Investigations into dual resonance models

Martin, David

INVESTIGATIONS INTO DUAL RESONANCE MODELS.

by

David Martin.

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CONTENTS

PREFACE

The work presented in this thesis was carried out between October 1971 and September 1974 in the Department of Mathematics, University of Durham under the supervision of Dr. D.B.Fairlie.

The material in this thesis claimed to be original has not been submitted for any other degree in this or any other University. The latter half of chapter three and the first half of chapter four are based on two papers by the author in collaboration with D.B. Fairlie and the latter half of chapter four is based on some unpublished work by the author, these parts are claimed to be original except where stated to the contary.

I would like to thank Dr. D.B.Fairlie most sincerely for his guidance and encouragement during the past three years when the work presented was done. I should also like to thank the S.R.C. for a research studentship.

ABSTRACT

This thesis is concerned with dual resonance models, especially the Neveu-Schwarz and Ramond Models. The first chapter is an introduction to the subject of dual models and is concerned with the concepts that lead to them and early ideas of dual models. Chapter two presents, in the operator formalism, the Conventional Dual Model and then the Neveu-Schwarz and Ramond Models and is meant to indicate the more important features of these models. The first part of chapter three deals with the string pictures of dual models and various ways of considering dual models which can be considered to be related to the string pictures. The latter half of chapter three deals with the formulation of the Neveu-Schwarz Model by the use of a finite Grassmann algebra which extends the Conventional Dual Model, in Koba-Nielsen variables, directly into the Neveu-Schwarz Model. The tree graph and one loop diagrams are calculated explicitly while the form for higher order terms is given in terms of automorphic functions. The first part of chapter four presents a method of obtaining the functions,involved in one loop meson and fermion diagrams with external mesons, by the use of Neumann functions on an annulus, the boundary conditions on the annulus giving the different loop diagrams. The second part of chapter four deals with the calculation of the Neumann function for one loop diagrams with external fermions and an attempt to obtain the partition function which is necessary to write down the complete amplitude at the one loop level. Chapter four is completed by the construction of the one loop amplitude for four external fermions.

CHAPTER ONE

INTRODUCTION.

In order to obtain an adequate description of elementary particles and their interactions the best approach appears to be the construction of a model which obeys a number of the criteria which seem essential to a correct theory, then hope that the model is sufficiently flexible to be able to be modified to describe elementary particles, or at least give new insights into the problem. Dual Models can be said to employ such an approach and although they give a far from complete description they can be considered useful as an attempt to increase our understanding of elementary particles.

DUALITY.

Duality is a phenomenological description of experimental data. At low energies the scattering of two particles can be described by the formation of resonances which subsequently decay. These resonances, when plotted on a Chew-Frautschi plot, take the form $J=A + B\, m^2$, where J is the spin of the resonance and m the mass. On the plot the resonances appear to lie on straight line trajectories. For high energies Regge theory provides a good description by assuming that the scattering amplitude is a analytic function of the angular momentum. The particles then lie on Regge trajectories $\alpha(s)= \alpha(0)+\alpha' s$, where $s= (p_1+ p_2)^2$ if the incoming particles have momenta p_1 and p_2. This gives a straight line trajectory and for a particle of mass m and spin J then $\alpha(m^2)=J$. The high energy behavior is then described in terms of Regge poles [1,2] $A(s,t) \sim s^{\alpha(t)}$ where $\alpha(t)$ is a Regge pole and where $A(s,t)$ is the scattering amplitude for fixed t, t being the momentum transfer squared. If there are an infinite number of resonances and an infinite number of Regge poles then, and only then, both methods may be equivalent. This is the idea of duality, that the resonances in the s-channel build up the Regge poles in the t-channel, that is the amplitude can be written as either a sum of resonances or a sum of Regge pole terms. This concept has been checked by the use of finite energy sum rules (F.E.S.R.) [3] and appears to be a reasonable assumption.

THE VENEZIANO MODEL .

Veneziano[4] constucted an amplitude to describe the scattering of $\pi\,\pi \to \pi\,\omega$ which was, because of the quantum numbers involved, a convenient choice. The amplitude he obtained has the above duality properties, is crossing symmetric and analytic. This amplitude gives an infinite number of Regge trajectories. The amplitude is written as

$$F(s,t,u) = A(s,t) + A(s,u) + A(t,u)$$

where $$A(s,t) = \frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))} = B(1-\alpha(s), 1-\alpha(t))$$

where $\alpha(s) = \alpha(0) + \alpha' s$ is the leading Regge trajectory and $B(a,b)$ is Euler's Beta function which can be written as

$$B(a,b) = \int_0^1 dx\; x^{a-1}(1-x)^{b-1}$$

The amplitude is meromorphic which corresponds to the narrow width resonance approximation and, hence, is not unitary. The amplitude can be written as a sum of pole terms

$$A(s,t) = \sum_{n=0}^{\infty} \frac{1}{(n-\alpha(s))} \frac{(\alpha(t))_n}{\Gamma(n+1)}$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer polynomial,

or, by using Stirling's formula, as a Regge term

$$A(s,t) \underset{\substack{s\to\infty \\ t \text{ fixed}}}{\sim} \Gamma(1-\alpha(t))\,[-\alpha(s)]^{\alpha(t)-1}$$

Veneziano also considered the scattering of scalar particles with vacuum quantum numbers constraining the leading trajectory to pass through the scalar particle and he concluded that the intercept

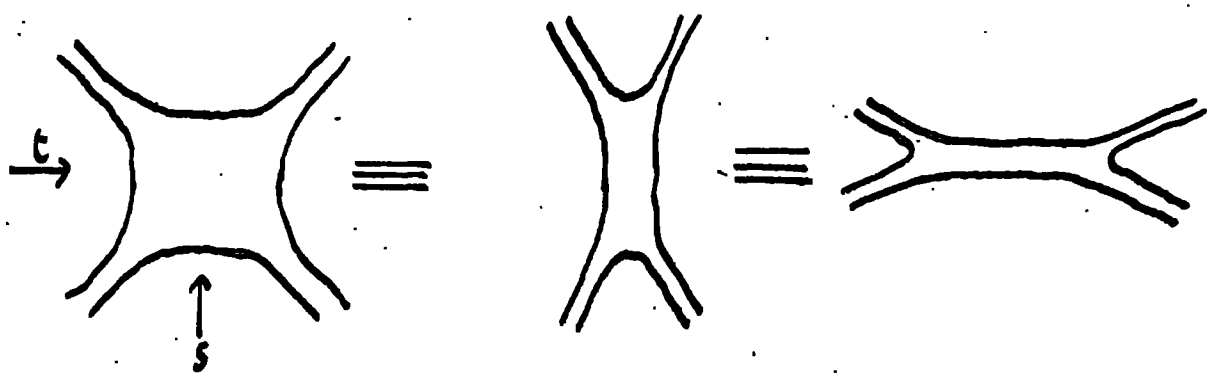
of the leading trajectory must be one, that is $\alpha(0)=1$.

Lovelace[5] and Shapiro[6] then produced an amplitude for $\pi\pi$ scattering where the amplitude for different isospin states were built up from the function

$$C(s,t) = \frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}$$

where $\alpha(s)$ represents the ρ trajectory. This amplitude possesses the same properties as the Veneziano amplitude and also, as Lovelace pointed out, certain properties which would be expected from chiral symmetry.

The duality properties of these amplitudes can be represented by quark diagrams.[7,8]



The second and third diagrams represent the poles in the s and t channels respectively, and the diagrams represent the scattering of bosons.

The Veneziano amplitude was then generalised so as to describe the scattering of N scalar particles, which is equivalent to generalising Euler's Beta function.[9,10,11] When the quark diagrams for N particles are drawn it can be seen that only (N-3) poles can occur similtaneously and that the amplitude is given by a sum of

((N-1)/2)! terms which are generalisations of the Beta function.

Isospin can be added in a trivial manner by the Chan - Paton procedure.[12] Due to the way the amplitudes can be represented diagrammatically this type of duality is known as planar duality.

KOBA-NIELSEN VARIABLES.

One convenient way of writing down the N point amplitude is in terms of Koba-Nielsen variables.[13,14] A variable z_i is associated with the i'th particle which has momentum k_i^μ and these variables lie on a unit circle. The N variables allow too much freedom when constructing the amplitude and so three are arbitarily fixed, this allows only (N-3) similtaneous poles. The amplitude is given by

$$A_N = \int \frac{\prod_{i=1}^{N} dz_i}{dV_{a,b,c}} \prod_{1\le i<j\le N} (z_i - z_j)^{-2k_i.k_j} \prod_{i=1}^{N} (z_i - z_{i+1})^{\alpha(0)-1}$$

where

$$dV_{a,b,c} = \frac{dz_a\, dz_b\, dz_c}{(z_a-z_b)(z_b-z_c)(z_c-z_b)}$$

The Regge trajectories are written down as $\alpha_{ij} = \alpha(0) + \alpha' s_{ij}$

where $s_{ij} = (k_i + k_{i+1} + \ldots + k_j)^2$

Poles then occur in the invariant mass s_{ij} when $z_i \to z_j$.

