SOME CLASSICAL SOLUTIONS OF YANG-MILLS EQUATIONS

by

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C O N T E N T S

A B S T R A C T

In this thesis some solutions to classical Euclidean Yang-Mills theory are considered and presented with emphasis on self-dual SU(2) solutions. Chapter one is a brief introduction to the subject. In chapter two the possibility of solution via inversion of the dynamic equation to obtain A_μ in terms of $F_{\mu\nu}$ and then imposing a self-consistency requirement is considered. Chapter three deals with the extension of Witten's method of solution through cylindrically symmetric ansätze to self-dual SU(3) theories. In Chapter four the invariances and Bäcklund-type transformations inherent in a self-dual SU(n) theory are investigated and these methods are used in Chapter five to present an analytic method for constructing all self-dual SU(2) solutions.

PREFACE

The work here presented was carried out between October 1975 and May 1978 in the Department of Mathematics, University of Durham under the supervision of Dr. D.B. Fairlie.

The material contained in this thesis has not been submitted previously for any degree in this or any other university. No claim of originality is made for Chapter one or the part of Chapter five so indicated; the remainder is claimed to be original except where otherwise indicated. Chapter two is based on a paper by the author in collaboration with D.B. Fairlie and E. Corrigan, Chapter three on a paper published by the author, Chapter four on a paper by the author in collaboration with Y. Brihaye, D.B. Fairlie and J. Nuyts and Chapter five on a paper by the author in collaboration with E. Corrigan, P. Goddard and D.B. Fairlie. I would like to thank D.B. Fairlie for his guidance and encouragement throughout the course of this work and E. Corrigan for helpful discussions. I would also like to thank the S.R.C. for a research studentship.

CHAPTER ONE

INTRODUCTION

Since Yang and Mills(1) and, independently, Shaw(2) discovered the basic equations and group invariance properties of the set of theories that bear the names of the former it has been the hope of theorists that they would shed some light at least on the problem of giving a sound theoretical model for the strong and weak interactions and more hopefully on the problem of unifying the forces experimentally observed in nature and give insight into the quantization of gravity.(3) With the realization that mathematical physicists had, in considering "Yang-Mills' Theories", been studying objects already known to pure mathematicians under the name of fibre bundles(4) and that gravity and electromagnetism were indeed both theories best described by fibre bundle language, though coming from different Lagrangians, these beliefs received both encouragement and assistance from what had been discovered by pure mathematical methods.

Thus given a gauge potential A_μ taking values in some Lie algebra with its equivalent field tensor

$$F_{\mu\nu} = \frac{\partial}{\partial x^\nu} A_\mu - \frac{\partial}{\partial x^\mu} A_\nu - [A_\mu, A_\nu] \tag{1.1}$$

with its dual

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu}{}^{\sigma\tau} F_{\sigma\tau} \tag{1.2}$$

satisfying the algebraic identity

$$D^{\mu} \tilde{F}_{\mu\nu} = \frac{\partial}{\partial x_{\mu}} \tilde{F}_{\mu\nu} + [A^{\mu}, \tilde{F}_{\mu\nu}] = 0 \tag{1.3}$$

it is hoped that

$$D^{\mu} F_{\mu\nu} = j_{\nu} \tag{1.4}$$

for some current j_{ν} describes the strong and weak interactions in the sense that the same equations with the Lie algebra being u(1) describe electromagnetism.

The path integral method of quantization has always relied on a Wick rotation of the time axis so pointing out the possible importance of classical, Euclidean solutions to any theory of physical interest. In Euclidean space-time the eigenvalues of the duality operator are ± 1 so for a sourceless theory it makes sense to consider the simpler equation of self-duality, namely

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \tag{1.5}$$

as the algebraic identity (1.3) then guarantees the truth of (1.4) with $j_{\nu} = 0$.

Moreover such solutions are most probably the most important in this sector as it is known that they minimise the action integral thus being the dominant contribution to the path integral.

Recently the works of Yang(5) and Atiyah and Ward(6) have shown that the self-duality equations in the correct coordinate systems take on essentially simple forms which can in principle be solved. This is due to the fact that if we complexify the space-time manifold coordinates can be defined such that the planes defined by pairs of these new coordinates are antiself-dual i.e. if y and z are two such coordinates the two form $dy \wedge dz$ is antiself-dual. An example of such co-ordinates is

$$y = \frac{(x_0 - ix_3)}{\sqrt{2}} \qquad \bar{y} = \frac{(x_0 + ix_3)}{\sqrt{2}} \tag{1.6}$$

$$z = \frac{(x_2 - ix_1)}{\sqrt{2}} \qquad \bar{z} = \frac{(x_2 + ix_1)}{\sqrt{2}}$$

Now if $F_{\mu\nu}$ is self-dual it follows that its components pojected down onto such planes are zero i.e.

$$F_{yz} = F_{\bar{y}\bar{z}} = 0 \tag{1.7}$$

and hence in these planes one can integrate the potential along paths in the plane to give a group element independant of the path chosen, or in other words the potentials may be written as

$$A_y = D^{-1}\frac{\partial D}{\partial y} \qquad A_z = D^{-1}\frac{\partial D}{\partial z}$$

$$A_{\bar{y}} = E^{-1}\frac{\partial E}{\partial \bar{y}} \qquad A_{\bar{z}} = E^{-1}\frac{\partial E}{\partial \bar{z}} \tag{1.8}$$

for some D and E belonging to the (complexified) Lie group and there only remains one equation to be solved which is easily checked to be

$$F_{y\bar{y}} + F_{z\bar{z}} = 0 \tag{1.9}$$

More generally Atiyah and Ward define a β - plane to be a plane such that if v_μ and w_μ are any two displacements in that plane then the tensor $v_\mu w_\nu - v_\nu w_\mu$ is antiself-dual. As displacements in such planes are antiself-dual it follows as before that the projection of a self-dual $F_{\mu\nu}$ onto all such planes must be zero and hence that the potentials in that plane must be gauge transforms of the vacuum. Using the twistor transformation of Penrose[7] Atiyah and Ward[6] then find a coordinate system for such β-planes and reduce the remaining self-duality equation to the problem of constructing an n-dimensional, analytic vector bundle over CP_3, a problem whose solution has been discovered by algebraic geometers for n = 2, enabling them in principle to give all SU(2) self-dual solutions. We return to their ideas in Chapter five providing an analytic way of producing these solutions. Before exhibiting these results we examine various other possible ways of solving both the self-dual and full equations of Yang-Mills' theory in Euclidean space-time.

CHAPTER TWO

A_μ AS DEFINED BY THE DYNAMICS EQUATION

Before turning specifically to the self-dual problem we mention an approach to the solution of Yang-Mills theories valid in other cases[8] which despite its complexity provides some insight into the problem of non-uniqueness of the gauge potentials for certain field strengths[9] and which may lead to a gauge-invariant formulation of the theory for the path integral formalism.[10] The method is at its most powerful for SU(2) where it shows that almost all solutions depend only on a symmetric, three by three matrix.

The three standard equations for a Yang-Mills gauge theory are

$$F^a_{\mu\nu} = \frac{\partial}{\partial x^\nu} A^a_\mu - \frac{\partial}{\partial x^\mu} A^a_\nu - f^{abc} A^b_\mu A^c_\nu \qquad (2.1)$$

$$\frac{\partial}{\partial x_\mu} F^a_{\mu\nu} + f^{abc} A^b_\mu F^{c\mu}{}_\nu = 0 \qquad (2.2)$$

$$\frac{\partial}{\partial x_\mu} \tilde{F}^a_{\mu\nu} + f^{abc} A^{b\mu} \tilde{F}^c_{\mu\nu} = 0 \qquad (2.3)$$

where, as usual,

$$\tilde{F}^a_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu}{}^{\sigma\tau} F_{\sigma\tau} \qquad (2.4)$$

(The presence of a source term j^a_μ on the left of 2.2 introduces non-essential complications to the following so we ignore that possibility.) The usual approach to the solution of these equations is to assume 2.1 whence 2.3 is a trivial algebraic identity and view 2.2 as the non-trivial equation to be solved. We prefer to reverse this, viewing 2.2 and 2.3 as equations to be solved for A^a_μ in terms of $F^a_{\mu\nu}$ and then to impose 2.1 as the non-trivial equation. Such an approach is, of course, only possible in non-Abelian groups where the non-vanishing of the structure constants gives an explicit mention of A^a_μ in 2.2 and 2.3.

Now any rank two antisymmetric tensor may be split up into a dual and antiself-dual part using the fundamental dual and antiself-dual tensors $\eta'^a_{\mu\nu}$ and $\eta^a_{\mu\nu}$ of 't Hooft which are defined by their values in the (x_0, x_1, x_2, x_3) coordinate system by

$$\eta^a_{\mu\nu} = \varepsilon_{0a\mu\nu} + \delta_{\mu a}\delta_{\sigma 0} - \delta_{\mu 0}\delta_{a\sigma} \qquad (2.5)$$

$$\eta'_{\mu\nu} = \varepsilon_{0a\mu\nu} - \delta_{\mu a}\delta_{\sigma 0} + \delta_{\mu 0}\delta_{a\sigma} \qquad (2.6)$$

and whose properties are listed in the appendix. Hence we may write

$$F^a_{\mu\nu} = R^{ai}\eta^i_{\mu\nu} + S^{ai}\eta'^i_{\mu\nu} \qquad (2.7a)$$

$$\tilde{F}^a_{\mu\nu} = -R^a{}_i \eta^i_{\mu\nu} + S^a{}_i \eta'^i_{\mu\nu} \tag{2.7b}$$

where R and S are $(n^2 - 1) \times 3$ matrices. Using these decompositions 2.2 and 2.3 may be rewritten as

$$\frac{\partial}{\partial x_\mu} R^a{}_i \eta^i_{\mu\nu} + f^{abc} A^{b\mu} R^c{}_i \eta^i_{\mu\nu} = 0 \tag{2.8a}$$

$$\frac{\partial}{\partial x_\mu} S^a{}_i \eta'^i_{\mu\nu} + f^{abc} A^{b\mu} S^c{}_i \eta'^i_{\mu\nu} = 0 \tag{2.8b}$$

If either of these, 2.8a say, can be solved for A^a_μ then we need only to impose a self-consistency condition via 2.1, namely

$$4R^a{}_i = F^a_{\mu\nu}\eta^{i\mu\nu} = \left[\frac{\partial}{\partial x^\nu} A^a_\mu - \frac{\partial}{\partial x^\mu} A^a_\nu - f^{abc} A^b_\mu A^c_\nu\right]\eta^{i\mu\nu} \tag{2.9}$$

which will then guarantee the validity of the other.

Now $-\eta^i_{\mu\nu}$ form a 4x4 matrix representation of the imaginary quaternionic units e_i say, as may be seen from their listed properties in the appendix, so we may view $f^{abc} R^c{}_i \eta^i_{\mu\nu}$ as a purely imaginary quaternionic matrix lying in the adjoint representation of the group in question. Hence the problem lies in finding a right inverse for this matrix.

Let this matrix be $F_i e_i$ where the F_i are real, adjoint representation matrices and let the inverse be $\alpha + \beta_i e_i$, α β_i real. Then the problem is to find α and β_i satisfying

$$F_i \alpha + \sum_{j,k} \varepsilon^{ijk} F_j \beta_k = 0 \tag{2.10a}$$

$$-\sum_{i=1}^{3} F_i \beta_i = I \tag{2.10b}$$

First we note that $\beta_i^T = -\beta_i$ and $\alpha^T = \alpha$, for letting $F = F_i e_i$ and $A = \alpha + \beta_i e_i$ we have that F is hermitian as F_i is antisymmetric. therefore A is also hermitian which implies the stated result.

Second it is clear that scaling R by a constant n will not affect A_μ^a implicitly defined by 2.8 and hence will leave the right-hand side of 2.9 unaffected but scale the left-hand side by n. Thus as $n \to o$ we find that $F_{\mu\nu}^a$ is self-dual, though whether this procedure would produce all self-dual solutions is not clear.

The solutions of 2.10 are not known for general SU(n) but are known for SU(2) where R is a three by three matrix. Then a solution exists and is unique up to a gauge transformation if and only if det R $\neq$ 0 when

$$[\alpha]_{ab} = R^a{}_i R^b{}_i \Big/ 2 \det R \tag{2.11a}$$

$$[\beta_i]_{ab} = \varepsilon_{ijk} R^a{}_j R^b{}_k \Big/ 2 \det R \tag{2.11b}$$

satisfy the given conditions. Hence if det R $\neq$ 0 A_μ^a is unique up to a gauge transformation and is given by

$$A^a_\mu = \frac{R^a{}_k}{2\det R}\frac{\partial M}{\partial x^\lambda}{}_{km}\,\eta_{m\mu\lambda} + \frac{\varepsilon^{aij}}{2}\frac{\partial R^i{}_m}{\partial x^\mu}R^{-1m}{}_j \qquad (2.12)$$

where $M = R^TR - \frac{1}{2}I\,\mathrm{tr}R^TR$. A gauge transformation is now $R \rightarrow OR$ where O is an SO(3) matrix, so M is a gauge invariant object. As any GL(3) matrix may be written as a product of an orthogonal matrix and a symmetric matrix we may choose a gauge where R is symmetric if we wish.

