The homotopy of q and classifying spaces

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THE HOMOTOPY OF Γ_q AND CLASSIFYING SPACES.

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March 1977.

THE HOMOTOPY OF Γ_q^r AND CLASSIFYING SPACES.

ABSTRACT : Let $r \geq 1$ and $\bar{\Gamma}_q^r$ be the topological groupoid of germs of orientation preserving ~~oriented~~ local C^r diffeomorphisms of $\mathbb{R}^q$. Then a E^2 spectral sequence is constructed with the E^2 terms computed from the homological properties of Γ_q^r, and E^∞ is the bigraded module associated to the filtration of the homology of the classifying space, $B\Gamma$ given by Haefliger in [HA3].

Let $S \geq 1$ and $\bar{\Delta}^s = \{1, 2, \ldots S+1\}$ be the objects of the category $C_{\bar{\Delta}^s}$ with morphisms $\leq$ and $\bar{\Gamma}_q^r(\bar{\Delta}^s)$ the space of functors from $C_{\bar{\Delta}^s}$ to $\bar{\Gamma}_q^r$ with the usual topology on $\bar{\Gamma}_q^r$. We prove that

a) $$H_t(\bar{\Gamma}_q^r(\bar{\Delta}^s)) = 0 \quad \text{FOR } t > q.$$

b) If $\overline{GL}_q$ is the group of linear transformations of $\mathbb{R}^q$ with positive determinant then if $\nu : \bar{\Gamma}_q^r(\bar{\Delta}^s) \longrightarrow (\overline{GL}_q)^s$ is the map obtained by taking derivatives of $\gamma(1 \leq i)$ for $2 \leq i \leq S+1$, $\gamma \in \bar{\Gamma}_q^r$

$$\nu_\# : \pi_t(\bar{\Gamma}_q^r(\bar{\Delta}^s)) \longrightarrow \pi_t((\overline{GL}_q)^s)$$

is an isomorphism for $t < q$.

These calculations go a long way towards calculating the E^2 terms of the above spectral sequence.

The spectral sequence is constructed for a large class of topological groupoids refered to as well formed topological groupoids, and the corresponding theorem on the high dimensional homologies of topological groupoids is proved for a special class of well formed topological groupoids which include the known topological groupoids associated with foliations.

I wish to thank the Science Research Council for financial support, my Supervisor Dr. J.Bolton and other members of the Durham University Mathematics Department for continuing encouragement, and my wife for typing the manuscript.

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March 1977.

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CHAPTER I

BACKGROUND.

1 Introduction:

Let M^m be a paracompact topological space such that for each $x \in M^m$ there exists an open neighbourhood U of x and embedding $\varphi : U \longrightarrow \mathbb{R}^m$ into the real vector space of dimension m, M^m is a topological manifold. A topological manifold is a space used to construct a host of geometries such as C^r manifolds and foliations. Haefliger in [HA1], [HA2] and [HA3] showed how, by taking germs of an atlas on topological manifolds, structures on M^m naturally corresponds to Γ_m^0 - structures on M^m where Γ_m^0 is the groupoid of germs of local homeomorphisms of $\mathbb{R}^m$ with the germ topology. Haefliger also showed how codimension r foliations give rise to Γ_r^0-structures on a topological manifold M^m. By specialising to subgroupoids of we restrict the type of foliation. In this way we get C^r foliations, analytic foliations and even PL foliations corresponding to the subgroupoids Γ_q^r, Γ_q^ω, Γ_q^{PL} of germs of local homeomorphisms of that are respectively germs of C^r differentiable, analytic and piece wise linear local homeomorphisms of $\mathbb{R}^r$.

The Γ- structure of a structure on a topological manifold is the homotopy version of the structure because it has the two complementary properties:

a) Γ- structure can be defined on an arbitary topological space and it has most of the properties of a principal G-bundle; Γ- structures have a classifying space $B\Gamma$ constructed in [HA3] which is constructed in the same way as the Milnor construction of a classifying space for principal G-bundles given in [MI4].

b) The Γ- structure contains most of the homotopy properties of foliations, for instance, Thurston [TH1] shows that for Γ_q^∞-structures concordance classes of foliations correspond 1-1 with homotopy classes of Γ_q^∞-structures together with concordance classes of monomorphisms of the normal bundle of Γ_q^∞-structures and the tangent bundle. Another well known property which illustrates the richness of the Γ_q^∞-structure is Haefliger's classifying theorem for foliations on open C^∞ manifolds given in [HA3].

The classifying space $B\Gamma$ for a subgroupoid of Γ_q^0 is the key to the homotopy properties of foliations.

Several interesting results on the homotopy properties of $B\Gamma_q^r$, $B\Gamma_q^\omega$ and $B\Gamma_q^0$ have been proved in [HA3], [MA2] and [TH2], But connected with $B\Gamma$ is the homotopy properties of topological groupoids and the relationship between the classifying space and the groupoid. This thesis develops some of these connections between $B\Gamma$ and Γ, and computes homotopy properties of Γ for a special class of topological groupoids Γ.

The connection between $B\Gamma$ and Γ has been shown in Haefliger's construcion of $B\Gamma$ from Γ but the construction does not give a useful spectral sequence. On the other hand Segal's classifying space constructed in [SE1] has a filtration which gives a spectral sequence, but because the topology it is difficult to show that Segal's classifying space is a classifying space for Γ- structures in the case Γ is not a topological group. We re-construct $B\Gamma$ in a way which is similar to the construction of Segal's classifying space and then use Segal's filtration to obtain a spectral sequence whichcorresponds to the spectral sequence of Segal's classifying space for Γ_N, where Γ is a special class of topological groupoids refered to as well formed.

In Chapter III we explore ways of calculating the terms of the spectral sequence showing that the $E^1_{s,t}$ terms vanish for open sub topological groupoids of Γ_q^0 when $t > q$, and for $\Gamma_q^\vee$ the homotopy groups up to $q-1$ are calculated for $\Gamma_q^\vee$. Given time it is felt that the terms of the spectral sequence will be sufficiently simplified to provide useful imformation on the homotopy properties of $B\Gamma$ or Γ for a large class of topological groupoids, for instance the C_s term in $E^1_{s,t}$ could be removed and the connection between $H_q(\Gamma_q^\vee)$ and the classifying space of the group of $C^\vee$ diffeomorphisms with compact support could be elucidated.

ORIGINALITY:

Some attempt has been made to exclude all proofs which are given elsewhere.

2 Definitions and Notation.

Let X, Y be topological spaces and $x \in X$. If $f: U \to Y$ is a map of an open neighbourhood U of x to Y then the germ of (x, f) denoted Germ (x, f) is the equivalence class of all maps $g: V \to Y$ where V is an open neighbourhood of x such that there exists an open neighbourhood W of x for which $f|W = g|W$. We will often talk of a germ of $f: U \to Y$ without refering to the particular point $x \in U$ and if $U' \subset U$ we put Germ $(U', f) = \{ \text{Germ}(x, f) \mid x \in U' \}$. A local homeomorphism from X to Y is a homeomorphism $h: U \to V$ of open subsets of X and Y respectively. We have the space of germs of local homeomorphisms of a topological space X. Such a space of germs is an example of a topological groupoid, given in [HA3]. For definition of category terminology see [SP1] Chapter 1 Section 1. A Topological Groupoid is a small category Γ such that every morphism is an equivalence, together with a topology on Γ (taken to be the space of morphisms) such that four structural maps, given in [MA2] for instance, are continuous when the set of objects, denoted $Ob\,\Gamma$, is identified with the identity elements. For a topological groupoid Γ and morphism $\gamma: X \to Y$ in Γ we put $R(\gamma) = X$ and $L(\gamma) = Y$. A topological groupoid gives a classifying space $B\Gamma$ constructed in [HA3]. We will refer to this particular construction as the Haefliger - Milnor approach. For the definition of manifolds and C^1 topology, the weak C^1 topology is used in our case to agree with [PH1], see [MU1]. For Tangent vector bundles and bundles in general see [MI2] and [ST1]. For definition of Automorphisms, Submersions and Regular maps see [PH1]. For Algebraic Topology terms see [SP1] and for CW Complexes see [LW1]. Most of the definitions used are given just before they are used and proofs of the elementary properties are omited if given elsewhere.

The Theorems and Lemmas are numbered in the order that they appear in each secticn; if more that one number is used the Theorem or Lemma being refered to is outside the section which referenced it. In this case the figures from left to right correspond to ~~number~~ chapter, section, ~~chapter~~ number respectively.

CHAPTER II

A SPECTRAL SEQUENCE FOR $B\Gamma$.

1. The alternative construction of $B\Gamma$

In constructing $B\Gamma$ using the Haefliger-Milnor approach [HA3] we start with a special set $A\Gamma$ construct the total space $E\Gamma$ as the equivalence classes of a relation and then give $E\Gamma$ a special topology. We then note that Γ acts from the left on $E\Gamma$ to give the orbit space $B\Gamma$. The alternative construction first chooses a topology on $A\Gamma$ which agrees with that on $E\Gamma$ and then carries out the intermediate constructions in reverse. The action on $A\Gamma$ by Γ provides the orbits $C\Gamma$ and the equivalence relation provides the quotient space $\overline{B\Gamma}$ from $C\Gamma$. $\overline{B\Gamma}$ is then shown to be homeomorphic to $B\Gamma$.

By using the alternative construction of $B\Gamma$ we can come closer to seeing the homotopy properties of the classifying space because $C\Gamma$ is easier than $E\Gamma$ to work with. Especially when evaluating spectral sequences.

In constructing the topology of $E\Gamma$ the notion of weak and strong topology is used, together with some other topology constructions given here:

Let X be a set then a topology for X is a special subset of the setof subsets of X. If τ and σ are topologies for X then τ is <u>weaker</u> than(<u>stronger</u> than) σ when $\sigma \subset \tau$ $(\tau \subset \sigma)$. If σ is a topology for X then a <u>basis</u> for σ is a subset $\beta \subset \sigma$ such that for $U \in \sigma$

$$\bigcup_{V \subset U \,\&\, V \in \beta} = U$$

A <u>subbasis</u> for σ is a subset $\beta' \subset \sigma$ such that the set of subsets of X generated by finite intersections of members of β' form a basis for σ. The topology is uniquely determined by its subbasis and a funtion $f: X \longrightarrow Y$ to a topological space Y with subbasis β is continuous if the inverse immage of the members of β are open in X.

1 LEMMA: If $\{X_i \mid i \in J\}$ is a collection of topological spaces, X is a set with subsets $\{U_i \mid i \in J\}$ and $\{f_i : U_i \longrightarrow X_i \mid i \in J\}$ is a collection of functions then there is a strongest topology τ for X which satisfies

$$U_i,\ i \in J \quad \text{is open in } X$$
$$f_i : U_i \longrightarrow X_i,\ i \in J \quad \text{is continuous.}$$

and it has a subbasis given by the following sets

$$f_i^{-1}(V) \quad \text{where } i \in J \text{ and } V \text{ is open in } X_i \ ; \ X \,\&\, \phi.$$

Normally X is included in the collection $f_i^{-1}(V)$ and ϕ can be also regarded as coming from $f_i^{-1}(\phi)$.

Before leaving the topic of lemma 1 it should be noted that in [HU1] it was mentioned that the meaning atributed to "strong topology" is ambiguous, however the context that it has been used in for $B\Gamma^1$ removes this ambiguity and the basis generated with our definition of basis for the lemma 2·1 agrees with that given in [MI4] by Milnor.

A special class of constructions similar to lemma 2·1 gives the following definitions collected from [HU1].

Let X, Y be topological spaces and $f: X \to Y$ a surjective function such that a

set $E \subset Y$ is open in Y iff $f^{-1}(E)$ is open in X.

Then f is an identification. This gives the following

2 LEMMA: If $f: X \to Y$ is an identification and $g: Y \to Z$ is a function of Y into a space Z, then a necessary and sufficient condition for the continuity of g is that of the composition $g \circ f$.

Let $f: X \to Y$ denote a surjective function from a space X onto a set Y. Then there exists a unique topology on Y the identification topology such that f is an identification. Let X be a topological space and Q a partition of X then if $P: X \to Q$ is the natural projection of X onto Q and we give Q the identification topology with respect to P then Q is a decomposition space of X. If X is a topological space with an equivalence relation $\sim$ on it, $\sim$ gives rise to a partition $X/\sim$ and the decomposition space $X/\sim$ is the quotient space over the equivalence relation $\sim$.

We recall that Γ is a small category and associated with each morphism is a left and right object. If $Ob\,\Gamma$ are the objects of Γ put $L: \Gamma \to Ob\,\Gamma$ and $R: \Gamma \to Ob\,\Gamma$ as the maps which assign the left and right objects respectively. From Γ we can construct $X^{\infty}\Gamma$ as the set of all $\gamma: N \to \Gamma$ such that for all $i, j \in N$ $L(\gamma(i)) = L(\gamma(j))$. Γ acts on $X^{\infty}\Gamma$ in the following way: if $\alpha \in \Gamma$ and $\gamma \in X^{\infty}\Gamma$ such that $R(\alpha) = L(\gamma(i))$ for $i \in N$ then put $(\alpha \circ \gamma)(i) = \alpha \circ \gamma(i)$ for $i \in N$.

To represent the action of Γ on sets such as $X^{\infty}\Gamma$ or Γ we introduce the following conventions. If A is a set with function $L: A \to Ob(\Gamma)$ then we put $\Gamma \hat{\times} A$ as the subset of $\Gamma \times A$ given by

$$\Gamma \hat{\times} A = \{ (\alpha, \gamma) \in \Gamma \times A \mid R(\alpha) = L(\gamma)$$

Also if A has a topology then $\Gamma \hat{\times} A$ is the topological space which has the topology induced by the inclusion into $\Gamma \times A$ (that is the inclusion map is an embedding). The above action of Γ on $X^{\infty}\Gamma$ now gives the map

$$\nu : \Gamma \hat{\times} X^{\infty}\Gamma \longrightarrow X^{\infty}\Gamma$$

where $\nu(\alpha, \gamma) = \alpha \circ \gamma$. Similarly we get the action $\nu : \Gamma \hat{\times} \Gamma \to \Gamma$ due to the composition of morphisms in the category Γ.

Let X be a set with map $L : X \to Ob\,\Gamma$ then we have the following abstract definition of an action of Γ on X : a function $\phi : \Gamma \hat{\times} X \to X$ is an <u>action</u> of Γ on X if

$$\begin{tikzcd}
\Gamma \hat{\times} (\Gamma \hat{\times} X) \arrow[r, "1 \times \phi"] \arrow[d, "\tilde{\phi}"'] & \Gamma \hat{\times} X \arrow[d, "\phi"] \\
\Gamma \hat{\times} X \arrow[r, "\phi"] & X
\end{tikzcd}$$

commutes where $\tilde{\phi}(\gamma, (\gamma', x)) = (\nu(\gamma, \gamma'), x)$.

If Γ acts on a space X by $\nu : \Gamma \hat{\times} X \to X$ then we have the equivalence relation $\sim$ on X defined by $x_1 \sim x_2$ iff there exists a $\gamma \in \Gamma$ such that $\nu(\gamma, x_1) = x_2$ the corresponding equivalence class of $x \in X$ is the orbit of x.

Put Δ^{∞} as the set of all maps $t : \mathbb{N} \to \mathbb{R}$ such that $t(i) \neq 0$ for only afinite number of $i \in \mathbb{N}$, $t(i) \geq 0$ for $i \in \mathbb{N}$ and

$$\sum_{i \in \mathbb{N}} t(i) = 1$$

put $T_i : \Delta^{\infty} \to \mathbb{R}$ as the function $t \longmapsto t(i)$. As a topological space Δ^{∞} will be given the strongest topology such that the maps

$$T_i : \Delta^{\infty} \longrightarrow \mathbb{R} \quad \text{for } i \in \mathbb{N}$$

are continuous.

In constructing $B\Gamma$ we will not use the topology on $A\Gamma$ so for the moment put $A\Gamma$ as a topological space with set $\Delta^\infty \times X^\infty\Gamma$.
For $t \in \Delta^\infty$ put

$$\sigma_t = \{ i \in \mathbb{N} \mid t(i) \neq 0 \}$$

σ_t is not empty and is finite. We can now introduce the equivalence relation on $A\Gamma$. For $(t_1, \gamma_1), (t_2, \gamma_2) \in \Delta^\infty \times X^\infty\Gamma$ put $(t_1, \gamma_1) \sim (t_2, \gamma_2)$ iff

$$t_1 = t_2$$

and

$$\gamma_1 | \sigma_{t_1} = \gamma_2 | \sigma_{t_2} .$$

$\sim$ is an equivalence relation and $E\Gamma$ is defined as the quotient

$$E\Gamma = A\Gamma/\sim$$

Let $P : A\Gamma \to E\Gamma$ be the natural projection to equivalence classes.

On Δ^∞ put $\bar{U}_i = \{ t \in \Delta^\infty \mid t(i) \neq 0 \}$ for $i \in \mathbb{N}$ and definethe map $X_i : \bar{U}_i \times X^\infty\Gamma \to \Gamma$ as, for $(t, \gamma) \in \bar{U}_i \times X^\infty\Gamma$, $X_i(t, \gamma) = \gamma(i)$.
If we put $U_i = P(\bar{U}_i \times X^\infty\Gamma)$ then T_i and X_i in $A\Gamma$ uniquely define maps t_i and x_i in $E\Gamma$ by the following conditions

$$\begin{tikzcd} A\Gamma \arrow[d, "P"] \arrow[dr, "T_i"] & \\ E\Gamma \arrow[r, "t_i"] & \mathbb{R} \end{tikzcd} \qquad \begin{tikzcd} \bar{U}_i \times X^\infty\Gamma \arrow[d, "P"] \arrow[dr, "X_i"] & \\ U_i \arrow[r, "x_i"] & \Gamma \end{tikzcd}$$

for $i \in \mathbb{N}$ commute. The maps t_i and x_i play a central role in proving the classifying properties of $B\Gamma$ so in the Haefliger-Milnor treatment the topology on $E\Gamma$ is set to the strongest topology which satisfies

a) the sets U_i are open in $E\Gamma$

b) the maps t_i are continuous for $i \in \mathbb{N}$

c) the maps $x_i : U_i \to \Gamma$ are continuous for $i \in \mathbb{N}$

Lemma 1·1 can be used to construct explicitly the usual subbasis for $E\Gamma$ and $U_i = t_i^{-1}(0,1]$.

The equivalence relation commutes with the map $L : \Delta^\infty \times X^\infty\Gamma \to \mathrm{Obs}\,\Gamma$ given by $(t,\gamma) \longmapsto L(\gamma(1))$ and with the action of Γ on $\Delta^\infty \times X^\infty\Gamma$ to give a map $L : E\Gamma \to \mathrm{Obs}\,\Gamma$ and an action $\bar{\nu}\,(\Gamma \hat{\times} E\Gamma) \to E\Gamma$ defined by the conditions

$$\begin{tikzcd} A\Gamma \arrow[d, "P"] \arrow[dr, "L"] & \\ E\Gamma \arrow[r, "L"] & \mathrm{Obs}\,\Gamma \end{tikzcd} \qquad \begin{tikzcd} \Gamma \hat{\times} A\Gamma \arrow[r, "\nu"] \arrow[d, "1\times P"] & A\Gamma \arrow[d, "P"] \\ \Gamma \hat{\times} E\Gamma \arrow[r, "\bar{\nu}"] & E\Gamma \end{tikzcd}$$

commute.

We put $B\Gamma$ as the quotient space of orbits of the action $\bar{\nu}$ on $E\Gamma$. Put $\pi : E\Gamma \to B\Gamma$ as the natural projection to orbits.

The alternative construction starts with choosing a topology on $A\Gamma$ such that $P : A\Gamma \to E\Gamma$ is an identification. The topology on $A\Gamma$ is given as the strongest topology on $A\Gamma$ which satisfies

a) $\bar{U}_i \times X^\infty\Gamma$ is open for $i \in \mathbb{N}$

b) the map $\bar{U}_i \times X^\infty\Gamma \xrightarrow{X_i} \Gamma$ is continuous for $i \in \mathbb{N}$

c) the map $A\Gamma \xrightarrow{T_i} \mathbb{R}$ is continuous for $i \in \mathbb{N}$.

We have that

3 LEMMA : The map $P : A\Gamma \to E\Gamma$ is an identification.

Proof : For $U \subset E\Gamma$ the correspondence $U \longmapsto P^{-1}(U)$ is bijective between the subbasis elements as given by Lemma 2·1 so it follows that U is open iff $P^{-1}(U)$ is.

Instead of constructing $E\Gamma$ from $A\Gamma$ we put $C\Gamma$ as a space which is homeomorphic to the quotient of $A\Gamma$ by its orbits under the action ν given above. The following gives a description of $C\Gamma$.
Let $A \subset \mathbb{N}$ be a non empty subset of $\mathbb{N}$ the pair $(A, \leq)$ gives rise to a small category C_A which has morphisms $a \leq b$ (where $a, b \in A$) and A as the set of objects. Put[†] $\Gamma(A)$ as the set of ~~morphisms~~ functors from C_A to the category Γ. $\Gamma(A)$ has the maps, for $a \leq b$, $a, b \in A$, $\pi_{a \leq b} : \Gamma(A) \longrightarrow \Gamma$ given by $\pi_{a \leq b}(\gamma) = \gamma(a \leq b)$. The topology on $\Gamma(A)$ is taken to be the strongest topology such that the functions $\pi_{a \leq b}$ are continuous.

The functors $\gamma \in \Gamma(\mathbb{N})$ are uniquely determined by the map $\psi_\gamma : \mathbb{N} \to \Gamma$, given by $\psi_\gamma(i) = \gamma(i \leq i+1)$, and conversely if $\alpha : \mathbb{N} \longrightarrow \Gamma$ is such that

$$\alpha(i+1) \circ \alpha(i) \quad \text{for all } i \in \mathbb{N}$$

is well defined, then there exists a (unique) $\gamma \in \Gamma(\mathbb{N})$ such that $\psi_\gamma = \alpha$. For $\gamma \in X^\infty\Gamma$ define the correspondence $\phi_\gamma : \mathbb{N} \to \Gamma$ by

$$\phi_\gamma(i) = \gamma(i+1)^{-1} \circ \gamma(i) \quad \text{for } i \in \mathbb{N}$$

then

$$\phi_\gamma(i+1) \circ \phi_\gamma(i) = \gamma^{-1}(i+2) \circ \gamma(i+1) \circ \gamma(i+1)^{-1} \circ \gamma(i)$$
$$= \gamma(i+2)^{-1} \circ \gamma(i)$$

is well defined. This by the above observation defines a unique map $r : X^\infty\Gamma \longrightarrow \Gamma(\mathbb{N})$ given by the property

$$\psi_{r(\gamma)}(i) = \phi_\gamma(i) \quad \text{for } i \in \mathbb{N},\ \gamma \in X^\infty\Gamma.$$

[†] Note that where the context removes the ambiguity A will be written in place of C_A ; hence the notion $\Gamma(A)$.

$1 \times \gamma : \quad A\Gamma \longrightarrow \Delta^{\infty}\times \Gamma(\mathbb{N})$ is surjective because γ has a left inverse $i : \Gamma(\mathbb{N}) \longrightarrow X^{\infty}\Gamma$ given by the definition : For $\gamma \in \Gamma(\mathbb{N})$, $i(\gamma)(K) = \gamma(1 \leq K)$ for $K \in \mathbb{N}$.

Define $C\Gamma$ as the topological space $\Delta^{\infty}_{\times}\Gamma(\mathbb{N})$ but with the topology given as the identification topology of $1\times\gamma : A\Gamma \longrightarrow \Delta^{\infty}_{\times}\Gamma(\mathbb{N})$

To construct $B\Gamma$ again we will construct a space $\overline{B\Gamma}$ from $C\Gamma$ and then show that $B\Gamma$ and $\overline{B\Gamma}$ are homeomorphic. To do the construction we need the following.

Let $F : A \longrightarrow B$ be a functor between categories A and B let C be a sub category of A then $F|C$ is the functor $F|C : C \longrightarrow B$ that assigns the object $F(a)$ to the object a of C and the morphism $F(m)$ to the morphism m of C. $F|C$ is the <u>restriction</u> of F to the subcategory C.

On $C\Gamma$ construct the following equivalence relation $\sim$. For $(t_1, \gamma_1), (t_2, \gamma_2) \in \Delta^{\infty}_{\times}\Gamma(\mathbb{N})$, $(t_1, \gamma_1) \sim (t_2, \gamma_2)$ iff

a) $t_1 = t_2$

b) $\gamma_1 | \sigma_{t_1} = \gamma_2 | \sigma_{t_2}$

where $|$ is taken to be the restriction to a subcategory. Put $\overline{B\Gamma}$ as the topological quotient $\overline{B}\Gamma \doteq C\Gamma/\sim$

Put $\overline{\pi} : C\Gamma \longrightarrow \overline{B\Gamma}$ as the natural projection to equivalence classes.

4 LEMMA: For $A \subset \mathbb{N}$; $A \neq \emptyset$; $\gamma_1, \gamma_2 \in X^{\infty}\Gamma$

$$\gamma(\gamma_1)|C_A = \gamma(\gamma_2)|C_A$$

$$\Updownarrow$$

$$\exists \gamma \text{ s.t. } \gamma_1|A = \gamma \circ \gamma_2|A .$$

Proof: Suppose $r(\gamma_1)|C_A = r(\gamma_2)|C_A$ We note that for $i, j \in \mathbb{N}$ and $i \leq j$

$$r(\gamma_1)(i \leq j) = \gamma_1(j)^{-1} \circ \gamma_1(i).$$

Since $A \neq \emptyset$ there exists an $k \in A$, so for $i \in \mathbb{N}$ we get $q = k \leq i$ or $i \leq k$ with

$$\gamma_1(q) = \gamma_2(q)$$

which gives

$$\gamma_1(k)\gamma_2(k)^{-1} = \gamma_1(i)\gamma_2(i)^{-1} \quad \text{for} \quad i \in A,$$

put $\gamma = \gamma_1(k) \circ \gamma_2(k)^{-1}$ then we get

$$\gamma_1(i) = \gamma \circ \gamma_2(i) \qquad \text{for all } i \in A.$$

which gives $\gamma_1|A = \gamma \circ \gamma_2|A$.

Conversely : suppose $\gamma_1|A = \gamma \circ \gamma_2|A$ then we get that for $i, j \in A$ with $i \leq j$

$$\begin{aligned} r(\gamma_1)(i \leq j) &= \gamma_1(j)^{-1} \circ \gamma_1(i) \\ &= \gamma_2(j)^{-1} \circ \gamma^{-1} \circ \gamma \circ \gamma_2(i) \\ &= \gamma_2(j)^{-1} \circ \gamma_2(i) \\ &= r(\gamma_2)(i \leq j) \end{aligned}$$

so $r(\gamma_1)|C_A = r(\gamma_2)|C_A$.

In the case that $A = \mathbb{N}$ the lemma shows that the inverse immages of points in CT^1 are infact orbits of the action of T^1 on AT^1.

If $X \xrightarrow{f} Y$ is a function between sets then we can define an equivalence relation on X by : for $a, b \in X$ $a \sim b$ iff $f(a) = f(b)$. The fibreset of f denoted by X/f is the set of equivalence classes and the equivalence classes are refered to as fibres of the map f. If X has a topology then X/f is given the identification topology.

5 LEMMA: $$A\Gamma/_{\pi\circ P} = A\Gamma/_{\bar{\pi}\circ(1\times r)}.$$

Proof: Suppose $\pi\circ P(t_2,\gamma_2) = \pi\circ P(t_1,\gamma_1)$

Then there exists an $\gamma\in\Gamma$ such that
$$\bar{\nu}(\gamma, P(t_1,\gamma_1)) = P(t_2,\gamma_2)$$
this means that, by definition of $\bar{\nu}$,
$$P(t_2,\gamma_2) = P(t_1,\gamma\circ\gamma_1)$$
$$\Longrightarrow \quad t_2 = t_1 \;\&\; \gamma_1|\sigma_{t_1} = \gamma_2|\sigma_{t_2}$$
and by lemma 4 this gives
$$t_2 = t_1 \;\&\; r(\gamma_1)|\sigma_{t_1} = r(\gamma_2)|\sigma_{t_2}$$
$$\Longrightarrow \quad \bar{\pi}\circ(1\times r)(t_1,\gamma_1) = \bar{\pi}\circ(1\times r)(t_2,\gamma_2)$$
Conversely suppose
$$\bar{\pi}\circ(1\times r)(t_1,\gamma_1) = \bar{\pi}\circ(1\times r)(t_2,\gamma_2)$$
Then we get there exists a $\gamma\in\Gamma$ such that
$$t_1 = t_2 \quad \text{and} \quad \gamma_1|\sigma_{t_1} = \gamma\circ\gamma_2|\sigma_{t_2}$$
from lemma 4. This in turn gives
$$\bar{\nu}(\gamma, P(t_2,\gamma_2)) = P(t_1,\gamma_1)$$
$$\Longrightarrow \quad \pi\circ P(t_2,\gamma_2) = \pi\circ P(t_1,\gamma_1)$$
Toconstruct the homeomorphism between $B\Gamma$ and $\overline{B\Gamma}$ we note that
$$B\Gamma = E\Gamma/_{\pi} = (A\Gamma/P)/_{\pi} \approx A\Gamma/_{\pi\circ P} =$$
$$A\Gamma/_{\bar{\pi}\circ(1\times r)} \approx (A\Gamma/_{1\times r})/\bar{\pi} = C\Gamma/_{\bar{\pi}} = \overline{B\Gamma}$$
where $\approx$ indicates the obvious homeomophism.

