Inelasticity and partial wave dispersion relations

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INELASTICITY AND

PARTIAL WAVE DISPERSION RELATIONS

by

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Abstract.

This thesis deals with some general work on the ND^{-1} equations and, more specifically, with two calculations which make use of them.

Chapter 1 commences with a brief outline of the subject, followed by a fairly detailed analysis of numerical methods of solution of the equations, including an analysis of cutoff and threshold problems. In particular we demonstrate a particular form of numerical solution which appears to be rather superior to any previous ones.

Chapter 2 discusses the problems that arise when a multi-channel calculation is approximated to be a single channel one: we show that in general the results are different, and discuss the conditions for them to agree.

Chapter 3 investigates what happens when N and D have simultaneous zeroes: it is shown that a potential that leads to this is necessarily singular and repulsive.

Chapter 4 opens with a general review of the successes and failures of the quark model in scattering theory.

It is shown that the quark model is necessarily inconsistent with what is commonly called bootstrap philosophy, and we investigate whether a reasonably convincing quark model may be constructed.

Chapter 5 outlines a calculation of the πN P_{11} phase shift. The previous work on this problem is discussed, and the computational method is outlined along with a discussion of what results can be expected from the calculation.

Partial Wave Amplitudes and the ND^{-1} Equations.

The study of partial wave amplitudes [1] derives from the solution of the radial Schrodinger equation in terms of eigenfunctions of the free equation.

$$\nabla^2\psi(\underline{x}) + V\psi(\underline{x}) = k^2\psi(\underline{x}) \qquad (1)$$

Making the substitution

$$\psi(\underline{x}) = {}_{l,m}\Sigma(2l + 1)\, Y_l^m(\theta,\phi)\, \phi_l(r)$$

and (2)

$$u_l(r) = r\phi_l(r)$$

gives the radial equation for the individual partial wave functions $u_l(r)$.

$$\frac{d^2}{dr^2} u_l + \left(k^2 - \frac{l(l+1)}{r^2} - V(r)\right) u_l(r) = 0 \qquad (3)$$

The 'free' solution of this equation is $u_l(r) = j_l(kr)$, a spherical Bessel function, which may be split into two parts which behave asymptotically as an ingoing part h^+ and an outgoing part h^-.

$$j_l(kr) = \frac{1}{2i}\left[h_l^+(kr) - h_l^-(kr)\right] \qquad (4)$$

The solution of (3) can then be written asymptotically

as

$$u_l(r) = \frac{1}{2i}\left[f_l^+(k)h_l^+(kr) - f_l^-(k)h_l^-(kr)\right] \qquad (5)$$

such that f^+ is the total ingoing amplitude, f^- the outgoing, and we can write the partial wave scattering amplitude a_l in the form

$$\frac{1}{2ik}\,\frac{\text{scattered wave}}{\text{ingoing wave}} = \frac{1}{2ik}\,\frac{f^- - f^+}{f^-} = \frac{e^{i\delta}\sin\delta}{k} = a_l(k) = \frac{1 - S_l(k)}{k} \qquad (6)$$

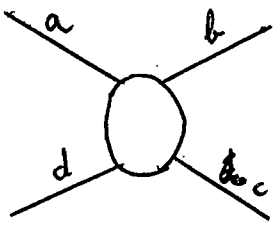
The functions $f^{\pm}$ are known as the Jost functions, and the study of their analytic properties and hence that of the a_l is of great importance. We describe them very briefly without proof.

If the potential is a superposition of Yukawas

$$V(r) = \int_{m_o}^{\infty} \sigma(m)\,\frac{e^{-mr}}{r}\,dm \qquad (7)$$

it can be shown that $f^+(k)$ has a cut from $-\frac{im_o}{2}$ to $-i\,\infty$ in the plane, and $f^-(k)$ has one from $\frac{im_o}{2}$ to $i\,\infty$. The only additional singularities in a_l occur because of zeroes of $f_l^-(k)$. These can be shown to lie in the lower half plane (corresponding to virtual states and resonances) and along the positive imaginary axis (corresponding to bound states of the amplitude).

This structure is paralled by that found in the relativistic case, which however requires more assumptions summarised in the Mandelstom hypothesis. The most important is that if crossing symmetry, which implies that one amplitude A(s,t,u) describes all scatterings which can be represented by diagram 1). Thus



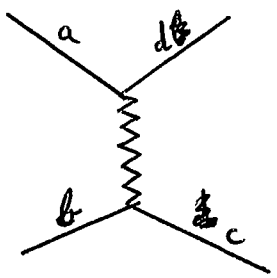
(1)

$a + b \to c + d$ could be considered with s as the (energy) variable and t = -(momentum transfer) . Then $a + \bar{d} \to b + \bar{c}$ is described by the same amplitude with the meanings of s and t interchanged. This allows us to derive the forces from the exchange of particles with the quantum numbers of the crossed channel; a bound state in the t channel corresponds to a force term in the s channel. Explicitly

$$A(s,t,u) = A(t,\cos t) = \Sigma(2l+1)a_l(t)P_l(\cos\theta t) \quad (8)$$

$$= A(s,\cos s) = \Sigma(2l+1)a_l(s)P_l(\cos\theta s) \quad (9)$$

where in general the two expansions have different regions of convergence.



A bound state in the t channel with spin l and mass m (a,b,c and d are presumed spinless and stable) allows us to approximate

$$A(t,\cos\theta t) \simeq \frac{P_l(\cos\theta t)}{t - m^2} \qquad (10)$$

From (9) and (10) we derive

$$a_{l'}(s) = \int_{-1}^{1} \frac{P_{l'}(\cos\theta s)\, P_l(\cos\theta t)\, d\cos\theta s}{t - m^2}$$

From this we can derive the discontinuity of $a_l(s)$ across a cut which turns out to occur along the negative real axis $-\infty<k^2<-\frac{m^2}{4}$, as in the potential case, where a Yukawa potential $\frac{e^{-mr}}{r}$ corresponds to the exchange of a particle of mass m. The discontinuity calculated in this way is identical to that derived from the corresponding Feynmann diagram.

Another cut is the so called unitarity cut, which derives from conservation of probability : $S_l\, S_l^{+} = 1$ which leads to $\mathrm{Im}\, a_l(k^2) = k|a_l(k^2)|^2 \qquad k^2>0$

This gives us a function with a known left hand cut (L.H.C) discontinuity and a known right hand cut (R.H.C) discontinuity. By Cauchys theorem, we can write a dispersion relation

$$\mathrm{Re}\, a_l(\nu) = \frac{P}{\pi}\int \frac{\mathrm{Im}\, a_l(\nu)}{\nu' - \nu} + \frac{P}{\pi}\int \frac{\sqrt{\nu'}\, |a_l(\nu')|^2 d\nu'}{\nu' - \nu} \qquad (11)$$

However this form is non-linear and hence rather

intractable. It was suggested by Chew and Mandelstom [2] that the amplitude could be split into two parts

$$a_1(\nu) = \frac{N_1(\nu)}{P_1(\nu)}$$

where N has only the L.H.C. and D the R.H.C.

On the L.H.C. $\text{Im } a_1(\nu) = \dfrac{\text{Im } N_1(\nu)}{D_1(\nu)}$ so $\text{Im } N_1(\nu) = \text{Im } a_1(\nu)\, D_1(\nu)$

and on the R.H.C. $\text{Im } \dfrac{1}{a} = \dfrac{\text{Im} D}{N} = -\rho$ or $\text{Im } D = -\rho N$.

This gives us the basic N/D equations

$$N(\nu) = \frac{P}{\pi} \int_L \frac{\text{Im } a_1(\nu')\, D(\nu')}{\nu' - \nu}\, d\nu' \qquad (12)$$

$$D(\nu) = 1 - (\nu - \nu_o)\frac{P}{\pi} \int_R \frac{\rho(\nu')N(\nu')}{(\nu' - \nu)(\nu' - \nu_o)}\, d\nu' \qquad (13)$$

One subtraction can be made in D to normalise it at an arbitrary point ν_o: N and D can be multiplied by $\nu - \nu_o$ without changing the amplitude a. The discussion of the purely mathematical features of the solution to these equations forms a large body of literature [3]. It can be shown that the equations are independent of the subtraction point [4] (though in general this is not true of approximate solutions): and that so long as the kernel of the equation obtained by substituting (13) in (12)

$$N(\nu) = B(\nu) + \frac{P}{\pi} \int_R \frac{(\nu' - \nu_o)\, B(\nu') - (\nu - \nu_o)B(\nu)}{(\nu' - \nu)(\nu' - \nu_o)}\, \rho(\nu')N(\nu')d\nu' \tag{14}$$

with $B(\nu) = \int_L \frac{\mathrm{Im}\, a_1(\nu')}{\nu' - \nu}\, d\nu'$

is $\mathcal{L}^2$, then the solution is unique under provisos to be discussed later, and analytic in the cut plane. In particular this is true for L.H.C. consisting solely of δ functions

$$\mathrm{Im}\, a_1(\nu) = \sum_i m_i \delta(\nu + \nu_i)$$

which leads to the simple closed form

$$D(\nu) = 1 + \Sigma m_i\, D(-\nu_i)F(\nu', \nu_o\ \nu_i) \tag{15}$$

$$F(\nu', \nu_o, \nu_i) \doteq \int_R \frac{\rho(\nu')\, d\nu'}{(\nu - \nu_o)(\nu' - \nu)(\nu' + \nu_i)}$$

At any point ν_B such that $D(\nu_B) = 0$, the amplitude $A(\nu)$ will have a pole, which corresponds to a bound state (or resonance).

The uniqueness of the solution is very important. The so-called C.D.D. ambiguity [4] concerns the addition of an arbitrary pole to either N or D.

If we take $D(\nu) \to D(\nu) + \frac{\alpha}{\nu - \nu_o}$

then $N(\nu) = N(\nu) + \alpha \frac{\nu B(\nu) - \nu_o B(\nu_o)}{\nu - \nu_o}$

and if α is small

$$a(\nu) \simeq \frac{N(\nu)}{D(\nu) + \alpha/\nu - \nu_o}$$

which will have a zero at $\nu = \nu_o$ and a nearby pole at ν_B such that $(\nu_B - \nu_o)\, D(\nu_B) = \alpha$

This implies a new resonance or bound state which is not produced by the forces of the problem, but is introduced at will. To find the effect of C.D.D. poles on the amplitude it is necessary to consider Levinsons theorem [5]: we outline a proof for elastic scattering.

$$a_l(s) = \frac{e^{2i\delta} - 1}{2ik} = \frac{N_l(s)}{D_l(s)}$$

We define the Omnes - Mushkelishvilli function

$$\mathcal{D}(s) = \exp \frac{-s}{\pi} \int_o^\infty \frac{\delta(s')}{s'(s' - s)}\, ds \qquad (16)$$

so that $\mathcal{D}$ has the same phase as D and has no poles or zeroes on the physical sheet. If two functions have the same cuts and same phase they can differ at most by a polynomial.

If we assume $\delta(s) \underset{s\to\infty}{\to} \delta(\infty) = \text{const} = m\,\pi$

where $\delta(0) = 0$ by definition

then $\mathcal{D}(s) \to \text{const}\ s^{-m}$.

Since D has no poles but only zeroes on the physical

sheet, and $D(\infty) = 1$

$$D(s) = \prod_{i=1}^{m} (s - s_i)\mathfrak{D}(s)$$

must be the relation between D and $\mathfrak{D}$; D will have ^{n}b zeroes, which should imply n bound states in a. If D has ^{n}c C.D.D. poles, it can be written

$$D(s) = \prod_{i}^{nb} (s - s_i)/\prod_{j}^{nc} (s - s_j) \qquad (17)$$

and Levinsons theorem takes the form

$$\delta(\infty) = (^{n}b - {}^{n}c)\pi$$

This or its multichannel generalisations should in principle enable one to decide whether a bound state was a C.D.D. pole or not. The situation is unfortunately complicated by the fact that a_l and D_l obey different Levinson's theorems: this because it is possible that a zero in D_l is exactly cancelled by a zero in N_l, and then Levinsons theorem for a_l becomes

$$\delta(\infty) \leq -(^{n}b - {}^{n}c)\pi$$
$$= -(^{n}b - n_c + n_l)\pi$$

where n_l is the number of superimposed zeroes, or extinct bound states (E.B.S). C.D.D. poles are discussed more fully in Chapter 2, and E.B.S. in Chapter 3.

The ND^{-1} method has been extended to multichannel calculation by Bjorken[6]: essentially all that is involved is writing (12) and (13) in matrix notation, with

$$a_l(\nu) = N_l(\nu)\, D_l(\nu)^{-1} \qquad (18)$$

$$\rho_{ij}(\nu) = \delta_{ij}\rho_j(\nu)$$

The $a_l(\nu)$ is symmetric, as required by time-reversal invariance. Levinson's theorem cannot be simply derived, because there is no multichannel analogue of (16).

In general, the ND^{-1} equations that we have arrived at are not analytically soluble. If the force function is simple e.g. $B(\nu) = \nu^{\alpha}$, then they can be solved by various ingenious transformations: Halpern[7] has used these to find solutions when $\alpha \geq 0$, in which case (12) and (13) are non-Fredholm equations.

These are an assortment of numerical methods for the solution of the equations: we discuss them in what appears to be an ascending order of merit. The simplest is what is usually known as the K matrix method: we take Re D = 1 in (12) and ignore (13), so

$$a(\nu) = \frac{N(\nu)}{1 - i\rho(\nu)N(\nu)}$$

This is at least unitary, but naturally very inaccurate

as it takes no account of rescattering corrections. An increase in accuracy is gained by including (13), which is the determinantal method [8]

$$D(\nu) = 1 + (\nu - \nu_o) \int_R \frac{\rho(\nu') B(\nu')}{(\nu' - \nu)(\nu' - \nu_o)} d\nu'$$

However this solution is not only subtraction point dependent, but does not give a symmetric a. Its accuracy has been discussed by Luming [9]: it is particularly poor, as might be expected, for a bound state.

Clearly this process can be continued: in fact the two solutions above form the first and second terms in the Neumann series for the equations. In many cases, however, the Neumann series is divergent while the correct solution, derived by Fredholm methods, is not: quite apart from this the work involved in a satisfactory iterative solution is enormous. This lead Blankenbecler and Roy [10] to point out that the poorness of the determinantal method lies in the approximation $D(\nu) = 1$ in a region near $D(\nu) = 0$: in other words near a bound state. The suggested a parametric form

$$D_o(\nu) = \frac{\nu - \nu_c}{\nu - \nu_b}$$

should be used in 12) giving

$$N(\nu) = \frac{(\nu_B - \nu_c)\, B(\nu_B) - (\nu - \nu_c)\, B(\nu)}{\nu_B - \nu}$$

$$D(\nu) = 1 - \frac{1}{\pi} \int_R \frac{\rho(\nu')}{} \quad \frac{(\nu_B - \nu_c)\, B(\nu_B) - (\nu' - \nu_c\, B(\nu')}{(\nu' - \nu)\,(\nu_B - \nu')}\, d\nu'$$

so

$$D(\nu) = 1 - (\nu_c - \nu_B)\, B(\nu_B) \quad \nu_B \frac{F(\nu_B) - \nu F(\nu)}{\nu_B - \nu}$$

$$+ \nu_B \frac{G(\nu_B, \nu_c) - \nu G(\nu, \nu_c)}{\nu_B - \nu}$$

where $F(a) = \int_R \frac{\rho(\nu')}{\nu'(\nu' - a)}$ $\qquad G(b, \nu_c) = \int_R \frac{\rho(\nu')(\nu' - \nu_c) B(\nu')}{\nu'(\nu' - b)}$

It is now possible to solve iteratively for ν_B and ν_c, as $D(\nu_c) = 0$ by hypothesis, and ν_B can be fitted by requiring $D(0) = D_o(0) = \frac{\nu_c}{\nu_B}$. This method is fairly accurate, provided that at least one bound state exists, but it is clumsy, as all iterative methods are.

