Some problems in algebraic group theory

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SOME PROBLEMS IN ALGEBRAIC GROUP THEORY

by

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STATEMENT

All results in this thesis, except those specifically attributed to others, are the product of my own research.

Peter Bardsley

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ABSTRACT

For smooth actions of compact Lie groups on differentiable manifolds, the existence of a smooth slice transversal to each orbit gives a clear description of the local structure. In 1973, D. Luna proved the existence of a slice in the étale topology at a closed orbit, for reductive algebraic groups acting on an affine variety, over an algebraically closed field of characteristic zero. This thesis explores the extent to which Luna's methods work over an arbitrary field. Conditions for the quotient of a morphism to be étale are given, necessary and sufficient conditions are given for the existence of a slice on a smooth affine scheme, and a new proof is given of the isomorphism of the unipotent variety of a split connected, simple, semisimple algebraic group with the nilpotent variety of its Lie algebra.

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LIST OF RESULTS

INTRODUCTION

If G is a compact Lie group, H a closed subgroup, and V a differentiable manifold on which H acts smoothly on the right, let $V *_H G = (V \times G)/H$, where H acts on $V \times G$ by $(v, g)\cdot h = (v\cdot h, h^{-1}g)$. Then $V *_H G$ is a differentiable manifold with a smooth right G action, which in the theory of group actions on manifolds plays the role of a non-linear analogue of induced representations. The geometric structure of $V *_H G$ is fairly simple. It is a H bundle over the base $H\backslash G$ with fiber V, and is associated to the principal bundle $G \to H\backslash G$.

Now if G acts smoothly on a differentiable manifold X and T is an orbit in X, then T is a smooth closed submanifold isomorphic to G/G_x, where G_x is the stabiliser of a point $x \in T$. It is well known [Palais 1960] that there is an invariant open neighbourhood U of T in X which is G-isomorphic to $S_x *_{G_x} G$, where S_x is a G_x stable smooth submanifold of X which cuts T transversally at x. Such a transversal S_x is called a slice at x. This effectively solves the local structure problem for smooth G actions on manifolds.

If now G is a reductive algebraic group over an algebraically closed field k acting on an affine variety X, or more generally if G is a reductive group scheme and X an affine algebraic group scheme over an arbitrary field k, then the question arises whether a similar theorem holds. The orbits on X are no longer necessarily closed, but the question arises only for closed orbits, and for technical reasons a slice cannot exist unless the orbit is separable. Results of Serre [Serre 58] show that slices may not exist in the Zariski topology (by which I mean there may be

no Zariski open neighbourhood of T of the form $S_x *_{G_x} G$), because Zariski topology is too coarse, but suggest that slices may exist in the etale topology. In [Luna 73, Luna a], D. Luna showed that a slice always exists in the étale topology if the base field is algebraically closed and of characteristic zero. The main result of this thesis is a necessary and sufficient condition for the existence of an étale slice over an arbitrary field (Proposition 6.4.1).

The proof basically follows the method of [Luna 73] and uses ideas from both papers, but allowing k to be an arbitrary field introduces a number of technical complications. One of these is that in prime characteristic reductive groups need no longer be linearly reductive. That is, not all linear representations are semisimple. Another is that over an arbitrary field the geometry is no longer adequately treated by the method of Serre varieties and it is necessary to work with schemes. Only algebraic schemes over a field are used, and these are quite well behaved. Chapter 1 contains a summary of the scheme theoretic framework I will need. An alternative would have been to use the rationality methods of [Borel], but this would have been awkward in places.

In Chapter 2, invariants are defined and are used in Chapter 3 to construct quotients for group actions on affine schemes. The correct definition of invariants for actions of group schemes over an arbitrary field (following [Seshadri]) is complicated and not immediately obvious. Chapter 3 also collects together definitions and simple properties of orbits, quasi orbits, and equivariant morphisms. Some of the results in §3.1 can be found scattered in various forms in [G.-D.], though with different proofs. Chapter 4 contains the heart of the proof, a criterion which may be used to show that the quotient of a morphism is étale (Proposition 4.1.1). Chapter 5 contains material on Grothendieck topologies, fiber bundles, and torseurs, which are principal fiber bundles

carrying a group action. Chapter 6 gives the construction of the etale slice, and Chapter 7 contains an application of these methods to give a new proof of the isomorphism of the unipotent and nilpotent varieties of a connected, simply connected semisimple algebraic group. There are two appendices. The first contains a proof of the representation theorem mentioned in Chapter 1 (which is implicit in [G.-D.] but not written down anywhere), and some formal properties of faithfully flat descent. The second contains a summary of some of the more important definitions and theorems about schemes which are used throughout the thesis.

One technical matter is worth further comment. In characteristic zero the operation of taking a quotient commutes with products, in the sense that $(X \times_Y Z)/G \cong (X/G) \times_Y Z$, provided that Y and Z have a trivial G action. This is proved easily using a "Reynolds operator" [Mumford]. But in prime characteristic G may not be linearly reductive and a Reynolds operator is not available. Special arguments are needed each time it is necessary to commute quotients and products (Proposition 3.2.1, Corollary 5.2.1).

The following conventions are in force throughout. All rings and algebras are commutative with 1 and all homomorphisms preserve 1. All schemes are separated and are over a fixed base ring k, which is almost always a field. If k is a field, $\overline{k}$ is an algebraic closure and $p : \mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$ is the canonical projection. The definition of a reductive group scheme is slightly wider than that of [SGA3, Demazure]: I allow a reductive group not to be connected. Standard properties of algebraic groups and Serre varieties [Borel, Humphreys, Dieudonné 74] are assumed without comment, whereas results peculiar to schemes are spelled out more explicitly. Cartesian squares or "pull-backs" are as defined in EGA0, §1.2.4. The dual concept is a co-cartesian square or "push-out". I will sometimes use the category theorist's convention of writing X for the

identity morphism of the object X when the context is clear. Proposition a.b.c means Proposition c in Chapter a, Section b; in Chapter a it is referred to as Proposition b.c, and in Section c as Proposition c.

References to the bibliography will be made by quoting the author, and if necessary the year of publication, thus [Luna 1973]. Unpublished articles will be identified by a letter, thus [Luna [a]]. Commonly used references will be abbreviated as follows (for details, see the bibliography).

[E.G.A.] Dieudonné and Grothendieck, *Eléments de Géométrie Algebrique.*

[S.G.A.] Grothendieck *et al*, *Séminaire de Géométrie Algebrique.*

[G.-D.] Gabriel and Demazure, *Groupes Algebriques.*

CHAPTER 1

1. Serre varieties

Serre varieties are as defined in [Dieudonné 74]. That is, an affine Serre variety over an algebraically closed field k is an algebraic set $X \subset k^n$, for some n, together with the coordinate ring $k[X]$ of polynomial functions on X, or equivalently it is a reduced k-algebra of finite type A together with its maximal ideal space $\mathrm{Max}(A)$. A variety is a neotherian ringed space which can be covered by open subsets which are affine varieties. All varieties are assumed separated, that is the diagonal morphism $X \to X \times X$ is a closed embedding. The category of affine varieties is contravariantly isomorphic to the category of reduced k-algebras of finite type. I will assume without comment properties of varieties in [Dieudonné 74, Humphreys 75, Chapter 1].

2. Schemes

Schemes are as defined in [EGA, Hartshorne]. All schemes are taken over a fixed base field k, and are assumed noetherian and separated. If $X = \mathrm{Spec}(A)$ is an affine scheme over k I will write $A = k[X]$ and call A the coordinate ring of X, just as for varieties. An affine scheme is algebraic if the coordinate ring is finitely generated as a k-algebra, and a scheme is algebraic if it has a finite covering by algebraic affine open subschemes. Most but not all schemes considered henceforth will be algebraic. For example, if $\bar{k}$ is the algebraic closure of k then $\mathrm{Spec}(\bar{k})$ may not be algebraic over k. The category of affine schemes over k is contravariantly isomorphic to the category of k-algebras.

If X is a scheme and $x \in X$ then the local ring $\mathcal{O}_x$ at x has a unique maximal ideal $\mathfrak{m}_x$ and $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ is called the residue field at

x . There is an evaluation map $ev_x : \mathcal{O}_x \to k(x) : f \mapsto f(x)$. If X is affine then m_x is the localisation of p_x , the prime at x , and there is an evaluation map $k[X] \to k(x)$.

$k(x)$ is an extension field of k . If $k(x) = k$ then x is a rational point and if $k(x)$ is a finite separable extension of k then x is a separable point. The set of rational points is written X_{rat} , the set of separable points X_{sep} , and the set of closed points X_0 . If X is affine then X_0 is just the set of maximal ideals in $k[X]$. Separable and rational points are closed so $X_{rat} \subset X_{sep} \subset X_0$. X_{rat}, X_{sep} and X_0 carry topologies induced from X .

If the base field k is algebraically closed and X is a reduced algebraic scheme, then X_0 is dense in X and every open set in X_0 is the restriction of a unique open set in X (see below). So there is a structure sheaf $\mathcal{O}_{X_0}$ induced on X_0 from the structure sheaf $\mathcal{O}_X$ on X , and X_0 with this structure sheaf is a Serre variety. Conversely every Serre variety over k can be embedded as the set of closed points in a unique reduced algebraic scheme. If X is an affine Serre variety then the scheme is $\mathrm{Spec}(k[X])$. Accordingly I will call reduced algebraic schemes over an algebraically closed field "Serre schemes". The category of "Serre schemes" is clearly isomorphic to the category of Serre varieties.

If X is a scheme then $|X|$ will be the underlying set (or sometimes the underlying topological space).

3. Jacobson schemes

These are a class of schemes with properties intermediate between those of "Serre" schemes and the most general schemes.

If X is a topological space then a subset $P \subset X$ is locally closed if it is an intersection of an open and a closed set in X, or equivalently if P is open in $\overline{P}$. P is constructible if it is a finite union of locally closed sets. Constructible sets are so called because constructible subsets of an algebraic set are those which are described by a finite number of polynomial equalities and inequalities. The collection of constructible subsets of X is written $\mathrm{Const}(X)$. A subset $P \subset X$ is "very dense" if every locally closed set in X contains a point of P. X is a Jacobson space if the closed points of X are very dense.

A Jacobson scheme is a noetherian scheme X such that X_0 is very dense in X. That is, X as a topological space is a Jacobson space. If X is affine then X is a Jacobson scheme iff $k[X]$ is a Jacobson ring, that is if every prime ideal is an intersection of maximal ideals. Since finitely generated algebras over a field are Jacobson rings [Bourbaki, Comm. Alg.], all algebraic schemes are Jacobson schemes.

If X is a Jacobson scheme then one can restrict attention to the set of closed points $X_0 \subset X$. If X, Y are Jacobson schemes and $\varphi : X \to Y$ is a morphism then φ maps X_0 into Y_0 (this is not true for more general schemes) and gives by restriction a map $\varphi_0 : X_0 \to Y_0$. Every subscheme of a Jacobson scheme is a Jacobson scheme. The map $\mathrm{Const}(X) \to \mathrm{Const}(X_0) : U \mapsto U \cap X_0$ is a bijection if X is a Jacobson scheme, and U is open or closed in X iff $U \cap X_0$ is open or closed in X_0. Thus constructible subsets of X are completely described by giving the closed points they contain. In particular if $p \in X$ is a non-closed point, then the set $\{\overline{p}\}$ is an irreducible closed set in X, and this is given in turn by $\{\overline{p}\} \cap X_0$, which is an irreducible closed set in X_0. p can be recovered from $\{\overline{p}\}$ as its generic point, the unique point which is

dense in $\{\overline{p}\}$. This geometric interpretation of the non closed points of X is similar to the way prime ideals are interpreted on an affine Serre variety. If $\mathcal{O}_X$ is the structure sheaf on X then $\mathcal{O}_X$ is a function defined on the open sets of X . Since the set of open sets of X is isomorphic to the set of open sets in X_0 , in a sense the sheaf $\mathcal{O}_X$ is "isomorphic" to the sheaf $\mathcal{O}_{X_0}$ on X_0 . Again,no information is lost by restricting to X_0 .

Thus in the study of Jacobson schemes one can almost always restrict attention to the behaviour of closed points. In this respect Jacobson schemes are similar to Serre varieties. They differ in that the local field $k(x)$ varies from point to point, and that the local rings may contain nilpotents. The schemes that will be used below will all be Jacobson schemes and it will rarely be necessary to consider non-closed points.

4. Base change

Let k be a ring and K be a k-algebra (commutative as always: in our case k and K will usually be fields). If X is a scheme over K then X is covered by affine open schemes of the form $\mathrm{Spec}(A)$, where A is a K algebra. But every K algebra is a k-algebra via the homomorphism $k \to K$, so X can be regarded as a scheme over k . When so regarded X will be written ${}_{(k)}X$ and called the scheme obtained from X by restriction of scalars.

If Y is a scheme over k then there is a natural morphism $Y \to \mathrm{Spec}(k)$ (if Y is affine then the comorphism is just the inclusion $k \to k[X]$ defining the k-algebra structure on $k[X]$). The scheme $Y_{(K)} = Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$ is a scheme over K and is called the scheme

obtained from Y by extension of scalars. If Y is affine then $Y_{(K)} = \mathrm{Spec}(k[Y] \otimes_k K)$, so by abuse of notation $Y_{(K)}$ is often written $Y \otimes_k K$. By a similar convention ${}_{(k)}X$ is often written as just X with an informal note that the base ring has been restricted from K to k.

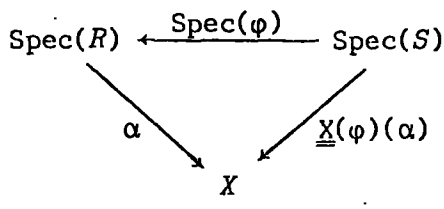
The functors $X \to {}_{(k)}X$ and $Y \to Y_{(K)}$ are a pair of adjoint functors. There is a natural isomorphism of sets $\mathrm{Mor}_k({}_{(k)}X, Y) \cong \mathrm{Mor}_K(X, Y_{(K)})$. These functors are called the base change functors associated with the homomorphism $k \to K$. They are also called the functors of ascent $(Y \to Y_{(K)})$ and descent $(X \to {}_{(k)}X)$. For further details see Appendices 1 and 2.

5. Functorial points

If X is a scheme or a Serre variety let $|X|$ be the underlying set of X. Consider for a moment Serre varieties over an algebraically closed field k. The affine variety $(\mathrm{Max}(k), k)$ consists of a single point, with the base field k attached as its local ring. If X is a Serre variety over k and $x \in X$, then there is a unique morphism $\varphi_x : \mathrm{Max}(k) \to X$ with image x. If X is affine then the comorphism of φ_x is evaluation at x. Conversely, any morphism $\mathrm{Max}(k) \to X$ has as its image a point in X. Thus, writing $X(k) = \mathrm{Mor}_k(\mathrm{Max}(k), X)$ there is an isomorphism of sets $|X| \cong X(k)$.

In the case of schemes over a base ring k (in our case a field), this is generalised as follows. Let Sch/k be the category of separated schemes over k, Alg/k the category of commutative k-algebras, S the category of sets, and $[Alg/k, S]$ the category whose elements are functors $Alg/k \to S$ and whose morphisms are natural transformations of functors. In

order that this makes sense I will assume that we are working in the set theoretic framework of [SGA ; see also G.-D.]. If X is a scheme over k define a functor $\underline{\underline{X}} \in [\mathcal{Alg}/k, S]$ as follows. If $R \in \mathcal{Alg}/k$ then $\mathrm{Spec}(R)$ is a scheme over k, so define $\underline{\underline{X}}(R) = \mathrm{Mor}_k(\mathrm{Spec}(R), X)$. The set $\underline{\underline{X}}(R)$ is called the set of points of X with coordinates in R. To distinguish such points from elements of $|X|$, elements of $\underline{\underline{X}}(R)$ will be called functorial points while elements of $|X|$ will be called set theoretic points. If $\varphi : R \to S$ is an algebra homomorphism then $\underline{\underline{X}}(\varphi) : \underline{\underline{X}}(R) \to \underline{\underline{X}}(S)$ is defined by $\underline{\underline{X}}(\varphi)(\alpha) = \alpha \circ \mathrm{Spec}(\varphi)$.



where $\mathrm{Spec}(\varphi)$ is the morphism whose comorphism is φ. $\underline{\underline{X}}(\varphi)$ is called the specialisation map associated with the homomorphism φ.

Each scheme X is now associated with a functor $\underline{\underline{X}} \in [\mathcal{Alg}/k, S]$. This can be used to define a functor $E : \mathcal{Sch}/k \to [\mathcal{Alg}/k, S]$ as follows. If X is a scheme $E(X) = \underline{\underline{X}}$. If $\varphi : X \to Y$ is a morphism of schemes $E(\varphi) = \underline{\underline{\varphi}} : \underline{\underline{X}} \to \underline{\underline{Y}}$ is the natural transformation defined by $\underline{\underline{\varphi}}_R : \underline{\underline{X}}(R) \to \underline{\underline{Y}}(R) : \alpha \mapsto \varphi \circ \alpha$.

E is important because it is a product preserving fully faithful embedding. For a proof see Appendix 1. That E is a fully faithful embedding means that the category of schemes can be identified with its image in the functor category. Functors in the image of this embedding are called representable functors. In fact some authors define a scheme as a functor on k-algebras satisfying certain axioms [G.-D.]. Given two schemes X, Y, every natural transformation $\underline{\underline{X}} \to \underline{\underline{Y}}$ comes from a unique morphism of the underlying schemes. So to define a morphism $X \to Y$ it is enough to

give for each $R \in Alg/k$ a map of sets $\varphi_R : \underline{\underline{X}}(R) \to \underline{\underline{Y}}(R)$ such that the maps φ_R are compatible with the specialisation maps of X and Y. That E preserves products means that for any schemes X, Y over a scheme Z, $(\underline{\underline{X \times_Z Y}})(R) = \underline{\underline{X}}(R) \times_{\underline{\underline{Z}}(R)} \underline{\underline{Y}}(R)$.

In future the notation $\underline{\underline{X}}(R)$ will be abbreviated to $X(R)$. It is essential to distinguish between the functorial points $X(R)$ and the set theoretic points of $|X|$. For example $\underline{\underline{X \times Y}}(R) = X(R) \times Y(R)$ but it is not true in general that $|X \times Y| = |X| \times |Y|$. For example if K is an extension field of k and $X = \mathrm{Spec}(K)$ then $|X|$ consists of a single point so $|X| \times |X|$ has just one point. But $X \times X = \mathrm{Spec}(K \otimes_k K)$ and $K \otimes_k K$ is not in general a field. This is one reason why functorial points behave better than set theoretic points.

There is one case where the set theoretic and functorial points are related. If the base ring k is a field then there is a natural isomorphism $X(k) \cong X_{\mathrm{rat}}$ sending $\varphi : \mathrm{Spec}(k) \to X$ to its image in X. This isomorphism will frequently be used to identify these two sets and to transfer a topology and a structure sheaf to $X(k)$. In particular if X is a "Serre" scheme over an algebraically closed field then $X(k) \cong X_{\mathrm{rat}} = X_0$ has a natural Serre variety structure.

6. Algebraic schemes

In this section let k be a field and $\overline{k}$ an algebraic closure of k. Let X be an algebraic scheme over k. Thus X is a Jacobson scheme and the closed points are very dense in X.

A point $x \in X$ is closed iff $k(x)$ is a finite extension of k. To see this it can be assumed that X is affine and $\mathfrak{p}_x$ is prime in $k[X]$.

If $k(x)$ is a finite extension of k, then every ring between k and $k(x)$ is a field, so $k[X]/\mathfrak{p}_x$ is a field, so $\mathfrak{p}_x$ is a maximal ideal. Thus x is closed. The converse is just the weak nullstellensatz [Atiyah-Macdonald].

Let $p : X \otimes_k \overline{k} \to X$ be the canonical projection. $X \otimes_k \overline{k}$ is algebraic over $\overline{k}$ but not necessarily over k so p is not a morphism of algebraic schemes. p is faithfully flat, since it comes by a base change from the faithfully flat morphism $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$. Thus p is a surjective open map and the topology on X is the quotient of the topology on $X \otimes_k \overline{k}$ by the equivalence relation defined by p [Dieudonné 1964]. p is also a closed map. As I have not been able to find a proof of this fact written down I include one here.

LEMMA 1. *The projection $p : X \otimes_k \overline{k} \to X$ is a closed map.*

Proof. Let V be a closed set in $X \otimes_k \overline{k}$. V is the underlying set of a reduced closed subscheme of $X \otimes_k \overline{k}$ which I will also call V [E.G.A.I, §4.6].

A field L between k and $\overline{k}$ is called a field of definition for V if there is a closed subscheme $V' \subset X \otimes_k L$ such that $V = V' \otimes_L \overline{k}$. By [E.G.A.IV, §4.9] V has a field of definition L which is a k algebra of finite type. By the nullstellensatz or directly L is a finite algebraic extension of k. Now factorise p :

$$X \otimes_k \overline{k} \xrightarrow[p_1]{} X \otimes_k L \xrightarrow[p_2]{} X .$$

Then $p_1(V) = V'$ which is closed in $X \otimes_k L$. But $p_2 = X \otimes_k \varphi$ where $\varphi : \mathrm{Spec}(L) \to \mathrm{Spec}(k)$ is the natural map. Since L is a finite extension of k, φ is a finite map. Finiteness is stable under base change, so p_2 is a finite map. Hence p_2 is a closed map and $p(V) = p_2\big(p_1(V)\big)$ is

closed. #

Thus p maps closed points of $X \otimes_k \overline{k}$ to closed points of X. The induced map $p_0 : (X \otimes_k \overline{k})_0 \to X_0$ is surjective, for if $x \in X_0$ is a closed point the fiber $p^{-1}(x)$ is a non-empty closed set, and it contains a closed point because $X \otimes_k \overline{k}$ is a Jacobson scheme. Since $(X \otimes_k \overline{k})_0 \cong (X \otimes_k \overline{k})(\overline{k}) \cong X(\overline{k})$ it can be seen that if $x \in X$ is a closed point, the number of closed points in the fiber $p^{-1}(x)$ is the number of ways of embedding $k(x)$ in $\overline{k}$ over k, which is the separable degree of $k(x)/k$, and this is always finite.

Let X be a geometrically reduced algebraic scheme. This means that $X \otimes_k \overline{k}$ is reduced, or equivalently that X is reduced, and for all the generic points x_i of X, $k(x_i)$ is a separable extension of k. Schemes of this type are used in [Borel], where they are called varieties defined over the subfield k of $\overline{k}$. $X \otimes_k \overline{k}$ is algebraic and reduced (over $\overline{k}$) so it is a "Serre" scheme and $(X \otimes_k \overline{k})_0 \cong (X \otimes_k \overline{k})(\overline{k}) \cong X(\overline{k})$ has a natural Serre variety structure. If $\varphi : X \to Y$ is a morphism of geometrically reduced algebraic schemes then $\varphi(\overline{k}) : X(\overline{k}) \to Y(\overline{k})$ is a morphism of Serre varieties. The functor

$$\left\{\begin{matrix}\text{geometrically reduced} \\ \text{algebraic schemes over } k\end{matrix}\right\} \to \left\{\begin{matrix}\text{Serre varieties} \\ \text{over } \overline{k}\end{matrix}\right\} : X \mapsto X(\overline{k})$$

is an embedding which is "faithful" but not "full". This means that not every morphism of Serre varieties $\psi : X(\overline{k}) \to Y(\overline{k})$ is of the form $\varphi(\overline{k})$ for some morphism φ of the underlying schemes (such morphisms are called in the language of Borel "defined over k"), but that if ψ is defined over k, then it comes from a unique morphism of the underlying schemes. Since not all the automorphisms of the variety $X(\overline{k})$ need be defined over k, it is possible that two non isomorphic geometrically reduced algebraic

schemes are associated with the same Serre variety. Isomorphisms defined over k partition the set of isomorphic Serre varieties into "k structures", which may be specified for example by giving the underlying scheme. There seems to be no simple geometric description of which morphisms are defined over k. For example it is possible that a morphism $\varphi : X(\overline{k}) \to Y(\overline{k})$ carries $X(l)$ into $Y(l)$ for every field l between k and $\overline{k}$ without φ being defined over k.

LEMMA 2. *If X is geometrically reduced and algebraic then X_{sep} is dense in X.*

Proof. Let k_s be the separable closure of k in $\overline{k}$ and let $p_s : X \otimes_k k_s \to X$ be the projection. Clearly X_{sep} is the image of $(X \otimes_k k_s)_{\text{rat}}$ under p_s. By [Borel, AG §13], $X(k_s)$ is dense in $X(\overline{k}) \cong X \otimes_k \overline{k}\ (\overline{k}) \cong (X \otimes_k \overline{k})_0$, so $(X \otimes_k k_s)_{\text{rat}}$ is dense in $(X \otimes_k k_s)_0$, so X_{sep} is dense in X_0. #

The following simple lemma will be used repeatedly.

LEMMA 3. *Let $\varphi : X \to Y$ be a morphism of algebraic schemes. Then $\varphi(k) : X(k) \to Y(k)$ is*

(i) surjective iff φ is surjective,

(ii) injective iff φ is a radical morphism.

Proof. *(i)* If $\varphi(\overline{k})$ is surjective then the image of $\varphi \otimes_k \overline{k}$ is a constructible set containing all the closed points of $Y \otimes_k \overline{k}$. Since $Y \otimes_k \overline{k}$ is a Jacobson scheme, $\varphi \otimes_k \overline{k}$ is surjective. But the projection $p_Y : Y \otimes_k \overline{k} \to Y$ is surjective, and $p_Y \circ \varphi \otimes_k \overline{k} = \varphi \circ p_X$, so φ is surjective. Conversely if φ is surjective then surjectivity is stable under ascent [EGAI, §3.5], so $\varphi \otimes_k \overline{k}$ is surjective. If $x \in Y \otimes_k \overline{k}$ is a closed point then the fiber $(\varphi \otimes_k \overline{k})^{-1}(x)$ is a non-empty closed set, so it

contains a closed point. Thus $\varphi(\overline{k})$ is surjective.

