Current algebra and superconvergence

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CURRENT ALGEBRA AND SUPERCONVERGENCE

by

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PREFACE

The work presented in this thesis was carried out in the Department of Mathematics, University of Durham during the period October 1965 to August 1967, under the supervision of Professor E.J.Squires. The author wishes to express his sincere gratitude to Professor Squires for continuous guidance and encouragement. Special thanks are also due to my colleagues, especially Mohammad Ahmed, Peter Watson and John Poston with whom the author had some very stimulating discussions.

Except where stated in the text, the work described is original and has not been submitted in this or any other University for a degree. It is based principally upon a paper by the author and one paper with the author as the collaborator. The author is grateful to Dr. Richard Roberts for his permission to include the joint work in this thesis.

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CONTENTS

ABSTRACT

We study the implication of the algebra of currents and the concepts of unitarity, analyticity and high energy behaviour of scattering amplitudes to investigate sum rules for some scattering processes.

In Chapter I we introduce the concept of current algebras and show how to derive sum rules starting from equal time commutators. In Chapter II we apply the formulation developed in Chapter I, to the equal time commutator of axial vector charge and the isovector electromagnetic current, and obtain a relation between the πN^*N^* p-wave coupling constant to s-wave πNN coupling constant.

In Chapter III we show how to derive sum rule for strongly interacting particles starting from commutator of weak currents. This leads us to examine whether we could derive the same sum rules without the help of current commutator , and using only the concepts of pure strong interaction, namely unitarity, analyticity, high energy behaviour. We find that this can be done. We then apply the formalism to derive sum rules for πD (N*1525) scattering, obtaining a value for the πDD coupling constant.

In Chapter IV, we develop and apply the helicity formalism to obtain superconvergence relation for $\pi N \rightarrow \pi N^*$ and $\pi N^* \rightarrow \pi N^*$. We obtain a relation between πNN coupling constant g_1 and πN^*N^* p-wave coupling constant g_3. We also find the value of g_3 in terms of g_2, the πNN^* coupling constant.

CHAPTER I

General Considerations and Basic Concepts in Current Algebra

1. One of the most powerful tools to investigate Elementary Particle Physics is Group Theory. The success with the classification of Elementary Particles based on SU(3)[1] and SU(6)[2] and other outstanding results obtained in this approach are well known. However we run into some difficulties when comparing the Group theoretical results with reality. The reason is that in nature all the proposed symmetries are more or less badly broken so that an estimate of the corrections due to symmetry breaking interactions is necessary. This is especially true in the case of groups like $SU(3) \times SU(3)$ where the symmetry limit is far from reality. Thus to achieve a complete understanding of the role of group theory we need a scheme where, on one hand the group theoretical results are reproduced, but on the other hand, a control on the "corrections" is maintained. To deal with these problems we will profit of the methods of quantum field theory and dispersion theory through the "current algebra approach".

An important feature which has come out of the study of weak interactions is the concept of current-current interaction. The weak currents have a vector and an axial character and show remarkable similarities to the electromagnetic currents.

For the weak currents involving leptons this similarity is easily seen by writing the well known* explicit forms of those currents in terms of lepton fields. For the hadron currents however the situation is not so simple. There are many possible ways of writing those currents, according to which object one regards as elementary, moreover since one cannot use the perturbation theory, it is extremely hard to distinguish between the various models of field theory.

In the case of hadrons currents the analogy between electromagnetic and weak currents has been extremely fruitful in suggesting the idea that all weak currents exhibit some kind of partial conservation, the amount of breaking depending on the type of currents. So, for a set of currents $J_\mu^\alpha(x)$ which can be considered divergenceless in a suitable symmetry limit

$$D_\alpha(x) = \partial_\mu J_\mu^\alpha(x) = 0 \tag{1.1}$$

we can define a set of charges

$$Q_\alpha(t) = \int J_o^\alpha(\underline{x},t)d\vec{x} \tag{1.2}$$

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*the weak current involving leptons is written as

$$J_\alpha(\text{lepton}) = \bar{\upsilon}_e\,\gamma_\alpha(1+\gamma_5)e + \bar{\upsilon}_\mu\,\gamma_\alpha(1+\gamma_5)\mu$$

where υ_e, υ_μ represent neutrino fields associated with electron and muon and e, μ represent the electron and muon field.

which are approximate constants of motion.

Thus the physical assumption of partially conserved currents provides us with new approximate constants of the motion represented by the generalized "charges", given by the space integrals of the time components of those currents.

In order to use those new constants of motion to deduce approximate symmetry properties of elementary particles one has to know the commutation relation between them. And this leads us to the fundamental idea of current algebra.

Many important properties of weak interactions do not depend on the specific form of the currents in terms of the fields but only on the commutation relations between the currents themselves.

The bridge between physical hadronic current* and symmetry operators of the theory is based on the fundamental suggestion by Gell-Mann(3) of identifying the physical charges with the generators (aside from a coupling constant) of a symmetry group, which is explicitly proposed to be SU(3) for vector currents and SU(3) $\times$ SU(3) for axial vector and vector currents taken together. In so doing the equal time commutation relation between charges and currents are taken to generate the algebra of the corresponding

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* Here we treat the strong interactions without approximation but consider the electromagnetic and weak and gravitational interaction only in first order.

group. For $SU(3) \times SU(3)$ following equal time commutation relations were proposed

$$\left.\begin{aligned} [\, Q_\alpha^V(t),\, Q_\beta^V(t)\,] &= i f_{\alpha\beta\gamma}\, Q_\gamma^V(t) \\ [\, Q_\alpha^V(t),\, Q_\beta^A(t)\,] &= i f_{\alpha\beta\gamma}\, Q_\gamma^A(t) \\ [\, Q_\alpha^A(t),\, Q_\beta^A(t)\,] &= i f_{\alpha\beta\gamma}\, Q_\gamma^V(t) \end{aligned}\right\} \qquad (1.3)$$

where $f_{\alpha\beta\gamma}$ are the structure constants of $SU(3)$. These relations follow from the simple quark model in which the axial and vector currents are given by

$$\left.\begin{aligned} J_\mu^{V,\alpha} &\sim \bar{\psi}\, \lambda^\alpha\, \gamma_\mu\, \psi \\ J_\mu^{A,\alpha} &\sim \bar{\psi}\, \lambda^\alpha\, \gamma_5\, \psi \end{aligned}\right\} \qquad (1.4)$$

where λ^α are the $SU(3)$ unitary spin matrices and ψ's are elementary spin 1/2 quark fields.

Equation (1.3) can be generalized to charge densities in the simplest ways by writing:

$$\left.\begin{aligned} [\, J_0^{V,\alpha}(\vec{x},t),\, J_0^{V,\beta}(0)\,] &= i\, \delta^3(x)\, f_{\alpha\beta\gamma}\, J_0^{V,\gamma}(0) \\ [\, J_0^{V,\alpha}(\vec{x},t),\, J_0^{A,\beta}(0)\,] &= i\, \delta^3(x)\, f_{\alpha\beta\gamma}\, J_0^{A,\gamma}(0) \\ [\, J_0^{A,\alpha}(\vec{x},t),\, J_0^{A,\beta}(0)\,] &= i\, \delta^3(x)\, f_{\alpha\beta\gamma}\, J_0^{V,\gamma}(0) \end{aligned}\right\} \qquad (1.5)$$

One could try to go on and to postulate simple commutation relations for the space components of the currents and also for different operators (e.g. the divergence of the current $D_\alpha = \frac{\partial J_\mu}{\partial x_\mu}$). However one runs into two difficulties. First of all it was shown by Schwinger[4] that in such cases simple forms like (1.4) and (1.5) as suggested by simple field theory as for example quark model lead in general to inconsistencies so that extra terms involving higher derivatives of delta functions become necessary. This has been further supported by a recent study of Johnson and Low[5], who obtain such extra terms in a simple quark model. Further sum rules obtained on the basis of these generalized commutators are very frequently divergent.

Going back to (1.4) and (1.5) and mixed commutator like

$$[\, Q^{A,\alpha}(t),\ J^{\beta}_{\mu}(\bar{x},t)] = i\, f_{\alpha\beta\gamma}\, J^{\gamma}_{\mu}(\bar{x},t) \qquad (1.6)$$

we can look at the group as underlying the structure of baryons and mesons, e.g. in the unitary symmetry scheme. Anyway the existence of algebra and the hypothesis that the related group is a symmetry group are two independent things. We can still exploit the group through its algebra without assuming invariance under it. In other words the commutation relations reflect the existence of the symmetry groups for the hadrons but they are supposed to be exact and not to be affected by the presence in the total Hamiltonian of a symmetry breaking part. Thus commutation relations can be the tool we are looking for.

Going back to equations (1.4) and (1.5) we wish to emphasise that commutator of the charges are taken at equal times, which is necessary because of the non-conservation of charges. This point is important because, this form, in the absence of some extra conditions, is not relativistically invariant.

The relativistic invariance of equations (1.4) and (1.5) stems from micro-causality which implies the vanishing of the commutator of two operators for space-like distances. This means that equations (1.5) are true in any frame of reference.

However the difficulty remains that when we sandwich equation (1.4) or (1.5) between particle states and use completeness, the procedure is not explicitly covariant[6] since each intermediate state contribution does not satisfy the causality requirement which is only satisfied by the sum as a whole. This means, although the final result is indeed invariant the separation between different intermediate state contributions depends on the choice of the frame of reference.

One can overcome the difficulty in two different ways:

First of all one can exploit the implicit invariance of the whole scheme in order to choose a special frame of reference; the

<u>infinite momentum</u> frame[5] when things becomes particularly simple and useful. We discuss one example of this method in the coming pages.

The second way is to make explicit use of micro-causality and give a general expression to the matrix elements of the commutator (1.5) by means of dispersion theory[7]. We will be concerned mainly with this method, and will use the method to investigate the equal time commutator of axial charge with the isovector electromagnetic current.

2. The infinite momentum frame of reference

Since we just want to illustrate the technique of using the infinite momentum frame of reference, we will give a particularly simple example[8] of that of the commutator of two charges

$$[\, Q_\alpha , Q_\beta \,] = i\, f_{\alpha\beta\gamma}\, Q_\gamma \tag{1.7}$$

and sandwich it between states of momenta $\vec{p}_1$ and $\vec{p}_2$. Introducing completeness we get

$$(2\pi)^3 \sum_n \left\{ < \vec{p}|J_{o\alpha}|n > < n|J_{o\beta}|\vec{p} > \delta^3(p_n - p) + \text{c.t.} \right\}$$

$$= i\, f_{\alpha\beta\gamma} < \vec{p}|J_{o\gamma}|\vec{p} > \tag{1.8}$$

overall three-momentum conservation gives $\vec{p}_1 = \vec{p}_2 = \vec{p}$.

Another useful form, equivalent to equation (1.8) can be obtained by making use of the identity

$$[\, Q_\alpha, H\,] = i \int D_\alpha \, d^3x \tag{1.9}$$

where

$$D_\alpha(x) = \frac{\partial j_\mu}{\partial x_\mu} \tag{1.10}$$

which, taken between states of <u>different energies</u>, gives

$$< p|Q_\alpha|n > = \frac{1}{E_p - E_n} < p|D_\alpha|n > \delta^3\,(\vec{p} - \vec{p}_n) \tag{1.11}$$

If one uses equation (1.11) for the intermediate state n, belonging to a different multiplet then p equation (1.7) becomes

$$< \vec{p}|J_{o\alpha}|\vec{p} > < \vec{p}|J_{o\beta}|\vec{p} > + (2\pi)^3 \sum_{n'} \left\{ \frac{< \vec{p}|D_\alpha|n' > < n'|D_\beta|\vec{p} >}{(E_p - E_{n'})^2} \delta^3(\vec{p} - \vec{p}_{n'}) \right.$$

$$\left. + \text{ crossed terms} \right\}$$

$$= i\, f_{\alpha\beta\gamma} < \vec{p}|J_{o\gamma}|\vec{p} > \tag{1.12}$$

We can characterize the summation in equation (1.8) or (1.12) by introducing the momentum transfer four-vector

$$K = P_n - P = \begin{pmatrix} E_n - E_p \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $E_n = \sqrt{m_n^2 + \vec{p}^{\,2}}$

$E_p = \sqrt{m^2 + \vec{p}^{\,2}}$

then $K_o = \sqrt{m_n^2 + \vec{p}^{\,2}} - \sqrt{m^2 - \vec{p}^{\,2}}$

The unpleasant non-vanishing time component of K, follows from the fact that we have only three-dimensional momentum conservation, and not the four-dimensional conservation of the energy-momentum. The value of this zeroeth component depends essentially on the frame we are using. We take advantage of the freedom in this choice by taking the limit in which also $K_o \to 0$.

Indeed for large p we can write

$$K_o \simeq \left(|\vec{p}| + \frac{m_n^2}{2|\vec{p}|} \right) - \left(|\vec{p}| + \frac{m^2}{2|\vec{p}|} \right)$$

in the limit $\vec{p} \to \infty$

$$K_o \simeq \frac{1}{|\vec{p}|}$$

and we obtain the four-conservation of the momentum. The important advantage of this choice is that K^2 is always zero and so we obtain a sum rule at constant momentum transfer.

We note that we have in general an infinite set of sum rules; taking $\vec{p} \to \infty$ we choose a particular one. The other choices however for example $\vec{p} = 0$ are not convenient because we obtain time-like values of K^2 in a region where poles in that variable can arise, making a simple evaluation of the sum rules quite hard.

Let us go back to equation (1.12) and introduce the notation:

$$\frac{1}{2E_p} W_{\alpha\beta}(n,p) = (2\pi)^3 \sum_n < p\ |D_\alpha| n > < n|D_\beta|p > \delta^3(p - p_n)$$
$$\times\ \delta(E_p - E_n + k_o) \qquad (1.13)$$

So that equation (1.12) becomes

$$< \vec{p}|\, J_{o\alpha}|\vec{p} > < \vec{p}|\, J_{o\beta}|\vec{p} > + \frac{1}{2E_p} \int \frac{dK_o}{K_o^2} \left\{ W_{\alpha\beta}(K_o\ \vec{p}) - W_{\beta\alpha}(K_o\ \vec{p}) \right\}$$
$$= i\, f_{\alpha\beta\gamma} < \vec{p}|\, J_{o\gamma}|\vec{p} > \qquad (1.14)$$

The auxiliary function $W_{\alpha\beta}$ defined in (1.13) is an invariant function of the four-vector K_μ and p_μ and therefore depends only on the invariants

$$\begin{aligned} K^2 &= \mu \\ K.p &= \upsilon \end{aligned} \qquad (1.15)$$

However such a function is treated in equation (1.14) in a <u>non-covariant manner</u> and each choice of the frame of reference corresponds to a different path of integration in the υ, μ plane. This can be seen by writing explicitly the four vectors

$$P = \begin{pmatrix} E_p \\ \vec{p} \end{pmatrix}, \quad K = \begin{pmatrix} K_o \\ \vec{0} \end{pmatrix}$$

So that

$$K_o = \frac{\upsilon}{E_p} \tag{1.16}$$

and the equation for the line of integration is the parabola

$$K_o^2 = u = \frac{\upsilon^2}{E_p^2} \tag{1.17}$$

As pointed out before in the infinite momentum frame $E_p \to \infty$ so that equation (1.17) reduces simply to $u \equiv 0$, so that the integral appearing in the L.H.S. of equation (1.14) becomes

$$\frac{1}{2} \int_{\upsilon_o}^{\infty} \frac{d\upsilon}{\upsilon^2} \left[W_{\alpha\beta}(\upsilon) - W_{\beta\alpha}(\upsilon) \right]$$

Let us now take a specific case by explicitly putting the SU(3) dependence of the amplitudes. Consider for example $[Q_A^+, Q_A^-]$ commutator sandwiched between proton states. Equation (1.14) then reads:

$$r^2 + \frac{1}{2\pi} \int \frac{d\upsilon}{\upsilon^2} \left[W^+(\upsilon) - W^-(\upsilon) \right] = 1 \tag{1.18}$$

where $r = \frac{g_A}{g_V}$ is the renormalization ratio appearing in the physical matrix element $< P|Q_A^+|N >$ and W^+ and W^- are the direct and crossed amplitudes corresponding to the scattering of positive and negative "spurions" from the target. If we are using commutator of axial charges equation (1.18) reduces to the famous Adler-Weisberger[9] sum rule for pion-nucleon scattering. As we can relate the functions $W^{\pm}(\nu)$; by virtue of the <u>Partial Conservation of Axial Vector Current</u> hypothesis[10]; to the pion-nucleon cross-section.

3. Dispersion theory of Current Algebra

In this section we describe the dispersion theory of current algebra. Consider an equal time commutator between two currents $J_0^{\alpha}(x)$ and $J_{\nu}^{\beta}(x)$

$$[\, J_0^{\alpha}(x),\ J_{\nu}^{\beta}(0)\,] = i\, C^{\alpha\beta\gamma}\, J_{\nu}^{\gamma}(0)\ \delta(\vec{x}\,) \qquad (1.19)$$

where α, β, γ are indices labelling the internal symmetry transformation properties of the current and μ, ν, are Lorentz indices. $C^{\alpha\beta\gamma}$ are the structure constants of the group.