This solution turns out to be much superior in practice, but it is still unnecessarily complex. To find more direct methods, we write the coupled equations

in the single integral equation form

$$N(s) = B(s) + \frac{1}{\pi} \int_R \frac{s'B(s') - sB(s)}{s'(s' - s)} \rho(s') N(s') ds' \qquad (19)$$

$$D(s) = 1 + \frac{s}{\pi} \int_L K(s',s) \, \mathrm{Im}B(s')D(s') ds' \qquad (20)$$

where $K(s',s) = \int_R \frac{\rho(s'')}{(s' - s'')(s'' - s)} ds''$

Either of these can obviously be solved by Neumann series. A method due to Shaw [11] puts $N(s) = B(s) C(s)$ in (19) and then approximates $C(s') = C(s)$.

This curious procedure gives

$$C(s) = B(s) \left[B(s) + \frac{1}{\pi} \int_R \frac{s'B(s') - sB(s)}{(s' - s)s'} \rho(s') ds' \right]^{-1}$$

$$N(s) = B(s) \left[1 + \frac{s}{\pi} \int \frac{\rho(s')}{s'(s'-s)} ds' + \frac{1}{\pi} \int \frac{B(s')}{B(s)} \frac{\rho(s')}{(s'-s)} ds' \right]^{-1}$$

and it is surprisingly satisfactory.

However all the preceding methods have one or more striking disadvantages: they suffer from one or more of the disadvantages of inaccuracy, dependence on the subtraction point, lack of time reversal invariance and require, in at least the last two cases, involved

numerical integrals. The most general numerical method is that of matrix inversion. Starting from

$$f(x) = h(x) + \int K(x,x') f(x')dx'$$

we replace the integral by a trapezoidal type sum

$$f(x) = h(x) + \Sigma K(x,x_i) f(x_i) (x_{i+1} - x_{i-1}) \quad (21)$$

This can be solved by taking

$$\Sigma(1 + K(x_j,x_i)\delta x_i) f(x_i) = h(x_j)$$

which is a set of simultaneous equations, which are solved for the $f(x_i)$ and substituted in (21).

This is precisely equivalent to replacing ImB by a series of δ functions, $m_i \delta(s' + s_i)$

$$m_i = \mathrm{ImB}(s_i)(s_{i+1} - s_{i-1})$$

This method can be made arbitrarily accurate (by taking a sufficient no. of mesh points), and is comparatively simple to solve on a computer, and it has been widely used as a result. Generally (17) is preferable to (18) because for unequal mass scattering the L.H.C. is of very complicated form, and (18) requires two integrations to be made over it, which (17) requires none.

It is, however, complicated to solve by hand, and this led Pagels [12] to suggest the following trick.

Equation (18) has the kernel

$$K(s',s) = \frac{s'F(s')}{s'-s} + \frac{sF(s)}{s-s'} \qquad (22)$$

where $$F(s) = \int_R \frac{\rho(s)}{s'^2(s'-s)}\, ds' \qquad (23)$$

which is a monotonically increasing function on the left hand cut. This suggests that it is a good approximation to write

$$F(s) \simeq \frac{C}{s-a} \qquad (24)$$

on the left. Combining equations (24), (18) and (12) allows us to derive

$$D(s) \simeq 1 + \frac{N(a)}{a-s} \quad C\,a \qquad (25)$$

Inserting this in (12) gives

$$N(s) = B(s) + \frac{aB(a) - sB(s)}{a-s} \quad CaN(a) \qquad (26)$$

which in turn gives

$$D(s) = 1 + s\,F(s)^{N(s)} + \frac{N(a) - N(s)}{a-s} \quad Ca \qquad (27)$$

Obviously the accuracy of the method depends on the goodness of fit (24), and this can be improved by taking more poles. Smith's [13] claim that the method

is not good is irrelevant; he considers a nonrelativistic potential model which leads to

$$F(\nu) = \frac{1}{\sqrt{-\nu}}$$

and this must be fitted by one pole in the range 0 to $-\infty$. This is clearly ridiculous: however, if more poles are added, even in this case a satisfactory solution can be found. Typically a five pole (ten parameter fit) to $F(\nu)$ $-20<\nu<0$ leads to an error of $D(\nu)$ of less than .001% for reasonable $\beta(\nu)$.

However it is clear that the Pagels method is not, as claimed, a method of solving the equation for D, as the actual numerical work is involved in solving (26). It is probable that it is equivalent to some Gaussian method and it's success is due to its use of the kinematic function as a weight function [14]. Rather than resort to ad hoc fitting procedures to find $C/s-a$, it is better to use this.

As an adaptation of this method will be used in Chapter 5 we describe it in some detail. Writing 19) more explicitly, and assuming equal mass elastic scattering, we have

$$N(s) = B(s) + \int_t^\infty \sqrt{\frac{s' - t}{s'}} \quad \frac{s'B(s') - sB(s)}{s'(s' - s)} N(s') \, ds' \tag{28}$$

$$\text{or} \quad I(s) = \int_t^a \sqrt{\frac{s' - t}{s'}} \quad F(s', s) \, ds' \tag{29}$$

where we have included an explicit cut-off at $s' = a$. This is required in most practical solutions of the equations because $B(s) \underset{s\to\infty}{\sim} s^{l-1}$ for the exchange of a spin l particle, which renders (28) divergent. The use of cut-off is justified by saying that since (presumably) Nature is not divergent and the first Born approximation is a cancellation with the higher Born terms must occur to produce a more or less well behaved force term. Since this occurs a long way from the physical region of interest, any reasonable behaviour will have negligible effect on the amplitude there. The methods used by Halpern [7] would render this unnecessary, but they would be very difficult to use in practice. In the original form of the Pagels method, the different asymptotic behaviour of the function F(s) and its approximation, (23) and (24), give an implicit cut-off.

Substituting $s' = \frac{t}{x}$ in (29) gives

$$I = t\int_{t/a} \sqrt{1 - x}\ \frac{F(x',x)}{x'^2}\ dx' \qquad (30)$$

$$= t(1 - \frac{t}{a})^{3/2} \sum_{i=1}^{n} \frac{w_i}{x_i^2} F(x_i,x) \qquad (31)$$

where [15]

$$x_i = \frac{t}{a} + (1 - \frac{t}{a})(1 - z_i^2)$$

$$w_i = 2z_i^2 m_i$$

where, in turn, z_i is the i^{th} zero of $Pn(z)$, and $m_i = 1/(1 - z_i)^2(P'_n(z_i))^2$ is the corresponding weight.

$$N(s) = B(s) + \sum_{i=1}^{n} \frac{s_i B(s_i) - sB(s)}{s_i - s}\ C_i s_i\ N(s_i) \qquad (32)$$

which is similar to (26) with an arbitrary number of poles. In turn

$$D(s) = 1 - \frac{s}{\pi} \int_t^\infty \sqrt{\frac{s' - t}{s'}}\ \frac{N(s')}{s'(s' - s)}$$

$$= 1 - \frac{s}{\pi} \int_t^\infty \sqrt{\frac{s' - t}{s'}}\ \left(\frac{N(s') - N(s)}{s'(s' - s)}\right) ds' - \frac{sN(s)}{\pi} \int_t^\infty \sqrt{\frac{s'-t}{s'}}\ \frac{ds'}{s'(s'-s)} \qquad (33)$$

$$= 1 - \Sigma\, s\, C_i s_i \left(\frac{N(s_i) - N(s)}{s_i - s}\right) - \frac{N(s)}{\pi t} \log\left(\frac{1 - \sqrt{\frac{s-t}{s}}}{1 + \sqrt{\frac{s-t}{s}}}\right) \qquad (34)$$

This method of solution appears to be the best: it gives reasonable results for a very few points in the integration (even two is satisfactory), it is fast and allows the accuracy to be arbitrarily improved.

Finally we briefly discuss the problem of threshhold conditions. Theory shows that partial waves behave like k^{2l+1} near threshold, which implies a $(2l + 1)$th order zero in the amplitude a_l (and hence in N_l) at threshold. The method frequently adopted is to write

$$a(s) = k^{2l+1}\, N/D$$

and write the dispersion relations for N and D, which obviously forces the correct behaviour. This seems unsatisfactory, not only because the basic failing is in the approximation to the L.H.C. discontinuity, but because this involves further trouble with asymptotic behaviour.

Therefore we use the Frye-Warnock method of enforcing threshold conditions: that is to multiply

the Born terms by a threshold factor $(W - W_E)/(W - a)$ (for a P wave), where a is arbitrary. This is equivalent to adding an extra pole on the left hand cut, as has been shown by Simmons [16].

$$\text{We replace } B(W) \rightarrow \bar{B}(W) = B(W) + \frac{C}{W - a} \qquad (35)$$

and solve the integral equation

$$N(W) = \bar{B}(W) + \int_R \frac{W'\bar{B}(W') - W\,\bar{B}(W)}{W' - W} \frac{\rho(W')N(W')}{W'}\, dW' \qquad (36)$$

Enforcing the condition $N(W_E) = 0$ leads to

$$C = \frac{W_E - a}{2 - D(a)} \frac{1}{\pi} \int_R \frac{B(W')}{W' - W_E} \rho(W')\, N(W')\, dW' \qquad (37)$$

Inserting (37) and (35) in (36), we find after some algebra

$$N(W) = B(W) + \frac{1}{\pi} \left(\frac{W - W_E}{W - a}\right) \int dW' \left(\frac{W'(W' - a)}{W' - W_E} B(W') - \frac{W(W - a)B(W)}{W - W_E}\right) \frac{\rho(W')N(W')}{W'(W' - W)}$$

which is the usual form.

Simmons demonstrates that there is some doubt about the validity: in particular he shows that the mass of the N^* in the πN P_{33} partial wave depends fairly strongly on a. We adopt this procedure, however, as it appears to be the only reasonable choice. Our

knowledge of the L.H.C. is limited to the first Born term: it is known that if S- matrix theory is to consistent, the many particle terms must be important, and our cavalier disregard of them must be compensated in some way. We do, however know the threshold behaviour on very general grounds, so to replace the former by a single pole to enforce the latter is conceptually reasonable.

CHAPTER 2

Inconcistency of the one and many-channel ND^{-1} calculations.

It was noticed by Squires[17] and independently by Bander, Coulter and Shaw[18] that the one channel calculation with inelasticity and the corresponding multichannel calculation are not necessarily equivalent, unless additional C.D.D. poles are inserted. This chapter forms a discussion of the two possible breakdowns, connected with the presence of zeroes in the one channel calculation and poles in the many-channel.

In general, zeroes cannot occur in a coupled channel problem because of the unitarity condition,

$$\text{Im } a_{11} = \rho_1 |a_{11}|^2 + \rho_2 |a_{12}|^2 \qquad (1)$$

and it is coincidental for a_{11} and a_{12} to have simultaneous zeroes. However, the one-channel calculation with inelasticity apparently permits a zero

$$\text{Im } a = R\rho |a|^2 \qquad (2)$$

$$R = 1 + \frac{\sigma \text{inel}}{\sigma \text{el}} = 1 + \frac{\rho_2 |a_{12}|^2}{\rho_1 |a_{11}|^2} \qquad (3)$$

and hence the methods may be inconsistent. An explicit

numerical calculation was carried[19] out, to see how this occurs in a model.

The two channel amplitude with a two pole input satifies the following equations

$$a(s) = N(s)\, D^{-1}(s)$$

$$N(s) = \frac{\mu_1 D(-m_1)}{s + m_1} + \frac{\mu_2 D(-m_2)}{s + m_2} \tag{4}$$

$$D(s) = 1 - F_1^{(1)} \mu_1 D(-m_1) - F^{(2)} \mu_2 D(-m_2)$$

$$F_{ij} = \delta_{ij} \frac{(s_i + m)^{\frac{1}{2}} + (s_i - s)^{\frac{1}{2}}}{s + m}$$

and the corresponding one channel amplitude satifies

$$D(s) = 1 - \frac{P}{\pi}\int \frac{\rho(s')R(s')N(s')}{s' - s}\, ds', \quad N(s) = \frac{\mu_{11}^{(1)} D(-m_1)}{s + m_1} + \frac{\mu_{11}^{(2)} D(-m_2)}{s + m_2}$$

$$= 1 - \mu_{11}^{(1)} D(-m_1) G^{(1)} s - \mu_{11}^{(2)} D(-m_2) G^{(2)}(s) \tag{5}$$

$$G^{(i)} = \int^{\infty} \frac{\rho_1(s')R(s')}{(s' - s)(s' + m_i)}$$

The method adopted was to calculate $\phi_{11}(s), \delta_{11}(s)$ and R(s) from (4), where ϕ_{11} and δ_{11} are defined by

$$a_{11} = e^{i\phi_{11}}|a_{11}| \qquad (6)$$

$$a_{11} = \frac{e^{2i\delta_{11}} - 1}{2ik}$$

and to use this R to calculate a(s) from (5) leading to two other parameters $\phi(s)$ and $\delta(s)$, similarly defined. The numerical values chosen were $s_1 = 8$, $s_2 = 10$, $m_1 = 1$, $m_2 = 8$ and

$$\text{i)} \quad \mu^{(1)} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 1.0 \end{bmatrix} \qquad \mu^{(2)} = \begin{bmatrix} -1.2 & 0.1 \\ 0.1 & 1.0 \end{bmatrix} \qquad (7)$$

chosen so that the elastic single channel amplitude has a zero at s = 20 with decreasing phase shift (figs 1,4)

ii) the same magnitudes, but with reversed sign, so that the phase shift is now increasing (fig 3,6).

The results of the calculations are shown in figs 4 and 6. As expected in case (1) the methods give different answers: in particular ϕ passes through 0 while ϕ_{11} passes through $\pi/2$, and as can be seen from fig 5, the calculated cross-sections are very different. Surprisingly, however, the results in case (ii) are identical, and this occurs because a zero forms in ReD which exactly

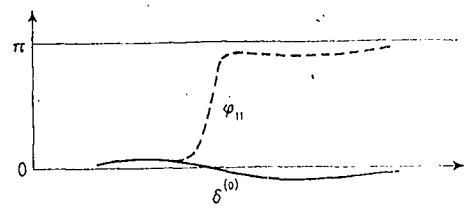


Fig. 1. – Showing the behaviour of φ_{11} for coupled channel problem when the uncoupled phase-shift ($\delta^{(0)}$) has a zero with a negative slope.

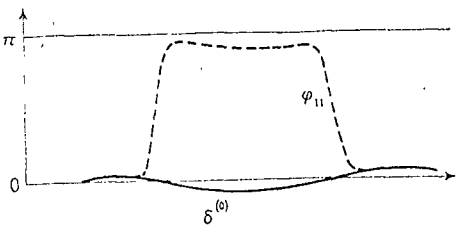


Fig. 2. – Showing an example where the «Levinson's theorem» is not violated, but where the one-channel method will break down.

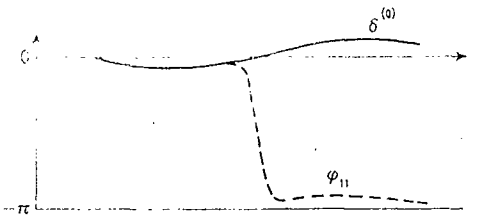


Fig. 3. – Showing the behaviour of φ_{11} when the uncoupled phase-shift ($\delta^{(0)}$) has a zero with a positive slope.

