Applications of differential geometry to computer curves and surfaces

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APPLICATIONS OF DIFFERENTIAL GEOMETRY TO COMPUTER CURVES AND SURFACES

by

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CONTENTS

ABSTRACT

This thesis realises the need for describing computer curves and surfaces in terms of intrinsic quantities and certain properties relative to the Euclidean space in which they are embedded.

Chapter 1 introduces some of the ideas and problems involved in what can be termed computational differential geometry.

Chapter 2 presents some analysis of the major types of computer curves in terms of a number of shape control parameters.

Chapter 3 gives a similar analysis of computer surfaces.

Chapter 4 considers the calculus of variations in connection with the minimal immersion and a particular invariantly defined functional analogous to energy.

Chapter 5 applies the energy functional to a class of computer curves.

Chapter 6 looks at a number of surfaces in relation to surface mappings and distortion. Some mappings are also derived. This generally involves the solution of non linear differential equations the linearisation of which will almost certainly remove the salient features of the theory.

A bibliography and a number of figures are provided following chapter 6.

Chapter 1

Introduction

In recent years the introduction of the computer as an additional tool in the industrial design process has stimulated considerable interest in the development of algorithms for curve and surface interpolation and approximation {8-19}. The purpose of this thesis has been to apply some of the fundamental ideas of differential geometry to this problem and to the problems associated with deriving two dimensional patterns from given three dimentional surface representations.

Traditionally the design process has involved a tremendous amount of repetitive work which has been time consuming, often irreversible and consequently expensive. This process of producing sketches and drawings, making models and applying numerous corrections and modifications has served to emphasise the need for what is now familiarly known as Computer Aided (Geometric) Design.

The basis of CAD could be stated briefly as the 'mathematical definition of shape'. More precisely one is interested in deriving for some shape - which may exist initially purely in the designers' mind - a mathematical representation as a surface in E^3 which will be of such form as can be held within a computer. The development of such computer hardware as 'real time' keyboards and graphics displays has meant that a designer can directly interact with the computer to produce this derivation by using the quick assimilation of computed results to make decisions and thereby efficiently guiding the computer until an acceptable representation has been obtained.

To achieve a surface representation it is clear that for all but the simplest of shapes geometrically an implicit equation would be unhandy if at all obtainable since we shall be mainly interested in local properties and having local control over the description. Consequently the technique generally adopted is that of considering the surface as being composed of a finite union of topological squares commonly known as 'patches' (11, 13, 14, 15) which are normally arranged to satisfy conditions which are analogous to a surface triangulation where:-

(i) no two patches have a common interior point,

(ii) to each side of a patch there correspond two and only two patches with this common side unless the side of the patch lies on the boundary of the surface in which case it belongs to only one patch,

(iii) any two patches can be joined by a sequence of patches where each patch in the sequence has one and only one side in common with the next one in the sequence,

(iv) all patches with a common vertex can be arranged in a definite order so that consecutive patches have a common side passing through the vertex.

An explicit form in which each point is expressed in terms of two curvilinear coordinates also enables us to describe a sense round the surface. However, such a patch representation is by no means unique and in addition to the above we insist that there are side smoothness conditions which will ensure that neighbouring patches are glued together differentiably to form the complete definition.

All functions are computed as truncated polynomials by digital computers and so it seems reasonable to use polynomials as the basis functions for both curves and for the surface patches. In practice for most purposes cubics for curves and bicubics for surfaces are optimal in that the degrees allow for sufficient shape flexibility and the necessary smoothness criteria.

The problem of interpolation reduces to one of determining suitable constraints to impose on properties such as arc length, surface area and smoothness in order to uniquely define the patch boundaries and interiors from the vertices. We insist however, that we do not impose restrictions which are peculiar to the particular parameterisation adopted and in particular we shall be interested in intrinsic properties. By using elements of intrinsic geometry we can use the results and ideas of Riemannian geometry which is mainly intrinsic and for which the literative is vast {3}. This would also enable ideas to proceed along more general lines looking at a particular field of tensors defined over differentiable manifolds.

In placing our constraints we must pay attention to two basic results from the theory of curves and surfaces in E^3. For curves, the fundamental existence and uniqueness theorem states that for arbitrary continuous functions $\kappa(s)$ and $\tau(s)$ defined on $a \leq s \leq b$ then there exists, up to a rigid motion, a unique curve for which κ is the curvature, τ is the torsion and s is the arc length. Correspondingly, for surfaces if the functions $E(u,v)$, $F(u,v)$, $G(u,v)$ are C^2 and $L(u,v)$, $M(u,v)$, $N(u,v)$ C^1 on some open set containing (u_o,v_o) and satisfy the compatibility equations of Gauss and Codazzi and if the quadratic differential form $Edu^2 + 2Fdudv + Gdv^2$ is positive definite then there exists a unique surface, up to Euclidean motion, defined on a neighbourhood of (u_0, v_0) having E, F, G and L, M, N as first and second fundamental coefficients. Length and area considerations force us to consider the theory of the calculus of variations to solve what amounts to a form of the general Plateau problem which can be stated as 'given an n-1 dimensional closed submanifold Γ in E^{n+k} then find a minimal submanifold M^n with boundary $\partial M^n = \Gamma$ '.

Having achieved a suitable mathematical representation which will act as the geometric data base we can then move on to consider other phases of design which could be embodied in an overall interactive system. The mass production of fashion articles for instance usually requires specially shaped tools such as jigs, moulds and patterns and it would be beneficial if these can be acquired as soon as possible so that the manufacturer can catch the market at its peak. Thus if we could use the computer held definition to determine shapes for patterns, which are normally two dimensional, this would for instance facilitate automation of the drawing process with the aid of digital plotters, and the cutting process with the aid of laser beams or high pressure water jets.

The optimum mapping of a particular surface into E^2 will obviously depend upon constraints imposed by such things as the elasticity of the material, the available machinery, the cost of the raw material and so on. We will need to know whether the optimum mapping is going to preserve some geometrical structure of the surface. This will not normally be possible because of the restrictions this would involve and so we will have to look at various norms for measuring the departure from the considered ideal. Such an ideal might be an isometry for example or more generally conformal. It is well known that every point on a surface has a neighbourhood which can be mapped conformally on some neighbourhood of any other surface, and this result has been widely exploited by cartographers. The natural generalisation of the conformal mapping is the quasiconformal mapping {7} which now plays an active role in the theory of analytic functions of a single complex variable.

For each chosen ideal we shall term the departure from that ideal the 'stretch' or 'distortion' of the particular mapping and the optimum solution will be the mapping having minimum stretch.

Although this thesis will be concerned with applying the fundamental ideas of differential geometry to computer aided design an interesting and useful observation is that the computer could become a very attractive tool in differential geometry. A set which admits a patch structure is a C^r manifold with metric for some r where r depends upon how the patches are joined together. Conversely if a manifold can be cubulated to give it a patch structure then the computer could be used to calculate quantities on the manifold. This would overcome one significant problem which has existed in the past in that for practical convenience many calculations have been confined to a small number of manifolds often only surfaces of revolution.

Considerable effort has gone into computer aided design in recent years but there is still a great need for research in many areas before a fully integrated CAD system could be considered as commercially viable. Even if CAD methods turn out to be no cheaper than traditional methods they will certainly be quicker, will allow more options to be considered and the mathematical compatibility of such things as moulds and patterns will mean that the various components making up the design should assemble together more accurately.

Chapter 2

Computer Curves in E^3

Definition 2.1

A Cubic curve in E^3 is one for which there exists some parameter u in terms of which the equation of the curve can be written as

$$\underline{r}(u) = \underline{a}_0 + \underline{a}_1 u + \underline{a}_2 u^2/2! + \underline{a}_3 u^3/3! \qquad (1)$$

where $\underline{a}_i \varepsilon E^3$ for $i=0,..3$

The parameter u is only indeterminate to within an affine transformation of the form $u \longrightarrow u' = \xi u + \eta$ where $\xi, \eta \varepsilon R$ and so if we apply the restriction that the curve contains two particular points in E^3 with specified parameter values $u = \alpha$ and $u = \beta$ say, then the parameter is uniquely determined. For convenience we normally take α and β to be 0 and 1 and let $0 \leqslant u \leqslant 1$. We will use primes to denote differentiation with respect to arc length s and a Natural representation of a curve is one for which s is the parameter.

Then $s(u) = \int_\alpha^u |\dot{\underline{r}}(u)| du = \int_\alpha^u f(u) du$ and so it is easy to evaluate s using Gauss Quadrature with $s = \sum_{i=1}^{N} w_i f(u_i)$, and verification that s is independent of the parametrization follows immediately from the rule for changing the variable of an integral. We shall first of all see that the cubic representation (1) places restrictions on both the parameter u and the curve itself as a geometric object.

Proposition 2.2

The only cubic curve having a Natural representation is the straight line.

To prove this, if

$\underline{r}(s) = \underline{a}_0 + \underline{a}_1 s + \underline{a}_2 s^2/2! + \underline{a}_3 s^3/3!$ then repeated differentiation gives

$\underline{r}^{(iv)}(s) = \{\kappa''(s) - \kappa(s)\tau^2(s) - \kappa^3(s)\}\underline{n} + \{2\kappa'(s)\tau(s) + \kappa(s)\tau'(s)\}\underline{b}$

$- 3\kappa(s)\kappa'(s)\underline{t} = \underline{0}$ where we have used the Serret-Frenet formulae :

$$\underline{t}' = \kappa\underline{n} \quad , \quad \underline{n}' = \tau\underline{b} - \kappa\underline{t} \quad , \quad \underline{b}' = -\tau\underline{n}$$

and where κ and τ denote the curvature and torsion respectively.

Hence $\kappa\kappa' = 2\kappa'\tau + \kappa\tau' = \kappa'' - \kappa\tau^2 - \kappa^3 = 0 \implies \kappa \equiv 0$

This means of course that we cannot take the arc length as parameter if we restrict ourselves to cubics.

Proposition 2.3

For given $\{\underline{r}_0, \underline{r}_1, \underline{t}_0, \underline{t}_1\} \subset E^3$ with $|\underline{t}_0| = |\underline{t}_1| = 1$. then there are ∞^2 cubic curves $\underline{r} : [0,1] \longrightarrow E^3$ such that

$$\underline{r}(i) = \underline{r}_i \text{ and } \underline{r}'(i) = \underline{t}_i \text{ for } i = 0,1$$

This follows since if $\underline{r}(u) = \underline{a}_0 + \underline{a}_1 u + \underline{a}_2 u^2/2! + \underline{a}_3 u^3/3!$ then $\dot{s}(u)\underline{t}(u) = \underline{a}_1 + \underline{a}_2 u + \underline{a}_3 u^2/2$ and the four coefficients satisfy

$$\begin{aligned} \underline{a}_0 &= \underline{r}_0 \\ \underline{a}_1 &= \dot{s}(0)\underline{t}_0 \\ \underline{a}_2 &= 2!\{3(\underline{r}_1-\underline{r}_0)-2\dot{s}(0)\underline{t}_0-\dot{s}(1)\underline{t}_1\} \\ \underline{a}_3 &= 3!\{2(\underline{r}_0-\underline{r}_1)+\dot{s}(0)\underline{t}_0+\dot{s}(1)\underline{t}_1\} \end{aligned} \qquad (*)$$

which means we have a doubly infinite family of curves parametrized by $\dot{s}(0)$ and $\dot{s}(1)$.

Corollary

For given $\{\underline{r}_0, \underline{r}_1, \underline{t}_0, \underline{t}_1\} \subset E^3$ with $|\underline{t}_0| = |\underline{t}_1| = 1$. and given $\kappa_0, \kappa_1 \in \mathbb{R}$ then there is a unique cubic curve $\underline{r}: [0,1] \longrightarrow E^3$ such that

$$\underline{r}(i) = \underline{r}_i, \quad \underline{r}'(i) = \underline{t}_i, \quad |\underline{r}''(i)| = \kappa_i \text{ for } i = 0,1$$

In fact since $\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u) = \kappa(u)\dot{s}^3(u)\underline{b}$ where $\underline{b} = \underline{t} \times \underline{n}$ then we obtain $\dot{s}(0)$ and $\dot{s}(1)$ from :

$$2\{3\underline{t}_0 \times (\underline{r}_1 - \underline{r}_0) - \dot{s}(1)\underline{t}_0 \times \underline{t}_1\} = \kappa_0\, \dot{s}^3(0)\underline{b}(0)$$
$$2\{3\underline{t}_1 \times (\underline{r}_0 - \underline{r}_1) - \dot{s}(0)\underline{t}_0 \times \underline{t}_1\} = \kappa_1\, \dot{s}^3(1)\underline{b}(1)$$

For the general curve $\underline{r} = \underline{r}(u)$ it is normally convenient to have the curvature κ and torsion τ expressed in terms of the derivatives of $\underline{r}$.

Since $\dot{\underline{r}}(u) = \dot{s}(u)\underline{t}(u)$ and

$\ddot{\underline{r}}(u) = \ddot{s}(u)\underline{t}(u) + \dot{s}^2(u)\kappa(u)\underline{n}(u)$ then

$\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u) = \dot{s}^3(u)\kappa(u)\underline{b}(u)$ where $\underline{b}(u) = \underline{t}(u) \times \underline{n}(u)$ and differentiation gives

$$\dot{\underline{r}}(u) \times \dddot{\underline{r}}(u) = \frac{d}{du}\{\dot{s}^3(u)\kappa(u)\}\,\underline{b}(u) - \dot{s}^4(u)\kappa(u)\tau(u)\underline{n}(u).$$

Therefore $\ddot{\underline{r}}(u).\{\dot{\underline{r}}(u) \times \dddot{\underline{r}}(u)\} = -s^6(u)\kappa^2(u)\tau(u)$

But $\dot{r}^2(u) = \dot{s}^2(u)$ and $\{\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u)\}^2 = \dot{s}^6(u)\kappa^2(u)$ and

so $\kappa^2(u) = \{\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u)\}^2/\{\dot{\underline{r}}^2(u)\}^3$ (*)

$\tau(u) = \left[\dot{\underline{r}}(u),\ \ddot{\underline{r}}(u),\ \dddot{\underline{r}}(u)\right]/\{\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u)\}^2$ (**)

Proposition 2.4

The curvature κ and torsion τ of the cubic curve $\underline{r}:[0,1] \longrightarrow E^3$ take the boundary values :

$\kappa(0) = \{(\underline{a}_1 \times \underline{a}_2)^2/(\underline{a}_1{}^2)^3\}^{\frac{1}{2}}$

$\kappa(1) = \left[\{\underline{a}_1 \times \underline{a}_2 + \underline{a}_1 \times \underline{a}_3 + \underline{a}_2 \times \underline{a}_3/2\}/\{(\underline{a}_1 + \underline{a}_2 + \underline{a}_3/2)^2\}^3\right]^{\frac{1}{2}}$

$\tau(0) = [\underline{a}_1,\ \underline{a}_2,\ \underline{a}_3]/(\underline{a}_1 \times \underline{a}_2)^2$

$\tau(1) = [\underline{a}_1,\ \underline{a}_2,\ \underline{a}_3,]/\{\underline{a}_1 \times \underline{a}_2 + \underline{a}_1 \times \underline{a}_3 + \underline{a}_2 \times \underline{a}_3/2\}^2$

These results follow immediately by substitution of

$\dot{\underline{r}}(u) = \underline{a}_1 + \underline{a}_2 u + \underline{a}_3 u^2/2$, $\ddot{\underline{r}}(u) = \underline{a}_2 + \underline{a}_3 u$ and $\dddot{\underline{r}}(u) = \underline{a}_3$

into equations (*) and (**). Also as a consequence of equations (*) and (**) we have

Proposition 2.5

The general curve $\underline{r} = \underline{r}(u)$ is planar iff $[\dot{\underline{r}}, \ddot{\underline{r}}, \dddot{\underline{r}}] = 0$.

This follows because $[\dot{\underline{r}}, \ddot{\underline{r}}, \dddot{\underline{r}}] = \kappa^2 \tau \dot{s}^6$ and so $\tau = 0$ implies that $[\dot{\underline{r}}, \ddot{\underline{r}}, \dddot{\underline{r}}] = 0$.

Conversely if $[\dot{\underline{r}}, \ddot{\underline{r}}, \dddot{\underline{r}}] = 0$ and $\tau \neq 0$ at some point then there is a neighbourhood of this point where $\tau \neq 0$ and so $\kappa = 0$ in this neighbourhood. But this means that the arc is a straight line and so $\tau = 0$ which is contrary to the hypothesis and so $\tau = 0$ at all points on $\underline{r}$.

Any discernible points along the curve will be of particular interest from the point of view of design.

Definition 2.6

A point of $\underline{r} = \underline{r}(u)$ at which $\underline{r}''(u) = \underline{0}$ is called a point of inflexion.

Proposition 2.7

If the cubic curve $\underline{r} : [0,1] \longrightarrow E^3$ has a point of inflexion then :

$$(i)\ [\underline{a}_1, \underline{a}_2, \underline{a}_3] = 0$$

$$\text{and}\quad (ii)\quad \dot{s}(0)\dot{s}(1)[\underline{t}_0, \underline{r}_1 - \underline{r}_0, \underline{t}_1] = 0$$

where $\underline{r}_i = \underline{r}(i)$ and $\underline{t}_i = \underline{r}'(i)$ for $i = 0,1$.

The proof follows because from (*) and (**)

$$\kappa^2(u)\tau(u) = [\,\dot{\underline{r}}(u),\ \ddot{\underline{r}}(u),\ \dddot{\underline{r}}(u)]/\dot{s}^6(u)$$

and so $\kappa = 0$ means that

$$[\,\dot{\underline{r}}(u),\ \ddot{\underline{r}}(u),\ \dddot{\underline{r}}(u)] = [\,\underline{a}_1,\ \underline{a}_2,\ \underline{a}_3\,] = 0$$

or in terms of boundary values this means that

$\dot{s}(0)\dot{s}(1)[\underline{t}_0,\ \underline{r}_1-\underline{r}_0,\ \underline{t}_1] = 0$ by substitution of equations (+).

This means we have the following :

Proposition 2.8

If the cubic curve $\underline{r} : [0,1] \longrightarrow E^3$ is such that either

(i) $[\,\underline{a}_1,\ \underline{a}_2,\ \underline{a}_3\,]$ is non zero

or (ii) $[\,\underline{t}_0,\ \underline{r}_1-\underline{r}_0,\ \underline{t}_1\,]$ is non zero

where $\underline{r}_i = \underline{r}(i)$, $\underline{t}_i = \underline{r}'(i)$, $i = 0,1$

then $\underline{r}$ has no point of inflexion.

The proof follows from Proposition 2.7 if $\dot{s}(0)$ and $\dot{s}(1)$ are non zero. And if $\dot{s}(0) = 0$ then

$$\dot{\underline{r}}(u) = 6u(1-u)(\underline{r}_1-\underline{r}_0) + (3u-2)\dot{s}(1)\underline{t}_1$$

$$\ddot{\underline{r}}(u) = 6(1-2u)(\underline{r}_1-\underline{r}_0) + 2(3u-1)\dot{s}(1)\underline{t}_1$$

$$\dddot{\underline{r}}(u) = -12(\underline{r}_1-\underline{r}_0) + 6\dot{s}(1)\underline{t}_1$$

and so $[\,\dot{\underline{r}},\ \ddot{\underline{r}},\ \dddot{\underline{r}}\,] = 0$ which means the curve is planar and $[\,\underline{t}_0,\ \underline{r}_1-\underline{r}_0,\ \underline{t}_1] = 0$

And similarly if $\dot{s}(1) = 0$

If the curve is planar then the point of inflexion of particular interest is the one where $\underline{n}$ changes sign, and control over such points is desirable since one is normally concerned in keeping the number to a minimum.

In the non planar case the analogy would be the point where $\underline{b}' = \underline{0}$ and $\underline{b}$ changes sign, for at this point the curve crosses its osculating plane.

To detect whether a curve $\underline{r} = \underline{r}(u)$ is to one side of or on a particular plane X then the obvious measure is given by $(\underline{r}(u)-\underline{P}).\underline{N}$ where $\underline{P}$ is some point on X and $\underline{N}$ is its normal.

In general we will only be able to define our curves by cubics locally. We would now like to consider the problem of linking together several curve segments to form an acceptable composite curve and to define in more precise terms what we mean by an acceptable curve.

Let $\underline{r} : [a,b] \longrightarrow E^3$ be a given space curve C and let us assume that C is smooth. The arc length of C can be used to provide a 'natural' cubic extension as follows :

We first assume that C is parametrized in what is the only geometrically significant way that is by means of arc length.

Since C is smooth we can write :

$$\underline{r}(s) = \underline{r}(c) + (s-c)\underline{r}'(c) + (s-c)^2/2!\ \underline{r}''(c) + \ldots\ldots + (s-c)^n/n!\,\underline{r}^{(n)}(c) + 0\,(s-c)^{n+1}$$

where a < c < b.

If we use the Serret-Frenet formulae then

$$\underline{r}(s) = \underline{r}(c) + (s-c)\underline{t} + (s-c)^2\kappa/2!\underline{n} + (s-c)^3/3! \{\kappa'\underline{n} + \kappa\tau\underline{b} - \kappa^2\underline{t}\} + (s-c)^4/4!\{(\kappa''-\kappa\tau^2-\kappa^3)\underline{n} + (2\kappa'\tau + \kappa\tau')\underline{b} - 3\kappa\kappa'\underline{t}\} + \ldots\ldots + 0(s-c)^{n+1}.$$

where $\underline{t}$, $\underline{n}$, $\underline{b}$, κ, τ are all evaluated at s = c.

Suppose now that we want to continue C be means of another curve Γ, which is cubic, with Γ given by $\hat{\underline{r}} : [c,d] \longrightarrow E^3$.

We take as parameter for Γ s the arc length of C, and we let $\hat{s}$ be the arc length of Γ. We write:

$$\hat{\underline{r}}(s) = \underline{r}(c) + (s-c)\underline{t} + (s-c)^2\kappa/2!\underline{n} + (s-c)^3/3! \{\kappa'\underline{n} + \kappa\tau\underline{b} - \kappa^2\underline{t}\}.$$

Then it is easy to show that at s = c we have :

$$\hat{\underline{t}} = \underline{t},\ \hat{\kappa} = \kappa,\ \hat{\underline{n}} = \underline{n},\ \hat{\tau} = \tau,\ \hat{\underline{b}} = \underline{b},\ \hat{\kappa}' = \kappa'$$

Thus we have found a smooth cubic continuation of the curve C from the point $\underline{r}(c)$, a < c < b. However we have assumed that there are no other restrictions on Γ, and this will not normally be the case. In general we will be looking to solve the following problem :

"Given $\{\underline{b}_1, \ldots\ldots \underline{b}_{n+1}\} \subset E^3$ we want to construct a curve Γ where :

(i) Γ passes through each $\underline{b}_i$ i = 1, ...n+1

(ii) Γ is cubic between $\underline{b}_i$ and $\underline{b}_{i+1}$ i = 1, ... n

(iii) $\Gamma \in C^k$, for a suitable k . "

Such a curve is known as a Piecewise Parametric Cubic or Parametric Cubic Spline {8,10,23}. Thus the i^{th} segment of Γ from $\underline{b}_i$ to $\underline{b}_{i+1}$ is of the form

$$\underline{\gamma}_i(u) = \underline{a}_0^i + \underline{a}_1^i u + \underline{a}_2^i u^2/2! + \underline{a}_3^i u^3/3!, \underline{a}_j^i \in E^3, u \in [0,1]$$

To determine Γ means we must determine for the n segments the 4n vector coefficients $\underline{a}_j^i$ for $i = 1, \ldots n$ and $j = 0, \ldots, 3$ or equivalently the 4n vector quantities $\underline{\gamma}_i(0)$, $\underline{\gamma}_i(1)$, $\dot{\underline{\gamma}}_i(0)$, $\dot{\underline{\gamma}}_i(1)$. Be requiring that $k \geqslant 0$ we are imposing 6n scalar conditions and to avoid corners we insist that $k \geqslant 1$, which places an additional 2n conditions. Jumps in the curvature vector are undesirable and so this means $k \geqslant 2$ and another 2n conditions. Now we could impose further continuity constraints to give us the final 2n conditions we now require. This would mean that in addition to continuity of $\underline{\gamma}_i$, $\underline{\gamma}_i'$, $\underline{\gamma}_i''$ we would also have continuity of torsion and rate of change of curvature. In fact if the points $\underline{b}_i$ $i = 1, \ldots n+1$ are coplanar continuity of τ is assured and so one might instead ask for continuity of κ''. However in practice Figs{1-4} the curves resulting from excessive continuity restrictions at this discrete set of points indicate that we must relax some of the conditions in order to gain control over the behaviour of the curve away from this set. This inevitably means we must consider restrictions in some way related to arc length. We have already shown (Proposition 2.3) that for certain fixed end conditions then there are ∞^2 cubics to choose from for each segment depending upon the values of the end speeds $\dot{s}(0)$ and $\dot{s}(1)$. Also there is no requirement for these speeds to be continuous across segments since C^1 continuity only concerns the tangent direction. Therefore these could be used as parameters for controlling the behaviour of the curve away from the segment endpoints. Manning {8} devised an algorithm which determines the speeds based on an approximation to a circular arc, and this has proved to be effective in practice. It is not clear however, that any geometic significance can be attached to this form of contraint in a non-planar situation or indeed when the segment end conditions are not symmetrical. Before looking at this problem we consider the implications of the continuity requirements.

Suppose that at the point $\underline{b}_i \in E^3$ we are going to continue the curve $\underline{\alpha}:[0,1] \longrightarrow E^3$ with $\underline{\beta}:[o,1] \longrightarrow E^3$ such that $\underline{\alpha}^{(k)}(1) = \underline{\beta}^{(k)}(0)$ for some k. Then if we express each condition in terms of the parameter u we find :

(i) k = 0

$$\underline{\beta}(0) = \underline{\alpha}(1) = \underline{b}_i$$

(ii) k = 1

Since, $\dot{\underline{\alpha}}(1) = \dot{s}(1)\underline{\alpha}'(1)$ and $\dot{\underline{\beta}}(0) = \dot{S}(0)\underline{\beta}'(0)$

then we require

$$\dot{\underline{\beta}}(0) = c\dot{\underline{\alpha}}(1) \text{ some scalar } c > 0$$

(iii) k = 2

For α, $\dot{\underline{\alpha}}(u) = \dot{s}(u)\underline{t}(u)$,

$\ddot{\underline{\alpha}}(u) = \dot{s}^2(u)\kappa(u)\underline{n}(u) + \ddot{s}(u)\underline{t}(u)$ and so

$$\kappa(u)\underline{n}(u) = \{\ddot{\underline{\alpha}}(u) - \ddot{s}(u)\underline{t}(u)\}/\dot{s}^2(u).$$

But $\ddot{s}(u) = \langle\underline{\dot\alpha}(u), \underline{\ddot\alpha}(u)\rangle/\dot{s}(u)$, $\dot{s}^2(u) = \langle\underline{\dot\alpha}(u), \underline{\dot\alpha}(u)\rangle$ and

$\underline{t}(u) = \underline{\dot\alpha}(u)/\dot{s}(u)$ and so

$\kappa(u)\underline{n}(u) = \{\underline{\ddot\alpha}(u)/\langle\underline{\dot\alpha}(u),\underline{\dot\alpha}(u)\rangle - \langle\underline{\dot\alpha}(u),\underline{\ddot\alpha}(u)\rangle\ \underline{\dot\alpha}(u)/\langle\underline{\dot\alpha}(u),\underline{\dot\alpha}(u)\rangle^2$

Thus we require :

$\{\underline{\ddot\alpha}(1)/\underline{\dot\alpha}^2(1)\} - \langle\underline{\dot\alpha}(1),\underline{\ddot\alpha}(1)\rangle\underline{\dot\alpha}(1)/\langle\underline{\dot\alpha}(1),\underline{\dot\alpha}(1)\rangle^2$

$= \{\underline{\ddot\beta}(0)/\underline{\dot\beta}^2(0)\} - \langle\underline{\dot\beta}(0),\underline{\ddot\beta}(0)\rangle\underline{\dot\beta}(0)/\langle\underline{\dot\beta}(0),\underline{\dot\beta}(0)\rangle^2$

But since $\underline{\dot\beta}(0) = c\underline{\dot\alpha}(1)$, $(k=1)$, we find that on rearranging we get

$$c^2\underline{\ddot\alpha}(1) - \underline{\ddot\beta}(0) = \langle\underline{\dot\alpha}(1),c^2\ \underline{\ddot\alpha}(1)-\underline{\ddot\beta}(0)\ \rangle\underline{\dot\alpha}(1)/\underline{\dot\alpha}^2(1) \qquad (*)$$

Thus if we have $c^2\underline{\ddot\alpha}(1) - \underline{\ddot\beta}(0) = d\underline{\dot\alpha}(1)$ equation (*) holds for all values of d.

