# T R A N S O N I C   F L O W

by

THOMAS EVANS

Thesis submitted to the University of Durham in the application for the degree of Doctor of Philosophy.

April 1967.

## Acknowledgements

The author wishes to express his appreciation of the guidance and encouragement given by his supervisor, Professor K. Stewartson. He also wishes to thank Mr. D. L. Woodcock and the members of Structures Department (D), R.A.E., Farnborough who introduced him to this branch of aerodynamics. A substantial part of the work was supported by Ministry of Aviation contracts with the Universities of Durham and London. Further expressions of thanks are due to Dr. N. Riley for helpful discussion at the final stage of thesis preparation, and to Miss M. Wing for the typing.

<u>CONTENTS</u> <u>Page</u>

C O N T E N T S (continued)

## ABSTRACT

The main original contributions of this thesis are presented in Chapters II, III and IV, each of which is largely self-contained, but which are all directed at particular aspects of the same problem, the prediction of the pressure on an airfoil in symmetrical transonic flow. Chapter I sets this problem in its relation to transonic aerodynamics and provides accounts of various studies which are used in subsequent chapters with only limited explanation. Chapters II and III are devoted to extending the theory of Cole and Royce for transonic flow past an axisymmetric body, first to two-dimensional transonic flow past a thin airfoil, and then to flow past a thin planar delta wing. Chapter IV examines a different type of approximation which has received considerable attention, the parabolic equation approximation. It is shown how to remove certain deficiencies of methods based on this approximation, but one of the consequences is to cast doubt on the reliability of the method which appeared to give closest agreement with experiment.

# CHAPTER I

## TRANSONIC AERODYNAMICS FOR STEADY FLOW.

In this chapter we review the small perturbation theory of transonic flow, and present in some detail accounts of those parts of it which are both used and extended in subsequent chapters. The intention is to set the subsequent work in context and also eliminate the need to give more than a brief reference when established methods are employed.

## §1. Introduction. The equations and boundary conditions

This work is concerned almost entirely with external aerodynamics and primarily with the flow past thin airfoils; internal aerodynamics enters only indirectly in the assessment of data derived from wind tunnels. The term transonic applies when the velocities in the flow field are in the neighbourhood of the local sound speed, the speed at which small disturbances propagate through the fluid, and the essential feature is the existence of both subsonic and supersonic regions. The typical situation is that of a body advancing into still air at nearly sonic speed, or as observed from a reference frame fixed in the body, a near sonic stream of air being perturbed by the presence of the body.

The equations of motion of a viscous, compressible, heat conducting gas, regarded as a continuum, are well established [1]. Almost certainly these describe the transonic situation indicated above, but they are of such complexity that drastic simplification is required to make progress towards a solution. It is usual to assume that viscosity and heat conduction are only significant in narrow layers, boundary layers and shocks, and to eliminate the corresponding terms in the equations. This is the inviscid model. In it, the boundary layers are accounted for by relaxing the no-slip condition, and the shocks, by allowing surfaces of discontinuity in the fluid. It seems likely that this model is not valid at large distances from the airfoil, and attempts have been made to include viscosity [2], [3]. However, experiment indicates that the model yields useful results in the neighbourhood of the airfoil, and it may be that the inviscid far field theories have some significance in an intermediate region. We confine ourselves to investigations within the inviscid theory.

The flow of an inviscid, non-conducting perfect gas is governed by:

the continuity equation

$$\frac{D}{Dt}\rho = -\rho \nabla \underset{\sim}{u} \qquad \underline{1.1},$$

the momentum equation

$$\frac{D}{Dt}\underset{\sim}{u} = -\frac{1}{\rho}\nabla p \qquad \underline{1.2},$$

and the energy equation

$$\frac{D}{Dt}S = 0 \qquad \underline{1.3},$$

where $\underset{\sim}{u}$ is the velocity, $\rho$ the density, p the pressure, S the entropy and $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{u}.\nabla$ .

These equations do not apply at shocks. Instead we have the corresponding jump relations. In terms of velocity components, these are accounted for by the continuity of the component tangential to the shock, and the shock polar, [4],

$$V_2^2 \left\{ \left[ 2/(\gamma+1) \right] U_1^2 - U_1 U_2 + c_*^2 \right\} = (U_1 - U_2)^2 \left\{ U_1 U_2 - c_*^2 \right\} \qquad \underline{1.4}$$

where $U_1$ is the incident velocity, $U_2$ , $V_2$ are respectively the parallel and perpendicular components of the velocity behind the shock, c is the speed of sound, and asterisks denote critical values.

For steady flow, equations (1.1) - (1.3) reduce to

$$\nabla(\rho \underset{\sim}{u}) = 0 \qquad \underline{1.5},$$

$$\underset{\sim}{u} \times \underset{\sim}{\omega} = \frac{1}{\rho}\nabla p + \frac{1}{2}\nabla \underset{\sim}{u}^2 \quad \text{with } \underset{\sim}{\omega} = \nabla \times \underset{\sim}{u} \qquad \underline{1.6},$$

$$(\underset{\sim}{u} \cdot \nabla) S = 0 \qquad \underline{1.7}.$$

From (1.6), (1.7) and the first law of thermodynamics, (or for (1.8) simply by conservation of energy)

$$I + \frac{1}{2}\underset{\sim}{u}^2 = H \qquad \underline{1.8}$$

and

$$\underset{\sim}{u} \times \underset{\sim}{\omega} = \nabla H - T\nabla S \qquad \underline{1.9}$$

where H is constant along streamlines, I is the enthalpy per unit mass and T temperature. $I = c^2/(\gamma - 1)$ for adiabatic flow of a perfect gas and from (1.5), taking $\underset{\sim}{u} \cdot \nabla$ of (1.8)

$$(\underset{\sim}{u} \cdot \nabla)\frac{1}{2}\underset{\sim}{u}^2 = c^2 \nabla \underset{\sim}{u} \qquad \underline{1.10}$$

For a small perturbation of a uniform stream, noting that from the energy derivation (1.8) is valid across shocks, we have that H is constant throughout the fluid and (1.9) reduces to

$$\underset{\sim}{u} \times \underset{\sim}{\omega} = -T\nabla S \qquad \underline{1.11}$$

Assuming the body causes relatively small departure from a uniform stream we write $\underset{\sim}{u} = \underset{\sim}{U}_\infty + \underset{\sim}{v}$ and take $|\underset{\sim}{v}| \ll U_\infty$. Choosing the direction of $U_\infty$ as the $x$-axis, substituting in (1.10) and using (1.8), after rearrangement we have

$$(1-M_\infty^2)\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = M_\infty^2\left[(\gamma+1)\frac{u}{U_\infty} + \frac{\gamma+1}{2}\frac{u^2}{U_\infty^2} + \frac{(\gamma-1)}{2}\frac{v^2+w^2}{U_\infty^2}\right]\frac{\partial u}{\partial x}$$

$$+ M_\infty^2\left[(\gamma-1)\frac{u}{U_\infty} + \frac{(\gamma+1)}{2}\frac{v^2}{U_\infty^2} + \frac{(\gamma-1)}{2}\frac{w^2+u^2}{U_\infty^2}\right]\frac{\partial v}{\partial y}$$

$$+ M_\infty^2\left[(\gamma-1)\frac{u}{U_\infty} + \frac{(\gamma+1)}{2}\frac{w^2}{U_\infty^2} + \frac{(\gamma-1)}{2}\frac{u^2+v^2}{U_\infty^2}\right]\frac{\partial w}{\partial z}$$

$$+ M_\infty^2\left[\frac{v}{U_\infty}\left(1+\frac{u}{U_\infty}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + \frac{w}{U_\infty}\left(1+\frac{u}{U_\infty}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) + \frac{vw}{U_\infty^2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\right] \tag{1.12}$$

where $\underset{\sim}{v} = (u, v, w)$ and $M_\infty = U_\infty / c_\infty$ .

Consider the two-dimensional case when there is no dependence on z, and suppose $M_\infty \to 1$ . Retaining only the linear terms, the equation (1.12) reduces to $\frac{\partial v}{\partial y} = 0$ . It is evident that higher order terms must be retained. However, the slow attenuation of the disturbance in the y direction, implied by this result, is a feature of the actual flow. The breakdown of the linearised theory is not so evident in the general case; nevertheless a non-linear equation is still required.

For $M_\infty$ near one, any shocks occuring are necessarily weak, and as entropy changes are of order $(M-1)^3$ , the flow is irrotational to second order, and we may introduce a potential, writing $\underset{\sim}{v} = U_\infty \nabla\varphi$ . To deduce the appropriate form of the transonic small disturbance equation one requires estimates of the relative orders of magnitude of the velocity components and their derivatives. In two-dimensional flow, one such estimate is provided by the characteristics of the system consisting of (1.10) and the condition of irrotationality. Taking the velocity in polar form $(q, \theta)$, the compatability relation

$$\frac{1}{q}\frac{dq}{d\theta} = \pm\sqrt{\frac{c^2}{q^2-c^2}}$$

may be simplified when $\left|\frac{q-c}{c}\right| \ll 1$ by writing

$$\Delta q = q - c^* \quad , \quad q^2 - c^2 = (\gamma+1)\, q^* . \Delta q \quad + O(\Delta q^2)$$

where an asterisk denotes the critical value, when $q = c$. It reduces to the form

$$\sqrt{\frac{(\gamma+1)\,\Delta q}{q^*}} \cdot \frac{1}{q^*} \quad \frac{d}{d\theta}\Delta q \quad = \quad \pm 1 \qquad .$$

Integrating we have that $v/U_\infty$ has the same order as $(u/U_\infty)^{3/2}$. Now the linearised form of the condition expressing no flow normal to the thin airfoil, indicates $v = O(\tau)$, where $\tau$ is the thickness ratio; taken with the irrotationality condition it suggests the following scaled variables

$$\tilde{x} = x \quad , \quad \tilde{y} = \tau^{1/3} y \quad , \quad \tilde{u} = \tau^{-2/3} u \quad , \quad \tilde{v} = \tau^{-1} v$$

The scaled variables are regarded as order one as $\tau \to 0$. These estimates, together with the assumption that in the three-dimensional case $w = O(v)$, are sufficient to show that on the right hand side of (1.12)

$$M_\infty^2\,(\gamma+1)/U_\infty \quad u\,\frac{\partial u}{\partial x}$$

is the only term which is as large as the left hand side terms. Consequently, the transonic small disturbance equation is

$$(1 - M_\infty^2)\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = M_\infty^2 (\gamma+1)/U_\infty \; u\frac{\partial u}{\partial x}$$

or in terms of the perturbation potential

$$(1 - M_\infty^2)\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = k\,\varphi_x\,\varphi_{xx} \qquad \underline{1.13}$$

with $k = M_\infty^2 (\gamma+1)$. Equation(1.13) is also valid for axisymmetric flow.

The axisymmetric case has been treated more systematically by Cole and Messiter [5] . They obtain the relation between longitudinal and transverse velocity components by assuming the non-degeneracy of the shock polar in scaled variables, and beyond this, only require that the resulting small perturbation equation should not degenerate. In the scaling, the body radius $\delta$ replaces $\tau^{1/3}$ used above. The vorticity and entropy changes arise automatically as part of the procedure. The equations are derived effectively for $M_\infty = 1$ and there is an element of indeterminacy in the value of k for $M_\infty \neq 1$ , which can only be settled by developing the theory for higher powers of $(M_\infty - 1)$ . The value of k arising from the above procedure gives a good approximation to the variation of the critical pressure coefficient with $M_\infty$ , as shown by Spreiter [6] . He also noted that the corresponding choice in the shock polar gives exact values for the velocity jump at normal shocks.

The transonic approximation to the shock polar follows from (1.4) by taking $c_*$ from (1.8) and retaining only the largest terms. It is

$$(1 - M_\infty^2)(\varphi_x^{(1)} - \varphi_x^{(2)})^2 + (\varphi_y^{(1)} - \varphi_y^{(2)}) + (\varphi_z^{(1)} - \varphi_z^{(2)}) = \frac{k}{2}(\varphi_x^{(1)} + \varphi_x^{(2)})(\varphi_x^{(1)} - \varphi_x^{(2)})^2 \qquad \underline{1.14}$$

where superscripts denote values on opposite sides of the shock. The relative merits of the different possible values of k are also discussed, from a physical standpoint, by Oswatitsch [7] in connection with different forms of similarity parameter, a concept introduced in §2.

In establishing the small perturbation equations, we have not treated explicitly the origin of the perturbation, the boundary condition of no flow normal to the thin airfoil or slender body. The approximation of this condition constitutes an essential part of the determination of higher approximations to the system of equations, as is apparent in [5] . However, for $\varphi$ it is sufficient to retain the leading term exactly as in subsonic aerodynamics. For the thin airfoil $y = \tau f(x, \frac{z}{\sigma})$ , the condition is

$$\varphi_y = \tau f(x, z/\sigma) \quad \text{on} \quad y = 0 \qquad \underline{1.15}$$

the error in transferring the boundary condition from the airfoil to $y = 0$ being higher order in $\tau$ . For a slender body of revolution, the condition takes the different form

$$2\pi r \varphi_r = \delta^2 Q'(x) \quad \text{on} \quad r = \delta \;, \text{ as } \delta \to 0 \qquad \underline{1.16}$$

where $\delta^2 Q(x)$ is the area of cross-section at the station $x$ , and $r = (y^2 + z^2)^{1/2}$ .

The local pressure coefficient $C_p = (p - p_\infty)/(\frac{1}{2}\rho_\infty U_\infty^2)$, also follows by retaining the largest terms in the Bernoulli equation. For a thin airfoil

$$C_p = -2u/U_\infty = -2\varphi_x \qquad \underline{1.17}$$

again evaluated on $y=0$, rather than on the airfoil. For a slender body of revolution

$$C_p = -2u/U_\infty - (v^2+w^2)/U_\infty^2 = -2\varphi_x - \varphi_r^2 \qquad \underline{1.18}$$

here evaluated on the body surface, the $\varphi_r^2$ term being retained because of the singularity in $\varphi_r$ on the axis.

## §2 The similarity and equivalence rules

The mathematical problem posed by the equations (1.13), (1.14), either of the boundary conditions (1.15), (1.16) and the condition $\nabla\varphi \to 0$ at large distances from the body, apparently depends on the parameters $M_\infty$, $\gamma$ and either $\sigma, \tau$ or $\delta$. However it is possible, by similarity rules, to relate flows having different parameter values, provided certain groupings of parameters are the same. One parameter, the chord length, has already been removed from the problem by choosing it as the unit of length; $\tau, \sigma$ and $\delta$ are ratios specifying thickness, span and slenderness respectively. For the thin wing, writing

$$\bar{y} = \{|1-M_\infty^2|\}^{1/2} y \;, \quad \bar{z} = \{|1-M_\infty^2|\}^{1/2} z \;, \quad \bar{x} = x$$

$$\varphi = \frac{\tau}{\{|1-M_\infty^2|\}^{1/2}} \bar{\varphi}(\bar{x}, \bar{y}, \bar{z}) \qquad \underline{2.1}\,,$$

then (1.13), (1.14) and (1.15) reduce to

$$\operatorname{sgn}(1-M_\infty^2)\,\bar{\varphi}_{\bar{x}\bar{x}} + \bar{\varphi}_{\bar{y}\bar{y}} + \bar{\varphi}_{\bar{z}\bar{z}} = \frac{(\gamma+1)M_\infty^2 \tau}{|1-M_\infty^2|^{3/2}} \bar{\varphi}_{\bar{x}} \bar{\varphi}_{\bar{x}\bar{x}} \;,$$

$$\operatorname{sgn}(1-M_\infty^2)(\bar{\varphi}_{\bar{x}}^{(1)} - \bar{\varphi}_{\bar{x}}^{(2)})^2 + (\bar{\varphi}_{\bar{y}}^{(1)} - \bar{\varphi}_{\bar{y}}^{(2)})^2 + (\bar{\varphi}_{\bar{z}}^{(1)} - \bar{\varphi}_{\bar{z}}^{(2)})^2$$
$$= \frac{(\gamma+1)M_\infty^2\tau}{2|1-M_\infty^2|^{3/2}} (\bar{\varphi}_{\bar{x}}^{(1)} + \bar{\varphi}_{\bar{x}}^{(2)})(\bar{\varphi}_{\bar{x}}^{(1)} - \bar{\varphi}_{\bar{x}}^{(2)})^2 \;,$$

$$\bar{\varphi}_{\bar{y}} = f\left(\bar{x}, \frac{\bar{z}}{\sigma|1-M_\infty^2|^{1/2}}\right) \quad \text{on} \quad y = 0 \;.$$

Thus $\varphi$ depends on two groups of parameters, and these are chosen as

$$\xi_\infty = \frac{M_\infty^2 - 1}{[M_\infty^2(\gamma+1)\tau]^{2/3}} \quad \text{and} \quad \bar{\sigma} = [M_\infty^2(\gamma+1)\tau]^{1/3}\sigma \qquad \underline{2.2}$$

Now the scaled variables (2.1) give no indication of magnitude; the magnitude of $\varphi_x$ was estimated in the derivation of (1.13) as $\tau^{2/3}$ as $M_\infty \to 1$, and this suggests writing

$$Y = \bar{y}/\sqrt{|\xi_\infty|} \quad , \quad Z = \bar{z}/\sqrt{|\xi_\infty|} \; , \; X = \bar{x} \; , \; \tilde{\bar{\Phi}} = \bar{\varphi}/\sqrt{|\xi_\infty|}$$

giving

$$-\xi_\infty \tilde{\bar{\Phi}}_{XX} + \tilde{\bar{\Phi}}_{YY} + \tilde{\bar{\Phi}}_{ZZ} = \tilde{\bar{\Phi}}_X \tilde{\bar{\Phi}}_{XX}$$

$$-\xi_\infty(\tilde{\bar{\Phi}}_X^{(1)} - \tilde{\bar{\Phi}}_X^{(2)})^2 + (\tilde{\bar{\Phi}}_Y^{(1)} - \tilde{\bar{\Phi}}_Y^{(2)})^2 + (\tilde{\bar{\Phi}}_Z^{(1)} - \tilde{\bar{\Phi}}_Z^{(2)})^2 = \tfrac{1}{2}(\tilde{\bar{\Phi}}_X^{(1)} + \tilde{\bar{\Phi}}_X^{(2)})(\tilde{\bar{\Phi}}_X^{(1)} - \tilde{\bar{\Phi}}_X^{(2)})^2 \qquad \underline{2.3}$$

From this form

$$C_p = \frac{\tau^{2/3}}{\{M_\infty^2(\gamma+1)\}^{1/3}} \tilde{C}_p(\xi_\infty, \bar{\sigma}) \qquad \underline{2.4}$$

where $\tilde{C}_p = -2\tilde{\bar{\Phi}}_X$ ; $\tilde{C}_p$ is the reduced variable normally used in the presentation of results. The terms of (2.3) are of order one, and $\xi_\infty = 0$ at sonic speed; in the two-dimensional case $\bar{\sigma} = \infty$, and disappears as a parameter.

For axisymmetric flow one needs to make assumptions about the behaviour of $\varphi$ near the axis, because of the singularity there. The derivation is given in [5] . The result is

$$\frac{1}{\delta^2} C_p + \frac{1}{\pi} Q''(x) \log[\delta^2 M_\infty^2(\gamma+1)^{1/2}] = \tilde{C}_p(\bar{\xi}_\infty)$$

where $\bar{\xi}_\infty = \dfrac{M_\infty^2 - 1}{M_\infty^2(\gamma+1)\delta^2}$ . $\qquad \underline{2.5}$

However its range of validity is more restricted than the thin wing result. A discussion of its deficiencies when $M_\infty \neq 1$ is presented in [8], [9].

For slender bodies it is possible to derive another relation between different flows, the equivalence rule. This is a direct consequence of the relation

$$\varphi \doteq \varphi_2(x; y, z) + g(x) \qquad \underline{2.6}$$

where $\varphi_2$ is a potential for the cross-flow (in the (y,z) plane at the station $x$ ) which satisfies $\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi_2 = 0$ and the boundary conditions implied by those on $\varphi$, and further $g(x)$ depends only on the cross-sectional area. Thus $g(x)$ may be determined from the axisymmetric flow past the so-called 'equivalent body'. For small departures from circular cross-section the relation (2.6) follows from the analysis of Cole and Messiter [5], but to treat thin airfoils an alternative approach seems necessary. That adopted by Oswatitsch [7], [10] and Heaslet and Spreiter [11], is to apply Green's theorem in a (y,z) plane at the station $x$ to equation (1.13), regarding the term on the right hand side as a distribution of sources. The analysis is carried out for $M_\infty = 1$, though Heaslet and Spreiter take care in showing this gives the limit for $M_\infty \to 1\pm$, and also make explicit the assumptions concerning shocks. Taking the difference between the equation for $\varphi$ and that for $\varphi^{(0)}$, the potential for the body of revolution:

$$\varphi - \varphi^{(0)} = \varphi_2 - \varphi_2^{(0)} + \frac{k}{2\pi}\iint\left[\varphi_x\varphi_{xx} - \varphi_x^{(0)}\varphi_{xx}^{(0)}\right]\log\rho_1 \; r_1\,dr_1\,d\theta_1 \qquad \underline{2.7}$$

where the integral is taken over the region external to the respective bodies, the subscript 2 denotes the two-dimensional cross-flow potential, and $\rho_1^2 = r^2 + r_1^2 - 2rr_1\cos(\theta-\theta_1)$ with $(r,\theta)$ the polar co-ordinates of (y,z), and $r_1, \theta_1$ current co-ordinates in the integration.

Oswatitsch argues that far from the body $\varphi_x - \varphi_x^{(0)}$ has an oscillatory character, and that it is $O(\sigma^2\varphi_x)$, so that the main contribution to the integral in (2.7) arises from the region $r_1 < \sigma$, and this is assessed by using $\varphi_2$, $\varphi_2^{(0)}$ for $\varphi$, $\varphi^{(0)}$. Comparison is made with the results of linearised theory, though this hardly tests the crucial question about whether the non-linearity has been successfully treated.

Heaslet and Spreiter adopt a similar approach, but attempt a more mathematical treatment. The method for a symmetrical air-foil is as follows. First express $\varphi_2$ as a Fourier series.

$$\varphi_2 = \frac{1}{2\pi}\int_{-s(x)}^{s(x)} \tau\,\Omega(x, z_1)\,\log\left(r^2 + z_1^2 - 2rz_1\cos\theta\right)^{1/2} dz_1$$

where $s(x)$ is the span at station $x$, $s(x) = O(\sigma)$, $\Omega = 2f$ with $f$ from (1.15) and

$$\log\left\{r^2 + r_1^2 - 2rr_1\cos(\theta-\theta_1)\right\}^{1/2} = \begin{cases} \log r_1 - \sum_{m=1}^{\infty}\left(\frac{r}{r_1}\right)^m \frac{\cos m(\theta-\theta_1)}{m} & r \leqslant r_1 \\[2ex] \log r - \sum_{m=1}^{\infty}\left(\frac{r_1}{r}\right)^m \frac{\cos m(\theta-\theta_1)}{m} & r > r_1 \end{cases}$$

Then for $r > s$,

$$\varphi_2 = \frac{1}{2\pi} S'(x)\log r + \sum_{m=1}^{\infty} a_{2m}(x)\left(\frac{s}{r}\right)^{2m}\cos 2m\theta \qquad \underline{2.8}$$

where

$$a_{2m} = -\frac{s\tau}{4\pi m}\int_{-S(x)}^{S(x)} \Omega(x, z_1)\left(\frac{z_1}{S}\right)^{2m}\frac{dz_1}{S}$$

and

$$S'(x) = \int_{-S(x)}^{S(x)} \tau\,\Omega(x, z_1)\,dz_1 ,$$

and for $r < S$

$$\varphi_2 = \frac{1}{2\pi} S'(x) \log s + F\left(x, \frac{r_1}{s}\right) + \sum_{m=1}^{\infty} \frac{1}{2m}\cos 2m\theta\, G_m\left(x, \frac{r_1}{s}\right) \qquad \underline{2.9}$$

where, by expanding the kernel as

$$\tau\,\Omega(x, z_1) = \sum_{n=0}^{\infty} \frac{\pi}{S} A_{2n}(x)\left(\frac{z_1}{S}\right)^{2n}$$

we have

$$F\left(x, \frac{r_1}{s}\right) = \sum_{n=0}^{\infty} \frac{A_{2n}(x)}{(2n+1)^2}\left[\left(\frac{r_1}{s}\right)^{2n+1} - 1\right] ,$$

$$G_m\left(x, \frac{r_1}{s}\right) = \sum_{n=0}^{\infty} A_{2n}(x)\left[\frac{1}{2m-2n-1}\left(\frac{r_1}{s}\right)^{2m} - \frac{1}{2m+2n+1}\left(\frac{r_1}{s}\right)^{2n+1} + \frac{1}{2m-2n-1}\right] .$$

Now, we may obtain an approximation to $J$, the integral on the right hand side of (2.7), by using the series expansions (2.8), (2.9) and replacing $\varphi$ by $\varphi_2 + \varphi^{(0)} - \varphi_2^{(0)}$.

$$J = \frac{k}{4\pi}\int_s^{\infty}\int_0^{2\pi} \frac{\partial}{\partial x}\left\{\left[\sum_{m=}^{\infty} a'_{2m}(x)\cos 2m\theta_1\left(\frac{s}{r_1}\right)^{2m}\right]^2 + 2\varphi_x^{(0)}\left[\sum_{m=1}^{\infty} a'_{2m}(x)\cos 2m\theta_1\left(\frac{s}{r_1}\right)^{2m}\right]\right\}\log\rho_1 \cdot r_1\,dr_1\,d\theta_1$$

$+$ (overleaf)

$$+ \frac{k}{4\pi} \int_0^s \int_0^{2\pi} \frac{\partial}{\partial x} \Big\{ \Big[ \frac{S''(x)}{2\pi} \log\left(\frac{s}{r_1}\right) + F'(x, \tfrac{r_1}{s}) + \sum_{m=1}^{\infty} \frac{\cos 2m\theta_1}{2m} G_m'(x, \tfrac{r_1}{s}) \Big]^2$$

$$+ 2\, \varphi_x^{(0)} \Big[ \frac{S''(x)}{2\pi} \log\left(\frac{s}{r_1}\right) + F'(x, \tfrac{r_1}{s}) + \sum_{m=1}^{\infty} \frac{\cos 2m\theta_1}{2m} G_m'(x, \tfrac{r_1}{s}) \Big] \Big\} \log \rho_1 \, . \, r_1 dr_1 d\theta.$$

In this last integral, it may be objected that we have included $\varphi_x^{(0)} \varphi_{xx}^{(0)}$ terms in the interior of the equivalent body, but their contribution is $O\left((s\tau)^3 (\log s\tau)^2\right)$ and may be neglected. By performing the $\theta_1$ integration it may be deduced that the $r_1$ integration in the first part of the expression for $J$ converges provided $\varphi^{(0)} \sim r^{-N}$ with $N > 0$ as $r \to \infty$.

At this stage, Heaslet and Spreiter, employ the following estimates:

$$a_n = O(\tau s) \;,\quad \varphi_x^{(0)} = O(\tau s) \;,\quad F = O(s\tau) \;,\quad G_m = O(s\tau) \qquad \underline{2.10}$$

and deduce

$$J = O(\tau^2 s^4 \log s) \qquad \underline{2.11}$$

As the terms of (2.7) apart from the integral, are $O(\tau s)$ it follows that as a first approximation $\varphi = \varphi_2 + \varphi^{(0)} - \varphi_2^{(0)}$. For points in the neighbourhood of the body

$$\varphi^{(0)} - \varphi_2^{(0)} \doteq \lim_{r \to 0} \varphi^{(0)} - \varphi_2^{(0)}$$

which gives a function $g(x)$ depending only on the cross-sectional area, and hence the result (2.6). Although the estimates (2.10), (2.11) seem reasonable, it should be noted that this stage of the argument falls short of rigorous justification.

## §3 Methods of solution of the equations

The similarity and equivalence rules enhance the value of particular solutions of the boundary value problems posed by (1.13) - (1.16), but do not provide them. One may seek self-similar solutions, but as the equations are non-linear, there is no possibility of building up general solutions by superposition.

An important solution of this type is that for axisymmetric flow at $M_\infty = 1$ provided by Guderley and Yoshihara [12], and further investigated in references [13], [14]. For this solution

$$\varphi = r^m f_m(\zeta)$$

where $\zeta = (\gamma+1)^{-1/3}(x/r^n)$, $m = 3n - 2$ and $n = 4/7$. It gives the asymptotic solution of the equations as $r \to \infty$, the far field. An account of other self-similar solutions is included in the article by Spreiter in [15].

Another approach is to note that (1. 13) is linear in the second derivatives, and does not contain x,y,z explicitly, so that for two-dimensions, application of the Legendre transformation [16] will result in a linear equation.

This idea is embodied in the hodograph transformation, which uses either $U = q\cos\theta$, $V = q\sin\theta$ or $q, \theta$ as the independent variables. The transformation may be applied to (1.5) before the assumptions of only small departure from a uniform stream, [17]. The dependent variable is taken as either the stream function $\psi$ or $\bar{\varphi} = xU + yV - \Phi(x,y)$, or simply as the velocity potential $\Phi(x, y)$. Approximations to the gas law may be used to simplify the analysis, the most notable being the Karman-Tsien approximation, and that of Tomotika and Tamada [18], though Chaplygin has solved the full equation by separation of variables, and an infinite set of solutions is available, [17]. Even with the possibility of superposing solutions, serious difficulty arises with the boundary conditions, and the technique adopted is either to find the solution which reduces to the flow past the given body as $M_\infty \to 0$, or as in [18], to produce a reasonable flow in the hodograph plane and deduce to what body it corresponds. For all cases, there is the restriction that the Jacobian of the transformation must not vanish. Lines along which the Jacobian vanishes are known as limit lines and these must not penetrate the flow field. In its simplest form the Jacobian is

$$D = -\frac{1}{\rho^2 q^2}\left\{\left(\frac{\partial\psi}{\partial q}\right)^2 + \left(1 - \frac{q^2}{c^2}\right)\frac{1}{q^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2\right\}$$

so that there is no possibility of limit lines in purely subsonic flow. Unfortunately, limit lines appear for quite low free stream Mach numbers; in the case of a circular cylinder in a uniform stream they are present at $M_\infty = 0{\cdot}6$. The above mentioned applications of the hodograph method are thus restricted to the lower transonic regime, which is basically high subsonic flow. This prompts comparison with the Rayleigh-Janzen expansion of $\Phi$ as a power series in $M_\infty^2$. Comparison of the Karman-Tsien approximation shows it is only correct to order $M_\infty^2$. For the Tomotika and Tamada solution, limit lines arise near $M_\infty = 0{\cdot}75$, but by this value, the predicted velocity distribution shows a sharp fall at about 60% chord, and presumably such a result is beyond the range of validity of a truncated $M_\infty^2$ expansion.

The middle transonic range, when shocks occur in the middle of the airfoil, is the most intractable of all. The method of recasting the small disturbance equation (1.13) as an integral equation seems the most promising approach, but there is still a wide gap between experiment and the theories of Oswatitsch [19] and Spreiter, Alksne and Hyett [20]. References to semi-empirical methods of bridging the gap are given in A. B. Tayler's article in [21].

For thin airfoils in a near sonic stream, the flow is in the upper transonic regime and hodograph methods again find application. The basic equation involved is Tricomi's

$$\tilde{\psi}_{\tilde{\eta}\tilde{\eta}} - \tilde{\eta}\,\tilde{\psi}_{\tilde{\theta}\tilde{\theta}} = 0$$

and it is derived by writing

$$\psi = \tilde{\psi}(\tilde{\eta},\tilde{\theta}) , \quad \eta = \tau\tilde{\eta} = (\gamma+1)^{1/3}\left(\frac{q-q^*}{q^*}\right) , \quad \tilde{\theta} = \theta\tau^{-3/2}$$

in the hodograph equation for $\psi$, and then retaining the largest terms as $\tau \to 0$. A similar equation may be derived for the Legendre transformed potential $\bar{\Phi}$. It corresponds to the transonic small disturbance equation (1.13).

The Tricomi equation has formed the basis for extensive investigations [22]. The separation of variables solutions, corresponding to the Chaplygin solutions, are discarded in favour of solutions of the form $\tilde{\eta}^n f_n(\tilde{\zeta})$, $\tilde{\zeta} = 9\tilde{\theta}^2/4\tilde{\eta}^3$ ; $f_n$ is a hypergeometric function, and it is possible to construct solutions by superposition. Care is still required to avoid limit lines within the flow field; the Jacobian now takes the form

$$D = -\frac{(\gamma+1)^{2/3}}{\rho^{*2} q^{*2}} \{ \psi_{\eta}^{2} - \eta \psi_{\theta}^{2} \}$$

where asterisks denote critical values. The difficulty with boundary conditions encountered at lower free stream Mach numbers persists, and prevents direct solution for a given profile. However, with sonic free stream, we need only consider the flow field up to the last characteristic that meets the sonic line, the limiting characteristic; the upstream flow field is independent, and the flow downstream of the limiting characteristic may be calculated by a different method. Questions of existence, uniqueness and whether the problem is properly posed are discussed in [23] . Certain problems with simple boundary conditions (i.e. straight boundaries or free streamlines) have been solved, and these fulfil two useful functions. First, we have accurate solutions with which to check more approximate methods, and secondly, solutions for simple wind-tunnel flows provide information on the nature of wind-tunnel interference effects.

A numerical method has been proposed by Dorodnicyn [24] , which is applied in the physical plane, and can in principle deal with three-dimensional flow past general profiles. However, in practice, it has only been applied to two-dimensional flows, and even then only to shapes for which a choice of 'natural' coordinates, depending on the boundary, reduces the computation involved. For two dimensions the method is as follows. The equations are recast in the form

$$\sum_{j=1}^{2} \frac{\partial P_i}{\partial x_j} (x_1, x_2 ; u_1 \ldots u_M) = F(x_1, x_2 ; u_1 \ldots u_M) , \quad i = 1 \ldots M ,$$

and integrated with respect to one variable, say $x_2$ ; the values of $u_i(x_1, x_2)$ are interpolated from a set of functions $u_{ik}(x_1)$ giving the values at certain stations $x_2 = x_{2k}$ . This leads to a system of ordinary differential equations for $u_{ik}(x_1)$ which may be solved by routine methods. For sonic flow the method is applied only to the region upstream of the limiting characteristic, whose location arises out of the method of solution. Downstream of the limiting characteristic it is suggested that the method of characteristics be employed.

Despite the impressive mathematics associated with the preceeding methods, they do not meet the engineers' need for a reasonable prediction of the pressure for three-dimensional flow past a typical airfoil. To make further progress, it seems at present, that we must relinquish the idea of solving for the whole flow field the boundary value problems posed by (1.13) - (1.16). Such an attitude has been adopted in a class of methods which we designate 'regional linearisation methods'. These assume that in the region near the airfoil, the equation (1.13) may be replaced by a linear approximate equation, and that this may be so selected that it provides the dominant contribution to the flow field near the airfoil. Basically, there are two ways of approximating the non-linear term of (1.13). The first way replaces $\varphi_{xx}$ by a known function, and gives a parabolic equation; the second replaces $\varphi_x$ and gives the possibility of a changing type, elliptic-hyperbolic, equation. In sections §4, §5 we give an account of investigations into each of the respective ways of making the approximation.

## § 4. The parabolic equation approximation

Oswatitsch and Keune [25] initiated the 'regional linearisation' method, after considering the experimental results for sonic flow past a slender, parabolic arc, body of revolution. They noted that over the front half of the body the acceleration was approximately constant, and suggested the equation

$$\varphi_{rr} + \frac{1}{r}\,\varphi_r = \lambda\,\varphi_x \qquad \underline{4.1}$$

as a model for equation (1.13) in the neighbourhood of the airfoil. As the parabolic equation allows no upstream influence, a condition of vanishing disturbance of the uniform stream at infinity, implies $\varphi_x = 0$ at $x = 0$, the origin being at the leading edge of the airfoil. Physically this might appear unreasonable. However by introducing at $x=0$ a typical distribution of velocity encountered in experiment, it was shown that the error was around 5% at a point midway between the tip and maximum thickness. Near the tip the error will be larger, but as the small disturbance equation is not uniformly valid, inaccuracy there is inevitable.

The preceeding approximation is involved in all the parabolic equation methods, and it may be noted that it is less justified in two-dimensional problems. The solution of (4.1) subject to boundary conditions of vanishing disturbance at infinity, and condition (1.16) is straightforward.

$$\varphi = -\frac{\delta^2}{4\pi}\int_0^x Q'(\xi)\;\sigma(x-\xi, r)\,d\xi \qquad (x>0) \qquad \underline{4.2}$$

where $\sigma(x-\xi, r) = \frac{1}{(x-\xi)} \exp\left(\frac{-\lambda r^2}{4(x-\xi)}\right)$ .

This leads to an expression for the perturbation speed at the body $r = \delta_1(x)$ :

$$\varphi_x(x, \delta_1) = \frac{Q''(x)}{4\pi} \log\left(\frac{\lambda \delta_1^2}{4x} 1.781\right) + \frac{1}{4\pi}\int_0^x \frac{Q''(x) - Q''(\xi)}{x-\xi} . d\xi + o(\delta_1^2) \qquad \underline{4.3}$$

The choice of λ requires care. Fortunately, at the maximum of $Q'(x)$ , λ disappears from the expression (4.3), which reduces the effect of errors in its choice. Two procedures are considered; both use the above theory to determine the flow up to the sonic line, and then employ a modification of the method of characteristics suitable for transonic flow. The first procedure determines the values of $r\varphi$ on the sonic line by the characteristics method and chooses $\lambda$ to give the best agreement with the values obtained from (4.2). The second determines $\varphi_x$ on the body by the characteristics method and chooses $\lambda$ to give the best agreement with the values from (4.3). The first was preferred on the grounds that the distribution along the sonic line is of decisive importance for the velocities downstream. It also seems more practical for body shapes different from the parabolic arc. The procedure is laborious and subsequent developments, having less regard for the behaviour off the body surface, determine $\lambda$ more simply, [26] . For the half parabolic arc body considered in [25] agreement with experiment is highly satisfactory.

Maeder and Thommen, employing the same approximations, extended the method to $M_\infty \neq 1$, and also treated the two-dimensional case, [27]. However their choice for $\lambda$ was dubious and, in the two dimensional case, the comparison with experimental results was unimpressive. In a subsequent paper [28], simultaneously with Hosokawa [29], they clarified the choice of $\lambda$. Maeder's account of this approach in [21] draws on both papers and relates them to the 'local linearisation' method of choosing $\lambda$, due to Spreiter [30]. The idea is to use the model equation

$$\beta_0^2 \varphi_{1xx} + \varphi_{1yy} + \varphi_{1zz} - \lambda \varphi_{1x} = 0 \qquad \underline{4.4}$$

choose $\beta_0$, $\lambda$ so that this best reproduces the $y,z$ variation of $\varphi$, and then calculate a correction term $q(x) \simeq \varphi - \varphi_1$, to give values of $\varphi_x$ on the body. Maeder expresses this in terms of the stream density $\rho u$. Writing $\varphi = \varphi_1 + I$, by using Green's theorem, we have

$$I = \iiint_{-\infty}^{\infty} T(\xi,\eta,\zeta) \; G(x-\xi, y-\eta, z-\zeta) \, d\xi . d\eta . d\zeta$$

where

$$T(\xi,\eta,\zeta) = \left[ (\gamma+1) M_\infty^2 \varphi_{\xi\xi} - \lambda \right] \varphi_\xi + \left[ \beta_0^2 - (1-M_\infty^2) \right] \varphi_{\xi\xi} ,$$

$G(x,y,z)$ satisfies (4.4) with the right hand side replaced by $\delta(x)\delta(y)\delta(z)$, and $\varphi_1$ satisfies the boundary conditions on $\varphi$.
Consequently

$$\beta_0^2 I_{xx} - \lambda I_x + (I_{yy} + I_{zz}) = T(x,y,z) \qquad \underline{4.5}$$

and the perturbation to the stream density may be written

$$(1 - M_\infty^2)\,\varphi_{xx} - (\gamma+)\,M_\infty^2\,\varphi_x\,\varphi_{xx} = \beta_0^2\,\varphi_{1xx} - \lambda\varphi_{1x} - (I_{yy} + I_{zz}) \qquad \underline{4.6}$$

This is an exact deduction from the transonic small disturbance equation (1.13), apart from possible contributions from weak shocks. Now if in (4.5), $T(x,y,z)$ is independent of $y, z$ then $I_{yy} + I_{zz} = 0$ and (4.6) becomes an ordinary differential equation for $\varphi_x$. Physically one would expect to minimise the dependence of I on $y,z$ for points near the body by making $T$ and its normal derivative vanish at the body. Unfortunately, with only two disposable constants, this can only be achieved at one point, and not on the whole body. Maeder, considering two-dimensional flow argues that this is best done at the sonic point. For a parabolic arc airfoil, the agreement with experiment is excellent, and the non-linear form of equation (4.6), neglecting $I_{yy} + I_{zz}$, gives the possibility of a jump in $\varphi_x$ analogous to a shock at the rear of the airfoil. However, as one changes the airfoil shape, agreement with experiment rapidly deteriorates.