The amplitude is invariant under a projective or Mobius transformation

$$z_i' = \frac{(az_i+b)}{(cz_i+d)} \qquad \text{where} \qquad ad-bc = 1$$

A convenient choice for the fixed z's is

$z_1 = 0$, $z_{N-1} = 1$, $z_N = \infty$,

then by writing the remaining variables as $x_i = \frac{z_i}{z_{i+1}}$

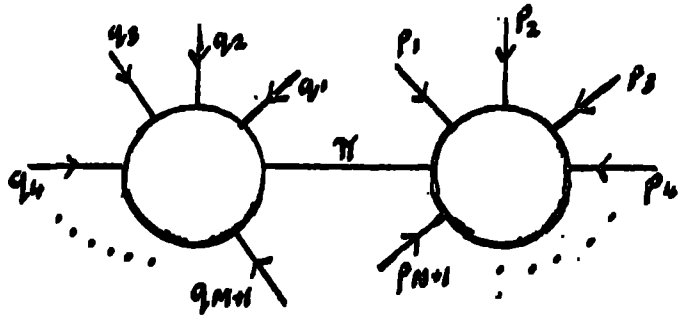
then the amplitude can be written in the Bardakci- Ruegg form.[8]

$$A_N = \int_0^1 \prod_{i=2}^{N-2} dx_i \; x_i^{-\alpha(s_i)-1} (1-x_i)^{\alpha(0)-1} \prod_{1\le i<j\le N} (1-x_{ij})^{-2k_i.k_j}$$

where $x_{ij} = x_i x_{i+1} \cdots x_{j-1}$ and $s_i = (k_1 + k_2 + \ldots + k_i)^2$

FACTORISATION.

It was shown, by using the Bardakci-Ruegg form, that the amplitude can be written in a factorisable form.[15,16] This can be represented diagrammatically as



Using the above variables the amplitude can be written in the form

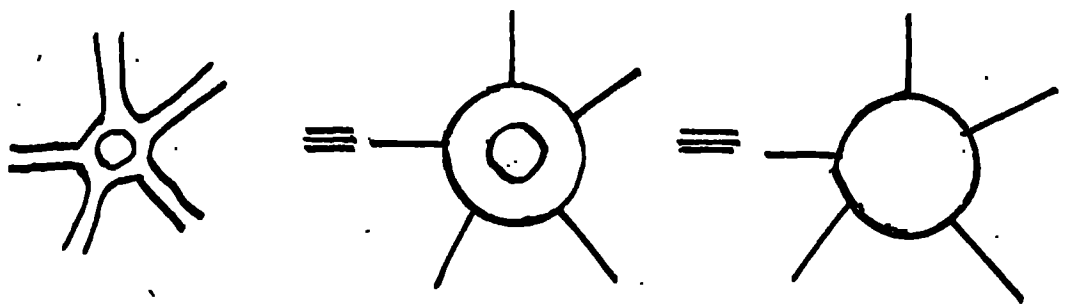
$$A_{N+M+2} = \sum_{J=0}^{\infty} \sum_{i=1}^{d(J)} \frac{g_i^{(J)}(q_1, q_2, \ldots q_{M+1})\, g_i^{(J)}(p_1, p_2, \ldots \ldots p_{N+1})}{\alpha(s) - J}$$

where $\alpha(s) = \alpha(0) + \pi^2 \alpha'$ and d(J) is the degeneracy of the level when $\alpha(s) = J$. As the integer d(J) is independent of the number of external particles this means that the amplitude is factorisable.

UNITARITY.

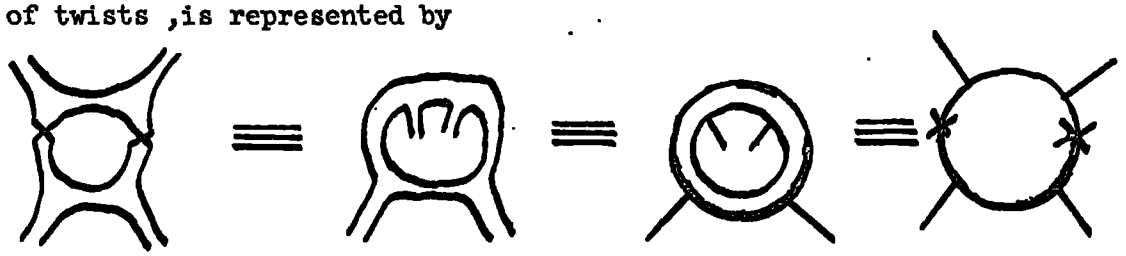
As the Veneziano amplitude uses the narrow width resonance approximation it violates unitarity but the violation takes the same form as in the Born term in the Feynman-Dyson expansion in Field Theory hence there exists the possibility of constructing a perturbation series to restore unitarity in a similar manner to Field Theory.[17,18] As the Veneziano amplitude seems a reasonable approximation it would appear likely that the perturbative corrections are small ,hence the expansion parameter is small making the perturbative approach a reasonable one to pursue. The terms can be represented diagrammatically by three types of diagram;planar,

orientable and non-orientable loop diagrams. These higher order amplitudes are constructed by taking the N point amplitude , joining together pairs of external legs and then Reggeizing the joined lines to produce a M loop graph with N-2M external particles.[19] To produce orientable and non-orientable terms an additional operation, called a twist, is needed. At the one loop level the planar loop is represented diagrammatically by

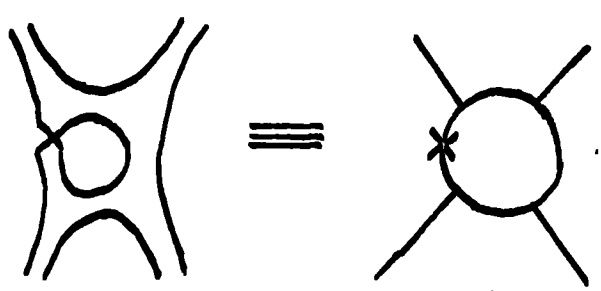


The twist is represented by

which means that the orientable diagram, which has an even number of twists ,is represented by



for the case of two twists and four external particles. The non-orientable diagram has an odd number of twists and for the case of four external particles with one twist it is represented by



It is possible to calculate the one loop amplitudes in terms of integrals, but unfortunately singularities exist at the end points of the integration. It may be possible to "renormalise" some of these by the subtraction of a counter term whereas others may be interpreted as due to an exchange involving vacuum quantum numbers, a pomeron. So in dual models it appears that it is not necessary to put in a pomeron trajectory as it is generated at the one loop level by Regge trajectories.

CONCLUSION.

At the stage it appears that the Veneziano or Conventional Dual Model can be used to investigate elementary particles as it has a number of features necessary for a description of strong interactions of elementary particles but it is by no means a complete theory. It is necessary to investigate further properties of the Conventional Dual Model and different ways of formulating it , and eventually extending it to more physical cases if possible. This is what we will discuss in the remainder of this thesis.

CHAPTER TWO

THE OPERATOR FORMALISM.

The operator formalism[20,21,22] is a useful method of understanding the structure of dual models. The N-point amplitude, which is obviously factorisable, is written in a completely Lorentz covariant manner. The spectrum of the model can be easily examined in this approach and this leads to the connection between the intercept of the leading trajectory, the dimension of space-time, and negative norm states, known as ghosts, though this is probably better understood by the consideration of a relativistic string, an approach we shall come to later.

This formalism introduces either harmonic oscillators a_n^μ and $a_n^{\mu\dagger}$ or operators α_n^μ and α_{-n}^μ which can be defined in terms of the harmonic oscillators. The harmonic oscillators obey the following commutation relations

$$[a_n^\mu, a_m^\nu] = [a_n^{\mu\dagger}, a_m^{\nu\dagger}] = 0$$

$$[a_n^\mu, a_m^{\nu\dagger}] = -g^{\mu\nu}\delta_{n,m} \quad , \; n,m = 1,2,\ldots\ldots$$

where $g^{\mu\nu}$ is given by $g^{00} = 1$, $g^{ii} = -1$, $i = 1,\ldots D-1$ therefore there are D-1 space dimensions rather than just three, an important point as we shall see later. The α's are defined by

$$\alpha_n^\mu = \sqrt{n}\, a_n^\mu \quad , \quad \alpha_{-n}^\mu = \sqrt{n}\, a_n^{\mu\dagger}$$

The α_n's act as destruction operators and the α_{-n}'s as creation operators on the vacuum state $|0\rangle$, therefore

$$\alpha_n^\mu |0\rangle = 0 \quad , \quad n = 1,2,\ldots\ldots$$

The commutation relations of the α's are given by

$$[\alpha_n^\mu, \alpha_m^\nu] = -g^{\mu\nu}\, n\, \delta_{n+m,0}$$

A momentum operator p^μ and a position operator q^μ are also introduced, these obey $$[q^\mu, p^\nu] = -i g^{\mu\nu}$$

then $\alpha_0^\mu = \sqrt{2}\, p^\mu$ where the slope of the Regge trajectory is taken to be one.