The starting point for the dispersion theoretic technique of deriving current algebraic sum rules from (1.1) is to consider the quantities[11]

$$T_{\mu\nu}^{\alpha\beta} = i \int d^4x\ e^{iq'\cdot x}\ \theta(x_0) < p' \,|\, [J_{\mu}^{\alpha}(x),\ J_{\nu}^{\beta}(0)\,] \,|\, p > \qquad (1.20)$$

$$\text{and} \quad t_{\mu\nu}^{\alpha\beta} = \tfrac{1}{2} \int d^4x\ e^{iq'\cdot x} < p' \,|\, [J_{\mu}^{\alpha}(x),\ J_{\nu}^{\beta}(0)\,] \,|\, p > \qquad (1.21)$$

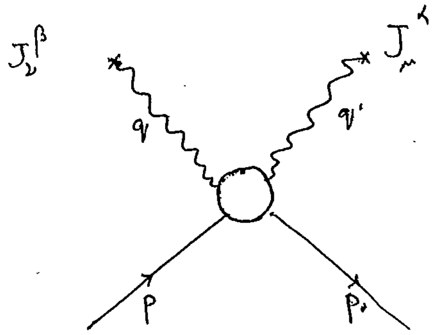
We then introduce the kinematic variables

$$P = \frac{p + p'}{2}, \qquad Q = \frac{q + q'}{2}$$

$$q = p' + q' - p, \qquad \nu = P.Q \tag{1.22}$$

$$\Delta = \frac{q' - q}{2} = \frac{p' - p}{2}$$

The quantities $T_{\mu\nu}$ and $t_{\mu\nu}$ are in a way related to the scattering process shown in the Figure below



Now for simplicity let us take p', p to be scalar partical states. Then we can decompose $T_{\mu\nu}$ and $t_{\mu\nu}$ on invariance grounds in the same set of elementary invariants, namely

$$T_{\mu\nu} = A_1 P_\mu P_\nu + A_2 Q_\mu P_\nu + A_3 \Delta_\mu P_\nu + \ldots \tag{1.23}$$

$$t_{\mu\nu} = a_1 P_\mu P_\nu + a_2 Q_\mu P_\nu + a_3 \Delta_\mu P_\nu + \ldots \tag{1.24}$$

the invariant functions A, a, ... depends on υ, t, q_1^2, q_2^2. The A functions are then Hilbert transform, with respect to the variable υ, of the corresponding a:

$$A_i(\upsilon,t,q_1^2,q_2^2) = \mathcal{H} a_i = \frac{1}{\pi}\int \frac{d\upsilon'}{\upsilon' - \upsilon} a_i(\upsilon',t,q_1^2,q_2^2) \tag{1.25}$$

or in general

$$T_{\mu\upsilon}^{\alpha\beta} = \mathcal{H} t_{\mu\upsilon}^{\alpha\beta} \tag{1.26}$$

Now let us consider the quantity $q'_\mu T_{\mu\upsilon}$. On partial integration we obtain

$$\begin{aligned} q'_\mu T_{\mu\upsilon}^{\alpha\beta} &= -\int d^4x\, e^{iq'.x}\, \theta(x_o) < p' | [\partial_\mu J_\mu^\alpha(x), J_\upsilon^\beta(0)] | p > \\ &\quad - \int \delta(x_o)\, e^{iq'.x} < p' | [J_o^\alpha(x), J_\mu^\beta(0)] | p > \\ &= D_\upsilon^{\alpha\beta} - i\, C^{\alpha\beta\gamma} < p' | J_\upsilon^\gamma(0) | p > \end{aligned} \tag{1.27}$$

where

$$D_\upsilon^{\alpha\beta} = -\int d^4x\, e^{iq'.x}\, \theta(x_o) < p' | [\partial_\mu J_\mu^\alpha(x), J_\upsilon^\beta(0)] | p > \tag{1.28}$$

and we have used e.t. commutator (1.19). We can now summarize the result of this operation as

$$q'_\mu T_{\mu\upsilon} - \mathcal{H} q'_\mu t_{\mu\upsilon} = (q'_\mu \mathcal{H} - \mathcal{H} q'_\mu) t_{\mu\upsilon}$$

$$= i C^{\alpha\beta\gamma} < p' | J^\gamma_\upsilon(0) | p > \qquad (1.29)$$

i.e. the following relation holds, even when the currents are not conserved,

$$(q'_\mu \mathcal{H} - \mathcal{H} q'_\mu) t_{\mu\upsilon} = i C^{\alpha\beta\gamma} < p' | J^\gamma_\upsilon(0) | p > . \qquad (1.30)$$

CHAPTER II

Dispersion Sum Rules for $\gamma + N \to \pi + N^*$

In this chapter we will use the dispersion theory of current algebra we developed in the previous chapter to investigate fully the equal time commutator between the isovector axial charge and the isovector electromagnetic current. This commutator has been used successfully by Fubini, Furlan and Rossetti[7,12] to derive sum rules for the anomalous magnetic moment of nucleon. We shall[12] consider the commutator

$$[\, A_\mu^3(x),\ J_\upsilon^{(3)}(0)\,] \tag{2.1}$$

(where we have adopted the SU(3) x SU(3) notation to label the isovector axial current (A_μ^3), the isovector J_υ^3 part of the electromagnetic current) sandwiched between N*(1236) and N states (we assume N* to be a stable particle). Following our general consideration in the previous chapter we define

$$T_\mu = i\,\epsilon_\upsilon \int \langle N^*(p')|\,[A_\mu^3(x),\ J_\upsilon^3(0)]\,|p\rangle\, e^{iq\cdot x}\,\theta(x_o)\, d^4x \tag{2.2}$$

and

$$t_\mu = \tfrac{1}{2}\epsilon_\upsilon \int \langle N^*(p')|\,[A_\mu^3(x),\ J_\upsilon^3(0)]\,|p\rangle\, e^{iq\cdot x}\, d^4x \tag{2.3}$$

where ϵ_ν is an arbitrary (for the moment) four vector introduced for convenience. As will be seen we will relate T_μ, t_μ with the photo pion production of N* on nucleon. So we shall decompose T_μ in the same set of invariants as we shall for the above mentioned process, i.e.

$$\gamma + N \rightarrow \pi + N^*$$

We define our set of kinematic variables for the above process as

$$k + p = q + p'$$

where k, q are the four momenta of the photon and the pion; p and p' are the four momenta of nucleon and the isobar $N_{33}(1236)$.

$$s = (p' + q)^2 = (p + k)^2$$

$$t = (p' - p)^2 = (k - q)^2 \; ; \quad P = \frac{p + p'}{2}$$

$$u = (p + q)^2 = (k + p')^2$$

and $$w = s + M^2 = u + m^2 = 2p'.q = 2p.q = P.q$$

where M, m are the N_{33} and N masses respectively. We further restrict ourselves to the simple kinematic configuration

$$t = 0$$

$$q^2 = k^2 = 0$$

and taking advantage of the freedom in our choice of ϵ we fix

$$k.\epsilon = 0$$

i.e. ϵ is assumed to be the photon polarization vector. Taking into account the gauge invariance and the relativistic invariance we can decompose the T-matrix element for the process under consideration as

$$\bar{\psi}_\mu(p')[T_\mu]u(p) = \bar{\psi}_\mu(p')[\sum_{i=1}^{11} \alpha_i \, a_i]\, u(p) \tag{2.4}$$

Here α_i's are the kinematic factors and a_i's are the invariant amplitudes, functions of w, t, q_1^2, q_2^2. In the appendix we give a full investigation of the decomposition of the amplitude T, but for the moment it suffices to know that in the limit of vanishing q, only following α's survive

$$\begin{aligned} \alpha_1 &= i\, P_\mu \, \gamma_a \, \gamma_b \, F_{ab} \\ \alpha_2 &= \delta_{\mu a} \, \gamma_b \, F_{ab} \end{aligned} \tag{2.5}$$

where $$F_{ab} = k_a \, \epsilon_b - \epsilon_a \, k_b \tag{2.6}$$

Thus we can decompose T_μ and t_μ of our equations (2.2) and (2.3) as

$$T_\mu = P_\mu(\alpha_1 A_1 + \alpha_2 B_1) + Q_\mu(\alpha_1 A_2 + \alpha_2 B_2) + \ldots \qquad (2.7)$$

and $$t_\mu = P_\mu(\alpha_1 a_1 + \alpha_2 b_1) + Q_\mu(\alpha_1 a_2 + \alpha_2 b_2) + \ldots \qquad (2.8)$$

Now we use the formulation we developed in the previous chapter which gives

$$(q_\mu \not{A} - \not{A} q_\mu)t_\mu = \langle p' | [A_0^3(\underline{x},0), J_\mu^3(0)] | p \rangle \epsilon_\mu \qquad (2.9)$$

in our cases. Now in quark model of Gell-Mann(12) where equations (1.4) of the previous chapter hold it can be shown (Appendix) that the equal time commutator on the R.H.S. of (2.9) vanishes so we have

$$(q_\mu \not{A} - \not{A} q_\mu)t_\mu = 0 \qquad (2.10)$$

Using (2.7) and (2.8), equation (2.10) reduces to

$$\int d\upsilon' (\alpha_1 a_1 + \alpha_2 a_2) = 0 \qquad (2.11)$$

Since $\alpha_{1,2} \neq 0$ we are led to the sum rules

$$\int d\upsilon \, a_1(\omega) = 0 \qquad (2.12)$$

$$\int d\upsilon \, a_2(\omega) = 0 \qquad (2.13)$$

The quantities a_1 and a_2 can be determined from the general quantities by selecting the coefficients of α_1 and α_2 terms respectively.

$$A_1 = \tfrac{1}{2}(2\pi)^4 \Big\{ \sum_n \delta^4(p' + q - p_n) < p'|\partial_\mu A^3_\mu|p_n >$$

$$\times < p_n|J^{(3)}.\epsilon|p > - \sum_{n'} \delta^4(p - q - p_{n'}) < p'|J^{(3)}.\epsilon|p_{n'} >$$

$$\times < p_{n'}|\partial_\mu A^3_\mu|p > \Big\} \tag{2.14}$$

Now from the partial conservation of axial vector current hypothesis

$$\partial_\mu A^3_\mu = C\, \phi_{\pi^0} \tag{2.15}$$

So (2.14) modifies to

$$A = \frac{c}{2} (2\pi)^4 \Big\{ \sum_n \delta^4(p' + q - p_n) < p'|\phi_{\pi^0}|p_n >$$

$$\times < p_n|J^{(3)}.\epsilon|p > - \sum_{n'} \delta^4(p - q - p_n) < p'|J^{(3)}.\epsilon|p_{n'} >$$

$$\times < p_{n'}|\phi_{\pi^0}|p > \Big\} \tag{2.16}$$

We saturate our sum rules (2.12) and (2.13) with low lying baryon states, i.e. the N and N*. The first contribution will come from the nucleon pole, and to deal with the trouble of polar term degenerate in mass with either of the external states we make use of the Fubini Furlan suggestion of keeping a fictitious mass difference between the initial and the final states in the following matrix elements

$$< N^*(p') | \phi_{\pi^0} | N^*(p_n) >$$

and $$< N\ (p_n) | \phi_{\pi^0} | N\ (p)\ >$$

Putting this mass difference to zero at the end of the calculation leads to unambiguous results. In order to calculate the various pole terms we define our Lagrangian for the various vertices

$$\begin{aligned}
\mathcal{L}_{\pi NN} &= g\ \bar{\psi}\ \gamma_5\ \psi \\
\mathcal{L}_{\pi NN^*} &= g^*\ \bar{\psi}_\mu\ q_\mu\ \psi \qquad (2.17) \\
\mathcal{L}_{\pi N^*N^*} &= g^{**}\ \bar{\psi}_\mu\ \gamma_5\ \psi_\mu + g'^{**}\ \bar{\psi}_\mu\ \gamma_5\ \gamma_\upsilon\ \psi_\upsilon
\end{aligned}$$

with this Lagrangian we define the following matrix elements

$$< N^*(p') | \phi_\pi(q) | N(p) > \ = \ i\, g^*\ \bar{\psi}_\mu(p')\ q_\mu\ u(p) \qquad (2.18)$$

and the electromagnetic vertex following Bjorken and Walecka[15]

$$< N^*(p')|J.\epsilon|N(p) > = \bar{\psi}_\mu(p')[\, q_\mu(\epsilon.p\, p.q - \epsilon.pq^2)\, g_1(q^2)$$

$$+ 2i\epsilon_{\mu\alpha\beta\gamma}\, p_\alpha\, q_\beta\, S_\upsilon\, g_2(q^2) + M\, q_\mu\, \gamma.S'\, \gamma_5(g_2(q^2) + g_3(q^2))\,]$$

$$\times\ \gamma_5\, u(p) \qquad (2.19)$$

where $\epsilon_{\mu\upsilon\lambda\sigma}$ is a completely antisymmetric metric tensor and the pseudovector S_μ is defined by

$$S_\mu = i\,\epsilon_{\mu\upsilon\lambda\sigma}\, p_\upsilon\, q_\lambda\, \epsilon_\sigma \qquad (2.20)$$

and $$q = p' - p$$

with a little algebra we can put the above matrix element in a form more useful for our purpose

$$< N^*(p')|J.\epsilon|N(p) > = \bar{\psi}_\mu(p')[\, q_\mu(q.\epsilon\ p.q - \epsilon.pq^2)\, g_1(q^2)$$

$$- 2\{[\,(p.q)^2 - p^2q^2\,]\epsilon_\mu + (p^2q.\epsilon - p.q\ p.\epsilon)q_\mu$$

$$+ (q^2\ p.\epsilon - p.q\ q.\epsilon)p_\mu\}\, g_2(q^2) + i\,Mq_\mu\{\, p.q\ \gamma.\epsilon$$

$$- p.\epsilon\ q.\gamma + i\,m\, q.\gamma\gamma.\epsilon - i\,m\,q \quad \}\, (g_2 + g_3)\,]\, \gamma_5\, u(p) \qquad (2.19')$$

In addition to the above we also define the following matrix element

$$< N(p')|J.\epsilon|N(p) > = i e \bar{u}(p') \Big[\gamma.\epsilon F_1(q^2) - \sigma_{ab} q_b \epsilon_a \times \frac{F_2(q^2)}{2M} \Big] u(p) \tag{2.21a}$$

and $$< N(p')|\phi_\pi|N(p) > = i g \bar{u}(p') \gamma_5 u(p) \tag{2.21b}$$

With these definitions the nucleon pole contribution to the a_1, a_2 in the limit $q \to 0$ is obtained to be

$$\lim_{q\to 0} a_1(\text{Nucleon}) = - e \frac{M(M + m)}{16 m^2} g(g_2 + g_3) \tag{2.22}$$

$$\lim_{q\to 0} a_2(\text{Nucleon}) = - e \frac{(M + m)(M^2 - m^2)}{8m^2} g g_2 \tag{2.23}$$

where g_2 and g_3 are evaluated at $q^2 = 0$ and are found to be

$$\begin{aligned} g_2(0) &= - \left\{ \frac{0.74}{(M - m)} - \frac{0.0086}{m_\pi} \right\} \frac{1}{m_\pi(M^2 - m^2)} \\ g_3(0) &= \left\{ \frac{0.74}{M} + \frac{0.74}{M - m} - \frac{0.0258}{m_\pi} \right\} \frac{1}{m_\pi(M^2 - m^2)} \end{aligned} \tag{2.24}$$

We are thus led to the following sum rules from a_1 and a_2

$$e \frac{M(M+m)}{16m^2} g(g_2 + g_3) - \frac{1}{\pi} \int_{\text{continuum}} a_1(w)dw = 0 \qquad (2.25a)$$

$$e \frac{(M+m)^2(M-m)}{8m^2} gg_2 - \frac{1}{\pi} \int_{\text{continuum}} a_2(w)dw = 0 \qquad (2.25b)$$

In the spirit of decouplet dominance hypothesis we will approximate the integral by the N*(1236) pole. To calculate its contribution to the sum rule we need to define the following matrix elements

$$< N^*(p')|\phi_{\pi^0}|N^*(p) > = i\, g^{**}\, \bar{\psi}_\mu(p')\, \gamma_5\, \psi_\mu\, (p) \qquad (2.26)$$

g^{**} is the πN^*N^* coupling constant

$$< N^*(p')|J.\epsilon|N^*(p) > = e\bar{\psi}_\mu(p') \left[\frac{P.\epsilon}{2M} \frac{1}{1+\eta} \left\{ G_0(q^2)\delta^{\mu\upsilon} \right.\right.$$

$$\left. + G_2(q^2) \frac{q_\mu\, q_\upsilon}{2M^2} \right\} + \frac{i}{8M^2} \frac{1}{1+\eta} (\gamma.\epsilon\; q.\gamma\; \gamma.P - \gamma.P\; \gamma.q\; \gamma.\epsilon)$$

$$\left. \left\{ G_1(q^2)\delta^{\mu\upsilon} + G_3(q^2) \frac{q_\mu\, q_\upsilon}{2M^2} \right\} \right] \psi_\upsilon(p) \qquad (2.27)$$