<u>(iv) k = 3</u>

For α, $\underline{\dddot\alpha}(u) = \{\underline{\dddot\alpha}(u) - 3\dot{s}(u)\ddot{s}(u)\kappa(u)\underline{n}(u) - \dddot{s}(u)\underline{t}(u)\}/\dot{s}^3(u)$

but $\dot{s}(u)\ddot{s}(u) = \langle\underline{\dot\alpha}(u),\underline{\ddot\alpha}(u)\rangle$ and so differentiating gives

$\dddot{s}(u) = \{\langle\underline{\dot\alpha}(u),\underline{\dddot\alpha}(u)\rangle + \underline{\ddot\alpha}^2(u) - \ddot{s}^2(u)\}/\dot{s}(u)$

or $\dddot{s}(u) = \{\langle\underline{\dot\alpha}(u),\underline{\dddot\alpha}(u)\rangle + \underline{\ddot\alpha}^2(u) - \langle\underline{\dot\alpha}(u),\underline{\ddot\alpha}(u)\rangle^2/$
$\langle\underline{\dot\alpha}(u),\underline{\dot\alpha}(u)\rangle\}/\dot{s}(u)$

Thus

$\underline{\dddot\alpha}(u) = \dot{s}^{-1}(u)\Big[\underline{\dddot\alpha}(u)-3\langle\underline{\dot\alpha}(u),\underline{\ddot\alpha}(u)\rangle\big[\{\underline{\ddot\alpha}(u)/\langle\underline{\dot\alpha}(u),\underline{\dot\alpha}(u)\rangle\} -$
$\langle\underline{\dot\alpha}(u),\underline{\ddot\alpha}(u)\rangle\underline{\dot\alpha}(u)/\langle\underline{\dot\alpha}(u),\underline{\dot\alpha}(u)\rangle^2\big] - (\underline{\dot\alpha}(u)/\langle\underline{\dot\alpha}(u),\underline{\dot\alpha}(u)\rangle)$
$\big[\langle\underline{\dot\alpha}(u),\underline{\dddot\alpha}(u)\rangle + \underline{\ddot\alpha}^2(u) - \langle\underline{\dot\alpha}(u),\underline{\ddot\alpha}(u)\rangle^2/\underline{\dot\alpha}^2(u)\big]\Big]/\underline{\dot\alpha}^2(u)$

and so we require

$$\dot{s}^{-1}(1)\Big[\dddot{\underline{\alpha}}(1) - 3<\dot{\underline{\alpha}}(1),\ddot{\underline{\alpha}}(1)>\big[(\ddot{\underline{\alpha}}(1)/\dot{\underline{\alpha}}^2(1)) - <\dot{\underline{\alpha}}(1),\ddot{\underline{\alpha}}(1)>\dot{\underline{\alpha}}(1)/<\dot{\underline{\alpha}}(1),\dot{\underline{\alpha}}(1)>^2\big] - (\dot{\underline{\alpha}}(1)/\dot{\underline{\alpha}}^2(1))\big[<\dot{\underline{\alpha}}(1),\dddot{\underline{\alpha}}(1)> + \ddot{\underline{\alpha}}^2(1) - <\dot{\underline{\alpha}}(1),\ddot{\underline{\alpha}}(1)>^2/\dot{\underline{\alpha}}^2(1)\big]\Big]/\dot{\underline{\alpha}}^2(1) =$$

$$\dot{S}^{-1}(0)\Big[\dddot{\underline{\beta}}(0) - 3<\dot{\underline{\beta}}(0),\ddot{\underline{\beta}}(0)>\big[(\ddot{\underline{\beta}}(0)/\dot{\underline{\beta}}^2(0)) - <\dot{\underline{\beta}}(0),\ddot{\underline{\beta}}(0)>\dot{\underline{\beta}}(0)/<\dot{\underline{\beta}}(0),\dot{\underline{\beta}}(0)>^2\big] - \{\dot{\underline{\beta}}(0)/\dot{\underline{\beta}}^2(0)\}\big[<\dot{\underline{\beta}}(0),\dddot{\underline{\beta}}(0)> + \ddot{\underline{\beta}}^2(0) - <\dot{\underline{\beta}}(0),\ddot{\underline{\beta}}(0)>^2/\dot{\underline{\beta}}^2(0)\big]\Big]/\dot{\underline{\beta}}^2(0).$$

But since $\dot{\underline{\beta}}(0) = c\dot{\underline{\alpha}}(1)$ (k = 1)

and $\ddot{\underline{\beta}}(0) = c^2\ddot{\underline{\alpha}}(1) - d\dot{\underline{\alpha}}(1)$ (k = 2)

we find that this reduces to

$$\{\dddot{\underline{\beta}}(0) - c^3\dddot{\underline{\alpha}}(1) - 3cd\ddot{\underline{\alpha}}(1)\} \times \dot{\underline{\alpha}}(1) = \underline{0}$$

Collectively these results mean that :

Proposition 2.9

If $\underline{\alpha} : [0,1] \longrightarrow E^3$ and $\underline{\beta} : [0,1] \longrightarrow E^3$ are two curves satisfying :

(i) $\underline{\beta}(0) = \underline{\alpha}(1)$

(ii) $\dot{\underline{\beta}}(0) = c\dot{\underline{\alpha}}(1)$

(iii) $\ddot{\underline{\beta}}(0) = c^2\ddot{\underline{\alpha}}(1) - d\dot{\underline{\alpha}}(1)$

(iv) $\dddot{\underline{\beta}}(0) = c^3\dddot{\underline{\alpha}}(1) - 3cd\ddot{\underline{\alpha}}(1) + e\dot{\underline{\alpha}}(1)$ where $c,d,e \in \mathbb{R}$ $c>0$

are uniquely determined, then $\underline{\beta}$ is a C^3 extension of $\underline{\alpha}$.

For comparison {Figures 7, 8, 9} with results from Chapter 5 we briefly describe two forms of curve representation already widely used in computer aided design {12, 17, 18}.

Let I denote the unit interval, $[0,1]$ and let $\mu\{f\}$ denote the number of sign changes of the function $f : I \longrightarrow \mathbb{R}$.

Definition 2.10

Let $A : C^r(I) \longrightarrow C^r(I)$ for some r such that

(i) $A(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 A(f_1) + \alpha_2 A(f_2)$ for $f_1, f_2 \in C^r(I)$, $\alpha_1, \alpha_2, \in \mathbb{R}$

(ii) $A(1) = 1$ where $1 : t \longmapsto 1 \ \forall \ t$

(iii) $A(\hat{I}) = \hat{I}$ where $\hat{I} : t \longmapsto t \ \forall \ t$

Then A is said to be Variation Diminishing if $\mu\{A(f)\} \leq \mu\{f\} \ \forall f \in C^r(I)$

This means that A(f) can only have as many points of inflexion as f.

Definition 2.11

Let $B_m : C^r(I) \longrightarrow C^r(I)$ for each m be given by

$$B_m(f) = \sum_{i=0}^{m} f(\tfrac{i}{m})\lambda_i$$

where $\lambda_i : t \longmapsto \binom{m}{i} t^i (1-t)^{m-i}$, $t \in I$

Then $B_m(f)$ is called the Bernstein approximation to f of degree m.

The Bernstein approximation has the following properties :

(i) $\mu\{B_m(f)\} \leqslant \dot{Z}\{B_m(f)\} \leqslant \mu\{f\}$

where $\dot{Z}\{B_m(f)\}$ denotes the zeros of $B_m(f)$ on I^0.

(ii) If $f \in C^n(I)$, $B_m^{(P)}(f) \longrightarrow f^{(P)}$ uniformly as $m \longrightarrow \infty$, $p = 0,1,\ldots\ldots n$

(iii) $$B_m^{(P)}(f)\Big|_{t=0} = m!/(m-P)! \sum_{j=0}^{P} (-1)^{P-j}\binom{P}{j} f\left(\frac{j}{m}\right)$$

$$B_m^{(P)}(f)\Big|_{t=1} = m!/(m-P)! \sum_{j=0}^{P} (-1)^{j}\binom{P}{j} f\left(\frac{m-j}{m}\right)$$

which means that these are simply functions of $f(0), f(\frac{1}{m})\ldots\ldots\ldots f(\frac{P}{m})$ and $f(1), f(\frac{m-1}{m}),\ldots\ldots f(\frac{m-P}{m})$

(iv) If $P:I \longrightarrow R$ is a polynomial of degree m then there is a unique set $\{ f(0), f(\frac{1}{m}),\ldots\ldots f(1) \}$ defined by

$$f\left(\frac{j}{m}\right) = \sum_{r=0}^{j} \binom{j}{r}\{(m-r)!/m!\}P^{(r)}\Big|_{t=0}$$

$$= \sum_{r=0}^{m-j} (-1)^r \binom{m-j}{r}\{(m-r)!/m!\}P^{(r)}\Big|_{t=1}$$

(v) If $\alpha_p \leqslant f^{(P)} \leqslant \beta_p$ then

$$\alpha_p \leqslant m^P(m-P)!/m!\{B_m^{(P)}(f)\} \leqslant \beta_p \text{ for } p = 1,\ldots m$$

and $\alpha_0 \leqslant \beta_m(f) \leqslant \beta_0$.

(vi) If $g:I \longrightarrow R$ is linear, then from (i)

$$\dot{Z}\{B_m(f)-g\} = \dot{Z}\{B_m(f-g)\} \leqslant \mu\{f-g\}$$

which means that $B_m(f)$ does not oscillate about f.

Definition 2.12

Let $\underline{r}:I \longrightarrow E^3$ be given by

$$\underline{r}(u) = \sum_{i=0}^{m} \lambda_i(u)\underline{d}_i \quad , \quad \underline{d}_i \in E^3 \quad i=0,\ldots\ldots m$$

Then $\underline{r}$ is the Bezier curve of degree m for $\{\underline{d}_i \epsilon E^3\}$.

Figure 5 shows typical Bezier curves for m = 3,5 and 6.

We see that $\underline{r}(0) = \underline{d}_0$ and $\underline{r}(1) = \underline{d}_m$, and since

$\lambda_i(u) > 0$, $u \epsilon I^0$ and

$\sum_{i=0}^{m} \binom{m}{i} u^i (1-u)^{m-i} = \left[(1-u)+u\right]^m = 1$ then if follows

that $\underline{r}$ lies within the convex hull formed by the $\underline{d}_i$.

We confine our attention now to Bezier Cubic curves. For any set of points $\left\{ \underline{d}_i \epsilon E^3 \right\}_{i=0}^{3}$ then the corresponding Bezier cubic curve is given by

$$r(u) = \sum_{i=0}^{3} \lambda_i(u)\underline{d}_i$$

where $\lambda_i(u) = \binom{3}{i} u^i (1-u)^{3-i}$, $u \epsilon I$.

We shall see that the points $\underline{d}_i$ control the geometry of $\underline{r}$ in a nice way. Firstly, the curve contains $\underline{d}_0 = \underline{r}(0)$ and $\underline{d}_3 = \underline{r}(1)$ and the tangent directions at $\underline{d}_0$ and $\underline{d}_1$ are

$$\underline{t}(0) = 3(\underline{d}_1 - \underline{d}_0)/\dot{s}(0) \quad \text{and}$$

$$\underline{t}(1) = 3(\underline{d}_3 - \underline{d}_2)/\dot{s}(1)$$

Then since the equation for the osculating plane of $\underline{r}$ is given by

$$\left[\underline{R}-\underline{r} , \dot{\underline{r}} , \ddot{\underline{r}} \right] = 0$$

at u = 0 we have

$$\left[\underline{R}-\underline{d}_0, 3(\underline{d}_1-\underline{d}_0), 6(\underline{d}_0-2\underline{d}_1+\underline{d}_2) \right] = 0$$

This means that the osculating plane at $\underline{r}(0)$ contains $\underline{d}_0$, $\underline{d}_1$ and $\underline{d}_2$.

Similarly the osculating place at $\underline{r}(1)$ contains $\underline{d}_1$, $\underline{d}_2$ and $\underline{d}_3$. Looking at this another way this means that $\underline{d}_2$ is the point of intersection of the osculating plane at $\underline{d}_0$ with the tangent line at $\underline{d}_3$. And similarly for $\underline{d}_1$. Unfortunately these are not unique if the curve is planar.

Further since $\kappa(u) = |\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u)|/\dot{s}^3(u)$ we see that (Figure 5) :

$$\kappa(0) = 2|(\underline{d}_0-\underline{d}_1) + (\underline{d}_2-\underline{d}_1)|\sin\theta/3|\underline{d}_1-\underline{d}_0|^2$$

$$\kappa(1) = 2|(\underline{d}_3-\underline{d}_2) + (\underline{d}_2-\underline{d}_1)|\sin\phi/3|\underline{d}_3-\underline{d}_2|^2$$

and since $\tau(u) = [\dot{\underline{r}}(u), \ddot{\underline{r}}(u), \dddot{\underline{r}}(u)]/(\dot{\underline{r}}(u) \times \ddot{\underline{r}}(u))^2$

$$\tau(0) = 2|(\underline{d}_3-\underline{d}_0) + 3(\underline{d}_1-\underline{d}_2)|\cos\alpha \,/\, \zeta_1 \quad \text{where}$$

$$\zeta_1 = |\underline{d}_1-\underline{d}_0||(\underline{d}_0-\underline{d}_1) + (\underline{d}_2-\underline{d}_1)|\sin\theta$$

$$\tau(1) = 2|(\underline{d}_3-\underline{d}_0) + 3(\underline{d}_1-\underline{d}_2)|\cos\beta \,/\, \zeta_2 \quad \text{where}$$

$$\zeta_2 = |\underline{d}_3-\underline{d}_2||(\underline{d}_3-\underline{d}_2) + (\underline{d}_2-\underline{d}_1)|\sin\phi$$

One other curve form of particular interest {Figure 6} is a spline analog of the Bezier form {17} and this provides a type of moving average formulation for a series of points in E^3 where in general the resulting curve does not contain the points but is generated from them in a predictable manner as far as the shape is concerned.

<u>Definition 2.13</u>

Let $t_0, t_1, \ldots, t_n$ be real numbers with $t_0 \leq t_1 \leq \ldots \leq t_n$.

Then a Polynomial Spline of order k is a map $\mathcal{S} : (t_0, t_n) \longrightarrow \mathbb{R}$ satisfying

(i) $\mathcal{S}\big|(t_i, t_{i+1})$ is a polynomial of degree k-1

(ii) $\mathcal{S} \in C^{k-2}$

The real numbers t_i are known as Knots and for fixed k and n such splines form a Vector Space over $\mathbb{R}$. A particular basis for this vector space is called the B-Spline basis and this was first introduced by Schoenberg {24}.

Definition 2.14

For reals t_0, t_1, ..., t_n the B-Spline basis of order k is given by the following recursive formulation.

$$B_{i,1}(t) = 1 \quad \text{for } t_i \leqslant t \leqslant t_{i+1}$$

$$B_{i,1}(t) = 0 \quad \text{for } t_i > t > t_{i+1}$$

$$B_{i,k}(t) = \{(t-t_i)/(t_{i+k-1}-t_i)\}B_{i,k-1}(t) + \{(t_{i+k}-t)/(t_{i+k}-t_{i+1})\}B_{i+1,k-1}(t)$$

There are a number of alternative formulations {17}.

B-Splines have the following properties :

(i) $B_{i,k}(t) > 0 \quad \text{for } t_i \leqslant t \leqslant t_{i+1}$

$B_{i,k}(t) = 0 \quad \text{for } t_i > t > t_{i+1}$

(ii) $\sum_i B_{i,k}(t) = 1$

(iii) If r knots are coincident, say

$t_\alpha = t_{\alpha+1} = \ldots\ldots = t_{\alpha+r-1}$ then continuity is reduced and $B_{j,k} \in C^{k-r-1}$, $j=\alpha, \alpha+1, \ldots, \alpha+r-1$.

(iv) Calculations greatly simplify if $t_m = m \in N$, for each m.

(v) In the particular case where there are just two knots at m = 0 and m = 1 each with multiplicity k then the B-Spline basis reduces to the Bernstein basis with

$$B_{i,k}(t) = \binom{k-1}{i} t^i (1-t)^{k-1-i}, \quad i=0,\ldots\ldots,k-1$$

(vi) The kth order B-Spline approximation to $f:[0,1] \longrightarrow \mathbb{R}$ is given by

$$B_k(f) = \sum_i f(\xi_i) B_{i,k} \quad \text{where the}$$

$$\xi_i = (t_{i+1} + t_{i+2} + \ldots\ldots + t_{i+k-1})/(k-1)$$

are termed the Nodes.

(vii) B-Spline approximation is variation diminishing.

(viii) In contrast to the Bernstein approximation, local changes in a function f produce only local changes in $B_k(f)$.

Definition 2.15

The B-spline curve form of order k for the set of points $\{\underline{P}_i\} \subset E^3$, i=0,.....,n is given by

$$\underline{r}(u) = \sum_{i=0}^{n} B_{i-\frac{k}{2},k}(u) \underline{P}_i$$

The B-spline curve has the following basic properties :

(i) If k is even it can be shown that the knots of the spline are 'opposite' the $\underline{P}_i$ whereas if k is odd then the knots are 'opposite' the points

$\underline{Q}_i = (\underline{P}_i + \underline{P}_{i+1})/2$

(ii) Since the curve can be regarded locally as a convex combination of $\underline{P}_\alpha, \ldots\ldots, \underline{P}_{\alpha+k-1}$ the curve lies within a series of local convex hulls.

(iii) For a given set $\{\underline{P}_i\}$, $i=0\ldots.n$ then if $k=2$, then $\underline{r} \epsilon C^0$ and interpolates the $\underline{P}_i$ linearly. As k tends to infinity then $\underline{r}$ tends to a single straight line.

(iv) If $\underline{r}$ is of order k and $\underline{P}_{\tau+1}, \underline{P}_{\tau+2}, \ldots \underline{P}_{\tau+k}$ are collinear for some τ then $\underline{r}$ contains a linear segment on the line through $\underline{P}_\alpha$ and $\underline{P}_\beta$ where $\tau+1 \leqslant \beta < \alpha \leqslant \tau+k$.

(v) If $\underline{r}$ is of order k and $\underline{P}_{\tau+1}, \underline{P}_{\tau+2}, \ldots\ldots, \underline{P}_{\tau+k-1}$ are collinear then $\underline{r}$ touches the line through $\underline{P}_\alpha$ and $\underline{P}_\beta$ where $\tau+1 \leqslant \beta < \alpha \leqslant \tau+k-1$.

The B-spline cubic curve for the sequence $\underline{P}_\alpha$ of points in E^3 is of the form

$$\underline{r}(u) = \sum_\alpha B(u-\alpha)\underline{P}_\alpha$$

where the B-spline scalar weighting function B(u) is defined by

$$B(u) = \left[\xi_3(u-2) - 4.\xi_3(u-1) + 6.\xi_3(u) - 4.\xi_3(u+1) + \xi_3(u+2)\right]/6$$

where $\xi_k(a) = a^k$ for $a \geqslant 0$

and $\xi_k(a) = 0$ for $a < 0$

Explicitly $\underline{r}$ is given by

$$\underline{r}(u) = \{\xi_3(u-i-1)-4.\xi_3(u-i)+6.\xi_3(u-i+1)-4.\xi_3(u-i+2)+ \xi_3(u-i+3)\}\underline{P}_{i-1}/6$$
$$+\{\xi_3(u-i-2)-4.\xi_3(u-i-1)+6.\xi_3(u-i)-4.\xi_3(u-i+1)+ \xi_3(u-i+2)\}\underline{P}_{i}/6$$
$$+\{\xi_3(u-i-3)-4.\xi_3(u-i-2)+6.\xi_3(u-i-1)-4.\xi_3(u-i)+ \xi_3(u-i+1)\}\underline{P}_{i+1}/6$$
$$+\{\xi_3(u-i-4)-4.\xi_3(u-i-3)+6.\xi_3(u-i-2)-4.\xi_3(u-i-1)+ \xi_3(u-i)\}\underline{P}_{i+2}/6$$
$$\text{for } i \leqslant u \leqslant i+1$$

which means that $\underline{r}(i) = \{\underline{P}_{i-1}+4\underline{P}_i+\underline{P}_{i+1}\}/6$

Thus if $\underline{P}_{i-1} = \underline{P}_i = \underline{P}_{i+1}$ then $\underline{r}(i) = \underline{P}_i$

Differentiation with respect to u gives

$$\dot{\underline{r}}(u) = \{\xi_2(u-i-1)-4.\xi_2(u-i)+6.\xi_2(u-i+1)-4.\xi_2(u-i+2)+ \xi_2(u-i+3)\}\underline{P}_{i-1}/2$$
$$+\{\xi_2(u-i-2)-4.\xi_2(u-i-1)+6.\xi_2(u-i)-4.\xi_2(u-i+1)+ \xi_2(u-i+2)\}\underline{P}_{i}/2$$
$$+\{\xi_2(u-i-3)-4.\xi_2(u-i-2)+6.\xi_2(u-i-1)-4.\xi_2(u-i)+ \xi_2(u-i+1)\}\underline{P}_{i+1}/2$$
$$+\{\xi_2(u-i-4)-4.\xi_2(u-i-3)+6.\xi_2(u-i-2)-4.\xi_2(u-i-1)+ \xi_2(u-i)\}\underline{P}_{i+2}/2$$

and so

$$\dot{\underline{r}}(i) = \{\underline{P}_{i+1}-\underline{P}_{i-1}\}/2$$

Thus the speed $\dot{s}(i) = 0$ if and only if $\underline{P}_{i+1} = \underline{P}_{i-1}$.
Also we can see that if $\underline{P}_{i-1}$, $\underline{P}_i$, $\underline{P}_{i+1}$, $\underline{P}_{i+2}$ are not coplanar then $\dot{s}(u) \neq 0$ in $i \leqslant u \leqslant i+1$.

Differentiating again gives

$$\underline{\ddot{r}}(u) = \{\xi_1(u-i-1)-4.\xi_1(u-i)+6.\xi_1(u-i+1)-4.\xi_1(u-i+2)+\xi_1(u-i+3)\}\underline{P}_{i-1}$$
$$+\{\xi_1(u-i-2)-4.\xi_1(u-i-1)+6.\xi_1(u-i)-4.\xi_1(u-i+1)+\xi_1(u-i+2)\}\underline{P}_i$$
$$+\{\xi_1(u-i-3)-4.\xi_1(u-i-2)+6.\xi_1(u-i-1)-4.\xi_1(u-i)+\xi_1(u-i+1)\}\underline{P}_{i+1}$$
$$+\{\xi_1(u-i-4)-4.\xi_1(u-i-3)+6.\xi_1(u-i-2)-4.\xi_1(u-i-1)+\xi_1(u-i)\}\underline{P}_{i+2}$$

and so

$$\underline{\ddot{r}}(i) = \underline{P}_{i-1}-2\underline{P}_i+\underline{P}_{i+1}$$

Since

$$\underline{\dddot{r}}(u) = \{\xi_0(u-i-1)-4.\xi_0(u-i)+6.\xi_0(u-i+1)-4.\xi_0(u-i+2)+\xi_0(u-i+3)\}\underline{P}_{i-1}$$
$$+\{\xi_0(u-i-2)-4.\xi_0(u-i-1)+6.\xi_0(u-i)-4.\xi_0(u-i+1)+\xi_0(u-i+2)\}\underline{P}_i$$
$$+\{\xi_0(u-i-3)-4.\xi_0(u-i-2)+6.\xi_0(u-i-1)-4.\xi_0(u-i)+\xi_0(u-i+1)\}\underline{P}_{i+1}$$
$$+\{\xi_0(u-i-4)-4.\xi_0(u-i-3)+6.\xi_0(u-i-2)-4.\xi_0(u-i-1)+\xi_0(u-i)\}\underline{P}_{i+2}$$

then

$\underline{\dddot{r}}(i) = 3(\underline{P}_{i-1}-\underline{P}_i)+\underline{P}_{i+1}$ and so we find that the curvature is

$$\kappa(i) = 8\sqrt{\{(\underline{P}_{i+1}\times\underline{P}_{i-1})+(\underline{P}_{i-1}\times\underline{P}_i)+(\underline{P}_i\times\underline{P}_{i+1})\}^2}/\{(\underline{P}_{i+1}-\underline{P}_{i-1})^2\}^3$$

and the torsion,

$$\tau(i) = \left[\underline{P}_{i-1},\underline{P}_i,\underline{P}_{i+1}\right]/\{(\underline{P}_{i+1}\times\underline{P}_{i-1})+(\underline{P}_{i-1}\times\underline{P}_i)+(\underline{P}_i\times\underline{P}_{i+1})\}^2$$

The cubic weighting function B(u), which satisfies

(i) $B(u) \geq 0 \;\forall\, u$

(ii) $\sum_{\alpha} B(u-\alpha) = 1$

(iii) there exists $\alpha \in R$ such that $B(u) = 0$ for all $|u| > \alpha$,

means that we can use the $\{\underline{P}_\alpha\}$ as a means of shape prediction and modification of the B-spline cubic curve and so this curve form is particularly attractive for a design system from the designers' point of view.

Chapter 3

Computer Surfaces in E^3

Definition 3.1

A Bicubic surface in E^3 is one for which there exist parameters u and v in terms of which the equation of the surface can be written as :

$$\underline{r}(u,v) = \sum_{i,j=0}^{3} \underline{a}_{ij} u^i v^j / i!j! \qquad \underline{a}_{ij} \in E^3$$

The parameters u and v are indeterminate to within transformations of the form $u \longmapsto \alpha u + \beta$ and $v \longmapsto \gamma v + \delta$ but if we wish the surface to pass through four particular points in E^3 with specified parameter values then the parameters are uniquely determined.