Generalised momentum and position operators can then be defined in terms of the α's. The generalised position operator is

$$Q^\mu(z) = \sqrt{2}\, q^\mu - i\alpha_0^\mu \ln z - i\sum_{n=1}^{\infty} \frac{\sqrt{2}}{n}(\alpha_{-n}^\mu z^n - \alpha_n^\mu z^{-n})$$

and the generalised momentum operator is given by

$$P^\mu(z) = \frac{iz}{\sqrt{2}} \frac{dQ^\mu(z)}{dz}$$
$$= p^\mu + \sum_{n=1}^{\infty}(\alpha_{-n}^\mu z^n + \alpha_n^\mu z^{-n})$$

These have the following commutation relations

$$[Q_\mu(z), Q_\nu(y)] = 2i\pi g_{\mu\nu}\, \epsilon(z-y)$$
$$[P_\mu(z), Q_\nu(y)] = -\sqrt{2}\,\pi z\, g_{\mu\nu}\, \delta(z-y)$$
$$[P_\mu(z), P_\nu(y)] = -i\pi z y\, g_{\mu\nu}\, \delta'(z-y)$$

The scattering amplitude for N particles can then be written as

$$\int \frac{\prod_{i=1}^{N} dz_i}{dV_{a,b,c}} \prod_{i=1}^{N} (z_i - z_{i+1})^{\alpha(0)-1} \langle 0 | \prod_{i=1}^{N} V(z_i, k_i) | 0 \rangle$$

where $V(z,k) = :e^{-ik.Q(z)}: z^{k^2}$ and is the Fubini-Veneziano vertex.[21,22]

$: \;:$ means that the operator $e^{-ik.Q(z)}$ is normal ordered

that is $$:e^{-ik.Q(z)}: = e^{-ik.Q^0(z)}\, e^{-ik.Q^-(z)}\, e^{-ik.Q^+(z)}$$

where $Q^{0\,\mu}(z) = \sqrt{2}\, q^{\mu} - i\alpha_0^{\mu} \ln z$

$$Q^{+\mu}(z) = i \sum_{n=1}^{\infty} \frac{\sqrt{2}}{n} \alpha_n^{\mu} z^{-n}$$

$$Q^{-\mu}(z) = -i \sum_{n=1}^{\infty} \frac{\sqrt{2}}{n} \alpha_{-n}^{\mu} z^{n}$$

These have the commutation relations

$$[Q_{\mu}^{0}(z), Q_{\nu}^{0}(y)] = 2 \ln\left(\frac{z}{y}\right) g_{\mu\nu}$$

$$[Q_{\mu}^{+}(z), Q_{\nu}^{-}(y)] = 2 \ln\left(1 - \frac{y}{z}\right) g_{\mu\nu} \qquad |z| \gg |y|$$

other combination being zero. It can easily be shown using these relations that

$$\langle 0 | \prod_{i=1}^{N} V(z_i, k_i) | 0 \rangle = \exp\left\{ - \sum_{1 \le i < j \le N} 2 k_i \cdot k_j \ln\left(1 - \frac{z_j}{z_i}\right) \right\} \prod_{i=1}^{N} z_i^{k_i^2}$$

$$= \prod_{1 \le i < j \le N} (z_i - z_j)^{-2 k_i \cdot k_j}$$

from which we obtain the Koba-Nielsen form of the amplitude.

The states of the model can now be written in terms of the creation operators α_{-n}^{μ} but this allows the existence of negative norm states, ghosts, due to the time components of the operators. By choosing the intercept of the leading trajectory to be one it is possible to construct an infinite number of operators, known as the Virasoro operators, which under certain conditions make it possible to show that the ghosts do not couple to the physical, positive norm, states.

VIRASORO OPERATORS.

The Virasoro operators [23] are defined by

$$L_m = -\oint_C \frac{dz}{4\pi i z} z^m : P^2(z): \qquad , m=0,\pm1,\pm2,\ldots$$

where : : stands for normal ordering and $P^\mu(z)$ is the generalised momentum operator. The contour integral is taken around the origin. These have the properties

$$L_n |0\rangle = 0 \qquad , \; n \geq -1$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12} n(n^2-1)\delta_{n+m,0}$$

$$[L_n, Q^\mu(z)] = z^{n+1} \frac{dQ^\mu(z)}{dz}$$

$$[L_n, P^\mu(z)] = z^n \left(z\frac{d}{dz} + n\right) P^\mu(z)$$

For a function X(z) if

$$[L_n, X(z)] = z^n \left(z\frac{d}{dz} + nJ\right) X(z)$$

then X(z) has conformal spin J, therefore $Q^\mu(z)$ has conformal spin zero and $P^\mu(z)$ has conformal spin one. The operators L_0 , L_{+1} and L_{-1} form a SU (1,1) sub-group[24] and have the useful properties

$$e^{\lambda L_0} X(z) e^{-\lambda L_0} = X(e^\lambda z)$$

$$e^{\lambda L_1} X(z) e^{-\lambda L_1} = (1-\lambda z)^J X\left(\frac{z}{1-\lambda z}\right) \qquad , |\lambda| \leq \frac{1}{|z|}$$

$$e^{\lambda L_{-1}} X(z) e^{-\lambda L_{-1}} = \left(\frac{z+\lambda}{z}\right)^J X(z+\lambda) \qquad , |\lambda| \leq |z|$$

By using the first of these relations it is possible to rewrite the amplitude in a convenient form. By choosing $z_N=0$, $z_2=1$, and $z_1=\infty$ then writing

$$\langle 0|V(z_1,k_1) = \langle 0|z_1^{k_1^2}\, e^{-ik_1\cdot q\sqrt{2}}$$

$$V(z_N,k_N)|0\rangle = e^{-ik_N\cdot q\sqrt{2}}|0\rangle$$

then the amplitude can be written as

$$\int_0^1\!\!\cdots\int_0^1 dz_3\cdots dz_{N-1}\; z_{N+1}^{\alpha(0)-1}\prod_{i=3}^{N-2}(z_i-z_{i+1})^{\alpha(0)-1}\langle 0|e^{-ik_1\cdot q\sqrt{2}}\prod_{i=2}^{N-1}V(z_i,k_i)\times e^{-ik_N\cdot q\sqrt{2}}|0\rangle$$

then using

$$V(z_i,k_i) = z_i^{L_0}V_0(1,k_i)z_i^{-L_0}z_i^{k_i^2}$$

$$V_0(z,k_i) = :e^{-ik_i\cdot Q(z)}:$$

and changing variables to $x_i = z_i z_{i-1}\cdots z_3$ the amplitude can be written as

$$\langle 0|e^{-ik_1\cdot q\sqrt{2}}V_0(1,k_2)\,P\,V_0(1,k_3)\cdots P\,V_0(1;k_{N-1})e^{-ik_N\cdot q\sqrt{2}}|0\rangle$$

$$P = \int_0^1 \frac{dx_i}{x_i}\; x_i^{L_0-\alpha(0)}(1-x_i)^{\alpha(0)-1}$$

When $\alpha(0)=1$, which is also necessary for the Virasoro operators to exist,

$$P = \frac{1}{L_0-1}$$

and the amplitude becomes

$$\langle 0;k_1|V_0(1,k_2)\frac{1}{L_0-1}V_0(1,k_3)\frac{1}{L_0-1}\cdots V_0(1,k_{N-1})|k_N;0\rangle$$

The amplitude is now written in terms of propagators, $\frac{1}{L_0-1}$ and vertices, $V_0(1,k)$.

Unfortunately the requirement that $\alpha(0)=1$ implies that the lowest state on the leading trajectory has mass squared of minus one and hence is a tachyon. As this is the ground state of the theory it is not possible to decouple it from the theory.

GHOSTS.

The states of the model may be written as

$$|\{\lambda\},n\rangle = C_n \prod_l (a_{-l})^{\lambda_l} |k:0\rangle$$

these are eigenstates of the operator L_0 with eigenvalues $-k^2+n$ where $\sum_l l\lambda_l = n$. The coefficients C_n are chosen such that

$$\langle\{\lambda\},n|\{\lambda\},n\rangle = \varepsilon_{\{\lambda\}n} \ , \quad \varepsilon = \pm 1$$

By writing

$$\langle\phi| = \langle 0:k_1|V_0(1,k_2)\frac{1}{L_0-1}\cdots V_0(1,k_{j-1})$$

and

$$|\psi\rangle = V_0(1,k_j)\frac{1}{L_0-1}\cdots V_0(1,k_{N-1})|k_N:0\rangle$$

then the amplitude can be written as

$$A_N = \langle\phi|\frac{1}{L_0-1}|\psi\rangle$$

A complete set of states $1 = \sum_{n\{\lambda\}} |\{\lambda\},n\rangle\langle\{\lambda\},n|$

can be inserted into the amplitude to give

$$\langle\phi|\sum_{n\{\lambda\}}|\{\lambda\},n\rangle\langle\{\lambda\},n|\frac{1}{L_0-1}\sum_{m\{\mu\}}|\{\mu\},m\rangle\langle\{\mu\},m|\psi\rangle$$

$$= \sum_{n\{\lambda\}} \langle\phi|\{\lambda\},n\rangle \frac{\varepsilon_{\{\lambda\}n}}{(-k^2+n-1)} \langle\{\lambda\},n|\psi\rangle$$

This then allows the existence of ghosts which appear at the pole when $\varepsilon = -1$ due to the existence of time components of the operators.