The astute reader will notice that $\overline{B\Gamma}$ is very much like the "classifying space" $B\Gamma_{\mathbb{N}}$ given in [SE1] . The difference being that $B\Gamma_{\mathbb{N}}$ is homeomorphic to $\overline{B\Gamma}$ with a weaker topology.

At this stage it should be noted that the construction is functorial in the sence that if we have topological groupoids Γ_1 and Γ_2 with a continuous functor $f : \Gamma_1 \longrightarrow \Gamma_2$ then f induces a continuous map $\tilde{f} : \overline{B\Gamma_1} \longrightarrow \overline{B\Gamma_2}$ and the correspondence $f \longmapsto \tilde{f}$ is a functor.

2 Spectral Sequences

In this section we will introduce the basic definitions and properties of spectral sequences, where the proof of the properties are ommited. For proofs not given here see Spanier [SP1].

Let R (usually taken as $\mathbb{Z}$) be a fixed principal ideal domain. A bigraded module E is an indexed collection of modules $\{E_{s,t} \mid s,t \in \mathbb{Z}\}$. A differential $d: E \to E$ is a collection of homomorphisms $d: E_{s,t} \longrightarrow E_{s-r,\, t+r-1}$ such that $d^2=0$ and r is a fixed integer. r is the degree of d. The homology is defined as the bigraded module $H(E)$ where

$$H(E)_{s,t} = \operatorname{Kern}(d: E_{s,t} \longrightarrow E_{s-r,\,t+r-1}) / d(E_{s+r,\,t-r+1})$$

An E^K spectral sequence is a sequence $\{E^r, d^r\}$ for $r \geq K$ such that

a) E^r is a bigraded module and $d^r: E^r \longrightarrow E^r$ is a differential of degree r.

b) For $r \geq K$ there is a prefered isomorphism

$$H(E^r) \approx E^{r+1}$$

A homomorphism $\varphi: E \to E'$ between E^K spectral sequences is a collection of homomorphisms $\varphi^r: E^r_{s,t} \longrightarrow E'^r_{s,t}$ for $r \geq K$ such that it commutes with differentials and such that the induced map between the cosets : $\varphi^r_*: H(E^r) \to H(E'^r)$ commutes with the prefered isomorphisms, ie:- the diagram

$$\begin{array}{ccc} H(E^r) & \xrightarrow{\varphi^r_*} & H(E'^r) \\ \wr\wr & & \wr\wr \\ E^{r+1} & \xrightarrow{\varphi^{r+1}} & E'^{r+1} \end{array}$$

commutes.

All the spectral sequences that we will be considering will satisfy $E^r_{s,t} = 0$ for $s < 0$ or $t < 0$. Such spectral sequences are refered to as first-quadrant spectral sequences. With this limitation in mind we have that for given s,t there exists an r' such that

$$E^{r'}_{s\,t} \approx E^{r'+1}_{s\,t} \approx \cdots\cdots$$

and we put $E^{\infty}_{s,t}$ as an isomorphic copy of one of these modules.

Let A be a graded module $\{A_t \mid t \in \mathbb{Z}\}$ such that $A_t = 0$ for $t < 0$. A filtration F of A is a sequence $F_s A$ of sub graded modules graded by $\{F_s A_t\}$ such that $F_s A \subset F_{s+1} A$ and $F_s A = 0$ for $s < 0$. F is convergent if $\bigcup F_s A = A$. Given a filtration F on A the associated bigraded module is $G(A)$ given by

$$G(A)_{s,t} = F_s A_{s+t} / F_{s-1} A_{s+t}.$$

s is the filtered degree and t is the complementary degree, and $s+t$ the total degree of an element of $G(A)_{s,t}$.

1 THEOREM Let F be a convergent filtration of a chain complex C which commutes with its differential, there is an E^1 spectral sequence which is first quadrant spectral sequence where

$$E^1_{s,t} \approx H_{s+t}(F_s C / F_{s-1} C),$$

d^1 corresponds to the boundary operator of the triple $(F_s C, F_{s-1} C, F_{s-2} C)$, and E^{∞} is isomorphic to the bigraded module $GH_*(C)$ (associated to the filtration

$$F_s H_*(C) = \operatorname{im}[H_*(F_s C) \to H_*(C)])$$

Also the spectral sequence construction is functorial.

2 THEOREM Let $\tau\colon C \to C'$ be a chain map preserving filtration between chain complexes having convergent filtrations. If for some

$r \geq 1$ the induced map $\tau^r : E^r \to E'^r$ is an isomorphism, then τ induces an isomorphism

$$\tau_* : H_*(C) \approx H_*(C')$$

in the homology of the chain complexes.

Let σ be a finite non empty subset of N then σ is an <u>ordered simplex</u> and members of σ are the <u>virtices</u> of σ. The <u>realisation</u> $|\sigma|$ of σ is the topological subspace of Δ^∞ given by

$$|\sigma| = \{t \in \Delta^\infty \mid t(i) \neq 0 \Rightarrow i \in \sigma\}$$

a <u>linear map</u> f between ordered simplexes σ_1 and σ_2 is a map such that for $t_1, t_2 \in |\sigma_1|$ and $\lambda \in I$, $\lambda f(t_1) + (1-\lambda) f(t_2) = f(\lambda t_1 + (1-\lambda) t_2)$. If $g : \sigma_1 \to \sigma_2$ is a function then if we put for vertex $v \in \sigma_1$, $v \in |\sigma_1|$ as $v(a) = 0$ if $a \neq v$ and $v(v) = 1$ we get a linear map $\bar{g}$ <u>induced</u> by g given by $\bar{g} : |\sigma_1| \to |\sigma_2|$ where $\bar{g}(v) = g(v)$. All linear maps of ordered simplexes are induced and are continuous.

Let $\Delta^q = |\{1, \dots, q+1\}|$ and $e^i_{q+1} : \Delta^q \longrightarrow \Delta^{q+1}$ be the linear map induced by the map

$$e^i_{q+1}(j) = \begin{cases} j & \text{for } j < i \\ j+1 & \text{for } j \geq i \end{cases}$$

Let X be a topological space, for $q \geq 1$ a <u>singular q - simplex</u> σ of X is defined to be a continuous map

$$\sigma : \Delta^q \longrightarrow X$$

For $q > 1$ and $1 \leq i \leq q+1$ the ith face of σ, denoted by $\sigma^{(i)}$, is defined to be the composite

$$\sigma(i) = \sigma \circ e^i_q : \Delta^{q-1} \longrightarrow \Delta^q \longrightarrow X$$

note that if $q > 1$ and $1 \leq j < i < q+1$ then

$$(\sigma^{(i)})^{(j)} = (\sigma^{(j)})^{i-1}$$

The <u>singular chain complex</u> of X, denoted by $\Delta(X)$, is defined as the non negative chain complex

$$\Delta(X) = \{\Delta_q(X), \partial_q\},$$

where $\Delta_q(X)$ is the free abelian group generated by singular q- simplexes for $q>0$ and $\Delta_q(X)=0$ otherwise, and for $q\geqslant 2$, ∂_q is defined by the equation

$$\partial_q(\sigma) = \sum_{1\leq i\leq q+1} (-)^{i+1}\sigma^{(i)}$$

If $f: X \to Y$ is continuous define $f: \Delta(X) \longrightarrow \Delta(Y)$ as the chain map $f(\sigma) = f\circ\sigma$ for a singular q- simplex σ. $f \longrightarrow f$ is a covarient functor from topological spaces to chain complexes. Composing this functor with the homology functor which assigns the homology of a chain complex to a chain complex we get the singular homology functor. The graded group $H(X)$ with

$$H(X)_q = H_q(\Delta(X))$$

is the singular homology of X.

We will have recource in the sequal to constructions which involve special sub complexes of $\Delta(X)$ which give special homology theories such as framed homology. However they require extra structure such as frame bundles to be introduced and will thus not be introduced until they are needed.

We also have the graded group $H(X,Y)$ for topological spaces $Y\subset X$ where

$$H(X,Y)_q = H_q(\Delta(X)/\Delta(Y))$$

in accordance with the usual notation we will write $H_q(X,Y)$ and $H_q(X)$ for $H(X,Y)_q$ and $H(X)_q$.

The treatment for singular homology can be extended to singular homology with coefficients in a module G. If C is a chain complex with differential ∂ then $C\otimes G$ is a chain complex with differential $\partial\otimes 1$ the homology of C with coefficients in the module G is the graded module $H(C\otimes G)$ and is denoted as $H(C;G)$ where the modules of $H(C;G)$ are written as $H_q(C;G)$. We will write

$$H_q(X,Y;G) = H_q\left(\left(\Delta(X)/\Delta(Y)\right)\otimes G\right).$$

From chain complex theory we have

3LEMMA Let $\tau: C\longrightarrow C'$ be a chain map between freely generated chain complexes such that $\tau_*: H(C)\approx H(C')$. For any R module G, τ induces an isomorphism $\tau_*: H(C;G)\approx H(C';G)$.

Using lemma 3 we can extend a lot of our results to homology with coefficients in a module G. But we will ommit the proofs for the general case coefficient module.

Let $A\subset B\subset C$ be topological spaces then (C,B,A) is a topological triple. The boundary operation of the triple (C,B,A) is a homomorphism $\partial_q: H_q(C,B)\longrightarrow H_{q-1}(B,A)$ induced by the boundary operator of the triple $(\Delta(C),\Delta(B),\Delta(A))$ infact if

$$0\longrightarrow \Delta(B)/\Delta(A)\xrightarrow{\alpha}\Delta(C)/\Delta(A)\xrightarrow{\beta}\Delta(C)/\Delta(B)\longrightarrow 0,$$

$z''\in\Delta(C)/\Delta(B)$, and $\{z''\}\in H(C,B)$ we have
$\partial_*\{z''\}=\{\alpha^{-1}\partial\beta^{-1}z''\}\in H(B,A)$.

A filtration $F: \phi\subset X_0\subset X_1\subset\cdots$ of subsets of a topological space is convergent if

a) $\cup X_i = X$

b) every compact subset of X is containedin some X_i.

Using Theorem 1 and putting $F_s(\Delta(X))=\Delta(X_s)$, we get

4 LEMMA: If $\{X_s\}$ is a filtration of the topological space X. There is an E^2 spectral sequence which is a first quadrant spectral sequence where

$$E^1_{s,t} \approx H_{s+t}(X_s, X_{s-1}),$$

d^1 corresponds to the boundary operator of the triple (X_s, X_{s-1}, X_{s-2}), and E^∞ is isomorphic to the bigraded module $GH_*(X)$ associated to the filtration

$$F_s H_*(X) = \operatorname{im}\left[H_*(X_s) \longrightarrow H_*(U\Delta(X_s))\right]$$

Also the spectral sequence construction is functorial and if the filtration $\{X_s\}$ is convergent

$$H_*(U\Delta(X_s)) = H_*(X)$$

The dimension of an ordered simplex σ is the number of vertices minus one and is written as $\mathrm{Dim}(\sigma)$. Put

$$(\Delta^\infty)^S = \bigcup_{\mathrm{Dim}(\sigma)=S} |\sigma|$$

then if $C_S = (\Delta^\infty)^S \times \Gamma(N) \subset C\Gamma$, where the topology on C_S is induced by $C\Gamma$, and $\mathcal{B}_S = \overline{\pi}(C_S)$

$$\emptyset \subset \mathcal{B}_0 \subset \mathcal{B}_1 \subset \cdots\cdots$$

gives a filtration of $\overline{\mathcal{B}\Gamma}$ such that

$$U\mathcal{B}_S = \overline{\mathcal{B}\Gamma}$$

$\mathcal{B}_S$ is the filtration, and hence spectral sequence, that we will be considering first, the spectral sequence is given by lemma 4.

There is one draw back and that is that the filtration is not convergent so the isomorphism

$$H_*(U\Delta(\mathcal{B}_S)) \approx H_*(\overline{\mathcal{B}\Gamma})$$

has to be proved by some other means.

To show that $\{\mathcal{B}_S\}$ is not convergent it is sufficient to show the corresponding filtration $\{(\Delta^{\infty})^s\}$ of Δ^{∞} is not convergent. This is easily done because consider the compact subset K of I given by

$$K = \{0, \tfrac{1}{n} \mid n \in \mathbb{N}\}$$

put $S_k \in \Delta^{\infty}$ as the map $S_k : \mathbb{N} \longrightarrow \mathbb{R}$ given by

$$S_k(i) = \begin{cases} 0 & \text{if } i \geq k+1 \\ 1/k^2 & \text{if } k = 2, 3, \dots, i+1. \\ 1-\frac{1}{k} & \text{if } i = 1 \end{cases}$$

then we get the map $\phi : K \longrightarrow \Delta^{\infty}$ given by

$$\phi(\tfrac{1}{n}) = S_n \quad \text{FOR} \quad n \in \mathbb{N}$$

and $\phi(0) = (1, 0, 0, \dots)$

is continuous, but the immage of ϕ is then compact.

However we can prove the isomorphism

$$H_*(U\Delta(B_S)) \approx H_*(\overline{B\Gamma})$$

by refering to Haefliger's constructions, used in [HA3] , to show that $B\Gamma$ is a classifying space. To provide motivation we will quote the following property of the singular homology theory:

5 LEMMA: Let X be a topological space and $z \in H_*(X)$ then there exists a compact subset K of X such that if $i : K \subset X$, $i_*(z') = z$ for some $z' \in H_*(K)$.

However we need a stronger condition than Lemma 5 we need that K is also hausdorff, however i need not be an inclusion, to construct members of $U\Delta(B_S)$ from $\Delta(\overline{B\Gamma})$. If $B\Gamma$ was a hausdorff topological space we would have no problems in constructing K , this is not so in general so we will have to resort to a "CW" version of K.

Given a set X and an indexed collection of topological spaces $\{X_i \mid i \in J\}$ and maps $f_i : X_i \longrightarrow X$ then the topology <u>coinduced</u> on X by the functions $\{f_i\}$ is the weakest topology such that the functions $\{f_i\}$ are continuous. Let $\{A\} = \mathcal{Q}$ be a collection of subsets of a topological space X then the topology on X is <u>coherent</u> with $\mathcal{Q}$ if the topology on X is coinduced from the subspaces $\{A\}$ by the inclusion maps $A \subset X$.

Put $\dot{\Delta}^q = \bigcup_{1 \le i \le q+1} e_q^i(\Delta^{q-1})$ with the topology induced from Δ^q. Note that $\dot{\Delta}^q$ has the topology which is coherent with $\{e_q^i(\Delta^{q-1})\}$ and is coinduced by the maps $\{e_q^i\}$. We now have the following definition Let A be a closed subset of a space X. X is said to be obtained from A by <u>adjoining n-cells</u> $\{\bar{e}_j^n\}$, where $n \geq 0$, if

a) For each j, $\bar{e}_j^n$ is a subset of X.

b) If $\dot{e}_j^n = \bar{e}_j^n \cap A$, then for $j \neq j'$, $\bar{e}_j^n - \dot{e}_j^n$ is disjoint from $\bar{e}_{j'}^n - \dot{e}_{j'}^n$.

c) X has a topology coherent with $\{A, \bar{e}_j^n\}$

d) For each j there is a map

$$f_j : (\Delta^n, \dot{\Delta}^n) \longrightarrow (\bar{e}_j^n, \dot{e}_j^n)$$

such that $f_j(\Delta^n) = \bar{e}_j^n$, f_j maps $\Delta^n - \dot{\Delta}^n (= \mathring{\Delta}_n)$ homeomorphicaly into $\bar{e}_j^n - \dot{e}_j^n$, and $\bar{e}_j^n$ has the topology coinduced by f_j and the inclusion map $\dot{e}_j^n \subset \bar{e}_j^n$. f_j is the <u>characteristic map</u> of $\bar{e}_j^n$. A(relative) CW complex (X, A) consists of a topological space X, a closed $A \subset X$ and a sequence of closed subspaces $(X)^K$ for $K \geq 0$ such that

a) $(X)^0$ is obtained from A by adjoining 0-cells.

b) For $K \geq 1$, $(X)^K$ is obtained from $(X)^{K-1}$ by adjoining

K -cells.

c) $X = \cup (X)^K$

d) X has a topology coherent with $\{(X)^K, A\}$.

$(X)^K$ is the K-skeleton of X, if $X = (X)^n$ for some n then the dimension of X is n, and X is constructed by adjoining a finite number of cells then X is a finite CW complex.

A pointed space is a pair (X, x_0) where X is a topological space and x_0 is a point in X. x_0 is a base point. Let $\pi_n(X, x_0)$, $n \geq 1$ be the homotopy groups of the pointed space (X, x_0) and $\pi_0(X)$ the path components of X. For a continuous map $f: (X, x_0) \longrightarrow (Y, y_0)$ between pointed topological spaces we put $f_\#: \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0)$ as the homomorphism induced by f. A map $f: X \longrightarrow Y$ is an n-equivalence if it induces a 1-1 and onto correspondence of the path components of X and Y, and for $x \in X$, $f_\#: \pi_q(X, x) \longrightarrow \pi_q(Y, f(x))$ is an isomorphism for $0 < q < n$ and an epimorphism for $q = n$. A map $f: X \longrightarrow Y$ is a weak homotopy equivalence if it is an n-equivalence for all $n \in \mathbb{N}$.

6 THEOREM: For a topological pair (X, A) there is a map $f: (CWX, A) \longrightarrow (X, A)$ such that f is a weak homotopy equivalence and (CWX, A) is a CW complex.

7 THEOREM : (Generalised Whitehead Theorem)
if $f: X \longrightarrow Y$ is an n-equivalence then

$$f_*: H_q(X) \longrightarrow H_q(Y)$$

is an isomorphism for $q < n$ and an epimorphism for $q = n$. The converse is true when X and Y are simply connected.

8 LEMMA: Every CW complex $(X, \emptyset)$ is a hausdorff topological space.

By using lemmas 5&8 and theorems 6&7 we get by noting that a subspace of a hausdorff space is hausdorff.

9 LEMMA: Let X be a topological space and $z \in H_*(X)$ then there exists a compact hausdorff space K and continuous function $f: K \to X$ such that there exists an $z' \in H_*(K)$ satisfying $f_*(z') = z$.

Let X be a topological space then a partition of unity on X is a collection of maps $\{\varphi_i : X \to I\}$ such that for $x \in X$ only a finite number of $\{\varphi_i(x)\}$ are not zero and $\sum \varphi_i(x) = 1$ for all $x \in X$.

Let $\overline{C}_s = \Delta^s \times X^{s}\Gamma \subset A\Gamma$ and put $\overline{B}_s = \pi \circ (1 \times v)(\overline{C}_s)$ for $s \geqslant 0$ then we see that $\overline{B}_s \subset B_s$.

10 LEMMA : Let K be a compact hausdorff topological space and $f : K \to \overline{B\Gamma}$ be a continuous map then there exists a continuous map $\hat{f} : K \to \overline{B}_s$ for some $s \geqslant 0$ such that $\hat{f}$ is homotopic to f.

Proof: $f : K \longrightarrow \overline{B\Gamma}$. We can identify $\overline{B\Gamma}$ with $B\Gamma$ via the homeomrophism given by Lemma 1•5 this gives $\overline{B\Gamma}$ the same classifying properties as $B\Gamma$ and the constructions on $A\Gamma$ are the same for $B\Gamma$ and $\overline{B\Gamma}$. $f : K \longrightarrow \overline{B\Gamma}$ pulls back the Γ-structure ω on $\overline{B\Gamma}$ induced from $B\Gamma$, to give a Γ - structure $f^*(\omega)$ on K. Since K is compact there exists a finite open cover $\{U_i \mid 1 \leq i \leq m\}$ of K and cocycle $\gamma_{ij} : U_i \cap U_j \longrightarrow \Gamma$ for $f^*(\omega)$ since the space K is hausdorff it is normal and by Urysohn's lemma we can construct a partition of unity $\{\varphi_i \mid 1 \leq i \leq m\}$ such that $U_i = \varphi_i^{-1}(0,1]$ In [HA3] Haefliger constructs a map into $B\Gamma$ which gives for $\overline{B\Gamma}$ a map $\tilde{f} : K \longrightarrow \overline{B}_s$ such that $\tilde{f}^*(\omega) = f^*(\omega)$. But by the classifying properties of $\overline{B\Gamma}$ we get $\tilde{f}$ and f are homotopic.

11 LEMMA: Let K be a compact hausdorff topological space and K' a compact (closed) subset of K then if $f: (K,K') \longrightarrow (\overline{B\Gamma}, \overline{B}_s)$ for some s then there exists an $r>s$ and map $\tilde{f}: (K,K') \to \overline{B}_r$ which is homotopic to f relative to $\overline{B}_r$ for some $v>s$.

Proof: is similar to that of lemma 10 as far as constructing $\tilde{f}$ from f where due care is taken to ensure that the cocycle chosen to represent $f^*(\omega)$ restricts to the cocycle on K' induced by $f^*(\omega)$ We then note that the canonical homotopy from f to $\tilde{f}$ construced by Haefliger in [HA3] for $B\Gamma$ gives a homotopy with the required properties, this is checked by noting that only a finite number of non zero compmnents of t in $(t,\gamma) \in im(f(K'))$ is used in the construction of the homotopy.

We need a "kernel" version of Lemma 5. Spanier provides it on page 204 of [SP1].

12 LEMMA: Let X be a topological space $K' \subset X$ compact and if $i: K' \subset X$, $z \in H_*(K')$ and $i_*(z)=0$ then there exists a compact subspace $K \subset X$ such that $K' \subset K$ and if $j: K' \subset K$ then $j_*(z)=0$.

For X a topological space and $z \in H_*(X)$ we have a map $f: K \to X$ constructed for lemma 9 such a map, together with $z' \in H_*(K)$ such that $f_*(z')=z$ is called a <u>canonical support</u> for z, we can use lemma 12 to prove the following:

13 LEMMA : Let (X,A) be a topological pair, $z \in H_*(A)$ $f: K \to A$ and z' be a canonical support for z. If $f_*(z') \in H_*(X)$ is zero then there exists a compact hausdorff space K' with $K \subset K'$

and extention $f': K' \longrightarrow X$ such that if $i: K \subset K'$ is the inclusion map then $i_*(z') = 0$.

Proof: Since $(f': K \to A, z')$ is canonical there exists a CW complex Q with $K \subset Q$ and a weak homotopy equivalence $\alpha: Q \to A$ such that $f = \alpha | K$. Let Z_α be the mapping cylinder of α and $\gamma: Z_\alpha \longrightarrow X$ Z_α's retract. Then Q is a closed subspace of Z_α and γ is an extention of α which is a weak homotopy equivalence. By Theorem 6 there exists a CW complex (P, Q) and weak homotopy equivalence $\bar{\alpha}: (P, Q) \longrightarrow (Z_\alpha, Q)$ Put $\alpha' = \gamma \circ \bar{\alpha}$ then α' is a weak homotopy equivalence which exdends α to P. Since $\alpha'_*(z') = 0$ we have if $j: K \subset P$ that $j_*(z') = 0$. We now apply lemma 12 to $z' \in H_*(K)$, $K \subset P$, to get, if $i: K \subset K'$, a compact subspace K' of P such that $i_*(z) = 0$. We now obtain the required $f': K' \longrightarrow X$ as $f' = \alpha' | K'$.

If (C, ∂) is a chain complex we will adopt the following convention: by $\{z\} \in H_*(C)$ we mean $\partial z = 0$ and $\{z\}$ is the cocycle class corresponding to z. If we need to distinguish between different homologies we will use different brackets.

14 THEOREM : The inclusion map

$$i: \bigcup_{s=0}^{\infty} \Delta(B_s) \subset \Delta(\overline{B\Gamma})$$

between chain complexes induces the isomorphism

$$H_*\left(\bigcup_{s=0}^{\infty} \Delta(B_s)\right) \approx H_*(\overline{B\Gamma})$$

Proof: We will first prove the theorem where B_s is replaced by $\overline{B}_s$ and i by j.

<u>Surjectivity</u> : Let $\{z\} \in H_*(\overline{B\Gamma})$ then by lemma9 there exists a canonical support $(f: K \to \overline{B\Gamma}, \{z'\})$ for $\{z'\}$. Let $\tilde{f}$ be the map $\tilde{f}: K \longrightarrow \overline{B}_s$ for some $s \geq 0$ given by lemma 10 then we have

$\hat{f}(z') \in \Delta(\bar{B}_s)$ and for $\{\hat{f}(z')\} \in H_*(\overline{BT})$

$$\{\tilde{f}(z')\} = \tilde{f}_*(\{z'\}) = f_*(\{z'\}) = \{z\}.$$

That is for $[\tilde{f}(z')] \in H_*(\bar{B}_s)$ $j_*([\tilde{f}(z')]) = \{z\}$.

Injectivity : Let $z \in \Delta(\bar{B}_s)$ and $[z] \in H_*(\bar{B}_s)$ such that $j_*([z]) = 0$. Choose a can onical support $(f : K \longrightarrow \bar{B}_s, \{z'\})$ with $\{z'\} \in H_*(K)$ then let $f' : K' \longrightarrow \overline{B\Pi}$ be the extention given by Lemma 13. $f' : (K', K) \longrightarrow (\overline{B\Pi}, \bar{B}_s)$ so there exists an $r > s$ and $\tilde{f} : K' \longrightarrow \bar{B}_r$ given by Lemma 11. $[z'] \in H_*(K')$ is zero and f is homotopic to $\tilde{f}/K : K \longrightarrow \bar{B}_r$, so we get

$$[z] = f_*(\{z'\}) = (\tilde{f}|K)_* \ (\{z'\}) = \tilde{f}_*([z']) = 0.$$

Now $\bar{B}_s \subset B_s$ so

$$K : \bigcup_{i=0}^{\infty} \Delta(\bar{B}_s) \subset \bigcup_{i=0}^{\infty} \Delta(B_s).$$

By a method similar to the above surjectivity proof we can show that K_* is surjective. We thus get a commutative diagram

$$\begin{tikzcd}
H_*\left(\bigcup_{i=0}^{\infty} \Delta(\bar{B}_s)\right) \arrow[rr, "j_*"] \arrow[dr, "K_*"'] & & H_*(\overline{B\Pi}) \\
& H_*\left(\bigcup_{i=0}^{\infty} \Delta(B_s)\right) \arrow[ur, "i_*"'] &
\end{tikzcd}$$

where j_* is an isomorphism and K_* is surjective. This means i_* must be an isomorphism.

Let $f : \sigma_1 \longrightarrow \sigma_2$ be an order preserving function between ordered simplexes. Then f gives a function $\Gamma(f) : \Gamma(\sigma_2) \longrightarrow \Gamma(\sigma_1)$ defined by

$$\Gamma(f)(\gamma)(i \leq j) = \gamma(f(i) \leq f(j))$$

The correspondance $f \longmapsto \Gamma(f)$ is a functor from order preserving maps between ordered simplexes, and $\Gamma(f)$ is always continuous.

Let σ be an ordered simplex, put $j_\sigma : \sigma \subset \mathbb{N}$, then j_σ induces an inclusion $j_\sigma : |\sigma| \subset \Delta^\infty$. Put $q_\sigma : \mathbb{N} \longrightarrow \sigma$ as the map which assigns to $i \in \mathbb{N}$ the smallest element j in σ which satisfies.

$$|i-j| = \min_{k \in \sigma} \{|i-k|\}$$

then q_σ is order preserving and satisfies $q_\sigma \circ j_\sigma = 1$ so $\Gamma(q_\sigma) : \Gamma(\sigma) \longrightarrow \Gamma(\mathbb{N})$ is injective. So we get an injective map

$$j_\sigma \times \Gamma(q_\sigma) : |\sigma| \times \Gamma(\sigma) \longrightarrow C\Gamma$$

Put $|\mathring{\sigma}| = \{t \in |\sigma| / \sigma_t = \sigma\}$ then for $\gamma \in \Gamma(\mathbb{N})$, $t \in |\mathring{\sigma}|$ we have $\gamma|\sigma_t = \Gamma(j_\sigma)(\gamma)$ and for $\gamma' \in \Gamma(\sigma)$

$$\Gamma(q_\sigma)\gamma'|\sigma_t = \gamma'.$$

This means that

$$\bar{\pi} \circ [j_\sigma \times \Gamma(q_\sigma)] : |\mathring{\sigma}| \times \Gamma(\sigma) \longrightarrow \bar{B}\Gamma$$

gives a bijective function from $|\mathring{\sigma}| \times \Gamma(\sigma)$ to $\bar{\pi}(|\mathring{\sigma}| \times \Gamma(\mathbb{N}))$. However this means that for $s \geqslant 0$

$$B_s - B_{s-1} \longleftrightarrow \bigcup_{\mathrm{Dim}(\sigma)=s} |\mathring{\sigma}| \times \Gamma(\sigma)$$

where $\longleftrightarrow$ means that there exists a bijective map. This suggests that

$$H_*(B_s, B_{s-1}) \simeq \bigoplus_{\mathrm{Dim}(\sigma)=s} H_*(|\mathring{\sigma}| \times \Gamma(\sigma))$$

In order to facilitate the proof of this type of identity we will adopt the expedient of simplifying the topology of the topological groupoids that we consider, but in doing so we must ensure that we can still apply the computations to the topological groupoids that interest us. In the next section we will introduce such a restricted class of topological groupoids and then compute the E^2 terms of their spectral sequences.

3 Well Formed Topological Groupoids

To help compute the E^2 spectral sequence we will restrict our treatment to a special class of topological groupoids, which still encompas the topological groupoids which interest us. A topological groupoid belonging to this special class will be refered to as well formed topological groupoid.