This matrix element has been given by Gourdin and Michilie[(16)] with

$$P = p' + p$$

$$\eta = q^2/4M^2$$

$$q = p' - p$$

$G_0(0) = z$, in units of e

$G_1(0) = \mu_1$, magnetic dipole moment of N* in units of e/2M

$G_2(0) = Q - Z$

$G_3(0) = \sqrt{6}\,\mu_3 - \mu_1$,

Q = electric quadrupole moment of N* in units of e/M^2

μ_3 = magnetic quadrupole moment of N* in units of e/M^3

We get after considerable manipulation in $q \to 0$ limit

$$\frac{1}{\pi}\int_{N^*} a_1(w)dw = \frac{e}{16M}\left[(M+m)g^{**}(g_2 + g_3) - \frac{m(m+5M)}{3m_\pi M^4}\, g^*\mu_1 \right] \tag{2.27a}$$

$$\frac{1}{\pi}\int_{N^*} a_2(w)dw = \frac{e(M-m)}{8M^2}\left[(M+m)^2 g^{**}g_2 - \frac{(M-m)}{3m_\pi M^2}\, g^*\mu_1 \right] \tag{2.27b}$$

Combining equations (2.25) and (2.27) we get the following relation from the sum rule for a_1

$$M(M+m)\left(\frac{g}{m^2} - \frac{g^{**}}{M^2}\right)(g_2 + g_3) = \frac{m(m+5M)}{3m_\pi M^5}\, g^*\mu_1 \tag{2.28}$$

from the sum rule for a_2

$$(M + m)^2 \left(\frac{g}{m^2} + \frac{g^{**}}{M^2} \right) g_2 = \frac{(M - m)}{3m_\pi M^4} g^* \mu_1 \qquad (2.29)$$

In the above expressions only g^{**} and μ_1 are unknown so we put in the numerical values for all the known factors in (2.28) and (2.29)

$M = 1.236$ GeV (units of energy being 1 Gev)

$m = 0.938$ GeV

The coupling constants ~~are~~ relative to the vertex $p\,\pi^0 N^*_{33}$ are

$$g^* = 1.81$$

$$g = 13.5$$

and from equation (2.24) at $q^2 = 0$

$$g_2(0) = -27.63\ (\text{Gev})^{-4}$$

$$g_3(0) = 34.59\ (\text{Gev})^{-4}$$

Substitution of these values in (2.28) and (2.29) and their subsequent solution yields the following values for the coupling constant g^{**} for the vertex $\pi^0 N^{*+} N^{*+}$ and μ_1 the N^* magnetic dipole moment

$$g^{**} = 22.77 = 1.69\ g$$

$$\mu_1 = -54.8$$

The value of μ_1 is in total disagreement with those obtained from quark model[17], SU(6) [18] etc.

But our value for g^{**} compares favourably with the one obtained from other methods. From U(6,6) [19], with mass splitting one obtains $g^{**} = 2.21\ g$, superconvergence relations for $\pi N \rightarrow \pi N^*$ [20] give

$$g^{**}_{11} = 1.8\ g$$

and
$$g^{**} = 1.94\ g$$

while pure dynamical N/D calculation[21] seem to give

$$g^{**}_{11} \approx 2.3\ g$$

while satic SU(6) [22] gives $g^{**} \sim g$.

Our value for g^{**} should also be compared with the one obtained by D.G.Sutherland[22] using Adler-Weisberger type calculation for πN^* elastic scattering. As well as with the one obtained by C. Michael[23] doing phenomenological multichannel scattering calculation .

In the following we give a table putting all the determination of g^{**} up to date using various techniques for ~~easy~~ comparison.

Table 1 Comparison of various determinations of $\frac{g^2_{\pi N^{*++} N^{*++}}}{4\pi}$

No.	Method	Value	Reference
1	Current algebra calculation for the process $\gamma + N \rightarrow \pi + N^*$	186.58	13
2	Current algebra calculation for Adler-Weisberger type for the process $\pi + N^* \rightarrow \pi + N^*$	136^{+41}_{-30}	22
3	Multichannel scattering in propagator formalism	170	23
4	Superconvergent relation for the process $\pi + N \rightarrow \pi + N^*$	211.39 and 226.187	20
5	U(6,6)	257.9	19
6	Multichannel N/D	81	21
7	Static SU(6)	14.5	24

CHAPTER III

Strong Interaction Sum Rules

1. Consider the equal time commutator

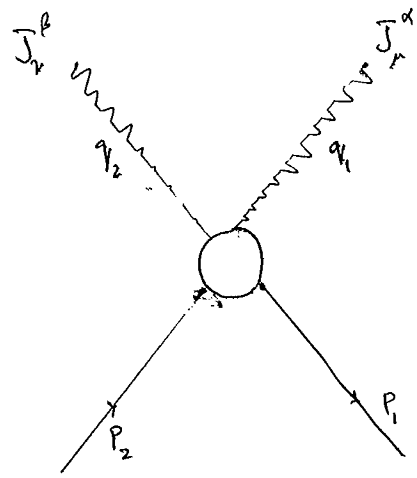
$$[J_o^\alpha(x),\ J_\upsilon^\beta(0)]_{x_o=0} = i\, f_{\alpha\beta\gamma}\ J_\upsilon^\gamma(0)\ \delta(\vec{x}) \qquad (3.1)$$

and define the quantities

$$T_{\mu\upsilon} = i \int d^4x\ e^{iq_1 \cdot x}\ \theta(x_o) < p_1 | [J_\mu^\alpha(x),\ J_\upsilon^\beta(0)] | p_2 > \qquad (3.2)$$

$$t_{\mu\upsilon} = \tfrac{1}{2} \int d^4x\ e^{iq_1 \cdot x} < p_1 | [J_\mu^\alpha(x),\ J_\upsilon^\beta(0)] | p_2 > \qquad (3.3)$$

which are in a way related to the scattering amplitude for the process shown in the figure below



(for simplicity p_1 and p_2 are taken to be spinless particle states). Then if we follow the procedure given in the first chapter we can obtain a sum rule of the form:

$$\frac{1}{2\pi}\int d\upsilon' \; a_1^{\alpha\beta}(\upsilon', t, q_1^2, q_2^2) = i\, f_{\alpha\beta\gamma}\, F_1^{\gamma}(t) \qquad (3.4)$$

where we have decomposed $T_{\mu\upsilon}$ and $t_{\mu\upsilon}$ in the same set of invariants, i.e.

$$T_{\mu\upsilon} = A_1 P_\mu P_\upsilon + A_2 P_\mu Q_\upsilon + A_3 \Delta_\mu P_\upsilon + \dots \qquad (3.5)$$

$$t_{\mu\upsilon} = a_1 P_\mu P_\upsilon + a_2 P_\mu Q_\upsilon + a_3 \Delta_\mu P_\upsilon + \dots \qquad (3.6)$$

where $P = \dfrac{p_1 + p_2}{2}$, $Q = \dfrac{q_1 + q_2}{2}$; $p_1 + q_1 = p_2 + q_2$

$$\upsilon = P.Q \;, \qquad \Delta = p_2 - p_1$$

$$t = \Delta^2 \qquad (3.7)$$

and the relation between $T_{\mu\upsilon}$ and $t_{\mu\upsilon}$ is expressed as

$$T_{\mu\upsilon} = \mathfrak{H}\, t_{\mu\upsilon} \qquad (3.8)$$

where $\mathfrak{H}$ means the "Hilbert transform" with respect to the variable υ.

In addition to all this we have expressed $< p_1|J^{\gamma}_{\upsilon}(0)|p_2 >$ as

$$< p_1|J^{\gamma}_{\upsilon}(0)|p_2 > \; = \; 2P_{\upsilon} \, F^{\gamma}_1(t) + \Delta_{\upsilon} \, F_2^{\gamma}(t) \qquad (3.9)$$

$$(F_2^{\gamma}(t) \; = \; 0 \text{ if } J^{\gamma}_{\upsilon} \text{ is conserved})$$

Now let us go back to the sum rule (3.4) and keeping t fixed, discuss its dependence on q_1^2, q_1^2. It is important to note that the right hand side is independent of q_1^2 and q_2^2. This leads us to assume that strong cancellations occur in the left hand side integral. To fully exploit this fact we multiply (3.4) by $(q_1^2 \to m_{\alpha}^2)$ $(q_2^2 \to m_{\beta}^2)$; m_{α}, m_{β} being the masses of the particles coupled to the currents J^{α}_{μ} and J^{β}_{υ} respectively. On extrapolation to $q_1^2 \to m_{\alpha}^2$ and $q_2^2 \to m_{\beta}^2$ the r.h.s. becomes zero while the l.h.s. involves the expression

$$\lim_{q^2_{1,2} \to m^2_{\alpha,\beta}} (q_1^2 - m_{\alpha}^2)\,(q_2^2 - m_{\beta}^2)\, a_1(\upsilon, q_1^2, q_2^2, t) \; = \; \text{const. Im } A_1(\upsilon,t) \qquad (3.10)$$

where A_1 is defined in (3.5), according to the decomposition of $\beta(q_2) + P_2 \to \alpha(q_1) + P_1$ scattering amplitude. Since the r.h.s. of (3.5) is not singular in q_1^2 and q_2^2. The sum rule (3.5) reduces to

$$\int d\upsilon \text{ Im } A_1(\upsilon,t) \; = \; 0 \qquad (3.11)$$

in the limit of $q^2_{1,2} \to m^2_{\alpha,\beta}$ which involves only the scattering amplitude of strongly interacting particles.

We want to stress the fact that equation (3.11) is actually independent of any detailed assumption on the current-current commutator. Indeed it can be obtained from the commutation relation between any couple of "vector currents", provided their commutator involves (because of locality) the presence of a δ-function or its finite order derivatives. Thus we are in pure-strong interaction regions, where there is no trace of weak currents, we started with. It is therefore natural to look for the possibility of deriving the equation (3.11) directly from the general properties of strong interactions namely Analyticity, Unitarity and High Energy behaviour of scattering amplitudes. V. De Alfaro et al.[25] showed that it is indeed possible to derive a general family of strong interaction sum rules from purely strong interaction quantities; analyticity, unitary and high energy behaviour of scattering amplitudes. The equation (3.11) being just one special case among this family.

In the following we discuss such a family of strong interaction sum rules or superconvergence relations as they have come to be called.

Superconvergence

2. Consider an analytic function $F(\nu)$ satisfying unsubtracted dispersion relation in the variable ν

momentum transfer they obtained two sum rules for $\rho\pi$ elastic scattering. They saturated the two "basic" sum rules with π, ω, ϕ intermediate states and obtained the relations

$$g^2_{\omega\rho\pi} = 4g^2_{\rho\pi\pi}/m^2$$

and (3.16)

$$g_{\rho\rho\pi} = 0$$

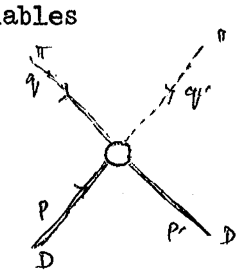
which agree with the relativistic SU(6) prediction[19],[26], since then a large number of sum rules have been obtained for various scattering processes[27].

We discuss in the following in detail the superconvergence relation[28] for πD (D is the isospin doublet of spin 3/2, mass 1525).

<u>πD Scattering</u>

3. Consider πD scattering with momenta as shown in the figure we define a set of kinematical variables

$$P = \tfrac{1}{2}(p + p')$$
$$Q = \tfrac{1}{2}(q + q')$$
$$\Delta = p' - p$$



where p, p' are the four-momenta of the incoming and outgoing D and q, q' are the four-momenta of the incoming and outgoing pion.

The usual Mandelstam variables are given as

$$s = (p + q)^2 = (p' + q')^2$$

$$t = (p' - p)^2 = (q' - q)^2$$

$$u = (p - q') = (p' - q)^2$$

$$\nu = P.Q = \tfrac{1}{4}(s - u)$$

and $\quad t = \Delta^2$ the momentum transfer.

We write the amplitude for this process as

$$T = \bar{\psi}_\mu (p') M_{\mu\nu} \psi_\nu(p) \tag{3.17}$$

where $\psi_\mu(p')$ and $\psi_\nu(p)$ are the Rarita-Schwinger spin 3/2 wave functions.

The covariant M function can be developed as[29]

$$M_{\mu\nu} = (A_{pp} + B_{pp}\not{Q})P_\mu P_\nu + (A_{QQ} + B_{QQ}\not{Q}) + (A_g + B_g Q)g_{\mu\nu} \tag{3.18}$$

where we have used time reversal to eliminate terms in $(P_\mu Q_\nu - Q_\mu P_\nu)$ $\times \begin{Bmatrix} 1 \\ Q \end{Bmatrix}$ and further because of the equivalence theorem, we have chosen to eliminate the "kinematical covariant" $(P_\mu Q_\nu + Q_\mu P_\nu) \begin{Bmatrix} 1 \\ \not{Q} \end{Bmatrix}$

in favour of the above six: because of the equivalence theorems we have

$$M(P_\mu Q_\upsilon + Q_\mu P_\upsilon) = 2P_\mu P_\upsilon \not{Q} + g_{\mu\upsilon}(M P.Q - P^2\not{Q})$$

$$M'(P_\mu Q_\upsilon + Q_\mu P_\upsilon)\not{Q} = -2P_\mu P_\upsilon(MQ^2 - 2P.Q\not{Q}) + \tfrac{1}{2}M\Delta^2 Q_\mu Q_\upsilon$$

$$+ g_{\mu\upsilon}\left[M[(P.Q)^2 - \tfrac{1}{4}\Delta^2 Q^2] - P.Q(P^2 - \tfrac{1}{4}\Delta^2)\not{Q}\right]$$

$$(3.19)$$

where M is the mass of D. These equalities are of course to be understood as holding only when sandwiched between appropriate spin 3/2 spinors,; we give a detailed treatment of these equivalence theorems in the appendix. The form of equation (3.19) shows that $(P_\mu Q_\upsilon + Q_\mu P_\upsilon)\begin{Bmatrix}1\\ \not{Q}\end{Bmatrix}$ can indeed be eliminated without introducing kinematic singularities into the scattering amplitudes.

<u>High energy behaviour of the amplitudes</u>

Assuming that for large s and fixed momentum transfer t the s-channel helicity amplitudes $T^{(\lambda',\lambda)}$ behave like $s^{\alpha(t)}$ for all λ',λ. Here λ', λ are the final and initial helicities of D and $\alpha(t)$ is the position of the leading Regge trajectory in the t-channel. We can deduce the asymptotic behaviour of the invariant amplitudes from the following observations. Writing (3.17) as

$$M_{\mu\upsilon} = \sum_i A_i K_{i;\mu\upsilon} \tag{3.20}$$

we have

$$T^{(\lambda',\lambda)} = \sum_i A_i K_i^{(\lambda',\lambda)} \sim s^{\alpha(t)} \tag{3.21}$$

where

$$K_i^{(\lambda',\lambda)} = \bar{\psi}_\mu^{(\lambda')}(p') K_i^{\mu\upsilon} \psi_\upsilon^{(\lambda)}(p) \tag{3.22}$$

Regarding equation (3.21) a set of constraints conditions on the high energy behaviour of the invariant amplitudes A_i, we see that this is suppressed by s^n where s^n is the highest behaviour of $K_i^{(\lambda',\lambda)}$ for various (λ',λ). This is easily obtained by expanding the spin 3/2 spinors as

$$\psi_\upsilon^{(\lambda)}(p) = \sum C(1/2\ 1\ 3/2;\ r\ \lambda-r)\ u^{(r)}(p)\epsilon_\upsilon^{(\lambda-r)}(p) \tag{3.23}$$

where ϵ_υ is a vector and $u^{(r)}$ is a Dirac spinor.

We note that all of the scalar products in (3.22) behave like s^0 except for $\epsilon^{(0)} . Q \sim s$, and $\bar{u}\, u \sim s^0$ where as $\bar{u}^{(\frac{1}{2})} \not{Q}\, u^{(\frac{1}{2})} \sim s$.

Hence the asymptotic behaviour of an invariant amplitude is improved by one power of s for every factor of Q or $\not{Q}$ which appears in its accompanying kinematical covariant. Thus we can immediately read off the high energy behaviour of the invariant amplitudes in equation (3.17), viz.

$$\left.\begin{aligned} A_{pp},\ A_g &\sim s^{\alpha} \\ B_{pp},\ B_g &\sim s^{\alpha-1} \\ A_{QQ} &\sim s^{\alpha-2} \\ \text{and}\quad B_{QQ} &\sim s^{\alpha-3} \end{aligned}\right\} \tag{3,24}$$

Now we specialize in the case of zero momentum transfer $t = 0$, and dispersing in υ, rather than in s for convenience in crossing, we find three superconvergent sum rules of the form

$$\int_0^\infty \mathrm{Im}\, A(\upsilon,\ t=0)\,d\upsilon = 0 \tag{3.25}$$

where

$$A = B^{(0)}_{QQ} \ ; \quad \upsilon\, B^{(1)}_{QQ} \ ; \quad A^{(1)}_{QQ} \tag{3.26}$$

where the (0), (1) tell the isospin of the Regge trajectory exchanged in t-channel.