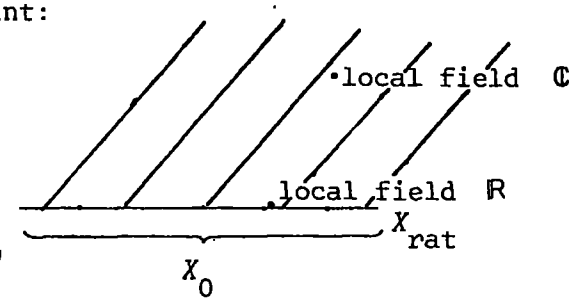
(ii) By [EGAI, §3.7] φ is radical iff the diagonal morphism $X \to X \times_Y X$ is surjective. The result then follows from part *(i)*. #

7. A simple example

To fix ideas, consider the following simple example. Let the base field be $\mathbb{R}$ and let $X = \mathrm{Spec}(\mathbb{R}[T])$ be the real affine line.

The prime ideals of $\mathbb{R}[T]$ are all maximal except for (0), so $X = X_0 \cup \{(0)\}$ where X_0 is the set of closed points and (0) is the generic point of X. Since $\mathbb{R}$ has characteristic zero every closed point is separable, and $X_{sep} = X_0$. The maximal ideals of $\mathbb{R}[T]$ fall into two classes. There are those of the form $\mathfrak{m}_\alpha = (T-\alpha)$, $\alpha \in \mathbb{R}$, and those of the form $\mathfrak{m}_{z,\overline{z}} = (T-z)(T-\overline{z})$, $z \in \mathbb{C}$, z not real. Let the point of X corresponding to the ideal $\mathfrak{m}_\alpha$ be written x_α, that corresponding to $m_{z,\overline{z}}$ be written $x_{z,\overline{z}}$. The local field at the point x_α is $\mathbb{R}$ and the point $x_{z,\overline{z}}$ it is $\mathbb{C}$, so $X_{rat} = \{x_\alpha : \alpha \in \mathbb{R}\}$. Thus X is parametrised by the upper half plane plus a point:

X = {generic point} $\cup$



Now consider $p : X \otimes_{\mathbb{R}} \mathbb{C} \to X$. $X \otimes_{\mathbb{R}} \mathbb{C}$ is the affine line $\mathrm{Spec}(\mathbb{C}[T])$ over $\mathbb{C}$. All the closed points are rational and are of the form y_z, $z \in \mathbb{C}$, which corresponds to the ideal $(T-z)$. The generic point (0) is the only non-closed point. The projection p sends y_z to x_z if z is

real, and to $x_{z,\bar{z}}$ if z is not real. The fibers of p over a rational point of X contains just one point, and the fiber over a non-rational point contains two points, parametrised by conjugate complex numbers.

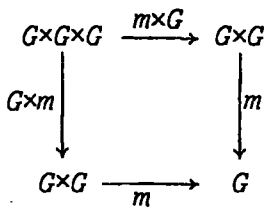
$\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, which is cyclic of order 2 generated by complex conjugation, acts naturally on $X \otimes_{\mathbb{R}} \mathbb{C}$ and X is the quotient of this action.

If S is an $\mathbb{R}$ algebra then $X(S) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}[T], S) \cong S$. In particular $X(\mathbb{R}) = \mathbb{R}$ is in one-to-one correspondence with X_{rat} and $X(\mathbb{C}) = \mathbb{C}$ corresponds with $(X \times_{\mathbb{R}} \mathbb{C})_{\mathrm{rat}}$.

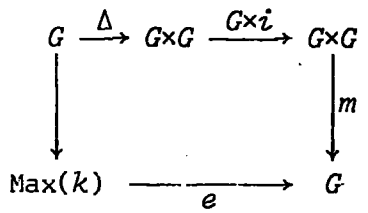
CHAPTER 2

1. Group varieties

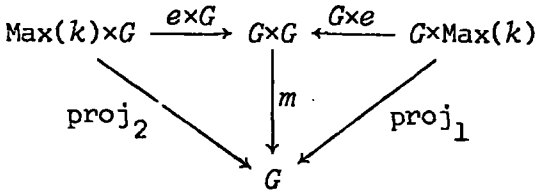
A group variety over an algebraically closed field k is a Serre variety G which is also a group and which is such that the multiplication, inversion and identity maps are variety morphisms. That is it is a variety G together with morphisms $m : G \times G \to G$, $i : G \to G$, $e : \mathrm{Max}(k) \to G$ such that the following diagrams commute. (Δ is the diagonal map.)



(1)

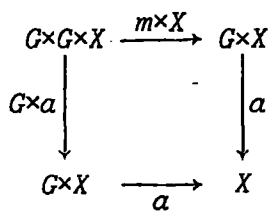


(2)

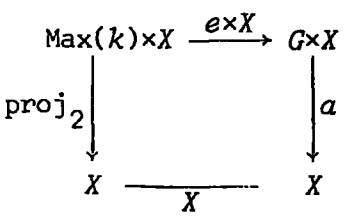


(3)

If G is a group variety and X is a variety an action of G on X is a morphism $a : G \times X \to X$ which is an action of the group $|G|$ on the set $|X|$. That is it is a morphism such that the following diagrams commute.



(4)



(5)

If G and X are affine then by the duality between affine varieties and reduced k-algebras of finite type a group structure on the variety G

is the same thing as a co-group structure, or Hopf algebra with identity and inversion structure on the coordinate ring $k[G]$. That is it is the same as giving algebra homomorphisms $m^* : k[G] \to k[G] \otimes_k k[G]$, $i^* : k[G] \to k[G]$, $e^* : k[G] \to k$, called respectively the comultiplication, co-inversion and co-identity, such that the duals of the diagrams (1) to (3) commute. Similarly an action a of G on X is the same thing as a homomorphism $a^* : k[X] \to k[G] \otimes_k k[X]$ called the co-action such that the duals of the diagrams (4) and (5) commute. This is also called a $k[G]$ co-module structure on $k[X]$.

If a group variety G acts on an affine variety X on the left then there is a right action of the group $G(k) \cong |G|$ on the algebra $k[X]$ called the action by left translation and written

$$k[X] \times |G| \to k[X] : (f, g) \to f^g .$$

It is defined by $f^g(x) = f(g \cdot x)$ for all $x \in X$. This equation is an adequate definition of the action because $k[X]$ is a ring of functions on X . An element $f \in k[X]$ is invariant if $f = f^g$ for all $g \in |G|$. The invariants form a subalgebra $k[X]^G \subset k[X]$.

2. Group schemes

A group scheme over a field k is a scheme G together with morphisms $m : G \times G \to G$, $i : G \to G$, $e : \mathrm{Spec}(k) \to G$ such that the scheme theoretic versions of the diagrams (1) to (3) commute. By this I mean that in the diagrams the varieties $\mathrm{Max}(k)$ are replaced by the schemes $\mathrm{Spec}(k)$. An action of a group scheme G on a scheme X is defined analogously. If G and X are affine, then by the duality between affine schemes and k-algebras this is the same as a co-group structure on $k[G]$ and a co-action of $k[G]$ on $k[X]$.

Since for a scheme G it may not be true that $|G \times G| \cong |G| \times |G|$, a group scheme structure does not generally define a group structure on the underlying set $|G|$. For this reason the functor $\underline{G}$ representing G is often more useful than the underlying set $|G|$.

If G is a group scheme then for each $R \in Alg/k$, $G(R)$ is a group whose multiplication, inversion and identity maps are $m(R)$, $i(R)$ and $e(R)$. If $\varphi : R \to S$ is an algebra homomorphism then the specialisation map $G(\varphi) : G(R) \to G(S)$ is a group homomorphism. Conversely, by the fully faithful embedding E, to give a group structure on a scheme G it is sufficient to give a group structure on $G(R)$ for each $R \in Alg/k$, such that the specialisation maps become group homomorphisms. Similarly, an action of G on X is the same thing as an action of $G(R)$ on $X(R)$ for each R, provided that the actions are consistent with the specialisation maps.

If G is a geometrically reduced algebraic group scheme, then $G(\overline{k})$, with the usual variety structure, is a group variety with a k structure such that the morphisms m, i, e are defined over k. This is the point of view of [Borel].

If G is a group scheme, then the functor $\underline{G}$ is a functor from k-algebras to groups. Such a functor is called a group functor. Clearly a group scheme is the same thing as a representable group functor.

As an example let μ be the affine scheme whose coordinate ring is $k[T, T^{-1}]$. This carries a group scheme structure whose comorphisms are $m^* : T \to T \times T$, $i^* : T \to T^{-1}$, $e^* : T \to 1$. If $R \in Alg/k$ then $\underline{\mu}(R) = R^{\times}$, the group of units in R. μ with this structure is a smooth one dimensional group scheme called the one-dimensional torus.

Other algebraic structures on schemes and functors, such as ring schemes and ring functors, or module schemes over a given ring scheme, can

be defined analogously. For example if R is a k-algebra, there is a ring functor $\underline{R}$ defined by $\underline{R}(S) = R \otimes_k S$ for each $S \in Alg/k$. In particular $\underline{k}$ is a ring functor with $\underline{k}(S) = S$ for each $S \in Alg/k$. If R is a k-algebra then for each $S \in Alg/k$, $\underline{R}(S)$ is a $\underline{k}(S)$ algebra, so $\underline{R}$ is a $\underline{k}$-algebra functor. If M is a k module, then there is a $\underline{k}$-module functor $\underline{M}$ defined by $\underline{M}(S) = M \otimes_k S$. It is clear that the functor

$$\{k\text{-modules}\} \to \{\underline{k}\text{-modules}\} : M \to \underline{M}$$

is a faithful embedding since M can be recovered from $\underline{M}$ as its module of "rational points": $M = \underline{M}(k)$.

These functors are not in general representable. However if M is a finite dimensional vector space over k then $\underline{M}$ is representable. Let M^* be the dual vector space and let $S(M^*)$ be the symmetric algebra on M^* over k. Then if $R \in Alg/k$,

$$\begin{aligned}
\underline{M}(R) &= M \otimes_k R \\
&\cong \mathrm{Hom}_{k\text{-mod}}(M^*, R) \\
&\cong \mathrm{Hom}_{k\text{-alg}}\big(S(M^*), R\big) \\
&\cong \mathrm{Mor}_k\big(\mathrm{Spec}(R), \mathrm{Spec}\big(S(M^*)\big)\big) \\
&= \underline{\mathrm{Spec}\big(S(M^*)\big)}(R)
\end{aligned}$$

so $\underline{M}$ is represented by the affine scheme $\mathrm{Spec}\big(S(M^*)\big)$. A functor formed like this from a finite dimensional vector space will be called a linear functor and the underlying scheme a linear scheme. In particular the functor $\underline{k}$ is representable, the underlying scheme being called the affine line $A^1_k = \mathrm{Spec}(k[T])$. If $T_1, \ldots, T_r$ is a basis for M^* then $S(M^*) = k[T_1, \ldots, T_r] = k[T_1] \otimes_k \ldots \otimes_k k[T_r]$ so

$$\mathrm{Spec}\big(S(M^*)\big) \cong A^r_k = A^1_k \times \ldots \times A^1_k .$$

If X is a linear scheme then the isomorphism of $k[X]$ with a

polynomial ring gives a natural grading on $k[X]$. For each $R \in Alg/k$, $X(R)$ is a free R module. A morphism $\varphi : X \to Y$ of linear schemes is linear if $\varphi(R)$ is R-linear for each $R \in Alg/k$.

LEMMA 1. *Let $\varphi : X \to Y$ be a morphism of linear schemes. Then φ is linear iff the comorphism $\varphi^* : k[Y] \to k[X]$ is homogeneous of degree zero.*

Proof. The linear structure on X is given by morphisms of addition and scalar multiplication, $a : X \times X \to X$ and $m : \mu \times X \to X$, where μ is the one dimensional torus group. The comorphisms $a^* : k[X] \to k[X] \otimes_k k[X]$ and $m^* : k[X] \to k[T, T^{-1}] \otimes_k k[X]$ are related to the grading on $k[X]$ as follows. Let the grading be written $k[X] = \bigoplus_n k[X]_n$. Then

$$f \in k[X]_1 \iff a^*(f) = f \otimes 1 + 1 \otimes f$$

$$\iff m^*(f) = T \otimes f .$$

Since a^* and m^* are algebra homomorphisms and $k[X]$ is the symmetric algebra on $k[X]_1$ these equations are sufficient to define m^* and a^*.

Now let $\varphi^* : k[Y] \to k[X]$ be the comorphism of φ. If $f \in k[Y]_1$, then $\varphi^*(a^*(f)) = \varphi^*(f \otimes 1 + 1 \otimes f) = \varphi^*(f) \otimes 1 + 1 \otimes \varphi^*(f)$ so $a^*(\varphi^*(f)) = \varphi^*(a^*(f))$ iff $\varphi^*(f) \in k[X]_1$. Thus φ commutes with a iff φ^* is homogeneous of degree zero. Scalar multiplication is treated similarly. #

A left action of G on a linear scheme X is a linear representation if $G(R)$ acts linearly on $X(R)$ for each R. By an argument like that in the lemma above, if G is affine, then $a : G \times X \to X$ is a linear representation iff the comorphism $a^* : k[X] \to k[G] \otimes_k k[X]$ is homogeneous of degree zero, when $k[G] \otimes_k k[X]$ is graded by giving $k[G] \otimes 1$ degree zero. If $V = X(k)$ is the vector space on which the linear scheme X is

modelled, then the homogeneous comorphism a^* restricts to give a $k[G]$ comodule structure on $k[X]_1 = V^*$, $V^* \to k[G] \times V^*$, and conversely a^* is determined by this comodule structure on V^* . A left comodule structure on V^* corresponds to a right comodule structure on V . To summarise, there is a one-to-one correspondence between left linear representations of G on the linear scheme X , and right $k[G]$ co-module structures on $V = X(k)$.

3. Left translation

If X is an affine scheme over k , then $k[X]$ is a k-algebra. The $\underline{k}$-algebra functor $\underline{\underline{k[X]}}$ defined as in §2 by $\underline{\underline{k[X]}}(S) = k[X] \otimes_k S$ will be abbreviated to $\underline{k}[X]$ and will be called the coordinate ring functor of X . $\underline{k}[X](S) = k[X] \otimes_k S$ will be abbreviated to $S[X]$. It is the coordinate ring of $X \otimes_k S$.

If a group scheme G acts on X on the left then left translation is a right action of G on the coordinate ring functor $\underline{k}[X]$. To define this action it is necessary to give for each $R \in Alg/k$ a right action of the group $G(R)$ on the R-algebra $R[X]$. Since $R[X]$ is the coordinate ring of $X \otimes_k R$ it will be enough to give a left action of $G(R)$ on $X \otimes_k R$ (as a scheme over R). There is a natural such action defined as follows. For $S \in Alg/R$, $X \otimes_k R(S) \cong X(S)$, and $G(R)$ acts on $X(S)$ via the specialisation map $G(R) \to G(S)$ and the action of $G(S)$ on $X(S)$. This action of $G(R)$ on $X \otimes_k R$ can also be described directly as follows. If $a : G \times X \to X$ is the action of G on X then $G \otimes_k R$ acts on $X \otimes_k R$ by $a \otimes R : (G \otimes R) \times_R (X \otimes R) \cong G \times_k X \otimes R \to X \otimes R$. If $g \in G(R)$ then g acts on $X \otimes R$ by the automorphism $T_g = (a \otimes R) \circ \left(g \times_R (X \otimes R)\right)$,

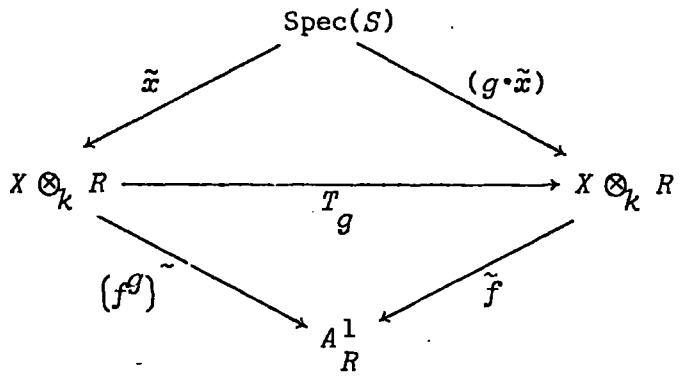
$$T_g : X \otimes R \cong \mathrm{Spec}(R) \times_R (X \otimes R) \xrightarrow{g \times_R (X \otimes R)} (G \otimes R) \times_R (X \otimes R) \xrightarrow{a \otimes R} X \otimes R .$$

It is easily checked that this action of $G(R)$ on $R[X]$ for each R is consistent with the specialisation maps, so this does describe a right action of G on the functor $\underline{\underline{k}}[X]$, and that G acts by $\underline{\underline{k}}$-algebra automorphisms.

Calling this action left translation is justified by the following alternative description. First, for each $S \in \mathcal{Alg}/R$ I will define a pairing $R[X] \times X(S) \to S$ which will be written $(f, x) \mapsto f(x)$. There is an isomorphism $R[X] \cong \mathrm{Hom}_{R\text{-alg}}(R[T], R[X]) \cong \mathrm{Mor}_R\left(X \otimes_k R, A^1_R\right)$ where $A^1_R = A^1_k \otimes_k R = \mathrm{Spec}(R[T])$. Let $f \in R[X]$ correspond to the morphism $\tilde{f} : X \otimes_k R \to A^1_R$ under this isomorphism. There is an isomorphism $X(S) \cong X \otimes_k R(S)$. Let $x \in X(S)$ correspond to $\tilde{x} \in X \otimes_k R(S)$ under this isomorphism. Then the pairing is given by

$$(f, x) \mapsto f(x) = \tilde{f} \circ \tilde{x} \in A^1_R(S) = S .$$

Then it is clear that for all $x \in X(S)$, $f \in R[X]$ and $g \in G(R)$, $f^g(x) = f(g \cdot x)$.



This equation is enough to define the action of $G(R)$ on $R[X]$. Given a morphism $\varphi : X \to Y$ of affine schemes, let $\underline{\underline{k}}(\varphi) : \underline{\underline{k}}[Y] \to \underline{\underline{k}}[X]$ be the obvious natural transformation. If G acts on X, Y on the left, and acts by left translation on $\underline{\underline{k}}[X]$, $\underline{\underline{k}}[Y]$, then φ is equivariant iff $\underline{\underline{k}}[\varphi]$ is equivariant.

LEMMA 1. *If G and X are affine and algebraic then the action of G on $\underline{k}[X]$ is rational. That is, $\underline{k}[X]$ as a G functor is the sum $\underline{k}[X] = \sum_i \underline{Y}_i$ of finite dimensional linear functors $\underline{Y}_i$ on which G acts linearly.* (*This means, that for each $R \in Alg/k$, $\underline{k}[X](R) = \sum_i \underline{Y}_i(R)$.*)

Proof. It is well known that the co-action of $k[G]$ on $k[X]$ is rational [Mumford]. So $k[X] = \sum_i V_i$ where the V_i are finite dimensional $k[G]$ stable subspaces. Let $\underline{Y}_i$ be the linear functor defined by $\underline{Y}_i(R) = V_i \otimes_k R$ and let Y_i be the linear scheme represented by $\underline{Y}_i$. Then as pointed out, §2, the left $k[G]$ comodule structure on V_i gives a right linear action of G on the scheme Y_i. It is easy to check that the morphism $j_i : Y_i \cong \underline{Y}_i \to \underline{k}[G]$ is equivariant. Since $k[G] = \sum_i V_i$,

$$\underline{k}[G](R) = R[X] = k[X] \otimes_k R = \sum_i V_i \otimes_k R = \sum_i \underline{Y}_i(R)$$

so $\underline{k}[G] = \sum_i \underline{Y}_i$. #

4. Invariants

Recall that if a group G acts on a ring R then the invariants form a subring $R^G = \{f \in R : \forall g \in G,\ f^g = f\}$. If a group scheme G acts on an affine scheme X, then G acts by left translation on $\underline{k}[X]$. Define

$$k[X]^G = \{f \in k[G] : \forall R \in Alg/k,\ f \otimes 1 \in R[X]^{G(R)}\}.$$

If $f \in k[X]$ let $\tilde{f} : X \to A^1_k$ be the morphism whose comorphism is $k[T] \to k[X] : T \mapsto f$. Let G act trivially on A^1_k. Then $\tilde{f}$ is equivariant iff $\underline{k}(f) : \underline{k}\left[A^1_k\right] \to \underline{k}[X]$ is equivariant. But for $R \in Alg/k$,

$\underline{k}(\tilde{f})(R) : R[T] \to R[X] : T \to f \otimes 1$, so $\tilde{f}$ is equivariant iff $f \in k[X]^G$.

LEMMA 1 [Seshadri]. *Let G and X be affine, let $a^* : k[X] \to k[G] \otimes_k k[X]$ be the co-action. Then there is an exact sequence of vector spaces*

$$0 \to k[X]^G \to k[X] \xrightarrow{a^*\text{-inj}_2} k[G] \otimes_k k[X] .$$

Proof. If $f \in k[X]$ then $f \in k[X]^G$ iff $\tilde{f} : X \to A^1_k$ is equivariant. But $\tilde{f}$ is equivariant iff the following diagram commutes:

$$\begin{array}{ccc} k[T] & \xrightarrow{\text{inj}_2} & k[G] \otimes k[T] \\ \downarrow{\scriptstyle (\tilde{f})^*} & & \downarrow{\scriptstyle k[G] \otimes (\tilde{f})^*} \\ k[X] & \xrightarrow{a^*} & k[G] \otimes k[X] \end{array} ;$$

that is, iff $a^*(f) = 1 \otimes f$. #

This exact sequence could of course be used to define $k[X]^G$, but it would not then be clear that $k[X]^G$ is a subring of $k[X]$.

PROPOSITION 1. *Let G be an algebraic affine group scheme over k acting on the left on an affine algebraic scheme X .*

(i) If R is a flat k-algebra, then $k[X \otimes_k R]^{G \otimes R} \cong k[X]^G \otimes_k R$.

(ii) If k is algebraically closed and G is reduced, then $k[X]^G = k[X]^{G(k)}$.

(iii) If G is geometrically reduced and if $\overline{k}$ is an algebraic closure of k , then $k[X]^G = k[X] \cap \overline{k}[X]^{G(\overline{k})}$, where $G(\overline{k})$ acts on $\overline{k}[X]$ by left translation.

Proof. In this and a following proposition I will use the facts that the image of, or fiber over, a stable subscheme, by an equivariant morphism is a stable subscheme, and that the intersection of stable subschemes is a

stable subscheme. These are proved in Chapter 3.

(i) Since R is flat, Seshadri's exact sequence stays exact when tensored with R :

$$0 \to k[X]^G \otimes_k R \to k[X] \otimes_k R = R[X] \to k[G] \otimes_k k[X] \otimes R \cong R[G] \otimes_R R[X] .$$

(ii) Clearly $k[X]^G \subset k[X]^{G(k)}$. Let $f \in k[X]^{G(k)}$. By Lemma 3.1 the action of G on $\underline{k}[X]$ is rational so there are linear schemes Y_i on which G acts linearly such that $\underline{k}[G] = \sum_i \underline{Y}_i$. $k[X] = \underline{k}[X](k) = \sum_i \underline{Y}_i(k)$ so $f = \sum f_i$ with $f_i \in \underline{Y}_i(k)$ and $f_i = 0$ except for a finite number of terms. By choosing a different decomposition, the Y_i such that $f_i \neq 0$ may be combined, so it can be assumed that $f \in \underline{Y}_i(k)$ for some i . Then there is a rational $G(k)$ stable point $x \in Y_i$ which is mapped onto f by the inclusion $j_i : Y_i \to \underline{k}[X]$. It will be enough to show that x is a G stable point of Y_i . Let $\varphi_x : \mathrm{Spec}(k) \to Y_i$ be the unique morphism with image x , and let θ be the orbit map

$$\theta : G \cong \mathrm{Spec}(k) \times G \xrightarrow{\varphi_x \times G} Y_i \times G \to Y_i .$$

The image of θ is a G-stable reduced subscheme of Y_i containing x [EGAI, §6.10]. But the only rational point in this image is x , since x is $G(k)$ fixed. k is algebraically closed, so a reduced subscheme is determined completely by its set of rational points. Thus the image of θ is $\{x\}$ and x is G stable.

(iii) Consider the following square

$$\begin{array}{ccc} \overline{k}[X]^{G \otimes \overline{k}} & \longrightarrow & \overline{k}[X] \\ \downarrow & & \downarrow \\ k[X]^G & \longrightarrow & k[X] . \end{array}$$

$$\overline{k}[X] = k[X] \otimes_k \overline{k} \cong k[X] \otimes_{k[X]^G} \left(k[X]^G \otimes_k \overline{k} \right) \cong k[X] \otimes_{k[X]^G} k[X \otimes \overline{k}]^{\alpha \otimes \overline{k}}$$

$$= k[X] \otimes_{k[X]^G} \overline{k}[X]^{\alpha \otimes \overline{k}} ,$$

so this square is cartesian. All the maps are injective so it is also co-cartesian (see Appendix 1, §2). Thus $k[X]^G = k[X] \cap \overline{k}[X]^{\alpha \otimes \overline{k}}$. Since $\overline{k}[X]^{\alpha \otimes \overline{k}} = \overline{k}[X]^{G(\overline{k})}$, the result is proved. #

5. Reductive groups

All group schemes will be assumed to be algebraic affine schemes over k.

Let G be a group scheme. G is a constant scheme if for every $R \in Alg/k$ the specialisation map $G(k) \to G(R)$ is an isomorphism. A zero dimensional constant group scheme is the same thing as a finite group. G is smooth if it is smooth as a scheme over k, and étale (= smooth and zero dimensional) if it is étale as a scheme over k. In characteristic zero every algebraic group is smooth. G is étale iff $|G|$ is a finite set of closed points and the local rings at these points are finite separable fields over k. A zero dimensional constant group is étale. If G is a zero dimensional group then G is étale iff $G \otimes_k k_s$ is constant (where k_s is the separable closure of k in $\overline{k}$), and G is constant iff every point is rational. [G.-D.I, §4.6, II, §4.2.]