To simplify notation we will put for subsets A, B of topological groupoid Γ

$$A \circ B = \{ \gamma_a \circ \gamma_b \in \Gamma / \gamma_a \in A, \gamma_b \in B \ \& \ R(\gamma_a) = L(\gamma_b) \}$$

1 LEMMA†: Let Γ be a topological groupoid then the map $v : X^{\infty}\Gamma \longrightarrow \Gamma(N)$ is continuous.

Proof : Let $a \leq b$ and $U \subset \Gamma$ be open in Γ. Then $\pi_{a \leq b}^{-1}(U)$ is a typical member of the subbasis for $\Gamma(N)$'s topology. Let $c \in v^{-1}(\pi_{a\leq b}^{-1}(U))$ this means $c(b)^{-1} \circ c(a) \in U$. By the continuity of composition and the inverse map there exist neighbourhoods U_a, U_b of $c(a)$ and $c(b)$ respectively such that $U_b^{-1} \circ U_a \in U$ but thus means that

$$v(\bar{X}_a^{-1}(U_a) \cap \bar{X}_b^{-1}(U_b)) \subset \pi_{a\leq b}^{-1}(U)$$

where $\bar{X}_a^{-1}(U_a) \cap \bar{X}_b^{-1}(U_b)$ is of course a neighbourhood of c. c is an arbitary member of $v^{-1}(\pi_{a\leq b}(U))$ so $v^{-1}(\pi_{a\leq b}(U))$ is open. $\pi_{a\leq b}^{-1}(U)$ is an arbitary subbasis element of $\Gamma(N)$'s topology so v is continuous.

Let Γ be a topological groupoid. A subset A of Γ is <u>tubular</u> if the maps $L, R : A \longrightarrow Ob\Gamma$ are injective open maps. The composition of morphisms in Γ is <u>tubular</u> if U, V are tubular open sets in Γ gives $U \circ V$ is a tubular open set.

† see next page for definition of the topology of $X^{\infty}\Gamma$.

Γ is called <u>tubular</u> if it has a subbasis consisting of tubular open sets, its composition of morphisms is tubular and taking inverses maps tubular sets to tubular sets.

A sub groupoid Γ' of Γ is <u>open</u> when Γ' is an open subset. It is easily checked that an open sub groupoid of a tubular groupoid is tubular.

2 EXAMPLE: Let $\Gamma^{\circ}X$ be the topological groupoid of germs of local homeomorphisms of a space X then $\Gamma^{\circ}X$ is tubular. In particular, if X is $\mathbb{R}^q$ then we get the topological groupoid Γ°_q and open sub groupoids of Γ°_q, such as Γ^r_q & Γ^{ω}_q, as examples of tubular topological groupoids.

3 EXAMPLE : Discrete groups; the open tubular sets being points.

It should be noted that in the case Γ is a topological group which is not discrete then it is not tubular. In order to include topological groups in our treatment we will use the class of well formed topological groupoids. This contains the class of tubular groupoids. Let $X^{\infty}\Gamma$ have the topology induced from the product space $\Gamma^{\mathbb{N}}$ that is, put $\overline{X}_i : X^{\infty}\Gamma \longrightarrow \Gamma$ as the projection $\overline{X}_i(\gamma) = \gamma(i)$ then the <u>topology on $X^{\infty}\Gamma$</u> is given as the strongest topology that satisfies for each $i \in \mathbb{N}$, $\overline{X}_i : X^{\infty}\Gamma \longrightarrow \Gamma$ is continuous. We now define our class of topological groupoids as follows:

Let Γ be a topological groupoid then it is <u>well formed</u> if

a) the maps , $R, L : \Gamma \longrightarrow \mathrm{Obs}\,\Gamma$ are open maps

b) the composition map $\nu : \Gamma \overset{\vee}{\times} \Gamma \longrightarrow \Gamma$

is open

c) the map

$$\gamma : X^{\infty}\Gamma \longrightarrow \Gamma(\mathbb{N})$$

is open.

4 LEMMA : Let Γ be a tubular topological groupoid, then for $\gamma \in X^{\infty}\Gamma$ and neighbourhood U of γ there exists an $n \in \mathbb{N}$ with open neighbourhood U_L of $L(\gamma)$ and tubular opensets $\omega_i \subset \Gamma$ FOR $i \leq n$ such that for $i \leq n$

$$U_L = L.(\omega_i) \quad \text{and}$$

$$\gamma \in \bigcap_{i=1}^{n} \overline{X}_i^{-1}(\omega_i) \subset \mathcal{U}$$

Proof: Since $\mathcal{U}$ is a neighbourhood of γ and Γ is tubular there exists an $n \in \mathbb{N}$ and tubular open sets $\overline{\omega}_i$ such that

$$\gamma \in \bigcap_{i=1}^{n} \overline{X}_i^{-1}(\overline{\omega}_i) \subset \mathcal{U}$$

Put $U_L = \bigcap_{i=1}^{n} L(\overline{\omega}_i)$, U_L is not empty and open so

$$\omega_i = L^{-1}(U_L) \cap \overline{\omega}_i \quad \text{FOR } i \in \mathbb{N}$$

is tubular, and we get

$$\gamma \in \bigcap_{i=1}^{n} \overline{X}_i^{-1}(\omega_i) \subset \mathcal{U}.$$

5 LEMMA : If Γ is a tubular topological groupoid, then the map

$$\gamma : X^{\infty}\Gamma \longrightarrow \Gamma(\mathbb{N})$$

is an open map.

Proof: Let $\mathcal{U} \subset X^{\infty}\Gamma$ be open . Then for $\gamma \in \mathcal{U}$, by Lemma 4, there exists an $n \in \mathbb{N}$, tubular open sets ω_i for $i = 1$ to n , open neighbourhood U_L of $L(\gamma)$ such that

$$\gamma \in \bigcap_{i=1}^{n} \bar{X}_i^{-1}(\omega_i) \subset U$$

Now, we claim that

$$r\left(\bigcap_{i=1}^{n} \bar{X}_i^{-1}(\omega_i)\right) = \bigcap_{i=1}^{n-1} \Pi_{i\leq i+1}^{-1}(\omega_{i+1}^{-1} \circ \omega_i)$$

(where $\Pi_{a\leq b}$ was given in section 2 to define the topology on $\Pi(N)$). To support the claim we note that if the '$=$' was replaced by '$\subseteq$' then it would be true, and then use the following: If $a \in \bigcap_{i=1}^{n-1} \Pi_{i\leq i+1}^{-1}(\omega_{i+1}^{-1} \circ \omega_i)$ then for $1 \leq i < n$ there exists $c(i) \in \omega_i$ and $b(i+1) \in \omega_{i+1}$ such that

$$a(i \leq i+1) = b(i+1)^{-1} \circ c(i)$$

However $a(i \leq i+1) \circ a(i-1 \leq i)$ is defined for $1 < i < n$ so for $1 < i < n$ $\quad R(c(i)) = R(b(i))$

and we get

$$a(i \leq i+1) = c(i+1)^{-1} \circ c(i)$$

for $1 \leq i < n$, because $c(i), b(i) \in \omega_i$. So

$$a \in r\left(\bigcap_{i=1}^{n} \bar{X}_i^{-1}(\omega_i)\right).$$

We now note that $\omega_{i+1}^{-1} \circ \omega_i$ for $1 \leq i < n$ are open subsets of Π so

$$r\left(\bigcap_{i=1}^{n} \bar{X}_i^{-1}(\omega_i)\right) \text{ is open in } \Pi(N).$$

It now follows that $r(U)$ is open, and since U is an arbitary open subset r is open. We thus get

6 EXAMPLE : A tubular topological groupoid is a well formed topological groupoid.

7 EXAMPLE : A topological group is a well formed topological groupoid.

It is easily seen that the topology on $\Gamma(\mathbb{N})$ in the case that Γ is a topological group G is the same as the product topology on $G \times G \times \cdots\cdots$ and $r : X^{\infty}\Gamma \longrightarrow \Gamma(\mathbb{N})$ is just a projection of $G \times G \times \cdots\cdots$ to $G \times G \times \cdots\cdots$ where $r(g_1, \ldots, g_n, \cdots) = (g_2^{-1}g_1, g_3^{-1}g_2, \cdots)$ So r is open.

7 LEMMA : Let σ be an ordered simplex and $j_\sigma : \sigma \subset \mathbb{N}$. Then the map $\Gamma(j_\sigma) : \Gamma(\mathbb{N}) \to \Gamma(\sigma)$ is an identification map.

Proof: If we use $q_\sigma : \mathbb{N} \longrightarrow \sigma$ given in the last part of section 2, we have q_σ is order preserving and $q_\sigma \circ j_\sigma = 1$ this means the diagram

$$\begin{tikzcd}
\Gamma(\sigma) \arrow[rr, "\Gamma(q_\sigma)"] \arrow[dr, "1"'] & & \Gamma(\mathbb{N}) \arrow[dl, "\Gamma(j_\sigma)"] \\
& \Gamma(\sigma) &
\end{tikzcd}$$

commutes. If $U \subset \Gamma(\sigma)$ is such that $\Gamma(j_\sigma)^{-1}(U)$ is open then by continuity $(\Gamma(j_\sigma) \circ \Gamma(q_\sigma))^{-1} U$ is open but by the commutative diagram this is exactly equal to U, so U is open. Because $\Gamma(j_\sigma)$ is continuous we get

$$U \subset \Gamma(\sigma) \text{ is open} \iff \Gamma(j_\sigma)^{-1}(U) \text{ is open.}$$

For $i \geq 1$ let $u_i : C\Gamma \longrightarrow \mathbb{R}$ be the map $(t, \gamma) \longmapsto t(i)$ and for ordered simplex σ

Let $\bar{U}_i = \{ t \in \Delta^{\infty} \mid t(i) \neq 0 \}$;

$$P_\sigma : \left(\bigcap_{i \in \sigma} \bar{U}_i\right) \times \Gamma(\mathbb{N}) \longrightarrow \Gamma(\sigma)$$

be the map $P_\sigma : (t, \gamma) \longmapsto \Gamma(j_\sigma)(\gamma)$, where $j_\sigma : \sigma \subset \mathbb{N}$.

8 LEMMA : If Γ is a well formed topological groupoid then the topology on $C\Gamma$ is the strongest topology which satisfies

a) for $i \geqslant 1$, $U_i : C\Gamma \longrightarrow \mathbb{R}$ is continuous

b) for $i \geqslant 1$, $\bar{U}_i \times \Gamma(N)$ is open

c) for an ordered simplex σ

$$P_\sigma : \Big(\bigcap_{i\in\sigma} \bar{U}_i\Big) \times \Gamma(N) \longrightarrow \Gamma(\sigma) \quad \text{is continuous.}$$

Also the map $1 \times v : A\Gamma \longrightarrow C\Gamma$ is open.

Proof: If we define the topology on $C\Gamma$ to be the one given in the hypothesis of the Lemma and show that $1 \times v$ is continuous and open as a result then it follows that $1 \times v$ is an identification and so $C\Gamma$ has the correct topology assigned to it. In our proof we will thus take the topology on $C\Gamma$ to be that given by conditions a), b) and c).

<u>$1 \times v$ is continuous:</u>

a) Let σ be an ordered simplex and U open in $\Gamma(\sigma)$ then if

$$W = P_\sigma^{-1}(U) \quad \text{we have} \quad W = \Big(\bigcap_{i\in\sigma} \bar{U}_i\Big) \times \Gamma(\partial\sigma)^{-1}(U)$$

but $\Gamma(\partial\sigma)$ is continuous and we have $W' = (\Gamma(\partial\sigma) \circ v)^{-1}(U)$ is open in $X^\infty\Gamma$. Choose for $i, j \in \sigma$, $i \leq j$ an open subset W_{ij} of Γ and put

$$U = \bigcap_{\substack{a \leq b \\ a, b \in \sigma}} \pi_{a\leq b}^{-1}(W_{ab})$$

such U's form a subbasis of the topology of $\Gamma(\sigma)$. Let $\gamma \in W'$. Since W' is open there exists an ordered simplex $\bar{\sigma}$ and open sets $U_i \in \Gamma$ for $i \in \bar{\sigma}$ such that

$$\gamma \in \bigcap_{i\in\bar{\sigma}} \bar{X}_i^{-1}(U_i) \subset W'$$

we note that we can replace $\mathcal{U}_i$ by $\hat{\mathcal{U}}_i$ where

$$\hat{\mathcal{U}}_i = \mathcal{U}_i \cap L^{-1}\Big(\bigcap_{j\in\bar{\sigma}} L(\mathcal{U}_i)\Big)$$

and still retain the above identity, and because L is open the sets $\hat{\mathcal{U}}_i$ for $i\in\bar{\sigma}$ are open. Also if

$$\gamma' \in \bigcap_{i\in\sigma\cap\bar{\sigma}} \bar{X}_i^{-1}(\hat{\mathcal{U}}_i)$$

then there exists an

$$\gamma \in \bigcap_{i\in\bar{\sigma}} \bar{X}_i^{-1}(\hat{\mathcal{U}}_i)$$

such that $\gamma'|_\sigma = \gamma|_\sigma$, but the condition for membership of γ' in ω' depends only on $\gamma'|_\sigma$. In fact

$$\gamma' \in \omega' \Longleftrightarrow \text{ for } a, b \in \sigma,\ a \leq b \quad \text{we have}$$

$$\gamma(b)^{-1} \circ \gamma(a) \in W_{ab}.$$

Hence $\displaystyle \gamma \in \bigcap_{i\in\sigma\cap\bar{\sigma}} \bar{X}_i^{-1}(\hat{\mathcal{U}}_i) \subset \omega'$

but without loss of generality we can choose $\sigma \subset \bar{\sigma}$ by putting

$$\hat{\mathcal{U}}_i = L^{-1}(\mathcal{U}_L) \quad \text{for} \quad i \in \sigma - \bar{\sigma} \ .$$

So we get that ω' is a union of sets of the type

$$\bigcap_{i\in\sigma} \bar{X}_i^{-1}(\hat{\mathcal{U}}_i)\ , \quad \hat{\mathcal{U}}_i \text{ open in } \Gamma.$$

This means that W is open.

b) Let $i \in \mathbb{N}$ and $\mathcal{U}$ be an open subset of $\mathbb{R}$ then

$$(\iota\times\nu)^{-1}(\mathcal{U}_i^{-1}(\mathcal{U})) = t_i^{-1}(\mathcal{U})$$

which is open .

From a) and b) above we can see that every subbasic set is mapped by $(1\times\nu)^{-1}$ to open sets in $A\Gamma$, so it follows that $1\times\nu$ is continuous.

$1\times\nu$ is open:

Let $\mathcal{U}$ be an open subset of $A\Gamma$ and $(t,\gamma)\in\mathcal{U}$ Then there exists an open subset W of Δ^∞ and an ordered simplex σ with open sets $U_i\subset\Gamma$ for $i\in\sigma$ such that

$$\begin{aligned} &W\times X^\infty\Gamma\cap\left(\bigcap_{i\in\sigma}X^{-1}(U_i)\right)\\ &=W\times X^\infty\Gamma\cap\left(\bigcap_{i\in\sigma}\bar{U}_i\times\bar{X}_i^{-1}(U_i)\right)\\ &=W\times X^\infty\Gamma\cap\left(\bigcap_{i\in\sigma}\bar{U}_i\right)\times\left(\bigcap_{i\in\sigma}\bar{X}_i^{-1}(U_i)\right)\\ &=W\cap\left(\bigcap_{i\in\sigma}\bar{U}_i\right)\times\bigcap_{i\in\sigma}\bar{X}_i^{-1}(U_i)=\bar{W}\ \text{SAY,}\end{aligned}$$

is an open neighbourhood of (t,γ) contained in $\mathcal{U}$. Applying $1\times\nu$ we get

$$W\cap\left(\bigcap_{i\in\sigma}\bar{U}_i\right)\times\nu\left(\bigcap_{i\in\sigma}\bar{X}_i^{-1}(U_i)\right)\quad\dots\dots(1)$$

Put $\quad W'=\{\gamma\in\Gamma(\sigma)\mid\exists\ \bar{\gamma}\in X^\infty\Gamma$ such that $\gamma(a\leq b)=\bar{\gamma}(b)^{-1}\circ\bar{\gamma}(a)$ and $\bar{\gamma}(a)\subset U_a$ for $a\leq b;\ a,b\in\sigma\}$

Then $\quad W''=\nu\left(\bigcap_{i\in\sigma}\bar{X}_i^{-1}(U_i)\right)=\Gamma(\partial\sigma)^{-1}(W')$

but W'' is open because ν is an open map which gives W' is open by Lemma 7. We see that (1) is $W\times\Gamma(N)\cap P_\sigma^{-1}(W')$ so is open in $C\Gamma$. Hence since γ is an arbtary member of $\mathcal{U}$ we get that the map $1\times\nu$ is open.

9 LEMMA : Let Γ be a well formed topological groupoid and σ be an ordered simplex. Then the map

$$\partial_\sigma\times\Gamma(q_\sigma):|\sigma|\times\Gamma(\sigma)\longrightarrow C\Gamma$$

is continuous and is an embedding when restricted to $|\mathring{\sigma}|$.(for notation see the end of section 2).

Proof: The topology for $C\Gamma$ is characterised by Lemma 8 and a topological subbasis is given by Lemma 1.1. First of all $(j_\sigma \times \Gamma(q_\sigma))^{-1}(\mathcal{U})$ need only be shown to be open for subbasis sets $\mathcal{U}$.

a) Let ω be open in $\mathbb{R}$ and put $\mathcal{U} = \mathcal{U}_i^{-1}(\omega)$ for some $i \in \mathbb{N}$, then

$$(j_\sigma \times \Gamma(q_\sigma))^{-1}(\omega) = \bar{t}_i^{-1}(\omega)$$

where $\bar{t}_i : (t, \gamma) \longmapsto \begin{cases} t(i) & \text{IF } i \in \sigma \\ 0 & \text{otherwise} \end{cases}$

and $\bar{t}_i$ is obviously continuous which gives $\bar{t}_i^{-1}(\omega)$ is open.

b)Let $\hat{\sigma}$ be an ordered simplex and ω be an open subset of $\Gamma(\hat{\sigma})$ then put $\mathcal{U} = P_{\hat{\sigma}}^{-1}(\omega)$, if $(j_\sigma \times \Gamma(q_\sigma))^{-1}(\mathcal{U}) \neq \emptyset$ then we have $\hat{\sigma} \subset \sigma$, because (t, γ) is in this set gives $\hat{\sigma} \subset \sigma_t$ and $\sigma_t \subset \sigma$.

Now $\mathcal{U} = \left(\bigcap_{i \in \hat{\sigma}} \bar{U}_i\right) \times \Gamma^{-1}(j_{\hat{\sigma}})(\omega)$

so $(j_\sigma \times \Gamma(q_\sigma))^{-1}(\mathcal{U}) = \left(|\sigma| \cap \left(\bigcap_{i \in \hat{\sigma}} \bar{\mathcal{U}}_i\right)\right) \times \Gamma(q_\sigma \circ j_{\hat{\sigma}})^{-1}(\omega)$

but if $e : \hat{\sigma} \subset \sigma$ then we get that

$q_\sigma \circ j_{\hat{\sigma}} = q_\sigma \circ (j_\sigma \circ e) = e$ since $q_\sigma \circ j_\sigma = 1$ so by the continuity of $\Gamma(e)$ we get that

$$(j_\sigma \times \Gamma(q_\sigma))^{-1}(\mathcal{U}) = \left(|\sigma| \cap \left(\bigcap_{i \in \hat{\sigma}} \bar{\mathcal{U}}_i\right)\right) \times \Gamma(e)^{-1}(\omega)$$

is open.

The $\mathcal{U}$'s given in a) and b) in the above constructions form a subbasis for topology of $C\Gamma$ so it follows that the map is continuous.

On the other hand we can show that the inverse images of $\mathcal{U}$'s form a subbasis for the topology of $|\sigma| \times \Gamma(\sigma)$. For instance if we set $\hat{\sigma} = \sigma$ then

$$(\dot{\gamma}_\sigma \times \Gamma(q_\sigma))^{-1}(\mathcal{U}) = |\mathring{\sigma}| \times \omega \quad \text{in the b) cases}$$

and $$(\dot{\gamma}_\sigma \times \Gamma(q_\sigma))^{-1}(\mathcal{U}) = \bar{t}_i^{-1}(\omega) \quad \text{in the a) cases.}$$

10 LEMMA : When Γ is well formed, open sets in $C\Gamma$ are fibred by the map $\bar{\pi}$ that is if $\mathcal{U}$ is open in $C\Gamma$ and $a, b \in \Gamma$ such that $\bar{\pi}(a) = \bar{\pi}(b)$ and $a \in \mathcal{U}$ then we get $b \in \mathcal{U}$.

Proof: it is sufficient to prove this for a subbasis of the topology on $C\Gamma$. Let σ be an ordered simplex and $\mathcal{U} = \bigcap_{i \in \sigma} \bar{U}_i \times \Gamma(\dot{\gamma}_\sigma)^{-1}(V)$ where V is open.

If $(t_1, \gamma_1) \in \mathcal{U}$ and $\bar{\pi}(t_1, \gamma_1) = \bar{\pi}(t_2, \gamma_2)$ for some $(t_2, \gamma_2) \in C\Gamma$ then $t_1 = t_2 = t$ say and $\gamma_1 | \sigma_t = \gamma_2 | \sigma_t$. Since $t \in \bigcap_{i \in \sigma} \bar{U}_i, \sigma \subset \sigma_t$ put e as this inclusion then we get

$$\begin{aligned} \Gamma(\dot{\gamma}_\sigma)(\gamma_1) &= \Gamma(e) \circ \Gamma(\dot{\gamma}_{\sigma_t})(\gamma_1) \\ &= \Gamma(e) \circ \Gamma(\dot{\gamma}_{\sigma_t})(\gamma_2) \\ &= \Gamma(\dot{\gamma}_\sigma)(\gamma_2) \end{aligned}$$

so $(t_2, \gamma_2) \in \mathcal{U}$.

Let $\mathcal{U} = \mathcal{U}_i^{-1}(V)$ where $i \in \mathbb{N}$ and V is open. Then by an analogous reasoning $\mathcal{U}$ is fibred by $\overline{\pi}$. The sets $\mathcal{U}$ so far considered form a subbasis for the topology of $C\Gamma$ so the lemma follows .

Recall from section 2 that for $s \geq 1$ $C_s = (\Delta^\infty)^s \times \Gamma(N)$ and $\overline{\pi}(C_s) = \mathcal{B}_s$. We have the following.

11 LEMMA: Let Γ be a well formed topological groupoid then $\overline{\pi}/C : C \longrightarrow \mathcal{B}$ is an open (continuous) map where $C \subset C\Gamma$ and $\mathcal{B} = \overline{\pi}(C)$.

Proof : Suppose $\mathcal{U}$ is open in $C\Gamma$. Then for $a \in \overline{\pi}(C) \cap \overline{\pi}(\mathcal{U})$ there exists $c \in C$ and $u \in \mathcal{U}$ such that $\overline{\pi}(c) = \overline{\pi}(u) = a$. But by Lemma 10 we get that $c \in \mathcal{U}$ and so $a \in \overline{\pi}(C \cap \mathcal{U})$. Hence $\overline{\pi}(C) \cap \overline{\pi}(\mathcal{U}) \subseteq \overline{\pi}(C \cap \mathcal{U})$ and we thus get $\overline{\pi}(C \cap \mathcal{U}) = \mathcal{B} \cap \overline{\pi}(\mathcal{U})$. But by Lemma 10 we have $\overline{\pi}^{-1}(\overline{\pi}(\mathcal{U})) = \mathcal{U}$ which is open so $\overline{\pi}(\mathcal{U})$ is open since the topology on $\mathcal{B}\Gamma$ is defined by the identification $\overline{\pi}$. The Lemma now follows from the arbitaryness of the open set $\mathcal{U}$.

12 LEMMA : Let Γ be a well formed topological groupoid, and σ be an s-dimensional ordered simplex. Then the map

$$Q_\sigma : |\sigma| \times \Gamma(\sigma) \xrightarrow{j_\sigma \times \Gamma(q_\sigma)} C\Gamma \xrightarrow{\overline{\pi}} \mathcal{B}\Gamma$$

is continuous and embeds $|\mathring{\sigma}| \times \Gamma(\sigma)$ onto an open subset of $\mathcal{B}_s$. Also the image is $\overline{\pi}(|\mathring{\sigma}| \times \Gamma(N))$.

Proof : The first part follows from the continuity of the maps. Consider $|\mathring{\sigma}| \times \Gamma(\sigma) \subset C\Gamma$. It is open in C_s because

$$|\mathring{\sigma}| \times \Gamma(\sigma) = C_S \cap \left(\bigcap_{i \in \sigma} \bar{U}_i \times \Gamma(N) \right)$$

so by Lemma 11, $\bar{\pi}\,(|\mathring{\sigma}| \times \Gamma(\sigma))$ is open in B_S. First of all we note that Q_σ maps onto $\bar{\pi}\,(|\mathring{\sigma}| \times \Gamma(N))$ because if $(t, \gamma) \in |\mathring{\sigma}| \times \Gamma(N)$ then $\sigma_t = \sigma$ and

$$\begin{aligned}
\Gamma(q_\sigma)(\Gamma(\dot{\gamma}_\sigma)(\gamma))\big|_\sigma &= \Gamma(\dot{\gamma}_\sigma) \circ \Gamma(q_\sigma) \circ \Gamma(\dot{\gamma}_\sigma)(\gamma) \\
&= \Gamma(q_\sigma \circ \dot{\gamma}_\sigma) \circ \Gamma(\dot{\gamma}_\sigma)(\gamma) \\
&= \Gamma(\dot{\gamma}_\sigma)(\gamma) = \gamma|\sigma
\end{aligned}$$

so $Q_\sigma(t, \Gamma(\dot{\gamma}_\sigma)(\gamma)) = \bar{\pi}\,(t, \gamma)$.

Q_σ is injective because if $(t_1, \gamma_1), (t_2; \gamma_2) \in |\mathring{\sigma}| \times \Gamma(\sigma)$ then $Q_\sigma(t_1, \gamma_1) = Q_\sigma(t_2, \gamma_2)$ gives $t_1 = t_2$ and

$$\gamma_1 = \Gamma(q_\sigma)(\gamma_1)|_\sigma = \Gamma(q_\sigma)(\gamma_2)\,|_\sigma = \gamma_2 .$$

Last of all the map is open because if A is image of $|\mathring{\sigma}| \times \Gamma(\sigma)$ in $C\Gamma$ then $\bar{\pi}/A$, by Lemma 11, is an open map. Using the result given in Lemma 9 we get that Q_σ is an open map.

Let e'_σ be the subset of $|\mathring{\sigma}|$ defined by

$$e'_\sigma = \left\{ t \in |\sigma| \;\middle|\; t(i) \geqslant \frac{1}{2(\mathrm{Dim}(\sigma)+1)} \text{ FOR } i \in \sigma \right\}$$

then by Lemma 12 we can regard $e'_\sigma \times \Gamma(\sigma)$ as a subset of B_S when $\mathrm{Dim}(\sigma) = S$, where the inclusion map is induced by Q_σ. We will in the following be interested in the homotopy properties of the inclusion

$$(B_S, B_{S-1}) \subset \left(B_S, B_S - \bigcup_{\mathrm{Dim}(\sigma)=S} \mathring{e}'_\sigma \times \Gamma(\sigma) \right)$$

where $\mathring{e}'_\sigma$ is the usual interior of a simplex given by

$$\mathring{e}'_\sigma = \left\{ t \in |\sigma| \;\middle|\; t(i) > \frac{1}{2(\mathrm{Dim}(\sigma)+1)} \text{ FOR } i \in \sigma \right\}$$

We will also need for an ordered simplex the notion of the boundary simplex $|\dot{\sigma}|$ given as

$$|\dot{\sigma}| = |\sigma| - |\mathring{\sigma}|$$

and similarly the boundary

$$\dot{e}'_\sigma = e'_\sigma - \mathring{e}'_\sigma$$

We note that the constructions for e'_σ are mapped to the corresponding constructions for $|\sigma|$ by the linear map which maps "vertices " to vertices. If a set is obviously a copy of $|\sigma|$ up to homeomorphism then we will freely use the constructions on to induce corresponding constructions on the copy without refering directly to the construction on the copy.

Consider C_S. Then we can see that C_S is covered by the collection

$$\{ |\sigma| \times \Gamma(N) \mid \mathrm{Dim}(\sigma) = S, \sigma \text{ is an ordered simplex} \}$$

and the $|\mathring{\sigma}| \times \Gamma(N)$ are disjoint and locally finite. Further more by using the topological subbasis for $C\Gamma$ we can see that $|\sigma| \times \Gamma(N)$ is a closed subset of $C\Gamma$ and the cover of C_S is locally finite(in other words the points of C_S have open neighbourhoods which only intersects a finite number of members of the collection.) Since the component in Δ^∞ of elements in $C\Gamma$ are mapped injectively by $\overline{\pi}$, that is $\overline{\pi}(t_1, \gamma_1) = \overline{\pi}(t_2, \gamma_2) \Rightarrow t_1 = t_2$ we can with the aid of Lemma 10 give the following

COROLLARY: If Γ is a well formed topological groupoid then

$$\{ Q_\sigma(|\sigma| \times \Gamma(N)) \mid \mathrm{Dim}\, \sigma = S \}$$

is a locally finite covering of B_S by closed subsets.

14 LEMMA : Let Γ be a well formed topological groupoid, then the inclusion, for $S>1$, given by

$$k : (B_S, B_{S-1}) \subset \Big(B_S, B_S - \bigcup_{\mathrm{Dim}(\sigma) = S} \mathring{e}'_\sigma \times \Gamma(\sigma)\Big)$$

is a homotopy equivalence.