Given 2.12 we now impose 2.1 and find that

$$F^i_{\mu\nu} = \eta_{m\mu\lambda}\frac{\partial}{\partial x^\nu}\left\{\frac{R}{2\det R}\frac{\partial M}{\partial x^\lambda}\right\}_{im} - \eta_{m\nu\lambda}\frac{\partial}{\partial x^\mu}\left\{\frac{R}{2\det R}\frac{\partial M}{\partial x^\lambda}\right\}_{im}$$

$$+ \frac{\varepsilon^{ira}}{4}\left\{\left[\frac{\partial R}{\partial x^\nu}\frac{\partial R^{-1}}{\partial x^\mu}\right]_{ar} - \left[\frac{\partial R}{\partial x^\mu}\frac{\partial R^{-1}}{\partial x^\nu}\right]_{ar}\right\}$$

$$- \frac{\varepsilon^{ipq}}{4\det R^2}\eta_{m\mu\lambda}\eta_{s\nu\alpha}R^p{}_k R^q{}_b\frac{\partial M}{\partial x^\lambda}{}_{km}\frac{\partial M}{\partial x^\alpha}{}_{bs}$$

$$+ \frac{\eta_{m\mu\lambda}}{4\det R}\left[R^{-1^T}\frac{\partial R^T}{\partial x^\nu}R\frac{\partial M}{\partial x^\lambda}\right]_{im}$$

$$- \frac{\eta_{m\nu\lambda}}{4\det R}\left[R^{-1^T}\frac{\partial R^T}{\partial x^\mu}R\frac{\partial M}{\partial x^\lambda}\right]_{im}$$

$$- \frac{\eta_{m\mu\lambda}}{4\det R}\left[\frac{\partial R}{\partial x^\nu}\frac{\partial M}{\partial x^\lambda}\right]_{im} + \frac{\eta_{m\nu\lambda}}{4\det R}\left[\frac{\partial R}{\partial x^\mu}\frac{\partial M}{\partial x^\lambda}\right]_{im}$$

$$+ \frac{\varepsilon^{ipa}}{4}\left[\frac{\partial R}{\partial x^\nu}R^{-1}R^{-1^T}\frac{\partial R^T}{\partial x^\mu} + R^{-1^T}\frac{\partial R^T}{\partial x^\nu}\frac{\partial R}{\partial x^\mu}R^{-1}\right]_{pa} \qquad (2.13)$$

Obviously the solution of the remaining equation 2.9

for a general symmetric R is not a simple problem, indeed only one solution is known: suppose R is of the form

$$R = \phi^2 I \qquad (2.14)$$

then it is easily seen that

$$A^i_\lambda = -\eta^i_{\lambda\mu} \frac{1}{\phi} \frac{\partial \phi}{\partial x_\mu} \qquad (2.15)$$

$$F^i_{\mu\nu} = -\eta^i_{\mu\nu} \frac{1}{\phi^2} \frac{\partial \phi}{\partial x^\lambda} \frac{\partial \phi}{\partial x_\lambda} - \eta^i_{\mu\lambda} \left\{ \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^\nu \partial x_\lambda} - \frac{2}{\phi^2} \frac{\partial \phi}{\partial x^\lambda} \frac{\partial \phi}{\partial x^\nu} \right\}$$

$$+ \eta^i_{\nu\lambda} \left\{ \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^\mu \partial x_\lambda} - \frac{2}{\phi^2} \frac{\partial \phi}{\partial x_\lambda} \frac{\partial \phi}{\partial x^\mu} \right\} \qquad (2.16)$$

and 2.9 becomes

$$2\phi^2 = -\frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^\nu \partial x_\nu} \qquad (2.17)$$

which has a solution

$$\phi = \frac{2a}{s^2 + a^2} \qquad (2.18)$$

where a is an arbitrary constant and $s^2 = x^2 + y^2 + z^2 + t^2$, the instanton solution of Belavin et al.[11]

Letting R i.e. ϕ^2 scale by n, a constant, and taking the limit as $n \to 0$ to get a self-dual solution we find that 2.17 becomes

$$0 = \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^\nu \partial x_\nu} \qquad (2.19)$$

with the 't Hooft solutions[12]

$$\phi = \sum_{i=1}^{n} \frac{\alpha_i}{(x-x_i)^2} \tag{2.20}$$

Even for what one would hope would be the next most simple case, namely R diagonal, solutions are not known, for then constraints arise from off-diagonal terms in 2.9. Letting

$$R = \begin{bmatrix} \sqrt{e} & 0 & 0 \\ 0 & \sqrt{f} & 0 \\ 0 & 0 & \sqrt{g} \end{bmatrix} \tag{2.21}$$

the equations to be solved are

$$\eta^3_{\lambda\nu}\left\{\frac{\partial f}{\partial x_\lambda}\frac{\partial e}{\partial x_\nu}\left[\frac{1}{f}+\frac{1}{e}\right]+\frac{\partial f}{\partial x_\nu}\frac{\partial g}{\partial x_\lambda}\left[\frac{1}{f}-\frac{1}{g}\right]+\frac{\partial g}{\partial x_\nu}\frac{\partial f}{\partial x_\lambda}\left[\frac{1}{e}-\frac{1}{g}\right]\right\}=0$$

$$\eta^2_{\lambda\nu}\left\{\frac{\partial f}{\partial x_\lambda}\frac{\partial e}{\partial x_\nu}\left[\frac{1}{e}-\frac{1}{f}\right]+\frac{\partial f}{\partial x_\nu}\frac{\partial g}{\partial x_\lambda}\left[\frac{1}{g}-\frac{1}{f}\right]+\frac{\partial g}{\partial x_\nu}\frac{\partial f}{\partial x_\lambda}\left[\frac{1}{g}+\frac{1}{e}\right]\right\}=0 \tag{2.22}$$

$$\eta^1_{\lambda\nu}\left\{\frac{\partial f}{\partial x_\lambda}\frac{\partial e}{\partial x_\nu}\left[\frac{1}{f}-\frac{1}{e}\right]+\frac{\partial f}{\partial x_\nu}\frac{\partial g}{\partial x_\lambda}\left[\frac{1}{f}+\frac{1}{g}\right]+\frac{\partial g}{\partial x_\nu}\frac{\partial e}{\partial x_\lambda}\left[\frac{1}{g}-\frac{1}{e}\right]\right\}=0$$

$$\frac{1}{2}\frac{\partial}{\partial x^\lambda}\left\{\frac{1}{\sqrt{fg}}\frac{\partial}{\partial x_\lambda}(e-f-g)\right\}-\frac{1}{8e\sqrt{fg}}\frac{\partial}{\partial x^\lambda}(f-e-g)\frac{\partial}{\partial x_\lambda}(g-e-f)=4\sqrt{e} \tag{2.23a}$$

$$\frac{1}{2}\frac{\partial}{\partial x^\lambda}\left\{\frac{1}{\sqrt{eg}}\frac{\partial}{\partial x_\lambda}(f-e-g)\right\}-\frac{1}{8f\sqrt{eg}}\frac{\partial}{\partial x^\lambda}(e-f-g)\frac{\partial}{\partial x_\lambda}(g-e-f)=4\sqrt{f} \tag{2.23b}$$

$$\frac{1}{2}\frac{\partial}{\partial x^\lambda}\left\{\frac{1}{\sqrt{ef}}\frac{\partial}{\partial x_\lambda}(g-e-f)\right\}-\frac{1}{8g\sqrt{ef}}\frac{\partial}{\partial x^\lambda}(e-f-g)\frac{\partial}{\partial x_\lambda}(f-e-g)=4\sqrt{g} \tag{2.23c}$$

a set of equations which apart from e = f = g have remained unsolved.

Why the equations are so much easier for the case when R is a multiple of the identity is linked up with the fact that for R symmetric M may be rewritten using the Cayley-Hamilton theorem which for a three by three matrix may be stated as

$$R^2 - \frac{1}{2}\operatorname{tr}(R^2)I = \operatorname{tr}R\left\{R - \frac{\operatorname{tr}R}{2}I\right\} + \det R\, R^{-1} \qquad (2.24)$$

and as R is symmetric the left-hand side is M. Thus one would be tempted to try the two simplifications $R = \operatorname{tr} R\frac{1}{3}I$ or $\operatorname{tr} R = 0$. Unfortunately the latter is not as powerful as the former leading to an expression for A^a_μ of

$$A^a_\mu = -\frac{\partial R^i{}_p}{\partial x^\lambda}\frac{R^{-1p}{}_j}{2}\left\{\delta^{ia}\eta^j_{\mu\lambda} + \delta^{ij}\eta^a_{\mu\lambda} - \varepsilon^{aij}\delta_{\lambda\mu}\right\} \qquad (2.25)$$

which although simpler than the general expression does not simplify 2.9 sufficiently for a solution to be spotted.

For SU(n) the condition for an inverse to the matrix $F_i e_i$ to exist takes the form of the non-vanishing of a trace of components of $F^a_{\mu\nu}$ but this inverse may no longer be unique. However as it is claimed, by a counting of parameters arguement, that all self-dual SU(n) fields arise from SU(2) imbeddings[13] we shall leave this problem.

Perhaps the most promising use of the above method is to be found following Halpern[10] who attempts to express all dynamic variables in the path integral formalism in terms of such gauge invariant objects as the matrix M mentioned earlier thus bypassing the usual problems in definition of a gauge and the resulting infinities which complicate other procedures in this approach.

CHAPTER THREE

SELF-DUAL SU(3) THEORIES

We now turn to the case of SU(3), adopting a procedure due to Witten,[14] to simplify the self-duality equations somewhat. We write an ansatz for A_μ having what Witten calls cylindrical symmetry, that is any rotation of the space coordinates can be compensated for by a gauge transformation. Although the resulting equations turn out to be in general intractable they can be solved for a special case which at first sight would seem to be an embedding of the SU(2) solution of (14) into SU(3). However there is an extra degree of freedom which results in an infinity of gauge inequivalent solutions labelled by $s > 0$ with $s = 1$ being the embedding of SU(2) into SU(3).

To construct the ansatz we need to construct skew-hermitian, traceless, three by three matrices thus lying in su(3) out of the rotational invariants x_i/r, δ_{ij} and ε_{ijk}, where $r^2 = x^2_1 + x^2_2 + x^2_3$. These are [15]

$$T_1^{ij} = i\left\{ \delta^{ij} - 3\frac{x^i x^j}{r^2} \right\} \qquad T_2^{ij} = \varepsilon_{ijl}\frac{x^l}{r}$$

$$T_{3k}^{ij} = \delta^{ik}\frac{x^j}{r} - \delta^{jk}\frac{x^i}{r} \qquad T_{4k}^{ij} = \varepsilon_{ikl}\frac{x^l x^j}{r^2} - \varepsilon_{jkl}\frac{x^l x^i}{r^2}$$

$$T_{5k}^{ij} = i\left\{ \delta^{ik}\frac{x^j}{r} + \delta^{jk}\frac{x^i}{r} - 2\frac{x^i x^j x^k}{r^3} \right\}$$

$$T_{6k}^{ij} = i\left\{ \varepsilon_{ikl}\frac{x^l x^j}{r^2} + \varepsilon_{jkl}\frac{x^l x^i}{r^2} \right\} \qquad (3.1)$$

(ε_{ijk} is not included as $\varepsilon_{ijk} = \frac{1}{2}\left(T_{4_j}^{ik} + T_{4i}^{kj} + T_{4k}^{ji}\right)$)

For instance putting $\Lambda_7 = J_1$, $-\Lambda_5 = J_2$, $\Lambda_2 = J_3$

where Λ_i are the Gellman representation of su(3) we find

$$T_2 = \frac{\underset{\sim}{r}\cdot\underset{\sim}{J}}{r} \qquad T_{3i} = \varepsilon_{ijk}\frac{r^j J_k}{r} \qquad T_{4i} = \frac{x_i\,\underset{\sim}{r}\cdot\underset{\sim}{J}}{r^2} - J_i \qquad (3.2)$$

By multiplying these with functions depending only on r and t = x_0 we can write down our cylindrically symmetric ansatz

$$A_k = -\frac{1}{3}\theta_1\frac{x^k}{r}T_1 - A_1\frac{x^k}{r}T_2 + \frac{(1+\phi_0)}{r}T_{3k} - \frac{\phi_1}{r}T_{4k}$$

$$+\frac{\psi_0}{r}T_{5k} - \frac{\psi_1}{r}T_{6k} \qquad (3.3a)$$

$$A_0 = -\frac{1}{3}\theta_0 T_1 - A_0 T_2 \qquad (3.3b)$$

where k = 1, 2, 3 and θ_i, A_i, ϕ_i, ψ_i are real functions of r and t. $\theta_i = \psi_i = 0$ gives Witten's su(2) ansatz[14] via 3.2.