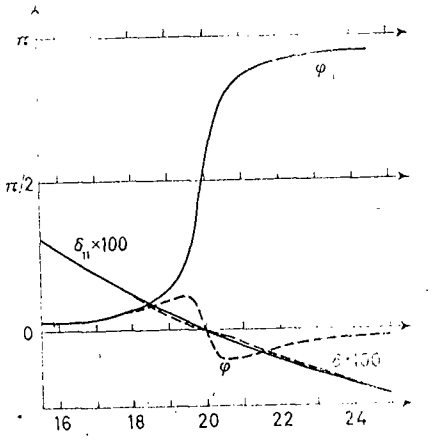


Fig. 4. Showing numerical results for Case I.

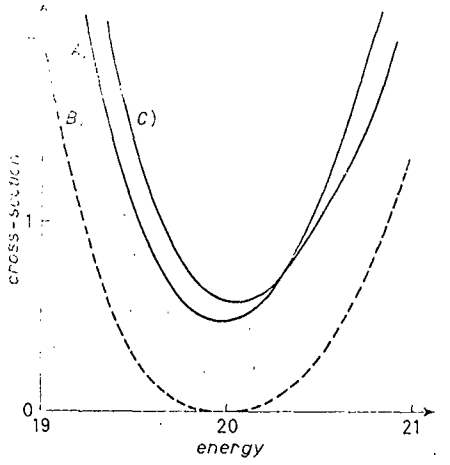


Fig. 5. – A comparison of the true partial-wave cross-section *A*) with the one-channel result *B*) for Case I. Curve *C*) is the corresponding curve for Case II, calculated by either method.

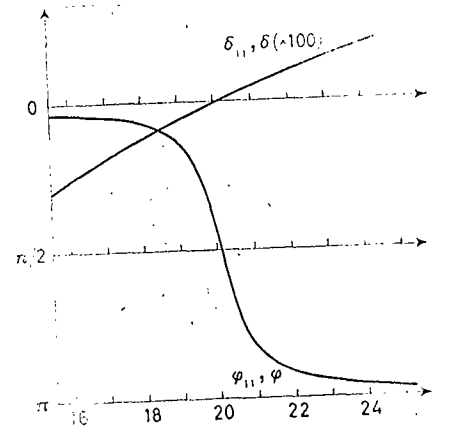


Fig. 6. – Showing numerical results for Case II. The two methods agree so that $\delta_{11} = \delta$ and $\varphi_{11} = \varphi$.

cancels that in N when calculated from (5). This phenomenon has curious consequences which we discuss in Chapter 3.

This anomalous result can be discussed in two ways. If we consider the function $\frac{N(s)}{a_{11}(s)}$, it is clear that this has the same phase as D on the R.H.C., and has the same asymptotic behaviour. It can only differ by having poles (since D is analytic) on the right hand physical sheet, which implies a zero in a_{11} since N is analytic: thus this is a necessary and sufficient condition for the failure of the method. From the following simple argument, it can be seen that the sign of $\frac{d\delta}{ds}$ is critical.

Assuming that $a_0(s_0) = 0$ where a_0 is the elastic single channel amplitude, then weakly coupling a second channel can serve only to move this zero slightly, so

$$\text{Re } a_{11}(\bar{s}) = 0 \quad a_{11}(\bar{s}') = 0 \qquad (8)$$

where $\bar{s}$ is real, $\bar{s}'$ complex. Hence

$$a_{11}(\bar{s}) \simeq (\bar{s} - \bar{s}') \left.\frac{da_{11}}{ds}\right|_{s=s'} + a_{11}(\bar{s}') \qquad (9)$$

so

$$\bar{s}' \simeq \bar{s} - a_{11} \left.\frac{da_{11}}{ds}\right|^{-1}_{s=\bar{s}'} \qquad (10)$$

Taking the imaginary part of this equation gives, using (8)

$$\text{Im } \bar{s}' \simeq - \text{Im } a_{11}(\bar{s}) \text{ Re } \frac{da_{11}}{ds}\Big|_{\bar{s}'} \Big/ \left|\frac{da_{11}}{ds}\right|^2 \quad (11)$$

Now $\text{Re } \frac{da_{11}}{ds}\Big|_{\bar{s}'} \simeq \text{Re } \frac{da_{11}}{ds}\Big|_{\bar{s}} \simeq \text{Re} \frac{da_0}{ds}\Big|_{\bar{s}} \simeq \frac{1}{k} \frac{d\delta_0}{ds}\Big|_{\bar{s}}$

and so finally

$$\text{Im } \bar{s}' \simeq - \frac{\text{Im } a_{11}(\bar{s})}{\left|\frac{da_{11}}{ds}\Big|_{\bar{s}}\right|^2} \frac{d\delta_0}{ds}\Big|_{\bar{s}} = - \left|\frac{a_{11}(\bar{s})}{\frac{da_{11}}{ds}\Big|_{\bar{s}}}\right|^2 \frac{d\delta_0}{ds}\Big|_{\bar{s}} \quad (12)$$

Hence Im $\bar{s}'$ is positive if $\frac{d\delta_0}{ds}$ is negative, and the zero lies on the physical sheet, causing the one channel method to fail (case(i)) and vice versa.

To derive a more specific condition, we consider the Omnes - Mushkelishvilli function

$$\mathcal{D}(s) = e^{-s/\pi \int \frac{\phi(s')}{s'(s'-s)} ds'}$$

As shown in Chapter 1, the relation between D and must be D(s) = (real polynomial)$\mathcal{D}(s)$

If the amplitude a $\to$ 0 as s $\to \infty$, then $\frac{\text{Im } a}{\text{Re } a} \to 0$ and

D(s) $\to$ 1, then $\phi(s) \to -m\pi$

$$\mathcal{D}(s) \to \text{const.} \quad s^{-m}$$

so the polynomial is of degree $m \geq 0$. Since a has no poles on the physical sheet away from the real axis, the

zeroes of the polynomial must lie on the physical sheet, and from (1), below the inelastic threshold. Hence D(s) must have m zeroes in this region, and if n_B is the number of bound states

$$n_B \leq m$$

where the inequality occurs because of the possibility of simultaneous zeroes of N and D. The multichannel method will give a similar relation for ϕ_{11}

$$\phi_{11} \underset{s\to\infty}{\to} - m_{11}\pi$$

Hence a sufficient condition for the failure of the one channel method is that the Levinson theorem $n_B \leq m_{11}$ is not satisfied: e.g. case (i). That it is not a necessary condition can be seen from fig 2, where the one channel method will clearly break down, although the inequality is satisfied. To see that the inequality is required, it is only necessary to consider the success of the method in case (2).

It is apparent that this trouble will not arise in the Frye-Warnock[3] method, because the unitary condition here is that $\text{Im } a = \rho|a|^2 + \frac{(1 - z^2)}{4}$

and the simultaneous satisfaction of $z = 1$ and $a = 0$ is

equivalent to $a_{11} = a_{12} = 0$.

This problem can be fairly simply overcome by adding a C.D.D. pole at or near the position of the zero, so that the zero is shifted onto the unphysical sheet. It is not obvious how the residue of this pole can be found. A further difficulty, discussed in the next chapter, is that in case (2), where N and D have a simultaneous zero, the equations are very ill-conditioned, and numerical solutions lead to spurious resonances. It is probably safe to say, therefore, that the R method is incorrect when the elastic amplitude has a zero.

The other problem is much more difficult to analyse, although paradoxically much simpler to cure. This occurs when a bound state occurs in a second channel when the problem is treated elastically: when the channels are coupled it may not occur in the first. An example of this would be in the reaction $\gamma N \to \gamma N$: there will be a resonance due to the photoproduction of the N^*_{1236}, but this can scarcely occur because of the forces in this reaction, which are purely electromagnetic: it will in fact occur because this reaction is coupled to $\pi N \to \pi N$. This can be seen slightly more qualitatively from

$$D = 1 - \frac{1}{\pi}\int_L K(s',s)\ \ \mathrm{Im}\ B(s')\ D(s')ds'$$

$$K(s,s') = \int_R \frac{R(s'')\rho(s'')}{(s''-s)(s'-s)}\,ds''$$

Clearly is Im B is identically zero, then $D(s) = 1$ independent of R. If a bound state occurs in another channel, this will show up as a zero of det D in the multichannel method, which cannot possibly be reflected by a zero in the D derived from (14).

Atkinson, Dietz and Morgan[20] have shown by a method of considerable elegance that the conditions for the breakdown of the one channel[F-W] method can be specified, exactly in certain cases. They show that the S matrix can be diagonalised

$$S = 1 + 2i\rho^{\frac{1}{2}}t\rho^{\frac{1}{2}} = 1 + 2iT$$

$$T_D = \begin{bmatrix} T^{(1)} & \\ & T^{(2)} \end{bmatrix} = OTO^{-1} \qquad (15)$$

where $O = \begin{bmatrix} \alpha & B \\ -B & \alpha \end{bmatrix}$

$$\alpha = \tfrac{1}{2}(1 + (1+K^2(s)))^{-\frac{1}{2}}$$

$$\beta = \tfrac{1}{2}(1 - (1+K^2(s)))^{-\frac{1}{2}} \qquad K = \frac{2\sqrt{\rho_1\rho_2}\ t_{12}}{\rho_1 t_{11} - \rho_2 t_{22}} \qquad (16)$$

so that one can write $T^{(i)} = e^{i\delta(i)}\ \sin\delta^{(i)}$, where $\delta^{(i)}$

is an "eigenphase-shift". The analytic structure of these eigenamplitudes is complex, but one can derive a Levinsons theorem for them.

$$D = t^{-1}N = \rho^{\frac{1}{2}}T^{-1}\rho^{\frac{1}{2}}N = \rho^{\frac{1}{2}}O^{-1}T_0^{-1}O\rho^{\frac{1}{2}}N \tag{17}$$

so Det $D = \prod_i \left[T^{(i)}\right]^{-1}$ det N

It is plausible that all zeroes of det D correspond to poles in particular $T^{(i)}$ and vice versa, and so by arguments analogous to those usually used to prove Levinson's theorem, one can write

$$\delta^{(i)}(\infty) = -\pi n_\beta^{(i)} \tag{18}$$

From (15), we can derive

$$\begin{aligned} z\, e^{2i\delta_{11}} &= \alpha^2 e^{2i\delta^{(1)}} + \beta^2 e^{2i\delta^{(2)}} \\ z\, e^{2i\delta_{22}} &= \beta\, e^{2i\delta^{(1)}} + \alpha^2 \alpha^{2i\delta^{(2)}} \end{aligned} \tag{19}$$

which can be geometrically interpreted as fig (6). Clearly if δ_{11} is to pass through $\pi/2$ it is necessary

1) that $\delta^{(1)}$ passes through $\pi/2$ and $\alpha^2 > \beta^2$
2) that $\delta^{(2)}$ passes through $\pi/2$ and $\beta^2 > \alpha^2$

in other words the vector $e^{2i\delta}$ must circle the origin. This is the so-called crank-shaft theorem.

If, for example, $\alpha^2 > \beta^2$ for $s > s_o$ the (presumed coincident) threshold and there is a bound state in eigenchannel 2 only, then

$$\delta^{(1)}(\infty) = 0 \quad \delta^{(2)}(\infty) = -\pi$$
$$\text{so} \qquad \delta_{11}(\infty) = 0 \quad \delta_{22}(\infty) = -\pi \tag{20}$$

The phase shift calculated by the F.W. method will agree with the multichannel one, so $\delta^{F.W.}(\infty) = 0$

The Levinson theorem in this case is

$$\delta(\infty) - \delta(0) = -\pi(\bar{n} - n_c)$$

where n_c is the number of C.D.D. poles, so obviously one C.D.D. pole must be inserted, and the C.D.D. pole-free calculation will not work. There result can be extended under certain conditions to many-channel distinct threshold problems.

This elegant result, however, is clearly of no use in a practical calculation, since it requires a full and accurate multichannel calculation to say whether the one channel solution will be correct. The following argument which presumably can be made rigorous for a L.H.C. cut consisting only of poles, appears to give a

more satisfactory criterion. Since any well behaved L.H.C. can be approximated to arbitrary accuracy by a sum of poles, the deduction is probably fairly valid.

A one channel D function, calculated without inelasticity will have n_{11} zeroes which in the limit of $g_{11} = 0+$ (the relevant coupling constant) will lie on the unphysical sheet close to the poles. However in the same limit, the many channel det D may have zeroes away from the cuts, corresponding either to bound states or resonances. Any R function calculated from these must be finite, and the inelastic D function will again have zeroes near the poles, because the form of the integral to be evaluated has not significantly altered. When the coupling is increased, these zeroes may move near or onto the physical sheet, and these will coincide with multichannel zeroes. However, the multichannel zeroes that start on or near the physical sheet will never have corresponding zeroes in the inelastic calculation, because this would mean discontinuity in $d\nu_0/dg_{11}$, where ν_0 is the position of the zero under consideration.

This argument must be slightly modified for the F.W. method, because a bound state could be produced even

in the limit of $g_{11} = 0$, since a generalised Born term

$$\bar{B}(s) = B(s) + \int_R \frac{1 - \eta^2(s')}{s' - s}\, ds'$$

is used in the equation for N

$$\frac{2}{1+\eta(s)} N(s) = \bar{B}(s) + \frac{1}{\pi} \int_R \frac{s'B(s') - sB(s)}{s'(s' - s)} \frac{2\rho(s')}{1+\eta(s')} N(s')ds'$$

and it is possible that this bound state is correct.

It is not obvious that any bound state produced by inelasticity is necessarily correct. Any criterion based on Levinson's theorem such as the crank-shaft theorem, is only a sufficient condition for breakdown, and so it is possible that a bound state calculated by either method could still be in the wrong position, so the amplitude would still require a C.D.D. pole.

A simple model which demonstrates these properties can be easily found. The simplest two channel model is considered: non-relativistic with coincident thresholds and a one pole L.H.C.

If $\quad \text{Im } B = \mu\delta(\nu + 1)$

we find $\quad D(\nu) = 1 - \mu D(-1)F(\nu)$ where $F(\nu) = \frac{1}{\sqrt{-\nu}+1}$

(for convenience the subtraction point is taken at ∞)

Hence $\underset{\sim}{D}(-1) = (1 + \frac{1}{2}\underset{\sim}{\mu})^{-1}$

and $$\underset{\sim}{a}(\nu) = \frac{\underset{\sim}{\mu}(1 + \frac{1}{2}\underset{\sim}{\mu})^{-1}\ 1 - \underset{\sim}{\mu}(1 + \frac{1}{2}\underset{\sim}{\mu})^{-1} F^{-1}}{1 + \nu}$$

$$= \underset{\sim}{\mu}(1 + \underset{\sim}{\mu}(\tfrac{1}{2} - F))^{-1}/(1 + \nu)$$

This gives an inelasticity function that can be directly integrated:

$$R = 1 + \left|\frac{a_{12}}{a_{11}}\right|^2$$

$$= 1 + \frac{4\ \mu_{12}{}^2}{(2\mu_{11}+\det\underset{\sim}{\mu})^2} \quad \frac{1 + \nu}{\nu + \left|\frac{2\mu_{11}-\det\underset{\sim}{\mu}}{2\mu_{11}+\det\underset{\sim}{\mu}}\right|^2} \qquad (24)$$

$$= 1 + a\,\frac{1 + \nu}{\nu + b}$$

Hence $$D(\nu) = 1 - \mu_{11} D(-1)\left(\frac{1}{\sqrt{-\nu+1}} + \frac{a}{\sqrt{-\nu+b}}\right) \qquad (25)$$

In the limit of $\mu_{11} = 0+$, $R \simeq 5$, so the effect of the inelasticity is to produce a second sheet pole at the position of the force pole, which gradually moves away as μ_{11} is increased. These two poles produce a zero, which moves up the negative real axis on the second sheet to threshold, where it passes through to the physical sheet and becomes a bound state. In the corresponding two channel det D, there are two zeroes, one of which may start on the physical sheet, while the

second corresponds exactly to that calculated by the one-channel method with inelasticity.