For convenience we will normally assume that $\underline{r}$ is defined on $[0,1] \times [0,1]$ and the four points chosen are the four distinguishable boundary points $\underline{r}(i,j)$ for $i,j \in \{0,1\}$. The four boundary curves $\underline{r}(u,i)$, $\underline{r}(i,v)$ $i \in \{0,1\}$ are cubic curves in the parameters u or v.

We will use $\underline{r}_1$ and $\underline{r}_2$ to denote the tangent vectors

$\partial \underline{r}(u,v)/\partial u$ and $\partial \underline{r}(u,v)/\partial v$ respectively and $\underline{N}$ for the unit surface normal where $\underline{N} = (\underline{r}_1 \times \underline{r}_2)/|\underline{r}_1 \times \underline{r}_2|$. We also use x,y to denote the usual dot product or inner product of vectors x,y.

If $\underline{\lambda} : [a,b] \longrightarrow S$ is a curve on a surface S then $\underline{\lambda}(t) = \underline{r}(u(t),v(t))$ and since the arc length s along $\underline{\lambda}$ is given by $s = \int_a^b \langle \dot{\underline{\lambda}}(t), \dot{\underline{\lambda}}(t) \rangle \, dt$ then along the parametric curves $v = v_c$ and $u = u_c$

$$s = \int_0^1 \sqrt{E(u,v_c)}\,du \quad \text{and} \quad s = \int_0^1 \sqrt{G(u_c,v)}\,dv.$$

where $E(u,v) = \langle \underline{r}_1, \underline{r}_1 \rangle$, $F(u,v) = \langle \underline{r}_1, \underline{r}_2 \rangle$,

$G(u,v) = \langle \underline{r}_2, \underline{r}_2 \rangle$ are the First Fundamental Coefficients.

Proposition 3.2

Given $\{ \underline{\alpha}_{ij}, \underline{\beta}_{ij}, \underline{\gamma}_{ij} \} \subset E^3$ where $i,j \in \{0,1\}$ then there are ∞^{12} bicubic surfaces $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ such that :

(i) $\underline{r}(i,j) = \underline{\alpha}_{ij}$

(ii) $\underline{r}_1(i,j) = \underline{\beta}_{ij}$

(iii) $\underline{r}_2(i,j) = \underline{\gamma}_{ij}$ $\quad i,j \in \{0,1\}$

This follows because there are twelve degrees of freedom remaining for specifying the sixteen coefficients of the surface equation.

Corollary 1

Given $\{\underline{\alpha}_{ij}, \underline{\beta}_{ij}, \underline{\gamma}_{ij}, \underline{n}_{ij}\} \subset E^3$, $i,j \in \{0,1\}$ then these determine a unique bicubic surface $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ satisfying :

(i) $\underline{r}(i,j) = \underline{\alpha}_{ij}$,
(ii) $\underline{r}_1(i,j) = \underline{\beta}_{ij}$,
(iii) $\underline{r}_2(i,j) = \underline{\gamma}_{ij}$,
(iv) $\underline{r}_{12}(i,j) = \underline{n}_{ij}$

The vector $\underline{r}_{12}$ is often called the 'twist' vector which is a misnomer because it is not an intrinsic property of the surface. To see this if we consider the sphere $\underline{r} : [0,\pi] \times [0,2\pi] \longrightarrow E^3$ where

$\underline{r}(u,v) = (a \sin u \cos v, a \sin u \sin v, a \cos v)$, then

$\underline{r}_{12} = \underline{0}$ if and only if $u = \pi/2$.

Obviously the above Corollary would also hold if we replaced condition (iv) by either $\underline{r}_{11}(i,j) = \underline{n}_{ij}$ or $\underline{r}_{22}(i,j) = \underline{n}_{ij}$.

Corollary 2

Given $\{\underline{\alpha}_{ij}, \underline{\beta}_{ij}, \underline{\gamma}_{ij}\} \subset E^3$ and reals L_{ij}, M_{ij}, N_{ij} where $i,j \in \{0,1\}$ then there is a unique bicubic surface $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ such that

(i) $\underline{r}(i,j) = \underline{\alpha}_{ij}$,
(ii) $\underline{r}_1(i,j) = \underline{\beta}_{ij}$,
(iii) $\underline{r}_2(i,j) = \underline{\gamma}_{ij}$

(iv) $\langle \underline{r}_{11}(i,j), \underline{N}(i,j) \rangle = L_{ij}$

$\langle \underline{r}_{12}(i,j), \underline{N}(i,j) \rangle = M_{ij}$

$\langle \underline{r}_{22}(i,j), \underline{N}(i,j) \rangle = N_{ij}$

Then L_{ij}, M_{ij}, N_{ij} are the values of the Second Fundamental Coefficients $L(u,v)$, $M(u,v)$, $N(u,v)$ at the four corners $\underline{r}(i,j)$.

Corollary 3

Given $\{\underline{\alpha}_{ij}, \underline{\beta}_{ij}, \underline{\gamma}_{ij}\} \subset E^3$ and reals H_{ij}, K_{ij} $i,j \varepsilon \{0,1\}$ then there are ∞^4 bicubic surfaces $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ such that :

(i) $\underline{r}(i,j) = \underline{\alpha}_{ij}$,

(ii) $\underline{r}_1(i,j) = \underline{\beta}_{ij}$,

(iii) $\underline{r}_2(i,j) = \underline{\gamma}_{ij}$

(iv) The mean $H(u,v)$ and Gaussian $K(u,v)$ curvatures take the values $H(i,j) = H_{ij}$ and $K(i,j) = K_{ij}$ for $i,j \varepsilon \{0,1\}$.

Corollary 4

Given $\{\underline{\alpha}_{ij}\} \subset E^3$ and reals E_{ij}, F_{ij}, G_{ij}, L_{ij}, M_{ij}, N_{ij} for $i,j \varepsilon \{0,1\}$ then there are ∞^{12} bicubic surfaces $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ such that

(i) $\underline{r}(i,j) = \underline{\alpha}_{ij}$

(ii) $E(i,j) = E_{ij}$, $F(i,j) = F_{ij}$, $G(i,j) = G_{ij}$,

$L(i,j) = L_{ij}$, $M(i,j) = M_{ij}$, $N(i,j) = N_{ij}$ are the values taken by the fundamental coefficients at the corners.

Each of the coefficients $\underline{a}_{ij}$ in the surface equation can be expressed in terms of $\underline{r}$ and its derivatives on the boundary. The boundary curves :

$$\underline{r}(u,o) = \underline{a}_{00} + \underline{a}_{10}u + \underline{a}_{20}u^2/2! + \underline{a}_{30}u^3/3! \quad \text{and}$$

$$\underline{r}(o,v) = \underline{a}_{00} + \underline{a}_{01}v + \underline{a}_{02}v^2/2! + \underline{a}_{03}v^3/3!$$

are cubic curves and so we can write

(i) $\underline{a}_{00} = \underline{r}(0,0)$

(ii) $\underline{a}_{10} = \underline{r}_1(0,0)$

(iii) $\underline{a}_{01} = \underline{r}_2(0,0)$

(iv) $\underline{a}_{20} = 2!\left[3\{\underline{r}(1,0)-\underline{r}(0,0)\} - 2\underline{r}_1(0,0)-\underline{r}_1(1,0)\right]$
$= \underline{r}_{11}(0,0)$

(v) $\underline{a}_{02} = 2!\left[3\{\underline{r}(0,1)-\underline{r}(0,0)\} - 2\underline{r}_2(0,0)-\underline{r}_2(0,1)\right]$
$= \underline{r}_{22}(0,0)$

(vi) $\underline{a}_{30} = 3!\left[2\{\underline{r}(0,0)-\underline{r}(1,0)\}+ \underline{r}_1(0,0)+\underline{r}_1(1,0)\right]$
$= \underline{r}_{111}(u,0)$

(vii) $\underline{a}_{03} = 3!\left[2\{\underline{r}(0,0)-\underline{r}(0,1)\}+ \underline{r}_2(0,0)+\underline{r}_2(0,1)\right]$
$= \underline{r}_{222}(0,v)$

Hence these coefficients have the significance of the cubic curve coefficients as far as the boundary curves $\underline{r}(u,0)$ and $\underline{r}(0,v)$ are concerned.

It is elementary but tedious to show that :

(viii) $\underline{a}_{11} = \underline{r}_{12}(0,0)$

(ix) $$\underline{a}_{21} = 2\left[3\{\underline{r}_2(1,0)-\underline{r}_2(0,0)\} - \underline{r}_{12}(1,0)-2\underline{r}_{12}(0,0)\right] = \underline{r}_{121}(0,0)$$

(x) $$\underline{a}_{12} = 2\left[3\{\underline{r}_1(0,1)-\underline{r}_1(0,0)\} - \underline{r}_{12}(0,1)-2\underline{r}_{12}(0,0)\right] = \underline{r}_{212}(0,0)$$

(xi) $$\underline{a}_{31} = 6\left[2\{\underline{r}_2(0,0)-\underline{r}_2(1,0)\} + \underline{r}_{12}(1,0)+\underline{r}_{12}(0,0)\right] = \underline{r}_{1112}(u,0)$$

(xii) $$\underline{a}_{13} = 6\left[2\{\underline{r}_1(0,0)-\underline{r}_1(0,1)\} + \underline{r}_{12}(0,1)+\underline{r}_{12}(0,0)\right] = \underline{r}_{2221}(0,v)$$

(xiii) $$\begin{aligned}\underline{a}_{22} &= 4\Big[9\{\underline{r}(0,0)+\underline{r}(1,1)-\underline{r}(1,0)-\underline{r}(0,1)\}+ \\ &\quad 3\{2(\underline{r}_1(0,0)-\underline{r}_1(0,1)) + \underline{r}_1(1,0)-\underline{r}_1(1,1)\} + \\ &\quad 3\{2(\underline{r}_2(0,0)-\underline{r}_2(1,0)) + \underline{r}_2(0,1)-\underline{r}_2(1,1)\} + \\ &\quad 4\underline{r}_{12}(0,0)+\underline{r}_{12}(1,1)+2(\underline{r}_{12}(1,0)+\underline{r}_{12}(0,1))\Big] \\ &= \underline{r}_{1212}(0,0)\end{aligned}$$

(xiv) $$\begin{aligned}\underline{a}_{32} &= -12\Big[6\{\underline{r}(0,0)+\underline{r}(1,1)-\underline{r}(1,0)-\underline{r}(0,1)\} - \\ &\quad 3\{\underline{r}_1(1,1)-\underline{r}_1(0,0)+\underline{r}_1(0,1)-\underline{r}_1(1,0)\} - \\ &\quad 2\{\underline{r}_2(1,1)-\underline{r}_2(0,1)+2(\underline{r}_2(1,0)-\underline{r}_2(0,0))\} + \\ &\quad \{\underline{r}_{12}(1,1)+2(\underline{r}_{12}(0,0)+\underline{r}_{12}(1,0))+\underline{r}_{12}(0,1)\}\Big] \\ &= \underline{r}_{11122}(u,0)\end{aligned}$$

(xv) $$\begin{aligned}\underline{a}_{23} &= -12\Big[6\{\underline{r}(0,0)+\underline{r}(1,1)-\underline{r}(1,0)-\underline{r}(0,1)\} - \\ &\quad 2\{\underline{r}_1(1,1)-\underline{r}_1(1,0)+2(\underline{r}_1(0,1)-\underline{r}_1(0,0))\} - \\ &\quad 3\{\underline{r}_2(1,1)-\underline{r}_2(0,0)+\underline{r}_2(1,0)-\underline{r}_2(0,1)\} + \\ &\quad \{\underline{r}_{12}(1,1)+2(\underline{r}_{12}(0,0)+\underline{r}_{12}(0,1))+\underline{r}_{12}(0,1)\}\Big] \\ &= \underline{r}_{22211}(u,0)\end{aligned}$$

(xvi) $$\begin{aligned}\underline{a}_{33} &= 36\Big[4\{\underline{r}(0,0)+\underline{r}(1,1)-\underline{r}(1,0)-\underline{r}(0,1)\} - \\ &\quad 2\{\underline{r}_1(1,1)-\underline{r}_1(0,0)+\underline{r}_1(0,1)-\underline{r}_1(1,0)\} - \\ &\quad 2\{\underline{r}_2(1,1)-\underline{r}_2(0,0)+\underline{r}_2(1,0)-\underline{r}_2(0,1)\} + \\ &\quad \{\underline{r}_{12}(1,1)+\underline{r}_{12}(0,0)+\underline{r}_{12}(1,0)+\underline{r}_{12}(0,1)\}\Big] \\ &= \underline{r}_{222111}(u,v)\end{aligned}$$

The surface equation coefficients and the second fundamental coefficients are related in the following manner :

(i) at u=0, v=0

$$L = [\underline{a}_{10}, \underline{a}_{01}, \underline{a}_{20}]/|\underline{a}_{10} \times \underline{a}_{01}|$$

$$M = [\underline{a}_{10}, \underline{a}_{01}, \underline{a}_{11}]/|\underline{a}_{10} \times \underline{a}_{01}|$$

$$N = [\underline{a}_{10}, \underline{a}_{01}, \underline{a}_{02}]/|\underline{a}_{10} \times \underline{a}_{01}|$$

(ii) at u=1, v=0

$$L = [\underline{\alpha}_{10}, \underline{\beta}_{10}, \underline{a}]/|\underline{\alpha}_{10} \times \underline{\beta}_{10}|$$

$$M = [\underline{\alpha}_{10}, \underline{\beta}_{10}, \underline{b}]/|\underline{\alpha}_{10} \times \underline{\beta}_{10}|$$

$$N = [\underline{\alpha}_{10}, \underline{\beta}_{10}, \underline{c}]/|\underline{\alpha}_{10} \times \underline{\beta}_{10}|$$

where

$$\underline{\alpha}_{10} = \underline{a}_{10} + \underline{a}_{20} + \underline{a}_{30}/2! = \underline{r}_1(1,0)$$

$$\underline{\beta}_{10} = \underline{a}_{01} + \underline{a}_{11} + \underline{a}_{21}/2! + \underline{a}_{31}/3! = \underline{r}_2(1,0)$$

$$\underline{a} = \underline{a}_{20} + \underline{a}_{30} = \underline{r}_{11}(1,0)$$

$$\underline{b} = \underline{a}_{11} + \underline{a}_{21} + \underline{a}_{31}/2! = \underline{r}_{12}(1,0)$$

$$\underline{c} = \underline{a}_{02} + \underline{a}_{12} + \underline{a}_{22}/2! + \underline{a}_{32}/3! = \underline{r}_{22}(1,0)$$

(iii) at u=0, v=1

$$L = [\underline{\alpha}_{01}, \underline{\beta}_{01}, \underline{d}]/|\underline{\alpha}_{01} \times \underline{\beta}_{01}|$$

$$M = [\underline{\alpha}_{01}, \underline{\beta}_{01}, \underline{e}]/|\underline{\alpha}_{01} \times \underline{\beta}_{01}|$$

$$N = [\underline{\alpha}_{01}, \underline{\beta}_{01}, \underline{f}]/|\underline{\alpha}_{01} \times \underline{\beta}_{01}|$$

where

$$\underline{\alpha}_{01} = \underline{a}_{10} + \underline{a}_{11} + \underline{a}_{12}/2! + \underline{a}_{13}/3! = \underline{r}_1(0,1)$$

$$\underline{\beta}_{01} = \underline{a}_{01} + \underline{a}_{02} + \underline{a}_{03}/2! = \underline{r}_2(0,1)$$

$$\underline{d} = \underline{a}_{20} + \underline{a}_{21} + \underline{a}_{22}/2! + \underline{a}_{23}/3! = \underline{r}_{11}(0,1)$$

$$\underline{e} = \underline{a}_{11} + \underline{a}_{12} + \underline{a}_{13}/2! = \underline{r}_{12}(0,1)$$

$$\underline{f} = \underline{a}_{02} + \underline{a}_{03} = \underline{r}_{22}(0,1)$$

(iv) at u=1, v=1

$$L = [\underline{\alpha}_{11}, \underline{\beta}_{11}, \underline{g}] / |\underline{\alpha}_{11} \times \underline{\beta}_{11}|$$

$$M = [\underline{\alpha}_{11}, \underline{\beta}_{11}, \underline{h}] / |\underline{\alpha}_{11} \times \underline{\beta}_{11}|$$

$$N = [\underline{\alpha}_{11}, \underline{\beta}_{11}, \underline{l}] / |\underline{\alpha}_{11} \times \underline{\beta}_{11}|$$

where $\underline{\alpha}_{11} = \sum_{i=1}^{3} \sum_{j=0}^{3} \underline{a}_{ij}/(i-1)!j! = \underline{r}_1(1,1)$

$$\underline{\beta}_{11} = \sum_{i=0}^{3} \sum_{j=1}^{3} \underline{a}_{ij}/i!(j-1)! = \underline{r}_2(1,1)$$

$$\underline{g} = \underline{a}_{20} + \underline{a}_{30} + \underline{a}_{21} + \underline{a}_{31} + (\underline{a}_{22}+\underline{a}_{32})/2! + (\underline{a}_{23}+\underline{a}_{33})/3! = \underline{r}_{11}(1,1)$$

$$\underline{h} = \underline{a}_{11} + \underline{a}_{21} + \underline{a}_{12} + \underline{a}_{22} + (\underline{a}_{31}+\underline{a}_{13}+\underline{a}_{32}+\underline{a}_{23})/2! + \underline{a}_{33}/4 = \underline{r}_{12}(1,1)$$

$$\underline{l} = \underline{a}_{02} + \underline{a}_{03} + \underline{a}_{12} + \underline{a}_{13} + (\underline{a}_{22}+\underline{a}_{23})/2! + (\underline{a}_{32}+\underline{a}_{33})/3! = \underline{r}_{22}(1,1)$$

We can now derive the expressions for the Gaussian and mean curvatures evaluated at the corners in terms of the surface equation coefficients.

(i) at u=0, v=0

$$K = (\lambda_{20}\lambda_{02}-\lambda_{11}^2)/(\xi_{00}|\underline{a}_{10}\times\underline{a}_{01}|^2)$$

$$H = (\lambda_{02} \langle\underline{a}_{10},\underline{a}_{10}\rangle + \lambda_{20} \langle\underline{a}_{01},\underline{a}_{01}\rangle - 2\lambda_{11}\langle\underline{a}_{10},\underline{a}_{01}\rangle) / (2\xi_{00}|\underline{a}_{10}\times\underline{a}_{01}|)$$

where $\lambda_{20} = [\underline{a}_{10},\underline{a}_{01},\underline{a}_{20}]$

$$\lambda_{02} = [\underline{a}_{10},\underline{a}_{01},\underline{a}_{02}]$$

$$\lambda_{11} = [\underline{a}_{10},\underline{a}_{01},\underline{a}_{11}]$$

$$\xi_{00} = \langle\underline{a}_{10},\underline{a}_{10}\rangle\langle\underline{a}_{01},\underline{a}_{01}\rangle - \langle\underline{a}_{10},\underline{a}_{01}\rangle^2$$

(ii) at u=1, v=0

$$K = (\lambda_a\lambda_c-\lambda_b^2)/(\xi_{10}|\underline{\alpha}_{10}\times\underline{\beta}_{10}|)$$

$$H = (\lambda_c<\underline{\alpha}_{10},\underline{\alpha}_{10}> + \lambda_a<\underline{\beta}_{10},\underline{\beta}_{10}> - 2\lambda_b<\underline{\alpha}_{10},\underline{\beta}_{10}>)/ (2\xi_{10}|\underline{\alpha}_{10} \times \underline{\beta}_{10}|)$$

where $\lambda_a = [\underline{\alpha}_{10},\underline{\beta}_{10},\underline{a}]$

$\lambda_b = [\underline{\alpha}_{10},\underline{\beta}_{10},\underline{b}]$

$\lambda_c = [\underline{\alpha}_{10},\underline{\beta}_{10},\underline{c}]$

$\xi_{10} = <\underline{\alpha}_{10},\underline{\alpha}_{10}><\underline{\beta}_{10},\underline{\beta}_{10}> - <\underline{\alpha}_{10},\underline{\beta}_{10}>^2$

(iii) at u=0, v=1

$$K = (\lambda_d\lambda_f-\lambda_e^2)/(\xi_{01}|\underline{\alpha}_{01}\times\underline{\beta}_{01}|^2)$$

$$H = (\lambda_f<\underline{\alpha}_{01},\underline{\alpha}_{01}> + \lambda_d<\underline{\beta}_{01},\underline{\beta}_{01}> - 2\lambda_e<\underline{\alpha}_{01},\underline{\beta}_{01}>)/ (2\xi_{01}|\underline{\alpha}_{01} \times \underline{\beta}_{01}|)$$

where $\lambda_d = [\underline{\alpha}_{01},\underline{\beta}_{01},\underline{d}]$

$\lambda_e = [\underline{\alpha}_{01},\underline{\beta}_{0i},\underline{e}]$

$\lambda_f = [\underline{\alpha}_{01},\underline{\beta}_{01},\underline{f}]$

$\xi_{01} = <\underline{\alpha}_{01},\underline{\alpha}_{01}><\underline{\beta}_{01},\underline{\beta}_{01}> - <\underline{\alpha}_{01},\underline{\beta}_{01}>^2$

(iv) at u=1, v=1

$$K = (\lambda_g\lambda_i-\lambda_h^2)/(\xi_{11}|\underline{\alpha}_{11}\times\underline{\beta}_{11}|^2)$$

$$H = (\lambda_i<\underline{\alpha}_{11},\underline{\alpha}_{11}> + \lambda_g<\underline{\beta}_{11},\underline{\beta}_{11}> - 2\lambda_h<\underline{\alpha}_{11},\underline{\beta}_{11}>)/ (2\xi_{11}|\underline{\alpha}_{11} \times \underline{\beta}_{11}|)$$

where $\lambda_g = [\underline{\alpha}_{11},\underline{\beta}_{11},\underline{g}]$

$\lambda_h = [\underline{\alpha}_{11},\underline{\beta}_{11},\underline{h}]$

$\lambda_i = [\underline{\alpha}_{11},\underline{\beta}_{11},\underline{i}]$

$\xi_{11} = <\underline{\alpha}_{11},\underline{\alpha}_{11}><\underline{\beta}_{11},\underline{\beta}_{11}> - <\underline{\alpha}_{11},\underline{\beta}_{11}>^2$

In Chapter Two we used the Serret-Frenet equations to reduce conditions involving the derivatives of $\underline{t}$, $\underline{n}$ and $\underline{b}$ to conditions involving $\underline{t}$, $\underline{n}$, $\underline{b}$ and the curvature and torsion scalars κ, τ. Analagously for surfaces, we can use the Gauss-Weingarten equations :

$$\underline{r}_{11}(u,v) = \Gamma^{1}_{11}(u,v)\underline{r}_1(u,v) + \Gamma^{2}_{11}(u,v)\underline{r}_2(u,v) + L(u,v)\underline{N}(u,v)$$
$$\underline{r}_{12}(u,v) = \Gamma^{1}_{12}(u,v)\underline{r}_1(u,v) + \Gamma^{2}_{12}(u,v)\underline{r}_2(u,v) + M(u,v)\underline{N}(u,v)$$
$$\underline{r}_{22}(u,v) = \Gamma^{1}_{22}(u,v)\underline{r}_1(u,v) + \Gamma^{2}_{22}(u,v)\underline{r}_2(u,v) + N(u,v)\underline{N}(u,v)$$
$$\underline{N}_1(u,v) = \beta^{1}_{1}(u,v)\underline{r}_1(u,v) + \beta^{2}_{1}(u,v)\underline{r}_2(u,v)$$
$$\underline{N}_2(u,v) = \beta^{1}_{2}(u,v)\underline{r}_1(u,v) + \beta^{2}_{2}(u,v)\underline{r}_2(u,v)$$

to express the derivatives of $\underline{r}_1$, $\underline{r}_2$ and $\underline{N}$ in terms of $\underline{r}_1$, $\underline{r}_2$, $\underline{N}$ and scalars Γ^{k}_{ij}, the Christoffel symbols of the second kind which depend upon the first fundamental coefficients and their derivatives, and β^{j}_{i}, the Weingarten coefficients which depend upon both the first and second fundamental coefficients. In fact

$$\Gamma^{1}_{11} = (GE_1 - 2FF_1 + FE_2)/(2[EG-F^2])$$
$$\Gamma^{1}_{12} = (GE_2 - FG_1)/(2[EG - F^2])$$
$$\Gamma^{1}_{22} = (2GF_2 - GG2 - FG_2)/(2[EG - F^2])$$
$$\Gamma^{2}_{11} = (2EF_1 - EE_2 + FE_1)/(2[EG - F^2])$$
$$\Gamma^{2}_{12} = (EG_1 - FE_2)/(2[EG - F^2])$$
$$\Gamma^{2}_{22} = (EG_2 - 2FF_2 + FG_1)/(2[EG - F^2])$$
$$\beta^{1}_{1} = (MF - LG)/(EG - F^2)$$
$$\beta^{2}_{1} = (LF - ME)/(EG - F^2)$$
$$\beta^{1}_{2} = (NF - MG)/(EG - F^2)$$
$$\beta^{2}_{2} = (MF - NE)/(EG - F^2)$$

For given functions E, F, G, L, M, N of u and v of sufficiently high class then in general there is no surface

$\underline{r} = \underline{r}(u,v)$ for which E, F, G, L, M, N are the first and second fundamental coefficients unless certain compatibility conditions are satisfied {1} . Using these conditions we can show in fact that K depends only upon the first fundamental coefficients and their derivatives and not upon the second fundamental coefficients.

The first fundamental coefficients have another important role to play and that is in calculating surface area.

Definition 3.3

The surface area of the bicubic surface $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ is given by

$$A = \int_0^1 \int_0^1 \sqrt{[E(u,v)G(u,v) - F^2(u,v)]}\, du dv$$

We can obtain A using Gauss Quadrature where if we write

$$\mu(u,v) = E(u,v)G(u,v) - F^2(u,v)$$

then $A = \sum_i \sum_j k_i\, l_j\, \mu(u_i, v_j)$

Hence we can evaluate A in terms of the coefficients $\underline{a}_{ij}$ or $\underline{r}$ and its derivatives at the corners.

We have now basically covered all of the geometric invariants which could conceivably be used in practice in defining our computer surfaces.

In general bicubics will not provide us with sufficient degrees of freedom to define each surface globally in terms of a single bicubic equation.

Definition 3.4

A Piecewise bicubic surface is a mapping $\underline{R} : [0,m] \times [0,n] \longrightarrow E^3$ where m, n ε N such that $\underline{R} \in C^0 [0,m] \times [0,n]$ and

$\underline{R}\big|[i-1,i] \times [j-1,j] = \underline{r}^{ij}$ $i,j \in N$ $0 < i \leqslant m$, $0 < j \leqslant n$ where $\underline{r}^{ij} : [0,1] \times [0,1] \longrightarrow E^3$ is a bicubic surface and where $\underline{R}(U,V) = \underline{r}^{ij}(U-i+1, V-j+1)$ for all $(U,V) \in [0,m] \times [0,n]$.