This expression for A_μ leads to a reinterpretation of the theory as a U(1) x U(1) gauge theory with a matrix of Higg's fields for if we define

$$\phi = \begin{bmatrix} \phi_1 & \phi_0 \\ \psi_1 & \psi_0 \end{bmatrix} \tag{3.4}$$

with a gauge action

$$\phi \longrightarrow o\phi u \qquad o, u \in SO(2) \tag{3.5}$$

so that

$$D_i \phi = \partial_i \phi + \begin{bmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{bmatrix} \phi + \phi \begin{bmatrix} 0 & -A_i \\ A_i & 0 \end{bmatrix} \tag{3.6}$$

$$i = 0, 1 \quad \partial_0 = \partial_t \quad \partial_1 = \partial_r \quad \varepsilon_{01} = -\varepsilon_{10} = 1$$

we find that

$$\frac{1}{2} F^a_{\mu\nu} F^a_{\mu\nu} = \frac{1}{3}(-\partial_1\theta_0 + \partial_0\theta_1)^2 + (\partial_0 A_1 - \partial_1 A_0)^2 + \frac{12}{r^4} \det \phi^2$$

$$+ \frac{2}{r^2} \operatorname{tr}\left[D_i\phi \{D_i\phi\}^T \right] + \frac{1}{r^4}(1 - \operatorname{tr}\phi^T\phi)^2 \tag{3.7}$$

where $F^a_{\mu\nu}\Lambda_a = F_{\mu\nu}$ and the Lagrangian is indeed invariant under 3.5. Further with $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\sigma\tau} F_{\sigma\tau}$ we have

$$F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = \frac{1}{r^2}\Big\{ 2(\partial_0\theta_1 - \partial_1\theta_0)\det\phi + (\partial_0 A_1 - \partial_1 A_0)(1 - \operatorname{tr}\phi^T\phi)$$

$$- \varepsilon_{ab}\varepsilon_{ij}\left[(D_i\phi)^T D_j\phi \right]_{ab} \Big\}$$

$$= \frac{1}{r^2}\Big\{ (\partial_0 A_1 - \partial_1 A_0) - \partial_i\left(\varepsilon_{ab}\varepsilon_{ij}\left[\phi^T D_j \phi\right]_{ab} \right) \Big\} \tag{3.8}$$

The self-duality equations $F^a_{0i} = \frac{1}{2}\epsilon_{ijk} F^a_{jk}$ are found to be

$$r^2(\partial_0\theta_1 - \partial_1\theta_0) = 6 \det \phi \tag{3.9a}$$

$$r^2(\partial_0 A_1 - \partial_1 A_0) = 1 - \mathrm{tr}\,\phi^T\phi \tag{3.9b}$$

$$[D_i\phi]_{ab} = [\epsilon_{ij} D_j \phi]_{ac}\,\epsilon_{cb} \tag{3.9c}$$

Corresponding to the U(1) x U(1) nature of the problem we impose the gauge conditions

$$\partial_i A_i = \partial_i \theta_i = 0 \tag{3.10}$$

That is

$$A_1 = -\partial_0\chi \qquad A_0 = \partial_1\chi \tag{3.11a}$$

$$\theta_1 = -\partial_0 w \qquad \theta_0 = \partial_1 w \tag{3.11b}$$

whence using the substitutions

$$\phi_0 + \psi_1 = e^{w-\chi}\lambda_1 \qquad \phi_0 - \psi_1 = e^{-w-\chi}\lambda_3 \tag{3.12a}$$

$$\psi_0 - \phi_1 = e^{w-\chi}\lambda_2 \qquad \phi_1 + \psi_0 = -e^{-w-\chi}\lambda_4 \tag{3.12b}$$

3.9c reduces to

$$\partial_r\lambda_1 = \partial_t\lambda_2 \qquad \partial_r\lambda_2 = -\partial_t\lambda_1 \tag{3.13a}$$

$$\partial_r \lambda_3 = \partial_t \lambda_4 \qquad \partial_r \lambda_4 = -\partial_t \lambda_3 \tag{3.13b}$$

that is $f = \lambda_1 + i\lambda_2$ and $g = \lambda_3 + i\lambda_4$ are arbitrary analytic functions of $z = r + it$. 3.9a and 3.9b now become

$$-r^2(\partial_0^2 \omega + \partial_1^2 \omega) = \tfrac{3}{2} e^{2\chi}(e^{-2\omega}|g|^2 - e^{2\omega}|f|^2) \tag{3.14a}$$

$$-r^2(\partial_0^2 \chi + \partial_1^2 \chi) = 1 - \tfrac{1}{2} e^{2\chi}(e^{2\omega}|f|^2 + e^{-2\omega}|g|^2) \tag{3.14b}$$

If neither f nor g are zero these may be simplified by making the transformations

$$\chi = \xi + \log r + \tfrac{1}{2}\log\left(\frac{1}{|fg|}\right) \qquad \omega = \eta + \tfrac{1}{2}\log\left(\left|\frac{g}{f}\right|\right) \tag{3.15}$$

when they become (ignoring for the time being any singularities introduced by this substitution)

$$\nabla^2 \eta = 3\, e^{2\xi} \sinh 2\eta \tag{3.16a}$$

$$\nabla^2 \xi = e^{2\xi} \cosh 2\eta \tag{3.16b}$$

Unfortunately these equations have proved to be intractable except for the case $\eta = 0$ when they reduce to Liouville's equation $\nabla^2 \xi = e^{2\xi}$ with the solution

$$\xi = -\log\left[\tfrac{1}{2}(1 - |h|^2)\right] + \tfrac{1}{2}\log\left|\frac{dh}{dz}\right|^2 \tag{3.17}$$

where $$h = \prod_{i=1}^{n} \frac{a_i - z}{\bar{a}_i + z} \qquad Re(a_i) > 0 \tag{3.18}$$

given by Witten.(14) To avoid singularities from the transformation 3.15 we require that $|dh/dz|^2 = |f||g|$ and that $\log |g/f|$ be harmonic. We fulfil these requirements by choosing

$$f = \alpha \frac{dh}{dz} \qquad g = \frac{1}{\alpha}\frac{dh}{dz} \qquad \alpha \text{ constant} \tag{3.19}$$

which is picked to make $D_i\phi = 0$ at the boundaries and because the topological quantum number, as is shown below, does not depend on α.

Returning to the general case we wish to find an expression for the topological quantum number $\int \frac{d^4x}{8\pi^2} F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}$ assuming $D_i\phi \rightarrow 0$ at the boundaries, for if this is so 3.8 tells us that

$$\frac{1}{8\pi^2}\int d^4x\, F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = \frac{1}{2\pi}\int dr\,dt\,(\partial_0 A_1 - \partial_1 A_0) \tag{3.20}$$

Now defining $\Phi = \phi_0 - i\phi_1 = \frac{1}{2}(fe^W + ge^{-W})$ and $\Psi = \psi_0 - i\psi_1 = \frac{1}{2}i(ge^{-W} - fe^W)$ we see that the covariant derivatives may be written as

$$D_j\Phi = \partial_j\Phi + iA_j\Phi - \theta_j\Psi \tag{3.21a}$$

$$D_j\Psi = \partial_j\Psi + iA_j\Psi + \theta_j\Phi \tag{3.21b}$$

which may be solved at the boundaries to give

$$A_j = \frac{i}{2}\partial_j \log(\Phi^2 + \Psi^2) = \frac{i}{2}\partial_j \log fg \tag{3.22}$$

so that we have

$$\frac{1}{2\pi}\int dr\,dt\,(\partial_0 A_1 - \partial_1 A_0) = \frac{1}{2\pi}\oint d\underset{\sim}{s}\cdot \underset{\sim}{A} = -\frac{1}{4\pi i}\oint d\underset{\sim}{s}\cdot\nabla \log fg$$

$$= \frac{1}{2}(N-P) \qquad (3.23)$$

where N is the number of zeros and P the number of poles of fg enclosed by the contour (the minus sign cancels as in the r,t plane the contour is clockwise). In general the form of fg is determined by the requirement that no singularities result from the log transformations needed to deduce 3.16 and therefore depend on their unknown solutions so we are not able to proceed any further. However for the solution 3.19 we have $fg = (dh/dz)^2$ independant of α so the number is n-1, the number of zeros of dh/dz in r>0.

The question now arises what effect does a specific choice of α have? First we note that $\alpha = 1$ is indeed Witten's SU(2) solution(14) embedded via 3.2 for then $\Psi = 0$. Let $\alpha = se^{i\beta}s, \beta \in \mathbb{R}$. If β changes by ε, ε small, then $\delta\Phi = \varepsilon\Psi$ and $\delta\Psi = -\varepsilon\Phi$ so the change in A_μ is given by

$$\delta A_k = \varepsilon\left(\frac{\psi_0}{r}T_{3k} - \psi_{1/r}T_{4k} - \frac{\phi_0}{r}T_{5k} + \frac{\phi_1}{r}T_{6k}\right) \qquad (3.24a)$$

$$\delta A_0 = 0 \qquad (3.24b)$$

and it is easily verified that this is a gauge transformation generated by $\frac{1}{3}\varepsilon T_1$. Therefore without loss of generality we may assume $\alpha = s \in \mathbb{R}$ $s > 0$. Let $s \to s + \varepsilon$.

Then $\delta\Phi = -i\frac{\varepsilon}{s}\Psi \qquad \delta\Psi = -i\frac{\varepsilon}{s}\Phi$ i.e.

$$\delta A_k = \frac{\varepsilon}{sr}\left(\Psi_1 T_{3k} - \Psi_0 T_{4k} + \phi_1 T_{5k} - \phi_0 T_{6k}\right) \qquad (3.25a)$$

$$\delta A_0 = 0 \qquad (3.25b)$$

Suppose this were a gauge transformation generated by $\frac{\varepsilon}{s}B$. First $\partial_0 B \neq 0$ for if it were we would require $[B, T_2] = 0$ which implies $B = aT_1 + bT_2$ but this will not work for 3.25a. Realising that B does not depend on Λ_7, Λ_5 or Λ_2 as these just rotate T_2, T_{3k} and T_{4k} into one another we may write

$$B = \alpha^1\Lambda_1 + \alpha^3\Lambda_3 + \alpha^4\Lambda_4 + \alpha^6\Lambda_6 + \alpha^8\Lambda_8 \qquad (3.26)$$

Now 3.25b $\delta A_0 = \frac{\varepsilon}{s}\{[B, A_0] + \partial_0 B\} = 0$ can be solved giving α^i in terms of $\cos\int A_0 dt$, $\sin\int A_0 dt$, $\cos 2\int A_0 dt$ and three arbitrary functions of x_1, x_2 and x_3. Putting this solution into 3.25a produces contradictions, showing that $s \to s + \varepsilon$ is not a gauge transformation.

These facts are mirrored in the alternative interpretation of the theory as a U(1) x U(1) gauge theory with a matrix of Higg's fields ϕ acted on by $\phi \to O\phi U$, O and U SO(2) matrices, so infinitesimal changes are $\phi \to \phi + A\phi + \phi B$, A and B so(2) matrices. Now $\beta \to \beta + \varepsilon$ is generated by

$$A = \begin{bmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{bmatrix} \qquad B = 0 \qquad (3.27)$$

However $s \rightarrow s + \varepsilon$ gives

$$\delta\phi = \varepsilon/s \begin{bmatrix} \psi_0 & -\psi_1 \\ -\phi_0 & \phi_1 \end{bmatrix} \tag{3.28}$$

which cannot be produced by any A and B in so(2). The invariants of ϕ under 3.5 are tr $\phi^T\phi$ and det ϕ and s may be expressed in terms of these. Let $\frac{1}{2}dh/dz = P - iQ$ then with $m = 1/s + s$ and $n = s - 1/s$ we find

$$\phi = \begin{bmatrix} Qm & Pm \\ Pn & Qn \end{bmatrix} \tag{3.20}$$

and putting $2w = \mathrm{tr}\,\phi^T\phi/\det\phi$ it can be verified that $w = (1 + s^4)/(s^4 - 1)$ independant of r and t.

Hence we have solutions for any topological quantum number k labelled by 2(k + 1) real numbers corresponding to the k + 1 complex numbers a_i in 3.18 and s, a total of 2k + 3 parameters, in contrast to the prediction of 12k - 8. Thus we expect that for k = 1 it must be possible to gauge away one of the parameters and for higher k 10k - 11 parameters are missing. More generally the alternative interpretation of these results in terms of a U(1) x U(1) Higg's theory suggests that invariants of the potential $V(\phi)$ under the gauge group may also be used to specify the theory.

CHAPTER FOUR

SOME PROPERTIES OF SELF-DUAL SU(n) THEORIES

Turning now to an SU(n) self-dual theory, we utilise the β-planes discovered by Yang[5] to show the symmetries implicit to such a situation.[16] With the definitions

$$\sqrt{2}\,y = x_0 - i x_3 \qquad \sqrt{2}\,z = x_2 - i x_1$$

$$\sqrt{2}\,\bar{y} = x_0 + i x_3 \qquad \sqrt{2}\,\bar{z} = x_2 + i x_1 \tag{4.1}$$

it is easily verified, using $\varepsilon_{y\bar{y}z\bar{z}} = 1$, $g_{y\bar{y}} = g_{\bar{y}y} = g_{z\bar{z}} = g_{\bar{z}z} = 1$ rest zero, that the self-duality equation

$$2F_{\mu\nu} = \varepsilon_{\mu\nu}{}^{\sigma\tau} F_{\sigma\tau} \tag{4.2}$$

reduces to the three matrix equations

$$F_{yz} = 0 \tag{4.3a}$$

$$F_{\bar{y}\bar{z}} = 0 \tag{4.3b}$$

$$F_{y\bar{y}} + F_{z\bar{z}} = 0 \tag{4.3c}$$

4.3a and b may at once be integrated, enabling us to write

$$A_y = D^{-1}\frac{\partial D}{\partial y} \qquad A_z = D^{-1}\frac{\partial D}{\partial z} \tag{4.4a}$$

$$A_{\bar{y}} = E^{-1}\frac{\partial E}{\partial \bar{y}} \qquad A_{\bar{z}} = E^{-1}\frac{\partial E}{\partial \bar{z}} \tag{4.4b}$$

As we have introduced complex coordinates the potentials do not now take values in su(n) but in its complexification sl(n,C) hence D and E are SL(n,c) matrices not SU(n) ones. However to ensure that on transforming back to real coordinates they lie in the correct Lie algebra the condition

$$[A_\mu]^\dagger = -A_{\bar{\mu}} \tag{4.5}$$

must be imposed, providing a relation between the derivatives of D and E.

It is easily seen that the relation

$$D^\dagger = E^{-1} \tag{4.6}$$

is sufficient for 4.5 to be true but it is not necessary as having imposed it the transformations

$$\begin{aligned} D &\longrightarrow AD \\ E &\longrightarrow BE \end{aligned} \tag{4.7}$$

where A and B are SL(n,C) matrices with A dependent on

$\bar{y}$ and $\bar{z}$ alone and B dependent on y and z alone will not change the potentials as defined by 4.4 so 4.5 will still be true. As such a mapping leaves the potentials and therefore all quantities associated with the theory invariant we ignore this possibility as irrelevant to our purposes and impose 4.6 on the generating matrices.

A gauge transformation now corresponds to

$$\begin{aligned} D &\longrightarrow DU \\ & \qquad\qquad\qquad U \in SU(n) \\ E &\longrightarrow EU \end{aligned} \tag{4.8}$$

However we may extend this and allow U to be SL(n,C) matrix at the risk of introducing singularities in the field strengths and potentials but not in such gauge invariant objects as the topological quantum number or the action.