The vertex $V_0(z,k)$ has conformal spin one

$$[L_n, V_0(z,k)] = z^n\left(z\frac{d}{dz}+n\right)V_0(z,k)$$

therefore

$$[L_0-L_n, V_0(1,k)] = nV_0(1,k)$$

then this can be used to show that the operator (L_0-L_n-1+n) on a vertex gives

$$(L_0-L_n-1+n)V_0(1,k) = V_0(1,k)(L_0-L_n-1)$$

Using the Virasoro algebra it is possible to show that

$$(L_0-L_n-1)\frac{1}{L_0-1} = \frac{1}{L_0-1}(L_0-L_n-1+n)$$

When (L_0-L_n-1+n) is passed through a vertex and then a propagator it reappears on the other side unaltered, therefore it can be passed through a number of vertices and propagators. Now $L_0|k;0\rangle$ just measures the momentum squared of the state $L_0|k;0\rangle = |k;0\rangle$

so

$$(L_0-L_n-1+n)V_0(1,k_{N-1})|k_N;0\rangle =$$

$$= V_0(1,k_{N-1})(L_0-L_n-1)|k_N;0\rangle = 0$$

Therefore (L_0-L_n-1+n) acts as a gauge on the state $|V\rangle$ [23,25]

i.e, $$(L_0-L_n-1+n)|V\rangle = 0$$

Now if a stste has the property that $L_0|\phi\rangle = (1-k)|\phi\rangle$ then the operator L_k at a residue of a pole gives

$$\langle\phi|L_k|V\rangle = \langle\phi|(L_0-1+k)|V\rangle$$

from the gauge conditions above this then gives

$$\langle\phi|L_k|V\rangle = \langle\phi|(1-k-1+k)|V\rangle = 0$$

The state $L_{-k}|\phi\rangle$ is then called spurious as is does not couple to $|\psi\rangle$, a tree state. If all the ghosts states are spurious states then they do not couple to the physical states and then are absent from the theory. This then gives the gauge conditions for physical states $L_m|\chi\rangle = 0$ and $(L_0 - 1)|\chi\rangle = 0$.

ABSENCE OF GHOSTS.

The absence of ghosts in the spectrum of the Conventional Dual Model can be proven under certain conditions [26,27] and this gives more insights into the structure of the model. The first step is to construct the D.D.F. states[28] which describe the emission of a spin one, zero mass particle, the "photon", from a tree graph. These operators, acting on the ground state, give physical states and can be considered as physical state operators. They are given by

$$A^i_{-n} = \oint \frac{dz}{2\pi i}\, \varepsilon^i . P(z)\, e^{-in\, k.Q(z)} \quad , i = 1, \ldots D-2$$

and are dependent on a particular reference frame where

$$\varepsilon^i . k = \varepsilon^i . p = 0$$

where k^μ is the momentum of the photon and p^μ is given by the tachyon state.

These operators then have the commutation relations

$$[A^i_n, A^j_m] = n\, \delta_{ij}\, \delta_{n+m,0}$$

which means that the states they create are orthogonal but as they only have transverse degrees of freedom they do not create all possible states. It is possible to construct operators that create the missing longitudinal physical states.[27,28]

As $[L_n, A^i_{-m}] = 0$, $n > 1$ then a state constructed from the A^i's have the property that $L_n|\phi\rangle = 1$. Due to a normal ordering problem this is not true for the longitudinal operators A^L_n

so that a correction term Φ_m is needed such that

$$[L_n, A_m^{(+)}] = 0 \qquad , A_m^{(+)} = A_m^{L} + \frac{m}{2}\Phi_m$$

These then have the algebra

$$[A_n^{(+)}, A_m^{(+)}] = (n-m)A_{n+m}^{(+)} + 2n^3\delta_{n+m,0}$$

$$[A_n^{i}, A_m^{(+)}] = n A_{n+m}^{i}$$

$$[\Phi_m, A_n^{j}] = 0$$

$$[A_n^{(+)}, \Phi_m] = m\Phi_{n+m}$$

These operators then give all possible states, both physical and spurious. The norm of these states can be calculated by using an isomorphism which has the same algebra as the above operators.

$$A_n^{i} \to \alpha_n^{i} \ , \ A_n^{(+)} \to \mathcal{L}_n \ , \ |p;0\rangle \to |0\rangle$$

where the $\mathcal{L}_n$'s are operators analogous to the L_n's but constructed from only D-2 spatial components of the α's . For $D \leq 26$ these give only positive norms , hence no ghosts exist in these dimensions. This is the proof due to Brower.[27] When $D > 26$ ghosts do exist but when D=26 the longitudinal operators give null states so that the D.D.F. states give all the physical states and there are only (D-2) sets of states, that is two sets of states have decoupled. D=26 is known as the critical dimension.

It is also possible to prove that ghosts do not exist for less than or equal to the critical dimension by another method due to

Goddard and Thorn,[28] A basis for all types of states can be constructed as follows .

$$\prod_{n=1}^{N} L_{-n}^{\lambda_n} \prod_{m=1}^{M} K_{-m}^{\mu_m} \prod_{l=1}^{K} A_{-l}^{\nu_l} |p:0\rangle \quad , K_n = \oint \frac{dz}{2\pi i z} z^n k.P(z)$$

It can then be shown that all physical states are given by a sum of D.D.F. states and other states which in twenty-six dimensions are null states . Thus in twenty-six dimensions the physical states are just the D.D.F. states, and ,therefore, have positive norm.

UNITARITY.

Unitary is, hopefully, implemented by a perturbative series the Born term being the amplitude discussed above. The one loop graphs are easily investigated in the operator formalism apart from the requirement that only physical states propagate around the loop. This does not provide a problem in the tree graph as the gauge conditions ensure only physical states propagate. Brink and Olive[30] constructed an on-mass shell physical state projection operator $\tau(k)$ where

$$\tau(k) = \oint \frac{dy}{2\pi i y} y^{\mathcal{L}_0 - H}$$

$$\mathcal{L}_0 = \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} A_{-n}^{i} A_{n}^{i}$$

$$H = L_0 - p^2$$

They make use of the fact that

$$\mathcal{L}_0 - H = (D_0 - 1)(L_0 - 1) + \sum_{n=1}^{\infty} (D_{-n} L_n + L_{-n} D_n)$$

$$D_n = \oint \frac{dz}{2\pi i z} z^n \frac{1}{k.P(z)}$$

for the critical number of dimensions and where k^μ is a light-like vector. This projection operator projects from the full space given by the α's onto the transverse space given by the A's, and when this operator is placed in a residue it is effectively unity for D=26 showing that in this case the physical states are just the D.D.F. states.

This on -mass-shell physical state operator can be used to ensure that only the physical states propagate around loops[31] though this is a rather difficult procedure. What is really required is a off-mass -shell physical state projection operator and one was constructed by Corrigan and Goddard[32] enabling the calculation to go through easier.

So the planar loop can be calculated by considering the integral

$$\int d^D k \; Trace\{V(k_1) P V(k_2) \ldots V(k_N) P\}$$

where the propagates contain the projection operators. To calculate non-planar loops it is necessary to have a twisting operator Ω [33]

$$\Omega = (-1)^H e^{-L_{-1}}$$

By combining the twisting operator with a propagator and making use of the gauge conditions it is possible to construct a hermitian twisting operator[34,35]

$$\theta = \int_0^1 dx \; x^{L_0-2} \Omega (1-x)^{L_0-L_1}$$

which is equivalent to using

$$\Theta = \Omega (1-x)^{L_0-L_1}$$

as the twisting operator. So by replacing the untwisted propagator by the twisted propagator non-planar loops can be constructed by taking the trace as indicated adove.[36,37] The results can be written in terms of Jacobi Theta functions. (See table one)

When the loop diagrams were originally investigated it was found that for the one loop orientable non-planar graph a new singularity appeared which had vacuum quantum numbers and so was termed a "pomeron." Unfortunately it appeared that the new singularity would violate unitarity but Lovelace[39] pointed out that when the dimension of space-time was twenty-six and if two dimensions of states decoupled then the pomeron would become a pole and therefore not violate unitarity. This can now be understood in terms of the D-2 sets of physical states in the critical dimension and the required results were obtained using the projection operators.[32,40] The only difference from the naive calculation using just a twisted or untwisted propagator is that the partion function is changed from $[f(\omega)]^{-D}$ to $[f(\omega)]^{2-D}$.

The pomeron can be best investigated by using the Jacobi imaginary transformation (see table two) so that the pomeron singularity is given when the variable $r \to 0$ ($\omega \to 1$). The pomeron singularities then are an infinite set of trajectories with a slope of half that of the Regge trajectories and with the intercept of the leading trajectory being two , and are a set of factorisable poles.[41,42] It is possible to show factorisation explicitly by introducing two sets of harmonic oscillators;[41] the pomeron states then coincide with the states of a non-planar dual model introduced by Virasoro[43] and Shapiro;[44] and no ghosts propagate around the pomeron loops.[45]

THE NEVEU-SCHWARZ MODEL.

The Neveu-Schwarz Model is an extension of the Convention Dual Model but has much more structure with half-integrally spaced Regge trajectories to which can be associated a "G-parity." The leading trajectory has intercept one and the tachyon at $m^2=-1$ decouples from the theory but there is still a tachyon, the ground state particle of the model, at $m^2=-\frac{1}{2}$. As the model is based on the Conventional Dual Model it has a very similar algebraic structure so that the methods used for the Conventional Dual Model can easily be adapted to the Neveu-Schwarz Model.

The model introduces additional operators which commute with the α operators but anticommute among themselves.

$$\{b_r^\mu, b_s^{\nu\dagger}\} = -g^{\mu\nu}\delta_{r,s} \quad , r,s = \tfrac{1}{2},\tfrac{3}{2},\ldots$$

A new field is defined in terms of these operators.

$$H^\nu(z) = \sum_{r=\frac{1}{2}}^{\infty} (b_r^\mu z^{-r} + b_r^{\mu\dagger} z^r)$$

The Virasoro operators are then given from the operators of the Conventional Dual Model plus an additional term given by

$$L_n^{(b)} = -\frac{1}{2}\oint \frac{dz}{2\pi i z}\, z^{n+1} :H(z).\frac{dH(z)}{dz}:$$

A further set of operators are defined as

$$G_r = \oint \frac{dz}{2\pi i z}\, z^r\, P(z).H(z)$$

which form an algebra with the Virasoro operators.