G is geometrically connected iff it is connected (since it is algebraic and contains a rational point [G.-D.I, §4.6]), and this happens iff $G \otimes_k \overline{k}$ is connected. Let G^0 be the connected component of the identity in G. Then G^0 is a connected open and closed algebraic normal subgroup scheme in G and $\dim G = \dim G^0$. If H is another group scheme

$(G \times H)^0 \cong G^0 \times H^0$, and $(G \otimes_k \overline{k})^0 \cong G^0 \otimes_k \overline{k}$ [G.-D.II, §5.1].

If G is smooth, then there exists an étale group scheme $\pi_0(G)$ and a surjective $\pi : G \to \pi_0(G)$ such that the sequence

$$1 \to G^0 \to G \to \pi_0(G) \to 1$$

is exact, in the sense that $G^0 \to G$ is a closed embedding, and $\pi_0(G)$ is the quotient of G by G^0 in the category of group schemes: if $\varphi : G \to H$ is a group scheme morphism such that $\varphi(G^0)$ is trivial then φ factorises uniquely through π . The fibers of π are the connected components of G . $\pi_0(G \times H) \cong \pi_0(G) \times \pi_0(H)$ and $\pi_0(G \otimes \overline{k}) \cong \pi_0(G) \otimes \overline{k}$. G is connected iff $G = G^0$ iff $\pi_0(G) = \mathrm{Spec}(k)$ [G.-D.I, §4.6, II, §5.2, III, §3.7].

I wish to use a slightly wider definition of reductive group than usual. I will define a reductive group scheme to be a smooth affine group scheme G such that $G^0(\overline{k})$ is a reductive group in the sense of [Borel, Humphreys]. The usual definition requires that G also be connected [SGAIII, Demazure]. G is reductive iff $G \otimes_k \overline{k}$ is reductive.

6. Some examples [G.-D.II, 5.2, SGAIII, Demazure]

(i) If G is an étale group scheme, then the identity is a rational point, so $G^0 \cong \mathrm{Spec}(k)$, so G is reductive by my definition.

(ii) The one dimensional torus μ defined in §2 is reductive.

(iii) Let $n \in \mathbb{N}^+$ and let μ_d be the subgroup scheme of μ defined by $\mu_d(R) = \{x \in R : x^d = 1\}$ for $R \in Alg/k$. μ_d is an affine group

scheme with coordinate ring $k[T]/(T^d-1)$. If k has characteristic zero, then μ_d is étale, while if k has characteristic p , μ_d is étale iff $p\nmid d$.

(iv) Let K be a finite extension field of k . Let X be the ring scheme defined by $X(R) = K \otimes_k R$. Forgetting the ring structure, $X \cong A_k^n$ where $n = [K : k]$. Define a group functor $\underline{G}_K$ by $\underline{G}_K(R) = \mathrm{Aut}_{R\text{-alg}}(K \otimes_k R)$. Then $\underline{G}_K$ is representable by a zero dimensional affine group scheme G_K which acts by ring automorphisms on X . G_K is étale iff K is separable over k . G_K is not constant if $[K : k] > 2$.

(v) The general linear group GL_n defined by $\mathrm{GL}_n(R) = \mathrm{GL}(n, R)$ is reductive.

(vi) For each type of Dynkin diagram there exist connected reductive groups of this type. These are constructed in [SGAII, Demazure].

CHAPTER 3

1. Images and fibers of G-morphisms

In this section let G be an affine algebraic group scheme acting on algebraic schemes X, Y. Let $\varphi : X \to Y$ be an equivariant morphisms. Y but not X will be assumed to be affine.

Recall that the closed set $\overline{\varphi(X)} \subset Y$ carries a unique closed subscheme of Y, which will also be written $\overline{\varphi(X)}$, called the scheme theoretic closure of the image of φ, characterised by the fact that if $f : X \to Y$ is any morphism such that the set $f(X)$ lies in the set $\overline{\varphi(X)}$, then f factorises uniquely via the inclusion morphism $\overline{\varphi(X)} \to Y$. If both X and Y are affine, then $\overline{\varphi(X)}$ is the subscheme defined by the ideal $\ker(\varphi^*) \subset k[Y]$, where φ^* is the comorphism [EGAI, §6.10]. If the set $\varphi(X)$ is locally closed, then $\varphi(X)$ is an open subset of $\overline{\varphi(X)}$, so it carries a unique open subscheme of $\overline{\varphi(X)}$. If φ has a locally closed image then $\varphi(X)$ will always be taken with this subscheme structure. If φ has a locally closed image then φ factorises via the inclusion morphism $\varphi(X) \to Y$. If X is reduced so is $\overline{\varphi(X)}$ and, if defined, so is $\varphi(X)$.

PROPOSITION 1. *Let G be an affine algebraic group scheme acting on algebraic schemes X, Y, let $\varphi : X \to Y$ be equivariant, and assume Y affine. Then*

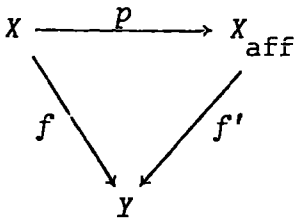
(i) $\overline{\varphi(X)}$ is a G-stable closed subscheme of Y, and

(ii) if φ has a locally closed image then $\varphi(X)$ is a G-stable subscheme of Y.

Proof. First two lemmas.

LEMMA 1. *Let G, $\varphi : X \to Y$ be as above. Let $X_{\text{aff}} = \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ be the "affinisation" of X. There is a canonical morphism $p : X \to X_{\text{aff}}$ and every morphism f from X to an affine scheme Y factorises uniquely*

via p .



Then X_{aff} *is a* G *scheme,* p *is equivariant, if* f *is equivariant so is* f' *, and* $\overline{f(X)} = \overline{f'(X_{\text{aff}})}$.

Proof. This follows easily from the universal characterisation of X_{aff} . Firstly $X \mapsto X_{\text{aff}}$ is clearly a functor from schemes to affine schemes. G is affine so $(G \times X)_{\text{aff}} \cong G \times (X_{\text{aff}})$. If $a : G \times X \to X$ is the action of G on X then $a_{\text{aff}} : G \times X_{\text{aff}} \to X_{\text{aff}}$ is the action of G on X_{aff} (that it is an action comes from the fact that $X \mapsto X_{\text{aff}}$ is a functor). The rest of the proof is similar. #

LEMMA 2. *Let* Y *be an affine* G *scheme,* V *a closed subscheme defined by the ideal* $\mathfrak{a} \subset k[Y]$. *Then* V *is* G *stable iff* $\mathfrak{a}$ *is stable under the co-action of* $k[G]$ *on* $k[X]$.

Proof. By the duality between affine schemes and k-algebras V is G stable iff there is a co-action $\alpha_V : k[V] \to k[G] \otimes_k k[V]$ compatible with the co-action of $k[G]$ on $k[Y]$. Since the tensoring is over a field both lines in the following diagram are exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a} & \longrightarrow & k[Y] & \longrightarrow & k[V] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_{\mathfrak{a}}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_V} & & \\ 0 & \to & k[G] \otimes \mathfrak{a} & \to & k[G] \otimes k[Y] & \to & k[G] \otimes k[V] & \to & 0 \end{array}$$

so such a co-action α_V exists iff $\mathfrak{a}$ is $k[G]$ stable. (The maps $\alpha_{\mathfrak{a}}$ and α_V are automatically co-actions if they exist because they are sub or quotient objects of the co-action α .) #

Proof of *(i)*. By Lemma 1, X can be assumed affine. Since $\overline{\varphi(X)}$ is the closed subscheme defined by the ideal $\ker(\varphi^*) \subset k[Y]$, by Lemma 2 it is enough to show that $\ker(\varphi^*)$ is $k[G]$ stable. From the exactness of the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker(\varphi^*) & \longrightarrow & k[Y] & \xrightarrow{\varphi^*} & k[X] \\ & & \vdots & & \downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle \alpha_X} \\ 0 & \to & k[G] \otimes \ker(\varphi^*) & \to & k[G] \otimes k[Y] & \xrightarrow{k[G]\otimes\varphi^*} & k[G] \otimes k[X] \end{array}$$

it follows that $\alpha_Y(\ker(\varphi^*)) \subset k[G] \otimes \ker(\varphi^*)$ so $\ker \varphi^*$ is $k[G]$ stable. #

Proof of *(ii)*. Let $U = \varphi(X)$. By *(i)* it can be assumed that U is a dense open subscheme of Y. It is to be shown that for all $R \in Alg/k$, $U(R)$ is a $G(R)$ stable subset of $Y(R)$. φ factorises $X \xrightarrow{\alpha} U \xrightarrow{\beta} Y$, where α is surjective and β is an open embedding.

$G(R)$ acts on $X \otimes R$, $U \otimes R$, $Y \otimes R$ in the usual way: if $S \in Alg/R$ then $G(R)$ acts on $X \otimes R(S) \cong X(S)$ via the specialisation map $G(R) \to G(S)$ and the action of $G(S)$ on $X(S)$, and similarly for $U \otimes R$ and $Y \otimes R$. By the properties of ascent (see Appendix 2), $\alpha \otimes R$ is surjective and $\beta \otimes R$ is an open embedding, so $U \otimes R$ is an open subscheme of $Y \otimes R$ and is the image of $\varphi \otimes R$. $\varphi \otimes R$ is $G(R)$ equivariant so the set underlying $U \otimes R$ is $G(R)$ stable. There is a unique open subscheme carried by each open set in $Y \otimes R$ so the subscheme $U \otimes R$ is $G(R)$ stable. In particular $U \otimes R(R) \cong U(R)$ is $G(R)$ stable, as was to be shown. This concludes the proof of Proposition 1. #

LEMMA 3. *Let G be an affine algebraic group scheme, acting on an algebraic scheme X. If G is smooth then X_{red} is a closed G stable subscheme of X.*

Proof. Since G is smooth and X_{red} is reduced, $G \times (X_{red})$ is

reduced [G.-D.I, §4.6] so $(G \times X)_{red} = G \times X_{red}$. The proof is then similar to the proof of Lemma 1. #

PROPOSITION 2. *Let G be an affine algebraic group scheme acting on algebraic schemes X, Y, and let $\varphi : X \to Y$ be equivariant. Let Z be a stable subscheme of Y and let $u : Z \to Y$ be the embedding. Then $u \times_Y X$ is an embedding, and $X_Z = Z \times_Y X$ is a G-stable subscheme of X.*

Proof. It is enough to show that for each $R \in Alg/k$, $X_Z(R)$ is a $G(R)$ stable subset of $X(R)$. Since $X_Z(R) = (X \times_Y Z)(R) = X(R) \times_{Y(R)} Z(R)$ this is clear. #

COROLLARY. *If Z_1, Z_2 are stable subschemes of X then so is $Z_1 \cap Z_2$, for by definition $Z_1 \cap Z_2$ is the product $Z_1 \times_X Z_2$.* #

PROPOSITION 3. *Let G be an affine algebraic group scheme acting on an affine algebraic scheme X over k. Let $\overline{k}$ be an algebraic closure of k, and let $p_X : X \otimes k \to X$ be the projection. G and $G \otimes \overline{k}$ act compatibly on X and $X \otimes \overline{k}$. Then*

(i) if V is a G stable subscheme of X, then the scheme theoretic fiber $p_X^{-1}(V)$ is $G \otimes \overline{k}$ stable;

(ii) if V is a $G \otimes \overline{k}$ stable subscheme of $X \otimes \overline{k}$ and the set $p_X(V)$ is locally closed, then the scheme theoretic image $p_X(V)$ is G stable.

Proof. *(i)* Let $j : V \to X$ be the embedding. Then $p_X^{-1}(V) = V \times_X (X \otimes \overline{k}) \cong V \otimes \overline{k}$ and the embedding $p_X^{-1}(V) \to X \otimes \overline{k}$ is $j \otimes \overline{k}$ so *(i)* is immediate.

(ii) G acts naturally on $X \otimes \overline{k}$ as a scheme over k. For this action p_X is G equivariant and $G \otimes \overline{k}$ stable subschemes are G stable,

so the result follows by Proposition 1. #

PROPOSITION 4. *Let G, X be as in Proposition* 3, *and assume that G is smooth. Let V be a reduced closed subscheme of X. Then V is G stable iff $V(\overline{k})$ is a $G(\overline{k})$ stable subset of $X(\overline{k})$.*

Proof. If V is G stable then $V(\overline{k})$ is $G(\overline{k})$ stable by definition. So assume that $V(\overline{k})$ is $G(\overline{k})$ stable. Let $\varphi : G \times V \to G \times X \to X$ be the natural morphism, and let Z be the closed scheme theoretic image of φ. Since V is reduced and G is smooth, $G \times V$ is reduced [G.-D.I, §4.6.3] and consequently Z is reduced. V is a closed subscheme of Z. Since both V and Z are reduced, and there is a unique reduced subscheme supported by each locally closed set in X, it is enough to show that the underlying sets of Z and V are the same.

Let $\psi : G \times V \to Z$ be the factorisation of $\varphi : G \times V \to X$ through Z. ψ is dominant, so $\psi \otimes \overline{k}$ is dominant, so $G(\overline{k})V(\overline{k})$ is dense in $Z(\overline{k})$. By assumption $G(\overline{k})V(\overline{k}) = V(\overline{k})$, so $V(\overline{k})$ is dense in $Z(\overline{k})$, which is dense in $Z \otimes \overline{k}$, so the embedding $V \otimes \overline{k} \to Z \otimes \overline{k}$ is dominant. By faithfully flat descent $V \to Z$ is dominant. V is closed in Z so both V and Z have the same underlying set. #

If V is a subscheme of X then the set theoretic boundary of V is a closed set which carries a unique closed reduced subscheme. This scheme theoretic boundary will be written ∂V. By definition, it is reduced.

PROPOSITION 5. *Let G, X be as in Proposition* 3, *and assume that G is smooth. Let V be a G stable subscheme of X. Then ∂V is G stable.*

Proof. Let $p : X \otimes \overline{k} \to X$ be the projection and let $W = p^{-1}(V)$ be the fiber over V. By Proposition 3, W is a $G \otimes \overline{k}$ stable subscheme of $X \otimes \overline{k}$. Now $|p^{-1}(\overline{V})| = \overline{|p^{-1}(V)|}$, because p is flat [Dieudonné 1964, III, Proposition 8], and $|\partial V| = |\overline{V}| \setminus |V|$ because V is locally closed, so

$|p(\partial W)| = |\partial p(W)| = |\partial V|$. ∂V and $p(\partial W)$ are both reduced so $\partial V = p(\partial W)$. Then by Proposition 3 it is sufficient to show that ∂W is $G \otimes \overline{k}$ stable.

Thus in proving Proposition 5 it may be assumed that k is algebraically closed. V is stable so by Proposition 1, $\overline{V}$, which is the scheme theoretic closed image of the embedding $V \to X$, is stable. Thus $\partial V(\overline{k}) = (\overline{V} \backslash V) \cap X(\overline{k}) = \overline{V}(\overline{k}) \backslash V(\overline{k})$ is $G(\overline{k})$ stable. By Proposition 4, ∂V is stable. #

2. Quotients and orbits

In this section let G be a reductive group scheme acting on an affine algebraic scheme X . Define $X/G = \mathrm{Spec}\left(k[X]^G\right)$ and let $\pi_X : X \to X/G$ be the morphism whose co-morphism is the inclusion $k[X]^G \to k[X]$.

PROPOSITION 1. *Let G be a reductive group scheme acting on an affine algebraic scheme X . Then*

(i) X/G is an algebraic scheme,

(ii) X/G is a categorical quotient of X by G in the category of schemes over k ,

(iii) π_X is surjective,

(iv) if V is a stable closed subscheme in X then $\pi_X(V)$ is closed,

(v) if V_1, V_2 are disjoint stable closed subschemes of X , then there exists an invariant F such that $F = 1$ on V_1 and $F = 0$ on V_2 . Hence $\pi_X(V_1) \cap \pi_X(V_2) = \emptyset$.

Proof. This has been proved in [Mumford] over a field k of characteristic zero. For algebraically closed fields of prime characteristic, a similar proof works for reductive algebraic groups acting on affine Serre

varieties. This uses Haboush's theorem [Haboush], that reductive groups are semi-reductive, and Nagata's proof of *(i)* for semi-reductive groups [Nagata]. The proposition has been proved by Seshadri [Seshadri], for connected reductive group schemes over an arbitrary field, and in fact over a very general class of base rings.

To obtain the result for not necessarily connected reductive group schemes over an arbitrary field it is enough to make only small changes in one of these proofs. The method I have chosen is to assume the result, for reductive groups acting on Serre varieties over an algebraically closed field, and to show that it then follows for schemes over an arbitrary field.

(i) X/G is algebraic. By Proposition 2.4.1, $k[X \otimes_k \overline{k}]^{G \otimes \overline{k}} = k[X]^G \otimes_k \overline{k}$ so by faithfully flat descent it can be assumed that k is algebraically closed. By the same proposition, if k is algebraically closed, $k[X \otimes \overline{k}]^{G \otimes \overline{k}} = \overline{k}[X]^{G(\overline{k})}$, which is a $\overline{k}$ algebra of finite type by assumption.

(iii) π_X is surjective. By Proposition 2.4.1, $\pi_{X \otimes \overline{k}} = \pi_X \otimes \overline{k}$ so $\pi_X(\overline{k}) = \pi_X \otimes \overline{k}(\overline{k}) = \pi_{X \otimes \overline{k}}(\overline{k})$. By assumption this is surjective so by Lemma 6.3, π_X is surjective.

(iv) If V is closed and stable then $\pi_X(V)$ is closed. Consider the following square:

$$\begin{array}{ccc} X & \xleftarrow{\;p_X\;} & X \otimes \overline{k} \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_{X \otimes \overline{k}} \cong \pi_X \otimes \overline{k}} \\ X/G & \xleftarrow[\;p_{X/G}\;]{} & X \otimes \overline{k}/G \otimes \overline{k} \cong X/G \otimes \overline{k} \end{array}$$

If $V \subset X$ is closed and G stable, then $p_X^{-1}(V)$ is closed and $G \otimes \overline{k}$

stable (Proposition 1.3). By assumption, $\pi_{X\otimes\overline{k}}\left(p_X^{-1}(V)\right)$ is closed, and $p_{X/G}$ is a closed map (Lemma 1.6.1) so $\pi_X(V)$ is closed.

(v) $\pi(V_1) \cap \pi(V_2) = \emptyset$. First note that if $\mathfrak{a}, \mathfrak{b}$ are ideals in a noetherian ring A , then $\mathfrak{a} + \mathfrak{b} = 1$ iff $r(\mathfrak{a}) + r(\mathfrak{b}) = 1$, where $r(\mathfrak{a})$ is the radical of $\mathfrak{a}$. If $\mathfrak{a} + \mathfrak{b} = 1$ then the conclusion is clear. To show the converse note that for some integer n , $r(\mathfrak{a})^n \subset \mathfrak{a}$ and $r(\mathfrak{b})^n \subset \mathfrak{b}$. Then $r(\mathfrak{a}) + r(\mathfrak{b}) = 1 \Rightarrow r(\mathfrak{a})^n + r(\mathfrak{b})^n = 1 \Rightarrow \mathfrak{a} + \mathfrak{b} = 1$.

Now let V_1 and V_2 be defined by ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $k[X]$. $\mathfrak{a}$ and $\mathfrak{b}$ are $k[G]$ stable (Lemma 1.2) and $\mathfrak{a} + \mathfrak{b} = 1$. Thus $\mathfrak{a} \otimes \overline{k} = \mathfrak{a}\overline{k}[X]$ and $\mathfrak{b} \otimes \overline{k} = \mathfrak{b}\overline{k}[X]$ are $G(\overline{k})$ stable, and $\mathfrak{a} \otimes \overline{k} + \mathfrak{b} \otimes \overline{k} = 1$. By the comment above the same applies to $r(\mathfrak{a} \otimes \overline{k})$ and $r(\mathfrak{b} \otimes \overline{k})$.

By assumption, $r(\mathfrak{a} \otimes \overline{k})^{G(\overline{k})} + r(\mathfrak{b} \otimes \overline{k})^{G(\overline{k})} = 1$ in $\overline{k}[X]^{G(\overline{k})} = \overline{k}[X]^{G\otimes\overline{k}}$. But $\left(r_{\overline{k}[X]}(\mathfrak{a} \otimes \overline{k})\right)^{G\otimes\overline{k}} = r_{\overline{k}[X]^{G\otimes\overline{k}}}\left((\mathfrak{a} \otimes \overline{k})^{G\otimes\overline{k}}\right)$, so $r\left((\mathfrak{a} \otimes \overline{k})^{G\otimes\overline{k}}\right) + r\left((\mathfrak{b} \otimes \overline{k})^{G\otimes\overline{k}}\right) = 1$ in $\overline{k}[X]^{G\otimes\overline{k}}$. Thus $(\mathfrak{a} \otimes \overline{k})^{G\otimes\overline{k}} + (\mathfrak{b} \otimes \overline{k})^{G\otimes\overline{k}} = 1$. But $(\mathfrak{a} \otimes \overline{k})^{G\otimes\overline{k}} = \mathfrak{a}^G \otimes \overline{k}$, so $\mathfrak{a}^G \otimes \overline{k} + \mathfrak{b}^G \otimes \overline{k} = 1$.

Now let $\{y_1, \ldots, y_k\}$ be a basis for $\mathfrak{a}^G$, and let $\{y_{k+1}, \ldots, y_r\}$ be a basis for $\mathfrak{b}^G$ as ideals in $k[X]^G$, and identify $k[X]^G$ with a subring of $k[X]^G \otimes \overline{k}$. Then the equation $\sum y_i x_i = 1$ has a solution with $x_i \in k[X]^G \otimes \overline{k}$. This ring is faithfully flat over $k[X]^G$ so by [Bourbaki, *Commutative Algebra*] the equation has a solution with $x_i \in k[X]^G$. That is, $\mathfrak{a}^G + \mathfrak{b}^G = 1$. Thus $\pi_X(V_1) \cap \pi_X(V_2) = \emptyset$. The existence of F follows from the Chinese remainder theorem.

(ii) X/G is a categorical quotient. By construction it is clear that

X/G is a quotient in the category of affine schemes. To show that it is a quotient in the bigger category, it is sufficnet, by [Mumford 0, §2.6], to show that if $\{W_i : i \in I\}$ is a set of stable closed subschemes of X, then $|\pi_X(\cap W_i)| = |\cap \pi_X(W_i)|$. $\subset$ is clear. To show the converse, assume that $|\pi_X(\cap W_i)| \not\supset |\cap \pi_X(W_i)|$. Then there exists a closed point $t \in X/G$ such that the fiber $X_t = \pi_X^{-1}(t)$, which is a closed stable subscheme, meets each W_i but does not meet $\bigcap_{i\in I} W_i$. Let $J \subset I$ be a maximal set such that $X_t \cap \left(\bigcap_{i\in J} W_i\right) \neq \emptyset$. J is a nonempty proper subset of I, so there exists $i_0 \in I\backslash J$. Then $W_{i_0} \cap X_t$ and $\left(\bigcap_{i\in J} W_i\right) \cap X_t$ are disjoint closed stable subschemes, so they are separated by an invariant. But this is impossible as they lie in the same fiber X_t of π_X. This concludes the proof of the proposition. #

PROPOSITION 2. *Let G be a reductive group scheme acting on an affine algebraic scheme X.*

(i) $X \otimes \overline{k}/G \otimes \overline{k} \cong X/G \otimes \overline{k}$ *and* $\pi_{X\otimes\overline{k}} \cong \pi_X \otimes \overline{k}$.

(ii) Let Y, Z be affine algebraic schemes with a trivial G action and let there be given morphisms $X \to Z$, $Y \to Z$. Assume that $X \to Z$ is equivariant and $Y \to Z$ is flat. Then G acts naturally on $X \times_Z Y$ and $(X \times_Z Y)/G \cong X/G \times_Z Y$.

(iii) Let G act on the left of X, and define a left action of G on $G \times X$ by the action of $G(R)$ on $G(R) \times X(R)$ given by $g\cdot(h, x) = (hg^{-1}, g\cdot x)$. *Then* $(G \times X)/G \cong X$.

(iv) Let $\varphi : X \to Y$ be an equivariant morphism and let $Z \to Y$ be faithfully flat, where X, Y, Z are affine algebraic schemes and G acts trivially on Y, Z. Then φ is a quotient map iff $\varphi \otimes_{k[Y]} k[Z]$ is a

quotient map.

Proof. Parts *(i)* and *(ii)* are just restatements of Proposition 2.4.1. To prove *(iii)* replace the schemes by the corresponding functors in $[Alg/k, S]$. It is clear that $\underline{X}$ is the categorical quotient in this larger category, so it must be the quotient in the smaller category of representable functors. The result follows by the uniqueness of categorical quotients.

(iv) It is to be shown that $\varphi^* : k[Y] \to k[X]^G$ is an isomorphism iff $\psi : k[Z] = k[Y] \otimes_{k[Y]} k[Z] \to \left(k[X] \otimes_{k[Y]} k[Z]\right)^{G \otimes k[Z]}$ is an isomorphism. By Proposition 2.4.1 (in which the base ring k need not be a field), $\left(k[X] \otimes_{k[Y]} k[Z]\right)^{G \otimes k[Z]} = k[X]^G \otimes_{k[Y]} k[Z]$, so $\psi = \varphi^* \otimes_{k[Y]} k[Z]$, and the result follows by faithful flatness. #

Let G, X be as in the propositions above, and let $x \in X$ be a rational point. Let $\mathrm{Spec}(k) \to X$ be the unique morphism whose image is x , and let $\sigma_x : G \to X$ be the morphism which is the composition $G \xrightarrow{\sim} G \times \mathrm{Spec}(k) \to G \times X \to X$. σ_x is called the orbit map at x .