The 'local linearisation' method (Spreiter [30] ) is a device to overcome the restriction of having only two disposable constants. To obtain $\varphi_x$ at a point of the airfoil, one uses the equation (4.4) and determines $\beta_0, \lambda$ to give the best approximation to the full equation (1.13) in the immediate neighbourhood of the point under observation. To follow the Maeder [21] method would be very laborious, and Spreiter adopted the choice $\beta_0^2 = 1 - M_\infty^2$, $\lambda = (\gamma+1)\,M_\infty^2\,\varphi_{xx}$, which is equivalent to the Maeder choice for a parabolic arc airfoil. Thus, applying the method for a two-dimensional thin airfoil in a sonic stream, we have:

$$\varphi_{1x}\,(z=0) \;=\; -\frac{1}{2\pi}\frac{\partial}{\partial x}\int_0^x d\xi \int_{-\infty}^{\infty} f(\xi,\eta)\,\sigma(x-\xi,\,y-\eta)\,d\eta \qquad \underline{4.7}$$

where $f(\xi,\eta)=f(\xi)$ is the local slope of the airfoil, and $\sigma$ is the source-type solution of (4.4) given in (4.2); now writing $u(x)$ for $\varphi_x\,(z=0)$ and replacing $\lambda$ by $(\gamma+1)\,u'(x)$ ,

$$u\,(x) \;=\; \frac{-1}{\{\pi(\gamma+1)\,u'(x)\}^{1/2}}\;\frac{d}{dx}\int_0^x \frac{f(\xi)}{\{x-\xi\}^{1/2}}\,d\xi \qquad ,$$

an ordinary differential equation which gives the velocity at the airfoil surface.

This 'local linearisation' approach is general, in the sense that it gives values of the reduced pressure coefficient $\tilde{C}_p$ in good agreement with experiment for a variety of shapes in both two-dimensional and axisymmetric flow, and even for 3-dimensional flow past planar wings agreement is fair. However, by using a simplification of the Maeder [21] choice of $\beta_0$ , $\lambda$ , one is paying less attention to flow off the body, and furthermore by using a local choice the implied model for the overall flow must be crude for the calculation at many points on the body. This last approximation is the difference between 'local' and 'regional' linearisation. Mathematically the method lacks a firm basis, a matter which we consider in Chapter IV.

The results of the parabolic equation approximation may be summarised as follows. For near sonic flow, the constant $\lambda$ methods [25], [28], [29] give good results for two-dimensional airfoils and bodies of revolution generated by a parabolic arc. For different shapes, to obtain good agreement with experiment one must use the 'local linearisation' method, [30], [9]. For three dimensional flow one may use the foregoing results with the equivalence rule § 2, or use the extension to thin planar wings of the 'local linearisation' method, [31]. This last method reduces to the two-dimensional case for infinite aspect ratio, and gives analytic results for a restricted class of small aspect ratio wings; otherwise it requires a considerable amount of computing for results of uncertain accuracy. One case treated analytically, the small aspect ratio, thin elliptic cone-cylinder has results significantly different from those predicted by use of the equivalence rule.

Finally we remark that Hosokawa has applied his method to lifting airfoils, and unsteady motion, while Teipel has applied Spreiter's ideas to unsteady motion. Accounts of these further applications of the parabolic equation approximation are available in [21].

## §5. The mixed-type equation approximation

The mixed-type equation approximation was proposed by Cole and Royce [32] . It is based on the model equation

$$\varphi_{yy} + \varphi_{zz} = a^2(x-b)\,\varphi_{xx} \qquad \underline{5.1}$$

so that in (1.13), $\{(\gamma+1)M_\infty^2\varphi_x - (1-M_\infty^2)\}$ has been replaced by $a^2(x-b)$ , where $x = b$ gives the sonic point, and the form of the approximation implies accelerated flow over the body. As with the other model equation, the first problem investigated was that of sonic flow past a body of revolution. To relate this problem to subsequent work we introduce a different choice of axes, using cylindrical polar co-ordinates, with $z$ along the axis of the body., $r = (x^2+y^2)^{1/2}$ and the origin of co-ordinates at the sonic point. Thus the position of the body relative to the axes is a matter for calculation. The equation (5.1) becomes

$$\varphi_{rr} + \frac{1}{r}\varphi_r - a^2 z\,\varphi_{zz} = 0 \qquad \underline{5.2}$$

and the boundary conditions are taken as vanishing disturbance at infinity upstream of the sonic line, and the condition on the body (1.16),

$$2\pi r\,\varphi_r = \delta^2 Q'(z+b) \quad \text{on} \quad r = \delta \quad \text{as} \quad \delta\to 0 . \qquad \underline{5.3}$$

The solution is sought in the form of an integral of a line of sources, but the change in type of the equation introduces complications. The sources may be derived by replacing the right hand side of equation (5.2) by $\frac{1}{\tau}\,\delta(\tau).\,\delta(z-\zeta)$ and using Hankel transforms; they take the following different forms in the hyperbolic $\left(z > \frac{a^2\tau^2}{4}\right)$ and elliptic $\left(z < \frac{a^2\tau^2}{4}\right)$ regions respectively:

$$\varphi_H = \begin{cases} -\frac{1}{2\zeta}\,\frac{\chi}{(\chi^2-1)^{1/2}} & , \zeta > 0 \\ 0 & , \zeta < 0 \end{cases}$$

$$\varphi_E = \begin{cases} 0 & , \zeta > 0 \\ -\frac{1}{4\zeta}\left(1 + \frac{\chi}{(\chi^2-1)^{1/2}}\right), & \zeta > 0 \end{cases} \qquad \underline{5.4}$$

where $\chi = \left(z+\zeta-\frac{a^2\tau^2}{4}\right)/(2\sqrt{\zeta z})$ and $\zeta$ gives the position of the source on the axis. This feature gives rise to a singularity in the potential formed by an integral of sources along the body axis, and a further term must be added so that the equation (5.2) is satisfied in the whole flow field. The potential satisfying equation (5.2) and the boundary conditions is

$$\varphi = \frac{1}{2}\,T(0)\,\log\left(\frac{a^2\tau^2}{4}-z\right) + \int_{-b}^{0} T(\zeta)\,\varphi_E(\tau,z\,;\zeta).d\zeta \qquad \text{for } z<\frac{a^2\tau^2}{4}$$

$$= \frac{1}{2}\,T(0)\,\log\left(z-\frac{a^2\tau^2}{4}\right) + \int_{-b}^{0} T(\zeta)\,\varphi_E(\tau,z;\zeta)\,d\zeta + \int_{0}^{(\sqrt{z}-\frac{a\tau}{2})^2} T(\zeta)\,\varphi_H(\tau,z\,;\zeta)\,d\zeta \qquad \text{for } z>\frac{a^2\tau^2}{4}$$

where $2\pi\,T(z) = Q'(z+b).$ $\underline{5.5}$

For both cases in (5.5), as $\tau\to 0$

$$\varphi_z(\tau,z) = -\frac{1}{2}\,T'(z)\,\log\left(\frac{b+z}{\frac{a^2\tau^2}{4}}\right) + \frac{1}{2}\int_{-b}^{z}\frac{T'(\zeta)-T'(z)}{\zeta-z}.d\zeta + \int_{0}^{z}\frac{T'(\zeta)-T'(z)}{\zeta-z}\,d\zeta + o(1)_{\tau\to 0} \qquad \underline{5.6}$$

It remains to choose $a^2$ and $b$ . As the flow near the body is dominated by the first term of (5.6), Cole and Royce chose $b$ such that $T'(o) = 0$, and then took $a^2 = (\gamma+1)\,\varphi_{zz}(r_o, b)$ where $r_o$ is the body radius at $z = 0$. This choice implies $b$ is independent of the thickness ratio of the thickness ratio $\delta$, as it should to satisfy the similarity law, but the choice of $a^2$ means $(\varphi_z)_{body}$ does not satisfy the similarity law (2.5). Another feature is that predicted values of the sonic point are slightly aft of $z = 0$, contrary to what one might expect on physical grounds.

For bodies of revolution given by:

$$r(z+b) = \begin{cases} \frac{\delta}{2}\left[(1-(z+b)) - (1-(z+b))^n\right] & r_{max} < \frac{1}{2} \\ \frac{\delta}{2}\left[(z+b) - (z+b)^n\right] & r_{max} > \frac{1}{2} \end{cases} \qquad \underline{5.7}$$

agreement between predicted values of $C_p$ with those from experiment is good for $n = 2$ and $r_{max} > \frac{1}{2}$, though it deteriorates as $r_{max}$ moves forward. However, considering the whole range of shapes, the results are superior to all the parabolic approximations except that of Spreiter.

The extension of the mixed type equation approximation to flow past thin wings, in both two-dimensional and finite aspect ratio situations, is presented in Chapters II and III.

REFERENCES (Ch. I)

[1] Howarth, L. Modern Developments in Fluid Dynamics, High Speed Flow. O.U.P. (1953). Ch. 2

[2] Miles, J. W., J. Aero. Sci. 23 (1956) 704 - 705

[3] Ryzhov, O. S. and Shefter, G. M. P. M. M. 28 (6) (1964) 996 - 1007

[4] Oswatitsch, K. Gas Dynamics, Academic Press (1956) 365

[5] Cole, J. D. and Messiter, A. F. ZAMP 8 (1957), 1 - 25

[6] Spreiter, J. R. J. Aero. Sci. 21 (1954), 70 - 72

[7] Oswatitsch, K. Advances in Appl. Mech. 6 (1960) 154 - 271

[8] Spreiter, J. R. J. Aero. Sci. 26 (1959) 465 - 485

[9] Spreiter, J. R. and Alksne, A. Y. NASA R-2 (1959)

[10] Oswatitsch, K. and Keune, F., Z. f. Flugwissensch. 3 (1955) 29 - 46

[11] Heaslet, M. A. And Spreiter, J. R. NACA Rep. 1318 (1957)

[12] Guderley, K. and Yoshihara, H. Quart. Appl. Maths. 8 (1951) 333 - 339

[13] Guderley, K. and Barish, D. T. J. Aero. Sci. 20 (1953) 491 - 499

[14] Randall, D. G. AIAA Journ. 3 (12) (1965) 2339 - 2341

[15] Flugge, W. (editor) Handbook of Engineering Mechanics, Art . 76 McGraw-Hill (1962)

[16] Courant, R. and Hilbert, D. Methods of Math. Physics II Academic Press (1962) Ch. 1

[17] Lighthill, M. J. Ch. 7 of [1] .

[18] Tomotika, S. and Tamada, K. Quart. Appl. Maths 9 (1951) 129 - 147

[19] Oswatitsch, K. Acta Physica Austriaca Bd. IV (1950) 228 - 271

[20] Spreiter, J. R. , Alksne, A. Y. and Hyett, B. J. NACA TN4148 (1958)

[21] Oswatitsch, K. (editor) Symposium Transsonicum. Springer (1964)

[22] Guderley, K. Theory of Transonic Flow (transl.) Pergammon (1962)

[23] Bers, L. Mathematical Aspects of Subsonic and Transonic Gas Dynamics. Wiley (1958)

[24] Dorodnicyn, A. A. Advances in Aero. Sci. Vol II. Pergammon (1959) 833 - 845

[25] Oswatitsch, K. And Keune, F. Proc. Conf. High Speed Aero., Polytech. Inst. Brooklyn (1955)

[26] Randall, D. G. R.A.E. TN Aero 2653 (1959)

[27] Maeder, P. F. and Thommen, H. U. J. Aero. Sci. 23 (1956) 187 - 188

[28] Maeder, P. F. and Thommen, H. U. Trans. ASME Ser. E. J. Appl. Mech. 83 (1961) 481 - 490

[29] Hosokawa, I. J. Phys. Soc. Japan 15 (1960) 149 - 157

[30] Spreiter, J. R. and Alksne, A. Y. NACA Rep. 1359 (1958)

[31] Spreiter, J. R. and Alksne, A. Y. NASA TR R-88 (1961)

[32] Cole, J. D. and Royce, W. W. In. Proc. VI. Midwest. Conf. Fluid Mech. (Austin, Texas, 1959).

# CHAPTER II

## AN APPROXIMATE SOLUTION FOR TWO-DIMENSIONAL TRANSONIC FLOW PAST THIN AIRFOILS.

In this chapter we develop the second type of regional linearisation approximation, outlined in Chapter I, §5. The work constitutes an extension of the method initiated by Cole and Royce, in their treatment of axisymmetric flow.

## § 1. INTRODUCTION.

The small disturbance equations for transonic flow are well established, but owing to their non-linearity, the derivation of exact solutions is a formidable task. A few such solutions have been obtained using the hodograph plane in which the equations become linear, but as the body profile cannot be specified a priori, this approach offers little scope for general airfoil shapes. Confronted with this situation, it is natural to ask whether it is possible to approximate the non-linear term, so that the equations become linear, and yet still retain sufficient accuracy in the prediction of surface pressure on typical airfoil shapes to give a useful engineering approximation.

To investigate this possibility, consider a uniform stream, flowing in the positive z direction of a rectangular cartesian system (x, y, z), past either a slender body lying along the z axis, or a thin airfoil which is everywhere close to the Oxz plane. For steady, isentropic, compressible flow we may introduce a potential

$$U_\infty \left( z + \varphi(x, y, z) \right)$$

and as $U_\infty$ approaches the speed of sound, the small disturbance equation, reduces to the transonic equation

$$\varphi_{rr} + \frac{1}{r}\varphi_r \equiv \varphi_{xx} + \varphi_{yy} = \left\{ (\gamma+1) M_\infty^2 \varphi_z - (1 - M_\infty^2) \right\} \varphi_{zz} \qquad \underline{1.1}$$

For the derivation of this equation, see for example, reference [1]. One has the choice of approximating either $\varphi_{zz}$ or $\left\{ (\gamma+1) M_\infty^2 \varphi_z - (1-M_\infty^2) \right\}$.

The first step was taken by Oswatitsch & Keune [2], who considered sonic ($M_\infty = 1$) flow past a slender, parabolic body of revolution, and noted that in practice, the acceleration was nearly constant for fluid near the surface of the forward half of the body. Accordingly the non-linear term $(\gamma + 1) M_\infty^2 \varphi_z \varphi_{zz}$ was replaced by $K \varphi_z$, giving the equation

$$\varphi_{xx} + \varphi_{yy} = K\varphi_z \qquad \underline{1.2}$$

which may be readily solved. The calculated pressure distribution was in promising agreement with experimental results. However, it should be noted that even in this favourable case, the approximate equation is valid only in the immediate neighbourhood of the body. It is this relative insensitivity to poor approximations away from the body which provides the basis for further development. Maeder & Thommen [3] employing the same approximation for $(\gamma+1)M_\infty^2 \; \varphi_z \; \varphi_{zz}$, retained the $(1-M_\infty^2)\varphi_{zz}$ term, thus extending the theory to cover the whole Mach number range. The method was applied to both two-dimensional thin airfoils and slender bodies of revolution, but for bodies with maximum thickness forward of centre, computed values differed considerably from experimental ones. Miles [4] made the following criticisms: (1) that at $M_\infty = 1$, a mixed type elliptic-hyperbolic equation had been replaced by a parabolic one; (2) that for bodies of revolution at large distances the solution differed from the asymptotic solution provided by Guderley & Yoshihara [5] ; (3) the method failed to account for a region at some distance from the body in which non-linear effects were of decisive importance. However, Cole pointed out that no finite drag could be associated with the asymptotic field, and suggested that there is no simple relation between the field near the body, and the far field, but that conditions near the body are of primary importance in determining the pressure. It would appear that the region of criticism 3 accounts for criticism 2. At least at the present stage, we must be content to assume this.

In [6], Cole & Royce initiated the second type of approximation to the non-linear term, by replacing $(\gamma+1)M_\infty^2 \varphi_z \varphi_{zz}$ by $a^2 z \varphi_{zz}$. The case considered was that of sonic flow ($M_\infty = 1$) past a slender body of revolution, for which the equation is

$$\varphi_{xx} + \varphi_{yy} = a^2 z\, \varphi_{zz} \qquad \underline{1.3}$$

This approach meets Miles' first criticism, in that it preserves the elliptic-hyperbolic type of the equation. The analysis is not so simple as for the first type of approximation, but gives quite good agreement with experiment for a variety of smooth slender bodies. However, the simpler analytic results of [3] have attracted all further development.

Maeder & Thommen [7] improved their earlier work, by showing how the choice of K might best be made, and also calculated a correction term. The sonic point z* is given by $\{(1-M_\infty^2)-(\gamma+1)\varphi_{1z}^*\}_{y=0} = 0$ , being the solution of the approximate linearised equation, while K was chosen as $\{(\gamma+1)\ \varphi_{1zz}^*\}_{y=0}$ . The correction term was simultaneously derived in a different manner by Hosokawa [8] . He assumed $\varphi = \varphi_1 + g$ , where $\varphi_1$ is the solution given in [3] . Having obtained the appropriate changing type equation for $g$ from 1.1, he argued that the behaviour of $g$ is largely one dimensional in the neighbourhood of the body, and that $g$ may be taken as a function of z only. With this assumption, the equation is readily integrated, and a quadratic equation for the velocity $g_z$ results. The constant of integration is chosen so that $g_z = 0$ at z*, and the choice of sign is made so that $\varphi_{1z} + g_z \gtrless 0$ according as the flow is supersonic or subsonic. For the case of a two-dimensional parabolic arc thin airfoil agreement with experiment is excellent, but as the position of maximum thickness moves forward it deteriorates rapidly. In the axisymmetric case at $M_\infty = 1$, even for the parabolic arc the results are little better than [3]. Maeder & Thommen's analysis [7] showed that the errors largely stemmed from the failure of $\varphi_1$ , to represent the solution sufficiently accurately, especially near the sonic point.

An alternative improvement on [3] which exploits the insensitivity to poor approximations away from the body has been developed by Spreiter & Alksne. For $M_\infty \doteq 1$, the choice of K in the expression for the velocity at the airfoil surface is made as $K = (\gamma+1) M_\infty^2 \varphi_{zz}$ at each point of the airfoil, so giving an ordinary differential equation for $\varphi_z \big|_{y=0}$. For $M_\infty$ away from one, a similar "local linearisation" technique is applied to the classical subsonic and supersonic theories. This method has been applied to two-dimensional thin airfoils [9], slender bodies of revolution [10] and wings of finite span [11]. It gives good agreement with experiment over a wide range of shapes, but there are a number of objections. The criticism concerning the use of 1.2 as an approximation to 1.1 has already been included. In addition, the method has the disadvantages that it requires a different treatment for different ranges of Mach number, so that the simple covering of the Mach range provided by [3] is lost, and also that for three dimensional problems it requires considerable numerical work. The principal objection to the method from a mathematical standpoint however, is that it is basically inconsistent. K is taken as a constant until after the formal solution to the differential equation 1.2 is obtained and then is allowed to vary with z. If one assumes at the outset that K is a function of z to be found, a formal solution may still be derived. For $M_\infty = 1$, the respective formal solutions are:

K constant: $$\varphi_z \Big|_{y=0} = -\frac{U_\infty}{\sqrt{\pi K}} \left\{ -\frac{F(0)}{\sqrt{z}} - \int_0^z \frac{F'(\xi)}{\sqrt{z-\xi}}\, d\xi \right\} \qquad \underline{1.4}$$

K = K(z) $$\varphi_z \Big|_{y=0} = -\frac{U_\infty}{\sqrt{\pi}} \frac{1}{K(z)} \left\{ -\frac{F(0)}{\sqrt{\eta}} - \int_0^z \frac{F'(\xi)\, d\xi}{\sqrt{\eta(z) - \eta(\xi)}} \right\} \qquad \underline{1.5}$$

where $$\eta(\xi) = \int_0^\xi \frac{dt}{K(t)} .$$

$F(z)$ is the local slope of the airfoil, and $z = 0$ at the leading edge. It is evident that although 1.5 reduces to 1.4 when $K(z)$ is a constant, if we regard them as equations for $u(z)$, where $u(z) = \varphi_z|_{y=0}$ , and $K(z) = (\gamma+1)\frac{\partial u}{\partial z}$ , then they are quite different. Consequently we cannot yet regard agreement with experiment as a justification for the Spreiter theory, because it may have arisen because of the cancellation of the physical error by the mathematical error, and it is desirable to find an alternative model for which a mathematically consistent theory may be developed. That proposed by Cole & Royce [6] has the attractive feature that it preserves the structure of the equation, and for the axisymmetric case, the results of the mathematically consistent theory, are as encouraging as those of the Spreiter & Alksne work, particularly when the maximum disturbance occurs towards the rear of the body. In this paper, we extend the Cole & Royce method to include two dimensional shapes in order to test the accuracy of the method for the extreme case of airfoils with finite aspect ratio.

The approach in this work is similar to that of Cole & Royce. Their model is one of accelerated flow, so that the flow is subsonic over the forward part of the airfoil, and supersonic behind, and the technique is to represent the body by sources in a manner analogous to subsonic theory. Extension to two-dimensions is, in principle, simple. The point sources are integrated to give line sources, and the same technique is applied. However, as the subsonic and supersonic sources have different forms, an additional term is necessary to give a continuous potential, and establishing that the two expressions for the potential join smoothly constitutes the main difficulty of the work. The general results are applied to the particular case of airfoils given by polynomials, and for powers up to the sixth, the tables provided make the calculation of pressure distributions on such airfoils quite easy.

## § 2. The approximate equation and the source solutions.

The Cole & Royce approximate transonic equation for axisymmetric flow at $M_\infty = 1$ is

$$\varphi_{rr} + \frac{1}{r}\varphi_r = a^2 z\, \varphi_{zz} \tag{2.1}$$

where z = 0 at the sonic point, and by use of Hankel transforms, the following point source solutions have been derived:

$$\text{for } \zeta > 0 \qquad \Phi(H) = \begin{cases} -\dfrac{z+\zeta-\frac{a^2r^2}{4}}{2\zeta\left\{\left[z-\zeta+\frac{a^2r^2}{4}\right]^2-a^2r^2\zeta\right\}^{1/2}}, & \sqrt{z} > \sqrt{\zeta}+\frac{ar}{2} \\ 0, & \sqrt{z} < \sqrt{\zeta}+\frac{ar}{2} \end{cases} \tag{2.2}$$

$$\text{for } \zeta < 0 \qquad \Phi(E) = -\frac{1}{4\zeta} - \frac{z+\zeta-\frac{a^2r^2}{4}}{4\zeta\left\{\left[z-\zeta-\frac{a^2r^2}{4}\right]^2-a^2r^2\zeta\right\}^{1/2}} \tag{2.3}$$

The two dimensional form of equation 2.1

$$\varphi_{yy} - a^2 z\, \varphi_{zz} = 0 \tag{2.4}$$

obtained by regarding $\varphi$ as independent of $x$, corresponds to choosing $\varphi_z$ as $\dfrac{a^2 z}{M_\infty^2(\gamma+1)}$ in the non-linear term $(\gamma+1)M_\infty^2\, \varphi_z\, \varphi_{zz}$ of the two-dimensional transonic equation, where z = 0 is the sonic point. Since $\Phi_x(E), \Phi_x(H) \rightarrow 0$ as $x \rightarrow \pm\infty$ then $\varphi(E) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(E)\, dx$ is a solution of 2.4 for $\zeta < 0$, and $\varphi(H) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(H)\, dx$ a solution for $\zeta > 0$. The factors $1/2\pi$ are introduced so that the potentials represent unit sources. Further it is convenient to define $\varphi$ (E) = 0 for $\zeta \geqslant 0$, and $\varphi$ (H) = 0 for $\zeta \leqslant 0$. Now $\varphi$ (E) and $\varphi$ (H) may be expressed in terms of elliptic integrals. For $\zeta < 0$, the integral for $\varphi$ (E) takes two forms according as $z \gtrless 0$.

For $z = -s < 0$,

$$\varphi(E) = \frac{-1}{4\pi a\xi}\left\{(\sqrt{\xi}+\sqrt{s})^2+\theta^2\right\}^{1/2}\left\{2(K-E)-k^2K\right\} \qquad \underline{2.5}$$

where K(k), E(k) are complete elliptic integrals, with $k = \frac{4\sqrt{\xi s}}{(\sqrt{\xi}+\sqrt{s})^2+\theta^2}$, $\theta = \frac{ay}{2}$

For $z > 0$

$$\varphi(E) = \frac{4z}{a\pi A^2(2q-A)}\left\{2E_* - 2\lambda_1 F_* - (1-\lambda_1)\right\} \qquad \underline{2.6}$$

where $E*$, $F*$ are the incomplete elliptic integrals

$$E* = E\left(k, \frac{1}{\sqrt{1+\lambda_1}}\right) = \int_0^{1/\sqrt{1+\lambda_1}}\left\{\frac{1-k^2u^2}{1-u^2}\right\}^{1/2}du, \qquad F_* = \int_0^{1/\sqrt{1+\lambda_1}}\left\{(1-u^2)(1-k^2u^2)\right\}^{-1/2}du$$

and $k^2 = 1-\lambda_1^2$, $q = (\Theta^2+\gamma^2)^{1/4}$, $A = \sqrt{2}\sqrt{q^2-\Theta}$, $\lambda_1 = \frac{2q-A}{2q+A}$,

$$\lambda_1\lambda_2 = 1, \qquad \gamma^2 = 4\xi z, \qquad \Theta = \xi+\theta^2-z.$$

The integral for $\varphi$ (H) takes the simpler forms

$$\varphi(H) = \begin{cases} -\frac{1}{a\pi\xi}\left[\frac{2\sqrt{\xi z}}{\{(\sqrt{z}+\sqrt{\xi})^2-\theta^2\}^{1/2}}K - \{(\sqrt{z}+\sqrt{\xi})^2-\theta^2\}^{1/2}E\right], & \sqrt{z} > \frac{ay}{2}+\sqrt{\xi} \\ 0, & \sqrt{z} < \frac{ay}{2}+\sqrt{\xi} \end{cases} \qquad \underline{2.7}$$

where K, E are complete elliptic integrals with $k^2 = \frac{(\sqrt{z}-\sqrt{\xi})^2-\theta^2}{(\sqrt{z}+\sqrt{\xi})^2-\theta^2}$, $\theta = \frac{ay}{2}$

In the Cole and Royce work, it was necessary to introduce a solution, singular on the surface $z = \frac{a^2r^2}{4}$. The significance of this is discussed later. Here, we simply note that the line $z = \frac{a^2y^2}{4}$ corresponds to the surface $z = \frac{a^2r^2}{4}$, and investigate the possibility of solutions singular on this line. Putting $h = \frac{a^2y^2}{4} - z$, and seeking solutions which are functions of h only, $\sqrt{|h|}$ is found as the solution. Though not singular itself, the derivatives are.

If we further seek solutions of the form $(z\zeta)^{1/4} f(\eta)$ , where $\eta = \frac{z + \zeta - \theta^2}{(z\zeta)^{1/2}}$ , of which $\sqrt{|h|}$ is a particular case, then two independent solutions ensue. For $z < 0$, $\zeta < 0$ the first takes the form 2.5, while for $z > 0$, $\zeta > 0$ the second takes the form 2.7.

It will be noted that the first gives a solution for $z > 0$, $\zeta > 0$, and likewise the second for $\zeta < 0$, $z < 0$, but these are of no help in satisfying boundary conditions.

## § 3 A potential derived by superposition of the two-dimensional sources.

Consider

$$\varphi_2 = \begin{cases} \dfrac{2\Omega(0)}{a}\sqrt{\dfrac{a^2y^2}{4}-z} + \displaystyle\int_{-b}^{0} 2\Omega(\zeta).\,\varphi(E).\,d\zeta \quad , \quad \dfrac{a^2y^2}{4} > z & \text{3.1a} \\ \displaystyle\int_{-b}^{(\sqrt{z}-\frac{ay}{2})^2} 2\Omega(\zeta)\{\varphi(E)+\varphi(H)\}\,d\zeta \quad , \quad \dfrac{a^2y^2}{4} < z & \text{3.1b} \end{cases}$$

Where in 3.1b, the Cauchy Principal value is taken at $\zeta = 0$. This potential may be regarded as due to a line of sources along $y = 0$, and by analogy with subsonic theory, we expect $\Omega(\zeta)$ to be the local slope of the airfoil. The additional term $\frac{2\Omega(0)}{a}\sqrt{\frac{a^2y^2}{4}-z}$ is introduced anticipating difficulty in establishing continuity across $z = \frac{a^2y^2}{4}$. It stems from the effective doubling in strength of sources in passing from the subsonic to supersonic regime. Any discontinuity at $y = 0$, $z = 0$ propogates along the characteristic through the point, i.e. along $z = \frac{a^2y^2}{4}$.

The proof of the existence of the integrals of 3.1 is straightforward when $z \neq \frac{a^2y^2}{4}$. For 3.1a consider the cases $z \quad 0$ separately; for $z < 0$, using 2.5, the integrand has an integrable singularity at $\zeta = 0$. For 3.1b using the expressions 2.6 and 2.7, it may be shown that the integrand behaves like $1/\zeta$ at the origin, so that the principal value is required. As $h = \frac{a^2y^2}{4} - z \rightarrow 0$, these considerations are no longer applicable. The examination of the behaviour of $\varphi$ and its derivatives is quite involved and is not given here.

The conclusion is that $\varphi_2$, $\frac{\partial\varphi_2}{\partial y}$ and $\frac{\partial\varphi_2}{\partial z}$ are continuous across $z = \frac{a^2y^2}{4}$ as required by the physical situation. Thus, the necessity for the addition of the term

$$\begin{cases} \dfrac{2\Omega(0)}{a}\sqrt{\dfrac{a^2y^2}{4}-z} & \dfrac{a^2y^2}{4} > z \\ 0 & \dfrac{a^2y^2}{4} < z \end{cases}$$

to the potential due to the sources alone is established. This is a natural extension of the Cole & Royce work, as it is the $x$ integral of the $\log\left|x-\frac{a^2r^2}{4}\right|$ term which they had to add.

We now examine the velocities, for $z \neq \frac{a^2y^2}{4}$ beginning with $\frac{\partial\varphi_1}{\partial y}$. It is convenient to consider separately the ranges $z < 0$, $\frac{a^2y^2}{4} > z \geqslant 0$, and $z > \frac{a^2y^2}{4}$.

1. $\underline{z < 0.}$ Interchanging the order of integration and differentiation, and using expressions for the derivatives of elliptic integrals (see [12] or [13] ), it may be shown

$$\frac{\partial\varphi_2}{\partial y} = \Omega(0)\frac{\theta}{\sqrt{\theta^2+s}} - \frac{1}{4\pi}\int_0^b \frac{\theta}{\{\theta^2+(\sqrt{\xi}+\sqrt{s})^2\}^{1/2}}\left\{2K-\left(\frac{2-k^2}{1-k^2}\right)E\right\}\frac{\Omega(-\xi)}{\xi}\,d\xi \qquad \underline{3.2}$$

where $k^2 = \dfrac{4\sqrt{\xi s}}{\{\theta^2+(\sqrt{\xi}+\sqrt{s})^2\}}$.

When $s \neq 0$, as $\xi \to 0$, $k^2 = O(\sqrt{\xi})$ and $2(K-E) - \left(\frac{k^2}{1-k^2}\right)E = O(\xi)$. Thus, for $s \neq 0$, $y \neq 0$ the integrand in 3.2 is bounded at $\xi = 0$; furthermore it is continuous for $0 \leqslant \xi \leqslant b$, so that the integral is defined. For $y = 0$, the integrand has a singularity at $\xi = s$, where $k = 1$. Consider the behaviour of 3.2 as $y \to 0$. Since $s \neq 0$, the first term vanishes. In the integrand K has a singularity like log y, but it is multiplied by y, and so makes no contribution; $E \to 1$. Thus

$$\frac{\partial\varphi_2}{\partial y} \sim \frac{\Omega(-s)}{2\pi}\int_0^b \frac{ay}{2}\cdot\frac{1}{\sqrt{s}}\,\frac{4s}{4s\left(\frac{ay}{2}\right)^2+(\xi-s)^2}\,d\xi = \frac{\Omega(-s)}{\pi}\left[\tan^{-1}\frac{\xi-s}{ay\sqrt{s}}\right]_0^b,$$

$$\frac{\partial\varphi_2}{\partial y} \to \Omega(-s) \quad \text{as} \quad y \to 0$$

When $s = 0$, the integral in 3.2 makes no contribution, as $y \to 0$, but the first term gives $\Omega(0)$. Thus for $z \leqslant 0$, $\frac{\partial\varphi_2}{\partial y} \to \Omega(z)$ as $y \to 0$. $\qquad \underline{3.3}$

2. $\frac{a^2y^2}{4} > z \geqslant 0$. An expression for $\frac{\partial \varphi_2}{\partial y}$ has been derived, by returning to the definition of $\varphi(E)$, for use in 3.1, and interchanging orders of integration and differentiation. Like the potential in this range, it may be reduced to a single integral, with incomplete elliptic integrals in the integrand, but it is unwieldy. However it is fairly easy to show that the only possibility of a singularity in the integrand occurs at $\zeta = 0$, and if it occurs, it is certainly integrable.

3. $z - \frac{a^2y^2}{4} = h' > 0$. Writing $\varphi_2 = \mathcal{A}_1(\varepsilon) + \mathcal{A}_2(\varepsilon)$, where

$$\mathcal{A}_1(\varepsilon) = \int_\varepsilon^b 2\Omega(\zeta)\,\varphi(E)\,d\zeta \quad \text{and} \quad \mathcal{A}_2(\varepsilon) = \int_\varepsilon^{(\sqrt{z}-\theta)^2} 2\Omega(\zeta)\,\varphi(H)\,d\zeta$$

we require
$$\frac{\partial \varphi_2}{\partial y} = \lim_{\varepsilon\to 0}\left(\frac{\partial \mathcal{A}_1(\varepsilon)}{\partial y} + \frac{\partial \mathcal{A}_2(\varepsilon)}{\partial y}\right)$$

Now, by the approach indicated in case 2, it may be shown

$$\frac{\partial \mathcal{A}_1}{\partial y} = -\frac{1}{\pi}\int_\varepsilon^b \Omega(\zeta)\,\frac{2z\theta}{4q^4(1-\lambda_1)A}\left[(3-2\lambda_1+3\lambda_1^2)2F_* - (6-\lambda_1-\lambda_2)(2E_* - 1+\lambda_1)\right].d\zeta$$

The only singularity of the integrand is at $\zeta = 0$. For small $\zeta$

$$q^2 \sim h', \quad A \sim 2\sqrt{h'}, \quad 2q - A \sim 2\sqrt{h'}\left(\frac{z}{2h'^2}\right)\zeta, \quad \lambda_1 \sim \frac{z}{4h'^2}\zeta$$

Thus the integrand $\sim \frac{\Omega(0)}{\pi\sqrt{h'}}\,\theta\cdot\frac{1}{\zeta}$ near $\zeta = 0$, and $\frac{\partial \mathcal{A}_1}{\partial y}$ has the form

$$\frac{\partial \mathcal{A}_1}{\partial y} = y\,G_1(y) + \frac{\Omega(0)}{\pi}\cdot\frac{ay}{2}\cdot\frac{1}{\sqrt{h'}}\log\varepsilon + O(\varepsilon) \qquad \underline{3.4}$$

where $G_1(y)$ is a bounded function of y, independent of $\varepsilon$.

Also, using the results for derivatives of elliptic integrals, as in case 1,

$$\frac{\partial \mathcal{A}_2}{\partial y} = \frac{\Omega([\sqrt{z}-\theta]^2)}{[1-\theta/\sqrt{z}]^{1/2}} + \frac{1}{2\pi}\int_\varepsilon^{(\sqrt{z}-\theta)^2}\frac{\Omega(\zeta)}{\zeta}\,\frac{\theta}{\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}}\left\{\left(\frac{1+k^2}{k^2}\right)E - \left(\frac{1-k^2}{k^2}\right)K\right\}.d\zeta$$

where $k^2 = \frac{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2}{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2}$.

Near $\zeta = 0$, $k^2 \sim 1$ the integrand $\sim \frac{1}{2\pi}\Omega(0)\frac{\theta z}{\sqrt{z-\theta^2}}\cdot\frac{1}{\zeta}$, and thus $\frac{\partial \varphi_2}{\partial y}$ has the form

$$\frac{\partial \varphi_2}{\partial y} = \frac{\Omega([\sqrt{z}-\theta]^2)}{[1-\theta/\sqrt{z}]^{1/2}} + y\, G_2(y) - \frac{\Omega(0)}{\pi}\cdot\frac{ay}{2}\log\varepsilon + O(\varepsilon) \qquad \underline{3.5}$$

where $G_2(y)$ is a bounded function of y independent of $\varepsilon$.

Adding 3.4 and 3.5 and taking the limit $\varepsilon \to 0$, we obtain a well defined expression for $\frac{\partial \varphi_2}{\partial y}$. Further, as $y \to 0$ it follows

$$\frac{\partial \varphi_2}{\partial y} \to \Omega(z) \qquad \underline{3.6}$$

Equations 3.3 and 3.6 indicate that $\Omega(z)$ must be taken as the local slope of the airfoil, in order to satisfy the boundary condition of tangential flow. The usual approximation of applying the condition at $y = 0$ rather than at the airfoil surface, is made, the error being at most of second order.

However, it should be noticed that although we have established the existence of a continuous velocity field which satisfies the boundary condition on the body, we have failed to satisfy the condition of vanishing disturbances at large distances. For consider $\frac{\partial \varphi_2}{\partial y}$ in the region of $\frac{a^2y^2}{4} > z$. The additional term in the potential gives a velocity $\frac{\Omega(0)\,\theta}{\sqrt{\theta^2-z}}$ which is non-zero as $y \to \infty$, for z finite, while the integral gives a velocity which makes no contribution as $y \to \infty$. This inability to satisfy boundary conditions need not be surprising, if we recall that the equation 2.4 is a valid approximation only in the neighbourhood of the body.

The examination of $\frac{\partial \varphi_2}{\partial z}$ follows a similar pattern to that for $\frac{\partial \varphi_2}{\partial y}$. However, some of the expressions are lengthy, so it is proposed to give only the values as $y \to 0$, these being the expressions of practical interest.

For $z < 0$,

$$\left.\frac{\partial \varphi_2}{\partial z}\right|_{y=0} = -\frac{\Omega(0)}{a\sqrt{z}} + \frac{1}{2\pi a}\int_0^b \frac{\Omega(-\zeta)}{\zeta}\left\{\frac{E}{(\sqrt{\zeta}-\sqrt{z})} + \frac{K}{(\sqrt{\zeta}+\sqrt{z})}\right\}d\zeta \qquad \underline{3.7}$$

where $k^2 = \frac{4\sqrt{\xi s}}{(\sqrt{\xi}+\sqrt{s})^2}$ and the Cauchy principal value is taken at $\xi = s$.

The range $0 < z < \frac{a^2y^2}{4}$ is not required as $y \to 0$. For $z > 0$, we consider the range $\frac{a^2y^2}{4} \ll z$, as $y \to 0$. $\frac{\partial \varphi_2}{\partial z}$ has the form $\lim_{\varepsilon \to 0} \left\{ \frac{\partial Q_1(\varepsilon)}{\partial z} + \frac{\partial Q_2(\varepsilon)}{\partial z} \right\}$, as there are $1/\zeta$ singularities in each of the parts, if we put $\varepsilon = 0$. The resulting expression for the velocity is

$$\left.\frac{\partial \varphi_2}{\partial z}\right|_{y=0}^{(z>0)} = \lim_{\varepsilon \to 0} \left\{ \frac{2}{a\pi} \int_\varepsilon^b \frac{\Omega(\xi)}{8z(\sqrt{z+\xi}-\sqrt{z})} \left[ \begin{array}{l} 2E_* - 2\lambda_1 F_* - (1-\lambda_1) \\ + \frac{\xi}{4(\xi+z)} \left\{ (3-2\lambda_1+3\lambda_1^2) 2F_* - (6-\lambda_1-\lambda_2)(2E_* - 1 + \lambda_1) \right\} \end{array} \right] d\xi \right.$$
$$\left. + \frac{2}{a\pi} \int_\varepsilon^z \frac{\Omega(\zeta)}{\zeta} \frac{1}{2(z-\zeta)} \left\{ 2\sqrt{\zeta}\, K - (\sqrt{\zeta}+\sqrt{z}) E \right\} d\zeta - \frac{\Omega(z)}{a\sqrt{z}} \right\} \qquad \underline{3.8}$$

where it is understood that $\lambda_1$ is evaluated at $y = 0$, and in the second integral $k^2 = \left( \frac{\sqrt{z} - \sqrt{\zeta}}{\sqrt{z}+\sqrt{\zeta}} \right)^2$.

