- p \rightarrow \pi^+ \pi^- n$ but to some background effects and in general the data below a value of the neutron kinetic energy T_N around 40 Mev should be treated with reservations.

FIGURE 3.1

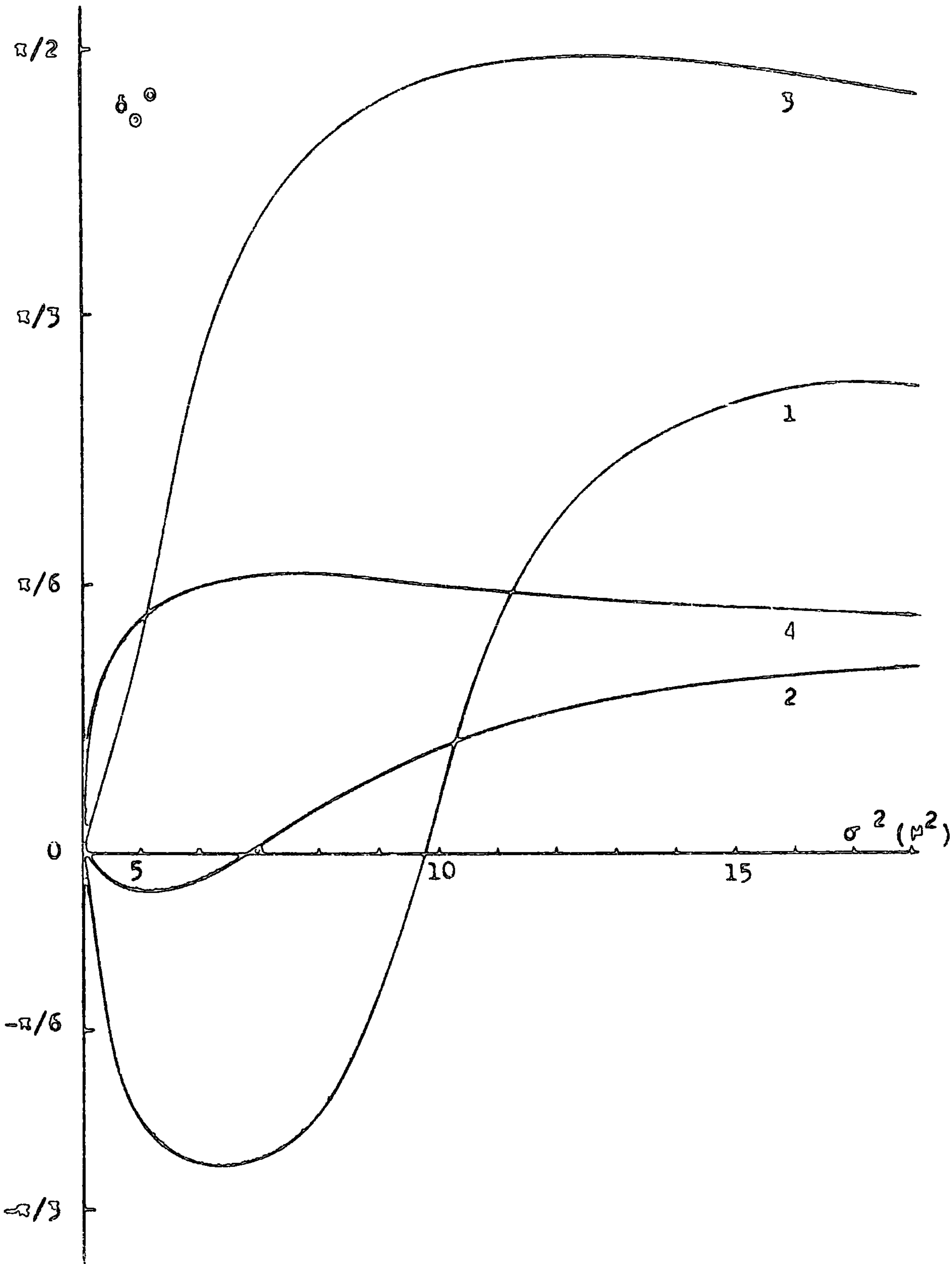
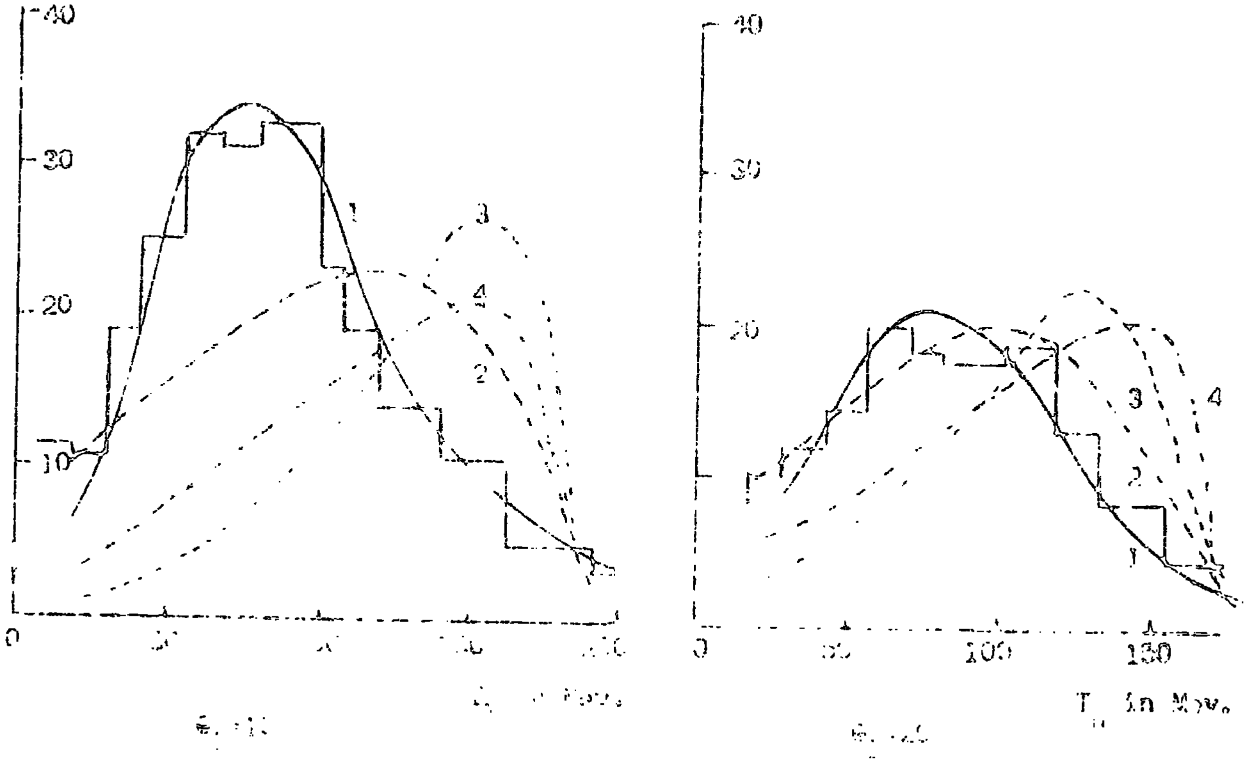


FIGURE 3.2



NEUTRON ENERGY DISTRIBUTIONS

T = 274 Mev.

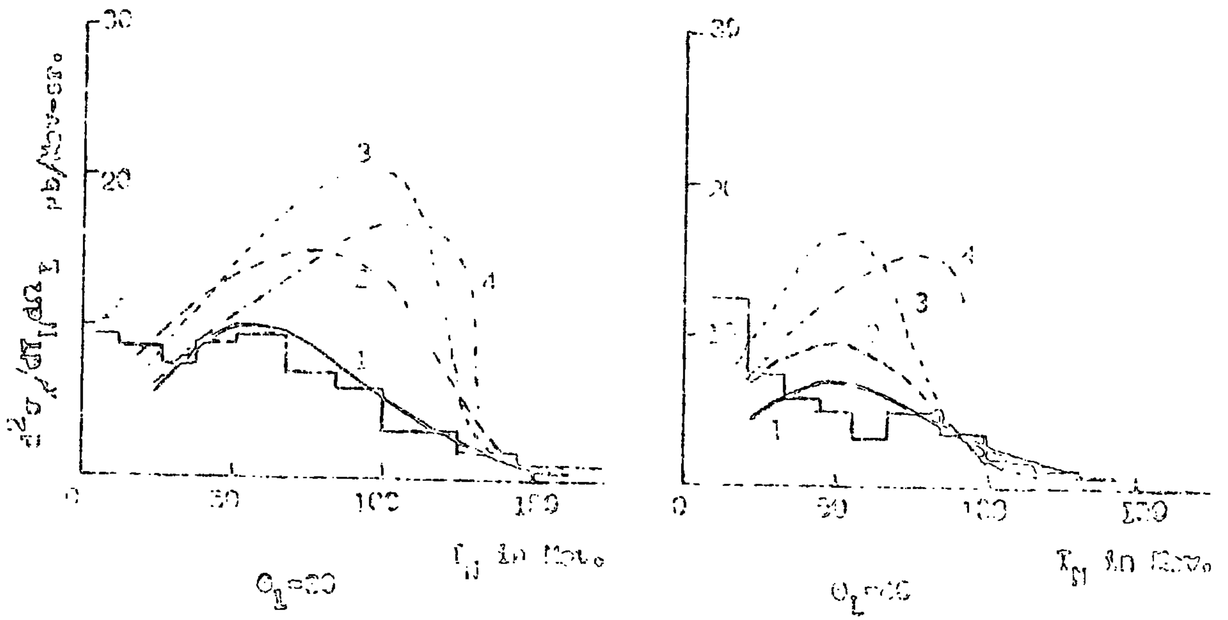
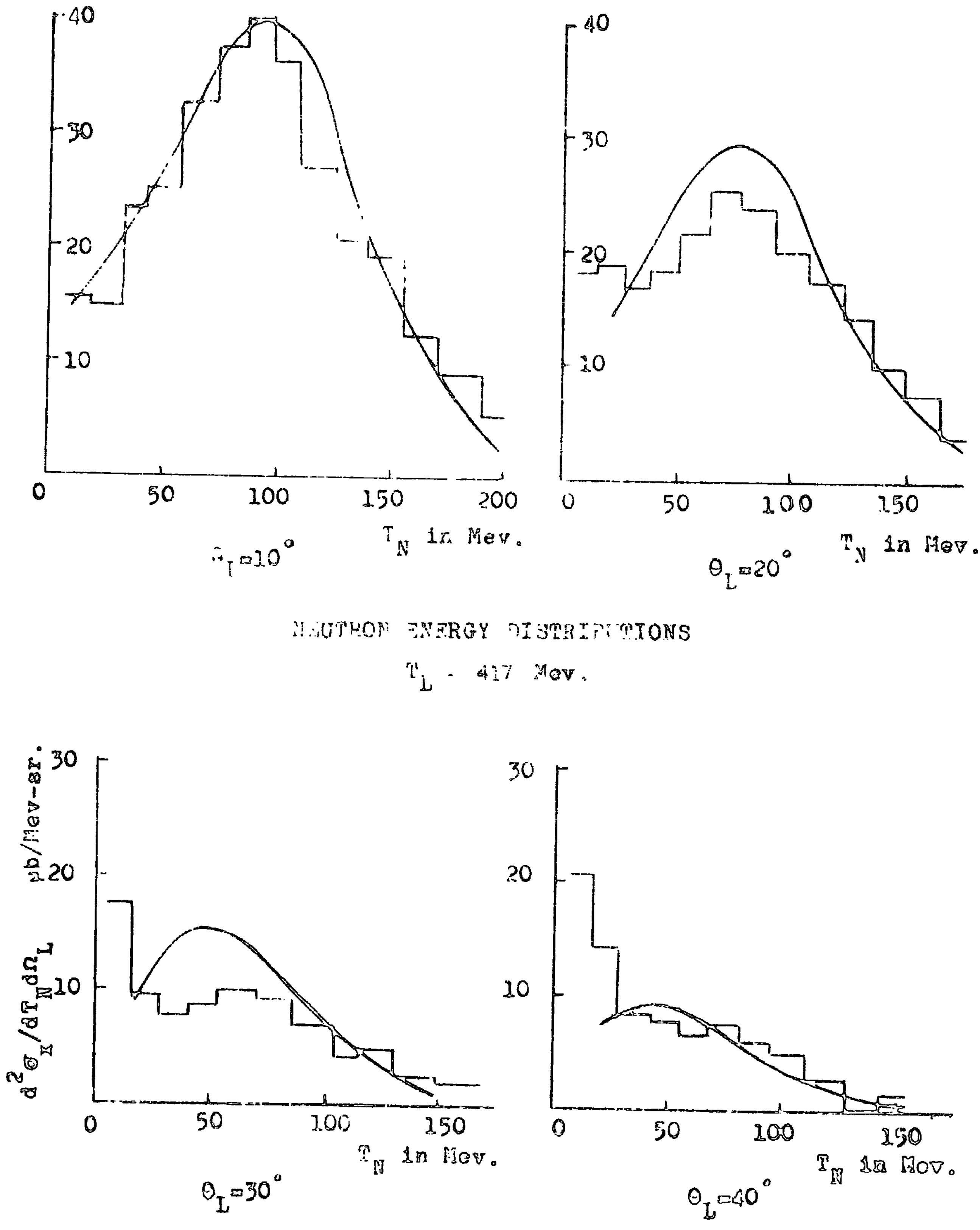


FIGURE 3.3



NEUTRON ENERGY DISTRIBUTIONS

T_L = 417 Mev.

Experimentally the peaks in the inelastic neutron distributions at low centre of mass momenta seem to disappear at incident pion energies above 450 Mev. It has been suggested that this behaviour is suggestive of a nucleon isobar threshold effect rather than a strong I = 0 pion-pion interaction. In the derivation of our model we have considered the final state pion-pion interaction in detail, but unlike the work of Goebel and Schnitzer[17], explicit consideration of the final state pion-nucleon interaction has been neglected. We believed this could be justified so long as we were considering values of the incident pion kinetic energy below 400 Mev, which is the threshold for N*(3,3) production. Even above this threshold, we find there is fair agreement of the data for inelastic neutron distributions at 417 Mev with the distributions computed from our model using the pion-pion parameters of solution 1. These distributions, shown in figure (3.3), appear to indicate therefore that in the context of our model, N* production is not an important contribution to these differential cross-sections at 417 Mev.

The modification of the peripheral model by parameterising the 'lowest partial wave' f_{1-}, which is the amplitude most likely to be poorly approximated by the peripheral interaction alone has enabled us to fit the low energy experimental data quite well using a specific pion-pion phase shift. The detailed consideration of the pion-pion interaction has allowed us to obtain information about the pion-pion phase shift over a range of values of σ^2. This is unlike the work of many authors who use the peripheral model to obtain only an average value of the pion-pion cross-section; and it also differs from the static model of Goebel and Schnitzer which in effect assumes that the S-wave pion-pion cross-section is proportional to λ_s^2/σ^2 and the P-wave cross-section to $\lambda_p^2(\sigma^2 - 4\mu^2)/\sigma^2$ with λ_s, λ_p constants.

It should be noted that because of the simple two pole solution for the pion-pion amplitude and the limited computing time available we are unable to say that the phase shift given by solution 1 is the only type of pion-pion interaction which is compatible with the data, even though from the large number of phase shifts considered this was the only satisfactory solution. Also the approximation of the f_{1-} partial wave in terms of the parameter C would have to be justified more fully before we could safely infer from the model that the differential pion-production cross-sections were consistent with one particular I = 0 S-wave pion-pion interaction.

A close study of the differential cross-sections computed from our model does however indicate some fairly general results. If, as has been supposed, all the $\pi N \rightarrow \tilde{\pi} N$ partial waves except f_{1-} are well approximated by the peripheral interaction then certain approximate bounds can be placed on $|\sin \delta_0^0|$ and $|f_{1-}|$ by fitting to the experimental data. For instance

$$0.3 \leq |\sin \delta_0^0| \leq 0.5 \ , \quad \sigma^2 \sim 5\mu^2 \qquad - (3.11)$$

$$0 \leq |\sin \delta_0^0| \leq 0.6 \ , \quad \sigma^2 \sim 9\mu^2$$

and $|f_{1-}|$ is an increasing function of σ^2 such that

$$2 \leq |f_{1-}(\sigma^2 = 8.8\mu^2)| / |f_{1-}(\sigma^2 = 5.2\mu^2)| \leq 2.5 \text{ at } T_L = 374 \text{ Mev} \qquad - (3.12)$$

These relations hold irrespective of the detailed parameterisation for f_{1-} but are a consequence of

i. assuming all partial waves except f_{1-} are reasonably approximated by the peripheral diagram; and

ii. f_{1-} has the phase δ_0^0 for physical values of σ^2.

The phase shift given by solution 1 is seen to satisfy these bounds and, as the solution which corresponds to the best fit to the differential cross-sections determined from our model (as formulated in the previous sections), it is worthy of some discussion. In the next section we shall compare this I = 0, S-wave phase-shift with results obtained from some other theoretical calculations, and also discuss phase-shifts which are compatible with modifications to the parameterisation of the model.