The general problem we are interested in solving is the following :

"Given $\{\underline{b}_{ij}\} \subset E^3$ where $i,j \in N$ $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$ then we would like to construct a unique piecewise bicubic surface $\underline{R}$ such that

(i) $\underline{R}(i,j) = \underline{b}_{ij}$

(ii) $\underline{R} \in C^k [0,m] \times [0,n]$ for some k "

and this means we have to find mn bicubic surfaces each satisfying certain boundary smoothness constraints.

Now any boundary curve of the bicubic surface $\underline{r}^{ij}$ is cubic and uniquely determined by contraints at the two endpoints. Hence adjacent bicubics sharing the same endpoint constraints will share the complete boundary. Thus at each point p of the common boundary U = const. of $\underline{r}^{ij}$ and $\underline{r}^{kl}$ we also have $\underline{r}_2^{ij}\big|_p = \underline{r}_2^{kl}\big|_p$. If in addition we constrain the $\underline{a}_{ij}$ so that at this boundary $\underline{r}_1^{ij}\big|_p = \underline{r}_1^{kl}\big|_p$ then we will have C^1 continuity across this boundary. We note that this last condition can be achieved by placing constraints at the boundary endpoints. For example if $\underline{r}^{ij}(1,v) = \underline{r}^{kl}(0,v)$ is a common boundary of $\underline{r}^{ij}$ and $\underline{r}^{kl}$ then if

(i) $\underline{r}_1^{ij}(0,0) = \underline{r}_1^{kl}(0,0)$

(ii) $\underline{r}_{21}^{ij}(1,0) = \underline{r}_{21}^{kl}(0,0)$

(iii) $\underline{r}_{221}^{ij}(1,0) = \underline{r}_{221}^{kl}(0,0)$

(iv) $\underline{r}_{2221}^{ij}(1,0) = \underline{r}_{2221}^{kl}(0,0)$

then $\underline{r}_1^{ij}(1,v) = \underline{r}_1^{kl}(0,v)$ for all $v \in [0,1]$.

We can continue in this way to build up conditions involving the surface equation coefficients which ensure that properties like fundamental coefficients, curvatures, normals and their derivatives are preserved across bicubic surface boundaries as required. Although this will enable us to select one from each of the mn 36-parameter families of bicubic surfaces which combined interpolate the $\underline{b}_{ij}$ the resulting piecewise bicubic will in general suffer a similar disease to the 'over' continuous piecewise cubic curve. Therefore we will have to relax some of these boundary constraints in favour of some form of restriction on the interior behaviour which inevitably means a restriction on the surface area.

In fact for some applications we may just required continuity of $\underline{N}$. At a common vertex p of four neighbouring bicubic surfaces conditions forcing continuity of the tangent directions of the boundary cubics are sufficient to ensure that all four surfaces have common normal at p. If $\underline{r}^{ij}$ and $\underline{r}^{kl}$ have common boundary $\underline{r}^{ij}(1,v) = \underline{r}^{kl}(0,v)$ then the normal will be continuous across the boundary if and only if $\underline{r}_1^{ij}(1,v)$, $\underline{r}_1^{kl}(0,v)$ and $\underline{r}_2^{kl}(0,v)$ $[= \underline{r}_2^{ij}(1,v)]$ are coplanar for each $v \in [0,1]$. Putting $\underline{r}_1^{ij}(1,v) = \underline{r}_1^{kl}(0,v)$ is sufficient to ensure this but it is not necessary since

$$\underline{r}_1^{kl}(0,v) \times \underline{r}_2^{kl}(0,v) = \mu[\underline{r}_1^{ij}(1,v) \times \underline{r}_2^{ij}(1,v)]$$

is satisfied if

$$\underline{r}_1^{kl}(0,v) = \mu\underline{r}_1^{ij}(1,v) + \lambda\underline{r}_2^{ij}(1,v)$$

where μ,λ are scalars with $\mu > 0$.

And similarly for the V = const. boundaries.

We consider briefly some forms of the bicubic representation which are currently in use in CAD systems.

Definition 3.5

Let $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ be given by

$$\underline{r}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i}\binom{n}{j} u^i(1-u)^{m-i} v^j(1-v)^{n-j} \underline{d}_{ij}$$

where $\{\underline{d}_{ij}\} \subset E^3$, m, n $\in$ N

Then $\underline{r}$ is called a Bezier surface {12}.

If m = n = 3 then $\underline{r}$ is the Bezier bicubic surface. We shall see that the coefficients $\underline{d}_{ij}$ bear nice relationships with the surface boundary properties, (Figure 10) :

(i) $\underline{d}_{00} = \underline{r}(0,0)$

(ii) $\underline{d}_{03} = \underline{r}(0,1)$

(iii) $\underline{d}_{30} = \underline{r}(1,0)$

(iv) $\underline{d}_{33} = \underline{r}(1,1)$

Thus the points to be interpolated are coefficients in the surface equation.

(v) $3(\underline{d}_{10} - \underline{d}_{00}) = \underline{r}_1(0,0)$

(vi) $3(\underline{d}_{13} - \underline{d}_{03}) = \underline{r}_1(0,1)$

(vii) $3(\underline{d}_{30} - \underline{d}_{20}) = \underline{r}_1(1,0)$

(viii) $3(\underline{d}_{33} - \underline{d}_{23}) = \underline{r}_1(1,1)$

(ix) $3(\underline{d}_{01} - \underline{d}_{00}) = \underline{r}_2(0,0)$

(x) $3(\underline{d}_{03} - \underline{d}_{02}) = \underline{r}_2(0,1)$

(xi) $3(\underline{d}_{31} - \underline{d}_{30}) = \underline{r}_2(1,0)$

(xii) $3(\underline{d}_{33} - \underline{d}_{32}) = \underline{r}_2(1,1)$

Thus these coefficients of the surface equation lie on the tangent planes to the surface at the interpolation points:

(xiii) $6(\underline{d}_{00} - \underline{d}_{10} + \underline{d}_{20} - \underline{d}_{10}) = \underline{r}_{11}(0,0)$

(xiv) $6(\underline{d}_{03} - \underline{d}_{13} + \underline{d}_{23} - \underline{d}_{13}) = \underline{r}_{11}(0,1)$

(xv) $6(\underline{d}_{10} - \underline{d}_{20} + \underline{d}_{30} - \underline{d}_{20}) = \underline{r}_{11}(1,0)$

(xvi) $6(\underline{d}_{13} - \underline{d}_{23} + \underline{d}_{33} - \underline{d}_{23}) = \underline{r}_{11}(1,1)$

(xvii) $9(\underline{d}_{00} - \underline{d}_{01} + \underline{d}_{11} - \underline{d}_{10}) = \underline{r}_{12}(0,0)$

(xviii) $9(\underline{d}_{13} - \underline{d}_{12} + \underline{d}_{02} - \underline{d}_{03}) = \underline{r}_{12}(0,1)$

(xix) $9(\underline{d}_{31} - \underline{d}_{30} + \underline{d}_{20} - \underline{d}_{21}) = \underline{r}_{12}(1,0)$

(xx) $9(\underline{d}_{33} - \underline{d}_{32} + \underline{d}_{22} - \underline{d}_{23}) = \underline{r}_{12}(1,1)$

(xxi) $6(\underline{d}_{00} - \underline{d}_{01} + \underline{d}_{02} - \underline{d}_{01}) = \underline{r}_{22}(0,0)$

(xxii) $6(\underline{d}_{01} - \underline{d}_{02} + \underline{d}_{03} - \underline{d}_{02}) = \underline{r}_{22}(0,1)$

(xxiii) $6(\underline{d}_{30} - \underline{d}_{31} + \underline{d}_{32} - \underline{d}_{31}) = \underline{r}_{22}(1,0)$

(xxiv) $6(\underline{d}_{31} - \underline{d}_{32} + \underline{d}_{33} - \underline{d}_{32}) = \underline{r}_{22}(1,1)$

Therefore at the interpolation point u=0, v=0 ($\equiv \underline{d}_{00}$) the second fundamental coefficients and the Gaussian and mean curvatures are :

$$L(0,0) = 6|(\underline{d}_{00} - \underline{d}_{10}) + (\underline{d}_{20} - \underline{d}_{10})|\cos\alpha$$

$$M(0,0) = 9|(\underline{d}_{00} - \underline{d}_{01}) + (\underline{d}_{11} - \underline{d}_{10})|\cos\gamma$$

$$N(0,0) = 6|(\underline{d}_{00} - \underline{d}_{01}) + (\underline{d}_{02} - \underline{d}_{01})|\cos\beta$$

$$K(0,0) = (4n_1n_2 - 9n_3^2)/9n_4$$

$$H(0,0) = (n_2|\underline{d}_{10}-\underline{d}_{00}|^2 + n_1|\underline{d}_{01} - \underline{d}_{00}|^2 - 3n_3|\underline{d}_{10}-\underline{d}_{00}||\underline{d}_{01}-\underline{d}_{00}|\cos\delta)/3n_4$$

where

$$n_1 = |(\underline{d}_{00}-\underline{d}_{10}) + (\underline{d}_{20}-\underline{d}_{10})|\cos\alpha$$

$$n_2 = |(\underline{d}_{00}-\underline{d}_{01}) + (\underline{d}_{02}-\underline{d}_{01})|\cos\beta$$

$$n_3 = |(\underline{d}_{00}-\underline{d}_{01}) + (\underline{d}_{11}-\underline{d}_{10})|\cos\gamma$$

$$n_4 = |\underline{d}_{10}-\underline{d}_{00}|^2|\underline{d}_{01}-\underline{d}_{00}|^2\sin^2\delta$$

with $\alpha,\beta,\gamma,\delta$ as shown in figure 10.

We can therefore see that the main advantage of this particular representation for CAD use is that properties of the surface are implied more directly by the designers' choice of the $\underline{d}_{ij}$.

In Chapter 2 we considered the Bezier cubic curve which we can write as:

$$\underline{r}(t) = t^3\underline{d}_3 + 3t^2\hat{t}\underline{d}_2 + 3t\hat{t}^2\underline{d}_1 + \hat{t}\underline{d}_0$$

where $t + \hat{t} = 1$.

The more obvious generalisation of the Bezier cubic curve is the triangular bicubic surface (Figure 11) :

$$\underline{r}(u,v) = u^3\underline{A} + 3u^2v\underline{B} + 3uv^2\underline{C} + v^3\underline{D} + 3u^2w\underline{E} +$$
$$3vw^2\underline{F} + 3uw^2\underline{G} + 3vw^2\underline{H} + w^3\underline{I} + 6uvw\underline{J}$$

where $u + v + w = 1$, $\{\underline{A},\underline{B},\underline{C}, \ldots\underline{J}\} \subset E^3$

For this particular representation

$$\underline{r}_1(u,v) = 3\left[u^2(\underline{A}-\underline{E}) + 2uv(\underline{B}-\underline{J}) + v^2(\underline{C}-\underline{F}) + 2uw(\underline{E}-\underline{G}) + 2vw(\underline{J}-\underline{H}) + w^2(\underline{G}-\underline{I})\right]$$

$$\underline{r}_2(u,v) = 3\left[u^2(\underline{B}-\underline{E}) + 2uv(\underline{C}-\underline{J}) + v^2(\underline{D}-\underline{F}) + 2uw(\underline{J}-\underline{G}) + 2vw(\underline{F}-\underline{H}) + w^2(\underline{H}-\underline{I})\right]$$

$$\underline{r}_{11}(u,v) = 6\left[u(\underline{A}-2\underline{E}+\underline{G}) + v(\underline{B}-2\underline{J}+\underline{H}) + w(\underline{E}-2\underline{G}+\underline{I})\right]$$

$$\underline{r}_{12}(u,v) = 6\left[u(\underline{B}-\underline{J}+\underline{G}-\underline{E}) + v(\underline{C}-\underline{F}+\underline{H}-\underline{J}) + w(\underline{J}-\underline{H}+\underline{I}-\underline{G})\right]$$

$$\underline{r}_{22}(u,v) = 6\left[u(\underline{C}-2\underline{J}+\underline{G}) + v(\underline{D}-2\underline{F}+\underline{H}) + w(\underline{F}-2\underline{H}+\underline{I})\right]$$

$$\underline{r}_{111}(u,v) = 6\left[\underline{A} - 3\underline{E} + 3\underline{G} - \underline{I}\right]$$

$$\underline{r}_{112}(u,v) = 6\left[\underline{B} - 2\underline{J} + \underline{H} - \underline{E} + 2\underline{G} - \underline{I}\right]$$

$$\underline{r}_{122}(u,v) = 6\left[\underline{C} - 2\underline{J} + \underline{G} - \underline{F} + 2\underline{H} - \underline{I}\right]$$

$$\underline{r}_{222}(u,v) = 6\left[\underline{D} - 3\underline{F} + 3\underline{H} - \underline{I}\right]$$

Along the boundary $u=0$,

$$\partial\underline{r}/\partial w = 3\left[v^2(\underline{F}-\underline{C}) + w^2(\underline{I}-\underline{G}) + 2vw(\underline{H}-\underline{J})\right]$$

$$\partial\underline{r}/\partial v = 3\left[v^2(\underline{D}-\underline{C}) + w^2(\underline{H}-\underline{G}) + 2vw(\underline{F}-\underline{J})\right]$$

Along $v=0$,

$$\partial\underline{r}/\partial u = 3\left|u^2(\underline{A}-\underline{B}) + w^2(\underline{G}-\underline{H}) + 2uw(\underline{E}-\underline{J})\right|$$

$$\partial\underline{r}/\partial w = 3\left|u^2(\underline{E}-\underline{B}) + w^2(\underline{I}-\underline{H}) + 2uw(\underline{G}-\underline{J})\right|$$

Along w=0,

$$\partial \underline{r}/\partial v = 3\left[u^2(\underline{B}-\underline{E}) + v^2(\underline{D}-\underline{F}) + 2uv(\underline{C}-\underline{J})\right]$$

$$\partial \underline{r}/\partial u = 3\left[u^2(\underline{A}-\underline{E}) + v^2(\underline{C}-\underline{F}) + 2uv(\underline{B}-\underline{J})\right]$$

Along the boundaries the normal is given by,

$$\begin{aligned}\underline{N}\Big|_{u=0} = \alpha(v)\Big[& v^4(\underline{F}-\underline{C}) \times (\underline{D}-\underline{C}) + w^4(\underline{I}-\underline{G}) \times (\underline{H}-\underline{G}) \\ & + v^2w^2\left[4(\underline{H}-\underline{J}) \times (\underline{F}-\underline{J}) + (\underline{I}-\underline{G}) \times (\underline{D}-\underline{C}) + (\underline{F}-\underline{C}) \times (\underline{H}-\underline{G})\right] \\ & + 2v^3w\left[(\underline{H}-\underline{J}) \times (\underline{D}-\underline{C}) + (\underline{F}-\underline{C}) \times (\underline{F}-\underline{J})\right] \\ & + 2vw^3\left[(\underline{I}-\underline{G}) \times (\underline{F}-\underline{J}) + (\underline{H}-\underline{J}) \times (\underline{H}-\underline{G})\right]\Big]\end{aligned}$$

where $\alpha(v) = 1/\left|\partial \underline{r}/\partial w \times \partial \underline{r}/\partial v\right|$

Similarly for $\underline{N}\Big|_{v=0}$ and $\underline{N}\Big|_{w=0}$

Thus continuity of normal $\underline{N}$ across u=0 puts severe constraints on the coefficients of the surface equation since we must already have $\underline{D}$, $\underline{F}$, $\underline{H}$, $\underline{I}$ as common coefficients in order to have a common boundary. For a surface with coefficients as above the continuity of $\underline{r}_1 \times \underline{r}_2$ across u=0 requires that for the neighbouring surface (see Figure 11) we position $\underline{P}$, $\underline{Q}$ and $\underline{R}$ such that $\{\underline{D}, \underline{C}, \underline{F}, \underline{Q}\}$, $\{\underline{F}, \underline{J}, \underline{H}, \underline{P}\}$ and $\{\underline{H}, \underline{G}, \underline{I}, \underline{K}\}$ form three skew parallelograms. This also means that exactly six triangular elements have an internal vertex in the system as common vertex. Thus at each vertex we have a semi-regular plane hexagon.

For the above representation of a surface, the continuity conditions are so restrictive that there are virtually no degrees of freedom left. This family of Bezier type surfaces is therefore not rich enough and we can conclude that bicubic triangles are unsuitable for surface fitting. This is not altogether surprising since in constructing smooth surfaces most of the difficulties arise from crossing the boundaries and in introducing the extra boundary we can only be adding to our troubles. We can contrast this with finite element analysis where the difficulties arise in jumping from one side of the element to the other, and so the addition of the extra boundary gives us an extra stepping stone.

The extension of B-spline curves to B-spline surfaces is exactly analogous to the development of Bezier surfaces from Bezier curves :

Definition 3.6

If the equation of the surface can be expressed in terms of B-spline functions as

$$\underline{r}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{i-\frac{k}{2},k}(u) \; B_{j-\frac{k}{2},k}(v) \, \underline{P}_{ij} \; , \; \{\underline{P}_{ij}\} \subset E^3$$

then $\underline{r}$ is called a B-spline surface {17}.

Surface representations of the types discussed above have been derived to ensure that most of the engineering shapes likely to be encountered can be defined in sufficient detail, albeit at a cost of smoothness in the sense of differential geometry. For other forms of analysis of the resulting surface however smoothness becomes the critical feature. One way of achieving a compromise would be to proceed from the bicubic definition to a smooth approximation taking over properties of the bicubic. This could obviously be done for instance by using higher order polynomials. In so doing instead of considering constraints on just one component of the tensor of second order partial derivatives $\partial^2 \underline{r}/\partial u^\alpha \partial u^\beta$ we could look at the tensor $x^i_{\alpha,\beta}$

where $x^i_{\alpha,\beta} = \partial^2 x^i/\partial u^\alpha \partial u^\beta - \Gamma^\gamma_{\alpha\beta}\, x^i_j = \Omega_{\alpha\beta} N^i$

where (N^i) is the unit normal and $(\Omega_{\alpha\beta})$ the second fundamental form.

Chapter 4

Extremal Mappings of Riemannian Manifolds

We make use of the following notation:

(i) $TM^n = \bigcup_{p \in M} T_pM^n$ is the tangent bundle of the differentiable n-manifold M.

(ii) $H(TM \times TM, \mathbb{R})$ is the bundle of tensors of M of type (2,0). In general $H(\underbrace{TM \times TM \ldots \times TM}_{r} \times \underbrace{(TM)^* \ldots \times (TM)^*}_{s}, \mathbb{R})$ is the bundle of tensors on M of type (r,s).

(iii) A section of $H(TM \times TM, \mathbb{R})$ is a (2,0) tensor.

Definition 4.1

A Riemannian manifold is a differentiable manifold together with a section g of $H(TM \times TM, \mathbb{R})$ satisfying

(i) $g(X,Y) = g(Y,X)$, (ii) $g(X,X) = 0$ iff $X = 0$

where X,Y are vector fields.

We write $g(X,Y)$ as $\langle X,Y \rangle$ and this is the first fundamental form on M. Thus for $M = E^n$ then $\langle X,Y \rangle = \sum_{i=1}^{n} x^i y^i$.

Definition 4.2

The second fundamental form on M is a section B of $H(TM \times TM, NM)$ given by $B(X,Y) = [\bar{\nabla}_X Y]^N$ the normal component of $\bar{\nabla}_X Y$ for the Riemannian connexion $\bar{\nabla}$.

Thus if $u^1, \ldots u^k$ are local coordinates at $x \in M \subset E^n$
$X = (x_1, \ldots x_n) = (x_1(u^1, \ldots u^k), \ldots, x_n(u^1, \ldots u^k))$ and
$(g_{ij}) = \left(\langle \frac{\partial X}{\partial u^i}, \frac{\partial X}{\partial u^j} \rangle\right)$. For a surface in E^3 then
$g_{ij} = \langle X_i, X_j \rangle$, $X_i = \partial X / \partial u^i$ $i=1,2$ and if

$N = X_1 \times X_2 / |X_1 \times X_2|$ is unit normal then

$\bar{\nabla}_{X_j} X_i = \partial X_i / \partial u^j = \sum \Gamma^k_{ij} X_k + b_{ij} N$ in terms of tangential and normal components. And for a curve γ on M with tangent $\sum \gamma^{r\prime} X_r$ then

$$\partial(\sum \gamma^i X_i)/\partial t = [\gamma^{i\prime\prime} + \gamma^{k\prime}\gamma^{r\prime}\Gamma^i_{kr}] X_i + \gamma^{k\prime}\gamma^{r\prime} b_{kr} N$$

Theorem 4.3

Given a Riemannian symmetric connexion ∇ and a curve $\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = p$ and $Y_0 \in T_p M$ then there is a unique $Y(t)$ along γ such that $\nabla_{\gamma'} Y = 0$.

To see this if $X^1, \ldots X^n$ are coordinates at p,

$$\nabla_{X_i} X_j = \sum_{k=1}^{n} \Gamma^k_{ij} X_k \quad , \quad X_i = \partial / \partial X_i$$

and $\gamma' = (X^{1\prime}(t), \ldots, X^{n\prime}(t))$ and

$$\begin{aligned} \nabla_{\gamma'} Y &= \nabla_{\gamma'} y^i X_i \\ &= y^{i\prime} X_i + y^i \nabla_{\sum X_j' X_j} X_i \\ &= y^{i\prime} X_i + y^i X_j' \Gamma^k_{ji} X_k \end{aligned}$$

Thus we have a linear system of differential equations

$y^{k'} + y^i X^{j'} \Gamma^k_{ij} = 0$ with initial condition $Y_0 = y^i(t)\big|_{t=0}$

Since $(|Y-\hat{Y}|^2)' = 2 <Y-\hat{Y}, \nabla_{\gamma'}(Y-\hat{Y})> = 0$

then $|Y-\hat{Y}|^2$ is constant and so since $Y(0) = \hat{Y}(0)$ we must have uniqueness.

Definition 4.4

A Geodesic is a curve γ with acceleration vector field $\nabla_{\gamma'}\gamma'$ zero.

Let Γ be a closed $(n-1)$ manifold in E^{n+k}. We say that X spans Γ if X is an n-submanifold and $\partial X = \Gamma$.

Let $X : D \longrightarrow E^{n+k}$, D closed ball in E^n and $X(\partial D) = \Gamma$.

Definition 4.5

A variation of X is a map $\bar{X} : D \times [-\varepsilon,\varepsilon] \longrightarrow E^{n+k}$ such that

(i) $\bar{X}_0(z) = X(z)$ for each $z \varepsilon D$

(ii) $\bar{X}_u(\partial D) = \bar{X}_0(\partial D) = X(\partial D)$ for each $u \varepsilon [-\varepsilon,\varepsilon]$

where $\bar{X}_u(z) = \bar{X}(z,u)$

Thus for fixed $z \varepsilon D$, $\bar{X}_u(z)$ is a curve with velocity vector $V(z) = d(\bar{X}_u(z))/du\big|_{u=0}$ called the Variation Vector Field associated with the variation $\bar{X}$. And $\bar{X}_u(D)$ is a surface with first fundamental form $g_{ij}(u) = <\bar{X}_{u,i}, \bar{X}_{u,j}>$ and has area $\int_D \sqrt{[\det g_{ij}(u)]}\, du^1 \ldots du^n = \int_D dA_u$.

If $(u^1,\ldots,u^n)$ are coordinates for a domain D in E^n and $X(u^1,\ldots,u^n)$ is a smooth map into E^{n+k} which is everywhere regular then X induces a Riemannian structure on D and if we take $<\ ,\ >$ to be the usual inner product on E^{n+k} then $X^*(<\ ,\ >)$ is a metric tensor on D.

Definition 4.6

$$g_{ij} = X^*(<\ ,\ >)(\partial/\partial u^i, \partial/\partial u^j) = <X_*(\partial/\partial u^i), X_*(\partial/\partial u^j)>$$
$$= \sum_1^{n+k} \partial X_k/\partial u^i \quad \partial X_k/\partial u^j.$$

The volume element of (g_{ij}) is defined by $dV = \sqrt{[\det(g_{ij})]}\,du^1\ldots du^n$, and $\int_D dV$ is the 'volume' of X.

If $\bar{G} : D \times [-\varepsilon.\varepsilon] \longrightarrow E^{n+k}$ is a smooth variation of X which leaves ∂D fixed, that is $\bar{G}(z,0) = X(z)$ and $\bar{G}(\partial D,s) = X(\partial D)$ then if $V(z) = d\bar{G}(z,s)/ds\Big|_{s=0}$ is the variation vector field of $\bar{G}$ then for small enough s, $\bar{G}_s$ is regular and $(g_{s\ ij}) = X_s^*(<\ ,\ >)$ is a metric tensor with volume element $dV_s = \sqrt{[\det(g_s]}\,du^1\ldots du^n$.

Definition 4.7

If M^n is a differentiable manifold and $X : M^n \longrightarrow E^{n+k}$ is regular then X induces a riemannian structure on M^n as above. Furthermore X_* is an isometry between each fibre on TM^n and its image in TE^{n+k}. Such a map is called an Isometric Immersion.

Definition 4.8

An immersion $X : M^n \longrightarrow E^{n+k}$ is called Minimal if $H \equiv 0$ where H is its mean curvature vector field.

We shall see that an isometric immersion $X : M^n \longrightarrow E^{n+k}$ is minimal if and only if the volume of M^n is stationary with respect to all variations with compact support in $M^n - \partial M^n$. We remark here that this carries over for immersions in $\bar{M}^{n+k}$ having $\langle\, ,\, \rangle$ as inner product, and in this case the induced metric from a regular map is

$g_{ij} = \sum \partial X_\alpha/\partial u^i \; \partial X_\beta/\partial u^j \; \bar{g}_{\alpha\beta}$ where $\bar{g}_{\alpha\beta}$ is just $\langle\, ,\, \rangle$ in local coordinates.

Proposition 4.9

With X, D, $\bar{X}$ as previously defined then there is a normal vector field H along X (defined locally) such that for any variation $\bar{X}$ which leaves ∂D fixed then

$V'(0) = -\int_D \langle H, V\rangle \, dV$ where V is the variation vector field of $\bar{X}$.

To see this one considers the varation $\bar{X}$ of X with variation vector field $V_{u_0}(z) = d(\bar{X}_u(z))/du\big|_{u=u_0}$ and by taking the area $A(u) = \int_D \sqrt{[\det g_{ij}(u)]}\, du_1 \ldots du_n$ one simply evaluates $dA/du\big|_{u=0}$, using the fact that V is zero on the boundary.

Suppose that $\gamma : [0,1] \longrightarrow M$ is a curve in M. The length of γ is given by

$$L_0^1(\gamma) = \int_0^1 \langle\gamma'(t),\gamma'(t)\rangle^{\frac{1}{2}}\,dt$$

If $\bar{\gamma} : [0,1] \times [-\varepsilon,\varepsilon] \longrightarrow M$ is a one paramoter variation of γ then $\bar{\gamma}_0(t) = \gamma(t)\ \forall\ t \in [0,1]$ and $\bar{\gamma}_u(0) = \gamma(0)$, $\bar{\gamma}_u(1) = \gamma(1)\ \forall\ u \in [-\varepsilon,\varepsilon]$. If we write $L_u = L_0^1(\bar{\gamma}_u)$ then $L_u = \int_0^1 \langle\bar{\gamma}_u'(t),\ \bar{\gamma}_u'(t)\rangle^{\frac{1}{2}}dt$ and using proposition 4.9 we have $d(L_u)/du\Big|_{u=0} = -\int_0^1 \langle V,H\rangle\langle\gamma'(t),\gamma'(t)\rangle^{\frac{1}{2}}\,dt$ where V is the variation vector field along γ, which depends upon the variation and $H = (\gamma'(t)/\langle\gamma'(t),\gamma'(t)\rangle^{\frac{1}{2}})'$ is the curvature vector which depends only on γ.