For analytic work, 3.8 is rather clumsy, and it is frequently better to return to the double integral implied in the definition. However, if we remove the contribution from $\Omega(0)$, so that the remaining integrands are no longer singular at $\zeta = 0$, it is quite ammenable to computation.

Some difficulty is experienced near $z = 0$, as can only be expected from our previous comments about behaviour at the sonic point.

## § 4 Behaviour in the neighbourhood of the sonic points, its prediction and the choice of the value of $a^2$.

The aim here, is to show that, near the origin, 3.7 has the form

$$a\frac{\partial\varphi_2}{\partial z}\bigg|_{y=0} = \left\{-\frac{\Omega(-b)}{b^{1/2}} - \int_0^b \frac{\Omega'(-\xi)}{\xi^{1/2}}\,d\xi\right\} - \frac{1}{4}\left\{\frac{\Omega(-b)}{b^{3/2}} - \frac{2\Omega'(-b)}{b^{1/2}} - \int_0^b \frac{2\Omega''(-\xi)}{\xi^{1/2}}\,d\xi\right\}s + o(s) \qquad \underline{4.1a}$$

and similarly that 3.8 has the form

$$a\frac{\partial\varphi_2}{\partial z}\bigg|_{y=0} = \left\{-\frac{\Omega(-b)}{b^{1/2}} - \int_0^b \frac{\Omega'(-\xi)}{\xi^{1/2}}\,d\xi\right\} + \frac{1}{4}\left\{\frac{\Omega(-b)}{b^{3/2}} - \frac{2\Omega'(-b)}{b^{1/2}} - \int_0^b \frac{2\Omega''(-\xi)}{\xi^{1/2}}\,d\xi\right\}z + o(z) \qquad \underline{4.1b}$$

so that we establish the continuity of $\frac{\partial\varphi_2}{\partial z}\big|_{y=0}$ at the origin, obtain its value $\frac{\partial\varphi_2}{\partial z}\big|_{\substack{y=0\\z=0}}$, and also the value of $\frac{\partial^2\varphi_2}{\partial z^2}\big|_{\substack{y=0\\z=0}}$.

For consistency, the value of $\frac{\partial\varphi_2}{\partial z}\big|_{\substack{y=0\\z=0}}$ should vanish, as we have taken the origin at the sonic point, and this gives the means of determining b, i.e. the equation

$$\frac{\Omega(-b)}{b^{1/2}} + \int_0^b \frac{\Omega'(-\xi)}{\xi^{1/2}}\cdot d\xi \qquad \underline{4.2}$$

The value of $\frac{\partial^2\varphi_2}{\partial z^2}\big|_{\substack{y=0\\z=0}}$ is required, because we choose to make $a^2 z$ the best approximation to $(\gamma+1)M_\infty^2\,\varphi_z$ at the sonic point. This is the choice made by Maeder & Thommen [7], to make their correction term minimum, except that they require to choose a point which gives the best value of $(\gamma+1)M_\infty^2\,\varphi_{zz}$, rather than $(\gamma+1)M_\infty^2\,\varphi_z$.

From this choice we obtain the equation for $a^2$:

$$a^2 = \frac{(\gamma+1)}{4a}\left\{\frac{\Omega(-b)}{b^{3/2}} - \frac{2\Omega'(-b)}{b^{1/2}} - \int_0^b \frac{2\Omega''(-\xi)}{\xi^{1/2}}\,d\xi\right\} \qquad \underline{4.3}$$

To proceed with our stated intention, consider equation 3.7. In the integral, split the range of integration at $\eta$. As we are considering $s \to 0$, we take $1 \gg \eta \gg s$.

$$a\frac{\partial\varphi_2}{\partial z}\bigg|_{y=0} = -\frac{\Omega(0)}{\sqrt{s}} + \frac{1}{2\pi}\int_0^\eta \frac{\Omega(-\xi)}{\xi}\left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})} + \frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\}d\xi + \frac{1}{2\pi}\int_\eta^b \frac{\Omega(-\xi)}{\xi}\left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})} + \frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\}d\xi \qquad 4.4$$

In the second term, for sufficiently small $\eta$ $\quad \Omega(-\xi) = \sum_{m=0}^{\infty} (-\xi)^m \frac{\Omega^m(0)}{m!}$ ,

over the range of integration.

Now define

$$E_o(s) = \frac{1}{2\pi}\left[\int_0^1 \left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})} + \frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\} d\xi - \frac{2\pi}{\sqrt{s}}\right]$$

$$E_n(s) = \frac{1}{2\pi}\int_0^1 \left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})} + \frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\} \xi^{n-1} d\xi \qquad , n > 0 . \qquad \underline{4.5}$$

and note also, that since $\eta \gg s$ , the third term of 4.4 may be written

$$\frac{1}{2\pi}\sum_0^{\infty} a_n s^{n/2} \int_\eta^b \frac{\Omega(-\xi)}{\xi^{\frac{1}{2}(3+n)}} \, d\xi$$

where $\quad \left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})} + \frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\} = \frac{1}{\sqrt{\xi}} \sum_{n=0}^{\infty} a_n \left(\frac{s}{\xi}\right)^{n/2}$

the coefficients $a_n$ being $\quad a_o = \pi \; , \; a_1 = 0 \; , \; a_2 = \frac{3\pi}{4} \; , \; \ldots$

Thus 4.4 may be written

$$a \left.\frac{\partial\varphi}{\partial z}\right|_{y=0} = \sum_{m=0}^{\infty} (-1)^m \frac{\Omega^m(0)}{m!} \eta^{m-\frac{1}{2}} E_m\!\left(\frac{s}{\eta}\right) + \frac{1}{2\pi}\sum_{n=0}^{\infty} a_n s^{n/2} \int_\eta^b \frac{\Omega(-\xi)}{\xi^{\frac{1}{2}(3+n)}} d\xi \qquad \underline{4.6}$$

Now it may be shown that

$$E_o\!\left(\frac{s}{\eta}\right) = -\frac{2}{\pi}\frac{\sqrt{\eta}}{(\sqrt{\eta}+\sqrt{s})} K \quad , \quad E_1\!\left(\frac{s}{\eta}\right) = \frac{2}{\pi}\left[\left(1+\sqrt{\frac{s}{\eta}}\right)E - \frac{s/\eta}{(1+\sqrt{s/\eta})} K\right]$$

where $k^2 = \frac{4\sqrt{\eta s}}{(\sqrt{\eta}+\sqrt{s})^2}$ , and K, E are complete elliptic integrals, modulus $k^2$ .

By using series expansions for the elliptic integrals, we obtain

$$E_o\!\left(\frac{s}{\eta}\right) = -1 + \frac{1}{4}\left(\frac{s}{\eta}\right) + O\left(s^{3/2}\right)$$

$$E_1\!\left(\frac{s}{\eta}\right) = 1 - 3/4\left(\frac{s}{\eta}\right) + O\left(s^{3/2}\right)$$

$$\Delta E_n\!\left(\frac{s}{\eta}\right) = E_n\!\left(\frac{s}{\eta}\right) - \left(\frac{s}{\eta}\right) E_{n-1}\!\left(\frac{s}{\eta}\right) = \frac{1}{2\pi}\int_0^1 \xi^{n-2}\left[\left(\sqrt{\xi}+\sqrt{\frac{s}{\eta}}\right)E + \left(\sqrt{\xi}-\sqrt{\frac{s}{\eta}}\right)K\right] d\xi$$

$$= \frac{1}{(2n-1)} - \frac{3s}{2\eta}\cdot\frac{1}{(2n-3)} + O\left(s^{3/2}\right)$$

and hence

$$E_n\left(\frac{s}{\eta}\right) = \frac{1}{2n-1} - \frac{3s}{2\eta}\cdot\frac{1}{(2n-3)} + O(s^{3/2})$$

Using these expansions in 4.6

$$a\left.\frac{\partial\varphi_2}{\partial z}\right|_{y=0} = \left[-\frac{\Omega(0)}{\eta^{1/2}} - \Omega'(0).\eta^{1/2} + \sum_{m=2}^{\infty}(-1)^m\frac{\Omega^m(0)}{m!}\frac{\eta^{m-\frac{1}{2}}}{(2m-1)} + \frac{1}{2}\int_\eta^b \frac{\Omega(-\xi)}{\xi^{3/2}}.d\xi\right]$$

$$+\frac{s}{4}\left[\frac{\Omega(0)}{\eta^{3/2}} + \frac{3\Omega'(0)}{\eta^{1/2}} + \sum_{m=2}^{\infty}(-1)^m\frac{\Omega^m(0)}{m!}.\frac{3.2\eta^{m-3/2}}{(2m-3)} + \frac{3}{2}\int_\eta^b\frac{\Omega(-\xi)}{\xi^{5/2}}d\xi\right]$$

$$+ O(s^{3/2})$$

Now, $\eta$ was chosen arbitrarily, so the right hand side must be independent of $\eta$ , and in particular, the coefficients of the powers of s, must be independent of $\eta$ . Thus

$$\left.\frac{\partial\varphi_2}{\partial z}\right|_{\substack{y=0\\z=0^-}} = \mathcal{F}\ \frac{1}{2}\int_0^b \frac{\Omega(-\xi)}{\xi^{3/2}}.d\xi = -\frac{\Omega(-b)}{b^{1/2}} - \int_0^b\frac{\Omega'(-\xi)}{\xi^{1/2}}.d\xi$$

$$\left.\frac{\partial^2\varphi_2}{\partial z^2}\right|_{\substack{y=0\\z=0^-}} = -\frac{3}{8}\mathcal{F}\int_0^b\frac{\Omega(-\xi)}{\xi^{5/2}}d\xi = \frac{1}{4}\left\{\frac{\Omega(-b)}{b^{3/2}} - \frac{2\Omega'(-b)}{b^{1/2}} - \int_0^b\frac{2\Omega''(-\xi)}{\xi^{1/2}}d\xi\right\}$$

where $\mathcal{F}$ denotes the finite part.

Having established the result for $z \to 0^-$ , we now consider the behaviour of 3.8, as $z \to 0^+$.

$$\left.\frac{\partial\varphi}{\partial z}\right|_{y=0} = \lim_{\varepsilon\to 0}\left\{\frac{\partial Q_1(\varepsilon)}{\partial z} + \frac{\partial Q_2(\varepsilon)}{\partial z}\right\}$$

In this case, for $\frac{\partial Q_1(\varepsilon)}{\partial z}$ it is convenient to return to the form provided in the definition of $Q_1(\varepsilon)$ , and from this we obtain

$$\frac{\partial Q_1(\varepsilon)}{\partial z} = \frac{2}{a\pi}\int_\varepsilon^b d\xi\ \Omega(-\xi)\int_0^\infty\frac{z+\xi+x^2\ .dx}{\{(z-\xi-x^2)^2+4\xi z\}^{3/2}}$$

Also

$$\frac{\partial Q_2(\varepsilon)}{\partial z} = \frac{2}{a\pi}\int_\varepsilon^z\frac{\Omega(\zeta)}{2\zeta(z-\zeta)}\left\{2\sqrt{\zeta}\,K - (\sqrt{\zeta}+\sqrt{z})E\right\}.d\zeta - \frac{\Omega(z)}{a\sqrt{z}}$$

The method is essentially the same as that for $z < 0$, in that we divide the range of integration $\left(\text{in } \frac{\partial Q_1(\varepsilon)}{\partial z}\right)$ so that we may employ a Taylor expansion of $\Omega(\zeta)$ near the origin, although the $\varepsilon$ limit alters the form a little.

For $1 \gg \eta \gg z$

$$\frac{2}{\pi}\int_{\varepsilon}^{\eta} d\xi.\Omega(-\xi)\int_0^{\infty}\frac{(z+\xi+x^2)}{\{(z-\xi-x^2)^2+4\xi z\}^{3/2}} = \sum_{m=0}^{\infty}(-1)^m\frac{\Omega^m(0)}{m!}\cdot\frac{2}{\pi}\int_{\varepsilon}^{\eta}d\xi.\xi^m\int_0^{\infty}\frac{(z+\xi+x^2).dx}{\{(z-\xi-x^2)^2+4\xi z\}^{3/2}}$$

and

$$\frac{2}{\pi}\int_{\varepsilon}^{z}\frac{\Omega(\zeta)}{2\zeta(z-\zeta)}\{2\sqrt{\zeta}K-(\sqrt{\zeta}+\sqrt{z})E\}.d\zeta - \frac{\Omega(z)}{\sqrt{z}} = \sum_{n=0}^{\infty}\frac{\Omega^n(0)}{n!}\left\{\frac{2}{\pi}\int_{\varepsilon}^{z}\frac{\{2\sqrt{\zeta}K-(\sqrt{\zeta}+\sqrt{z})E\}}{2(z-\zeta)}\zeta^{m-1}d\zeta - \frac{z^n}{\sqrt{z}}\right\}$$

Now putting

$$V_0(z) = \lim_{\varepsilon\to 0}\left\{\frac{2}{\pi}\int_{\varepsilon}^{1}d\xi\int_0^{\infty}\frac{(z+\xi+x^2)}{\{(z-\xi-x^2)^2+4\xi z\}^{3/2}}.dx + \frac{2}{\pi}\int_{\varepsilon}^{z}\frac{2\sqrt{\zeta}K-(\sqrt{\zeta}+\sqrt{z})E}{2\zeta(z-\zeta)}.d\zeta - \frac{1}{\sqrt{z}}\right\}$$

$$V_n(z) = \frac{2}{\pi}\int_0^1 d\xi.\xi^n\int_0^{\infty}\frac{(z+\xi+x^2).dx}{\{(z-\xi-x^2)^2+4\xi z\}^{3/2}} \qquad n \geq 1$$

<u>4.7</u>

$$U_m = \frac{2}{\pi}\int_0^1\frac{\{2\sqrt{\zeta}K-(\sqrt{\zeta}+1)E\}}{2(1-\zeta)}\zeta^{m-1}.d\zeta - 1 \quad , \quad m \geq 0$$

$$U_0 = 0$$

<u>4.8</u>

we may now write

$$a\frac{\partial\varphi}{\partial z}\bigg|_{y=0} = \frac{2}{\pi}\int_{\eta}^{b}d\xi.\Omega(-\xi)\int_0^{\infty}\frac{(z+\xi+x^2)\,dx}{\{(z-\xi-x^2)^2+4\xi z\}^{3/2}} + \sum_{m=0}^{\infty}(-1)^m\frac{\Omega^m(0)}{m!}\eta^{m-\frac{1}{2}}V_m\left(\frac{z}{\eta}\right)$$

$$+ \sum_{n=0}^{\infty}\frac{\Omega^n(0)}{n!}z^{n-\frac{1}{2}}U_n$$

<u>4.9</u>

Now it may be shown

$$V_0\left(\frac{z}{\eta}\right) = -\frac{2}{\pi}\frac{2\sqrt{\eta}}{(\sqrt{\eta+z}+\sqrt{z})}F_*$$

$$V_1\left(\frac{z}{\eta}\right) = \frac{2}{\pi}\frac{1}{\sqrt{\eta}}\left\{(\sqrt{z+\eta}+\sqrt{z})2E_* - 4\sqrt{z}\right\}$$

$$U_1 = -4/\pi$$

where $F_*$ , $E_*$ are the incomplete elliptic integrals, defined in 2.6, with

$$\lambda_1 = \frac{\sqrt{\eta+z}-\sqrt{z}}{\sqrt{\eta+z}+\sqrt{z}} .$$

As $\eta > z$ , by expanding the integrand, it is easily established that

$$\frac{2}{\pi}\int_\eta^b d\xi\, \Omega(-\xi) \int_0^\infty \frac{(z+\xi+x^2)\, dx}{\{(z-\xi-x^2)^2+4\xi z\}^{3/2}}$$

$$= \left[\frac{1}{2}\int_\eta^b \frac{\Omega(-\xi)}{\xi^{3/2}}\, d\xi\right] - \left[\frac{3}{8}\int_\eta^b \frac{\Omega(-\xi)}{\xi^{5/2}}\, d\xi\right] z + O(z^2) \qquad \underline{4.10}$$

Further, by employing expansions for incomplete elliptic integrals [13] when $\lambda_1 = \frac{\sqrt{\eta+z}-\sqrt{z}}{\sqrt{\eta+z}+\sqrt{z}}$

$$E_* = \frac{\pi}{4} + \left(\frac{z}{\eta}\right)^{1/2}\left(1-\frac{\pi}{4}\right) + \left(\frac{z}{\eta}\right)\left(\frac{5\pi}{16}-1\right) + O\left(\left(\frac{z}{\eta}\right)^{3/2}\right)$$

$$F_* = \frac{\pi}{4} + \left(\frac{z}{\eta}\right)^{1/2}\cdot\frac{\pi}{4} + \left(\frac{z}{\eta}\right)\cdot\frac{\pi}{16} + O\left(\left(\frac{z}{\eta}\right)^{3/2}\right)$$

and hence

$$V_0\left(\frac{z}{\eta}\right) = -1 + \frac{z}{4\eta} + O\left(\left(\frac{z}{\eta}\right)^{3/2}\right)$$

$$V_1\left(\frac{z}{\eta}\right) = 1 - \frac{4}{\pi}\left(\frac{z}{\eta}\right)^{1/2} + \frac{3}{4}\left(\frac{z}{\eta}\right) + O\left(\left(\frac{z}{\eta}\right)^{3/2}\right) .$$

If in 4.9, we now take the two infinite series, and write them as

$$\sum_{m=0}^{\infty} (-1)^m \frac{\Omega^m(0)}{m!}\, I_m \quad , \text{ where } \quad I_m = \left(\eta^{m-\frac{1}{2}} V_m\left(\frac{z}{\eta}\right) + (-1)^m z^{m-\frac{1}{2}} U_m\right)$$

then using the above approximations

$$I_0 = -\frac{1}{\eta^{1/2}} + \frac{z}{4\eta^{3/2}} + O(z^{3/2})$$

$$I_1 = \eta^{1/2} + \frac{3}{4}\frac{z}{\eta^{1/2}} + O(z^{3/2}) \qquad \underline{4.11}$$

Now consider $I_m$ $(m > 1)$. Expanding the integrand in $V_m\left(\frac{z}{\eta}\right)$ , for small z, for $m > 1$ it may be shown

$$I_m = \frac{1}{(2m-1)}\eta^{m-\frac{1}{2}} - \frac{3}{4}\frac{1}{(2m-3)}\eta^{m-3/2} z + o(z) \qquad \underline{4.12}$$

Using the expansions 4.10, 4.11, 4.12 in 4.9

$$a\left.\frac{\partial \varphi_2}{\partial z}\right|_{y=0} = \left[-\frac{\Omega(0)}{\eta^{1/2}} - \Omega'(0)\,\eta^{1/2} + \sum_{m=2}^{\infty}(-1)^m\frac{\Omega^m(0)}{m!}\frac{\eta^{m-\frac{1}{2}}}{(2m-1)} + \frac{1}{2}\int_{\eta}^{b}\frac{\Omega(-\xi)}{\xi^{3/2}}\,d\xi\right]$$

$$+\frac{z}{4}\left[\frac{\Omega(0)}{\eta^{3/2}} - \frac{3\Omega'(0)}{\eta^{1/2}} + 3\sum_{m=2}^{\infty}(-1)^m\frac{\Omega^m(0)}{m!}\frac{\eta^{m-3/2}}{(2m-3)} - \frac{3}{2}\int_{\eta}^{b}\frac{\Omega(-\xi)}{\xi^{5/2}}\,d\xi\right]$$

$$+\quad o(z)$$

Now $\eta$ was chosen arbitrarily, so the right hand side must be independent of $\eta$ and in particular the coefficients of the powers of z must be independent of $\eta$

Thus

$$\left.\frac{\partial \varphi_2}{\partial z}\right|_{\substack{y=0\\ z=0+}} = \mathcal{F}\,\frac{1}{2}\int_0^b\frac{\Omega(-\xi)}{\xi^{3/2}}\,d\xi = -\frac{\Omega(-b)}{b^{1/2}} - \int_0^b\frac{\Omega'(-\xi)}{\xi^{1/2}}\,.d\xi$$

$$\left.\frac{\partial^2 \varphi_2}{\partial z^2}\right|_{\substack{y=0\\ z=0+}} = -\mathcal{F}\,\frac{3}{8}\int_0^b\frac{\Omega(-\xi)}{\xi^{5/2}}\,d\xi = \frac{1}{4}\left\{\frac{\Omega(-b)}{b^{3/2}} - 2\frac{\Omega'(-b)}{b^{1/2}} - \int_0^b\frac{2\Omega''(-\xi)}{\xi^{1/2}}\,d\xi\right\}$$

which completes the proof of equations 4.1a, 4.1b, and consequently of 4.2, which gives b, and 4.3 which gives $a$ .

## §5. The calculation of pressure distribution on the airfoil.

In this section, we collect the results required for the calculation of pressure distributions on thin airfoils having small slope, and then apply the method to the particular class of airfoils whose slopes are given by polynomials

The non-dimensional pressure distribution $C_p = \dfrac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}$ is given by $C_p = -2 \left.\dfrac{\partial \Phi_2}{\partial z}\right|_{y=0}$, to the usual linear approximation.

According to the transonic similarity rules, the quantity

$$\tilde{C}_p = \frac{(\gamma+1)^{1/3}}{\tau^{2/3}} C_p$$

should be independent of the thickness ratio $\tau$, and in our approximation this is the case. For the purpose of comparison with other work, results are given in terms of $\tilde{C}_p$.

Suppose the local slope of the airfoil $\Omega(z) = \tau A \,\omega(z)$, then equation 4.2 becomes

$$\frac{\omega(-b)}{b^{1/2}} + \int_0^b \frac{\omega'(-\xi)}{\xi^{1/2}}\, d\xi = 0 \qquad \underline{5.1}$$

and the value of $a$, from 4.3, is given by

$$a^3 = (\gamma+1)\,\tau\, B^3$$

where

$$B^3 = \frac{A}{4}\left\{ \frac{\omega(-b)}{b^{3/2}} - \frac{2\,\omega'(-b)}{b^{1/2}} - \int_0^b \frac{2\,\omega''(-\xi)}{\xi^{1/2}}\, d\xi \right\} \qquad \underline{5.2}$$

Using these results in 3.7 and 3.8, we obtain

$$\tilde{C}_p(z<0) = -2\frac{A}{B}\left\{ -\frac{\omega(0)}{\sqrt{s}} + \frac{1}{2\pi}\int_0^b \frac{\omega(-\xi)}{\xi}\left\{\frac{E}{(\sqrt{\xi}-\sqrt{s})} + \frac{K}{(\sqrt{\xi}+\sqrt{s})}\right\} d\xi \right\} \qquad \underline{5.3}$$

$$\tilde{C}_p(z>0) = -2\frac{A}{B}\left\{ D(\varepsilon) - \frac{\omega(z)}{\sqrt{z}} + \frac{2}{\pi}\int_{\varepsilon}^{z} \frac{\omega(\zeta)}{2\zeta(z-\zeta)}\left\{2\sqrt{\zeta}\,K - (\sqrt{\zeta}+\sqrt{z})E\right\} d\zeta \right\} \qquad \underline{5.4}$$

where lim $\varepsilon \to 0$ is understood, and $D(\varepsilon)$ takes the alternative forms

$$\frac{2}{\pi}\int_{\varepsilon}^{b} \omega(-\zeta) \int_{0}^{\infty} \frac{(z+\zeta+x^2)}{\{(z-\zeta-x^2)^2+4\zeta z\}^{3/2}} \, .\, dx$$

5.5a

$$\frac{2}{\pi}\int_{\varepsilon}^{b} \frac{\omega(-\zeta)}{8z(\sqrt{z+\zeta}-\sqrt{z})}\left[\begin{array}{c} 2E_* - 2\lambda_1 F_* - (1-\lambda_1) \\ + \frac{\zeta}{4(\zeta+z)}\{(3-2\lambda_1+3\lambda_1^2)2F_* - (6-\lambda_1-\lambda_2)(2E_*-1+\lambda_1)\} \end{array}\right] d\zeta$$

5.5b

The forms 5.3, 5.4 are not immediately useful for numerical calculation. However, if the case $\omega(z) = z^m$, $0 \leq m < 1$ can be treated, then the remaining numerical integrations are straightforward. We illustrate this by the example of a polynomial for $\omega(z)$, the case $\omega(z)$ = constant being the extreme case from the point of view of singular integrands.

Suppose then, that $\omega(z) = \sum_{n=0}^{N} \omega^n(0) \frac{z^n}{n!}$

First consider 5.3

$$\tilde{C}_p(z<0) = -\frac{2A}{B}\left\{-\frac{\omega(0)}{\sqrt{s}} + \frac{1}{2\pi}\sum_{n=0}^{N}(-1)^n \frac{\omega^n(0)}{n!}\int_0^b\left\{\frac{E}{(\sqrt{\zeta}-\sqrt{s})}+\frac{K}{(\sqrt{\zeta}+\sqrt{s})}\right\}\zeta^{n-1}.d\zeta\right\}$$

This may be written

$$\tilde{C}_p(z<0) = -\frac{2A}{B}\sum_{n=0}^{N}\frac{\omega^n(0)}{n!}(-1)^n b^{n-\frac{1}{2}} E_n\left(\frac{s}{b}\right) \qquad 5.6$$

where
$$E_0(s) = -\frac{1}{\sqrt{s}} + \frac{1}{2\pi}\int_0^1\left\{\frac{E}{(\sqrt{\zeta}-\sqrt{s})}+\frac{K}{(\sqrt{\zeta}+\sqrt{s})}\right\}\frac{d\zeta}{\zeta}$$

$$E_n(s) = \frac{1}{2\pi}\int_0^1\left\{\frac{E}{(\sqrt{\zeta}-\sqrt{s})}+\frac{K}{(\sqrt{\zeta}+\sqrt{s})}\right\}\zeta^{n-1}.d\zeta \qquad 5.7$$

Next consider 5.4

$$\frac{2}{\pi}\int_{\varepsilon}^{z}\frac{\omega(\zeta)}{2\zeta(z-\zeta)}\{2\sqrt{\zeta}K-(\sqrt{\zeta}+\sqrt{z})E\}.d\zeta - \frac{\omega(z)}{\sqrt{z}}$$
$$= \omega(0)\left[\frac{2}{\pi}\int_{\varepsilon}^{z}\frac{\{2\sqrt{\zeta}K-(\sqrt{\zeta}+\sqrt{z})E\}}{2\zeta(z-\zeta)}d\zeta - \frac{1}{\sqrt{z}}\right] + \sum_{n=1}^{\infty}\frac{\omega^n(0)}{n!}(-1)^n\left[\frac{2}{\pi}\int_0^z\frac{\{2\sqrt{\zeta}K-(\sqrt{\zeta}+\sqrt{z})E\}}{2(z-\zeta)}\zeta^{n-1}.d\zeta - z^{n-\frac{1}{2}}\right],$$

$$D(\varepsilon) = \omega(0).\frac{2}{\pi}\int_\varepsilon^b d\zeta\int_0^\infty dx\,\frac{z+\zeta+x^2}{\{(z-\zeta-x^2)^2+4\zeta z\}^{3/2}} + \sum_{n=1}^{N}\frac{\omega^n(0)}{n!}(-1)^n\int_0^b d\zeta\,\zeta^n\int_0^\infty dx.\frac{z+\zeta+x^2}{\{(z-\zeta-x^2)^2+4\zeta z\}^{3/2}}$$

Define

$$b^{-1/2}V_0\left(\frac{z}{b}\right) = \lim_{\varepsilon\to 0}\left[\frac{2}{\pi}\int_\varepsilon^b d\zeta\int_0^\infty dx.\frac{z+\zeta+x^2}{\{(z-\zeta-x^2)^2+4\zeta z\}^{3/2}} + \frac{2}{\pi}\int_\varepsilon^z\frac{\{2\sqrt{\zeta}K-(\sqrt{\zeta}+\sqrt{z})E\}}{2\zeta(z-\zeta)}d\zeta - \frac{1}{\sqrt{z}}\right] \quad \underline{\mathbf{5.8}}$$

$$V_n(z) = \frac{2}{\pi}\int_0^1\frac{\zeta^n}{8z(\sqrt{z+\zeta}-\sqrt{z})}\left[\begin{array}{l} 2E_* - 2\lambda_1 F_* - (1-\lambda_1) \\ +\dfrac{\zeta}{4(\zeta+z)}\{(3-2\lambda_1+3\lambda_1^2)2F_* - (6-\lambda_1-\lambda_2)(2E_*-1+\lambda_1)\}\end{array}\right].d\zeta$$

<u>5.9</u>

$$U_n = \left[-1+\frac{2}{\pi}\int_0^1\frac{t^{2n-1}}{(1-t^2)}\{2tK-(t+1)E\}.dt\right],\ n>0$$

$$U_0 = 0$$

<u>5.10</u>

Then substituting in 5.4

$$\tilde{C}_p(z>0) = -\frac{2A}{B}\sum_{n=0}^{N}\frac{\omega^n(0)}{n!}\left[(-1)^n b^{n-\frac{1}{2}}V_n\left(\frac{z}{b}\right) + z^{n-\frac{1}{2}}U_n\right] \quad \underline{5.11}$$

Now $E_0$ (s), $V_0$ (z) may be evaluated analytically. The results have been quoted in §4 (near 4.6, 4.9), together with those for $E_1$ (s), $V_1$ (z), $U_1$ . The actual evaluation is straightforward except for $U_1$ which requires a little ingenuity, but it is tedious and is not included.

$V_n$ (z), n $\geqslant 1$ , may be readily computed, and the analytic evaluation of $V_1$ (z) serves as a check on accuracy. In 5.11, the argument is $\left(\frac{z}{b}\right)$, so that for accurate work, $V_n$ (z) must be generated for each particular case. However, a table is provided for n = 1 ... 5, and for typical values of the argument.

$U_n$ may be derived by considering $\Delta U_n = U_n - U_{n-1}$ , which may be readily computed. Values of $U_1$ ...$U_5$ are given.

$E_n(s)$ may be derived by considering $\Delta E_n(s) = E_n(s) - s\,E_{n-1}(s)$.

Then $$\Delta E_n(s) = \frac{1}{2\pi}\int_0^1 \xi^{n-2}\{(\sqrt{\xi}+\sqrt{s})E + (\sqrt{\xi}-\sqrt{s})K\}\,d\xi$$

and it is convenient to consider the form

$$\Delta E_n(s) = s^{n-\frac{1}{2}}\Big\{\frac{1}{2\pi}\int_0^1 \{(\sqrt{\xi}+1)E + (\sqrt{\xi}-1)K\}\,\xi^{n-2}\,d\xi + \frac{1}{2\pi}\int_0^{1/s}\{(\sqrt{\xi}+1)E + (\sqrt{\xi}-1)K\}\,\xi^{n-2}\,d\xi$$

The situation for $E_n(s)$ is the same as that for $V_n(z)$. For accurate work it must be generated for each particular case, but for $n = 1 \ldots 5$, and typical values of $\left(\frac{s}{b}\right)$ a table is provided.

Finally, we use the expressions for $\widetilde{C}p$ in 5.6, 5.11, and the tables to consider members of a class of airfoils investigated experimentally by Michel, Marchaud & Le Gallo [14].

The airfoils are given by

$$y = A\,2\ell\left[\left(\frac{\tilde{z}}{2\ell}\right) - \left(\frac{\tilde{z}}{2\ell}\right)^n\right] \qquad \underline{5.12}$$

where $\tilde{z} = 0$ at the leading edge, and also for the reversed form

$$y = A\,.\,2\ell\left[\left(1-\frac{\tilde{z}}{2\ell}\right) - \left(1-\frac{\tilde{z}}{2\ell}\right)^n\right]$$

The chord is $2\ell$, and the thickness ratio $\tau$ is given by

$$A = \tau\,\frac{n^{n/(n-1)}}{2(n-1)}$$

In terms of the notation of this work

$$\Omega(z) = \tau\left(\frac{n^{n/(n-1)}}{2(n-1)}\right)\begin{cases}\left[1 - n\left(\frac{z+b}{2\ell}\right)^{n-1}\right] & \text{for maximum thickness aft.}\\[2ex] \left[n\left(1-\frac{z+b}{2\ell}\right)^{n-1} - 1\right] & \text{for maximum thickness forward.}\end{cases}$$

The experimental work was performed for values of n = z, 3.38, 6.05 which give maximum thickness at 50%, 60%, 70% chord respectively, and 40%, 30% chord when reversed. Spreiter [9] and Thommen [15] also considered these airfoils. For simplicity, so that we may use the tables we only consider integer values of n.

Example

When n = 2, $\Omega(z) = 2\tau'\left[1 - 2\left(\frac{z+b}{2\ell}\right)\right] \equiv A\tau\,\omega(z)$

From 5.1, b = $\ell/2$ . From 5.2 B = $(2/\ell)^{1/2}$

Then 5.6 gives $\tilde{C}_p(z<0) = -2\left(\frac{\ell}{2}\right)^{1/2}\left\{\left(\frac{2}{\ell}\right)^{1/2}E_0\left(\frac{s}{b}\right) + \left(\frac{2}{\ell}\right)\left(\frac{\ell}{2}\right)^{1/2}E_1\left(\frac{s}{b}\right)\right\}$

and hence $\tilde{C}_p(z<0) = \frac{4}{\pi}\left\{\left(\frac{1+s/b}{1+\sqrt{s/b}}\right)K - \left(1+\sqrt{\frac{s}{b}}\right)E\right\}$

Also 5.11 gives $\tilde{C}_p(z>0) = -2(\ell/2)^{1/2}\left\{(2/\ell)^{1/2}V_0\left(\frac{z}{b}\right) - \left(\frac{2}{\ell}\right)\left[-\left(\frac{\ell}{2}\right)^{1/2}V_1\left(\frac{z}{b}\right) + z^{1/2}U_1\right]\right\}$

and hence $\tilde{C}_p(z>0) \doteq -\frac{4}{\pi}\left\{\left(\sqrt{1+z/b}+\sqrt{\frac{z}{b}}\right)2E_* - \left(\sqrt{1+\frac{z}{b}}-\sqrt{\frac{z}{b}}\right)2F_* - 2\sqrt{\frac{z}{b}}\right\}$

The resulting distribution is plotted in Fig.1.

For other values of n, the calculation is equally straightforward, and although the numerical work becomes heavier, it is certainly a matter for a desk machine. The cases n = 6 , n = 3 have been evaluated, and the resulting distributions are plotted in Figs 2...5.

## §6 Conclusion.

This work considers the possibility of using the equation

$$\varphi_{yy} = a^2 z\, \varphi_{zz}$$

as an approximation to the transonic small disturbance equation for a sonic stream flowing past two dimensional thin wings.

It has been shown that it is possible to derive a unique $\varphi$ simply by satisfying the equation, and the boundary conditions on the body. The value of $a^2$ is chosen so that $a^2 z$ constitutes the best linear approximation to $\{(\gamma+1)M_\infty^2 \varphi_z\}_{y=0}$ in the neighbourhood of the sonic point. However, this potential unlike the three dimensional solution of which it is the integral, does not give vanishing disturbances at large distances. This feature draws attention to the assumption implicit in all the linearised approximations, that it is sufficient to approximate the equation correctly in the immediate neighbourhood of the body to derive surface pressures. It is not regarded as a serious defect. Physically, in the transonic regime, the influence of a body does extend to large distances in the direction normal to the flow.

The pressure distributions obtained by the present method are given in Figs 1-5, together with the theoretical distributions of Spreiter [9] and Maeder & Thommen [3] , and the experimental distributions of Michel [14] . Agreement with experiment is not so good as that of Spreiter's work [9] for maximum thickness forward, but for maximum thickness towards the rear, it is at least equal if not better. In [16] some doubts have been raised about the experimental results but they do not seem to justify regarding differences between theory and experiment for maximum thickness aft as due to experimental error, while ignoring the error in other cases. However, as explained in the introduction, the theoretical basis of Spreiter's method is suspect. The present results are better than those of the mathematically consistent theory of Maeder & Thommen [3] , though the method lacks the ease of application of [3] . The improvements on [3] given in [7] and [8] provide better agreement with experiment for the parabolic-arc thin airfoil and neighbouring profiles. However, in the axisymmetric case [8] , the results constitute little improvement on [3] , and are inferior to those of Cole & Royce [6] .

It is considered that, in the two dimensional case, in view of the case of application of [3] and its improvements, the advantage in using the present method is perhaps marginal. However, the reasonable agreement with experiment when the maximum disturbance (i.e. maximum thickness) is towards the rear, together with the excellent results of [6] for the axisymmetric case with maximum disturbance towards the rear, suggest that for three-dimensional shapes satisfying this condition, fairly accurate results may be obtained. One such shape is the delta, and it is hoped to consider, in a later paper, the application of the present approach to this shape.