Proof : Consider the map $d_\sigma : (|\sigma|, |\sigma| - \overset{\circ}{e}'_\sigma) \to (|\sigma|, |\dot\sigma|)$ given by, for $t \in |\sigma|$, $i \in \sigma$, $D = 1 + \mathrm{Dim}(\sigma)$

$$\begin{cases} \dfrac{t(i) - \mu}{1 - D\mu} & \text{FOR } \mu \leq \frac{1}{2D} \\[2ex] \dfrac{t(i) - \frac{1}{2D}}{1 - \frac{1}{2}} & \text{FOR } \mu \geq \frac{1}{2D} \end{cases}$$

where $\mu = \min\limits_{i \in \sigma} \{t(i)\}$. It has the property that if K_σ is the inclusion $K_\sigma : (|\sigma|, |\dot\sigma|) \subset (|\sigma|, |\sigma| - \overset{\circ}{e}'_\sigma)$

then $d_\sigma \circ K_\sigma = 1$ and there exists a homotopy relative to $|\dot\sigma|$

$$H_\sigma : I \times (|\sigma|, |\sigma| - \overset{\circ}{e}'_\sigma) \longrightarrow (|\sigma|, |\sigma| - \overset{\circ}{e}'_\sigma)$$

such that $(H_\sigma)_0$ is the identity map and $(H_\sigma)_1$ is $K_\sigma \circ d_\sigma$. These maps are usually used to show that the inclusion K_σ is a homotopy equivalence, but they can also be used to prove the lemma. Consider $|\sigma| \times \Gamma(N)$ in $C\Gamma$. H_σ gives the map

$$\hat{H}_\sigma : I \times (|\sigma|, |\sigma| - \overset{\circ}{e}'_\sigma) \times \Gamma(N) \longrightarrow (|\sigma|, |\sigma| - \overset{\circ}{e}'_\sigma) \times \Gamma(N)$$

given by $\hat{H}_\sigma(v, (t, \gamma)) = (H_\sigma(v, t), \gamma)$. We now get that there exists a unique map

$$\bar{H} : I \times (B_s, B_s - \bigcup_{\mathrm{Dim}(\sigma)=s} \overset{\circ}{e}'_\sigma) \longrightarrow (B_s, B_s - \bigcup_{\mathrm{Dim}(\sigma)=s} \overset{\circ}{e}'_\sigma)$$

that satisfies, for $\mathrm{Dim}(\sigma) = s$ the diagram

$$\begin{array}{ccc} I \times (|\sigma| \times \Gamma(N)) & \xrightarrow{\hat{H}_\sigma} & |\sigma| \times \Gamma(N) \\ \big\downarrow 1 \times \bar\pi & & \big\downarrow \bar\pi \\ I \times B_s & \xrightarrow{\bar{H}} & B_s \end{array}$$

commutes, $\hat{H}_\sigma$ obviously preserves fibres of $\bar\pi$ and the maps $\hat{H}_\sigma$ agree on $(\Delta^\infty)^{s-1} \times \Gamma(N)$ being constant on this subset of C_s. $\bar{H}$ is continuous because it is continuous on the locally finite closed cover $\{ Q_\sigma(|\sigma| \times \Pi(N)) \mid \mathrm{Dim}\,\sigma = s \}$ given by Corollary 13. Now put $D : B_s - \bigcup_{\mathrm{Dim}(\sigma)=s} \mathring{e}_\sigma \times \Pi(\sigma) \longrightarrow B_{s-1}$ as the map

$D(x) = \bar{H}_1(x)$ Then $D \circ K = 1$ and $K \circ D = \bar{H}_1$ is homotopic relative to B_{s-1} to the identity map, so K is an homotopy equivalence.

15 LEMMA : Let Π be a well formed topological groupoid. Then the map

$$H_*\Big(\bigcup_{\mathrm{Dim}(\sigma)=s} (e'_\sigma, \dot{e}'_\sigma) \times \Pi(\sigma)\Big) \longrightarrow H_*\Big(B_s, B_s - \bigcup_{\mathrm{Dim}(\sigma)=s} \mathring{e}'_\sigma\Big)$$

induced by inclusion is an isomorphism.

Proof : By Lemma 12 we can identify $|\mathring{\sigma}| \times \Pi(\sigma)$, by using Q_σ, with an open subset of B_s. Let $\bar{e}_\sigma$ be a slightly larger simplex than e'_σ say given by

$$\bar{e}_\sigma = \Big\{ t \in |\mathring{\sigma}| \;\Big|\; t(i) > \frac{1}{4(\mathrm{Dim}(\sigma)+1)} \text{ FOR } i \in \sigma \Big\}$$

put $Z = B_s - \bigcup_{\mathrm{Dim}(\sigma)=s} \bar{e}_\sigma \times \Pi(\sigma)$. Now $\bar{\pi}^{-1}(\bar{e}_\sigma \times \Pi(\sigma)) =$ $\bar{e}_\sigma \times \Pi(N)$ which is closed in $C\Pi$ and by choosing a slightly larger simplex $\bar{\bar{e}}_\sigma$ than $\bar{e}_\sigma$ we can include the closed set $\bar{e}_\sigma \times \Pi(N)$ in $\mathring{\bar{\bar{e}}}_\sigma \times \Pi(N)$ which is open in C_s and such open sets are dijoint in C_s. It now follows that the union of closed sets $\bigcup_{\mathrm{Dim}(\sigma)=s} \bar{e}_\sigma \times \Pi(N)$ is closed in C_s, so by using the fact that $\bar{\pi}^{-1}\Big(\bigcup_{\mathrm{Dim}(\sigma)=s} \bar{e}_\sigma \times \Pi(\sigma)\Big) = \bigcup_{\mathrm{Dim}(\sigma)=s} \bar{e}_\sigma \times \Pi(N)$ and Lemma 11 we get

that Z is open in B_s. By a similar argument we can construct a closed set Z_1, and open set Z_2 such that $Z \subset Z_1 \subset Z_2 \subset$ $B_s - \bigcup_{\mathrm{Dim}(\sigma)=s} \mathring{e}'_\sigma \times \Pi(\sigma)$ by putting

$$Z_1 = B_s - \bigcup_{\mathrm{Dim}(\sigma)=s} \mathring{e}^*_\sigma \times \Pi(\sigma)$$

$$Z_2 = B_s - \bigcup_{Dim(\sigma)=s} \mathring{e}^{**}_\sigma \times \Gamma(\sigma)$$

where $e^{*}_\sigma = \{ t \in |\mathring{\sigma}| \mid t(i) > 5/16 \frac{1}{Dim(\sigma)+1} \}$

$e^{**}_\sigma = \{ t \in |\mathring{\sigma}| \mid t(i) > 6/16 \frac{1}{Dim(\sigma)+1} \}$

so $Z \subset$ interior$\left(B_s - \bigcup_{Dim(\sigma)=s} \mathring{e}'_s \times \Gamma(\sigma) \right)$ in B_s

and $\bigcup_{Dim(\sigma)=s} (\bar{e}_\sigma, \bar{e}_\sigma - \mathring{e}'_\sigma) \times \Gamma(\sigma) \subset$

$\left(B_s, B_s - \bigcup_{Dim(\sigma)=s} \mathring{e}'_\sigma \times \Gamma(\sigma) \right)$ is an excision map which induces

isomorphisms. However

$$\bigcup_{Dim(\sigma)=s} (e'_\sigma, \dot{e}'_\sigma) \times \Gamma(\sigma) \subset \bigcup_{Dim(\sigma)=s} (\bar{e}_\sigma, \bar{e}_\sigma - \dot{e}'_\sigma) \times \Gamma(\sigma)$$

is a homotopy equivalence, which induces isomorphisms in homology. By composing these isomorphisms we get the required isomorphism.

16 LEMMA: If Γ is a well formed topological groupoid, then the maps $\{ Q_\sigma \mid Dim(\sigma) = s \}$ induce the direct sum representation

$$\{Q_{\sigma *}\} : \bigoplus_{Dim(\sigma)=s} H_*((|\sigma|, |\dot{\sigma}|) \times \Gamma(\sigma)) \approx H_*(B_s, B_{s-1})$$

Proof: consider the commutative diagram

$$\begin{tikzcd}
\bigoplus_\sigma H_*((|\sigma|, |\dot{\sigma}|) \times \Gamma(\sigma)) \arrow[r, "\{Q_{\sigma *}\}"] \arrow[d] & H_*(B_s, B_{s-1}) \arrow[d] \\
\bigoplus_\sigma H_*((|\sigma|, |\sigma| - \mathring{e}'_\sigma) \times \Gamma(\sigma)) & H_*(B_s, B_s - \bigcup_\sigma \mathring{e}'_\sigma \times \Gamma(\sigma)) \\
\bigoplus_\sigma H_*((e'_\sigma, \dot{e}'_\sigma) \times \Gamma(\sigma)) \arrow[r] \arrow[u] & H_*(\bigcup_\sigma (e'_\sigma, \dot{e}'_\sigma) \times \Gamma(\sigma)) \arrow[u]
\end{tikzcd}$$

where the horizontal maps are induced by the maps Q_σ and the vertical maps by inclusion. By using Lemma 14 the two top vertical maps are isomorphisms induced by homotopy equivalences, the bottom left vertical map is an isomorphism being induced by a suitable excision and by Lemma 15 the bottom right map is an isomorphism. The bottom map is an isomorphism because the $e'_\sigma \times T(\sigma)$'s in B_s are dijoint closed subsets. We thus get that $\{Q_{\sigma *}\}$ is an isomorphism.

Before continuing to a calculation of d^1 we will look at $H_*((|\sigma|, |\dot\sigma|) \times T(\sigma))$ in more detail.

The following Eilenberg - Zilber theorem [EZ!] is usually used to calculate the homotopy of a product of topological spaces.

17 THEOREM : On the category of ordered pairs of topological spaces X and Y there is a natural chain equivalence of the functor $\Delta(X \times Y)$ with the functor $\Delta(X) \otimes \Delta(Y)$, where is the tensor product.

In our treatment we will restrict ourselves to the cases that interest us when defining the relative version and the "homology cross product".

Let (X, B) be a topological pair and Y a topological space then their product is $(X, B) \times Y = (X \times Y, B \times Y)$ we note that since the complexes $\Delta(X)$ & $\Delta(B)$ are free the natural equivalence in Theorem 17 gives the natural chain equivalence

$$\Delta(X)/\Delta(B) \otimes \Delta(Y) \simeq \frac{\Delta(X) \otimes \Delta(Y)}{\Delta(B) \otimes \Delta(Y)} \longrightarrow \frac{\Delta(X \times Y)}{\Delta(B \times Y)}$$

this gives the following <u>homology cross product</u>, where G is a $\mathbb{Z}$ module, given by

$$\mu' : H_p(X,B) \otimes H_q(Y;G) \longrightarrow H_{p+q}((X,B)\times Y;G)$$

as the cross product

$$H_p(\Delta(X)/\Delta(B)) \otimes H_q(\Delta(Y) \otimes G)$$

$$H_{p+q}(\Delta(X)/\Delta(B) \otimes \Delta(Y) \otimes G)$$

followed by the functional homomorphism of the chain complex to

$$H_{p+q}(\Delta(X\times Y)/\Delta(B\times Y) \otimes G)$$

we have the usual Künneth formula :

18 THEOREM : The homomorphism $\mu' : H_p(X,B) \otimes H_q(Y:G) \longrightarrow H_{p+q}((X,B)\times Y;G)$ is an isomorphism, if $H_*(X,B)$ is a free abelian group.

The proof is by direct application of the Künneth formula given in Spanier [SPI] and the properties of the torsion product. Let σ be an ordered simplex and $\xi_\sigma : \Delta^S \longrightarrow |\sigma|$, where $s = \mathrm{Dim}(\sigma)$ be the map defined by $\xi_\sigma(t)(v_i) = t(i)$ where the vertices of σ are given by $v_1 < v_2 < \cdots\cdots < v_{s+1}$ Then ξ_σ is a generator for $\Delta(|\sigma|)/\Delta(|\dot\sigma|)$

19 LEMMA : Let $\{\xi_\sigma\} \in H_*(|\sigma|,|\dot\sigma|)$. be the class corresponding to the cocycle ξ_σ . Then for $q \geq 0$

$$H_q(|\sigma|,|\dot\sigma|) = \begin{cases} 0 & \text{if } q \neq s \\ \mathbb{Z}[\{\xi_\sigma\}] & \text{for } q = s \end{cases}$$

where $\mathbb{Z}[\{\xi_\sigma\}]$ is the free group generated by $\{\xi_\sigma\}$.

The proof is standard; $H_q(|\sigma|,|\dot\sigma|)$ can be computed as the ordered homology of the simplicial complex pair $(K, \dot{K})$ and then note that the generator is mapped to $\{\xi_\sigma\}$ by the natural equivalence between homology theories.

The map ξ_σ has another useful property which we will use; ξ_σ is induced by an order preserving map in the sense that ξ_σ is linear and if $\bar{\Delta}_S$ are the vertices of Δ_S then $\xi_\sigma | \bar{\Delta}_S : \bar{\Delta}_S \dashrightarrow \sigma$ is order preserving, so we get the homeomorphism

$$\Pi(\xi_\sigma | \bar{\Delta}^S) : \Gamma(\Delta^S) \longrightarrow \Gamma(\sigma)$$

Another property of interest is that if Π is a single point e, then Γ is a tubular topological groupoid with Δ^{∞} homeomorphic to $\overline{B\Gamma}$ and the subsets B_S correspond to $(\Delta^\infty)^S$. So by using Theorem 18 and Lemmas 16, 19 we get in a some what round about way that

$$C_S = H_S((\Delta^\infty)^S, (\Delta^\infty)^{S-1}) \simeq \bigoplus_{\mathrm{Dim}(\sigma)=S} H_S(|\sigma|, |\dot{\sigma}|)$$

with the generators given by $\hat{\xi}_\sigma : \Delta^S \xrightarrow{\xi_\sigma} |\sigma| \subset (\Delta^\infty)^S$.
For an ordered simplex σ let z be the inclusion

$$z : \bigcup_{\mathrm{Dim}(\sigma)=S} (|\sigma|, |\dot{\sigma}|) \subset ((\Delta^\infty)^S, (\Delta^\infty)^{S-1})$$

then we get:

20 LEMMA : Let Γ be a well formed topological groupoid and $S > 1$, $t \geq 0$ then the following maps are isomorphisms:

$$C_S \otimes H_t(\Gamma(\Delta^S); G) \xleftarrow{z_* \otimes 1} \bigoplus_{\mathrm{Dim}(\sigma)=S} H_S(|\sigma|, |\dot{\sigma}|) \otimes H_t(\Gamma(\bar{\Delta}^S); G)$$

$$\xrightarrow{1 \otimes \Gamma(\xi_\sigma | \bar{\Delta}_S)_*} \bigoplus_{\mathrm{Dim}(\sigma)=S} H_S(|\sigma|, |\dot{\sigma}|) \otimes H_t(\Gamma(\sigma); G)$$

$$\xrightarrow{\mu'} \bigoplus_{\mathrm{Dim}(\sigma)=S} H_{S+t}((|\sigma|, |\dot{\sigma}|) \times \Gamma(\sigma))$$

$$\{Q\sigma_*\} \; H_{s+t}(B_s, B_{s-1}) \approx E^1_{s,t}.$$

and thus $C_s \otimes H_t(\Gamma(\Delta^s); G) \approx E^1_{s,t}$

We shall now construct a boundary map $\partial_1 : C_s \otimes H_t(\Gamma(\bar{\Delta}^s); G) \longrightarrow C_{s-1} \otimes H_t(\Gamma(\Delta^{s-1}); G)$ which commutes with d^1 given in Lemma 2·4 when $X_s \equiv B_s$. Let σ be an ordered simplex, represent the vertices of σ by $v_1 < v_2 < \cdots\cdots v_{q+1}$ where q is the dimension of σ. Put $\sigma_{(i)} = \{v_1, \cdots v_{i-1}, v_{i+1} \cdots v_{q+1}\}$ as the <u>i th face</u> of σ, and $i_{(j)}\sigma$ as the inclusion

$$i_{(j)}\sigma : \sigma_{(j)} \subset \sigma \qquad \text{for } 1 \leq j \leq q+1$$

This gives the diagram (where $s = q$)

$$\begin{array}{ccccc}
\Gamma(\sigma) & & \xrightarrow{\quad \Gamma(i_{(j)}\sigma) \quad} & & \Gamma(\sigma_{(j)}) \\
\Big\downarrow{\scriptstyle \Gamma(\xi_\sigma | \bar{\Delta}^s)} & & & & \Big\downarrow{\scriptstyle \Gamma(\xi_{\sigma(j)} | \bar{\Delta}^{s-1})} \\
\Gamma(\bar{\Delta}^s) & \xrightarrow{\Gamma(i_{(j)}\bar{\Delta}^s)} & \Gamma(\bar{\Delta}^s_{(j)}) & \xleftarrow{\Gamma(\xi_{\bar{\Delta}^s_{(j)}} | \bar{\Delta}^{s-1})} & \Gamma(\bar{\Delta}^{s-1})
\end{array}$$

Put μ_j as the composite along the bottom row. We now have the following commutative diagram for an S dimensional ordered simplex σ

$$\begin{array}{ccc}
(|\sigma_{(i)}|, |\dot{\sigma}_{(i)}|) \times \Gamma(\bar{\Delta}^{s-1}) & \xleftarrow{1 \times \mu_i} & (|\sigma_{(i)}|, |\dot{\sigma}_{(i)}|) \times \Gamma(\bar{\Delta}^s) \\
 & & \cap \\
\Big\downarrow{\scriptstyle 1 \times \Gamma(\xi_{\sigma(i)} | \bar{\Delta}^{s-1})^{-1}} & & (|\sigma|, |\dot{\sigma}|) \times \Gamma(\bar{\Delta}^s) \\
 & & \Big\downarrow{\scriptstyle 1 \times \Gamma(\xi_\sigma | \Delta^s)^{-1}} \\
(|\sigma_{(i)}|, |\dot{\sigma}_{(i)}|) \times \Gamma(\sigma_{(i)}) & & (|\sigma|; |\dot{\sigma}|) \times \Gamma(\sigma) \\
\Big\downarrow{\scriptstyle Q\sigma_{(i)}} & & \Big\downarrow{\scriptstyle Q\sigma} \\
(B_{s-1}, B_{s-2}) & \subset & (B_s, B_{s-1})
\end{array}$$

Let μ be the natural chain equivalence $\mu : \Delta(X)\otimes\Delta(Y) \to \Delta(X\times Y)$ for topological spaces X, Y given by the Eilenberg - Zilber theorem,

$$\bar{K}_\sigma : (|\sigma|, |\dot\sigma|)\times\Gamma(\bar\Delta^s) \longrightarrow (B_s, B_{s-1})$$

be the composition

$$(|\sigma|, |\dot\sigma|)\times\Gamma(\bar\Delta^s) \xrightarrow{1\times\Gamma(\xi_\sigma|\bar\Delta^s)^{-1}} (|\sigma|, |\dot\sigma|)\times\Gamma(\sigma) \xrightarrow{Q\sigma} (B_s, B_{s-1})$$

and

$$0 \longrightarrow \frac{\Delta(B_{s-1})}{\Delta(B_{s-2})} \xrightarrow{\alpha} \frac{\Delta(B_s)}{\Delta(B_{s-2})} \xrightarrow{\beta} \frac{\Delta(B_s)}{\Delta(B_{s-1})} \longrightarrow 0$$

If $\bar\omega$ is a cocycle of $\Delta(B_s)$ then put $\{\bar\omega\}_s$ as the corresponding class in $\Delta(B_s)/\Delta(B_{s-1})$ and $[\bar\omega]_s$ as the corresponding class in $\Delta(B_s)/\Delta(B_{s-2})$

Let ω be a cocycle of $\Delta(\Gamma(\bar\Delta^s))$ and ∂^* the boundary operator of the above exact sequence. Then we get

$$\begin{aligned}
\partial^*\{\bar{K}_\sigma\mu(\xi_\sigma\otimes\omega)\}_s &= \alpha^{-1}\partial[\bar{K}_\sigma\mu(\xi_\sigma\otimes\omega)]_s \\
&= \alpha^{-1}[\bar{K}_\sigma\mu(\textstyle\sum_i(-)^i\xi_{\sigma(i)}\otimes\omega)]_s \\
&= \alpha^{-1}[\textstyle\sum_i(-)^i\bar{K}_{\sigma(i)}(1\times\mu_i)\mu(\xi_{\sigma(i)}\otimes\omega)]_s \\
&= \alpha^{-1}[\textstyle\sum_i(-)^i\bar{K}_{\sigma(i)}\mu(\xi_{\sigma(i)}\otimes\mu_i(\omega))]_s \\
&= \{\textstyle\sum_i(-)^i\bar{K}_{\sigma(i)}\mu(\xi_{\sigma(i)}\otimes\mu_i(\omega))\}_{s-1}
\end{aligned}$$

hence we get the following diagram commutes

$$\begin{array}{ccc}
C_s\otimes H_t(\Gamma(\bar\Delta^s)) & \xrightarrow{\partial_1} & C_{s-1}\otimes H_t(\Gamma(\bar\Delta^{s-1})) \\
\downarrow\{\hat{K}_{\sigma*}\} & & \downarrow\{\hat{K}_{\sigma*}\} \\
H_{s+t}(B_s, B_{s-1}) & \xrightarrow{d^1} & H_{s+t-1}(B_{s-1}, B_{s-2})
\end{array}$$

where $\{\hat{K}_{\sigma*}\}$ is the comosite isomorphism given in Lemma 20, and ∂_1 is the boundary operator

$$\partial_1(\xi_\sigma \otimes \omega') = \sum (-)^i \xi_{\sigma(i)} \otimes \mu_{i*}(\omega')$$

we thus get the:

21 THEOREM : Let Γ be a well formed topological groupoid and $B\Gamma$ the classifying space for Γ. There is a convergent E^2 spectral sequence with $E^2_{s,t} \approx H_s(C_s \otimes H_t(\Gamma(\bar{\Delta}^s); G : \partial_1)$ where the boundary operator is given by

$$\partial_1(\xi_\sigma \otimes \omega) = \sum_i (-)^i \xi_{\sigma(i)} \otimes \mu_{i*}(\omega)$$

and E^∞ the bigraded module associated to the filtration of $H_*(B\Gamma, G)$ defined by

$$F_s H_*(B\Gamma; G) = \mathrm{im}\left[H_*(B_s; G) \longrightarrow H_*(B\Gamma; G)\right].$$

CHAPTER III

THE HOMOTOPY OF $\Gamma_q^{\vee}$.

Tubular Topological Groupoids.

This section will introduce a useful general construction for tubular topological groupoids when the tubular topological groupoid is "realisable" a property that will be defined. The construction is refered to as lifting Γ to a shift of embeddings and provides a local homotopy lifting of maps of compact hausdorff spaces from the object space of Γ to Γ itself so we can choose the nature of the image in the objects of Γ produced by the map $L : \Gamma \longrightarrow \mathrm{Obs}\,\Gamma$ as long as we stay in a specified neighbourhood of the image. But to do this we will look at a characterisation of realisable tubular topological groupoids.

Let Γ from now on be a tubular topological groupoid.

Let $\gamma \in \Gamma$ and U be a neighbourhood of γ which is tubular. $L(U)$ and $R(U)$ are open neighbourhoods of $L(\gamma)$ and $R(\gamma)$ and the maps $L|U$ & $R|U$ are homeomorphisms put $H(U)$: $R(U) \longrightarrow L(U)$ as the homeomorphism $L|U \circ (R|U)^{-1}$. Then to γ we can associate the germ Germ $(R(\gamma), H(U))$. where, to refresh the readers memory, this germ is the germ of the map $H(U)$ with domain $R(\gamma)$ and range $L(\gamma) = H(U)(R(\gamma))$. It is interesting to note that the correspondence $\gamma \longmapsto$ Germ $(R(\gamma), H(U))$ is independent of the neighbourhood U.

1 LEMMA: Let $\gamma \in \Gamma$ and U, V be open tubular neighbourhoods of then Germ $(R(\gamma), H(U)) =$ Germ $(R(\gamma), H(V))$.

Proof : since Γ is tubular there exists a tubular neighbourhood W which is open and $\gamma \in W \subset U \cap V$ so $H(W) = L|W \circ (R|W)^{-1} =$ $H(U)|L(W) = H(V)|L(W)$ since $L(W)$ is an open neighbourhood of $L(\gamma)$ we get the required identity.

The map $\gamma \longrightarrow$ germ $(R(\gamma), H(u))$ is thus well defined. It is the canonical map F from Γ to the groupoid of germs of local homeomorphisms of $\mathrm{Ob}\,\Gamma$, $\overline{\mathrm{Germ}}\,(\mathrm{Ob}\,\Gamma)$. To shorten the notation we will write $\overline{G}(\Gamma)$ for $\overline{\mathrm{Germ}}\,(\mathrm{Ob}\,\Gamma)$.

Let X be a topological space and $h : U_1 \longrightarrow U_2$ a local homeomorphism then put Germ (U_1, h) as a set of germs Germ(x, h) where $x \in U_1$. The groupoid of germs of local homeomorphisms is converted into a topological groupoid by using the topology which has as basis the set of all sets Germ (U_1, h) where $h : U_1 \to U_2$ is a local homeomorphism.

2 LEMMA : The map $F : \Gamma \longrightarrow \overline{G}(\Gamma)$ is open continuous, and a functor between topological groupoids when Γ is a tubular topological groupoid.

Proof: the proof is routine being split into parts.

i) for $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1 \circ \gamma_2$ defined we will show that $F(\gamma_1 \circ \gamma_2) = F(\gamma_1) \circ F(\gamma_2)$ Let U_1 and U_2 be tubular neighbourhoods of γ_1 and γ_2 respectively now $U_1 \circ U_2$ is tubular and it is easily checked that

$$H(U_1 \circ U_2) = H(U_1) | R(U_1) \cap L(U_2) \circ H(U_2)^{-1} | R(U_1) \cap L(U_2)$$

from the definition of H, so since $R(U_1) \cap L(U_2)$ is an open neighbourhood of $R(\gamma_1) = L(\gamma_2)$ we have the required identity.

ii) units are mapped to units.

iii) the map F is continuous. To show this let $\gamma \in \Gamma$ and N be an open neighbourhood of $F(\gamma) \in \bar{G}(\Gamma)$. Since N is a neighbourhood there exists a local homeomorphism $h : U_1 \longrightarrow U_2$ of $OG\Gamma$ such that $F(\gamma) \in$ Germ $(U_1, h) \subset N$. This means that $R(\gamma) \subset U_1$, R is continuous so there exists an open tubular neighbourhood U of γ such that $R(U) \subset U_1$, also by the definition of F there exists an open set U_3 such that $R(\gamma) \in U_3 \subset R(U)$ and $H(U) \mid U_3 = h \mid U_3$. But again by continuity of R there exists a tubular neighbourhood $\bar{U}$ such that $R(\bar{U}) \subset U_3$ and $\gamma \in \bar{U} \subset U$ .This means that $H(\bar{U}) = H(U) \mid R(\bar{U}) = h \mid R(\bar{U})$ and $R(\gamma) \in R(\bar{U})$ where $R(\bar{U})$ is open. Hence $F(\bar{U}) \subset$ Germ $(R(\bar{U}), H(\bar{U})) \subset N$ and is a neighbourhood of $R(\gamma)$.

iv) F is open. This follows from the fact that for a tubular open set V, Germ $(R(U), H(U))$ is open in $\bar{G}(\Gamma)$.

We are interested only in topological groupoids such as Γ_2^0, Γ_2^r and Γ_2^ω which have F as an injection.

A tubular topological groupoid is realisable if the cannonical map is also injective .

It is interesting to note that not all tubular topological groupoids are realisable and a study of such topological groupoids would provide an interesting study in topological groupoids.

3 EXAMPLE Let X be a topological space and $\mathbb{Z}$ be the integers with discrete topology then on $\mathbb{Z} \times X$ define the comosition rule

$$(n, a) \circ (m, a) = (n+m, a).$$

$\mathbb{Z} \times X$ is then a topological groupoid with objects X . which is not realisable.

In the case that Γ is realisable we can identify Γ with an open subgroupoid of $\bar{G}(\Gamma)$. In the following treatment we will assume that this has been done.

Let X and Y be topological spaces and $i_1 : X \longrightarrow X \times Y$ be a topological embedding such that $i_1(x) = (x, \alpha(x))$ for some continuous $\alpha : X \longrightarrow Y$ then i_1 is a level preserving embedding which is the graph of the map $\alpha : X \longrightarrow Y$. Note that every continuous map has a graph.

Let Γ be a realisable topological groupoid and $\mathrm{Obs}\,\Gamma$ its space of objects. Let P be a topological space. Then a Γ-shift of P is a tuple (i_1, i_2, W_1, W_2, d) with a pair of level preserving embeddings $i_1, i_2 : P \longrightarrow P \times \mathrm{Obs}\,\Gamma$ and a homeomorphism $d : W_1 \longrightarrow W_2$ of nieghbourhoods W_1, W_2 of $i_1(P)$ and $i_2(P)$ respectively which is locally of the form $1 \times \hat{d}$, $\hat{d}$ being a homeomorphism $\hat{d} : U \longrightarrow V$ of open subsets of $\mathrm{Obs}\,\Gamma$ such that $\mathrm{Germ}\,(U, \hat{d}) \subset F(\Gamma)$. Also we require that $i_2 = d \circ i_1$.