Thus $d(L_u)/du\Big|_{u=0} = 0$ if and only if $H \equiv 0$ and so $d(L_u)/du\Big|_{u=0} = 0$ for all variations fixing endpoints if and only if γ is a geodesic. And if $M = E^n$ then $H = (\gamma'(t)/\langle\gamma'(t),\gamma'(t)\rangle^{\frac{1}{2}})' = 0$ means that $\gamma''(t) = 0$ which on integration just gives a linear equation.

If $X(s)$ is a curve in E^n parameterized by arc length then $X''(s)$ is a vector normal to X and $X''(s) = 0$ if and only if X is locally the shortest curve between points on X. In this case $H = X''(s) = B(X'(s),X'(s))$ and if the curve is moved in the direction of $X''(s)$ then its length will be decreased.

Suppose that Γ is a closed curve in E^3 and let $X : D \longrightarrow E^3$ be a surface which spans Γ. Then X has area $A_D(X) = \int_D \sqrt{[\det(g_{ij})]}\,du_1du_2$ and if $\bar{X} : D \times [-\varepsilon,\varepsilon] \longrightarrow E^3$ is a smooth varation of X leaving ∂D fixed with variation

vector field $V(z) = \partial\bar{X}(z,u)/\partial u\Big|_{u=0}$ and $\bar{X}(z,u) \equiv \bar{X}_u(z)$ has metric $g_{ij}(u)$ then $d\,A(u)/du\Big|_{u=0} = -\int \langle V,H\rangle dA_0 = 0$ if and only if X is a minimal surface, where
$A(u) = \int_D \sqrt{[\det g_{ij}(u)]}\;du_1 du_2$.

If $X : M^2 \longrightarrow E^{2+k}$ is given locally in terms of conformal coordinates then at each point (g_{ij}) is of the form $\begin{pmatrix}\lambda^2 & 0\\ 0 & \lambda^2\end{pmatrix}$ and the Laplacian $\Delta X = (\Delta X_1, \ldots \Delta X_{2+k}) = H/\lambda^2$.
This means that X is minimal if and only if $\Delta X = 0$ if and only if all the coordinate functions X_i are harmonic.
If we put $f_\alpha = \partial X_\alpha/\partial u_1 - \partial X_\alpha/\partial u_2$, $\alpha = 1, \ldots, 2+k$ then from the Cauchy Riemann equations each f_α is analytic as a complex function of $z = u_1 + iu_2$ and also

(i) $\sum_\alpha f_\alpha^2 = 0$
(ii) $\sum_\alpha |f_\alpha|^2 = 2\lambda^2$

Proposition 4.10

Given that $f_1, f_2, \ldots, f_{2+k}$ are analytic functions of $z = u_1 + iu_2$ on a simply connected domain $D \subset \mathbb{C}$ such that they satisfy (i) and (ii) above then there exists $X : D \longrightarrow E^{2+k}$ which is minimal and in conformal coordinates.

This follows by taking $X_\beta = \mathrm{Re}\int f_\beta \qquad \beta = 1, \ldots, 2+k$

Definition 4.11

Let $\gamma : [a,b] \longrightarrow M$ be a curve in M which is piecewise smooth and continuous. The energy of γ, {4,6} is given by
$E_a^b(\gamma) = \int_a^b \langle\gamma',\gamma'\rangle dt.$

Proposition 4.12

$$d\, E(\bar{\gamma})/du\Big|_{u=0} = -2\left[\sum <U(t),\Delta_t V> + \int_a^b <U(t),\nabla_V V>dt\right]$$

where $\bar{\gamma} : [a,b] \times [-\varepsilon,\varepsilon] \longrightarrow M$ is a one parameter variation of γ, $\Delta_t V = V_{t+} - V_{t-}$ is the jump in the velocity vector $V(t) = \gamma'(t)$ at the corner t, $U(t) = \partial\bar{\gamma}/\partial u\Big|_{u=0}$ and $\nabla_V V$ is the acceleration vector of γ.

This result follows since

$$\begin{aligned} d\, E(\bar{\gamma})/du &= d\left[\int_a^b <\bar{\gamma}',\bar{\gamma}'>dt\right]/du \\ &= \int_a^b\left[<\nabla_u\bar{\gamma}',\bar{\gamma}'> + <\bar{\gamma}',\nabla_u\bar{\gamma}'>\right]dt \\ &= 2\int_a^b <\nabla_u\bar{\gamma}',\bar{\gamma}'>\, dt \\ &= 2\int_a^b <\nabla_{\bar{\gamma}'} U,\bar{\gamma}'>\, dt \end{aligned}$$

If $\bar{\gamma} \in C^1[t_{i-1},t_i] \times [-\varepsilon,\varepsilon]$ $i=1,\ldots,m$, $t_0 = a$, $t_m = b$

then since

$$\partial<\partial\bar{\gamma}/\partial u,\gamma'>/\partial t = <\nabla_{\gamma'} U,\gamma'> + <\partial\bar{\gamma}/\partial u,\nabla_{\gamma'}\gamma'>$$

we have

$$\int_{t_{i-1}}^{t_i} <\nabla_{\gamma'} U,\gamma'>\, dt = <\partial\bar{\gamma}/\partial u,\gamma>\Big|_{t=t_{i-1}}^{t=t_i} - \int_{t_{i-1}}^{t_i} <\partial\bar{\gamma}/\partial u,\nabla_{\gamma'}\gamma'>\, dt.$$

Now $\partial\bar{\gamma}/\partial u = 0$ at $t=0$ or $t=1$ and so

$$\int_a^b <\nabla_{\gamma'} U,\gamma'>\, dt = -\sum_{i=1}^{m-1} <\partial\bar{\gamma}/\partial u,\Delta_{t_i}\bar{\gamma}'> - \int_a^b <\partial\bar{\gamma}/\partial u,\nabla_{\gamma'}\gamma'>\, dt$$

Corollary

Stationary points of the energy function are geodesics.

This follows because if γ is a geodesic then γ is smooth

and $\nabla_V V = 0$ and so $d\,E(\bar{\gamma})/du\big|_{u=0} = 0$ $\forall$ variations $\bar{\gamma}$.
Conversely if $d\,E(\bar{\gamma})/du\big|_{u=0} = 0$ for all variations $\bar{\gamma}$ then $\nabla_V V = 0$ where defined and V is continuous so from the uniqueness theorem for differential equations since there is only one continuous solution then γ is an unbroken geodesic.

<u>Definition 4.13</u>

Let $\gamma : [a,b] \longrightarrow M$ be a curve in M. Then $\bar{\gamma}(t,u,w)$ is a two parameter variation of γ if
$\bar{\gamma} : [a,b] \times [-\varepsilon,\varepsilon] \times [-\delta,\delta] \longrightarrow M$ with
$\bar{\gamma}(t,0,0) = \gamma(t)$

If we write $\gamma'(t) = V$, $U(t) = \partial\bar{\gamma}/\partial u$, $W(t) = \partial\bar{\gamma}/\partial w$ then $U(t)$ is a vector field along the curve $\bar{\gamma}(t,u_0,w_0)$ for fixed u_0, w_0 and $W(t) = \partial\bar{\gamma}(t,u_0,w)/\partial w\big|_{w=w_0}$.

<u>Proposition 4.14</u>

For a smooth geodesic γ defined on $[a,b]$ and a piecewise smooth variation $\bar{\gamma}(t,u,w)$ then

$$\partial^2 E/\partial u\partial w\Big|_{u=w=0} = -2\Big[\sum_t \langle U_t, \Delta_t \nabla_V W\rangle + \int_a^b \langle U, R(W,V)V + \partial^2 W/\partial t^2\rangle \; dt$$

for variation vector fields U,W of the variation $\bar{\gamma}$ of γ.

This follows from straightforward differentiation where
$R(W,V)V = \nabla_W\nabla_V V - \nabla_V\nabla_W V - \nabla_{[W,V]}V$ and where
$[W,V] = \partial/\partial w \;\; \partial/\partial t - \partial/\partial t \;\; \partial/\partial w = 0.$

Corollary

If $\bar{\gamma}(t,u,w)$ is a smooth variation with $\bar{\gamma}(a,u,w) = p$ and $\bar{\gamma}(b,u,w) = q$ that is $U(a) = U(b) = W(a) = W(b) = 0$

then

$$\partial^2 E/\partial u \partial w = -2\int_a^b \langle U, R(W,V)V + \partial^2 W/\partial t^2 \rangle \, dt.$$

We shall use the following notation :

$$\Omega = \{ \gamma \mid \gamma : [a,b] \longrightarrow M \}$$
$$\Omega_p^q = \{ \gamma \varepsilon \Omega \mid \gamma(a) = p, \gamma(b) = q \}$$
$$T_\gamma \Omega = \{ U \mid U \text{ is vector field along } \gamma \}$$
$$T_\gamma \Omega_p^q = \{ U \varepsilon T_\gamma \Omega \mid U(p) = U(q) = 0 \}$$

We can therefore think of E, E_*, E_{**} as mappings :

$E : \Omega \longrightarrow \mathbb{R}$, $E_* : T\Omega \longrightarrow \mathbb{R}$, $E_{**} : T_\gamma \Omega \times T_\gamma \Omega \longrightarrow \mathbb{R}$

such that if $W \varepsilon T_\gamma \Omega$ then $E_*(W) = \partial E/\partial w \big|_{w=0}$

and if $U, W \varepsilon T_\gamma \Omega$ then $E_{**}(U,W) = \partial^2 E/\partial u \partial w \big|_{u=w=0}$

Propostion 4.15

If $\gamma \varepsilon \Omega_p^q$ is a piecewise smooth curve of shortest length on M then γ is a smooth geodesic and

$E_*(U) = 0$, $E_{**}(U,U) \geq 0 \quad \forall \; U \varepsilon T_\gamma \Omega_p^q$.

To see this, since geodesics are stationary points of E then $E_*(U) = 0$.

Also
$$[L_a^b(\gamma)]^2 = [\int_a^b \langle \gamma', \gamma' \rangle^{\frac{1}{2}} dt]^2$$
$$\leq \int_a^b \langle \gamma', \gamma' \rangle dt \int_a^b 1 \, dt \quad \text{(Schwarz Inequality)}$$
$$= E_a^b(\gamma)\,(b-a)$$

But if γ is a geodesic $L_a^b(\gamma) < L_a^b(\alpha) \quad \forall\, \alpha \in \Omega_p^q$ and so

$\left[L_a^b(\gamma)\right]^2 \leqslant \left[L_a^b(\alpha)\right]^2 \leqslant E_a^b(\alpha)(b-a)$

or $E_a^b(\gamma)(b-a) \leqslant E_a^b(\alpha)(b-a)$

Thus for any variation $\bar{\gamma}$ of γ, $E_a^b(\gamma) = E_a^b(\bar{\gamma}(0)) \leqslant E_a^b(\bar{\gamma})$

and $d^2E(\bar{\gamma})/du^2\Big|_{u=0} \geqslant 0$

Physically we can interpret 'Energy' as follows. If a rubber band describes a curve Γ when stretched between two points of a frictionless curved surface S then the potential energy arising from tension is proportional to the energy of Γ. In particular if we have an equilibrium state then the energy is minimised and Γ is a geodesic on S.

Let p, q ε E^3 and let $\lambda : [0,1] \longrightarrow E^3$ be any curve in E^3 with $\lambda(0) = p$ and $\lambda(1) = q$.

Then $\lambda \varepsilon \Omega_p^q$ and the Energy of λ is given by

$$E(\lambda) = \int_0^1 <\lambda'(t), \lambda'(t)> dt$$

We now consider $(\Omega_c^{12})_p^q \subset \Omega_p^q$ where

$(\Omega_c^{12})_p^q = \left\{ \lambda \varepsilon \Omega_p^q \;\middle|\; \lambda \text{ is a cubic curve} \right\}$.

From this 12 parameter family of curves we can extract a 2 parameter subfamily $(\Omega_c^2)_p^q$ as follows :

Let T_p, T_q ε E^3 with $|T_p| = |T_q| = 1$

then

$$(\Omega_c^2)_p^q = \left\{ \lambda \varepsilon (\Omega_c^{12})_p^q \;\middle|\; \lambda'(t)\Big|_p = \alpha_\lambda T_p,\ \lambda'(t)\Big|_q = \beta_\lambda T_q \right\}$$

for some α_λ, $\beta_\lambda \in \mathbb{R}$.

If $\tilde{E}(\lambda)$ is $E(\lambda)$ restricted to the set $(\Omega_c^2)_p^q$ we will show in Chapter 5 that there is just one $\lambda \in (\Omega_c^2)_p^q$ which minimizes $\tilde{E}$.

Definition 4.16

Let $f : M^m \longrightarrow N^n$ be a smooth mapping of Riemannian manifolds M (compact) and N. If $(x^1,\ldots\ldots,x^m)$ and $(y^1,\ldots\ldots,y^n)$ are local coordinates at $p \in M$ and $f(p) \in N$ respectively then the energy of f is given by

$$E(f) = \int_M (g^{ij}\hat{g}_{\alpha\beta} f_i^\alpha f_j^\beta) * 1$$

where $ds^2 = g_{ij}dx^i dx^j$ and $d\hat{s}^2 = \hat{g}_{\alpha\beta}dy^\alpha dy^\beta$ are the metrics in local coordinates, and $f_i^\alpha = \partial f^\alpha/\partial x^i$, and $*1$ is the volume element of M.

The problem of deforming any given mapping into one for which $\delta E = 0$ is considered in {6}. For this the Euler equations turn out to be $\tau^\alpha = 0$ where

$$\tau^\alpha = g^{ij}\left[\partial f_i^\alpha/\partial x^j - \Gamma_{ij}^h f_h^\alpha + \hat{\Gamma}_{\beta\gamma}^\alpha \; f_j^\gamma f_i^\beta\right]$$

where $\Gamma, \hat{\Gamma}$ are the Christoffel symbols on M, N and any map for which $\tau^\alpha = 0$ is a Harmonic map. If we take $M = S^1$ then f is harmonic if and only if $f(S^1)$ is a closed geodesic on N.

Chapter 5

An Extremal curve in E^3

Using definition 2.1 for the cubic curve equation we can write any member of $(\Omega_c^2)_p^q$ in the form

$$\underline{r}(u) = \underline{a}_0 + \underline{a}_1 u + \underline{a}_2 u^2/2! + \underline{a}_3 u^3/3! \qquad \underline{a}_i \in E^3$$

where $u \in [0,1]$

Definition 5.1

The energy of the cubic curve $\underline{r}$ is given by

$$\bar{E}(\underline{r}) = \int_0^1 \langle \dot{\underline{r}}(u), \dot{\underline{r}}(u) \rangle \, du$$

Then

$$\underline{r}(0) = \underline{a}_0 = p$$

$$\dot{\underline{r}}(0) = \dot{s}(0)\underline{t}(0)$$

$$\dot{\underline{r}}(1) = \dot{s}(1)\underline{t}(1)$$

and we recall from chapter 2 that each coefficient in the cubic equation can be written in terms of these as

$$\underline{a}_1 = \dot{s}(0)\underline{t}(0)$$

$$\underline{a}_2 = 6(\underline{r}(1)-\underline{r}(0)) - 4\dot{s}(0)\underline{t}(0) - 2\dot{s}(1)\underline{t}(1)$$

$$\underline{a}_3 = 12(\underline{r}(0)-\underline{r}(1)) + 6\dot{s}(0)\underline{t}(0) + 6\dot{s}(1)\underline{t}(1)$$

Writing

$$|\underline{r}(1)-\underline{r}(0)| = W$$

$$<\underline{r}(1)-\underline{r}(0),\underline{t}(0)> = W \cos \theta$$

$$<\underline{r}(1)-\underline{r}(0),\underline{t}(1)> = W \cos \phi$$

then

$$<\dot{\underline{r}}(u),\dot{\underline{r}}(u)> = 9\alpha u^4 + 12\beta u^3 + 2\gamma u^2 + 4\delta u + \dot{s}^2(0)$$

where

$$\alpha = 4W(W-\dot{s}(0)\cos\theta - \dot{s}(1)\cos\phi) + \dot{s}^2(0) + \dot{s}^2(1) + 2\dot{s}(0)\dot{s}(1)\cos(\theta+\phi)$$

$$\beta = W(-6W+7\dot{s}(0)\cos\theta + 5\dot{s}(1)\cos\phi) - 2\dot{s}^2(0) - \dot{s}^2(1) - 3\dot{s}(0)\dot{s}(1)\cos(\theta+\phi)$$

$$\gamma = 6W(3W-5\dot{s}(0)\cos\theta - 2\dot{s}(1)\cos\phi) + 11\dot{s}^2(0) - 2\dot{s}^2(1) + 11\dot{s}(0)\dot{s}(1)\cos(\theta+\phi)$$

$$\delta = 3\dot{s}(1)W\cos\theta - 2\dot{s}^2(0) - \dot{s}(0)\dot{s}(1)\cos(\theta+\phi)$$

And the energy of $\underline{r}$ is given by

$$\bar{E}(\underline{r}) = W(6W-\dot{s}(0)\cos\theta - \dot{s}(1)\cos\phi)/5 + (2\dot{s}^2(0) + 2\dot{s}^2(1) - \dot{s}(0)\dot{s}(1)\cos(\theta+\phi))/15$$

from which it can be shown that $\bar{E}$ takes on its minimum when

$$\dot{s}(0) = 3W\left[4\cos\theta + \cos\phi\cos(\theta+\phi)\right]/\left[16 - \cos^2(\theta+\phi)\right]$$

$$\dot{s}(1) = 3W\left[4\cos\phi + \cos\theta\cos(\theta+\phi)\right]/\left[16 - \cos^2(\theta+\phi)\right] \quad (*)$$

In a symmetrical condition where $\theta = \phi$ then

$$\dot{s}(0) = \dot{s}(1) = 3W\cos\theta/(5-2\cos^2\theta)$$

Figures 12-15 show some examples of minimum energy cubic curves. The boundary speeds (*) satisfy :

$$\dot{s}(i) > 0 \quad \text{iff} \quad <\underline{r}(1)-\underline{r}(0),\underline{t}(i)> > 0 \qquad i=0,1$$

$$\dot{s}(i) = 0 \quad \text{iff} \quad <\underline{r}(1)-\underline{r}(0),\underline{t}(i)> = 0 \qquad i=0,1$$

Using proposition 2.4 the curvature and torsion of these cubics take the boundary values :

$\kappa(0) = (2\sqrt{f(\theta,\phi,\dot{s}(1))}/\dot{s}^2(0)$

$\kappa(1) = (2\sqrt{f(\phi,\theta,\dot{s}(0))}/\dot{s}^2(1)$

$\tau(0) = 3\dot{s}(1)\left[\underline{t}(0),\underline{r}(1)-\underline{r}(0),\underline{t}(1)\right]/\dot{s}(0)f(\theta,\phi,\dot{s}(1))$

$\tau(1) = 3\dot{s}(0)\left[\underline{t}(0),\underline{r}(1)-\underline{r}(0),\underline{t}(1)\right]/\dot{s}(1)f(\phi,\theta,\dot{s}(1))$

where $\dot{s}(0)$ and $\dot{s}(1)$ are given by equations (*) and where

$$f(a,b,\alpha) = 3W\left[3W \sin^2 a - 2\alpha(3 \cos b - \cos a \cos (a+b))\right] + \alpha \sin^2(a+b)$$

and for the minimum energy planar cubic the curvature takes the boundary values

$\kappa(0) = 2\left[3W \sin \theta - \dot{s}(1) \sin (\theta + \phi)\right]/\dot{s}^2(0)$

$\kappa(1) = 2\left[3W \sin \phi - \dot{s}(0) \sin (\theta + \phi)\right]/\dot{s}2(1)$

where again $\dot{s}(0)$ and $\dot{s}(1)$ satisfy (*).

And if $\theta = \phi$ then we have

$\kappa(0) = \kappa(1) = 2 \sin \theta(5-4 \cos^2\theta)(5-2 \cos^2\theta)/3W \cos^2\theta$

Let $\xi = \{ p1, p2, \ldots\ldots p_n \} \subset E^3$ and let $\underline{r}^i \in (\Omega_c^2)_{pi}^{pi+1}$

Proposition 5.2

Given ξ there is a unique piecewise cubic curve Γ which satisfies

(i) Γ contains p_i $i = 1,\ldots\ldots n$

(ii) Γ is represented by $\underline{r}^i$ between p_i and p_{i+1} and

$\underline{r}^1(0) = \underline{r}^n(1)$

(iii) $\Gamma \in c^2$

(iv) $\underline{r}^i$ minimizes $\bar{E}$ on $(\Omega_c^2)_{p_i}^{p_{i+1}}$ for $i=1,\ldots n$

This follows because at p_i continuity of tangent direction and curvature vector means (proposition 2.9) that we must have

$$\dot{\underline{r}}^i\Big|_{u=0} = c\,\dot{\underline{r}}^{i-1}\Big|_{u=1} \quad \text{for some } c > 0$$

and

$$\ddot{\underline{r}}^i\Big|_{u=0} = c^2\ddot{\underline{r}}^{i-1}\Big|_{u=1} - d\,\dot{\underline{r}}^{i-1}\Big|_{u=1} \quad \text{for some } d.$$

By writing $\ddot{\underline{r}}^i\big|_{u=0}$ and $\ddot{\underline{r}}^{i-1}\big|_{u=1}$ in terms of the cubic curve coefficients and substituting in turn for these using the equations of proposition 2.3 then we find that the second condition for the continuity of curvature vector at p_i reduces to {8} :

$$\begin{aligned}\mu_i\underline{t}_i = 3\Big[&\langle\dot{\underline{r}}^i,\dot{\underline{r}}^i\rangle\Big|_{u=0}(\underline{r}^{i-1}(1)-\underline{r}^{i-1}(0)) \\ &+ \langle\dot{\underline{r}}^{i-1},\dot{\underline{r}}^{i-1}\rangle\Big|_{u=1}(\underline{r}^i(1)-\underline{r}^i(0))\Big] \\ &- \langle\dot{\underline{r}}^{i-1},\dot{\underline{r}}^{i-1}\rangle^{\frac{1}{2}}\Big|_{u=0}\langle\dot{\underline{r}}^i,\dot{\underline{r}}^i\rangle\Big|_{u=0}\underline{t}_{i-1} \\ &- \langle\dot{\underline{r}}^{i-1},\dot{\underline{r}}^{i-1}\rangle\Big|_{u=1}\langle\dot{\underline{r}}^i,\dot{\underline{r}}^i\rangle\Big|_{u=1}\underline{t}_{i+1}\end{aligned}$$

where μ_i is an arbitrary scalar and where

$$\underline{t}_{i-1} = \dot{\underline{r}}^{i-1}/\langle\dot{\underline{r}}^{i-1},\dot{\underline{r}}^{i-1}\rangle^{\frac{1}{2}}\Big|_{u=0}$$

$$\underline{t}_{i+1} = \dot{\underline{r}}^i/\langle\dot{\underline{r}}^i,\dot{\underline{r}}^i\rangle^{\frac{1}{2}}\Big|_{u=1}$$

Hence these together with the equations (*) amount to 5n conditions involving

$$\langle\dot{\underline{r}}^i,\dot{\underline{r}}^i\rangle^{\frac{1}{2}}\Big|_{u=1},\ \langle\dot{\underline{r}}^i,\dot{\underline{r}}^i\rangle^{\frac{1}{2}}\Big|_{u=0},\ \mu_i,\ \underline{t}_i$$

which are sufficient in number to uniquely define the curve Γ.

Definition 5.3

The energy $\bar{E}$ of the bicubic surface $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ is given by

$$\bar{E}(\underline{r}) = \int_0^1\int_0^1 (E(u,v)+G(u,v))dudv$$

where E and G are the two first fundamental coefficients.

The bicubic surface which minimizes $\bar{E}$ on some class of surfaces satisfying specified boundary conditions will be the surface which minimizes the Dirichlet integral {29} within the class.

Chapter 6

Surface Mappings and Distortion

Definition 6.1

A Ruled surface is a two dimensional submanifold M of E^3 with the property that each point $p \in M$ lies on a straight line, called the generator at p, which also lies on M.

It is easy to show that {1} on a ruled surface the Gaussian curvature K is non positive. Further :

Definition 6.2

If M has the property that the tangent plane is constant along generators then M is called Developable.

On a developable, the mean curvature is constant along generators and a theorem due to Massey {2} states that a closed connected surface is developable if and only if the Gaussian curvature $K \equiv 0$.

If we consider the ruled surface $\underline{R} : [0,1] \times [0,1] \longrightarrow E^3$ defined by

$$\underline{R}(u,v) = \underline{R}(u,0) + v[\underline{R}(u,1) - \underline{R}(u,0)]$$

then this has v as the parameter along the generators.

And $\underline{R}$ is developable if

$$[\underline{R}(u,1) - \underline{R}(u,0), \underline{R}_1(u,0), \underline{R}_1(u,1)] = 0 \quad \forall u.$$

If we restrict ourselves to bicubics then the ruled bicubic $\underline{r} : [0,1] \times [0,1] \longrightarrow E^3$ with parameter v along generators is given by

$$\underline{r}(u,v) = \underline{a}_{00} + \underline{a}_{10}u + \underline{a}_{20}u^2/2! + \underline{a}_{30}u^3/3!$$
$$+ v[\underline{a}_{01} + \underline{a}_{11}u + \underline{a}_{21}u^2/2! + \underline{a}_{31}u^3/3!]$$

And this is developable if $[\underline{a}(u),\underline{b}(u),\underline{c}(u)] = 0$

where

$$\underline{a}(u) = \underline{a}_{01} + \underline{a}_{11}u + \underline{a}_{21}u^2/2! + \underline{a}_{31}u^3/3!$$
$$\underline{b}(u) = \underline{a}_{10} + \underline{a}_{20}u + \underline{a}_{30}u^2/2!$$
$$\underline{c}(u) = \underline{a}_{11} + \underline{a}_{21}u + \underline{a}_{31}u^2/2!$$

<u>Proposition 6.3</u>

Given $\{\underline{\alpha}_{ij}\} \subset E^3$ and real numbers E_{ij}, F_{ij} then there are ∞^4 bicubic surfaces which are ruled, have v as parameter along generators and which satisfy

(i) $\underline{r}(i,j) = \underline{\alpha}_{ij}$

(ii) $E(i,j) = E_{ij}$, $F(i,j) = F_{ij}$ $i,j \in \{0,1\}$

are the first fundamental coefficients E, F evaluated at the corners.

Proposition 6.4

Given $\{\underline{\alpha}_{ij}, \underline{\beta}_{ij}\} \subset E^3$ then there is a unique ruled bicubic surface having v as parameter along generators such that

$$\text{(i)} \quad \underline{r}(i,j) = \underline{\alpha}_{ij}$$

$$\text{(ii)} \quad \underline{r}_1(i,j) = \underline{\beta}_{ij} \quad i,j \in \{0,1\}$$

And of course similar results hold for surfaces which are bicubics with u parameter generators.