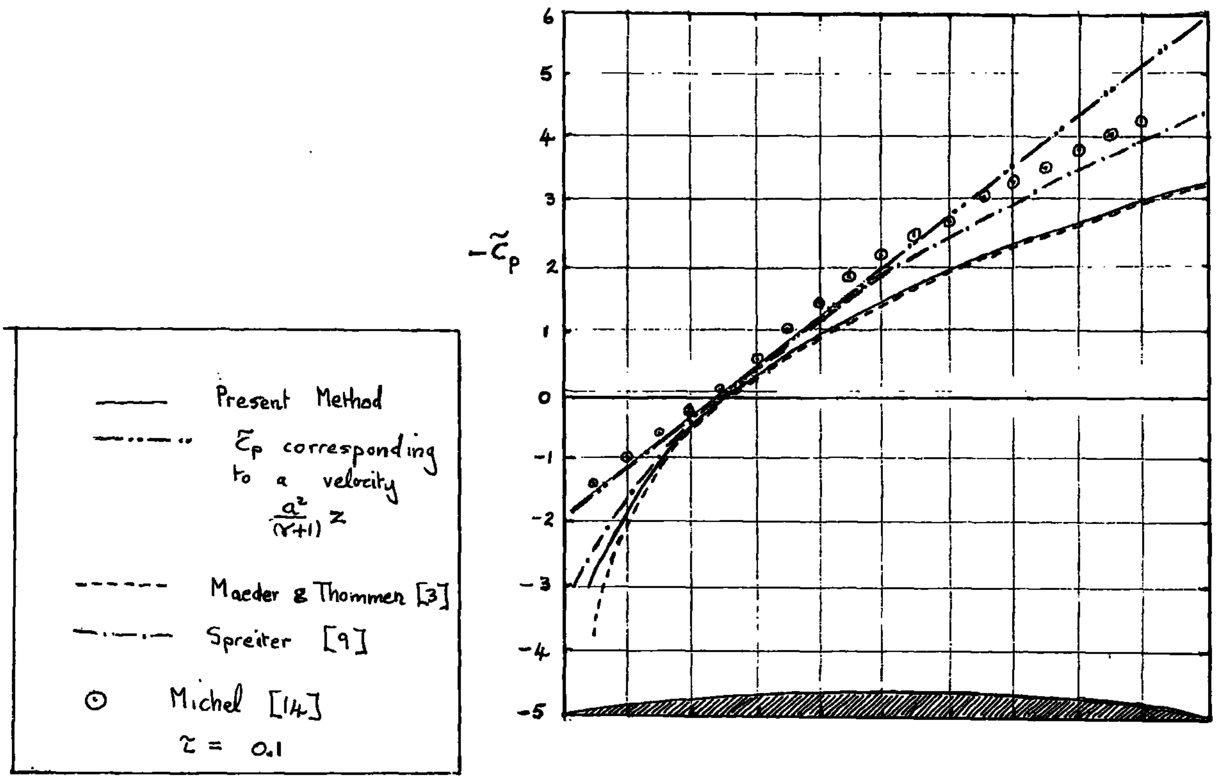



Fig 1. Pressure distribution $\tilde{c}_p$

Airfoil maximum thickness 50% chord

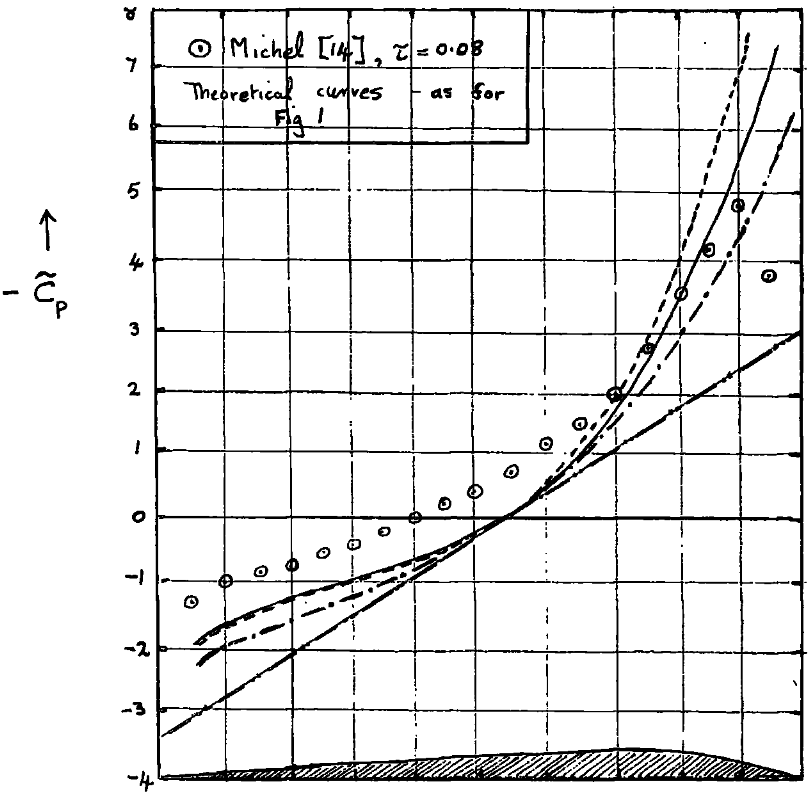



Fig 2. Pressure distribution $\tilde{C}_p$

Airfoil maximum thickness ≏ 70% chord

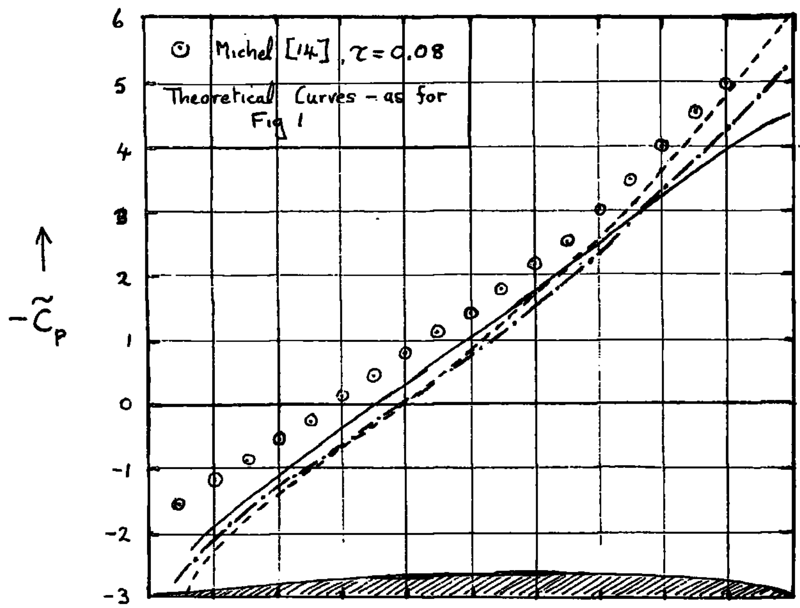



Fig 3 Pressure distribution $\tilde{C}_p$

Airfoil maximum thickness ≃ 60% chord

N.B. - in the notation of 5.12 - present results are for $n = 3, 6$
Spreiter, Maeder, Michel for $n = 3.38$, $n = 6.05$

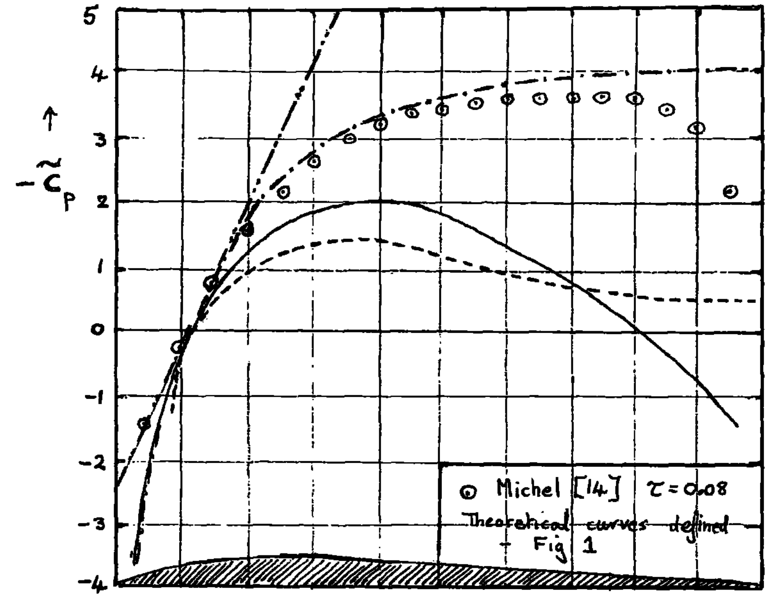



Fig 4 Pressure distribution $\tilde{C}_p$

Airfoil maximum thickness ≏ 30% chord.

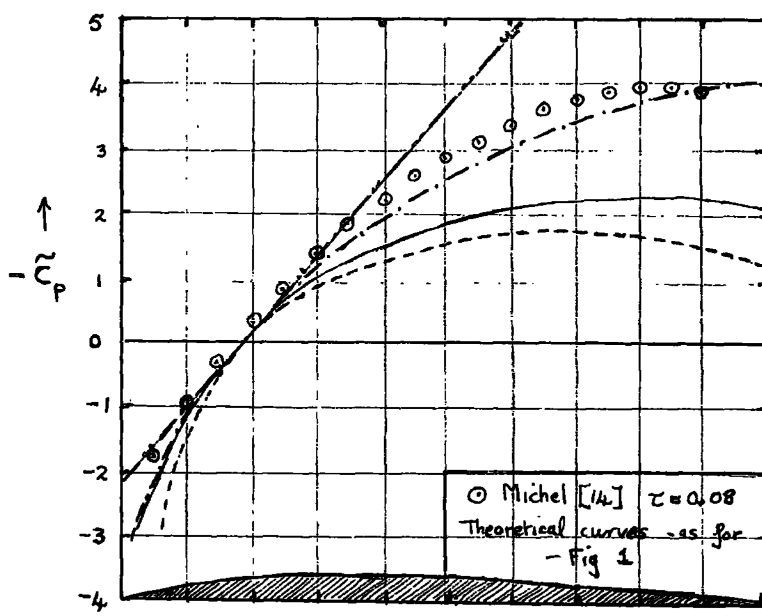



Fig 5 Pressure distribution $\tilde{C}_p$

Airfoil maximum thickness ≏ 40% chord.

---

N.B. – in the notation of 5.12, theoretical results are for $n = 3, 6$, and experiments for $n = 3.38$, $n = 6.05$

Table 1. $U_n$ *integrals*

$U_1 = -1{\cdot}27324$
$U_2 = -1{\cdot}13177$
$U_3 = -1{\cdot}08650$
$U_4 = -1{\cdot}06433$
$U_5 = -1{\cdot}05119$

Table 2. $E_n(s)$ *integrals*

| $s$ | $E_0(s)$ | $E_1(s)$ | $E_2(s)$ | $E_3(s)$ | $E_4(s)$ | $E_5(s)$ |
|---|---|---|---|---|---|---|
| 0·0 | −1·00000 | 1·00000 | 0·33333 | 0·20000 | 0·14286 | 0·11111 |
| 0·1 | −1·02651 | 0·92251 | 0·40103 | 0·23131 | 0·16082 | 0·12356 |
| 0·2 | −1·05655 | 0·83935 | 0·45311 | 0·27223 | 0·18655 | 0·14085 |
| 0·3 | −1·09110 | 0·74920 | 0·48750 | 0·31759 | 0·22130 | 0·16570 |
| 0·4 | −1·13160 | 0·65016 | 0·50137 | 0·36107 | 0·26384 | 0·20026 |
| 0·5 | −1·18034 | 0·53935 | 0·49073 | 0·39471 | 0·30963 | 0·24449 |
| 0·6 | −1·24113 | 0·41208 | 0·44927 | 0·40753 | 0·34916 | 0·29356 |
| 0·7 | −1·32122 | 0·25973 | 0·36615 | 0·38294 | 0·36470 | 0·33343 |
| 0·8 | −1·43698 | 0·06352 | 0·21898 | 0·29078 | 0·32106 | 0·32859 |
| 0·9 | −1·64126 | −0·23462 | −0·05734 | 0·05356 | 0·12859 | 0·18091 |

Table 3. $V_n(z)$ *integrals*

| $z$ | Analytic integration $V_0(z)$ | Analytic integration $V_1(z)$ | Numerical integration $V_1(z)$ | Numerical integration $V_2(z)$ | Numerical integration $V_3(z)$ | Numerical integration $V_4(z)$ | Numerical integration $V_5(z)$ |
|---|---|---|---|---|---|---|---|
| 0·2 | −0·95493 | 0·57219 | 0·57130 | 0·25809 | 0·16387 | 0·11956 | 0·09403 |
| 0·4 | −0·91769 | 0·46410 | 0·46384 | 0·21906 | 0·14165 | 0·10428 | 0·08244 |
| 0·6 | −0·88611 | 0·40031 | 0·40018 | 0·19309 | 0·12605 | 0·09333 | 0·07404 |
| 0·8 | −0·85867 | 0·35657 | 0·35651 | 0·17420 | 0·11440 | 0·08499 | 0·06755 |
| 1·0 | −0·83461 | 0·32417 | 0·32410 | 0·15960 | 0·10523 | 0·07837 | 0·06239 |
| 1·2 | −0·81322 | 0·29883 | 0·29883 | 0·14795 | 0·09785 | 0·07302 | 0·05819 |
| 1·4 | −0·79393 | 0·27839 | 0·27839 | 0·13840 | 0·09174 | 0·06850 | 0·05462 |
| 1·6 | −0·77642 | 0·26152 | 0·26152 | 0·13038 | 0·08652 | 0·06468 | 0·05163 |
| 1·8 | −0·76044 | 0·24720 | 0·24720 | 0·12350 | 0·08212 | 0·06143 | 0·04908 |
| 2·0 | −0·74574 | 0·23491 | 0·23491 | 0·11758 | 0·07824 | 0·05857 | 0·04679 |
| 2·2 | −0·73218 | Identical | 0·22422 | 0·11236 | 0·07480 | 0·05602 | 0·04482 |
| 2·4 | −0·71957 | with | 0·21480 | 0·10772 | 0·07181 | 0·05379 | 0·04304 |
| 2·6 | −0·70779 | numerical | 0·20646 | 0·10364 | 0·06907 | 0·05182 | 0·04144 |
| 2·8 | −0·69678 | result to | 0·19894 | 0·09994 | 0·06665 | 0·04997 | 0·04000 |
| 3·0 | −0·68647 | 5 decimal | 0·19220 | 0·09658 | 0·06443 | 0·04832 | 0·03864 |
| 3·2 | −0·67673 | places | 0·18602 | 0·09352 | 0·06245 | 0·04686 | 0·03750 |
| 3·4 | −0·66750 | | 0·18035 | 0·09072 | 0·06061 | 0·04545 | 0·03635 |
| 3·6 | −0·65877 | | 0·17526 | 0·08817 | 0·05889 | 0·04418 | 0·03533 |
| 3·8 | −0·65050 | | 0·17049 | 0·08582 | 0·05730 | 0·04304 | 0·03444 |
| 4·0 | −0·64267 | | 0·16609 | 0·08359 | 0·05583 | 0·04195 | 0·03355 |
| 4·2 | −0·63516 | | 0·16202 | 0·08155 | 0·05449 | 0·04093 | 0·03272 |
| 4·4 | −0·62796 | | 0·15820 | 0·07964 | 0·05322 | 0·03998 | 0·03196 |
| 4·6 | −0·62115 | | 0·15463 | 0·07786 | 0·05201 | 0·03909 | 0·03126 |
| 4·8 | −0·61459 | | 0·15132 | 0·07620 | 0·05093 | 0·03826 | 0·03062 |
| 5·0 | −0·60829 | | 0·14821 | 0·07461 | 0·04985 | 0·03743 | 0·02998 |
| 5·2 | −0·60224 | | 0·14521 | 0·07315 | 0·04889 | 0·03673 | 0·02941 |
| 5·4 | −0·59645 | | 0·14241 | 0·07175 | 0·04794 | 0·03603 | 0·02884 |
| 5·6 | −0·59085 | | 0·13980 | 0·07041 | 0·04705 | 0·08533 | 0·02833 |
| 5·8 | −0·58550 | | 0·13732 | 0·06914 | 0·04622 | 0·03470 | 0·02782 |
| 6·0 | −0·58028 | | 0·13490 | 0·06793 | 0·04545 | 0·03412 | 0·02731 |
| 6·2 | −0·57525 | | 0·13267 | 0·06685 | 0·04469 | 0·03355 | 0·02687 |
| 6·4 | −0·57041 | | 0·13051 | 0·06576 | 0·04393 | 0·03298 | 0·02642 |
| 6·6 | −0·56570 | | 0·12847 | 0·06468 | 0·04323 | 0·03247 | 0·02604 |
| 6·8 | −0·56118 | | 0·12650 | 0·06373 | 0·04259 | 0·03202 | 0·02559 |
| 7·0 | −0·55672 | | 0·12459 | 0·06277 | 0·04195 | 0·03151 | 0·02521 |
| 7·2 | −0·55246 | | 0·12280 | 0·06190 | 0·04138 | 0·03107 | 0·02489 |
| 7·4 | −0·54832 | | 0·12109 | 0·06099 | 0·04074 | 0·03062 | 0·02451 |
| 7·6 | −0·54431 | | 0·11943 | 0·06016 | 0·04023 | 0·03018 | 0·02419 |
| 7·8 | −0·54043 | | 0·11784 | 0·05933 | 0·03966 | 0·02979 | 0·02387 |
| 8·0 | −0·53661 | | 0·11631 | 0·05857 | 0·03915 | 0·02941 | 0·02355 |
| 8·2 | −0·53291 | | 0·11485 | 0·05781 | 0·03864 | 0·02903 | 0·02324 |
| 8·4 | −0·52929 | | 0·11345 | 0·05710 | 0·03820 | 0·02871 | 0·02298 |
| 8·6 | −0·52578 | | 0·11205 | 0·05640 | 0·03775 | 0·02833 | 0·02266 |
| 8·8 | −0·52235 | | 0·11071 | 0·05577 | 0·03731 | 0·02801 | 0·02241 |
| 9·0 | −0·51904 | | 0·10944 | 0·05513 | 0·03686 | 0·02769 | 0·02215 |
| 9·2 | −0·51579 | | 0·10823 | 0·05449 | 0·03641 | 0·02737 | 0·02190 |
| 9·4 | −0·51261 | | 0·10702 | 0·05389 | 0·03603 | 0·02706 | 0·02165 |
| 9·6 | −0·50949 | | 0·10587 | 0·05329 | 0·03565 | 0·02674 | 0·02145 |
| 9·8 | −0·50643 | | 0·10479 | 0·05271 | 0·03527 | 0·02648 | 0·02120 |

## REFERENCES


[1] Guderley, K. G. — Theory of transonic flow (translation) Pergamon (1962)

[2] Oswatitsch, K. & Keune, F. — Proc. Conf. High Speed Aeronautics, Polytechnic Institute; Brooklyn, 1955.

[3] Maeder, P. F. & Thommen, H. U. — J. Aero. Sci. 23 (1956) 187 - 188.

[4] Miles, J. W. — J. Aero. Sci. 23 (1956), 704 - 705.

[5] Guderley, K. G. & Yoshihara, H. — Quart. Appl. Math. 8 (1951) 333 - 339.

[6] Cole, J. D. & Royce, W. W. — Proc. Sixth Midwest. Conf. Fluid Mech., Austin, Texas (1959).

[7] Maeder, P. F. & Thommen, H. U. — Trans. ASME Ser. E., J. Appl. Mech. 83 (1961), 481 - 490.

[8] Hosokawa, I. — J. Phys. Soc. Japan, 15 (1960) 149 - 157.

[9] Spreiter, J. R. & Alksne, A. Y. — NACA Rep. 1359 (1958)

[10] Spreiter, J. R. & Alksne, A. Y. — NASA TR R-2 (1959)

[11] Spreiter, J. R. & Alksne, A. Y. — NASA TR R-88 (1961)

[12] Whittaker, E. T. & Watson, G. N. — Modern Analysis, Cambridge (1920)

| | | |
|---|---|---|
| [13] | Byrd, P. F. & Friedman, M. D. | Handbook of elliptic integrals for engineers and physicists. Springer (1954). |
| [14] | Michel, R., Marchaud, F. & Le Gallo, J. | ONERA Pub. 65 (1953), 72 (1954). |
| [15] | Oswatitsch, K. (editor) | Symposium Transsonicum, Aachen 1962. Springer (1964). |
| [16] | Carrol, J. B. & Anderson, G. F. | J. Aero. Sci. 23 (1956), 604 - 605. |

## §7 Appendix

The behaviour of the potential (3.1) as $\left|\frac{a^2y^2}{4} - z\right| \to 0$.

First consider $h = \frac{a^2y^2}{4} - z > 0$ ; the integral in (3.1a) may be written:

$$\int_{-b}^{0} 2\,\Omega(\zeta)\,\phi(\varepsilon)\,d\zeta = \frac{8z}{\pi a}\left(\int_{\delta}^{b} + \int_{0}^{\delta}\right) d\xi .\, \Omega(-\xi) \int_{0}^{\infty} \frac{x^2 .\, dx}{\{(x^2+\xi+h)^2+4\xi z\}^{3/2}} \tag{7.1}$$

where $1 \gg \delta \gg \sqrt{h}$ . Replacing $\Omega(-\xi)$ by its Taylor series for the range $(0, \delta)$, the second part of the integral takes the form

$$\sum_{n=0}^{\infty} (-1)^n\, \Omega^{(n)}(0)\; A_n(\delta, h) \tag{7.2}$$

where

$$A_n(\delta,h) = \frac{8z}{\pi a}\int_0^{\delta} d\xi .\, \xi^n \int_0^{\infty} \frac{x^2\, dx}{\{(x^2+\xi+h)^2+4\xi z\}^{3/2}}$$

Interchanging the order of integration, performing the $\xi$ -integration and expanding for small $h$ , it may be shown that

$$\begin{aligned} A_0 &= -\tfrac{2}{a}\, h^{1/2} + \alpha_{00}(\delta) + \alpha_{01}(\delta).h + O(h^2) \\ A_1 &= \alpha_{10}(\delta) + \alpha_{11}(\delta).h + O(h^2) \\ (n>1) \quad A_n &= \alpha_{n0}(\delta) + \alpha_{n1}(\delta).h + O(h^2) \end{aligned} \tag{7.3}$$

where $\alpha_{mn}(\delta)$ depend only on $\delta$ , being definite integrals. The absence of powers of $h^{1/2}$ in $A_n\,(n>1)$ is established by induction; by comparison with $A_1$ it may be shown

$$0 \leqslant \alpha_{n0} \leqslant \alpha_{10} \qquad 7.4$$

Now consider $h' = z - \frac{a^2y^2}{4} > 0$ ; the expression (3.1b) may be recast in the form

$$\phi_2 = \lim_{\varepsilon \to 0} \left\{ \frac{8z}{\pi a} \int_\varepsilon^b d\xi . \Omega(-\xi) . \int_0^\infty \frac{x^2 \, dx}{\{(x^2 + \xi - h')^2 + 4\xi z\}^{3/2}} \right.$$

$$\left. + \int_\varepsilon^{(\sqrt{z}-\theta)^2} \frac{2\Omega(\zeta)}{\pi a \zeta} \left\{ \frac{2\sqrt{\zeta z}}{\{(\sqrt{z}+\sqrt{\zeta})^2 - \theta^2\}^{1/2}} K - \{(\sqrt{z}+\sqrt{\zeta})^2 - \theta^2\}^{1/2} E \right\} . d\zeta \right\}$$

7.5

Reasoning as for the case $h > 0$ , the first of these integrals may be written

$$\frac{8z}{\pi a} \int_\delta^b d\xi . \Omega(-\xi) \int_0^\infty \frac{x^2 \, dx}{\{(x^2 + \xi - h')^2 + 4\xi z\}^{3/2}}$$

$$+ \sum_{n=0}^{\infty} (-1)^n \Omega^{(n)}(0) . B_n(\delta, h, \varepsilon) \qquad 7.6$$

where

$$B_n(\delta, h', \varepsilon) = \frac{8z}{\pi a} \int_\varepsilon^\delta d\xi . \xi^n \int_0^\infty \frac{x^2 . dx}{\{(x^2 + \xi - h')^2 + 4\xi z\}^{3/2}} .$$

Consider $B_0$ : interchanging the order of integration and performing the $\xi$ -integration,

$$B_0 = \frac{2}{\pi a}\left( I_0^{(1)}(\varepsilon) - I_0^{(2)}(\delta)\right)$$

where

$$I_0^{(1)} = \frac{\pi}{2}\sqrt{z-h'} - 2\sqrt{h'} - 2\sqrt{z-h'}\,\tan^{-1}\left(\frac{\sqrt{h'}}{\sqrt{z-h'}}\right) + \sqrt{h'}\,\log\frac{16\,h'^2}{z\varepsilon}$$

and by expanding $I_0^{(2)}$ in powers of $h'$

$$\frac{\pi}{2}\sqrt{z-h'} - I_0^{(2)} = \alpha_{00}(\delta) - \alpha_{01}(\delta)\,.\,h' + O(h'^2).$$

By integration and expansion the contribution $J(\varepsilon)$ from the hyperbolic $(\zeta > 0)$ sources may be reduced to

$$\left\{ \Omega(0)\,\frac{2}{\pi a}\sqrt{h'}\,\log\left(\frac{z\varepsilon}{16\,h'^2}\right) + O\left(h'^{3/2}\log h'\right)\right\}\left\{1 + O(\varepsilon)\right\}$$

so that

$$\lim_{\varepsilon\to 0}\left( J(\varepsilon) + B_0(\delta, h', \varepsilon)\right) = \alpha_{00}(\delta) - \alpha_{01}(\delta)\,h' + O(h'^2) \qquad \underline{7.7\,i}$$

For $B_n$ $(n > 1)$ we may take the $\varepsilon$ limit, and use $B_n(\delta, h', 0)$ . The integral involved in $B_n(\delta, h', 0)$ is the same as that for $A_n(\delta, h)$ , so that

$$B_1\,(\delta, h', 0) = \alpha_{10}(\delta) - \alpha_{11}(\delta)\,.\,h' + O(h'^2) \qquad \underline{7.7\,ii}$$

The induction argument ensures a regular expansion in powers of $h'$ for $B_n$ $(n > 1)$ , so that

$$B_n(\delta, h', 0) = \alpha_{n0}(\delta) - \alpha_{n1}(\delta)\, h' + O(h'^2) \qquad \underline{7.7\ \text{iii}}$$

We can now deduce the required properties of $\phi_2$ as $|h| \to 0$. From (7.3) we see the necessity for the addition of the term $\frac{2}{a}\,\Omega(0).\sqrt{h}$ in the expression for $\phi_2$ in the region $h > 0$, (equation (3.1a)); it removes the singularity as $h \to 0$.

From (7.1), (7.2), (7.3) and (7.5), (7.6), (7.7) we have

$$\lim_{|h|\to 0} \phi_2 = \frac{8z}{\pi a}\int_\delta^b d\xi.\Omega(-\xi)\int_0^\infty \frac{x^2.\ dx}{\{(x^2+\xi)^2 + 4\xi z\}^{3/2}}$$

$$+ \sum_{n=0}^{\infty} (-1)^n\, \Omega^{(n)}(0)\ \alpha_{n0}(\delta) \ .$$

One expects this value to be independent of $\delta$, a point which can in principle be verified by partial integration. The result is valid for all $\delta > 0$, and it happens that $\alpha_{00}(\delta)$, $\alpha_{10}(\delta) \to 0$ as $\delta \to 0^+$, so using (7.4) we have that

$$\lim_{|h|\to 0} \phi_2 = \frac{8z}{\pi a}\int_0^b d\xi.\Omega(-\xi)\int_0^\infty \frac{x^2.dx}{\{(x^2+\xi)^2 + 4\xi z\}^{3/2}} \ .$$

Thus the potential is continuous across $z = \frac{a^2y^2}{4}$. Now consider $\frac{\partial\varphi}{\partial y}$ as $|h| \to 0$.

From (7.1), (7.2), (7.3) and (7.5), (7.6), (7.7), differentiating we have

$$\lim_{|h|\to 0} \frac{\partial \phi_2}{\partial y} = -\frac{8z}{\pi a} \cdot 2\,\frac{a^2 y}{4} \int_\delta^b d\xi . \Omega(-\xi) . \int_0^\infty \frac{3\,(x^2+\xi)\,x^2 . dx}{\{(x^2+\xi)^2 + 4\xi z\}^{5/2}}$$

$$+ 2\,\frac{a^2 y}{4} \sum_{n=0}^{\infty} (-1)^n\, \Omega^{(n)}(0) . \alpha_{n1}(\delta) .$$

Thus $\frac{\partial \phi}{\partial y}$ is continuous across $z = \frac{a^2 y^2}{4}$. However $\alpha_{01}(\delta)$ is no longer bounded as $\delta \to 0$ ; the limit is independent of $\delta$ and may be written

$$\lim_{|h|\to 0} \frac{\partial \phi_2}{\partial y} = -\frac{8z}{\pi a}\; 2\,\frac{a^2 y}{4}\; \mathcal{Y} \int_0^b d\xi . \Omega(-\xi) . \int_0^\infty \frac{3\,(x^2+\xi)\,x^2\, dx}{\{(x^2+\xi)^2 + 4\xi z\}^{5/2}} ,$$

where $\mathcal{Y}$ denotes the finite part of the integral.

The continuity of $\frac{\partial \phi_2}{\partial z}$ across $z = \frac{a^2 y^2}{4}$ may be established from the continuity of $\frac{\partial \phi_2}{\partial y}$ by the following argument. $\phi_2$ satisfies the differential equation (2.4) away from the line $z = \frac{a^2 y^2}{4}$. The characteristics of (2.4) are real in the region $z > 0$, and we may draw the neighbouring characteristics to $z = \frac{a^2 y^2}{4}$ for both $h \gtrless 0$. Along the neighbouring characteristics, considering $y > 0$, the following compatability condition applies:

$$u - \ell = \text{constant}$$

where $\ell = \int_{\substack{\text{along} \\ \text{characteristic}}} \frac{dv}{a\sqrt{z}}$ , $u = \frac{\partial \phi_2}{\partial z}$ , $v = \frac{\partial \phi_2}{\partial y}$ .

For finite $\tau$ , it is possible to obtain the constants of integration from the values on the airfoil, and as the airfoil is smooth the difference in constants for neighbouring characteristics on either side of $z = \frac{a^2 y^2}{4}$ may be made arbitrarily small. Thus the continuity of $v$ across the curve $z = \frac{a^2 y^2}{4}$ implies the continuity of $u$ .

# CHAPTER III

## AN APPROXIMATE SOLUTION FOR TRANSONIC FLOW PAST THIN SYMMETRICAL DELTA WINGS.

In this chapter, we extend the method of approximation used in Chapter II, to treat a three-dimensional problem, the flow past a planar wing.

## §1. Introduction

The prediction of the pressure distribution on an airfoil in transonic flow is of obvious engineering interest. With supersonic aircraft it is no longer desirable to simply design to reduce transonic effects to a minimum. The design must be essentially for supersonic flow, with only sufficient attention to transonic effects to avoid drastic pressure changes; for this one needs to calculate the pressure on a given airfoil in the transonic regime.

The ultimate aim must be an unsteady theory, taking account of non-linear effects. However, even the steady, symmetrical flow problem has proved difficult, and most of the work has been confined to either two-dimensional or axisymmetric flow. Hodograph methods [1] have provided some accurate solutions in the two-dimensional case. The computational method of Dorodnicyn , employed by Chushkin [3] for two-dimensional flow, also takes account of the whole flow field, but for a general, three-dimensional profile, the demands on the machine appear prohibitive. The methods more suited to practical application involve a greater degree of approximation, and only aim to produce reasonable results for the flow field in the immediate neighbourhood of the airfoil. A review of various approximate methods for symmetrical flow was given in Chapter I, and some discussion of their relative merits was under taken in Chapter II, § 1. Reference [4] gives further detail of these methods, and also considers lifting airfoils and unsteady flow. It is now proposed to employ the second type of regional linearisation approximation, [5], [6] in a three-dimensional situation.

One standard form of the transonic, small disturbance equation is

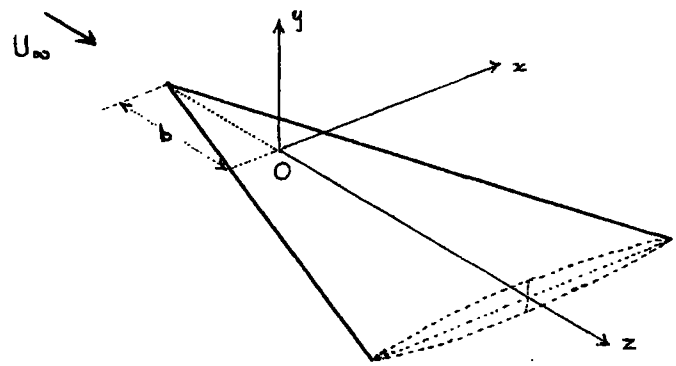
$$(1 - M_\infty^2)\,\Phi_{XX} + \Phi_{YY} + \Phi_{ZZ} = M_\infty^2\,(\gamma + 1)\,\Phi_X\,\Phi_{XX} \qquad \underline{1.1}$$

where $X, Y, Z$ are rectangular co-ordinates, $X$ is in the direction of the mainstream, $\Phi(X,Y,Z)$ gives the small disturbance term in the flow potential $U_\infty(X + \Phi)$ and $U_\infty$, $M_\infty$ are respectively the velocity and Mach number of the undisturbed flow upstream of the airfoil. The shocks which occur are weak and only appear as surfaces of discontinuity of velocity separating regions of potential flow. The simplified transonic shock relation and this form of the small disturbance equation are available in the review, Spreiter [7] .

The present approach considers $M_\infty = 1$ and assumes accelerated flow over the airfoil. At $M_\infty = 1$ shocks are expected to be confined to the rear of the airfoil and their effect is disregarded. The approximation, for points in the neighbourhood of the airfoil, is to replace the $\Phi_X$ in the non-linear term of (1.1) by a linear function of $X$ . This then is the model situation investigated; for convenience a different choice of co-ordinates is adopted with the origin on the centre chord of the airfoil , and $z$ in the direction of the mainstream.

## §2. The model boundary value problem.

The velocity field for flow past the airfoil is derived from a potential $U_\infty(z + \varphi(x,y,z))$ . The delta wing lies in the plane $Oxz$ , $Oz$ lies along the centre chord, and the origin is a distance $b$ along the centre chord. The wing is symmetrical




about the plane $y = 0$ ; suppose its upper surface is given by

$$y - \tau f(x,z) = 0$$

where $\tau$ is the thickness ratio, and the leading edge by $x = \pm\sigma(b+z)$ .

The equation modelling the transonic flow is taken as

$$\varphi_{xx} + \varphi_{yy} = a^2 z\, \varphi_{zz} \qquad \underline{2.1}$$

For boundary conditions on $\varphi$ , we use the condition of no velocity normal to the airfoil surface, and a tentative condition that the perturbation velocities derived from $\varphi$ , become small at large distances for $z < 0$ . Thus

$$\left.\begin{array}{ll} l\varphi_x + m\varphi_y + n(1+\varphi_z) = 0 & \text{on the airfoil,} \\ \varphi_x \;, \varphi_y \;, \varphi_z \to 0 \quad \text{at } \infty & \text{for } z < 0 . \end{array}\right\} \qquad \underline{2.2a}$$

where $(\ell, m, n)$ are the direction cosines of the normal to the airfoil surface.

Equations (2.1), (2.2a) are suitable for slender bodies as well as thin wings; the restriction to thin wings enters as we take

$$(\ell, m, n) = \left(-\tau \frac{\partial f}{\partial x}, 1, -\tau\frac{\partial f}{\partial z}\right)$$

and then consider $\tau$ small. Regarding the derivatives of $\varphi$ as small, and retaining only the largest terms, conditions (2.2a) become

$$\left.\begin{aligned} \varphi_y &= \tau \frac{\partial f}{\partial z} \quad \text{on} \quad y = \tau f(x,z), \\ \varphi_x, \varphi_y, \varphi_z &\to 0 \quad \text{at } \infty \quad \text{for} \quad z<0. \end{aligned}\right\} \qquad \underline{2.2\,b}$$

Note that $\frac{\partial f}{\partial x} \propto \frac{1}{\sigma}$ so that we require $\frac{\tau}{\sigma} \ll 1$ .

Point source solutions of (2.1) situated on the $y=0$ plane are

$$\varphi(H) = \frac{-\left(z + \zeta - \frac{a^2r^2}{4}\right)}{2\zeta\left\{\left[z-\zeta-\frac{a^2r^2}{4}\right]^2 - a^2r^2\zeta\right\}^{1/2}} \qquad (\zeta > 0),$$

$$\varphi(E) = -\frac{1}{4\zeta} - \frac{\left(z + \zeta - \frac{a^2r^2}{4}\right)}{4\zeta\left\{\left[z-\zeta-\frac{a^2r^2}{4}\right]^2 - a^2r^2\zeta\right\}^{1/2}} \qquad (\zeta<0) \qquad \underline{2.3}$$

where $r^2 = (x-\eta)^2 + y^2$ , and the source is situated at $z = \zeta$ , $x = \eta$ .

Adopting a similar approach to Ch. II , as $\tau \ll 1$ , we seek an approximation to the solution of (2.1), (2.2) in the form :

for $z < \frac{a^2y^2}{4}$,

$$\varphi = \frac{1}{2\pi}\int_{\mathcal{D}_1} \bar{X}(\zeta,\eta)\,\varphi(E)\,d\zeta.d\eta + \frac{1}{2\pi}\int_{-\sigma b}^{\sigma b} \bar{X}(0,\eta)\,\tfrac{1}{2}\log\left|z - \tfrac{a^2r^2}{4}\right|.d\eta ,$$

for $z > \frac{a^2y^2}{4}$

$$\varphi = \frac{1}{2\pi}\int_{\mathcal{D}_1} \bar{X}(\zeta,\eta).\,\varphi(E).d\zeta.d\eta + \frac{1}{2\pi}\int_{\mathcal{D}_2} \bar{X}(\zeta,\eta)\,\varphi(H)\,d\zeta\,d\eta$$
$$+ \frac{1}{2\pi}\int_{-\sigma b}^{\sigma b} \bar{X}(0,\eta)\,\tfrac{1}{2}\log\left|z - \tfrac{a^2r^2}{4}\right|.d\eta \qquad \underline{2.4}$$

where

$\sigma b$ denotes the semi-span of the airfoil at $z = 0$,

$\mathcal{D}_1$ denotes that region of the planform for which $\zeta < 0$

and $\mathcal{D}_2$ denotes the intersection of the cone of dependence of the field point $(x,y,z)$ with that region of the planform for which $\zeta > 0$.

(We take planform to mean the projection of the airfoil surface on the plane $y = 0$ ). The log terms are included to make the potential continuous across the line $z = \frac{a^2y^2}{4}$, as in Ch. II.

The cone of dependence of a point $(x,y,z)$ may be deduced from the vanishing of the denominator of $\varphi(H)$, which indicates the cone of influence of a source. A source at $(x_0, y_0, z_0)$ influences downstream points $(x,y,z)$ satisfying

$$\sqrt{z} - \sqrt{z_0} = \frac{a^2}{4}\left\{(x-x_0)^2 + (y-y_0)^2\right\} \qquad \underline{2.5}$$

and conversely all points $(x_0,y_0,z_0)$ upstream of $(x,y,z)$ and satisfying (2.5), influence $(x,y,z)$. It may readily be verified that (2.5) defines a characteristic surface of equation (2.1). We only use sources for which $y_0 = 0$. The cone of dependence of $(x, y, z)$ intersects the plane $y = 0$ in points $(\eta, 0, \zeta)$ satisfying

$$\left\{ \frac{4}{a^2}(\sqrt{z} - \sqrt{\zeta})^2 - y^2 \right\}^{1/2} = \eta - x \qquad \text{right hand branch}$$

$$\left\{ \frac{4}{a^2}(\sqrt{z} - \sqrt{\zeta})^2 - y^2 \right\}^{1/2} = x - \eta \qquad \text{left hand branch}$$

and the region of dependence $\mathcal{D}$ is defined by

$$\left\{ \frac{4}{a^2}(\sqrt{z} - \sqrt{\zeta})^2 - y^2 \right\}^{1/2} \geqslant | x - \eta | . \qquad \underline{2.6}$$

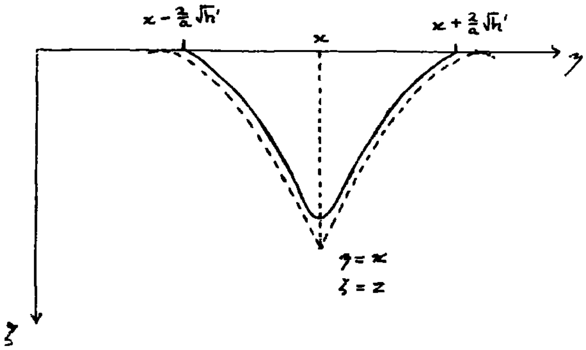



Figure 1

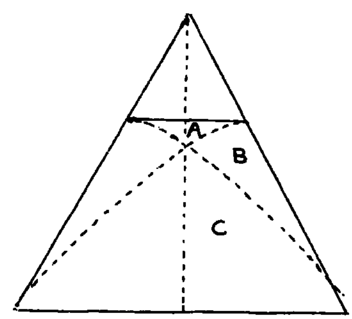



Figure 2

The situation for small $y$ is illustrated in Fig. 1; $h' = z - \frac{a^2y^2}{4}$

The domain of integration in (2.4), $\mathcal{D}_2$, varies according to the intersection of $\mathcal{D}$ and the planform. For a delta wing, the field points with $z, x > 0$ give rise to 3 cases (A) $x + \frac{2}{a}\sqrt{h'} < \sigma b$ (B) $\frac{2}{a}\sqrt{h'} - x < \sigma b < x + \frac{2}{a}\sqrt{h'}$ (C) $\frac{2}{a}\sqrt{h'} - x > \sigma b$

(see Fig. 2, drawn for simplicity for $y = 0$ ).

The quantity of engineering interest is the non-dimensional pressure distribution $C_p \simeq -2 \left.\frac{\partial \varphi}{\partial z}\right|_{airfoil}$. This is derived from the Bernoulli equation, on the assumption that the derivatives of $\varphi$ are small, so that quadratic terms may be neglected. This assumption is involved in the form of the boundary conditions (2.2b). In linearised aerodynamics, it is the practice to replace quantities which should be evaluated at the airfoil surface, by their values at $y=0$. In the present problem, as $\alpha^2$ depends on $\tau$, $\tau$ cannot be simply scaled out of the boundary value problem for $\varphi$, and it is not obvious that the classical approximation remains valid. Accordingly we evaluate quantities required on the airfoil, at $y=\tau$, and so retain an indication of the errors involved, when using values at $y=0$.

Although in principle, once $\bar{\mathcal{X}}(\zeta,\eta)$ is found in terms of $\mathcal{f}(x,z)$ we may obtain the pressure distribution from (2.4), in practice, the singularities in the integrands make it desirable to simplify the expressions by analytical methods. Accordingly we are interested in the asymptotic behaviour as $\tau \to 0$.

It is convenient to introduce the following notation for the various limit processes involved :

$$\mathcal{L}_\zeta,\ \mathcal{L}_\eta \quad : \quad \zeta \text{ and } \eta \quad \text{integrations}, \mathcal{L}_z \ \text{z-differentiation}$$

$$\mathcal{L}_\tau : \lim \tau \to 0\ ,\quad \mathcal{L}_\sigma : \lim \sigma \to \infty \quad \text{and} \quad \mathcal{L}_\tau \mathcal{L}_\sigma \varphi = \mathcal{L}_\tau(\mathcal{L}_\sigma \varphi) \ \text{etc.} \qquad \underline{2.7}$$

The order is important, since some processes do not commute. In this notation Ch. II was concerned with $\mathcal{L}_\tau \mathcal{L}_z \mathcal{L}_\zeta \mathcal{L}_\sigma \mathcal{L}_\eta$ in evaluating the surface pressure. In practice $\mathcal{L}_z \mathcal{L}_\zeta \mathcal{L}_\tau \mathcal{L}_\sigma \mathcal{L}_\eta$, and similar orders were used, and this merely necessitated the use of principal values, the penalty for the interchange of non-commutative processes. We do not escape so lightly in the present case.