Now we have from experimental information about high energy scattering that

$$\begin{aligned} \alpha^{(0)}(t = 0) &= 1 \\ \alpha^{(1)}(t = 0) &\cong 0.5 \end{aligned} \tag{3.27}$$

Saturation

Now we try to saturate the sum rules (3.25) with the N, N*(1236) and D intermediate states. We use the effective Lagrangians

$$\mathcal{L}_{DN\pi} = \frac{g_{DN\pi}}{m_\pi} \psi_\mu \gamma_5 \partial_\mu \phi_\pi u \tag{3.28}$$

$$\mathcal{L}_{DN^*\pi} = g_{DN^*\pi} \bar{\psi}_\mu \psi_\mu \phi_\pi \tag{3.29}$$

We neglect the derivative couplings

and

$$\mathcal{L}_{DD\pi} = g_{DD\pi} \bar{\psi}_\mu \gamma_5 \psi_\mu \phi_\pi \tag{3.30}$$

We further use the relation

$$\Gamma_{D\to \pi N^*} = \frac{g^2_{DN^*\pi}}{4\pi} \frac{p(E + m_{N^*})}{9\, m_D\, m^2_{N^*}} [4E^2 + 4m_{N^*} E + 10m^2_{N^*}] \tag{3.31}$$

which gives

$$\frac{g^2_{DN^*\pi}}{4\pi} = 0.04 .$$

We get the sum rules for $A^{(1)}_{QQ}$, $\upsilon B^{(1)}_{QQ}$ and $B^{(0)}_{QQ}$ after saturating them with N, N*(1236) and D as

$$\frac{1}{m^2_\pi} g^2_{DN\pi} - \frac{4}{3m^2_{N^*}} g^2_{DN^*\pi} + \frac{2}{3m^2_D} g^2_{DD\pi} = 0 \tag{3.32}$$

for $B^{(I=0)}_{QQ}$

$$\frac{m_D - m_\pi}{m_\pi^2}\, g^2_{DN\pi} + \frac{2(m_D + 2m_{N^*})}{3m^2_{N^*}}\, g^2_{DN^*\pi} - \frac{2}{3m_D}\, g^2_{DD\pi} = 0 \qquad (3.33)$$

for $A^{(I=1)}_{QQ}$

and finally for $\upsilon\, B^{(I=1)}_{QQ}$

$$\frac{m_D^2 + m_\pi^2 - m_N^2}{m_\pi^2}\, g^2_{DN\pi} + \frac{2(m_D^2 + m_\pi^2 - m^2_{N^*})}{3m^2_{N^*}}\, g^2_{DN^*\pi} + \frac{2m_\pi^2}{m_D^2}\, g^2_{DD\pi} = 0 \qquad (3.34)$$

Let us now put in the experimental values of the known quantities and see what results we get. We put

$$m_N = 938 \text{ Mev}$$

$$m_{N^*} = 1236 \text{ Mev}$$

$$m_D = 1525 \text{ Mev}$$

$$m_\pi = 138 \text{ Mev}$$

$$g^2_{DN^*\pi} = 0.04$$

for $g_{DN\pi}$ we note that the width of D $(N^*(1525))$ is given by

$$\Gamma_{N^*(1525)\to \pi N} = \frac{g^2_{DN\pi}}{4\pi}\, \frac{P^3(E_N - m_N)}{3m_D\, m_\pi^2} \qquad (3.35)$$

where the Lagrangian is that given by (3.28).

Then the phenomenological value of the elastic width[30] is 73 Mev, giving $\frac{g^2_{DN\pi}}{4\pi} = 0.70$. The same phase shift analysis gives the inelastic width as 29 Mev and there is good evidence to support the belief that the entire inelastic mode is πN^* [31].

Putting these parameter we find that relations (3.32) and (3.34) give $g^2_{DD\pi} < 0$. So without giving any plausible argument we neglect them. Equation (3.33) however gives

$$\frac{g^2_{DD\pi}}{4\pi} = 48 \tag{3.36}$$

However the error in N*(1525) width are quite large, actually about 20 Mev, so that we can reasonably put

$$\frac{g^2_{DD\pi}}{4\pi} \sim 48 \pm 13 \tag{3.37}$$

The faith one has in this value, depends of course on how strong one's belief is that the $A^{(I=1)}_{QQ}$ sum rule is in some sense a good sum rule. All we can say is that it is the same sum rule in πN^* scattering[29], which when saturated with N and N* intermediate states gives in $m_N = m_{N^*}$ limit results which are consistent with U(6,6) theory results, thereby earning some degree of distinction as a sum rule of good repute.

It is now interesting to compare this result with the one derived by another, perhaps less dubious, method. Gatto et al.[32]

have examined the consequences of saturating the algebra of chiral $SU(3) \times SU(3)$ with baryon resonances of positive and negative parity. The positive parity states are assumed all to belong to the usual 56-fold of $SU(6)$ while the $(20\ L = 1)$ multiplet is supposed to incorporate in it $\frac{1}{2}^-$, $\frac{3}{2}^-$, $\frac{5}{2}^-$ resonances. These authors thus obtain values for the various possible axial vector coupling constants at zero momentum transfer in terms of one real parameter a where $0 \leqslant a \leqslant 1$.

We now proceed to relate these weak coupling constants to the strong interaction one by using the hypothesis of pole dominance of the divergence of axial current (PDDAC). We assume $a = \frac{1}{2}$ (which implies $G_A/G_V = 1$ and $g_{\pi NN} = 13.5$). We then obtain at zero pion momentum

$$g_{pN^{*+}\pi^0} = g_{NN\pi} \frac{m_\pi}{m_N} \sqrt{\frac{3}{2}} \frac{3\sqrt{a}}{1 + 4a} = 1.87 \tag{3.38}$$

corresponding to $N^*(1236)$ width of 118 Mev.

Furthermore one gets

$$g_{N^{*+}N^{*+}\pi^0} = g_{NN\pi} \frac{3m_{N^*}}{2m_N(1 + 4a)} = 9.6 \tag{3.39}$$

which compares favourably with the value 10.2 from $U(6,6)$ [19] .
We are thus tempted to apply this procedure to D and thereby obtain

$$g_{DD\pi} = g_{NN\pi} \frac{3m_D}{m_N(1 + 4a)} = 23.4$$

or
$$\frac{g^2_{DD\pi}}{4\pi} = 44 \qquad (3.49)$$

to be compared with the value we obtain (3.36). This needs some comments. We note that the possible SU(6) assignment for the negative parity baryon resonances has been a method of speculation for sometime. The 70-fold was originally suggested to accommodate the $\frac{1}{2}^-$, $\frac{3}{2}^-$ and $\frac{5}{2}^-$ baryons. However, under the larger symmetry of U(6,6), definite parity predictions could be made concerning SU(6) multiplets(33). Delbourgo and Rashid(34) pointed out that one SU(6) 70-fold belonged to the 572 of U(6,6) with positive parity and although it was possible to incorporate a 70 in U(6,6) 5720 fold, the resulting widths were too small. One of the most attractive schemes was that of Gatto et al.(35), who introduced the idea of kinetic supermultiplets which have proved remarkably successful in classifying the higher boson resonances. It turns out that the most economical classification places the $\frac{1}{2}^-$, $\frac{3}{2}^-$, $\frac{5}{2}^-$ baryons states in the 220 of U(6,6), which corresponds to the 20 of SU(6) with orbital angular momentum $L = 1$. As the resulting mass formulae are not testable there is no direct evidence in favour of the (20 $L = 1$) assignment, but we like to like to feel that the striking similarity of equations (3.36) and (3.40) is at least a point in favour of this kinetic super-multiplet schemes.

CHAPTER IV

The Use of Helicity Amplitudes in Superconvergent Sum Rules

1.

In the previous chapter we demonstrated the existence of a wide class of strong interaction sum rules. If $A(s,t)$ is some scattering amplitude and if it fulfils the two requirements:

a) $A(s,t) \sim \frac{1}{s^b}$ as $s \to \infty$ with $b > 1$

and

b) $A(s,t)$ has no kinematical singularity in s at fixed t,

then the imaginary part of $A(s,t)$ obeys the sum rule:

$$\int ds \, \mathrm{Im}\, A(s,t) = 0 \qquad (4.1)$$

where the integral is taken over the left and right hand cuts of $A(s,t)$.

In order to get amplitudes satisfying the requirement (b) one may decompose the complete scattering amplitude in covariants, following the prescription of Hall-Wightman theorem[38]. However this procedure is somewhat involved. On the other hand we can decompose the whole scattering amplitude in the covariants, which come out from its perturbative expansion[39]. This was the method we employed in finding out the superconvergent sum rules for $\pi N^*(1525)$ scattering. This is also the method employed by almost all the authors given in ref. (27). But this method, besides being on a not very firm foundation, i.e. the perturbation theory, is affected by ambiguities in its application even for low spin particle [(40)].

Fortunately there exists another method proposed by Trueman[(40)] and also by Odorico[(41)] using helicity amplitudes, through which we can find very easily superconvergent relations for arbitrary spin particle scattering. In the following we describe this method in detail.

II. Kinematic Singularities in the Cross-Channel Energy Variable

The partial wave expansion of a general helicity amplitude for a reaction $a + b \rightarrow c + d$, with t being the energy squared and s being the square of the energy in the direct channel $(\bar{d}+b \rightarrow c+\bar{a})$ is*

Our normalization of the scattering amplitude differs from that of Jacob and Wick[(42)]. The two are related by

$$f_{cd;ab}(s,t) = 2\pi \left(\frac{sp_{ab}}{p_{cd}} \right)^{\frac{1}{2}} f^{J.W}_{cd;ab}(s,t)$$

$f_{cd;ab}(s,t)$ is related to the S-matrix by

$$S_{cd;ab}(s,t) - \delta_{cd;ab} = (2\pi)^4 i\, \delta^4(p_c+p_d-p_a-p_b)(p_a^0 p_b^0 p_c^0 p_d^0)^{-\frac{1}{2}} f_{cd;ab}(s,t)$$

$$f^{s}_{cd;ab}(s,t) = \sum_{J} (2J+1)\, F^{J}_{cd;ab}(t)\, d^{J}_{\lambda\mu}(\theta_t) \qquad (4.2)$$

$$\lambda = a - b \; ; \qquad \mu = c - d$$

θ_t is the scattering angle in the t-channel, which is taken to be the angle between a and c; and $d^{J}_{\lambda\mu}$ is the d-function of the rotation matrix element. In the t-channel centre of mass system

$$\cos\theta_t = \left[2st + t^2 - t\sum_{i} m_i^2 + (m_a^2 - m_b^2)(m_c^2 - m_d^2)\right] \Big/ T_{ab}\, T_{cd} \qquad (4.3)$$

where

$$T_{ab} = \left[t - (m_a - m_b)^2\right]\left[t - (m_a + m_b)^2\right] = 4tP_{ab}$$

$$T_{cd} = \left[t - (m_c - m_d)^2\right]\left[t - (m_c + m_d)^2\right] = 4t\,P_{cd} \qquad (4.4)$$

where P_{ab} and P_{cd} are the initial and the final momenta in the t-channel c.m. system. We see that cos θ_t is an analytic function

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while $f^{J.W}_{cd;ab}(s,t)$ has a simpler relation to the differential cross-section, i.e.

$$\frac{d\sigma}{d\Omega} = |f^{J.W}_{cd;ab}(s,t)|^2$$

Also we set equal to zero the azimuthal angle, which is independent of the invariant quantities s and t. We also use a, b, c and d as notations for particles as well as the helicity states of the of s. In general, the d-function is related to the Jacobi polynomial by [43]

$$d^J_{\lambda\mu}(\theta_t) = \pm\left[\frac{(J+M)!\,(J-M)!}{(J+N)!\,(J-N)!}\right]^{\frac{1}{2}} (1-z_t)^{\frac{1}{2}|\lambda-\mu|}\;(1+z_t)^{\frac{1}{2}|\lambda+\mu}$$

$$\times\; P^{(|\lambda-\mu|,|\lambda+\mu|)}_{(J-M)}\;(z_t) \qquad (4.5)$$

where $M \equiv$ maximum of $(|\lambda|,|\mu|)$

$N \equiv$ minimum of $(|\lambda|,|\mu|)$ and $z_t = \cos\theta_t$

so that equation (4.2) becomes

$$f^t_{cd;ab}(s,t) = (1-z_t)^{\frac{1}{2}|\lambda-\mu|}\;(1+z_t)^{\frac{1}{2}|\lambda+\mu|}\sum_J (2J+1)$$

$$\times\; F^J_{cd;ab}(t)\;P^{(|\lambda-\mu|,|\lambda+\mu|)}_{(J-M)}(z_t) \qquad (4.6)$$

where we have put the other constant factors in $F^J_{cd;ab}(t)$. We now see that the presence of spins has introduced into the helicity amplitudes a definite set of s zeros and singularities through the factors $(1+z_t)^{\frac{1}{2}|\lambda+\mu|}$ and $(1-z_t)^{\frac{1}{2}|\lambda-\mu|}$. It has been shown by many authors[44] that these s-zeros and singularities are the only kinematic ones. The remaining s-singularities are associated

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corresponding particles. Which meaning they take can be easily understood from the context. We use A, B, C, D for the corresponding antiparticles and their helicity states.

with the failure of the Jacobi expansion to converge, a dynamical effect unrelated to particle spins. And it can be shown that the kinematic s-singularities of $f^{(t)}_{cd;ab}$ are all on the boundary of the physical region.

Thus the new amplitudes, defined by

$$\bar{f}^{(t)}_{cd;ab} \equiv f^{t}_{cd;ab}\,[1 - z_t]^{-\frac{1}{2}|\lambda-\mu|}[1 + z_t]^{-\frac{1}{2}|\lambda+\mu|}$$

$$= \sum_J (2J + 1)\, F^J_{cd;ab}(t) \times P^{(|\lambda-\mu|,|\lambda+\mu|)}_{(J-M)}(z_t) \tag{4.7}$$

contain only dynamical s-singularities. And by assumption of maximal analyticity in s-matrix theory, $\bar{f}^{(t)}_{cd;ab}$ satisfies a fixed-t dispersion relation in s.

Let us now concentrate on the factor $\chi_{\lambda,\mu}(s,t)$ defined as

$$\chi_{\lambda,\mu}(s,t) = (1 + z_t)^{-\frac{1}{2}|\lambda+\mu|}\,(1 - z_t)^{-\frac{1}{2}|\lambda-\mu|} \tag{4.8}$$

and let us study the asymptotic behaviour of $\chi_{\lambda,\mu}$ for very large s.

From equation (4.3) for fixed t and large s we have

$$z_t \underset{\substack{s\to\infty \\ t \text{ fixed}}}{=} 2st/\{[t - (m_a+m_b)^2][t - (m_a-m_b)^2]\,[t - (m_c-m_d)^2] \times [t - (m_c+m_d)^2]\}^{\frac{1}{2}} \tag{4.9}$$

so that we have

$$\chi_{\lambda\mu}(s,t) \xrightarrow[\substack{s\to\infty \\ t \text{ fixed}}]{} \frac{s^{n(|\lambda|,|\mu|)}}{c(t)} \tag{4.10}$$

where $n(\lambda,\mu)$ is the maximum of $|\lambda|$ or $|\mu|$,

where $c(t)$ can be determined from (4.9) and (4.3).

Thus, from each $f^{(t)}$ which describes a process involving helicity flip, an amplitude $\bar{f}^{(t)}$ can be constructed which is more convergent as $s \to \infty$, the higher the helicity flip the better the convergence, as seen in (4.11)

$$\bar{f}^{(t)}_{cd;ab}(s,t) = \frac{1}{\chi_{\lambda,\mu}(s,t)} f^{(t)}_{cd;ab}(s,t) = c(t)\, s^{-n(|\lambda|,|\mu|)}$$

$$\times\; f^{(t)}_{cd;ab}(s,t) \qquad \text{as} \quad s \to \infty \tag{4.11}$$

The next problem is to determine the bounds on $f^{(t)}_{cd;ab}(s,t)$. For this we appeal to the Regge theory which gives the behaviour of $f^{(t)}_{cd;ab}(t,s)$ for large s to be

$$f^{(t)}_{cd;ab}(t,s) \sim s^{\alpha(t)} \tag{4.12}$$

where $\alpha(t)$ is the position of the Regge trajectory exchanged in the t-channel. So we can write the overall asymptotic behaviour of $\bar{f}^{(t)}_{cd;ab}(t,s)$ for large s as

$$\bar{f}^{(t)}_{cd;ab}(t,s) \underset{s\to\infty}{\sim} s^{\alpha(t)-n(|\lambda|,|\mu|)} \tag{4.13}$$

So the amplitude $\bar{f}^{(t)}_{cd;ab}(t,s)$ will satisfy the sum rule:

$$\int_{-\infty}^{+\infty} ds \ \mathrm{Im}\ (s^{\beta-1}\ \bar{f}^{(t)}_{cd;ab}(t,s)\ ) = 0 \tag{4.14}$$

where

$$\beta = \alpha(t) - n(|\lambda|,|\mu|) < 0 \tag{4.15}$$

Our next problem is to convert the integral in equation (4.14) into one extended only over the region with $z_t \geqslant 0$. This problem we tackle in the next section.