Using this extended form of the gauge freedom the remaining equation 4.3c may now be written in a particularly concise form. Define

$$P = DE^{-1} \tag{4.9}$$

then from 4.6 P is hermitian and of determinant one. Clearly from 4.8 P is gauge invariant. Conversely given an hermitian P of determinant one which is positive definite it can be expressed as DD^{+} where det D = 1 uniquely up to an SU(n) matrix, for there exists a unitary matrix U diagonalising P

$$U P U^{\dagger} = \mathrm{diag}(b_1, \ldots\ldots, b_n) \qquad (4.10)$$

where the b_i are real and positive with $\prod_{i=1}^{n} b_i = 1$. Then let

$$d = \mathrm{diag}(\sqrt{b_1}, \ldots\ldots\ldots, \sqrt{b_n}) \qquad (4.11)$$

so det d = 1 and

$$P = U^{\dagger} d d^{\dagger} U = (e^{ia/n} U^{\dagger} d)(d^{\dagger} U e^{-ia/n}) \qquad (4.12)$$

where det $U = e^{ia}$. Clearly this gives $D = e^{ia/n} U^{+} d$ which is unique up to post multiplication by a special unitary matrix.

Now do the gauge transformation (a) $U = D^{-1}$ or (b) $U = E^{-1}$. The gauge potentials then reduce to

$$\text{(a)} \quad A_y = A_z = 0 \qquad A_{\bar{y}} = P\frac{\partial P^{-1}}{\partial \bar{y}} \qquad A_{\bar{z}} = P\frac{\partial P^{-1}}{\partial \bar{z}} \qquad (4.13a)$$

$$\text{(b)} \quad A_y = P^{-1}\frac{\partial P}{\partial y} \qquad A_z = P^{-1}\frac{\partial P}{\partial z} \qquad A_{\bar{y}} = A_{\bar{z}} = 0 \qquad (4.13b)$$

and 4.3c becomes

$$\text{(a)} \quad \frac{\partial}{\partial y}\left\{P\frac{\partial P^{-1}}{\partial \bar{y}}\right\} + \frac{\partial}{\partial z}\left\{P\frac{\partial P^{-1}}{\partial \bar{z}}\right\} = 0 \qquad (4.14a)$$

(b) $$\frac{\partial}{\partial \bar{y}}\left\{ P^{-1}\frac{\partial P}{\partial y}\right\} + \frac{\partial}{\partial \bar{z}}\left\{ P^{-1}\frac{\partial P}{\partial z}\right\} = 0 \qquad (4.14b)$$

By the hermiticity of P 4.14a can be deduced from 4.14b by taking the hermitian conjugate so we will concentrate on the latter equation.

4.14 may be written in a relativistically invariant form using the projection operator onto the y - z plane $\frac{1}{2}(\delta_{\mu}^{\ \nu} + i\eta^{3}_{\ \mu}{}^{\nu})$(17) to write 4.13 as

$$2A_{\mu} = P^{-1}\frac{\partial P}{\partial x^{\mu}} + i\eta^{3}_{\ \mu}{}^{\nu} P^{-1}\frac{\partial P}{\partial x^{\nu}} \qquad (4.15)$$

Then (with subscripts on P denoting differentiation)

$$F_{\mu\nu} = \frac{P^{-1}_{\nu}P_{\mu}}{4} - \frac{P^{-1}_{\mu}P_{\nu}}{4} + \frac{i}{4}\eta^{3}_{\ \mu}{}^{\lambda}\left\{ \frac{\partial}{\partial x^{\lambda}}(P^{-1}P_{\nu}) + \frac{\partial}{\partial x^{\nu}}(P^{-1}P_{\lambda})\right\} - \frac{i}{4}\eta^{3}_{\ \nu}{}^{\lambda}\left\{ \frac{\partial}{\partial x^{\lambda}}(P^{-1}P_{\mu}) + \frac{\partial}{\partial x^{\mu}}(P^{-1}P_{\lambda})\right\} - \frac{\eta^{3}_{\ \mu}{}^{\lambda}\eta^{3}_{\ \nu}{}^{\sigma}}{4}\left\{ P^{-1}_{\lambda}P_{\sigma} - P^{-1}_{\sigma}P_{\lambda}\right\} \qquad (4.16a)$$

$$\tilde{F}_{\mu\nu} = \frac{\rho^{-1}_{\nu}\rho_{\mu}}{4} - \frac{\rho^{-1}_{\mu}\rho_{\nu}}{4} + \frac{i}{4}\eta^{3}{}_{\mu}{}^{\lambda}\left\{\frac{\partial}{\partial x^{\lambda}}(\rho^{-1}\rho_{\nu}) + \frac{\partial}{\partial x^{\nu}}(\rho^{-1}\rho_{\lambda})\right\}$$

$$-\frac{i}{4}\eta^{3}{}_{\nu}{}^{\lambda}\left\{\frac{\partial}{\partial x^{\lambda}}(\rho^{-1}\rho_{\mu}) + \frac{\partial}{\partial x^{\mu}}(\rho^{-1}\rho_{\lambda})\right\} + \frac{\eta^{3}_{\mu\nu}}{4}\left\{\eta^{3\tau\lambda}\rho^{-1}_{\tau}\rho_{\lambda}\right.$$

$$\left. - 2i\frac{\partial}{\partial x^{\lambda}}(\rho^{-1}\rho_{\lambda})\right\} + \frac{\varepsilon_{\mu\nu}{}^{\lambda\tau}}{4}\rho^{-1}_{\lambda}\rho_{\tau} \qquad (4.16b)$$

The three equations resulting from $F_{\mu\nu} - \tilde{F}_{\mu\nu} = 0$ may now be obtained from the fact that $F_{\mu\nu} - \tilde{F}_{\mu\nu}$ is antiself-dual and hence all information about it is contained in the equations deduced from contracting it with $\eta^{a\mu\nu}$ a = 1, 2, 3. Performing this we see that contracting with $\eta^{1\mu\nu}$ and $\eta^{2\mu\nu}$ give zero and the contraction with $\eta^{3\mu\nu}$ gives

$$\frac{\partial}{\partial x^{\tau}}\left\{[i g^{\tau\lambda} - \eta^{3\tau\lambda}]\rho^{-1}\frac{\partial\rho}{\partial x^{\lambda}}\right\} = 0 \qquad (4.17)$$

which is the desired result. The invariant form of 4.14a can now be obtained by considering a real co-ordinate frame where $\eta^{3\tau\lambda}$ is real and taking hermitian conjugates to deduce that

$$\frac{\partial}{\partial x^{\tau}}\left\{[i g^{\tau\lambda} + \eta^{3\tau\lambda}]\rho\frac{\partial\rho^{-1}}{\partial x^{\lambda}}\right\} = 0 \qquad (4.18)$$

As mentioned before although we are in a possibly singular gauge the topological quantum number being gauge invariant will be non-singular and has a particularly simple expression in terms of P.

$$\mathrm{tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}] = 4\,\mathrm{tr}[F_{y\bar{y}}F_{z\bar{z}} + F_{\bar{y}z}F_{y\bar{z}}]$$

$$= 4\,\mathrm{tr}\Big[\frac{\partial}{\partial\bar{z}}\Big(P^{-1}\frac{\partial P}{\partial z}\Big)\frac{\partial}{\partial\bar{y}}\Big(P^{-1}\frac{\partial P}{\partial y}\Big)$$

$$- \frac{\partial}{\partial\bar{y}}\Big(P^{-1}\frac{\partial P}{\partial z}\Big)\frac{\partial}{\partial\bar{z}}\Big(P^{-1}\frac{\partial P}{\partial y}\Big)\Big] \qquad (4.19)$$

Being able to express so much so elegantly in terms of P it is disappointing that there does not appear to exist a Lagrangian variation of which with respect to the elements of P would give 4.14. To construct a Lagrangian it is necessary to work in a different gauge, discussion of which we leave to later; first we enumerate the algebraic invariances implied by 4.14.

Let A and B be constant GL(n,C) matrices then $P \longrightarrow APB$ is an invariance of 4.14. To preserve the hermiticity of P and det P = 1 we require $B = A^+$ and $|\det A| = 1$. Let $\det A = e^{ia}$, then $A = e^{ia/n}\tilde{A}$ where $\tilde{A} = e^{-ia/n}A$ is an SL(n,C) matrix. Clearly $APA^+ = \tilde{A}P\tilde{A}^+$ so without loss of generality we have the following invariance transformations of P

$$P \longrightarrow APA^{+} \qquad A \in SL(n, C) \tag{4.20}$$

The other obvious invariance of 4.14 is

$$P \longrightarrow P^{-1^{T}} \tag{4.21}$$

For n = 2 this is included in 4.20 being generated by

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{4.22}$$

However it was pointed out to the writer by Dr. D Olive that for n$>$2 the group automorphism 4.21 is necessarily outer and hence cannot be contained in the inner automorphism 4.20.

In fact 4.20 may be extended by letting A be functionally dependant on $\bar{y}$, $\bar{z}$ alone. For by the uniqueness, up to a gauge dependance, of the decomposition of P into D and $E^{-1} = D^{+}$ 4.20 is equivalent to

$$D \longrightarrow AD \qquad E^{-1} \longrightarrow E^{-1}A^{+} \tag{4.23}$$

and as shown previously this does not alter the potentials in any way provided A is independant of y and z.

By the same type of arguements 4.21 is equivalent to

$$D \longrightarrow D^{-1^{T}} \qquad E^{-1} \longrightarrow E^{T} \tag{4.24}$$

i.e.

$$D \longrightarrow E^* \qquad E^{-1} \longrightarrow D^{*-1} \tag{4.25}$$

(* denotes complex conjugation). For SU(2) this is a gauge transformation as it is included in 4.23, but for SU(n) $n>2$ this is not true as if it were there would exist a matrix $W(\bar{y},\bar{z})$ and $U \in SU(n)$ such that

$$W(D^{-1})^T U = D \tag{4.26}$$

but this is not possible as 4.26 implies

$$E^{-1} = D^+ = U^+ E^T W^+ \tag{4.27}$$

so

$$P = DE^{-1} = W(P^{-1})^T W^+ \tag{4.28}$$

which is false.

All that can be said of 4.21 is that it leaves invariant the topological quantum number and action as can be seen from 4.19 using $\mathrm{tr}\, Q = \mathrm{tr}\, Q^T$. That this is the most important transformation can be guessed at from the fact that even for SU(2) where it is essentially trivial it plays a crucial role in the construction of solutions.

We now wish to discuss the Lagrangian and Bäcklund-type transformations associated with 4.14.

To do this we use the Iwasaro decomposition of SL(n,C) to choose a particularly important gauge called by Yang[5] the R gauge. The Iwasaro decomposition may be stated for any SL(n,C) matrix D as

$$D = LFU \tag{4.29}$$

where L is lower triangular matrix with ones along the diagonal, F is a real diagonal matrix with det F = 1 and U is a SU(n) matrix. From 4.6 we have

$$E = (D^{+})^{-1} = (L^{+})^{-1}F^{-1}U \tag{4.30}$$

Hence we may choose a gauge where

$$D = LF \qquad E^{-1} = FL^{+} \tag{4.31}$$

In this gauge consider the expression

$$-\frac{1}{2}\,\mathrm{tr}\left\{ (A_y + A_y^T)(A_{\bar{y}} + A_{\bar{y}}^T) + (A_z + A_z^T)(A_{\bar{z}} + A_{\bar{z}}^T) \right\}$$

$$= 2\,\mathrm{tr}\left[\frac{\partial F}{\partial y}\frac{\partial F}{\partial \bar{y}} F^{-2} + \frac{\partial F}{\partial z}\frac{\partial F}{\partial \bar{z}} F^{-2} \right] + \mathrm{tr}\left[L^{-1}\frac{\partial L}{\partial y} F^{2} \frac{\partial L^{+}}{\partial \bar{y}} L^{+^{-1}} F^{-2} \right.$$

$$\left. + \, L^{-1}\frac{\partial L}{\partial z} F^{2} \frac{\partial L^{+}}{\partial \bar{z}} L^{+^{-1}} F^{-2} \right] \tag{4.32}$$

Using the properties of L and F it may be verified that variation of this Lagrangian with respect to L, F and L^+ yield 4.14. It is now clear why no simple expression for a Lagrangian with a variational principle depending on P exists: if it did one would expect it to be in the gauge used previously with $A_{\bar{y}} = A_{\bar{z}} = 0$ and in that gauge 4.32 tends to zero.

To construct the Bäcklund transformations we define new coordinates by

$$\sqrt{2}p = \bar{y} + z \qquad \sqrt{2}q = y - \bar{z} \qquad (4.33a)$$

$$\sqrt{2}\,\bar{p} = y + \bar{z} \qquad \sqrt{2}\,\bar{q} = \bar{y} - z \qquad (4.33b)$$

Then

$$\sqrt{2}A_p = A_{\bar{y}} + A_z \qquad \sqrt{2}A_q = A_y - A_{\bar{z}} \qquad (4.34a)$$

$$\sqrt{2}A_{\bar{p}} = A_y + A_{\bar{z}} \qquad \sqrt{2}A_{\bar{q}} = A_{\bar{y}} - A_z \qquad (4.34b)$$

Hence in the R gauge a knowledge of $A_{\bar{p}}$ and $A_{\bar{q}}$ allows us to construct A_p and A_q since the diagonal entries come only from terms like $F^{-1}\partial_y F$ and are thus perfect differentials enabling us to construct F. Then the fact that terms such as $F^{-1}L^{-1}\partial_y LF$ are strictly lower diagonal and terms such as $FL^+\partial_y(L^+)^{-1}F^{-1}$ are strictly upper diagonal enable us to extract them from $A_{\bar{p}}$ and $A_{\bar{q}}$ for instance leading to complete expressions for A_p and A_q.