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{8} n(n^2-1)\delta_{n+m,0}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}(r^2-\tfrac{1}{4})\delta_{r+s,0}$$

$$[L_n, G_r] = (\tfrac{n}{2}-r)G_{n+r}$$

The vertex function for the emission of a tachyon is given by

$$V(z,k) = k.H(z)\,V_0(z,k)$$

where $V_0(z,k)$ is the vertex function for the Conventional Dual Model. The tachyon in the theory is usually called a "pion" but obviously is not the physical pion. The amplitude for N of these pions can be written down as

$$A_N^{(1)} = \langle 0:k_1|k_1.b_{\frac{1}{2}}\,V_1(1,k_2)\frac{1}{L_0-1}\ldots.V(1,k_{N-1})k_N.b_{\frac{1}{2}}^{+}|k_N:0\rangle$$

which is zero if N is odd as the b operators are contracted in pairs. This amplitude is written in what is known as the $\mathcal{F}_1$ formalism.

It is possible to rewrite the amplitude in another formalism, the $\mathcal{F}_2$ formalism.[4/] The advantage of this formalism is that there are a lot less states than the $\mathcal{F}_1$ formalism but, of course, the same number of physical states. The ground state is the pion so that the tachyon at $m^2=-1$ is not present in this formalism. Also in this formalism the G operators act as gauge operators. The N point amplitude is written as

$$A_N^{(2)} = \langle 0:k_1|V(1,k_2)\frac{1}{L_0-\frac{1}{2}}\ldots.V(1,k_{N-1})|k_N:0\rangle$$

The physical states in the $\mathcal{F}_2$ formalism must satisfy the gauge conditions

$$(L_0 - \tfrac{1}{2})|\phi\rangle = 0$$

$$L_n|\phi\rangle = 0 \qquad n = 1, 2, \ldots$$

$$G_r|\phi\rangle = 0 \qquad r = \tfrac{1}{2}, \tfrac{3}{2}, \ldots$$

As each state has either an odd or even number of b operators it is possible to associate a"G-parity" to each state depending on the number of b operators in the state. The G-parity operator is given by $G = (-1)^{\sum_{m=\frac{1}{2}}^{\infty} b_m^\dagger b_m}$

The inclusion of the b operators means that more ghosts are included due to the time components of these operators but there exists a larger set of gauges due to the G operators and, therefore, the possibility of the ghosts decoupling from the model.

The proof of the absence of ghosts depends on constructing D.D.F.. type states for the model which in the critical number of dimensions, in this case ten, give all the physical states. The proofs all use the $\mathcal{F}_2$ formalism and are given by direct extensions from the proofs of the absence of ghosts in the Convention Dual Model.[26,30,48,49] When the dimension is less than ten longitudinal operators are needed to give the complete set of physical states.

THE RAMOND FERMION MODEL.

The free particle feactures of the Conventional Dual Model can be produced from the Klein-Gordon equation for a free boson by introducing the generalised momentum and position operators and using the correspondence principle as proposed by Ramond.[50] This method can then be used to extend the free Dirac equation into a free fermion dual model.[51]

For the boson case the starting point is

$$-p^2 + m^2 = 0$$

The momentum is replaced by the generalised momentum

$$p^\mu \rightarrow \langle p^\mu \rangle$$

where $\langle p^\mu \rangle = -\oint \frac{dz}{2\pi i z} p^\mu(z) = p^\mu$

This procedure then gives

$$\langle p^\mu \rangle . \langle p^\mu \rangle + m^2 = 1$$

and by the correspondence principle this is transformed to

$$\langle p^2 \rangle + m^2 = 0$$

which as $m^2=-1$ and from the definition of the Virasoro operators can be written , when acting on a physical state $|V\rangle$, as

$$(L_0 - 1)|V\rangle = 0$$

From considering the usual ghost eliminating conditions

$$p . a^{(n)} = 0$$

which is rewritten as $\langle p^\mu \rangle . \langle z^n p^\mu \rangle = 0 \quad , \langle z^n p^\mu \rangle = \alpha_n^\mu$

then by the correspondence principle

$$\langle p^\mu \rangle . \langle z^n p^\mu \rangle \rightarrow \langle z^n p^2 \rangle = 0$$

which just gives the remaining gauge conditions. $L_n|V\rangle = 0$

For the fermion case it is necessary to introduce generalisations of the Dirac matrices such that[51]

$$\langle \Gamma_\mu \rangle = \gamma_\mu$$

where Γ_μ are the generalisations of the Dirac matrices γ_μ and obey

$$\{\Gamma_\mu(z), \Gamma_\nu(z')\} = 2 g_{\mu\nu} \delta(z-z') \quad , \Gamma_\mu^\dagger = \gamma_0 \Gamma_\mu \gamma_0$$

which are just generalisations of the properties of the Dirac matrices. These conditions give

$$\Gamma^{\mu}(z) = \gamma^{\mu} + i\sqrt{2}\,\gamma_5 \sum_{n=1}^{\infty} \left(d_n^{\mu\dagger} z^{n} + d_n^{\mu} z^{-n}\right)$$

$$\{d_m^{\mu}, d_n^{\nu\dagger}\} = -g^{\mu\nu}\delta_{n,m}$$

The form of the Γ's is very similar to the form of the H's in the Neveu-Schwarz Model and so an algebra can be similarly constructed.

$$L_n^{(d)} = -\frac{1}{4}\oint \frac{dz}{2\pi i z}\, z^{n+1} :\Gamma(z).\frac{d\Gamma(z)}{dz}: \quad , F_n = \frac{i}{\sqrt{2}}\oint \frac{dz}{2\pi i z}\, z^{n} :\Gamma(z).P(z):$$

By adding the $L_n^{(d)}$'s to the Virasoro operators of the Conventional Dual Model the following algebra is given

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{8} n(n^2-1)\delta_{n+m,0}$$

$$[L_n, F_m] = \left(\frac{n}{2} - m\right)F_{n+m}$$

$$\{F_n, F_m\} = 2L_{n+m} + \frac{D}{2}\left(n^2 - \frac{1}{4}\right)\delta_{n+m,0}$$

Starting from the Dirac equation and using the correspondence principle

$$\gamma.p - m = 0 \rightarrow \langle \Gamma.P \rangle - m = 0$$

which gives on a physical state

$$\left(F_0 - \frac{im}{\sqrt{2}}\right)|\Psi\rangle = 0$$

The gauge conditions $L_n|\Psi\rangle = 0$ are obtained from the square of the generalised Dirac equation. Ramond introduces further gauge conditions as follows

$$F_n\left(F_0 - \frac{im}{\sqrt{2}}\right)|\Psi\rangle = 0$$

$$\therefore \quad \left(2L_n - \left(F_0 - \frac{im}{\sqrt{2}}\right)F_n\right)|\Psi\rangle = 0$$

$$L_n|\Psi\rangle = 0 \qquad \therefore F_n|\Psi\rangle = 0 \,.$$

The states $|\Psi\rangle$ can be written as

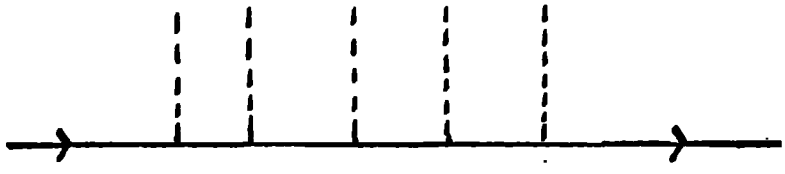
$$|\Psi\rangle = \prod_n \left(d_n^{\dagger}\right)^{\lambda_n} \prod_m \left(\alpha_m^{\dagger}\right)^{\mu_m} |k;0\rangle u(k)$$

where u(k) is a spinor.

INTERACTIONS OF FERMIONS.

To see if a dual theory exists for fermions it is necessary to consider the interactions of fermions. Though in the operator formalism there is a difficulty in calculating the amplitude for fermion - fermion scattering and this was approached gradually as the calculations became more complex. The difficulties of the calculations can be considered a major shortcoming of the operator formalism.

The simplest form of interaction is that for a fermion which emits mesons as it is propagated along. This can be represented diagrammatically by



where the solid line represents a ground state fermion and the dotted lines the emitted mesons. The emitted mesons are described by the Neveu-Schwarz Model and the formulation of the Neveu-Schwarz Model is used to produce the amplitude.[52,53] It is necessary to define a generalised γ^5 by $\Gamma^5 = \gamma^5 (-1)^{\sum_{n=1}^{\infty} d_n^\dagger d_n}$

such that $\{\Gamma^5, \Gamma^\mu(z)\} = 0$

(keeping the notation Γ^5 even when the dimension of space-time is greater than four.)

The amplitude is given by

$$A_N^{F_2} = \bar{u}(k_1)\langle 0:k_1|V_2(1,k_2)\frac{1}{F_0}\ldots\ldots V_2(1,k_{N-1})|k_N:0\rangle u(k_N)$$

in the $\mathcal{F}_2$ formalism and where $V_2(1,k) = \Gamma^5 V_0(1,k)$

As this involves the Neveu-Schwarz Model the critical dimension must be ten, and because of the similarity of the fermion model

to the Neveu-Schwarz Model it is not difficult to prove that this is so. Schwarz deduced this by considering the algebra of the model[54] and Corrigan and Goddard proved this by using fermion physical state operators.[55] A necessary condition is that the mass of the ground state fermion must be zero.