A curious feature of this particular model is that if the negative square root of b^2 is taken in (25), the calculations are identical: in other words this is precisely the correct form for the C.D.D. pole. This is presumably a coincidence due to the very simple form of the model, and does not seem to carry through to more complex cases.

In view of the forgoing it seems safe to make the following statements:

1) The R method will always break down when a zero occurs in the amplitude: in principle for case 1) and in practice for case 2).
2) The R method will always break down when a bound state occurs in the multichannel calculation with $g_{11} = 0$, but will work when the bound state occurs in the elastic calculation. It is difficult to generalise about intermediate cases, when the bound state is produced by the interchannel coupling.
3) The η method will not work when the crank-shaft theorem indicates it will not, which will usually be when the bound occurs in a second channel.

These observations are of little practical use, because we must always solve the multichannel problem to decide if the one-channel will work. We may summarise this chapter by saying that if the single channel ND^{-1} method does not have the correct characteristics, then it is unlikely that adding inelasticity will make an essential difference.

CHAPTER 3.

Anomalous Solutions of the ND^{-1} equations.

In general, the ND^{-1} equations have solutions which have the following properties:

1) $D(\nu) \to 1$ as $\nu \to \infty$

2) The same Levinson's theorem is satisfied by the D function and the amplitude, at least when C.D.D. poles areaabsent.

3) The N and D functions are unique, and stable with respect to small perturbations of the input.

However, it is possible to find force functions for which none of these statements hold: in particular one can find solutions which have a simultaneous zero in N and D, which is the previously mentioned extinct bound state (E.B.S.). This is of interest because of Chew's suggestion that Regge trajectories may have vanishing residues at what would otherwise be physically important angular momentum values.

We consider, for simplicity, a two pole non-relativistic model, with the solution (eqn, 4 of chap. 2)

$$D(\nu) = 1 - \frac{\mu_1 D(-\nu_1)}{\sqrt{-\nu}+\sqrt{\nu_1}} - \frac{\mu_2 D(-\nu_2)}{\sqrt{-\nu}+\sqrt{\nu_2}} \qquad (1)$$

If solving this for $D(-\nu_1)$ and $D(-\nu_2)$, we find

$$1 + \frac{\mu_1}{2\sqrt{\nu_1}} D(-\nu_1) + \frac{\mu_2 D(-\nu_2)}{\sqrt{\nu_1}+\sqrt{\nu_2}} = 1 \qquad (2)$$

$$\frac{\mu_1}{\sqrt{\nu_1}+\sqrt{\nu_2}} D(-\nu_2) + 1 + \frac{\mu_2}{2\sqrt{\nu_2}} D(-\nu_2) = 1$$

with the solution

$$D_1 = \frac{A + \mu_2\sqrt{\nu_1}B}{C} \qquad D_2 = \frac{A - \mu_1\sqrt{\nu_2}B}{C} \qquad (3)$$

where $D_i = D(-\nu_i)$

$$A = 4\sqrt{\nu_1\nu_2}\,(\sqrt{\nu_1} + \sqrt{\nu_2}) \qquad (4)$$

$$B = 2(\nu_1 - \nu_2)$$

$$C = (\mu_1 + 2\sqrt{\nu_1})(\mu_2 + 2\sqrt{\nu_2})(\sqrt{\nu_1} + \sqrt{\nu_2}) - 4\mu_1\mu_2\sqrt{\nu_1\nu_2}$$

There is apparently no solution if $C = 0$ (5)

corresponding to the vanishing of the determinant for eqn.(2) unless also

$$\mu_i = \frac{2\sqrt{\nu_i}\,(\sqrt{\nu_1} + \sqrt{\nu_2})}{2\sqrt{\nu_i} - \sqrt{\nu_1} - \sqrt{\nu_2}} \qquad (6)$$

which implies the eqns. (2) are identical. Without any significant loss of generality, and a great increase in numerical convenience, we take $\nu_1 = 4$, $\nu_2 = 9$.

Then (5) gives

$$\mu_1 = \frac{-100(\mu_2 + 6)}{\mu_2 + 150} \qquad (7)$$

Consider, for example, $\mu_2 = -\mu_1 = 10$. If we take a small

perturbation ε of the μ_i, we find that a_1 is well behaved as $\varepsilon \to 0$, although the individual N and D functions are not. In fact

$$D(\nu) = 1 + \frac{300(10+\varepsilon)}{\varepsilon(\varepsilon+250)(\sqrt{\nu}+2)} - \frac{200(20-\varepsilon)}{\varepsilon(\varepsilon+250)(\sqrt{-\nu}+3)}$$

$$N(\nu) = -\frac{300(10+\varepsilon)}{\varepsilon(\varepsilon+250)(\nu+4)} + \frac{200(20-\varepsilon)}{\varepsilon(\varepsilon+250)(\nu+9)}$$

$$\text{while} \quad a(\nu) \underset{\varepsilon\to 0}{=} \frac{\nu-2}{\nu+1} \frac{(\sqrt{-\nu}+2)(\sqrt{-\nu}+3)(1-\sqrt{-\nu})}{(\nu+4)(\nu+9)}$$

which is an apparently well behaved amplitude with the expected force poles and a bound state at $\nu = -1$. However, asymptotically

$$a(\nu) \sim \frac{(-\nu)^{3/2} + 4\nu}{\nu^2}$$

$$\text{so Im } a(\nu) \sim \frac{1}{\nu^{\frac{1}{2}}} \qquad \text{Re } a(\nu) \sim \frac{1}{\nu}$$

and so $\delta(\infty) = \pi/2$: in other words, the normal Levinson's theorem does not hold.

If we now satisfy (6), (in this case $\mu_1=-20$, $\mu_2=30$) which is, of course, a special case of (5), an E.B.S. is produced. If we write $J_i = \mu_i D_i$, then we can choose J_1 and J_2 such that $N(\nu_0) = D(\nu_0) = 0$ for some ν_0. If

we then calculate μ_1 and μ_2 from (1), and then feed these back in through (2), we find that (5) is satisfied, and the equations (2) are identical, and give

$$D_2 = (1 + 4D_1)/6$$

Hence we find

$$D(\nu) = \frac{((-\nu)+20D_1-4)}{(\sqrt{-\nu}+2)(\sqrt{-\nu}+3)} \qquad (8)$$

$$N(\nu) = \frac{5((-\nu)+20D_1-4)}{(\nu+4)(\nu+9)} \qquad (9)$$

and D and N have a simultaneous zero at the totally arbitrary point $\nu = -4+20D_1$. Clearly D obeys a Levinson's theorem

while a_1 obeys $\delta(\infty) = -\pi = -(n_e + n_b)\pi$

$$\delta(\infty) = 0 = -n_b\pi. \qquad (10)$$

where n_e is the number of extinct bound states

Auberson and Wanders show that the point in the $\mu_1\mu_2$ plane given by (6) is an instability point in the following sense: small changes in μ_1 and μ_2 from these critical values lead to seven different combinations of bound states, virtual states and resonances. An amusing (though useless) observation is that if only one of the poles satisfies (6), the bound state is automatically produced at the position of the other: the second pole becomes irrelevant.

Atkinson and Halpern find the general condition that an E.B.S. is produced. If

$$\text{Im } B(\nu) = - \text{Im } z(\nu) \frac{1}{\pi} \int_0^\infty \frac{\rho(\nu')z(\nu)}{\nu'-\nu} \, d\nu' \qquad (11)$$

where $z(\nu)$ is an arbitrary function analytic but for an L.H.C, satisfying

$$\frac{1}{\pi}\int_0^\infty \rho(\nu')z(\nu')d\nu' = \mu \qquad (12)$$

and $\rho(\nu')$ may include an inelasticity factor, than a_1 has a E.B.S. The proof is simple: writing

$$N(\nu) = \frac{\nu - \nu_0}{\mu} \, z(\nu)$$

gives

$$D(\nu) = 1 - \frac{1}{\pi\mu} \int_0^\infty \rho(\nu)z(\nu') \frac{\nu'-\nu_0}{\nu'-\nu} \, d\nu' \qquad (13)$$

$$= \frac{\nu-\nu_0}{\pi\mu} \int_0^\infty \frac{\rho(\nu')z(\nu')}{\nu'-\nu} \, d\nu'$$

so $D(\nu_0) = N(\nu_0) = 0$. Another way of stating the same result is to say that the homogenous equations for N and D are solved, which again gives a one parameter arbitrariness in the solution. It is trivial to see that the input $\mu_1\delta(\nu+\nu_1)+\mu_2\delta(\nu+\nu_2)$ with (6) satifies (11).

Another curious but useless relation for extinct bound states involves the Omnes-Muskhelishvilli function.

If we write

$$D(\nu) = (\nu - \nu_0)\mathfrak{D}(\nu)$$

and require that $N(\nu_0) = 0$ also, we find

$$\int_L \text{Im } B(\nu')\mathfrak{D}(\nu')d\nu' = 0 \text{ or } \int_L B(\nu) \frac{\mathfrak{D}(\nu)}{d\nu} d\nu = 0 \qquad (14)$$

The simultaneous zeroes found in the previous chapter imply that R must satisfy a condition on a certain Fredholm determinant. (11) leads to an integral equation for z that is the homogeneous equation for N

$$z(\nu) = \int_R \frac{B(\nu)-B(\nu')}{\nu - \nu'} \rho(\nu')R(\nu')z(\nu')d\nu' \qquad (15)$$

and this must have a non-trivial solution: in other words $\Delta = 0$ where (16)

$$\Delta = 1 - \frac{(-1)^n}{n} \int_R \det|K(\nu_i,\nu_j)| \prod_{i=1}^{n} d\nu_i d\nu_j \qquad (17)$$

and $K(\nu,\nu') = \frac{B(\nu)-B(\nu')}{\nu - \nu'} \rho(\nu')R(\nu')$. In fact the $R(\nu)$ found numerically satisfies a simpler condition, analagous to (2), which can be derived from (11), to a high degree of accuracy. However it is very hard to see how one can show that any R, derived from a multichannel calculation with the usual unitarity conditions and the condition on the phase shift, will satisfy (16). R is not an analytic

function, and this appears to make any statements of this form impossible.

Gross and Kayser show that an E.B.S. cannot occur for a 'normal' potential. If a potential satisfies

$$\int_0^\infty r|V(r)|dr < \infty \text{ and } \int_0^\infty e^{-\mu r}|V(r)|\,dr < \infty \qquad (18)$$

and we write

$$D(k) = f(-k),\ N(k) = \frac{f(k) - f(-k)}{2ik} \qquad (19)$$

then $f(\pm k) = u(\pm k, 0)$ where $u(\pm k, r)$ are the solutions of the Schrodinger equation behaving asymptotically as $e^{\pm ikr}$.

Then the Wronskian $W\left[u(k,r), u(-k,r)\right]$ can be formed and shown to be independent of r and equal to 2ik at $r = \infty$. However, if $f(k) = f(-k) = 0$, corresponding to an E.B.S., then the Wronskian for $r = 0$ is zero. Hence conditions (18) cannot be satisfied.

They then proceed to show that the potential in fact behaves like $\frac{1}{r^2}$ at the origin. The Gelfand-Levitan-Marchenko (G-L-M) method is used, which enables one to express the L.H.C. discontinuity in terms of an equivalent potential: the following is a summary of their method. Marchenko showed that the potential may be written

$$V(r) = \frac{d}{dr} A(r,r) \tag{20}$$

where $A(x,y)$ is given by

$$A(x,y) + F(x+y) + \int_x^\infty dt\, A(x,t)F(t+y) = 0$$

and $F(r)$ is given by the Fourier transform of A,

$$A(x,t) = \int a(\nu,x)e^{-t\sqrt{-\nu}}\, d\nu \tag{21}$$

one may derive

$$a(\nu,x) - \int_L d\nu' \frac{f(\nu)\, e^{-x(\sqrt{-\nu}+\sqrt{-\nu'})}\, a(\nu'x)}{\sqrt{-\nu} + \sqrt{-\nu'}} = f(\nu)e^{-x\sqrt{-\nu}} \tag{22}$$

where $f(\nu) = \frac{1}{\pi} \operatorname{Im} a_1(\nu)$ on the L.H.C.

When the L.H.C. is a sum of poles, (21) cannot be solved directly. However (20) is equivalent to

$$V(r) = \frac{(\Delta'(r))^2 - \Delta(r)\ \Delta''(r)}{(\Delta(r))} \tag{23}$$

where $\Delta(R)$ is the Fredholm determinant of (21). We may then solve explicitly for the potential: if the L.H.C. consists of n poles, the first n terms in the Fredholm series are non-zero, so corresponding to a two pole input

$$\Delta_1(x) = -\int_L f(\nu)\ \frac{e^{-2x\sqrt{-\nu}}}{2\sqrt{-\nu}}\ d\nu$$

$$\Delta_2(x) = \tfrac{1}{2} \iint_{LL} f(\nu)\ \frac{e^{-2x\sqrt{-\nu}}}{2\sqrt{-\nu}}\ f(\nu')\ \frac{e^{-2x\sqrt{-\nu}}}{2\sqrt{-\nu'}}$$

$$-f(\nu)\, f(\nu') \frac{e^{-2x(\sqrt{-\nu}+\sqrt{-\nu'})^2}}{\sqrt{-\nu}+\sqrt{-\nu'}}\, d\nu'\, d\nu$$

leading to

$$\Delta(x) = 1 + \frac{\mu_1 e^{-2p_1 x}}{2p_1} + \frac{\mu_2 e^{-2p_2 x}}{2p_2} + \frac{(p_1-p_2)^2 \mu_1 \mu_2 e^{-2x(p_1+p_2)}}{4p_1p_2(p_1+p_2)^2}$$

(writing $p_i = \sqrt{\nu_i}$)

Now if $\Delta(x)$ = constant at $x = 0$, the resulting potential will be well behaved. However, if μ_1 and μ_2 are chosen such that $\Delta(x) \approx x$ as $x \to 0$, it is easy to see that $V(x) \sim 1/x$. This corresponds to the dondition

$$1 + \frac{\mu_1}{2p_1} + \frac{\mu_2}{2p_2} + \frac{(p_1-p_2)^2 \mu_1 \mu_2}{4p_1p_2(p_1+p_2)^2} = 0 \qquad (26)$$

which is identical to (5). If we further demand that $\Delta'(x) \sim x$ as $x \to 0$, then we find a further condition

$$\mu_1 + \mu_2 + \frac{(p_1-p_2)^2 \mu_1 \mu_2}{2p_1p_2(p_1+p_2)} = 0 \qquad (27)$$

(note that $1/x^2$ behaviour of $V(x)$ is retained, since if $\Delta'(x) \sim x$, then $\Delta(x) \sim x^2$). These two conditions turn out to be equivalent to (6). As the second order vanishing of this quantity $\Delta(x)$ has no particular physical significance, an E.B.S. cannot be regarded as having any significance in potential theory. A plot of the potential satisfying (26) and (27) shows that it is

purely repulsive.

It is a fact that is not generally realised that although presumably any L.H.C. discontinuity corresponds to a potential, the connection is not analytic. In the μ space there is a line (or in general a surface) given by C = 0 in eqn. (4), or its generalisations, which behave like $1/r^2$ at the origin. However artbitrarily small perturbations lead to well behaved potentials. There is a one to one correspondence between the vanishing of the Fredholm determinant for the D equation, the vanishing of the Fredholm determinant for the Gelfand-Levitan equation and the production of an E.B.S.