2. <u>Comparison with some other studies of the Pion-Pion Interaction</u>

The pion-pion I = 0 S-wave phase shift given by solution 1 in the last section has some interesting features.

i. It has a scattering length $a_0^0 = -1.7$ (in natural units).

ii. δ_0^0 passes through zero near $\sigma^2 = 10\mu^2$

and iii. δ_0^0 reaches a maximum of 50° near $\sigma^2 = 16\mu^2$.

The negative scattering length of -1.7 is in agreement with some values which have been obtained from forward pion-pion elastic scattering dispersion relations. For instance L J Rothe[20] evaluated the high energy contribution to the dispersion integral by assuming that the high energy behaviour of the forward scattering amplitude is dominated by a few leading crossed-channel Regge poles and used the available experimental information on the total cross-sections to compute the low energy contribution. He found the scattering length a_0^0 to have a value of $-1.7\,^{+1.3}_{-0.5}$. As a check on his method he evaluated the I = 1 S-wave amplitude at threshold and found a_0^1 to have the small value of -0.4 compared to the value of zero imposed by the Pauli Principle.

Pisut, Bona and Lichard[19] have also computed the S- and P- wave pion-pion scattering lengths from dispersion relations for forward scattering. Their method damps the high energy contribution to the dispersion integral but instead requires a knowledge of the amplitude at $\sigma^2 = 2$ (i.e. a subtraction constant). This constant was calculated from dispersion relations using the pion-pion phase shifts of Wolf[22] which, it should be noted, contain the ε^0(720 Mev) - an S-wave resonance. They obtained the results $a_0^0 = -1.3 \pm 0.6$, $a_0^2 = 0.38 \pm 0.2$ and $a_1^1 = 0.037 \pm 0.004$. Pisut, Bona and Lichard also considered the sum rules proposed by Adler[28] from Current Algebra considerations which relates the g_A/g_V ratio to the $\pi\pi$ cross-section, i.e.

$$1/g_{A}/g_{V} = \left(\frac{2m}{g_r K(0)}\right)^2 \frac{1}{4\pi} \int_{4\mu^2}^{\infty} \frac{d\sigma'^2}{\sigma'^2 - \mu^2} \left(\sigma_0^{+-}(\sigma'^2) - \sigma_0^{++}(\sigma'^2) \right) \qquad - (3.13)$$

where σ_0^{+-} (σ_0^{++}) is the total cross-section for scattering of a zero mass π^- (π^+) on a physical π^+ meson, and g_r is the rationalised renormalised πN coupling constant. They observed that if the above values for the scattering lengths are used this sum rule may be satisfied with an I = 0 S-wave phase shift which is initially negative then turns over and becomes large and positive in the region $\sigma^2 \approx 25\mu^2$. This is similar to the phase shift given by our solution 1.

It should also be said that there are certain similarities between properties i. and ii. of this solution 1 and the phase-shifts obtained by Lovelace, Heinz and Donnachie[21] using dispersion relations for πN scattering in the backward direction. They used a method due to Atkinson[29] and considered the relation

$$\mathrm{Re}\, F(\nu) = \frac{g^2}{\nu - \nu_0} + \frac{P}{\pi} \int_0^{\infty} \frac{\mathrm{Im}\, F(\nu')}{\nu' - \nu} d\nu' + 1/\pi \int_{-\infty}^{-\mu^2} \frac{G(\nu')\, d\nu'}{\nu' - \nu} \qquad - (3.14)$$

where $\nu = q^2$, the square of the relative three momenta and $\nu_o = -\mu^2 + \mu^2/4m^2$ is the position of the single nucleon pole. Im $F(\nu')$ in the second term is the imaginary part of the backward scattering πN amplitude and $G(\nu')$ is the backward amplitude for the process $\pi\pi \rightarrow N\bar{N}$. The 'discrepancy' which is the last term on the right hand side of (3.14), is obtained by using the experimental values for the other terms.

Lovelace, Heinz and Donnachie used the data on the πN phase shifts up to 600 Mev and the backward differential cross-sections for higher energies to put bounds on the amplitudes. Using the (+) isospin combination which gives the I = 0 $N\bar{N} \rightarrow \pi\pi$ amplitude on the left hand cut, the phase between the $\pi\pi$ and $N\bar{N}$ thresholds may be identified with $\delta_o^{\,o}$ if one ignores inelastic effects and the contributions from the higher pion-pion partial wave phase-shifts. They found from this that two types of phase-shift $\delta_o^{\,o}$ satisfy the experimental information: one with a negative scattering length which turns up through zero, and the other with a positive scattering length. The former phase-shift, therefore, has properties similar to i. and ii. of our solution 1. However they also found that both types of phase shift resonate (i.e. pass through $\pi/2$) which is contrary to property iii. of our solution 1. This they conclude is evidence 'beyond all reasonable doubt' for the S-wave σ-resonance.

In connection with the above it should be noted that the property iii. of our solution 1 is not really determined by fitting the pion-production cross-sections, which uses values of δ_0^0 from threshold to σ = 420 Mev, but comes from our parameterisation of the pion-pion amplitude in terms of a simple two-pole N function. It is possible that by using a more sophisticated analysis for the pion-pion S-wave amplitude rather than this two pole approximation there could exist a resonance above 420 Mev but below this value the phase-shift and D-function would be similar to the two pole solution. There is the further possibility that if the σ-resonance does exist and the phase-shift is of the 'turn-over' type then by analogy with the πN-P_{11} amplitude there may exist a CDD pole in the pion-pion scattering amplitude . This CDD pole would affect the parameterisation, and perhaps the results, of our model and we shall consider this possibility in the next section.

There have been numerous other theoretical predictions of the I = 0 S-wave pion-pion interaction, many of them in disagreement with our solution 1. We shall discuss these in some detail later. However, the corroboration of the results obtained from our model for pion-production and the results of forward pion-pion dispersion relations encourages us to take another look at two important methods of obtaining information on the pion-pion amplitude, namely the pion-nucleon partial wave dispersion relation 'discrepancy' analysis and the ABC effect in the ^{3}He distributions in proton-deuteron scattering, to see whether the results of these analyses could be compatible with the solution 1.

For the 'discrepancy' analysis we use the dispersion relation derived by T D Spearman[11] for S-wave pion-nucleon scattering amplitudes. These relations emphasise the better known low energy pion-nucleon data (in particular the scattering lengths) and stress the low energy contribution from the two pion exchange term. This is done by defining the function

$$g_0^I(s) = f_0^I(s) / B(s) \qquad - (3.15)$$

where

$$B(s) = \left[\left(s - (m-1)^2\right)\left((m+1)^2 - s\right) \right]^{1/2} \qquad - (3.16)$$

and f_0^I is the πN S-wave scattering amplitude with isospin I in which case by considering the singularities of $f_0^I(s)$ and B(s) we can write

$$\mathrm{Re}\, g_0^I(s) - \frac{1}{\pi}\int_{(m+1)^2}^{\infty} ds' \frac{\mathrm{Re}\, f_0^I(s')}{|B(s')|\,(s'-s)} - \frac{1}{\pi}\int_{(m-1/m)^2}^{m^2+2} ds' \frac{\mathrm{Im}\, f_0^I(s')}{|B(s')|\,(s'-s)} + \frac{1}{\pi}\int_{0}^{(m-1)^2} ds' \frac{\mathrm{Re}\, f_0^I(s')}{|B(s')|\,(s'-s)}$$
$$= -\frac{1}{\pi}\int_{-\infty}^{0} ds' \frac{\mathrm{Re}\, f_0^I(s')}{|B(s')|\,(s'-s)} + \frac{R_0^I}{s} + \frac{1}{2\pi i}\int_{\text{circle}} ds' \frac{\Delta f_0^I(s')\, e^{-i\phi/2}}{|B(s')|\,(s'-s)} = \mathcal{D}^I(s) \text{ (say)} \qquad - (3.17)$$

where $\Delta f_0^I(s')$ is the discontinuity in $f_0^I(s')$ across the circle $s = (m^2 - 1)e^{i\phi}$, and R_0 is the residue of a possible pole at the origin. Since

$$\mathrm{Re}\, g_0^I = -\mathrm{Im}\, f_0^I / B \quad \text{for real } s \geq (m+1)^2$$
$$= +\mathrm{Im}\, f_0^I / B \quad \text{for real } s \leq (m-1)^2 \qquad - (3.18)$$

all the terms on the left hand side of (3.17) can be evaluated in terms of the low energy pion-nucleon data for values of s above $(m + 1)^2 = 59.6$ to 76 (say) and by using the crossing relations for values of s below $(m - 1)^2 = 32.2$ to 20 (say). The left hand side of (3.17) therefore can be evaluated in the regions

$$20 \leq s \leq 32.2$$

$$\text{and} \quad 59.6 \leq s \leq 76 \qquad - (3.19)$$

The circle cut arises from the crossed channel process $\pi\pi \to N\bar{N}$ in such a way that low energies in this channel correspond to a range of values on the front of the circle. Thus the front of the circle $|\phi| \leq 60^\circ$ is restricted to low energy crossed channel reactions and arises primarily from the two pion exchange. The contribution to the discrepancies $\mathcal{D}^{\pm}(s)$ given by this part of the circle to the third term on the right hand side of (3.17), $\mathcal{D}^{\pm}_{\pi\pi}(s)$, may be written as

$$\mathcal{D}^{\pm}_{\pi\pi}(s) = \int_{4\mu^2}^{t_{max}} dt' \sum_i K_i^{\pm}(s,t') \, \text{Im}\, f_i^{\pm}(t') \qquad - (3.20)$$

where $K_i^{\pm}(s,t')$ are appropriate kernels and $f_i^{\pm}(t')$ are the relevant helicity amplitudes for the 'crossed' process $\pi\pi \to N\bar{N}$. It can be argued that this term $\mathcal{D}_{\pi\pi}(s)$ should contain the predominant energy dependent effect of the discrepancies $\mathcal{D}(s)$ in the regions (3.19) corresponding to the nearby singularities, i.e. the long range forces. It may be reasonable therefore to approximate the other terms on the right of equation (3.17) by a constant and the energy dependence of the left hand side may be equated to $\mathcal{D}_{\pi\pi}(s)$. In turn this may provide information on the pion-pion phase-shift δ_J^I since by unitarity, the phase of the helicity amplitude $f_{\pm}^{J}(t)$ in the region $4\mu^2 \leq t \leq 16\mu^2$ will be $\delta_J^I(t)$. This is also the phase of

$(D_J^{\ I})^{-1}$ where $D_J^{\ I}(t)$ is the D-function arising in an N over D solution for the pion-pion scattering amplitude. $\mathcal{D}^+(s)$ corresponds to isospin I = 0 in the $\pi\pi \to N\bar{N}$ channel in which, from the Pauli Principle only even values of the angular momentum J occur. For low energies in this channel, therefore, one might expect only the S-wave, $f_+^{\ 0}$, should be important so we attempt to fit the data for $\mathcal{D}^+(s)$ using only this S-wave term. For this case $f_+^{\ 0}(t)$ and $D_0^{\ 0}(t)^{-1}$ have the same phase on the right hand cut so that the product $f_+^{\ 0}.D_0^{\ 0}$ has only the left hand singularities. Writing a dispersion relation for the product f.D with two subtractions to improve the convergence we obtain

$$f_+^0 = {}^1/D_0^0(t) \left[\begin{array}{l} D_0^0(0)\,\mathrm{Re}\,f_+^0(0) + t\,\frac{\partial}{\partial t''}\left[D_0^0(t'')\,\mathrm{Re}\,f_+^0(t'')\right]_{t''=0} \\ + \frac{1}{\pi}\,t^2\,\partial/\partial t'' \displaystyle\int_{-\infty}^{4-1/m^2} dt' \left.\frac{D_0^0(t')\,\mathrm{Im}\,f_+^0(t')}{(t'-t)(t'-t'')}\right|_{t''=0} \end{array} \right] \quad - (3.21)$$

where the subtraction constants have been calculated by Menotti,[30] and Im $f_+^{\ 0}(t)$ is determined for $0 \leqslant t \leqslant 4 - 1/m^2$ by the Born term and for $t < 0$ by the analytic continuation of the pion-nucleon data. We can now compare the right hand side of equation (3.17) with the left hand side given by a constant and equation (3.20) using (3.21) for different forms of the pion-pion D-function.

When a one pole approximation to the N/D pion-pion dispersion relations was used reasonable agreement was found for a pion-pion interaction corresponding to solution 4 of the last section. However, the one pole approximation is too restrictive to produce any but the simplest type of phase-shifts. It cannot, for instance, produce a phase-shift which passes through zero. Therefore, in repeating Spearman's analysis we use a two pole approximation to the pion-pion amplitude similar to that considered in the last section. With this extra parameterisation of the pion-pion amplitude, three general shapes for the phase shift $\delta_0^{\,o}$ are found to give agreement with the discrepancy data which is as good, if not better, than that found by Spearman. The pole positions and residues for these satisfactory phase shifts (which are sketched in figure (3.4)) are given in table (3.2). Figure (3.5) shows the discrepancies $\mathcal{D}^{+}(s)$ generated by these three pion-pion solutions.

Note that solution (a) is similar to solution (1) of the last section so that a phase-shift which starts negative (with a negative scattering length) and then becomes positive could be compatible with both the pion-production data and the forward pion-nucleon dispersion relation analysis. This 'turn-over' type of phase-shift was previously suggested by Hamilton et al who employed the conformal mapping

$$\eta = \frac{1 - (v + 1)^{\frac{1}{2}}}{1 + (v + 1)^{\frac{1}{2}}}$$

to transform the physical sheet of the v (= q^2) plane into the interior of the unit circle $|\eta| \leq 1$. By approximating $N_0^{\,o}(v)$ by $\alpha_0 + \alpha_1\eta$ they found two phase-shifts similar to our solutions (a) and (b) satisfied their discrepancy analysis. Spearman, in his