To proceed with the evaluation of the asymptotic behaviour of $\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau}$, we must make further assumptions about $\bar{X}$. The two simplest alternatives are

$$\bar{X}(\zeta,\eta) = \tau X(\zeta,\eta) = \tau \sum_{n=0}^{\infty} X_{n\zeta}\,\eta^n \qquad \underline{2.8}$$

$$\bar{X}(\zeta,\eta) = \tau X(\zeta,\eta) = \tau \sum_{m=0}^{\infty} X_{\eta m}\,\zeta^m \qquad \underline{2.9}$$

We have introduced $\tau$ anticipating the relation between $X(\zeta,\eta)$ and the local airfoil slope; $X_{n\zeta}$ is independent of $\tau, \eta$, and similarly $X_{\eta m}$ is independent of $\tau, \zeta$. We also write $X(\zeta,\eta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} X_{nm}\eta^n\zeta^m$ where $X_{nm}$ are constants.

If the choice of the form for $X$ is made on physical grounds, (2.8) must be used. The reason for this stems from the wish to truncate the series after a few terms. Truncating the $\eta$ series implies having only simple spanwise thickness distributions available as approximations. This seems preferable to having simple chordwise approximations, since apart from the wing tip, the practical chordwise thickness distributions are more complicated than the spanwise ones.

Mathematically, the choice effectively determines the order of limiting processes, and the physically preferable choice leads to the more involved analysis, since $\mathcal{L}_\zeta\,\varphi(E)$ may be evaluated in terms of elementary functions, whereas $\mathcal{L}_\eta\,\varphi(E)$ requires elliptic integrals. Accordingly, we first examine the case $\mathcal{X}(\zeta,\eta)=$ constant, aiming to replace $\mathcal{L}_\tau \mathcal{L}_z \mathcal{L}_\eta \mathcal{L}_\zeta$ by $\mathcal{L}_\eta \mathcal{L}_\tau \mathcal{L}_\zeta \mathcal{L}_z$ wherever possible. Then, with this as guide, we treat the more general shapes given by (2.8), by aiming for the order $\mathcal{L}_\zeta \mathcal{L}_\tau \mathcal{L}_\eta \mathcal{L}_z$ .

## §3. Verification of the formal solution.

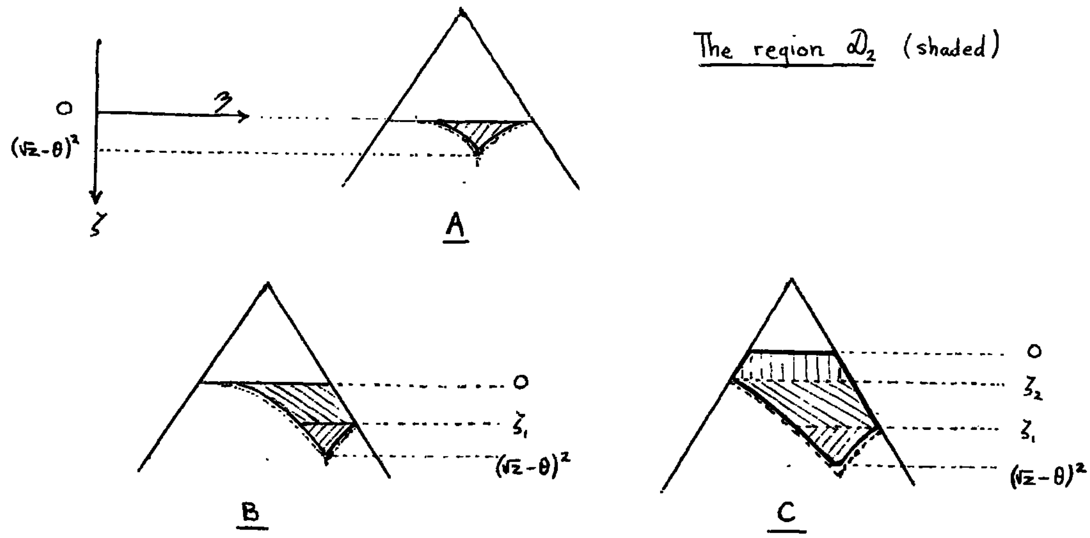
Before getting involved in the main task of developing asymptotic forms of $\frac{\partial \varphi}{\partial z}$, a number of points must be cleared up. We need to establish that the integrals in (2.4) converge, and that the potential so defined gives a continuous velocity field. We also need to check that this potential satisfies the boundary conditions, and to obtain the relation between $\mathcal{X}$ and $\frac{\partial f}{\partial z}$. The choice of values for $a^2$ and $b$ is deferred until the asymptotic forms of the pressure distribution are available. However, it is necessary to have some estimate of the relation between $a$ and $\tau$. We assume $a \gg \tau$, but that $a \to 0$ as $\tau \to 0$; the two-dimensional relation is $a \sim \tau^{1/3}$.

First then we examine the convergence of the integrals in (2.4). For $z < \frac{a^2 y^2}{4}$

$$\int_{\mathcal{D}_1} \bar{\mathcal{X}}(\zeta,\eta)\, \varphi(E)\, d\zeta . d\eta = \int_{-b}^{0} d\zeta \int_{-\sigma(b+\zeta)}^{\sigma(b+\zeta)} \bar{\mathcal{X}}(\zeta,\eta)\, \varphi(E) . d\eta \qquad \underline{3.1}$$

As $\bar{\mathcal{X}}(\zeta,\eta)$ is continuous in the closed region $\mathcal{D}_1$, it is sufficient to consider the case $\bar{\mathcal{X}}(\zeta,\eta) = 1$. One may express $\int_\eta \varphi(E)$ in terms of standard elliptic integrals, and thus show that this provides an integrable function for $\int_\zeta$.

For $z > \frac{a^2 y^2}{4}$, the integral over $\mathcal{D}_1$ in (2.4) has the same form as (3.1). The only possible singularity of $\varphi(E)$ in this case arises from near $\zeta = 0$. By subtracting a term $\frac{1}{\zeta}\left[\zeta\, \mathcal{X}(\zeta,\eta)\, \varphi(E)\right]_{\zeta \to 0}$ from the integrand, the resulting double integral is made convergent. However, we must add the integral of this term, and this is divergent unless we modify $\mathcal{D}_1$ . Now consider the integral over $\mathcal{D}_2$ . As mentioned in § 2, there are three cases to consider.




The various limits of integration may be deduced from (2.6) and the equation of the edges of the delta $\eta = \pm\sigma(b+\zeta)$ . The figures are drawn for a field point $(x, \tau, z)$ ;$\tau$ is small and $\theta = \frac{a\tau}{2}$ ; $\zeta_1$ , $\zeta_2$ are given by

$$\sqrt{\zeta_1} = \sqrt{z} - \{\theta^2 + \tfrac{a^2\sigma^2}{4}(b+\zeta_1-x')^2\}^{1/2} \quad \text{with } x' = \tfrac{x}{\sigma}$$
$$\sqrt{\zeta_2} = \sqrt{z} - \{\theta^2 + \tfrac{a^2\sigma^2}{4}(b+\zeta_2+x')^2\}^{1/2} \qquad \underline{3.2}$$

The only possible singularities of $\mathcal{L}_\eta\, \chi(\zeta,\eta)\,\varphi(H)$ are at $\zeta = 0$ , and on the characteristic surface. For $\chi(\zeta,\eta)$ in the form (2.8), $\mathcal{L}_\eta\, \chi(\zeta,\eta)\,\varphi(H)$ may be evaluated in terms of elliptic integrals. This provides an integrand for $\mathcal{L}_\zeta$ which is integrable, except at $\zeta = 0$ . By using $\mathcal{L}_\zeta \mathcal{L}_\eta \{ \chi(\zeta,\eta)\varphi(H) - \frac{1}{\zeta}\left[\zeta\, \chi(\zeta,\eta)\,\varphi(H)\right]_{\zeta\to 0}\}$ the singularity is removed. The term we must add to compensate gives a divergent integral. However, by modifying the regions of integration $\mathcal{D}_1$, $\mathcal{D}_2$ so that for $\mathcal{D}_1$, $-b \leq \zeta \leq -\varepsilon$ and for $\mathcal{D}_2$, $\varepsilon \leq \zeta \leq (\sqrt{z}-\theta)^2$, the integrals with $1/\zeta$ behaviour are defined. Furthermore, it transpires that $\mathcal{L}_\eta [\zeta\, \mathcal{X}(\zeta,\eta)\, \varphi(\varepsilon)]_{\zeta\to 0} = \mathcal{L}_\eta [\zeta\, \mathcal{X}(\zeta,\eta)\, \varphi(H)]_{\zeta\to 0}$ for each of the cases $A$, $B$, $C$ so that in (2.4), if we use modified regions $\mathcal{D}_1(\varepsilon)$, $\mathcal{D}_2(\varepsilon)$ the two contributions combine to give a finite value as $\varepsilon \to 0$. This corresponds to the Cauchy principal value employed in the two dimensional case.

We have not yet considered the terms

$$I = \frac{1}{2\pi} \int_{-\sigma b}^{\sigma b} \mathcal{X}(0,\eta)\, \frac{1}{2} \log \left| z - \frac{a^2\tau^2}{4} \right| . d\eta \qquad \underline{3.3}$$

The addition of these terms ensures the continuity of the potential and its first derivatives across $z = \frac{a^2y^2}{4}$ ; the proof is similar to the corresponding two-dimensional analysis, Ch. II, §7. Considering $z - \frac{a^2y^2}{4} \to 0+$, for sufficiently small values of $z - \frac{a^2y^2}{4}$ the region of integration $\mathcal{D}_2$ must be as in case A. Thus the main difference from the two-dimensional case is the finite range of $\eta$-integration required for $\mathcal{D}_1$. The remainder of the proof is routine and is omitted.

Now consider the boundary conditions. To satisfy these

$$\left.\frac{\partial \varphi}{\partial y}\right|_{y=\tau} \sim \tau \frac{\partial f}{\partial z} \qquad \text{as} \quad \tau \to 0 .$$

By analogy with classical aerodynamics we expect

$$\mathcal{X}(z,x) = 2 \frac{\partial f}{\partial z} \qquad \underline{3.4}$$

and this is indeed the case. We outline the derivation; the asymptotic methods are considered in greater detail in subsequent sections. Denote $\frac{\partial}{\partial y}$ by $\mathcal{L}_y$. Then for $\frac{a^2y^2}{4} - z < 0$

$$\frac{1}{\tau}\left[\mathcal{L}_y \varphi\right]_{y=\tau} = \int_{s-\varepsilon}^{s+\varepsilon} d\zeta \left(\mathcal{L}_\gamma \mathcal{L}_y \varphi(E)\right) + o(1) \quad \text{as } \tau \to 0.$$

Now

$$\mathcal{L}_\gamma \mathcal{L}_y \varphi(E) \sim \frac{1}{s^{1/2}} \frac{\theta}{\{(\sqrt{s}-\sqrt{\xi})^2+\theta^2\}} \sum_{i=3,4} \frac{X_i}{\{X_i^2 + \theta^2 + (\sqrt{s}-\sqrt{\xi})^2\}^{1/2}}$$

where

$$s = -z \quad , \quad X_3 = \frac{a\sigma}{2}(b+\zeta-x') \quad , \quad X_4 = \frac{a\sigma}{2}(b+\zeta+x').$$

To perform the $\zeta$ integration, note that although $X_i \to 0$ as $\tau \to 0$ , $X_i \gg \theta$ and for points on the airfoil $X_i > 0$ $(i=3,4)$ , so that the essential contribution comes from an integrand

$$\frac{2}{s^{1/2}} \frac{\theta}{\{(\sqrt{s}-\sqrt{\xi})^2+\theta^2\}}$$

Integrating and letting $\theta \to 0$ it follows

$$\frac{1}{\tau}\left[\frac{\partial\varphi}{\partial y}\right]_{y=\tau} \to \frac{1}{2}\,\chi(-s,x) \qquad \text{as} \quad \tau \to 0 .$$

For $z > \frac{a^2y^2}{4}$ , in the integrals for $\frac{1}{\tau}\left[\mathcal{L}_y \varphi\right]_{y=\tau}$ first take out those parts of the integrands which give rise to the singularities at $\zeta = 0$ for the unmodified regions $\mathcal{D}_1$ , $\mathcal{D}_2$ . This need only be done for a region independent of $y$ in the immediate neighbourhood of $\zeta = 0$ . The remaining integrals are defined over $\mathcal{D}_1$ and $\mathcal{D}_2$ separately. The term which needs to be added to compensate makes $o(1)$ contribution as $\tau \to 0$. The integrand in the $\mathcal{D}_1$ integral has no singularities in the domain of integration, and applying $\mathcal{L}_y$ produces a factor $\theta$, which means there is only an $o(1)$ contribution to $\frac{1}{\tau}\,[\,\mathcal{L}_y\,\varphi\,]_{y=\tau}$ as $\tau \to 0$. The $\mathcal{D}_2$ integral, with modified integrand has no singularities in its integrand apart from that on the characteristic surface. First consider $\mathcal{X}(\zeta,\eta) = \Omega(\zeta)$; $\mathcal{L}_\eta\,\varphi(H)$ may be expressed in terms of elliptic integrals in very similar manner to the two-dimensional case, Ch. II. Dividing the region $\mathcal{D}_2$ by the line $\zeta = c < (\sqrt{z}-\theta)^2$, the $\zeta$ integrals over the part $\zeta < c$ make only $o(1)$ contributions to $\frac{1}{\tau}\left[\frac{\partial\varphi}{\partial y}\right]_{y=\tau}$ as $\tau \to 0$. The main contribution in each of the cases A, B, C comes from a term

$$\frac{1}{2\pi}\,\frac{\partial}{\partial y}\int_c^{(\sqrt{z}-\theta)^2} d\zeta\,.\int_{x-\frac{2}{a}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}}^{x+\frac{2}{a}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}} \Omega(\zeta)\,\varphi(H).d\eta \qquad \underline{3.5}$$

The introduction of $\mathcal{X}(\zeta,\eta)$ for $\Omega(\zeta)$ will not alter the order of the contribution for $\zeta < c$. But $c$ can be made arbitrarily close to $z$, $\left(c<(\sqrt{z}-\theta)^2 \text{ as } \tau\to 0\right)$; this means the value of $\mathcal{X}(\zeta,\eta)$ must be correspondingly close to $\mathcal{X}(\zeta, x)$ for the range $\left(c\,,\,(\sqrt{z}-\theta)^2\right)$. Consequently the value of $\frac{1}{\tau}\left[\frac{\partial\varphi}{\partial y}\right]_{y=\tau}$ as $\tau\to 0$ is given by (3.5), with $\Omega(\zeta) = \mathcal{X}(\zeta, x)$; and this value is known from the two-dimensional case. Thus

$$\frac{1}{\tau}\left[\frac{\partial\varphi}{\partial y}\right]_{y=\tau} \to \frac{1}{2}\,\mathcal{X}(z,x) \quad \text{as} \quad \tau\to 0 \;.$$

This result is now established for both $z>0$ and $z<0$.

Consequently to satisfy the boundary condition on the airfoil $\mathcal{X}(z, x)$ must satisfy (3.4).

There remains the behaviour of the velocities at infinity. As the potential is an integral over a finite range of the axisymmetric case, these still become vanishingly small. The finite term in the two-dimensional case stems from the infinite range of integration. However, as the equation (2.1) is not a reasonable model to the transonic equation at large distances, it is questionable to regard this behaviour as having any significance.

## § 4. Asymptotic behaviour of $\left.\frac{\partial \varphi}{\partial z}\right|_{y=\tau}$ as $\tau \to 0$ for $\mathcal{X} = \mathcal{X}_{00}$

(order of operations $\mathcal{L}_\eta \, \mathcal{L}_\tau \, \mathcal{L}_\xi \, \mathcal{L}_{z}$ )

Having established that the integrals (2.4) provide a solution to the boundary value problem, we now undertake the programme indicated at the close of § 2, and begin with the pilot case $\mathcal{X} = \mathcal{X}_{00}$.

The region $z < \frac{a^2 y^2}{4}$ .

First consider the term added to ensure continuity

$$I_0 = \tau \frac{\mathcal{X}_{00}}{2\pi} \int_{-\sigma b}^{\sigma b} \tfrac{1}{2} \log \left| z - \tfrac{a^2 r^2}{4} \right| . d\eta \quad ,$$

$$\mathcal{L}_z I_0 = \frac{\tau \mathcal{X}_{00}}{4\pi} \int_{-\sigma b}^{\sigma b} \frac{d\eta}{\{ z - \frac{a^2}{4}[(x-\eta)^2 + y^2]\}} = \frac{-\tau \mathcal{X}_{00}}{2\pi a \sqrt{h}} \left[ \tan^{-1} \frac{t}{\sqrt{h}} \right]_{t=-\frac{a\sigma}{2}(b+x')}^{t=\frac{a\sigma}{2}(b-x')} \qquad \underline{4.1}$$

where $s = -z$ , $x = \sigma x'$ and $h = s + \frac{a^2 y^2}{4}$ .

Now consider

$$\mathcal{D} = \frac{\tau}{2\pi} \mathcal{X}_{00} \frac{\partial}{\partial z} \int_0^b d\xi \int_{-\sigma(b-\xi)}^{\sigma(b-\xi)} \varphi(E) . d\eta$$

where $\xi = -\zeta$ . Changing to order of operations $\mathcal{L}_\eta \, \mathcal{L}_\xi \, \mathcal{L}_z$ , and performing the $\xi$ integration $(\mathcal{L}_\xi)$ ,

$$\mathcal{D} = \frac{\tau}{4\pi} \mathcal{X}_{00} \left\{ \frac{2}{a\sqrt{h}} \left[ \tan^{-1} \frac{t}{\sqrt{h}} \right]_{t=-\frac{a\sigma}{2}(b+x')}^{t=\frac{a\sigma}{2}(b-x')} - \mathcal{D}_1 - \mathcal{D}_2 \right\} \qquad \underline{4.2}$$

where

$$\mathcal{D}_1 = \int_0^{\sigma b} \left\{ \left[ b + s + \theta^2 - \eta/\sigma + \tfrac{a^2}{4}(\eta - x)^2 \right]^2 - 4s\,(b - \eta/\sigma) \right\}^{-1/2} . d\eta \quad ,$$

$$\mathcal{D}_2 = \int_0^{\sigma b} \left\{ \left[ b + s + \theta^2 - \eta/\sigma + \tfrac{a^2}{4}(\eta + x)^2 \right]^2 - 4s\,(b - \eta/\sigma) \right\}^{-1/2} d\eta \ .$$

A straightforward attempt to expand the integrands to determine $\mathcal{L}_\tau$ fails; there is a non-uniformity arising from the neighbourhood of $\eta = \sigma(b-s)$. Consider $\mathcal{B}_1$, and write $\eta = \sigma\eta'$, $\bar{a} = \frac{a\sigma}{2}$, $\tau' = \tau/\sigma$ :

$$\mathcal{B}_1 = \sigma \int_0^b \{ [b+s-\eta' + \bar{a}^2((\eta'-x')^2+\tau'^2)]^2 - 4s(b-\eta')\}^{-1/2} d\eta' .$$

Now split the range of integration at $b-s \pm \bar{a}^p$, $0<p<1$. The contribution from the outer part may be obtained by direct expansion of the integrand:

$$\left( \int_{b-s+\bar{a}^p}^{b} + \int_0^{b-s-\bar{a}^p} \right) |b-\eta'-s|^{-1} \left\{ 1 + \frac{2\bar{a}^2((\eta'-x')^2+\tau'^2)+\bar{a}^4((\eta'-x')^2+\tau'^2)}{(b-s-\eta')^2} \right\}^{-1/2} d\eta'$$

$$= \log\left(\frac{s(b-s)}{\bar{a}^{2p}}\right)\{ 1 + O(\bar{a}^{2(1-p)}).[1+O(\tau'^2)]\} .$$

For the contribution from the inner part, put $b-s-\eta' = \bar{a}\tilde{\eta}$, $b-s-x' = A_1$. Thus the contribution from the range $(b-s-\bar{a}^p, b-s+\bar{a}^p)$ becomes

$$\sigma \int_{-\bar{a}^{p-1}}^{\bar{a}^{p-1}} \{ \tilde{\eta}^2 + 4s(A_1^2+\tau'^2) + 2\bar{a}\tilde{\eta}(A_1^2+\tau'^2 - 2\bar{a}\tilde{\eta}A_1 + \bar{a}^2\tilde{\eta}^2)$$
$$+ 4s(-2\bar{a}\tilde{\eta} + \bar{a}^2\tilde{\eta}^2) + \bar{a}^2(\tau'^2 + A_1^2 - 2\bar{a}\tilde{\eta}A_1 + \bar{a}^2\tilde{\eta}^2)^2 \}^{-1/2} d\tilde{\eta} .$$

Now using the assumption $a \gg \tau$, and expanding the integrand, we have

$$\sigma \int_{-\bar{a}^{p-1}}^{\bar{a}^{p-1}} \{ \tilde{\eta}^2 + 4sA_1^2 \}^{-1/2} \{ 1 + O(\bar{a}) \} . d\tilde{\eta}$$

$$= \log\left(\frac{\bar{a}^{2p}}{A_1^2 s}\right) \{ 1 + O(\bar{a}^{2(1-p)}) + O(\bar{a}) \} .$$

One expects the terms in $\bar{a}^{(1-p)}$ to cancel, since $p$ is arbitrary. Certainly, taking $p = \frac{1}{2}$, on adding the contributions we have

$$\mathcal{B}_1 = \sigma \log\left(\frac{b-s}{\bar{a}^2 A_1^2}\right) + O(\bar{a}\log\bar{a}) .$$

We introduce the following notation

$$A_1(\zeta) = b+\zeta-x' \quad , \qquad A_2(\zeta) = b+\zeta+x' \qquad \underline{4.3}$$

and omit the argument whenever its value can be deduced from the context. Thus in the expression for $\mathcal{B}_1$, $A_1$ denotes $A_1(-s)$.

The $\mathcal{L}_\tau$ behaviour of $\mathcal{B}_2$ is obtained in a similar manner to $\mathcal{B}_1$,

$$\mathcal{B}_2 = \sigma \log\left(\frac{b-s}{\bar{a}^2 A_2^2}\right) + O(\bar{a}\log\bar{a}) .$$

Now $\mathcal{L}_\tau \left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau} = \mathcal{L}_\tau \mathcal{L}_z I_o + \mathcal{L}_\tau \mathcal{B}$, so that from (4.1) and (4.2)

$$\frac{2\pi a}{\tau} \left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau} = \mathcal{X}_{oo}\left\{ \bar{a}\, 4\log\bar{a} - 2\bar{a}\log\left(\frac{b-s}{(b-s)^2-x'^2}\right) + O(\bar{a}^2\log\bar{a})\right\} . \qquad \underline{4.4}$$

<u>The region $z > \frac{a^2y^2}{4}$</u> Again consider

$$\mathcal{L}_z I_o = \frac{\tau}{4\pi}\mathcal{X}_{oo}\int_{-\sigma b}^{\sigma b} \frac{d\eta}{\{z - \frac{a^2}{4}[(x-\eta)^2+y^2]\}}$$

$$= \frac{\tau\,\mathcal{X}_{oo}}{4\pi a\sqrt{h'}} \log\left|\frac{[\sqrt{h'}+\frac{a\sigma}{2}(b-x')][\sqrt{h'}+\frac{a\sigma}{2}(b+x')]}{[\sqrt{h'}-\frac{a\sigma}{2}(b+x')][\sqrt{h'}-\frac{a\sigma}{2}(b-x')]}\right| \qquad \underline{4.5}$$

where $h' = z - \frac{a^2y^2}{4}$ .

In case C, $\sqrt{h'} > \frac{a\sigma}{2}|b \pm x'|$, the integrand is non-singular, and the interchange of $\mathcal{L}_z$ , $\mathcal{L}_\eta$ is justified. However in other cases, another limit process is required to define the $I_0$ integral, and the result is only formal. The result may be verified by actually evaluating $I_0$ before applying $\mathcal{L}_z$ .

We now treat $\mathcal{B}(\varepsilon) = \frac{\tau}{2\pi} \mathcal{X}_{oo} \frac{\partial}{\partial z} \int_\varepsilon^b d\xi \int_{-\sigma(b-\xi)}^{\sigma(b-\xi)} \varphi(E) . d\eta$ .

There is involved here an additional limit process, $\mathcal{L}_\varepsilon$ because in (2.4) we must take the contributions from $\varphi(E)$ and $\varphi(H)$ together to form a principal value. Proceeding formally, we change the order of operations to $\mathcal{L}_\varepsilon \mathcal{L}_\eta \mathcal{L}_\xi \mathcal{L}_z$ and carry out $\mathcal{L}_\xi \mathcal{L}_z$ :

$$\mathcal{B}(\varepsilon) = \frac{\tau \mathcal{X}_{oo}}{4\pi} \left\{ \int_0^{\sigma b} \left[ - \left\{ \left( \xi - h' + \tfrac{a^2}{4}(\eta - x)^2 \right)^2 + 4\xi z \right\}^{-1/2} \right]_\varepsilon^{b-\eta/\sigma} . d\eta \right.$$

$$\left. + \int_{-\sigma b}^{0} \left[ - \left\{ \left( \xi - h' + \tfrac{a^2}{4}(\eta - x)^2 \right)^2 + 4\xi z \right\}^{-1/2} \right]_\varepsilon^{b+\eta/\sigma} . d\eta \right. .$$

The contribution for the upper limits of the $\xi$ integration (for $y = \tau$ )

$$= \frac{\tau \mathcal{X}_{oo}}{4\pi} \left\{ -\sigma \int_0^b (b - \eta' + z)^{-1} d\eta' - \sigma \int_{-b}^0 (b + \eta' + z)\, d\eta' + O(\bar{a}^2) \right\}$$

$$= \frac{\tau \mathcal{X}_{oo}}{2\pi} \sigma \left\{ \log\left( \frac{z}{b+z} \right) + O(\bar{a}^2) \right\}$$

where we have expanded the integrand, and used the assumption $a \gg \tau$ .

The contribution from the lower limits of the $\xi$-integration ,

$$\frac{\tau \mathcal{X}_{oo}}{4\pi} \int_{-\sigma b}^{\sigma b} \left\{ (t^2 - h' + \varepsilon)^2 + 4z\varepsilon \right\}^{-1/2} . d\eta \qquad \text{with } t = \tfrac{a}{2}(\eta - x) ,$$

is different for the cases A , B , C depending on whether the integrand with $\varepsilon = 0$ has singularities within the range of integration. In the absence of such singularities one may expand the integrand in powers of $\varepsilon$ ; when present they indicate sources of non-uniform behaviour. The cases $A, B$ have the points $t = \pm\sqrt{h'}$ within the range of integration. A typical contribution involving a source of non-uniformity is

$$I = \int_{\sqrt{h'}}^{\frac{a\sigma}{2}(b-x')} \{ (t^2 - h' + \varepsilon)^2 + 4 z \varepsilon \}^{-1/2} . dt$$

To obtain the behaviour as $\varepsilon \to 0$, put $u = t - \sqrt{h'}$, and split the range at $u = \varepsilon^{1/2-\delta}$. For the inner part of the range, i.e. near $u = 0$, putting $u = \varepsilon^{1/2} U$, and then expanding the integrand, one obtains the contribution

$$\frac{1}{2\sqrt{h'}} \log\left(2\sqrt{\frac{h'}{z}}\,\frac{1}{\varepsilon^{\delta}}\right) + O(\varepsilon^{2\delta}) + O(\varepsilon^{1/2-\delta}\log\varepsilon) \quad , \quad 0 < \delta < \tfrac{1}{2} .$$

For the outer part of the range, expanding the integrand directly, one obtains a contribution

$$\frac{1}{2\sqrt{h'}} \log\left(\frac{\frac{a\sigma}{2}(b-x') - \sqrt{h'}}{\frac{a\sigma}{2}(b-x') + \sqrt{h'}}\right) - \frac{1}{2\sqrt{h'}} \log\left(\frac{\varepsilon^{1/2-\delta}}{2\sqrt{h'}}\right) + O(\varepsilon^{\frac{1}{2}-\delta}) + O(\varepsilon^{2\delta}) .$$

Thus

$$I = \frac{1}{2\sqrt{h'}} \log\left(\frac{4h'}{\sqrt{z}\,\varepsilon^{1/2}}\right) + \frac{1}{2\sqrt{h'}} \log\left(\frac{\frac{a\sigma}{2}(b-x') - \sqrt{h'}}{\frac{a\sigma}{2}(b-x') + \sqrt{h'}}\right) + o(1) \quad \text{as } \varepsilon \to 0.$$

Using this approach, it may be shown:

$$\int_{-\sigma b}^{\sigma b} \left\{ (t^2 - h' + \varepsilon)^2 + 4z\varepsilon \right\}^{-1/2} . d\eta$$

$$= \frac{2}{a} \left\{ \frac{2}{\sqrt{h'}} \log \left( \frac{4h'}{\sqrt{z}} \frac{1}{\varepsilon^{1/2}} \right) + \frac{1}{2\sqrt{h'}} \log \left( \frac{(\sigma b - \frac{2}{a}\sqrt{h'})^2 - x^2}{(\sigma b + \frac{2}{a}\sqrt{h'})^2 - x^2} \right) \right\}$$

$$+ \ o(1) \quad , \text{as} \quad \varepsilon \to 0 .$$

in case $A$, which has both $t = \pm\sqrt{h'}$ in the range,

$$= \frac{2}{a} \left\{ \frac{1}{\sqrt{h'}} \log \left( \frac{4h'}{\sqrt{z}} \frac{1}{\varepsilon^{1/2}} \right) + \frac{1}{2\sqrt{h'}} \log \left( \frac{\sigma^2 b^2 - (\frac{2}{a}\sqrt{h'} - x)^2}{(\frac{2}{a}\sqrt{h'} + x)^2 - \sigma^2 b^2} \right) \right\}$$

$$+ \ o(1) \quad \text{as} \quad \varepsilon \to 0$$

in case $B$, which has $t = -\sqrt{h'}$ in the range, and

$$= \frac{2}{a} \left\{ \qquad + \frac{1}{2\sqrt{h'}} \log \left( \frac{(\frac{2}{a}\sqrt{h'} + \sigma b)^2 - x^2}{(\frac{2}{a}\sqrt{h'} - \sigma b)^2 - x^2} \right) \right\} + o(1) \quad \text{as } \varepsilon \to 0$$

in case $C$, which has no source of non-uniformity in the range.

Collecting terms, for order of operations $\mathcal{L}_\tau \mathcal{L}_\varepsilon \mathcal{L}_\eta \mathcal{L}_\zeta \mathcal{L}_z$, we have

in case A

$$\mathcal{B}(\varepsilon) = \frac{\tau X_{oo}}{2\pi a} \left\{ \frac{2}{\sqrt{h'}} \log \left( \frac{4h'}{\sqrt{z}} \frac{1}{\varepsilon^{1/2}} \right) + \frac{1}{2\sqrt{h'}} \log \left( \frac{(\sigma b - \frac{2}{a}\sqrt{h'})^2 - x^2}{(\sigma b + \frac{2}{a}\sqrt{h'})^2 - x^2} \right) + o(1)_{\varepsilon \to 0} \right.$$

$$\left. + \frac{a\sigma}{2} \, 2 \log \left( \frac{z}{z+b} \right) + O(\bar{a}^2) \right\} \qquad \underline{4.6A} ,$$

in case B

$$\mathcal{B}(\varepsilon) = \frac{\tau \mathcal{X}_{oo}}{2\pi a}\left\{ \frac{1}{\sqrt{h'}} \log\left(\frac{4h'}{\sqrt{z}}\frac{1}{\varepsilon^{1/2}}\right) + \frac{1}{2\sqrt{h'}}\log\left(\frac{\sigma^2 b^2 - (\frac{2}{a}\sqrt{h'}-x)^2}{(\frac{2}{a}\sqrt{h'}+x)^2-\sigma^2 b^2}\right) + o(1)_{\varepsilon\to 0} \right.$$

$$\left. + \frac{a\sigma}{2}\cdot 2\log\left(\frac{z}{z+b}\right) + O(\bar{a}^2)\right\} \qquad \underline{4.6B},$$

in case C

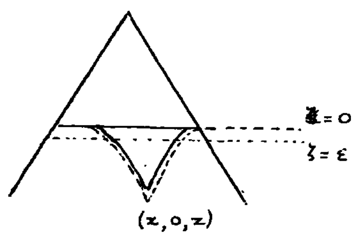
$$\mathcal{B}(\varepsilon) = \frac{\tau \mathcal{X}_{oo}}{2\pi a}\left\{ \frac{1}{2\sqrt{h'}}\log\left(\frac{(\frac{2}{a}\sqrt{h'}+\sigma b)^2 - x^2}{(\frac{2}{a}\sqrt{h'}-\sigma b)^2-x^2}\right) + o(1)_{\varepsilon\to 0} + \bar{a}\cdot 2\log\left(\frac{z}{z+b}\right) \right.$$

$$\left. + O(\bar{a}^2)\right\}. \qquad \underline{4.6C}$$

Now we treat the remaining contribution to $\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau}$ ,

$$\mathcal{H}(\varepsilon) = \frac{\tau\mathcal{X}_{oo}}{2\pi}\frac{\partial}{\partial z}\int_{\mathcal{D}_2}\varphi(H)\,.d\zeta\,.d\eta \quad .$$

Case A




$$\mathcal{H}(\varepsilon) = \frac{\tau\mathcal{X}_{oo}}{2\pi}\frac{\partial}{\partial z}\int_{-\{(\sqrt{z}-\sqrt{\varepsilon})^2-\theta^2\}^{1/2}}^{\{(\sqrt{z}-\sqrt{\varepsilon})^2-\theta^2\}^{1/2}} dt \int_{\varepsilon}^{(\sqrt{z}-\sqrt{\theta^2+t^2})^2} d\zeta\,.\left\{\frac{(-1)(h'+\zeta-t^2)}{2\zeta\{(h'+\zeta-t^2)^2-4\zeta z\}^{1/2}}\right\}$$

where $\theta = \frac{ay}{2}$ is evaluated at $y=\tau$ ,

$t = \frac{a}{2}(\eta - x)$ , and the limits of integration have been obtained using (2.6).

To simplify $\mathcal{H}(\varepsilon)$ we first note the $t$ - integrand is even in $t$ , so that $t$ integration need only be taken over the range $0,\ \{(\sqrt{z}-\sqrt{\varepsilon})^2-\theta^2\}^{1/2}$. We now interchange $\frac{\partial}{\partial z}$, and the $t$ integration; there is no contribution from differentiating the upper limit, since $(\sqrt{z}-\sqrt{\theta^2+t^2})^2=\varepsilon$ at $t=\{(\sqrt{z}-\sqrt{\varepsilon})^2-\theta^2\}^{1/2}$ and the singularity in the $\zeta$ integrand is weak. By performing the $\zeta$ integration, and then differentiating we reduce $\mathcal{H}(\varepsilon)$ to

$$\mathcal{H}(\varepsilon) = -\frac{2\tau X_{00}}{\pi a}\int_0^{\{(\sqrt{z}-\sqrt{\varepsilon})^2-\theta^2\}^{1/2}} \frac{dt}{\{(z-\theta^2-t^2+\varepsilon)^2-4\varepsilon z\}^{1/2}}$$

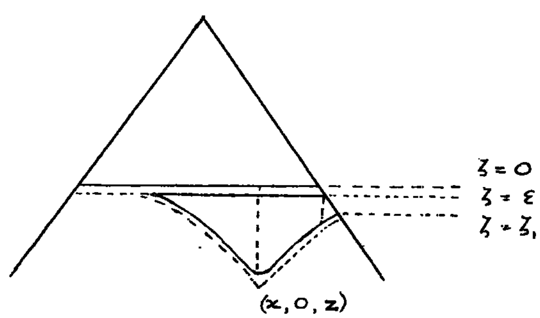
The asymptotic behaviour of the integral as $\varepsilon\to 0$ may be deduced by splitting the range of integration at $\Upsilon-\varepsilon^{\delta}$, where $\Upsilon=\{(\sqrt{z}-\sqrt{\varepsilon})^2-\theta^2\}^{1/2}$. Away from $\Upsilon$, the outer part of the integral, write $T=\Upsilon-t$, and expand the integrand for small $\varepsilon$; near $\Upsilon$, put $\sqrt{\varepsilon}\,\tilde{t}=\Upsilon-t$ and then expand the integrand; $0<\delta<\tfrac{1}{2}$ for the validity of the expansions.

Thus

$$\mathcal{H}(\varepsilon) = -\frac{2\tau X_{00}}{\pi a}\,\frac{1}{2\sqrt{h'}}\,\log\frac{4h'}{\sqrt{z}}\,\frac{1}{\varepsilon^{1/2}} + o(1)_{\varepsilon\to 0} \qquad \underline{4.7A}$$

Case B. First consider $x'<b$.

B $(x'<b)$



$$\mathcal{H}(\varepsilon) = \frac{\tau X_{00}}{a\pi} \frac{\partial}{\partial z} \left\{ \int_{-\{(\sqrt{z}-\sqrt{\varepsilon})^2 - \theta^2\}^{1/2}}^{\frac{a\sigma}{2}A_1(\varepsilon)} dt \int_{\varepsilon}^{(\sqrt{z}-\sqrt{\theta^2+t^2})^2} d\zeta \right.$$

$$\left. + \int_{\frac{a\sigma}{2}A_1(\varepsilon)}^{\frac{a\sigma}{2}A_1(\zeta_1)} dt \int_{(\mathfrak{z}'-b)}^{(\sqrt{z}-\sqrt{\theta^2+t^2})^2} d\zeta \right\} \left( \frac{(-1)(h'+\zeta-t^2)}{2\zeta\{(h'+\zeta-t^2)^2 - 4\zeta z\}^{1/2}} \right),$$

where $\zeta_1$ satisfies

$$\sqrt{\zeta_1} = \sqrt{z} - \left\{ \theta^2 + \frac{a^2\sigma^2}{4} A_1^2(\zeta_1) \right\}^{1/2} \qquad \underline{4.8}$$

and $A_1$ was defined in (4.3).

As in case A, one may interchange $\frac{\partial}{\partial z}$ and the $t$ integration, since at $t = \{(\sqrt{z}-\sqrt{\varepsilon})^2 - \theta^2\}^{1/2}$, $(\sqrt{z}-\sqrt{\theta^2+t^2})^2 = \varepsilon$ and at $t = \frac{a\sigma}{2}A_1(\zeta_1)$, $(\sqrt{z}-\sqrt{\theta^2+t^2})^2 = \zeta_1 = \left(\frac{2t}{a\sigma} + x' - b\right) = \mathfrak{z}' - b$ so that the contributions from differentiating the limits of $t$ integration are zero. Performing the $\zeta$ integration, and then $\mathcal{L}_z$ one obtains.

$\int_{-\{(\sqrt{z}-\sqrt{\varepsilon})^2-\theta^2\}^{1/2}}^{0} dt\,(\;)$ (evaluated in case A), $\int_{0}^{\frac{a\sigma}{2}A_1(\varepsilon)} dt\,(\;)$ which may be expanded in $\varepsilon$ in a straightforward manner, and then integrated, and $\int_{\frac{a\sigma}{2}A_1(\varepsilon)}^{\frac{a\sigma}{2}A_1(\zeta_1)} dt\,\{\;\}$ which again may be expanded in $\varepsilon$ in a straightforward manner.