3. <u>Crossing s $\leftrightarrow$ u</u>

We start by changing the variable in (4.14) from s to z_t, i.e.

$$\int_{-\infty}^{+\infty} ds \ \mathrm{Im}\ (s^{\beta-1}\ \bar{f}^{(t)}_{cd;ab}(t,s) \equiv \int_{-\infty}^{+\infty} dz_t \ \mathrm{Im}\ (s^{\beta-1}\ \bar{f}^{(t)}_{cd;ab}(t,z_t)) = 0 \tag{4.16}$$

We have simply to observe that in $\bar{f}^{(t)}_{cd;ab}(t,s)$ essentially nothing changes if instead of considering a and b as the first and second initial particles, we reverse their order. Limiting, for simplicity, to the cases where a,b are both bosons or both fermions, we have that the inversion of the order of two particles introduces a negative sign in the second case and leaves it unaffected in the first case. Therefore we may write

$$\bar{f}^{(t)}_{cd;ab}(t,s) = \langle JM\ cd|T^J(t)|JM\ ab\rangle$$
$$= \langle JM\ cd|T^J(t)(-1)^{2S_b}\rho_{ab}|JM\ ba\rangle \tag{4.17}$$

where ρ_{ab} is the operator which changes the order of a and b. On the other side:

$$(-1)^{2S_b}\rho_{ab}|JM\ ab\rangle = (-1)^{J+\lambda}|JM\ ba\rangle \tag{4.18}$$

$$\text{with}\quad \lambda = a - b$$

One also has: (43)

$$P^J_{\mu\lambda}(z_t) = (-1)^{J+\mu}P^J_{\mu,-\lambda}(-z_t) \tag{4.19}$$

Then:

$$(-1)^{2S_b}\rho_{ab}|JM\ ab\rangle P^J_{\mu\lambda}(z_t) = (-1)^{\lambda+\mu}|JM\ ba\rangle P^J_{\mu,-\lambda}(-z_t) \tag{4.20}$$

and therefore we may conclude

$$\bar{f}_{cd;ab}(t,-|z_t|) = (-1)^{\lambda+\mu}\,\bar{\bar{f}}_{cd;ba}(t,|z_t|) \tag{4.21}$$

where $\bar{\bar{f}}_{cd;ba}(t,z_t)$ is defined like $\bar{f}_{cd;ab}(t,z_t)$. So for instance

z_t in $\bar{\mathcal{F}}_{cd;ba}(t,z_t)$ is the cosine of the angle between the momenta of b and c, whereas in $\bar{f}_{cd;ab}(t,z_t)$, z_t is the cosine of the angle between the momenta of a and c. Consequently the sum rule in equation (4.14) can be written as

$$\int_0^\infty ds \; \mathrm{Im}(s^{\beta-1}[\, \bar{f}_{cd;ab}(t,s) - (-1)^{\lambda+\mu} \; \bar{\mathcal{F}}_{cd;ba}(t,s)]) = 0 \tag{4.22}$$

The above sum rule is further modified if we consider the isospin of the particles taking part in the reaction. Suppose for instance that a and b have got the same isospin T_o and let us consider the amplitude $\bar{f}^{(T)}_{cd;ab}(t,s)$ corresponding to their coupling to some isospin T. Then for this amplitude the equation (4.22) has to be modified to

$$\int_0^\infty ds \; \mathrm{Im}(s^{\beta-1}[\bar{f}^{(T)}_{cd;ab}(t,s) - (-1)^{\lambda+\mu+T+2T_o} \, \bar{\mathcal{F}}^{(T)}_{cd;ba}(t,s)]\,) = 0 \tag{4.23}$$

4. Identical Particles

Next we look at the restriction imposed by the statistics obeyed by the particles concerned in the reaction. If the particles a and b (or c and d), are identical, the sum rule (4.22) may become trivial. Suppose for instance that a and b are identical. If one also has a = b and μ is even the sum rule is trivial. In general triviality may occur when the helicities of the two identical particles (both ingoing or outgoing) are equal; the cases in which it really occurs may be read directly from equation (4.22) or (4.23) if the particles carry isospin.

5. Regge Pole Contribution

The knowledge of what Regge Poles contribute to a given superconvergent sum rule is a useful indication about the convergence of the sum rule itself. In the following we restrict ourselves to Meson-Baryon scattering so that the baryon number exchanged in t-channel is zero.

Let us denote by $[\bar{f}_{cd;ab}(t,s)]_\alpha$ the contribution of a given Regge trajectory $\alpha(t)$ to the amplitude $\bar{f}_{cd;ab}(t,s)$. One has:

$$[\bar{f}_{cd;ab}(t,s)]_\alpha = K_{cd;ab}(t)[1 + (-1)^c e^{-i\pi\alpha(t)}]\, s^{\alpha(t)-n(|\lambda|,|\mu|)} \tag{4.24}$$

where c is the signature (even or odd) and $K_{cd;ab}(t)$ is a real function of t,

One has then:

$$[\bar{f}_{cd;ab}(t,-|s|)]_\alpha = K_{cd;ab}(t)[1 + (-1)^c e^{-i\pi\alpha(t)}]e^{i\pi(\alpha(t)-n)^*}|s|^{\alpha(t)-n}$$

$$= (-1)^c (-1)^n K_{cd;ab}(t)[1 + (-1)^c e^{i\pi\alpha(t)}]|s|^{\alpha(t)-n} \tag{4.25}$$

* we have dropped $(|\lambda|,|\mu|)$ from $n(|\lambda|,|\mu|)$ to keep the notation simple.

and consequently:

$$[\bar{f}_{cd;ab}(t,-|s|)]^{*}_{\alpha} = (-1)^{c}\ (-1)^{n}[\bar{f}_{cd;ab}(t,|s|)]_{\alpha} \qquad (4.26)$$

The contribution of the given Regge trajectory to the (possible) sum rule which $\bar{f}_{cd;ab}(t,s)$ satisfies, contains a factor $[1 - (-1)^{c}\ (-1)^{n}]$. So for instance sum rules corresponding to even n never have any contribution from the Pomeranchuk trajectory.

In the following we specialize to the special case of pseudoscalar meson-baryon (octet and decouplet only) scattering.

6. <u>Meson (Pion) - Baryon (Nucleon, N*(1236) Scattering</u>

We define our channels to be

$$a + b \rightarrow c + d \qquad \text{(t-channel)}$$

$$a + \bar{c} \rightarrow \bar{b} + d \qquad \text{(s-channel)}$$

$$b + \bar{c} \rightarrow \bar{a} + d \qquad \text{(u-channel)}$$

where particles a and b represent pions and c and d baryon (N or N*).

Now the most general sum rule we could write for $a + b \rightarrow c + d$ scattering in t-channel is as given in equation

$$\int_{0}^{\infty} ds\ \mathrm{Im}(s^{\beta-1}[\bar{f}^{(T)}_{cd;ab}(t,s) - (-1)^{\lambda+\mu+T+2T_{0}}\ \bar{f}^{(T)}_{cd;ba}(t,s)]\ ) = 0$$

where T_o is the isospin carried by either a or b, T is the isospin exchanged in t-channel. Now in our specialized case

$$a = b = 0 = \lambda = a - b$$

$$T_o = 1$$

so that the above equation reduces to

$$\int_0^\infty ds \; \mathrm{Im}(s^{\beta-1}[1 - (-1)^{\mu+T}] \, \overline{F}^{(T)}_{cd;oo}(t,s)) = 0 \qquad (4.27)$$

where now $\qquad \beta = \alpha^{(T)}(t) - |\mu| < 0 \qquad (4.28)$

Now in the t-channel the isospin exchanged can be T = 0,1,2, and if we restrict ourselves to forward direction only, i.e. $t = 0$ we know that

$\alpha^{T=0}(t=0) = 1)$ (corresponding to Pomeranchuk trajectory)

$\alpha^{T=1}(t=0) \cong 0.5$ (corresponding to ρ-trajectory)

$\alpha^{T=2}(t=0) < 0$ Since no I = 2 particles have been discovered it is reasonable to assume that this is so.

However this neglects the presence of a cut which can be there because of the exchange of two ρ's as pointed out by Philips and Muzinich[45]. In which case $\alpha^{T=2}(t=0)$ can be positive. Anyway without giving any argument we neglect this possibility, and as pointed out in Section 5, sum rules corresponding to even μ have no contribution from the $T = 0$, i.e. Pomeranchuk trajectory. In the following we will consider the sum rule for two processes

a) $\pi + N \rightarrow \pi + N^*$ (s-channel)

and

b) $\pi + N^* \rightarrow \pi + N^*$ (s-channel)

7. $\underline{\pi + N \rightarrow \pi + N^*}$ Scattering Sum Rules:

We define the various reaction channels as follows

$$\pi + N \rightarrow \pi + N^* \qquad \text{(s-channel)}$$

$$\pi + \bar{\pi} \rightarrow \bar{N} + N^* \qquad \text{(t-channel)}$$

$$\bar{\pi} + N \rightarrow \bar{\pi} + N^* \qquad \text{(u-channel)}$$

We label π and $\bar{\pi}$ as pions and $\bar{N}$ and N^* as c and d respectively. a, b, c, d also denote the helicities of corresponding particles as pointed out in section 6.

$$a = b = 0 = \lambda$$

and our sum rules are of the form

$$\int ds \ \mathrm{Im}(s^{\beta-1}[1 - (-1)^{\mu+T}] \ \bar{f}_{cd;oo}(t,s) = 0$$

$$\beta = \alpha^T(t) - |\mu| < 0$$

We further fix $t = 0$

for $T = 0$ and 2 the non-trivial sum rules are for odd μ , i.e.

$\mu = 1$ corresponding to $c = 1/2$; $d = 3/2$

$c = -1/2$; $d = 1/2$

but $\beta = \alpha^{T=0}(0) - 1 \nless 0$ so no superconvergent sum rule corresponding to $T = 0$ for $T = 2$; $\alpha^{T=2}(0) < 0$ so that $\beta > -1$ and we have the superconvergent sum rules

$$\int_0^\infty ds \, \mathrm{Im} \, \bar{f}^{(T=2)}_{1/2 \; 3/2;00}(t=0,s) = 0 \tag{4.29}$$

and

$$\int_0^\infty ds \, \mathrm{Im} \, \bar{f}^{(T=2)}_{-1/2 \; 1/2} (t=0,s) = 0 \tag{4.30}$$

for $T = 1$ only non-trivial sum rules correspond to even μ, i.e.

$\mu = 0,2$ corresponding to $c = 1/2$; $d = 1/2$

$c = -1/2$; $d = 3/2$

but for $T = 1$, $\mu = 0$

$\beta \nless 0$ so no superconvergent sum rule.

On the other hand for $T = 1$, $\mu = 2$

$\beta \cong -1.5$

so we have got a superconvergent sum rule:

$$\int_0^\infty ds \ \mathrm{Im}\ \bar{f}^{T=1}_{-1/2\ 3/2}(t=0,s) = 0 \qquad (4.31)$$

So, for the process under study we have three non-trivial superconvergent sum rules (4.29 - 4.31).

8. $\underline{\pi + N^* \rightarrow \pi + N^*}$ Scattering Sum Rule

As in section 7 we define our direct and crossed channels as

$$\pi + N^* \rightarrow \pi + N^* \qquad \text{(s-channel)}$$

$$\pi + \bar{\pi} \rightarrow \bar{N}^* + N^* \qquad \text{(t-channel)}$$

$$\bar{\pi} + N^* \rightarrow \bar{\pi} + N^* \qquad \text{(u-channel)}$$

We label π and $\bar{\pi}$ as a and b and denote by c and d $\bar{N}^*$ and N^* respectively. a, b, c, d also denote the helicities of the particles a, b, c and d respectively.

Now $a = b = 0 = \lambda = a - b$

So for fixed $t = 0$ our sum rules are of the type

$$\int ds \ \mathrm{Im}(s^{\beta-1}[1 - (-1)^{\mu+T}]\ \bar{f}_{cd;oo}(t=0,s)) = 0$$

$$\beta = \alpha^T(0) - |\mu| < 0$$

for $T = 0, 2$ the non-trivial sum rules correspond to odd value of μ , i.e.

$$c = -3/2 \; ; \quad d = 3/2 \qquad \mu = 3$$

$$\left.\begin{array}{l} c = +1/2 \; ; \quad d = 3/2 \\ c = -1/2 \; ; \quad d = 1/2 \end{array}\right\} \quad \mu = 1$$

Since for $\mu = 1$, $T = 0$ $\beta \nless 0$ there is no superconvergent sum rule corresponding to $T = 0$. While for $T = 2$ $\beta > -1$ we have two superconvergent relations:

$$\int_0^\infty ds \; \mathrm{Im} \; \bar{f}^{(T=2)}_{1/2 \; 3/2;00}(t=0,s) = 0 \tag{4.32}$$

$$\int_0^\infty ds \; \mathrm{Im} \; \bar{f}^{(T=2)}_{-1/2 \; 1/2;00}(t=0,s) = 0 \tag{4.33}$$

but for $\mu = 3$ and $T = 0$ $\beta \approx -2$

which gives us a moment superconvergent relation

$$\int_0^\infty ds \; \mathrm{Im} \; (s \; \bar{f}^{(T=0)}_{-3/2 \; 3/2;00}(t=0,s)) = 0 \tag{4.34}$$

and an ordinary sum rule

$$\int_0^\infty ds \; \mathrm{Im}(\; \bar{f}^{(T=0)}_{-3/2 \; 3/2;00}(t=0,s)) = 0 \tag{4.35}$$

and for $T = 2$ $\mu = 3$ we have $\beta > -3$

which gives us three sum rules

$$\int_0^\infty ds \; \mathrm{Im} \; (s^2 \bar{f}^{(T=2)}_{-3/2\;3/2;00}(t=0,s)) = 0 \qquad (4.36)$$

$$\int_0^\infty ds \; \mathrm{Im} \; s \, \bar{f}^{(T=2)}_{-3/2\;3/2;00}(t=0,s) = 0 \qquad (4.37)$$

and

$$\int_0^\infty ds \; \mathrm{Im} \; \bar{f}^{(T=2)}_{-3/2\;3/2;00}(t=0,s) = 0 \qquad (4.38)$$

For $T = 1$ the only non-trivial sum rule can be obtained for even values of μ, i.e.

$\mu = 0$ corresponding to $c = 3/2$; $d = 3/2$

$c = 1/2$; $d = 1/2$

and $\mu = 2$ corresponding to $c = -1/2$; $d = 3/2$

But for $T = 1$ and $\mu = 0$ $\beta \nless 0$ so no superconvergent sum rule can be obtained. On the other hand for $\mu = 2$ and $T = 1$ $\beta \cong -1.5$ and we have a superconvergent relation

$$\int_0^\infty ds \; \mathrm{Im}(\bar{f}^{(T=1)}_{-1/2\;3/2}(t=0,s)) = 0 \qquad (4.39)$$

Thus we find eight superconvergent sum rules for the process $\pi + N^* \to \pi + N^*$.

Our next problem is to extract any useful information from these sum rules (4.32 - 4.39). This we can do by saturating the sum rules with low lying baryon states in the s-channel. Before we do that we require the necessary framework to enable us to do just that. We develop this formalism in the next section.

9. Direct Channel Pole Contribution to "t" channel amplitude

Suppose that a particle X of mass m, spin s and parity $\eta = (-1)^L \eta_a \eta_b$ is an intermediate state in the s-channel $a + b \to c + d$ where η_a, η_b are the intrinsic parities of particles a and b respectively. We wish to calculate the pole term it induces into the modified t-channel helicity amplitudes $\bar{T}_n(t,s)^*$, which are free from kinematic s-singularities.

The partial wave expansion of the s-channel helicity amplitude is:

$$G_n(s,t) = \sum_J (2J + 1)\, a_n^J(s)\, d_{\lambda\mu}^J(\theta_s) \qquad (4.40)$$

where $$a_n^J = < cd|T^J|ab >$$

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* n denotes all the helicity indices

From helicity representation we go to the JM representation of Jacob and Wick(42)

$$< JM,\ LS|JM;\lambda_1\lambda_2 > \ = \left[\frac{2L+1}{2J+1}\right]^{\frac{1}{2}} C(L\,S\,J;0,\lambda_1\ -\lambda_2)\ C(s_1 s_2 s;\lambda_1,-\lambda_2) \tag{4.41}$$

C's being the Clebsch-Gordon coefficients.

Equation (4.40) modifies to

$$G_n(s,t) = \sum_J (2J+1)\ d^J_{\lambda\mu}(\theta_s) < JM;cd|JM,L_f,S_f >$$

$$< JM,L_f,S_f|T^J|JM;L_i\ S_i > < JM,\ L_i\ S_i|JM,ab > \tag{4.42}$$

$$= \sum_{J,\ell} (2J+1)\ d^J_{\lambda\mu}(\theta_s)\ C^J_{n\ell}\ a^T_\ell(s) \tag{4.43}$$

where

$$C^J_{n\ell} = < JM;cd|JM,L_f\ S_f > < JM,L_i\ S_i|JM,ab > \tag{4.44}$$

$$a^J_\ell = < JM,L_f\ S_f|T^J|JM,L_i\ S_i > \tag{4.45}$$

This expansion is useful since particle X can be an intermediate state only in those amplitudes for which $J = S$ and $|L_f - L_i|$ is limited by parity. Let Σ_ℓ designate the sum over such amplitudes.