PROPOSITION 3. *Let G be a reductive group scheme, acting on an affine algebraic scheme X . Let $x \in X$ be a rational point. Then the orbit map σ_x has a locally closed image so the scheme theoretic image O_x exists. O_x is a G-stable smooth subscheme of X called the orbit of G through the rational point x .*

Proof. Let O_x for the moment be just the set theoretic image of σ_x . It is to be shown that O_x is locally closed. Assume first that k is algebraically closed. Let $O_x(k) = O_x \cap X(k)$. Then $O_x(k) = \sigma_x(k)G(k)$, since the inclusion $\subset$ is clear and the fiber of σ_x over points of $O_x(k)$ must contain a rational point. Thus $G(k)$ acts transitively on $O_x(k)$.

O_x is the image of a morphism so it is a constructible set. Thus it contains a non-empty set U which is open in $\overline{O_x}$. U must contain a rational point so $G(k)U$ is a $G(k)$ stable subset of O_x which contains all the closed points of O_x and which is open in $\overline{O_x}$. $G(k)U$ is constructible so $G(k)U = O_x$. Thus O_x is locally closed.

If k is not algebraically closed, let $\overline{x}$ be the unique rational point of $X \otimes \overline{k}$ over x and consider the diagram

$$\begin{array}{ccccc} G \otimes \overline{k} & \xrightarrow{\sigma_x \otimes \overline{k}} & X \otimes \overline{k} & & \overline{x} \\ \downarrow{\scriptstyle p_G} & & \downarrow{\scriptstyle p_X} & & \downarrow \\ G & \xrightarrow[\sigma_x]{} & X & & x \end{array}$$

It is easy to see that $\sigma_x \otimes \overline{k} = \sigma_{\overline{x}}$. Thus $O_x = p_X(O_{\overline{x}})$ and

$$O_{\overline{x}} = \sigma_{\overline{x}}(G \otimes \overline{k}) = (\sigma_x \otimes \overline{k})(G \otimes \overline{k}) = p_X^{-1}(O_x) .$$

p_X is flat so $p_X^{-1}(\overline{O_x}) = \overline{p_X^{-1}(O_x)} = \overline{O_{\overline{x}}}$ [Dieudonné 1964, III, B.8]. Then $p_X|_{\overline{O_{\overline{x}}}}$ is faithfully flat hence open, so $p_X(O_{\overline{x}}) = O_x$ is open in $\overline{O_x}$.

Thus O_x is locally closed, so the scheme theoretic image of σ_x exists. It will be written O_x and is G-stable by Proposition 2.1.1. Let x, $\overline{x}$ be as in the paragraph above. By generic flatness $\sigma_{\overline{x}} : G \otimes \overline{k} \to O_{\overline{x}}$ is flat on a non empty open subset U of $G \otimes \overline{k}$. Since $G(\overline{k})$ acts transitively on $G(\overline{k})$ in $G \otimes \overline{k}$, $\sigma_{\overline{x}} : G \otimes \overline{k} \to O_{\overline{x}}$ is faithfully flat. By faithfully flat descent $\sigma_x : G \to O_x$ is faithfully flat. But faithfully flat morphisms preserve smoothness so O_x is smooth. #

Introduce a partial order among the stable subschemes of X, by $U \leq V$

if U is a subscheme of $\overline{V}$. By Lemma 3.1.3 and Corollary 3.1.2 the minimal elements of this partial order are closed reduced stable subschemes of X. By analogy with the algebraically closed case I will call these minimal elements closed pseudo-orbits. By Proposition 3.2.1 if $x \in X/G$ is a closed point, then the fiber $X_x = \pi_X^{-1}(x)$ contains a unique closed pseudo-orbit, and conversely every pseudo-orbit lies in such a fiber. Thus the closed pseudo-orbits of G on X are parametrised by the closed points of X/G.

PROPOSITION 4. *Closed orbits are pseudo-orbits. A closed pseudo-orbit is an orbit iff it contains a rational point.*

Proof. If T is a closed orbit then T is closed and reduced. Let S be a non empty stable subscheme of T. S may be assumed closed and reduced. By Lemma 1.6.3, since the orbit map $G \to T$ is surjective $G(\overline{k})$ acts transitively on $T(\overline{k})$. Thus $S(\overline{k}) = T(\overline{k})$. By the same lemma $S = T$. Thus T is minimal in the partial order.

Every orbit contains a rational point (for example the image of the identity in G under the orbit map). Conversely if T is a closed pseudo-orbit containing a rational point x, then the orbit O_x is a stable subscheme of X, and $O_x \cap T$ is a stable subscheme of T. By minimality, $T \subset O_x$ so $T = O_x$. #

NOTE. If T is a closed orbit in X then T is parametrised by a rational point of X/G. I do not know if the converse is true. (If the field k is finite, then the converse is true [Borel V, 16.5].

PROPOSITION 5. *Let G be a reductive group scheme acting on an affine algebraic scheme X. Let B be a closed reduced subscheme of X. The following are equivalent:*

(i) V is a closed pseudo-orbit;

(ii) *there exists a closed* $G \otimes \overline{k}$ *orbit in* $X \otimes \overline{k}$ *which maps onto* V *under* p_X *;*

(iii) *if* $p_X^{-1}(V)_r$ *is the reduced fiber over* V *, then* $p_X^{-1}(V)_r$ *is a finite union of closed* $G \otimes \overline{k}$ *orbits, each of which maps onto* V *;*

(iv) $V(\overline{k})$ *is a finite union of closed* $G(\overline{k})$ *orbits, each of which maps onto* V_0 *(the set of closed points in* V *).*

Proof. *(i)* $\Rightarrow$ *(iii).* $p_X^{-1}(V)_r$ is a closed reduced $G \otimes \overline{k}$ stable subscheme of $X \otimes \overline{k}$, hence it contains a closed $G \otimes \overline{k}$ orbit T, and $p_X(T) \subset V$. Since V is minimal, $p_X(T) = V$.

By dimension theory (regarding both X and $X \otimes \overline{k}$ as schemes over k for the moment), $\dim T \geq \dim V$. But T is a closed subscheme of $p_X^{-1}(V)_r$ so

$$\dim T \leq \dim p_X^{-1}(V)_r = \dim p_X^{-1}(V) = \dim(V \otimes \overline{k})$$

$$= \dim V + \operatorname{trans\,deg}(\overline{k}/k) = \dim V .$$

Thus $\dim T = \dim V = \dim p_X^{-1}(V)_r$.

There are plenty of rational points so $G(\overline{k})$ permutes the components of T, which must thus be equidimensional. Thus T is a union of components of $p_X^{-1}(V)_r$. If C is a component of $p_X^{-1}(V)_r$ such that $C \not\subset T$ but $C \cap T \neq \emptyset$, then there exists a rational point of T which lies in two distinct components of $p_X^{-1}(V)_r$. But $G(\overline{k})$ acts transitively on $T(\overline{k})$ so this is absurd, for then every rational point would have this property. Thus either $C \subset T$ or C is disjoint from T. Then $p_X^{-1}(V)_r \backslash T$ is closed, reduced and G-stable and the result follows by an obvious induction.

(iii) $\Rightarrow$ *(ii)* is trivial.

(ii) $\Rightarrow$ *(i)*. If $V = p_X(T)$ where T is a closed $G \otimes \overline{k}$ orbit then by Proposition 3.1.1, V is closed, reduced and G stable. It must be shown that V is minimal among G-stable subschemes. If V properly contains a stable subscheme W then W may be assumed closed and reduced, and the set W must be a proper subset of V. Then $p_X^{-1}(W)_r$ is a $G \otimes \overline{k}$ stable proper subscheme of $X \otimes \overline{k}$ which meets T in a proper subscheme of T, contradicting the minimality of T.

(iii) $\Leftrightarrow$ *(iv)*. Reduced closed subschemes of $X \otimes \overline{k}$ are in one-to-one correspondence with closed subsets of $X(\overline{k})$ because $X \otimes \overline{k}$ is a Jacobson scheme. By Proposition 3.1.4 reduced closed $G \otimes \overline{k}$ stable subschemes of $X \otimes \overline{k}$ correspond to $G(\overline{k})$ stable closed subsets of $X(\overline{k})$. By a standard result on algebraic groups, minimal stable subsets of $X(\overline{k})$ are the closed $G(\overline{k})$ orbits. Finally $p_X(T) = V$ iff $p_X\big(T(\overline{k})\big) = V_0$ because V_0 is dense in V, and p_X is a closed map. #

3. Equivariant étale morphisms

LEMMA 1. *Let G be a reductive group scheme acting on an affine algebraic scheme X. Let $F \in k[X]^G$ be an invariant, let $X_F = \{x \in X \mid F \notin \mathfrak{p}_x\}$ be the principal open subscheme defined by F, and let $j : X_F \to X$ be the embedding. Then G acts naturally on X_F, j is equivariant, and j/G is an open embedding of $(X_F)/G$ onto $(X/G)_F$ in X/G.*

Proof. Since $X_F = X \times_{(X/G)} (X/G)_F$, there is a natural action of G on X_F and $\pi_X(X_F) = (X/G)_F$ which is an open subscheme of X/G. j/G maps $(X_F)/G$ onto $\pi_X(X_F) = (X/G)_F$ so to show that j/G is an embedding it

is enough to show that $(X_F)/G \cong (X/G)_F$. This follows from Proposition 3.2.2 since $(X/G)_F \to X/G$ is flat. #

PROPOSITION 1. *Let G be a reductive group scheme acting on affine algebraic schemes X, Y, and let $\varphi : X \to Y$ be an equivariant morphism. Let T be a closed pseudo-orbit in X and let $x \in T$. The following are equivalent:*

(i) φ is étale at x;

(ii) φ is étale at every point of T;

(iii) there is a G stable open affine neighbourhood U of T such that φ is étale at every point of U.

In case (iii), U may be chosen so that $\pi_X(U) \cong U/G$ and $\pi_U = \pi_X|_U$.

Proof. Recall first that if $p_X : X \otimes \overline{k} \to X$ is the usual projection, φ is étale at x iff $\varphi \otimes \overline{k}$ is étale at one point of the fiber $p_X^{-1}(x)$ iff $\varphi \otimes \overline{k}$ is étale at every point of the fiber $p_X^{-1}(x)$ [Appendix 2].

(i) $\Rightarrow$ (ii). First consider the algebraically closed case. If φ is étale at x then φ is étale on a non-empty open set U which contains x. $U \cap T$ is non-empty and constructible, so it contains a rational point. Thus x may be assumed rational. $G(k)$ acts transitively on $T(k)$, so φ is étale at every rational point of T. The set of points at which φ is étale is open and contains $T(k)$, so it contains T.

Now if k is not algebraically closed, let $\overline{k}$ be an algebraic closure. If φ is étale at x then $\varphi \otimes \overline{k}$ is étale at each point of the fiber $p_X^{-1}(x)$. But $p_X^{-1}(T)$ contains a $G \otimes \overline{k}$ orbit O which maps surjectively onto T. O meets the fiber $p_X^{-1}(x)$, so it contains a point at which $\varphi \otimes \overline{k}$ is étale. Thus $\varphi \otimes \overline{k}$ is étale at every point of O, and φ is étale at every point of $p_X(O) = T$.

(ii) $\Rightarrow$ (iii). Let Z be the set of points at which φ is not étale.

Z is closed and can be assumed non-empty. Let Z_r be the reduced closed subscheme carried by Z. I claim that Z_r is G stable. By Proposition 3.1.4 it is sufficient to show that $Z_r(\overline{k})$ is $G(\overline{k})$ stable. Now $Z_r(\overline{k}) = Z_r \otimes \overline{k}(\overline{k}) = p_X^{-1}(Z) \cap X(\overline{k})$. But $x \in p_X^{-1}(Z)$ iff φ is not étale at $p_X(x)$, iff $\varphi \otimes \overline{k}$ is not étale at x. So $Z_r(\overline{k})$ is the set of rational points in $X \otimes \overline{k}$ at which $\varphi \otimes \overline{k}$ is not étale, and this is clearly $G(\overline{k})$ stable.

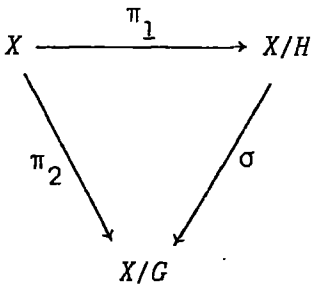
T and Z_r are disjoint closed G-stable subschemes so they are separated by an invariant F. Let F vanish on Z_r but not on T and let $U = X_F$. Then U is an affine open neighbourhood of T on which φ is étale. U satisfies the conditions of the proposition by Lemma 1.

(iii) $\Rightarrow$ *(i)*. This is trivial. #

CHAPTER 4

1. Etale morphisms and quotients

LEMMA 1 [Luna 1973]. *Let X be a normal algebraic affine scheme and let G be a finite group of automorphisms of X. Let $H < G$ be a subgroup. Consider*



where π_1, π_2 are the quotient morphisms and σ is the factorisation of π_2 through π_1. Let $x \in X$ be a rational point, $y = \pi_1(x)$ and $z = \pi_2(x)$, and let G_x be the stabiliser of x. Then σ is étale at y iff $G_x \subset H$.

Proof. First note that X/H and X/G are normal, and $\sigma_z^* : \mathcal{O}_{X/G,z} \to \mathcal{O}_{X/H,y}$ is injective.

If $G_x \subset H$ then by [Bourbaki, *Commutative Algebra*, V, §2.2.4] σ is unramified at y. Then by E.G.A.0, 23.2.1 and E.G.A.IV, 18.10.1, σ is étale at y.

If σ is étale at y then it is unramified at y. Both y and z are rational so $k(y) = k(z)$. So by [Bourbaki, V, §2.2.4], $k[X]^H \subset k[X]^{G_x}$. Thus $k(X/H) \subset k\left(X/G_x\right) \subset k(X)$, where by $k(X)$ is meant the field of rational functions of X, and similarly for X/H, X/G_x. But $k(X/G) = k(X)^G$ [Bourbaki, V, §1.9, Proposition 23] and likewise for H and G_x. Thus $k(X)$ is a Galois extension of $k(X/G)$ with group G.

Therefore $G_x \subset H$. #

PROPOSITION 1. *Let G be a reductive group scheme acting on geometrically normal algebraic affine schemes X and Y . Let $\varphi : X \to Y$ be an equivariant morphism. Let T be a closed pseudo-orbit in X and let $t = \pi_X(T)$ be the point of X/G parametrising T . Assume*

(i) φ is étale at a point $x \in T$,

(ii) $\varphi(T)$ is closed,

(iii) φ maps T isomorphically onto its scheme theoretic image $\varphi(T)$.

Then φ/G is étale at t .

If T is an orbit then (iii) may be weakened to

(iii)' φ is injective on T .

Proof. This proposition is a generalisation of the "Lemme fondamental" of [Luna 1973].

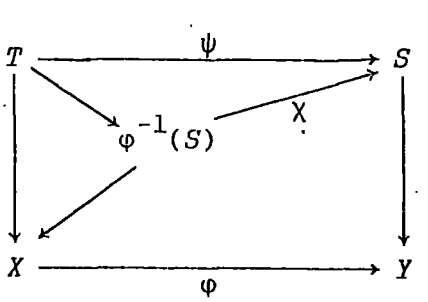
To abbreviate, write S for $\varphi(T)$, the scheme theoretic image of T . S is a closed reduced subscheme of Y . Let $\psi : T \to S$ be defined by

$$\begin{array}{ccc} T & \xrightarrow{\psi} & S \\ \cap & & \cap \\ X & \xrightarrow{\varphi} & Y \ . \end{array}$$

The proof proceeds by a series of reductions. In the first place, by Proposition 3.3.1, φ may be assumed everywhere étale.

I will now show that if T is an orbit then

(iii)' $\Rightarrow$ (iii). Let $\varphi^{-1}(S)$ be the scheme theoretic fiber of φ over S and let $\chi : \varphi^{-1}(S) \to S$ be the induced morphism:



Since χ comes from φ by a base change χ is étale. By EGAI, §4.3.6, T is a closed subscheme of $\varphi^{-1}(S)$.

S is an orbit so it is normal, and χ is étale so it has normal fibers (in fact they are geometrically regular). So by EGAIV, 6.5.4, $\varphi^{-1}(S)$ is normal. In particular $\varphi^{-1}(S)$ is reduced and the connected components are the same as the irreducible components.

$\dim T = \dim S = \dim \varphi^{-1}(S)$, so T is a reduced closed subscheme of $\varphi^{-1}(S)$ of maximal dimension, and clearly the components of T are equidimensional. Thus T is a union of irreducible components of $\varphi^{-1}(S)$. Thus T is an open subscheme of $\varphi^{-1}(S)$ and $\psi = \chi|_T$ is étale.

For $y \in S$, let $n(y)$ be the number of geometric points in the fiber $\psi^{-1}(y)$ over y (considered as a scheme over $k(y)$). From EGAI, §6.5,

$$n(y) = \#\left(\psi^{-1}(y) \otimes_{k(y)} K\right)$$

$$= \sum_{x \in \psi^{-1}(y)} [k(x) : k(y)]_s$$

where K is an algebraic closure of $k(y)$ and $[\ :\ ]_s$ is the separable degree of the field extension. Since ψ is étale all the field extensions are separable so

$$n(y) = \sum_{x \in \psi^{-1}(y)} [k(x) : k(y)] .$$

I claim that $n(y) = 1$ for all $y \in S$.

Consider first the closed points of S. Let $y \in S$ be a closed point, and let y' be a closed point of $S \otimes \overline{k}$ over y. Let $n(y')$ be the number of geometric points in the fiber of $\psi \otimes \overline{k}$ over y'. Then $n(y) = n(y')$. To see this, regard the closed point y as a closed subscheme with local ring $k(y)$ (and treat other closed points similarly) and compute the number $N = \#\left(T \otimes \overline{k} \times_{S \otimes \overline{k}} y \otimes \overline{k}\right)$ in two different ways.

First of all $y \otimes \overline{k}$ is the union of $[k(y) : k]_s$ closed points, and $T \otimes \overline{k} \times_{S \otimes \overline{k}} y \otimes \overline{k}$ is the union of the $\psi \otimes \overline{k}$ fibers over these points. Since $G(\overline{k})$ permutes the points of $y \otimes \overline{k}$, these fibers all have the same number of points, so $N = [k(y) : k]_s n(y')$. On the other hand $T \otimes \overline{k} \times_{S \otimes \overline{k}} y \otimes \overline{k} \cong (T \times_S y) \otimes \overline{k} = \psi^{-1}(y) \otimes \overline{k}$. Each point $x \in \psi^{-1}(y)$ splits into $[k(x) : k]_s$ points in $\psi^{-1}(y) \otimes \overline{k}$. So

$$N = \sum_{x \in \psi^{-1}(y)} [k(x) : k]_s$$

$$= \left(\sum_{x \in \psi^{-1}(y)} [k(x) : k(y)]_s \right) [k(y) : k]_s = n(y)[k(y) : k]_s .$$

Thus $n(y) = n(y')$.

Now let $x_0 \in T$ be a rational point and let $y_0 = \psi(x_0)$. Then y_0 is rational and $n(y_0) = [k(x_0) : k(y_0)]_s = [k : k]_s = 1$. Since $G(\overline{k})$ acts transitively on the closed points of $S \otimes \overline{k}$, $n(y)$ is constant on the closed points of S. Thus $n(y) = 1$ for all closed points $y \in S$.

$n(y)$ is lower semicontinuous (EGAIV, 18.2.8) so $\{y \in X : n(y) \leq 1\}$ is a closed set containing all the closed points of S. S is a Jacobson space so $n(y) \leq 1$ for all $y \in S$. But ψ is surjective by definition, so $n(y) = 1$ for all $y \in S$.

In particular, evaluating $n(y)$ at the generic points of the connected components of S (which are irreducible) shows that ψ is birational. Thus ψ is a bijective birational morphism between normal schemes so ψ is an isomorphism by Z.M.T. [EGAIV, 8.12.11]. This completes the proof that *(iii)' ⇒ (iii)*.

I will now show that it is sufficient to prove the proposition under the assumption that k is algebraically closed. To do so, assume that the

proposition is proved when k is algebraically closed and let k be arbitrary.

Let $\overline{k}$ be an algebraic closure of k and consider

$$\begin{array}{ccc} T \otimes \overline{k} & \xrightarrow{\psi \otimes \overline{k}} & S \otimes \overline{k} \\ \downarrow & & \downarrow \\ X \otimes \overline{k} & \xrightarrow[\varphi \otimes \overline{k}]{} & Y \otimes \overline{k} \end{array}$$

Then $\varphi \otimes \overline{k}$ is étale, $\psi \otimes \overline{k}$ is an isomorphism, $T \otimes \overline{k}$ and $S \otimes \overline{k}$ are smooth closed subschemes and all the maps are $G \otimes \overline{k}$ equivariant, as these properties are preserved under base change. $T \otimes \overline{k}$ contains a closed $G \otimes \overline{k}$ orbit T' such that $p_X : X \otimes \overline{k} \to X$ maps T' onto T (see Proposition 3.2.5). Let $t' \in X \otimes \overline{k}/G \otimes \overline{k}$ be the closed point parametrising the $G \otimes \overline{k}$ orbit T'. Then under the projection $X \otimes \overline{k}/G \otimes \overline{k} \xrightarrow{\sim} X/G \otimes \overline{k} \xrightarrow{p_{X/G}} X/G$ the point t' is sent to t. The morphism $\varphi \otimes \overline{k}$ satisfies the conditions of the proposition at a point in the orbit T', so by assumption $\varphi \otimes \overline{k}/G \otimes \overline{k} \cong \varphi/G \otimes \overline{k}$ is étale at the point t'. Then by faithfully flat descent φ/G is étale at t. Thus in proving the proposition k may be assumed algebraically closed.

I now show that φ can be assumed to be a finite morphism, though one which is not necessarily everywhere étale. As the field k is now assumed algebraically closed the action of the group scheme G is adequately described by the action of the group $G(k)$ and G invariants, and so on, are the same as $G(k)$ invariants.

For $x \in X/G$ let $X_x = \pi_X^{-1}(x)$ be the fiber over x and, for x closed, let T_x be the unique closed orbit in X_x. Let $\tilde{X}$ be the normalisation of Y in the field $k(X)$. By Z.M.T. [Dieudonné 1964, p. 136, Corollary 1] there is a factorisation of φ :

$$X \xrightarrow{i} \tilde{X} \xrightarrow{\tilde{\varphi}} Y$$

where i is a dominant open embedding, and $\tilde{\varphi}$ is a finite morphism. Clearly $G(k)$ acts rationally on $k[\tilde{X}]$, so G acts on $\tilde{X}$, and i and $\tilde{\varphi}$ are equivariant.

LEMMA 2. *For $x \in X/G$ a closed point, let T_x be the closed orbit parametrised by x and let $\tilde{T}_x = i(T)$. The following are equivalent:*

(i) $\tilde{T}_x$ is closed in $\tilde{X}$;

(ii) $\varphi(T_x)$ is closed in Y;

(iii) $i/G(x) \notin \pi_{\tilde{X}}(\tilde{X} \setminus i(X))$.

Proof. *(i)* $\Rightarrow$ *(ii)* because $\tilde{\varphi}$ is finite.

(ii) $\Rightarrow$ *(i)*. Regarding $X(k)$ as a Serre variety, $\tilde{\varphi}^{-1}\varphi(T_x) \cap X(k)$ is a union of $G(k)$ orbits, which are finite in number and all of the same dimension since $\tilde{\varphi}$ is finite. One of these is $\tilde{T}_x(k)$. Since orbits of minimal dimension are closed $\tilde{T}_x(k)$ is closed in $\tilde{\varphi}^{-1}\varphi(T_x)(k)$ so $\tilde{T}_x$ is closed in $\tilde{X}$.

(i) $\Longleftrightarrow$ *(iii)*. There is only one closed orbit in $\overline{\tilde{T}_x}$ because if there were more than one they could not be separated by invariants. Let S be the closed orbit in $\overline{\tilde{T}_x}$. Then $\tilde{T}_x$ is closed iff $S = T_x$ iff $S \subset i(X)$ iff S and $\tilde{X} \setminus i(X)$ are disjoint $G(k)$ stable closed subsets of $\tilde{X}$ iff S and $\tilde{X} \setminus i(X)$ are separated by an invariant iff $\pi_{\tilde{X}}(S) \notin \pi_{\tilde{X}}(\tilde{X} \setminus i(X))$. But $\pi_{\tilde{X}}(S) = i/G(x)$. #

COROLLARY. *Let $V \subset X/G$ be the constructible set defined by $V(k) = \{x \in X/G(k) : \varphi(T_x)$ is closed$\}$. Then V is open and $i/G|_V$ is an open embedding.*

Proof. That V is open follows from part *(iii)* of the lemma and from

part (i) it follows that i/G is injective on rational points of V. By Lemma 1.6.3 or directly i/G is injective on V. It will be sufficient to show that i/G is a local isomorphism [EGAI, §4.2.2].

Since this is a local property it can be assumed that V is affine, and in fact by replacing X by $\pi_X^{-1}(V)$ it can be assumed that $V = X/G$. Let $x \in X/G$ be a closed point and let $F \in k[\tilde{X}]^G$ be an invariant which separates $\tilde{T}_x$ and $\tilde{X}\backslash i(X)$. Since $k[\tilde{X}] \subset k[X]$, F will be considered an element of $k[X]$. Then $i|_{X_F} : X_F \to \tilde{X}_F$ is an isomorphism so $i/G|_{(X_F)/G} : (X_F)/G \to (\tilde{X}_F)/G$ is an isomorphism. But by Lemma 3.3.1, $(X_F)/G$ and $(\tilde{X}_F)/G$ are open neighbourhoods of x and $i/G(x)$, so i/G is a local isomorphism at x. #

Now returning to the factorisation of φ,

$$\begin{array}{ccccc} X & \xrightarrow{\ i\ } & \tilde{X} & \xrightarrow{\ \tilde{\varphi}\ } & Y \\ \downarrow & & \downarrow & & \downarrow \\ X/G & \xrightarrow{\ i/G\ } & \tilde{X}/G & \xrightarrow{\ \tilde{\varphi}/G\ } & Y/G \end{array}$$

by if necessary replacing X by $\pi_X^{-1}(U)$ where U is an affine open neighbourhood of t in V it can be assumed that i/G is an open embedding. Then $\varphi = \tilde{\varphi} \circ i$ is étale at T iff $\tilde{\varphi}$ is étale at $i(T)$ and $\varphi/G = \tilde{\varphi}/G \circ i/G$ is étale at t iff $\tilde{\varphi}/G$ is étale at $i/G(t)$. The pair $(\tilde{\varphi},\ i(T) = \tilde{T})$ satisfy the hypotheses of the proposition but in addition $\tilde{\varphi}$ is finite.