Definition 6.5

The Exponential map $\exp : TM \longrightarrow M$ is defined as follows. If $(p,V) \in T_pM$ then $\exp(p,V) = \exp_p V$ is the value at $|V|$ of the unique geodesic $\gamma(t)$ in M such that

$$\text{(i)} \ \gamma(0) = p \quad \text{(ii)} \ \gamma'(t)\Big|_{t=0} = V/|V|$$

That is $\exp_p V = \gamma(|V|)$

Theorem 6.6

Let $\lambda : [a,b] \longrightarrow M \subset E^3$ be a cubic curve on the developable surface M such that $\lambda(a) = \lambda(b)$ and suppose that λ is contractible on M. Then M is a plane.

The proof follows since if λ is contractible on M then there is some closed disc inside λ and we can find a generator G of M which cuts λ twice. Thus we construct the map $\mu : [a,b] \longrightarrow [a,b]$ such that

$$\lambda(\mu(t)) \in G \quad \text{if} \quad \lambda(t) \in G.$$

Let $[c,d] \subset [a,b]$. Then we have smooth diffeomorphisms $\alpha : [c,d] \longrightarrow [c,d] \times \mathbb{R}$ and $\beta = \lambda \times \exp : [c,d] \times \mathbb{R} \longrightarrow M$ where β is given by

$$\beta : (t,r) \longmapsto p \in M$$

where p is the point of distance r along the generator through $\lambda(t)$. Consider now just Im β on λ. Let $V \subset M$ be a neighbourhood containing this image. Since λ is an embedding there is a diffeomorphism $\gamma : V \longrightarrow U \subset E^2$. Now take the projection $\delta : U \longrightarrow [e,f] \subset \mathbb{R}$. Then $\mu = \delta\gamma\beta\alpha : [c,d] \longrightarrow [e,f]$ is smooth.

Now since λ is cubic we can write

$$\lambda(\mu(t)) = \lambda(t) + [\mu(t)-t]\lambda'(t) + [\mu(t)-t^2]/2!\ \lambda''(t) + [\mu(t)-t]^3/3!\ \lambda''(t).$$

Since M is developable the tangent plane is constant along generators and so

$$[\lambda(\mu(t)) - \lambda(t), \lambda'(t), \lambda'(\mu(t))] = 0 \qquad (+)$$

But

$$\lambda'(\mu(t))\mu'(t) = \lambda' + (\mu'-1)\lambda' + (\mu-t)\lambda'' + (\mu-t)(\mu'-1)\lambda'' + (\mu-t)^2/2\ \lambda''' + (\mu-t)^2(\mu'-1)/2\ \lambda'''$$

Therefore (+) reduces to

$$(\mu(t)-t)[\lambda'', \lambda', \lambda''']/\alpha = 0 \qquad \alpha = \text{constant}$$

Thus λ is a plane curve on each open subset of S where S contains all members t of $[a,b]$ such that a generator through $\lambda(t)$ cuts λ again not tangentially.

Consider two curves $\lambda : [a,b] \longrightarrow E^3$ and $\mu : [c,d] \longrightarrow E^3$ which are not coplanar. Then we can generate a developable M which contains both λ and μ as follows :

If $\lambda(r)\Big|_p$ and $\mu(t)\Big|_q$ are points of λ and μ which lie on a generator of M then

$$[\lambda-\mu, \lambda', \mu'] = 0 = f(r,t) \quad \text{say.}$$

And $\frac{\partial f}{\partial r}\delta r + \frac{\partial f}{\partial t}\delta t = 0$ and so

$$\delta r/\delta t = -[\lambda-\mu, \lambda', \mu'']/[\lambda-\mu, \lambda'', \mu']$$
$$= -\phi(r,t)$$

Hence we just need to solve

$$\frac{dr}{dt} = -\phi(r,t)$$

giving the solutions $r = \psi(t)$ where

$$f(r,\psi(t)) \equiv 0$$

Suppose now that we want to find some mapping f of a portion of a curved surface M where $f : M \longrightarrow E^2$. We assume that f is defined on $U \subset M$ where U contains at least two points of M.

Let $x,y \in U$. Then we given the following quantitive definition for the distortion of f.

<u>Definition 6.7</u>

The Scale of the map f for each pair $x,y \in U$ is the ratio

$$s(f) = d_E(f(x),f(y))/d_M(x,y)$$

where $d_E(f(x),f(y))$ is the Euclidean distance between $f(x),f(y) \epsilon E^2$ and $d_M(x,y)$ is the geodesic distance between $x,y \epsilon M$.

Ideally we would like $s(f)$ to be constant for all pairs of points in U, but this will not usually be possible.

Definition 6.8

The minimum scale $s_1(f)$ is given by

$$s_1(f) = \inf_{x,y \epsilon U} s(f)$$

and the maximum scale $s_2(f)$ is given by

$$s_2(f) = \sup_{x,y \epsilon U} s(f)$$

Then $s_1(f)$ and $s_2(f)$ are the best possible constants such that

$$s_1 d_s(x,y) \leq d_E(f(x),f(y)) \leq s_2 d_M(x,y) \quad \forall\ x,y \epsilon U$$

This leads to the following definition for distortion which measures the extent to which $s(f)$ is non constant.

Definition 6.9

The distortion δ_0 of f is given by

$$\delta_0 = \log s_2/s_1$$

Then $0 \leq \delta_0 \leq \infty$ and δ_0 is finite if and only if both $s_1, s_2 > 0$.

Definition 6.10

The map f_0 on U has minimum distortion if $\delta_0(f_0) \leq \delta_0(f)$ for all maps f on U.

If we take $M = S^3$ and $U = D_\alpha$ is the closed disc of geodesic radius $r\alpha$. Thus $D_\alpha = \{ x \in S^3 : d_s(x, x_0) \leqslant r\alpha \}$ for some fixed point $x_0 \in S^3$. Then {25} there is one and only one minimum distortion map f_0 on D_α. This map is C^∞ and is the 'azimuthal equidistant projection', which preserves both distances and directions from x_0. And in this case $\delta_0(f_0) = \log(\alpha/\sin\alpha)$. This map is in fact the inverse of the exponential map.

If M is smooth with Riemannian metric in terms of local coordinates given by $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ then we can consider Δ, the associated Laplace - Beltrami operator, and a second Riemannian metric on M of the form $\sigma^2 ds^2$ where σ is positive and in C^2. The Gaussian curvature $\hat{K}$ associated with this new metric is given by {26} the formula $\hat{K} = (K - \Delta\log\sigma)/\sigma^2$. If $f : M \longrightarrow E^2$ is conformal and σ is the infinitesimal scale function $\sigma(x) = \lim_{y \to x} s(f)$ then $\hat{K}\big|_x$ is the Gaussian curvature of E^2 at $f\big|_x$. But $\hat{K} \equiv 0$ and so the differential equation $\Delta\log\sigma = K$ must be satisfied.

Conversely given any solution σ to the differential equation $\Delta\log\sigma = K$ the Riemannian metric $\sigma^2 ds^2$ has curvature $\hat{K} \equiv 0$. Thus {26} any sufficiently small connected open subset of M with the metric $\sigma^2 ds^2$ can be mapped isometrically onto an open subset of the plane, which is unique up to rigid motions of the plane.

We consider now the construction of a particular mapping $f : M \longrightarrow E^2$ which arises from the inverse of the exponential map. If M is a surface defined by $\underline{r} = \underline{r}(u^1, u^2)$ where $(u^1, u^2) \in [0,1] \times [0,1]$ then we first of all apply a linear transformation to u^1, u^2 so that we have $g_{ij}(1/2, 1/2) = \delta_{ij}$.

Now the differential equations for geodesics on M are

$$\frac{d^2u^1}{ds^2} + \Gamma^1_{ij} \frac{du^i}{ds} \frac{du^j}{ds} = 0$$

$$\frac{d^2u^2}{ds^2} + \Gamma^2_{ij} \frac{du^i}{ds} \frac{du^j}{ds} = 0$$

where the Γ^k_{ij} are the Christoffel symbols.

By imposing the initial conditions that:

(i) $u^1(0) = 1/2$, $u^2(0) = 1/2$.

and by taking a^1, a^2 such that

(ii) $du^1/ds\Big|_{s=0} = a^1$ and $du^2/ds\Big|_{s=0} = a^2$

this will give solutions of the form $u^1 = u^1(s, a^1, a^2)$ and $u^2 = u^2(s, a^1, a^2)$ and if we write

$$u^i = \sum A^i_{\lambda\mu\nu} (a^1)^\mu (a^2)^\nu s^\lambda$$

and solve for the A's then at s = 1 this gives

$$f^{-1} = u^i(1, a^1, a^2)$$

Let M^m and N^n be Riemannian manifolds with M compact and let $f : M^m \longrightarrow N^n$ be C^1. At $p \in M$ take local coordinates x^i and at f(p) y^i so that locally $y^\alpha = f^\alpha(\chi^1, \ldots . \chi^m)$ and

The differential f_* has matrix representation $(\partial f^\alpha/\partial x^i)$ relative to these coordinates where $u^i\Big|_p \longmapsto \left(\frac{\partial f^\alpha}{\partial x^i}u^i\right)\Big|_{f(p)}$

Definition 6.11

(i) The distortion δ_1 of f at $p\varepsilon M$ is given by

$$\delta_1(f,p) = \sup_{u^i} \|f_*(u^i)\|^2 = \sup_{u^i} g'_{\alpha\beta}f^\alpha_i f^\beta_j u^i u^j$$

where g and g' are the metrics on M and N and $\|u^i\| = 1$

(ii) The distortion of f is given by

$$\delta_1(f) = \sup_{p\varepsilon M} \delta_1(f,p)$$

Now if f is an isometry then $g_{ij} = g'_{\alpha\beta}f^\alpha_i f^\beta_j$ and so as a measure of non-isometry we can take

$$\delta_1(p,u^i) = g'_{\alpha\beta}f^\alpha_i f^\beta_j u^i u^j - g_{ij}u^i u^j$$

Then the problem is to find the maximum of

$$g'_{\alpha\beta}\ f^\alpha_i f^\beta_j u^i u^j - 1 \text{ subject to } g_{ij}u^i u^j - 1 = 0$$

If we write

$$F(u) = g'_{\alpha\beta}f^\alpha_i f^\beta_j u^i u^j - 1 - \lambda(g_{ij}u^i u^j - 1)$$

then

$$\partial F/\partial u^i = g'_{\alpha\beta}f^\alpha_i f^\beta_j u^j - \lambda g_{ij} u^j = 0$$

and so the distortion at p will come from the characteristic roots of the equation

$$\det\left[g'_{\alpha\beta}f^\alpha_i f^\beta_j - \lambda g_{ij}\right] = 0$$

Definition 6.12

If $p\varepsilon M$ and $f : M \longrightarrow N$ is such that for each $p\varepsilon M$ then

$\|f_*(u)\|/\|u\|$ is constant on T_pM, then f is called conformal.

Every smooth oriented 2-manifold has a conformal structure. If M and N are oriented C^2 surfaces immersed in E^3 the Euclidean metric imposes a specific conformal structure upon M and N. Conformal parameters may be introduced on M for instance by means of isothermal coordinates.

At each point $p \epsilon M^m$ let A be the matrix representation of the differential f_* relative to orthonormal bases of T_pM^m and $T_{f(p)}N^n$ and let A^t be the transpose of A. If rank f_* = rank A = r at each point then $r \leq \min(m,n)$ and rank $A^tA = r$.

If $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots\ldots \geq \lambda_r > \lambda_{r+1} = \lambda_{r+2} \ldots\ldots = \lambda_m = 0$ are the characteristic roots of A^tA then we consider the following ratio function $\|\Lambda^r f_*\|$ of volume elements of M and N {27} , where

$$\|\Lambda^r f_*\|^2 = \sum_{i_1 < \ldots < i_r}^{r} \lambda_{i_1} \ldots\ldots \lambda_{i_r}$$

Thus $\|\Lambda^1 f_*\| = \|f_*\|$ is the ratio of distances.

At each $p\epsilon M$ let S^{r-1} be a unit (r-1) sphere in T_pM. If rank $f_* = r = \min(m,n)$ the image of S^{r-1} under f_* is an ellipsoid of dimension r-1.

<u>Definition 6.13</u>

$f : M \longrightarrow N$ is called K-quasiconformal if at each point

$p \in M$ then $a/b \leq K$ where a and b are the largest and smallest axes of the ellipsoid $f_*(S^{r-1})$ in $T_{f(p)}N$.

One may verify that f is K-quasiconformal if and only if $\lambda_1/\lambda_r \leq K^2$ at each point.

Let M and N be oriented C^2 surfaces immersed in E^3.

Definition 6.14

$f : M \longrightarrow N$ is called a Teichmüller mapping if isothermal coordinates u,v may be chosen in the neighbourhood of all but a discrete set of exceptional points on M so that the first fundamental forms at corresponding points of M and N are given by

$$ds^2 = \mu(u,v)(du^2 + dv^2)$$

$$d\hat{s}^2 = \hat{\mu}(u,v)(K^2du^2 + dv^2)$$

In fact a Teichmuller mapping is either a conformal mapping or else it is the K quasiconformal mapping characterised by a particular Beltrami differential {7}. It is known that {28} the mapping minimising K amongst all homeomorphisms between closed Riemann surfaces R_1 and R_2 of the same finite genus ≥ 2, which are homotopic to some fixed homeomorphism between R_1 and R_2, always exists is unique and is Teichmüller. Thus a Teichmüller mapping always exists between any two closed oriented C^2 surfaces immersed in E^3 and each one will give the most nearly conformal mapping between the surfaces among homeomorphisms homotopic to it.

Let $\underline{r} : D \longrightarrow S \subset E^3$ be a representation of some surface S in E^3. We consider three mappings f_1, f_2, f_3 of S into E^2.

(1) A mapping f_1 which has the property that $f_1\big|_{\partial S}$ has a distortion equal to zero and is such that on $S-\partial s$ the distortion is a minimum.

Consider first of all the mapping of S onto its parameter domain D. If the metrics on S and D are $d\sigma^2$ and ds^2 then

$d\sigma^2 = Edu^2 + 2Fdudv + Gdv^2$ and $ds^2 = du^2 + dv^2$

This mapping will produce stretching on ∂S and so $d\sigma^2 \neq ds^2$ on the boundaries. Therefore we look for a mapping $h : D' \longrightarrow D$ such that the metric ds'^2 on D' satisfies $ds'^2 = d\sigma^2$ on the boundaries. If $z \in D'$ let h(z) be given by $u = u(u', v')$, $v = v(u', v')$. Then

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \partial u/\partial u' & \partial u/\partial v' \\ \partial v/\partial u' & \partial v/\partial v' \end{pmatrix} \begin{pmatrix} du' \\ dv' \end{pmatrix} = J_h \begin{pmatrix} du' \\ dv' \end{pmatrix}$$

And

$$d\sigma^2 = (du\ dv) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$$

$$= (du'dv')\ J_h^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} J \begin{pmatrix} du' \\ dv' \end{pmatrix}$$

So the new first fundamental matrix is $J_h^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} J$

with parameters u', v'.

For a small displacement $\begin{pmatrix} du' \\ dv' \end{pmatrix}$ along $\partial D'$ we require that $ds'^2 = d\sigma^2$ which means on $\partial D'$

$$(du')^2 + (dv')^2 = (du'dv')\, J_h^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} J \begin{pmatrix} du' \\ dv' \end{pmatrix}$$

We in fact find a conformal $h: D \longrightarrow D'$ which achieves this. For a small displacement dz on ∂D we have displacement $h'(z)dz$ on $\partial D'$ and so to unstretch the boundary we require

$$|h'(z)||dz| = d\sigma = \phi(z)|dz|$$

where $\phi(z) = d\sigma/|dz|$ is the stretch factor at the boundary point z. We therefore require $h'(z)$ analytic in D with $|h'(z)| = \phi(z)$ on ∂D. If there is such an analytic $h'(z)$ then $\log|h'(z)| = u(z)$ is harmonic in D and is therefore the solution of Dirichlets problem for the given boundary function $\log \phi(z)$. Having obtained this unique $u(z)$ we can find the conjugate harmonic $v(z)$ from the Cauchy Riemann equations ($u_x = v_y$, $u_y = -v_x$) by integration. This will determine $v(z)$ up to an arbitrary additive constant, and

$$h'(z) = e^{u(z)+iv(z)}$$

and integrating gives us $h(z)$ which satisfies

$$|h'(z)| = \phi(z)$$

Then $h(z)$ certainly maps the boundary onto a curve with $ds'^2 = d\sigma^2$ but unfortunately we do not know that $\partial D'$ is a simple closed curve.

Now consider $\bar{h} : D' \longrightarrow D''$ such that $\bar{h}\big|_{\partial D'}$ has distortion zero and where $\bar{h}$ reduces the distortion at the point of maximum distortion in $D'-\partial D'$.

Now $d\sigma^2 - ds^2 = (E-1)du^2 + 2Fdudv + (G-1)dv^2$.

At $z \in D-\partial D$ the distortion in the two principal directions is given by the characteristic roots of (g_{ij}).

Taking $\det(g_{ij} - \lambda\delta_{ij}) = 0$ we have

$$\lambda^2 - (E+G)\lambda + EG-F^2 = 0$$

and the solutions λ_1, λ_2 are the magnitudes of the principal axes of the infinitesimal ellipse about $\underline{r}\big|_z$.

If we choose $\bar{h}$ such that

$$\delta = (\lambda_1-1)^2 + (\lambda_2-1)^2$$

is minimised this will give the required solution.

And in fact we can write

$$\delta = (\lambda_1+\lambda_2)^2 - 2\lambda_1\lambda_2 - 2(\lambda_1+\lambda_2) + 2$$

which is conveniently in terms of the trace and determinant of the matrix.

(2) A mapping $f_2 : S \longrightarrow \bar{S} \subset E^2$ having the property that f is conformal and is locally an isometry on ∂S.

We assume also that S is bicubic represented in terms of parameters s and t by the equation

$$\underline{r}(s,t) = \sum_{j,k=0}^{3} \underline{a}_{jk} s^j t^k \qquad \underline{a}_{jk} \in E^3$$

with $D = [0,1] \times [0,1]$.

On S let the metric $d\sigma^2$ be given by

$$d\sigma^2 = Eds^2 + 2Fdsdt + Gdt^2$$

and if $h : D' \longrightarrow D$ is given by $s=s(u,v), t=t(u,v)$

then

$$d\sigma^2 = E[s_u du+s_v dv]^2 + 2F[s_u du+s_v dv][t_u du+t_v dv] + G[t_u du+t_v dv]^2$$

$$= [Es_u^2+2Fs_u t_u+Gt_u^2]du^2 + 2[Es_u s_v+F(s_u t_v+s_v t_u)+Gt_u t_v]dudv + [Es_v^2+2Fs_v t_v+Gt_v^2]dv^2$$

Then $f_2 : S \longrightarrow D'$ is conformal if and only if

$$Es_u s_v + F(s_u t_v+s_v t_u) + Gt_u t_v = 0 \quad \text{and}$$

$$E(s_u^2-s_v^2) + 2F(s_u t_u-s_v t_v) + G(t_u^2-t_v^2) = 0$$

Solving these non linear differential equations will give a class of solutions from which we choose the one giving the local isometry property on the boundary.

Writing these equations in matrix form we have

$$(s_u t_u) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} s_v \\ t_v \end{pmatrix} = 0 \quad \text{and}$$

$$(s_u t_u) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} s_u \\ t_u \end{pmatrix} = (s_v t_v) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} s_v \\ t_v \end{pmatrix}$$

which combined give

$$\begin{pmatrix} s_u & t_u \\ s_v & t_v \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} s_u & s_v \\ t_u & t_v \end{pmatrix} = \begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix}$$

where ϕ is a scalar.

Writing $\underline{x} = \begin{pmatrix} s \\ t \end{pmatrix}$, $\underline{w} = \begin{pmatrix} u \\ v \end{pmatrix}$, $M^T M = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$

we have

$$(\partial \underline{x}/\partial \underline{w})^T M^T M (\partial \underline{x}/\partial \underline{w}) = \phi I$$

Then $\phi^{-1/2} M(\partial \underline{x}/\partial \underline{w})$ is orthogonal and so we can write

$$(\partial \underline{w}/\partial \underline{x}) = \begin{pmatrix} \theta & \Psi \\ -\Psi & \theta \end{pmatrix} M \qquad (+)$$

Now since

$$M = \begin{pmatrix} \sqrt{E} & F/\sqrt{E} \\ 0 & \sqrt{EG-F^2}/\sqrt{E} \end{pmatrix}$$

Then (+) becomes

$$\begin{pmatrix} u_s & u_t \\ v_s & v_t \end{pmatrix} \begin{pmatrix} 1/\sqrt{E} & -F/\sqrt{E(EG-F^2)} \\ 0 & \sqrt{E}/\sqrt{EG-F^2} \end{pmatrix} = \begin{pmatrix} \theta & \Psi \\ -\Psi & \theta \end{pmatrix}$$

Hence we have the two differential equations

$$E\partial v/\partial t = \sqrt{EG-F^2}\,\partial u/\partial s + F\partial v/\partial s$$

$$-E\partial u/\partial t = \sqrt{EG-F^2}\,\partial v/\partial s - F\partial u/\partial s$$

The uniqueness comes by insisting that f is locally an isometry on the boundary.

If the metric on D' is $d\sigma'^2$ with $d\sigma'^2 = du^2+dv^2$ then since along constant t, $d\sigma^2 = Eds^2$ and along constant s, $d\sigma^2 = Gdt^2$ then the required boundary conditions are

$$(\partial u/\partial s)^2 + (\partial v/\partial s)^2 = E \quad \text{for } t\varepsilon\}0,1\}$$

$$(\partial u/\partial t)^2 + (\partial v/\partial t)^2 = G \quad \text{for } s\varepsilon\{0,1\{$$

which will give uniqueness up to rotation and translation.

(3) A mapping $f_3 : S \longrightarrow \bar{S} \subset E^2$ which minimises a given norm for the departure from isometry.

Let the metric ds^2 on S be given by

$$ds^2 = Edu^2+2Fdudv+Gdv^2$$

and let $h : D \longrightarrow D' = \bar{S}$ be given by

$x=x(u,v)$, $y=y(u,v)$ with metric $d\bar{s}^2$ on $\bar{S}$ given by

$$d\bar{s}^2 = \bar{E}du^2+2\bar{F}dudv+\bar{G}dv^2$$

where

$\bar{E} = x_1{}^2+y_1{}^2$, $\bar{F} = x_1x_2+y_1y_2$, $\bar{G} = x_2{}^2+y_2{}^2$

Let $\mu = d\bar{s}^2/ds^2$. Then

$(E\mu-\bar{E})du^2 + 2(F\mu-\bar{F})dudv + (G\mu-\bar{G})dv^2 = 0$

and the maximum and minimum values μ_1,μ_2 of μ arise from the condition

$$(F\mu-\bar{F})^2 = (E\mu-\bar{E})(G\mu-\bar{G})$$

Writing

$H^2 = EG - F^2$, $\bar{H}^2 = \bar{E}\bar{G} - \bar{F}^2$, $J = (E\bar{G}+\bar{E}G-2F\bar{F})/2$

then these values are

$$\mu = \left[J \pm \sqrt{(J^2-H^2\bar{H}^2)}\right]/H^2$$

If f_3 is an isometry then $\mu_1 = \mu_2 = 1$ and

$J = H^2 = \bar{H}^2$, $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$

Two obvious norms for measuring the amount by which f_3 distorts S are :

(a) at each $p\varepsilon S$

$$\delta_1\ (f_3,p) = \log(\mu_1/\mu_2)$$

Then $\delta_1 \geqslant 0$ and $\delta_1 = 0$ if and only if $\mu_1 = \mu_2$

Also $\delta_1 \longrightarrow \infty$ as $\mu_1 \longrightarrow \infty$ or $\mu_2 \longrightarrow 0$

In terms of the fundamental coefficients,

$$\delta_1\ (f_3,p) = 2\log\ (J+\sqrt{C})/H\bar{H}$$

where $C = J^2 - H^2\bar{H}^2$.

And a measure of the overall distortion of f_3

is $\delta_1(f_3) = 2\iint_D \log(J+\sqrt{C})/\bar{H}\ dudv$

(b) at each $p\varepsilon S$

$\delta_2(f_3,p) = (\mu_1+\mu_1^{-1}+\mu_2+\mu_2^{-1}-4)/2$

Then $\delta_2 \geq 0$ and $\delta_2 = 0$ if and only if $\mu_1 = \mu_2 = 1$

Also $\delta_2 \to \infty$ as $\mu_1 \longrightarrow \infty$ or $\mu_2 \to 0$.

In terms of the fundamental coefficients,

$\delta_2(f,p) = (J/\bar{H}^2 + J/H^2) - 2$

The overall distortion of f_3 in this case is

$$\delta_2(f_3) = \iint_D \left[(J/\bar{H}^2 + J/H^2) - 2\right] H\ dudv$$

To obtain the mapping which minimizes the chosen norm we can follow the normal methods of the calculus of variations. If $x = \alpha(u,v)$ and $y = \beta(u,v)$ is the extremal mapping f_0 say then we consider $\hat{f}$ given by

$$\hat{\alpha}(u,v) = \alpha(u,v) + \delta\xi(u,v)$$

$$\hat{\beta}(u,v) = \beta(u,v) + \varepsilon\eta(u,v)$$

Then

$$\hat{E} = \bar{E} + 2\delta\alpha_1\xi_1 + 2\varepsilon\beta_1\eta_1 + O(\delta^2) + O(\varepsilon^2)$$

$$\hat{F} = \bar{F} + \delta(\alpha_1\xi_2 + \alpha_2\xi_1) + \varepsilon(\beta_1\eta_2 + \beta_2\eta_1) + O(\delta^2) + O(\varepsilon^2)$$

$$\hat{G} = \bar{G} + 2\delta\alpha_2\xi_2 + 2\varepsilon\beta_2\eta_2 + O(\delta^2) + O(\varepsilon^2)$$

$$\hat{H} = \bar{H}^2 + 2\varepsilon\bar{H}(\alpha_1\eta_2 - \alpha_2\eta_1) + 2\delta\bar{H}(\beta_2\xi_1 - \beta_1\xi_2) + O(\delta^2) + O(\varepsilon^2)$$

Then

$$\delta_2(f) = \iint_D \left[(\hat{E}G+E\hat{G}-2F\hat{F})(1/\hat{H}^2+1/H^2)-2\right]Hdudv$$

$$= \iint_D \left[(J/\bar{H}^2+J/H^2)-2\right]Hdudv$$

$$+ 2\delta\iint_D (\gamma_a\xi_1 + \gamma_b\xi_2)\ dudv$$

$$+ 2\varepsilon\iint_D (\gamma_c\eta_1 + \gamma_d\eta_2)\ dudv + O(\delta^2) + O(\varepsilon^2)$$

where $\gamma_a,\gamma_b,\gamma_c,\gamma_d$ are functions of $E,F,G,H,\bar{E},\bar{F},\bar{G},\bar{H}$ and $\partial\alpha/\partial u$, $\partial\alpha/\partial v$, $\partial\beta/\partial u$, $\partial\beta/\partial v$.