FIGURE 3.4

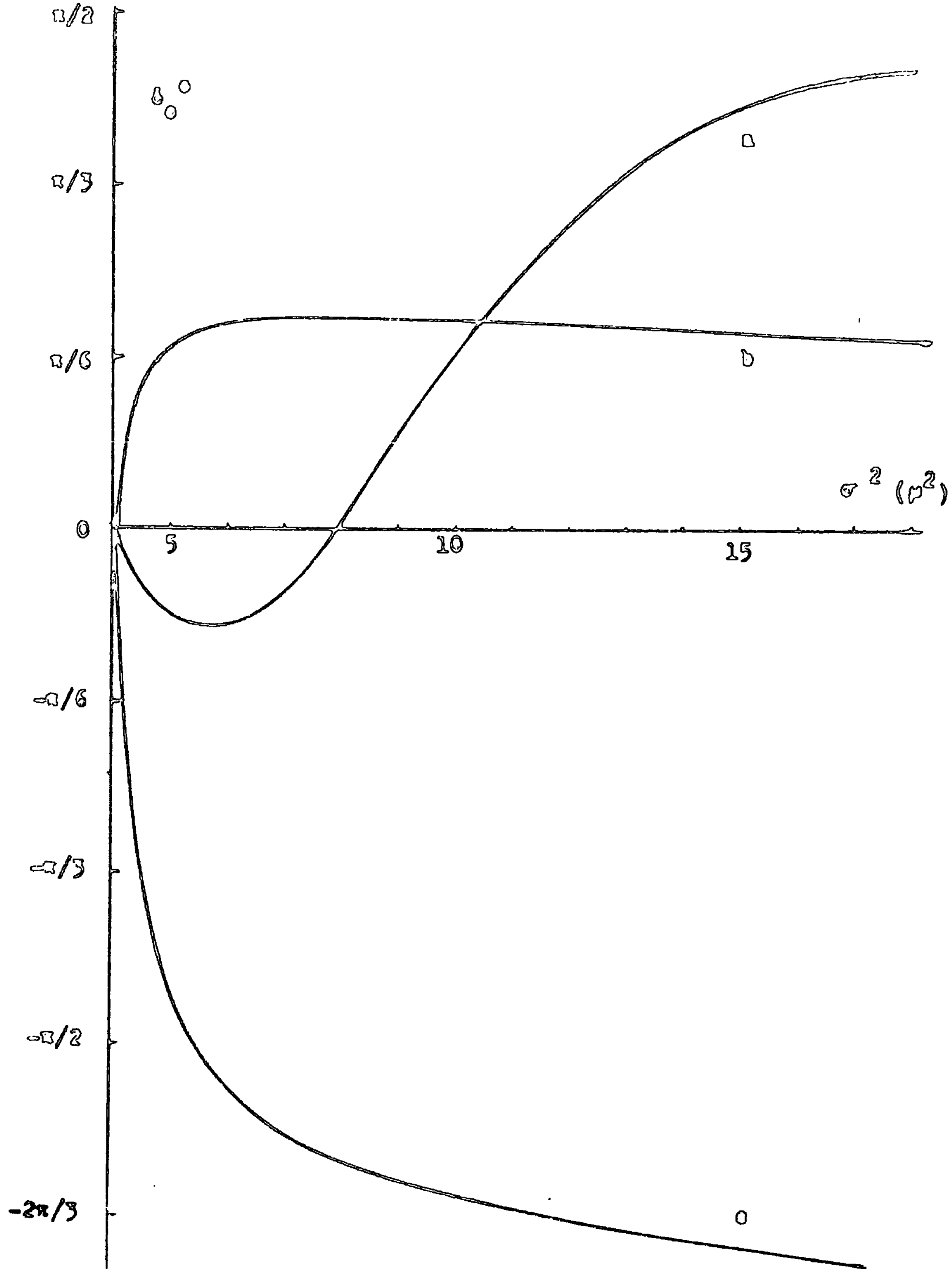


FIGURE 3.5

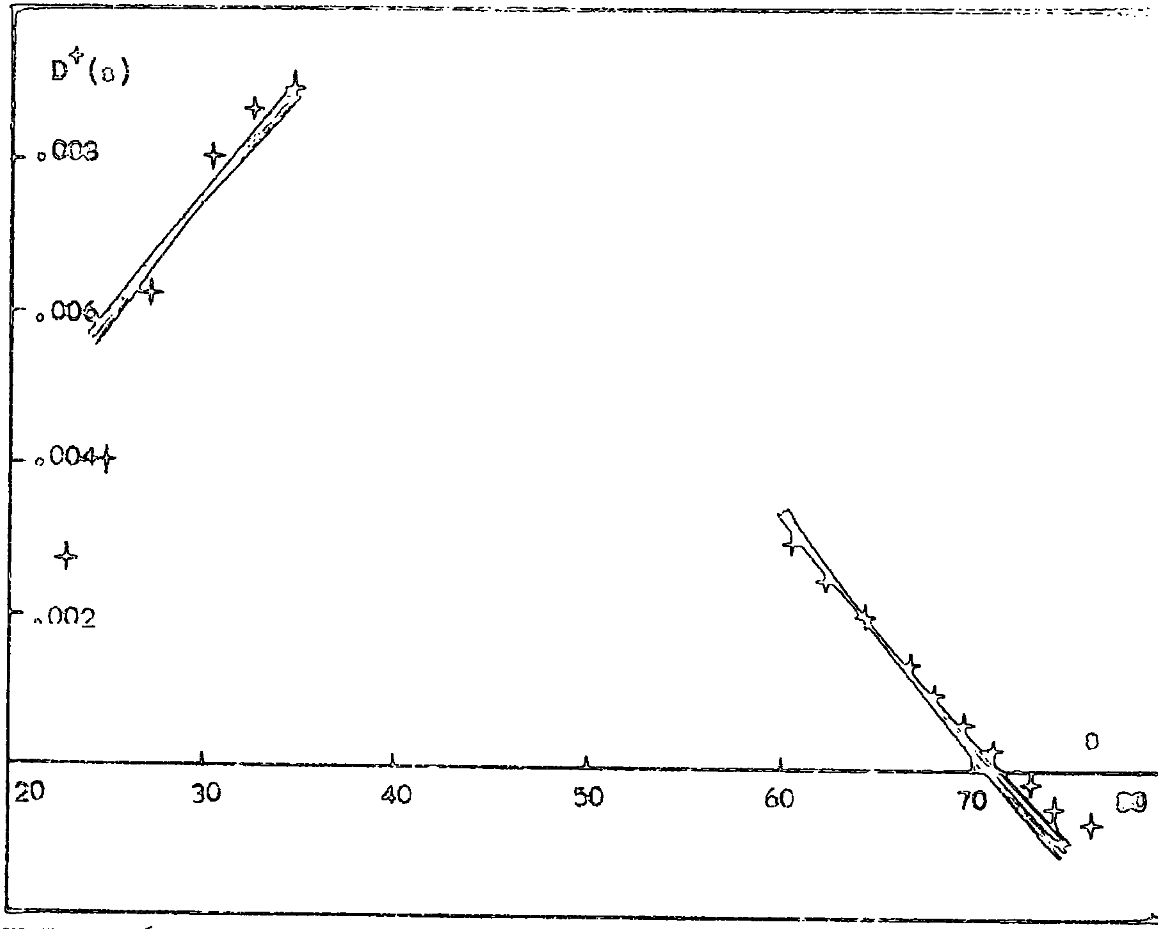
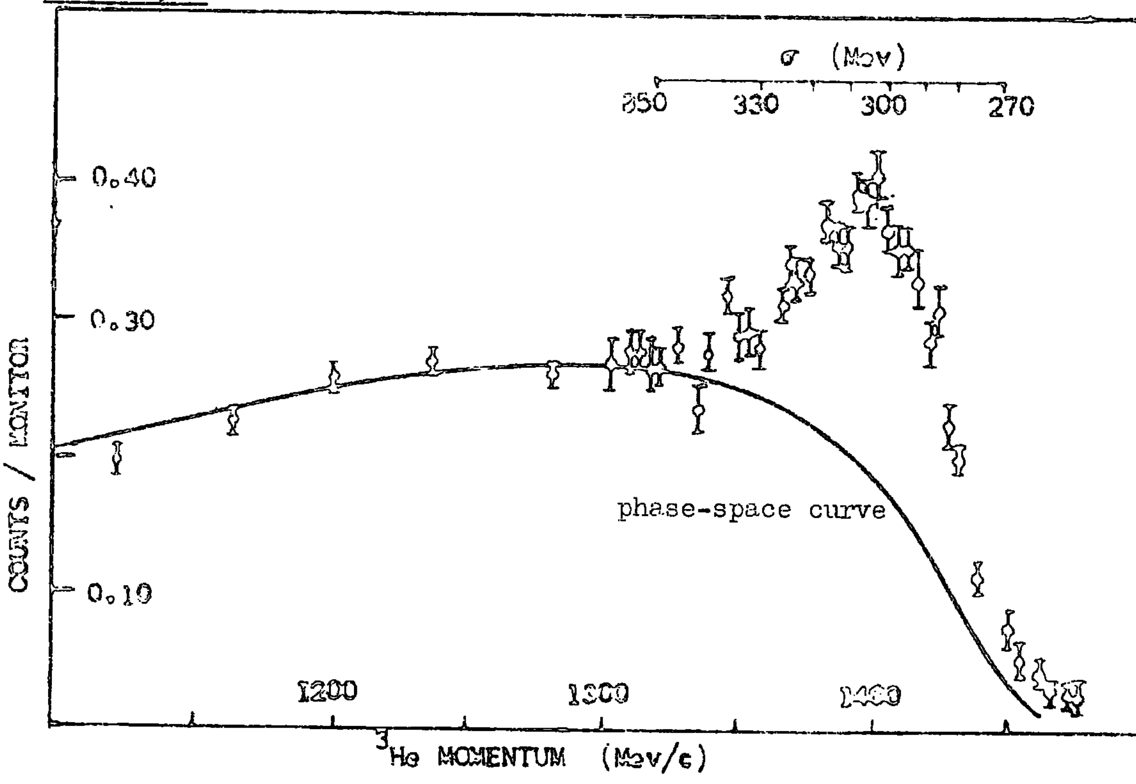


FIGURE 3.6



analysis on the other hand, rejected the 'turn-over' solution since it did not give such a close fit to the discrepancy data as the one pole N-function approximation. However, by increasing the parameterisation of the pion-pion amplitude all three solutions (a), (b) and (c) are seen to give satisfactory agreement with the data.

TABLE 3.2

Solution Number	Pole Positions σ_1^2	Pole Positions σ_2^2	Residues Γ_1/σ_1^2	Residues Γ_2/σ_2^2	Scattering length a_0^0
a	-42	-450	0.7	-6	-0.4
b	-42	-450	0.7	-2.5	1.9
c	-42	-450	0.8	-2	-6.8

The other evidence for the I = 0 S-wave pion-pion interaction which we shall consider in this section is that coming from the experiments of Abashian et al on $p + d \rightarrow {}^3He + 2\pi$ reactions. They measured the recoil spectum of ${}^3He$ ions produced at a fixed angle and fixed energy of the incident proton and found there was a pronounced peak above the phase-space curve in the momentum range corresponding to energies just above the two pion threshold. No such peak was observed in the analogous experiment for $p + d \rightarrow 3H + 2\pi$ implying that if the peak is caused by a pion-pion interaction, this interaction is in an I = 0 state.

To analyse the data many authors have expressed the final state interaction by an enhancement factor modifying the phase-space distribution[31,32]. This enhancement factor is obtained in the same way as we derived a phenomenological form for the $\pi N \to \tilde{\sigma} N$ partial wave $h_{1-}(\sigma^2)$. The amplitude for the process $p + d \to {}^3He + 2\pi$ considered as a function of σ^2, $A(\sigma^2)$, will have a branch cut for values of $\sigma^2 \geq 4\mu^2$, where as before σ^2 is the square of the total centre of mass energy of the two pions. If the only final state interaction considered is the two pion interaction then the phase of this amplitude for $4\mu^2 \leq \sigma^2 \leq 16\mu^2$ from unitarity will be δ_0^0; where $16\mu^2$ is the first inelastic threshold. By defining the function

$$D(\sigma^2) = \exp\left[-\frac{(\sigma^2-\sigma_s^2)}{\pi}\int_{4\mu^2}^{\infty} d\sigma'^2 \frac{\delta_0^0(\sigma'^2)}{(\sigma'^2-\sigma^2)(\sigma'^2-\sigma_s^2)}\right] \qquad - (3.22)$$

and ignoring inelastic effects, the product A.D has only the left hand singularities in σ^2, since D has the phase $-\delta_0^0$ on the right, so that we can write

$$A(\sigma^2) = \frac{1}{D(\sigma^2)}\int_{L.H.} \frac{D(\sigma'^2)\, \mathrm{Im}\, A(\sigma'^2)}{\sigma'^2-\sigma^2}\, d\sigma'^2 \qquad - (3.23)$$

with possible subtractions to ensure the convergence of the integral.

In deriving a phenomenological form for the $\pi N \to \tilde{\sigma} N$ partial wave h_{1-} the integral in equation (2.45) which is analogous to (3.23) was replaced by a constant. Also in the 'discrepancy' analysis equation (3.23) was discussed in the form (3.21). In that case, however, the integral could be evaluated by making an analytic continuation of the experimental pion-nucleon data to determine the $\pi\pi \rightarrow N\bar{N}$ helicity amplitude Im $f_+^{\,o}(t')$. For the process $p + d \rightarrow {}^3He + 2\pi$ the usually assumed forms for the phase-space enhancement factors are

i. $|A_o^{\,o}|^2$ where $A_o^{\,o}$ is the pion-pion I = 0 S-wave amplitude

or ii. $|C/D_o^{\,o}(\sigma^2)|^2$ where $D_o^{\,o}(\sigma^2)$ is the D-function arising in an N/D solution for the amplitude $A_o^{\,o}$ and C is a constant. These enhancement factors are equivalent to approximately the integral in (3.23) by $N_o^{\,o}(\sigma^2)$ in i. (where $N_o^{\,o}(\sigma^2)$ is the N-function corresponding to $D_o^{\,o}(\sigma^2)$) and by a constant C in ii. It has been pointed out by Spearman" that since $A_o^{\,o} = N_o^{\,o}/D_o^{\,o}$, so long as $N_o^{\,o}$ is a slowly varying function there will be little difference between i. and ii., i.e. between N/D and C/D. However, if the N-function is not slowly varying i. and ii. will provide very different types of enhancement factor, and there is no way of telling if one type is preferable to the other. Unfortunately if the phase shift $\delta_o^{\,o}$ passes through zero (such as solution 1 of the last section and solution (a) of this) then assuming the absence of CDD poles, the N-function will pass through zero also, and hence is not a slowly varying function to be well approximated by a constant. Thus there is no way of telling which of the enhancement factors i. or ii. - if either - we should use to examine the compatibility of the 'turn-over' type of phase-shift with the ABC effect.

One remark might be made here about the ABC effect. The experimental data sketched in figure (3.6) seem to show a 'dip' at a value of $\sigma = 340$ Mev. This 'dip' as well as the threshold peak could be reproduced by an enhancement factor of type i. for a phase-shift which has a large negative scattering length and passes through zero at $\sigma = 340$ Mev, when one 'folds in' the experimental resolution. Thus if one favours the enhancement factor of type i. it should be possible for the 'turn-over' type of phase-shift to reproduce even the small details of the data. However it must be stressed that the enhancement factor must be justified much more fully before any information on the pion-pion interaction can be deduced.

This method of deducing enhancement factors for final state effects by equations of the form (3.22) and (3.23) always has this arbitrariness unless the integral over the left hand singularities is known, as in the case of equation (3.21) for the discrepancies analysis.

Presumably a better approximation to the h_{1-} $\pi N \rightarrow \tilde{\sigma} N$ partial wave amplitude and the $p + d \rightarrow {}^3He + 2\pi$ amplitude therefore would be to replace the integral over the left hand singularities in (2.45) and (3.23) by one or more poles. In this case we would obtain

$$h_{1-}(\sigma^2) = \frac{1}{D_0^0(\sigma^2)} \sum_i \left(\frac{\Gamma_i'}{\sigma^2 - \sigma_i'^2} \right)$$

$$\text{and} \qquad A(\sigma^2)_{p+d \to {}^3He + 2\pi} = \frac{1}{D_0^0(\sigma^2)} \sum_i \left(\frac{\Gamma_i''}{\sigma^2 - \sigma_i''^2} \right) \qquad - (3.24)$$

In the absence of information on these left hand singularities we can consider the residues and positions of these poles to be arbitrarily chosen to give good agreement with the experimental data for any of the three types of phase shift compatible with the discrepancy analysis, i.e. solutions (a), (b) or (c).

Note that the phase-shifts (a) and (b) satisfy equation (3.11), the bounds on the pion-pion phase-shift $\delta_0^{\,0}$ obtained from the pion-production data which are independent of the detailed parameterisation of h_{1-}, the P_{11} $\pi N \to \tilde{\pi} N$ partial wave. The third phase-shift (c) however, because of its rapidly decreasing behaviour would produce a sharp peak in the $\pi^- p \to \pi^+ \pi^- n$ differential cross-sections near $\sigma^2 = 5\mu^2$. This would occur since all the $\pi N \to \tilde{\pi} N$ partial waves other than the P_{11} wave are approximated by the peripheral contributions which are proportional to Sin $\delta_0^{\,0}$. As $\delta_0^{\,0}$ passes through $-\pi/2$ and approaches $-\pi$ these partial waves would oscillate between zero and some maximum value, thus producing a peak in the differential distributions. This peak is not observed experimentally. Of course, if the phase-shift decreases <u>very</u> rapidly this peak may be so sharp that it would be difficult to detect within the experimental resolution. However this would correspond to an unrealistically large scattering length and would probably not satisfy the Wigner[33] condition on the slope of the phase-shift. We conclude therefore that this type of phase shift (c) is not compatible with the pion-production data.

3. A CDD Pole in the I = 0 S-Wave Pion-Pion Interaction?

In the last two sections we discussed at some length the information on the low energy pion-pion interaction which may be inferred from

i. the pion production data using the model proposed in Chapter II

ii. pion-nucleon partial wave dispersion relation analysis

and iii. the ^{3}He spectrum in proton-deuteron collisions.

By making the simplest parameterisation of the P_{11} $\pi N \rightarrow \sigma N$ partial wave in terms of the final state pion-pion interaction, a 'turn-over' type of phase-shift was seen to give reasonable fits to the energy distributions of the inelastic neutron in the $\pi^- p \rightarrow \pi^+ \pi^- n$ reaction. Also this type of phase-shift was one of three which were seen to give satisfactory agreement with the 'discrepancy' analysis of the pion-nucleon scattering data. However it was noted that no information could be deduced from the ABC effect in the process $p + d \rightarrow {}^3He + 2\pi$ unless assumptions were introduced about the form of the enhancement factor used to modify the phase-space contribution. It was pointed out that these assumptions were similar to those made in the parameterisation of the P_{11} $\pi N \rightarrow \sigma N$ partial wave in Section 4 of Chapter II, and that since we have no information about the integrals over the left hand singularities in equations (2.45) and (3.23) we may arbitrarily assume their forms to be those required to fit the data.

We see therefore that the two phase-shifts (a) and (b) given by the discrepancy analysis which satisfy the farily general bounds (3.11) could be compatible with the $\pi^- p \rightarrow \pi^+ \pi^- n$ differential cross-sections and the ABC effect.

So far in this discussion of a two-pion final state interaction in the various processes

$$\begin{aligned} &\pi^- p \rightarrow \pi^+ \pi^- n \\ &N\bar{N} \rightarrow \pi \pi \\ &pd \rightarrow {}^3He + 2\pi \end{aligned} \qquad - (3.25)$$

we have ignored the possibility of CDD poles[34] in the I = 0 S-wave pion-pion amplitude. This possibility would affect the various mathematical models used to extract information on the pion-pion interaction from these processes and may also produce rather different results. The effect that a CDD pole would have on the various models may be demonstrated in the following way.

For the case of elastic scattering of a state $|a\rangle$ to a state $|a\rangle$ the D-function in an N over D decomposition for the amplitude, is constructed to have only the right hand singularities (arising from elastic unitarity) and the N-function has the remaining singularities - necessarily on the left. However when the state $|a\rangle$ can scatter into other states $|b\rangle$ the unitarity condition implies the existence of other right hand branch points (and branch cuts) corresponding to the thresholds for inelastic processes. Bjorken[35] has extended the N over D decomposition to construct the coupled scattering amplitude for these processes so that they satisfy analyticity and unitarity. He writes the amplitude for the scattering process $i \rightarrow j$, $A_{ij}(\sigma^2)$, in terms of matrix N and D functions as

$$A_{ij}(\sigma^2) = N_{ik}(\sigma^2)\left\{D(\sigma^2)\right\}^{-1}_{kj} \qquad - (3.26)$$

where the N-matrix elements N_{ij} have only the left hand singularities and the D-matrix elements have only the right hand singularities such that

$$D_{ij}(\sigma^2) = \delta_{ij} - \frac{\sigma^2-\sigma_s^2}{\pi}\int_{\sigma_{T_i}^2}^{\infty} \frac{N_{ij}(\sigma'^2)\,\rho_j(\sigma'^2)\,d\sigma'^2}{(\sigma'^2-\sigma^2)(\sigma'^2-\sigma_s^2)} \qquad - (3.27)$$

where $\rho_j(\sigma^2)$ are kinematical factors (such as $+\frac{1}{16\pi}\sqrt{\frac{\sigma^2-4}{\sigma^2}}$ for pion-pion scattering). The subscripts i and j range over all the coupled channels, but for the present example it will be sufficient to consider only two channels.