Let $Z = (i_1, i_2, W_1, W_2, d)$ be a Γ-shift of P then define $\vec{F}(Z)$ as $f : P \longrightarrow \Gamma$ where for $x \in P$ and some local representation $A \times B_1 \xrightarrow{1 \times \hat{d}} A \times B_2$ of d, $f(x) = \mathrm{Germ}\,(\alpha(x), \hat{d})$ where i_1 is the graph of α, and $(x, \alpha(x)) \in A \times B_1$. f is continuous and well defined. f is called the Germ of the Γ-shift Z. Denote the set of Γ-shifts of P by $A\Gamma(P)$ and the set of continuous maps $g : P \longrightarrow \Gamma$ by $\mathrm{Fun}(P, \Gamma)$

4 THEOREM : If P is a compact hausdorff topological space and Γ is realisable then the map

$$\vec{F} : A\Gamma(P) \longrightarrow \mathrm{Fun}(P, \Gamma)$$

which assigns the germ of a Γ-shift Z to Z is surjective.

Proof : If $P \xrightarrow{f} \Gamma$ is continuous $f(P)$ is compact. So there exists an integer n such that there exists n local homeomorphisms $d_i : U_i \longrightarrow V_i$ of $\mathrm{Obs}\,\Gamma$ such that

$$f(P) \subset \bigcup_{i=1}^{n} \mathrm{Germ}(U_i, d_i) \subset F(\Gamma).$$

During the proof this cover will be refered to as an <u>n-cover.</u>

We will use induction on n where the following hypothesis is used.
<u>Hypothesis $H(n)$</u> : if $f : P \longrightarrow \Gamma$ has an n-cover and is continuous then f is a germ of a Γ— shift of P, when P is compact and hausdorff.

Suppose Germ (U_1, d_1) is a 1—cover of $f : P \longrightarrow \Gamma$. Put α as the composite

$$P \xrightarrow{f} \Gamma \xrightarrow{R} \mathrm{Obs}\,\Gamma$$

and β as

$$P \xrightarrow{f} \Gamma \xrightarrow{L} \mathrm{Obs}\,\Gamma$$

then put $i_1 = 1\times\alpha$, $i_2 = 1\times\beta$, $\omega_1 = P\times U_1$, $\omega_2 = P\times U_2$ (where $U_2 = d_1(U_1)$) and $d = 1\times d_1$. $\Xi = (i_1, i_2, \omega_1, \omega_2, d)$ is obviously a Γ-shift of P and $F(\Xi) = f$.
This shows that hypothesis $H(1)$ is true.

Suppose $H(m)$ is true for $m \geq 1$. We will show that this gives $H(m+1)$ is true. Let $f : P \to \Gamma$ be a continuous map from a compact hausdorff topological space P, and the $m+1$ homeomorphisms

$$\{d_i : U_i \longrightarrow V_i \mid 1 \leq i \leq m+1\}$$

be an $m+1$-cover of $f(P)$. Put $P' = \bigcup_{i=1}^{m} f^{-1}(\mathrm{Germ}(U_i, d_i))$ and $T = f^{-1}(\mathrm{Germ}(U_{m+1}, d_{m+1}))$. P' and T are open so $S = P' - P'\cap T$ $= P - T$ is closed and compact and $S \subset P'$ so f/S satisfies the hypothesis of $H(m)$. Now $T - T\cap P' = P - P'$ is closed and compact and $(P-P') \cap S' = \emptyset$ so, since P is compact and hausdorff, P is a normal topological space so by Urysohn's Lemma there exists a function $\varphi: P \to [0,1]$ which is continuous, takes the value 0 on S and the value 1 on $P-P'$, $\varphi^{-1}([0,\frac{1}{2}])$ is closed in P and hence compact and hausdorff, further more $\varphi^{-1}([0,\frac{1}{2}]) \subset P'$ so $f|\varphi^{-1}([0,\frac{1}{2}])$ satisfies the hypothesis of $H(m)$. Hence there exists a Γ-shift of $\varphi^{-1}([0,\frac{1}{2}])$, $z = (i_1, i_2, W_1, W_2, d)$. We shall extend z to a "shift" on an open neighbourhood of W_1, in $P \times OGs\Gamma$. For $x \in \varphi^{-1}([0,\frac{1}{2}])$ there exists a homeomorphism $d_x: B_{1x} \to B_{2x}$ of open subsets of $OGs\Gamma$, and an open set A_x in $\varphi^{-1}([0,\frac{1}{2}])$ such that $1 \times d_x = d | A_x \times B_{1x}$ and $i_1(x) \in A_x \times B_{1x}$.

Since A_x is open in $\varphi^{-1}([0,\frac{1}{2}])$ there exists an open subset A'_x in P such that $A'_x \cap \varphi^{-1}([0,\frac{1}{2}]) = A_x$. Put

$$B'_{1x} = \{ y \in B_{1x} \cap U_{m+1} | \text{ Germ}(y, d_x) = \text{Germ}(y, d_{m+1}) \}$$

and $B'_{2x} = d_x(B'_{1x})$. B'_{1x} is open so $d'_x = d_x | B'_{1x}$ is a homeomorphism $d'_x: B'_{1x} \to B'_{2x}$ between open subsets of $OGs\Gamma$. Also Germ $(B'_{1x}, d'_x) \subset F(\Gamma)$. Put $W'_1 = \cup \{ A'_x \times B'_{1x} | x \in \varphi^{-1}([0,\frac{1}{2}]) \}$ and construct W'_2 in a similar way. Define $d': W'_1 \to W'_2$ as $1 \times d'_x$ on $A'_x \times B'_{1x}$. d' is well defined because if $(a,b) \in A'_x \times B'_{1x} \cap A'_y \times B'_{1y}$,

$$1 \times d_x(a,b) = 1 \times d_{m+1}(a,b) = 1 \times d_y(a,b).$$

$d'_x | \varphi^{-1}([0,\frac{1}{2}])$ is a restriction of d_x, so d and d' agree on $W_1 \cap W'_1$ so can be extended to a map $d'': W_1 \cap W'_2 \to W_2 \cap W'_2$. Put $\operatorname{int} W_1$, $\operatorname{int} W_2$ as the interiors of W_1 and W_2 respectively in $P \times OGs\Gamma$. Then put

$$W_3 = \operatorname{int} W_1 \cup W'_1 \cup \varphi^{-1}((\tfrac{1}{2},1]) \times U_{m+1}$$

$$W_4 = \operatorname{int} W_2 \cup W'_2 \cup \varphi^{-1}((\tfrac{1}{2},1]) \times V_{m+1}$$

and define $\bar{d}: W_3 \to W_4$ as, for $(a,b) \in W_3$

$$\bar{d}((a,b)) = \begin{cases} d''(a,b) & \text{for } a \in \varphi^{-1}([0,\frac{1}{2}]) \\ (a, d_{m+1}(b)) & \text{otherwise.} \end{cases}$$

W_3 and W_4 are open, $\bar{d}$ is surjective and since $W_3 \cap \varphi^{-1}((\frac{1}{2},1]) \times \mathrm{Obs}\,T = \varphi^{-1}((\frac{1}{2},1]) \times U_{m+1}$, $W_3 \cap \varphi^{-1}([0,\frac{1}{2}]) \subset W_1$ and $\bar{d}$ is level preserving we get that $\bar{d}$ is injective. $\bar{d}$ restricted to $\mathrm{int}\,W_1$, W_1', or $\varphi^{-1}((\frac{1}{2},1]) \times U_{m+1}$ is a homeomorphism of open sets in $P \times \mathrm{Obs}\,T$ so we conclude that since it is also bijective $\bar{d}$ is a homeomorphism of open subsets of $P \times \mathrm{Obs}\,T$. $\bar{d}$ is locally of the form $1 \times \hat{d}$ where $\hat{d} : U \longrightarrow V$ is a homeomorphism of open subsets of $\mathrm{Obs}\,T$ and Germ $(U, \hat{d}) \subset F(T)$. So $\bar{d}$ is a candidate for a T-shift of P construction.

Put $\alpha : P \longrightarrow \mathrm{Obs}\,T$ as $\alpha = R \circ i$ then the graph of α, say j_1 is contained in W_3. This is so because:

i) if $x \in \varphi^{-1}([0,\frac{1}{2}])$ then $j_1(x) = i_1(x)$ and is contained in $W_1 \cap \varphi^{-1}([0,\frac{1}{2}]) \times \mathrm{Obs}\,T$ which is open so $j_1(x) \in \mathrm{int}(W_1)$. We also note that $\bar{d}$ and d agree on some neighbourhood of $j_1(x)$.

ii) if $x \in \varphi^{-1}((\frac{1}{2},1])$ then, since $x \in T$, $f(x) =$ Germ(y, d_{m+1}) for some $y \in U_{m+1}$ so $j_1(x) \in \varphi^{-1}((\frac{1}{2},1]) \times U_{m+1}$.

iii) if $x \in \varphi^{-1}(\{\frac{1}{2}\})$ then as for ii) $f(x) =$ Germ $(j_1(x), d_{m+1})$. Also from the definition of d we have Germ $(\alpha(x), d_x) = f(x)$, this means that B'_{1x} is not empty and $j_1(x) = (x, \alpha(x)) \in A_x \times B'_x \subset W_1'$.

We conclude that if $j_2 = \bar{d} \circ j_1$, then $Z' = (j_1, j_2, W_3, W_4, \bar{d})$ is a T-shift of P and further more by inspecting the proofs given in i) to iii) we get that also $F(Z') = f$. Hence proposition $H(m)$ implies $H(m+1)$, and since $H(1)$ is true by induction $H(r)$ is true for all $r \in \mathbb{N}$.

Let X and Y be topological spaces and put Y^X as the set of continuous maps from X to Y .Define for $A \subset X$, $B \subset Y$ the subset

$$\langle A ; B \rangle = \{ f \in Y^X \mid f(A) \subset B \}$$

of Y^X. Y^X is usually given the compact open topology which has as subbasis the collection of sets $\langle K ; U \rangle$ where K is compact and U is open. For our purpose we will use an alternative topology the <u>uniform topology</u> on Y^X given by the subbasis $\{\langle U \rangle\}$, U open in $X \times Y$ where

$$\langle U \rangle = \{ f \in Y^X \mid (1 \times f) \circ \Delta \in \langle X ; U \rangle \}$$

and Δ is the diagonal map. It is easily checked that $\{\langle U \rangle\}$ is a basis for the topology and is infact the collection of open sets in the uniform topology.

5 LEMMA: If Γ is a realisable topological groupoid and P is a compact hausdorff topological space then the map $\alpha : \Gamma^P \longrightarrow Ob\Gamma^P$ given by $\alpha(f) = R \circ f$ is continuous when uniform topologies are used.

Proof : This lemma comes from a general property of continuous maps: $P \times \Gamma \xrightarrow{1 \times R} P \times Ob\Gamma$ is continuous so if $U \subset P \times Ob\Gamma$ is open we get $(1 \times R)^{-1}(U)$ is open and $\alpha^{-1}\langle U \rangle = \langle (1 \times R)^{-1}(U) \rangle$, which is open.

Let $f : X \longrightarrow Y$ be a continuous map between topological spaces then a <u>system of local sections of f</u> is given by the set of pairs

$$\{ (U_x, S_x) \mid x \in X \}$$

such that U_x is open in Y and contains $f(x)$ and S_x is a continuous map $S_x : U_x \longrightarrow X$ which satisfies for $y \in U_x$, $f \circ S_x = 1_{U_x}$.

6 LEMMA : If Γ is a realisable topological groupoid then, using the uniform topology for $\mathrm{Obs}\,\Gamma^P$ and Γ^P we get that for P a compact hausdorff topological space

$$\alpha : \Gamma^P \longrightarrow \mathrm{Obs}\,\Gamma^P$$

has a system of local sections.

Proof:Let $f \in \Gamma^P$ then by Theorem 4 there exists a Γ— shift for P, $z = (i_1, i_2, \omega_1, \omega_2, \alpha)$ such that $F(z) = f$. Put $\hat{U}_f = W_1$, for $y \in \hat{U}_f$ there exists open sets $U_1 \subset P$ $V_1 \subset \mathrm{Obs}\,\Gamma$ such that $y \in U_1 \times V_1 \subset W_1$ and $\alpha | U_1 \times V_1$ is of the form $1 \times \alpha'$,where $\alpha' : V_1 \to V_2$ is a local homeomorphism. Put $\pi_2 : P \times \mathrm{Obs}\,\Gamma \longrightarrow \mathrm{Obs}\,\Gamma$ as the projection $\pi_2(\bar{x}, \bar{y}) = \bar{y}$ then Germ $(\pi_2(y), \alpha') \in \Gamma$,define $\hat{S}_f(y) =$ Germ $(\pi_2(y), \alpha)$. This gives a map $\hat{S}_f : \hat{U}_f \longrightarrow \Gamma$ which is continuous. We have $S_f : \langle \hat{U}_f \rangle \longrightarrow \Gamma^P$ given by for $g \in \langle \hat{U}_f \rangle$

$$S_f(g) = \hat{S}_f \circ (1 \times g) \circ \Delta$$

Now $R \circ S_f(g)(x) = R \circ \hat{S}_f(x, g(x)) = g(x)$ FOR $x \in P$ so $\{ (\langle \hat{U}_f \rangle, S_f) \mid f \in \Gamma^P \}$ is a candidate for a system of local sections.

To complete the proof all we have to show is that the sections $S_f : \langle \hat{U}_f \rangle \longrightarrow \Gamma^P$ are continuous. Let $g \in \langle \hat{U}_f \rangle$ and $\langle V \rangle$ be an open neighbourhood of $S_f(g)$. Let $\pi_1 : P \times \mathrm{Obs}\,\Gamma \to P$ be the projection $\pi_1(\bar{x}, \bar{y}) = \bar{x}$, then $W = (\pi_1 \times \hat{S}_f)^{-1}(V)$ is open and for $x \in P$

$$\begin{aligned}(\pi_1 \times \hat{S}_f)(x, g(x)) &= (x, \hat{S}_f(x, g(x))) \\ &= (1 \times \hat{S}_f \circ (1 \times g) \circ \Delta) \circ \Delta(x) \\ &= (1 \times S_f(g)) \circ \Delta(x) \\ &\in V\end{aligned}$$

so $g \in \langle W \rangle$ and in particular $g \in \langle W \cap \hat{U}_f \rangle$. By using the above formula we now get

$$S_f(\langle W \cap \hat{U}_f \rangle) \subset \langle V \rangle$$

so since $\langle V \rangle$ and g are arbitary choices we get S_f is continuous

From now on we shall restrict our attention to the case when $Ob_s\Gamma = \mathbb{R}^q$ for some $q \in \mathbb{N}$ and Γ is realisable. However it should be noted that a lot of the results can be extended to manifolds in general.

Another constraint that we will apply to our treatment is that we will consider only those P's which are hausdorff compact and locally compact. For such spaces P and for arbitary topological spaces X the uniform topology and the compact open topology of X^P are the same.

Let P be a compact hausdorff locally compact topological space and Γ be an open subtopological groupoid of Γ_q^0, where $q > 0$, then Γ is realisable, $Ob_s\Gamma \subset \mathbb{R}^q$. Γ is a <u>complete</u> sub topological groupoid of Γ_q^0 iff $Ob_s\Gamma = \mathbb{R}^q$.

In view of Lemma 6, we know that the neighbourhood structure of $(\mathbb{R}^q)^P$ is important. $(\mathbb{R}^q)^P$ has the usual vector space structure induced from that on $\mathbb{R}^q$. In fact for $f, g \in (\mathbb{R}^q)^P$, $\lambda \in \mathbb{R}$ we have

$$(-g)(x) = -g(x) \quad \text{for } x \in P$$
$$(f+g)(x) = f(x) + g(x) \quad \text{for } x \in P$$
$$(\lambda g)(x) = \lambda . g(x) \quad \text{for } x \in P$$
$$O(x) = 0 \quad \text{for } x \in P$$

Let V be a vector space over $\mathbb{R}$ then a <u>norm</u> on V is a real valued function

$$\nu : V \longrightarrow \mathbb{R}$$

with $\nu(x)$ denoted by $\|x\|$ for every $x \in X$ which satisfies the three conditions:

For any two vectors a and b in V, we have

$$\|a+b\| \leq \|a\| + \|b\|$$

For every vector x in V and any real number α, we have

$$\|\alpha x\| = |\alpha| . \|x\|$$

For any vector x in X, $\|x\| = 0$ implies $x = 0$.

$\mathbb{R}^2$ has several norms, an example of a norm is the sum of the moduli of a vector's components. Let $\| \ \|$ be a norm on $\mathbb{R}^2$ then this induces a norm on $(\mathbb{R}^2)^P$ in the usual way. In fact since P is compact and for $x_0 \in \mathbb{R}^2$, $\delta > 0$

$$B_\delta(x_0) = \{ x \in \mathbb{R}^2 \mid \|x - x_0\| < \delta \}$$

is always open the functions on P are bounded when continuous, so we get the norm for $f \in (\mathbb{R}^2)^P$

$$\|f\| = \sup_{x \in P} \{ \|f(x)\| \} .$$

For $\delta > 0$, $f \in (\mathbb{R}^2)^P$ put

$$\overline{B}_\delta(f) = \{ g \in (\mathbb{R}^2)^P \mid \|g - f\| < \delta \}$$

Then we get the following :

7 LEMMA: Let P be locally compact, compact and hausdorff topological space then

$$\{ \overline{B}_\delta(f) \mid f \in (\mathbb{R}^2)^P , \delta > 0 \}$$

is a basis for the topological space $(\mathbb{R}^2)^P$.

Proof: For $f \in (\mathbb{R}^q)^P$ and $\delta > 0$ let $g \in \overline{B}_\delta(f)$ then since P is compact there exists a $\delta_1 > 0$ satisfying $\delta > \delta_1$ such that for all $x \in P$ $\|f(x) - g(x)\| < \delta_1$. So if we choose $\delta_2 = \frac{1}{4}(\delta - \delta_1)$, $\delta_2 > 0$, and $\overline{B}_{\delta_2}(g) \subset \overline{B}_\delta(f)$. Since g is continuous, for $x \in P$, we can choose an open neighbourhood U_x of x in P which satisfies $g(U_x) \subset B_{\delta_2/2}(g(x))$.

Since P is compact we can choose a finite sequence $x_1, \dots x_m$ of members of P such that the graph $1 \times g$ of g has the immage covered by

$$W = \bigcup_{i=1}^{m} U_{x_i} \times B_{\delta_2/2}(g(x_i))$$

Now W is open and so $\langle W \rangle$ is an open neighbourhood of g. Also we can note that if $h \in \langle W \rangle$ then for all $x \in P$

$$\|g(x) - h(x)\| \leq \|g(x) - g(x_i)\| + \|h(x) - g(x_i)\| \text{ FOR SOME } i$$

$$\leq \delta_2/2 + \delta_2/2 = \delta_2$$

So $\quad g \in \langle W \rangle \subset \overline{B}_{\delta_2}(g) \subset \overline{B}_\delta(f)$

and since g was an arbitary member of $\overline{B}_\delta(f)$ we get $\overline{B}_\delta(f)$ is open. Let N be an open subset of $P \times \mathbb{R}^q$ such that $f \in \langle N \rangle$ where f is an arbitary member of $(\mathbb{R}^q)^P$. For $x \in P$ we have, since f is continuous, that there exists an open neighbourhood U_x of x AND $\delta_x > 0$ such that

$$(1 \times f)(U_x) \subset U_x \times B_{\delta_x/2}(f(x)) \subset U_x \times B_{\delta_x}(f(x)) \subset N.$$

P is compact so there exists a finite sequence $x_1, \dots x_m$ of members of P such that

$$U = \bigcup_{i=1}^{n} U_{x_i} \times B_{\delta_{x_i}/2}$$

and $f \in \langle U \rangle$.

Put $\delta = \min_{1 \le i \le m} \{ \delta_{x_i}/2 \}$, then $\delta > 0$ and if $h \in \overline{B}_\delta(f)$,

for $x \in U_{x_i}$

$$\| h(x) - f(x_i) \| \le \| h(x) - f(x) \| + \| f(x) - f(x_i) \|$$
$$\le \delta + \delta_{x_i}/2$$
$$< \delta_{x_i}$$

Since for $x \in P$ there exist an $1 \le i \le m$ such that $x \in U_{x_i}$ we can conclude that $h \in \langle N \rangle$. Now h was an arbitary member of $\overline{B}_\delta(f)$ so $\overline{B}_\delta(f) \subset \langle N \rangle$. The collection of sets of the form $\langle N \rangle$ where N is open in $P \times \mathbb{R}^q$ forms a topological basis for $(\mathbb{R}^q)^P$. Hence summarising we get that the collection of sets of the form $\overline{B}_\delta(f)$ where $f \in (\mathbb{R}^q)^P$ and $\delta > 0$, give a topological basis for $(\mathbb{R}^q)^P$.

Combining Lemmas 6 and 7 we get the following :

8 Collary : Let Γ be a complete sub topological groupoid of Γ^0_q where $q > 0$ then for each $f \in \Gamma^P$ there exists a $\delta_f > 0$ and continuous map $S : \overline{B}_{\delta_f}(R \circ f) \longrightarrow \Gamma^P$ such that $\alpha \circ S = 1$. Where of course

$$\overline{B}_{\delta_f}(R \circ f) \subset (\mathbb{R}^q)^P.$$

We note that $\overline{B}_\delta(R \circ f)$ is <u>convex</u> in the following sense. Let $g_1, g_2 \in \overline{B}_\delta(R \circ f)$ then $(1-t)g_1 + t g_2 \in \overline{B}_\delta(R \circ f)$ for $0 \le t \le 1$, as can be checked by using the difinition of a norm. Also if I is the unit interval (closed) then the map $h : I \longrightarrow \overline{B}_\delta(R \circ f)$ given by $h : t \longmapsto (1-t)g_1 + t g_2$ is continuous and we have h gives rise to a homotopy $H : I \times P \longrightarrow \Gamma^q$ In this way we can construct homotopies in Γ^P which connect a given member m of Γ^P with a member n which is close to m and has a specified map $R\circ n$ in $\mathbb{R}^q$.

Our first application of the above constructions will be the proof of the following theorem, which will be proved in stages but is quoted here to provide motivation.

9 THEOREM : Let Γ be a complete sub topological groupoid of Γ_q^0, the topological groupoid of germs of homeomorphisms of open subsets of $\mathbb{R}^q$, then

$$H_t(\Gamma(\bar{\Delta}^s)) = 0 \quad \text{for } s>0,\ t>q.$$

We notice that if $\delta>0$ and $f\in(\mathbb{R}^q)^P$ where P is compact locally compact and hausdorff topological space then first of all

$$\bar{B}_\delta(f) = \langle U_\delta(f)\rangle$$

where

$$U_\delta(f) = \{(x,y)\in P\times\mathbb{R}^q \mid \|f(x)-y\|<\delta\}$$

and is open. In the proof of Lemma 6 the system of local sections were of a special type which gives

10 LEMMA : Let Γ be a complete sub topological groupoid of Γ_q^0 where $q>0$ and $f\in\Gamma^P$ there exist a system of local sections

$$S : \bar{B}_{\delta f}(R\circ f) \longrightarrow \Gamma^P$$

(as given by Corollory 8) which have the two following properties:

a) there exists a continuous map $\hat{S} : U_{\delta f}(R\circ f)\longrightarrow\Gamma$ such that $S(g) = \hat{S}\circ(1\times g)\circ\Delta$.

b) if $g : I\to P$ is a continuous ark in P and $y\in\mathbb{R}^q$ such that for $t\in I$, $(g(t),y)\in U_{\delta f}(R\circ f)$ then $\hat{S}(g(0),y) = \hat{S}(g(1),y)$.

Proof :property a) comes directly from the construction given in Lemma 6, and Lemma 7 where we take $\hat{S}$ as a restriction of $\hat{S}_f$. For property b) we note that for each $t \in I$, since g is continuous, there exists an $\epsilon > 0$ such that for $s \in I$, $|s-t| < \epsilon$ we get $\hat{S}(g(s), y) = \hat{S}(g(t), y)$ from the construction of $\hat{S}_f$. Put $V_t = \{ s \in I \mid \hat{S}(g(s), y) = \hat{S}(g(t), y) \}$ then we have shown that V_t is always open and non empty so there exists a set $T \subset I$ such that $I = \bigcup_{t \in T} V_t$ and for $t_1, t_2 \in T$,

$V_{t_1} \cap V_{t_2} \neq \emptyset \Rightarrow t_1 = t_2$. However I is connected so we conclude that $V_t = I$ for all $t \in I$.

Given the system of local sections provided by Corollory 8 and an $f \in \Gamma^P$ we can construct an equivalence relation $\sim$ on $U_{\delta f}(R \circ f)$ given by for $(a_1, b_1), (a_2, b_2) \in U_{\delta f}(R \circ f)$ let $(a_1, b_1) \sim (a_2, b_2)$ when

a) $b_1 = b_2$ and

b) there exists a continuous arc $g . I \longrightarrow P$ such that for $t \in I$ $(g(t), b_1) \in U_{\delta f}(R \circ f)$ and $a_1 = g(0)$, $a_2 = g(1)$.

The relation $\sim$ is obviously an equivalence relation. Put

$$Q_f = U_{\delta f}(R \circ f)/\sim$$

and give Q_f the quotient topology and let $\Pi_f : U_{\delta f}(R \circ f) \to Q_f$ be the projection of members of $U_{\delta f}(R \circ f)$ to their equivalence class. From Lemma 10 we get that there exists a map $q_f : Q_f \longrightarrow \Gamma$ such that

$$\begin{tikzcd}
U_{\delta f}(R \circ f) \arrow[rr, "\hat{S}"] \arrow[dr, "\Pi_f"'] & & \Gamma \\
& Q_f \arrow[ur, "q_f"'] &
\end{tikzcd}$$

commute. By Lemma 2·1·2 q_f is continuous since $\hat{S}$ is.

A convenient way of summarising our efforts so far is the following Lemma.

11 LEMMA : Let Γ be a complete sub topological groupoid of Γ_q^o where $q>0$ and $f\in\Gamma^P$ then the following diagram commutes and is a diagram of continuous functions, when P is compact, locally compact and hausdorff topological space.

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "(1\times R\circ f)_{\Delta}"'] & \Gamma \\ U_{\delta f}(R\circ f) \arrow[ur, "\hat{S}"] \arrow[r, "\pi_f"] & Q_f \arrow[u, "q_f"'] \end{tikzcd}$$

It can be seen that if Q_f satisfies $H_{q+n}(Q_f)=0$ for $n>0$ then we have a good chance of showing that Theorem 9 holds. We will show that a special subset of Q_f can replace Q_f in Lemma 11 which satisfies this condition:, in order to enable us to calculate the higher dimentional homologies we will resort to using a special class of spaces to represent Q_f called compact polyhedra.

The following will give the definitions and properties in brief. For further details see for instance [SP1].

A <u>simplicial complex</u> consists of a set $\{v\}$ of <u>virtices</u> and a set $K=\{S\}$ of finite non empty subsets of $\{v\}$ called simplexes such that

a) Any set consisting of exactly one vertex is a simplex.

b) Any non empty subset of a simplex is a simplex.

We shall identify the simicial complex with K.

The <u>dimension</u> denoted by $\mathrm{Dim}(S)$ of a simplex S is the number of vertices in S minus 1 and the dimension of a simplicial complex K is

$$\sup \{ \mathrm{Dim}(S) \mid S \text{ is a simplex of } K \}.$$

<u>A simpicial map</u> $\varphi : K_1 \longrightarrow K_2$ from simplicial complex K_1, to simplicial complex K_2 is a function φ from the vertices of K_1, to the vertices of K_2 which maps simlexes to simplexes. The simplicial complexes and simplicial maps forms a category of simplicial complexes. A <u>sub complex</u> L of a simplicial complex K is a subset of K which is a simplicial complex: in this case we write $L \subset K$.

For a simplicial complex let $|K|$ be the set of all functions α from the set of vertices of K to I such that

a) For any α, $\{ v \in K \mid \alpha(v) \neq 0 \}$ is a simplex of K

b) For any α, $\sum_{v \in K} \alpha(v) = 1$.

If $K = \phi$ we define $|K| = \phi$.

$|K|$ has a metric d defined by

$$d(\alpha, \beta) = \sqrt{\sum_{v \in K} [\alpha(v) - \beta(v)]^2}$$

for $\alpha, \beta \in |K|$. This gives a topological space $|K|_d$. For $S \in K$ the <u>closed simplex</u> $|S|$ is defined by

$$|S| = \{ \alpha \in |K| \mid \alpha(v) \neq 0 \Rightarrow v \in S \}$$

We use the metric to define a topology on $|S|$ and choose a topology on $|K|$ which is coherent with $\{|S| \mid S \in K\}$ to give the topological space $|K|$. A triangulation (K, f) of a topological space X is a simplicial complex K and homeomorphism $f: |K| \longrightarrow X$. If X has a triangulation, X is called a polyhedron. A finite polyhedron is a polyhedron which can be triangulated by a triangulation with a finite simplicial complex. Note that a finite polyhedron is a compact and locally compact hausdorff topological space.

if $\varphi: K_1 \longrightarrow K_2$ is a simplicial map then the function $|\varphi|: |K_1| \longrightarrow |K_2|$ given by

$$|\varphi|(\alpha)(v') = \sum_{\varphi(v)=v'} \alpha(v) \qquad v' \in K_2$$

is continuous and gives a covarient functor from the category of simplicial complexes to the category of topological spaces.