So now assume in addition to the hypotheses of the proposition that k is algebraically closed and that φ is finite. From this point the proof follows that of Luna fairly closely. All points mentioned below will be assumed rational.

$G^0(k)$ is a normal subgroup of $G(k)$ and by looking at rings of

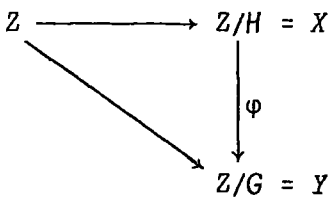
invariants it is easy to see that $X/G \cong (X/G^0)/(G/G^0)$. So it is sufficient to consider separately the cases G connected and G finite.

First assume that G is connected and hence $G(k)$ is connected as a Serre variety. The orbits T and S are then connected and lie completely inside a single connected component of X and Y . Thus it may be assumed that both X and Y are connected and hence irreducible.

$\varphi : X \to Y$ is étale on a dense open set, so $k(X)$ is a finite separable extension of $k(Y)$, so $k(X)$ lies in a finite Galois extension F . Let Z be the normalisation of Y in F . Clearly $F = k(Z)$. Let G be the Galois group of $k(Z)$ over $k(Y)$ and let H be the subgroup which fixes the elements of $k(X)$. Then

$$k[Y] = k[Z] \cap k(Y) = k[Z] \cap k(Z)^G = k[Z]^G$$

(using the normality of Y) and similarly $k[X] = k[Z]^H$. So $X = Z/H$ and $Y = Z/G$.

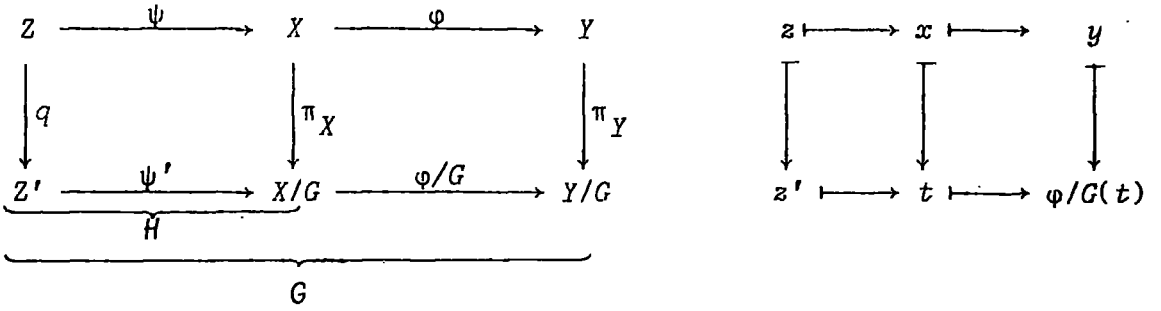


Let $x \in T \subset X$ and let $z \in Z$ lie over x . Then by Lemma 1, φ is etale at x iff $G_z \subset H$.

Now let Z' be the normalisation of Y/G in $k(Z)$. G acts on $k(Z)$ leaving fixed the elements of $k(Y/G)$ so G acts on $k[Z']$ by algebra automorphisms. Thus G acts on Z' . I claim $Z'/G \cong Y/G$, that is $k[Z']^G = k[Y]^G$. $\supseteq$ is clear, so let $f \in k[Z']^G$. Then f is integral over $k[Y]^G$ and $f \in k(Z)^G = k(Y)$. But $k[Y]^G$ is integrally closed in $k(Y)$ for if $f \in k(Y)$ is integral over $k[Y]^G$ the action of $G(k)$ preserves the integral equation of f (the coefficients are invariants) and permutes the roots of this equation. Since $G(k)$ is connected and the set

of conjugates of f is finite, $G(k)$ fixes f. That is, $f \in k(Y)^G$. Since Y is normal and f is integral over $k[Y]^G \subset k[Y]$, $f \in k[Y]$. Thus $f \in k[Y]^G$.

Similarly $k[X]^G = k[Z']^H$ so $Z'/H \cong X/G$. If $z' \in Z'$ lies over $t \in X/G$, then by Lemma 1, φ/G is étale at t iff $G_{z'} \subset H$.



Let $z \in Z$ lie over $x \in X$ and let $z' = q(z)$. Then it is sufficient to show that $G_z \subset H$ implies $G_{z'} \subset H$.

$$\psi\left(G_{z'} \cdot (z)\right) \subset \pi_X^{-1}(t) \cap \varphi^{-1}(y) \subset \pi_X^{-1}(t) \cap \varphi^{-1}(S).$$

But $\varphi^{-1}(S)$ is a finite union of orbits all of the same dimension so $\varphi^{-1}(S)$ is a union of closed orbits. However $\pi_X^{-1}(t)$ contains just one closed orbit, namely T, so $\psi\left(G_{z'} \cdot (z)\right) \subset T$. But $\varphi|_T$ is injective so $\psi\left(G_{z'} \cdot (z)\right) = x$. H acts transitively on the fiber of ψ over x so $G_{z'} \cdot (z) \subset H \cdot z$. Thus $HG_{z'} \subset G_z$ and $G_{z'} \subset HG_z$. But by assumption $H \subset G_z$. This completes the proof when G is connected.

Now assume that G is finite. Clearly it may be assumed that X/G and Y/G are connected but not necessarily that X and Y are connected. Let $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$ be the connected components of X and Y and assume that $x \in X_1$ and $y = \varphi(x) \in Y_1$. G permutes the X_i, otherwise X could be partitioned into disjoint closed G-stable subsets and there would be an invariant which disconnects X/G. If $g \in G$ then

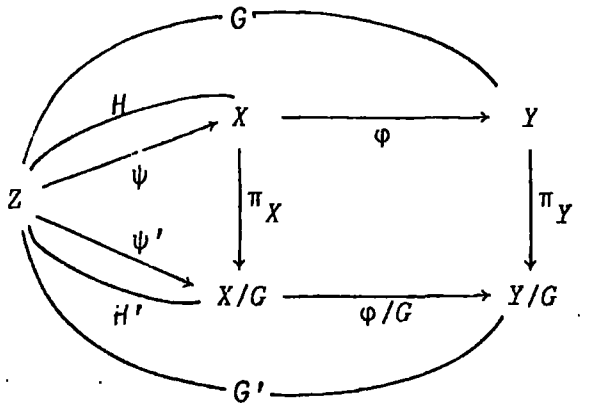
for each i, j either $gX_i = X_j$ or $gX_i \cap X_j = \emptyset$. Let G_1 be the subgroup which preserves the component X_1 (that is $g \cdot X_1 = X_1$). Then each G orbit O on X intersects X_1 in a unique G_1 orbit. $O \cap X_1$ contains at least one G_1 orbit because G permutes the X_i, and it contains at most one because if $O \cap X_1$ contains two points p, q then there exists $g \in G$ such that $g \cdot p = q$. But $g \cdot X_1 \cap X_1 \neq \emptyset$ so $g \cdot X_1 = X_1$ and $g \in G_1$. Thus each G_1 invariant on X_1 extends uniquely to a G invariant on X. $k[X_1]^{G_1} \cong k[X]^G$ and $X_1/G_1 \cong X/G$.

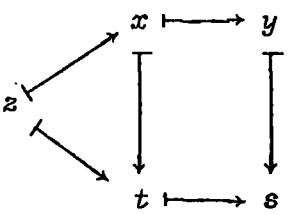
Similarly if $G(Y_1)$ is the subgroup which preserves the component Y_1 of Y then $Y_1/G(Y_1) \cong Y/G$. Then φ/G can be factorised

$$X/G \cong X_1/G_1 \xrightarrow{\alpha} Y_1/G_1 \xrightarrow{\beta} Y_1/G(Y_1) \cong Y/G$$

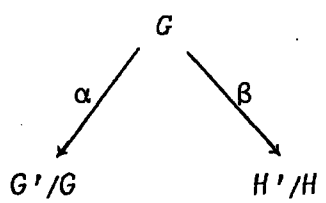
where $\alpha = \varphi'/G_1$, $\varphi' : X_1 \to Y_1$ being the restriction of φ to X_1, and $\beta : Y_1/G_1 \to Y_1/G(Y_1)$ is the natural map. By Lemma 1, β is étale at $\alpha(t)$ iff $G_y \subset G_1$. But if g fixes y then g fixes x because φ is injective on the orbit through x so $g \in G_1$. Thus β is étale at $\alpha(t)$.

Thus it can be assumed that X and Y are connected, and hence irrediculble. $k(Y)$ is a Galois extension of $k(Y/G)$ [Bourbaki, *Commutative Algebra*] and by assumption $k(X)$ is a finite separable extension of $k(Y)$ so $k(X)$ lies in a Galois extension F of $k(Y/G)$. Let Z be the normalisation of Y/G in F. As in the connected case $F = k(Z)$.





Let $z \in Z$ be a point bying over x . Let $G' = \mathrm{Gal}\big(k(Z) : k(Y/G)\big)$, $G = \mathrm{Gal}\big(k(Z) : k(Y)\big)$, $H' = \mathrm{Gal}\big(k(Z) : k(X/G)\big)$, $H = \mathrm{Gal}\big(k(Z) : k(X)\big)$. Note that $H \subset H' \subset G'$ and $H \subset G \subset H'$. Since $k(Y)$ is a Galois extension of $k(Y/G)$, G is a normal subgroup of G' and $\mathrm{Gal}\big(k(Y) : k(Y/G)\big) = G'/G$. Similarly H is normal in H' and $\mathrm{Gal}\big(k(X) : k(X/G)\big) = H'/H$. By [Bourbaki, *Commutative Algebra*], $k(Y/G) = k(Y)^G$ so the action of G on $k(Y)$ gives a surjection $\alpha : G \to \mathrm{Gal}\big(k(Y) : k(Y/G)\big) = G'/G$. There is a similar surjection $\beta : G \to H'/H$.



As in the connected case it is enough to show that $G_z \subset H \Rightarrow G'_z \subset H'$. Define a relation $\sim$ between G' and H' as follows. If g, h are in G', H' , and $[g], [h]$ are their cosets in G'/G and H'/H respectively, then $g \sim h$ if there exists $\sigma \in G$ such that $\alpha(\sigma) = [g]$ and $\beta(\sigma) = [h]$. If $g \in G'_z$ then there exists $h \in H'_x$ such that $g \sim h$, for let $\sigma \in G$ be such that $\alpha(\sigma) = [g]$. Then σ and g both give the same automorphism of Y so σ fixes y . φ is injective on the orbit T through x so σ fixes x , and any element h of the coset $\beta(\sigma)$ will do.

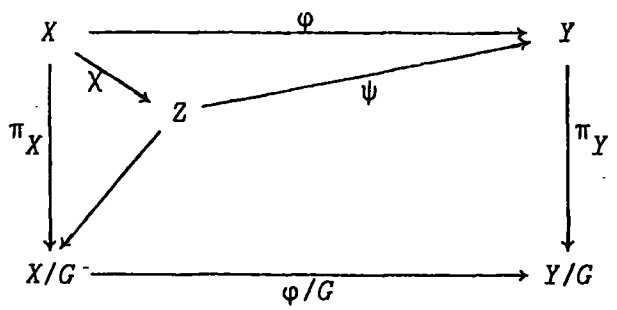
So assume $G_z \subset H$, let $g \in G'_z$ and let $h \in H'_x$ be such that $g \sim h$. h stabilises the fiber $\psi^{-1}(x)$, but H acts transitively on this fiber so $h \in HG_z$. $g \sim h$ so hg^{-1} acts trivially on Y. Thus $hg^{-1} \subset HG'_z$ $G = HG_z \subset H \subset H'$. Thus $g \in H'$. This completes the proof for G finite, and so completes the proof of the proposition. #

2. Some consequences

LEMMA 1. *Let G be a reductive group scheme acting on geometrically reduced affine algebraic schemes X, Y. Let $\varphi : X \to Y$ be an equivariant morphism. Assume that φ and φ/G are étale, that φ maps closed pseudo-orbits in X onto closed pseudo-orbits in Y, and that φ is injective on at least one closed orbit lying over each connected component of X/G.*

Then $X \cong X/G \times_{(Y/G)} Y$.

Proof. I can assume X/G is connected. Let $Z = Y \times_{X/G} Y/G$ and let $\chi : X \to Z$ be the natural map.



It is to be shown that χ is an isomorphism. Both φ and ψ are étale so χ is étale [SGAIV, 3.3]. I will show that χ preserves closed pseudo-orbits.

Let $T \subset X$ be a closed pseudo-orbit, $S = \varphi(T)$ and let $t = \pi_X(T)$, $s = \pi_Y(T)$ be the points in X/G, Y/G parametrising T, S. Consider the

fibers X_t, Z_t and Y_s over t and s. By definition

$$Z_t = Z \times_{X/G} k(t) = \left(Y \times_{Y/G} X/G\right) \times_{X/G} k(t)$$

$$\cong Y \times_{X/G} k(t) \cong \left(Y \times_{Y/G} k(s)\right) \otimes_{k(s)} k(t) = Y_s \otimes_{k(s)} k(t) .$$

$$\begin{array}{ccc} T \subset X_t & \xrightarrow{\varphi_t} & Y_s \supset S \\ \downarrow{\scriptstyle \chi_t} & \nearrow{\scriptstyle \psi_t} & \\ Z_t = Y_s \otimes_{k(s)} k(t) & & \end{array}$$

Since $k(t)$ is a finite algebraic extension of $k(s)$ the morphism ψ_t is finite and the fiber of ψ_t over s is the union of a finite number of closed pseudo-orbits. (They are closed in $\psi_t^{-1}(s)$, and hence in Z_t, because they all have the same dimension.) $\chi_t(T)$ is one of these pseudo-orbits so $\chi_t(T)$ is closed in Z_t and hence in Z because Z_t is closed in Z. Thus $\chi(T)$ is closed in Z.

φ/G is flat so by Proposition 3.2.2,

$$Z/G = \left(Y \times_{X/G} Y/G\right)/G \cong Y/G \times_{Y/G} X/G \cong X/G ,$$

so χ/G is an isomorphism. Thus the pair $(\chi, \chi/G)$ satisfies the conditions of Lemma 1 and to show that χ is an isomorphism it will be enough to prove the lemma under the assumption that φ/G is an isomorphism.

Thus it is necessary to show that if φ/G is an isomorphism so is φ. As in the proof of Proposition 4.1 the hypotheses are stable under the base change $k \to \overline{k}$ and φ is an isomorphism iff $\varphi \otimes \overline{k}$ is an isomorphism because the base change is faithfully flat [Appendix 1]. Thus it can be assumed that k is algebraically closed.

Now let $X \xrightarrow{i} \tilde{X} \xrightarrow{\tilde{\varphi}} Y$ be the Z.M.T. factorisation of φ, which exists because φ is quasi-finite (it is étale). i is a dominant open embedding, $\tilde{\varphi}$ is finite, $\tilde{X}$ is algebraic and G acts, and i and $\tilde{\varphi}$ are

equivariant. By the argument of Lemma 1.2, i preserves closed orbits and $\tilde{\varphi}$ preserves closed orbits because it is finite. Furthermore, i/G and $\tilde{\varphi}/G$ are isomorphisms. To see this, consider first i/G. The comorphism $(i/G)^* : k[\tilde{X}]^G \to k[X]^G$ is the restriction to $k[\tilde{X}]^G$ of the comorphism i^*, and this is injective because i is dominant. Thus $(i/G)^*$ is injective. But $(i/G)^* \circ (\tilde{\varphi}/G)^* = (\varphi/G)^*$ which is by assumption an isomorphism so $(i/G)^*$ is surjective. Thus i/G is an isomorphism. $\tilde{\varphi}/G$ is an isomorphism because $\tilde{\varphi}/G \circ i/G = \varphi/G$ and φ/G is an isomorphism. Thus in proving Lemma 1 it can be further assumed that φ is either an open embedding or a finite morphism.

If φ is an open embedding then $\varphi(X)$ is a stable open subscheme of Y and $Y\backslash\varphi(X)$ a stable closed subscheme. If $Y\backslash\varphi(X)$ is non empty then it contains a closed orbit. But $\varphi/G : X/G \to Y/G$ is an isomorphism so every closed orbit in Y lies in $\varphi(X)$. Thus φ is an isomorphism.

If φ is finite and étale, then it is an étale covering, and $n(y)$, the number of geometric points in the fiber $\varphi^{-1}(y)$ is constant on connected components of Y. X/G is connected so $G(k)$ permutes the components of X and hence also the components of Y, so $n(y)$ is constant on Y. Let T be the orbit in X on which φ is injective, and let $S = \varphi(T)$. $\varphi^{-1}(S)$ is a union of closed orbits, but $\varphi^{-1}(S)$ lies in $X_t = \pi_X^{-1}(t)$, where $t = \pi_X(T)$, because φ/G is an isomorphism. Since X_t contains a unique closed orbit, $\varphi^{-1}(S) = T$. Then evaluating $n(y)$ at a rational point $y \in S$ shows that $n(y) = 1$. Thus φ is a 1-sheeted étale covering, which is an isomorphism. #

PROPOSITION 1. *Let G be a reductive group shceme acting on geometrically normal affine algebraic schemes X and Y. Let $\varphi : X \to Y$ be equivariant. Then*

(i) if φ is an open embedding which preserves closed pseudo-orbits then φ/G is an open embedding;

(ii) assume that there is a closed orbit in X lying over every connected component of X/G. If φ is an étale covering which is an isomorphism on closed pseudo-orbits then φ/G is an étale covering.

Proof. *(i)* φ is an open embedding iff it is étale and radical [EGAIV, 17.9]. If φ is an open embedding which preserves closed pseudo-orbits then by Proposition 1.1, φ/G is étale. It remains only to show that φ/G is radical. By Lemma 1.6.3, φ/G is radical iff $\varphi/G(\overline{k})$ is injective, and this is so iff $\varphi \otimes \overline{k}$ sends closed $G \otimes \overline{k}$ orbits to closed orbits. If S is a closed $G \otimes \overline{k}$ orbit in $X \otimes \overline{k}$ then $p_X(S) = T$ is a closed pseudo-orbit in X and S is a closed subscheme of $p_X^{-1}(T) = T \otimes \overline{k}$. By assumption $T \to X \xrightarrow[\varphi]{} Y$ is a closed embedding so by ascent $T \otimes \overline{k} \to X \otimes \overline{k} \xrightarrow[\varphi \otimes \overline{k}]{} Y \otimes \overline{k}$ is a closed embedding. Thus $\varphi \otimes \overline{k}(S)$ is closed in $Y \otimes \overline{k}$. So $\varphi/G(\overline{k})$ is injective and φ/G is an open embedding.

(ii) By Proposition 1.1, φ/G is étale and by Lemma 1, $X \cong Y \times_{Y/G} X/G$. For $y \in Y/G$ a closed point, let $n(y)$ be the number of geometric points in the fiber $\varphi/G^{-1}(y)$, regarded as a scheme over $k(y)$. By EGAIV, 18.2.9, φ/G is an étale covering iff $n(y)$ is constant on connected components of Y/G. Let $y' \in Y$ be a closed point lying over $y \in Y/G$, and let $n(y)$ be the number of geometric points in the fiber $\varphi^{-1}(y)$, regarded as a scheme over $k(y)$. Then

$$(\varphi/G)^{-1}(y) \otimes_{k(y)} \overline{k} = \left(X/G \times_{Y/G} k(y)\right) \otimes_{k(y)} \overline{k} \cong X/G \times_{Y/G} \overline{k},$$

and

$$\varphi^{-1}(y') \otimes_{k(y')} \overline{k} = \left(X \times_Y k(y')\right) \otimes_{k(y')} \overline{k}$$

$$\cong X \times_Y \overline{k} \cong \left(X/G \times_{Y/G} Y\right) \times_Y \overline{k} \cong X/G \times_{Y/G} \overline{k},$$

so $n(y) = n(y')$. Since φ is an étale covering $n(y)$ is constant on the closed points of components of Y/G . By semicontinuity $n(y)$ is constant on components of Y/G . Thus φ/G is an étale covering. #

CHAPTER 5

1. Grothendieck topologies

A Grothendieck topology on the category of schemes over k, which I will only define informally, is a collection T of morphisms, called the open neighbourhoods of the topology, satisfying certain axioms modelled on those of ordinary topology [SGA3]. If T is a Grothendieck topology, and $\varphi : U \to X$, $\psi : V \to X$ are "open neighbourhoods on X in the topology T", then the fiber product $U \times_X V \to X$ plays the role of intersection in ordinary topology, and base change by φ corresponds to restricting attention to the local behaviour on the "neighbourhood" U. The most important topologies are the following (I only mention some of these because I wish to quote theorems framed in this language.)

1. The Zariski Topology. T is the set of open embeddings. If X is a scheme the T-neighbourhoods on X may be identified with the Zariski open subsets of X, giving the ordinary Zariski topology on X.

2. The Etale Topology. T is the set of étale morphisms.

3. The Etale (f) Topology. T is the set of étale coverings, that is the finite étale morphisms.

4. The f.p.q.c. Topology. T is the set of morphisms which are flat and quasi-compact.

5. The f.p.p.f. Topology. T is the set of morphisms which are flat and of finite presentation.

6. The Flat Topology. T is the set of flat morphisms.

The most important of these are 1 and 2. 4 and 5 will be mentioned only in passing. On the category of noetherian schemes 4, 5 and 6 coincide.

If $k = \mathbb{C}$ then nice schemes (say nonsingular Serre varieties) over $\mathbb{C}$ carry a topology induced from the ordinary topology on $\mathbb{C}$. Since étale

maps over $\mathbb{C}$ are locally analytic isomorphisms, properties of the complex topology are closely related to properties of the étale topology (rather than the Zariski topology, whose open sets are too large to describe some local behaviour). In general, the étale topology can be regarded as a substitute for the complex topology, which is not available over an arbitrary field.

2. Fiber bundles

All schemes are assumed affine and algebraic. Let T be one of the above Grothendieck topologies.

A trivial fiber bundle with fiber F and base B is a morphism $\pi : X \to B$ and a morphism $X \to F$ such that the square

$$\begin{array}{ccc} X & \to & F \\ \downarrow & & \downarrow \\ B & \to & \mathrm{Spec}(k) \end{array}$$

is cartesian. That is, $X = B \times F$. A fiber bundle in the topology T, with base B and fiber F, is a morphism $\pi : X \to B$ which is locally trivial in T. That is, for every closed point $x \in B$, there is a morphism $\sigma : C \to B$ in T, such that $x \in \sigma(C)$, and $X_C = X \times_B C \to C$ is a trivial bundle with fiber F. σ is called a T-trivialisation of X around x.

If $\pi : X \to B$ is a bundle, it is easy to see that for any $B' \to B$, $\pi_{B'} : X_{B'} \to B'$ is a bundle with the same fiber. If $\pi : X \to B$ is a bundle, then since B is neotherian, it is covered by a finite collection of trivialisations $C_i \xrightarrow{\sigma_i} B$. (That is, the union of the $\sigma_i(C_i)$ covers B.) By taking the disjoint union of the C_i, there is a trivialisation $C = \coprod_i C_i \to B$ which is flat and surjective (though it may not be in T). Clearly C may be constructed to be affine. Thus every bundle has an affine

faithfully flat trivialisation.

Let G be a reductive group scheme and let $\pi : X \to B$ be a bundle with fiber F. X is an equivariant bundle if G acts on X and F, and acts trivially on B, π is equivariant, and the trivialisations of X can be chosen to be equivariant. An equivariant bundle has, by the construction above, an equivariant faithfully flat trivialisation. A torseur [SGA3, G.-D.III, §4] is an equivariant bundle whose fiber F is isomorphic to G acting on itself by translation. If $\pi : X \to B$ is a torseur, and $C \to B$ is any morphism, then it is easy to see that $\pi_C : X_C = X \times_B C \to C$ is also a torseur.

LEMMA 1. *Let G be a reductive group scheme, and let $\pi : X \to B$ be an equivariant affine algebraic fiber bundle, with fiber F. If $F/G = \mathrm{Spec}(k)$, then $B = X/G$, and π is a quotient map. In particular, a torseur is a quotient map.*

Proof. Let $\sigma : C \to B$ be an equivariant, faithfully flat, affine trivialisation of π. Since being a quotient map is stable under faithfully flat descent (by Proposition 3.2.2), if π_C is a quotient then so is π. So π can be assumed trivial. Then

$$X/G = (B \times F)/G = B \times (F/G) = B \times \mathrm{Spec}(k) = B ,$$

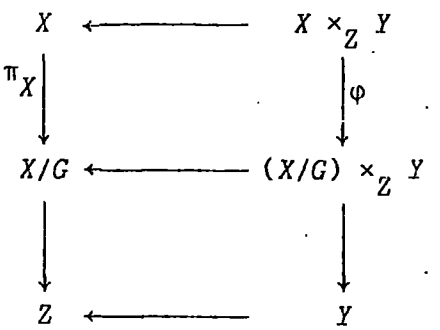
by the same proposition, since B is flat over $\mathrm{Spec}(k)$. #

Thus if $\pi : X \to B$ is a torseur, then π is a quotient map. It can be shown that π is an orbit map in the following generalised sense: for any $R \in \mathcal{A}lg/k$, the fibers of $\pi(R) : X(R) \to B(R)$ are the orbits of $G(R)$ on $X(R)$. $\pi(R)$ need not however be surjective, so π may not be a quotient in the category $[\mathcal{A}lg/k, S]$.