Thus

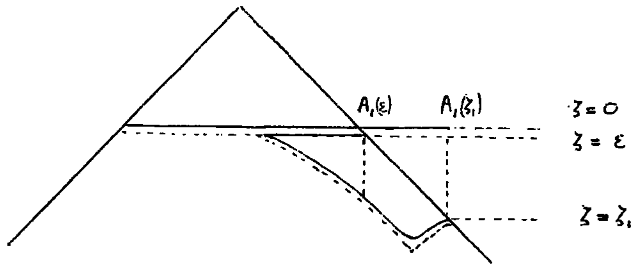
$$\mathcal{H}(\varepsilon) = \frac{\tau X_{00}}{a\pi} \left\{ -\frac{1}{2\sqrt{h'}} \log\left(\frac{4h'}{\sqrt{z}\,\varepsilon^{1/2}}\right) - \frac{1}{2\sqrt{h'}} \log\left(\frac{\frac{a\sigma}{2}(b-x')+\sqrt{h'}}{\sqrt{h'} - \frac{a\sigma}{2}(b-x')}\right) \right.$$

$$- \int_{\mathfrak{z}'-b}^{(\sqrt{z}-\sqrt{\theta^2+t^2})^2} \left\{ \left(z-\theta^2-t^2+\frac{2t}{a\sigma}+x'-b\right)^2 - 4z\left(\frac{2t}{a\sigma}+x'-b\right) \right\}^{1/2}$$

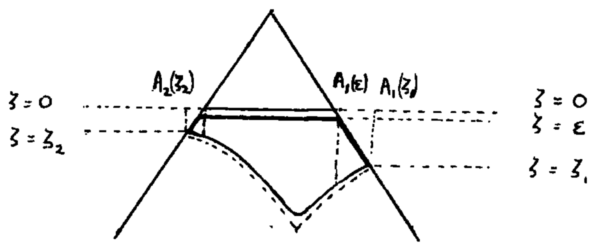
$$\left. + \; o(1)_{\varepsilon\to 0} \right\} \qquad \underline{4.7\,B}$$

Now consider $x' > b$ ; the ranges of integration are the same, though now $A_1(\varepsilon)$ is negative. Thus $\mathcal{H}(\varepsilon)$ is given by (4.7B).

B $(x' > b)$




Case C . Consider the case $0 < x' < b$ ; the other case $x' > b$ gives the same result for $\mathcal{H}(\varepsilon)$ .




$$\mathcal{H}(\varepsilon) = \frac{\tau \mathcal{X}_{00}}{a\pi} \frac{\partial}{\partial z} \left\{ \left( \int_{-\frac{a\sigma}{2}A_2(\zeta_2)}^{-\frac{a\sigma}{2}A_2(\varepsilon)} dt + \int_{\frac{a\sigma}{2}A_1(\varepsilon)}^{\frac{a\sigma}{2}A_1(\zeta_1)} dt \right) \int_{\zeta'-b}^{(\sqrt{z}-\sqrt{\theta^2+t^2})^2} d\zeta \right.$$

$$\left. + \int_{-\frac{a\sigma}{2}A_2(\varepsilon)}^{\frac{a\sigma}{2}A_1(\varepsilon)} dt \int_0^{(\sqrt{z}-\sqrt{\theta^2+t^2})^2} d\zeta \right\} \frac{(-1)(h'+\zeta-t^2)}{2\zeta\{(h'+\zeta-t^2)^2-4\zeta z\}^{1/2}}$$

The expression may be simplified in a similar way to cases A, B. There is no source of non-uniformity, and to obtain $\lim_{\varepsilon \to 0} \mathcal{H}(\varepsilon)$ one may put $\varepsilon = 0$ .

Thus

$$\mathcal{H}(\varepsilon) = -\frac{\tau\,\mathcal{X}_{oo}}{a\pi}\Bigg\{ \int_{\frac{a\sigma}{2}(b+x')}^{\frac{a\sigma}{2}A_2(\zeta_2)} \left\{ \left(z-\theta^2-t^2+\frac{2t}{a\sigma}-x'-b\right)^2 - 4\left(\frac{2t}{a\sigma}-x'-b\right)z\right\}^{-1/2} dt$$

$$+\int_{\frac{a\sigma}{2}(b-x')}^{\frac{a\sigma}{2}A_1(\zeta_1)} \left\{ \left(z-\theta^2-t^2+\frac{2t}{a\sigma}+x'-b\right)^2 - 4\left(\frac{2t}{a\sigma}+x'-b\right)z\right\}^{-1/2} dt$$

$$+\frac{1}{2\sqrt{h'}}\log\left(\frac{\left(\sqrt{h'}+\frac{a\sigma}{2}b\right)^2-\frac{a^2x^2}{4}}{\left(\sqrt{h'}-\frac{a\sigma}{2}b\right)^2-\frac{a^2x^2}{4}}\right) + o(1)_{\varepsilon\to 0}\Bigg\} \qquad \underline{4.7C}$$

One may now assemble the contributions to $\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau}$ from (4.5), (4.6) and (4.7). Letting $\varepsilon\to 0$ gives a well defined velocity in each case. However, we are interested in the behaviour as $\tau\to 0$, and in this limit the regions in which cases A, B apply become vanishingly small. Consequently we need only assemble $\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau}$ in case C, for which

$$\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau} = \frac{\sigma\tau}{2\pi}\mathcal{X}_{oo}\Bigg\{ \log\left(\frac{z}{b+z}\right) + O(\bar{a}^2)$$

$$-\frac{1}{\bar{a}}\int_{\bar{a}(b+x')}^{\bar{a}A_2(\zeta_2)} \left\{ \left(z-\theta^2-t^2+t/\bar{a}-x'-b\right)^2 - 4z\left(t/\bar{a}-x'-b\right)\right\}^{-1/2} dt$$

$$-\frac{1}{\bar{a}}\int_{\bar{a}(b-x')}^{\bar{a}A_1(\zeta_1)} \left\{ \left(z-\theta^2-t^2+t/\bar{a}+x'-b\right)^2 - 4z\left(t/\bar{a}+x'-b\right)\right\}^{-1/2} dt \Bigg\}.$$

We need the asymptotic forms of the integrals. First note that the integrands are singular at $\bar{a}A_1(\zeta_1)$ and $\bar{a}A_2(\zeta_2)$ ; an attempt at a straightforward expansion in powers of $\bar{a}$ fails due to a non-integrable singularity arising at $\bar{a} A_i(\zeta_i)$, $(i = 1, 2)$. Changing the integration variable to $T = t/\bar{a} + x' - b$, the limits of integration become $0$, $\zeta_1$. Now split the range of integration at $\zeta_1 - \bar{a}^p$, $0 < p < 1$. For the outer part $\int_0^{\zeta_1 - \bar{a}^p} (\ ) \, dT$, expand directly for small $\bar{a}$; for the inner part, put $\bar{a} \tilde{T} = \zeta_1 - T$, and then expand the integrand. Thus the last integral has the asymptotic form

$$\sigma \log\left(\frac{\sqrt{z}}{\bar{a} A_1(z)}\right) + O(\bar{a} \log \bar{a})$$

and a similar result applies for the other. The term $O(\bar{a} \log \bar{a})$ was obtained taking $p = 1/2$; it is not based on an assumption that terms involving $p$ must cancel.

Thus

$$\left.\frac{\partial \varphi}{\partial z}\right|_{y=\tau} = \frac{\sigma \tau}{2\pi} \chi_{oo} \left\{ \log\left(\frac{z}{b+z}\right) - \log\left(\frac{z}{\bar{a}^2 A_1(z) A_2(z)}\right) + O(\bar{a} \log \bar{a}) \right\},$$

$$\frac{2\pi a}{\tau} \left.\frac{\partial \varphi}{\partial z}\right|_{y=\tau} = \chi_{oo} \left\{ 4 \bar{a} \log \bar{a} - 2\bar{a} \log\left(\frac{b+z}{(b+z)^2 - x'^2}\right) + O(\bar{a}^2 \log \bar{a}) \right\}, \qquad \underline{4.9C}$$

since $\bar{a} = \frac{a\sigma}{2}$. Note that this result is the same as that for $z < \frac{a^2\tau^2}{4}$, equation (4.4).

## § 5. Expressions for $\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau}$ as integrals of standard elliptic integrals. ($\chi = \chi_{0\xi}$, order of operations $\mathcal{L}_\eta \mathcal{L}_\tau \mathcal{L}_\xi \mathcal{L}_z$)

We now turn to the more practical order of operations, and first deal with the case when $\chi$ is a function of $\xi$ only, $\chi(\xi,\eta) = \chi_{0\xi}$.

Applying $\mathcal{L}_z$ to the expression for $\varphi$, (2.4), in the case $z < \theta^2$ we have

$$\mathcal{L}_z \varphi = \mathcal{L}_z I_0 + \mathcal{B} \qquad \underline{5.1}$$

where $\mathcal{L}_z I_0$ is given by (4.1), and

$$\mathcal{B} = \frac{\tau}{2\pi} \frac{\partial}{\partial z} \int_0^b d\xi \,.\, \chi_{0\xi} \int_{-\sigma(b-\xi)}^{\sigma(b-\xi)} \varphi(\varepsilon)\,.\,d\eta \,.$$

Changing to order of operations $\mathcal{L}_\xi \mathcal{L}_\eta \mathcal{L}_z$, $\mathcal{L}_\eta \mathcal{L}_z \varphi(\varepsilon)$ may be expressed in terms of elliptic integrals. Define

$$X_3(-\xi) = \tfrac{a\sigma}{2} A_1(-\xi) \;,\quad X_4(-\xi) = \tfrac{a\sigma}{2} A_2(-\xi) \;,\quad \mu_i = X_i / \{\theta^2 + (\sqrt{s}-\sqrt{\xi})^2\}^{1/2} \qquad \underline{5.2\text{ i}}$$

and

$$F_i = F\left(k, \frac{\mu_i}{(1+\mu_i^2)^{1/2}}\right) = \int_0^{\mu_i/\sqrt{1+\mu_i^2}} \{(1-u^2)(1-k^2u^2)\}^{-1/2}.du$$

$$\text{with } k^2 = \frac{4\sqrt{\xi s}}{\{\theta^2 + (\sqrt{s}+\sqrt{\xi})^2\}^{1/2}}$$

$$E_i = E\left(k, \frac{\mu_i}{(1+\mu_i^2)^{1/2}}\right) = \int_0^{\mu_i/\sqrt{1+\mu_i^2}} (1-u^2)^{-1/2}(1-k^2u^2)^{1/2}.du \qquad \underline{5.2\text{ ii}}$$

$F, E$ being incomplete elliptic integrals.

Then

$$\mathcal{B} = \frac{\tau}{2\pi a} \int_0^b \chi_{03}\,(G_3 + G_4).\,d\xi$$

where

$$G_i = \frac{1}{4\xi}\left\{ \frac{F_i}{\{\theta^2 + (\sqrt{s}+\sqrt{\xi})^2\}^{1/2}} + \frac{\xi - s - \theta^2}{\{(\xi+s+\theta^2)^2 - 4\xi s\}^{1/2}}\,\frac{E_i}{\{\theta^2 + (\sqrt{s}-\sqrt{\xi})^2\}^{1/2}} \right.$$

$$\left. + \frac{2\sqrt{\xi}\,(\sqrt{\xi}+\sqrt{s})}{\{\theta^2 + (\sqrt{s}+\sqrt{\xi})^2\}}\;\frac{X_i}{\{[s+\theta^2+\xi+X_i^2]^2 - 4\xi s\}^{1/2}} \right\} . \qquad \underline{5.3}$$

If in (5.3) we let $\sigma \to \infty$, and then put $\theta = 0$, we recover the two dimensional result, equation (3.7) of Ch. II. Strictly, it is not permissible to put $\theta = 0$, since we need the value of $\frac{\partial\varphi}{\partial z}$ at $y = \tau$ and $a \to 0$ with $\tau$. However the consequences are not serious. They are accounted for by taking the principal value of the integral.

For $z > \theta^2$,

$$\mathcal{L}_z\,\varphi = \mathcal{L}_z\,I_0 + \mathcal{L}_\varepsilon\left(\mathcal{B}(\varepsilon) + \mathcal{H}(\varepsilon)\right) \qquad \underline{5.4}$$

where $\mathcal{L}_z\,I_0$ is given by (4.5), $\mathcal{L}_\varepsilon$ is the limit associated with taking the principal value and $\mathcal{B}(\varepsilon)$, $\mathcal{H}(\varepsilon)$ are the contributions arising from the integrals over the modified regions $\mathcal{D}_1$, $\mathcal{D}_2$ respectively in (2.4). $\mathcal{L}_\varepsilon$ and $\mathcal{L}_z$ have been interchanged.

Now

$$\mathcal{B}(\varepsilon) = \frac{\tau}{2\pi}\frac{\partial}{\partial z}\int_\varepsilon^b d\xi\,.\,X_{03}\int_{-\sigma(b-\xi)}^{\sigma(b-\xi)} \varphi(\varepsilon)\,d\eta .$$

Again changing to order of operations $\mathcal{L}_\xi\,\mathcal{L}_\eta\,\mathcal{L}_z$, after some tedious manipulation, $\mathcal{L}_y \mathcal{L}_z \varphi(\varepsilon)$ may be expressed in terms of standard elliptic integrals.

Define

$$Y_i = \frac{X_i - q}{X_i + q} \quad (i = 3, 4) \quad \text{and} \quad Y_i = 1 \quad (i = 1, 2)$$

$$E_i^* = E(k, \tau_i) \quad , \quad F_i^* = F(k, \tau_i) \quad , \quad \tau_i = \frac{Y_i}{\sqrt{\lambda_1 + Y_i^2}}$$

5.5

where $k^2 = 1 - \lambda_1^2$ , $q = (\Theta^2 + \gamma^2)^{1/4}$ , $A = \sqrt{2}\sqrt{q^2 - \Theta}$ , $\lambda_1 = \dfrac{2q - A}{2q + A}$

$$\lambda_1 \lambda_2 = 1 \quad , \quad \gamma^2 = 4 \xi z \quad , \quad \Theta = \xi + \theta^2 - z \ .$$

The notation differs only slightly from Ch II . There, we treat only the case $Y_i = 1$ , so the suffix $i$ is redundant and the asterisk is written as a subscript

Then

$$\mathcal{B}(\varepsilon) = \frac{\tau}{2\pi a} \int_{\varepsilon}^{b} \mathcal{X}_{o\xi} \sum_{i=1}^{4} \frac{1}{2A^2(2q - A)} (H_i + M_i) \, d\xi \qquad \underline{5.6}$$

where

$$H_i = 2E_i^* - 2\lambda_1 F_i^* - \frac{(1 - \lambda_1^2) Y_i}{\{\lambda_1 + Y_i^2\}^{1/2} \{1 + \lambda_1 Y_i^2\}^{1/2}}$$

and

$$M_i = \frac{(q^2 + \Theta)(z + \xi + \theta^2)}{8q^4} \Big\{ (3 - 2\lambda_1 + 3\lambda_1^2) 2F_i^* + (-1)(6 - \lambda_1 - \lambda_2)\Big(2E_i^* - \frac{(1 - \lambda_1^2) Y_i}{\{(\lambda_1 + Y_i^2)(1 + \lambda_1 Y_i^2)\}^{1/2}}\Big)\Big\}$$

If we let $\sigma \to \infty$ , and put $\theta = 0$ (5.6) reduces to the expression in equation (3.8) of Ch II .

Now consider the contribution from the sources $\varphi(H)$ ; as in § 4 there are three cases A , B , C .

Case A

$$\mathcal{H}(\varepsilon) = \frac{\tau}{2\pi} \frac{\partial}{\partial z} \int_{\varepsilon}^{(\sqrt{z}-\theta)^2} d\zeta . \mathcal{X}_{o\zeta} \int_{x-\frac{2}{a}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}}^{x+\frac{2}{a}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}} \varphi(H).d\eta$$

Putting $t = \frac{a}{2}(\eta - x)$ , and differentiating

$$\mathcal{H}(\varepsilon) = \frac{\tau}{2\pi}\left\{ \Theta_1 + \int_{\varepsilon}^{(\sqrt{z}-\theta)^2} \mathcal{X}_{o\zeta} \, (\Lambda_1 + \Lambda_2).d\zeta \right. \qquad \underline{5.7A}$$

where

$$\Theta_1 = -\frac{2\pi}{a} \frac{\mathcal{X}_{o\zeta}}{\{4\sqrt{z}(\sqrt{z}+\theta)\}^{1/2}} \qquad \text{with} \quad \zeta = (\sqrt{z}-\theta)^2$$

and

$$\Lambda_1 = \frac{\partial}{\partial z} \frac{2}{a} \int_0^{\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}} \varphi(H).dt$$

$$= \frac{1}{2a\zeta\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}} \left\{ \frac{(\sqrt{z}-\sqrt{\zeta})}{\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}} 2\sqrt{\zeta}\, F_1^{(1)} - \left(\frac{z-\zeta-\theta^2}{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2}\right) E_1^{(1)} \right\},$$

$$\Lambda_2 = \frac{\partial}{\partial z} \frac{2}{a} \int_{-\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}}^{0} \varphi(H).dt = \Lambda_1 \qquad \underline{5.8}$$

with

$$F_1^{(1)} = F(k,1) \quad , \quad E_1^{(1)} = E(k,1) \quad \text{and} \quad k^2 = \frac{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2}{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2} .$$

Note that (5.7A) is independent of $\sigma$ . If we put $\theta = 0$ it reduces to the corresponding expression in equation (3.8) of Ch.II .

Case B

$$\mathcal{H}(\varepsilon) = \frac{\tau}{2\pi}\frac{\partial}{\partial z}\left\{ \int_{\varepsilon}^{\zeta_1} d\zeta\ \mathcal{X}_{o\zeta} \int_{x-\frac{2}{a}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}}^{\sigma(b+\zeta)} \right.$$

$$\left. + \int_{\zeta_1}^{(\sqrt{z}-\theta)^2} d\zeta\,.\,\mathcal{X}_{o\zeta} \int_{x-\frac{2}{a}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}}^{x+\frac{2}{a}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}} \right\} \varphi(H).\,d\eta$$

Differentiating

$$\frac{2\pi}{\tau}\mathcal{H}(\varepsilon) = \Theta_1 + \int_{\zeta_1}^{(\sqrt{z}-\theta)^2} \mathcal{X}_{o\zeta}\,(\Lambda_1+\Lambda_2)\,d\zeta$$

$$+ \quad \Theta_2 - \Theta_3 \quad + \int_{\varepsilon}^{\zeta_1} \mathcal{X}_{o\zeta}\,(\Lambda_2+\Lambda_3)\,d\zeta \qquad \underline{5.7B}$$

where $\Theta_1$ , $\Lambda_1$ , $\Lambda_2$ are given by (5.8), and it may be shown that :

$$\Theta_2 - \Theta_3 = 0 \ ,$$

$$\Lambda_i = \frac{1}{2a\zeta\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}}\left\{ \frac{(\sqrt{z}-\sqrt{\zeta})}{\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}}\,2\sqrt{\zeta}\,F_i^{(1)} - \frac{z-\zeta-\theta^2}{\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}}\,E_i^{(1)} \right.$$

$$\left. + \frac{\{(z-\zeta-\theta^2)(z+3\zeta-\theta^2-X_i^2)+4\zeta\theta^2\}\ X_i}{\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}\{(z+\zeta-\theta^2-X_i^2)^2-4\zeta z\}^{1/2}} \right\}$$

with $F_i^{(1)} = F(k,\varpi_i)$ , $E_i^{(1)} = E(k,\varpi_i)$ , $\varpi_i = \dfrac{X_i}{\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}}, (i=3,4)$.

$\underline{5.9}$

Case C

$$\mathcal{H}(\varepsilon) = \frac{\tau}{2\pi}\frac{\partial}{\partial z}\left\{ \int_{\varepsilon}^{\zeta_2} d\zeta \int_{-\sigma(b+\zeta)}^{\sigma(b+\zeta)} + \int_{\zeta_2}^{\zeta_1} d\zeta \int_{x-\frac{2}{a}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}}^{\sigma(b+\zeta)} \right.$$

$$\left. + \int_{\zeta_1}^{(\sqrt{z}-\theta)^2} d\zeta \int_{x-\frac{2}{a}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}}^{x+\frac{2}{a}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}} \right\} \mathcal{X}_{o\zeta}\,\varphi(H)\,.\,d\zeta \ ,$$

and in a similar manner to the previous cases it may be shown

$$\frac{2\pi}{\tau}\mathcal{H}(\varepsilon) = \Theta_1 + \int_{\zeta_1}^{(\sqrt{\tau}-\theta)^2} \chi_{0\zeta}(\Lambda_1 + \Lambda_2)\,d\zeta + \int_{\zeta_1}^{\zeta_2} \chi_{0\zeta}(\Lambda_2 + \Lambda_3)$$
$$+ \int_{\varepsilon}^{\zeta_2} \chi_{0\zeta}(\Lambda_3 + \Lambda_4)\,.\,d\zeta \qquad \underline{5.7c}$$

It remains to show that $\mathcal{B}(\varepsilon)$ given by (5.6), and $\mathcal{H}(\varepsilon)$ given by (5.7) tend to a finite value when $\mathcal{L}_\varepsilon$ is taken, as indicated in (5.4). The analysis is straightforward;

$$\mathcal{B}(\varepsilon) = \frac{\tau}{2\pi a}\,\chi_{0\zeta}\Big|_{\zeta=0} \sum_{i=1}^{4} \operatorname{sgn}(Y_i)\left(\frac{-1}{4\sqrt{h'}}\right)\log\varepsilon + O(1)$$
as $\varepsilon \to 0$;

and from $\mathcal{H}(\varepsilon)$, only $\Lambda_1$, $\Lambda_2$ give rise to singular terms as $\varepsilon \to 0$, so that the $\log\varepsilon$ terms cancel in each of the cases $A$, $B$, $C$.

## §6. Expressions for $\frac{\partial\varphi}{\partial z}\Big|_{y=\tau}$ as integrals of standard elliptic integrals ( $\chi = |\eta|\,\chi_{13}^{*}$ , $\chi = \eta^2 \chi_{23}$ , order of operations $\mathcal{L}_\zeta\, \mathcal{L}_\tau\, \mathcal{L}_\eta\, \mathcal{L}_z$ . )

The analysis of § 5 may be repeated for other forms of the function $\chi(\zeta,\eta)$. The work is laborious. The cases (a) $\chi(\zeta,\eta) = |\eta|\,\chi_{13}^{*}$ and (b) $\chi(\zeta,\eta) = \eta^2\,\chi_{23}$ have been considered, and the results are presented below. Case (a) is needed for a diamond spanwise cross-section, and (b) for a parabolic arc, as may be seen from (3.4).

For $z < \theta^2$,

$$\mathcal{L}_z\,\varphi = \mathcal{B} + \mathcal{L}_z\, I \qquad \underline{6.1}$$

where, in case (a)

$$\mathcal{L}_z\, I = \mathcal{L}_z\, I_1^{*} = -\frac{\tau\,\chi_{10}^{*}}{2\pi a^2}\left\{ \log \frac{\left[h + \frac{a^2\sigma^2}{4}(b-x')^2\right]\left[h + \frac{a^2\sigma^2}{4}(b+x')^2\right]}{\left[h + \left(\frac{a\sigma}{2}x'\right)^2\right]^2} + \frac{ax}{\sqrt{h}}\left(\tan^{-1}\frac{\frac{a\sigma}{2}(b-x')}{\sqrt{h}} - \tan^{-1}\frac{\frac{a\sigma}{2}(b+x')}{\sqrt{h}} + 2\tan^{-1}\frac{\frac{a\sigma}{2}x'}{\sqrt{h}}\right)\right\} \qquad \underline{6.2a}$$

and

$$\mathcal{B} = \frac{\tau}{2\pi a}\int_0^b \chi_{13}^{*}\left\{ x\,(G_3 - G_4 + 2G_5) + \frac{1}{a}(W_3 + W_4 - 2W_5)\right\} d\xi \, . \qquad \underline{6.3a}$$

In this last expression $G_i$ is given by (5.3) with $X_5 = \frac{ax}{2}$ (definition), and

$$W_i \equiv 0 \quad (i = 1, 2) \quad , \quad W_i = \frac{s - \xi + \theta^2 + X_i^2}{2\xi\left\{(s + \xi + \theta^2 + X_i^2)^2 - 4\xi s\right\}^{1/2}} \quad (i = 3, 4, 5). \qquad \underline{6.4}$$

In case (b)

$$\mathcal{L}_z\, I = \mathcal{L}_z\, I_2 = -\frac{\tau\,\chi_{20}}{2\pi a}\left\{ \frac{4\sigma b}{a} + \frac{2x}{a}\log\left|\frac{h + \frac{a^2\sigma^2}{4}(b-x')^2}{h + \frac{a^2\sigma^2}{4}(b+x')^2}\right| + \frac{x^2 - \frac{4}{a^2}h}{\sqrt{h}}\left(\tan^{-1}\frac{a\sigma(b-x')}{2\sqrt{h}} + \tan^{-1}\frac{a\sigma(b+x')}{2\sqrt{h}}\right)\right\} \qquad \underline{6.2b}$$

and

$$\mathcal{B} = \frac{\tau}{2\pi a}\int_0^b \chi_{20}\left\{\frac{4}{a^2}\left(G_3^{(2)} + G_4^{(2)}\right) + x^2(G_3+G_4) + \frac{2x}{a}(W_3+W_4-2W_5)\right\}d\xi \qquad \underline{6.3b}$$

In this last expression $G_i$, $W_i$ have been defined already and

$$G_i^{(2)} = -\frac{1}{2\xi}\left\{\frac{s-\xi+\theta^2}{\{(\sqrt{s}+\sqrt{\xi})^2+\theta^2\}^{1/2}}F_i - \{(\sqrt{s}+\sqrt{\xi})^2+\theta^2\}^{1/2}E_i + \frac{X_i \quad 2\sqrt{\xi}\,(\sqrt{\xi}+\sqrt{s})}{\{[s+\xi+\theta^2+X_i^2]^2-4\xi s\}^{1/2}}\right\}. \qquad \underline{6.5}$$

For $z > \theta^2$,

$$\mathcal{L}_z\,\varphi = \mathcal{L}_z\,I + \mathcal{L}_\varepsilon\left(\mathcal{B}(\varepsilon) + \mathcal{H}(\varepsilon)\right) \qquad \underline{6.6}$$

where in case (a).

$$\mathcal{L}_z\,I = \mathcal{L}_z\,I_1^* = \frac{\tau\chi_{10}^*}{2\pi a^2}\left\{-\log\left|\frac{[h'-\frac{a^2\sigma^2}{4}(b-x')^2][h'-\frac{a^2\sigma^2}{4}(b+x')^2]}{[h'-\frac{a^2\sigma^2}{4}x'^2]^2}\right| + \frac{ax}{2\sqrt{h'}}\log\left|\frac{[(\sqrt{h'}-\frac{a\sigma}{2}x')^2-\frac{a^2\sigma^2}{4}b^2][\sqrt{h'}+\frac{a\sigma}{2}x']^2}{[(\sqrt{h'}+\frac{a\sigma}{2}x')^2-\frac{a^2\sigma^2}{4}b^2][\sqrt{h'}-\frac{a\sigma}{2}x']^2}\right|\right\} \qquad \underline{6.7.a}$$

$$\mathcal{B}(\varepsilon) = \frac{\tau}{2\pi a}\int_\varepsilon^b \chi_{13}^*\left\{\frac{x}{2A^2(2q-A)}\left[(2H_1+H_3-H_4+2H_5)+(2M_1+M_3-M_4+2M_5)\right] + \frac{1}{a}\left[W_3+W_4-2W_5\right]\right\}d\xi \qquad \underline{6.8a}$$

and $\mathcal{H}(\varepsilon)$ is the contribution from the $\varphi(H)$ sources, being different for cases $A, B, C$. Note $W_i$ is still defined by (6.4), since $s = -z$, and $\xi = -\zeta$.

In case (b)

$$\mathcal{L}_z\, I = \mathcal{L}_z\, I_2 = \frac{\tau\, \mathcal{X}_{o2}}{2\pi a}\left\{ -\frac{4\sigma b}{a} + \frac{\frac{4}{a^2}h' + x^2}{2\sqrt{h'}} \log\left| \frac{(\sqrt{h'}+\frac{a\sigma}{2}b)^2 - \frac{a^2\sigma^2}{4}x'^2}{(\sqrt{h'}-\frac{a\sigma}{2}b)^2 - \frac{a^2\sigma^2}{4}x'^2} \right| \right.$$

$$\left. - \frac{2x}{a}\log\left| \frac{h' - \frac{a^2\sigma^2}{4}(b-x')^2}{h' - \frac{a^2\sigma^2}{4}(b+x')^2} \right| \right\} , \qquad \underline{6.7b}$$

$$\mathcal{B}(\varepsilon) = \frac{\tau}{2\pi a}\int_\varepsilon^b \mathcal{X}_{2z}\left\{ \frac{4}{a^2}\sum_{i=1}^{4}\left[ \frac{4q}{(4q^2-A^2)^{1/2}} F_i^* - \frac{\theta^2+\xi-3z-\frac{a^2x^2}{4}}{2A^2(2q-A)} H_i \right.\right.$$

$$\left.\left. - \frac{\{(\theta^2+\xi-z)^2 - 4\xi z - \frac{a^2x^2}{4}\}}{2A^2(2q-A)(z+\xi+\theta^2)} M_i \right] + \frac{2x}{a}(W_3 - W_4)\right\} d\xi , \qquad \underline{6.8b}$$

and for $\mathcal{H}(\varepsilon)$ there are again 3 cases to consider.

Consider $\mathcal{H}(\varepsilon)$ in case (a). The regions of integration are unchanged from the $\mathcal{X}_{oz}$ case, but the factor $|\gamma|$ causes a slight modification.

Case A $\left(\frac{ax}{2} > \sqrt{h'}\right)$

$$\frac{2\pi}{\tau}\mathcal{H}(\varepsilon) = \frac{-4x}{a} \cdot \frac{\mathcal{X}_{13}^*[\zeta = (\sqrt{z}-\theta)^2]}{\{4\sqrt{z}(\sqrt{z}+\theta)\}^{1/2}}\,\frac{\pi}{2} + \int_\varepsilon^{(\sqrt{z}-\theta)^2} \mathcal{X}_{13}^* \,.\, 2x\Lambda_1 \,.\, d\zeta ,$$

Case A $\left(\frac{ax}{2} < \sqrt{h'}\right)$

$$\frac{2\pi}{\tau}\mathcal{H}(\varepsilon) = \frac{-4x}{a}\,\frac{\mathcal{X}_{13}^*[\zeta = (\sqrt{z}-\theta)^2]}{\{4\sqrt{z}(\sqrt{z}+\theta)\}^{1/2}}\,\frac{\pi}{2}$$

$$+ \int_{(\sqrt{h}-\frac{ax}{2})^2}^{(\sqrt{z}-\theta)^2} \mathcal{X}_{13}^* \,.\, 2x\Lambda_1\, d\zeta + \int_\varepsilon^{(\sqrt{h}-\frac{ax}{2})^2} \mathcal{X}_{13}^* \left(2x\Lambda_5 - \frac{4}{a^2}W_5\right) d\zeta . \qquad \underline{6.9A}$$

Case B $\left(\frac{ax}{2} > \sqrt{h'}\right)$

$$\frac{2\pi}{\tau}\mathcal{H}(\varepsilon) = \frac{-4x}{a}\,\frac{\mathcal{X}_{13}^*[\zeta = (\sqrt{z}-\theta)^2]}{\{4\sqrt{z}(\sqrt{z}+\theta)\}^{1/2}}\,\frac{\pi}{2} + \int_{\zeta_1}^{(\sqrt{z}-\theta)^2} \mathcal{X}_{13}^* \; 2x\Lambda_1 . d\zeta$$

$$+ \int_\varepsilon^{\zeta_1} \mathcal{X}_{13}^* \left[x(\Lambda_2+\Lambda_3) + \frac{2}{a^2}W_3\right] d\zeta ,$$

Case B $(\sqrt{\zeta_1} > \sqrt{h'} - \frac{ax}{2} > 0)$

$$\frac{2\pi}{\tau} H(\varepsilon) = \frac{-4x}{a} \frac{\chi_{13}^{*}[\zeta = (\sqrt{z}-\theta)^2]}{\{4\sqrt{z}(\sqrt{z}+\theta)\}^{1/2}} \frac{\pi}{2} + \int_{\zeta_1}^{(\sqrt{z}-\theta)^2} \chi_{13}^{*} \; 2x\Lambda_1 \, d\zeta$$

$$+ \int_{(\sqrt{h}-\frac{ax}{2})^2}^{\zeta_1} \chi_{13}^{*} \left[ x(\Lambda_3 + \Lambda_2) + \frac{2}{a^2} W_3 \right] d\zeta$$

$$+ \int_{\varepsilon}^{(\sqrt{h}-\frac{ax}{2})^2} \chi_{13}^{*} \left[ x(\Lambda_3 - \Lambda_2 + 2\Lambda_5) + \frac{2}{a^2}(W_3 - 2W_5) \right] . d\zeta \quad .$$

Case B $(\sqrt{h'} - \frac{ax}{2} > \sqrt{\zeta_1})$

$$\frac{2\pi}{\tau} H(\varepsilon) = \frac{-4x}{a} \frac{\chi_{13}^{*}[\zeta = (\sqrt{z}-\theta)^2]}{\{4\sqrt{z}(\sqrt{z}+\theta)\}^{1/2}} \frac{\pi}{2} + \int_{(\sqrt{h}-\frac{ax}{2})^2}^{(\sqrt{z}-\theta)^2} \chi_{13}^{*} \cdot 2x\Lambda_1 \, d\zeta$$

$$+ \int_{\sqrt{\zeta_1}}^{(\sqrt{h}-\frac{ax}{2})^2} \chi_{13}^{*} \left[ 2x\Lambda_5 - \frac{4}{a^2} W_5 \right] . d\zeta$$

$$+ \int_{\varepsilon}^{(\sqrt{h}-\frac{ax}{2})^2} \chi_{13}^{*} \left\{ x(\Lambda_3 - \Lambda_2 + 2\Lambda_5) + \frac{2}{a^2}(W_3 - 2W_5) \right\} d\zeta \qquad \underline{6.9B}$$

Case C $(\sqrt{\zeta_1} > \sqrt{h'} - \frac{ax}{2})$

$$\frac{2\pi}{\tau} H(\varepsilon) = \frac{-4x}{a} \frac{\chi_{13}^{*}[\zeta = (\sqrt{z}-\theta)^2]}{\{4\sqrt{z}(\sqrt{z}+\theta)\}^{1/2}} \frac{\pi}{2} + \int_{\zeta_1}^{(\sqrt{z}-\theta)^2} \chi_{13}^{*} \; 2x\Lambda_1 \, d\zeta$$

$$+ \int_{(\sqrt{h}-\frac{ax}{2})^2}^{\zeta_1} \chi_{13}^{*} \left[ x(\Lambda_3 + \Lambda_2) + \frac{2}{a^2} W_3 \right] d\zeta + \int_{\zeta_2}^{(\sqrt{h}-\frac{ax}{2})^2} \chi_{13}^{*} \left[ x(\Lambda_3 - \Lambda_2 + 2\Lambda_5) + \frac{2}{a^2}(W_3 - 2W_5) \right] d\zeta$$

$$+ \int_{\varepsilon}^{\zeta_2} \chi_{13}^{*} \left[ x(\Lambda_3 - \Lambda_4 + 2\Lambda_5) + \frac{2}{a^2}(W_3 + W_4 - 2W_5) \right] . d\zeta \quad .$$

Case C $(\sqrt{h'} - \frac{ax}{2} > \sqrt{\zeta_1})$

$$\frac{2\pi}{\tau} H(\varepsilon) = \frac{-4x}{a} \cdot \frac{\chi_{13}^{*}[\zeta = (\sqrt{z}-\theta)^2]}{\{4\sqrt{z}(\sqrt{z}+\theta)\}^{1/2}} \frac{\pi}{2} + \int_{(\sqrt{h'}-\frac{ax}{2})^2}^{(\sqrt{z}-\theta)^2} \chi_{13}^{*} \; 2x\Lambda_1 . d\zeta + \int_{\zeta_1}^{(\sqrt{h'}-\frac{ax}{2})^2} \chi_{13}^{*} \left[ x \, 2\Lambda_5 - \frac{4}{a^2} W_5 \right] d\zeta$$

$$+ \int_{\zeta_2}^{\zeta_1} \chi_{13}^{*} \left[ x(\Lambda_3 - \Lambda_2 + 2\Lambda_5) + \frac{2}{a^2}(W_3 - 2W_5) \right] d\zeta$$

$$+ \int_{\varepsilon}^{\zeta_2} \chi_{13}^{*} \left[ x(\Lambda_3 - \Lambda_4 + 2\Lambda_5) + \frac{2}{a^2}(W_3 + W_4 - 2W_5) \right] . d\zeta \quad .$$

$$\underline{6.9C}$$

In these expressions $\Lambda_i$ is given by (5.9), and $W_i$ by (6.4).

Now consider $\mathcal{H}(\varepsilon)$ in case (b) :

Case A

$$\frac{2\pi}{\tau}\mathcal{H}(\varepsilon) = -x^2\frac{\sqrt{z}-\theta}{\sqrt{z}}\frac{\mathcal{X}_{23}[\zeta=(\sqrt{z}-\theta)^2]}{\{4\sqrt{z}(\sqrt{z}+\theta)\}^{1/2}}\frac{\pi}{2} + \int_{\varepsilon}^{(\sqrt{z}-\theta)^2}\mathcal{X}_{23}\{\tfrac{4}{a^2}(L_1+L_2)+x^2(\Lambda_1+\Lambda_2)\}d\zeta \qquad \underline{6.10A},$$

where

$$L_1 = L_2 = \frac{1}{2\zeta a}\left\{\frac{2\sqrt{\zeta}(\sqrt{z}+\sqrt{\zeta})}{\{(\sqrt{z}+\sqrt{\zeta})^2+\theta^2\}^{1/2}}F_1^{(1)} - \{(\sqrt{z}+\sqrt{\zeta})^2+\theta^2\}^{1/2}E_1^{(1)}\right\} \qquad \underline{6.11A},$$

Case B

$$\frac{2\pi}{\tau}\mathcal{H}(\varepsilon) = -x^2\frac{\sqrt{z}-\theta}{\sqrt{z}}\frac{\mathcal{X}_{23}[\zeta=(\sqrt{z}-\theta)^2]}{\{4\sqrt{z}(\sqrt{z}+\theta)\}^{1/2}}\frac{\pi}{2}$$

$$+\int_{\zeta_1}^{(\sqrt{z}-\theta)^2}\mathcal{X}_{23}\{\tfrac{4}{a^2}(L_1+L_2)+x^2(\Lambda_1+\Lambda_2)\}.d\zeta$$

$$+\int_{\varepsilon}^{\zeta_1}\mathcal{X}_{23}\{\tfrac{4}{a^2}(L_2+L_3)+\tfrac{4x}{a^2}W_3+x^2(\Lambda_2+\Lambda_3)\}d\zeta \qquad \underline{6.10B},$$

where

$$L_i = \frac{1}{2\zeta a}\left\{\frac{2\sqrt{\zeta}(\sqrt{z}+\sqrt{\zeta})}{\{(\sqrt{z}+\sqrt{\zeta})^2+\theta^2\}^{1/2}}F_i^{(1)} - \{(\sqrt{z}+\sqrt{\zeta})^2+\theta^2\}^{1/2}E_i^{(1)} - \frac{x_i(\zeta-z+\theta^2+X_i^2)}{\{(z+\zeta-\theta^2-X_i^2)^2-4\zeta z\}^{1/2}}\right\} \qquad (i=3,4,5)$$

$$\underline{6.11B},$$

Case C

$$\frac{2\pi}{\tau}\mathcal{H}(\varepsilon) = -x^2\frac{\sqrt{z}-\theta}{\sqrt{z}}\frac{\mathcal{X}_{22}[\zeta=(\sqrt{z}-\theta)^2]}{\{4\sqrt{z}(\sqrt{z}+\theta)\}^{1/2}}\frac{\pi}{2}$$

$$+\int_{\zeta_1}^{(\sqrt{z}-\theta)^2}\mathcal{X}_{23}\{\tfrac{4}{a^2}(L_1+L_2)+x^2(\Lambda_1+\Lambda_2)\}.d\zeta$$

$$+\int_{\zeta_2}^{\zeta_1}\mathcal{X}_{23}\{\tfrac{4}{a^2}(L_3+L_2)+\tfrac{4x}{a^2}W_3+x^2(\Lambda_3+\Lambda_2)\}d\zeta + \int_{\varepsilon}^{\zeta_2}\mathcal{X}_{23}\{\tfrac{4}{a^2}(L_3+L_4)+\tfrac{4x}{a}(W_3-W_4)+x^2(\Lambda_3+\Lambda_4)\}d\zeta$$

$$\underline{6.10C}.$$

The behaviour as $\varepsilon \to 0$, required for $\mathcal{L}_z \varphi$ in (6.6), is similar to that for $\mathcal{X}(\zeta,\eta) = \mathcal{X}_{o\zeta}$. Thus as $\varepsilon \to 0$, (6.8a) and (6.9) give a finite velocity for case (a), and (6.8b) and (6.10) give a finite velocity for case (b).