A semi simplicial complex X consists of a sequence $\{X_n \mid n=1,\dots\}$ of disjoint sets together with a collection of maps in each dimension n:

$d_i: X_{n+1} \longrightarrow X_n$, $i = 1, 2, \dots, n+2$ the i^{th} face operator;

$S_j: X_n \longrightarrow X_{n+1}$, $j = 1, 2, \dots, n+1$ the j^{th} degeneracy operator;

which satisfy the semisimplicial identities:

a) $d_i d_j = d_{j-1} d_i \quad , \quad i < j;$

b) $d_i S_j = S_{j-1} d_i \quad , \quad i < j;$

c) $d_i S_j = 1 \quad , \quad i = j, j+1;$

d) $d_i S_j = S_j d_{i-1} \quad i > j+1$

e) $s_i s_j = s_{j+1} s_i \qquad i \leq j$

The elements of X_n are called the <u>n- simplexes of</u> X .

This is the definition given by A.T. Lundel and S.Weingram in [LW 1] chapter III. We will use the results on SSC (semisimplicial complex) given in [LW1] to construct our finite polyhedron in the usual way.

Let Δ^n be the topological space $\Delta^n = \{(t_1, \dots, t_{n+1}) \in \mathbb{R}^{n+1} | t_i \geq 0$ and $\sum t_i = 1$ with the topology induced from $\mathbb{R}^n\}$.
First put $d_i^* : \Delta^n \longrightarrow \Delta^{n+1} : (t_1, \dots, t_{n+1}) \longmapsto (t_1, \dots t_i, 0, t_{i+1}, \dots t_{n+1})$
and $s_i^* : \Delta^n \longrightarrow \Delta^{n-1} : (t_1, \dots t_{n+1}) \longmapsto (t_1, \dots, t_j, t_{j+1} + t_{j+2}, \dots t_{n+1})$
and then construct a topological space $|X|$ from a $SSC\, X$.

Given a SSC X to each member $x_n \in X_{n+1}$ associate a copy (Δ^n, x_n) of Δ^n , and let $M(X)$ be the disjoint topological union of such copies. Generate an equivalence relation $\sim$ on $M(X)$ by defining the elementary equivalences:

$$(d_i^* t, x_n) \sim (t, d_i x_n) \qquad \text{for all } t \in \Delta^{n-1}$$
$$(s_i^* t, x_n) \sim (t, s_i x_n) \qquad \text{for all } t \in \Delta^{n+1}$$

Put $|X|$ as the quotient space of $M(X)$ by $\sim$. with quotient map $\eta : M(X) \longrightarrow |X|$ Put $[t, x] = \eta(t, x)$, the topological space $|X|$ constructed from the $SSC\, X$ is the <u>geometric realisation</u> of X .

12 LEMMA : Let X be a $SSC\, X$ then $|X|$ is a polyhedron, further more if $|X|$ is a finite CW complex $|X|$ is a finite polyhedron.
Proof : by [LW1] construct the SSC SdX from X and show that

a) $|SdX|$ is homeomorphic to $|X|$

b) SdX is regulated SSC which in turn gives that $|SdX|$ is a regular CW complex

c) a regular CW complex has a triangulation.

a), b) and c) show that $|X|$ is a polyhedron. The statement on finiteness of the polyhedron in the special case that $|X|$ is a finite CW complex can be checked by following through the construction of the triangulation of $|X|$.

Let X be a topological space and for $n \geq 1$, $S_n(X)$ be the collection of continuous maps from Δ^n to X. Then $S(X) = \{ S_n \mid n = 1, \cdots \}$ has a SSC structure with i^{th} face operator $d_i(\sigma^n) = \sigma^n \circ d^*_i$ and j^{th} degeneracy operator $S_j(\sigma^n) = \sigma^n \circ S^*_j$ where $\sigma^n \in S_n(X)$.

Let $z \in \Delta_n(X)$ and $\partial z = 0$ then there exists an $r \geq 1$ and integers n_i together with continuous maps $x_i : \Delta^n \longrightarrow X$ such that

$$z = \sum_{i=1}^{r} n_i x_i$$

By applying degeneracy and face operators to the collection $\{ \{x_i \mid i = 1 \text{ to } r\}$ we can generate a sub SSC $S(z)$ of $S(X)$. Furthermore $|S(z)| \subset |S(X)|$ and $|S(z)|$ is a finite polyhedron, because the number of nongenerate simplexes is finite and hence $|S(z)|$ is a finite CW complex which by Lemma 12 makes it a finite polyhedron.

Put $y_i : \Delta^n \longrightarrow |S(z)|$ as the composite $t \longmapsto (t, x_i) \longmapsto \eta(t, x_i)$. y_i is obviously continuous. Also it is well known that the map $j_X : |S(X)| \longrightarrow X$ given by $j_X(\eta(t, x_n)) = x_n(t)$ is continuous so put $g = j_X \,|\, |S(z)|$ to get that

$$\begin{tikzcd} \Delta^n \arrow[r, "y_i"] \arrow[dr, "x_i"'] & {|S(z)|} \arrow[d, "g"] \\ & X \end{tikzcd}$$

commutes. Check : $(g \circ y_i)(t) = d_X([t, x_i]) = x_i(t)$ put $z' = \sum_{i=1}^{r} n_i y_i$ then we get $g_*(z') = z$. Also

$$\partial z' = \sum_{i=1}^{r} n_i \partial y_i = (-)^j \sum_{i=1}^{r} \sum_{j} n_i d_j(y_i)$$

where $d_j(y_i)(t) = \eta(d^*_j(t), x_i)$

$$= \eta(t, d_j x_i)$$

Let $\Delta(z)$ be the free abelian group generated by $S(z)$ then the homomorphism $F : \Delta(z) \to \Delta|S(z)|$ given by $F(x)(t) = \eta(t, x)$ gives $\partial z' = \partial F(z) = F(\partial z) = 0$

We thus get the following

13 LEMMA : Let X be a topological space and z be a cocycle in $\Delta(X)$ then there exists a finite polyhedron $|S(z)|$ and a map $g : |S(z)| \to X$ such that $g_* z' = z$ for some cocycle $z' \in \Delta(|S(z)|)$.

Such a polyhedron for the cocycle z is a <u>carrier polyhedron with coclcle z'</u> for z .

Lemma 13 shows that in proving Theorem 9 we need only consider the cases when P is a finite polyhedron, which should simplify Q_f.

We will now go into a development of this observation which will show that the diagram in Lemma 11 can be restricted to highly linear maps between simplicial complexes. This will show that Q_f can without loss of generality be replaced by a fimite polyhedron of dimension less than or equal to q . To do this we will need cell complexes, polyhedra, and their properties.

Given $\mathbb{R}^m$ a cell is a sub set S of $\mathbb{R}^m$ which satisfies and is defined by a set of linear equations and inequalities,

$$\ell_i(x) = C_i \quad \text{FOR} \quad i = 1 \text{ to } s$$

$$L_j(x) \geq C_j \quad \text{FOR} \quad j = 1 \text{ to } t$$

for $x \in S$. The space $\mathbb{R}^m$ is the support space for S.

We will be interested in only compact cells. A finite cell is a cell S which is bounded; A cell S is a closed subspace of its support space so a finite cell is compact.

A face S_1 of a cell S is a cell obtained from S by setting some of the inequalities that defines S to equalities. A cell, since it is defined by a finite set of linear expressions has only a finite number of faces and the set of faces of a cell S depends only upon the set S.

A cell complex is a collection K of finite cells in some $\mathbb{R}^m$ which satifies

i) K is finite

ii) if $\sigma, \sigma' \in K$ then $\sigma \cap \sigma'$ is a face of σ and σ' or $\sigma \cap \sigma' = \phi$.

iii) all faces of cells of K are members of K.

The space $|K| = \bigcup_{\sigma \in K} \sigma$ is called a euclidean cell complex.

The product of two cell complexes K, K' is the set

$$K \times K' = \{\sigma \times \sigma' \mid \sigma \in K, \sigma' \in K'\}$$

$K \times K'$ is a cell complex and $|K \times K'| = |K| \times |K'|$.

14 EXAMPLE : If K is the collection of faces of a cell then K is a

cell complex and $|K|$ is a cell.

Let K be a simplicial complex then the map $\ell : |K| \longrightarrow \mathbb{R}^m$ is <u>linear</u> if for each simplex $\sigma \in K$ with virtices $v_1, \cdots v_n$ $\ell \backslash |\sigma|$ maps

$$\sum_i \lambda_i v_i \text{ to } \sum \lambda_i \ell(v_i) ,$$

A <u>geometric realisation</u> of a simplicial complex K is a linear map $\ell : |K| \longrightarrow \mathbb{R}^m$ which is an embedding we have the well known ;

15 PROPOSITION : Every finite simplicial complex has a geometric realisation.

If $\ell : |K| \longrightarrow \mathbb{R}^m$ is a geometric realisation then $K' = \{ \ell(|S|) \mid S \in K \}$ is a cell complex the <u>immage</u> of K .
If a cell complex is an immage of a simplicial complex it is a <u>Euclidean simplicial complex</u>, we will also refer to $|K|$ as a Euclidean polyhedron.

16 COROLLARY : A finite polyhedron is homoemorphic to a Euclidean polyhedron.

Another example of a Euclidean polyherdon is obtained by suitably subdividing a cell complex. A <u>sub division</u> of a cell complex K is a cell complex K' such that

i) $|K| = |K'|$

ii) if S' is a cell of K' , there is some cell S of K such that $S' \subset S$

(this is similar to the definition of simplicial sub division, see [SP1]).

17 PROPOSITION : Every cell complex has a sub division which is a Euclidean simplicial complex.

In [SP1] Chapter 3 Section 3 the notion of simplicial sub division is introduced ; it is easily checked that a simplicial sub division of a simplicial complex gives a corresponding sub division of a cell complex when it is a Euclidean simplicial complex.
This correspondence gives the following properties borrowed from simplicial sub division theory.

Let K be a cell complex put mesh K as

$$\operatorname{mesh} K = \sup \{ \operatorname{diam} S \mid S \in K \}$$

where we have chosen some metric for the real vector space in which $|K|$ is embedded and for a compact subset S of this space the diam (S) is the largest distance between two points in S .

18 PROPOSITION : Let K be a cell complex and $\delta > 0$ then there exists a sub division K' of K such that mesh $K' < \delta$, and the sub division K' can be chosen to be a Euclidean polyhedron.

This is proved in the usual way by barycentric subdivision after using Proposition 17 to triangulate $|K|$.
We will use the following convention: if K is a cell complex which is also an Euclidean polyhedron then we will use K to designate the corresponding simplicial complex and $|K|_c$ the topological space for the cell complex and $|K|_s$ the topological space for simplicial complex when these spaces need to be destnguished, otherwise the subscript will be ommited.

Let K_1 and K_2 be cell complexes then a <u>cellular map</u> is a continuous map $\varphi : |K_1|_c \longrightarrow |K_2|_c$ which is linear on each cell of K_1 and maps cells to cells, in the sense that if $\sigma \in K_1$, then $\varphi(\sigma) \in K_2$. Also if K_1 and K_2 are Euclidean simplicial complexes then there exists a simplicial map φ for f such that $|\varphi|$ is identified with f by the homeomorphisms of the geometric realisations associated with the Euclidean polyhedra. So we can translate the simplicial approximation theorems to approximation by cellular maps theorems. We will not do this here but content ourselves with the definition of cellular approximation to make our treatment viable.

A <u>cellular approximation</u> $\varphi : |K_1| \longrightarrow |K_2|$ of a continuous map $f : |K_1| \longrightarrow |K_2|$ for cell complexes K_1 and K_2 is a cellular map such that if $\sigma \in K_2$ and $f(\alpha) \in \sigma$ then $\varphi(\alpha) \in \sigma$.

19 LEMMA : Let the hypothesis of Lemma 11 be satisfied then the diagram, by suitable restriction and homotopy of f, can be replaced by the diagram

$$\begin{tikzcd}
P = |J| \arrow[r, "f^{\#}"] \arrow[d, "\bar{f}"'] & T \\
|K_f| \arrow[ur, "\hat{S}"] \arrow[r, "\pi_f"'] & Q_f \arrow[u, "q_f"']
\end{tikzcd}$$

when P is a Euclidean polyhedron, where $|K_f|$ is a Euclidean polyhedron $\subset P \times \mathbb{R}^2$, $\bar{f}$ is a cellular map and $f^{\#}$ is homotopic to f, furthermore $|K_f|$ is a neighbourhood of the immage of $\bar{f}$ and J, K_f can be Euclidean simplicial complexes.

Proof: Since P is compact subspace of (say) $\mathbb{R}^m$, $V_{Sf}(R\circ f)$ is bounded in $\mathbb{R}^m \times \mathbb{R}^q$ so there exists a copy of Δ^q in $\mathbb{R}^q$, $|S|$, which is a Euclidean polyhedron in $\mathbb{R}^q$ that contains the immage of $U_{\delta f}(R \circ f)$ under the projection π of $P \times \mathbb{R}^q$ into $\mathbb{R}^q$. This means $U_{\delta f}(R \circ f)$ is contained in $P \times |S|$. Put $P = |\bar{P}|$, the product of two cell complexes is a cell complex so by proposition 17 $|\bar{P}| \times |S|$ is a Euclidean polyhedron in $\mathbb{R}^{m+q}$. By proposition 18 there is a subdivision of $\bar{P} \times S$ which has a mesh less than $\delta f/2$ which is a Euclidean cell complex, say K_1, Since P is a Euclidean polyhedron there exists a Euclidean simplicial complex J such that $|J| = P$ and a subdivision K_1' of K_1 for which there exists a cellular map $\bar{f} : |\delta| \longrightarrow |K_1'|$ that is a cellular approximation of f. Since the mesh of K_1' is less than $\delta_f/2$ we have the immage of $\bar{f}$ is in $U_{\delta f}(R \circ f)$ and in fact $U_{\delta f/2}(\bar{f})$ is in $U_{\delta f}(R \circ f)$. Consider the union K_f of all cells of K_1' that have a point in common with the immage of $\bar{f}$ then K_f is a Euclidean simplicial complex. Now

$$K_f \subset U_{\delta f/2}(\bar{f})$$

because the mesh of K_f is less than $\delta f/2$. Put $f^{\#} = \hat{S} \circ f$ and restrict $\hat{S}$ to $\hat{S}$ to get the required diagram.

If we look at the construction of Q_f from $U_{\delta f}(R \circ f)$ we see that $\pi_f(K_f)$ is homeomorphic to $K_f/\sim$ where $\sim$ is the equivalence relation used to define Q_f. We will now prove some properties of $K_f/\sim$. K_f is a Euclidean simplicial complex. Let $\mathbb{R}^m \times \mathbb{R}^q$ be the support space for K_f. Put $\ell : \mathbb{R}^m \times \mathbb{R}^q \longrightarrow \mathbb{R}^q$ as the projection $(a, b) \longmapsto b$ and $\pi : |K_f| \longrightarrow |K_f|/\sim$ the projection to equivalence classes.

Let $S \in K_f$ then $\ell(S)$ is a cell. From the definition of the equivalence relation we have there exists an $\bar{\ell}_S : \Pi(S) \longrightarrow \ell(S)$ such that

$$\begin{tikzcd} S \arrow[rr, "\ell"] \arrow[dr, "\Pi"'] & & \ell(S) \\ & \Pi(S) \arrow[ur, "\bar{\ell}_S"'] & \end{tikzcd}$$

commutes, Π is an identification so $\bar{\ell}_S$ is continuous.

20 LEMMA : For $S \in K_f$ and $a \in \Pi(S)$

$$\Pi^{-1}(a) \cap S = \ell^{-1}(\bar{\ell}_S(a)) \cap S$$

when $\Pi^{-1}(a) \cap S \neq \phi$.

Proof: There exists an $x \in \Pi^{-1}(a) \cap S$ if $y \in |K_f|$ and $x \sim y$ then $\ell(x) = \ell(y)$ so $\Pi^{-1}(a) \cap S \subset \ell^{-1}(\ell(x)) \cap S = \ell^{-1}(\bar{\ell}_S(a)) \cap S$. Also if $z \in S \cap \ell^{-1}(\ell(x)) \cap S$ then $h : I \longrightarrow K$ given by $t \longmapsto tz + (1-t)x$ gives for $t \in I$, $\ell \circ h(t) = t\ell(z) + (1-t)\ell(x) = \ell(x)$ and has z and x as end points so $z \sim x$ (since I maps into K). Hence $z \in \Pi^{-1}(a) \cap S$ and we get $\Pi^{-1}(a) \cap S \supset \ell^{-1}(\bar{\ell}_S(a)) \cap S$.

21 LEMMA : For $S \in K_f$ the map $\bar{\ell}_S : \Pi(S) \longrightarrow \ell(S)$ is a homeomorphism.

Proof: Since $\bar{\ell}_S$ is continuous $\Pi(S)$ is compact and $\ell(S)$ is hausdorff it is sufficient to prove that $\bar{\ell}_S$ is bijective. Since $\ell : S \longrightarrow \ell(S)$ is surjective the diagram for $\bar{\ell}_S$ gives $\bar{\ell}_S$ is surjective. For injectivity we note that if $a, b \in \Pi(S)$ such that $\bar{\ell}_S(a) = \bar{\ell}_S(b)$ we have from Lemma 20 that

$$\Pi^{-1}(a) \cap S = \ell^{-1}(\bar{\ell}_S(a)) \cap S = \ell^{-1}(\bar{\ell}_S(b)) \cap S = \Pi^{-1}(b) \cap S \neq \phi$$

so $\exists\ y \in \Pi^{-1}(a) \cap \Pi^{-1}(b) \Rightarrow \Pi(y) = a = b$.

22 PROPOSITION : Let K and K' be cell complexes with carrier

space $\mathbb{R}^n$. Then $K \cap K'$ is a cell complex and $|K \cap K'| = |K| \cap |K'|$ also there exists a cell complex J (which can be chosen to be a Euclidean simplicial complex) for which $|J| = |K| \cup |K'|$ and J restricted to $|K|$ and $|K'|$ gives subdivisions of K and K'.

Let A be the family of complexes produced by taking finite intersections of the cells $e(s)$, $s \in K_f$. A is a finite family and by Proposition 22 we get a triangulation T of

$$\bigcup_{a \in A} |a|$$

into a Euclidean polyhedron which gives subdivisions of members of A. We can use this fact to triangulate $K_f/\sim$. That is there exists a Euclidean simplicial complex T such that

$$|T| = \bigcup_{a \in A} |a|$$

and for each $a \in A$, $T \big| |a|$ is a subdivision of a.

For each $y \in \mathbb{R}^q$ when it is not empty $e^{-1}(y) \cap K_f$ is a Euclidean cell complex with complex given by

$$J_y = \{ s \cap e^{-1}(y) \mid s \in K_f \}$$

if $a \in J_y$ and $s \in K_f$ is such that for all $s' \in K_f$ which satisfy $a = s' \cap e^{-1}(y)$ is a face of s', and s satisfies $a = s \cap e^{-1}(y)$ then s is a canonical representation of a.

23 LEMMA : For $a \in J_y$ given above there exists a canonical representation, s of a and if $b \in J_y$ is a face of a and t is a canonical representation of b then t is a face of s.

Proof : If $s, s' \in K_f$ and $s \cap \ell^{-1}(y) = s' \cap \ell^{-1}(y) = a$ then $(s \cap s') \cap \ell^{-1}(y) = a$ but $s \cap s'$ is a face of s and s' so since K is finite.

$$\cap \{ s \mid s \cap \ell^{-1}(y) = a,\ s \in K \}$$

is the required canonical representation of a. For the second part of the Lemma we note that $b \subset a$ so $b \cap a = b$ which gives $(s \cap t) \cap \ell^{-1}(y) = t \cap \ell^{-1}(y)$, but t is a canonical representaion of b so $t \subset s \cap t \Rightarrow t = s \cap t$ so t is a face of s by the definition of cell complexes.

The Euclidean simplicial complex T and the map $\bar{\ell}_s^{-1}$ gives a triangulation (T_s, k_s) of $\pi(s)$.

24 LEMMA : Given the cells $s_1, s_2 \in K_f$ and the above triangulations of $\pi(s_1)$ and $\pi(s_2)$ then if $\pi_1(s_1) \cap \pi(s_2) \neq \phi$, $k_{s_1}^{-1}(\pi(s_1) \cap \pi(s_2))$ and $k_{s_2}^{-1}(\pi(s_1) \cap \pi(s_2))$ are sub simplicial complexes of T_{s_1} and T_{s_2} respectively.

Proof : First note that if $s \in K_f$ and t is a face of ~~f~~ s then $\bar{\ell}_t^{-1} = \bar{\ell}_s^{-1} | \ell(t)$. This is true because for $a \in \ell(t)$ there exists an $x \in t$ such that $\ell(x) = a$ which gives

$$\bar{\ell}_s^{-1}(a) = \pi(x) = \bar{\ell}_t^{-1}(a).$$

Now consider the hypothesis of the Lemma, if $b \in \pi(s_1) \cap \pi(s_2)$ there exists an $x_1 \in s_1$ and $x_2 \in s_2$ such that $x_1 \sim x_2$ and $\pi(x_1) = b$. We thus have an arc $h : I \longrightarrow J_{\ell(x_1)}$ with end points x_1 and x_2. This gives, by using the simplicial approximation theorem, a sequence $\sigma_1, \ldots, \sigma_n$ of cells in $J_{\ell(x_1)}$ such that the faces $\sigma_i \cap \sigma_{i+1} \neq \phi$ for $i = 1$ to $n-1$ and $x_1 \in \sigma_1$, $x_2 \in \sigma_n$. Let $\hat{\sigma}_i$ be the canonical representation of σ_i in K_f and f_i the canonical representation of $\sigma_i \cap \sigma_{i+1}$. Since $\hat{\sigma}_1 \ni x_1$, $\hat{\sigma}_n \ni x_2$ and f_1, f_{n-1} are faces of $\hat{\sigma}_1$ and $\hat{\sigma}_n$ respectively, f_1 and f_{n-1} are respectively faces of S_1 and S_n and $\ell(x_1) \in \ell(f_1) \cap \ell(f_{n-1})$. We thus have $\bar{e}_{f_1}^{-1} | \ell(f_1) \cap \ell(f_{n-1}) = \bar{e}_{S_1}^{-1} | \ell(f_1) \cap \ell(f_{n-1})$ etc.

But consider

$$\mathcal{B} = \bigcap_{i=1}^{n-1} \ell(f_i) \quad .$$

we have $f_i \cap \ell^{-1}(x_1) \neq \phi$ so $\ell(x_1) \in \mathcal{B}$, f_2 is a face of $\hat{\sigma}_2$ as well as f_1 so

$$\bar{e}_{S_1}^{-1} | \mathcal{B} = \bar{e}_{f_1}^{-1} | \mathcal{B} = \bar{e}_{\hat{\sigma}_2}^{-1} | \mathcal{B} = \bar{e}_{f_2}^{-1} | \mathcal{B}$$

and using this as an inductive step, by induction

$$\bar{\ell}_{S_1}^{-1} | \mathcal{B} = \bar{e}_{f_{n-1}}^{-1} | \mathcal{B} = \bar{\ell}_{S_2}^{-1} | \mathcal{B} \, ,$$

so we have $\bar{\ell}_{S_1}^{-1} (\mathcal{B}) \subset \Pi(S_1) \cap \Pi(S_2)$ and since $\mathcal{B}$ is a sub complex of $|T|$ in the sense that $\mathcal{B}$ is a union of cells of T. Hence $k_{S_1}^{-1} (\bar{\ell}_{S_1}^{-1} (\mathcal{B}))$ and $k_{S_2}^{-1} (\bar{\ell}_{S_2}^{-1} (\mathcal{B}))$ are sub simplicial complexes of T_{S_1} and T_{S_2} respectively. Since $\sigma \in \bar{e}_{S_1}^{-1} (\mathcal{B})$ and σ was an arbitary member of $\Pi(S_1) \cap \Pi(S_2)$ we have - noting that a union of arbitary family of sub simplicial complexes is a sub simplicial complex - the required result.

Lemma 24 gives the following:

25 COROLLARY: $K_{f/\sim}$ is a finite polyhedron with a triangulation which has dimension ≤ 2 .

The proof is by direct application of Lemma 24.

We will now indicate, before proving theorem 9, how the above results can be extended to $\Gamma(\bar{\Delta}^s)$.

26 LEMMA : Given for $i=1$ to S the continuous maps $h_i : X \longrightarrow \Gamma^1$ which satisfy $R \circ h_i = R \circ h_j$ for all $i,j = 1$ to S then for $x \in X$ and

$$g(x)(i \leq j) = h_j(x) \circ h_i^{-1}(x) \;,\; h_1(x) = R \circ h_S(x)$$

$g(x)$ is a functor from $C_{\bar{\Delta}^{s-1}}$ to Γ^1 (that is $g(x) \in \Gamma^1(\bar{\Delta}^{s-1})$) and $g : X \longrightarrow \Gamma^1(\bar{\Delta}^{s-1})$ defined by $x \longmapsto g(x)$ is continuous.

Proof : Let $x \in X$ and N be a neighbourhood of $g(x)$ in $\Gamma^1(\bar{\Delta}^{s-1})$, then by the definition of the topology on $\Gamma^1(\bar{\Delta}^{s-1})$ given in Chapter 2 Section 1 there exists open sets $N(i \leq j)$ in Γ^1 such that

$$g(x) \in \bigcap_{i \leq j} \pi_{i \leq j}^{-1} \left(N(i \leq j)\right) \subset N$$

Γ^1 is a complete sub topological groupoid of Γ_q^0 and the h_i's are continuous, so there exists open subsets V_{1i} and V_{2i} of $\mathbb{R}^q$, neighbourhood U of x, and homeomorphisms $d_i : V_{1i} \longrightarrow V_{2i}$ which satisfy, for $y \in U$, $h_i(y) = \text{Germ}\,(R \circ h_i(y), d_i)$ and $\text{Germ}\,(V_{1i}, d_i) \subset N(1 \leq i)$,(note that $g(1 \leq i) = h_i$). By suitable restrictions (such as putting $V = \bigcap V_{1i}$) we can make the V_{1i} independent of i and equal to say V . Germ $(V_{2i}, d_j \circ d_i^{-1})$ is a neighbourhood of $g(x)(i \leq j)$ so by restricting V further we can choose V so that Germ $(V_{2i}, d_j \circ d_i^{-1}) \subset N(i \leq j)$ for all i and j and V is an open neighbourhood of $R \circ h_i(x)$. Hence for $U' = U \cap (R \circ h_1)^{-1}(V)$; $x \in U'$,and U' is open and for $y \in U'$ $g(i \leq j)(y) \in N(i \leq j)$ so $g(U') \subset N$, and since x is an arbitary member of X , g is continuous.

If the maps h_i satisfy the hypothesis of Lemma 26 for a topological space then g is the <u>derived</u> map into $\Gamma^1(\bar{\Delta}^{s-1})$.

27 LEMMA : Let P be a compact locally compact hausdorff topological space and for $s \geq 0$ $f : P \longrightarrow \Gamma(\bar{\Delta}^s)$ be continuous then there exists an open neighbourhood W of $(1 \times \pi_{1 \leq 1} \circ f)(\Delta(P))$ in $P \times \mathbb{R}^2$, where $\Delta : P \longrightarrow P \times P$ is the diagonal map, and a continuous map $g : W \longrightarrow \Gamma(\bar{\Delta}^s)$ such that

a) $g \circ (1 \times \pi_{1 \leq 1} \circ f) \circ \Delta = f$

b) if $h : I \longrightarrow W \cap \ell^{-1}(x)$ for some $x \in \mathbb{R}^2$ is continuous then $g(h(0)) = g(h(1))$

Proof : For $i = 1$ to $s+1$ the maps $\pi_{1 \leq i} : \Gamma(\bar{\Delta}^s) \longrightarrow \Gamma$ are continuous so there exists by Lemma 10 continuous maps

$$h_i : U_{\delta(\pi_{1 \leq i} \circ f)}(\pi_{1 \leq 1} \circ f) \longrightarrow \Gamma$$

such that

a) $h_i \circ (1 \times \pi_{1 \leq 1} \circ f) \circ \Delta = \pi_{1 \leq i} \circ f$

b) if $h : I \longrightarrow U_{\delta(\pi_{1 \leq i} \circ f)}(\pi_{1 \leq 1} \circ f) \cap \ell^{-1}(x)$ for some $x \in \mathbb{R}^2$ is continuous, then $h_i(h(0)) = h_i(h(1))$.

Now if we put $\delta = \min_i \{ \delta(\pi_{1 \leq i} \circ f) \}$ and $W = U_\delta(\pi_{1 \leq 1} \circ f)$ then by applying Lemma 26 we get by putting g equal to the derived map of $\{h_i\}$ the required result.

Using Lemma 27, and putting $R \circ f = \pi_{1 \leq 1} \circ f$ for $f : P \longrightarrow \Gamma(\bar{\Delta}^s)$ we can generalise Lemma 10 to apply to $\Gamma(\bar{\Delta}^s)$ in place of Γ. This gives the corresponding generalisation of Lemmas 11, 19, 20, 21, 23 and 24 and lastly of Corollary 25.

Proof of Theorem 9

Let $\{z\} \in H_{n+q}(\Gamma(\bar{\Delta}^s))$ where $n \geq 1$ then by Lemma 13 there exists a carrier polyhedron P with cocycle z' for z ; we have there exists a continuous $f : P \longrightarrow \Gamma(\bar{\Delta}^s)$ such that $f_*(\{z'\}) = \{z\}$. By the generalisation of Lemma 19 Corollary 25 there exists a simplicial complex K_f and maps $\bar{f}, f^{\#}, \hat{s}_1$ such that

$$\begin{tikzcd} P \arrow[rr, "f^{\#}"] \arrow[dr, "\bar{f}"'] & & \Gamma(\bar{\Delta}^s) \\ & {|K_f|/\sim} \arrow[ur, "\hat{s}_1"'] & \end{tikzcd}$$

commutes,

dimension of $K_f/\sim \leq q$, and $f^{\#}$ is homotopic to f . Hence

$$\begin{aligned} \{z\} &= f_*(\{z'\}) \\ &= f^{\#}_*(\{z'\}) \\ &= \hat{s}_{1*}(\bar{f}_*\{z'\}) \\ &= \hat{s}_*(0) \\ &= 0 \end{aligned}$$

Since z was an arbitary cocycle the theorem now follows.