If we denote the two-pion state by the subscript "1" then the pion-pion amplitude $A(\pi\pi \rightarrow \pi\pi)$ with inelasticity may be written as

$$A_{11} = \frac{N_{11}D_{22} - N_{12}D_{21}}{D_{11}D_{22} - D_{12}D_{21}} \qquad - (3.28)$$

and by identifying "2" with either $\pi^- p\ \overline{n}$ or $N\bar{N}$ or pd '^{3}He' we may denote any of the amplitudes for the processes (3.28) as a function of σ^2 by

$$A_{21} = \frac{N_{21}D_{22} - N_{22}D_{21}}{D_{11}D_{22} - D_{12}D_{21}} \qquad - (3.29)$$

In most cases if the coupling between channels is very weak (i.e. if to a good approximation one can ignore inelastic effects) D_{12} and D_{21} will be small so that

$$A_{11} \approx N_{11}/D_{11}$$

$$\text{and} \quad A_{21} \approx N_{21}/D_{11} \qquad - (3.30)$$

so long as D_{22} is of the order of unity. (Note that in this case N_{21} is approximately the left hand integral in (2.45), (3.21) or (3.23) depending on which of the three processes (3.25) we are considering, and the two enhancement factors for the ABC effect discussed in the last section are found by taking N_{21} to be equal to N_{11} or a constant.) However even in the case of weak coupling between channels if there is a zero in D_{22} at some point σ_a^2, then the small terms D_{12}, D_{21} etc. near σ_a^2 must be of the same order as D_{22}. Using a linear form for D_{22} we can write

$$A_{11} \approx \frac{N_{11} - \frac{n_{11}}{\sigma^2 - \sigma_a^2}}{D_{11} - \frac{d}{\sigma^2 - \sigma_a^2}} \qquad - (3.31)$$

$$A_{21} \approx \frac{N_{21} - \frac{n_{21}}{\sigma^2 - \sigma_a^2}}{D_{11} - \frac{d}{\sigma^2 - \sigma_a^2}} \qquad - (3.32)$$

In other words pole terms - known as CDD poles - are introduced into the one channel N and D functions.

It is perhaps more usual to call a pole a CDD pole if it is inserted arbitrarily into the single channel N or D functions rather than into both the N and D functions[36,37]. However in the fuller discussion of this problem which we give in the Appendix it is shown that all three ways of inserting CDD poles into the one channel N over D equations may be equivalent under certain conditions.

From the above simple illustration using the two channel N over D equations it may be seen that if there exists a CDD ambiguity in pion-pion I = 0 S-wave scattering, the D-function we must use is

$$\tilde{D}_0^{\,0} = D_1 - \frac{d}{\sigma^2 - \sigma_a^2} \qquad - (3.33)$$

where D_1 is given by

$$D_1 = 1 - \frac{\sigma^2 - \sigma_s^2}{\pi} \int_4^{\infty} \frac{d\sigma'^2}{16\pi} \sqrt{\frac{\sigma'^2 - 4}{\sigma'^2}} \frac{N_1 - \frac{n}{\sigma^2 - \sigma_a^2}}{(\sigma'^2 - \sigma^2)(\sigma'^2 - \sigma_s^2)} \qquad - (3.34)$$

and N_1 may be approximated by one or more poles as before. Also we see from (3.32) that the phase-space enhancement factor for the ABC effect in this case will be given by

$$\frac{N_{21} - \frac{n_{21}}{\sigma^2 - \sigma_a^2}}{\tilde{D}_0^{\,0}} \qquad - (3.35)$$

and the phenomenological form which we use for the parameterisation of the P_{11} $\pi N \to \sigma N$ partial wave h_{1-} will be

$$h_{1-} = \frac{\bar{N}_{21} - \frac{\bar{n}_{21}}{\sigma^2 - \sigma_a^2}}{\tilde{D}_0^{\,0}} \qquad - (3.36)$$

where the residues $n_{21}, \bar{n}_{21}$ and the functions N_{21} and $\bar{N}_{21}$ may be chosen to give good agreement with the experimental data.

The possibility of a CDD pole in the I = 0 S-wave pion-pion interaction has been suggested by several authors recently. In particular the 'turn-over' shape of the phase-shift $\delta_o^{\,o}$ found by Lovelace, Heinz and Donnachie from a dispersion relation analysis of the pion-nucleon scattering data in the backward direction may be thought to indicate the existence of a CDD pole. To see this let us consider the one channel N over D functions (3.31) in the form

$$A_{11} = \frac{N}{D + \dfrac{d^1}{\sigma^2 - \sigma_o^{\,2}}} \qquad - (3.37)$$

where again we refer the reader to the Appendix in which we show the equivalence of the two forms (3.31) and (3.37). Clearly A_{11} will have a zero at $\sigma^2 = \sigma_o^{\,2}$ and if d^1 is small the denominator of (3.37) will pass through zero near $\sigma_o^{\,2}$. Thus a corresponding phase-shift which passes through zero and near-by through $\pi/2$ may be the result of a CDD pole, although of course this shape could also occur because of the detailed dynamics of the scattering process without the existence of such a pole in the single channel N over D equations. It is pertinent to mention however that the only case in which a known physical phase-shift seems to have this type of behaviour is the P_{11} $\pi N \rightarrow \sigma N$ partial wave and that from the calculations performed by Coulter and Shaw[36] it would appear that 'free parameters' in the form of a CDD pole have to be introduced in order to produce the experimentally observed phase-shift.

Reasons for a CDD pole in the I = 0 S-wave pion-pion partial wave $A_0^{\,0}$ have also been put forward by Atkinson and Halpern[38]. Their arguments assume that the observed nonet of $2^+(=J^P)$ particles is the Regge recurrence of a nonet of 0^+ extinct bound states, and that these latter can be calculated in a dynamical 0^-0^- (3-channel calculation). In the limit of exact SU(3) symmetry they find that $A_0^{\,0}$ contains a CDD pole which they believe may well survive SU(3) symmetry breaking effects. It is also interesting to note that if these 0^+ extinct bound states or ghost bound states (i.e. poles with zero residues) do 'exist' they have the same effect as proper bound states in determining the asymptotic behaviour of the phase-shift from Levinson's theorem. In this case, from the arguments given by Rothleitner and Stech[39] and later by Squires[40], if the phase-shift is positive at the first inelastic threshold then it is known that a CDD pole must be inserted into the one channel N over D equations, without any additional assumptions about SU(3) symmetry or breaking effects. The first inelastic threshold is the four pion threshold, $\sigma^2 = 16\mu^2$. Our results and those of most authors favour a phase-shift which is positive here, so that the 'existence' of such a bound state would imply a CDD pole.

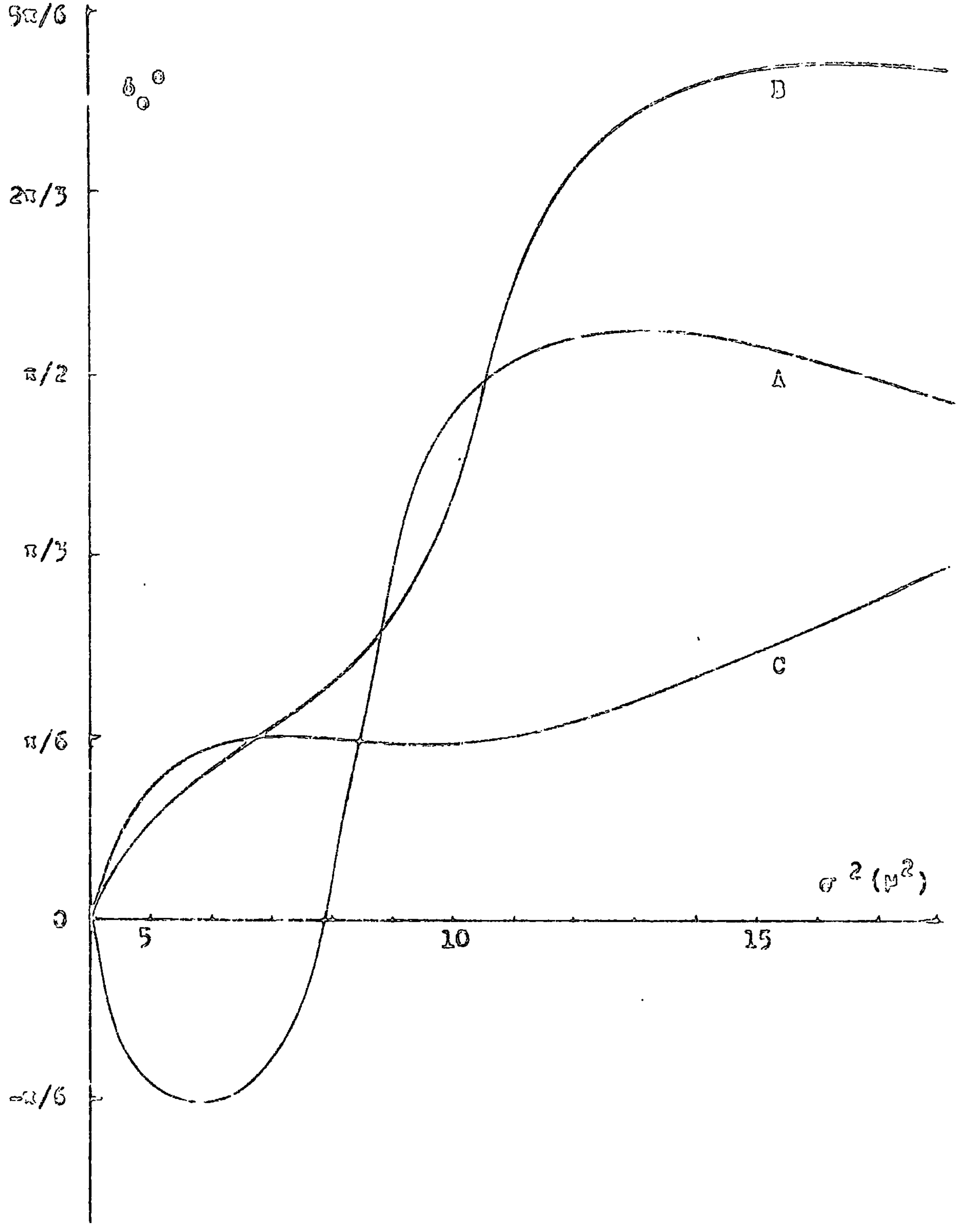
For these reasons we compute the results obtained from the pion-nucleon partial wave discrepancy analysis assuming the existence of a CDD pole in the I = 0 S-wave pion-pion scattering amplitude. Using the D-function $\tilde{D}_0^{\,0}$ given by equations (3.33) and (3.34) in the discrepancy analysis - equations (3.21) and (3.20) - at least three types of phase-shifts are found to give good agreement with the experimental data for this simple form. The three phase-shifts shown in figure (3.7) use a two pole approximation to the function N_1 given in (3.34). The parameters of these solutions are given in table (3.3).

TABLE 3.3

Solution Number	Pole positions (N_1) σ_1^2	Pole positions (N_1) σ_2^2	Residues (N_1) Γ_1/σ_1^2	Residues (N_1) Γ_2/σ_2^2	CDD pole position σ_a^2	CDD pole residues n	CDD pole residues d
A	-36	-156	1.9	-2.1	11.6	1.50	0.02
B	-36	-156	5.1	-4.6	12.4	2.30	1.61
C	-36	-156	-15.4	14.1	27.6	-82.5	-16.0

From the preceding discussion in Section 2 it is clear that because of our lack of knowledge of the detailed dynamics of the processes $\pi^- p \rightarrow \pi^+ \pi^- n$ and $p + d \rightarrow {}^3He + 2\pi$ - exhibited by our arbitrary parameterisation of the 'left hand' singularities - any of the three phase-shifts could satisfy the pion-production data and the ABC

FIGURE 3.7



effect. (In fact using (3.36) and (3.35) for the $\pi N \to \tilde{\sigma} N$ partial wave h_{1-} and the ABC enhancement factor respectively, the extra parameters enable us to fit the present data with $\tilde{N}_{21}$ and N_{21} approximated by constants.) Thus the phase-shifts (A) and (B) which are the two 'σ-resonance' solutions found by Heinz Lovelace and Donnachie with negative and positive scattering lengths respectively may also be compatible with the ABC effect, pion-production cross-sections and the low energy pion-nucleon data used in the discrepancy analysis. This is also true for phase-shift (C) which possesses a resonance at a much higher energy - the ε^{o}(720 Mev) perhaps.

Let us summarise our results. In the case that there exists no 'free parameters', i.e. CDD poles, in pion-pion S-wave scattering we find two types of phase-shift which could be compatible with the pion-production cross-section data, the ABC effect and the low energy pion-nucleon data. If, on the other hand, we assume the existence of a CDD pole we find at least three resonating types of phase-shift which could be compatible with these experimental observations, and that two of these are similar to the solutions found by Lovelace, Heinz and Donnachie from a dispersion relation analysis of pion-nucleon data in the backward direction. It is perhaps useful to stress again that the reason we are unable to find a unique solution is that our lack of knowledge of the dynamics of the various processes allows us to introduce parameters which we can vary to fit the available data. Only if various assumptions are made about these processes, which may or may not be justified, could any more restrictive information be obtained about the I = 0 S-wave pion-pion phase-shift. However not all the available evidence on the pion-pion interaction has been considered here. In the next chapter we shall give a survey of other studies which have been made to obtain information on this interaction and compare the results of these studies with the results given here.

C H A P T E R   F O U R

SURVEY OF THE INFORMATION AVAILABLE ON THE PION-PION INTERACTION

IV - SURVEY OF THE INFORMATION AVAILABLE ON THE PION-PION INTERACTION

1. Dispersion Relation Analyses and Current Algebra Techniques

Our aim in this concluding chapter is to give a survey of the different information available on the pion-pion interaction pointing out possible sources of error made in its interpretation. In the light of this survey, together with the results and discussion given earlier in this thesis, it may be possible to draw some general conclusions about the pion-pion I = 0, S-wave amplitude. It must be stressed that, because of the large amount of apparently conflicting evidence, any conclusions that are made must, of necessity, be only tentative.

In any survey of the pion-pion interaction, the pion-nucleon dispersion relation analyses must be given some prominence since the large amount of 'reasonably accurate' data on the pion-nucleon system allows one, at least in principle, to derive more detailed information on the pion-pion interaction than is usually possible. The partial wave dispersion relation discrepancy analysis has already been discussed at some length, but one or two remarks about this analysis may be in order here. In the derivation of the discrepancy $\mathcal{D}(s)$ in equation (3.17) it was supposed that the energy dependent contribution was that coming from the front of the circle, i.e. $\mathcal{D}_{\pi\pi}(s)$, and that the other contributions (from the short range forces) could be well approximated by a constant. This may be in error, although it was found[11] that giving these other contributions some simple energy dependence (such as a distant pole), made little difference to the fits to the data. Also the inclusion in the analysis of the pion-pion $\ell = 2$ partial waves, which have so far been neglected, may provide rather different results. For instance it is usually assumed that the inclusion of these D-waves would improve the fit of the discrepancies to the experimental values in the higher energy range[10]. However, it is probably fair to say that the inclusion of $\ell = 2$ partial waves at low energies where they are damped considerably by the centrifugal barrier should have little effect.

The dispersion relation analysis of the backward pion-nucleon scattering amplitude has been studied by Lovelace, Heinz and Donnachie[21]. Here again the better known pion-nucleon data provides a means of obtaining an estimate of the low energy pion-pion interactions. As in the case of the partial wave dispersion relation analysis, the partial waves with $\ell \geq 2$ are neglected as are any four-pion inelastic effects. The pion-pion phase shift is given by the phase of the backward scattering amplitude on the left hand cut. Lovelace et al. admit that while there is sufficient experimental data, the calculation of the pion-pion phase-shift from this analysis is delicate, and they sought to reduce the possible errors by using dispersion relations for the F^+ and F^- pion-nucleon isospin amplitudes together. As a check on the numerical errors, they fitted the data in 184 different ways. By fitting the F^+ data up to 600 Mev, two types of pion-pion I = 0, S-wave phase-shift were found, one with a positive scattering length ($a_0^0 \sim 0.7\,\mu^{-1}$) with a resonance at 430 ± 70 Mev of width 400^{+400}_{-150} Mev and the other with a negative scattering length which turns over, passing back through zero near 350 Mev and resonates at 680 ± 85 Mev with a width 750 ± 50 Mev. Lovelace et al state that the evidence favouring a resonance rather than a large scattering length interaction seems to come chiefly from π p data on the upper slope of the N* (1236) resonance which would explain why it had not been obtained in earlier analyses using only low energy data. They believe that there is evidence beyond all reasonable doubt from this work for the existence of an I = 0, S-wave resonance and suggest that the very large width would make it difficult to pick out from phase space in most other calculations.