By using $$[F_0, V_0(1,k)] = k.\Gamma(1)V_0(1,k) = V_1(1,k)$$

and $$\frac{1}{F_0} = \frac{F_0}{F_0^2} = -\frac{F_0}{L_0}$$

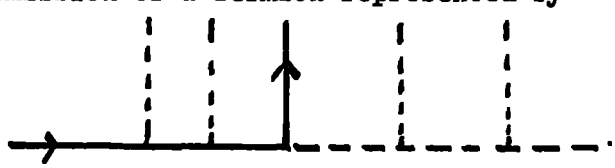
the amplitude can be written in the $\mathcal{F}_1$ formalism[55] as

$$A_N^{F_1} = \tilde{u}(k_1)\langle 0;k_1|V_1(1,k_2)\frac{1}{L_0}\ldots V_1(1,k_{N-1})|k_N;0\rangle \tilde{u}(k_N)$$

where $$u(k_N) = \gamma.k_N \tilde{u}(k_N)$$

the zero mass of the fermion means that it is easier to work in the $\mathcal{F}_2$ formalism.[56]

By dualizing the above amplitude it is possible to consider the emission of a fermion represented by



This involves the difficulty of finding a vertex function which changes the d operators to the b operators. The result, obtained by Corrigan and Olive,[57] is very complicated. In the $\mathcal{F}_1$ formalism the vertex is given by

$$V_F(z,k) = V_0(z,k)\, z^{k^2}\, e^{z L_{-1}^{(d)}}\, W(z)$$

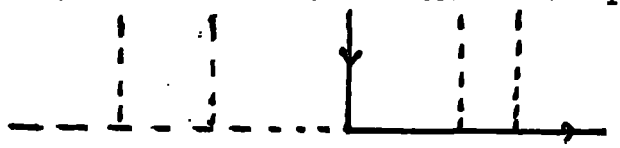
where

$$W(z) = {}_b\langle 0|\exp\Big(\sum_{r=\frac{1}{2}}^{\infty}\gamma^5\gamma.b_r B_r(z) + \sum_{n=1}^{\infty}\sum_{m=\frac{1}{2}}^{\infty} d_n^{+}.b_m B_{nm}(z) + \frac{1}{2}\sum_{s=\frac{1}{2}}^{\infty}\sum_{t=\frac{1}{2}}^{\infty} b_s^{\mu} A_{st}(z) b_{t\mu}\Big)|0\rangle_d$$

$$B_r(z) = \frac{-i}{\sqrt{2}}(-z)^{-r}\binom{-\frac{1}{2}}{r-\frac{1}{2}}, \quad B_{nm}(z) = (-z)^{n-m}\binom{n-\frac{1}{2}}{m-\frac{1}{2}}$$

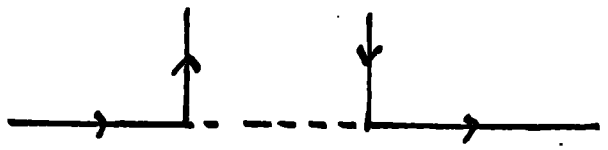
$$A_{st}(z) = \frac{1}{2}(-z)^{-t-s}\binom{-\frac{1}{2}}{t-\frac{1}{2}}\binom{-\frac{1}{2}}{s-\frac{1}{2}}\frac{(s-t)}{(s+t)}$$

This vertex can be used to construct the amplitude represented by



by using $\overline{V}_F(z,k) = \gamma^0 V_F^\dagger(z,k)\gamma^0$

By putting the two diagrams together the vertex allows the amplitude for fermion-antifermion scattering to be considered.



For this amplitude to be calculated using the operator formalism it is necessary to make sure that only physical meson states propagate and because the mass of the fermion is zero the $\mathcal{F}_2$ formalism is used. The meson propagator can be constrained to contain only physical states by the use of the projection operator but as this is not a Lorentz invariant procedure it is not obvious that the amplitude is Lorentz covariant. If suitable gauge conditions exist then it is possible that Lorentz covariance can be restored. Brink, Olive, Rebbi and Scherk[58] introduced gauges which are a sum of either the F's or the G's and when they act on a fermion emission vertex give infinite sums of the F's and G's on either side of the vertex. One of the gauges makes it possible to write the fermion emission vertex in the $\mathcal{F}_2$ formalism. So for a fermion tree changing into a meson tree the amplitude is written as

$$\frac{-1}{\sqrt{2}m}\langle\Psi_F|V_F(1,k)u(k)|\Psi_{NS}\rangle_1 = {}_2\langle\Psi_F|V_F(1,k)\gamma_5 u(k)|\Psi_{NS}\rangle_2$$

where the subscripts 1 and 2 stand for the different formalisms and $|V_F\rangle$ and $|V_{NS}\rangle$ stand for the fermion and meson trees. By using this vertex and the new gauges Olive and Scherk[58] found that it was necessary to modify the meson propagator to produce a Lorentz covariant result such that only physical states coupled. Instead of using

$$\int_0^1 \frac{dx}{x}\, x^{L_0-\frac{1}{2}}$$

the modified propagator

$$\int_0^1 \frac{dx}{x}\, \frac{x^{L_0-\frac{1}{2}}}{\Delta(x)}$$

is used where

$$\Delta(x) = \det(1 - A(x)^2)$$

$$A_{rs}(x) = x^r d_{rs} \quad , \quad d_{rs} = \frac{1}{2}\frac{(-1)^{r+s+1}}{(r+s)}\binom{-\frac{3}{2}}{r-\frac{1}{2}}\binom{-\frac{1}{2}}{s-\frac{1}{2}}$$

This still leaves the problem of actually calculating the amplitude which is not easy due to the matrices involved. Progress towards the result was made by Corrigan[59] and by Schwarz and Wu[60] and the amplitude was eventually calculated by Corrigan, Goddard, Olive and Smith[61] and by Scharz and Wu.[62]

LOOPS IN THE NEVEU-SCHWARZ-RAMOND MODEL.

The one loop graphs in the Neveu-Schwarz Model can be calculated,[63] in the $\mathcal{F}_2$ formalism, as a direct extension of the one loop graphs in the Convention Dual Model and corrections can be made to the partition function so that only physical states are propagated in the loops.[31] In the Neveu-Schwarz Model there are two types of pomeron depending whether or not there are an even number or odd number of emitted particles between the twists, producing even or odd "G-parity" pomeron. The pomeron is a pole in ten dimensions and also factorises.[41] The leading pomeron trajectory has intercept two and even G-parity.

Green[64] has calculated the planar loop diagram in the Ramond Fermion Model which involves a fermion propagating in a loop emitting mesons. This gives the result that the leading divergence cancels with the leading divergence of the Neveu-Schwarz planar loop. The question then arises, what happens to the pomeron loops,is there a similar cancellation ? For this to be answered in the operator formalism it is necessary to have a twisting operator for the fermion line. We shall discuss such loops further on but using a different formalism.

CHAPTER THREE.

RELATIVISTIC FREE STRING.

Although the operator formalism is useful for the construction of the Conventional Dual Model it is not the only method of understanding the structure of the model. By considering particles as objects extended in one dimension, that is strings, it is possible to produce the results of the operator formalism by considering the interaction of strings.[65] The harmonic modes of the vibration of the string give the spectrum of the model and constraints on the string give the Virasoro gauge conditions.[66,67,68] A detailed investigation into a free relativistic string[69] leads to a deeper understanding of many of the features of the Convention Dual Model. Goldstone, Goddard, Rebbi and Thorn[69] started with a classical system for a one dimensional relativistic string and proceeded to quantize the system , the quantization procedure revealing the structure of the system and the connection with the Conventional Dual Model.

The relativistic string sweeps out a two dimensional surface in space-time which is parameterised by $x_\mu = x_\mu(\sigma,\tau)$, $0 \le \sigma \le \pi$, $-\infty \le \tau \le \infty$ The action for the system is proportional to the area of the surface [67,68] and is given by

$$S = \frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \int_0^{\pi} d\sigma \sqrt{\left(\frac{\partial x}{\partial \tau}\cdot\frac{\partial x}{\partial \sigma}\right)^2 - \left(\frac{\partial x}{\partial \tau}\right)^2\left(\frac{\partial x}{\partial \sigma}\right)^2}$$

By using the variational principle the equations of motion of the classical relativistic string are given by

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial x^\mu}{\partial \tau}\right)}\right) + \frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial x^\mu}{\partial \sigma}\right)}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial x^\mu}{\partial \sigma}\right)} = 0 \quad , \sigma = 0, \pi \quad , \mathcal{L} = \sqrt{\left(\frac{\partial x}{\partial \tau}\cdot\frac{\partial x}{\partial \sigma}\right)^2 - \left(\frac{\partial x}{\partial \tau}\right)^2\left(\frac{\partial x}{\partial \sigma}\right)^2}$$

By choosing an orthonormal co-ordinate system

$$\left(\frac{\partial x}{\partial \sigma}\right)^2 = -\left(\frac{\partial x}{\partial \tau}\right)^2 \quad , \quad \left(\frac{\partial x}{\partial \sigma}\right)\cdot\left(\frac{\partial x}{\partial \tau}\right) = 0$$

the equations are simplified to

$$\frac{\partial^2 x^\mu}{\partial \sigma^2} - \frac{\partial^2 x^\mu}{\partial \tau^2} = 0 \, , \quad \frac{\partial x^\mu}{\partial \sigma} = 0 \quad , \sigma = 0, \pi$$

the conditions of orthonormality acting as constraints on the system. This is possible because the surface the string sweeps out can be reparameterised as a function of σ' and τ' where $\sigma' = \sigma'(\sigma,\tau), \tau' = \tau'(\sigma,\tau)$. The action is invariant under a group of reparameterisations which can be considered as a gauge group of the string , the above choice of co-ordinate system being a partucular choice of gauge.