Theorem 9 shows a property of tubular topological groupoids which can be realised in the case that Γ is an open sub topological groupoid of Γ°_q : the proof of the theorem being easily extended to such topological groupoids in general. The next section will concentrate on computing the low dimensional homotopy groups of $\Gamma(\bar{\Delta}^s)$ when Γ is the topological groupoid of germs of local diffeomorphisms.

2 The Low Dimentional Homotopy Groups of Γ_q^r.

Γ_q^r the groupoid of germs of C^r diffeormorphisms has two components corresponding to whether the germs were derived from orientation preserving or orientation reversing diffeomorphisms. We shall cencentrate on $\bar{\Gamma}_q^r$ the orientation preserving component of Γ_q^r, to simplify our treatment. In $\bar{\Gamma}_q^r$ we will define once and for all the base point γ_0 as $\gamma_0 = \text{Germ}(0, 1)$ where $1 : \mathbb{R}^q \to \mathbb{R}^q$ is the identity map.

Let $e \in GL_q$ be the identity element of the general linear group GL_q of $\mathbb{R}^q$, then we can define a map $\nu : (\bar{\Gamma}_q^r, \gamma_0) \to (GL_q, e)$ by essentially taking the differential of the germs: For $\gamma \in \bar{\Gamma}_q^r$, let $h : U \to V$ be a C^r diffeomorphism of the open sets U, V in $\mathbb{R}^q$ such that $R(\gamma) \in U$ and $\gamma = \text{Germ}(R(\gamma), h)$. Let $dh|_{R(\gamma)}$ be the differential of h at $R(h)$. Now put $\nu(\gamma)$ as the non singular matrix with entries $\nu(\gamma)_{ij}$ such that for co-ordinate maps

$$\pi_i : (x_1, \dots x_q) \longmapsto x_i$$

$$dh|_{R(\gamma)} \left(\frac{\partial}{\partial \pi_i}\right)\Big|_{R(\gamma)} = \sum_j \nu(\gamma)_{ji} \frac{\partial}{\partial \pi_j}\Big|_{L(\gamma)}$$

it is easily checked that if $\gamma' \in \bar{\Gamma}_q^r$ is such that $\gamma \circ \gamma'$ is well defined then

$$\nu(\gamma \circ \gamma') = \nu(\gamma)\nu(\gamma')$$

and that $\nu : (\bar{\Gamma}_q^r, \gamma_0) \to (GL_q, e)$ is continuous for $r \geq 1$.

Let $\overline{GL}_q$ be the component of GL_q which is arc wise connected to the identity element. We will prove:

1 THEOREM : Given $q \geq 1$ and the map ν described above,

$$\nu_\# : \pi_t(\bar{\Gamma}_q^r) \longrightarrow \pi_t(\overline{GL}_q)$$

is an isomorphism for $r \geq 1$, $t < q$.

Where $\pi_t(X)$ is the t^{th} homotopy group of a path connected space; If X is not path connected then the homotopy group will depend upon which component the base point is in.

The definition of ν can be extended to Γ_q^r and the above theorem gives the Corollary:

2 Corollary : For $x_0 \in \Gamma_q^r$

$$\nu_{\#} : \pi_t\,(\Gamma_q^r, x_0) \longrightarrow \pi_t\,(GL_q, \nu(x_0))$$

is an isomorphism for $r \geq 1$, $t < q$

Proof: Γ_q^r has exactly two components, as GL_q does. Let $r \in GL_q$ be a reflection say

$$r : (x_1, \dots x_q) \longmapsto (-x_1, x_2, \dots x_q)$$

then define $\bar{r} : \Gamma_q^r \longrightarrow \Gamma_q^r$ as the map

$$\bar{r} : \gamma \longmapsto \mathrm{Germ}(r(R(\gamma)), r) \circ \gamma$$

obviously $\bar{r}$ is continuous and $\bar{r}^2 =$ identity.

Also define $\hat{r} : GL_q \longrightarrow GL_q$ as the map

$$\hat{r} : g \longmapsto rg$$

then we have

$$\begin{tikzcd} \Gamma_q^r \arrow[r, "\bar{r}"] \arrow[d, "\nu"] & \Gamma_q^r \arrow[d, "\nu"] \\ GL_q \arrow[r, "\hat{r}"] & GL_q \end{tikzcd}$$

commutes and $\hat{r}^2 =$ identity.

Now $\bar{r}$ and $\hat{r}$ map the components homeomorphicaly to each other so if $\overline{GL_q}$ is the component containing e we have

$$\begin{tikzcd} \pi_t(\bar{\Gamma}_q^r) \arrow[r, "\nu_*"', "\simeq"] \arrow[d, "\simeq"', "\bar{r}_*"] & \pi_t(\overline{GL_q}) \arrow[d, "\simeq"', "\tilde{r}_*"] \\ \pi_t(\bar{r}_*(\bar{\Gamma}_q^r)) \arrow[r, "\nu_*"] & \pi_t(\tilde{r}(\overline{GL_q})) \end{tikzcd}$$

is a diagram of isomorphisms by applying Theorem 1, for $t < q$, $r \geq 1$.

In proving Theorem 1 we will rely heavily upon the use of submersion and imersion theory which has been developed for the C^{∞} case, but the C^{r} case can be included as a corollary of the C^{∞} case by using a smooth Lemma. So we will first concentrate upon the C^{∞} case.

When dealing with submersions we will make use of the usual constructions such as C^{∞} differentiable manifolds, bundles A over M, spaces of regular maps and the exponential map given in [MI2]. We will also use the notation given in A. Phillips' paper on submersions of open manifolds [PH1].

Let $S_q^r = \mathrm{Sub}(S^r \times \mathring{D}^{q-r}, \mathbb{R}^q)$ be the space (with C^1 topology) of submersions of ~~the disc~~ $S^r \times \mathring{D}^{q-r}$ into $\mathbb{R}^q$, that is $a \in S_q^r$ is a smooth map $a: S^r \times \mathring{D}^{q-r} \to \mathbb{R}^q$ such that the differential da has maximum rank and is thus nonsingular. Also put $i: S^r \longrightarrow S^r \times \mathring{D}^{q-r}$ as the inclusion $i: x \longmapsto (x, 0)$. Then we can construct a , not necessarily continuous, map

$$G: S_q^r \times S_q^r \longrightarrow \mathrm{Fun}(S^r, \Gamma_q^{\infty})$$

(where $\mathrm{Fun}(S^r, \Gamma_q^{\infty})$ is the space of maps from S^r to Γ_q^{∞}) as follows: Let $(a, b) \in S_q^r \times S_q^r$ then for $x \in S^r$ there exists an open neighbourhood U of $i(x)$ in $S^r \times \mathring{D}^{q-r}$ such that $a|U$ and $b|U$ are C^{∞} embeddings of U to the open sets $a(U)$ and $b(U)$ respectively. Put $D: a(U) \to b(U)$ as the map $D: y \longmapsto b|U \circ (a|U)^{-1}(y)$ and put $G(a,b) = \mathrm{Germ}(a(i(x)), D)$. $G(a,b)$ by its construction and the germ topology on Γ_q^r is continuous.

We are interested in the groups $\pi_t(\Gamma_q^{\infty}, \gamma_0)$ where γ_0 is a base point which maps to e under ν.

$$S^n = \{(x_1, \dots x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$$

put $* \in S^v$ as the base point $* = (1,0,0,\cdots 0)$, and reguard $\pi_n(X, x_0)$ for a topological space X as the homotopy classes of continuous maps $f:(S^n,*) \longrightarrow (X, x_0)$ then we can restrict Fun $(S^v, \overline{T}_q^\infty)$ to these maps which generate $\pi_n(\overline{T}_q^\infty)$, to give $\overline{\text{Fun}}\,(S^v, \overline{T}_q^\infty)$. We need a corresponding version for S_q^v. We have a special member $e \in S_q^v$ given by mapping to an open tubular neighbourhood of the canonical embedding of S^v into $\mathbb{R}^q$ for $v<q$ as follows: Let $P_1 : \overset{\circ}{D}{}^{q-v} \to \mathbb{R}$ be the projection into the first component of $\mathbb{R}^{q-v}$, $P_2 : \overset{\circ}{D}{}^{q-v} \longrightarrow \mathbb{R}^{q-v-1}$ the projection $(x_1, \cdots x_{q-v}) \longrightarrow (x_2, \cdots x_{q-v})$ and $\mathbb{R}^q$ be identified with $\mathbb{R}^{v+1} \times \mathbb{R}^{q-v-1}$ then for $(x,y) \in S^v \times \overset{\circ}{D}{}^{q-v}$ put

$$e(x,y) = (x(1+P_1(y)),\ P_2(y))$$

Let, for $\epsilon > 0$, $j_\epsilon : D^q \to \mathbb{R}^q$ be the smooth embedding $j_\epsilon : x \longmapsto \epsilon x + e(*)$ There exists an $\zeta_1 > 0$ such that $j_\epsilon(D^q) \subset e(S^v \times \overset{\circ}{D}{}^{q-v})$ for $\zeta_1 \geq \epsilon > 0$. For such an ϵ there exists a unique smooth embedding $J_\epsilon : D^q \longrightarrow S^v \times \overset{\circ}{D}{}^{q-v}$ such that $j_\epsilon = e \circ J_\epsilon$.
For $\epsilon > 0$ such that $\zeta_1 > \epsilon$. Let

$$\overline{S}{}_q^v(\epsilon) = \{ a \in S_q^v \mid a | J_\epsilon(D^q) = e | J_\epsilon(D^q) \}$$

For convenience later on ,let $\{\zeta_n\}$ be the sequence

$$\{\zeta_n > 0 \mid \zeta_i = \zeta_1/n,\ n \in \mathbb{N}\}$$

then we get a sequence of inclusions

$$\overline{S}{}_q^v(\zeta_1) \subset \overline{S}{}_q^v(\zeta_2) \subset \cdots\cdots$$

Put $\overline{S}{}_q^v = \bigcup_{i=1}^\infty \overline{S}{}_q^v(\zeta_i)$, $\overline{S}{}_q^v$ is the set of all $a \in S_q^v$ which agree with e on some neighbourhood of $(*, 0) \in S^v \times \overset{\circ}{D}{}^{q-v}$. However $\overline{S}{}_q^v$ with the topology induced from S_q^v is too weak for the desired properties that we will need so we will adopt the expedient of slightly changing the topology of $\overline{S}{}_q^v$. $\overline{S}{}_q^v(\zeta_i)$ is closed because if $a \notin \overline{S}{}_q^v(\zeta_i)$ then there exists an $x \in J_{\zeta_i}(D^q)$ such that $a(x) \neq e(x)$, but $N = \{ b \in S_q^v \mid b(x) \neq e(x) \}$ is a neighbourhood of a which does not intersect $J_{\zeta_i}(D^q)$ Define the new topology on $\overline{S}{}_q^v$ as follows:

$C \subset \bar{S}^v_q$ is closed if and only if $C = \bar{S}^v_q$ or there exists an $i \in \mathbb{N}$ such that $C \subset \bar{S}^v_q(\varepsilon_i)$ and is closed in $\bar{S}^v_q(\varepsilon_i)$. Note that $\bar{S}^v_q(\varepsilon_k) \subset \bar{S}^v_q$ is an embedding in the new topology so $\bar{S}^v_q$ is Hausdorff. Furthermore if K is compact and $f : K \longrightarrow \bar{S}^v_q$ is continuous then there exists an $i \in \mathbb{N}$ such that $f(K) \subset \bar{S}^v_q(\varepsilon_i)$ This is the required property for $\bar{S}^v_q$ in the weaker topology $\bar{S}^v_q$ does not have this nice property Let $\bar{D}^v = \{(x_1, \ldots x_{v+1}) \in S^v \mid x_1 \geq 0\}$ then we will choose ε_1 small enough so that

$$\bar{J}_{\varepsilon_1}(D^q) \subset \bar{D}^v \times \mathring{D}^{q-v} .$$

This condition will be used later to simplify a proof.

If $\bar{G} = G \mid \bar{S}^v_q \times \bar{S}^v_q$ we get that

$$\bar{G} : \bar{S}^v_q \times \bar{S}^v_q \longrightarrow \overline{Fun}\,(S^v, \bar{T}^\infty_q)$$

We shall now work towards showing some properties of $\bar{G}$ that will be used in the proof of Theorem 1.

Let $i : S^v \longrightarrow S^v \times \mathring{D}^{q-v}$ be the map $i : x \longmapsto (x, 0)$. Then we have

3 LEMMA : Let $f \in$ Fun $(S^v, \mathbb{R}^q)$, $\epsilon > 0$, and $v < q$. Then there exists an $a \in S^v_q$ such that

a) a is connected to e by an arc $H : I \longrightarrow S^v_q$

b) for $(x, y) \in S^v \times \mathring{D}^{q-v}$ $(x, a(x, y)) \in U_\epsilon(f)$

Proof: $\mathbb{R}^q$ is contractable so by using the proof of Theorem 5·10 in [HI1] there exists an arc $h_0 : I \longrightarrow Reg(S^v, \mathbb{R}^q)$ into the space of smooth regular maps of S^v into $\mathbb{R}^q$ with the C^1 topology such that $h_0(0) = e \circ i$ and $h_0(1) \in \bar{B}_{\epsilon/2}(f)$. Now for each $x \in Reg(S^v, \mathbb{R}^q)$ there is associated a normal bundle given as follows; using standard fibre bundle terminology [ST1] there exists an embedding which is a fibre bundle homomorphism $e'_x : TS^v \longrightarrow x^*(T\mathbb{R}^q)$ of the tangent bundle of S^v into the pullpack of $T\mathbb{R}^q$ (the tangent bundle of $\mathbb{R}^q$) by x , given as follows: For $y \in S^v$ and $v \in TS^v$ in the fibre of y we put $e'_x(v) = (x, d_x(v))$ where d_x is the differential of x . The Normal bundle N_x for $x \in \mathrm{Reg}(S^v, \mathbb{R}^q)$ is the quotient bundle $x^*(T\mathbb{R}^q)/e'_x(TS^v)$. If we use the standard metric tensor in $T\mathbb{R}^q$ given by identifying each fibre of $T\mathbb{R}^q$ with $\mathbb{R}^q$ and lifting up the scalar product of vectors in $\mathbb{R}^q$, then we can pull the metric back to a unique metric on $x^*(T\mathbb{R}^q)$ and identify N_x with the subbundle $N_{\perp x}$ perpendicular to $e'(TS^v)$. Arcs in a space give rise to homotopies and vica versa by using the exponential correspondence theorem [SP1] so we will use the same symbol to denote either case. Because h_0 is continuous in the C^1 topology we can construct a bundle homomorphism which is an embedding $e'' : TS^v \times I \longrightarrow h_0^*(T\mathbb{R}^q)$ and for fixed $t \in I$ restricts to the above embedding of TS^v for $h_0(t)$. Again using a metric induced from $T\mathbb{R}^q$ we have a bundle N over $S^v \times I$ which is the set of vectors perpendicular to $TS^v \times I$ in $h_0^*(T\mathbb{R}^q)$. Since I is contractible there exists an isomorphism of bundles

$$j : N_{\perp h_0(0)} \times I \dashrightarrow N$$

such that j_0 is the identity on $N_{\perp h(0)}$. N is a manifold with boundary with the usual differential structure derived from S^v; [ST1]. It can be seen by inspecting the isomorphism that it can be chosen to be smooth. Take j as smooth. Then for fixed $t \in I$ we can construct a submersion $h_t : S^v \times \mathring{D}^{q-v} \longrightarrow \mathbb{R}^q$ which restricts to $h_0(t)$ in the following way. Let DX be the disc bundle for a vector bundle X given by $DX = \{v \in X \mid g(v, v) < 1\}$ for a specified metric g on X . Then since S^v is compact and $h_0(t)$ is a local embedding there exists an δ' satisfying $\epsilon/2 > \delta' > 0$ such that, for $x \in S^v$ and $v \in DN$ in the fibre of x, $\phi : (x, v) \longmapsto \exp_x \delta' v$ gives a submersion of DN into $\mathbb{R}^q$, where $\exp$ is the exponential map for the standard metric on $T\mathbb{R}^{q}$. Now $j_t : DN_{\perp h_0(t)} \to N$ is a smooth embedding so $\bar{h}_t = \phi' \circ j_t$ is a submersion of . Since j and $\exp$ are smooth maps we have $\bar{h} : DN_{\perp h_0(0)} \times I \to \mathbb{R}^{q}$ gives an arc in $\mathrm{Reg}(DN_{\perp h_0(0)}, \mathbb{R}^{q})$. Now h_0 gives a submersion which has the property: There exists a smooth isomorphism

$$\bar{r} : S^{r} \times \overset{\circ}{D}{}^{q-r} \longrightarrow DN_{\perp h_0(0)}$$

such that if $\delta_1 : S^{r} \times \overset{\circ}{D}{}^{q-r} \longrightarrow S^{r} \times \overset{\circ}{D}{}^{q-r}$ is the map $(x, v) \longmapsto (x, \delta' v)$ then the diagram

$$\begin{tikzcd} S^{r} \times \overset{\circ}{D}{}^{q-r} \arrow[r, "\delta_1"] \arrow[d] & S^{r} \times \overset{\circ}{D}{}^{q-r} \arrow[d, "e"] \\ DN_{\perp h_0(0)} \arrow[r, "\bar{h}_0"] & \mathbb{R}^{q} \end{tikzcd}$$

commutes so we get the arc $h \in S_q^{r}$ given by $h_t = \bar{h}_t \circ \bar{r}$. By "expanding" $e \circ \delta_1$, to e we can extend h to an arc H that connects e with $\bar{h}_1 \circ \bar{r}$ in S_q^{r}. It can now be checked that $\bar{h}_1 \circ \bar{r}$ is the required $\alpha \in S_q^{r}$.

4 LEMMA : In Lemma 3 S_q^{r} can be replaced by $\bar{S}_q^{r}$, if $f(*) = e(*)$.

Proof : For the moment let $\bar{\varepsilon} > 0$ be given. Let $D^{q} = \{x \in \mathbb{R}^{q} \mid \|x\| \leq 1\}$, $T_q D^{q}$ be the bundle of q-frames, that is, the ordered sets $(v_1, \ldots v_q)$ of linearly independent vectors in TD^{q}, and $\mathrm{Sect}(T_q D^{q})$ be the space of sections with the compact open topology. Put $j_{\bar{\varepsilon}} : D^{q} \to \mathbb{R}^{q}$ as the smooth embedding $y \longmapsto \bar{\varepsilon} y + e(*)$ Since j_{ε} is an embedding there exists a feild of q-frames $(v_1, \ldots, v_q)$ given by $dj_{\varepsilon}(v_1, \ldots v_q)_x = \left(\frac{\partial}{\partial \pi_1}, \ldots, \frac{\partial}{\partial \pi_q}\right)_x$ where π_i is the projection of the i^{th} co-ordinate to $\mathbb{R}$ from $\mathbb{R}^{q}$ and the subscript x means " at the point x." If $s : x \longmapsto (u_1, \ldots u_q)_x$ is a section of $T_q D^{q}$ then there exists a unique element $g \in GL_q$ such that

$$(v_j)_0 = \bar{\epsilon}\, Ad(g)\,(u_j)_0 \ ; \ Ad(g)_{ik} = g_{ki}$$

where 0 is the centre of the disc D^q. This gives a correspondence θ: Sect $(T_q D^q) \longrightarrow GL_q$ which is a homotopy equivalence; the homotopy inverse being

$$\theta' : g \longmapsto (u_j)$$

where $(u_j)_x = Ad(g)^{-1}(v_j)_0$ Let $\psi : GL_q \longrightarrow Sub(D^q, \mathbb{R}^q)$ be the map that assigns to $g \in GL_q$ the submersion $y \longmapsto e(x) + \bar{\epsilon} g y$ and $\overline{Sub}(D^q, \mathbb{R}^q)$ the immage of ψ.

$$\psi : GL_q \longrightarrow \overline{Sub}(D^q, \mathbb{R}^q)$$

is a homeomorphism and hence a homotopy equivalence. Let M be an open manifold of dimension $m = q$. For $f \in Sub(M, \mathbb{R}^q)$ the space of submersions of M to $\mathbb{R}^q$, put $\nabla f \in$ Sect $(T_q M)$ as given by

$$(\nabla f)_x = \left(\frac{\partial}{\partial \pi_1 \circ f}, \ldots, \frac{\partial}{\partial \pi_q \circ f}\right)_x$$

then by [PHI] Theorem B, ∇ is a weak homotopy equivalence, where for topological spaces X, Y and map $f : X \to Y$ which is continuous f is a weak homotopy equivalence if it induces a bijection between path components and for each $x \in X$, $\pi_k(X, x) \longrightarrow \pi_k(Y, f(x))$ is an isomorphism for $k > 0$.

The diagram

$$\begin{array}{ccc} Sub(D^q, \mathbb{R}^q) & \xrightarrow{\nabla} & Sect(T_q D^q) \\ \cup & & \uparrow \theta' \\ \overline{Sub}(D^q, \mathbb{R}^q) & \xleftarrow{\psi} & GL_q \end{array}$$

is a commutative diagram of weak homotopy equivalences, so we get

$$\overline{Sub}(D^q, \mathbb{R}^q) \subset Sub(D^q, \mathbb{R}^q)$$

is a weak homotopy equivalence for each $\bar{\epsilon} > 0$. Now there exists an embedding $J_{\bar{\epsilon}} : D^q \longrightarrow S^v \times B^{q-v}$ such that $j_{\bar{\epsilon}} = e \circ J_{\bar{\epsilon}}$ if $\bar{\epsilon}$ is sufficiently small. We will assume that $\bar{\epsilon}$ has been chosen for this to be true- this always being possible. Then we have a continuous map $\pi : S^v_q \longrightarrow Sub(D^q, \mathbb{R}^q)$ given by $\pi : a \longmapsto a \circ J_{\bar{\epsilon}}$, put $\hat{S}^v_q = \pi^{-1}(\overline{Sub}(D^q, \mathbb{R}^q))$ then by [PH1] Lemma 4•1 $\pi : S^v_q \longrightarrow Sub(D^q, \mathbb{R}^q)$ has the covering homotopy property because $S^v \times \mathring{D}^{q-v}$ is constructed from $J_{\bar{\epsilon}}(D^q)$ by thickening up an added handle of index $\leq q-1$ .Consider the diagram

$$\begin{array}{ccc} \hat{S}^v_q & \subset & S^v_q \\ \downarrow \pi & & \downarrow \pi \\ \overline{Sub}(D^q, \mathbb{R}^q) & \subset & Sub(D^q, \mathbb{R}^q) \end{array}$$

The vertical maps have the covering homotopy property and the base map is a weak homotopy equivalence. Also the fibres are mapped homeomorphically to fibres so applying [PH1] Lemma 1 in Appendix I, the inclusion $\hat{S}^v_q \subset S^v_q$ is a weak homotopy equivalence. By Lemma3 there exists an $b \in S^v_q$ and arc h connecting b to e in S^v_q such that for $(x,y) \in S^v \times \mathring{D}^{q-v}$, $(x, b(x,y)) \in U_{\bar{\epsilon}/21}(f)$.

Now let

$$V_p = \{x \in \mathbb{R}^q \mid \|x - f(*)\| \leq \bar{\epsilon}/p\} \quad \text{for } p>0,\ V = V_{21}$$

and $\bar{V} = \mathbb{R}^q - V$. V is a compact manifold with boundary and $b(*) \subset V - \partial V$. Let $Aut(\mathbb{R}^q, \bar{V})$ be the space of diffeomorphisms with the C^1 topology that leaves $\bar{V}$ fixed. Then by constructing a vector field that agrees with the vector $f(*) - b(*)$ in a disc containing $b(*)$ and $f(*)$ in V and is zero in some neighbourhood of ∂V and intergrating it we can, since V can be extended to a compact manifold without boundary, construct a continuous map $\lambda : I \longrightarrow Aut(\mathbb{R}^q, \bar{V})$ such that $\lambda(0)$ is the identity and $\lambda(1)(b(*)) = f(*) = e(*)$. Now consider the map $b' : I \longrightarrow S^v_q$ given by $b'(t) = \lambda(t) \circ h(t)$. b' is continuous, $b'(0) = e$, $b'(1)(*) = e(*)$, and for $(x,y) \in S^v \times \mathring{D}^{q-v}$ $(x, b'(1)(x,y)) \in U_{\bar{\epsilon}/q}(f)$ Let $Sub_*(D^q, \mathbb{R}^q)$ be the subspace of submersions of D^q to $\mathbb{R}^q$ which maps 0 to $e(*)$

then we will need the following fact which will be used twice:

5 SUBLEMMA : If $h_2, h_2' : I \longrightarrow \mathrm{Sub}_*(D^q, \mathbb{R}^q)$ are arcs, $h_2(0) = h_2'(0)$, there exists an arc $\hat{h}_2 : I \longrightarrow \mathrm{Sub}_*(D^q, \mathbb{R}^q)$ and a δ satisfying $1 > \delta > 0$ such that for $t \in I$

a) $\hat{h}_2(t)(x) = h_2(t)(x)$ for $|x| > \delta$, $x \in D^q$

b) there exists a neighbourhood U of 0 in D^q such that $\hat{h}_2(t)(x) = h_2'(x)$ for $x \in U$.

c) $\hat{h}_2(0) = h_2(0)$

Proof of sublemma : Put $D^q_{\delta_1} = \{x \in \mathbb{R}^q \mid \|x\| \leq \delta_1\}$. By the compactness of I, the fact that the space of smooth embeddings is an open subset of the space of submersions [MU1] and a submersion is locally an embedding there exists a $\delta_1 > 0$ and a continuous arc $\phi : I \longrightarrow \mathrm{Emb}(D^q_{\delta_1}, \mathring{D}^q)$ into the space of smooth embeddings of $D^q_{\delta_1}$ to $\mathring{D}^q$ with the C^1 topology such that

$$h_2'(t) \mid D^q_{\delta_1} = h_2(t) \circ \phi(t)$$

Put $\bar{\delta} = \sup_{t \in I, x \in D^q_{\delta_1}} \|\phi(t)(x)\|$ then by compactness $1 > \bar{\delta} > 0$. Put $\delta = \frac{1+\bar{\delta}}{2}$ then $1 > \delta > 0$ and $\phi : I \longrightarrow \mathrm{Emb}(D^q_{\delta_1}, \mathring{D}^q_\delta)$. By [PH1] Sublemma 3·3 there exists for each $t \in I$ a neighbourhood D_t of $\phi(t) \in \mathrm{Emb}(D^q_{\delta_1}, \mathring{D}^q_\delta)$ and continuous map $\mu_t : D_t \longrightarrow \mathrm{Aut}(D^q, D^q - \mathring{D}^q_\delta)$ to the space of smooth diffeomorphisms of D^q to D^q that leave $D^q - \mathring{D}^q_\delta$ fixed such that for $g \in D_t$

$$\mu_t(g) \circ \phi(t) = g \quad , \quad \mu_t(\phi(t)) = \text{identity}$$

By continuity for $t \in I$, there exists an $\epsilon_t > 0$ such that for $|t - s| < \epsilon_t$, $\phi(s) \in D_t$. Multiplying by a member of $\mathrm{Aut}(D^q, D^q - \mathring{D}^q_\delta)$ maps $\mathrm{Aut}(D^q, D^q - \mathring{D}^q_\delta)$ homeomorphically to $\mathrm{Aut}(D^q, D^q - \mathring{D}^q_\delta)$ so for a, b satisfying $a < b$, $a, b \in (t - \epsilon_t, t + \epsilon_t)$ we have for $a \leq s \leq b$

$$\mu_{a,b} : s \longmapsto \mu_t(\phi(s)) \circ \mu_t(\phi(a))^{-1}$$

gives an arc in $Aut(D^q, D^q - \mathring{D}^q_\delta)$ from the identity which satisfies

$$\mu_{a,b}(s) \circ \phi(a) = \phi(s)$$

Since I is compact, with the usual metric $d(a,b) = |a-b|$, let δ be the Lebesgue number of the covering $\{(t-\epsilon_t, t+\epsilon_t) | t \in I\}$ then if we choose $n > \frac{1}{\delta}$ we can construct for $i = 1$ to n

$$\mu_{\frac{i-1}{n}, i/n} : [\tfrac{i-1}{n}, i/n] \longrightarrow Aut(D^q, D^q - \mathring{D}^q_\delta)$$

which have the properties of $\mu_{a,b}$ above. The composition of members of $Aut(D^q, D^q - \mathring{D}^q_\delta)$ is continuous, so we can construct

$$\mu : I \longrightarrow Aut(D^q, D^q - \mathring{D}^q_\delta)$$

such that $\mu(t) \circ \phi(0) = \phi$ and $\mu(0) =$ identity by combining the above automorphisms: Define $\mu(t)$ inductively as follows, $\mu(0)$=identity and for $t \in [\frac{i-1}{n}, \frac{i}{n}]$, $\mu(t) = \mu_{\frac{i-1}{n}, i/n}(t) \circ \mu(\frac{i-1}{n})$.
Put $\hat{h}_2 : I \longrightarrow Sub_*(D^q, \mathbb{R}^q)$ as the arc $\hat{h}_2(t) = h_2(t) \circ \mu(t)$ then $\hat{h}_2$ is the required arc.