At first sight it might appear that dispersion relation techniques could also be usefully applied to the pion-pion partial wave amplitudes. The first inelastic threshold is relatively distant from the elastic threshold and crossed channel reactions which provide the 'left hand forces' may themselves be written as a sum involving pion-pion partial waves. This sum is the analytic continuation of a Legendre expansion to regions well outside the formal region of convergence so that cut-off parameters must be introduced into the integral equations. When one approximates the Legendre expansion by the P-wave term alone and attempts to calculate the P-wave amplitude assuming that the ρ-meson is the dominant force term, one has the simple bootstrap type of mechanism, the ρ-meson in the crossed channel providing, in principle, the force for the creation of a ρ-resonance in the direct channel. This was successfully accomplished by Chew and Mandelstam[41]. However, when a more rigorous treatment of the integral equation was attempted by considering the S- and P-waves together,[42] the ρ-meson was produced only when a bound state was produced in the I = 0, S-wave amplitude. This has been re-investigated recently by Kyle, Martin and Pagels[43] who found that a P-wave resonance was accompanied by unphysically large S-wave amplitudes. On the other hand, Bransden and Moffat[44] performed a similar analysis using dispersion relations for the inverse partial wave amplitudes. They obtained a P-wave resonance near 700 Mev and an S-wave solution with scattering lengths in agreement with Schnitzer, i.e. $a_o^o = 0.5$, $a_o^2 = 0.16$.

The main difficulty with these pion-pion partial wave dispersion relations is the determination of the discontinuity across the left hand cuts. An understanding of the short-range forces which for S-wave amplitudes are not damped by the centrifugal barrier is required before this type of analysis could be considered adequate for a detailed study of S-wave pion-pion scattering.

A dispersion relation for the forward pion-pion scattering amplitude however does not suffer from the above difficulty. The discontinuity across both the left and right hand cuts from the optical theorem can be written in terms of the total pion-pion cross-sections. The calculations of this dispersion relation by Rothe[20] and by Pisut, Bona and Lichard[19] have also been discussed earlier. When Rothe introduced a large I = 0, S-wave pion-pion interaction into the dispersion integral by assuming a constant phase-shift of $\pi/4$ in the energy range $\sigma^2 = 5\mu^2$ to $40\ \mu^2$, the additional contribution to the scattering length $a_0^{\,0}$ was found to be as large as $+0.9\ \mu^{-1}$, this, together with an estimated error of $+0.1\ \mu^{-1}$ arising from possible effects in the I = 2 amplitude, led Rothe to suggest that

$$a_0^{\,0} = -1.7 {}^{+\,1.3}_{-\,0.5}\ \mu^{-1} \qquad - (4.1)$$

These large errors, in particular that coming from the assumed form of the low energy I = 0, S-wave phase shift, indicate the approximate nature of this result. Nevertheless, the calculation of the I = 1, S-wave scattering length $a_0^{\,1}$ (which should be zero) using similar approximations led to the quite acceptably small value of $-0.4\ \mu^{-1}$.

Pisut, Bona and Lichard used a subtracted form of the dispersion relation for the forward pion-pion scattering amplitude which greatly depressed the high energy behaviour. For the low energy form of the total pion-pion cross-sections they employed a scattering length approximation and obtained

$$a_0^{\,0} = -1.3 \pm 0.6\ \mu^{-1}$$

$$\text{and} \quad a_0^{\,2} = +0.38 \pm 0.2\ \mu^{-1} \qquad - (4.2)$$

where the errors were due only to the errors of the subtraction constant. The errors due to inaccuracies in the cross-sections were said to be much smaller than these.

Although neither of the above analyses used a detailed shape for the pion-pion phase-shift which was consistent with the negative scattering length, the interesting point is the measure of agreement obtained for the value of $a_0^{\,0}$ using rather different methods and input data. On the other hand, another recent calculation of the pion-pion scattering lengths using forward pion-pion dispersion relations by Meiere and Sugawara[45] suggests that $a_0^{\,0}$ should be small and positive. This calculation required that the high energy limit of the pion-pion cross-sections be the same in all three isospin channels and that the scattering becomes asymptotic farily rapidly above the ρ and f^0 resonances in the respective channels. Meiere and Sugawara used a once subtracted dispersion relation and the phase representation for the crossing symmetric forward pion-pion amplitude and an unsubtracted dispersion relation for the crossing anti-symmetric amplitude. The only pion-pion resonances considered were the ρ and f^0; and the S-wave interactions, described by effective-range expansions - with effective ranges between 0 and 2 μ^{-1}, were assumed to dominate the low energy region. From this analysis they obtained the values

$$a_0^0 = 0.25 \pm 0.08 \mu^{-1}; \quad a_0^2 = 0.00 \pm 0.03 \mu^{-1} \qquad - (4.3)$$

and suggested that the unknown details of the high energy scattering are relatively unimportant in their determination of these scattering lengths. It should be noted, however, that their dispersion integrals are not so heavily damped at high energies as that of Pisut et al.

Both sets of authors, Pisut et al and Meiere and Sugawara have also considered the sum-rule derived by Adler[28] from current algebras which relates the g_A/g_V ratio to the pion-pion total cross-sections. Using their values for the scattering lengths Pisut et al found this sum-rule to be satisfied with a phase-shift δ_0^0 which is initially negative becoming positive and possessing the ε^0 resonance. However, this is far from being uniquely determined. Only the square of the I = 0, S-wave amplitude is used in the sum rule so that a resonating phase-shift with a positive scattering length and subsequent 'dip' such as that obtained by Wolf[22] would give very similar results. So too would a large phase-shift instead of an ε^0 resonance. Also Meiere and Sugawara re-writing the Adler sum-rule in a different form obtained the result

$$2a_0^0 - 5a_0^2 \approx 0.7 \mu^{-1} \qquad - (4.4)$$

which is compatible with their values of the scattering lengths. The sum rule is for pion-pion scattering with one pion off its mass shell. Both sets of authors have made some corrections for this, but they are corrections rather more to the kinematics than to the detailed dynamics. This is probably adequate for the approximate discussion of the integral over the cross-sections given by the sum-rule. There is, however, another method for deriving the pion-pion scattering lengths from current algebras in which the off-the-mass shell corrections must be considered more closely.

Current algebra techniques, together with the hypothesis of a partial conserved axial-vector current (PCAC) have been employed by Weinberg[46] and others[47] to derive the scattering lengths for the scattering of a 'pion' off any target particle in the limit of the initial and final state 'pions' having zero four-momentum (and therefore zero mass). When the target particle is much heavier than the physical pion-mass this limit might be assumed to make little change in the low energy scattering amplitude, and simple approximations for the extrapolation to the physical threshold in this case lead to scattering lengths in remarkable agreement with the experimental data. Unfortunately, in the case of the pion-pion scattering, this extrapolation may make a significant difference to the results and it is not clear how it should be performed. Weinberg has assumed that the scattering amplitude can be expanded in powers of s, t and u (in the standard notation) and that one can neglect all powers higher than the first. By fitting the co-efficients of this expansion to the values of the amplitude given by current algebras and Adler's self-consistency conditions,[48] Weinberg obtained the S-wave scattering lengths

$$a_0^0 = 0.2\ \mu^{-1}; \quad a_0^2 = -0.06\ \mu^{-1} \qquad - (4.5)$$

It has been pointed out that the assumption of this power series expansion holding up to and somewhat beyond threshold, clearly violates unitarity[49]. Weinberg has remarked that his small scattering lengths could be due to the fact that by writing such an expansion one has already assumed them to be negligible. Iliopoulos[49] has used a more flexible parameterisation of the pion-pion amplitude which allows for the elastic unitarity branch cut. In doing this he introduced further co-efficients which cannot all be determined and obtained a family of solutions for a_0^0 and a_0^2. We connect these solutions by the relation

$$(1.18 + a_0^2)^2 = (0.74 + 0.62 \times a_0^0)^2 + 0.52 \qquad - (4.6)$$

similar conclusions have also been reached by J Sucher and C H Woo[50]. This problem of the pion-mass extrapolation has been considered in an alternative way by the use of dispersion relation techniques[51]. These suggest that corrections to the current algebra results are not likely to be significant as long as the S-wave pion-pion interaction is relatively weak. Such dispersion relation analyses must necessarily be rather approximate. We conclude therefore that while current algebra techniques may give useful results for meson baryon scattering, no reliable predictions have yet been obtained for pion-pion scattering lengths.

2. Pion Production and Final State Interactions in Decay Processes

We shall review in this section the information on the pion-pion interaction which may be obtained from the pion production process

$$\pi + N \rightarrow \pi + \pi + N \qquad - (4.7)$$

and the decay processes

$$\begin{aligned} K &\rightarrow 2\pi + e + \nu \\ K &\rightarrow 3\pi \\ \eta &\rightarrow 3\pi \\ K_1^0 &\rightarrow 2\pi \end{aligned} \qquad - (4.8)$$

We have discussed at some length the interpretation of the process $\pi N \rightarrow \pi\pi N$ by the peripheral diagram and modifications to it, and we noted in Chapter II that such a diagram could be the dominant interaction in certain regions of the variables but that these regions were poorly defined. It is worthwhile mentioning that while care must be exercised in its application the peripheral model has been successful in determining the ρ and f^0 mesons. Wolf [22] has extended this use of the peripheral model (with the modifications suggested by Ferrari and Selleri [13]) to perform a pion-pion phase-shift analysis from the pion production data taken at several Gev. incident pion energy. This analysis gave phase-shifts which contain both the ρ-resonance with a mass of 760 Mev and a full width at half maximum of 170 Mev, and the f^0-resonance (an $I = 0$, D-wave resonance) with a mass of 1250 Mev and a width of 140 Mev. The results for the I = 0, S-wave phase-shift, however, are not so clear cut. At low values of the di-pion mass an effective range formula was used with a scattering length $a_0^{\,0}$ assumed from the ABC data of $2 \pm 1\ \mu^{-1}$. This parameterisation for $\delta_0^{\,\sigma}$ gave satisfactory agreement with the experimental data on the $\pi^- p \rightarrow \pi^+ \pi^- n$ cross-sections although this is not very accurate in the low di-pion mass region. It should be noted that, since only the pion-pion cross-sections are used in this peripheral calculation, a negative scattering length of the same order of magnitude would probably have provided similar results; hence the phase-shift $\delta_0^{\,0}$ at low energies is hardly determined from this analysis. At higher energies, Wolf determined $\delta_0^{\,0}$ by fitting to the forward-backward assymetry parameter $R_{\pi^+\pi^-}$. This parameter is defined as

$$R = \frac{\sigma_x(\theta_\pi < \pi/2) - \sigma_x(\theta_\pi > \pi/2)}{\sigma_x(\theta_\pi < \pi/2) + \sigma_x(\theta_\pi > \pi/2)} = \frac{F - B}{F + B} \qquad - (4.9)$$

the ratio of the forward to backward scattering of the two final state pions, and is found to be large ($\sim$ 0.4 to 0.6) in the di-pion energy range from 600 to 900 Mev. This effect, Wolf and others[52] claim, can only be explained by assuming the existence of a resonance in the I = 0, S-wave pion-pion amplitude with a mass of about 740 Mev and a width of 90 Mev. Recently Bander and Shaw[53] considered this effect using a peripheral model with absorption and found that a phase-shift of $\sim +60^\circ$ gave as good a fit as did a resonance in this energy range. They ruled out a negative phase-shift of $\sim -60^{\circ}$ by examining the distributions in θ_{π} as a function of the di-pion mass.

Goebel and Schnitzer [17] have considered the pion-production process using a static model calculation which includes both a direct 'knock-on' single pion exchange diagram and a re-scattering diagram, where the re-scattering is due to the N*(3.3) resonance. By fitting this model to the π^+ angular distribution in the process $\pi^- p \rightarrow \pi^+ \pi^- n$ at 430 Mev they found

$$2a_0^{\,0} + a_0^{\,2} = 1.16 \text{ and } a_1^{\,1} = 0.07\,\mu^{-1} \qquad - (4.10)$$

From the total $\pi^+ p$ inelastic cross-section at 470 Mev they found two acceptable values for $a_0^{\,2}$, i.e. 0.16 and -0.14 which gave values for $a_0^{\,0}$ of 0.50 and 0.65 respectively. The model seems to reproduce reasonably well the $\pi^- p \rightarrow \pi^+ \pi^- n$ angular distributions at 370 Mev and the total cross-section up to about 500 Mev, although it should be said that this data is not particularly accurate (an earlier calculation with this model, using earlier data, produced a negative value for $a_0^{\,0}$) and that in the range 200 to 300 Mev the fit to the total cross-section is not much better than the unmodified peripheral calculation. It would be interesting to see if similar results are obtained from a relativistic treatment of this model; and certainly until such a treatment is performed the values quoted above must be only tentative.

Let us now turn to the meson decay processes. Various, rather approximate, methods have been introduced to derive information on the pion-pion interaction from the processes $K \rightarrow 3\pi$ and $\eta \rightarrow 3\pi$. For instance the final state interactions in the three pion decay modes of the K-meson have been studied by Khuri and Treiman[54] using dispersion relation techniques. By adopting certain approximations they are led to a set of linear integral equations for the $K^{\pm} \rightarrow 3\pi$ decay amplitudes in which the kernels depend on the pion-pion S-wave scattering amplitudes. By assuming a scattering length interaction for these pion-pion amplitudes, and fitting the shape of the decay spectra by an iterated solution for these integral equations, they found $a_0^2 - a_0^0$ was positive and of the order of $0.7\ \mu^{-1}$. Similar results were obtained by Sawyer and Wali[55] also using dispersion relation methods.

The three pion decay modes of K and η mesons have also been studied using a model in which these decays proceed through a resonant $I = 0$, S-wave pion-pion interaction. Brown and Singer[56] found that good agreement could be obtained with all the available data on the K and η spectra and branching ratios if the di-pion resonance (sigma) has a mass of 400 Mev and a width of 75 to 100 Mev.

Fadeev equations have also been used to obtain information on the pion-pion interaction from a study of these three pion decay modes. Unfortunately no clear results have emerged. For instance Prasad[57] obtained a good fit to the data only with a σ-type resonance in the I = 0, S-wave pion-pion amplitude while Dunn and Ramachandran[58] found a scattering length interaction was satisfactory with scattering lengths of $a_o^{\,o} = 0.3$, $a_o^{\,2} = 1.5$ for K decays and $a_o^{\,o} = 0.1$ $a_o^{\,2} = 1.75$ for η decays. A large I = 2, S-wave scattering length, $a_o^{\,2}$, was also found by Barbour and Schult[59] who obtained a fit to the data with $(a_o^{\,2})^2 - (a_o^{\,o})^2 \sim 2$.

The only decay process of the type meson $\to$ mesons + leptons for which any sizeable amount of data is available is the so called $K_{\ell 4}$ decays, i.e. $K \to \pi\pi e \nu$ where e and ν represent the electron (or positron) and neutrino respectively. Jacob, Mahoux and Omnes[60] and later Maksymowicz[61] have treated the final state interaction of this process by assuming that the final state interaction can be 'factored off' from the decay mechanism, i.e. that the only singularity the amplitude has as a function of the two-pion invariant mass is the right hand unitarity branch cut. This 'factoring off' of the final state interaction is similar to the assumption made in deriving the phenomenological form for the P_{11} $\pi N \to \gamma N$ partial wave in Section 4 of Chapter II and means that in writing a dispersion relation for the amplitude the integral over the left hand singularities may be approximated by a constant. With this assumption the $K \rightarrow \pi\pi e\nu$ invariant amplitude is of the form C/D_0^0 where D_0^0 is the D-function arising in an N over D solution for the pion-pion scattering amplitude.

A somewhat different approach for deriving the decay amplitude has been to consider the analytic continuation of the amplitude for the process $K + (e\nu) \rightarrow \pi\pi$. Kacser, Singer and Truong[62] have assumed an unsubtracted dispersion relation for this amplitude in which the discontinuity across the right hand cut is given by unitarity and that across the left hand cut is given by the crossed process $K + \pi \rightarrow \pi + (e\nu)$. After neglecting various terms which they believe to be small, they finally obtained a form for the $K \rightarrow \pi\pi e\nu$ amplitude in terms of D_0^0 and D_1^1, the D-functions, arising in an N over D solution for the I = 0, S-wave and the I = 1, P-wave pion-pion amplitudes. The best fits for this amplitude to the available experimental data on the two-pion spectra were obtained with a pion-pion scattering length a_0^0 of 1 ± 0.3; although this solution gave a decay rate which is approximately five times larger than that observed experimentally. Thus the amplitudes derived could have the right energy dependence but are then wrong in their absolute value by over a factor of two.

Finally, in this survey of the available information on the pion-pion interaction, we must mention the K_1° - K_2° mass difference. Recent experimental results have indicated that this mass difference is $-0.5/\tau_1$, where τ_1 is the lifetime of the K_1°-meson[63]. A theoretical study of this difference has been made by Kang and Land[64] who assume that it is due primarily to the self-energy of the K_1° arising from the two pion state with I = 0. The mass difference in this case is related to the $K_1^{\circ}\pi\pi$ form factor which can be written in terms of the pion-pion D-function, D_0°. This was calculated by obtaining self-consistent solutions to the N over D equations for the pion-pion amplitude A_0° in which the driving forces were taken to be the exchange of a ρ-meson and an assumed S-wave interaction. Kang and Land obtained the value of $\Delta M = -0.5\ \tau^{-1}$ for a negative scattering length a_0°, the phase-shift turning up through zero with a value of $0^{\circ} \pm 10^{\circ}$ near 500 Mev; and found that positive scattering lengths led to positive mass differences.