## § 7. Asymptotic behaviour of $\frac{\partial\varphi}{\partial z}\Big|_{y=\tau}$ as $\tau\to 0$ ( $\mathcal{X} = \mathcal{X}_{03}$ , order of operations $\ell_\xi \ell_\tau \ell_y \ell_z$ ).

We now examine the behaviour of the expressions for $\frac{\partial\varphi}{\partial z}\Big|_{y=\tau}$ as $\tau\to 0$ . This involves no further physical assumption, since we assumed $\tau$ small to derive the approximate form of the boundary condition (2.2b). Further, since $a^2$ represents an estimate of the velocity perturbation on the airfoil, $\frac{a\sigma}{2} = \bar{a} \to 0$ as $\tau\to 0$.

First consider $\mathcal{X}(\xi,y) = \mathcal{X}_{03}$ ; for $z<\theta^2$ we need the behaviour of $\int_0^b G_i\, d\xi$ in order to deduce that of $\mathcal{B}$ given by (5.3). There is a source of non-uniformity near $\xi = -s$ , and a direct expansion of the integrand for small $\tau$ is not adequate.

We split the range of integration at $s\pm\delta$ , where $\delta\to 0$ as $\tau\to 0$ in such a way that $\bar{a}/\delta \to 0$ . Thus

$$\int_0^b \mathcal{X}_{03}\, G_i\, d\xi = \left( \int_0^{s-\delta} + \int_{s+\delta}^{b} + \int_{s-\delta}^{s+\delta} \right) \mathcal{X}_{03}\, G_i\, d\xi \qquad \underline{7.1}$$

For the range $(s-\delta,\ s+\delta)$ ;

$$1-k^2 = O(\delta^2)\ , \qquad w_i = \frac{\mu_i}{\{1+\mu_i^2\}^{1/2}} = \frac{X_i}{\{(\sqrt{\xi}-\sqrt{s})^2+\theta^2+X_i^2\}^{1/2}} = O(1)$$

$$E_i = w_i\,(1+O(\bar{a}^2))\ , \qquad F_i = \tfrac{1}{2}\log\left|\frac{1+w_i}{1-w_i}\right|(1+O(\bar{a}^2))$$

$$G_i = \frac{[1+O(\bar{a})]}{k\,\xi\,(\sqrt{\xi}+\sqrt{s})}\left\{\left(\tfrac{1}{2}\log\left|\frac{1+w_i}{1-w_i}\right| - w_i\right) + \frac{2\sqrt{\xi}(\sqrt{\xi}-\sqrt{s})}{(\sqrt{\xi}-\sqrt{s})^2+\theta^2}\,w_i + \frac{2\sqrt{\xi}}{\sqrt{\xi}+\sqrt{s}}\,w_i\right\}.$$

For the ranges $(0,\ s-\delta)$ and $(s+\delta,\ b)$ ;

$$k^2 = O(1)\ , \qquad w_i = O(\bar{a}/\delta)\ ,$$

$$E_i = w_i + O\left(\left(\tfrac{\bar{a}}{\delta}\right)^2\right)\ , \qquad F_i = w_i + O\left(\left(\tfrac{\bar{a}}{\delta}\right)^2\right)$$

$$G_i = \frac{[1 + O((\frac{\bar{a}}{\delta})^2)]}{4\xi\,(\sqrt{\xi} + \sqrt{s})} \left\{ \qquad + \frac{2\sqrt{\xi}\,(\sqrt{\xi} - \sqrt{s})}{(\sqrt{\xi} - \sqrt{s})^2 + \theta^2} W_i + \frac{2\sqrt{\xi}}{\sqrt{\xi} + \sqrt{s}} W_i \right\}$$

Thus

$$\int_0^b \mathcal{X}_{0z}\, G_i\, d\xi = g_i^{(1)} + g_i^{(2)} \qquad \underline{7.2},$$

where

$$g_i^{(1)} = \int_{s-\delta}^{s+\delta} \frac{\mathcal{X}_{0z}}{4\xi\,(\sqrt{\xi} + \sqrt{s})} \left(\frac{1}{2} \log\left(\frac{1 + W_i}{1 - W_i}\right) - W_i\right) d\xi\ (1 + O(\bar{a})),$$

$$g_i^{(2)} = \int_0^b \mathcal{X}_{0z} \frac{W_i}{2\sqrt{\xi}\,(\sqrt{\xi} + \sqrt{s})} \left(\frac{\sqrt{\xi} - \sqrt{s}}{(\sqrt{\xi} - \sqrt{s})^2 + \theta^2} + \frac{1}{\sqrt{\xi} + \sqrt{s}}\right) d\xi\,\left(1 + O\left(\frac{\bar{a}}{\delta}\right)\right).$$

Now by expanding and integrating

$$g_i^{(1)} = \mathcal{X}_{0z}(\zeta = -s)\ \frac{\bar{X}_i}{2s}\ (1 + O(\delta)) + O(\bar{a}^2), \quad \text{where } \bar{X}_i = X_i(\xi = -s),$$

and

$$g_i^{(2)} = \frac{a\sigma}{2}\left\{ - \mathcal{X}_{0z}(\zeta = -s) \log\left(\frac{4(\sqrt{b} - \sqrt{s})\sqrt{s}}{\bar{X}_i^2}\right) + \int_0^b \frac{\mathcal{X}_{0z}(-s) - \mathcal{X}_{0z}(-\xi)}{|\sqrt{s} - \sqrt{\xi}|} \frac{d\xi}{2\sqrt{\xi}} \right.$$

$$\left. - \int_0^b \frac{\mathcal{X}_{0z}(-\xi)}{(\sqrt{\xi} + \sqrt{s})} \operatorname{sgn}(\xi - s) \frac{d\xi}{2\sqrt{\xi}} + g_i^{(3)} + O\left(\frac{\bar{a}}{\delta} \log \bar{a}\right)\right\} + g_i^{(4)}$$

where

$$g_i^{(3)} = \int_0^b \frac{\mathcal{X}_{0z}(-\xi)}{\{(\sqrt{\xi} + \sqrt{s})^2 + \theta^2\}^{1/2}} \frac{\theta^2}{\{(\sqrt{\xi} - \sqrt{s})^2 + \theta^2\}} \frac{1}{\{(\sqrt{s} - \sqrt{\xi})^2 + \theta^2 + X_i^2\}^{1/2}} \frac{d\xi}{2\sqrt{\xi}},$$

$$g_i^{(4)} = \bar{X}_i \int_0^b \frac{1}{(\sqrt{\xi} + \sqrt{s})} \cdot \frac{\mathcal{X}_{0z}(-\xi)}{\{(\sqrt{\xi} + \sqrt{s})^2 + \theta^2\}^{1/2}} \frac{2\sqrt{\xi}\,(\sqrt{\xi} - \sqrt{s}) + \theta^2}{\{(\sqrt{\xi} - \sqrt{s})^2 + \theta^2\}\{(\sqrt{s} - \sqrt{\xi})^2 + \theta^2 + X_i^2\}^{1/2}} \frac{d\xi}{2\sqrt{\xi}}.$$

To deal with $g_i^{(4)}$, write

$$\mathcal{X}_{0z}(-\xi) = \mathcal{X}_{0z}(-s) - (\xi - s)\,\mathcal{X}_{0z}'(-s) + (\xi - s)^2\, \mathcal{H}(-\xi)$$

Now if $\mathcal{X}_{0z} = 1$,

$$g_i^{(4)} = (1 + O(\theta^2))\,\bar{X}_i \left\{ \int_0^b \frac{\sqrt{\xi} - \sqrt{s}}{(\sqrt{\xi}+\sqrt{s})\{(\sqrt{\xi}-\sqrt{s})^2+\theta^2\}\{(\sqrt{s}-\sqrt{\xi})^2+\theta^2+X_i^2\}^{1/2}} \frac{d\xi}{2\sqrt{\xi}} \right.$$

$$\left. + \int_0^b \frac{1}{(\sqrt{\xi}+\sqrt{s})^2\{(\sqrt{s}-\sqrt{\xi})^2+\theta^2+X_i^2\}^{1/2}} \frac{d\xi}{2\sqrt{\xi}} \right\}$$

The second integral $= \int_0^b \left( \frac{1}{(\sqrt{\xi}+\sqrt{s})^2} - \frac{1}{4s} \right) \frac{1}{|\sqrt{s}-\sqrt{\xi}|} \frac{d\xi}{2\sqrt{\xi}} + \frac{1}{4s} \log\left( \frac{4(\sqrt{b}-\sqrt{s})\sqrt{s}}{\bar{X}_i^2} \right)$

The first integral may be expressed $+ O(\bar{a})$

$$\int_0^b \frac{1}{2\sqrt{s}} \frac{\sqrt{\xi}-\sqrt{s}}{\{(\sqrt{\xi}-\sqrt{s})^2+\theta^2\}\{(\sqrt{s}-\sqrt{\xi})^2+\theta^2+X_i^2\}^{1/2}} \frac{d\xi}{2\sqrt{\xi}}$$

$$- \frac{1}{4s} \log\left( \frac{16s(\sqrt{b}-\sqrt{s})}{X_i^2(\sqrt{b}+\sqrt{s})} \right) + o(1)_{\tau\to 0} .$$

To evaluate this, the integral must have its range split twice; at $s \pm \delta_1$ where $\delta_1 \to 0$, $\delta_1 \gg \bar{a}$, as $\tau \to 0$, and within this at $s \pm \delta_2$, where $\delta_2 \to 0$, $\tau \ll \delta_2 \ll \bar{a}$ as $\tau \to 0$.

Using this device, this last integral may be shown to be

$$\mathcal{Y} \int_0^b \frac{1}{2\sqrt{s}\,(\sqrt{\xi}-\sqrt{s})\,|\sqrt{\xi}-\sqrt{s}|} \frac{d\xi}{2\sqrt{\xi}} + \frac{1}{2\sqrt{s}} \cdot \frac{4\sqrt{s}}{A_i(-s)} + o(1)_{\tau\to 0}$$

so that collecting terms

$$g_i^{(4)} = (1 + O(\theta^2))\,\bar{X}_i \left\{ \int_0^b \left( \frac{1}{(\sqrt{\xi}+\sqrt{s})^2} - \frac{1}{4s} \right) \frac{1}{|\sqrt{s}+\sqrt{\xi}|} \frac{d\xi}{2\sqrt{\xi}} + \frac{1}{4s} \log \frac{\sqrt{b}+\sqrt{s}}{4s} \right.$$

$$\left. + \frac{2}{A_i(-s)} + \mathcal{Y} \int_0^b \frac{1}{2\sqrt{s}\,(\sqrt{\xi}-\sqrt{s})|\sqrt{\xi}-\sqrt{s}|} \frac{d\xi}{2\sqrt{\xi}} \right\} .$$

The device of splitting the range of integration twice may be employed to show $g_i^{(3)} = O(\tau)$. The expression for $g_i^{(4)}$ was only derived for $\mathcal{X}_{o\xi} = 1$ ; for $\mathcal{X}_{o\xi} = (\xi - s)$ it may be shown by similar analysis

$$g_i^{(4)} = \bar{X}_i \log\left(\frac{b-s}{\bar{X}_i^2}\right)(1 + O(\bar{a}^2)) + o(\bar{a})$$

and for $\mathcal{X}_{o\xi} = (\xi - s)^2 \mathcal{H}(-\xi)$

$$g_i^{(4)} = \bar{X}_i \int_0^b \mathcal{H}(-\xi) \operatorname{sgn}(\xi - s)\, d\xi + o(\bar{a}) \quad .$$

Consequently for general $\mathcal{X}_{o\xi}$ we have

$$g_i^{(4)} = \mathcal{X}_{o\xi}(-s)\, \bar{X}_i \left\{ \frac{2}{A_i(-s)} + \frac{1}{s} - \frac{1}{2}\left(\frac{1}{(\sqrt{b}+\sqrt{s})^2} + \frac{1}{\sqrt{s}(\sqrt{b}-\sqrt{s})}\right)\right\}$$

$$- \mathcal{X}_{o\xi}'(-s)\, \bar{X}_i \log\left(\frac{b-s}{X_i^2}\right)$$

$$+ \bar{X}_i \int_0^b \mathcal{H}(-\xi) . \operatorname{sgn}(\xi - s)\, d\xi .$$

Thus collecting terms, using (4.1), (5.1), (5.3), (6.2), and simplifying, we obtain

$$\frac{2\pi a}{\tau} \left.\frac{\partial \varphi}{\partial z}\right|_{y=\tau} = \mathcal{X}_{o\xi}(-s) . 2\bar{a} \log \frac{\bar{a}^2[(b-s)^2 - x'^2]}{(b-s)}$$

$$+ \mathcal{X}_{o\xi}'(-s) . 2\bar{a}\left[-2s + (b-s)\log\frac{\bar{a}^2[(b-s)^2 - x'^2]}{(b-s)} + x' \log\left(\frac{b-s+x'}{b-s-x'}\right)\right]$$

$$+ \bar{a}\,\mathcal{H}(0)(-2bs) + 2\bar{a}\int_0^b (b-\xi)\mathcal{H}(-\xi)\operatorname{sgn}(\xi - s) . d\xi + o(\bar{a}) . \qquad \underline{7.3}$$

This agrees with the result (4.4) derived using the other order of operations.

Now consider $\mathcal{X}(\zeta,\eta) = \mathcal{X}_{0\zeta}$ for $z > \theta^2$. $\mathcal{L}_z\varphi$ is given by (5.4), so that we need the behaviour of $\mathcal{B}(\varepsilon)$ given by (5.6) and $\mathcal{H}(\varepsilon)$ given by (5.7) as $\tau \to 0$. As explained in §4 we need only consider case C.

Although the expression for $\mathcal{B}$ looks fierce it is really quite tame, because there are no sources of non-uniformity as $\tau \to 0$. One can derive asymptotic forms of $M_i$, $H_i$, but it is simpler to return to the definition of $\mathcal{B}$ as a double integral. Straightforward expansion quickly yields

$$\frac{2\pi a}{\tau}\mathcal{B}(0) = a\int_0^b \mathcal{X}_{0\xi}\frac{\sigma(b-\xi)}{(z+\xi)^2}d\xi + o(\bar{a}) . \tag{7.4}$$

We now write (5.7c) in the form

$$\frac{2\pi a}{\tau}\mathcal{H}(0) = \frac{2\pi\,\mathcal{X}_{0\zeta}(z)}{\{4\sqrt{z}(\sqrt{z}-\theta)\}^{1/2}} + \int_{\zeta_1}^{(\sqrt{z}-\theta)^2}\mathcal{X}_{0\zeta}\,a\Lambda_1\,d\zeta + \int_{\zeta_2}^{(\sqrt{z}-\theta)^2}\mathcal{X}_{0\zeta}\,a\Lambda_2\,d\zeta$$
$$+ \int_0^{\zeta_1}\mathcal{X}_{0\zeta}\,a\Lambda_3\,d\zeta + \int_0^{\zeta_2}\mathcal{X}_{0\zeta}\,a\Lambda_4\,d\zeta . \tag{7.5}$$

Straightforward expansion gives

$$\int_{\zeta_i}^{(\sqrt{z}-\theta)^2}\mathcal{X}_{0\zeta}\,.\,a\Lambda_i\,.d\zeta = -\mathcal{X}_{0\zeta}(z)\,\bar{a}\,\frac{\pi}{2}\,\frac{A_i(z)}{2z} + O(\bar{a}^2) , \quad (i=1,2).$$

We now examine

$$\int_0^{\zeta_1}\mathcal{X}_{0\zeta}\,a\Lambda_3\,d\zeta , \qquad (\Lambda_3 \text{ given by } (5.9)) .$$

There is a non-uniformity arising from near $\zeta_1$. Accordingly we split the range of integration at $z-\delta$, where

$$\delta \to 0 \quad \text{as} \quad \tau \to 0 \quad , \text{ but } \quad \delta \gg \bar{a} \quad \text{and} \quad \zeta_1 - z + \delta > 0 \ .$$

For the range $(z-\delta, \zeta_1)$,

$$\varpi_3 = O(1) \quad , \quad k^2 = O(\delta) \quad ,$$

$$F_3^{(1)} = \sin^{-1}\varpi_3 + O(\delta) \quad , \quad E_3^{(1)} = \sin^{-1}\varpi_3 + O(\delta) \quad ,$$

$$a\Lambda_3 = -a\Lambda_3^{(0)} + a\Lambda_3^{(1)} + O(\delta)$$

where

$$\Lambda_3^{(0)} = \frac{1}{2\zeta(\sqrt{z}+\sqrt{\zeta})}\left(\sin^{-1}\varpi_3 - \varpi_3\right)$$

and

$$\Lambda_3^{(1)} = \frac{\varpi_3}{2\zeta\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}}\left\{-1 + \frac{(z-\zeta-\theta^2)(z+3\zeta-\theta^2-X_i^2)-4\zeta\theta^2}{\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}\{(z+\zeta-\theta^2-X_i^2)^2-4\zeta z\}^{1/2}}\right\}$$

For the range $(0, z-\delta)$,

$$\varpi_3 = \frac{X_3}{\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}} = O\left(\frac{\bar{a}}{\delta}\right) \quad , \quad k^2 = O(1)$$

$$E_3^{(1)} = F_3^{(1)} + O\left(\left(\frac{\bar{a}}{\delta}\right)^2\right) = \varpi_3 + O\left(\left(\frac{\bar{a}}{\delta}\right)^2\right) \quad , \quad a\Lambda_3 = a\Lambda_3^{(1)} + O\left(\left(\frac{\bar{a}}{\delta}\right)^2\right).$$

Then, retaining terms larger than $o(\bar{a})$,

$$\int_0^{\zeta_1} \chi_{0\zeta}\, a\Lambda_3 . d\zeta = -\int_{z-\delta}^{\zeta_1} \chi_{0\zeta}\, a\Lambda_3^{(0)} . d\zeta + \int_0^{\zeta_1} \chi_{03}\, a\Lambda_3^{(1)} . d\zeta + o(\bar{a})$$

<u>7.6</u>

Now $$\int_{z-\delta}^{\zeta_1} \mathcal{X}_{0\zeta}\, a\Lambda_3^{(0)}.d\zeta = \mathcal{X}_{0\zeta}(z)\int_{z-\delta}^{\zeta_1} a\Lambda_3^{(0)}.d\zeta + O(\delta)$$

and with some effort the integral may be evaluated:

$$\int_{z-\delta}^{\zeta_1} \mathcal{X}_{0\zeta}\, a\Lambda_3^{(0)}.d\zeta = \frac{\mathcal{X}_{0\zeta}(z)}{2z}\left\{-\frac{a\sigma}{2}\bar{A}_1\frac{\pi}{2} + \bar{A}_1\frac{a\sigma}{2} + \frac{a\sigma}{2}\bar{A}_1\log 2\right\}$$
$$+ O(\delta) + O\left(\left(\bar{a}/\delta\right)^2\right)$$

where $\bar{A}_1 = A_1(z)$ .

For the other contribution to (7.6) write

$$\int_0^{\zeta_1} \mathcal{X}_{0\zeta}\, a\Lambda_3^{(1)}.d\zeta = \sum_{n=2}^{5}\int_0^{\zeta_1} \mathcal{X}_{0\zeta}\, a\Lambda_3^{(n)}.d\zeta ,$$

where

$$\int_0^{\zeta_1} a\Lambda_i^{(2)}\,d\zeta = \int_0^{\zeta_1}\frac{\mathcal{X}_{0\zeta}\,\varpi_i}{\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}}\left(-1 + \frac{\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2-X_i^2\}^{1/2}}{\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2-X_i^2\}^{1/2}}\right)\frac{d\zeta}{2\zeta} ,$$

$$\int_0^{\zeta_1} a\Lambda_i^{(3)}.d\zeta = \int_0^{\zeta_1}\frac{\mathcal{X}_{0\zeta}\,\varpi_i}{\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}}\,\frac{(\sqrt{z}-\sqrt{\zeta})\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2-X_i^2\}^{1/2}}{\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2-X_i^2\}^{1/2}}\,\frac{d\zeta}{\sqrt{\zeta}} ,$$

$$\int_0^{\zeta_1} a\Lambda_i^{(4)}.d\zeta = \int_0^{\zeta_1}\frac{\mathcal{X}_{0\zeta}\,\varpi_i}{\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}}\,\frac{\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}\,2\sqrt{\zeta}\,(\sqrt{z}+\sqrt{\zeta})\,d\zeta}{\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2-X_i^2\}^{1/2}\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2-X_i^2\}^{1/2}} ,$$

$$\int_0^{\zeta_1} a\Lambda_i^{(5)}\,d\zeta$$
$$= \int_0^{\zeta_1}\frac{\mathcal{X}_{0\zeta}\,\varpi_i}{\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}}\,\frac{2(z-\zeta-\theta^2)\,d\zeta}{\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2\}^{1/2}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2\}^{1/2}\{(\sqrt{z}-\sqrt{\zeta})^2-\theta^2-X_i^2\}^{1/2}\{(\sqrt{z}+\sqrt{\zeta})^2-\theta^2-X_i^2\}^{1/2}} .$$

The $\Lambda_1^{(2)}$, $\Lambda_i^{(3)}$ integrals may be evaluated by straightforward expansion and integration but the $\Lambda_i^{(4)}$, $\Lambda_i^{(5)}$ integrals require the ranges split near $\zeta_1$ to evaluate certain items.

Collecting terms we obtain the value of $\int_0^{\zeta_1} \mathcal{X}_{0\zeta}\, a\Lambda_3\, d\zeta$ from (7.6). A similar result holds for $\int_0^{\zeta_1} \mathcal{X}_{0\zeta}\, a\Lambda_4 . d\zeta$ ; and collecting the various contributions and simplifying, (7.5) gives $\mathcal{H}(0)$ . Finally we add the contribution $\mathcal{B}(\varepsilon)$ from (7.4) and $\ell_z\, I_0$ from (4.5), to obtain

$$\frac{2\pi a}{\tau} \left.\frac{\partial \varphi}{\partial z}\right|_{y=\tau} = \bar{a}\, \mathcal{X}_{0\zeta}(z)\; 2 \log \frac{\bar{a}^2 [(b+z)^2 - x'^2]}{(b+z)}$$

$$+ 2\bar{a}\, \mathcal{X}'_{0\zeta}(z) \Big\{ (b+z) \log \frac{\bar{a}^2 [ (b+z)^2 - x'^2]}{(b+z)}$$

$$+ x' \log \left( \frac{b+z+x'}{b+z-x'} \right) + 2z \Big\}$$

$$+ \bar{a}\, \mathfrak{X}(0) . 2bz + \bar{a} \int_0^z \mathfrak{X}(\zeta)\, 4(b+\zeta) . d\zeta + \bar{a} \int_0^b \mathfrak{X}(-\zeta) . 2(b-\zeta) d\zeta$$
$$+ o(\bar{a}) . \qquad \underline{7.7}$$

where $\mathfrak{X}(\zeta) . (\zeta - z)^2 = \mathcal{X}_{0\zeta} - \mathcal{X}_{0\zeta}(z) - (\zeta - z)\, \mathcal{X}'_{0\zeta}(z)$ .

Note that this is the same expression as for $z < 0$ , (7.3) and it agrees with (4.9c), a particular case derived using a different order of limiting processes.

## § 8. Asymptotic behaviour of $\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau}$ as $\tau \to 0$ .

## ( $\mathcal{X} = \mathcal{X}_{0\zeta} + |\eta|\, \mathcal{X}_{1\zeta}^{*}$ , $\mathcal{X} = \mathcal{X}_{0\zeta} + \eta^2 \mathcal{X}_{2\zeta}$ )

One may next proceed, in a similar manner, to derive the asymptotic values as $\tau \to 0$ for cases (a) and (b) treated in § 6. However, it is possible to use the order of operations $\mathcal{L}_\eta\ \mathcal{L}_\tau\ \mathcal{L}_\zeta\ \mathcal{L}_z$ to obtain these expressions, and this reduces the effort required. The approach is similar to that in § 4 and most of the detail is omitted.

First consider $\mathcal{X}(\zeta,\eta) = (\mathcal{X}_{00} + \mathcal{X}_{01}\zeta) + (\mathcal{X}_{10}^{*} + \zeta\mathcal{X}_{11}^{*})|\eta|$ . 8.1

The value of $\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau}$ has already been derived for $\mathcal{X}_{00} + \mathcal{X}_{01}\zeta$ in § 7. We present all the results together for convenience, and begin with those for $z < 0$ ;

$$\mathcal{X}_{00} : \frac{2\pi a}{\tau}\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau} = \mathcal{X}_{00}\,\bar{a}\left\{2\log\frac{\bar{a}^2[(b-s)^2-x'^2]}{b-s}\right\} + o(\bar{a}) ,$$

$$\mathcal{X}_{10}^{*} : \frac{2\pi a}{\tau}\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau} = \mathcal{X}_{10}^{*}\,\bar{a}\left\{2\sigma(b-s)\log\frac{\bar{a}^2[(b-s)^2-x'^2]}{b-s} + 2\sigma(b-2s)\right\} + o(\bar{a}) ,$$

$$\mathcal{X}_{01} : \frac{2\pi a}{\tau}\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau} = \mathcal{X}_{01}\,\bar{a}\left\{2(b-2s)\log\frac{\bar{a}^2[(b-s)^2-x'^2]}{b-s} + 2x'\log\left(\frac{b-s+x'}{b-s-x'}\right)\right\} + o(\bar{a}) ,$$

$$\mathcal{X}_{11}^{*} : \frac{2\pi a}{\tau}\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau} = \mathcal{X}_{11}^{*}\,\bar{a}\left\{\sigma(b-s)^2\log\frac{\bar{a}^2[(b-s)^2-x'^2]}{b-s} + \sigma x'^2\log\frac{x'^2}{[(b-s)^2-x'^2]} - \sigma 2s(b-s)\log\frac{\bar{a}^2[(b-s)^2-x'^2]}{b-s} - \sigma\left[2s(b-2s) - \tfrac{3}{2}b^2 - 4s^2 + 7sb\right]\right\} + o(\bar{a}) .$$

8.2

For $z > 0$, as in § 4, the analysis is quite different from that for $z < 0$, but the results are the same if one recalls $s = -z$.

If we now regard (8.1) as a particular case of

$$\chi(\zeta,\eta) = \chi_{0\zeta} + |\eta|\,\chi^*_{1\zeta} \qquad \underline{8.3}$$

we may put the results (8.2) in the form:

$$\begin{aligned}\frac{2\pi a}{\tau}\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau} = 2\bar{a}\Big\{ & \chi_{0\zeta}(z).\log\frac{\bar{a}^2[(b+z)^2-x'^2]}{b+z} \\ & + \chi'_{0\zeta}(z)\left((b+z)\log\frac{\bar{a}^2[(b+z)^2-x'^2]}{b+z} + x'\log\left(\frac{b+z+x'}{b+z-x'}\right)+2z\right) \\ & + \chi^*_{1\zeta}(z)\left(\sigma(b+z)\log\frac{\bar{a}^2[(b+z)^2-x'^2]}{b+z} + \sigma(b+2z)\right) \\ & + \chi^{*\prime}_{1\zeta}(z)\left(\frac{\sigma}{2}(b+z)^2\log\frac{\bar{a}^2[(b+z)^2-x'^2]}{b+z} - \frac{\sigma x'^2}{2}\log\frac{(b+z)^2-x'^2}{x'^2}\right. \\ & \left. + \frac{\sigma}{2}\left(\tfrac{3}{2}b^2+7bz+4z^2\right)\right)\Big\} + o(\bar{a}) .\end{aligned} \qquad \underline{8.4}$$

The expression is valid for all $z$ on the airfoil. It is now a relatively simple matter to extend (8.4) to the general case (8.3). The reason for this is that it is sufficient to find $\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau}$ for

$$\chi(\zeta,\eta) = (z-\zeta)^2\left(\mathcal{X}_0(\zeta;z) + |\eta|\,\mathcal{X}^*_1(\zeta;z)\right)$$

where

$$(z-\zeta)^2\,\mathcal{X}_0(\zeta,\eta) = \chi_{0\zeta} - \chi_{0\zeta}(z) - (\zeta-z)\chi'_{0\zeta}(z) ,$$

and

$$(z-\zeta)^2\,\mathcal{X}^*_1(\zeta,\eta) = \chi^*_{1\zeta} - \chi^*_{1\zeta}(z) - (\zeta-z)\,\chi^{*\prime}_{1\zeta}(z) .$$

Now the factor $(z-\zeta)^2$ eliminates the non-uniform behaviour to the order of approximation in $\bar{a}$ required so we may use $\mathcal{L}_\zeta \mathcal{L}_\eta \mathcal{L}_\tau \mathcal{L}_z$. The outcome is that we must add to (8.4).

$$\bar{a} \left\{ \mathcal{X}_0(0;z)(-2bs) + 2\int_0^b (b-\xi)\, \mathcal{X}_0(-\xi,z)\, \mathrm{sgn}(\xi-s)\, d\xi \right.$$
$$\left. - \mathcal{X}_1^*(0;z)\, \sigma b^2 s + \sigma \int_0^b \mathcal{X}_1^*(-\xi,z)(b-\xi)^2 \mathrm{sgn}(\xi-s)\, d\xi \right\}$$
when $z = -s < 0$.

and

$$\bar{a} \left\{ \mathcal{X}_0(0;z).\, 2bz + 2\int_0^b (b-\xi)\, \mathcal{X}_0(-\xi,z) + 4\int_0^z (b+\zeta)\, \mathcal{X}_0(\zeta;z).d\zeta \right.$$
$$\left. + \mathcal{X}_1^*(0,z)\, \sigma b^2 z + \sigma \int_0^b \mathcal{X}_1^*(-\xi,z)(b-\xi)^2 d\xi + \sigma \int_0^z \mathcal{X}_1^*(\zeta,z)\, 2(b+\zeta)^2 d\zeta \right\}$$
when $z > 0$. 8.5

The two expressions are different forms of the same function of $z$, and they agree with the particular cases (7.3), (7.7) obtained using the different order $\mathcal{L}_\zeta \mathcal{L}_\tau \mathcal{L}_\eta \mathcal{L}_z$.

Equations (8.4) and (8.5) give the velocity for a delta wing of diamond spanwise section, but fairly general chordwise section: there are limitations in that we assume $\mathcal{X}_{0\zeta}' = \frac{d}{d\zeta}\mathcal{X}_{0\zeta}$ and $\mathcal{X}_{1\zeta}^{*\prime} = \frac{d}{d\zeta}(\mathcal{X}_{1\zeta}^*)$ are finite.

A delta wing with parabolic arc spanwise section

$$\mathcal{X}(\zeta,\eta) = \mathcal{X}_{0\zeta} + \eta^2 \mathcal{X}_{2\zeta} \qquad 8.6$$

may be treated in a similar manner. The resulting expression for the velocity is .

$$
\begin{aligned}
\frac{2\pi a}{\tau}\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau} = {} & 2\bar{a}\Big\{ \mathcal{X}_{0\zeta}(z)\log\frac{\bar{a}^2[(b+z)^2-x'^2]}{b+z} \\
& + \mathcal{X}'_{0\zeta}(z)\Big((b+z)\log\frac{\bar{a}^2[(b+z)^2-x'^2]}{b+z} + x'\log\frac{b+z+x'}{b+z-x'} + 2z\Big) \\
& + \mathcal{X}_{2\zeta}(z)\,\sigma^2\Big((b+z)^2\log\frac{\bar{a}^2[(b+z)^2-x'^2]}{b+z} + \tfrac{1}{2}(3b^2+10bz+6z^2)\Big) \\
& + \mathcal{X}'_{2\zeta}(z)\,\frac{\sigma^2}{3}\Big((b+z)^3\log\frac{\bar{a}^2[(b+z)^2-x'^2]}{b+z} + x'^3\log\frac{b+z+x'}{b+z-x'} - 2x'^2(b+z) \\
& \qquad + \big(\tfrac{13}{6}b^3 + \tfrac{25}{2}b^2z + 14bz^2 + 5z^3\big)\Big)\Big\} \\
& + \bar{a}\Big\{ \mathfrak{X}_0(0,z)\cdot 2bz + 2\int_0^b \mathfrak{X}_0(-\xi,z)(b-\xi)\,d\xi + 4\int_0^z \mathfrak{X}_0(\zeta,z)(b+\zeta)\,d\zeta \\
& + 2\sigma^2\frac{zb^3}{3}\mathfrak{X}_2(0,z) + \frac{2}{3}\int_0^b \mathfrak{X}_2(-\xi,z)(b-\xi)^3\,d\xi + \frac{4}{3}\int_0^z \mathfrak{X}_2(\zeta,z)(b+\zeta)^3\,d\zeta \Big\}. \qquad \underline{8.7}
\end{aligned}
$$

The equation (8.7) is valid for all $z$ on the airfoil, and it checks with the expression obtained using the analysis for $z < 0$ .

As the results (8.4) - (8.6) apply for small $\tau$ it seems natural to relate them to the transonic equivalence rule [8] which applies to the non-linear transonic small disturbance equation and is valid for $\tau^{1/3}\sigma$ small.

At this stage it seems worthwhile to point out that although the expressions for $\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau}$ have been derived for $\tau \to 0$ , our expansions are essentially in terms of $\bar{a} = \frac{a\sigma}{2}$ .

The equivalence rule relates the potential $\varphi(x,y,z)$ to a harmonic cross-flow $\varphi_{2D}(x,y\,;z)$ in the $(x,y)$ plane at $z$ as follows:

$$\varphi(x,y,z) = \varphi_{2D}(x,y\,;z) + g(z)$$

and it further asserts that $g(z)$ is the same function for all slender bodies.

We first calculate the respective cross-flows. The boundary value problem is

$$\nabla^2 \varphi_{2D} = 0 \ , \quad \varphi_r \to 0 \quad \text{as} \quad r \to \infty$$

On $\quad y = 0 \qquad \varphi_y = 0 \quad |x| > \sigma(b+z) \ , \quad \varphi_y = \tau \frac{\partial f}{\partial z} \quad |x| < \sigma(b+z)$

Thus

$$\frac{2\pi}{\tau} \left.\frac{\partial \varphi_{2D}}{\partial z}\right|_{y=0} = \frac{\partial}{\partial z} \int_{-\sigma(b+z)}^{\sigma(b+z)} \mathcal{X}(z,\eta) \log|x-\eta| . d\eta \ .$$

For the diamond cross-section

$$\frac{2\pi}{\tau} \left.\frac{\partial \varphi_{2D}}{\partial z}\right|_{y=0} = \sigma\, \mathcal{X}_{0\zeta}(z) \log\left[\sigma^2(b+z)^2 - x^2\right] + \sigma^2 \mathcal{X}^*_{1\zeta}(b+z) \log\left[\sigma^2(b+z)^2 - x^2\right]$$

$$+ \mathcal{X}'_{0\zeta}(z)\left\{\sigma(b+z)\log\left[\sigma^2(b+z)^2 - x^2\right] - 2\sigma(b+z) - x \log\left|\frac{b+z-x'}{b+z+x'}\right|\right\}$$

$$+ \mathcal{X}^{*\prime}_{1\zeta}(z)\left\{\frac{\sigma^2}{2}(b+z)^2 \log\left[\sigma^2(b+z)^2 - x^2\right] - \frac{x^2}{2}\log\frac{(b+z)^2 - x'^2}{x'^2} - \frac{3}{2}\sigma^2(b+z)^2\right\} \qquad \underline{8.8a}$$

Also if $\quad y = \tau\, \ell(z)\left(1 - \frac{|x|}{\sigma(b+z)}\right) \quad$ gives the shape of the airfoil ,

$$\mathcal{X}_{0\zeta} = 2\, \ell'(z) \qquad , \qquad \sigma\, \mathcal{X}^*_{1\zeta} = -2 \frac{d}{dz}\left(\frac{\ell(z)}{b+z}\right)$$

and the area of cross-section $\qquad S(z) = 2\tau\sigma\, \ell(z) . (b+z) .$

Note that $\ell(z)$ must have a factor $(b+z)$ to ensure finite slope at the tip.

For the parabolic arc section

$$\frac{2\pi}{\tau} \left.\frac{\partial \varphi_{2D}}{\partial z}\right|_{y=0} = \sigma\left(\mathcal{X}_{0\zeta}(z) + (b+z)\,\mathcal{X}'_{0\zeta}(z) + \sigma^2(b+z)^2 \mathcal{X}_{2\zeta}(z) + \frac{\sigma}{3}(b+z)^3 \mathcal{X}'_{2\zeta}(z)\right) \log\left[\sigma^2(b+z)^2 - x^2\right]$$

$$+ \mathcal{X}'_{0\zeta}(z) . \left(-2\sigma(b+z) - x \log\frac{b+z-x'}{b+z+x'}\right)$$

$$+ \mathcal{X}'_{2\zeta}(z)\left(-\frac{x^3}{3}\log\frac{b+z-x'}{b+z+x'} - \frac{1}{3}\left(\frac{2}{3}\sigma^3(b+z)^3 + x^2\sigma\, 2(b+z)\right)\right) . \qquad \underline{8.8b}$$

Also if $y = \tau\, \ell(z) \left(1 - \frac{x^2}{\sigma^2 (b+z)^2}\right)$ gives the shape of the airfoil

$$\mathcal{X}_{0z} = 2\, \ell'(z) \quad , \qquad \sigma^2 \mathcal{X}_{2z} = -2 \frac{d}{dz}\left(\frac{\ell(z)}{(b+z)^2}\right)$$

and the area of cross-section $S(z) = \frac{8}{3}\sigma\tau\, \ell(z)\cdot(b+z)$.

$\ell(z)$ must now have a factor $(b+z)^2$ to ensure finite slope at the tip.