$b'(1)$ is locally an embedding so there exists an $\epsilon' > 0$, such that $b'(1) \circ J_{\epsilon'}$ is an embedding and $b'(1) \circ J_{\epsilon'}(D^q) \subset V_q$. By using a construction used in the proof of [PH1] Lemma2·1 there exists an arc h_3 in $Sub_*(D^q, \mathbb{R}^q)$ which connects $b'(1) \circ J_{\epsilon'}$ with $\overline{Sub}(D^q, \mathbb{R}^q)$. By using Sublemma 5 we can construct an arc h_4 in $Sub_*(D^q, \mathbb{R}^q)$ that for some neighbourhood of ∂D^q agrees with $b'(1) \circ J'_\epsilon$ for all $t \in I$, $b'(1) \circ J_{\epsilon'} = h_4(0)$, and for some neighbourhood U of 0 agrees with h_3.
Select $\bar{\epsilon}$ so that

$$D^q_{\bar{\epsilon}/\epsilon} \subset U$$

For given $\alpha > 0$ let $R_\alpha : D^q \to D^q$ be the map $x \longmapsto \alpha x$. Construct an arc $b'' : I \longrightarrow S^r_q$ as follows, if $J_{\bar{\epsilon}}^{-1} : J_\epsilon(D^q) \to D^q$ be the inverse then put

$$G''(t)(x) = \begin{cases} G'(1)(x) & \text{FOR } x \notin J_{\epsilon'}(D^q) \\ h_4(t) \circ J_{\epsilon'}^{-1}(x) & \text{FOR } x \in J_{\epsilon'}(D^q) \end{cases}$$

$J_{\bar{\epsilon}}(D^q) = J_{\epsilon'} \circ k_{\bar{\epsilon}/\epsilon'}(D^q) \subset J_{\epsilon'}(U)$ so $G''(1) \in \hat{S}^v_q$

Also $G''(0) = G'(1)$ and because $G' \circ J_{\epsilon'}(D^q) \subset V_q$ and G'' is "constant" except on perhaps $J_{\epsilon'}(D^q)$ for $(x,y) \in S^v \times \mathring{D}^q$

$$(x, G''(1)(x,y)) \in U_{\epsilon/3}(f) .$$

$G''(1)$ as well as being in $\hat{S}^v_q$ is connected by an arc in S^v_q to e which is in $\hat{S}^v_q$. Recall that $\hat{S}^v_q \subset S^v_q$ is a weak homotopy equivalence. so there exists an arc $\bar{G}$ in $\hat{S}^v_q$ that connects e to $G''(1)$. Applying Sublemma 5 again there is an arc $\hat{G}$ in $SuG_*(D^q, \mathbb{R}^q)$ which agrees with $\bar{G} \circ J_{\bar{\epsilon}}$ on some neighbourhood of ∂D^q and agrees with $e \circ J_{\bar{\epsilon}}$ on some neighbourhood V of 0 so define the arc $H : I \longrightarrow S^v_q$ by

$$H(t)(x) = \begin{cases} \bar{G}(t)(x) & \text{FOR } x \notin J_{\bar{\epsilon}}(D^q) \\ \hat{G} \circ J_{\bar{\epsilon}}^{-1}(x) & \text{FOR } x \in J_{\bar{\epsilon}}(D^q) \end{cases}$$

From the construction of $\hat{G}$ we see that H is an arc in $\bar{S}^v_q$ from e and furthermore for $(x,y) \in S^v \times \mathring{D}^{q-v}$, $(x, H(1)(x,y)) \in U_\epsilon(f)$. This gives the required construction for H .

We can now use Lemma 3 and 4 to prove:

6 LEMMA : Given $f \in \overline{Fun}(S^v, \overline{\Gamma}^\infty_q)$ then there exists an $(a, G) \in \bar{S}^v_q \times \bar{S}^v_q$ such that $\bar{G}(a,G)$ is connected to f by an arc in $\overline{Fun}(S^v, \overline{\Gamma}^\infty_q)$, and a is connected to e by an arc in $\bar{S}^v_q$.

Proof : By Lemma 1·10 we have the continuous map $\hat{S} : U_{\delta f}(R \circ f) \to \Gamma$ which has the properties a) and b) of Lemma 1·10. Now by Lemmas 3

and 4 there exists an arc H in $\bar{S}^r_q$ such that $H(0) = e$ and $H(1)$ is such that for $(x,y) \in S^r \times \mathring{D}^{q-r}$, $(x, H(1)(x,y)) \subset U_{\delta f}(R\circ f)$. Define $b : S^r \times \mathring{D}^{q-r} \longrightarrow \mathbb{R}^q$ as the map $b : (x,y) \longmapsto L \circ \hat{S}(x, H(1)(x,y))$ where L is the map obtained by taking the right units of T.
Now $R \circ \hat{S}(x, H(1)(x,y)) = H(1)(x,y)$. Since S constructed from $\hat{S}$ in Lemma 1·10 is a local section, $\hat{S}$ is continuous and T^∞_q has the germ topology; for $(x,y) \in S^r \times \mathring{D}^{q-r}$ there exists some open neighbourhood U of (x,y) in $S^r \times \mathring{D}^{q-r}$ and a smooth diffeomorphism $d : W_1 \longrightarrow W_2$ of open subsets of $\mathbb{R}^q$ such that $H(1)(U) \subset W_1$ and for $(x',y') \in U$, $S(x', H(1)(x',y')) = \mathrm{Germ}(H(1)(x',y'), d)$, but this gives $b|U = d \circ H(1)|U$. Since (x,y) was an arbitary member of $S^r \times \mathring{D}^{q-r}$ we get that $b \in S^r_q$ and $G(H(1), b) = \hat{S} \circ (1 \times H(1)) \circ \Delta \circ i$ which by the convexity of $U_{\delta f}(R \circ f)$ is homotopic to f. Furthermore, since $\hat{S} \circ (1 \times H(1)) \circ \Delta \circ i(*) = \hat{S} \circ (*, e(*)) = \hat{S}(*, f(*)) = \gamma_0$, $b \in \bar{S}^r_q$, because $H(1) \in \bar{S}^r_q$. If we put $a = H(1)$, $(a,b) \in \bar{S}^r_q \times \bar{S}^r_q$ is the required pair.

7 LEMMA : Let $a : I \longrightarrow \bar{S}^r_q \times \bar{S}^r_q$ be an arc then for $t \in I$ $G(a(t))$ is connected to $G(a(0))$ by an arc in $\overline{\mathrm{Fun}}(T^\infty_r)(S^r, \Gamma^\infty_q)$

Proof : Let $N = S^r \times D^{q-r}_{1/2}$, $M = S^r \times D^{q-r}_{1/4}$, $j : S^r \times D^q_{1/4} \longrightarrow S^r \times D^q_{1/2}$ be the inclusion, and $\mathrm{Aut}(N)$ be the space of smooth diffeomorphisms of N that leave, ∂N the boundary of N, fixed. Put for $t \in I$, $(a_1(t), a_2(t)) = a(t)$. By using [PH I] Lemma 3·1 on stability of submersions we get that for each $t \in I$ there exists an $\epsilon_t > 0$ and continuous maps

$$v_1(t), v_2(t) : (t-\epsilon_t, t+\epsilon_t) \longrightarrow \mathrm{Aut}(N) \quad \text{such that}$$

a) $v_1(t)(t) = v_2(t)(t) =$ identity

b) $a_i(s) \circ j = a_i(t) \circ v_i(t)(s) \circ j$ for $i = 1$ or 2, $s \in (t-\epsilon_t, t+\epsilon_t)$

Let $(b_1, b_2) \in \bar{S}^{\nu}_q \times \bar{S}^{\nu}_q$ then we have a continuous map $Q(b_1,b_2) : S^{\nu} \times \overset{\circ}{D}{}^{q-\nu} \longrightarrow \bar{\Gamma}^{\infty}_q$ given by the following construction: For $x \in S^{\nu} \times \overset{\circ}{D}{}^{q-\nu}$ there exists an open neighbourhood U of x such that $b_1 | U : U \to b_1(U)$ and $b_2 | U : U \to b_2(U)$ are diffeomorphisms to open subsets of $\mathbb{R}^q$. Put $Q(b_1,b_2)(x) = \text{Germ}\,(b_1(x), b_2 \circ b_1^{-1})$. $Q(b_1,b_2)$ is obviously continuous and $G(b_1,b_2) = Q(b_1,b_2) \circ i$

By using a relative version of [PHI] Sublemma 3·3 there exists, by suitable adjustment of $\epsilon_t > 0$, a neighbourhood U_t of the disc $\{*\} \times \overset{\circ}{D}{}^{q-\nu}$ and modified v_1, v_2 such that $v_1(t)(s)|U_t$ and $v_2(t)(s)|U_t$ are identity maps on U_t – since a_1 and a_2 are constant on such a neighbourhood. Let $h : I \times S^{\nu} \longrightarrow S^{\nu} \times \overset{\circ}{D}{}^{q-\nu}$ be a homotopy with $h(0) = i$ which collapses a disc of S^{ν} about $*$ into $\{*\} \times \overset{\circ}{D}{}^{q-\nu}$ and moves the rest of S^{ν} to $S^{\nu} \times \overset{\circ}{D}{}^{q-\nu} - N$ then put h_1 and h_2 as the arcs in $\bar{\Gamma}^{\infty}_q$ given by, for $\bar{t} \in I$, $x \in S^{\nu}$, $s \in [t - \epsilon_t, t + \epsilon_t]$

$$h_1(\bar{t})(x) = Q(a_1(t), a_2(t)) \circ h(\bar{t})(x)$$
$$h_2(\bar{t})(x) = Q(a_1(t) \circ v_1(t)(s), a_2(t) \circ v_2(t)(s)) \circ h(\bar{t})(x).$$

Now $h_1(1) = h_2(1)$, $h_1(0) = G(a(t))$, and $h_2(0) = G(a(s))$ so by joining h_1 to h_2 we construct an arc in $\bar{\Gamma}^{\infty}_q$ that connects $G(a(t))$ with $G(a(s))$. We have shown that if $\sim$ is the equivalence relation on I given by $s \sim t$ if there exists an arc connecting $G(a(s))$ with $G(a(t))$ in $\bar{\Gamma}^{\infty}_q$ then the equivalence classes are open subsets of I, But I is connected so there is only one equivalence class and $0 \sim t$.

To shorten the notation put $F^{\nu} = S^{\nu} \times \overset{\circ}{D}{}^{q-\nu}$. Let $T_q F^{\nu}$ be the bundle of q-frames of the tangent space of F^{ν}; $(v_1, \ldots, v_q)_x \in T_q F^{\nu}$ where $v_1, \ldots v_q$ are linearly independent tangent vectors at x.

Let $\text{Sect}(E)$ be the space of sections of a bundle then we have a map $\nabla : S^{\nu}_q \longrightarrow \text{Sect}(T_q F^{\nu})$ given by, for $x \in S^{\nu} \times \overset{\circ}{D}{}^{q-\nu}$, $a \in S^{\nu}_q$

$$(\nabla a)_x = \left(\frac{\partial}{\partial \pi_1 \circ a}, \frac{\partial}{\partial \pi_2 \circ a}, \cdots \frac{\partial}{\partial \pi_q \circ a} \right)_x$$

where $\Pi_1 \circ a, \Pi_2 \circ a, \cdots \Pi_q \circ a$ is reguarded as a set of co-ordinate maps in some neighbourhood of x. Now if the metric on F^v is chosen so that it depends upon the submersion $a \in F^v$ by making it induced by a from $\mathbb{R}^q$ then ∇ defined above coincides with the gradient map used in [PH1] Theorem B. So, because the induced metric is a continuous function of a in the compact open topology, by applying Theorem B, $\nabla : S^v_q \longrightarrow Sect(T_q F^v)$ is a weak homotopy equivalence. For $\epsilon > 0$ such that $\zeta_1 \geq \epsilon$ we have the maps

$$\Pi_\epsilon : S^v_q \longrightarrow Sub(D^q, \mathbb{R}^q), \ \bar{\Pi}_\epsilon : Sect(T_q F^v) \longrightarrow Sect(T_q D^q),$$

where $T_q D^q$ is the bundle of q-frames of tangent vectors of D^q, given by, for $a \in S^v_q$, $\bar{a} \in Sect(T_q F^v)$, $\Pi_\epsilon(a) = a \circ J_\epsilon$ and

$$d J_\epsilon (\bar{\Pi}_\epsilon(\bar{a})) = \bar{a} \mid J_\epsilon(D^q)$$

where dJ_ϵ is the map obtained by taking the differential of J_ϵ and applying it to the vectors of the section $\bar{\Pi}(\bar{a})$. We have the commutative diagram

$$\begin{array}{ccc} S^v_q & \xrightarrow{\nabla} & Sect(T_q F^v) \\ \downarrow \Pi & & \downarrow \bar{\Pi} \\ Sub(D^q, \mathbb{R}^q) & \xrightarrow{\nabla} & Sect(T_q D^q) \end{array}$$

where ∇ on the bottom row is defined in the same way as the top row ∇. By [PH1] Theorem B, [PH1] Lemmas 4·1 5·1 ∇ on the bottom row is a weak homotopy equivalence and the maps Π and $\bar{\Pi}$ have the covering homotopy property. So since ∇ is a weak homotopy equivalence ∇ induces a weak homotopy equivalence between fibres; if we put $Sect_\epsilon(T_q F^v) = \bar{\Pi}^{-1}(\nabla \circ J_\epsilon \circ e)$ then $\nabla : S^v_q(\epsilon) \rightarrow Sect_\epsilon(T_q F^v)$ is a weak homotopy equivalence. Note that $Sect_\epsilon(T_q F^v)$ is the set of sections that agree with ∇e on $J_\epsilon(D^q)$. Let $Fun_\epsilon(S^v, GL_q)$ be the set of all continuous maps of S^v into GL_q that map $J_\epsilon(D^q)$ to the identity element e in GL_q. For each $a \in Sect_\epsilon(T_q F^v)$ and $x \in F^v$ there exists a unique $\mu(a)(x) \in GL_q$ such that

$$\mu(a)(x)\,(\nabla e)_x = (a)_x$$

so by continuity of $\mu(a)(x)$ construction we have a map $\mu : Sect_\epsilon(T_q F^r) \longrightarrow Fun(F^r, GL_q)$ which is continuous. Let $\hat{\mu} : Sect_\epsilon(T_q F^r) \longrightarrow \overline{Fun_\epsilon}(S^r, \overline{GL_q})$ be the map $\hat{\mu}(a)(x) = \mu(a)(i(x))$ for $a \in Sect(T_q F^r)$ and $x \in S^r$.

8 LEMMA : The map $\hat{\mu} : Sect_\epsilon(T_q F^r) \longrightarrow \overline{Fun_\epsilon}(S^r, \overline{GL_q})$ is a homotopy equivalence for $\zeta_1 \geq \epsilon > 0$, $r < q$.

Proof: Let $D : F^r \longrightarrow F^r$ be a diffeomorphism which leaves $S^r \times \{0\}$ fixed point wise and is such that if $U = i^{-1}(J_\epsilon(D^q)) \cap S^r \times \{0\})$ then $D(U \times D^{q-r})$ contains $J_\epsilon(D^q)$. Let $\bar{\mu} : \overline{Fun_\epsilon}(S^r, \overline{GL_q}) \longrightarrow Sect_\epsilon(T_q F^r)$ be the map defined by for $g \in Fun_\epsilon(S^r, \overline{GL_q})$ and $(x, y) \in F^r$

$$(\bar{\mu}(g))_{(x,y)} = g(x)\,(\nabla e)_{(x,y)}$$

$\hat{\mu} \circ \bar{\mu} =$ identity, construct a homotopy $H : I \times Sect_\epsilon(T_q, F^r) \to Sect_\epsilon(T_q, F^r)$ given by for $(x, y) \in F^r$, $t \in I$ and $a \in Sect_\epsilon(T_q F^r)$

$$H_t(a)_{D(x,y)} = \mu(a)(D(x, ty))(\nabla e)_{D(x,y)}$$

Then $H_1 =$ identity and $H_0 = \bar{\mu} \circ \mu$. So $\bar{\mu}$ is a homotopy inverse of $\hat{\mu}$.

10 LEMMA : For $\zeta_1 \geq \epsilon > 0$ the inclusion $k : \overline{Fun_\epsilon}(S^r, \overline{GL_q}) \subset \overline{Fun}(S^r, \overline{GL_q})$ is a homotopy equivalence.

Proof: First construct the homotopy $G : I \times [-1, 1] \longrightarrow [-1, 1]$ given by

$$G_t(x) = \begin{cases} 2t - 1 + (1-t)(1-x) & \text{for } -1 \leq x \leq 0 \\ xt + (1-t) & \text{for } 0 \leq x \leq 1 \end{cases}$$

using G we construct the homotopy $H : I \times S^r \longrightarrow S^r$ by

$$H_t(x_1,\dots,x_{r+1}) = \begin{cases} (x_1,\dots x_{r+1}) & \text{FOR } |x_1|=1 \\ (G_t(x_1), R_t(x_1)x_2,\dots R_t(x_1)x_{r+1}) & \text{for } |x_1|\neq 1 \end{cases}$$

where $R_t(x) = \sqrt{\dfrac{1-[G_t(x)]^2}{1-x^2}}$ for $|x|\neq 1$.

Let $\bar{k} : \overline{Fun}(S^r, \overline{GL_q}) \longrightarrow \overline{Fun}_\epsilon(S^r, \overline{GL_q})$ be the map such that for $a \in \overline{Fun}(S^r, \overline{GL_q})$, $x \in S^r$

$$\bar{k}(a)(x) = a \circ H_1(x) .$$

We will show that $\bar{k}$ is the homotopy inverse of k. Consider $\bar{H} : I \times \overline{Fun}(S^r, \overline{GL_q}) \longrightarrow \overline{Fun}(S^r, \overline{GL_q})$ given by for $a \in \overline{Fun}(S^r, \overline{GL_q})$, $x \in S^r$, $t \in I$

$$\bar{H}_t(a)(x) = a \circ H_t(x)$$

now since $J_\epsilon(D^q) \subset J_{\xi_1}(D^q) \subset \bar{D}^r \times \mathring{D}^{q-r}$ we have that $\bar{H}_t(\overline{Fun}_\epsilon(S^r, \overline{GL_q})) \subset \overline{Fun}_\epsilon(S^r, \overline{GL_q})$ and $\bar{H}_0 =$ identity, hence by using $\bar{H}$ we have $\bar{k} \circ k$ is homotopic to identity map and again by using $\bar{H}$, $k \circ \bar{k}$ is homotopic to the identity.

Now the diagrams

$$\begin{array}{ccccc} \bar{S}^r_q(\xi_i) & \xrightarrow{\nabla} & Sect_{\xi_i}(T_q F^r) & \xrightarrow{k \circ \hat{\mu}} & \overline{Fun}(S^r, \overline{GL_q}) \\ \cap & & \cap & & \| \\ \bar{S}^r_q(\xi_j) & \xrightarrow{\nabla} & Sect_{\xi_j}(T_q F^r) & \xrightarrow{k \circ \hat{\mu}} & \overline{Fun}(S^r, \overline{GL_q}) \end{array}$$

for $i \leq j$ commute and the horizontal maps are weak homotopy equivalences. Because of the way in which compact sets map into $\bar{S}^r_q$ we get a map $\bar{S}^r_q \xrightarrow{k \circ \hat{\mu} \circ \nabla} \overline{Fun}(S^r, \overline{GL_q})$ which is $\bar{S}^r_q(\xi_i) \xrightarrow{k \circ \hat{\mu} \circ \nabla} \overline{Fun}(S^r, \overline{GL_q})$ when restricted to $\bar{S}^r_q(\xi_i)$ and we have the following .

11 LEMMA : If $\bar{e} : \bar{S}^r_q \longrightarrow \bar{S}^r_q$ is the map $\bar{e} : a \longmapsto e$ and Δ is the diagonal map $\Delta : \bar{S}^r_q \longrightarrow \bar{S}^r_q \times \bar{S}^r_q$ then the diagram

$$\begin{tikzcd} \bar{S}_q^r \arrow[r, "R\circ\hat{\mu}\circ\nabla"] \arrow[d, "\bar{G}\circ(\bar{e}\times 1)\circ\Delta"'] & \overline{Fun}(S^r, \bar{G}l_q) \arrow[d, "A"] \\ \overline{Fun}(S^r, \overline{\Pi}_r^{\infty}) \arrow[r, "\nu"] & \overline{Fun}(S^r, \bar{G}l_q) \end{tikzcd} \qquad A(g) = [Ad(g)]^{-1}$$

commutes and the top line $R\circ\hat{\mu}\circ\nabla$ is a weak homotopy equivalence.

Proof : The fact that $R\circ\hat{\mu}\circ\nabla$ is a weak homotopy equivalence follows directly from above and the fact that if K is compact and $f: K \to \bar{S}_q^r$ is continuous then $f(K) \subset \bar{S}_q^r(\zeta_i)$ for some $i \in \mathbb{N}$. The diagram commutes :if $a \in \bar{S}_q^r$, $x \in S^r$ there exists an open neighbourhood U of $i(x)$ in F^r such that $a|U$ is an embedding. $e|U$ is automaticaly an embedding and

$$da^{-1}\left(\frac{\partial}{\partial\Pi_1}, \dots, \frac{\partial}{\partial\Pi_q}\right) = \left(\frac{\partial}{\partial\Pi_1\circ a}, \dots, \frac{\partial}{\partial\Pi_q\circ a}\right)$$

Now $(\hat{\mu}\circ\nabla a)_x$ is defined by

$$da^{-1}\left(\frac{\partial}{\partial\Pi_1}, \dots \frac{\partial}{\partial\Pi_q}\right)_{a(i(x))}$$

$$= (\hat{\mu}\circ\nabla_a)_x \, de^{-1}\left(\frac{\partial}{\partial\Pi_1}, \dots \frac{\partial}{\partial\Pi_q}\right)_{e(i(x))}$$

and
$$= de^{-1}\left[(\hat{\mu}\circ\nabla_a)_x \left(\frac{\partial}{\partial\Pi_1}, \dots \frac{\partial}{\partial\Pi_q}\right)_{e(i(x))}\right]$$

because de^{-1} is linear. Also $\nu(\bar{G}\circ(\bar{e}\times 1)\circ\Delta(a))(i(x)) = \nu(\bar{G}(e,a))(i(x))$ so we have on applying de^{-1} to both sides of the equation that defines ν that

$$de^{-1}\left(\frac{\partial}{\partial\Pi_1}, \dots \frac{\partial}{\partial\Pi_q}\right)_{e(i(x))}$$

$$= da^{-1}\left[Ad\, \nu(\bar{G}\circ(\bar{e}\times 1)\circ\Delta(a))(i(x)) \left(\frac{\partial}{\partial\Pi_1}, \dots \frac{\partial}{\partial\Pi_q}\right)_{a(i(x))}\right]$$

but this means that $A(\hat{\mu}\circ\nabla_a) = \nu(\bar{G}\circ(\bar{e}\times 1)\circ\Delta(a))$ and the diagram commutes .

Proof of Theorem 1 FOR C^∞ :

Surjectivity: Let $\alpha \in \Pi_t(\overline{GL_q})$ then there exists an $a \in \overline{S}^v_q$ such that the homotopy class $[A \circ k \circ \hat{\mu} \circ \nabla(a)]$ of $A \circ k \circ \hat{\mu} \circ \nabla(a)$ is α, since $A \circ k \circ \hat{\mu} \circ \nabla$ is a weak homotopy equivalence. But by the second half of Lemma 11 this means that the homotopy class $[\overline{G}(a, e)] \in$ $[\overline{G}(a, e)] \in \Pi_t(\overline{\Gamma}^\infty_q)$ is mapped by $\nu_\#$ to α.

Injectivity: Suppose $\gamma \in \overline{Fun}(S^t, \overline{\Gamma}^\infty_q)$ such that the homotopy class $[\gamma]$ is mapped by $\nu_\#$ to the identity element in $\Pi_t(\overline{GL_q})$. By Lemma 6 there exists an $(a, b) \in \overline{S}^t_q$ such that $\overline{G}(a, b)$ is connected to γ by an arc in $\overline{Fun}(S_v, \overline{\Gamma}^\infty_q)$, and a is connected by an arc C to e in $\overline{S}^v_q$. First we note that $\nu_\#[\overline{G}(a, b)] = \nu_\#[\gamma] = 0$. Also by Lemma 7 $[\overline{G}(e, b)] = [\overline{G}(a, b)] = [\gamma]$ But by Lemma 11 $[A \circ k \circ \hat{\mu} \circ \nabla(b)] = \nu_\#[\overline{G}(e, b)] = \nu_\#[\gamma] = 0$ and there thus exists an arc in $\overline{S}^v_q$ that connects e with b. Applying Lemma 7 again we get $[\overline{G}(e, b)] = [\overline{G}(e, e)]$, but if $1 : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ is the identity map $\overline{G}(e, e)(x) = Gen(e(x), 1)$. The inclusion $S^t \subset \mathbb{R}^q$ is homotopic, keeping the base point, to a constant map. We thus get an arc in $\overline{Fun}(S^t, \overline{\Gamma}^\infty_q)$ which connects $\overline{G}(e, e)$ with the constant map so we get

$$[\gamma] = [\overline{G}(e, b)] = [\overline{G}(e, e)] = 0 .$$

Proof of Theorem when C^s :

To prove this we will need the following smoothing Lemma.

12 LEMMA: Let $f : S^v \times \overset{\circ}{D}{}^{q-v} \longrightarrow \mathbb{R}^q$ be a C submersion which is C^∞ in some neighbourhood W of $\{*\} \times D^{q-v}_{1/2}$ then there exists a C^s submersion $f_1 : S^v \times \overset{\circ}{D}{}^{q-v} \longrightarrow \mathbb{R}^q$ such that $f_1 | W' = f | W'$ for some closed neighbourhood W' of $L = \{*\} \times D^{q-v}_{1/2} \cup S^v \times \{(\tfrac{1}{2}, 0, 0, \cdots)\}$.

Proof : Reguard $S^v \times \overset{\circ}{D}{}^{q-v}$ as identified by e with $e(S^v \times \overset{\circ}{D}{}^{q-v})$ and

S^v identified with $i(S^v)$, then $S^v \times \mathring{D}^{q-v}$ is an open subset of $\mathbb{R}^q$. Let D be an open disc in S^v such that D is in W. $B = S^v - D$ is a compact subset of $\mathbb{R}^q$ and there exists an open neighbourhood V of B such that $\bar{V} \subset S^v \times \mathring{D}^{q-v}$ and $V \cap L = \phi$. By using the Smoothing Lemma 4·1 in [MUT] and the fact that there exists an $\delta > 0$ such that δ close C^s maps to f are submersions we get the required result. By using a proof similar to that given in Lemma 6 and if we replace $\bar{\Gamma}^\infty_q$ by $\bar{\bar{\Gamma}}^s_q$, the component of $\bar{\Gamma}^s_q$ that contains the identity element γ_0, then for $f \in \overline{Fun}(S^v, \bar{\Gamma}^s_q)$ there exists an $a \in \bar{S}^v_q$ and a C^s submersion $b : S^v \times \mathring{D}^{q-v} \longrightarrow \mathbb{R}^q$ which agrees with e on some neighbourhood of $*$ such that $G(a, b)$ is connected by an arc in $\overline{Fun}(S^v, \bar{\Gamma}^s_q)$ to f. By using Lemma 12 there exists an $b' \in \bar{S}^v_q$ and an arc in $\overline{Fun}(S^v, \bar{\Gamma}^s_q)$ that connects $G(a, b)$ with $G(a, b')$ where the arc is constructed in a similar way to the construction given towards the end of Lemma 7. Since f is an arbitary member we get that if $i_s : \bar{\Gamma}^\infty_q \longrightarrow \bar{\Gamma}^s_q$ is the inclusion map

$$i_{s\#} : \pi_v(\bar{\Gamma}^\infty_q) \longrightarrow \pi_v(\bar{\Gamma}^s_q)$$

is surjective for $v < q$. But we have the commutative diagram

$$\begin{tikzcd}
\pi_t(\bar{\Gamma}^\infty_q) \arrow[rr, "i_{s\#}"] \arrow[dr, "\nu_\#"'] & & \pi_t(\bar{\Gamma}^s_q) \arrow[dl, "\nu_\#"] \\
& \pi_t(\overline{GL}_q) &
\end{tikzcd}$$

and for $t < q$, $\nu_\#$ on the left is an isomorphism so $i_{s\#}$ is injective and by the surjectivity of $i_{s\#}$, $i_{s\#}$ is an isomorphism and $\nu_\#$ on the right is an isomorphism. This completes the proof of Theorem 1.

Let $S \geq 1$ then define a map $\nu : \bar{\Gamma}^t_q(\bar{\Delta}^s) \rightarrow (\overline{GL}_q)^s$ where $(GL_q)^s$ is the S-fold product of $\overline{GL}_q$, by $\nu : \gamma \longrightarrow (\nu \circ \pi_{1 \le 2}(\gamma), \ldots, \nu \circ \pi_{1 \le s}(\gamma))$

then it can be easily seen by using Lemma 1·27 and a similar proof to that given in Theorem 1 that we get the following extention.

13 THEOREM : Let $r \geq 1$, $0 \leq t < q$, $q \geq 1$, $s \geq 1$. , then

$$\nu_{\#} \pi_t(\Gamma_q^r(\bar{\Delta}^s)) \longrightarrow \pi_t((\overline{GL}_q)^s)$$

is an isomorphism.

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