We make the following pole approximation for the amplitudes assuming all poles to correspond to stable particles, even if they are not (i.e. N*).

$$a_{\ell}^{J=s}(s) = \sum_{\ell} m_r \, \Gamma_{\ell} \, \delta(s - m_r^2) \tag{4.46}$$

where Γ_{ℓ} is related to the couplings at the two vertices. In this way we isolate the contribution of particle X from the remainder. Now we are interested in the contribution of the exchange of the particle X to the t-channel helicity amplitudes. This leads us to analytically continue equation (4.46) into the physical domain of t-channel and relate it to $\bar{f}_n(s\ t)$ by means of crossing-relations:

$$\bar{f}_{n'}(t\ s) = \sum_{n,\ell} \tilde{B}_{n'n}(t\ s)\ (2s+1) d^{s}_{\lambda\mu}(\theta_s)\ C^{s}_{n\ell}\ m_r \Gamma_{\ell}\ \delta(s - m_r^2) \tag{4.47}$$

where $\tilde{B}_{n'n}(t\ s)$ is the crossimg matrix between s-kinematic singularities free t-channel helicity amplitudes and t-kinematic singularities free s-channel helicity amplitudes .

Having developed this formalism we apply it to saturate sum rules for $\pi N \rightarrow \pi N^*$ and $\pi N^* \rightarrow \pi N^*$.

Let us first consider $\pi N \rightarrow \pi N^*$.

We define

$n' \equiv (cd;ab)$

$1 \equiv (1/2\ 1/2;00)$

$2 \equiv (-1/2\ 1/2;00)$

$3 \equiv (1/2\ 3/2;00)$

$4 \equiv (-1/2\ 3/2;00)$

a,b,c,d being t-channel helicity labels $a = b = 0$ always for the cases under consideration.

while

$n \equiv (\bar{b}d;a\bar{c})$

$1 \equiv (0\ 1/2;0\ 1/2)$

$2 \equiv (0\ 1/2;0\ -1/2)$

$3 \equiv (0\ 3/2;\ 0\ 1/2)$

$4 \equiv (0\ 3/2;\ 0\ -1/2)$

again $a = \bar{b} = 0$

$\bar{c}$ represent nucleon as well as its helicity label, d represent N* and also its helicity label

at $t = 0$ the three superconvergent sum rule (4.29-4.31) reduce to

$$\int_0^\infty \left[\sum_{\ell,J} \tilde{B}_{31}\ (t=0,s=m_r^2)\ (2J+1)\ C^J_{1\ell}\ m_r\ \Gamma_\ell\ \delta(s-m_r^2)\right]^{T=2} ds = 0 \tag{4.29'}$$

$$\int_0^\infty \left[\sum_{\ell,J} \tilde{B}_{21}(t=0,s=m_r^2)\ (2J+1)\ C^J_{1\ell}\ m_r\ \Gamma_\ell\ \delta(s-m_r^2)\right]^{T=2} ds = 0 \tag{4.30'}$$

and

$$\int_0^\infty \left[\sum_{\ell,J} \tilde{B}_{41}(t=0, s=m_r^2)\ (2J+1)\ C^J_{1\ell}\ m_r\ \Gamma_\ell\ \delta(s - m_r^2) \right]^{T=1} ds = 0 \tag{4.31'}$$

We truncate the sum with J = 1/2 (Nucleon) and J = 3/2 (N*(1236)) only. We further allow p-wave and f-wave couplings at πN^*N^* vertex. The sum rules thus reduce to

$$g_1 = \frac{\tilde{B}_{31}(t=0, s=M^2)}{\tilde{B}_{31}(t=0, s=m^2)} \left(-\frac{4}{5}\sqrt{\frac{2}{5}}\right)(3g_4 - g_3)\left(\frac{M}{m}\right) \tag{4.29"}$$

$$g_2 = \frac{\tilde{B}_{21}(t=0, s=M^2)}{\tilde{B}_{21}(t=0, s=m^2)} \left(-\frac{4}{5}\sqrt{\frac{2}{5}}\right)(3g_4 - g_3)\left(\frac{M}{m}\right) \tag{4.30"}$$

and

$$g_3 = \frac{\tilde{B}_{41}(t=0, s=M^2)}{\tilde{B}_{41}(t=0, s=m^2)} \left(-\frac{2}{5}\sqrt{\frac{2}{5}}\right)(3g_4 - g_3)\left(\frac{M}{m}\right) \tag{4.31"}$$

with the following definitions:

g_1 = πNN coupling constant

g_2 = πNN^* coupling constant

g_3 = p-wave πN^*N^* coupling constant

g_4 = f-wave πN^*N^* coupling constant for nucleon pole we have put

$$\Gamma_\ell = g_1 g_2$$

and for N* pole we have put

$$\sum_\ell \Gamma_\ell = + \frac{2}{5} g_2(-3g_4 + g_3)$$

For non-degenerate N, N* masses it is difficult to make any reasonable prediction, because of the ambiguities in the determination of the crossing matrix elements B's, which are complex below threshold. For degenerate mass case however the situation is much simpler. The first two sum rules reduce in the limit of very weak f-wave coupling ($g_4 \ll g_3$) to the result

$$g_3 = -1.976\ g_1 \tag{4.49}$$

and the last gives in the same limit

$$g_3 = 3.952\ g_1$$

Next we consider the superconvergent relation for $\pi N^* \rightarrow \pi N^*$. We saturate the sum rules (4.32 - 4.39) with N and N* pole in the same way as we did for $\pi N \rightarrow \pi N^*$ case. These sum rules after some simplification reduce to except for (4.32) which is trivially satisfied:

$$-3(3g_3 + g_4)^2 + (g_3 - 3g_4)^2 = \frac{55}{8}\left(\frac{m}{M}\right)g_2^2 \tag{4.50}$$

$$(3g_3 + g_4)^2 - 3(g_3 - 3g_4)^2 = \frac{399}{20}\left(\frac{m}{M}\right)g_2^2 \tag{4.51}$$

$$(3g_3 + g_4)^2 - 3(g_3 - 3g_4)^2 = \frac{399}{20}\left(\frac{m}{M}\right)^3 g_2^2 \tag{4.52}$$

$$-(3g_3 + g_4)^2 + 3(g_3 - 3g_4)^2 = \frac{399}{16}\left(\frac{m}{M}\right)^5 g_2^2 \tag{4.53}$$

$$-(3g_3 + g_4)^2 + 3(g_3 - 3g_4)^2 = \frac{399}{16}\left(\frac{m}{M}\right)^3 g_2^2 \tag{4.54}$$

$$-(3g_3 + g_4)^2 + 3(g_3 - 3g_4)^2 = \frac{399}{16}\left(\frac{m}{M}\right)g_2^2 \tag{4.55}$$

and

$$(3g_3 + g_4)^2 - (g_3 - 3g_4)^2 = \frac{121}{8}\left(\frac{m}{M}\right)g_2^2 \tag{4.56}$$

at zero momentum transfer.

Let us discuss these sum rules in detail. Sum rules 50, 53, 54, 55 all correspond to exchange of I = 2 trajectory in the t-channel, and all give f-wave coupling to be much stronger than the p-wave couplings. This is unacceptable on already known facts in various scattering processes. We therefore nelgect these sum rules. If we accept the suggestion of Muzinich and Philips[45], these sum rules may not exist at all. Because the above authors prove that in the I = 2 state Regge cuts corresponding to double ρ-exchange may occur for which $\alpha^{I=2}(t=0) > 0$ and hence no superconvergent relations. Our results about I = 2 sum rule seem to uphold the view of Philips and Muzinich.

The sum rule (4.56) which corresponds to $f^{T=1}_{-1/2\ 3/2}$ together with (4.52) which corresponds to $\left(s f^{T=0}_{-3/2\ 3/2} \right)$ are consistent with the following results

$$\left.\begin{aligned} g_3 &= 1.197\ g_2 \\ g_4 &= 0.009\ g_2 \end{aligned}\right\} \qquad (4.57)$$

which give for the f-wave to p-wave coupling ratio and πN^*N^* vertex

$$\frac{g_3}{g_4} \cong 133$$

a result which seems quite reasonable. Thus sum rules corresponding to ρ- and Pomeranchuk exchange seem to be the only good ones. The other sum rule for Pomeranchuk exchange, i.e. $\int \mathrm{Im}\ f^{T=0}_{-3/2\ 3/2}(t=0,s)ds = 0$ is not compatible with the result of (4.57). This tempts us to assume that the only sum rules for πN^* scattering which are saturated with N and N* are the ones which correspond to the exchange of ρ-trajectory in the t-channel. This is a conclusion Jones and Scadron[29] reached in their study of superconvergent relations for πN^* scattering invariant amplitudes.

Thus (4.57) gives under the assumption of $g_4 \ll g_3$ the result

$$g_3 = 1.194\ g_2 \qquad (4.58)$$

In the equal mass case, i.e. $m = M$, we get from (4.57)

$$g_3 = 1.375\ g_2 \qquad (4.59)$$

In equal mass case however sum rules (4.50 - 4.55) are not saturated with N and N*.

In conclusion we may say that although the superconvergent sum rules written in terms of t-channel helicity amplitudes are true; the question of saturation of these sum rules is still open. We still have no general criteria to determine which sum rule is saturated with low lying states and which is not.

It is however obvious how simple it is to write superconvergent relation for any scattering process in terms of t-channel helicity amplitudes compared to those in terms of invariant amplitudes.

APPENDIX

a) Notations:- we use the metric such that

$$V_\mu = (\underline{V},\ iV_o), \quad a_\mu b_\mu = \underline{a}.\underline{b} - a.b$$

our γ-matrices are Hermitian and satisfy the relations

$$\{\gamma_\mu, \gamma_\upsilon\} = 2\delta_{\mu\upsilon}, \quad \gamma_5\gamma_\mu + \gamma_\mu\gamma_5 = 0, \quad \gamma_5\gamma_5 = 1,$$

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \frac{1}{4!}\epsilon_{\mu\upsilon\lambda\rho}\gamma_\mu\gamma_\upsilon\gamma_\lambda\gamma_\rho \qquad \text{(A.1)}$$

We use the Dirac equation

$$(\gamma.P - im)u(P) = 0 \qquad \text{(A.2)}$$

and the normalization

$\bar{u}(p,\lambda')\, u(p,\lambda) = \delta_{\lambda\lambda'}$ where λ, λ' are the helicities of the Dirac particle. (A.3)

The propagator for spin 1/2 particles is

$$\sum_{\text{spins}} u(p,\lambda)\, \bar{u}(p,\lambda) = \frac{\gamma.p + im}{2im} \qquad \text{(A.4)}$$

For spin 3/2 particles we use the Rarita-Schwinger formalism. The wave function $\psi_\mu(p)$ satisfying the following three conditions

i) $(\gamma\cdot p - im)\psi_\mu(p) = 0$

ii) $p_\mu\cdot\psi_\mu(p) = 0$ (A.5)

iii) $\gamma_\mu\cdot\psi_\mu(p) = 0$

the projection operator being

$$\sum_\lambda \psi_\mu(p,\lambda)\bar{\psi}_\upsilon(p,\lambda) = \left\{ \delta_{\mu\upsilon} - \tfrac{1}{3}\gamma_\mu\gamma_\upsilon + \frac{i}{3m}(p_\mu\gamma_\upsilon - p_\upsilon\gamma_\mu) + \frac{2}{3m^2}p_\mu p_\upsilon \right\} \frac{\gamma\cdot p + im}{2im} \qquad (A.6);$$

b) <u>Decomposition of total amplitude for $\gamma + N \to \pi + N^*$ in Lorentz and Gauge invariant amplitudes:</u>

We define

$$k + p = q + p' \qquad (A.7)$$

p, p' being the four-momenta for proton and N* respectively, q and k being those of pion and photon respectively. We expand the T(q) as

$$T(q) = \bar{\psi}_\mu(p')\sum_{i=1}^{11} \alpha_i\, a_i\, u(p) \qquad (A.8)$$

the various α_i's are

$$p_\mu \gamma_\lambda \gamma_\rho F_{\lambda\rho}, \quad q_\mu p_\lambda q_\rho F_{\lambda\rho}, \quad \delta_{\mu\lambda} p_\rho F_{\lambda\rho}$$

$$p_\mu p_\lambda \gamma_\rho F_{\lambda\rho}, \quad q_\mu p_\lambda \gamma_\rho F_{\lambda\rho}, \quad \delta_{\mu\lambda} q_\rho F_{\lambda\rho}$$

$$p_\mu p_\lambda q_\rho F_{\lambda\rho}, \quad q_\mu q_\lambda \gamma_\rho F_{\lambda\rho}, \quad \delta_{\mu\lambda} \gamma_\rho F_{\lambda\rho}$$

$$p_\mu q_\lambda \gamma_\rho F_{\lambda\rho}, \quad q_\mu \gamma_\lambda \gamma_\rho F_{\lambda\rho},$$

where $$F_{\lambda\rho} = k_\lambda \epsilon_\rho - k_\rho \epsilon_\lambda$$

and we have used the subsidiary conditions (i) - (iii) to reduce the choice of kinematic factor α_i's to the above eleven.

In the limit of zero momentum pion, i.e. $q \to 0$ only two α's $p_\mu \gamma_\lambda \gamma_\rho F_{\lambda\rho}$ and $\delta_{\mu\lambda} \gamma_\rho F_{\lambda\rho}$ survive because of the identities

$$\bar{\psi}_\mu(p')[\gamma\cdot q \, \gamma_\lambda \gamma_\rho F_{\lambda\rho}] u(p) = \bar{\psi}_\mu(p')[i(M-m) \gamma_\lambda \gamma_\rho F_{\lambda\rho} - 4p_\lambda \gamma_\rho F_{\lambda\rho}] u(p) \qquad (A.9)$$

and

$$\bar{\psi}_\mu(p')[\gamma\cdot q \, \delta_{\mu\lambda} \gamma_\rho F_{\lambda\rho}] u(p) = \bar{\psi}_\mu(p')[p_\mu \gamma_\lambda \gamma_\rho F_{\lambda\rho} + i\delta_{\mu\lambda} \gamma_\rho F_{\lambda\rho}] u(p) \qquad (A.10)$$

Thus in the limit of $q \rightarrow 0$ the amplitude $T_{q\rightarrow 0}(q)$ reduces to

$$T_{q\rightarrow 0}(q) = \bar{\psi}_{\mu}(p')[p_{\mu}\,\gamma_{\lambda}\,\gamma_{\rho}\,F_{\lambda\rho}a_1 + i\delta_{\mu\lambda}\,\gamma_{\rho}\,F_{\lambda\rho}\,a_2]\,u(p) \tag{A.11}$$

describing the process $\gamma + N \rightarrow \pi + N^*$

c) <u>Quark Model</u>

In terms of quark fields we construct the vector and axial vector currents of weak interactions by

$$J^{\alpha}_{\mu}(x) = \bar{q}(x)\,\gamma_{\mu}\,\lambda^{\alpha}\,q(x)$$

$$J_{5\mu}(x) = \bar{q}(x)\,\gamma_{\mu}\,\gamma_5\,\lambda^{\alpha}\,q(x) \tag{A.12}$$

where λ^{α} are the Gell Mann matrices obeying the SU(3) algebra

$$[\lambda^{\alpha},\lambda^{\beta}] = i\,f_{\alpha\beta\gamma}\,\lambda^{\gamma}$$

and (A.13)

$$\{\lambda^{\alpha},\lambda^{\beta}\} = d_{\alpha\beta\gamma}\,\lambda^{\gamma}$$

where α, β, run from one to eight. Using the basic commutation properties of Dirac fields, i.e.

$$\{q^{a+}_{\mu}(x), q^{b}_{\upsilon}(y)\}_{x_o=y_o} = \delta_{ab}\,\delta_{\mu\upsilon}\,\delta^3(\underline{x}-y)$$

$$\{q^{a}_{\mu}(x), q^{b}_{\upsilon}(y)\}_{x_o=y_o} = \{q^{+a}_{\mu}, q^{+b}_{\upsilon}(y)\} = 0 \tag{A.14}$$

(a,b being particle label and q^+, q denoting the creation and annihilation operators) together with the relation (A.13). We find with the help of the identity

$$[A \otimes B, A \otimes B'] = \tfrac{1}{2}\{[A,A'] \otimes \{B,B'\} + \{A,A'\} \otimes [B,B']\} \tag{A.15}$$

that following commutation relation hold

$$\delta(x_o - y_o)[J_o^\alpha(x), J_\mu^\beta(y)] = i f_{\alpha\beta\gamma} J_\mu^\gamma(x)\ \delta^4(x-y) \tag{A.16}$$

$$\delta(x_o - y_o)[J_o^\alpha(x), J_{5\mu}^\beta(y)] = i f_{\alpha\beta\gamma} J_{5\mu}^\gamma(x)\ \delta^4(x-y) \tag{A.17}$$

$$\delta(x_o - y_o)[J_{50}^\alpha(x), J_{5\mu}^\beta(x)] = i f_{\alpha\beta\gamma} J_\mu^\gamma(x)\ \delta^4(x-y) \tag{A.18}$$

we further define the vector and axial changes in the following way

$$Q^\alpha(t) = \int_{x_o = t} d^3x\ J_o^\alpha(x) \tag{A.19}$$

$$Q_5^\alpha(t) = \int_{x_o = t} d^3x\ J_{50}^\alpha(x) \tag{A.20}$$

With these definitions the following commutation relations are derived

$$[\,Q^\alpha(t), J_\mu^\beta(x)\,]_{x_o = t} = i f_{\alpha\beta\gamma} J_\mu^\gamma(x) \tag{A.21}$$

$$[\, Q_5^{\alpha}(t),\, J_{\mu}^{\beta}(x)\,]_{x_o = t} = i\, f_{\alpha\beta\gamma}\, J_5^{\gamma}\,(x) \qquad \text{(A.22)}$$

along with a few more. We will explicitly evaluate the last commutator with $\alpha = 3,\ \beta = 3$.