Kang and Land also considered the model proposed by Barger and Kazes[65], later developed by Nishijima[66] and considered recently in an approximate form by Truong[67], which uses the analytic properties of the function $[N(\sigma^2)D(\sigma^2)]^{-1}$ to obtain the K_1° - K_2° mass difference as

$$\Delta M = -\tfrac{1}{2}\text{Cot}\ \delta_0^{\ 0}(M^2) + \text{correction terms due to an integral over the left hand singularities}$$

Taking into account the correction terms and possible pole contributions from zeros in the N and D functions, Kang and Land found this relation gave the same numerical results as before. Rockmore and Yao[68] on the other hand, also using self-consistent solutions to the N and D functions, found the zero in the N function to be below the physical threshold and obtained a pion-pion I = 0, S-wave amplitude which has a positive scattering length. The sign of the scattering length, of course, is dependent on the sign of the N function so that the only difference in these two calculations is the position of the zero in the N function which is determined by the detailed description of the driving forces assumed for the pion-pion interaction.

This concludes our survey of the available information on the low-energy pion-pion interaction. The large volume of literature has by no means been exhausted, but we hope that at least the more 'reliable' techniques for extracting information have been covered[69]. The contradictory results obtained by different authors only serve to show how difficult is this problem and how unwise we should be in summarising these results if we were to make any but rather tentative statements about the form of the low-energy I = 0, S-wave pion-pion phase-shift. Perhaps the only point on which the various studies described in this Chapter are in agreement is the rather basic one that a pion-pion interaction <u>does</u> exist. It would be extremely difficult to account for so many of the observed effects in the various scattering processes unless some kind of pion-pion interaction were operative.

3. Conclusions

In this concluding section we should like to compare the results presented in the earlier Chapters of this thesis with the various analyses sketched in the preceeding two sections. To do this, let us first of all make one or two fairly general remarks about the results presented in the above survey. It is interesting to note, for instance, that where the method allows a detailed study of the I = 0, S-wave pion-pion phase-shift δ_0^0 over a range of values of the di-pion energy, it appears that Sin δ_0^0 is positive and perhaps farily large, somewhere in the range above 400 Mev. We have seen that the I = 0, S-wave amplitude A_0^0 is frequently presumed to possess either the σ - or the ε^0-resonance in which case the phase-shift passes upwards through an odd multiple of $\pi/2$ - probably $+1.\pi/2$. Where such resonances are not required to fit the data, a scattering length interaction has usually been assumed, with the scattering lengths sometimes found to be positive and at other times negative. When a_0^0 is positive the phase-shift is taken to be steadily increasing, while negative values for a_0^0 are obtained either when the high energy effects are heavily damped or when there is sufficient parameterisation of the amplitude for the phase-shift to turn up through zero and become positive. Thus the available information on the pion-pion interaction seems to support a phase-shift which is positive somewhere in the range above 400 Mev. In connection with this it is worth mentioning that if the phase-shift δ_0^0 is indeed initially negative and thereafter turns up through zero then a scattering length approximation to this type of interaction could, in certain calculations, quite mistakenly lead to positive values for the scattering length a_0^0 by averaging the effect of the interaction over a range of energies.

With this proposition that the phase-shift δ_0^0 is eventually positive, we should like to suggest that the available information on the pion-pion interaction could be compatible with either of two shapes for the <u>low-energy</u> I = 0, S-wave pion-pion phase-shift. These are:

i. a positive phase-shift characterised by a positive scattering length of $1 \pm 0.3\ \mu^{-1}$

ii. a negative phase-shift of $-1.7 {+1.3 \atop -0.5}$ which soon turns over and becomes positive.

The first of these types of phase-shifts could be in agreement with all the different calculations considered in the survey (because of their approximate nature) except perhaps those by Rothe[20] and by Pisut et al[19] using forward pion-pion dispersion relations and by Kang and Land[64] on the $K_1^0 - K_2^0$ mass difference. These three calculations however, as we have seen, are by no means free from criticism. For instance, we noted that Rothe found a large additional contribution to the scattering length a_0^0 when he assumed a large I = 0, S-wave interaction. A more detailed consideration of the phase-shift δ_0^0 therefore might lead to a positive rather than a negative value for this scattering length. On the other hand, Pisut et al used a heavily damped form of the forward pion-pion dispersion relation which was, therefore, less dependent on the shape of the phase-shift so that their negative value for a_0^0 should perhaps be more convincing. However, in damping the integral, they were forced to introduce a subtraction constant which was evaluated using the phase-shifts derived by Wolf. In this highly damped form the equation is particularly sensitive to the value of the subtraction constant and it is not impossible to believe that errors in the Wolf phase-shifts could produce large errors in their value for the scattering length. Also the calculation by Kang and Land of the $K_1^0 - K_2^0$ mass difference may well be in error. Their calculation involved self-consistent solutions of the N over D equations for the pion-pion amplitude A_0^0, and found that N had a zero above threshold and that a_0^0 is negative. On the other hand, as we have already pointed out, Rockmore and Yao[68], using similar self-consistent solutions for A_0^0, found the zero in N to be below threshold and a_0^0 to be positive. Thus these calculations for the $K_1^0 - K_2^0$ mass difference indicate that the pion-pion phase-shift is either of type i. or ii. but cannot really support one rather than the other.

The other low-energy phase-shift, ii., which we suggest could be compatible with the available data is one which is initially negative, but which then turns over and becomes positive. This type of phase-shift we have seen is compatible with the backward pion-nucleon dispersion relation analysis of Lovelace et al[21] and also with our re-calculation for the S-wave pion-nucleon dispersion relation 'discrepancies'. Because of the large amount of reasonably accurate pion-nucleon scattering data, these analyses, at least in principle, should provide somewhat more reliable results than other more tenative methods. Of these other methods, we have already mentioned how some form of scattering length approximation for this type of phase-shift might mistakenly lead to positive values for a_0^0 in certain cases. These could presumably include the model for the pion-production amplitude of Goebel and Schnitzer[17] and the forward pion-pion dispersion relation analysis of Meiere and Sugawara[45] in which the dispersion integrals are poorly convergent. We should like to propose, therefore, that some authors have not obtained this type of solution for δ_0^0 because their parameterisation of the amplitude A_0^0 was insufficient to allow for it. We could perhaps go further and suggest that whenever the parameterisation of the amplitude has been sufficient, this type of phase-shift, ii., is almost always found as a possible solution. If this is the case, then only the current algebra predictions of Weinberg and others[46,47] need to be seriously considered here as evidence against a negative scattering length; but even for these, if one assumes the pion-pion interaction to be non-negligible, so that the unitarity branch cut becomes important, we have seen that a negative solution for a_0^0 is just as possible as a positive one. Thus we conclude that either type of phase-shift i. or ii. could be compatible with the available information on the low-energy pion-pion interaction.

The value of $1 \pm 0.3\ \mu^{-1}$ for the scattering length of solution i. is suggested by the dispersion relation analyses of pion-nucleon backward scattering amplitude and the partial wave amplitudes. The approximate nature of the other 'less reliable' calculations is such that we believe this is quite a realistic value for the positive solution. For the other solution, ii., it is more difficult to put bounds on the size of the negative scattering length. However, for a negative phase-shift which soon turns over and becomes positive, the scattering length is no longer a very good guide to the low-energy interaction. We would suggest that the value found by Rothe of $-1.7\ {}^{+\,1.3}_{-\,0.5}$ is a fair measure of the size of a_0^0 for this solution, ii., where the errors admit both the values obtained from the turn-over solutions of the two pion-nucleon dispersion relation analyses.

We have not yet considered the question of either a σ - or an ε^{0}-resonance in the pion-pion I = 0, S-wave partial-wave amplitude. The existence of the ε^{0}-resonance was suggested in order to account for the size of the ratio of forward to backward scattering of pions in the process $\pi N \rightarrow \pi\pi N$[52]. However it has recently been shown in a peripheral model which includes absorption effects, that the phase-shift δ_0^0 need be no larger than 60^0 to account for this ratio . Hence the usefulness of postulating the existence of an ε^0 resonance is not now very great. Also the success of the postulated σ-resonance in accounting for the pion spectra in $K \rightarrow 3\pi$ and $\eta \rightarrow 3\pi$ decays[56] is perhaps not that significant. Three body interactions are notoriously difficult to handle, and a fuller analysis of these decays very possibly could lead to non-resonating solutions for the I = 0, S-wave pion-pion amplitude. Nevertheless the existence of a pion-pion I = 0, S-wave resonance from the work of Lovelace, Heinz and Donnachie on backward pion-nucleon dispersion relations is difficult to refute and the evidence one way or another from other calculations is very inconclusive. We would summarise this situation, therefore, as follows. It seems quite possible that a pion-pion I = 0, S-wave resonance does exist but, before this matter is really put beyond any reasonable doubt, strong confirmation should be provided both from better analyses of other processes and by repeated analyses of the data on the pion-nucleon scattering amplitude in the backward direction taken to even higher energies.

Finally let us consider these conclusions in the context of our phenomenological analysis of the process $\pi^- p \rightarrow \pi^+ \pi^- n$ taken near the pion-production threshold. The basis of this analysis consisted of treating formally the two pion system in the final state as though it were a single system σ with zero isospin and angular momentum but with a continuous mass spectrum. In this case, the production process becomes $\pi^- p \rightarrow \sigma n$. A partial wave decomposition was made of the amplitude for this process and all the partial wave amplitudes, except that for which the σ n state is in the S-wave, were given the values obtained from the single pion exchange graph. The remaining amplitude, with the S-wave σ n final state (the f_{1-} in the notation of Chapter II) was calculated by assuming that of the three particles in the final state, only the two pions provided an important final state interaction since the energies considered were below the N* production threshold. Thus the only right hand cut of this amplitude is associated with this two pion interaction (which is assumed to be in an $I = 0$, S-wave state). By multiplying by the D-function arising in an N over D solution for the $I = 0$, S-wave pion-pion amplitude, this right hand cut was removed and the effect of the left hand cuts was approximated by a constant parameter C. All the partial waves were then combined and the differential cross-section for the production process was calculated in terms of the various parameters involved in the calculation.

The calculated values for the differential cross-sections were compared with the experimental data, and the parameters defining the pion-pion amplitude and the value of C were adjusted to give an optimum fit. Using this quite reasonable model for the low energy pion-production amplitude, it was found that the differential pion-production cross-sections at 374 Mev could be reproduced satisfactorily with a non-resonating pion-pion $I = 0$, S-wave phase-shift only of type ii., i.e. initially negative, turning over and becoming positive. Two other processes, low-energy pion-nucleon S-wave scattering and the ABC effect were also examined and were shown to be consistent with a phase-shift of this turn-over type.

Of the two types of low energy phase-shift i. and ii., given by the analyses considered in the previous two sections, we would suggest therefore that the low energy pion production data perhaps favours type ii., at least for non-resonating solutions. Unfortunately, we are unable to say anything stronger than this because, although it seems a quite reasonable approximation to replace the effect of the left hand cuts in (2.45) by a constant parameter at least over the small range of σ^2 we are considering, it would be unwise to draw any hard and fast conclusions from a model in which there is even an element of doubt.

It has been suggested from a study of the possible driving forces that a pion-pion resonance can only be produced in the $I = 0$, S-wave partial wave amplitude by the introduction of free parameters, i.e. a CDD pole[70]. An alternative parameterisation of the pion-pion phase-shift was made, therefore, in which the effect of including a CDD pole was taken into account. In Section 3 of Chapter III, we discussed how such a CDD pole would alter the various analyses of the data on the low energy pion-production cross-sections, proton-deuteron scattering and the low energy S-wave pion-nucleon scattering amplitude. Here again, a solution of the turn-over type was obtained which was consistent with all three processes, but in this case equally acceptable solutions with positive scattering lengths were also found without any further alteration of our model. Thus, for a resonating amplitude which contains a CDD pole, we conclude that either type of phase-shift $\delta_0^{\,0}$ found by Lovelace, Heinz and Donnachie from pion-nucleon scattering data in the backward direction could be compatible with the low energy pion-production data. We also note that such phase-shifts could be compatible with the ABC effect, the pion-nucleon 'discrepancy' analysis and probably most of the other methods discussed in this Chapter for deriving the pion-pion interaction.

Finally, we must mention a suggestion due to Chew[71] that if there exists one or two ghost bound states in the I = 0, S-wave pion-pion amplitude (due to the backward intercepts of the P and P′ Regge trajectories at negative values of σ^2) the phase-shift δ_0^0 might be expected to decrease rapidly to $-\pi$ or -2π. Cook[72] has proposed that this could be compatible with the data for various processes such as $K_{\ell 4}$ decays and the three pion decays of the K and η mesons. However, the work of Atkinson and Halpern[38] shows that the existence of ghost bound states could also imply the existence of CDD poles which would affect the assymptotic behaviour of δ_0^0. Also from our analysis of the low energy pion-production data in terms of the modified peripheral model, this type of phase-shift would produce one or more 'bumps' in the differential production cross-sections at low values of the squared di-pion energy (σ^2) which are not observed experimentally. This result is independent of the detailed parameterisation of the P_{11} $\pi^- p \rightarrow \tilde{\sigma}$ n partial wave and we would suggest therefore that this type of phase-shift could be ruled out.

We have seen that it is not yet possible to predict with any certainty the form of the low-energy pion-pion interaction. Perhaps useful information could be obtained from a much refined version of the model proposed by Goebel and Schnitzer. Also it might prove possible in the future to produce accurate predictions on the scattering lengths by providing some means of reducing the pion-mass extrapolation difficulties in current algebra techniques. A more likely prospect is that by better analyses of the various processes discussed in this thesis, together with more accurate and plentiful data, one specific type of pion-pion phase-shift will be suggested. However, the conclusions reached in this thesis are, we believe, the extent of the present knowledge on the low-energy pion-pion interaction.

A P P E N D I C E S

APPENDICES

1. The Pion-Pion Amplitude

The S-matrix element for the pion-pion scattering process sketched in Figure (A1.1)

$$q_1(\alpha) + q_2(\beta) \rightarrow q_1'(\gamma) + q_2'(\delta) \qquad \text{- (A1.1)}$$

is given in terms of the invariant amplitude A by the relation

$$\langle \vec{q}_1', \gamma; \vec{q}_2', \delta \,|\, S \,|\, \vec{q}_1, \alpha; \vec{q}_2, \beta \rangle = \delta_{\gamma \vec{q}_1', \delta \vec{q}_2'}^{\alpha \vec{q}_1, \beta \vec{q}_2} + i(2\pi)^4 \delta^4(q_1 + q_2 - q_1' - q_2') \frac{A_{\alpha\beta\gamma\delta}}{\sqrt{16\, \omega_1 \omega_2 \omega_1' \omega_2'}} \qquad \text{- (A1.2)}$$

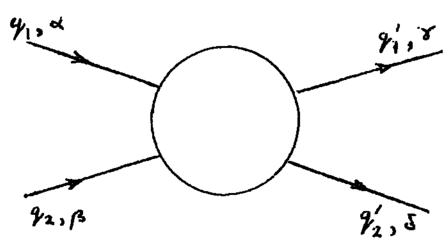
where the pseudo-scalar pi-mesons with isotopic spin one, are denoted by their three-momenta $\vec{q}_i$ and isospin indices $\alpha, \beta, \gamma, \delta$.

$A_{\alpha\beta\gamma\delta}$ can be written in terms of three independent scalar functions as

$$A_{\alpha\beta\gamma\delta} = A_I \delta_{\alpha\beta} \delta_{\gamma\delta} + A_{II} \delta_{\alpha\gamma} \delta_{\beta\delta} + A_{III} \delta_{\alpha\delta} \delta_{\beta\gamma} \qquad \text{- (A1.3)}$$

which is related to the fact that, assuming charge independence one has three possibilities for the total isotopic spin I = 0,1,2.