Trivially

$$F_{p\bar{p}} + F_{q\bar{q}} = F_{\bar{y}\bar{z}} - F_{yz} \tag{4.35a}$$

$$2F_{qp} = F_{y\bar{y}} + F_{z\bar{z}} + F_{\bar{y}\bar{z}} + F_{yz} \tag{4.35b}$$

$$2F_{\bar{q}\bar{p}} = -(F_{y\bar{y}} + F_{z\bar{z}}) + F_{\bar{y}\bar{z}} + F_{yz} \tag{4.35c}$$

Hence equations 4.3 are equivalent to

$$F_{qp} = 0 \tag{4.36a}$$

$$F_{\bar{q}\bar{p}} = 0 \tag{4.36b}$$

$$F_{p\bar{p}} + F_{q\bar{q}} + 0 \tag{4.36c}$$

The idea behind the following construction is to perform a gauge-type transformation of $A_{\bar{p}}$ and $A_{\bar{q}}$ alone and try to set this equal to an $A'_{\bar{p}}$ and $A'_{\bar{q}}$ coming from different D and E matrices but being in an R gauge. Thus as noted before this implies that we know what A'_p and A'_q are. We are thus left, after separating out components of the matrices, differential equations between the elements of D' and E' and D and E. To check that they are integrable equations and give a self-dual solution is equivalent to verifying 4.3 and thus also to the set 4.36. As the new diagonal element of $A'_{\bar{p}}$ and $A'_{\bar{q}}$ must again be perfect differentials it is reasonable to at least at first work with elements of the permutation group to ensure this condition.

Before presenting our results we give some notation. H is the nxn matrix with elements $H_{i,i+1} = 1$, $i \neq n$, $H_{n,1} = 1$, all other entries zero. $M^{(r)}$ is the matrix $\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ with r ones and n - r minus ones. $S = M^{(n-1)}H$. $H^TS = J = \mathrm{diag}(-1,1,\ldots,1)$.

<u>THEOREM</u>: Let D and E be generating matrices in an R gauge satisfying 4.3 then D' and E' also in an R gauge, defined from $F' = HFH^T$, $A'_{\bar p} = HA_{\bar p}H^T$ and $A'_{\bar q} = HA_{\bar q}H^T$ also satisfy 4.3

<u>Proof</u>: From 4.34 and the definition of H if $A'_{\bar p}$ and $A'_{\bar q}$ are to define D' and E' we must have

$$A'_p = SA_pS^T \qquad A'_q = SA_qS^T \tag{4.37}$$

This is easily seen as $A'_{pij} = A'_{\bar q ij}$, $i < j$ and A'_{pij}, $= -A'_{\bar q ij}$, $i > j$ and similarly for A'_q. Hence we need to check 4.3 or equivalently 4.36 hold. But

$$F'_{qp} = SF_{qp}S^T = 0 \tag{4.38a}$$

$$F'_{\bar q \bar p} = HF_{\bar q \bar p}H^T = 0 \tag{4.38b}$$

as $SS^T = HH^T = I$, so only $F'_{p\bar p} + F'_{q\bar q}$ remains.

Now

$$
\begin{aligned}
2H^T(F'_{p\bar{p}} + F'_{q\bar{q}})H = 2\Big(& \frac{\partial}{\partial \bar{p}} JA_pJ - \frac{\partial}{\partial p} A_{\bar{p}} - [JA_pJ, A_{\bar{p}}] + \frac{\partial}{\partial \bar{q}} JA_qJ \\
& - \frac{\partial A_{\bar{q}}}{\partial q} - [JA_qJ, A_{\bar{q}}] \Big) \\
= {} & \left(-\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{y}}\right) J(A_y - A_{\bar{z}})J - \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial \bar{z}}\right)(A_{\bar{y}} - A_z) \\
& - [J(A_y - A_{\bar{z}})J, A_{\bar{y}} - A_z] + \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial \bar{z}}\right) J(A_{\bar{y}} + A_z)J \\
& - \left(\frac{\partial}{\partial \bar{y}} + \frac{\partial}{\partial z}\right)(A_y + A_{\bar{z}}) - [J(A_{\bar{y}} + A_z)J, A_y + A_{\bar{z}}] \\
= {} & \frac{\partial}{\partial \bar{z}} JA_zJ - \frac{\partial}{\partial \bar{z}} A_z - \frac{\partial}{\partial z} A_{\bar{z}} + \frac{\partial}{\partial z} JA_{\bar{z}}J + \frac{\partial}{\partial \bar{y}} JA_yJ - \frac{\partial A_y}{\partial \bar{y}} \\
& - \frac{\partial}{\partial y} A_{\bar{y}} + \frac{\partial}{\partial y} JA_{\bar{y}}J - [JA_yJ, A_{\bar{y}}] + [A_y, JA_{\bar{y}}J] \\
& - [JA_{\bar{z}}J, A_z] + [A_{\bar{z}}, JA_zJ] + [JA_yJ - A_y, JA_zJ - A_z] \\
& + [A_{\bar{y}} - JA_{\bar{y}}J, A_{\bar{z}} - JA_{\bar{z}}J] \qquad (4.39)
\end{aligned}
$$

where we have used $\partial_y A_z - \partial_z A_y = [A_z, A_y]$ and $\partial_{\bar{z}} A_{\bar{y}} - \partial_{\bar{y}} A_{\bar{z}} = [A_{\bar{y}}, A_{\bar{z}}]$ i.e. $F_{yz} = F_{\bar{y}\bar{z}} = 0$.

As both $JA_yJ - A_y$ and $JA_zJ - A_z$ are zero except for the $i,1^{th}$ component $(i \neq 1)$ they commute. Similarly $A_{\bar{y}} - JA_{\bar{y}}J$ and $A_{\bar{z}} - JA_{\bar{z}}J$ are zero except for the $1,i^{th}$ $(i \neq 1)$ component so they commute.

We now use

$$[JA_{\bar{z}}J]_{ij} = [A_{\bar{z}}]_{ij} \qquad i,j \neq 1 \text{ or } i=j=1$$

$$[JA_{\bar{z}}J]_{1j} = -[A_{\bar{z}}]_{1j} \qquad j \neq 1$$

$$[JA_{z}J]_{ij} = [A_{z}]_{ij} \qquad i,j \neq 1 \text{ or } i=j=1$$

$$[JA_{z}J]_{i1} = -[A_{z}]_{i1} \qquad i \neq 1$$

together with $(A_z)_{ij} = 0$ $i<j$, $(A_{\bar{z}})_{ij} = 0$ $i>j$ to check that the remainder of 4.39 is zero by taking the i,j^{th} components.

a) $i,j \neq 1$: all differentiations vanish and the commutators become

$$(s = \min[i,j] \qquad t = \max[i,j])$$

$$-\sum_{k=2}^{s} A_{yik}A_{\bar{y}kj} + A_{yi1}A_{\bar{y}1j} + \sum_{k=t}^{n} A_{\bar{y}ik}A_{ykj} + \sum_{2}^{s} A_{yik}A_{\bar{y}kj} - A_{yi1}A_{\bar{y}1j}$$

$$-\sum_{t}^{n} A_{\bar{y}ik}A_{ykj} - \sum_{2}^{s} A_{zik}A_{\bar{z}kj} + A_{zi1}A_{\bar{z}1j} + \sum_{t}^{n} A_{\bar{z}ik}A_{zkj} + \sum_{2}^{s} A_{zik}A_{\bar{z}kj}$$

$$-A_{zi1}A_{\bar{z}1j} - \sum_{t}^{n} A_{\bar{z}ik}A_{zkj} = 0$$

b) $i = 1 \quad j \neq 1$

$$- 2\frac{\partial}{\partial z}A_{\bar z 1j} - 2\frac{\partial}{\partial y}A_{\bar y 1j} - A_{y11}A_{\bar y 1j} + \sum_1^j A_{\bar y 1k}A_{ykj} - A_{y11}A_{\bar y 1j} + \sum_1^j A_{\bar y 1k}A_{ykj}$$

$$- A_{z11}A_{\bar z 1j} + \sum_1^j A_{\bar z 1k}A_{zkj} - A_{z11}A_{\bar z 1j} + \sum_1^j A_{\bar z 1k}A_{zkj}$$

$$= 2\Big\{ -\frac{\partial}{\partial z}A_{\bar z 1j} - A_{z11}A_{\bar z 1j} + \sum_1^j A_{\bar z 1k}A_{zkj} - \frac{\partial}{\partial y}A_{\bar y 1j} - A_{y11}A_{\bar y 1j}$$

$$+ \sum_1^j A_{\bar y 1k}A_{ykj}\Big\} = 2\Big\{ -\frac{\partial A_{\bar z}}{\partial z} - [A_z, A_{\bar z}] - \frac{\partial}{\partial y}A_{\bar y}$$

$$- [A_y, A_{\bar y}]\Big\}_{1j} = 2\,[F_{z\bar z} + F_{y\bar y}]_{1j} = 0$$

c) $i \neq 1 \quad j = 1$

As, using $(A_{\bar z})^+ = -A_z$ and $(A_{\bar y})^+ = -A_y$, we have that the lefthand side of 4.39 is skew-hermitian

$$[H^T(F'_{p\bar p} + F'_{q\bar q})H]_{i1} = -[H^T(F'_{p\bar p} + F'_{q\bar q})H]^*_{1i} = 0 \quad \text{from b.}$$

d) $i = j = 1$: the derivative terms vanish and the commutators are

$$- A_{y11}A_{\bar y 11} - \sum_2^n A_{\bar y 1k}A_{yk1} + A_{\bar y 11}A_{y11} + A_{y11}A_{\bar y 11} + \sum_2^n A_{\bar y 1k}A_{yk1}$$

$$- A_{\bar y 11}A_{y11} - A_{z11}A_{\bar z 11} - \sum_2^n A_{\bar z 1k}A_{zk1} + A_{\bar z 11}A_{z11} + A_{z11}A_{\bar z 11}$$

$$+ \sum_2^n A_{\bar z 1k}A_{zk1} - A_{\bar z 11}A_{z11} = 0$$

Hence $F'_{p\bar{p}} + F'_{q\bar{q}} = 0$ Q.E.D.

There are two ways of interpreting the above transformations. First one can ignore the fact that it is possible to use them to define new generating matrices and simply look on them as a way of renaming the functions occuring in the original D and E. This is not as simplistic an approach as it at first appears ; as we shall see in the next chapter the 't Hooft solutions for SU(2) are most succinctly written using this way of changing variables.

If one wishes, however, to use them to construct new D and E matrices and new potentials then calling the transformation β it can be seen that β^r does not necessarily give an SU(n) solution. To show this we first establish the following lemma.

Lemma: $0 \leq p \leq n-1$ then $H^p (S^T)^p = M^{(n-p)}$

Proof: It is clearly true for p = 1 as it then reduces to $HS^T = HH^TM^{(n-1)} = M^{(n-1)}$. Assume it is true for $p \leq j$. Then $H^{j+1}(S^T)^{j+1} = H(H^j(S^T)^j)S^T = HM^{(n-j)}S^T = HM^{(n-j)}H^TM^{(n-1)}$. Now take the i,k^{th} component.

a) $i \neq n$

$H_{i,i+1} M^{(n-j)}_{i+1,i+1} H_{i+1,i} M^{(n-1)}_{i,k} = 0$ unless $i = k$. Clearly we have

$$i \geq n-j-1 \quad [H^{j+1}(S^T)^{j+1}]_{i,i} = 1$$

$$i < n-j-1 \quad [H^{j+1}(S^T)^{j+1}]_{i,i} = -1$$

b) $i = n$

$$H_{n,1} \; M_{11}^{(n-j)} \; H_{1n} \; M_{n,k}^{(n-1)} = 0 \text{ unless } n' = k \text{ when it is } -1 .$$

Hence $H^{j+1} (S^T)^{j+1} = M^{(n-j-1)}$ Q.E.D.

We now prove our previous assertion. First let $0 \leq r \leq n-1$. The condition that the potentials give an $SU(n-r,n)$ theory is $(A'_{\mu})^+ = -M^{(n-r)} A'_{\bar{\mu}} M^{(n-r)}$. Applying this to $A'_{\bar{p}}$ and using $(A_{\bar{p}})^+ = -A_p$ by hypothesis we have

$$(A'_{\bar{p}})^{\dagger} = (H^r A_{\bar{p}} (H^T)^r)^{\dagger} = H^r (A_{\bar{p}})^{\dagger} (H^T)^r = -H^r A_p (H^T)^r$$

$$= -H^r (S^T)^r A'_p S^r (H^T)^r = -M^{(n-r)} A'_p M^{(n-r)} \tag{4.40}$$

with a similar result for $A'_{\bar{q}}$. For $r = n$ $M^{(0)} A'_p M^{(0)} = A'_p$ so we return to $SU(n)$. Hence we have proved that

<u>Lemma</u>: If we start with $SU(n)$ potentials and apply β^r to them then if $r \equiv p \bmod n$, $0 \leq p \leq n-1$, we end up with self-dual potentials for an $SU(n-p,p)$ theory.

Although we have not been able to prove that all Bäcklund transformations are generated by H in this manner the result seems likely; for instance for $n = 3$ one might try to construct a transformation based on the transposition (1,2) but this turns out not to be possible, with similar results in n=4 for (1,2) (4,3).

The only relevance of the $p,q,\bar{p},\bar{q}$ coordinate system seems to be that it is a different pair of β-planes. However the choice of yet another pair does not give an essentially different Bäcklund transformation but is equivalent to the one defined from $p,q,\bar{p},\bar{q}$ combined with an algebraic transformation of the type 4.23.

Now algebraic transformations of the type 4.20 or equivalently 4.23 are gauge transformations in the R gauge formalism for starting with D in any gauge there exists a U_1 given by the Iwasaro decomposition taking us to an R gauge and given AD there exists a U_2 taking us from the same potentials to an R gauge. Hence starting in one R gauge $U_1(U_2)^{-1}$ takes us to the other.

For transformations of type 4.21 or 4.24 these are not gauge transformations between R gauges except for SU(2). This is unfortunate as judging from the case n = 2 it would seem that the most important transformation of the self-duality equations is β composed with 4.21 and, except for n = 2, the latter is not easily expressible in terms of the elements of D and E in the necessary gauge and the former is not simply written as equations between P and P'.