An interesting, speculation suggests itself at this point. The multichannel N/D equations have a multichannel Schrodinger analogue, and presumably their solutions are identical: similarly the one channel N/D equations, with inelasticity, have a corresponding Schrodinger equation with an equivalent (presumably complex, energy dependent) potential. It is plausible that the connection between E.B.S. and $1/r^2$ potentials carries through to the inelastic case. We have seen in Chapter 2 how an E.B.S. must be produced if the method with inelasticity is to be correct when the amplitude has a zero: hence the

equivalent potential should have a $1/r^2$ singularity. Now it is known that in elastic potential scattering, singular repulsive potentials lead to Regge trajectories that are straight as $s \to \infty$ (because the behaviour of the trajectory at infinite energy is linked to the curvature of the potential at the origin.) Hence the empirical straightness of the trajectories may be simply explained by the fact that all physical calculations should really be considered as coupled channel problems.

It is apparently very difficult to proceed further, because equivalent potentials for coupled channels are energy-dependent (and hence non-local) and complex. The method outlined above is unsuccesful, because an implicit assumption in the derivation of the G-L-M equations is that the ν-plane may be 'unfolded' along the unitarity cut: in other words there is no unitarity cut in the $k = \sqrt{\nu}$ plane. It is probably impossible to solve the resulting equations if the transformation $k = R\sqrt{\nu}$ is made, because R is not an analytic function. The method of Bargmann cannot be used, because this assumes that the S- matrix may be written in the form

$$S(k) = \frac{f(k)}{f(-k)} \qquad (28)$$

where $f(k)$ is of the form $f(k) = \prod_{n=1}^{N} \frac{k - \alpha n}{k - \beta n}$ (29)

Although one can consider non-unitary S-matrices (and hence complex potentials) in this method, which is equivalent to, but less convenient than, the G-L-M formalism with a pole input, it is not true that any complex potential corresponds to a real multichannel one, in the same way that R must satisfy a very subtle condition if it is to represent a 'genuine' multichannel effect. For a 'genuine' R, such as that calculated at the end of the preceding chapter, it is impossible to make the decomposition in (28) and (29).

CHAPTER 4.

A Dynamical Quark Model.

It is apparent from the discussion in Chapter 2 that, if the major forces in a second channel, then it is in general improbable that a multichannel bound state will be found in a one-channel calculation. In this and the next chapter we consider two cases: one in which the major forces must occur in a second channel and one in which they may do so. The first deals with the so-called 'naive' quark model, and the second with a two channel calculation of the P_{11} wave in πN scattering.

The great success of SU(3) as a classification scheme led to many attempts to obtain a less empirical basis for it. The most successful of these is the quark model[24,25], which assumes a triplet of basic spin $\frac{1}{2}$ particles with the following quantum numbers

	I	I_3	S	Y	B	Q
p	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
n	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
λ	0	0	1	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$

(1)

Then the mesons may be constructed from quark-anti-quark pairs

e.g. $$\pi^+ = \frac{1}{\sqrt{2}} (p \uparrow \bar{n} \downarrow - p \downarrow \bar{n} \uparrow)$$

$$\rho^+_{\uparrow} = p \uparrow p \uparrow$$

and the baryons from symmetrised quark triplets

e.g. $$\uparrow = \frac{1}{\sqrt{18}} (B_{115} - 2B_{124})$$

where the B_{ijk} are SU(6) invariant tensors which describe the wave functions: e.g. B_{114} is the fully symmetrised combination of $p \uparrow p \uparrow n \downarrow$

It is fairly simple to show that if baryon number and charge is conserved, then permutation symmetry on the quarks is equivalent to SU(6) symmetry[25]. However, very much more can be deduced from the quark model than from the symmetry; for example, the quark model predicts that mesons will occur in nonets (corresponding to the 9 possible $q\bar{q}$ combinations in the various possible states of angular momentum). Plausible potentials lead to reasonable intermultiplet splittings, and the following nonets have be tentatively identified[27] (the notation is the standard spectroscopic one).

State	L	S	J^{PG}	Particles
$^1S_0$	0	0	0^{--}	$\pi K \eta \eta'$
$^3S_1$	0	1	1^{-+}	$\rho K^*_{891} \omega \phi$
$^3P_2$	1	1	2^{+-}	$A_2 K^*_{1120} ff'$
$^3P_1$	1	1	1^{+-}	$A_1 K^*_{1320} DE$
$^3P_0$	1	1	0^{+-}	$S_{460}, \pi_{1003}, K_{725}, \pi_\nu(1050)$
$^1P_1$	1	0	1^{++}	δ_{960}, K_{1080},
$^3D_3$	2	1	3^{-+}	$R_3(1640)$
$^3D_2$	2	1	2^{-+}	$R_2(1650)$
$^3D_1$	2	1	1^{-+}	$R_1(1700)$
$^1D_2$	2	0	2^{--}	
$^3F_4$	3	1	4^{+-}	S
$^3G_5$	4	1	5^{-+}	T
$^3H_6$	5	1	6^{+-}	U

The assignments in the first three nonets are almost certain, as is the existence of most of the other particles: however it must be emphasised that the scheme is still fairly speculative. For example, the spin of the R,S,T and U mesons is a guess based on the assumption of a linear Regge trajectory; and the existence of the whole scalar nonet is in doubt. It appears that the observed resonances may tend to split into two or three (this has already happened for the R, and may well happen for the A_1 and

A_2) when better experimental resolution becomes available, and there will, if anything, be a surfeit of particles. The only particle at present that cannot be fitted into a multiplet is the I=3/2 K^*_{1175}, and the existence of this is very much in doubt.

For baryons, the model is less successful, because of the much greater complexity of 3 quark systems and the requirement of parastatistics to give a symmetric S-wave wave-function. The octet and decuplet are satisfactorily described, and it has been shown by Dass and Ross[2] that a large number of members of higher SU(6) multiplets will be coupled either very weakly or very strongly to presently available particles, so that they are unlikely to be experimentally observed for some time.

The quark model gives a satisfactory physical interpretation[25] of the Gell-Mann-Okubo mass formula, by assuming that $m_n = m_p$, $m_\lambda = m_p + \delta$ and an isospin splitting $\propto I(I+1)$; and the Coleman-Glashow formula, by assuming that $m_n = m_p + \delta_E$ and an electrostatic interaction $\propto Q_q$. The famous SU(6) result $\mu_n/\mu_p = -2/3$ follows, by assuming that the magnetic moments of the quarks are equal to their charge, and sum to give that of the nucleon. However there is some difficulty in

explaining the fact that mesons obey a $(\text{mass})^2$ formula, while baryons do not: this can be explained by assuming certain types of potential[27].

From other assumptions, further mass relations can be derived: if the wave function is the same for all members of a multiplet then the mass splitting depends solely on the additional mass of the λ quark

$$m_K^2 - m_\pi^2 = m_{K^*_{891}}^2 - m_\rho^2 = m_{K^*_{1405}}^2 - m_{A2}^2 \quad (.220:.214:.218) \tag{4}$$

However, such rationalisation of SU(6) results is a comparatively small success. The greatest success of the quark model has been in predicting relations between cross-sections. This was originally done by Lipkin[29] and collaborators and extended to a large number of other interactions[30]. The basic assumption is the impulse approximation; the particles may be envisaged as being bound by a comparatively long-range energy-dependent force, and interacting via a short-range one. (The model may be likened to hard balls attached by elastic rods; it is amusing to note that a relativistic rotator gives rise to an energy level formula $M^2 = kl$[31], which

is apparently satisfied by the ρ-A_2-R-S-T-U chain of mesons). Hence in any given collision, the amplitude is the sum of two-quark amplitudes while the other quarks are regarded as spectators. Thus, ignoring spin, we can write the $\pi^+ p \to \pi^+ p$ amplitude as

$$\langle \pi^+ p | T | \pi^+ p \rangle = \langle (p\bar{n})(ppn) | T | (p\bar{n})(ppn) \rangle$$

$$= 2\langle \bar{n}p | T | \bar{n}p \rangle + 2\langle pp | T | pp \rangle + \langle pn | T | pn \rangle + \langle \bar{n}n | T | \bar{n}n \rangle \quad (5)$$

By summing the similar amplitudes, we can derive

$$\frac{2}{3} (\sigma_{pN} + \sigma_{\bar{p}N} + \sigma_{pp} + \sigma_{\bar{p}p}) = \sigma_{\pi^+ p} + \sigma_{\pi^- p} \quad (6)$$

Here we have used only the optical theorem and isospin invariance. The further assumption of the Pomeranchuk theorem gives the well known

$$\frac{\sigma_{pp}}{\sigma_{\pi p}} = \frac{3}{2} \quad (7)$$

Difficulty arises when (7) is compared with experiment because it is not clear whether or not we should compare the cross-sections at the same energy. The discrepancy is about 20% if both are compared at 18 G.e.v. Van Hove suggested they should be compared at the same C.o.m. energy per quark;

i.e. $$\frac{Epp}{E\pi p} = \frac{3x3}{2x3} = \frac{3}{2}$$

which gives the ratio in [7] as .6. HDD Watson and P James[32] have argued that the cross-sections should be compared at an energy such that the relative velocity of the quarks should be the same. Considering the proton in each case as stationary, we find

$$E_\pi = \frac{m_\pi}{\sqrt{1-v_\pi^2}} \qquad E_p = \frac{m_p}{\sqrt{1-v_p^2}}$$

and since the velocity of the quark is obviously equal to that of the particle, $\frac{E\pi}{E_p} = \frac{m\pi}{m_p}$. The relation (7) now requires a plausible extrapolation, but the agreement now seems excellent. More recently Van Hove and Kokkedee have argued that, since the quark model specifically assumes conservation of quarks and anti-quarks individually, we must exclude the annihilation cross-section from $\sigma p\bar{p}$ in equation 6), which brings it into satisfactory agreement with experiment. In fact the error in 6) is then less than the experimental error. It is an interesting observation that no other model of high energy scattering predicts even that

The basic idea has been extended to non-forward and inelastic reactions, which require additional

assumptions to replace the optical theorem. Kokkedee and Van Hove consider

$$T_{AB}(s,t) = \sum_{i,j} f_i^A(t) f_j^B(t) \langle ij|T|ij\rangle$$

as an ansatz for the scattering amplitude: f_i^A and f_j^B are the formfactors (corresponding to the Fourier transform of the wave-function of quark i in particle A etc) and $\langle ij|T|ij\rangle$ the matrix element for the quark scattering reactions. These last are probably more or less equal at high energy, so we can write

$$T_{AB}(s,t) = A(t) \Sigma_i f_i^A(t) \Sigma_j f_j^B(t) \qquad (9)$$

which illustrates a property similar to the Regge property of factorisation.

For simple inelastic reactions which only involve the change of spin, charge or strangeness of one pair of quarks, the amplitude can be written as

$$\langle AB|T|CD\rangle = \langle ij|T|kl\rangle \qquad (10)$$

where quark i is in particle A and scatters off quark j from particle B. Hence $P\bar{P} \to \Lambda\bar{\Lambda}$ is described by (ignoring spin)

$$\langle (ppn)(\bar{p}\bar{p}\bar{n})|T|(pn\lambda)(\bar{p}\bar{n}\bar{\lambda})\rangle = \langle p\bar{p}|T_Q|\lambda\bar{\lambda}\rangle \qquad (11)$$

and the pn, $\bar{p}\bar{n}$ pairs are assumed not to enter into the reaction. To achieve any quantitative results one must assume that t dependence is the same for any i,j,k, l in (10). If the analysis is carried out with the inclusion of spin these requirements can be relaxed but the predictions are rather more complex.

The simplest predictions tend to be negative: for example

$$\langle P\bar{P}|T|\ \equiv \rangle = \langle P\bar{P}|T|\Sigma^{-}\bar{\Sigma}^{+}\rangle = \langle P\bar{P}|T|N^{*-}\bar{N}^{*-}\rangle = 0 \qquad (12)$$

because all require two colisions of the type $p\bar{p} \to \lambda\bar{\lambda}$

However $\langle P\bar{P}|T|\Sigma^{+}\bar{\Sigma}^{+}\rangle \neq 0$ (13)

because this requires only $n\bar{n} \to \lambda\bar{\lambda}$: experimentally the cross-section from (13) is at least 30 times as large as those in from (12). An additional encouraging result is that one would expect the angular distribution given by (13) to be largely forward; this is in fact so, while the distributions (such as they are) from the first two reactions in (12) are isotropic.

Calculations have been done involving spin, and these too have satisfactory consequences. For example, $\pi^{+}p \to \pi^{+}P$ with spin-flip and $\pi^{+}p \to \rho^{+}P$ are clearly associated with the two amplitudes

$$\langle p\uparrow p\uparrow|T|\ p\uparrow p\downarrow\rangle \text{ and } \langle \bar{n}\uparrow p\uparrow|T|\ \bar{n}\uparrow p\downarrow\rangle$$

which might be expected to vanish at the same rate. Experimentally ρ production decreases rapidly with energy, while spin-flip remains fairly large even at high energy. In fact it turns out that the amplitudes have coefficients such that ρ production vanishes in the Pomeranchak limit while spin flip does not.

Further relations can be derived between production amplitudes for particles in different multiplets. In this case, the wave function must be split into a spatial part which will be the same for every particle in a multiplet, and a part which depends solely on the quark interaction. Thus

$$\begin{aligned} A_{\pi^- p \to K^- \Sigma^+} &= f_1(\underline{\mathbf{Y}}, k) \langle \bar{p}p | T | \bar{\lambda}\lambda \rangle \\ A_{\pi^- p \to K^*_{1405}\, \Sigma^+} &= f_2(\underline{\mathbf{Y}}, k) \langle \bar{p}p | T | \bar{\lambda}\lambda \rangle \end{aligned} \tag{14}$$

so that

$$\frac{A_{\pi^- p \to K^*_{891}\, \Sigma^+}}{A_{\pi^- p \to \rho^- p}} = \frac{A_{\pi^- p \to K^*_{1320}\, \Sigma^+}}{A_{\pi^- p \to A_2^-\, p}} = \frac{A_{\pi^- p \to K^- Y^*_{1385}}}{A_{\pi^- p \to \pi^- N^*_{1238}}} \tag{15}$$

Backward scattering can be qualitatively described by the quark model. We must assume now that quarks have a tendency to pair inside a baryon, in much the same way as nucleons tend to form α particles inside a nucleus.

Then meson-baryon backward scattering will proceed by the backward scattering of the meson antiquark from the baryon quark pair (dence), while the remaining quarks continue in their original direction. Thus $\pi^+ p$ backward scattering may be written

$$(p_1\bar{n})+(p_2p_3n)\rightarrow(p_2,_3\bar{n})+(p_1p_3,_2n)$$

with a matrix element $2\langle\bar{n}\ (p\ n)|T|\bar{n}\ (p\ n)\rangle$. This has several very simple experimental consequences: for example

$$\frac{d\sigma}{dt}\pi^-p\rightarrow\pi^-p\ \Big|_{\theta=\frac{\pi}{2}} = \frac{d\sigma}{dt}K^-p\rightarrow\pi^-\Sigma^+\ \Big|_{\theta=\frac{\pi}{2}} \qquad (16)$$

$$\frac{d\sigma}{dt}\pi^-p\rightarrow K^-\Sigma^+\ \Big|_{\theta=\frac{\pi}{2}} = \frac{d}{dt}K^-p\rightarrow K^+\Xi^-\ \Big|_{\theta=\frac{\pi}{2}} \qquad (17)$$

These results are independent of spin or energy considerations: the first has L.H.S. and R.H.S. of 7.0μb and 10 ± 5μb respectively at 3.5 G.e.v.