FIGURE A1.1



Defining the invariants $\bar{s}$, $\bar{t}$, $\bar{u}$ by the relations

$$\bar{s} = -(q_1 + q_2)^2 = -(q_1' + q_2')^2$$

$$\bar{t} = -(q_1 - q_1')^2 = -(q_2 - q_2')^2 \qquad - (A1.4)$$

$$\bar{u} = -(q_1 - q_2')^2 = -(q_2 - q_1')^2$$

such that $\bar{s} + \bar{t} + \bar{u} = 4\mu^2$ - (A1.5)

where μ is the mass assumed from independence to be the same for each pion; then it follows from the property of crossing symmetry for the amplitude A that

$$A_I(\bar{s},\bar{t},\bar{u}) = A_{II}(\bar{t},\bar{s},\bar{u}) = A_{III}(\bar{u},\bar{t},\bar{s}) = A_I(\bar{s},\bar{u},\bar{t}) = \qquad - (A1.6)$$

$$A_{II}(\bar{u},\bar{s},\bar{t}) = A_{III}(\bar{t},\bar{u},\bar{s})$$

i.e. A is invariant under the interchange of any of the external particles. The three amplitudes A_I, A_{II}, A_{III} can be expressed in terms of the amplitudes T^I which corresponds to a definite value of total isotopic spin I by means of the projection operators for the various isospin states

$$P^0 = \tfrac{1}{3}\,\delta_{\gamma\delta}\,\delta_{\alpha\beta}$$

$$P^1 = \tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) \qquad - (A1.7)$$

$$P^2 = \tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) - \tfrac{1}{3}\,\delta_{\alpha\beta}\,\delta_{\gamma\delta}$$

so that $A = \sum_{I=0}^{2} T^I P^I$ - (A1.8)

$$\text{where } T^0 = 3A_I + A_{II} + A_{III}$$
$$T^1 = A_{II} - A_{III} \qquad \text{- (A1.9)}$$
$$T^2 = A_{II} + A_{III}$$

For the coupling of two particles the wave function is given by

$$\Psi_{I,I_3} = \sum_{I_3^1, I_3^2} C(I^1, I^2, I;\ I_3^1, I_3^2, I_3)\ \psi_{I^1, I_3^1}\ \psi_{I^2, I_3^2} \qquad \text{- (A1.10)}$$

and for the case of both particles being pions where $I^1 = I^2 = 1$ and I_3^1, I_3^2 range over +1,0,-1 the Clebsch-Gordon co-efficients have the values given in (A1.11)

	$I_3^2 = 1$	$I_3^2 = 0$	$I_3^2 = -1$
$I = 2$	$\sqrt{\frac{(1+I_3)(2+I_3)}{3.4}}$	$\sqrt{\frac{(2-I_3)(2+I_3)}{2.3}}$	$\sqrt{\frac{(1-I_3)(2-I_3)}{2.2.3}}$
$I = 1$	$-\sqrt{\frac{(1+I_3)(2-I_3)}{2.2}}$	$\sqrt{\frac{I_3 . I_3}{2}}$	$\sqrt{\frac{(1-I_3)(2+I_3)}{2.2}}$
$I = 0$	$\sqrt{\frac{(1-I_3)(2-I_3)}{2.3}}$	$-\sqrt{\frac{(1-I_3)(1+I_3)}{3}}$	$\sqrt{\frac{(2+I_3)(1+I_3)}{2.3}}$

- (A1.11)

Therefore associating π^+ with the wave function $-\psi_{1,1}$, π^0 with $\psi_{1,0}$ and π^- with $\psi_{1,-1}$ (where the phase convention of Condon and Shortley[73] has been assumed) the amplitudes for the various physical processes can be written in terms of the isotopic spin amplitudes T^0, T^1, T^2 as

$$A(\pi^{\pm}\pi^{\pm} \rightarrow \pi^{\pm}\pi^{\pm}) = T^2$$
$$A(\pi^{\pm}\pi^{0} \rightarrow \pi^{\pm}\pi^{0}) = \tfrac{1}{2}T^2 + \tfrac{1}{2}T^1$$
$$A(\pi^{0}\pi^{0} \rightarrow \pi^{0}\pi^{0}) = \tfrac{2}{3}T^2 + \tfrac{1}{3}T^0 \qquad \text{- (A1.12)}$$
$$A(\pi^{+}\pi^{-} \rightarrow \pi^{+}\pi^{-}) = 1/6T^2 + \tfrac{1}{2}T^1 + \tfrac{1}{3}T^0$$
$$A(\pi^{+}\pi^{-} \rightarrow \pi^{0}\pi^{0}) = -\tfrac{1}{3}T^2 + \tfrac{1}{3}T^0$$

In the mass centre the three invariants (A1.4) can be written in terms of k, the magnitude of the three momentum of each particle, and θ_π the angle of scattering as

$$\bar{s} \equiv \sigma^2 = 4(k^2 + \mu^2)$$
$$\bar{t} = -2k^2(1 - \cos\theta_\pi) \qquad \text{- (A1.13)}$$
$$\bar{u} = -2k^2(1 + \cos\theta_\pi)$$

which are the same equations as derived in Chapter I for the example of equal mass scattering. However, since pions obey Bose statistics, when the pion-pion amplitude is expanded in partial waves the imposition of the Pauli Principle, which demands that the composite wave function must be symmetric under the interchange of the constituent pions, limits the summation to include states of only odd angular momentum and odd isospin or even angular momentum and even isospin; that is to say

$$T^{I}(k^{2},\cos\theta_{\pi}) = \sum_{\substack{\ell \text{ even, } I \text{ even} \\ \ell \text{ odd, } I \text{ odd}}} (2\ell + 1)\, A_{\ell}^{I}(k^{2}) P_{\ell}(\cos\theta_{\pi}) \quad - \text{(A1.14)}$$

where each partial wave can be written in terms of a partial wave phase-shift as

$$A_{\ell}^{I}(k^{2}) = \frac{1}{16\pi}\sqrt{\frac{k^{2}+\mu^{2}}{k^{2}}}\; e^{i\delta_{\ell}^{I}} \sin\delta_{\ell}^{I} . R_{\ell}^{I}(k^{2}) \quad - \text{(A1.15)}$$

where $R_{\ell}^{I}(k^{2})$ is the ratio of the total partial-wave cross-section to the elastic cross-section.

2. Kinematics of the Inelastic Processes

In this section we shall give some useful relations between variables defined in the overall centre of mass system and quantities defined in the laboratory system for the process

$$\pi(q) + N(p) \rightarrow \pi(q_1') + \pi(q_2') + N(p') \qquad - (A2.1)$$

and the associated process

$$\pi(q) + N(p) \rightarrow \tilde{\sigma}(q') + N(p') \qquad - (A2.2)$$

where $\tilde{\sigma}(q')$ represents $\pi(q_1') + \pi(q_2')$ considered as a single system of 'mass' σ. Only those results which are required in the thesis are presented here. For a full discussion of the kinematics of such production processes as (A2.1) we refer the reader to an excellent exposition given by Ferrari and Selleri[13].

If p, q, p', q_1', q_2', q' are taken to be the four momenta of the particles as indicated in (A2.1) and (A2.2) the usual Lorentz invariant quantities can be written as

$$\begin{aligned} s &= -(p + q)^2 = -(p' + q_1' + q_2')^2 = -(p' + q')^2 \\ t &= -(p - p')^2 = -(q - q_1' - q_2')^2 = -(q - q')^2 \\ u &= -(q - p')^2 = -(q_1' + q_2' - p)^2 = -(q' - p)^2 \end{aligned} \qquad - (A2.3)$$

where use has been made of the energy-momentum conservation conditions

$$p + q = p' + q'$$
$$q' = q_1' + q_2' \qquad \text{- (A2.4)}$$

If μ is taken as the mass of each pion and m the mass of ~~nucleon~~ then in the centre of mass system ($\vec{p} + \vec{q} = 0$) for process (A2.2)

$$p = (iE, \vec{p}); \quad p' = (iE', \vec{p}')$$
$$q = (iw, -\vec{p}); \quad q' = (iw', -\vec{p}') \qquad \text{- (A2.5)}$$

so that

$$E^2 = m^2 + \bar{p}^2; \qquad E'^2 = m^2 + \bar{p}'^2$$
$$w^2 = \mu^2 + \bar{p}^2; \qquad w'^2 = \sigma^2 + \bar{p}'^2 \qquad \text{- (A2.6)}$$

where

$$\bar{p} = |\vec{p}| \; ; \qquad \bar{p}' = |\vec{p}'| \qquad \text{- (A2.7)}$$

Therefore, if

$$\cos\theta = \frac{\vec{p}.\vec{p}'}{\bar{p}.\bar{p}'} \qquad \text{- (A2.8)}$$

we can write

$$s \equiv W^2 = (E + w)^2 = m^2 + \mu^2 + 2\bar{p}^2 + 2Ew$$
$$= (E' + w')^2 = m^2 + \sigma^2 + 2\bar{p}'^2 + 2E'w' \qquad \text{- (A2.9)}$$

and

$$t \equiv -\Delta^2 = 2(m^2 + \overline{p}\,\overline{p}' \cos\theta - E.E') \qquad \text{- (A2.10)}$$

From energy conservation

$$E + w = E' + w' \qquad \text{- (A2.11)}$$

it follows that

$$w' = (W^2 + \sigma^2 - m^2)/2W \qquad \text{- (A2.12)}$$

and hence

$$\overline{p}'^2 = \left[(W+\sigma)^2 - m^2\right]\left[(W-\sigma)^2 - m^2\right]/4W^2 \qquad \text{- (A2.13)}$$

from which it may be seen that σ, the pion-pion centre of mass energy has a range of values from the two pion threshold 2μ up to its maximum value $(W - m)$ corresponding to $\overline{p}'^2 = 0$.

In the laboratory frame of reference in which the nucleon N(p) is at rest ($\vec{p} = 0$) it can be deduced that

$$\begin{aligned} &\text{the kinetic energy of } \pi(q),\ T_L = (W^2 - (m+\mu))^2/2m \\ &\text{the three momentum of } \pi(q),\ q_L = (T_L(T_L + 2m))^{\frac{1}{2}} \\ &\text{the kinetic energy of } N(p'),\ T_N = \Delta^2/2m \\ \text{and}\ &\text{the three-momentum of } N(p'),\ p_N = (T_N(T_N + 2m))^{\frac{1}{2}} \end{aligned} \qquad \text{- (A2.14)}$$

The maximum value of the pion-pion centre of mass energy $\sigma_{max.}$ can be expressed in terms of T_L by the relation

$$\sigma_{max} = (2mT_L + (m + \mu)^2)^{\frac{1}{2}} + m \qquad - (A2.15)$$

so that σ max is an increasing function of T_L. Also in the laboratory frame Θ_L, the scattering angle between $\vec{p}'$ and $\vec{q}$ is defined by the equation

$$2p_N q_L \, \mathrm{Cos}\Theta_L = \sigma^2 - \mu^2 + 2T_N(T_L + m + \mu) \qquad - (A2.16)$$

from which it follows that

$$\begin{aligned} d\sigma^2 d\Omega \equiv d\sigma^2 d(\mathrm{Cos}\Theta)\, d\phi &= 2p_N q_L \, d(\mathrm{Cos}\Theta_L)\, d\phi_L \, d(\mathrm{Cos}\Theta) \\ &= -2p_N q_L \, d\Omega_L \, d(\mathrm{Cos}\Theta) \end{aligned} \qquad - (A2.17)$$

where, because of the definition of Θ_L, $d\Omega_L = -d(\mathrm{Cos}\Theta_L)d\phi_L$. Therefore from (A2.10)

$$d\sigma^2 d\Omega = \frac{p_N q_L}{\bar{p}\,\bar{p}'} \, d\Omega_L \, d\Delta^2 \qquad - (A2.18)$$

and from (A2.14)

$$d\sigma^2 d\Omega = 2m \frac{p_N q_L}{\bar{p}\,\bar{p}'} \, d\Omega_L \, dT_N \qquad - (A2.19)$$

Hence the differential cross-sections for the process $\pi N \rightarrow \pi\pi N$ defined in the centre of mass system, $d^2\sigma_x/d\sigma^2 d\Omega$, can be expressed in terms of $d^2\sigma_x/dT_N d\Omega_L$, the differential cross-sections measured in the laboratory frame of reference as

$$\frac{d^2\sigma_x}{d\sigma^2 d\Omega} = \frac{\bar{p}\,\bar{p}'}{2m\, p_N q_L} \cdot \frac{d^2\sigma_x}{dT_N d\Omega_L} \qquad - (A2.20)$$

3. Kinematic Singularities of the $\pi N \to \sigma N$ Partial Wave Amplitudes[27]

For a fixed value of the energy σ the differential cross-section for the process $\pi N \to \sigma N$ is defined in the centre of mass system by the relation

$$\frac{d\sigma_*}{d\Omega} = \frac{1}{(8\pi W)^2} \sum \frac{\bar{p}'}{\bar{p}} \left| F_{\lambda_a \lambda_c} \right|^2 \qquad - (A3.1)$$

where the sum represents an average over the initial spin states and a summation over the final spin states. $F_{\lambda_a \lambda_c}$ is related to two invariant amplitudes A and B by the equation

$$F_{\lambda_a \lambda_c} = \bar{u}_{\lambda_c}(p') \left[A(s,t,u)\gamma_5 + i(q+q')_\mu \gamma_5 \gamma^\mu B(s,t,u) \right] u_{\lambda_a}(p) \qquad - (A3.2)$$

However, we may also write the differential cross-section in the centre of mass system in the form

$$\frac{d\sigma_*}{d\Omega} = \sum \frac{\bar{p}'}{\bar{p}} \left| f_{\lambda_a \lambda_c} \right|^2 \qquad - (A3.3)$$

where

$$f_{\lambda_a \lambda_c} = \chi_c^+ \left(f_1(\theta) \frac{\vec{\sigma}\cdot\vec{p}'}{|\vec{p}'|} + f_2(\theta) \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} \right) \chi_a \qquad - (A3.4)$$

and $\vec{\sigma}$ are the Pauli spin matrices, χ_a, χ_c are two-spinors. We may therefore relate F to f by the convention

$$-\frac{1}{8\pi W} F_{\lambda_a \lambda_c} = f_{\lambda_a \lambda_c} \qquad - (A3.5)$$

and by a little manipulation, f_1 and f_2 may be related to the invariant functions A and B in the following manner

$$f_1 = \frac{1}{8\pi W}\left(A.\sqrt{\frac{E+m}{E'+m}}\,\bar{p}' - B\sqrt{\frac{E+m}{E'+m}}\,2W\,\bar{p}'\right)$$

$$f_2 = \frac{1}{8\pi W}\left(-A\sqrt{\frac{E'+m}{E+m}}\,\bar{p} - B\sqrt{\frac{E'+m}{E+m}}\,2W\,\bar{p}\right) \qquad - \text{(A3.6)}$$

From the Mandelstam hypothesis we will assume that these invariant functions satisfy spectral representations such that

$$A(s,t,u) = 1/\pi^2\int dt'\int du'\,\frac{a_{23}(t',u')}{(t'-t)(u'-u)} + 1/\pi^2\int ds'\int dt'\,\frac{a_{12}(s',t')}{(s'-s)(t'-t)}$$

$$+ 1/\pi^2\int du'\int ds'\,\frac{a_{31}(u',s')}{(u'-u)(s'-s)} + (\text{pole terms}) \qquad - \text{(A3.7)}$$

and there exists a similar expression for B. Equation (A3.7) can be re-written in the form

$$A(s,t,u) = 1/\pi\int dt'\,\frac{\rho_1(s,t')}{t'-t} + 1/\pi\int du'\,\frac{\rho_2(s,u')}{u'-u} + (\text{pole terms}) \qquad - \text{(A3.8)}$$

and similarly for B(s,t,u), where

$$\rho_1(s,t') = 1/\pi\int ds'\,\frac{a_{12}(s',t')}{s'-s} + 1/\pi\int du'\,\frac{a_{23}(t',u')}{u'+t'+s-\Sigma}$$

$$\rho_2(s,u') = 1/\pi\int ds'\,\frac{a_{31}(u',s')}{s'-s} + 1/\pi\int dt'\,\frac{a_{23}(t',u')}{u'+t'+s-\Sigma} \qquad - \text{(A3.9)}$$

and $\Sigma = 2m^2 + \mu^2 + \sigma^2$

Note that from crossing symmetry

$$\rho_1(s,t) = \pm\rho_2(s,t) \qquad - \text{(A3.10)}$$

depending on whether A(B) is symmetric or antisymmetric under crossing.

From equations (2.30) and (A3.6) the partial wave amplitudes $f_{\ell\pm}$ are defined in terms of A and B by the equation

$$f_{\ell\pm} = \frac{1}{8\pi W}\left[p'\sqrt{\frac{E+m}{E'+m}}\left(A_\ell - 2W B_\ell\right) - \bar{p}\sqrt{\frac{E'+m}{E+m}}\left(A_{\ell\pm1} + 2W B_{\ell\pm1}\right)\right] \qquad - \text{(A3.11)}$$

where

$$\left(A_\ell\,;\,B_\ell\right) = \int_{-1}^{1} dx\, P_\ell(x)\,\left(A\,;\,B\right) \qquad - \text{(A3.12)}$$

In order to determine the kinematic singularities of these partial waves, we consider the functions $A_\ell/(\bar{p}\,\bar{p}')^\ell$, $B_\ell/(\bar{p}\,\bar{p}')^\ell$. From equations (A3.8), (A3.9) and (A3.10) it follows that we can write $A_\ell/(\bar{p}\,\bar{p}')^\ell$ as

$$\frac{A_\ell(s)}{(\bar{p}\bar{p}')^\ell} = 1/\pi \int dt'\,\rho_1(s,t')\,I_\ell(t',s) \qquad - \text{(A3.13)}$$

where

$$I_\ell(t',s) = \frac{1}{(\bar{p}\bar{p}')^\ell}\int_{-1}^{1} d(\cos\theta)\,P_\ell(\cos\theta)\left(\frac{1}{t'+\bar{p}^2+\bar{p}'^2-2\bar{p}\bar{p}'\cos\theta} + \frac{(-1)^\ell}{t'+\bar{p}^2+\bar{p}'^2+2\bar{p}\bar{p}'\cos\theta}\right) \qquad - \text{(A3.14)}$$

It will be seen that I_ℓ contains no singularities other than those arising from vanishing denominators in the Mandelstam representation.