In the next chapter we use this composition for the case $n = 2$ to construct solutions of the equation 4.3 from the 't Hooft solutions derived in chapter two.

CHAPTER FIVE

SELF-DUAL SU(2) THEORIES

We now apply the results of the previous chapter to the case when the group is SU(2)[18]. This enables us to provide an induction theorem which by virtue of the work of Atiyah and Ward[6] in algebraic geometry constructs all self-dual SU(2) solutions. There remains however the problem of where such solutions are singular and what type of sigularity exists. Work needs to be done on this problem.

First we present some notation. Matrices D and E^{-1} of equations 4.4 are

$$D = \frac{1}{\sqrt{f}}\begin{bmatrix} 1 & 0 \\ e & f \end{bmatrix} \qquad E^{-1} = \frac{1}{\sqrt{f}}\begin{bmatrix} 1 & -g \\ 0 & f \end{bmatrix} \qquad (5.1)$$

so for SU(2) solutions we requie f real and

$$-e = g^{*} \qquad (5.2)$$

and the potentials are

$$A_y = \begin{bmatrix} -\frac{f_y}{2f} & 0 \\ \frac{e_y}{f} & \frac{f_y}{2f} \end{bmatrix} \qquad A_z = \begin{bmatrix} -\frac{f_z}{2f} & 0 \\ \frac{e_z}{f} & \frac{f_z}{2f} \end{bmatrix}$$

$$A_{\bar{y}} = \begin{bmatrix} \frac{f_{\bar{y}}}{2f} & \frac{g_{\bar{y}}}{f} \\ 0 & -f_{\bar{y}}/2f \end{bmatrix} \qquad A_{\bar{z}} = \begin{bmatrix} f_{\bar{z}}/2f & \frac{g_{\bar{z}}}{f} \\ 0 & -f_{\bar{z}}/2f \end{bmatrix} \qquad (5.3)$$

(subscripts on e, f and g denote differentiation)

The self-duality equations 4.3 are now

$$f^{-1}(f_{y\bar{y}}+f_{z\bar{z}})-f^{-2}f_yf_{\bar{y}}-f^{-2}f_zf_{\bar{z}}-f^{-2}e_yg_{\bar{y}}-f^{-2}e_zg_{\bar{z}}=0 \qquad (5.4a)$$

$$f^{-1}(e_{y\bar{y}}+e_{z\bar{z}})-2f^{-2}e_yf_{\bar{y}}-2f^{-2}e_zf_{\bar{z}}=0 \qquad (5.4b)$$

$$f^{-1}(g_{y\bar{y}}+g_{z\bar{z}})-2f^{-2}g_{\bar{y}}f_y-2f^{-2}g_{\bar{z}}f_z=0 \qquad (5.4c)$$

For completeness we mention here the alternative ansätze for A_μ given by a Bäcklund transformation which gives SU(2) solutions with e,f and g still satisfying 5.4 but with 5.2 replaced by

$$e = g^* \qquad (5.5)$$

It is

$$A_y=\begin{bmatrix} f_y/2f & 0 \\ -g_{\bar{z}}/f & -f_y/2f \end{bmatrix} \qquad A_z=\begin{bmatrix} f_z/2f & 0 \\ g_{\bar{y}}/f & -f_z/2f \end{bmatrix}$$

$$A_{\bar{y}}=\begin{bmatrix} -f_{\bar{y}}/2f & e_z/f \\ 0 & f_{\bar{y}}/2f \end{bmatrix} \qquad A_{\bar{z}}=\begin{bmatrix} -f_{\bar{z}}/2f & -\frac{e_y}{2f} \\ 0 & f_{\bar{z}}/2f \end{bmatrix} \qquad (5.6)$$

For this ansatz 5.4 is not just a consequence of $F_{y\bar{y}}+F_{z\bar{z}}=0$: 5.4a follows from this but 5.4b and c result from imposing $F_{\bar{y}\bar{z}} = F_{yz} = 0$ respectively. The important point is that if e and g satisfy either 5.2 or 5.5 we can still construct a real SU(2) solution from them and the induction we use later alternates these conjugation relations because it uses the Bäcklund transformations.

The algebraic transformations 4.23, including as mentioned previously $P \to (P^{-1})^T$, can be expressed in terms of e, f and g in the following form:

<u>Lemma</u> : Let

$$S = \begin{bmatrix} e & f \\ f & g \end{bmatrix} \qquad (5.7)$$

be a solution matrix of 5.4, then so is S' where S' $= (aS + b)(cS + d)^{-1}$ if a, b, c and d are diagonal matrices satisfying ad - bc = I.

<u>Proof</u> : by inspection.

In particular $P \to (P^{-1})^T$ corresponds to $S \to S^{-1}$ i.e.

$$\begin{bmatrix} e & f \\ f & g \end{bmatrix} \longrightarrow \frac{1}{eg-f^2}\begin{bmatrix} g & -f \\ -f & e \end{bmatrix} \qquad (5.8)$$

The Bäcklund transformation β can be expressed as

$$f' = f^{-1} \qquad (5.9a)$$

$$e'_z = -f^{-2} g_{\bar{y}} \qquad e'_y = f^{-2} g_{\bar{z}} \qquad (5.9b)$$

$$g'_{\bar{z}} = f^{-2} e_y \qquad g'_{\bar{y}} = -f^{-2} e_z \tag{5.9c}$$

and we note that unlike 5.8, 5.9 does reverse any conjugation relation between e and g so either taking us from ansatz 5.3 to 5.6 (and remaining in SU(2)) or staying in one ansatz and taking us from SU(2) to SU(1,1).

The combination of 5.8 with 5.9 will be crucial to our main result so we give it in full below

$$f' = -(eg - f^2)/f \tag{5.10a}$$

$$\frac{\partial}{\partial \bar{y}}\left\{\frac{e}{eg-f^2}\right\} = -\frac{1}{f'^2}\frac{\partial e'}{\partial z} \qquad \frac{\partial}{\partial \bar{z}}\left\{\frac{e}{eg-f^2}\right\} = \frac{1}{f'^2}\frac{\partial e'}{\partial y} \tag{5.10b}$$

$$\frac{\partial}{\partial z}\left\{\frac{g}{eg-f^2}\right\} = \frac{-1}{f'^2}\frac{\partial g'}{\partial \bar{y}} \qquad \frac{\partial}{\partial y}\left\{\frac{g}{eg-f^2}\right\} = \frac{1}{f'^2}\frac{\partial g'}{\partial \bar{z}} \tag{5.10c}$$

We can now prove the following theorem :

<u>Theorem</u> Let Δ_r be a set of 2m-1 fields $|r| \leq m-1$ satisfying

$$\frac{\partial \Delta_r}{\partial y} = -\frac{\partial \Delta_{r+1}}{\partial \bar{z}} \qquad \frac{\partial \Delta_r}{\partial z} = \frac{\partial \Delta_{r+1}}{\partial \bar{y}} \qquad \Delta_0 \text{ real} \qquad \Delta_1 = -\Delta_{-1}^{*} \tag{5.11}$$

so that $\Box \Delta_r = 0$. For m = 1 we impose $\Box \Delta_0 = 0$. Then we have an infinite hierarchy of ansätze giving self-dual solutions constructed out of Δ_r by the following method.

$A^{(1)}$ is given by $e = f = g = 1/\Delta_0$

For $m \geqslant 2$ let the mxm matrix $D^{(m)}$ be $D^{(m)}_{r,s} = \Delta_{r+s-m-1}$ and let $\delta^{(m)} = D^{(m)-1}$, then $A^{(m)}$ is given by

$$e = \delta^{(m)}_{11} \qquad f = \delta^{(m)}_{1m} = \delta^{(m)}_{m1} \qquad g = \delta^{(m)}_{mm}$$

<u>Proof</u> : By inspection $A^{(1)}$ does indeed provide a solution to 5.4 (namely the known 't Hooft solutions(12) when put into ansatz 5.6). To give an inductive proof we must also check the m = 2 ansatz.
This gives

$$\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} \Delta_{-1} & \Delta_0 \\ \Delta_0 & \Delta_1 \end{bmatrix}^{-1} \tag{5.12}$$

but by invariance 5.8 it is sufficient to prove

$$\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} \Delta_{-1} & \Delta_0 \\ \Delta_0 & \Delta_1 \end{bmatrix} \tag{5.13}$$

provides a solution. As Δ_0 real and $\Delta_1 = -\Delta^*_{-1}$ we use ansatz 5.3 and from the differential relations between Δ_r we have

$$e_z = f_{\bar{y}} \qquad e_y = -f_{\bar{z}} \tag{5.14a}$$

$$g_{\bar{z}} = -f_y \qquad g_{\bar{y}} = f_z \tag{5.14b}$$

Putting these into 5.4 reduces them to

$$f^{-1}\Box f = 0 \tag{5.15}$$

i.e. the 't Hooft solutions.

Assume the result is true for m-1. Let (e,f,g) be the functions associated with $A^{(m-1)}$ and (s,t,u) be constructed out of $\delta^{(m)}$. We aim to show that (s,t,u) are related to (e,f,g) by 5.10 and so also provide a solution. Now

$$su - t^2 = \left[\tilde{D}^{(m)}_{11}\tilde{D}^{(m)}_{mm} - \tilde{D}^{(m)}_{1m}\tilde{D}^{(m)}_{m1}\right] / (\det D^{(m)})^2$$

$$= \det D^{(m-2)}/\det D^{(m)} \tag{5.16}$$

where $\tilde{D}^{(m)}$ is the adjugate matrix to $D^{(m)}$, and we have used a result of Jacobi which states that if $\tilde{M}^{(r)}$ is an rxr submatrix of $\tilde{M}$, the adjugate of the nxn matrix M, and $M^{(n-r)}$ is the (n-r)x(n-r) submatrix of M obtained by stricking out similarly placed rows and columns in M to those of $\tilde{M}^{(r)}$ in $\tilde{M}$ then $\det \tilde{M}^{(r)} = (\det M)^{r-1} \det M^{(n-r)}$.

$$t = (-)^{m+1}\det D^{(m-1)}/\det D^{(m)} \tag{5.17}$$

so

$$(su - t^2)/t = (-)^{m+1}\det D^{(m-2)}/\det D^{(m-1)} = -f \tag{5.18}$$

We now use the result

$$\tilde{D}^{(m)}_{1m}\frac{\partial\tilde{D}^{(m-1)}_{11}}{\partial y} - \tilde{D}^{(m-1)}_{11}\frac{\partial\tilde{D}^{(m)}_{1m}}{\partial y} = \tilde{D}^{(m)}_{11}\frac{\partial\tilde{D}^{(m-1)}_{1\,m-1}}{\partial z} - \tilde{D}^{(m-1)}_{1\,m-1}\frac{\partial\tilde{D}^{(m)}_{11}}{\partial z} \quad (5.19)$$

which to avoid interruptions in the arguement is proved later. 5.19 may be written as

$$\left[\tilde{D}^{(m)}_{1m}/\tilde{D}^{(m-1)}_{1\,m-1}\right]^2\frac{\partial}{\partial y}\left[\tilde{D}^{(m-1)}_{11}/\tilde{D}^{(m)}_{1m}\right] = -\frac{\partial}{\partial z}\left[\tilde{D}^{(m)}_{11}/\tilde{D}^{(m-1)}_{1\,m-1}\right] \quad (5.20)$$

But

$$s = \tilde{D}^{(m)}_{11}/\det D^{(m)} \quad (5.21)$$

so

$$s/(us-t^2) = \tilde{D}^{(m)}_{11}/\det D^{(m-2)} = (-)^m\tilde{D}^{(m)}_{11}/\tilde{D}^{(m-1)}_{1\,m-1} \quad (5.22)$$

and

$$e = (-)^{m+1}\tilde{D}^{(m-1)}_{11}/\tilde{D}^{(m)}_{1m} \qquad f = (-)^{m+1}\tilde{D}^{(m-1)}_{1\,m-1}/\tilde{D}^{(m)}_{1m} \quad (5.23)$$

so 5.19 implies

$$f^{-2}\frac{\partial e}{\partial y} = \frac{\partial}{\partial z}\left(s/(su-t^2)\right) \quad (5.24)$$

Similarly a slight variation of 5.19 implies

$$f^{-2}\frac{\partial e}{\partial z} = -\frac{\partial}{\partial \bar{y}}\left[s/(su-t^2)\right] \tag{5.25}$$

and

$$\tilde{D}^{(m)}_{1m}\frac{\partial \tilde{D}^{(m-1)}_{m-1\,m-1}}{\partial \bar{z}} - \tilde{D}^{(m-1)}_{m-1\,m-1}\frac{\partial \tilde{D}^{(m)}_{1m}}{\partial \bar{z}} = \tilde{D}^{(m)}_{mm}\frac{\partial \tilde{D}^{(m-1)}_{1\,m-1}}{\partial \bar{y}} - \tilde{D}^{(m-1)}_{1\,m-1}\frac{\partial \tilde{D}^{(m)}_{mm}}{\partial \bar{y}} \tag{5.26}$$

and a comparable equation give via

$$u/(su-t^2) = (-)^m\,\tilde{D}^{(m)}_{mm}\,/\,\tilde{D}^{(m-1)}_{1\,m-1} \tag{5.27}$$

and

$$g = (-)^{m+1}\,\tilde{D}^{(m-1)}_{m-1\,m-1}\,/\,\tilde{D}^{(m)}_{1m} \tag{5.28}$$

$$\frac{\partial}{\partial z}\left[u/(su-t^2)\right] = -f^{-2}\frac{\partial g}{\partial \bar{y}} \tag{5.29a}$$

$$\frac{\partial}{\partial y}\left[u/(su-t^2)\right] = f^{-2}\frac{\partial g}{\partial \bar{z}} \tag{5.29b}$$

Hence as (e,f,g) are a solution so are (s,t,u).