COROLLARY. *Let X, Y, Z be affine algebraic G spaces, where G is a reductive group scheme. Let $X \to Z$, $Y \to Z$ be equivariant, and assume that G acts trivially on Z and Y. If $\pi_X : X \to X/G$ is a torseur,*

then $(X \times_Z Y)/G \cong (X/G) \times_Z Y$.

Proof. $X \to Z$ factorises via $\pi_X : X \to X/G$.



Since π_X is a torseur and φ comes from π_X by a base change, φ is a torseur. By the lemma φ is a quotient map. #

This corollary will be useful in cases where Proposition 3.2.2 does not apply.

3. Associated bundles

From now on I will be dealing with both left and right actions so I will begin writing X/G for the quotient of a right action and $G\backslash X$ for the quotient of a left action.

Let $X \to B$ be a torseur (with G acting on the right), and let F be an affine algebraic scheme with G acting on the left. Let G act on $X \times F$ as follows: if $R \in Alg/k$, $G(R)$ acts on $(X \times F)(R) = X(R) \times F(R)$ by $g\cdot(x, f) = (x\cdot g^{-1}, g\cdot f)$. The quotient of this action is written $X *_G F$. The map $X \times F \xrightarrow{\text{proj}} X \to B$ is equivariant, so there is an induced map $\pi : X *_G F \to B$. π is a fiber bundle with fiber F . To see this, let $x \in B$ be a closed point and let $U \to B$ be an affine trivialisation of X around x . Thus $X_U = X \times_B U \cong U \times G$. Then

$$(X *_G F)_U = (X \times F)/G \times_B U \cong (X \times F \times_B U)/G \cong (X_U \times F)/G$$

$$\cong (G \times U \times F)/G \cong U \times F$$

by Proposition 3.2.2 *(ii)* and *(iii)*. Thus U is a trivialisation of $X *_G F$ around $x \in B$.

$X * F \to B$ is called the bundle associated to the torseur $X \to B$ and the fiber F. It can be thought of intuitively as the bundle constructed from X by replacing the fiber G by F, while retaining the global structure of the bundle X over B.

4. Induced bundles

Let G be a reductive group scheme, and H a reductive subgroup scheme. Assume that the topology T is such that $G \to G/H$ is a H torseur. If V is an affine algebraic scheme, on which H acts on the left, then G acts on the left of $G *_H V$ by acting by left translation on the left hand factor, and the bundle map $G *_H V \to G/H$ is equivariant. I will call $G *_H V$ the G bundle induced from the H scheme V. The justification for calling it this is in the following proposition.

PROPOSITION 1. *Let G be a reductive group scheme, and H a reductive subgroup scheme. Let T be a Grothendieck topology from among those mentioned in §1, such that $G \to G/H$ is a H torseur. Let V be an algebraic affine scheme, with a left H action, and let $\pi : G *_H V \to G/H$ with the left action of G by left translation be the induced G-bundle. Let $i : H \to G$ be the inclusion and let $j = H\backslash(i \times V) : V \to G *_H V$.*

$$\begin{array}{ccc} H \times V & \xrightarrow{i \times V} & G \times V \\ \downarrow & & \downarrow \\ V & \xrightarrow[j]{} & G *_H V \end{array}$$

Then

(1) *j has the following universal characterisation. $j : V \to G *_H V$ is H-equivariant, and if $\varphi : V \to X$ is any H equivariant morphism to an*

affine algebraic G *scheme* X, *then* φ *factorises uniquely through* j.

(2) $G *_H V \xrightarrow{\pi} G/H$ *is a bundle with fiber* V.

(3) j *is a closed embedding, which maps* V *isomorphically onto the fiber of* $G *_H V$ *over the distinguished point of* G/H.

(4) *Let* x_0 *be a rational* H*-fixed point on* V, *which will be identified with its image in* $G *_H V$. *Let* T *be the* G *orbit through* x_0 *in* $G *_H V$. *Then* T *is a closed orbit which maps isomorphically onto* G/H *under* π. *If* V *is smooth, then so is* $G *_H V$, *and* T *is transversal to* V *at* x_0.

(5) $G\backslash(G *_H V) \cong H\backslash V$.

(6) *Let* $U \to H\backslash V$ *the embedding of an affine subscheme. Then* $(G *_H V)_U \cong G *_H (V_U)$, *and* $G *_H (V_U) \to G *_H V$ *is an embedding.*

Proof. (1) follows immediately from the definitions.

(2) This follows from the properties of associated bundles.

(3) First consider the case $V = H$. Then $G *_H H = G$ and $j : H \to G$ is just the inclusion map. All algebraic subgroups are closed [G.-D.II, §5.5.1] so j is a closed embedding. It must be shown that the following square is Cartesian.

$$\begin{array}{ccc} H & \to & G \\ \downarrow & & \downarrow \\ e & \to & G/H \end{array}$$

where e is the distinguished point of G/H (e is rational). By assumption, $G \to G/H$ is a torseur, so there is a faithfully flat affine trivialisation $B \to G/H$. By faithful flatness, it is enough to check that the following square, obtained by base change, is Cartesian [see Appendix 1, §2].

$$\begin{array}{ccc} B \times_{G/H} H & \to & B \times H \\ \downarrow & & \downarrow \\ B \times_{G/H} e = B_e & \to & B \end{array}$$

But $B \times_{G/H} H = (B \times_{G/H} e) \times_e H = B_e \times_e H = B_e \times H$, since $k(e) = k$, and $B_e \times H = B_e \times_B (B \times H)$, so the second square is Cartesian.

Now return to the general case. Consider the following diagram

$$\begin{array}{ccccc} H \times V & \longrightarrow & G \times V & \xrightarrow{\text{proj}} & G \\ \downarrow & \text{I} & \downarrow & \text{II} & \downarrow \\ V & \xrightarrow{j} & G *_H V & \longrightarrow & G/H \\ \downarrow & \text{III} & \downarrow & & \\ e & \longrightarrow & G/H & & \end{array}$$

I will show that all squares are cartesian. First consider square II. Since both $G \to G/H$ and $G *_H V \to G/H$ are fiber bundles, they have a common faithfully flat trivialisation $B \to G/H$. By faithful flatness, it will be enough to show that the base-changed square is cartesian:

$$\begin{array}{ccc} (B \times H) \times V & \longrightarrow & B \times H \\ \downarrow & & \downarrow \\ B \times V & \longrightarrow & B \end{array}$$

which is clear. So square II is cartesian. A consequence of this which will be needed below is that $G \times V \to G *_H V$ is a H-torseur.

Now consider the square made by combining I and III.

$$e \times_{G/H} (G \times V) = (e \times_{G/H} G) \times V = H \times V ,$$

by the special case treated first. Now consider square III.

$$e \times_{G/H} (G *_H V) = e \times_{G/H} (G \times V)/H = (e \times_{G/H} G \times V)/H ,$$

by the corollary of Lemma 2.1, since $G \times V \to G *_H V$ is a torseur, and $(e \times_{G/H} G \times V)/H = (H \times V)/H \cong V$ by Proposition 3.2.2. Thus square III is

cartesian, and j is an isomorphism onto the fiber of $G *_H V$ over e. Since e is closed j is a closed embedding.

(4) Consider $H \times V \to G \times V \underset{i}{\rightleftarrows} G$, where $G \times V \to G$ is the projection and i is the embedding of G onto the slice through x_0. Let H act, "in the centre" of $H \times V$ and $G \times V$, and by right translation on G. Then the diagram is H-equivariant i is an H map, because x_0 is a fixed point of V, and passing to the quotient gives

$$V \underset{j}{\longrightarrow} G *_H V \underset{i/H}{\rightleftarrows} G/H .$$

Clearly i/H is an isomorphism of G/H onto the orbit T through x_0. i/H is a closed embedding, as may be seen by going to the dual category of rings, where $(i/H)^*$ is surjective. Thus T is a closed orbit.

If V is smooth, then $G *_H V$ is smooth, by local triviality (smoothness is preserved by faithfully flat descent). Since x_0 is a rational point, the tangent space as defined in EGAIV is the same as the usual definition for Serre varieties. It is the dual vector space to $\mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of the local ring at x_0, and it is a vector space over $k(x_0) = k$. So the diagram above gives the following commuting sequence of tangent spaces

$$0 \to T_{x_0}(V) \xrightarrow[T_{x_0}(V)]{} T_{x_0}\left(G *_H V\right) \to T_{x_0}(G/H) \to 0 .$$

I claim that this sequence is exact. Exactness at $T_{x_0}(G/H)$ is clear. j is a closed embedding so $T_{x_0}(j)$ is injective. Exactness in the middle follows from the fact that $\dim\left(G *_H V\right) = \dim V + \dim G/H$.

The latter fact follows from local triviality as follows. Let $y = \pi(x_0)$. These are both rational points. Let $\sigma : B \to G/H$ be a

faithfully flat trivialisation, and let τ be its pull back.

$$\begin{array}{ccccc} y & (G *_H V) & \xleftarrow{\tau} & (G *_H V)_B = B \times V \\ \downarrow & \downarrow & & \downarrow \\ x_0 & G/H & \xleftarrow[\sigma]{} & B \end{array}$$

Since σ and τ are flat,

$$\dim B = \dim G/H + \dim \sigma^{-1}(x_0)$$

and

$$\dim(G *_H V)_B = \dim B + \dim V = \dim(G *_H V) + \dim \tau^{-1}(x_0) .$$

But

$$\tau^{-1}(x_0) = (G *_H V)_B \times_{(G*_H V)} x_0 = B \times_{G/H} x_0 \cong B \times_{G/H} y ,$$

since both x_0 and y are rational. Thus the sequence is exact.

Clearly $T_{x_0}(i/H) : T_{x_0}(G/H) \to T_{x_0}(G *_H V)$ is an isomorphism onto $T_{x_0}(T)$. Thus $T_{x_0}(G *_H V) = T_{x_0}(V) \oplus T_{x_0}(T)$. That is, T and V are transversal at x_0.

(5) Let H act on $G \times V$ "in the middle", and let G act by left translation on the left. Since these actions commute, the quotients may be calculated in any order. Thus

$$G\backslash(G *_H V) = G\backslash[(G \times V)/H) \cong (G\backslash(G \times V))/H = H\backslash V ,$$

using Proposition 3.2.2.

(6) Since $(G *_H V)_U \to G *_H V$ is an embedding, it will be sufficient to show that $(G *_H V)_U \cong G *_H (V_U)$. But

$$(G *_H V)_U = (G \times V)/H \times_{G/H} U = (G \times V \times_{G/H} U)/H = (G \times V_U)/H = G *_H V_U ,$$

because $G \times V$ is an H torseur. This completes the proof of the proposition. #

From (1) it can be seen that the induced bundle construction may be

regarded as a non-linear analogue of induced representations. For example there is a "Frobenius Reciprocity": for any H scheme V and G scheme X,

$$\mathrm{Hom}_G(G *_H V, X) \cong \mathrm{Hom}_H(V, X) .$$

5. Flat and étale torseurs

Let G be an affine algebraic group scheme, and let H be an algebraic subgroup scheme. H is automatically closed and affine. Then a categorical quotient G/H exists in the category of schemes over k, even if H is not reductive [SGA3, G.-D.III, §3.5]. G/H may not be affine, if H is not reductive. If H is reductive, then G/H may of course be constructed as usual by the method of invariants.

The purpose of this section is to prove the following proposition.

PROPOSITION 1. *Let G be an affine algebraic group scheme, H an algebraic subgroup scheme, and $\pi : G \to G/H$ the quotient map. Assume that G/H is affine. Let T be a Grothendieck topology. Then*

(i) if T is the flat topology, then π is an H torseur in T;

(ii) if T is the étale (f) topology, and H is reductive, then π is an H torseur in T.

Proof. Before beginning the proof, I will summarise for the convenience of the reader some results which I will want to quote, and the terminology in which they are written.

In [D.-G.], a torseur means a torseur for the f.p.p.f. topology, and a torseur dur means a torseur for the f.p.q.c. topology. In [SGA3 and Demazure 1964], all schemes are taken over a given base scheme S, and a bundle means a bundle for the f.p.q.c. topology with base S. Statements about bundles over a fixed base S translate immediately into statements

about bundles over an arbitrary base as follows. If $X \to B$ is a G bundle, say over a base field k, then by the isomorphism $(G \times B) \times_B X \cong G \times B$, X is a $G \times B$ bundle over B, where $G \times B$ is a group scheme over B. I collect together the following facts:

FACT 1. If G, H are as in the proposition, then $G \to G/H$ is a torseur for the f.p.p.f. topology [G.-D.III, §4.1.8].

FACT 2. If G is a connected reductive group scheme, if $\pi : X \to B$ is a G-torseur for the f.p.q.c. topology, and if B is normal, then π is a G-torseur for the étale (f) topology [Demazure 1964, SGA3].

FACT 3. If G is a finite group, acting without fixed points on an affine scheme X, then $X \to X/G$ is a G-torseur for the étale (f) topology [SGAI, §V, 2.6].

Since all our schemes are noetherian, the f.p.p.f. f.p.q.c. and flat topologies coincide. Part *(i)* of Proposition 1 follows immediately from these facts, as does part *(ii)* if H is either connected or finite.

Before proceeding, consider the following lemma.

LEMMA 1. *Let H be a reductive group scheme, acting on the right on an affine algebraic scheme X. Assume that $X \to X/H^0$ is a torseur in the flat topology. Then there is an action of the group scheme H/H^0 on X/H^0, and $(X/H) \cong (X/H^0)/(H/H^0)$.*

Proof. The action of H/H^0 on X/H^0 is a little difficult to describe because $H(R) \to H/H^0(R)$ is not necessarily surjective for all $R \in Alg/k$. First note that $(X \times H)/(H^0 \times H^0) \cong X/H^0 \times X/H^0$, because $X \times H \to X/H^0 \times H/H^0$ is a $H^0 \times H^0 = (H \times H)^0$ torseur for the flat topology. Using the fact that H^0 is normal in H, the map $X \times H \to X \to X/H^0$ is $H^0 \times H^0$ equivariant, when $H^0 \times H^0$ acts on $X \times H$ by the product of the H^0 actions on X and H, and $H^0 \times H^0$ acts

trivially on X/H^0 . So this map factors through the quotient $(X \times H)/H^0 \times H^0$:

$$\begin{array}{ccc} X \times H & \longrightarrow & X \\ \downarrow & & \downarrow \\ (X \times H)/(H^0 \times H^0) & \longrightarrow & X/H^0 \\ \| & \nearrow_{\beta} & \\ X/H^0 \times H/H^0 & & \end{array}$$

β is the required action. β is easier to describe using the SAG3 type construction of quotients. In that picture it is obvious that it is in fact an action.

Now let $\overline{k}$ be an algebraic closure of k and consider the sequence of groups

$$1 \to H^0(\overline{k}) \to H(\overline{k}) \to H/H^0(\overline{k}) \to 1 \ .$$

I claim that this is exact. First of all, $H^0 \to H$ is an open embedding, so it is a monomorphism so $H^0(\overline{k}) \to H(\overline{k})$ is injective. Furthermore, H^0 is the scheme theoretic fiber of H over the identity in H/H^0 by Proposition 5.4.1. Thus $H^0(\overline{k})$ is the kernel of $H(\overline{k}) \to H/H^0(\overline{k})$, and the sequence is exact in the middle. Finally, $H \to H/H^0$ is surjective, so by Lemma 1.6.3, $H(\overline{k}) \to H/H^0(\overline{k})$ is surjective.

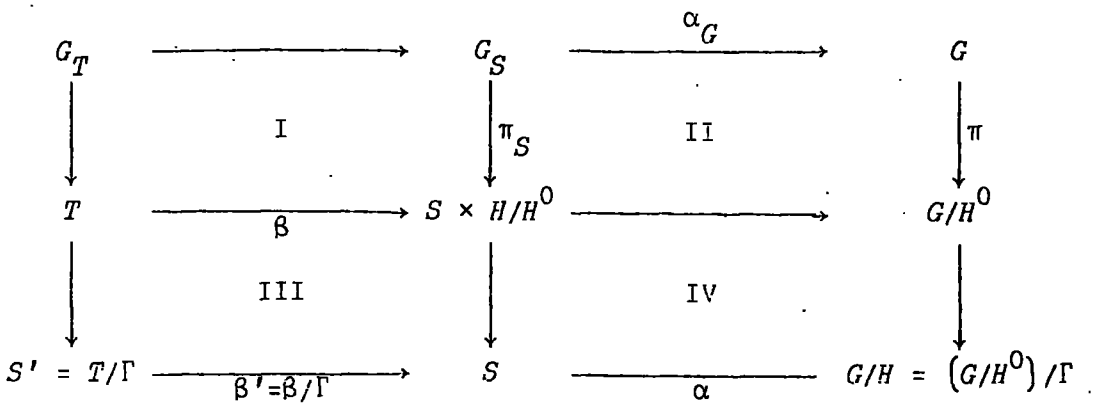
Now

$$k[X]^H = k[X] \cap \overline{k}[X]^{H(\overline{k})} = k[X] \cap \left(\overline{k}[X]^{H^0(\overline{k})}\right)^{H/H^0(\overline{k})} = \left(k[X]^{H^0}\right)^{H/H^0} \ . \qquad \#$$

Now returning to the proof of the proposition, assume that H/H^0 is a constant finite group (we know it is étale). Let

$$\Gamma = H/H^0(k) = H/H^0(\overline{k}) = H(\overline{k})/H^0(\overline{k})$$

be the associated finite group. Consider the following diagram (squares I, II and IV are cartesian):



where S is a finite, etale, affine trivialisation of the Γ-torseur $G/H^0 \to G/H$, and T is a finite, étale, affine trivialisation of the H^0-torseur π_S. I will show that for an appropriate choice of T, $S' = T/\Gamma$ is a finite, etale trivialisation of $G \to G/H$. Let S_0 be the open subscheme of $S \times H/H^0$ which projects to the identity in H/H^0: $S \times H/H^0$ consists of the disjoint union of $|\Gamma|$ copies of S_0, and Γ acts by permuting these. Let $(G_S)_0$ be the sub-bundle lying over S_0. It is a H^0-torseur, so let T_0 be a finite, étale, affine trivialisation of $(G_S)_0$. Let $T = T_0 \times H/H^0$, and let Γ act on T by permuting copies of T_0. Then $\beta : T \to S \times H/H^0$ is the required trivialisation. Clearly, $T/\Gamma \cong T_0$ and $\beta' = \beta/\Gamma$ is isomorphic to β restricted to T_0. Thus β/Γ is an étale covering, and square III is cartesian.

So S' is an etale covering of G/H, and

$$G_{S'} = G_T = T \times H^0 = \left(S' \times_S (S \times H/H^0)\right) \times H^0 = S' \times (H/H^0 \times H^0) .$$

Thus $G_{S'}$ is a trivial bundle, with fiber $H/H^0 \times H^0$. But $G_{S'}$ is the pullback of $G \to G/H$, which is an H-torseur in the flat topology, so $G_{S'} \to S'$ is an H torseur. Thus the fiber $H/H^0 \times H^0$ is isomorphic to to H, as an H space.

Now consider the general case. By [EGAIV, §4.5.11], there is a finite separable extension K of k such that the connected components of $H/H^0 \otimes_k K$ are geometrically connected. Since H/H^0 is étale over k, $H/H^0 \otimes_k K$ is étale over K so is a constant scheme. Thus $(H \otimes_k K)/(H \otimes_k K)^0 = (H \otimes_k K)/\left(H^0 \otimes_k K\right) = H/H^0 \otimes_k K$ is a finite constant group. Thus by what is shown above, $(G \otimes_k K) \to (G \otimes_k K)/(H \otimes_k K)$ is a $H \otimes_k K$ torseur in the étale (f) topology. Let β be a finite étale trivialisation:

$$\begin{array}{ccccc} G & \longleftarrow & G \otimes_k K & \longleftarrow & S \times_K (H \otimes_k K) = S \times_k H \\ \downarrow & & \downarrow & & \downarrow \\ G/H & \longleftarrow & G/H \otimes_k K = (G \otimes_k K)/(H \otimes_k K) & \xleftarrow{\beta} & S \end{array}$$

Since $G/H \otimes_k K \to G/H$ is an étale covering, S is a finite étale trivialisation of $G \to G/H$, and the proposition is proved. #

CHAPTER 6

1. Stabilisers and orbits

Let G be an affine algebraic group scheme, acting on an affine algebraic scheme X, and let $x \in X(k)$ be a rational point. Let $\theta : G \to X$ be the orbit map through x. θ is the composition $G \to G \times \mathrm{Spec}(k) \to G \times X \to X$, and is given in the functorial picture by: if $R \in Alg/k$, $\theta_x(R) : G(R) \to X(R) : g \mapsto g \cdot x_R$, where x_R is the image of x under $X(k) \to X(R)$. By Proposition 3.2.3 the image is locally closed, and carries a natural subscheme structure called the orbit through x.

The stabiliser of x is the fiber of θ_x over x: $G_x = G \times_X k(x) = G \times_X k$, as x is rational. G_x is an algebraic closed subgroup scheme of G. Fibers may be computed by passing to the functor category. In this picture, if $R \in Alg/k$, then $G_x(R) = \{g \in G(R) : g \cdot x_R = x_R\} = G(R)_{x_R}$. This definition agrees with that of [D.-G.III, §3.5, II, §1.3]. If T is the orbit through x, then the natural map $G/G_x \to T$ is an isomorphism [G.-D.III, §3.5].

LEMMA 1. *Let G, X, x, θ, T be as above, with G reductive.*

(1) *The following are equivalent:*

(i) G_x *is geometrically reduced;*

(ii) G_x *is smooth;*

(iii) θ *is separable at* x *;*

(iv) θ *is smooth at* x *;*

(v) θ *is separable;*

(vi) θ *is smooth.*

If any of these equivalent conditions holds I will say that T is a

separable orbit.

(2) *If T is a closed separable orbit, then G_x is reductive.*

Proof. First a note on definitions. An algebraic scheme X is geometrically reduced iff $X \otimes_k \bar{k}$ is reduced, iff it is reduced and the local fields at the generic points of X are separable extensions of the base field. X is smooth if it is geometrically regular, that is, if $X \otimes_k \bar{k}$ is regular. A morphism $f : X \to Y$ is separable (respectively smooth) at $y \in Y$ if it is flat at y, and the fiber over y is geometrically reduced (respectively smooth) [EGAIV, §6.8.1].

(i) $\Leftrightarrow$ *(ii).* *(ii)* $\Rightarrow$ *(i)* by definition (regular $\Rightarrow$ reduced). For algebraic group schemes, geometrically reduced implies smooth, by [G.-D.II, §5.2.1].

(i) $\Leftrightarrow$ *(iii)* and *(ii)* $\Leftrightarrow$ *(iv).* Since θ is flat, these are true by definition.

(iv) $\Leftrightarrow$ *(vi)* $\Leftarrow$ is trivial. So assume θ is smooth at x. Then $\theta \otimes \bar{k}$ is smooth at the rational point of $T \otimes \bar{k}$ lying over x. $G(\bar{k})$ acts transitively on $T(\bar{k})$, so θ is smooth over every closed point of T. The set where θ is smooth is open [EGAIV, §6.8.7] so θ is smooth.

(iii) $\Leftrightarrow$ *(v)* $\Leftarrow$ is trivial. But *(iii)* $\Leftrightarrow$ *(iv)* $\Rightarrow$ *(vi)* $\Rightarrow$ *(v).*

Proof of 2. This is well known if k is algebraically closed [Richardson 1977]. Let $x \in T$ be rational, and let x' be the rational point of $X \otimes \bar{k}$ lying over x, and let T' be the orbit in $X \otimes \bar{k}$ lying over T. Clearly T is closed and separable iff T' is, so $(\theta_{x'} = \theta_x \otimes \bar{k})$. Thus $G_x \otimes \bar{k}$ is reduced iff $(G \otimes \bar{k})_{x'}$ is reduced.

Now $(G_x \otimes \bar{k})(\bar{k}) = G_x(\bar{k}) = G(\bar{k})_{x'}$, by definition, and $G(\bar{k})_{x'} = (G \otimes \bar{k})_{x'}(\bar{k})$. Thus $G_x \otimes \bar{k}$ and $(G \otimes \bar{k})_{x'}$ have the same rational points. If T is separable then $G_x \otimes \bar{k} = (G \otimes \bar{k})_{x'}$.

If T is closed then T' is closed so $(G \otimes \overline{k})_{x'} = G_x \otimes \overline{k}$ is reductive. Thus G_x is reductive. #

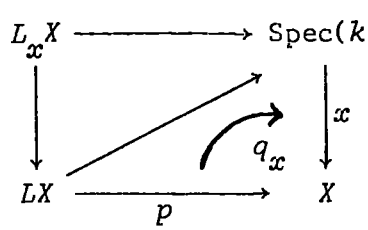
2. Tangent schemes

If R is a ring, let $R[\varepsilon]$ be the R algebra generated by an element ε such that $\varepsilon^2 = 0$. There are natural R-algebra homomorphisms $p : R[\varepsilon] \to R : \varepsilon \mapsto 0$, and $\sigma : R \to R[\varepsilon]$. Multiplication by ε is an endomorphism of the R-module $R[\varepsilon]$, mapping $R + 0\varepsilon$ isomorphically onto $R\varepsilon$.

If X is a scheme, define a functor L_X by $L_X(R) = X(R[\varepsilon])$, and, if $\varphi : R \to S$ is a homomorphism, $L_X(\varphi) = X(\varphi')$, where $\varphi' : R[\varepsilon] \to S[\varepsilon]$ is the unique extension of φ sending $\varepsilon \mapsto \varepsilon$. Then the functor L_X is representable, and defines a scheme LX, called the scheme theoretic tangent bundle of X [G.-D.II, §4.4].