There are two methods of quantizing the system by the use of Poisson brackets because of the existence of the constraints. The first method, the covariant method, considers all co-ordinates and momenta to be independent, these are then quantised and the constraints are applied afterwards. The surface is described as

$$x^\mu(\sigma,\tau) = \sum_{n=0}^{\infty} q_n^\mu(\tau) \cos n\sigma$$

where the q_n are taken as independent co-ordinates and the momenta are given by $p_n^\mu = \frac{1}{2}\frac{\partial q_n^\mu}{\partial \tau}$.

By defining $2\alpha_n^\mu = 2p_n^\mu - in q_n^\mu$ and $\alpha_n^\mu = \alpha_{-n}^{\mu\,+} \quad , n>0$

this gives the variables $\alpha_n^\mu , \alpha_{-n}^\mu , p_0^\mu , q_0^\mu$

which are, of course, just the operators from which the Conventional Dual Model is constructed and so these give the spectrum of the dual model but including ghosts. The constraint equations

$$\left(\frac{\partial x}{\partial \sigma}\right)^2 - \left(\frac{\partial x}{\partial \tau}\right)^2 = 0 \quad , \quad \left(\frac{\partial x}{\partial \sigma}\right)\cdot\left(\frac{\partial x}{\partial \tau}\right) = 0$$

when calculated give, in the classical case, $L_n = 0$ where

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha_{n-m}\alpha_m$$

which in the quantum mechanical case goes to $L_n|\phi\rangle=0$ where $|\phi\rangle$ is a physical state , and, because of the normal ordering problem, $(L_0-a(0))|\phi\rangle=0$

The second method of quantization is the canonical non-covariant method where the constraints are used to eliminate two degrees of freedom, the remaining independent variables are quantized. In this approach light cone co-ordinates are used where $a^{\pm}=\frac{a_0\pm a_1}{\sqrt{2}}$. Due to the freedom of reparameterising the surface a particular choice can be made such that $\tau=\frac{x^+}{2p^+}$

then similarly to the first method

$$x^+(\sigma,\tau)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{\alpha_n^+}{in}\cos n\sigma+\alpha_0^+\tau+q_0^+$$

which implies

$$\alpha_n^+=0\ ,n\neq 0\ ,\qquad \alpha_0^+=2p_0^+\ ,\ q_0^+=0$$

the constraint equations then give

$$L_n=\frac{1}{2}\sum_{m=-\infty}^{\infty}\left(\alpha_{-m}^+\alpha_{n+m}^-+\alpha_{-m}^-\alpha_{n+m}^+-\underline{\alpha}_{-m}\underline{\alpha}_{n+m}\right)$$

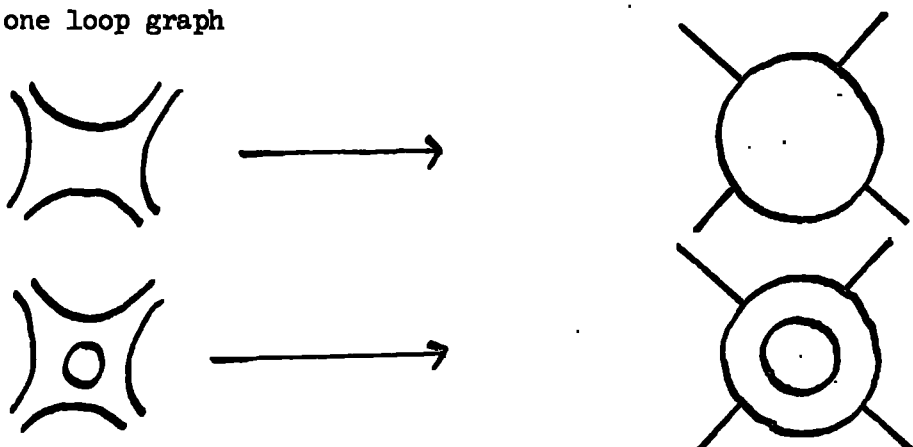
writing $\alpha_n=\frac{1}{2}\sum_m\underline{\alpha}_{-m}\cdot\underline{\alpha}_{n+m}$ these imply $\alpha_n^-=\frac{\alpha_n}{\alpha_0^+}=\frac{\alpha_n}{2p_0^+}$

so the independent variables are $\underline{\alpha}_n\ ,\ q_0^-\ ,\ p_0^+$ and these can be quantized using Possion brakets. This method produces only transverse states of the string, the $\underline{\alpha}_n^i$ s, and this ensures that the spectrum is always positive definite and we can now see why two sets of states decouple in the Conventional Dual Model. This method is non-covariant but is self-consistent if an algebra for the generators of Lorentz transformations can be constructed. This is only possible when D=26 and the intercept of the Regge trajectory

is one. So this gives us an understanding of the critical dimension and the necessity of $\alpha(0)=1$ for the Conventional Dual Model, but as yet interactions have not been built in.

THE ANALOGUE MODEL.

This method of describing interactions of the Conventional Dual Model gives an interpretation of the amplitude in terms of quark diagrams. The Analogue Model[70] introduces an interaction region from the quark diagrams as follows, for the Born term and the planar one loop graph



The amplitude for each component of momentum is given by a functional integral over all configurations such that the energy loss is minimised

$$A = \frac{\int d(\text{configurations}) \exp(-E)}{\int d(\text{configurations})}$$

This is just the same as the heat loss of a uniform plate, representing the interaction region, due to electric currents, representing the momenta of the particles, entering the boundary of the plate.

The amplitude for the Born term is given in terms of the Koba-Nielsen variables which represent the positions of the electric currents on the interaction region. For the planar one loop graph the variables are the same as in the operator formalism after the Jacobi transformation has been used.

This method can be reformulated in terms of the Neumann function $N_\beta(z_i, z_j)$ for the surface under consideration, where z_i and z_j represent the positions of the momenta p_i and p_j, then the amplitude for that particular surface is given by

$$A = \int \exp\left(-2 \sum_{i<j} p_i \cdot p_j N_\beta(z_i, z_j)\right)$$

the Neumann function being such that the normal derivative is a constant on the boundaries.[71] For the Born term the Neumann function is given by $N_\beta(z_i, z_j) = \ln(z_i - z_j)$ which gives the usual amplitude.

The Analogue Model can be extended to non-planar diagrams and generally to all orders of diagrams by finding the Neumann function associated with a interaction region by using functionals on Riemann surfaces.[71,72,73] The Riemann surface is given by the interaction region and the double of the Riemann surface is constructed from this and is closed and orientable, on this double automorphic functions can be constructed and the amplitudes are obtained in terms of these. At the one loop level the different diagrams have as their double a torus but differ by the paths along which the z's are defined. The amplitude in general is given by

$$\exp(-2p_i \cdot p_j \Omega(z_i, z_j)) = \left[\prod_\alpha \frac{(z_i - T_\alpha(z_j))(z_j - T_\alpha(z_i))(z_i - z_j)}{(z_i - T_\alpha(z_i))(z_j - T_\alpha(z_j))(z_i z_j)^{\frac{1}{2}}}\right]^{-2p_i \cdot p_j}$$

where T_α is a member of the group of projective transformations. This in fact just gives the Neumann function for the interaction region.

THE FUNCTIONAL INTEGRATION APPROACH.

The amplitudes can also be written in a functional integration form but are solved by the Neumann function for the surface under consideration.[74,75] The functional integration is formulated in terms of a Lagrangian density

$$\mathcal{L}(\Phi) = -\frac{1}{2}\left(\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2\right)$$

which is the Lagrangian density of a free string but with imaginary time. The amplitude for a region D is defined by the rudimentary amplitude

$$\langle \exp\left(i\sqrt{2\pi}\sum_{l}\int_{0}^{\pi} d\xi\, k_l(\xi)\,\Phi(q_l,\sigma_0[\xi])\right)\rangle_0$$

where k_1 are the momenta of the external particles, and the functional average is defined by

$$\langle A(\Phi)\rangle_0 = \frac{1}{n_0}\int\cdots\int \mathcal{D}^{(0)}\Phi(x,y)A(\Phi)\exp\left(\int_{\mathcal{D}} dx\,dy\,\mathcal{L}(\Phi)\right)$$

n_0 being a normalisation constant. This can be calculated by changing variables

$$\Phi(x,y)=\Phi'(x,y)-\frac{i}{\sqrt{2\pi}}\sum_{j}\int_{0}^{\pi} d\xi\, k_j(\xi)\,N(z,q_j\,\sigma[\xi])$$

this then gives the rudimentary amplitude

$$(2\pi)^4\,\delta^0\Big(\sum_i\int_0^\pi d\xi\, k_i(\xi)\exp\Big(\tfrac{1}{2}\sum_{i,j}\int_0^\pi d\xi\int_0^\pi d\xi'\, k_i(\xi)k_j(\xi')N(q_i\sigma[\xi],q_j\sigma[\xi']$$

The difficulty with this method and also the approach using automorphic functuins is that the partition funtion for loops is not obtained directly. By starting from the operator formalism and rewriting the amplitude in either the functional form or interms of automorphic functions it is possible to obtain a term which gives the partition function.[73,75]

THE PARTITION FUNCTION.