Q.E.D.

To complete the exposition we prove 5.19

and 5.26. As the proofs are similar we give a proof of 5.26 and only indicate that for 5.19. Define the mxm matrix $C^{(m)}$ by $C^{(m)}_{r,s} = \Delta_{r+s-m-2}$. Then

$$c^{(m)} = \det C^{(m)} = \tilde{D}^{(m+1)}_{m+1\, m+1} \tag{5.30}$$

$$d^{(m)} = \det D^{(m)} = (-)^m \tilde{D}^{(m+1)}_{1\, m+1} = \tilde{C}^{(m+1)}_{11} \tag{5.31}$$

$$\tilde{C}^{(m)}_{1r} = (-)^{m-1} \tilde{D}^{(m)}_{rm} \qquad \tilde{C}^{(m)}_{r1} = (-)^{m-1} \tilde{D}^{(m)}_{mr} \tag{5.32}$$

Then 5.26 is equivalent to proving

$$d^{(m)} \frac{\partial c^{(m-1)}}{\partial \bar{z}} - c^{(m-1)} \frac{\partial d^{(m)}}{\partial \bar{z}} = d^{(m-1)} \frac{\partial c^{(m)}}{\partial y} - c^{(m)} \frac{\partial d^{(m-1)}}{\partial y} \tag{5.33}$$

Now

$$d^{(m-1)} \frac{\partial c^{(m)}}{\partial y} + c^{(m-1)} \frac{\partial d^{(m)}}{\partial \bar{z}} = d^{(m-1)} \sum_{rs}^{m} \tilde{C}^{(m)}_{rs} \frac{\partial C^{(m)}_{rs}}{\partial y}$$

$$+ c^{(m-1)} \sum_{rs}^{m} \tilde{D}^{(m)}_{rs} \frac{\partial D^{(m)}_{rs}}{\partial \bar{z}}$$

$$= \sum_{rs}^{m} \frac{\partial C^{(m)}_{rs}}{\partial y} \left[d^{(m-1)} \tilde{C}^{(m)}_{rs} - c^{(m-1)} \hat{D}^{(m)}_{rs} \right]$$

(from 5.11)

$$= \sum_{rs}^{m} \frac{\partial c_{rs}^{(m)}}{\partial y} \left[\tilde{C}_{11}^{(m)} \tilde{C}_{rs}^{(m)} - \tilde{D}_{mm}^{(m)} \tilde{D}_{rs}^{(m)} \right]$$

$$= \sum_{rs}^{m} \frac{\partial c_{rs}^{(m)}}{\partial y} \left\{ \left(\tilde{C}_{11}^{(m)} \tilde{C}_{rs}^{(m)} - \tilde{C}_{r1}^{(m)} \tilde{C}_{1s}^{(m)} \right) + \left(\tilde{D}_{rm}^{(m)} \tilde{D}_{ms}^{(m)} - \tilde{D}_{mm}^{(m)} \tilde{D}_{rs}^{(m)} \right) \right\}$$

$$= \sum_{rs}^{m-1} \frac{\partial c_{r+1\,s+1}^{(m)}}{\partial y} c^{m} \tilde{D}_{rs}^{(m-1)} - \sum_{rs}^{m-1} \frac{\partial c_{rs}^{(m)}}{\partial y} d^{(m)} \tilde{C}_{rs}^{(m-1)}$$

(by Jacobi)

$$= c^{(m)} \sum_{rs}^{m-1} \frac{\partial D_{rs}^{(m-1)}}{\partial y} \tilde{D}_{rs}^{(m-1)} + d^{(m)} \sum_{rs}^{m-1} \frac{\partial c_{rs}^{(m-1)}}{\partial \bar{z}} \tilde{C}_{rs}^{(m-1)}$$

$$= c^{(m)} \frac{\partial d^{(m-1)}}{\partial y} + d^{(m)} \frac{\partial c^{(m-1)}}{\partial \bar{z}} \qquad (5.34)$$

as claimed. To prove 5.19 we make similar manipulations involving the matrix $B^{(m)}$ defined by $B_{rs}^{(m)} = \Delta_{r+s-m}$ so that 5.19 is equivalent to

$$d^{(m)} \frac{\partial b^{(m-1)}}{\partial y} - b^{(m-1)} \frac{\partial d^{(m)}}{\partial y} = d^{(m-1)} \frac{\partial b^{(m)}}{\partial \bar{z}} - b^{(m)} \frac{\partial d^{(m-1)}}{\partial \bar{z}} \qquad (5.35)$$

where $b^{(m)} = \det B^{(m)}$ and proceed as above.

To interpret the solutions given by the theorem we use the work of Atiyah and Ward[6] and its analytic interpretation due to Goddard[18]; the latter shows how to relate the claim of the former to the above ansätze. According to Atiyah and Ward each ansätze starting from a given Δ_0 preserves the topological quantum number k say but increases the parameters on which the solution depends until, for a given m some (unknown) function of k, the maximum number of parameters namely 8k-3 is reached[19]. Hence as solutions at the m = 1 level for all k are known, the 't Hooft solutions

$$\Delta_0 = 1 + \sum_{i=1}^{k} \frac{\lambda_i^2}{(x - x_i)^2} \tag{5.36}$$

the above construction provides all self-dual SU(2) solutions.

We recall the definition of a β-plane given in chapter one, namely if v_μ and w_μ are any displacements in that plane $v_\mu w_\nu - v_\nu w_\mu$ is antiself-dual. To see Atiyah and Ward's construction we must find a way of labelling these planes which the authors of reference 6 do using the Penrose twistor notation[7]. Let X be the matrix given by

$$X = x^0 I - i \underset{\sim}{x} \cdot \underset{\sim}{\sigma} = \sqrt{2} \begin{bmatrix} y & -\bar{z} \\ z & \bar{y} \end{bmatrix} \tag{5.37}$$

where $\underset{\sim}{\sigma}$ are the Pauli spin matrices. Then for fixed 2-spinors w and π the equation

$$\omega = X\pi \tag{5.38}$$

determines a β-plane. Now let θ stand for the 4-spinor formed by

$$\theta = \begin{pmatrix} \pi \\ \omega \end{pmatrix} \qquad (5.39)$$

Clearly θ and $\lambda\theta$ define the same plane for $\lambda \neq 0$, so β-planes are labelled by points in CP_3, complex projective 3-space. If $[\theta] = \{\lambda\theta : \lambda \in C \ \lambda \neq 0\}$ is such a point let the corresponding β-plane be $\beta_{[\theta]}$. As explained in chapter one Atiyah and Ward point out that in such a plane the equation

$$A_\alpha dx^\alpha = g^{-1} \frac{\partial g}{\partial x^\alpha} dx^\alpha \qquad dX\pi = 0 \qquad (5.40)$$

may be integrated between $x, y \in \beta_{[\theta]}$ to give $g_{[\theta]}(x,y) \in Sl(2,C)$, the complexification of SU(2), if A_α is a self-dual potential. Hence over $[\theta]$ we can construct a two dimensional fibre $V_{[\theta]}$ given by 2-spinor fields $\psi_{[\theta]}(x)$ defined over $\beta_{[\theta]}$ and related at different points of $\beta_{[\theta]}$ by

$$\psi_{[\theta]}(x) = g_{[\theta]}(x,y)\,\psi_{[\theta]}(y) \qquad x, y \in \beta_{[\theta]} \qquad (5.41)$$

thus arriving at a two dimensional analytic vector bundle over CP_3.

To define coordinates on this bundle it is necessary to pick an $x_{[\theta]} \in \beta_{[\theta]}$ for each $[\theta]$ and specify the value of $\Psi_{[\theta]}(x_{[\theta]})$ so determining $\Psi_{[\theta]}(x)$ everywhere given $g_{[\theta]}(x,y)$. However such a choice of $x_{(\theta)}$ cannot be done smoothly; at least two such choices are necessary, say

$$x^1_{[\theta]} = \begin{bmatrix} \omega_1/\pi_1 & 0 \\ \omega_2/\pi_1 & 0 \end{bmatrix} \quad \pi_1 \neq 0 \tag{5.42a}$$

$$x^2_{[\theta]} = \begin{bmatrix} 0 & \omega_1/\pi_2 \\ 0 & \omega_2/\pi_2 \end{bmatrix} \quad \pi_2 \neq 0 \tag{5.42b}$$

the two coordinate systems being related by the transition function $g(\omega,\pi) = g_{[\theta]}(x^1_{[\theta]}, x^2_{[\theta]})$ by

$$\Psi_{[\theta]}(x^1_{[\theta]}) = g_{[\theta]}(x^1_{[\theta]}, x^2_{[\theta]})\,\Psi_{[\theta]}(x^2_{[\theta]}) \tag{5.43}$$

$g(\omega,\pi)$ defining the isomorphism class of the bundle and being homogenous: $g(\omega,\pi) = g(\lambda\omega,\lambda\pi)$.

Ward[20] points out that A_μ may be regained from $g(\omega,\pi)$ using the fact that

$$g(\omega,\pi) = g_{[\theta]}(x^1_{[\theta]}, x)\, g_{[\theta]}(x, x^2_{[\theta]}) \tag{5.44}$$

for each $x \in \beta_{[\theta]}$. So writing $h(x,\zeta) = g_{[\theta]}(x^1_{[\theta]}, x)$ and $k(x,\zeta) = g_{[\theta]}(x^2_{[\theta]}, x)$ where $\zeta = \pi_1/\pi_2$

$$g(x\pi, x) = h(x,\zeta)\, k^{-1}(x,\zeta) \tag{5.45}$$

$h(x,\zeta)$ being analytic away from $\zeta = 0$ and $k(x,\zeta)$ being analytic away from $\zeta = \infty$, for fixed ζ. With these conditions Liouville's theorem implies that 5.45 uniquely determines h and k up to a gauge transformation

$$h \longrightarrow h\,u(x) \qquad k \longrightarrow k\,u(x) \qquad u(x) \in SU(2) \qquad (5.46)$$

From equation 5.40

$$A_{i1}(x) - \zeta A_{i2}(x) = h^{-1}(x,\zeta)\left\{\frac{\partial}{\partial x_{i1}} - \zeta\frac{\partial}{\partial x_{i2}}\right\}h(x,\zeta) \qquad (5.47a)$$

$$= k^{-1}(x,\zeta)\left\{\frac{\partial}{\partial x_{i1}} - \zeta\frac{\partial}{\partial x_{i2}}\right\}k(x,\zeta) \qquad (5.47b)$$

as $(dx)\pi = 0$ is satisfied if $dx_{i2} = \zeta dx_{i1}$ where $A_{ij}dx_{ij} = A_\mu dx^\mu$. Hence the isomorphism class determines the potential up to a gauge equivalence.

Conversely Ward[20] shows that any $g(\omega,\pi)$ homogenous in its variables with a suitable domain of analyticity yields a self-dual potential provided g may be split up as in 5.45, as then 5.47a and 5.47b follow from

$$D_i g = 0 \qquad D_i = \frac{\partial}{\partial x_{i1}} - \zeta\frac{\partial}{\partial x_{i2}} \qquad (5.48)$$

the lefthand side of 5.47 coming from Liouville's theorem. In reference 6 it is argued that it is sufficient to take

$$g = \begin{bmatrix} \zeta^m & \rho(x,\zeta) \\ 0 & \zeta^{-m} \end{bmatrix} \qquad \text{m an integer} \tag{5.49}$$

with ρ depending on x and ζ through the variables $x_{11}\zeta + x_{12}$, $x_{21}\zeta + x_{22}$ and ζ. Hence

$$D_i \rho = 0 \tag{5.50}$$

In the same paper the authors argue that the hierarchy of ansätze arising from different choices of m preserve the topological quantum number k but increase the number of parameters in the solution till saturation is reached for some m(k).

To tie in this result with the construction previously given we use the work of Goddard[18] presented below. Let

$$h = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \qquad k = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{5.51}$$

$\alpha\delta - \beta\gamma$ = ad - bc = 1, α, β, γ and δ regular as functions of ζ except at ζ = 0, a, b, c and d regular as functions of ζ except at $\zeta = \infty$. Then from 5.45 and 5.49 we have

$$c = \gamma\zeta^m \qquad d = \delta\zeta^m \tag{5.52a}$$

$$a\zeta^m + \rho c = \alpha \qquad b\zeta^m + \rho d = \beta \tag{5.52b}$$

From 5.52a we deduce that c and d are polynomials of at most degree m and write

$$c(x,\zeta) = \sum_0^m c_r(x)\zeta^r \qquad d(x,\zeta) = \sum_0^m d_r(x)\zeta^r \tag{5.53}$$

Further as the coefficient of ζ^r vanishes in the Laurent series of $\alpha - a\zeta^m$ and $\beta - b\zeta^m$ $0 < r < m$ 5.52b implies

$$\oint \frac{d\zeta}{\zeta} \rho c \zeta^{-r} = \oint \frac{d\zeta}{\zeta} \rho d \zeta^{-r} = 0 \qquad 0 < r < m \tag{5.54}$$

the contours encircling the origin. Letting

$$\Delta_r(x) = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \rho(x,\zeta)\zeta^r \tag{5.55}$$

these may be written as

$$\sum_0^m c_s \Delta_{s-r} = \sum_0^m d_s \Delta_{s-r} = 0 \qquad 0 < r < m \tag{5.56}$$

5.50 implies that Δ_r satisfy

$$\frac{\partial \Delta_r}{\partial y} = -\frac{\partial \Delta_{r+1}}{\partial \bar{z}} \qquad \frac{\partial \Delta_r}{\partial z} = \frac{\partial \Delta_{r+1}}{\partial \bar{y}} \tag{5.57}$$