The homomorphisms $R \underset{p}{\overset{\sigma}{\rightleftarrows}} R[\varepsilon]$ define morphisms, which I will also call $p,\sigma : LX \underset{\sigma}{\overset{p}{\rightleftarrows}} X$, a projection $p = X(p) : LX(R) = X(R[\varepsilon]) \to X(R)$ onto X, and $\sigma = X(\sigma) : X(R) \to X(R[\varepsilon]) = LX(R)$, which is a section of p.

If $x \in X$ is a point, then the fiber of p over x is written L_xX and called the tangent scheme at x. If x is a rational point, then there is a morphism $q_x : LX \to \mathrm{Spec}(k) \xrightarrow{x} X$ (where $LX \to \mathrm{Spec}(k)$ is the canonical map), and L_xX is the kernel of the pair $(p, q_x) : L_xX \to LX \underset{q_x}{\overset{p}{\rightrightarrows}} X$:



If $x \in X$ is rational, and we assume as usual that X is affine, then if $\mathfrak{m}$ is the maximal ideal of $k[X]$ at x, $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over $k(x) = k$, and its dual $(\mathfrak{m}/\mathfrak{m}^2)^*$ is called the tangent space T_xX at x. T_xX is a vector space over k, and must not be confused with L_xX, which is a scheme. Now $L_xX(R)$ is the kernel of the pair $p(R), q_x(R) : X(R[\varepsilon]) \to X(R)$. That is, it is the set of k algebra homomorphisms $\varphi : k[X] \to R[\varepsilon]$ such that the following diagram commutes:

$$\begin{array}{ccc} k[X] & \longrightarrow & R[\varepsilon] \\ \downarrow{\scriptstyle ev_x} & & \downarrow{\scriptstyle p} \\ k & \longrightarrow & R \end{array} .$$

Writing $\varphi = \alpha + \beta\varepsilon$, with $\alpha, \beta : k[X] \to R$, this just says that α is evaluation at x and β is a derivation of $k[X]$ at x. Thus there is an isomorphism

$$\begin{aligned} L_xX(R) &\cong \mathrm{Der}_x(k[X], R) \\ &\cong \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, R) \\ &\cong (\mathfrak{m}/\mathfrak{m}^2)^* \otimes_k R = T_xX \otimes_k R . \end{aligned}$$

Thus L_xX is the linear scheme associated with the vector space T_xX, and $T_xX = L_xX(k)$.

If X is itself a linear scheme, say $X(R) = V \otimes_k R$, where V is a finite dimensional vector space over k, then for each R, $X(R)$ is an R module, and $LX(R) = X(R[\varepsilon])$ is an R module via $R \xrightarrow{\sigma} R[\varepsilon]$. $p(R)$ and $\sigma(R)$ are then R module homomorphisms, and multiplication by ε gives an R module endomorphism of $LX(R)$. Then $\sigma : X \to LX$ embeds X linearly as a closed subscheme in LX; if ε is the endomorphism of LX

given by "multiplication by ε", then $\varepsilon \circ \sigma$ maps X isomorphically onto L_0X, the fiber of p over 0. If $x \in X$ is a rational point, then there is a scheme automorphism τ_x of X, "translation by x", given by $\tau_x(R) : X(R) \to X(R) : y \mapsto y + x_R$, where x_R is the image of x under the specialisation map $X(k) \to X(R)$. X also acts by translation on LX: first lift x to LX by the section σ, then translate. τ_x then gives an isomorphism $L_0X \cong L_xX$.

If a group scheme G acts on X then G acts on LX as follows. For each R, $G(R[\varepsilon])$ acts on $X(R[\varepsilon]) = LX(R)$, so $G(R)$ acts, via the specialisation map $G(R) \to G(R[\varepsilon])$. The morphisms p, σ are then equivariant. If $x \in X$ is a rational fixed point (that is, the reduced subscheme supported by x is G stable), then q_x is equivariant, so T acts on L_xX.

If T is a G orbit on X, and $x \in T$ is a rational point then G_x fixes x so G_x acts on $L_x(T)$ and $L_x(X)$. Clearly $L_x(T)$ is a G_x stable subscheme of $L_x(X)$. Looking at rational points, $T_x(T)$ is a $G_x(k)$ stable subspace of $T_x(X)$.

If X is linear, and G acts linearly, then the endomorphism ε and translation by x (x a rational fixed point) are G morphisms, so the isomorphism $X \cong L_xX$ is a G isomorphism when G fixes x.

3. Transversality

If X is an algebraic scheme and $x \in X$ is a regular rational point, and Z, W are closed subschemes of X passing through x, such that $\dim_x X = \dim_x Z + \dim_x W$, then Z and W are said to be transversal at x

if Z and W are regular at x and $T_x(X) = T_x(Z) \oplus T_x(W)$ [EGAIV, 17.13.8].

LEMMA 1. *Let $\varphi : X \to Y$ be a morphism of algebraic schemes. Let $x \in X$ be a rational point, and assume that X and Y are regular at x and $y = \varphi(x)$ respectively. Let Z, W be subschemes of X such that $\dim_x X = \dim_x Z + \dim_x W$, which meet transversally at x. Let $Z' = \varphi(Z)$, $W' = \varphi(W)$ be the images of W and Z, assumed to be closed subschemes of Y, and assume that $\dim_y Y = \dim_y Z' + \dim_y W'$ and Z' and W' meet transversally at y. Let $\varphi_Z : Z \to Z'$ and $\varphi_W : W \to W'$ be the restrictions of φ. If φ_Z and φ_W are étale at x then φ is étale at x.*

Proof. Let $\mathcal{O}_{X,x}$ be the local ring at x with the $\mathfrak{m}_x$-adic filtration. Let $\hat{\mathcal{O}}_{X,x}$ be the completion, and let $\mathrm{Gr}(\mathcal{O}_{X,x}) = \bigoplus_r \mathfrak{m}_x^r/\mathfrak{m}_x^{r+1}$ be the associated graded ring. Since x is a regular point, $\mathrm{Gr}(\mathcal{O}_{X,x})$ is the polynomial ring on the tangent space $T_x(X)$. Because x is rational, φ is étale at x iff $\hat{\varphi}_x : \hat{\mathcal{O}}_{Y,y} \to \hat{\mathcal{O}}_{X,x}$ is an isomorphism [EGAIV, 17.6.3]. By [Atiyah-Macdonald, p. 122] this is so iff $\mathrm{Gr}(\varphi_{X,x}) : \mathrm{Gr}(\mathcal{O}_{Y,y}) \to \mathrm{Gr}(\mathcal{O}_{X,x})$ is an isomorphism.

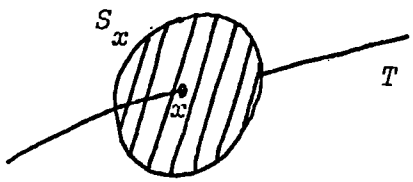
Since $T_x(X) = T_x(Z) \oplus T_x(W)$, $\mathrm{Gr}(\mathcal{O}_{X,x}) = \mathrm{Gr}(\mathcal{O}_{Z,x}) \otimes_k \mathrm{Gr}(\mathcal{O}_{W,x})$ and similarly for $\mathrm{Gr}(\mathcal{O}_{Y,y})$, so $\mathrm{Gr}(\varphi_{X,x}) = \mathrm{Gr}(\varphi_{Z,x}) \otimes_k \mathrm{Gr}(\varphi_{W,x})$. But it follows from the assumptions that $\mathrm{Gr}(\varphi_{Z,x})$ and $\mathrm{Gr}(\varphi_{W,x})$ are isomorphisms. #

4. Etale slices

Let G be a reductive group, acting on the right on an affine

algebraic scheme X, let T be a closed orbit in X, and let $x \in T$ be a rational point. Let S_x be a G_x stable subscheme of X passing through x such that $\dim_x(X) = \dim_x(S_x) + \dim_x(T)$. The embedding $S_x \to X$ induces a G equivariant morphism $\varphi : S_x *_{G_x} G \to X$. S_x is said to be a slice at x if φ and φ/G are étale and the square

$$\begin{array}{ccc} S_x *_{G_x} G & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ (S_x *_{G_x} G)/G \cong S_x/G_x & \xrightarrow{\varphi/G} & X/G \end{array}$$



is cartesian. If a slice S_x exists, then in a neighbourhood (in the etale sense) of T, X looks like the induced bundle $S_x *_{G_x} G$.

PROPOSITION 1. *Let G be a reductive group scheme acting on a smooth affine scheme X. Let T be a closed separable orbit, $x \in T$ a rational point, and let V_x be a G_x stable smooth subscheme of X passing through x such that $\dim_x(X) = \dim_x(T) + \dim_x(V_x)$. Then V_x contains an open subscheme S_x which is a slice at x iff V_x is transversal to T at x.*

Proof. Let V_x' be the copy of V_x in $V_x *_{G_x} G$, let x' be the point in V_x' corresponding to $x \in V_x$, and let T' be the closed orbit through x' in $V_x *_{G_x} G$. Let $\psi : V_x *_{G_x} G \to X$ be the induced map. By definition ψ maps V_x' isomorphically onto V_x, and ψ maps T' isomorphically onto T because x' and x have the same stabiliser.

If V_x contains a slice S_x, then without loss of generality $S_x = V_x$. Since $\varphi = \psi$ is étale at x', $T_{x'}(\varphi) : T_{x'}(V_x *_{G_x} G) \to T_x(X)$ is an isomorphism. But V_x' and T' are transversal at x' by Propositions

5.4.1 and 5.5.1, so V_x and T are transversal. Conversely, if V_x and T are transversal, then by Lemma 3.1, ψ is étale at x'. By Lemma 1.1, $G_{x'} = G_x$ is reductive, because T is closed and separable, so Proposition 4.1.1 applies. Thus ψ/G is étale at the point $t \in (V_x *_{G_x} G)/G = V_x/G_x$ parametrising T'. Let U be an affine open subscheme of V_x/G_x containing t, such that ψ/G is étale on U, and ψ is étale on $\pi^{-1}(U)$. (Such a U exists by Proposition 3.3.1.) Let $S_x = V_x \times_{V_x/G_x} U$, which is a G_x stable open subscheme of V_x. Then by Propositions 5.4.1 and 5.5.1,

$$\pi^{-1}(U) = (V_x *_{G_x} G) \times_{V/G_x} U = (V_x \times_{V_x/G_x} U) *_{G_x} G = S_x *_{G_x} G .$$

Then $S_x *_{G_x} G$ is a G stable open sub-bundle of $V_x *_{G_x} G$ and $\varphi = \psi|_{S_x *_{G_x} G}$ is étale, as is φ/G. Then by Lemma 4.2.1 the square

$$\begin{array}{ccc} S_x *_{G_x} G & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ S_x/G_x & \xrightarrow{\varphi/G} & X/G \end{array}$$

is cartesian. Thus S_x is a slice at x. #

If an affine algebraic group scheme G acts linearly on a linear scheme V, then a linear subscheme L (which corresponds to a linear subspace $L(k)$ of the vector space $V(k)$) is G stable iff $L(\bar{k})$ is $G(\bar{k})$ stable. V is simple if it has no non-trivial stable linear subspaces and it is semisimple if it is a direct sum of simple subspaces. G is a linearly reductive group scheme if every linear representation is semisimple. If $\mathrm{char}(k) = 0$, then G is linearly reductive iff G is reductive, while if $\mathrm{char}(k) = p$ then G is linearly reductive iff G^0 is a torus and

$[G(\overline{k}) : G^0(\overline{k})]$ is prime to p [Nagata, G.-D.IV, §3.3.2, §3.3.6].

PROPOSITION 2. *Let G be a reductive group scheme acting on an affine algebraic scheme X, let T be a closed separable orbit, and let $x \in T$ be a rational point.*

(i) If X is linear and G acts linearly, then there is a slice at x iff the tangent scheme $L_x(X)$ is the direct sum of $L_x(T)$ and a G_x stable complementary linear subscheme. This happens iff

$$0 \to T_x(T) \otimes_k \overline{k} \to T_x(X) \otimes_k \overline{k} \to T_x(X)/T_x(T) \otimes_k \overline{k} \to 0$$

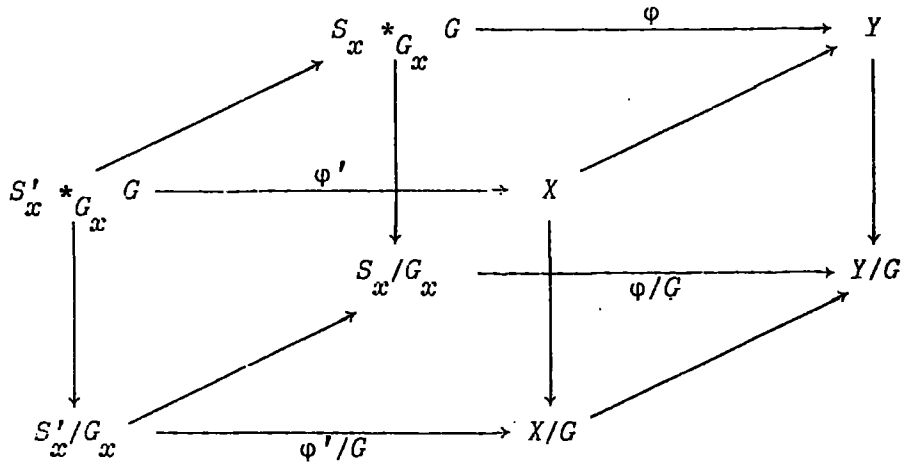
has a $G_x(\overline{k})$ equivariant splitting defined over k.

(ii) If X can be equivariantly embedded in an affine algebraic G scheme Y which has a slice at x, then X has a slice at x.

(iii) If G is linearly reductive then X has a slice at x.

Proof. *(i)* Since X is smooth, there is a slice at x iff there is a transversal at x. If there is a smooth transversal S_x then $L_x(X) = L_x(T) \oplus L_x(S_x)$. Conversely if $L_x(X)$ splits, let W be a G_x stable complement of $L_x(T)$. It was shown in §2 that there is a G_x equivariant isomorphism $L_x(X) \to X$ sending $0 \mapsto x$. The image of W under this isomorphism is a transversal at x.

(ii) Let $\varphi : S_x *_{G_x} G \to Y$ be a slice in Y at x, and let $S'_x = S_x \times_Y X$, which is a G_x stable subscheme of X, and consider the cube



I claim that S'_x is a slice in X. I must show that φ' and φ'/G are étale, and that the front face of the cube is cartesian.

First of all

$$S'_x *_{G_x} G = (S \times_Y X) *_{G_x} G = (S *_{G_x} G) \times_Y X$$

by Proposition 5.4.1, so the top square is cartesian and hence φ' is étale. Thus

$$S'_x *_{G_x} G = X \times_Y (S_x *_{G_x} G) = X \times_Y (Y \times_{Y/G} S_x/G_x) = X \times_{Y/G} S_x/G_x .$$

So

$$S'_x/G_x = (S'_x *_{G_x} G)/G = X/G \times_{Y/G} S_x/G_x$$

(because φ/G is flat), and the bottom square is cartesian, making φ'/G étale.

Finally

$$S'_x *_{G_x} G = X \times_{Y/G} S_x/G_x = X \times_{X/G} (X/G \times_{Y/G} S_x/G_x) = X \times_{X/G} S'_x/G_x ,$$

so the front square is cartesian.

(iii) Since $k[G]$ acts rationally on $k[X]$, there is an equivariant embedding of X in a linear G scheme Y. (The proof is the same as in [Borel].) By *(ii)* it is sufficient to assume X linear. Then the result follows from *(i)*. #

5. An example

This is an example of an action where a slice fails to exist. Let k be a field of characteristic 2 which may be taken to be algebraically closed, and I will use the language of varieties rather than schemes. Let P_2 be the vector space of polynomials of degree two in two variables X, Y. P_2 is a vector space of dimension three. As a basis, take the polynomials X^2, XY, Y^2. Let $SL(2, k)$ act on P_2 in the usual way, that is by the second symmetric power of the canonical representation on the two dimensional space spanned by X, Y. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, k)$ then the action on P_2 is given by the matrix

$$\begin{bmatrix} a^2 & c^2 & ac \\ b^2 & d^2 & bd \\ 0 & 0 & 1 \end{bmatrix}.$$

The two dimensional subspace spanned by X^2, Y^2 is stable but it has no stable complement, for it is easy to see that the only line stable under the subgroup $\left\{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} : t \in k\right\}$ is the line spanned by X^2, and this is not stable under the whole of $SL(2, k)$.

Let V be the dual space to P_2, with a basis α, β, γ dual to X^2, Y^2, XY. Then $SL(2, k)$ acts on V, leaving stable the line through γ but stabilising no complementary hyperplane. By Haboush's theorem there is a homogeneous invariant φ which does not vanish on this line. In fact, it is easy to see that $\varphi = X^2 \cdot Y^2 - (XY) \cdot (XY)$ will do, identifying P_2 with V^*. φ cuts out an $SL(2, k)$ stable cone in V.

Let $X = V_\varphi = \{x \in V : \varphi(x) \neq 0\}$ be the affine open subvariety defined

by φ . Let T^1 be the one dimensional torus and let $G = \mathrm{SL}(2, k) \times T^1$. $\mathrm{SL}(2, k)$ acts on X by the restriction of the action on V , and T^1 acts by scalar multiplication. Together these give an action of G on X . Let L be the line through γ . Then L is a closed G orbit in X . The orbit map is $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, t\right) \mapsto t \cdot \gamma$, which is separable, so L is a separable orbit. The stabiliser of γ is $\mathrm{SL}(2, k)$ and the tangent space at γ is isomorphic to V with the same $\mathrm{SL}(2, k)$ action that we started with. Since there exists no infinitesimal transversal at γ there exists no transversal, and hence there is no slice at γ .

If we take the same space X and the group $G = \mathrm{GL}(2, k)$ with the obvious action then, the picture is similar except that the closed orbit L is not separable. (The orbit map is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (ad-bc)^2 \cdot \gamma$.)

6. Some corollaries

COROLLARY 1. *Let G be a reductive group, acting on an affine algebraic scheme X . Assume that $X(k)$ is dense in X . Then $\pi : X \to X/G$ is an étale torseur iff G_x is trivial for all $x \in X(k)$.*

Proof. If S_x is a slice at $x \in X(k)$ then $S_x *_{G_x} G \to X$ is an étale trivialisation of π at x . #

COROLLARY 2. *Let G be a reductive group acting on an affine algebraic scheme X . Assume that X contains a single closed pseudo-orbit T , and that T is a closed separable orbit. Let $x \in T$ be rational. Assume that G_x is linearly reductive. Then*

(i) X is isomorphic to an induced bundle $V *_{G_x} G$ for some G_x scheme V ;

(ii) if X is smooth then V may be assumed linear;

(iii) if X is smooth, and T consists of a single point then X is linear and G acts linearly.

Proof. *(i)* Let S_x be a slice at x, and consider

$$\begin{array}{ccc} S_x *_{G_x} G & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ S_x/G_x & \xrightarrow{\varphi/G} & X/G \end{array} .$$

X/G contains just one closed point, which is rational because it lies under x. Since X/G is a Jacobson scheme $X/G \cong \mathrm{Spec}(k)$. Since S_x/G_x is étale over the base field k, S_x/G_x is a union of closed, separable points. Clearly S_x/G_x can be assumed connected and it contains a rational point so $S_x/G_x = \mathrm{Spec}(k)$. Thus φ/G is an isomorphism. Since the square is cartesian, φ is an isomorphism and *(i)* is proved with $V = S_x$.

(ii) Let V be as in *(i)*. It can be assumed that V is a smooth G_x stable subscheme embedded in a linear scheme Y and that the only closed G_x pseudo-orbit in V is the rational fixed point x. It is enough to show that some open neighbourhood $U \subset V$ containing x is isomorphic to a linear scheme, for such a neighbourhood will also be a slice.

Identify Y and the tangent scheme $L_x(Y)$. Then $L_x(V)$ is identified with a G_x stable linear subscheme Z of Y. Since G_x is linearly reductive, there is a G_x equivariant projection $Y \to Z$. Let $\psi : V \to Z$ be the restriction of this projection. ψ is G_x equivariant, and maps $L_x(V)$ isomorphically onto $L_0(Z)$. Since V and Z are smooth, ψ is étale at x [EGAIV, §17.11]. By Proposition 4.1.1 and Proposition 4.2.1 there is a G_x stable affine open subscheme $U \subset V$ containing x such that

U/G_x is an affine open subscheme of V/G_x, $\psi|_U$ is étale, $\psi/G_x|_{U/G_x}$ is étale, and $Z = U \times_{U/G_x} Z/G_x$. Then by the same argument as in *(i)*, $\psi/G_x|_{U/G_x}$ is an isomorphism so $\psi|_U$ is an isomorphism. Then $U \subset V$ is the required subscheme.

(iii) This follows from *(i)* and *(ii)*, since $G_x = G$ and $V *_G G \cong V$. #

COROLLARY 3. *Assume that the base field k is algebraically closed. Let G be a reductive group acting on an affine algebraic scheme X. Let T be a closed separable orbit and let $x \in T$ be rational. If there is a slice at x, then for all rational points y in a neighbourhood of x, G_y is conjugate under $G(k)$ to a subgroup of G_x.*

Proof. Let

$$\begin{array}{ccc} S_x *_{G_x} G & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ S_x/G_x & \xrightarrow{\varphi/G} & X/G \end{array}$$

be a slice at x. Let $U = \operatorname{im} \varphi$, which is an open subscheme of X, and let y be a rational point in U. Let y' be a closed point of $S_x *_{G_x} G$ lying over y. Let V be the G fiber of X containing y, and let V lie over $v \in X/G$, and let V' be the G_x fiber of $S_x *_{G_x} G$ containing y' and let V' lie over $v' \in S_x/G_x$. Then $V' \cong V \otimes_{k(y)} k(y')$. But y is rational and k is algebraically closed so $k(y) = k(y') = k$. Thus $V' \cong V$, and $G_{y'} = G_y$, and $G_{y'}$ is conjugate to a subgroup of G_x because $(S_x *_{G_x} G)/G_x \cong G/G_x \cong T$ which is an orbit. #

CHAPTER 7

1. Semisimple group schemes

An affine algebraic group scheme G is said to be semisimple if $G \otimes \overline{k}$ is reduced, and the underlying Serre variety is a semisimple algebraic group in the sense of [Borel]. Equivalently, G is semisimple if it is geometrically reduced, and $G(\overline{k})$ is a semisimple algebraic group. A torus T in G is a connected abelian subgroup scheme such that $T \otimes \overline{k}$ is diagonalisable over $\overline{k}$. The torus T is split if it is diagonalisable over k, and the semisimple group G is split if it contains a maximal torus which splits. If G is semisimple, then there is a finite, separable extension field K of k such that $G \otimes_k K$ is split, so over an algebraically closed field every semisimple group splits.

If G is an affine, algebraic group scheme, let $\mathfrak{g} = L_e(G)$ be the tangent scheme at e, and let $g = T_e(G) = \mathfrak{g}(k)$ be the tangent space at e. g is a Lie algebra over k, and $\mathfrak{g}$ is a "Lie algebra scheme" [G.-D.II, §4.4]. If G is semisimple, let $\Delta(G)$ be the Dynkin diagram of the semisimple Lie algebra $\mathfrak{g}(\overline{k})$. Let p be the characteristic of the field k. If $\Delta(G)$ is connected, then the characteristic p is "good" for G provided that:

Type of $\Delta(G)$	Good p
A_n	p arbitrary
B_n	$p \neq 2$
C_n	$p \neq 2$
D_n	$p \neq 2$
G_2	$p \neq 2, 3$
F_4	$p \neq 2, 3$
E_6	$p \neq 2, 3$
E_7	$p \neq 2, 3$
E_8	$p \neq 2, 3, 5$

If $\Delta(G)$ is not connected, p is good for G if p is good for each component of $\Delta(G)$ [Steinberg 1970].

If G is a split semisimple group scheme, let $C(G)$ be the center of $G(\bar{k})$ (I will call $C(G)$ the geometric center of G), and let $\pi(G)$ be the fundamental group of G. $\pi(G)$ is the quotient of the character group of a maximal split torus T by the root subgroup. $\pi(G)$ and $C(G)$ are finite abelian groups. G is simply connected if $\pi(G) = 1$, and G is of adjoint type if $C(G) = 1$.

Split semisimple groups are classified by their Dynkin diagrams and fundamental groups. For each Dynkin diagram Δ, there is a finite number of split groups G over k with $\Delta(G) = \Delta$. Of these, one is simply connected (call it G_1), and one is of adjoint type (call it G_0). $\pi(G_1) = C(G_0) = 1$, and $\pi(G_0) \cong C(G_1)$. The other possible fundamental groups for this Dynkin diagram are the subgroups of $\pi(G_0)$. If k is algebraically closed, then all the groups with Dynkin diagram Δ may be realised as quotients of G_1 by finite central subgroups, but this is not true over an arbitrary field. If G is simply connected, then it is a direct product of simple group schemes (that is, of semisimple group schemes with connected Dynkin diagrams). If G is simple and of type A_ℓ, $\pi(G)$ can be any subgroup of $\mathbb{Z}_{\ell+1}$, while if G is simple of other types, $\pi(G)$ must be one of $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, or $\mathbb{Z}_2 \times \mathbb{Z}_2$. See [Tits] for the classification of semisimple group schemes over k, and see [Steinberg 1967] for the existence of split groups.

If G is split semisimple, and G_1, G_0 are the simply connected and adjoint group schemes of the same type, then there exist unique central isogenies (that is, surjective equivariant morphisms π, such that $\ker \pi(\bar{k})$ is finite and central)

$$G_1 \xrightarrow{\pi_1} G \xrightarrow{\pi_2} G_0 .$$

These isogenies are separable (that is, étale) iff $\mathrm{char}(k)$ is relatively prime to $|\pi(G)|$ and $|C(G)|$. If $\Delta(G)$ has no component of type A and $\mathrm{char}(k)$ is good for G, then π_1 and π_2 are separable [Tits].