We start with

$$[Q_5^3(t),\, J_{\mu}^3(x)]_{x_o = t} = \int_{x_o = t} d^3x\, \bar{q}\, [\, \gamma_o\, \gamma_5\, \lambda^3,\, \gamma_{\mu}\, \lambda^3]\, q$$

$$= \int_{x_o = t} d^3x\, \frac{\bar{q}}{2} \Big\{ [\, \gamma_o\, \gamma_5, \gamma_{\mu}]\ \{\lambda^3, \lambda^3\} + \{\, \gamma_o\, \gamma_5, \gamma_{\mu}\}\ [\lambda^3, \lambda^3] \Big\}\, q$$

$$= \int_{x_o = t} d^3x\, \frac{\bar{q}}{2} \Big\{ 0 \times d_{338}\, \lambda^8 + 2\gamma_o\, \gamma_5\, \gamma_{\mu} \times 0 \Big\}\, q$$

$$= 0 \qquad \text{(A.23)}$$

We have used (A.15)

(A.23) is the result we used in deriving equation (2.10)

d) <u>$t \longleftrightarrow s$ crossing matrix for $\pi N^* \to \pi N^*$</u>

In the following we calculate the crossing matrix between t-channel and s-channel helicity amplitudes for $\pi N^* \to \pi N^*$. Following Wang(44) the crossing matrix elements for two body elastic scattering is given by

$$M^{cd;ab}_{c'A';D'b'} = [1 - z_t]^{-\frac{1}{2}|\lambda-\mu|} \; [1 + z_t]^{-\frac{1}{2}|\lambda+\mu|}$$

$$\times \sum \left\{ d^{J_a}_{A'a}(\chi_a) \; d^{J_b}_{b'b}(\chi_b) \; d^{J_c}_{c'c}(\chi_c) \; d^{J_d}_{D'd}(\chi_d) \right.$$

$$\left. + \eta_s \; d^{J_a}_{A'-a}(\chi_a) \; d^{J_b}_{b'-b}(\chi_b) \; d^{J_c}_{c'-c}(\chi_c) \; d^{J_d}_{D'-d}(\chi_d) \right\}$$

$$\times [1 - z_s]^{\frac{1}{2}|\lambda'-\mu'|} \; [1 + z_s]^{\frac{1}{2}|\lambda'+\mu'|} \qquad \text{(A.24)}$$

z_t, z_s being the c.m. scattering angle in t and s-channels respectively, where

$$\lambda = a - b \; ; \qquad \lambda' = D' - b'$$

$$\mu = c - d \; ; \qquad \mu' = c' - A' \qquad \text{(A.25)}$$

and $$\eta_s = \frac{\eta_A \; \eta_c}{\eta_D \; \eta_b} \, (-)^{J_c+J_a-J_d-J_b} \, (-)^{\lambda'-\mu'}$$

where η_A, η_c, η_D, η_b are the intrinsic parities of particles a, b, c and d. Now for πN^* elastic scattering we have to identify c, d with $\bar{N}^*$ and N^* respectively and b, a with the pions. For this case we have

$$J_a = J_b = 0 \quad \text{and} \quad \lambda = 0$$

$$\lambda' = D', \quad \mu' = c'$$

$$\eta_s = (-)^{D'-c'}$$

Further if we specialize to forward scattering, i.e. $t = 0$ and $z_s = 1$ then the non-zero crossing matrix elements are for $\lambda' = \mu'$ i.e. $c' = D'$.

Thus the crossing matrix element for $\pi N^* \to \pi N^*$ at $t = 0$ are

$$M^{cd;00}_{c'0;D'0} = (\mathrm{Sin}\ \theta_t)^{-|c-d|}$$

$$\times \left\{ d^{3/2}_{c'c}(\chi_c)\ d^{3/2}_{D'd}(\chi_d) + d^{3/2}_{c'-c}(\chi_c)\ d^{3/2}_{D'-d}(\chi_d) \right\}$$

$$\mathrm{Cos}\ \chi_c = -t(s + \mu^2 - M^2) \Big/ \left\{ t(t - 4\mu^2)\ [s - (M+\mu)^2] \right.$$

$$\left. [s - (M-\mu)^2] \right\}^{\frac{1}{2}}$$

$$= 0 \quad \text{at} \quad t = 0$$

Similarly we find

$$\mathrm{Cos}\ \chi_d = 0$$

So $\qquad \mathrm{Sin}\ \chi_a = \mathrm{Sin}\ \chi_d = 1$

Now the only permitted configuration for $c' = D'$ in $\pi N^* \to \pi N^*$ scattering is for $c' = D' = 3/2$ and $1/2$.

Let us determine the crossing matrix element for particle helicities we are interested in

$$M^{-3/2\ 3/2;00}_{3/2\ 0;3/2\ 0}\ (st = 0) = (\mathrm{Sin}\ \theta_t)^{-3} \left\{ d^{3/2}_{3/2\ -3/2}(\chi_c)\ d^{3/2}_{3/2\ 3/2}(\chi_d) \right.$$

$$\left. + d^{3/2}_{3/2\ 3/2}(\chi_c)\ d^{3/2}_{3/2\ -3/2}(\chi_d) \right\}$$

$$= -\frac{1}{8}\,(\text{Sin}\ \theta_t)^{-3}\left\{ (1 - \cos\ \chi_c)^{3/2}(1 - \cos\ \chi_d)^{3/2} \right.$$

$$\left. + (1 - \cos\ \chi_c)^{3/2}\ (1 - \cos\ \chi_d)^{3/2} \right\}$$

$$= -\frac{1}{4}\,(\text{Sin}\ \theta_t)^{-3}$$

$$M^{-3/2\ 3/2;00}_{1/2\ 0;1/2\ 0}(st=0) = (\text{Sin}\ \theta_t)^{-3}\left\{ d^{3/2}_{1/2\ -3/2}(\chi_c)\ d^{3/2}_{1/2\ 1/2}(\chi_d) \right.$$

$$\left. + d^{3/2}_{1/2\ 3/2}(\chi_c)\ d^{3/2}_{1/2\ -3/2}(\chi_d) \right\}$$

$$= \frac{3}{8}\,(\text{Sin}\ \theta_t)^{-3}\left\{ \text{Sin}^2\frac{\chi_c}{2}\ \text{Cos}\frac{\chi_c}{2}\ \text{Cos}^2\frac{\chi_d}{2}\ \text{Sin}\frac{\chi_d}{2} \right.$$

$$\left. + \text{Cos}^2\frac{\chi_c}{2}\ \text{Sin}\frac{\chi_c}{2}\ \text{Sin}^2\frac{\chi_d}{2}\ \text{Cos}\frac{\chi_d}{2} \right\}$$

$$= \frac{3}{8}\,(\text{Sin}\ \theta_t)^{-3}\left\{ \text{Sin}\ \chi_c\ \text{Sin}\ \chi_d(1 - \text{Cos}\ \chi_c)^{\frac{1}{2}} \right.$$

$$(1 + \text{Cos}\ \chi_d)^{\frac{1}{2}} + \text{Sin}\ \chi_c\ \text{Sin}\ \chi_d(1 + \text{Cos}\ \chi_c)^{\frac{1}{2}}$$

$$\left. (1 - \text{Cos}\ \chi_d)^{\frac{1}{2}} \right\}$$

$$= \frac{3}{4}\,(\text{Sin}\ \theta_t)^{-3}$$

$$M^{-1/2\ 3/2;00}_{3/2\ 0;3/2\ 0}(st=0) = (\mathrm{Sin}\ \theta_t)^{-2} \left\{ d^{3/2}_{3/2\ -1/2}(\chi_c)\ d^{3/2}_{3/2\ 3/2}(\chi_d) \right.$$

$$\left. + d^{3/2}_{3/2\ 1/2}(\chi_c)\ d^{3/2}_{3/2\ -3/2}(\chi_d) \right\}$$

$$= \frac{\sqrt{3}}{8} (\mathrm{Sin}\ \theta_t)^{-2} \left\{ \mathrm{Sin}\ \chi_c (1 - \mathrm{Cos}\ \chi_c)^{1/2} \right.$$

$$(1 + \mathrm{Cos}\ \chi_d)^{3/2} + \mathrm{Sin}\ \chi_c (1 + \mathrm{Cos}\ \chi_c)^{1/2}$$

$$\left. (1 - \mathrm{Cos}\ \chi_d)^{3/2} \right\}$$

$$= \frac{\sqrt{3}}{8} (\mathrm{Sin}\ \theta_t)^{-2} \left\{ 2 \right\}$$

$$= \frac{\sqrt{3}}{4} (\mathrm{Sin}\ \theta_t)^{-2}$$

$$M^{-1/2\ 3/2;00}_{1/2\ 0;1/2\ 0}(st=0) = (\mathrm{Sin}\ \theta_t)^{-2} \left\{ d^{3/2}_{1/2\ -1/2}(\chi_c)\ d^{3/2}_{1/2\ 3/2}(\chi_d) \right.$$

$$\left. + d^{3/2}_{1/2\ 1/2}(\chi_c)\ d^{3/2}_{1/2\ -3/2}(\chi_d) \right\}$$

$$= (\mathrm{Sin}\ \theta_t)^{-2} \frac{\sqrt{3}}{8} \left\{ -(3\mathrm{Cos}\ \chi_c + 1)(1 - \mathrm{Cos}\ \chi_c)^{\frac{1}{2}} \right.$$

$$\mathrm{Sin}\ \chi_d (1 + \mathrm{Cos}\ \chi_d)^{\frac{1}{2}} + (3\ \mathrm{Cos}\ \chi_c - 1)$$

$$\left. (1 + \mathrm{Cos}\ \chi_c)^{\frac{1}{2}}\ \mathrm{Sin}\ \chi_d\ (1 - \mathrm{Cos}\ \chi_d)^{\frac{1}{2}} \right\}$$

$$= (\text{Sin } \theta_t)^{-2} \frac{\sqrt{3}}{8} \{ -1 -1 \}$$

$$= -\frac{\sqrt{3}}{4} (\text{Sin } \theta_t)^{-2}$$

$$M^{1/2\ 3/2;00}_{3/2\ 0;3/2\ 0}(st=0) = (\text{Sin } \theta_t)^{-1} \left\{ d^{3/2}_{3/2\ 1/2}(\chi_c)\, d^{3/2}_{3/2\ 3/2}(\chi_d) \right.$$

$$\left. + d^{3/2}_{3/2\ -1/2}(\chi_c)\, d^{3/2}_{3/2\ -3/2}(\chi_d) \right\}$$

$$= \frac{\sqrt{3}}{8} (\text{Sin } \theta_t)^{-1} \left\{ - \text{Sin } \chi_c (1 + \text{Cos } \chi_c)^{1/2} \right.$$

$$(1 + \text{Cos } \chi_d)^{3/2} - \text{Sin } \chi_c (1 - \text{Cos } \chi_c)^{1/2}$$

$$\left. (1 - \text{Cos } \chi_d)^{3/2} \right\}$$

$$= -\frac{\sqrt{3}}{4} (\text{Sin } \theta_t)^{-1}$$

$$M^{1/2\ 3/2;00}_{1/2\ 0;1/2\ 0}(st=0) = (\text{Sin } \theta_t)^{-1} \left\{ d^{3/2}_{1/2\ 1/2}(\chi_c)\, d^{3/2}_{1/2\ 3/2}(\chi_d) \right.$$

$$\left. + d^{3/2}_{1/2\ -1/2}(\chi_c)\, d^{3/2}_{1/2\ -3/2}(\chi_d) \right\}$$

$$= \frac{\sqrt{3}}{8} (\text{Sin } \theta_t)^{-1} \left\{ (3 \text{ Cos } \chi_c - 1)(1 + \text{Cos } \chi_c)^{\frac{1}{2}} \right.$$

$$\text{Sin } \chi_d (1 + \text{Cos } \chi_d)^{\frac{1}{2}} - (3 \text{ Cos } \chi_c + 1)$$

$$(1 - \text{Cos } \chi_c)^{\frac{1}{2}} \text{ Sin } \chi_d (1 - \text{Cos } \chi_d)^{\frac{1}{2}}$$

$$= -\frac{\sqrt{3}}{4} (\mathrm{Sin}\ \theta_t)^{-1}$$

$$M^{-1/2\ 1/2;00}_{3/2\ 0;3/2\ 0}(st=0) = (\mathrm{Sin}\ \theta_t)^{-1} \Big\{ d^{3/2}_{3/2\ -1/2}(\chi_c)\ d^{3/2}_{3/2\ 1/2}(\chi_d)$$

$$+ d^{3/2}_{3/2\ 1/2}(\chi_c)\ d^{3/2}_{3/2\ -1/2}(\chi_d) \Big\}$$

$$= \frac{3}{8} (\mathrm{Sin}\ \theta_t)^{-1} \Big\{ - \mathrm{Sin}\ \chi_c\ \mathrm{Sin}\ \chi_d (1 - \mathrm{Cos}\ \chi_c)^{\frac{1}{2}}$$

$$(1 + \mathrm{Cos}\ \chi_d)^{\frac{1}{2}} - \mathrm{Sin}\ \chi_c\ \mathrm{Sin}\ \chi_d (1 + \mathrm{Cos}\ \chi_c)^{\frac{1}{2}}$$

$$(1 - \mathrm{Cos}\ \chi_d)^{\frac{1}{2}} \Big\}$$

$$= -\frac{3}{4} (\mathrm{Sin}\ \theta_t)^{-1}$$

$$M^{-1/2\ 1/2;00}_{1/2\ 0;1/2\ 0}\ (st=0) = (\mathrm{Sin}\ \theta_t)^{-1} \Big\{ d^{3/2}_{1/2\ -1/2}(\chi_c)\ d^{3/2}_{1/2\ 1/2}(\chi_d)$$

$$+ d^{3/2}_{1/2\ 1/2}(\chi_c)\ d^{3/2}_{1/2\ -1/2}(\chi_d) \Big\}$$

$$= - (\mathrm{Sin}\ \theta_t)^{-1} \frac{1}{8} \Big\{ (3\ \mathrm{Cos}\ \chi_c + 1)(\ 1 - \mathrm{Cos}\ \chi_c)^{\frac{1}{2}}$$

$$(3\ \mathrm{Cos}\ \chi_d - 1)(1 + \mathrm{Cos}\ \chi_d)^{\frac{1}{2}} + (3\ \mathrm{Cos}\ \chi_c - 1)$$

$$(1 + \mathrm{Cos}\ \chi_c)^{\frac{1}{2}}(3\ \mathrm{Cos}\ \chi_d + 1)^{\frac{1}{2}}(1 - \mathrm{Cos}\ \chi_d)^{\frac{1}{2}}$$

$$= \frac{1}{4} (\mathrm{Sin}\ \theta_t)^{-1}$$

$$M^{3/2\ 3/2;00}_{1/2\ 0;1/2\ 0}(st=0) = \left\{ d^{3/2}_{1/2\ 3/2}(\chi_c)\ d^{3/2}_{1/2\ 3/2}(\chi_d) + d^{3/2}_{1/2\ -3/2}(\chi_c) \right.$$

$$\left. d^{3/2}_{1/2\ -3/2}(\chi_d) \right\}$$

$$= \frac{3}{8} \left\{ \mathrm{Sin}\ \chi_c\ \mathrm{Sin}\ \chi_d\ (1 + \mathrm{Cos}\ \chi_c)^{\frac{1}{2}}\ (1 + \mathrm{Cos}\ \chi_d)^{\frac{1}{2}} \right.$$

$$\left. + \mathrm{Sin}\ \chi_c\ \mathrm{Sin}\ \chi_d (1 - \mathrm{Cos}\ \chi_c)^{\frac{1}{2}} (1 - \mathrm{Cos}\ \chi_d)^{\frac{1}{2}} \right\}$$

$$= \frac{3}{4}$$

$$M^{3/2\ 3/2;00}_{3/2\ 0;3/2\ 0}(st=0) = d^{3/2}_{3/2\ 3/2}(\chi_c)\ d^{3/2}_{3/2\ 3/2}(\chi_d) + d^{3/2}_{3/2\ -3/2}(\chi_c)$$

$$d^{3/2}_{3/2\ -3/2}(\chi_d)$$

$$= \frac{1}{8} \left\{ (1 + \mathrm{Cos}\ \chi_c)^{3/2} (1 + \mathrm{Cos}\ \chi_d)^{3/2} \right.$$