We may express a and b in terms of c and d

$$a(x,\zeta) = -\frac{1}{2\pi i}\oint \frac{d\lambda}{\lambda-\zeta}\lambda^{-m}\rho(x,\lambda)\,c(x,\lambda) \tag{5.58a}$$

$$b(x,\zeta) = -\frac{1}{2\pi i}\oint \frac{d\lambda}{\lambda-\zeta}\lambda^{-m}\rho(x,\lambda)\,d(x,\lambda) \tag{5.58b}$$

where the contours positively encircle the origin in $|\lambda| > |\zeta|$. Equations 5.56 leave four of the 2m+2 functions c_r and d_r arbitrary and the constraint ad - bc = 1 reduces this to three, mirroring the gauge freedom. From a,b,c and d satisfying these constraints potentials can be constructed from 5.47a.

$$A_{i_1}(x) - \zeta A_{i_2}(x) = \begin{bmatrix} dD_ia - bD_ic & dD_ib - bD_id \\ aD_ic - cD_ia & -dD_ia + bD_ic \end{bmatrix} \tag{5.59}$$

Goddard shows that(18)

$$dD_ia - bD_ic = d_0\frac{\partial a_0}{\partial x_{i_1}} - b_0\frac{\partial c_0}{\partial x_{i_1}} - \zeta\left(d_m\frac{\partial \alpha_0}{\partial x_{i_2}} - \beta_0\frac{\partial c_m}{\partial x_{i_2}}\right) \tag{5.60a}$$

$$aD_ic - cD_ia = \alpha_0\frac{\partial c_m}{\partial x_{i_1}} - c_m\frac{\partial \alpha_0}{\partial x_{i_1}} - \alpha_0\frac{\partial c_{m-1}}{\partial x_{i_2}} + c_{m-1}\frac{\partial \alpha_0}{\partial x_{i_2}} - \alpha_{-1}\frac{\partial c_m}{\partial x_{i_2}}$$

$$+ c_m\frac{\partial \alpha_{-1}}{\partial x_{i_2}} - \zeta\left(\alpha_0\frac{\partial c_m}{\partial x_{i_2}} - c_m\frac{\partial \alpha_0}{\partial x_{i_2}}\right) \tag{5.60b}$$

$$d D_i b - b D_i d = d_0 \frac{\partial b_0}{\partial x_{i_1}} - b_0 \frac{\partial d_0}{\partial x_{i_1}} - \zeta \Big\{ d_0 \frac{\partial b_0}{\partial x_{i_2}} - b_0 \frac{\partial d_0}{\partial x_{i_2}} - d_0 \frac{\partial b_1}{\partial x_{i_1}} + b_1 \frac{\partial d_0}{\partial x_{i_1}}$$

$$- d_1 \frac{\partial b_0}{\partial x_{i_1}} + b_0 \frac{\partial d_1}{\partial x_{i_1}} \Big\} \qquad (5.60c)$$

where α_s, β_s, a_s and b_s stand for the coefficent of ζ^s in the Laurent series of α, β , a and b respectively and may be expressed in terms of Δ_s, c_s and d_s using equations 5.52b. The constraint that ad - bc $= \alpha\delta - \beta\gamma = 1$ become

$$\alpha_0 d_m - \beta_0 c_m = a_0 d_0 - b_0 c_0 = 1$$

To recover the previous results we need only to choose the gauge

$$c_m = d_0 = 0 \qquad (5.61)$$

and

$$c_0^2 = d_m^2 = f \qquad (5.62)$$

Then from 5.59 and 5.60 we recover 5.6 with

$$e = c_0 d_1 \qquad g = c_{m-1} d_m \qquad (5.63)$$

5.61 reduces equations 5.56 to two sets of m-1 homogenous equations in m unknowns enabling us to verify that e, f and g are indeed given by the formulae stated in the previous theorem, providing the necessary link-up enabling us to claim that the analytic method provides all solutions from the 't Hooft solutions at the $m = 1$ level.

The fact that the $m = 2$ ansatz merely reproduces these is an accident due to the use of 5.8 : for $m > 2$ this invariance obviously cannot be used to reduce the non-linear dependance of (e,f,g) on the Δ_r to a linear one.

We should mention here that the explicit form for $A^{(2)}$ given in reference 6 is incorrect: instead of using ansatz 5.3 those authors used 5.6 so their form for A_μ gives SU(1,1) solutions. If we think of Δ_{-1}, Δ_0 and Δ_1 as being the components of an antiself-dual Maxwell field $\phi_{\mu\nu}$ given by

$$\phi_{zy} = \Delta_1$$

$$\phi_{\bar{z}\bar{y}} = \Delta_{-1} \tag{5.64}$$

$$\phi_{\bar{y}y} = \phi_{\bar{z}z} = \Delta_0$$

then $A_\mu = \frac{1}{2} i A_\mu^j \sigma_j$ can be written

$$A_\mu^j = R_j^{\mu\nu\alpha\beta} \frac{\phi^2}{\eta_3^{\lambda\rho}\phi_{\lambda\rho}} \frac{\partial}{\partial x^\nu} \left\{ \frac{\phi_{\alpha\beta}}{\phi^2} \right\} \tag{5.65}$$

where $\phi^2 = \phi_{\mu\nu}\phi^{\mu\nu}$ or, after doing a gauge transformation to get to 5.13

$$A^j_\mu = S_j^{\mu\nu\alpha\beta} \frac{1}{\eta_3^{\lambda\rho}\phi_{\lambda\rho}} \frac{\partial}{\partial x^\nu} \{ \phi_{\alpha\beta} \} \tag{5.66}$$

where

$$R_1^{\mu\nu\alpha\beta} = i g^{\mu\nu} \eta_1^{\alpha\beta} + \eta_3^{\mu\nu} \eta_2^{\alpha\beta}$$

$$R_2^{\mu\nu\alpha\beta} = g^{\mu\nu} \eta_2^{\alpha\beta} - i \eta_3^{\mu\nu} \eta_1^{\alpha\beta} \tag{5.67}$$

$$R_3^{\mu\nu\alpha\beta} = \eta_3^{\mu\nu} \eta_3^{\alpha\beta}$$

and

$$S_1^{\mu\nu\alpha\beta} = i g^{\mu\nu} \eta_1^{\alpha\beta} - \eta_3^{\mu\nu} \eta_2^{\alpha\beta}$$

$$S_2^{\mu\nu\alpha\beta} = - g^{\mu\nu} \eta_2^{\alpha\beta} - i \eta_3^{\mu\nu} \eta_1^{\alpha\beta} \tag{5.68}$$

$$S_3^{\mu\nu\alpha\beta} = \eta_3^{\mu\nu} \eta_3^{\alpha\beta}$$

The reality conditions on $\phi_{\alpha\beta}$ are, from 5.64,

$$[\phi_{zy}]^* = -\phi_{\bar{z}\bar{y}} \qquad \phi_{\bar{y}y} \text{ real} \tag{5.69}$$

The first order partial differential equations satisfied by Δ_r namely relations 5.11, imply

$$\Box \Delta_r = 0 \tag{5.70}$$

In general this cannot be satisfied without either Δ_r not tending to zero at infinity or a source on the lefthand-side of 5.70 i.e. Δ_r and therefore $D^{(m)}$ will have a set of singularity points. What this implies about the singularity structure of A_μ or $F_{\mu\nu}$ is not apparent and needs further investigation. For $m = 1$ $F_{\mu\nu}$ is non singular and hence also for $m = 2$ as we have shown that this is a repitition of the former. Reference 6 claim, from algebraic geometric considerations, that the A_μ can always, by a suitable choice of gauge, be written as rational functions with poles lying on a hypersurface labelled by a polynomial of degree equal to the instanton number in CP_3. Unfortunately the only case they consider in detail is the $m = 2$ ansatz for SU(1,1). Hence the exact nature of the singularity structure for non-'t Hooft-like solutions remains an open problem, and the above construction while in principle giving all solutions in practice is so tedious, involving the inversion of an mxm matrix, as to be of little help in this problem. Perhaps the alternative method of solution of the theory[21][22] reducing it to a non-linear algebraic problem as opposed to a non-linear differential one will cast light on the structure of singularities, though again the only solutions that have been found so far are the 't Hooft ones.

CHAPTER SIX

CONCLUSION

After their initial introduction[1][2] gauge theories suffered a long period of neglect by theoretical physicists partly because of the difficulty of their non-linearity and partly due to the fact that because of this property it was difficult to conceive of a quantization scheme. With the introduction of the path intergral method the latter problem showed signs of eventually being solved and the former was simplified by the new emphasis on self-dual solutions in Euclidean space.

Gauge theories also recovered popularity with the increasing belief in the dogma that quantum chromodynamics would explain everything and it was fortunate that this upsurge in interest among theoretical physicists coincided with a burst of activity on the part of such pure mathematicians as Atiyah, Ward, Drinfeld and Manin which helped to solve, at least for some of the simpler types of theories, the Yang-Mills equations. Using the methods of algebraic geometry and vector bundle theory a difficult, second order, non-linear set of differential equations were reduced to a solvable set of first order equations for the case of SU(2) providing the set of self-dual solutions presented in chapter five.

Whether, via the similarities between the theories exhibited in chapter four, such techniques can be applied to general SU(n) self-dual theories to produce a complete solution to that problem is an open question :

as explained in chapter four the difficulty seems to lie in the fact that the most important algebraic invariance $P \longrightarrow (P^{-1})^T$ is not easily written in terms of D and E matrices in an R gauge so being difficult to combine with the Bäcklund transformations to produce an inductive type proof of an infinite sequence of ansätze as given in chapter five for SU(2). Perhaps a more fruitful approach would be to use the SU(2) embedding theorems of Bernard et al.[13] and then use the various transformations presented in this thesis to generate the required parameters; certainly more work needs to be done both on this aspect of the theory and on the singularity problem for the SU(2) solution.

Later papers[21][22] have used similar algebraic geometric techniques to reduce the self-dual problem from a differential one to a linear algebra problem but with the matrices satisfying non-linear constraints. So far the imposition of these constraints have proved intractable and only the 't Hooft type solutions have been cast in this mould though using this approach it has proved possible to construct a Green's function for the theory.[22]

Recent work by Witten[24] suggests that geometric language may be used to cast the complete Yang-Mills theory, not just the self-dual problem, into a question of the construction of fibre bundles over $CP_3 \times CP_3$ though the algebra now has to be extended to a supersymmetric one. This raises such problems as what is meant by a graded Lie group and the principal fibre bundle constructed therefrom, not to mention the meaning of a connection on such an object together with its resulting curvature. While ad hoc local methods can be used to point the way to a future rigorous theory[25] it is not immediately clear, to the author at least, how one would set up such an extension of classical fibre bundle theory to handle such graded Lie algebras.

In conclusion while many intriguing and important questions remain to be answered the fruitful co-operation of theoretical physics and pure mathematics has so far cast new light on gauge theories enabling their complete solution in admittedly restricted circumstances and providing hope that what was once thought to be an intractable problem may in future be seen to be one that, though hard, is capable of solution.

APPENDIX

Euclidean, four dimensional space-time is given by (x_0, x_1, x_2, x_3) with metric tensor $\delta_{\mu\nu}$. The coordinate system $(y, \bar{y}, z, \bar{z})$ is also used with

$$\sqrt{2}\, y = x_0 - ix_3 \qquad \sqrt{2}\, \bar{y} = x_0 + ix_3 \tag{A.1a}$$

$$\sqrt{2}\, z = x_2 - ix_1 \qquad \sqrt{2}\, \bar{z} = x_2 + ix_1 \tag{A.1b}$$

when the metric tensor is

$$g_{y\bar{y}} = g_{\bar{y}y} = g_{z\bar{z}} = g_{\bar{z}z} = 1$$
$$\text{all other components zero} \tag{A.2}$$

In all chapters except three the curvature tensor $F_{\mu\nu}$ is defined from the potentials A_μ by

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} - [A_\mu, A_\nu] \tag{A.3}$$

with gauge transformations

$$A_\mu \longrightarrow g^{-1} A_\mu g + g^{-1} \frac{\partial g}{\partial x^\mu} \tag{A.4}$$

following Yang's conventions[5].

In chapter three we follow Witten's convention[14] as the work is based on his.

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} + [A_\mu, A_\nu] \qquad \text{(A.5)}$$

with gauge transformations being

$$A_\mu \longrightarrow g A_\mu g^{-1} + g \frac{\partial g^{-1}}{\partial x^\mu} \qquad \text{(A.6)}$$

The antiself-dual tensors $\eta^a_{\mu\sigma}$ can be defined in the (x_0, x_1, x_2, x_3) reference frame by

$$\eta^a_{\mu\sigma} = \varepsilon_{0a\mu\sigma} + \delta_{\mu a}\delta_{\sigma 0} - \delta_{\mu 0}\delta_{a\sigma} \qquad \text{(A.7)}$$

and satisfy the relations[23]

$$\eta^a_{\nu\sigma}\eta^a_{\mu\kappa} = g_{\mu\nu}g_{\sigma\kappa} - g_{\mu\sigma}g_{\nu\kappa} - \varepsilon_{\nu\sigma\mu\kappa} \qquad \text{(A.8a)}$$

$$\eta_{a\mu\sigma}\eta_b{}^\mu{}_\kappa = \varepsilon_{abc}\eta_{c\sigma\kappa} + \delta_{ab}g_{\kappa\sigma} \qquad \text{(A.8b)}$$

$$\varepsilon_{abc}\eta^b_{\mu\kappa}\eta^c_{\nu\sigma} = \eta_{a\mu\nu}g_{\kappa\sigma} - \eta_{a\mu\sigma}g_{\kappa\nu} + \eta_{a\kappa\sigma}g_{\mu\nu} - \eta_{a\kappa\nu}g_{\mu\sigma} \qquad \text{(A.8c)}$$

$$\eta^a_{\mu\delta}\varepsilon^\delta{}_{\nu\kappa\sigma} = \eta^a_{\nu\kappa}g_{\mu\sigma} + \eta^a_{\sigma\nu}g_{\mu\kappa} + \eta^a{}_{\kappa\sigma}g_{\mu\nu} \qquad \text{(A.8d)}$$

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