2. The unipotent and nilpotent varieties

Let G be a connected semisimple group scheme over k, let $\mathfrak{g}$ be the Lie algebra scheme of G, and let G act on itself by conjugation, and on $\mathfrak{g}$ by the adjoint representation [G.-D.II, §4.4.1]. I will write the quotient of these actions $\pi_G : G \to G/G$ and $\pi_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}/G$. Let $V(G) = \pi_G^{-1}([e])$ and $V(\mathfrak{g}) = \pi_{\mathfrak{g}}^{-1}([0])$, where $[e] = \pi_G(e)$ and $[0] = \pi_{\mathfrak{g}}(0)$. $V(G)$ and $V(\mathfrak{g})$ are called respectively the unipotent variety of G, and the nilpotent variety of $\mathfrak{g}$. If G is simply connected, then G/G is an affine space A^r [Steinberg 1970], and in good characteristic $\mathfrak{g}/G$ is an affine space [Demazure 1973]. So in good characteristic, if G is simply connected, then $V(G)$ and $V(\mathfrak{g})$ are complete intersections, and are hence geometrically normal. It is known [Springer] that in good characteristic, if G is simply connected and connected then $V(G)$ is isomorphic to $V(\mathfrak{g})$ as G spaces. I will give a proof of this fact for split groups.

LEMMA 1. *Let G_1, G_2 be connected semisimple group schemes and let $\varphi : G_1 \to G_2$ be a separable isogeny. G_1 acts on $V(G_1)$ by conjugation and on $V(\mathfrak{g}_1)$ by the adjoint action, and G_1 acts on $V(G_2)$, $V(\mathfrak{g}_2)$ via φ. As G_1 spaces, $V(G_1) \cong V(G_2)$ and $V(\mathfrak{g}_1) \cong V(\mathfrak{g}_2)$.*

Proof. For the given actions, φ is an equivariant etale morphism which satisfies the conditions of Proposition 4.1.1 at $e \in G_1$. So φ/G is étale at $\pi_1(e)$ and the following square is, after restricting to a

neighbourhood of $\pi_1(e)$, cartesian.

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ G_1/G_1 & \xrightarrow[\varphi/G_1]{} & G_2/G_1 \end{array} .$$

Thus the fiber $V(G_1) = \pi_1^{-1}([e])$ is isomorphic, as G space, to the fiber $\pi_2^{-1}([e])$. But φ surjective, so $G_2/G_1 \cong G_2/G_2$, and π_2 is the quotient map for the action of G_2 on itself by conjugation. Thus $\pi_2^{-1}([e]) \cong V(G_2)$.

The proof that $V(\mathfrak{g}_1) \cong V(\mathfrak{g}_2)$ is similar. #

LEMMA 2. *Let Δ be a simple Dynkin diagram, not of type A , and assume that* char(k) *is good for Δ . There exists a split group G , of type Δ , over k , and a faithful representation $\rho : G \to \mathrm{GL}(V)$ of G , such that $\mathfrak{g}$ has a linear, G stable complement $\mathfrak{m}$ in $\mathfrak{gl}(V)$. For such a group G there exists a morphism $\varphi : G \to \mathfrak{g}$, which is equivariant, which sends e to 0 , and which is étale at e .*

Proof. The following proof from [Richardson 1967] works over an arbitrary field.

First consider the groups of type B_l . Let G be the subgroup scheme of $\mathrm{GL}(2l+1)$ cut out by the polynomial equations $\det(X) = 1$ and ${}^tXSX = S$, where S is the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{bmatrix} .$$

By [G.-D.II, 5.2.7] G is smooth, so $G \otimes \overline{k}$ is reduced, and $G \otimes \overline{k}$ is cut out by the same equations over $\overline{k}$. Thus $G \otimes \overline{k} = \mathrm{SO}(2l+1, \overline{k})$, and G

is a "k-form" of $SO(2l+1, \bar{k})$. It is easily seen that G is split [Humphreys 1970, 34.6].

By the usual calculation with the defining polynomials, the Lie algebra $g = \mathfrak{g}(k)$ is defined by the equations $SX + {}^tXS = 0$ in $\mathfrak{gl}(2l+1, k)$. Let $\mathfrak{m}$ be the linear subscheme of $\mathfrak{gl}(2l+1)$ whose rational points are defined by $X \in \mathfrak{m}(k)$ iff $SX - {}^tXS = 0$. Clearly $\mathfrak{m}$ is a complementary linear subscheme to $\mathfrak{g}$ in $\mathfrak{gl}(2l+1)$. It must be shown that $\mathfrak{m}$ is G stable. But $\mathfrak{m}(\bar{k}) = \{X \in \mathfrak{gl}(2l+1, \bar{k}) : SX - {}^tXS = 0\}$, and clearly this is $G(\bar{k}) = SO(2l+1, \bar{k})$ stable.

Since $\mathfrak{gl}(2l+1) = \mathfrak{g} \oplus \mathfrak{m}$, there is a G-equivariant projection $\psi : \mathfrak{gl}(2l+1) \to \mathfrak{g}$ with kernel $\mathfrak{m}$. Let $\varphi : G \to \mathfrak{g}$ be the restriction of ψ to G (which is embedded as a closed subscheme of $\mathfrak{gl}(2l+1)$). Clearly $L_0(\varphi)$ maps $L_0(G) = \mathfrak{g}$ isomorphically onto $L_0(\mathfrak{g}) = \mathfrak{g}$. Since e and 0 are both smooth rational points, φ is étale at e [EGAIV, 17.11.2].

The proof for the other classical groups (types C, D) is similar.

For the exceptional types, let G be a split group of adjoint type. By construction, G is a smooth closed subscheme of $GL(g)$ [Steinberg 1967, Chapter 5]. The inclusion map ρ is the adjoint representation of G . The trace form $\langle X, Y \rangle = \mathrm{tr}(XY)$ on $\mathrm{End}(g)$ is nondegenerate and restricts to give the Killing form on g . Since the characteristic is good, the discriminant of the Killing form is nonzero and the form is nondegenerate on g . Thus $\mathfrak{gl}(g) = \mathrm{End}(g) = g \oplus g^{\perp}$. Let $\mathfrak{m}$ be the linear subscheme such that $\mathfrak{m}(k) = g^{\perp}$. The rest of the proof is as for the classical case. #

PROPOSITION 1. *Let G be a connected semisimple group scheme over k , with Dynkin diagram Δ . Assume that* $\mathrm{char}(k)$ *is good for G , and that either Δ has no components of type A , or that G is simply connected. Then $V(G)$ is equivariantly isomorphic to $V(\mathfrak{g})$.*

Proof. If Δ has no component of type A, then there is a separable isogony $\varphi : G_1 \to G$, where G_1 is the simply connected group of type Δ. By Lemma 1, it can be assumed that G is simply connected. Thus G is the direct product of simple groups G_i, and $V(G) = \prod_i V(G_i)$, $V(\mathfrak{g}) = \prod_i V(\mathfrak{g}_i)$. Thus it can be assumed that G is simple. Now consider separately the two following cases.

CASE 1. Δ is of type A_l. Then G is the subscheme of $GL(l+1)$ cut out by the equation $\det(X) = 1$. $V(G)$ is the reduced subscheme corresponding to the subvariety of unipotent matrices in $SL(l+1, \bar{k})$, and $V(\mathfrak{g})$ is the reduced subscheme corresponding to the nilpotent matrices in $\mathfrak{sl}(l+1, \bar{k})$. The required isomorphism $V(\mathfrak{g}) \to V(G)$ is given by the morphism $V(\mathfrak{g})(\bar{k}) \to V(G)(\bar{k}) : X \mapsto X + 1$, which is defined over k.

CASE 2. Δ is not of type A. Then Lemma 2 applies, and there is an equivariant morphism $\varphi : G \to \mathfrak{g}$ which is étale at e. Then φ/G is étale at $\pi_G(e)$, and, after restricting to a neighbourhood of $\pi_G(e)$ in G/G, the following square is cartesian.

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & \mathfrak{g} \\ \downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi_{\mathfrak{g}}} \\ G/G & \xrightarrow[\varphi/G]{} & \mathfrak{g}/G \end{array} .$$

Thus $V(G)$, the fiber of π_G over $\pi_G(e)$, is isomorphic to $V(\mathfrak{g})$, the fiber of $\pi_{\mathfrak{g}}$ over $\pi_{\mathfrak{g}}(0)$. #

APPENDIX 1

1. Representing schemes by functors

Let k be a fixed base ring, which in our case is a field, though this is not essential, and let all schemes be over k. All schemes are assumed separated. Let Sch be the category of (separated) schemes over k, Aff the category of affine schemes over k, Alg the category of commutative k-algebras with 1, S the category of sets. If C, D are categories then C^0 is the opposite category to C (all arrows are reversed), and $[C, D]$ is the category whose objects are (covariant) functors $C \to D$, and whose morphisms are natural transformations of these functors. $[C^0, D]$ can then be regarded as the category of contravariant functors from C to D. In order that this makes sense I will assume the set theoretic framework of [SGAI]. I will follow the category theorists' convention of writing X for the identity morphism of the object X when the context is clear. Note that Aff^0 is isomorphic to Alg. I will regard these two categories as identical.

If C is a category, let $Y = Y_C : C \to [C^0, S]$ be the embedding functor defined by $Y(P) = \underline{P}$, where $\underline{P} : C \to S$ is the contravariant functor $\underline{P}(Q) = \mathrm{Hom}_C(Q, P)$, and for u a morphism of C, $Y(u) = \underline{u}$ is the natural transformation $\underline{u}(f) = u \circ f$. Recall from [Maclane] the

YONEDA LEMMA. *$Y : C \to [C^0, S]$ is a fully faithful embedding.*

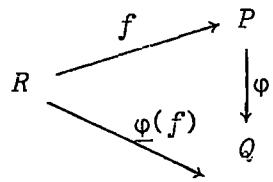
Proof. To say that Y is a fully faithful embedding means that for each pair $P, Q \in C$,

$$Y : \mathrm{Hom}_C(P, Q) \to \mathrm{Hom}_{[C^0, S]}(\underline{P}, \underline{Q})$$

is an isomorphism of sets. So let $\alpha : \underline{P} \to \underline{Q}$ be a natural transformation,

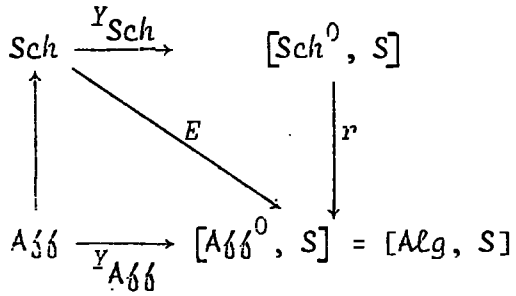
and let $\varphi = \alpha_P(P) \in \underline{Q}(P) = \mathrm{Hom}_{\mathcal{C}}(P, Q)$. If $R \in \mathcal{C}$ and $f \in \underline{P}(R)$ then $f = P \circ f = \underline{P}(f)(P)$. So

$$\alpha_R(f) = \alpha_R\big(P(f)(P)\big) = \underline{Q}(f)\big(\alpha_P(P)\big) = \underline{Q}(f)(\varphi) = f \circ \varphi = \underline{\varphi}(f) = Y(\varphi)(f) .$$



Thus α is the image of a unique element $\varphi \in \mathrm{Hom}_{\mathcal{C}}(P, Q)$. #

Now let $r : [Sch^0, S] \to [Aff^0, S]$ be the restriction functor, and let $E = r \circ Y_{Sch}$. If $X \in Sch$ I will write $\underline{X}$ for $E(X)$.



PROPOSITION 1. *E is a fully faithful embedding.*

Proof. I will write Y for Y_{Sch} . If $X \in Sch$, then it is easy to see that $Y(X) = \underline{X}$ is a sheaf for the Zariski topology on Sch . That is, if T Sch and $\Gamma = \left\{T_i \xrightarrow{\varphi_i} T\right\}$ is a family of open embeddings which cover T , then the square

$$\begin{array}{ccc} \underline{X}(T) & \xrightarrow{\alpha} & \coprod_i \underline{X}(T_i) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ \coprod_j \underline{X}(T_j) & \xrightarrow{\delta} & \coprod_{i,j} \underline{X}(T_i \cap T_j) \end{array}$$

is cartesian, where $\alpha, \beta, \gamma, \delta$ are the obvious restriction maps. For example $\alpha = \coprod_i \underline{X}(\varphi_i) \circ \Delta$, where $\Delta : \underline{X}(T) \to \coprod_i \underline{X}(T)$ is the diagonal map.

If $X, Y \in Sch$, and $\alpha : \underline{X} \to \underline{Y}$ is a natural transformation, then

there is a unique natural transformation $\alpha' : \underline{X} \to \underline{Y}$. It is defined as follows. For $T \in Sch$ let $\Gamma = \left\{T_i \xrightarrow{\varphi_i} T\right\}$ be a Zariski covering by affine open subsets. Since T is separated, $T_{ij} = T_i \cap T_j$ are also affine [EGAI, §5.3.6]. Then in the diagram below, the vertical lines are exact, and the lower square commutes because of the naturality of α, so there exists a unique $\alpha'_{T,\Gamma}$ extending α_{T_i}, $\alpha_{T_{ij}}$.

$$\begin{array}{ccc}
\underline{X}(T) & \xrightarrow{\alpha'_{T,\Gamma}} & \underline{Y}(T) \\
\downarrow & & \downarrow \\
\prod_i \underline{X}(T_i) & \xrightarrow[\prod_i \alpha_{T_i}]{} & \prod_i \underline{Y}(T_i) \\
\downarrow\downarrow & & \downarrow\downarrow \\
\prod_{i,j} \underline{X}(T_{ij}) & \xrightarrow[\prod_{i,j} \alpha_{T_{ij}}]{} & \prod_{i,j} \underline{Y}(T_{ij})
\end{array}$$

If $\Delta = \{S_i \to T\}$ is another affine Zariski covering then by comparing these coverings with the covering $\{S_i \cap T_j \to T\}$, it can be assumed that Δ is a refinement of Γ. Then by the uniqueness of the extension $\alpha'_T = \alpha'_{T,\Gamma} = \alpha'_{T,\Delta}$ is independent of the covering used. The naturality of α'_T follows by a similar uniqueness argument.

But Y is fully faithful, so there exists a unique morphism $\varphi : X \to Y$ such that $Y(\varphi) = \alpha'$ and hence a unique φ such that $E(\varphi) = \alpha$. #

PROPOSITION 2. *E preserves products. More precisely, if S is a cartesian square in Sch then $\underline{S} = Y(S)$ is a cartesian square in [Alg, S] and for any $R \in Alg$, $\underline{S}(R)$ is a cartesian square of sets.*

Proof. Simple diagram chasing. #

2. Formal properties of faithfully flat base change

Let A be an abelian category, or what is the same thing by the embedding theorem, a category of modules over a ring k.

LEMMA 1. *Consider the following diagram in* A *:*

$$\begin{array}{ccc} N & \xleftarrow{\delta} & P \\ {\scriptstyle\gamma}\uparrow & & \uparrow{\scriptstyle\beta} \\ Q & \xleftarrow[\alpha]{} & R \end{array}$$

Then this is a commuting, co-cartesian square, iff the following sequence is exact:

$$R \xrightarrow[\varphi]{} P \oplus Q \xrightarrow[\psi]{} N \to 0$$

where $\varphi(r) = \big(\beta(r), -\alpha(r)\big)$ *and* $\psi(p, q) = \delta(p) + \gamma(q)$.

Proof. $\operatorname{Im}\varphi \subset \ker\psi$ iff for all $r \in R$,

$$\psi \circ \varphi(r) = \delta \circ \beta(r) - \gamma \circ \alpha(r) = 0 .$$

That is, iff the square commutes. ψ is surjective and $\operatorname{im}\varphi = \ker\psi$ iff ψ is the co-kernel of φ. That is, if the following condition holds: if $\theta : P \oplus Q \to X$ is a morphism such that $\theta \circ \varphi = 0$, then θ factors uniquely through ψ. But this is precisely the condition that the square is co-cartesian. #

COROLLARY 1. *In the category* Alg, *co-cartesian squares are preserved under faithfully flat ascent and descent. In the dual category of affine schemes cartesian squares are preserved under faithfully flat ascent and descent.*

COROLLARY 2. *Consider the commuting square of affine schemes:*

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle q} \\ Z & \xrightarrow[s]{} & W \end{array} .$$

If this square is cartesian, and either s *or* q *is scheme theoretically*

dominant, then the square is also co-cartesian.

Proof. Passing to the dual category, assume that the square in Lemma 1 is the square of comorphisms. Then either α or β is injective so φ is injective. Thus φ is the kernel of ψ and by the dual of Lemma 1 the square of comorphisms is cartesian. #

APPENDIX 2

All schemes in this appendix are assumed affine and algebraic over a field k. General references for this material are [EGA], [Dieudonné 1964], [G.-D.], [Hartshorne].

1. Dimension

If X is a scheme, then $\dim(X)$ is the maximum length of ascending chains of irreducible closed subsets in the underlying topological space $|X|$. If $x \in X$, then $\dim_x(X)$ is the minimal dimension of an open neighbourhood of x.

If A is a ring, let $K.\mathrm{Dim}(A)$ be the Krull dimension of A, and if K is an extension field of k, let $\mathrm{Trans.Deg.}(K)$ be the transcendence degree of K over k. If X is a scheme, then $\dim(X) = K.\mathrm{Dim}(k[X])$, and if $x \in X$ and X_i are the irreducible components of X passing through x, then

$$\dim_x(X) = \sup_i \dim(X_i) = K.\mathrm{Dim}(\mathcal{O}_x) + \mathrm{Trans.Deg.}(k(x)) .$$

If K is an extension field of k, then $\dim(X) = \dim(X \otimes_k K)$. If $x \in X$, and $y \in X \otimes_k K$ is a point lying over x, then $\dim_x(X) = \dim_y(X \otimes_k K)$. If $f : X \to Y$ is a morphism, $x \in X$ and $y = f(x)$, then

$$\dim_x(X) \leq \dim_y(Y) + \dim_x(X_y) .$$

Equality holds if f is flat at x.

2. Flatness and generic flatness

Let $f : X \to Y$ be a morphism. f is flat if for every $x \in X$,

$y = f(x)$, $O_{x,X}$ is a flat $O_{y,Y}$ module. f is faithfully flat if f is flat and surjective. f is (faithfully) flat iff $k[Y]$ is a (faithfully) flat $k[X]$ module.

If f is flat, and Z is a constructible subset of Y , then $f^{-1}(\overline{Z}) = \overline{f^{-1}(Z)}$. If f is flat, then f is an open map. If f is faithfully flat, then the topology on Y is the quotient of the topology on X by the equivalence relation defined on X by f .

If $f : X \to Y$ is a dominant morphism and Y is reduced, then there exists a dense open set $U \subset Y$ such that f is flat over U , and the set of points of X where f is flat is open.

3. Ascent and descent

Consider the cartesian square

$$\begin{array}{ccc} X_Z & \xrightarrow{f_Z} & Z \\ \downarrow & & \downarrow g \\ X & \xrightarrow[f]{} & Y \end{array} .$$

Then f_Z is obtained from f by ascent by the morphism g , and f is obtained by descent by g . The process of ascent and descent is also called base change. Let P be a property of morphisms. P is said to be stable under ascent if $P(f) \Rightarrow P(f_Z)$, and P is stable under descent if $P(f_Z) \Rightarrow P(f)$. If these implications hold only if g is flat, faithfully flat, etc, then P is stable under flat ascent, faithfully flat ascent, etc. A similar language is used for properties of the schemes X and X_Z .

The following properties of morphisms are stable under ascent:

1. Surjective,

2. (Open, closed) embedding,

3. Finite,

4. Algebraic,

5. Affine,

6. Flat,

7. Smooth,

8. Etale.

The following property is preserved under flat ascent. (This only works for noetherian schemes.)

9. Dominant.

All the above properties are stable under faithfully flat descent, as are the following:

10. Radical,

11. Open,

12. Closed.

If a property P is not stable under ascent, then a scheme X or morphism f has the stronger property "universally P" if, under all base changes $g : Z \to Y$, X_Z or f_Z has property P. Examples are universally open, universally closed, and universally injective (which is the same thing as radical).

If X is a scheme over k, and f a morphism of schemes over k, then X (or f), has the property "geometrically P" if $X \otimes_k K$ (or $f \otimes_k K$), has the property P for all extension fields K of k. Examples are geometrically reduced, geometrically normal, geometrically connected, and geometrically regular. To establish these properties, it is sufficient to check property P for K an algebraic closure of k. X is geometrically reduced iff it is reduced and the local fields at the generic points of X are separable.

If X is a scheme and K is a field, then elements of $X(K)$ may be called "geometric points" of X (with coordinates in K). If $f : X \to Y$ is a morphism, and $y : \mathrm{Spec}(K) \to Y$ is a geometric point of Y, then the scheme $X_y = X \times_Y \mathrm{Spec}(K)$ is called the geometric fiber over y. It is a scheme over K. There is a natural morphism $X_y \to X$, but this will not in general be an embedding. To say that the geometric fibers of f have property P means the same as to say that the ordinary fibers of f are geometrically P.

4. Smooth and étale morphisms

A scheme X is regular if the local rings at all points are regular. Since algebraic schemes are "excellent" [EGAIV], X is regular if the local rings at closed points are regular.

A morphism $f : X \to Y$ is smooth at $x \in X$ if f is flat at x, and the geometric fiber $X \otimes_k \overline{k(y)}$, $\left(y = f(x)\right)$, is regular. f is étale at x if it is smooth and the fiber dimension is zero at x. If f is étale at a point $x \in X$, then it is étale in a neighbourhood of x. Basic examples of étale morphisms are open embeddings and finite separable field extensions.

A scheme X is étale if the canonical morphism $X \to \mathrm{Spec}(k)$ is étale. X is étale iff X is the disjoint union of a finite number of schemes of the type $\mathrm{Spec}(K)$, where K is a finite separable field extension of k.

If $f : X \to Y$ is a morphism, $x \in X$ and $f(x) = y$, and x is a rational point, then f is étale at x iff $\hat{f}_y : \hat{O}_{y,Y} \to \hat{O}_{x,X}$ is an isomorphism, where the completion is by the $\mathfrak{m}_y$-adic and $\mathfrak{m}_x$-adic filtrations.

LEMMA. *Let* $f : X \to Y$ *be a morphism. The following are equivalent:*

(i) f *is étale;*

(ii) f *is flat and unramified;*

(iii) f *is flat, the fiber dimension is zero, the fibers of* f *over the closed points of* Y *are reduced, and for all closed points* $x \in X$, $k(x)$ *is separable over* $k(y)$ $\big(y = f(x)\big)$.

If Y *is normal, these are equivalent to*

(iv) f *is flat, and for all* $x \in X$ *the comorphism* $f_x^* : \mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$ *is injective* $\big(y = f(x)\big)$.

Here f *is unramified at* $x \in X$ *means that* $k(x)$ *is a finite separable extension of* $k(y)$ $\big(y = f(x)\big)$, *and* $\mathfrak{m}_y \mathcal{O}_{x,X} = \mathfrak{m}_x$.

Proof. The equivalence of *(i)* and *(ii)* is standard, and of *(i)* and *(iv)* is in [EGAIV, §18.10.1]. *(i)* $\Rightarrow$ *(iii)* is clear.

To show the converse, it is enough to show that f is étale at the closed points of X . Let $y \in Y$ be a closed point. It must be shown that X_y is geometrically regular of dimension zero, that is, that X_y is étale over $k(y)$. X_y is algebraic and zero dimensional, so it is artinian. That is, it is a disjoint union of local artinian schemes. But a reduced local artinian ring is a field, so the points of X_y are all closed. By assumption the local fields of X_y are separable over $k(y)$, and they are clearly finite extensions of $k(y)$, so X_y is étale over $k(y)$. #

5. Zariski's main theorem

Let $f : X \to Y$ be a morphism of affine algebraic schemes, and assume that f has finite discrete fibers. For simplicity, assume that X and Y

are normal. Then exists a unique factorisation of f,

$$X \xrightarrow{i} \tilde{Y} \xrightarrow{\tilde{f}} Y$$

where $\tilde{Y}$ is a normal affine algebraic scheme, i is a dominant open embedding, and $\tilde{f}$ is finite. The uniqueness of such a factorisation is clear, for $k[\tilde{Y}]$ must be the integral closure of $k[Y]$ in $k[X]$. For existence, see the proof of Remark 18.12.14 in [EGAIV]. This factorisation will be called the Zariski factorisation of f.

Assume that a smooth affine group scheme G acts on X, Y, and that f is equivariant. Then there is an action of G on $\tilde{Y}$ for which i, $\tilde{f}$ are equivariant.

$$\begin{array}{ccccc} G \times X & \xrightarrow{i\times G} & G \times \tilde{Y} & \xrightarrow{\tilde{f}\times G} & G \times Y \\ \downarrow & & \vdots & & \downarrow \\ X & \xrightarrow{i} & \tilde{Y} & \xrightarrow{\tilde{f}} & Y \end{array}$$

To see this, pass to the dual category.

$$\begin{array}{ccccc} k[G] \otimes k[X] & \supset & k[G] \otimes k[\tilde{Y}] & \leftarrow & k[G] \otimes k[Y] \\ \uparrow & \overset{\beta}{\nwarrow} & \uparrow \alpha & & \uparrow \\ k[X] & \supset & k[\tilde{Y}] & \leftarrow & k[Y] \end{array}$$

It will be enough to show that there exists a homomorphism α completing the diagram. By base change, $G \times i$ is a dominant open embedding, and $G \times \tilde{f}$ is finite. $G \times \tilde{Y}$ is normal because G is smooth [EGAIV, §6.5.4]. Let $g \in k[\tilde{Y}]$. Clearly $\beta(g)$ is integral over $k[G] \otimes k[Y]$, so $\beta(g) \in k[G] \otimes k[\tilde{Y}]$, so β factorises uniquely through $k[G] \otimes k[\tilde{Y}]$. This factorisation defines α.

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