However, the most elegant results obtained to date are the relations between density matrices for high spin particles produced in collisions[34]. Some density matrices are given as simple numbers by the quark model, without any assumptions about spin or energy considerations. For example, the reaction

$$p\bar{p}\rightarrow\Delta^{++}\ \Delta^{++}$$

leads to the values

	Theor	Expt. at 5.7 Gev/c
$\rho_{\frac{1}{2}\frac{1}{2}}$	0.5	$0.41 \pm .02$
$\rho_{\frac{3}{2}-\frac{1}{2}}$	0	0.0 ± 0.01
$\rho_{\frac{3}{2}\frac{1}{2}}$	0	-0.05 ± 0.02

The results are clearly good, even though their energy is not really sufficiently high. Similar results seem to provide the cleanest possible test of the model.

There have been several objections to the quark model. The most basic is simply that quarks have never been observed: this could be due [redacted] to their very high mass: the pair production cross-section is estimated to go down by a factor of 10^5 for every increment of one proton mass[35]. An aesthetically more satisfactory solution is Schiffs suggestion that free quarks cannot exist because of some fundamental conservation law, of electric of baryonic charge, which only allows physical states such that Q = 0 modulo 3.

Secondly there is the problem of parastatistics[36]. The lowest state for baryons (the 56 of SU(6)) requires a totally symmetric wave function. Since the unitary spin part of the wave-function is symmetric, then if the

spatial part is s-wave the quarks must obey parastatistics to avoid violating the Pauli principle. The alternative is that the lowest energy state is P-wave[37], which is unexpected on simple potential arguments (unless the dominant force between quarks is a three body one). However neither solution is impossible.

Thirdly there is the objection that the relations given by the quark model are simply wrong. In particular Barger and Durand[38] have claimed that results similar to (6), (7) and (13) are incorrect: notably the Johnson-Trieman relations

$$\sigma_{K^-p} - \sigma_{K^+p} = 2(\sigma_{\pi^-p} - \sigma_{\pi^+p}) \qquad (20)$$

However, this requires an additional assumption in that full SU3 symmetry is required, e.g.

$$\langle \bar{\lambda} p | \bar{\lambda} p \rangle = \langle \bar{n}\ p | \bar{n}\ p \rangle \qquad (21)$$

However, this is not part of the quark model, which assumes specifically that λ amplitudes are in general different from p and n: expirically there is a 20% difference, which carries through to backward scattering

$$\sigma^B_{K^+p\to K^+p} \simeq .8\ \sigma^B_{\pi^+p\to\pi^+p} \qquad (22)$$

whereas SU3 would predict them to be equal. Their second serious objection is that the relation

$$2\frac{d\sigma}{dt}\pi^- p \to \pi^0 n = \frac{d\sigma}{dt}K^- p \to \bar{K}^0 n = \frac{d\sigma}{dt}\bar{p}p \to \bar{N}N \qquad (23)$$

is seriously inaccurate. The first relation does not follow from the quark model, but, even using the Van-Hove or James-Watson energy prescriptions, the second is in poor experimental agreement. However, if the annihilation contribution is subtracted from the $\bar{p}p \to \bar{N}N$ amplitude, the agreement becomes fairly good. This is also the type of process which will be very sensitive to rescattering corrections, which differ for BB and MB processes.

A third significant failure of the quark model is in $\bar{P}P$ annihilation at rest[39]. The original model of Rubinstein and Stern which merely rearranges the quarks and antiquarks into pairs to form mesons is hopelessly wrong. About a dozen clear experimental comparisons can be made, and only two are correct: the branching ratios into $\pi^+\pi^-\eta$ and into $\pi^+\pi^-\pi^0$ final states (and the second for the wrong reasons, in that this in fact proceeds mainly through $\rho\pi$). The average number of mesons produced is 5 (not 3) and strange decays (forbidden on this model), form 10% of the total.

An attempt has been made to allow for creation and annihilation of $q\bar{q}$ pairs: the simplest assumption is that the pairs are produced in a $^1S_0$ state, and $p\bar{p}$, $n\bar{n}$ and $\lambda\bar{\lambda}$ production are equally probable. This is equivalent to producing an SU6 singlet. The rearrangement model used here is hardly any more successful: it predicts

$$\frac{A p\bar{p}\to\pi\pi}{A p\bar{p}\to\rho\pi} \sim 9$$

theoretically even before the unfavourable phase space has been allowed for, whereas the experimental ratio is 1/10. Since the annihilation is at rest, the mesons will be comparatively slowly moving and so will interact fairly strongly: in other words the flaw is in the assumption that there are no final state interactions. Experimental comparison should only be made after all strong decays have occurred, and even so this is probably not a good test of the model.

These are empirical objections: a theoretical one is that the model ought to be poor because it ignores the kinetic energy of the quarks inside the hadrons. This would be particularly bad for high spin particles: any extension to (say) spin 5/2 particles would be suspect on these grounds. Deloff[40] has shown that in a reason-

able model rescattering corrections will be of the order of 10%. This is irrelevant for comparisons of MB or BB amplitudes separately, but further confuses the comparison of the two together.

Finally there is the objection that the quark model tells nothing that cannot be obtained from the peripheral (or Regge pole) model plus some symmetry. For example the predictions (12) are found by the peripheral model because there is no doubly charged or double strange meson that can be exchanged: at high energy they must proceed by two particle or pole exchange, which makes no predictions about angular distribution. However, it is still difficult to obtain relations between MB and BB amplitudes on any other model. This is not so much an objection as a matter of taste.

It is apparent that to understand the quark model more fundamentally we must know something about the dynamics. An interesting suggestion is that quarks interact by a vector particle coupled to the baryon current[41]: this immediately gives SU3 symmetry. When we investigate the model more seriously, however, a paradox comes to light. If, for example, the ρ is a $q\bar{q}$ bound state then

it occurs only incidentally (i.e. as a C.D.D. pole) in the $\pi\pi$ channel. This implies that any low energy dynamical model, for example the ρ bootstrap, cannot hope to work, as the bound state is in the wrong channel. In other words the quark model, which assumes the ρ is basically a $q\bar{q}$ bound state, and conventional bootstrap theory, which makes it a $\pi\pi$ bound state, are incompatible. Incidentally it is impossible for it to be a bound state in both channels simultaneously, because degenerate perturbation theory tells one that the combined amplitude will contain two poles. It is known that a believable ρ can be produced by a bootstrap: therefore it is meaningful to ask whether it can possibly be made of quarks.

A model calculation[42] was performed to investigate this possibility. Because of our complete ignorance of the $q\bar{q}$ force, it is probably not unreasonable to ignore spin and isospin. We therefore consider two heavy (~5 G.e.v) quarks scattering in an S-state producing a deeply bound ρ: the force being provided by an X meson (again scalar). If we then couple in a second channel, corresponding to $\pi\pi$, and adjust the parameters to produce the correct $\rho \to 2\pi$ width, the position of the ρ pole will move. This displacement will give a rough idea of how

good or bad the model can be.

The first proposals for a realistic quark model[43] suggested that non-relativistic quantum mechanics could be used despite the very deep binding. The range due to the exchange of a meson of mass m_x is

$$R \sim \frac{h}{m_x c}$$

so from the uncertainty principal $p \sim \frac{h}{R} \sim m_x$

so that the kinetic energy $T \sim \frac{p}{2m_q} \sim \frac{m_x}{m_q}$

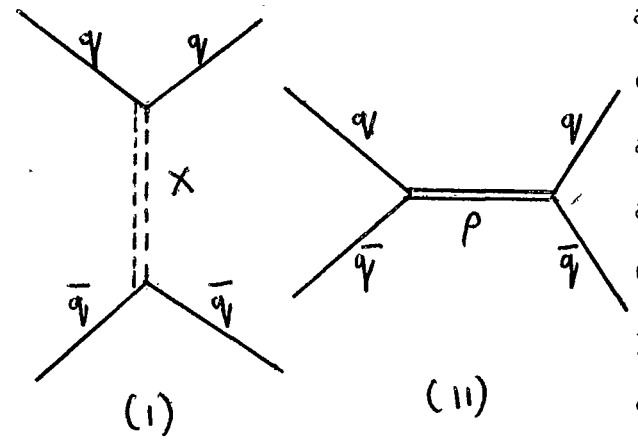
If $m_x \sim 1$ B.e.v. and $m_q \sim 5$ B.e.v., then $T \sim 200$ M.e.v. $<< m_q$, and the non-relativistic approximation is justified. From arguments about 'reasonable' potentials, one may show that the energy differences between states of different orbital angular momentum are likely to be of the order of the ground state energy, which is physically reasonable and can be compared to the multiplet structure described earlier. However, Greenberg has indicated that the non-relativistic approximation may be no good even for a superposition of Yukawas.

An alternative possibility is that the $q\bar{q}$ force is predominantly short range[44], corresponding to $m_x \sim m_q$. This would suggest very different predictions: the

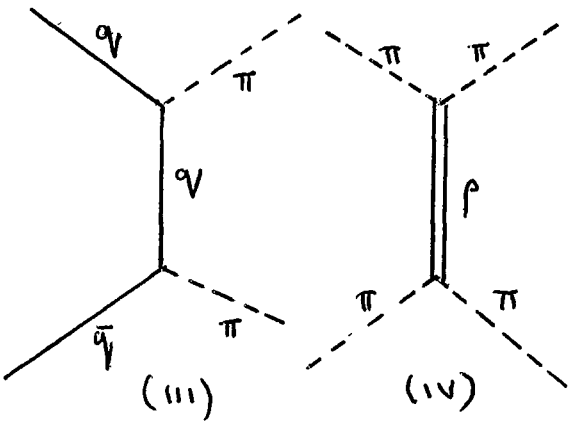
kinetic energy of the quarks would now be $\sim m_q$, and one would expect the energy difference between the s and p states to be similar. Thus only the $I = 0$ states in the conventional quark model would be genuine $q\bar{q}$ resonances: the higher ones would be bound states of pseudo-scalar or vector mesons. This would no longer require the baryon excited states to be C.D.D. poles: the N^*_{1236} would be a genuine πN bound state; this model has the pleasing result that only the lowest lying particles, the pseudo-scalar and vector meson nonets and the baryon octet are fundamentally quark bound states, but unfortunately it requires an extra singlet spin 3/2 baryon, which has not yet been observed (in other words, the baryons form a 20 representation of SU(6)). An even more extreme model might assume that quarks couple almost entirely by spin: this would give only a pseudo-scalar nonet and a baryon octet as fundamental, and has no C.D.D. problems as a result. However, although these models are appealing for dynamical reasons, they are far less rich in experimental predictions.

To test both models, we carry out a dynamical calculation with a variable mass for the exchanged meson.

For a given value of m_x, we use the Feynmann diagram (i)



as an input to the ND^{-1} equations. The ρ appears as a bound state in this amplitude (ii), and we adjust $G_{xq\bar{q}}$ until m_ρ is near the physical value. We then couple the ππ channel by (iii), introducing a second parameter $G_{\pi q\bar{q}}$ which is adjusted until $\mu_{\rho\to 2\pi}$ has the experimental value. There is an additional force due to (iv), but this turns out to be comparatively unimportant. Because most of the dynamics of the calculation comes from the $q\bar{q}$ channel, it should be possible to treat the ππ channel as a perturbing influence, and we now derive a formula which describes perturbature influences on ND^{-1} equations, similar to that of Dashen and Frautschi[45].



Both unperturbed and perturbed amplitudes obey unitarity on the right,

$$\text{Im } a^{-1} = -P$$

$$\text{Im } (a + \delta a)^{-1} = -P$$

so $$\text{Im } (a^{-1}\, \delta a\, a^{-1}) = 0 \qquad \text{in the R.H.C.}$$

Writing $a = ND^{-1} = (D^T)^{-1} N^T$ gives

$$\text{Im } (D\, \delta a\, D^T) = 0 \tag{24}$$

while on the L.H.C.

$$\text{Im } (D\, \delta a\, D^T) = D\, \text{Im}\, \delta a\, D^T \tag{25}$$

If we consider (24) and (25) specialised to the case where initially $a_{12} = a_{21} = 0$ (so the channels are decoupled), we obtain a dispersion relation for δa_{12}

$$\delta a \quad (=a_{12}) = \frac{1}{\pi D_{11}(s) D_{22}(s)} \int_L ds' \frac{D_{11}(s')\, \text{Im}\, a_{12}(s')\, D_{22}(s')}{s' - s} \tag{26}$$

Now if we consider the one channel case, initially

$$\text{Im } a_{11} = \rho_1 |a_{11}|^2 \tag{27}$$

and finally $$\text{Im}(a_{11} + \delta a_{11}) = \rho_1 |a_{11} + \delta a_{11}|^2 + \rho_2 |a_{12}|^2 \tag{28}$$

If the first channel has a high threshold, we may ignore the first term so that

$$\text{Im } \delta a_{11} = \rho_2 |a_{12}|^2 \qquad t_2 < s < t_1$$

or, since D_{11} is real in this region

$$\text{Im } (D_{11} \delta a_{11} D_{11}) = \rho_2 |a_{12}|^2 |D_{11}|^2 \tag{29}$$

Again dispersing this gives

$$\delta a_{11} = \frac{1}{D_{11}^2} \int_R \frac{ds'}{\pi} \rho_2 \frac{|a_{12}|^2 D_{11}|^2}{s' - s} \qquad (30)$$

Although (24) suggests that $D_{11}\delta a_{11} D_{11}$ has no R.H.C. this is only in first order, (30) is clearly a second order expression, which suggests that the approximation should be good. Near the ρ pole we have

$$a_{11} = \frac{-f_1^2}{s - s\rho}$$

so

$$\delta a_{11} = -\frac{f_1^2 \delta s\rho}{(s-s\rho)^2} + \frac{2f_1 \delta f_1}{(s-s\rho)} \qquad (31)$$

(30) and (31) give

$$\delta s\rho = -\frac{1}{f_1^2 |D_{11}'(s\rho)|^2} \int_R \frac{ds'}{\pi} \frac{|D_{11}|^2 |a_{12}|^2}{s' - s\rho} \qquad (32)$$

However, we also have

$$a_{12} = \frac{-f_1 f_2}{s - s\rho}$$

near the ρ pole, and this combined with (26) gives

$$f_1 f_2 = \frac{g}{D_{11}'(s\rho) D_{22}(s\rho)} \int_R \frac{D_{11}(s') \text{Im } a_{12}(s') D_{22}(s')}{s' - s\rho} ds' \qquad (33)$$

$$= \frac{g\, I(s\rho)}{D_{11}'(s\rho) D_{22}(s\rho)}$$

where g is some coupling constant.

From (33) and (26), g can be eliminated to give

$$a_{12}(s) = -f_1 f_2 \frac{D_{11}'(s\rho)D_{22}(s\rho)}{D_{11}(s)D_{22}(s)} \frac{I(s)}{I(s\rho)} \qquad (34)$$

so finally

$$\delta s\rho = \frac{-f_2^2}{\pi} \int_R^{\rho_2} \frac{I(s')}{I(s\rho)}^2 \frac{D_{22}(s\rho)}{D_{22}(s')} \frac{ds'}{s'-s\rho} \qquad (35)$$

which is an expression for $\delta s\rho$ in terms of f_2^2, the coupling of the bound state in the second channel. A very simple form occurs if we take

$$\text{Im } a_{12} = g\,\delta(s + \varepsilon) \qquad (36)$$

when $$I(s) = \frac{D_{11}(\varepsilon)D_{22}(\varepsilon)}{s + \varepsilon}$$

and (35) becomes

$$\delta s\rho = \frac{-f_2^2}{\pi} \int_R^{\rho_2} \frac{D_{22}(s\rho)}{D_{22}(s)}^2 \frac{s_\rho + \varepsilon}{s + \varepsilon}^2 \frac{ds}{s - s\rho} \qquad (37)$$

Several points about (35) and (37) are worth noting. If t_2, the channel 2 threshold, is larger than $s\rho$, then all the quantities in (35) are positive definite, and so the mass-shift is negative definite. Even for $s\rho > t_2$, the mass shift turns out to be negative in the example considered: however this is not true of (37). Secondly if $s\rho > t_2$, (35) gives an expression for the imaginary part of the mass

$$\text{Im } \delta\rho s\rho = f_2^2 \rho(s\acute{\rho})$$

which is the usual coupling constant-width relation, so long as $s\acute{\rho}$ is taken to the perturbed mass (i.e. $s\acute{\rho} = s\rho + \delta s\rho$). Otherwise the expression is clearly wrong if the bound state moves below t_2. Finally, (37) is independent of D_{11}: the binding in channel 1 has no effect on the mass shift due to channel 2. We would therefore expect this to be approximately true of (35).