If f_0 is to minimise δ_2 then we require

$$\iint_D (\gamma_a\partial\xi/\partial u + \gamma_b\partial\xi/\partial v)\ dudv = 0$$

$$\iint_D (\gamma_c\partial\eta/\partial u + \gamma_d\partial\eta/\partial v)\ dudv = 0$$

for arbitrary functions $\xi(u,v)$ and $\eta(u,v)$.

If $D = [0,1] \times [0,1]$ then we require

$$\partial\gamma_a/\partial u + \partial\gamma_b/\partial v = 0$$

$$\partial\gamma_c/\partial u + \partial\gamma_d/\partial v = 0$$

with the boundary conditons

$$\gamma_a\Big|_{u=0} = \gamma_c\Big|_{u=1} = \gamma_b\Big|_{v=0} = \gamma_d\Big|_{v=1} = 0$$

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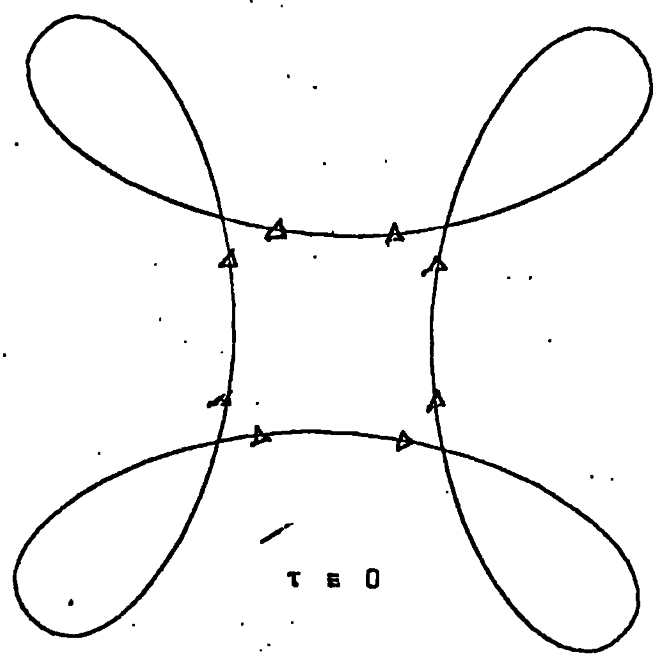
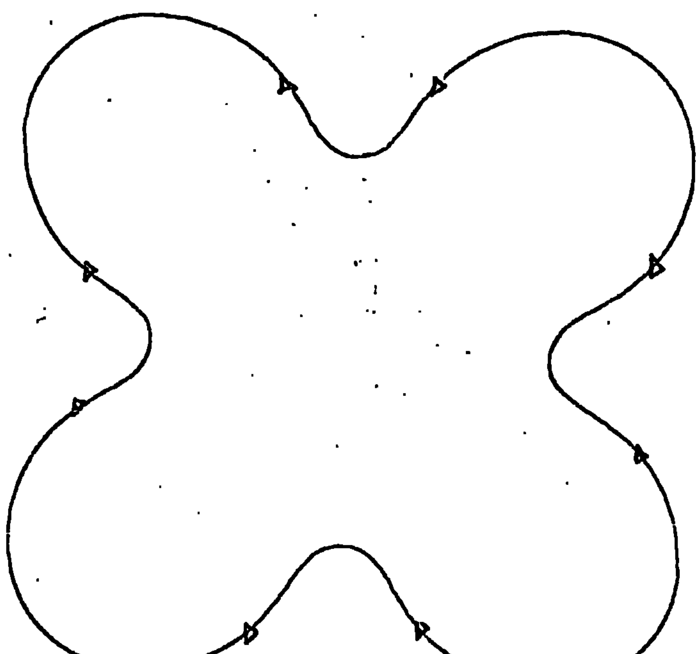


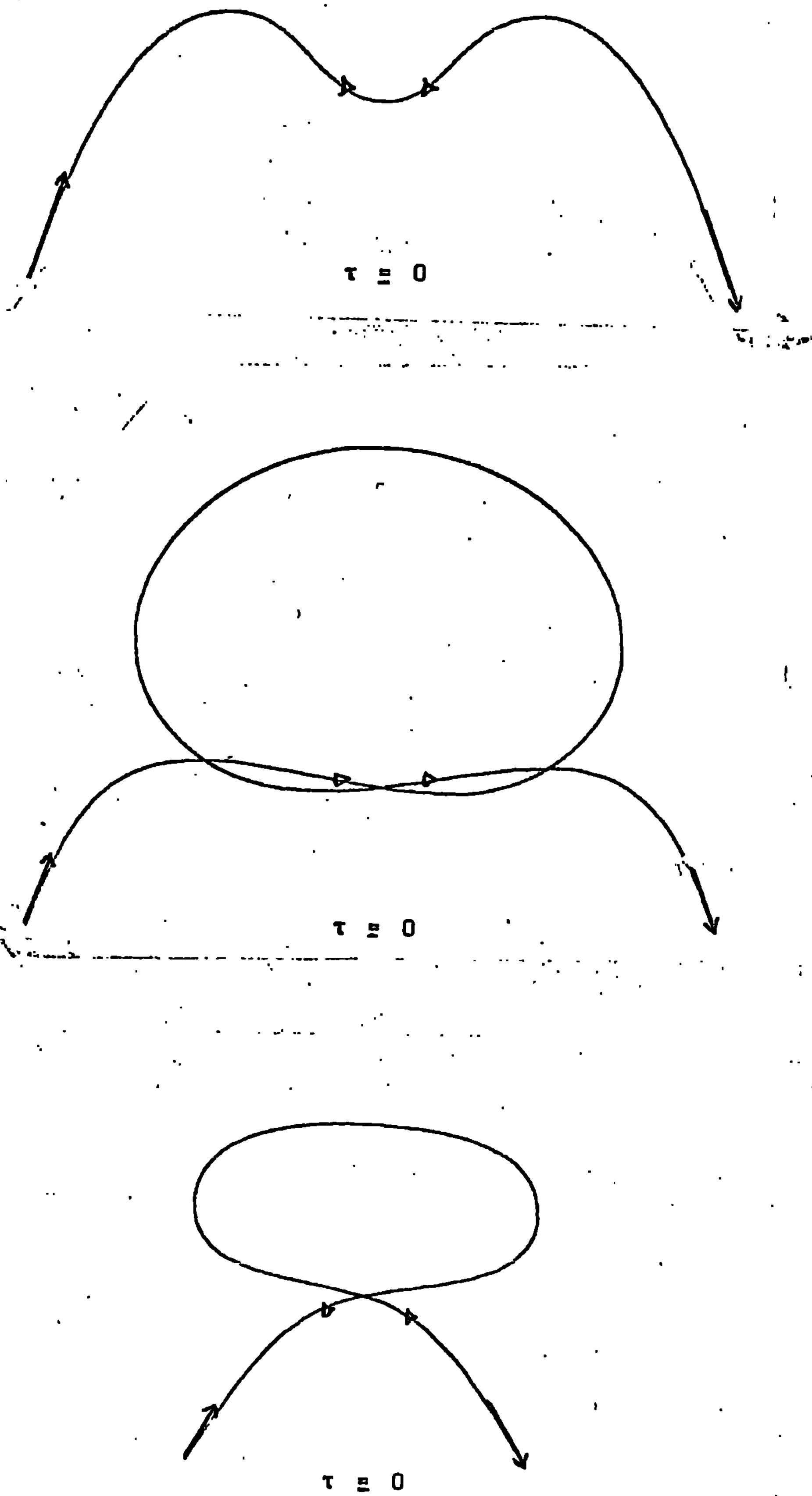
τ ≡ 0



τ ≡ 0

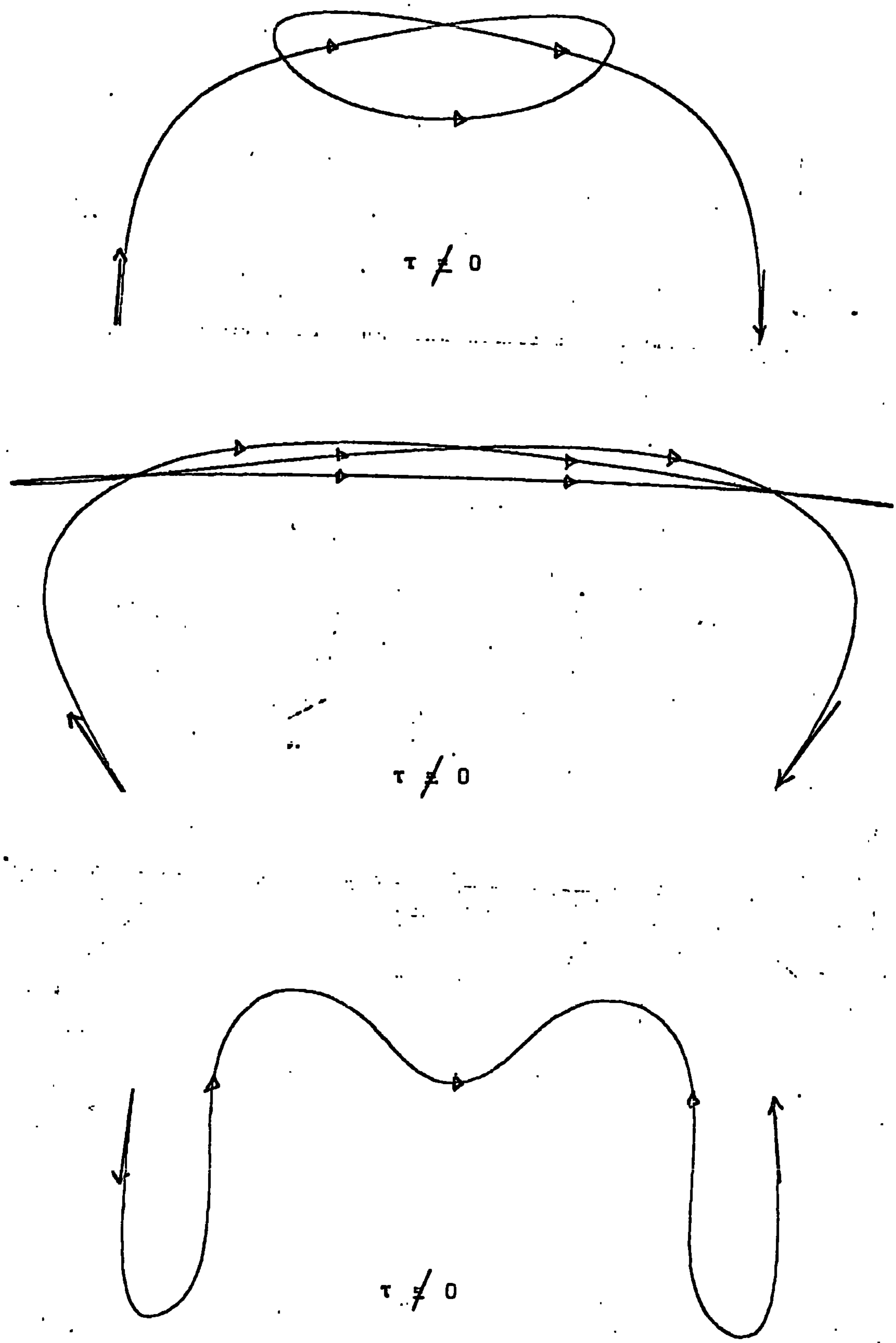
C^3 Piecewise Cubics

Figure 1



C^3 Piecewise Cubics

Figure 2



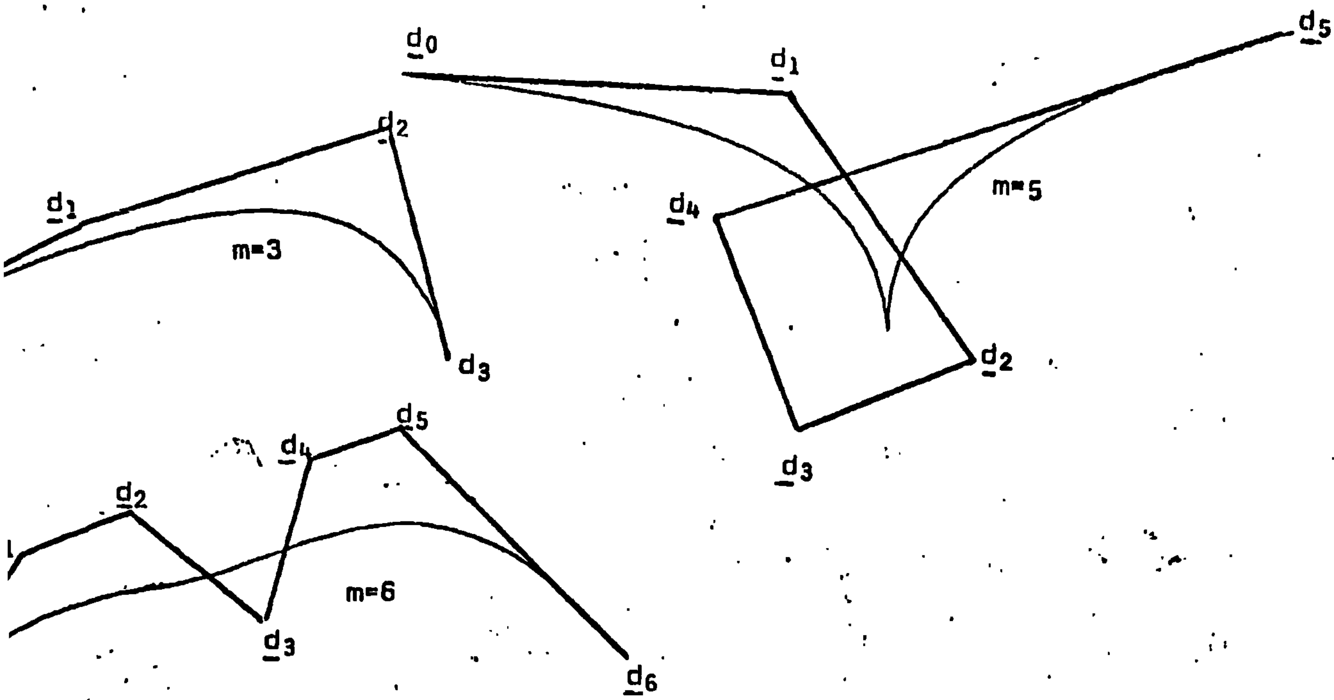
C^3 Piecewise Cubics

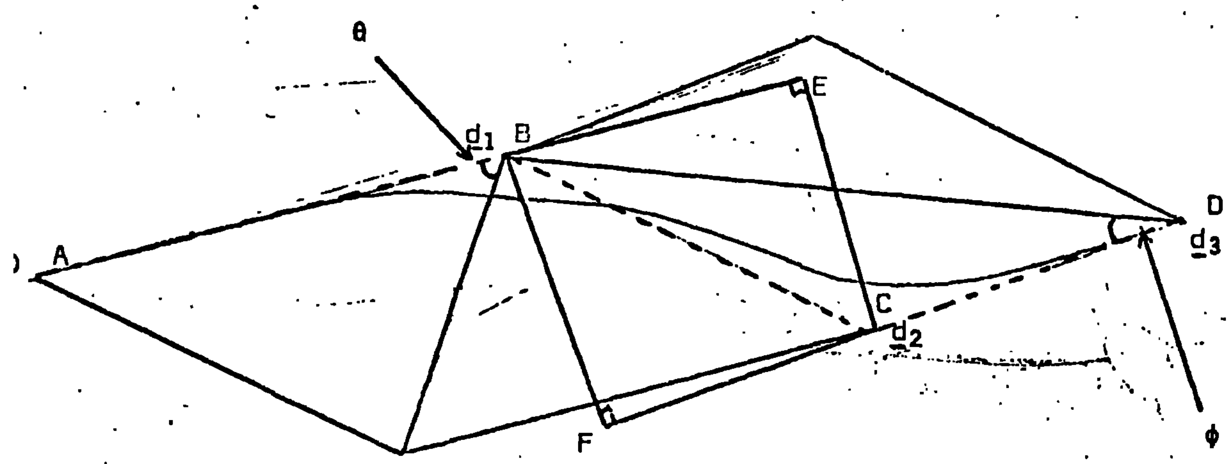
Figure 3

$\tau \equiv 0$

C^3 Piecewise Cubics

Figure 4





$\kappa(0) = 2|\overline{CE}| / 3|\overline{AB}|^2$ $\kappa(1) = 2|\overline{BF}| / 3|\overline{CD}|^2$

$\tau(0)=2|\overline{CH}| \cos\alpha \;/\; |\overline{AB}||\overline{CE}|$

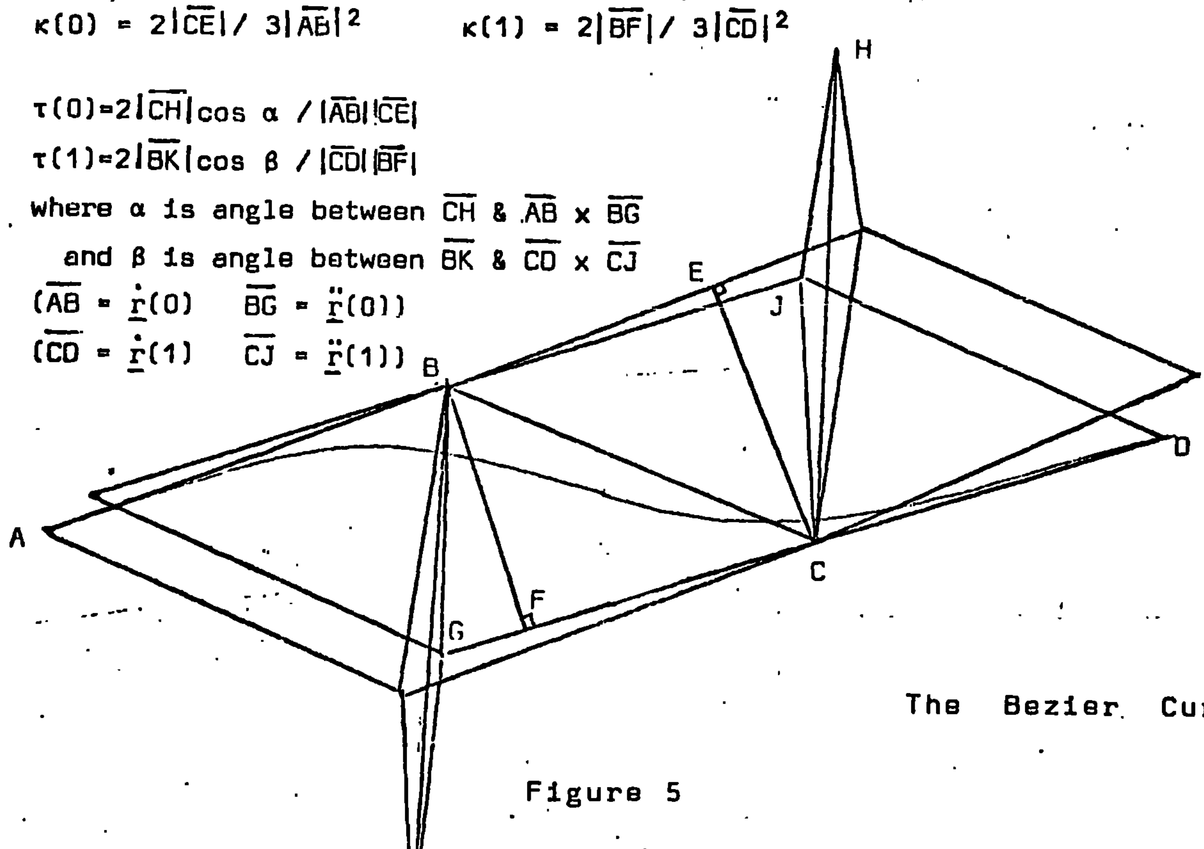
$\tau(1)=2|\overline{BK}| \cos\beta \;/\; |\overline{CD}||\overline{BF}|$

where α is angle between $\overline{CH}$ & $\overline{AB} \times \overline{BG}$

and β is angle between $\overline{BK}$ & $\overline{CD} \times \overline{CJ}$

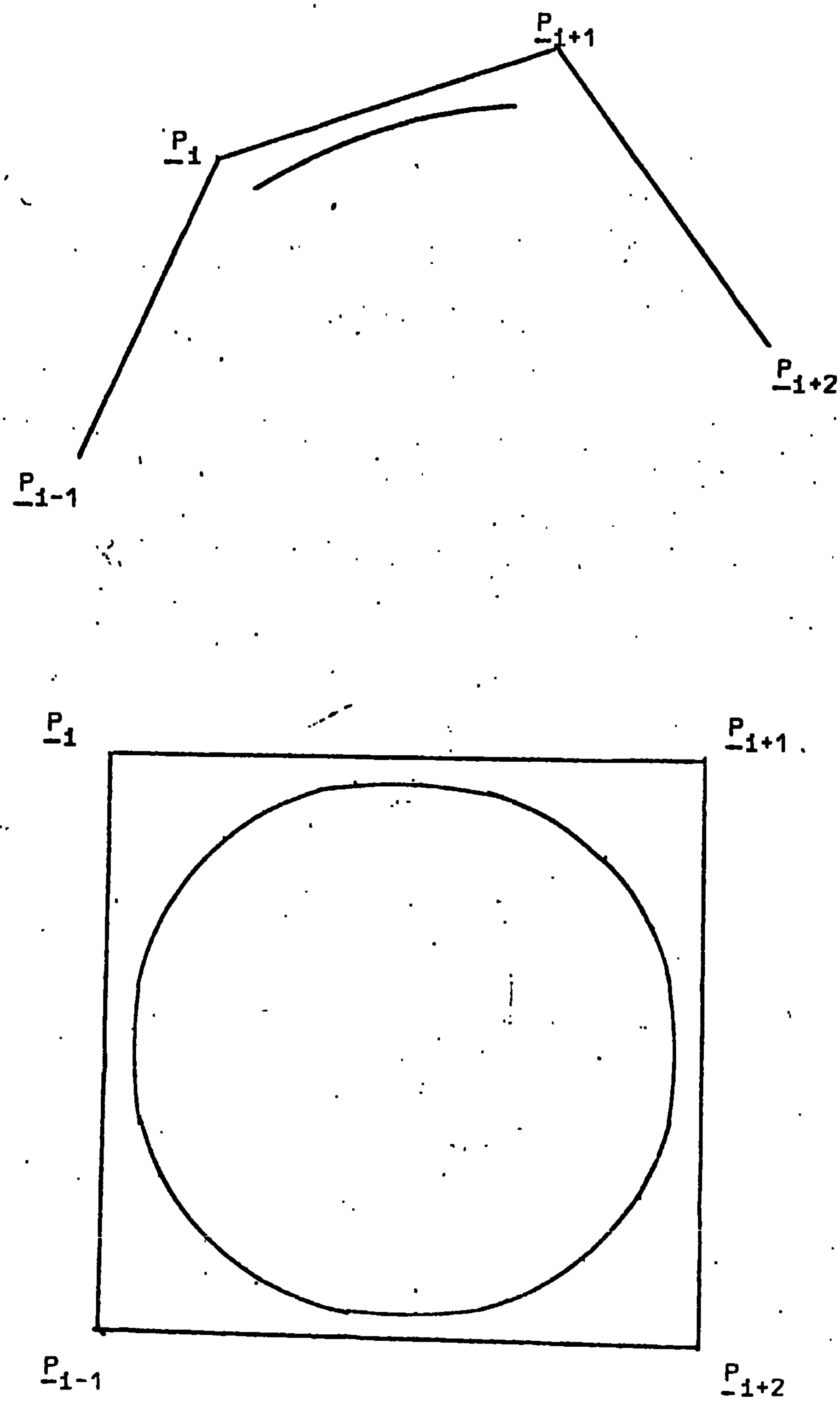
$(\overline{AB} = \dot{\underline{r}}(0)$ $\overline{BG} = \ddot{\underline{r}}(0))$

$(\overline{CD} = \dot{\underline{r}}(1)$ $\overline{CJ} = \ddot{\underline{r}}(1))$