Now integrating by parts

$$bz\left(2\,\mathcal{H}_0(0;z) + \sigma b\, \mathcal{H}_1^*(0;z)\right) + \int_0^b (b-\xi)\left(\mathcal{H}_0(-\xi,z) + \sigma(b-\xi)\,\mathcal{H}_1^*(-\xi,z)\right) d\xi$$
$$+ 2\int_0^z (b+\zeta)\left(2\,\mathcal{H}_0(\zeta,z) + \sigma(b+\zeta)\,\mathcal{H}_1^*(\zeta,z)\right) d\zeta$$

$$= -\frac{1}{\sigma\tau}\int_0^b \frac{S''(-\xi) - S''(z)}{z+\xi} - \frac{2}{\sigma\tau}\int_0^z \frac{S''(\zeta) - S''(z)}{z-\zeta}\cdot d\zeta$$

$$-4(b+2z)\,\mathcal{X}_{0z}' - 2(b+2z)\,\sigma\,\mathcal{X}_{1z}^* - 2(b^2+4bz+2z^2)\,\sigma\,\mathcal{X}_{1z}^{*\prime},$$

and

$$2bz\left(\mathcal{H}_0(0,z) + \tfrac{1}{3}\sigma^2 b^2\,\mathcal{H}_2(0,z)\right) + 2\int_0^b \left\{(b-\xi)\,\mathcal{H}_0(-\xi,z) + \frac{(b-\xi)^3}{3}\sigma^2\,\mathcal{H}_2(-\xi,z)\right\} d\xi$$
$$+ 4\int_0^z \left\{(b+\zeta)\,\mathcal{H}_0(\zeta,z) + \frac{(b+\zeta)^3}{3}\sigma^2\,\mathcal{H}_2(\zeta,z)\right\} d\zeta$$

$$= -\frac{1}{\sigma\tau}\int_0^b \frac{S''(-\xi) - S''(z)}{z+\xi} - \frac{2}{\sigma\tau}\int_0^z \frac{S''(\zeta) - S''(z)}{z-\zeta}\, d\zeta + 4\,\mathcal{X}_{0z}'\cdot(b+z)$$

$$+ \sigma^2 \mathcal{X}_{2z}\,(3b^2 + 10bz + 6z^2) + \frac{2}{3}\sigma^2\,\mathcal{X}_{2z}'\left(\frac{2}{3}(b+z)^3 + \frac{3}{2}b^3 + \frac{19}{2}b^2 z + 14bz^2 + \frac{11}{3}z^3\right).$$

If we now insert these results, and (8.8) in the expressions (8.4), (8.5), (8.7) we obtain

$$\left.\frac{\partial\varphi}{\partial z}\right|_{y=\tau} = \left.\frac{\partial\varphi_{2D}}{\partial z}\right|_{y=0} + \frac{\partial\varphi^{\circ}}{\partial z} - \frac{\partial\varphi^{\circ}_{2D}}{\partial z} + \frac{\tau\sigma}{4\pi}\mathcal{E} \qquad \underline{8.9}$$

for both cross-sections. In this equation,

$$\frac{\partial\varphi^{\circ}_{2D}}{\partial z} = \frac{1}{2\pi} S''(x) \log \tau \quad ,$$

$$\frac{\partial\varphi^{\circ}}{\partial z} = \frac{1}{4\pi}\left\{ S''(z) \log\frac{a^2\tau^2}{4(b+z)} - \int_0^b \frac{S''(-\xi)-S''(z)}{\xi+z}\,d\xi - 2\int_0^z \frac{S''(\zeta)-S''(z)}{\zeta-z}\,d\zeta \right\} \qquad \underline{8.10},$$

and

for case (a), diamond

$$\mathcal{E}_a = \sigma X^{*\prime}_{1z}\left(\frac{5}{2}b^2 - 5bz + 3z^2\right) \quad \text{with} \quad \sigma X^{*\prime}_{1z} = -2\frac{d^2}{dz^2}\left(\frac{\mathcal{E}(z)}{b+z}\right) \qquad \underline{8.11a}$$

for case (b), parabolic arc

$$\mathcal{E}_b = \sigma^2 X'_{2z}\ \frac{2}{3}\left(\frac{2}{3}b^3 + \frac{4}{3}z^3\right) \quad \text{with} \quad \sigma^2 X'_{2z} = -2\frac{d^2}{dz^2}\left(\frac{\mathcal{E}(z)}{(b+z)^2}\right) \qquad \underline{8.11b}$$

Now (8.10) is the solution of (2.1) for axisymmetric flow derived by Cole and Royce [6] . Consequently if $\mathcal{E} = 0$ , as it does for a wedge or parabolic arc chordwise section, with diamond spanwise section, (8.9) corresponds to the equivalence rule developed for the nonlinear transonic small disturbance equation. However, in general $\mathcal{E} \neq 0$ , and as the terms in (8.4) and (8.7) are each derived by two substantially independent ways, which check, one is prompted to examine the proof of the equivalence rule to see why it does not carry over to the equation (2.1). Before embarking on this, we note that the choice of $a$ and $b$ in (8.9) has yet to be made. If we choose $b$ such that $\left.\frac{\partial\varphi}{\partial z}\right|_{\substack{y=0\\z=0}} = 0$ which is the consistent procedure, then we lose transonic similarity because $b$ will depend on $a$ and hence on $\tau$. An alternative is to take the sonic point where $S'' = 0$, corresponding to the Cole and Royce procedure for the equivalent axisymmetric body. A reasonable choice for $a^2$ would then be

$$a^2 = (\gamma+1) \left.\frac{\partial^2\varphi}{\partial z^2}\right|_{\substack{y=0\\z=0}}$$

which implies that $a\sigma$ is a function of $\sigma^3(\gamma+1)\tau$. The Cole and Royce choice of $a^2$ does not seem appropriate as it is associated with the flow on the axisymmetric body, and not with the flow at some distance where one might expect some relation to the planar case. The suggested choice leads to an expression for the reduced pressure distribution of the form

$$\tilde{C}_p = \frac{(\gamma+1)^{1/3}}{\tau^{2/3}} \left( -2 \left.\frac{\partial\varphi}{\partial z}\right|_{y=0} \right) = \bar{\sigma} f(z;\bar{\sigma}) + o(\bar{\sigma})$$

where $\bar{\sigma} = (\gamma+1)^{1/3}\tau^{1/3}\sigma$, which is consistent with the similarity rule. With the Cole and Royce choice of $b$ one would expect to reproduce transonic equivalence, apart from a different value of $a^2$. Why, then, does it fail to appear?

One of the main difficulties of transonic flow is to assess the cumulative effect of the small non-linear terms at large distances from the body. In their accounts of the equivalence rule, Oswatitsch [9] and Guderley [1] use physical arguments.

Heaslet and Spreiter [8] proposed to give the rule a better basis. Retaining the nonlinear term, and taking account of shocks, they produce an integral equation (equation (2.7) of Chapter I). Then, by using an iterative method, similar to that used in linearised theory, they produce a term which is expected to be the start of an asymptotic expansion. But does this straightforward iteration procedure give a solution of the non-linear equation? To set up the transonic small disturbance equation it is necessary to scale the co-ordinates, Cole and Messiter [10], and it seems likely that this should be done when assessing the contribution of sources over an infinite region of integration. The problem of assessing the terms in the integral equation of [8] is then rather more difficult.

An illustration of the possible mechanism arises if we apply the Heaslet and Spreiter method to the model equation (2.1). The integral equation is obtained as before, and we have to assess the value of the integral. For simplicity consider a rectangular planform, wedge chordwise section, and rectangular spanwise section;

$$\varphi_W = \tau \int_0^1 d\zeta \int_{-\sigma}^{\sigma} d\eta \; \varphi(E), \quad \text{with } \varphi(E) \text{ given by } (2.3).$$

The equivalent-body potential is $\varphi_B = 2\sigma\tau \int_0^1 d\zeta\, \varphi(E).$ , with $r$ now simply $\sqrt{x^2+y^2}$ . The integral term of the equation is an integral over the region of the plane $z$ = constant, external to the respective bodies:

$$T = a^2 z \iint \frac{\partial^2}{\partial z^2}(\varphi_W - \varphi_B)\, \frac{1}{2} \log\left(\tau^2 + \tau_1^2 - 2\tau\tau_1 \cos(\theta - \theta_1)\right)\tau_1\, dr_1\, d\theta_1$$

where $(\tau, \theta, z)$ are cylindrical co-ordinates corresponding to $(x, y, z)$ . This is required to be smaller than the terms retained, i.e. $\ll \sigma\tau$ . Consider $\varphi_W - \varphi_B$ for large $\tau_1$ ;

$$\varphi_W - \varphi_B$$

$$= 2\tau\sigma \int_0^1 \frac{d\zeta}{4\zeta}\, \frac{1}{2\sigma} \int_{-\sigma}^{\sigma} \frac{a^2 \varepsilon}{\left\{\left(z + \zeta - \frac{a^2\tau_1^2}{4}\right)^2 - 4\zeta z\right\}^{1/2}} \left(1 + \frac{a^2\varepsilon\left(z + \zeta - \frac{a^2 r^2}{4}\right)}{\left\{\left(z + \zeta - \frac{a^2 r^2}{4}\right)^2 - 4\zeta z\right\}}\right) + \cdots d\eta$$

where $\varepsilon = -2\eta x_1 + \eta^2$ . Then as $\tau_1 \to \infty$ there is a term $\propto 1/a^2$ , and this gives the possibility that $T \propto \sigma\tau$ which would alter the iteration.

It is, of course, no proof that the range of validity suggested in [8] is wrong. The model equation does not describe the physical situation at large distances from the body, even if the above analysis were made rigorous.

## § 9. Conclusion

The original aim in developing this method was to provide a relatively simple means of predicting pressure distributions for wings whose aspect ratio, though fairly small, might be beyond the range of validity of the transonic equivalence rule, [8] . Two particular spanwise cross-sections for a delta wing have been considered in detail. Rather surprisingly, for small aspect ratio, the results do not agree with the equivalence rule, and it is far from certain that the disagreement stems from the use of the model equation (2.1). Arguments have been put forward which bring into question previous estimates of the range of validity of the equivalence rule.

In an experimental study of the equivalence rule applied to elliptic cone-cylinders Page [11] obtained small differences between measured values and equivalence predictions, but there was no striking variation with chordwise station. However, consideration of this one particular shape seems insufficient to decide the issue, especially as special conditions apply at the shoulder and the tip.

There is a lack of published experimental work for direct comparison of the results of the present method. This is understandable if one assumes the transonic equivalence rule has a fairly large range of validity. However, it should be noted that the equivalent-body of revolution may have a shape outside the range for which axisymmetric theories have been checked against experiment. This factor considerably reduces the value of indirect comparison, even assuming the airfoil lies within the range of validity of the equivalence rule.

Although there are a number of theories which in principle can be extended to three dimensional flow, published work seems limited to that of Spreiter and Alksne [12] . This method is quite involved, except in very particular cases; for the elliptic cone cylinder it disagrees with the equivalence rule results, and in the other cases there is a lack of comparison with experiment. There is no case suitable for comparison with the present method.

A further point of discussion is the relation of the three-dimensional theory to the two extreme cases, two-dimensional flow, and axisymmetric flow. Starting with the model equation (2.1), there are two stages of approximation. The first is the assumption $\tau \ll 1$ , $\sigma \gg \tau$ required to develop the planar, or thin wing, theory, and it excludes the possibility of recovering the axisymmetric case for which $\sigma \sim \tau$ . The second stage occurs much later when we assume $a\sigma \to 0$ as $\tau \to 0$ . Immediately prior to this we may let $\sigma \to \infty$ and recover the two dimensional case [5] . At this stage we could also employ computational methods for finite values of $a\sigma$ , and allow small variation from the delta planform by modifying $\sigma$ . Different planforms could be treated using similar analysis. However for airfoils intermediate between the two extreme cases, the limitations of the model equation would be pronounced.

As a final comment, one would expect the assumption that the transonic equation changes type at the plane $z = 0$ , imposes severe restrictions on the accuracy of the method, and even for small aspect ratio progress probably depends on the refinement of the model equation, rather than the calculation of further terms of the asymptotic expansion.

REFERENCES

[1] Guderley, K. G. — Theory of Transonic Flow Pergamon (1962)

[2] Dorodnicyn, A. A. — Advances in Aero, Sci. Vol II 833 - 45 Pergamon (1959)

[3] Chushkin, P. I. — P.M.M. (1957) 21 253 - 60

[4] Oswatitsch, K. (editor) — Symposium Transsonicum, Springer (1964).

[5] Evans, T. — Proc. Camb. Phil. Soc. (1965) 61 573 - 593 .

[6] Cole, J. D. & Royce, W. W. — Proc. VI th Midwest. Conf. Fluid Mech. Austin, Texas (1959)

[7] Spreiter, J. R. — J. Aero. Sci. (1959) 26 465 - 486

[8] Heaslet, M. A. & Spreiter, J. R. — NACA Report 1318 (1957) .

[9] Oswatitsch, K. — Advances in Applied Mech. IV, 154 - 269. Academic Press (1960).

[10] Cole, J. D. & Messiter, A. F. — Z.A.M.P. 8 (1957) .

[11] Page, W. A. — NACA TN 4233 (1958) .

[12] Spreiter, J. R. & Alksne, A. Y. — NASA R-88 (1961) .

# CHAPTER IV

## THE PARABOLIC EQUATION APPROXIMATION IN TRANSONIC FLOW

In this chapter, we return to the first type of regional linearisation approximation mentioned in Chapter I. The aim is to determine the reason for the quite good agreement with experiment of Spreiter's 'local linearisation' method, (I §4). We take up the suggestion made in Chapter II §1, and reduce a fairly general form of the parabolic approximation, to a particular boundary-value problem encountered by Maeder and Thommen in the method described in Chapter I §4. This provides a mathematical basis from which to assess Spreiter's method.

## 1. INTRODUCTION

Among the various methods for predicting the surface pressure on two-dimensional airfoils for near sonic flow, the method of Spreiter and Alksne [1] seems at first sight the most attractive. For typical non-lifting airfoils it gives the best agreement with experiment, but unfortunately it has the serious drawback that its mathematical basis is suspect.

The method may be described as follows. The transonic small disturbance equation for sonic flow

$$\Phi_{yy} = (\gamma + 1)\, \Phi_z\, \Phi_{zz} \tag{1.1}$$

is approximated by the parabolic equation

$$\varphi_{yy} = \lambda\, \varphi_z \tag{1.2}$$

$\lambda$ being a constant to be chosen. The boundary conditions are taken as:

$$\varphi_z\,,\ \varphi_y \to 0 \quad \text{as} \quad z \to -\infty\,,\ \text{and as } y \to \infty \qquad ,$$

$$\varphi_y \to \begin{cases} F(z) & 0 \leqslant z \leqslant 2 \\ 0 & z < 0 \end{cases} \qquad \text{as } y \to 0 \qquad . \tag{1.3}$$

Here $(z,y)$ is a coordinate system, fixed relative to the leading edge of the airfoil $(0,0)$, with $z$ positive in the downstream direction, and parallel to the undisturbed mainstream. The flow is assumed to be inviscid, and the velocity at any point of the fluid is derived from the potential

$U_\infty(z + \Phi)$, where $U_\infty$ is the speed of the airfoil relative to the undisturbed fluid upstream. All lengths have been made non-dimensional with l, where 2l is the length of the airfoil, and F(z) is the slope of the airfoil. The parabolic nature of equation (1.2) allows no upstream influence, so only the region $z \leqslant 2$ is of interest, and furthermore it means that the flow in the region $y \geqslant 0$ is independent of that for $y \leqslant 0$. From this independence, it follows that one may treat lifting airfoils by the theory developed for symmetrical airfoils as noticed by Randall [2], and this is the reason for the boundary condition $\varphi_y \to 0$ for $z < 0$. However, such a condition is physically unrealistic for non-symmetrical airfoils, and one hopes to be able to relax it in later work.

Suppose $\varphi(z,y;\lambda)$ is the solution of equation (1.2) satisfying the boundary conditions (1.3), then there remains the choice of a suitable value of the constant $\lambda$. Maeder and Thommen [3] take its value as $(\gamma + 1)\varphi_{zz}(z^*,0;\lambda)$, where asterisk * denotes the value at the sonic point, determined from $\varphi_z(z^*,0;\lambda) = 0$. In a sense, this is the best choice and one cannot obtain good agreement with experiment simply by choosing another value. To overcome this difficulty, Spreiter [1] takes $\lambda$ as $(\gamma + 1)\ \varphi_{zz}(z,0;\lambda)$, and then uses this in the formal expression for $\varphi_z(y = 0)$ to obtain a simple differential equation for $\varphi_z(y = 0)$. Thus $\lambda$ is held constant until a formal solution is obtained, but is then allowed to vary with z.

The justification of this technique consists of rewriting equation (1.1) as

$$\Phi_{yy} - \lambda \Phi_z = \left\{(\gamma + 1) \Phi_{zz} - \lambda\right\} \Phi_z \tag{1.4}$$

and then applying generalised forms of Green's theorem to obtain

$$\Phi = - 2\lambda^{-1} \int_0^z F(\xi) \sigma(\zeta = 0) \, .d\xi - 1/\lambda \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{z} \sigma \, f.d\xi \quad , \tag{1.5}$$

where $\sigma = \lambda^{1/2} \left\{4\pi(z - \xi)\right\}^{-1/2} \exp[-\lambda(y - \zeta)^2 / \left\{4(z - \xi)\right\}]$

and $f = \left\{(\gamma + 1) \Phi_{zz} - \lambda\right\} \Phi_z$ .

Apart from neglecting possible contributions from shock waves, equation (1.5) follows from equation (1.1) without approximation. The argument is now that at any particular station z, by choosing $\lambda = (\gamma + 1) \Phi_{zz}(y = 0)$ we may neglect the double integral, and still obtain a reasonable value for $\Phi_z(y = 0)$, a device which is most successful when applied to the z differentiated form of equation (1.5). Applying this argument at each station z, one obtains the differential equation for $\Phi_z(y = 0)$ mentioned in the description of Spreiter's technique.

Although this justification avoids the obvious inconsistency displayed in the initial description, the validity of the assumption that the contribution from the double integral may be neglected, is mathematically no more justifiable than holding $\lambda$ constant and then varying it when convenient. It would be more satisfying if $\lambda$ could be treated as a function of z throughout the analysis. Then the only issue would be whether equation (1.2) with $\lambda$ a function of z, provides a reasonable model of the physical situation, and this could be assessed by comparison with experiment. Such a treatment is presented below.

## 2. GENERAL ANALYSIS

We consider only symmetrical airfoils, and formulate the problem for $-\infty < y < \infty$ so that we may take over the analysis of Maeder and Thommen. We take as the equation modelling (1.1)

$$\varphi_{yy} = K(z)\,\varphi_z \tag{2.1},$$

where $K(z) > 0$, so that we assume accelerated flow in the region under consideration. The boundary conditions are :

$$\varphi_z\,,\ \varphi_y \to 0 \quad \text{as } z \to -\infty\ , \quad \text{and also as } |\,y\,| \to \infty\ ,$$

$$\varphi_y \to F(z) \qquad \text{as } y \to 0\ , \text{ for } 0 \leqslant z \leqslant 2\ . \tag{2.2}$$

$K(z)$ is to be determined by requiring

$$K(z) = (\gamma + 1)\, u'(z) \tag{2.3}$$

where $u(z)$ denotes the perturbation velocity at the airfoil surface $\varphi_z\big|_{y=0}$, and prime denotes the derivative.

If we take $z$ as a function of some variable $\xi$ , then equation (2.1) may be written

$$\varphi_{yy} = \varphi_\xi \tag{2.4}$$

provided $\dfrac{dz}{d\xi} = K(z)$. For monotonic distributions $u(z)$, $K(z) \neq 0$, and the condition inverts to give

$$\xi(z) = \int_0^z \frac{dt}{K(t)}\ . \tag{2.5}$$

The boundary value problem posed by equation (2.4), together with the boundary conditions (2.2) written in terms of $\xi$ , is that solved by Maeder and Thommen [3], in the particular case of a sonic free stream. Thus

$$\varphi \Big|_{y\,=\,0} = - \frac{1}{\pi^{1/2}} \int_0^{\xi} F(z(t)) \left\{ \xi - t \right\}^{-1/2} dt$$

and

$$\frac{d\varphi}{d\xi}\Big|_{y=0} = -\pi^{-1/2} \left\{ F(0)\ \xi^{-1/2} + \int_0^{\xi} F'(z(t)) \left\{ \xi - t \right\}^{-1/2} z'(t)dt = \frac{\partial\varphi}{\partial z}\Big|_{y=0} \left(\frac{dz}{d\xi}\right). \right. \qquad (2.6)$$

In the particular case $\frac{dz}{d\xi} = \lambda$ this reduces to the result given in [3]. If we now use the condition (2.3) in equation (2.6), and write the whole in terms of $u(z)$, an integro-differential equation for $u$ results. Alternatively, we may regard $z$ as a function of $\xi$ . From equation (2.3)

$$(\gamma + 1)u = \int_{\xi^*}^{\xi} \left(\frac{dz}{d\xi}\right)^2 d\xi \qquad (2.7)$$

and substituting in equation (2.6) we obtain

$$\left(\frac{dz}{d\xi}\right) \int_{\xi^*}^{\xi} \left(\frac{dz}{d\xi}\right)^2 .d\xi = -(\gamma + 1)\pi^{-1/2} \left\{ F(0).\,\xi^{-1/2} + \int_0^{\xi} F'(z(t)) \left\{\xi - t\right\}^{1/2} z'(t)\ dt \right\}.$$

Writing $\xi = c_1 \eta$ , $\frac{dz}{d\xi} = c_2 g'(\eta)$ it is possible to absorb any constant factor on the right hand side, and obtain the equation in the form

$$g'(\eta) \int_{\eta^*}^{\eta} [g'(\theta)]^2 .d\theta = - \frac{\Omega(0)}{\eta^{1/2}} - \int_0^{\eta} \frac{\Omega'(g(t))}{\{\eta - t\}^{1/2}}\, g'(t).dt \quad , \qquad (2.8)$$

where we have taken $F(z) = \tau\,\Omega(z)$. In this case $c_2^{3/2} = (\gamma + 1)\tau\,\pi^{-1/2}$ and $c_1 c_2 = 1$. For equations such as (2.8) questions of existence and uniqueness have yet to be settled. It is proposed to obtain numerical solutions by an iterative method similar to that used in the proof of existence for a linear Volterra equation.

For later reference we express $u$ and $z$ in terms of $g(\eta)$ :

$$(\gamma + 1)u = c_2 \int_{\eta^*}^{\eta} [g'(t)]^2 .dt \quad , \quad z = g(\eta) \quad . \qquad (2.9)$$

## 3. THE WEDGE AND PARABOLIC ARC AIRFOILS

First consider the case of the wedge,† for which $F(z) = F(0)$. The solution may be obtained directly from the integro-differential equation for u, using the conditions that the sonic point is at the shoulder ($u = 0$ at $z = 1$) and the stagnation point is at the leading edge. ($u = -U_\infty$ at $z = 0$). However, to lead on to the method in the general case, we proceed from equation (2.8). Putting $g'(\eta) = H(\eta)/\eta^{1/2}$ it may be easily shown that $H(\eta) = -\left\{3 \log (\eta/\eta^*)\right\}^{-1/3}$, and if we denote $\int_{\eta^*}^{\eta} [g'(t)]^2 \, dt$ by $Y(\eta)$, then $Y = \left\{3 \log (\eta/\eta^*)\right\}^{1/3}$. Using the boundary conditions, it soon follows that

$$(1 - z) = \frac{\int_0^{-Y} \exp(-v^3/6) v \, dv}{\int_0^{\infty} \exp(-v^3/6) v \, dv}, \quad (0 < z < 1). \qquad (3.1)$$

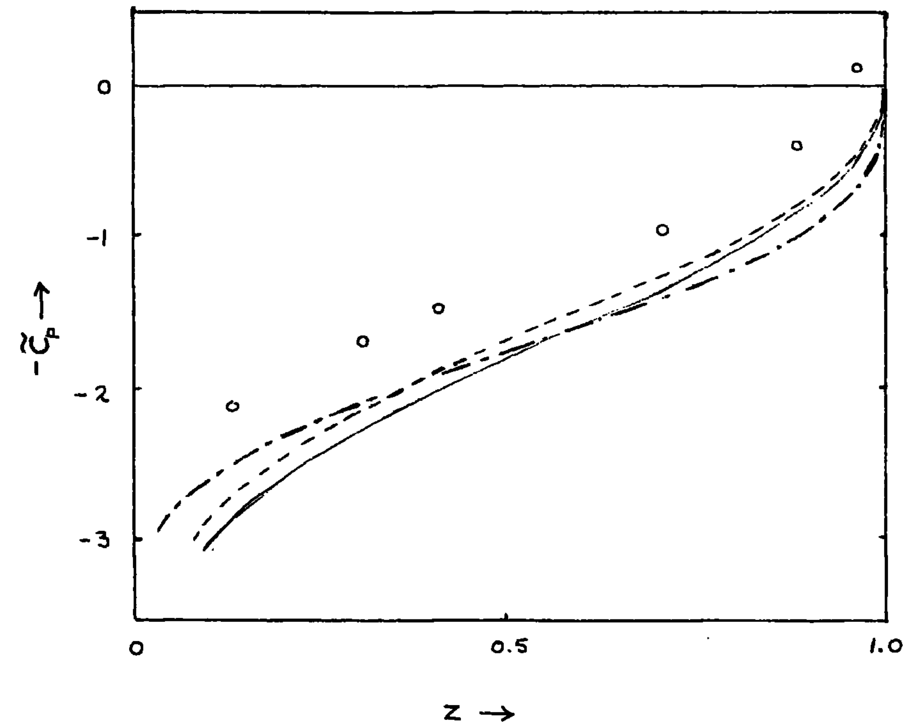
This relation has been derived by Thommen (private communication) but it was regarded as the solution to a different problem. The upper limit of integration $\infty$ is strictly only valid as the thickness ratio $\tau \to 0$, and Y, which is negative, is related to u by equation (2.9). The right hand side of equation (3.1) may be expressed in terms of the incomplete gamma function. Our interest lies in the pressure distribution p(z), and for ease of comparison with other work, we use the conventional reduced pressure coefficient $\tilde{C}_p = (\gamma + 1)^{1/3} \tau^{-2/3} C_p$, where $C_p = p/\frac{1}{2}\rho_\infty U^2_\infty$. Now $\tilde{C}_p = -2u(\gamma+1)^{1/3} \tau^{-2/3}$ $= -2\pi^{-1/3} Y$, so that equation (3.1) becomes

$$(1 - z) = I\left(\frac{\pi}{48} \tilde{C}_p^{\,3}, -\frac{1}{3}\right) \qquad (3.2)$$

where I denotes the I(x,p) function of Pearson's Tables [4]. The plot of $\tilde{C}_p$ against z is displayed in Fig 1.

† The wedge is regarded as the forward half of an airfoil, length $2\ell$.

Figure 1 . $\tilde{C}_p(z)$ for a thin wedge profile .




—·—·—·— Spreiter [1]

- - - - - - Guderley & Yoshihara [6]

———— Present method .

o Experiment , Knechtel [13] , $\tau = 0{\cdot}079$

Now consider the parabolic arc airfoil for which $F(z) = 2\tau(1 - z)$. Equation (2.8) reduces to

$$g' \int_{\eta^*}^{\eta} [g'(\theta)]^2 . d\theta + {}^1/\eta^{1/2} = \int_0^{\eta} \frac{g'(\theta)}{\{\eta - \theta\}^{1/2}} . d\theta \tag{3.3}$$

where in this case $c_2^{\ 3} = 2^2(\gamma + 1)^2\tau^2/\pi$. As mentioned already, we proceed by iteration, regarding the right hand side of equation (3.3) as a perturbation term. The successive approximations $\{g_n\}$ are given by

$$g'_{n+1} \int_{\eta_n^*}^{\eta} [g'_{n+1}(\theta)]^2 . d\theta = -\eta^{-1/2} + \int_0^{\eta} \frac{g'_n(\theta)}{\{\eta - \theta\}^{1/2}} \, d\theta \quad , \tag{3.4}$$

and $\{\eta^*_n\}$ by

$$-\{\eta^*\}^{-1/2} + \int_0^{\eta^*_n} g'_n(\theta) \ \{\eta - \theta\}^{-1/2} \, d\theta \quad . \tag{3.5}$$

It might be thought that the wedge solution would provide a suitable $g_o$, but the position of the sonic point precludes this. It is better to use a value of $g'_o(\eta)$ corresponding to the Maeder and Thommen approximation [3], which is simply $g'_o(\eta) = 2$. Then, from equation (3.4)

$$g'_1(\eta) \int_{\eta^*}^{\eta} g'^{\,2}_1(\theta), d\theta = -\{\eta\}^{-1/2} + \int_0^{\eta} \beta \{\eta - \theta\}^{-1/2} . d\theta$$

where $\beta = 2$. In terms of $\{Y_n\}$

$$\left(\frac{dY_1}{d\eta}\right)^{1/2} Y_1 = -\eta^{-1/2} + 2\beta\,\eta^{1/2} \quad .$$

Integrating and writing $w = \beta\eta$,

$$Y_1^{\ 3}/3 = \log\left(\frac{w}{w^*}\right) - 4w + 2w^2 + 4w^* - 2w^{*2} \quad .$$

Since $\beta = 2$, $w^* = 1/2$ and so

$$g'_1(\eta) = (4\eta - 1)\eta^{-1/2}\left\{3(\log 4\eta - 8\eta + 8\eta^2 + 3/2)\right\}^{-1/3}$$

Note the formal resemblance of $Y_1$ to Spreiter's expression for $\tilde{C}_p$ on the same profile, equation (53) of [1]. If we adopt the unsystematic approach of using $g'_o$ rather than $g'_1$ to obtain z in terms of $\eta$, then we recover Spreiter's expression. A similar relation exists for the case of the wedge.

To continue the iteration, we must resort to numerical integration, and for this it is convenient to work in terms of $H_n(\eta) = \eta^{1/2} g'_n(\eta)$. Equation (3.4) gives

$$H_{n+1}(\eta)\int_{\eta^*}^{\eta} H^2_{n+1}(\theta).\frac{d\theta}{\theta} = -1 + \eta^{1/2}\int_o^{\pi/2} 2H_n(\eta\sin^2\alpha)d\alpha$$

Denoting $\int_o^{\pi/2} 2H_n(\eta\sin^2\alpha).d\alpha$ by $Q_n(\eta)$ we have

$$Y^3_{n+1} = \int_{\eta^*_n}^{\eta}\left\{-1 + \theta^{1/2}Q_n(\theta)\right\}^2 \frac{d\theta}{\theta} = \log(\eta/\eta^*) + P_n(\eta) \tag{3.6}$$

where

$$P_n = \int_{\eta^*_n}^{\eta}\left\{Q_n{}^2(\theta) - 2\theta^{-1/2}Q_n(\theta)\right\} d\theta \quad . \tag{3.7}$$

In terms of Y and Q,

$$H_{n+1} = \left(\frac{dY_{n+1}}{d\eta}\right)^{1/2}\eta^{1/2} = \left\{-1 + \eta^{1/2}Q_n(\eta)\right\} / Y_{n+1} \tag{3.8}$$

This is the basic formula for computing. At each stage the value of $\zeta^*$ must be determined from

$$-1 + (\zeta_n^*)^{1/2} Q_n(\zeta^*) = 0 \quad . \qquad (3.9)$$

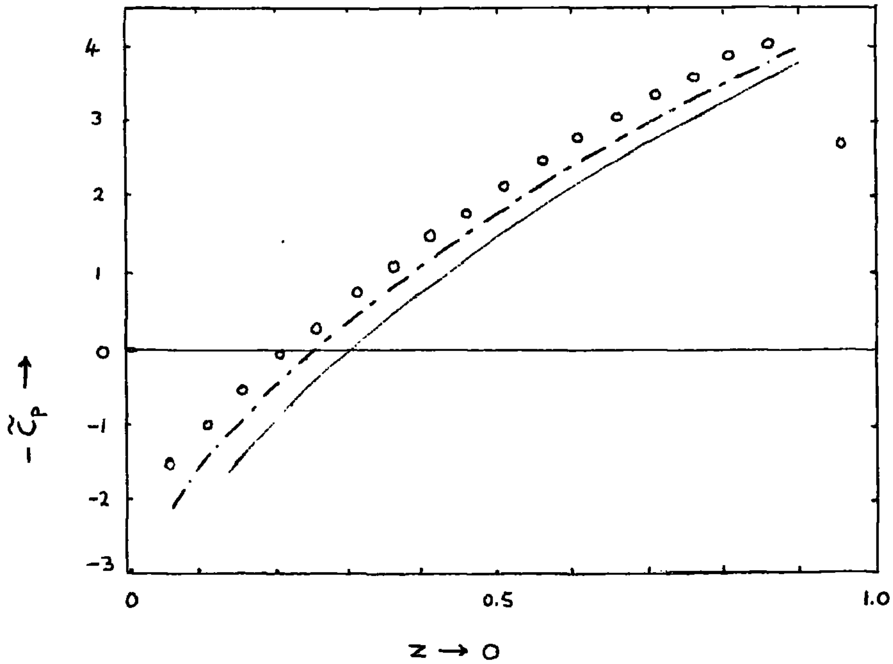
It is impracticable to generate all approximations previous to the $n^{th}$ to obtain each value of $H_n$, and the scheme adopted was to interpolate in tables of $Q_n$, $P_n$ and use equation (3.8) to obtain the $H_{n+1}$ required to evaluate $H_{n+2}$. The number of stations of $\zeta$ employed was varied as a check, and the results are expected to be accurate to 1°/o . The reduced pressure distribution is given by $C_p = -2.2^{2/3}\pi^{-1/3}Y$ and the plot of $\tilde{C}_p$ against z is presented in Fig 2.

TABLE 1

The Wedge Profile

| z | $\tilde{C}_p$(Spreiter) | $\tilde{C}_p$(Present) |
|---|---|---|
| 0.092 | 2.63 | 3.00 |
| 0.147 | 2.45 | 2.75 |
| 0.219 | 2.26 | 2.50 |
| 0.307 | 2.08 | 2.25 |
| 0.408 | 1.90 | 2.00 |
| 0.518 | 1.71 | 1.75 |
| 0.628 | 1.53 | 1.50 |
| 0.732 | 1.34 | 1.25 |
| 0.825 | 1.14 | 1.00 |
| 0.900 | 0.93 | 0.75 |
| 0.955 | 0.71 | 0.50 |
| 0.989 | 0.44 | 0.25 |

Figure 2. $\tilde{C}_p(z)$ for a parabolic arc profile.




—·—·— Spreiter [1].

——— Present method.

o Experiment, Michel et al. [5]
$\tau = 0.1$.

TABLE 2

The Parabolic Arc Profile

| z | $-\tilde{C}_p$(Spreiter) | $-\tilde{C}_p$(Present) |
|---|---|---|
| 0.256 | -1.24 | -1.73 |
| 0.394 | -0.49 | -0.91 |
| 0.614 | 0.46 | 0.12 |
| 0.799 | 1.15 | 0.85 |
| 0.964 | 1.69 | 1.44 |
| 1.118 | 2.16 | 1.93 |
| 1.264 | 2.59 | 2.38 |
| 1.401 | 2.96 | 2.78 |
| 1.533 | 3.31 | 3.14 |
| 1.661 | 3.64 | 3.48 |
| 1.784 | 3.94 | 3.80 |

## 4. OTHER AIRFOIL PROFILES

For general profiles we must return to equation 2.8. In terms of $H_n$ the iterative equation becomes

$$H_{n+1}(\zeta)\int_{\zeta^*}^{\zeta} H_{n+1}^2(\theta)\;\frac{d\theta}{\theta} = -\Omega(0) - \zeta^{1/2}\int_0^{\pi/2}\Omega'[g_n(\zeta\sin^2\alpha)].2H_n(\zeta\sin^2\alpha)d\alpha \tag{4.1}$$

The equations for computation take the same form as for the parabolic arc airfoil (3.7, 3.8, 3.9) but the value of $Q_n(\zeta)$ is now given by

$$Q_n = -\int_0^{\pi/2}\Omega'[g_n(\zeta\sin^2\alpha)].\;2H_n(\zeta\sin^2\alpha).\;d\alpha \tag{4.2}$$

We consider the Michel, Marchaud and Le Gallo airfoils, given by

$$\Omega(z) = \begin{cases} \tilde{A}\,[1 - n(\frac{z}{2})^{n-1}] & \text{maximum thickness aft} \qquad (4.3a) \\ \tilde{A}\,[n(1 - \frac{z}{2})^{n-1} - 1] & \text{maximum thickness forward} \;, \qquad (4.3b) \end{cases}$$

where $\tilde{A} = \dfrac{n^{n/(n-1)}}{2(n - 1)}$ .

If, as for the parabolic arc case, we use a linear initial approximation $g_o'(\eta) = \beta$, we have $Y = \beta(z - z^*) = (\gamma + 1)u/c_2$ , and a suitable value of $\beta$ may be determined by the Maeder and Thommen theory [3].

For the airfoil with maximum thickness aft (4.3a)

$$Y_1^3 = 3\tilde{A}^2 \int_{\eta^*}^{\eta} \left\{ - \frac{1}{\eta^{1/2}} + \frac{n}{2}(n-1) \int_0^{\pi/2} \left(\frac{\beta\eta \sin^2\alpha}{2}\right)^{n-2} . \, 2\beta\,\eta^{1/2} \sin\alpha . d\alpha \right\}^2 d\eta \qquad (4.4)$$

This may be integrated analytically, and using equation (3.9) to determine a value for $\eta^*$ , after some manipulation we obtain

$$Y_1^3 = 3A^2[\log(\eta/\eta^*) - \frac{2\,\Gamma(\frac{1}{2})\,\Gamma(n+1)}{(n-1)\Gamma(n-\frac{1}{2})}\left(\frac{\beta\eta}{2}\right)^{n-1} + \frac{1}{2(n-1)}\left\{\frac{\Gamma(\frac{1}{2})\,\Gamma(n+1)}{\Gamma(n-\frac{1}{2})}\right\}^2 + \frac{3}{2(n-1)}\,] \qquad (4.5)$$

Now $\tilde{C}_p = 2/\pi^{1/3}\, Y_1$ , so that this again is the same as Spreiter's result, if $\beta\eta$ is replaced by z. The systematic approximation to z follows readily using equation (3.8).

For airfoils with maximum thickness forward (4.3b) we obtain results having the same relation to Spreiter's as those for maximum thickness aft. Fortunately equation (3.9) has only one root in the length of the airfoil, so that although the algebra is heavier, the method carries through in a straightforward manner. However, the linear initial approximation is not very suitable, and to obtain accurate values many iterations would be needed.

## 5. DISCUSSION

The principal object of this work was to examine the reasons for the quite good agreement between the theoretical work of Spreiter[1] and the experimental results due to Michel, Marchaud and Le Gallo [5]. By eliminating certain assumptions whose influence could not be assessed, it was hoped that the theoretical method could be put on a sounder basis. The consequence of this approach may be seen in Figs. 1 and 2; the degree of agreement between theory and experiment is less satisfactory, so that the agreement achieved in [1] is to some extent fortuitous.

The relationship between the present method and Spreiter's is quite simple. The latter may be regarded as a first approximation to the present method, since it corresponds to taking the velocity as given by the first iteration in the solution of the integral equation (2.8), while using the initial or zeroth approximation for the associated value of z. However, it is not a very satisfactory approximation, as can be seen from Tables 1 and 2. The excellent agreement between values of pressure drag obtained from Refs. [1] and [6] is due to a cancelling of positive and negative errors, and this cannot be expected for every profile.

There still remains the question of the usefulness of the parabolic equation model in simulating the physical situation. There are three possible sources of disagreement between theory and experiment: (i) failure of the transonic small disturbance equation to represent the important features of the flow; (ii) failure of the parabolic equation to provide a good approximation to the transonic equation; (iii) failure of the experimental model tests to give values of pressure corresponding to free flight. That the second of these is not solely responsible for the major part of the differences is established by the case of the wedge.

Assessment of the theory must involve comparison with exact solutions of the transonic flow equations and experimental results. Only for the case of the wedge, is there a theory which may be regarded as close to the exact solution, the hodograph plane solution of Guderley and Yoshihara [6] mentioned above. For this case, the present theory is an improvement on [1], although complete agreement is not obtained. The remaining difference must be attributed to the implicit assumptions concerning the dependence of $\varphi$ on y, which are common to all the parabolic approximations so far developed. For other cases, we can only compare with experiment, and as mentioned already, the present theory gives worse agreement with the experimental results [5] than the Spreiter theory [1]. However, the reliability of the experimental results has not gone unchallenged. A full discussion of the experimental results is not attempted here, but for completeness we indicate certain difficulties.

The Michel [5] airfoils were simulated by a tunnel wall bump technique, and compared with a mid-tunnel test case. Further experiments by Carrol and Anderson [7] indicate errors due to the wall boundary layer. The agreement with the mid-tunnel test case could be due to more pronounced tunnel wall reflection errors in this case. A systematic study of tunnel wall reflection errors has been undertaken by Spreiter, Smith and Hyett [8]. An application of Guderley [9] and Barish [10] theories predicts a pattern of errors similar to the differences between the theoretical results [1] and experiment. However the physical explanation advanced earlier in [8] would only account for an opposite pattern, and it seems a pity that the Barish form of the correction, equation (11) of [8] was not checked experimentally.

Questions of a different type are raised by the case of the wedge. Spreiter has discussed this matter in [11] and [8]. Briefly, the problem is that there are differences between theoretical results which may be regarded as accurate (Guderley and Yoshihara [6], Helliwell and Mackie [12]) and the experimental values of Knetchel [13], and these differences cannot be accounted for by the quantitative theories of wind tunnel interference of Marschner [14] and Morioka [15]. The differences may be due to the sharp shoulder, but Fig 2 suggests a more general effect.

In view of these difficulties, we cannot yet regard the difference between theory and experiment as largely due to either the failure of the parabolic equation as a model of the full transonic equation, or the failure of the transonic equation to represent the physical situation for free flight.

REFERENCES

[1] Spreiter, J.R. and Alksne, A.Y. NACA Rep. 1359 (1958).

[2] Randall, D.G. A.R.C. C.P.456 London (1959).

[3] Maeder, P.F. and Thommen, H.U. J. Aero. Sci. 23 (1956), 187-188.

[4] Pearson, K. Tables of the incomplete gamma function (Cambridge, 1946).

[5] Michel, R., Marchaud, F. and Le Gallo, J. O.N.E.R.A. Publ.65 (1953), 72 (1954).

[6] Guderley, K.G. and Yoshihara, H. J. Aero. Sci. 17 (1950), 723-735.

[7] Carrol, J.B. and Anderson, G.F. J. Aero. Sci. 23 (1956), 604-605.

[8] Spreiter, J.R., Smith D.W. and Hyett, B.J. NACA R-73 (1960).

[9] Guderley, K.G. Theory of transonic flow (translation; Pergammon, London, 1962).

[10] Barish, D.T. USAF WADC Tech. Rep. 52-88 (1952).

[11] Spreiter, J.R. In Symposium transsonicum, Aachen 1962 (Springer; Berlin, 1964)

[12] Helliwell, J.B. and Mackie, A.G. J. Fluid Mech. 3 (1957) 93-109.

[13] Knechtel, E.D. NASA TN D-15 (1959).

[14] Marschner, B.W. J. Aero. Sci. 23 (1956) 368-376.

[15] Morioka, S. J. Phys. Soc. Japan 14 ( 1959) 1098-1101.

---