Computer programs were written to solve this problem, both exactly, by calculating the full two channel amplitude corresponding initially to a one pole input and secondly to scalar particle exchange, as well as by the perturbative expressions (37) and (35). The ND^{-1} equations were solved by matrix inversion of the integral equation for D: this is simple because the L.H.C. is straight, and the Born term for scalar exchange has a simple imaginary part.

The one pole approximation, where the channel 1 input is taken to be $\text{Im } B = g_{q\bar{q}x} \ \delta(s + 4m_q^2 - m_x^2)$ (38)

shows a remarkable agreement between (37) and the exact expression. The position of the pole (36) was

taken to be at the end of the cut due to quark exchange (diagram (iii): i.e. at $\varepsilon = -(4m_\pi^2 - \frac{m_\pi^4}{m_q^2})$. This weights (37) towards the lower end of the integral, and gives a positive shift. To make the approximation (38) reasonable, m is taken to be large, so the force pole is distant. This produces very linear D and N functions, and in turn the mass shift as a function of f_2^2 is highly linear. The agreement between (37) and the exact expression is excellent; from $s_\rho = m_\rho^2 = 1(\text{Bev})^2$ right up to $s_\rho = 4m_q^2 = 100(\text{Bev})^2$ (the channel 1 threshold), the discrepancy between the perturbative and the exact solution is about 1%.

However, for a rather more realistic input, the shift turns out to be negative. Unfortunately $\text{Re}D_{22}$ has a cusp at $s = t_2$, and this destroys the linearity. A relation between s and g can be found, similar to (35), and this remains linear: the residue

$$f_2 = \frac{N_{11}D_{12} - N_{12}D_{11}}{f_1 \frac{d}{ds}(\det D(s))} \qquad (39)$$

has a non-linear behaviour because of the cusp in det D. For this reason the mass of the ρ is initially taken to

be rather higher than the physical value. The results for two different values of m_x are shown in figs (1), (2) and (3).

A poor feature of this calculation is the very large value we obtain for $\frac{G^2}{4\pi} q\bar{q}\rho(\sim 100)$ while $\frac{G^2}{4\pi} q\bar{q}\pi \sim 5$. One would hope that they would be roughly similar, as the only difference between the ρ and π in this calculation is the mass. This is also the fault in another dynamical quark calculation[46], where an attempt is made to bootstrap the ρ and π in $q\bar{q}$ scattering (the above calculation, with $m_x = m_\rho$, could be considered as a very crude bootstrap). The reason for this is qualitatively clear. The equation for D may be written

$$D(s) = 1 + \int_2^\infty K(s,s') \text{ Im } B(s')D(s')ds' \quad (40)$$

For the exchange of a particle of mass M_x, the L.H.C. starts at $s = 4M_q^2 - M_x^2$, and has a logarithmic singularity at this point, whereas for the rest of the cut it is fairly slowly varying. This rapid variation will affect D in turn, and so $D'(s)$ will be large only near the end of the cut. (This argument is similar to the Ball-Frazer mechanism for the production of resonances).

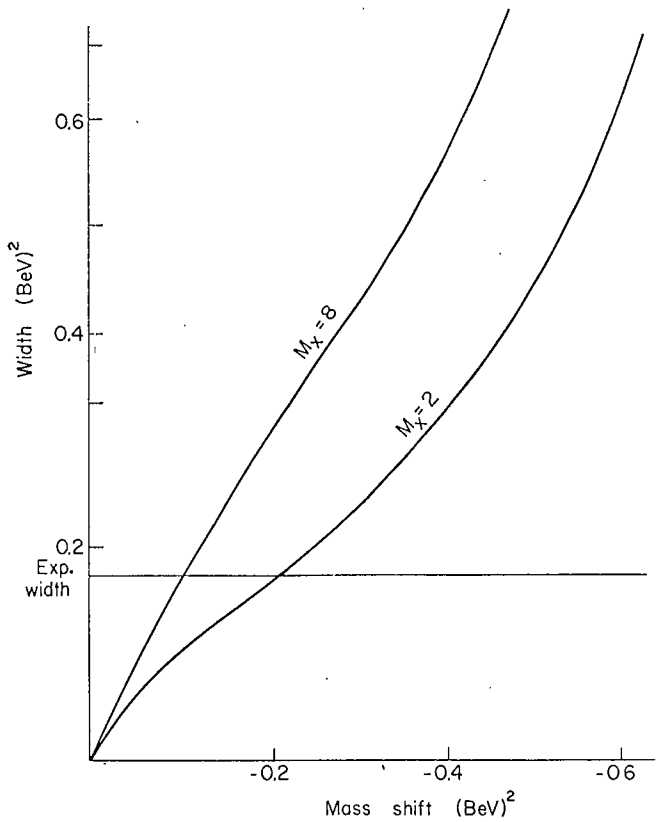


FIG. 1. The variation of the $\rho \rightarrow \pi\pi$ coupling with the shift in the ρ mass squared, for two values of m_X . The line marked "experimental width" is actually that for which the dimensionless $\rho\pi\pi$ coupling in our model problem agrees with the experimental value.

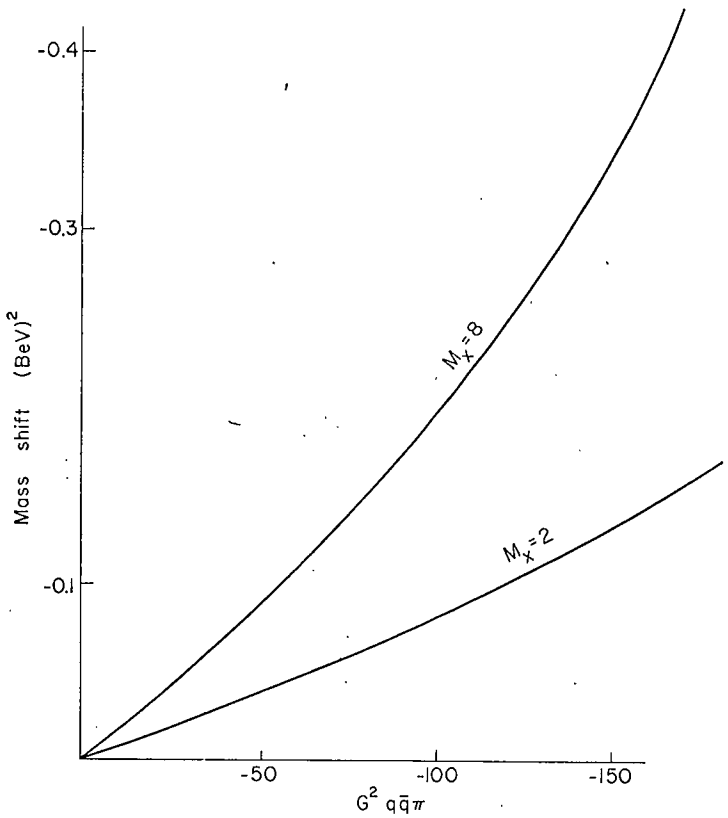


FIG. 2. The variation of the shift of the ρ mass with $G^2_{q\bar{q}\pi}$

Hence if $M_x = m_\rho$, or alternatively $m_x > 2mq$, D' will be small at $s = m_\rho^2$, and the coupling will be large. Hence it appears that one must have

$$m_x^2 \sim 4m_q^2 - m_\rho^2$$

to produce a reasonable ρ. It is interesting to note that the ρ now has the mass to bind the X: however it is not obvious that this new form of reciprocal bootstrap can work. Speculations as to the form of the X can be made: in the short range quark model it could be the genuine $l = 1$, $J = 1$ bound state of two quarks. The bootstrap would probably require $g_{q\bar{q}\rho} \sim g_{q\bar{q}x}$, so

$$\frac{g_{x\pi\pi}}{g_{\rho_{\pi\pi}}} \sim \frac{m_\rho^2}{m_x^2} \sim \frac{1}{100}$$

so that the width of the X in the $\pi\pi$ channel could be fairly small. A second possibility is that the X corresponds to the excitation of the principal quantum number n. On the basis of potential models, one may argue that the gap between ($l = 0$, $n = 0$) and ($l = 0$, $n = 1$) states will be very much greater than that between ($l = 0$, $n = 0$) and ($l = 1$, $n = 0$) states.

The results of this calculation are reasonably favourable to the quark model. Taking $m_\rho = 1$ Bev,

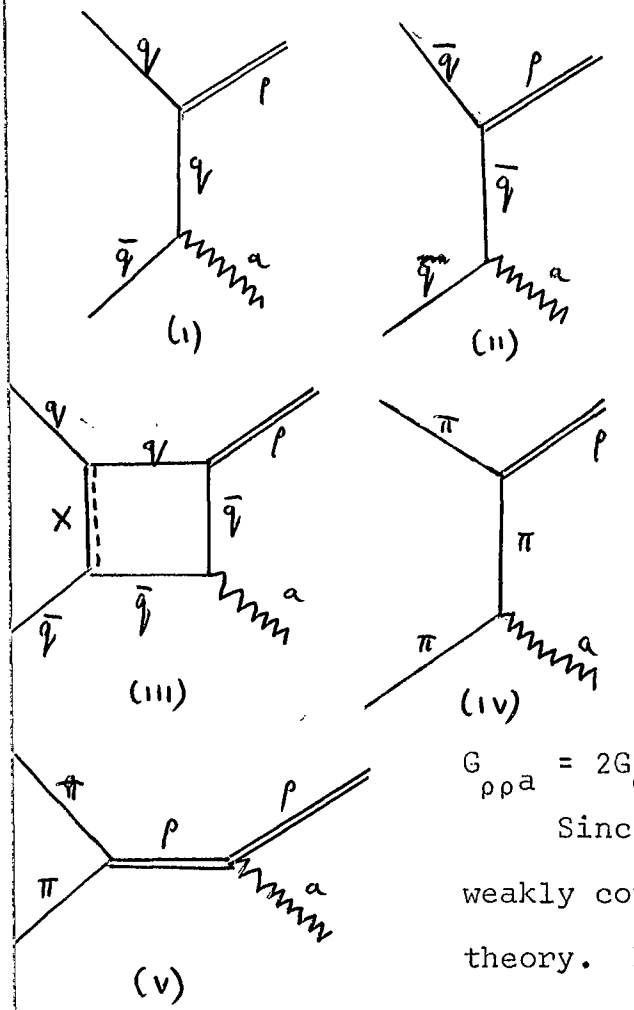
m_q = 5 Bev and m_π = .139 Bev, we find that for the physical values of $g_{\rho\pi\pi}$

m_x	$-\Delta m_\rho^2$
8	.11(Bev)^2
2	.20 "
.5	.20 "

so that the shift is reasonably small.

The second dynamical model investigated attempts to estimate the effect of the strong interactions on weaker ones. One of the most celebrated SU(6) calculations is that for the magnetic moments, which assumes, among other things, that the electromagnetic features of a bound state of quarks is independent of the strong interaction effects. We therefore consider a (rather unrealistic) calculation of the effects of a low mass channel on a weak coupling.

We introduce a fictitious photon-like particle a. For reasons of kinematic convenience it is much simpler to assume that the particle is spinless and has mass m_ρ. Then the a ρ channel is coupled to the $q\bar{q}$ channel, but has essentially no forces: the dominant ones come from the other channels.



In the pure quark model, the $q\bar{q} \to \rho a$ amplitude will have forces given by (i), (ii) and (iii). The coupling of the a to the quark is assumed known, and the additive quark model would obviously suggest that

$$2G_{qqa} = G_{\rho\rho a} \qquad (40)$$

whereas the effects of the strong forces will be to renormalise this so

$$G_{\rho\rho a} = 2G_{qqa}A_o \qquad (41)$$

Since the ρa channel is (by definition) weakly coupled, we may again use perturbation theory. From (24) and (25)

$$D^T{}_{\delta a}\, D = \frac{1}{\pi}\int_L \frac{D\ \mathrm{Im}_{\delta a}\ D^T}{s' - s}\, ds' \qquad (42)$$

Now we assume that there is essentially no force in $\rho a \to \rho a$, so considering only the $q\bar{q}$ and ρa channels, we find that (26) becomes

$$\delta a_{12} = \frac{1}{\pi D_{11}} \int_L \frac{D_{11}\mathrm{Im}\ \delta a_{12}}{s' - s} \qquad (43)$$

The force terms that give rise to Im a are taken from diagrams (i) and (ii). In principle (iii) should be of much the same strength, though possibly further from the physical region: it is rather complicated, however, and as this is an order of magnitude calculation we ignore it. Hence we find

$$\text{Im } \delta a_{12} = \frac{2\pi}{(s-4m_q^2)^{\frac{1}{2}}(s-4m_\rho^2)^{\frac{1}{2}}} g_{q\bar{q}\rho} g_{q\bar{q}a} = g_{q\bar{q}\rho} g_{q\bar{q}a} f(s, m_q^2, m_\rho^2) \tag{44}$$

Analagously to (33), we find at the ρ pole

$$g_{\rho\rho a} = \frac{2g_{q\bar{q}a}}{D_{11}(s_\rho)} \frac{1}{\pi} \int_R \frac{ds'}{(s'-s_\rho)} \frac{D_{11}(s')}{(s'-4m_q^2)^{\frac{1}{2}}(s'-4m_\rho^2)^{\frac{1}{2}}} \tag{45}$$

$$= 2g_{q\bar{q}a} A_o$$

Annumerical evaluation of the integral A_o in (45) shows that $A_o \simeq 1$, which is very encouraging as it indicates that the original additivity assumption is sound. However, it is clear that the $\pi\pi$ channel will have an influence on $g_{\rho\rho a}$, due to diagram (v). We therefore repeat the calculation, now including the $q\bar{q}$, $\pi\pi$ and ρa channels. (42) becomes

$$\delta a = \frac{1}{(\det D)^2} \oint_R D^{T-1} \int_R \frac{D^T \, \text{Im}\delta a \, D}{s' - s} ds' \; D^{-1} \tag{46}$$

$$\delta a = \frac{1}{(\det D)^2}\,\frac{1}{\pi}\begin{bmatrix} D_{22}(s)-D_{12}(s) & . \\ -D_2(s)D_{11}(s) & . \\ . \quad . & \det D \end{bmatrix}\int_L \begin{bmatrix} D_{11}(s')D_{21}(s') & . \\ D_{12}(s')D_{22}(s') & . \\ . \quad . & 1 \end{bmatrix}$$

$$\begin{bmatrix} . & . & \mathrm{Im}\delta a_{13} \\ . & . & \mathrm{Im}\delta a_{23} \\ \mathrm{Im}\delta a_{13} & \mathrm{Im}\delta a_{23} & \end{bmatrix}\begin{bmatrix} D_{11}(s')-D_{12}(s') & . \\ D_{21}(s\ ) \quad D_{22}(s') & . \\ . \quad . & 1 \end{bmatrix}\frac{ds'}{s'-s_\rho}\begin{bmatrix} D_{22}(s)-D_2\ (s) & . \\ -D_{12}(s)D_{11}(s) & . \\ . \quad . & \det D \end{bmatrix}$$

which gives

$$\Delta a_{13} = \frac{1}{\det D}\left(D_{22}I_1 - D_{12}I_2\right) \tag{47}$$

where

$$I_1 = \frac{1}{\pi}\int_L \frac{D_{11}\mathrm{Im}_\Delta a_{13} + D_{21}\mathrm{Im}_\Delta a_{23}}{s'-s}\,ds' \tag{48}$$

$$I_2 = \frac{1}{\pi}\int_L \frac{D_{12}\mathrm{Im}_\Delta a_{13} + D_{22}\mathrm{Im}_\Delta a_{23}}{s'-s}\,ds' \tag{49}$$

$$g_{\rho\rho a} = \frac{1}{\det' D(s_\rho)}\left[2g_{q\bar{q}a}\left(D_{22}\int_L \frac{D_{11}(s')f(s',m_q^2,m_\rho^2)}{s'-s}\,ds' - D_{12}\int_L \frac{D_{12}(s')f(s',m_q^2,m_\rho^2)}{s'-s}\,ds'\right) + 2g_{\pi\pi a}\frac{g_{\pi\pi\rho}}{g_{q\bar{q}\rho}}\left(D_{22}\int_L \frac{D_{21}(s')f(s',m_\pi^2,m_\rho^2)}{s'-s}\,ds' - D_{12}\int_L \frac{D_{22}(s')f(s',m_{\pi\pi}^2,m_\rho^2)}{s'-s}\,ds\right)\right] \tag{50}$$

$$\text{i.e. } g_{\rho\rho a} = 2g_{q\bar{q}a}A + 2g_{\pi\pi a}B \tag{51}$$

Note that in (50), all the quantities have been determined in the first calculation. We find the following values for A_o, A and B

	m_x = 8 Gev	m_x = .5 Gev
A_o	0.91	1.33
A	1.00	1.50
B	0.02	0.57

These results are interesting in that the short-range quark model is very clearly less susceptible to perturbing influences than the long-range. Since presumably $G_{\pi\pi a} \simeq 2G_{q\bar{q}a}$, naive quark model would give a result about 2/5 of the correct one. This probably explains why the predictions of the quark model which involve vector meson magnetic moments are not very good.