$$\left. + (1 - \mathrm{Cos}\ \chi_d)^{3/2} (1 - \mathrm{Cos}\ \chi_c)^{3/2} \right\}$$

$$= \frac{1}{4}$$

$$M^{1/2\ 1/2;00}_{1/2\ 0;1/2\ 0}(st=0) = d^{3/2}_{1/2\ 1/2}(\chi_c)\, d^{3/2}_{1/2\ 1/2}(\chi_d) + d^{3/2}_{1/2\ -1/2}(\chi_c)\, d^{3/2}_{1/2\ -1/2}(\chi_d)$$

$$= \frac{1}{8}\Big\{ (3\ \mathrm{Cos}\ \chi_c - 1)(1 + \mathrm{Cos}\ \chi_c)^{\frac{1}{2}}(1 + \mathrm{Cos}\ \chi_d)^{\frac{1}{2}} (3\ \mathrm{Cos}\ \chi_d - 1) + (3\ \mathrm{Cos}\ \chi_c + 1)(1 - \mathrm{Cos}\ \chi_c)^{\frac{1}{2}} (1 - \mathrm{Cos}\ \chi_d)^{\frac{1}{2}}(3\ \mathrm{Cos}\ \chi_d + 1) \Big\}$$

$$= \frac{1}{4}$$

$$M^{1/2\ 1/2;00}_{3/2\ 0;3/2\ 0}(st=0) = d^{3/2}_{3/2\ 1/2}(\chi_c)\, d^{3/2}_{3/2\ 1/2}(\chi_d) + d^{3/2}_{3/2\ -1/2}(\chi_c)\, d^{3/2}_{3/2\ -1/2}(\chi_d)$$

$$= \frac{3}{8}\Big\{ \mathrm{Sin}\ \chi_c\ \mathrm{Sin}\ \chi_d(1 + \mathrm{Cos}\ \chi_c)^{\frac{1}{2}}(1 + \mathrm{Cos}\ \chi_d)^{\frac{1}{2}} + \mathrm{Sin}\ \chi_c\ \mathrm{Sin}\ \chi_d(1 - \mathrm{Cos}\ \chi_c)^{\frac{1}{2}}(1 - \mathrm{Cos}\ \chi_d)^{\frac{1}{2}} \Big\}$$

$$= \frac{3}{4}$$

Because of time reversal we have to take

$$M^{-1/2\ 3/2}_{c'0;D'0} + M^{-3/2\ 1/2}_{c'0;D'0} \qquad \text{for} \qquad M^{-1/2\ 3/2}_{c'0;D'0}$$

and

$$M^{1/2\ 3/2}_{c'0;D'0} - M^{3/2\ 1/2}_{c'0;D'0} \qquad \text{for} \qquad M^{1/2\ 3/2}_{c'0;D'0}$$

So that the $t \leftrightarrow s$ crossing matrix for πN^* scattering is given by

$$M = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2}\cdot\frac{1}{Y^2} & 0 & 0 & 0 & -\frac{\sqrt{3}}{2}\cdot\frac{1}{Y^2} & 0 \\ -\frac{1}{4}\cdot\frac{1}{Y^3} & 0 & 0 & 0 & \frac{3}{4}\cdot\frac{1}{Y^3} & 0 \\ \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 \\ -\frac{3}{4}\cdot\frac{1}{Y} & 0 & 0 & 0 & \frac{1}{4}\cdot\frac{1}{Y} & 0 \end{bmatrix}$$

where we have put $\sin\theta_t = Y$

e) <u>$t \leftrightarrow s$ crossing matrix element of $\pi N \to \pi N^*$</u>

As before the crossing matrix element for pion-baryon scattering is given by at $t = 0$

$$M^{cd;00}_{c'0;D'0}(st=0) = (\text{Sin}\ \theta_t)^{-|c-d|} \left\{ d^{J_c}_{c'c}(\chi_c)\ d^{J_d}_{D'd}(\chi_d) + \eta_s\ d^{J_c}_{c'-c}(\chi_c)\ d^{J_d}_{D'-d}(\chi_d) \right\}$$

with $c' = D'$ always

For the special case of $\pi N \to \pi N^*$ we have to identify c with $\bar{N}$ and d with N^*. So that

$$\eta_s = (-)^{J_d - J_c} = (-)^{3/2\ -1/2} = -1$$

so that

$$M^{cd;00}_{c'0;D'0}(st=0) = (\text{Sin}\ \theta_t)^{-|c-d|} \left\{ d^{1/2}_{c'c}(\chi_c)\ d^{3/2}_{D'd}(\chi_d) - d^{1/2}_{c'-c}(\chi_c)\ d^{3/2}_{D'-d}(\chi_d) \right\}$$

at $t = 0$ the only non-zero crossing matrix elements are the following

$$M^{1/2\ 1/2;00}_{1/2\ 0;1/2\ 0}(st=0) = d^{1/2}_{1/2\ 1/2}(\chi_c)\ d^{3/2}_{1/2\ 1/2}(\chi_d) - d^{1/2}_{1/2\ -1/2}(\chi_c)\ d^{3/2}_{1/2\ -1/2}(\chi_d)$$

$$= \text{Cos}\ \frac{\chi_c}{2}\ \tfrac{1}{2}(3\ \text{Cos}\ \chi_d - 1)\ \text{Cos}\ \frac{\chi_d}{2} - \text{Sin}\ \frac{\chi_c}{2}\ \tfrac{1}{2}\ (3\ \text{Cos}\ \chi_d) + 1)\ \text{Sin}\ \frac{\chi_d}{2}$$

$$= \frac{1}{4}(1 + \text{Cos }\chi_c)^{\frac{1}{2}}(3 \text{ Cos }\chi_d - 1)(1 + \text{Cos }\chi_d)^{\frac{1}{2}}$$

$$- \frac{1}{4}(1 - \text{Cos }\chi_c)^{\frac{1}{2}}(3 \text{ Cos }\chi_d + 1)(1 - \text{Cos }\chi_d)^{\frac{1}{2}}$$

$$= a_{11}$$

$$M^{-1/2\ 1/2;00}_{1/2\ 0;1/2\ 0}(st=0) = (\text{Sin }\theta_t)^{-1}\left\{ d^{1/2}_{1/2\ -1/2}(\chi_c)\ d^{3/2}_{1/2\ 1/2}(\chi_d)\right.$$

$$\left. - d^{1/2}_{1/2\ 1/2}(\chi_c)\ d^{3/2}_{1/2\ -1/2}(\chi_d)\right\}$$

$$= (\text{Sin }\theta_t)^{-1}\left\{ - \text{Sin }\frac{\chi_c}{2}\ \tfrac{1}{2}(3 \text{ Cos }\chi_d - 1)\right.$$

$$\left. \text{Cos }\frac{\chi_d}{2} + \text{Cos }\frac{\chi_c}{2}\ \tfrac{1}{2}(3 \text{ Cos }\chi_d + 1)\ \text{Sin }\frac{\chi_d}{2}\right\}$$

$$= -\frac{(\text{Sin }\theta_t)^{-1}}{4}\left\{ (1 - \text{Cos }\chi_c)^{\frac{1}{2}}(3 \text{ Cos }\chi_d - 1)\right.$$

$$(1 + \text{Cos }\chi_d)^{\frac{1}{2}} - (1 + \text{Cos }\chi_c)^{\frac{1}{2}}(3 \text{ Cos }\chi_d + 1)$$

$$\left.(1 - \text{Cos }\chi_d)^{\frac{1}{2}}\right\}$$

$$= a_{21}$$

$$M^{1/2\ 3/2;00}_{1/2\ 0;1/2\ 0}(st=0) = (\sin\theta_t)^{-1}\Big\{ d^{1/2}_{1/2\ 1/2}(\chi_c)\, d^{3/2}_{1/2\ 3/2}(\chi_d)$$

$$- d^{1/2}_{1/2\ -1/2}(\chi_c)\, d^{3/2}_{1/2\ -3/2}(\chi_d) \Big\}$$

$$= (\sin\theta_t)^{-1}\Big\{ \sqrt{3}\cos\frac{\chi_c}{2}\sin\frac{\chi_d}{2}\cos^2\frac{\chi_d}{2}$$

$$+ \sqrt{3}\sin\frac{\chi_c}{2}\sin^2\frac{\chi_d}{2}\cos\frac{\chi_d}{2} \Big\}$$

$$= \frac{\sqrt{3}}{21}(\sin\theta_t)^{-1}\Big\{ (1+\cos\chi_c)^{\frac{1}{2}}\sin\chi_d$$

$$(1+\cos\chi_d)^{\frac{1}{2}} + (1-\cos\chi_c)^{\frac{1}{2}}\sin\chi_d(1-\cos\chi_d)^{\frac{1}{2}}\Big\}$$

$$= a_{31}$$

$$M^{-1/2\ 3/2;00}_{1/2\ 0;1/2\ 0}(st=0) = (\sin\theta_t)^{-1}\Big\{ d^{1/2}_{1/2\ -1/2}(\chi_c)\, d^{3/2}_{1/2\ 3/2}(\chi_d)$$

$$- d^{1/2}_{1/2\ 1/2}(\chi_c)\, d^{3/2}_{1/2\ -3/2}(\chi_d) \Big\}$$

$$= \frac{\sqrt{3}}{4}(\sin\theta_t)^{-1}\Big\{ -(1-\cos\chi_c)^{\frac{1}{2}}\sin\chi_d$$

$$(1+\cos\chi_d)^{\frac{1}{2}} - (1+\cos\chi_c)^{\frac{1}{2}}\sin\chi_d$$

$$(1-\cos\chi_d)^{\frac{1}{2}} \Big\}$$

$$= -\frac{\sqrt{3}}{4}\,(\text{Sin }\theta_t)^{-1}\Big\{(1-\text{Cos }\chi_c)^{\frac{1}{2}}(1+\text{Cos }\chi_d)^{\frac{1}{2}}$$

$$\text{Sin }\chi_d + (1+\text{Cos }\chi_c)^{\frac{1}{2}}(1-\text{Cos }\chi_d)^{\frac{1}{2}}\ \text{Sin }\chi_d\Big\}$$

$$= a_{41}$$

For equal mass case: $m = M$

$$\text{Cos }\chi_c = \text{Cos }\chi_d = 0$$

$$\text{Sin }\chi_c = \text{Sin }\chi_d = 1$$

and we can show that

$$a_{11} = -\frac{1}{2}$$

$$a_{21} = +\frac{1}{2}\cdot\frac{1}{Y}$$

$$a_{31} = +\frac{\sqrt{3}}{2}\cdot\frac{1}{Y}$$

$$a_{41} = -\frac{\sqrt{3}}{2}\cdot\frac{1}{Y}$$

So that a $t = 0$ and $M = m$ the crossing matrix $t \leftrightarrow s$ channels for $\pi N \rightarrow \pi N^*$ is given as:

$$B^{(ts)}_{n'n}(st{=}0) = \begin{bmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ +\frac{1}{2}\cdot\frac{1}{Y} & 0 & 0 & 0 \\ +\frac{\sqrt{3}}{2}\cdot\frac{1}{Y} & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2}\cdot\frac{1}{Y} & 0 & 0 & 0 \end{bmatrix} \quad ; \quad Y = \text{Sin }\theta_t$$

References

1) Y. Ne'eman, Nuclear Physics, 26, 222 (1961);

M. Gell-Mann, CTS L-20 (1961) unpublished, Phys. Rev. 125, 1067 (1962).

The Eightfoldway, Edit. Y. Ne'eman and M. Gell-Mann, Benjamin Inc.

2) F. Gürsey, L.A.Radicati, Phys. Rev. Lett. 13, 173 (1964);

A. Pais, Phys. Rev. Lett. 13, 175 (1964);

B. Sakita, Phys. Rev. 136, B1756 (1964).

3) M. Gell-Mann, Phys. Rev. 125, 1067 (1962).

4) J. Schwinger, Phys. Rev. Lett. 3, 296 (1959).

5) K. Johnson, F.E.Low, Supplements of Prog. Theoretical Phys. 37 and 38, 74 (1966).

6) S. Fubini, G. Furlan, C. Rossetti, Nuovo Cimento 40, 1171 (1965).

8) S. Fubini, Lectures on Current Algebra, Istanbul Summer School, August, 1966.

9) S.L.Adler, Phys. Rev. Lett. 14, 1051 (1965);

W.I.Weisberger, Phys. Rev. Lett. 14, 1047 (1965).

10) M. Gell-Mann, M. Levy, Nuovo Cimento 16, 703 (1960);

J. Nambu, Phys. Rev. Lett. 4, 380 (1960).

11) S. Fubini, Nuovo Cimento, 43, 475 (1966).

12) S. Fubini, G. Furlan, C. Rossetti, Nuovo Cimento 43, 161 (1965).

13) M.A.Alam, Nuovo Cimento 46, 752 (1966).

14) M. Gell-Mann, Phys. Lett. 8, 214 (1964).

15) J.D.Bjorken, J.D. Walecka, Ann. Phys. 38, 35 (1966).

16) M. Gourdin, J. Micheli, Nuovo Cimento 40, 225 (1965).

17) G. Morpurgo, Physics 2, 95 (1965).

18) M.A.B.Beg, B.W.Lee, A. Pais, Phys. Rev. Lett. 13, 514 (1964).

19) B. Sakita, H.C.Wali, Phys. Rev. 139, B1355 (1965).

20) H.F.Jones, M.D.Scadron, Nuovo Cimento 48A, 545 (1967).

21) P.R.Auvil, JJ. Brehm, Phys. Rev. 140, B135 (1965).

22) D.G.Sutherland, Nuovo Cimento 68A, 188 (1967).

23) C. Michael, Phys. Rev. 156, 1677 (1957).

24) R.H.Capps, Phys. Rev. Lett. 13, 299 (1964).

25) V. Alfaro, S. Fubini, G. Furlan, C. Rossetti, Phys. Lett. 21 576 (1966).

26) A. Salam, R. Delbourgo, J.Strathdee, Proc. Roy. Soc.(London) A284, 146 (1965).

27) B. Sakita, K. Wali, Phys. Rev. Lett. 18, 29 (1967);
G. Altarelli, F. Buccella, R. Gatto, Phys. Lett. 24B, 57 (1967);
P. Babu, F.J.Gilman, M. Suzuki, Phys. Lett. 24B, 65 (1967);
H.F.Jones, M.D.Scadron, Nuovo Cimento 68A, 546 (1967);
H.F.Jones, M.D.Scadron, ICTP/67/17 (Preprint);

R.J.Rivers, ICTP/67/11 (Preprint);

P.H.Frampton, J.C.Taylor, Nuovo Cimento 49, 152 (1967).

P.H.Frampton, Oxford University Preprint;

F.J.Gilman, H. Harari, Phys. Rev. Lett. 18, 1150 (1967).

R. D'Auria, V. de Alfaro, Nuovo Cimento 48A, 284 (1967);

H Harari, Phys. Rev. Lett. 18, 316 (1967);

H. Pagals, Phys. Rev. Lett. 18, 319 (1967);

L.K.Pande, Nuovo Cimento XLVIII 48, 839 (1967).

28) M.A.Alam, R.G.Roberts, Durham University Preprint (August 1967).

29) H.F.Jones, M.D.Scadron, Preprint, Imperial College ICTP/67/17.

30) C. Lovelace, International Conference on Elementary Particles, Berkeley, California, 1966.

31) R.G.Roberts, Ann. Phys. (to be published).

32) R. Gatto, L. Miani, G. Preparata, Phys. Rev. Lett. 16, 377 (1966).

33) A. Salam, J. Strathdee, P.T.Matthews, J. Charap, Phys. Rev. Lett. 15, 184 (1965).

34) R. Delbourgo, M.A.Rashid, Proc. Roy. Soc. (London) 286A, 412 (1965).

35) R. Gatto, L. Miani, G. Preparata, Phys. Rev. 142, 1135 (1966).

38) D. Hall, A.S.Wightman, Dan. Vid. Sebsk. Mat. Phys. Medd. 31 No. 5 (1957).

39) A.C.Hearn, Nuovo Cimento 21, 333 (1961).

40) T.L.Trueman, Phys. Rev. Lett. 17, 1198 (1966).

41) R. Odorico, University of Trieste Preprint (Nov. 1966); April (1967).

42) M. Jacob, G.C.Wick, Ann. of Phys. 7, 404 (1959).

43) G. Szego, in "Orthogonal Polynomials", Am. Math. Soc., New York (1959).

44) F. Calogero, J.M.Charap and E.J.Squires, Ann. of Phys. 25, 325 (1963).

M. Gell-Mann, M. Goldberger, F.E.Low, E. Marx, F. Zachariasen, Phys. Rev. B133, 145 (1964).

Y. Hara, Phys. Rev. 136, B507 (1964).

L.C.Wang, Phys. Rev. 142, 1187 (1966).

45) R.J.N.Philips, Phys. Lett. 24B , 342 (1967).

I.J.Muzinich, Phys. Rev. Lett. 18, 381 (1967).

46) P.A.Carruthers, J.P.Krisch, Ann. of Phys. 33, 1 (1965).

There are no references numbered (6), (36), (37).