Since

$$\frac{1}{2}\int_{-1}^{1}\frac{P_\ell(x)\,dx}{y-x} = Q_\ell(y)$$

the integral in (A3.14) will vanish at $\bar{p} = 0$, $\bar{p}' = 0$ like $(\bar{p}\,\bar{p}')^\ell$ so that no pole is introduced by dividing by this factor. Also, since I_ℓ is a function only of even powers of $(\bar{p}\,\bar{p}')$ no branch points occur arising from these kinematical factors. Hence $A_\ell/(\bar{p}\,\bar{p}')^\ell$ and similarly $B_\ell/(\bar{p}\,\bar{p}')^\ell$ are free from kinematical singularities and therefore from equation (A3.11) we can define functions $h_{\ell\pm}$ such that

$$h_{\ell\pm} = \sqrt{\frac{E+m}{E'+m}}\;\frac{W}{\bar{p}'\pm 1}\;\frac{f_{\ell\pm}}{(\bar{p}\bar{p}')^\ell}$$

which are analytic in p^2 and p'^2 except for branch cuts from $-m^2$ to $-\infty$ and dynamical singularities.

4. The N over D Equations and CDD Poles[74]

We wish to discuss the partial wave scattering amplitude A(s) in a one channel formalism. Frye and Warnock[75] have developed coupled integral equations in terms of N and D functions to calculate the amplitude A and have introduced a complex phase shift $\delta = \alpha + i\beta$ to denote the inelastic contributions to the unitarity equation arising above the first inelastic threshold. Chew and Mandelstam have also devised integral equations for A(s), but in their case they introduce the inelasticity by the function R(s) which is the ratio of the total to the elastic cross-sections. We shall denote by $\delta(s)$ the phase-shift of the scattering amplitude if we are concerned with an elastic system, the real part of the phase-shift if we are using the Frye-Warnock equations, and the phase of the amplitude in the Chew-Mandelstam equations.

Let us write

$$A(s) = N(s)/D(s) \qquad - (A4.1)$$

where N(s) has only the left hand cut and D(s) has only the right hand cut. We shall construct the D-function so that it tends to a real constant at infinity; hence D(s) must satisfy the following conditions:

i. It must have the phase $\delta(s)$ for $s_T \leq s < \infty$.

ii. It must have a zero corresponding to each particle pole of A whether the particle is 'elementary' or not.

iii. It must tend to a constant at infinity. (By the Riemann-Schwartz principle this constant must be real and it may be normalised to one.)

Condition i. is satisfied if we write D in terms of $\mathcal{D}$ where

$$\mathcal{D}(s) = \exp\left\{-s/\pi \int_{s_T}^{\infty} \frac{\delta(s')\,ds'}{s'(s'-s)}\right\} \qquad - (A4.2)$$

However $\mathcal{D}(s)$ has no zeros so we must multiply $\mathcal{D}(s)$ by the factor $\prod_{i=1}^{n_B}(s - p_i)$ where n_B denotes the number of particle poles (on the physical sheet) whether elementary or not. If we assume that $\delta(\infty) = m\pi$ then it is seen that in order to satisfy condition ii. we must divide the product $\mathcal{D}(s).\prod_{i=1}^{n_B}(s - p_i)$ by a polynomial of degree $m + n_B = n_c$ (say). Thus, we can write the D-function which satisfies condition i. to iii. as

$$D(s) = \frac{\prod_{i=1}^{n_B}(s-p_i)}{\prod_{j=1}^{n_c}(s-s_j)}\,.\,\mathcal{D}(s) \qquad - (A4.3)$$

where the s_j will be taken to be arbitrary. Note that since $N = A.D$, $N(s)$ will in general share the same poles as $D(s)$, (CDD poles).

It is easily seen that these pole positions s_j are in fact arbitrary. For example, if $n_c = 1$, N and D have one pole each. Since D tends to real constant at infinity, ImD $\to$ 0 and it follows by a suitable normalisation of D that an unsubtracted dispersion relation can be written for D - 1. Also, since ImD = ρ N for $s > s_T$ from unitarity, where ρ is a certain kinematical term (e.g. $\rho = -\frac{1}{16\pi}\sqrt{\frac{s-4m^2}{s}}$ for equal mass spinless particles). N(s) $\to$ 0 for $s \to \infty$ and therefore N(s) satisfies an unsubtracted dispersion relation. From the Cauchy representations of D and N, we obtain

$$D(s) = 1 + \frac{a_1}{s - s_1} + \frac{1}{\pi}\int_{s_T}^{\infty} \frac{\rho(s')\,N(s')}{s' - s}\,ds'$$

$$N(s) = \frac{b_1}{s - s_1} + \frac{1}{\pi}\int_{-\infty}^{0} \frac{\mathrm{Im}\,A(s')\,D(s')}{s' - s}\,ds' \qquad \text{- (A4.4)}$$

where a_1 and b_1 are the CDD parameters and s_1 is the arbitrary pole position. Now if one forms

$$D_2(s) = \frac{s - s_1}{s - s_2}\,D(s)$$

$$N_2(s) = \frac{s - s_1}{s - s_2}\,N(s) \qquad \text{- (A4.5)}$$

by direct substitution into (A4.5) we see that

$$D_2(s) = 1 + \frac{a_2}{s - s_2} + \frac{1}{\pi}\int_{s_T}^{\infty} \frac{\rho(s')\,N_2(s')}{s' - s}\,ds'$$

$$N_2(s) = \frac{b_2}{s - s_2} + \frac{1}{\pi}\int_{-\infty}^{0} \frac{\mathrm{Im}\,A(s')\,D_2(s')}{s' - s}\,ds' \qquad \text{- (A4.6)}$$

where

$$a_2 = a_1 + (s_2 - s_1) + \frac{s_2 - s_1}{\pi} \int_{s_T}^{\infty} \frac{\rho(s')N_2(s')}{s' - s_1} ds'$$

$$\text{and} \quad b_2 = b_1 + (s_1 - s_2)/\pi \int_{-\infty}^{0} \frac{\text{Im } A(s')D_2(s')}{s' - s_1} ds' \qquad - (A4.7)$$

so that s_1 is truly arbitrary.

However if we <u>define</u> s_2 so that either a_2 or b_2 is zero then we can write equations (A4.6) either as

$$D_2(s) = 1 + \frac{a}{s - s_o} + \frac{1}{\pi} \int_{s_T}^{\infty} \frac{\rho(s')N_2(s')}{s' - s} ds'$$

$$N_2(s) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\text{Im } A(s')D_2(s')}{s' - s} ds' \qquad - (A4.8)$$

where we have written s_o for s_2; or as

$$D_2(s) = 1 + \frac{1}{\pi} \int_{s_T}^{\infty} \frac{\rho(s')N_2(s')}{s' - s} ds'$$

$$N_2(s) = \frac{b}{s - s_B} + \frac{1}{\pi} \int_{-\infty}^{0} \frac{\text{Im } A(s')D_2(s')}{s' - s} ds' \qquad - (A4.9)$$

where s_B is written for s_2 ($s_B < s_T$).

In (A4.8) a and s_o are the two CDD pole parameters, and in (A4.9) b and s_B are the two parameters. Note that s_B is the position of a particle pole. If $s_B < s_T$, the two forms (A4.4) and (A4.9) are equivalent. However, if the effect of a_1 and b_1 is to produce a zero in $D_1(s)$ at $\overline{\text{Re}}\, s_B > s_T$, the particle pole is that of a resonance and necessarily has s_B complex (on the second Riemann sheet). In this case, equations (A4.9) would have to be modified.

From the above discussion it can be seen that the three forms of introducing CDD parameters into the N and D equations, i.e. (A4.4), (A4.8) and (A4.9) are generally equivalent. The arbitrariness of s_1 in (A4.4) means that only <u>two</u> parameters are introduced by a CDD pole in this form, just as in (A4.8) and (A4.9).

If we reconsider the decomposition of A(s) in terms of n/d where d has no poles but can have any number of subtractions, then we must write

$$d(s) = D(s) \prod_{j=1}^{n_c} (s - s_j) \qquad \text{-- (A4.10)}$$

It follows, therefore, that Re $d(s) \sim s^{n_c}$ and Im $d/s^{n_c} \to 0$ as $s \to \infty$, so that there exists a dispersion relation for d with n_c subtractions. From the equations for N and D it may be seen directly that n and d satisfy the equations

$$d(s) = 1 + \alpha_1(s - s_o) + \ldots + \alpha_{n_c}(s - s_o)^{n_c} + \frac{(s - s_o)^{n_c}}{\pi} \int_{s_T}^{\infty} ds' \frac{\rho(s')n(s')}{(s' - s)(s' - s_o)^{n_c}} \tag{A4.11}$$

$$n(s) = \beta_1 + \beta_2(s - s_o) + \ldots + \beta_{n_c}(s - s_o)^{n_c - 1} + \frac{(s - s_o)^{n_c}}{\pi} \int_{-\infty}^{0} ds' \frac{\mathrm{Im}\, A(s')d(s')}{(s' - s)(s' - s_o)^{n_c}}$$

One may also show that s_o, the subtraction point, is arbitrary and that the s_{n_c} parameters α_i, β_i are determined by the residues of the CDD poles, e.g. a_1, b_1 in (A4.4).

R E F E R E N C E S

REFERENCES

1. H Yukawa, Proc. Phys. Math. Soc. Japan, 17 48 (1935)

2. H Yukawa, S Sakata, Proc. Phys. Math. Soc. Japan, 19 1084 (1937)

3. C M G Lattes, H Muirhead, G P S Occhialini, C F Powell, Nature 159 694 (1947)

   C M G Lattes, G P S Occhialini, C F Powell, Nature 160 453, 486 (1947)

4. See for example A H Rosenfeld et al Revs. Mod. Phys. 37 633 (1965)

5. R J Eden, P V Landshoff, D I Olive, J.C Polkinghorne, 'The Analytic S-Matrix', Cambridge Univ. Press (1966)

6. S S Schweber, 'Introduction to Relativistic Quantum Field Theory', Harper and Row, New York (1961)

7. Throughout the thesis four-vectors are $A_\mu = (A, A_4 = iA_o)$ so that scalar products are $A.B = A.B - A_oB_o$

8. G F Chew, S Mandelstam, Phys. Rev. 119 467 (1960)

9. J Hamilton, Review Article in 'Strong Interactions and High Energy Physics', Scottish Summer School (1963)

10. J Hamilton, P Menotti, T D Spearman, Annals of Physics 12 172 (1961)

    J Hamilton, P Menotti, T D Spearman, W S Woolcock, Nouvo Cimento 20 519 (1961)

    J Hamilton, T D Spearman, W S Woolcock, Annals of Physics 17 1 (1962)

    J Hamilton, P Menotti, G C Oades, L L J Vick, Phys. Rev. 128 1881 (1962)

11. T D Spearman, Phys. Rev. 129 1847 (1963)

12. A Abashian, N E Booth, K M Crowe, Phys. Rev. Letters 5 258 (1960); Phys. Rev. Letters 7 35 (1961); Phys. Rev. 132 2309, 2314 (1963)

13. For a full discussion see E Ferrari, F Selleri, Nouvo Cimento Supplement 24 453 (1962)

14. C.Ceolin, R.Stroffolini, Nouvo Cimento 22 437 (1961)

15. B C Barish, UCRL - 10470 (1962) (unpublished)
R J Kurz, UCRL - 10564 (1962) (unpublished)
J Solomon, UCRL - 10585 (1963) (unpublished)
B C Barish, R J Kurz, V Perez-Mendez, J Solomon, Phys. Rev. 135 B416 (1964)

16. J Kirz, J Schwartz, R D Tripp, Phys. Rev. 130 2481 (1963)

Also, T D Blockhintseva et al, J E T P 17 80 (1963)

17. C J Goebel, H J Schnitzer, Phys. Rev. 123 1021 (1961)

H J Schnitzer, Phys. Rev. 125 1059 (1962)

18. M Olsson, G B Yodh, Phys. Rev. Letters 10 353 (1963)

19. J Pisut, Phys. Letters 21 569 (1966)

J Pisut, P Bona, P Lichard, Univerzita Komenskehe, Bratislava, Czech. preprint

20. H J Rothe, Phys. Rev. 140 B1421 (1965)

21. C Lovelace, R M Heinz, A Donnachie, Phys. Letters 22 332 (1966)

22. G Wolf, Phys. Letters 19 329 (1964)

23. G F Chew, F E Low, Phys. Rev. 113 1640 (1959)

24. C Moller, Kgl. Danske Videnskab. Selskab. Mat-fys. Medd. 23 No.1 (1945); 24 No.19 (1946)

25. M Jacob, G C Wick, Annals of Physics 7 404 (1959)

26. L S Rodberg, Phys. Rev. Letters 3 58 (1959)

V V Anisovich, JETP 12 71 and 946 (1961)

B C Barish et al, Phys. Rev. Letters 6 297 (1961)

27. A somewhat similar analysis was performed for $\pi\pi \to N\bar{N}$ partial waves by W R Frazer, J R Fulco, Phys. Rev. 117 1609 (1960)

28. S L Adler, Phys. Rev. 140 B736 (1965)

29. D Atkinson, Phys. Rev. 128 1908 (1962)

30. P Menotti, Nouvo Cimento, 23 931 (1961)

31. K M Watson, Phys. Rev. 88 1163 (1952)

32. M Jacob, G Mahoux, R Omnes, Nouvo Cimento 23 838 (1962)

33. E P Wigner, Phys. Rev. 98 145 (1955)

34. L Castillejo, R H Dalitz, F J Dyson, Phys. Rev. 101 453 (1956)

35. J D Bjorken, Phys. Rev. Letters 4 473 (1960)

36. P W Coulter, G L Shaw, Phys. Rev. 141 1419 (1966)

37. S Frautschi, 'Regge poles and S-matrix Theory', Benjamin (1963)

38. D Atkinson, M B Halpern, Phys. Rev. 150 1377 (1966)

39. J Rothleitner, R Stech, Zeitshrift fur Physik 180 377 (1964)

40. E J Squires, Nouvo Cimento 34 1751 (1964)

41. G F Chew, S Mandelstam, Nouvo Cimento 19 752 (1961)

42. See for example, F Zachariasen, Phys. Rev. Letters 7 122 (1961)
J S Ball, D Y Wong, Phys. Rev. Letters 7 390 (1961)
L A P Balazs, Phys. Rev. 128 1939 (1962)

43. C F Kyle, A W Martin, H R Pagels, Stanford Univ. preprint ITP - 229.

44. B H Bransden, J W Moffat, Nouvo Cimento 21 505 (1961); 23 598 (1963)

45. F T Meiere, M Sugawara, Phys. Rev. 153 1702 (1967)

46. S Weinberg, Phys. Rev. Letters 17 616 (1966)

47. See for example, A P Balachandran, M G Gundzik, F Nicodemi, Syracuse Univ. preprint.
N N Khuri, Phys. Rev. 153 1477 (1967)

48. S L Adler, Phys. Rev. 137 B1022 (1965); 139 B1638 (1965)

49. J Iliopoulos, CERN preprint TH 775

50. J Sucher, C-H Woo, Phys. Rev. Letters 18 723 (1967)

51. F T Meiere, M Sugawara, Phys. Rev. 153 1707 (1967)

52. See for example, L Durand, Y Chiu, Phys. Rev. Letters 14 329 (1965); erratum 14 680 (1965)
V Hagopian, W Selove, Phys. Rev. Letters 10 533 (1963)
M Islam, R Pinon, Phys. Rev. Letters 12 310 (1964)

53. M Bander, G L Shaw, Phys. Rev. 155 1675 (1967)

54. N N Khuri, S B Treiman, Phys. Rev. 119 1115 (1960)

55. R F Sawyer, K C Wali, Phys. Rev. 119 1429 (1960)

56. L M Brown, P Singer, Phys. Rev. 133 3812 (1964)

57. R Prasad, Nouvo Cimento 35 682 (1965)

58. W A Dunn, R Ramachandran, Phys. Rev. 153 1558 (1967)

59. I M Barbour, R L Schult, Univ. of Illinois preprint

60. M Jacob, G Mahoux, R Omnes, Nouvo Cimento 23 838 (1962)

61. A T Maksymowicz, UCRL - 16026 (1965)

62. C Kacser, P Singer, T N Truong, Phys. Rev. 137 B1605 (1965); 139 AB5(E) (1965)

63. Proceedings of the Thirteenth International Conference on High Energy Physics, Berkely, California 1966 (to be published)

64. K Kang, D J Land, Phys. Rev. Letters 18 503 (1967)

65. V Barger, E Kazes, Phys. Rev. 124 279 (1961)

66. K Nishijima, Phys. Rev. Letters 12 39 (1964)

67. T N Truong, Phys. Rev. Letters 17 1102 (1966)

68. R Rockmore, T Yao, Phys. Rev. Letters 18 501 (1967)

69. For further references see P Singer, Finnish Summer School (1966) (to be published)

70. V V Serebryakov, D V Shirkiv, Zh-Eksp. i Teor. Fiz. 42 610 (1962)

71. G F Chew, Phys. Rev. Letters 16 60 (1966)

72. L F Cook, Phys. Rev. Letters 17 212 (1966)

73. E U Condon, G H Shortley, 'Theory of Atomic Spectra', Cambridge Univ. Press (1964)

74. We follow the discussion given by D Atkinson, K Dietz, D Morgan, Annals of Physics 37 77 (1966)

75. G Frye, R L Warnock, Phys. Rev. 130 478 (1963)