The quark model, then, has a large number of attractive features. It seems to be empirically correct for a very large number of strong, electromagnetic and weak interactions and it has the overwhelming virtue of simplicity. Against this must be set the objections previously outlined. Obviously a satisfactory dynamical theory must be found to explain the pecularities of the model.

CHAPTER 5.

The P_{11} Partial Wave in πN Scattering.

A very large number of papers have been written on πN scattering at low energy[47),50),51),52)]. A fairly large proportion of these deals partly or entirely with the P_{11} partial wave: this chapter describes a further attempt to understand the quantitative features.

It is simple to consider the S_{11} and the P_{11} waves together because of the Macdowell symmetry principle[48)] which states

$$A_{S_{11}}(W) = A_{P_{11}}(-W)$$

The P_{11} amplitude contains the following phenomena: the nucleon pole below threshold, a rapidly increasing inelasticity at low energy (~1080 Mev) and a phase shift which is initially negative, but rises through 0^0 at about 1100 Mev and increases to a resonant or near resonant value at about 1400 Mev. The 'experimental'situation is still slightly confused, but most phase shift[49)] analyses agree reasonably well. The S_{11} phase shift starts positive and gradually climbs up to about 30^0 at around 1380 Mev, where it appears to have a cusp: inelasticity very suddenly increases and it resonates

at 1570 Mev. This resonance almost overlaps with a second at 1700 Mev, which is, however, fairly elastic.

So far N/D calculations have been rather unsuccessful[50,51,52]. Taking the Born terms from N, N* and possibly ρ exchange, and adjusting any free parameters (usually a cutoff) so that the position of the nucleon pole is correct, one finds that the P_{11} phase shift decreases monotonically while the S_{11} has the incorrect sign, if elastic unitarity is used[52]. Considerably better results are obtained if the nucleon is inserted as a C.D.D. pole (not surprisingly, as this introduces three further parameters): clearly the position and coupling constant of the nucleon can be fitted by the pole and the cutoff can be used to fit the zero of the P_{11} wave, for example. The inclusion of inelastic, unitarity makes no significant change, although the calculation with the C.D.D. pole included is marginally improved.

Two channel calculations have been fairly primitive so far; two have been performed[53,54], both of which use the σ-N channel as the second. These are both successful, in the sense that when fitted to the nucleon

pole the adjustable parameters can be adjusted to produce a zero in the phase shift, but both are suspect on numerical grounds: that by Coulter, Shaw and Wong because the L.H.C. is reduced to two poles and that by Bender ET AL because the one pole Pagels method is used. A two channel calculation using the πN^* as the second channel has been performed[55], but this deals with the P_{33} wave, and again is of doubtful value for numerical reasons. (They have also taken the curious step of producing the N^* independently in the πN and πN^* channel, which naturally gives two resonances when the channels are coupled).

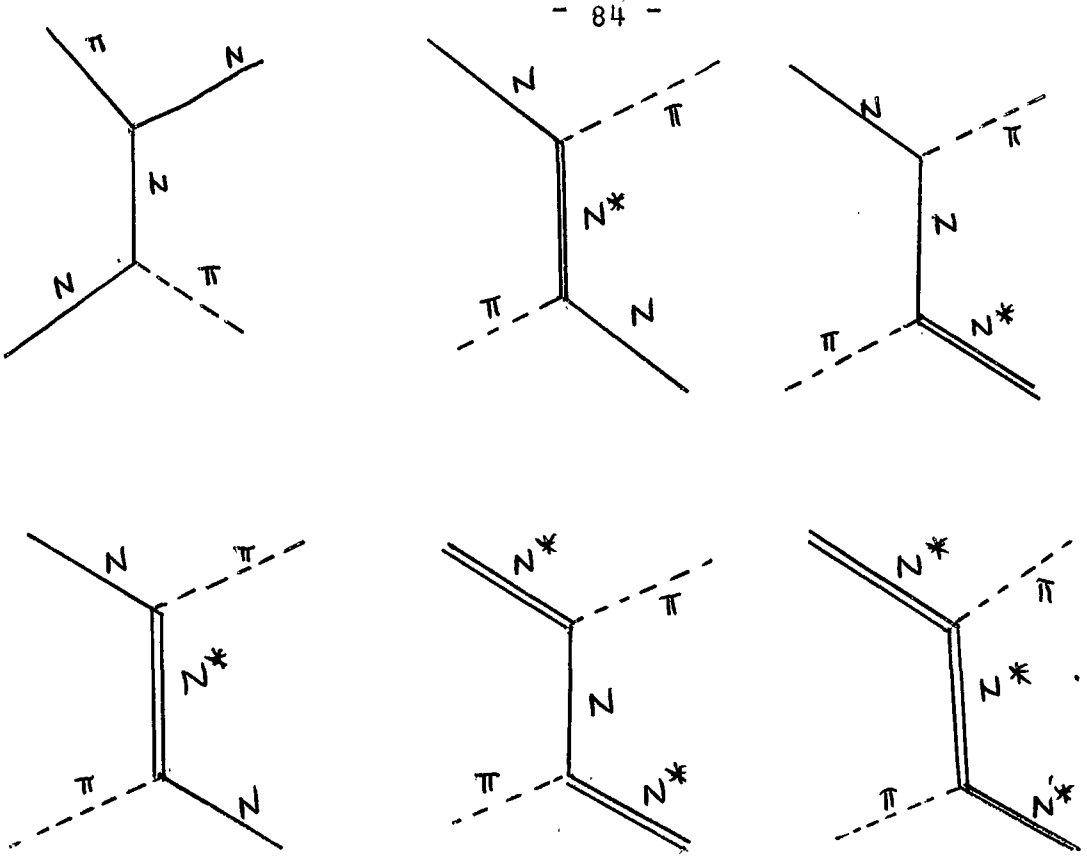
We have attempted this calculation considering the N^* as the second channel. This has been done not so much to produce a convincing fit to the phase shifts as to provide a basis for a proposed calculation of the proton-neutron mass difference. It seems fairly clear from the preceding discussion that a C.D.D. pole is required in the one channel calculation, which implies that the nucleon is at least partly a bound state in a second channel[56]. The natural candidate, from an SU(6) point of view, is the πN^* channel: a simple calculation gives

$|B\rangle = \sqrt{2/5}|PB\rangle + \sqrt{4/9}|PD\rangle$ + SMALL CONTRIBUTIONS FROM VECTOR MESON STATES

in an obvious notation. We see, therefore, that the nucleon is quite largely made up of decuplet states: this leads to the natural conclusion that πN^* is the most important second channel. This leads to a 'plausible' explanation of why the celebrated Dashen-Frautschi calculation leads to the wrong sign for the n-p mass difference: the most important single state in the proton wave function (in terms of Clebsch-Gordan coefficients) is the $\pi^- N^{*++}$.

On the assumption that for some reason the dynamical symmetry breaking accentuates the πN^* contribution, it is possible that the $\pi^- N^{*++}$ could outweigh the other contributions. As this state has the largest (negative) Coulomb energy, it could lead to the proton being the lighter particle. This calculation is beset by a number of technical difficulties: the infra-red divergence of the photon exchange diagram (which gives rise to the mass-splitting) and the complexity of the numerical work when the feedback terms (due to the mass-splitting of the N^* and π as well as that of the N) are taken into account are only two.

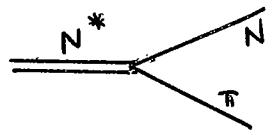
The force input to this problem consists of N and N^* exchange in all channels [58].



ρ exchange has been ignored because it is generally beleived to be less important than N and N*: it is also harder to calculate.

The numerical method adopted to solve the ND^{-1} equation is the modified Pagels method, outlined in Chapter 1). Additional complications are added by the different masses of the particles, by the presence of Macdowell symmetry, by the required P-wave threshold conditions, and by the instability of the N*. Threshold conditions are enforced in principle by adding a pole at some point on the force cut and in fact by multiplying

the Born terms by (W-t)/W. A point which we have so far ignored is the presence of overlapping cuts. The Born term from (v) (for example) has a cut which overlaps the unitarity cut in πN^* scattering: this is because the process



is a physically allowed one. This is probably not a serious problem; for example the Frye-Warnock method[3] assumes the existence of overlapping cuts. However, if the problem is treated 'realistically' by considering the N* mass as complex, the cuts will no longer be superimposed. We therefore consider the force cuts as being slightly displaced, whereupon the formalism goes through exactly as before. Trouble may arise near the end of the cuts, where the real part of the Born term has a logarithmic singularity: however this is a numerical rather than a physical difficulty[59]. We note the problem would be virtually insoluble using the integral equation for D, due to the peculiar contours of integration required.

The calculation contains three free parameters: a cutoff and two coupling constants $g_{\pi N^* N^*}$, the so called 'electric' and 'magnetic' couplings. These are in

principle given by SU(6): we have preferred to treat them as free parameters. These are adjusted by a minimization routine in the program to fit the position and residues of the nucleon pole, which are related to g_{NN} and g_{NN^*}.

By following an analysis analagous to that on pages 16 and 17 we find the equation for N may be written

$$\underset{\sim}{N}(W) = \underset{\sim}{B}(W) + \frac{1}{\pi} + \int_{t}^{b} + \int_{-b}^{-t} \frac{(W'-t)\underset{\sim}{B}(W')-(W-t)\underset{\sim}{B}(W)}{W'(W'-W)} \underset{\sim}{\rho}(W')\underset{\sim}{N}(W')\, dW' \quad (3)$$

where $\rho_{ij}(W') = \delta_{ij} \left(\frac{(W'^2-(M_i+\mu)^2)(W'^2-(M_i-\mu)^2)}{4W'^4} \right)^{1/2}$ (4)

M_1 = nucleon mass; M = N* mass; μ = π mass

t is the relevant threshold

b is the cutoff

Hence we find

$$\underset{\sim}{N}(W) = \underset{\sim}{B}(W) + \sum_{i=1}^{n} \frac{(W_i-t)\underset{\sim}{B}(W_i)-(W-t)\underset{\sim}{B}(W)}{(W_i - W)} C_i W_i \underset{\sim}{N}(W_i) \quad (5)$$

where $W_i = t/(t/b \pm (1 - t/b)(1 - x_i^2))$

$$C_i = \frac{2 x_i^2 m_i}{\pi t} (1 - \frac{t}{a})^{3/2} \left(\frac{(W_i+t)(W_i^2-t^2)}{W^3} \right)^{\frac{1}{2}}$$

and similarly

$$D(W) = 1 - \sum_{i=1}^{n} \frac{C_i W_i (N(W_i) - N(W))}{W_i - W} - N(W)$$

$$\int_t^b + \int_{-b}^{-t} \frac{\rho(W') - \rho(W)}{W'(W'-W)}\, dW' + \rho(W) \left(\int_t^b + \int_{-b}^{-t} \right) \frac{dW'}{W'(W'-W)} \qquad (8)$$

With the advantages of hindsight, we will discuss the flows in this calculation. It can, at best, only give one resonance in the S_{11} wave because the lower one is patently an ηN resonance[60]. It is possible that the S_{11} scattering length will still have the wrong sign because the ηN channel may have an influence even below its threshold: however the fact that the calculation with a C.D.D. pole included gives the correct sign is encouraging[62]. Clearly it should give the nucleon pole correctly, as well as the P_{11} scattering length. It could give an increasing phase shift around the πN* threshold, due to the Ball-Frazer mechanism, but the phase shift zero and the Roper resonance are probably due to the σ-N channel: phenomenological analyses of low energy pion production suggest that this is the important channel up to at least 1350 Mev.

The calculation is unsatisfactory in that at least two more channels should be included: the $_{\sigma}N$ and the $_{\eta}N$. It is also unsatisfactory in that the cutoff is not a very believable one. We should really exclude the distant part of the force cut, which is what really leads to the divergences rather than the unitarity cut which we believe is well known. We should make some allowance for the complex mass of the N* so as to avoid an unphysical cusp as its threshold. A method of doing this is as follows: in an experiment to observe an N*, the probability of a mass M being observed is

$$P(M') = \frac{C\Gamma^2}{(M'-W_0)^2+\frac{\Gamma^2}{4}}$$

in the narrow width approximation where C is normalisation constant so

$$\int_{m+2\mu}^{\infty} P(M')\,dM' = 1$$

Hence the 'average' phase space factor should be

$$\rho_0(W) = \int_{W-\mu}^{W_{max}} P(M')\rho(M',W)\,dM'$$

$$= \int_{M+2\mu}^{W-\mu} \frac{C_\Gamma{}^2}{(M'-M_0)^2+\Gamma^2/4} \left(\frac{(W^2-(M'-\mu)^2)(W^2-(M'+\mu)^2)}{4W^4} \right)^{1/2} dM'$$

$$= 0 \text{ if } W < M + \mu$$

This has the correct behaviour in that it has a double zero at $W = M + \mu$, so there is no cusp, and for μ small this reduces to the usual expression.

All these corrections are only attempts to patch up what is clearly an unsatisfactory theory: quite clearly the problem is a three-body one and should be dealt with as such. It is to be hoped that no strange particle channels are required.

APPENDIX - Notation

The notation which is generally regarded as standard has been used throughout

$S = W^2$ where W is the C.o.m. energy

$t = -\Delta^2$ Δ is the momentum transfer

so that in the equal mass case

$s = 4k^2 - 4m^2$

$t = -2k^2(1 - \cos \theta_s)$

We have also used $\nu = k^2$ and $z_s = \cos \theta_s$: in the examples of Chapters 1,2 and 3 we have used ν and s interchangeably

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