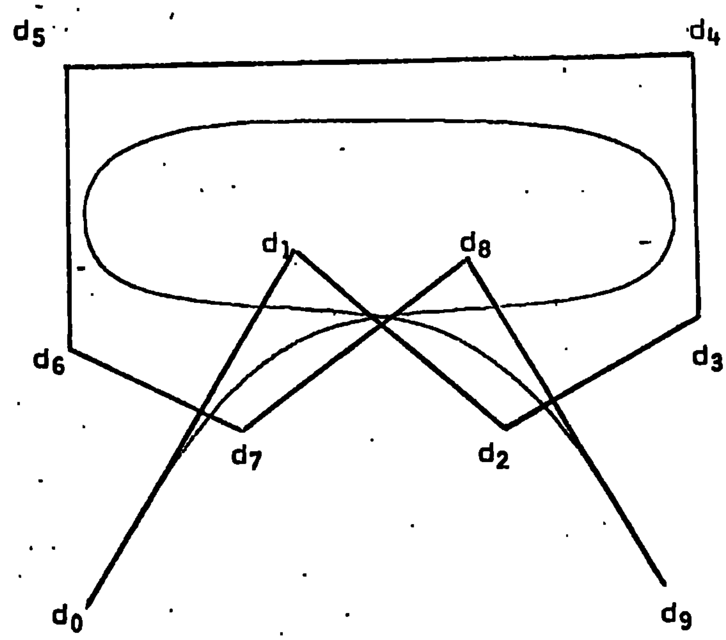
The Bezier Curve

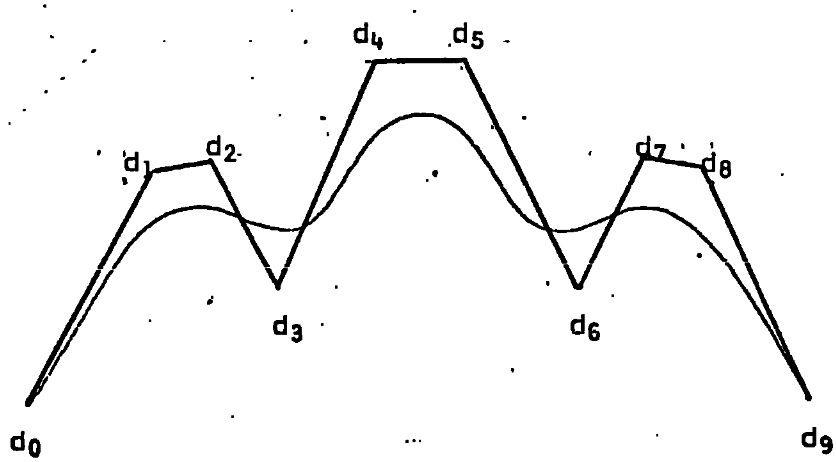
Figure 5



Typical B-Spline Cubic Curves

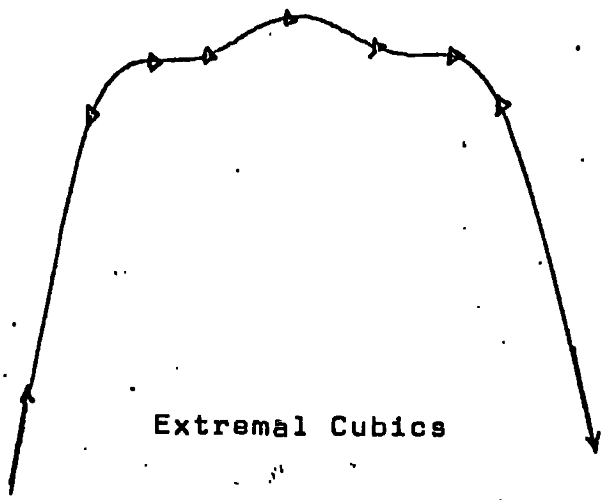
Figure 6



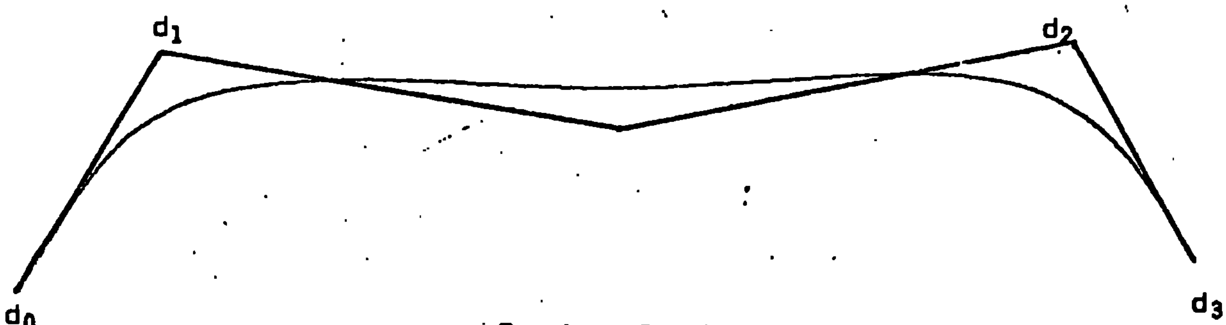


Bezier Curves

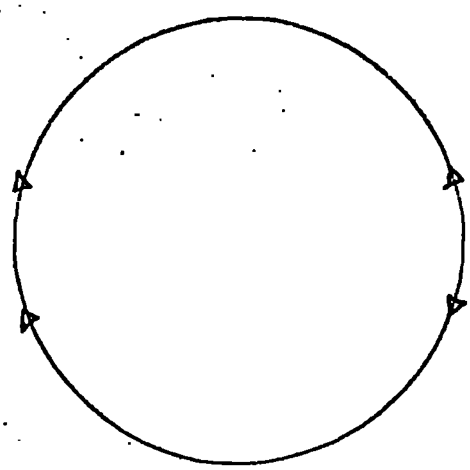
Figure 7



Extremal Cubics



Bezier Cubic



Manning Cubics

Figure 8

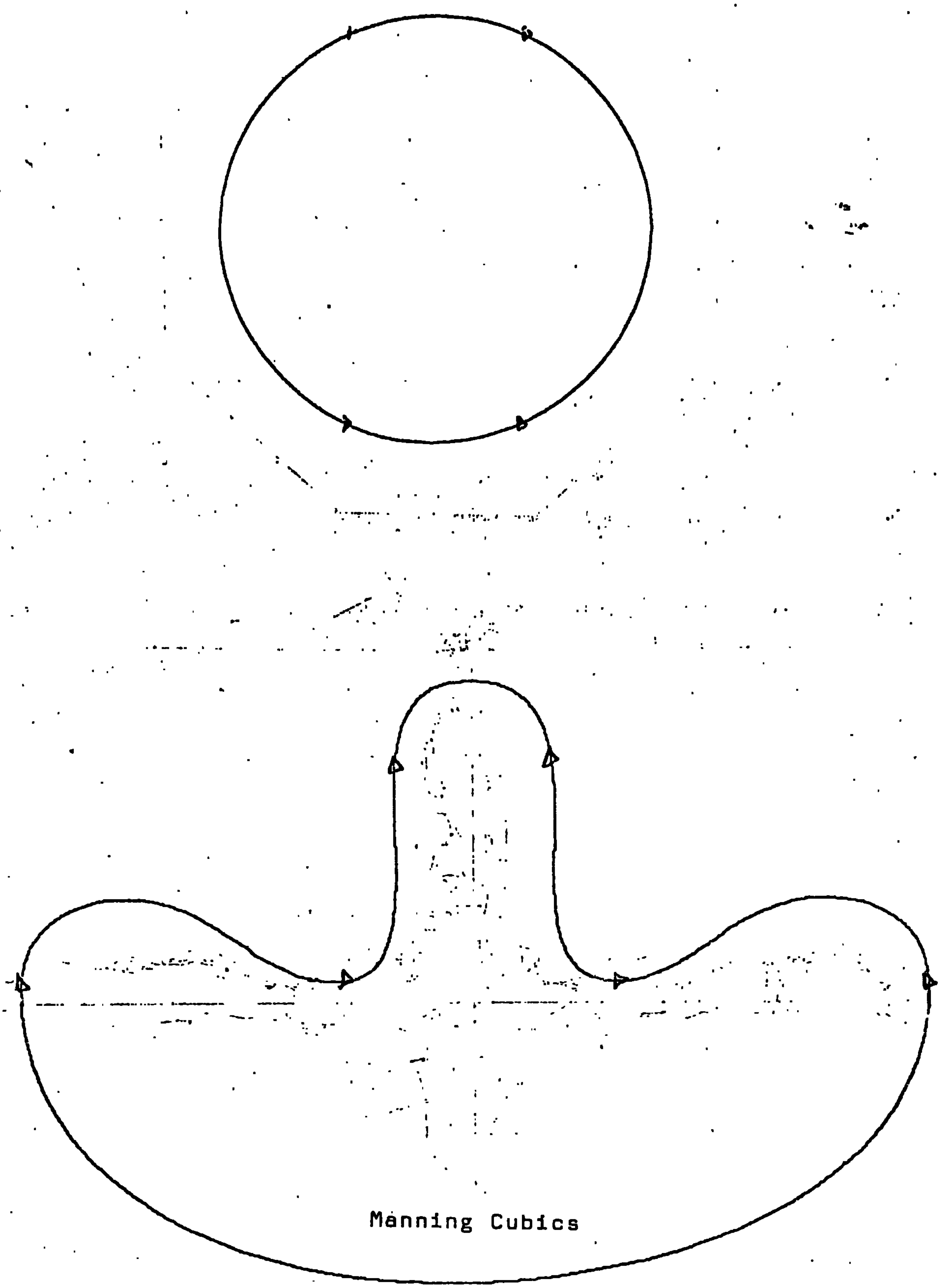
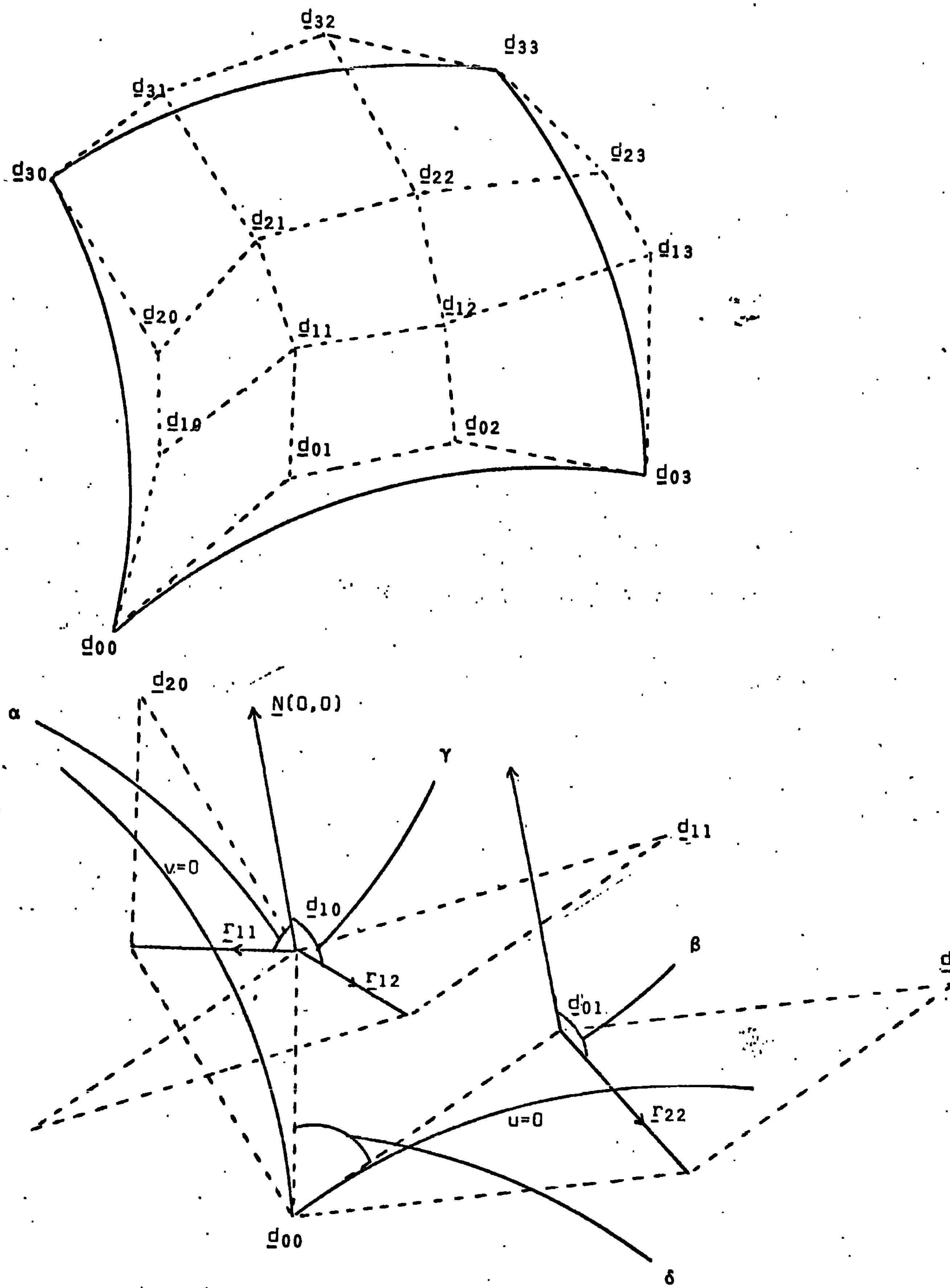
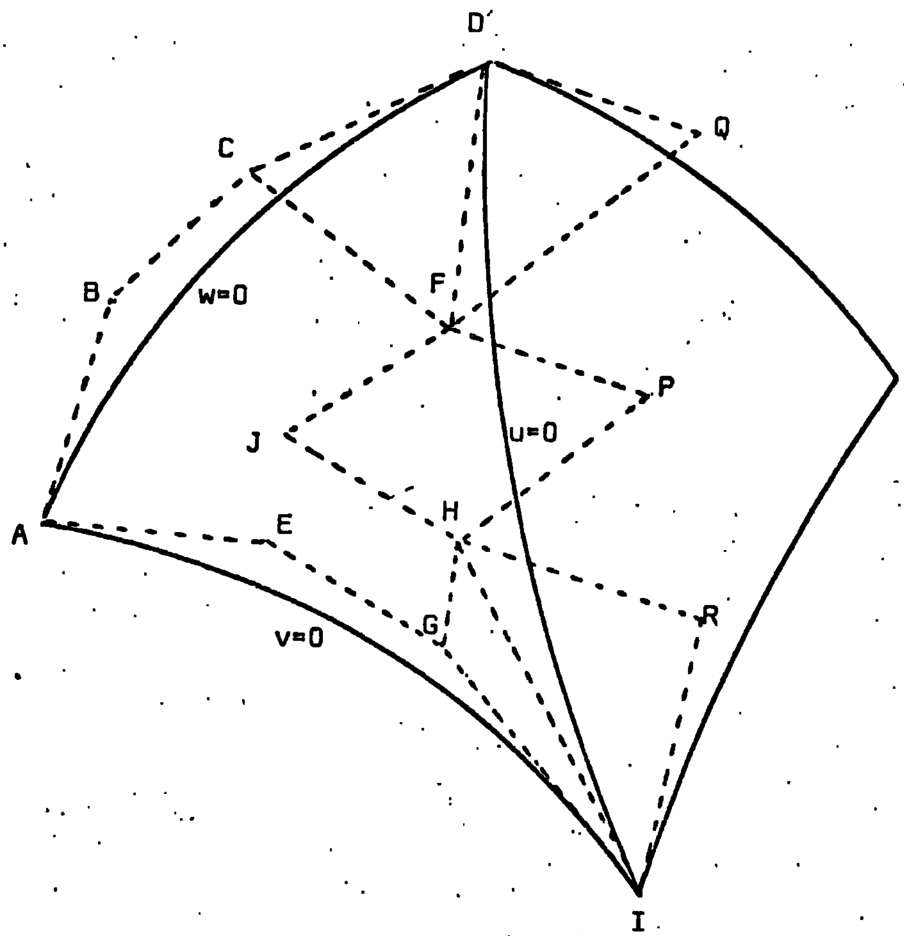


Figure 9



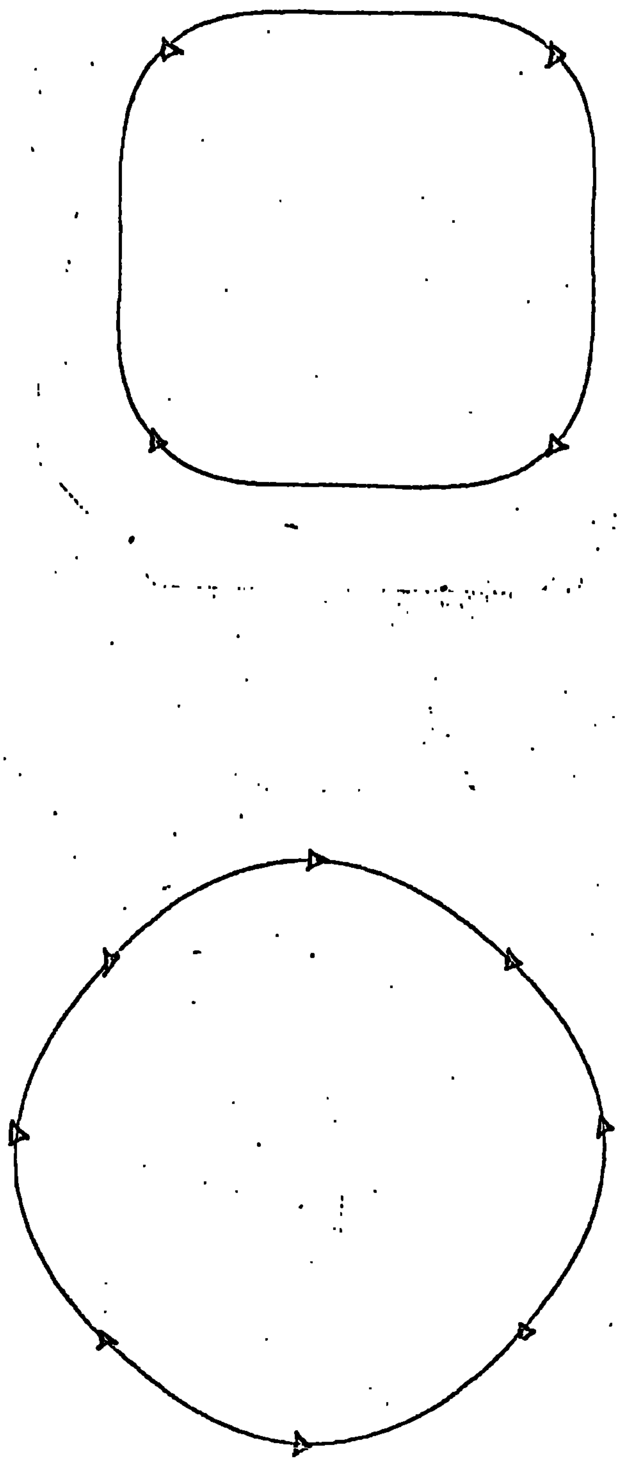
The Bezier Bicubic Surface

Figure 10



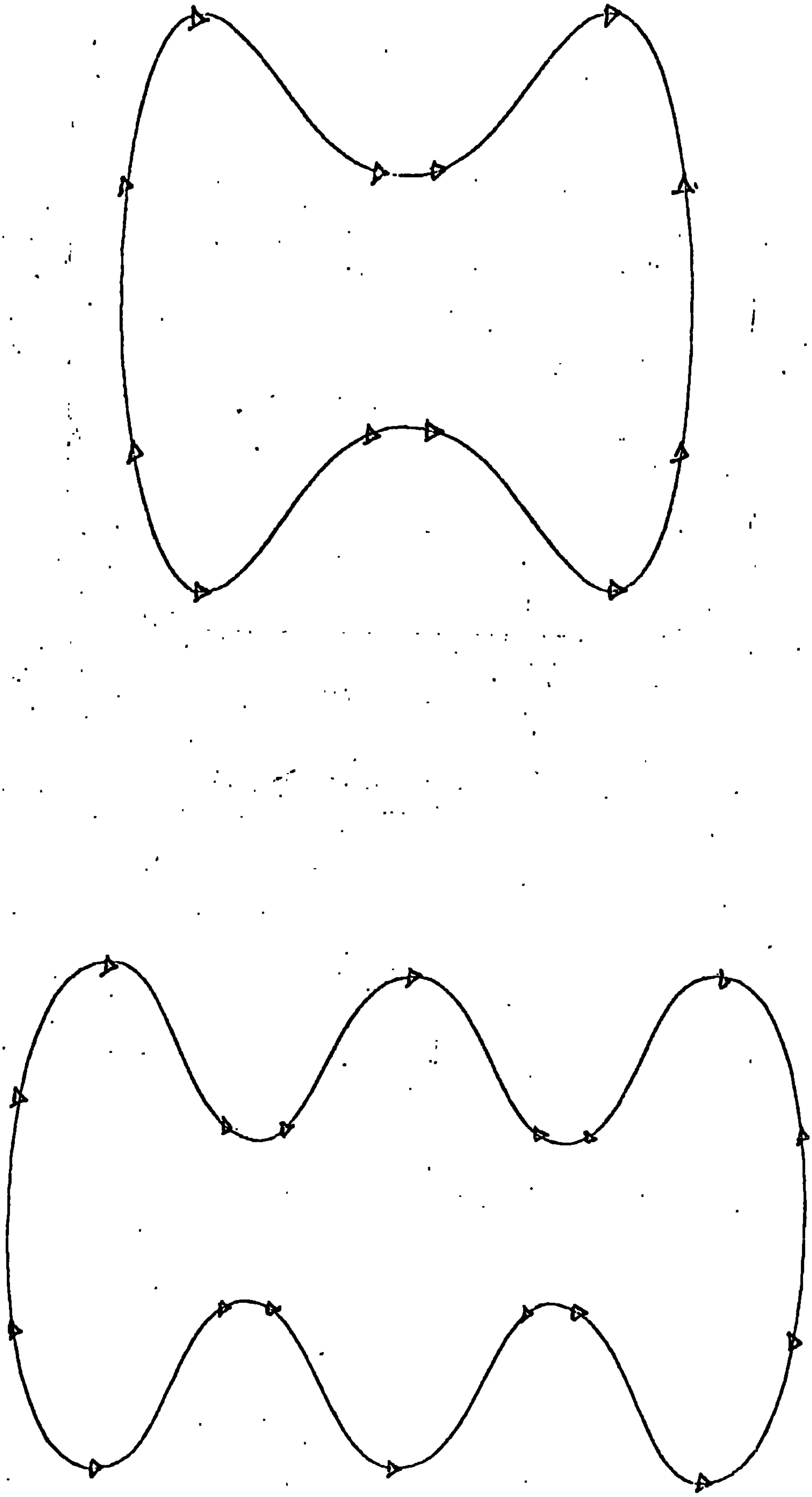
Triangular Surface Boundary Continuity

Figure 11



Extremal Piecewise Cubics (τ = 0)

Figure 12



Extremal Piecewise Cubics ($\tau \equiv 0$)

Figure 13

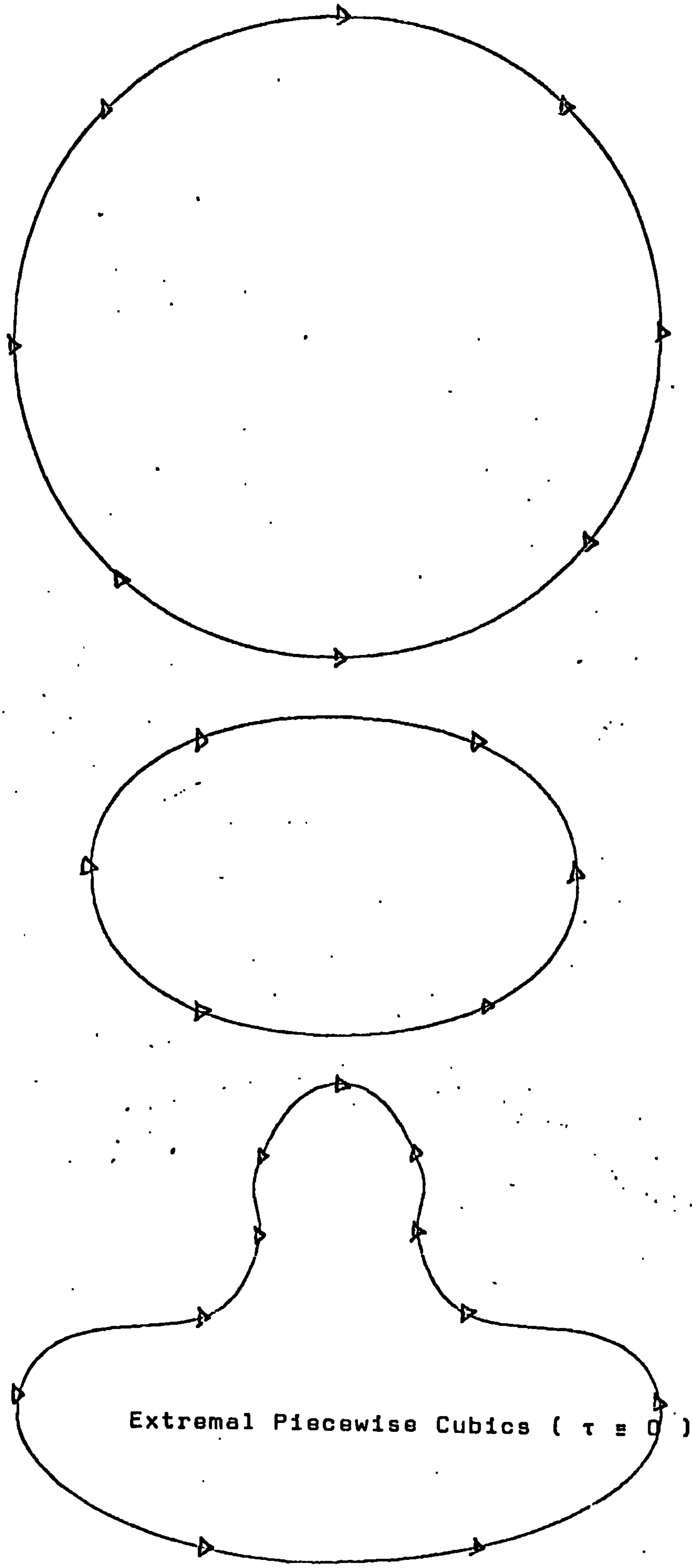
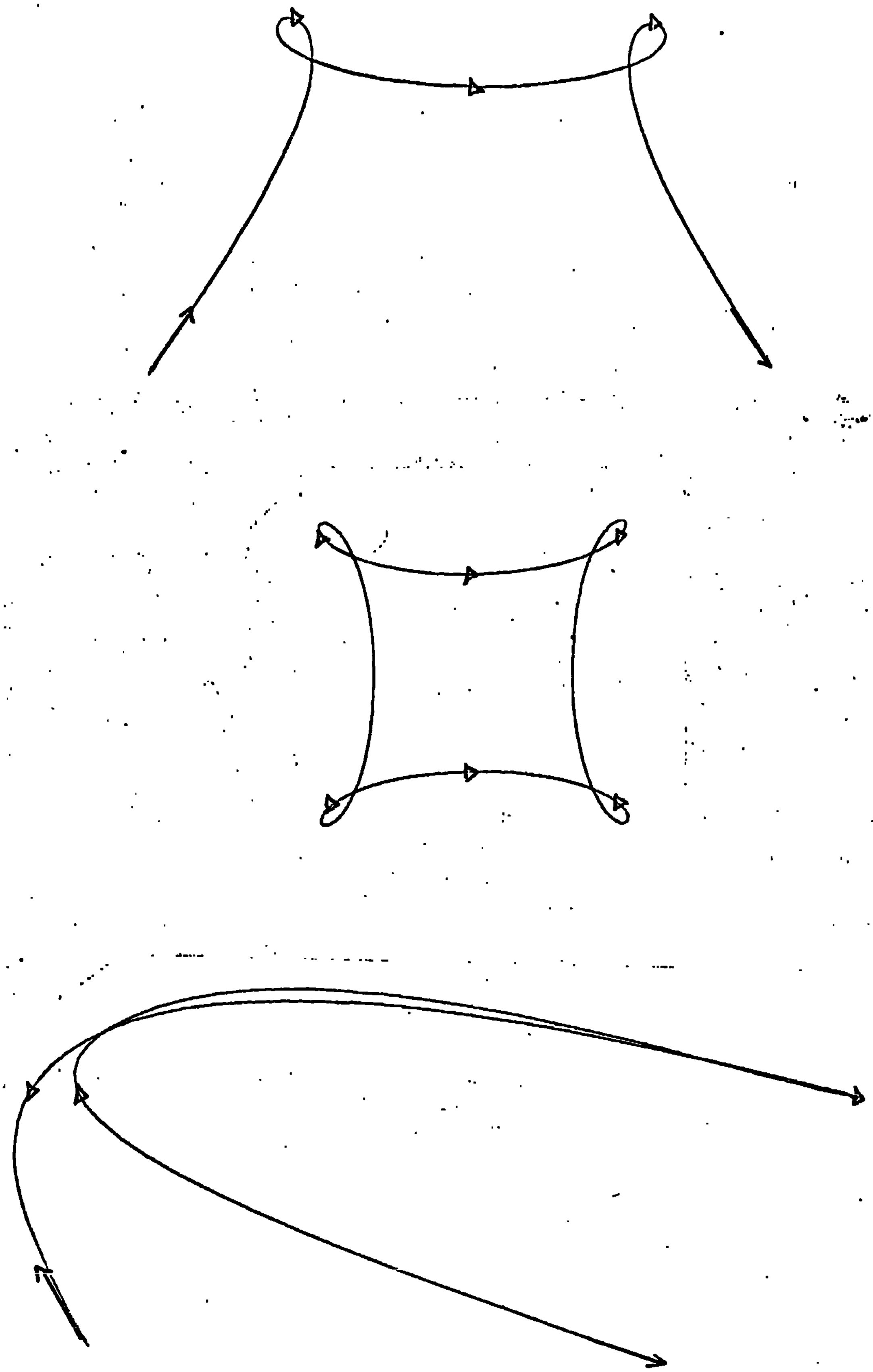


Figure 14



Extremal Piecewise Cubics ($\tau \neq 0$)

Figure 15