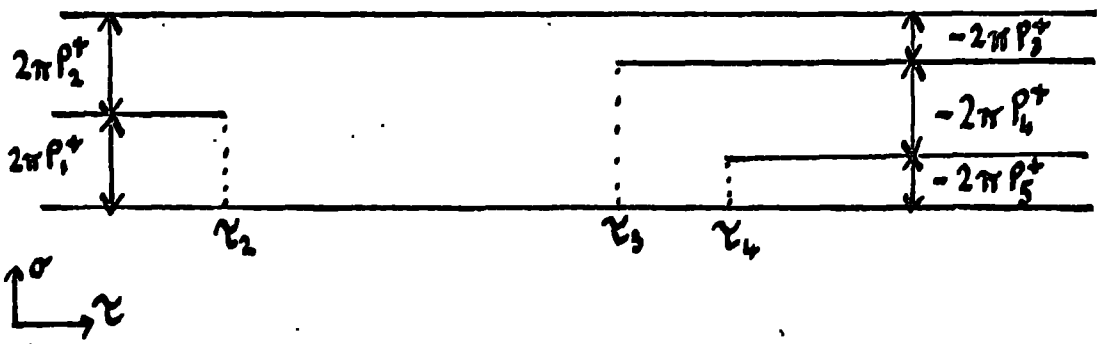
One method of reproducing the results of the functional integration form is by considering particles to be made up of partons and describing the interactions by a network of propagators and vertices of these partons. Then in the limit of the number of partons tending to infinity the dual amplitude is given.[76,77] This method can also be formulated

in terms of an electrical analogue which reproduces the correct results in the limit of the network becoming dense, that is continuous. This then allows a method of calculating the partition function for single loop diagrams by taking an electrical network with appropriate boundary conditions to give a cylinder, then by considering the energy dissapated and carefully taking the limit as the network becomes dense.[78] This then reproduces the results of the operator formalism.

THE INTERACTING STRING PICTURE:

As we have seen there appears to be a connection between free strings and the functional integration approach to the Conventional Dual Model. These two approaches were brought together by Mandelstam.[79] The particles are described by strings in terms of the physical state operators, that is in the non-covariant method which means that only positive norm states are used. The interactions are built up by strings joining together then separating, an integration being taken over all possible paths, the Feynman Path Integral. This produces the amplitude in terms of a functional integration. Lorentz covariance is not obvious in this approach and it is necessary that D=26 and $\alpha(0)= 1$. This approach uses imaginary time for the calculation, giving an elliptical differential equation, at the end of the calculation it can be changed back to real time.

The equations are obtained in terms of slightly modified variables of Goldstone, Goddard, Rebbi and Thorn.[61] The interaction can be represented by



A variable ρ is defined by $\rho = \tau + i\sigma$ and x^+ is choosen to be $x^+ = i\tau$. The lengths of the strings are given by $2\pi p^+$. The functional integral is invariant under conformal transformations, so this means that the variables can be transformed onto the upper half plane. This is achieved by the transformation

$$\rho = \sum_{r=1}^{N} 2 p_r^+ \ln(z - Z_r)$$

where z is the variable which corresponds to ρ . The Z_r's are given by the initial and final positions of the strings. If $Z_r > Z_s$ for $r<s$ then as $z \to Z_1$ $\rho \to -\infty$, when z passes through Z_1 the variable ρ gains an imaginary part equal to the length of the first string. As the variable z goes from plus infinity to minus infinity through the points Z_r the variable ρ goes along the boundaries of the strings in a clock-wise manner. Maxima in the variable z gives ,in ρ, the positions where the strings join together and separate. The function integral can now be solved in terms of the Neumann functions of the upper half plane as this is equivalent to the ρ plane. It is possible to describe the scattering of excited states using this method by writing the Neumann function in terms of a Fourier series. This approach has the advantage of only dealing with physical states but Lorentz invariance has to be proven. This method can be said to justify the Analogue model.

THE NEVEU-SCHWARZ-RAMOND MODEL AND INTERACTING STRINGS.

The string approach can be extended to the Neveu-Schwarz and Ramond Models and this brings out the connection between the two models and, in fact, they are described by one model.

The free relativistic string for the Neveu-Scharz Model[80] consists of the free relativistic string for the Conventional Dual Model but with an additional two component Lorentz vector field

which represents a continuously distributed spin along the string. These additional variables have conformal spin one-half

$$S^\mu = \begin{pmatrix} S_1^\mu \\ S_2^\mu \end{pmatrix}, \quad S_1^\mu(\rho') = \left(\frac{\partial \rho'}{\partial \rho}\right)^{-\frac{1}{2}} S_1^\mu \quad , \quad S_2^\mu(\rho') = \left(\left(\frac{\partial \rho'}{\partial \rho}\right)^*\right)^{\frac{1}{2}} S_2^\mu(\rho)$$

Similarly as before the string can be quantized in both covariant and non-covariant methods , the latter method allowing the model to be written in terms of transverse positive norm states, but only when the dimension of space-time is ten and the ground state has $m^2=-1/2$ .The quantization procedure is similar to that for the string in the Conventional Dual Model and it is not worthwhile going into further detail. The Ramond Model can be obtained by suitably changing the boundary conditions of the string.

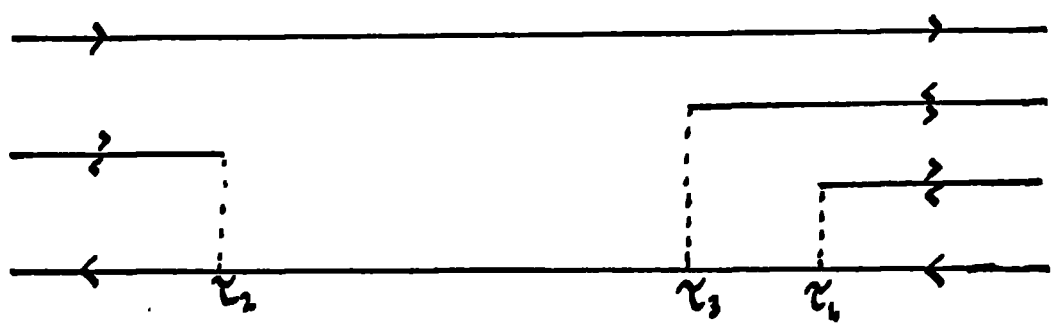
The functional integration method has also been given for this model[81,82] and the Neumann functions are given by

$$\left(\frac{\partial}{\partial \tau} + i\frac{\partial}{\partial \sigma}\right) K_{1b}(\rho,\rho') = 2\pi\, \delta^2(\rho-\rho')\, \delta_{1,b}$$

$$\left(\frac{\partial}{\partial \tau} - i\frac{\partial}{\partial \sigma}\right) K_{2b}(\rho,\rho') = 2\pi\, \delta^2(\rho-\rho')\, \delta_{2,b}$$

where b=1 or 2 corresponds to the fields S_1^i or S_2^i where and where $\rho = \tau+i\sigma$. The correct boundary conditions have to be imposed on the Neumann functions, and imaginary time is used.

It is possible to put the two approaches together to produce an interacting string picture for the Neveu-Schwarz-Ramond Model.[83,84] Interactions are represented diagrammatically by



The arrows represent the different boundary conditions on the ends of the strings, the forward arrow means

$$S_1 = -S_2 \quad , \quad \frac{\partial S_1}{\partial \sigma} = \frac{\partial S_2}{\partial \sigma}$$

and the backward arrow

$$S_1 = S_2 \quad , \quad \frac{\partial S_1}{\partial \sigma} = -\frac{\partial S_2}{\partial \sigma}$$

Two forward arrows on the ends of a string represent a fermion, two backwards arrows an antifermion and arrows in opposite directions a meson. There are two types of meson, those with a forward arrow at the top which Mandelstam denotes as a quark-antiquark pair, and those with a backward arrow at the top which he denotes as a zero quark meson. The parity of the former is opposite to that of the corresponding zero quark meson, as the mass of the ground state fermion is zero there is no way of determining the absolute parity.

The Feynman path integral technique can be used to produce a functional integration which contains the Lagrangian

$$\mathcal{L}(\sigma,\tau) = \frac{1}{2\pi}\left(-\frac{1}{2}\left(\frac{\partial x}{\partial \sigma}\right)^2 - \frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 - S_1\left(\frac{\partial}{\partial \tau} + i\frac{\partial}{\partial \sigma}\right)S_1 - S_2\left(\frac{\partial}{\partial \tau} - i\frac{\partial}{\partial \sigma}\right)S_2\right)$$

The fermion ground states are written in terms of generalised Pauli matrices σ^i where the zero mode operator is
The σ's are paired as $-i\sigma^1\sigma^2$, $-i\sigma^3\sigma^4$, etc. and these form the basis states of the fermions. The solution is given by

$$K^{+-}(z,z') = -K^{-+}(z',z) = \frac{1}{(z-z')}\left\{\prod{}'\left(\frac{z-z_r}{z'-z_r}\right)^{\frac{1}{2}} \prod{}''\left(\frac{z'-z_r}{z-z_r}\right)^{\frac{1}{2}}\right\}$$

where the product Π' is over incoming fermion states and the product

Π'' is over incoming antifermion states, the plus and minus signs are the helicity combinations of paired indices of the σ's

$$b_0^{\pm} = \frac{1}{\sqrt{2}}(b_0^i \pm i b_0^j)$$

All other Neumann functions are zero. The Neumann functions can also be written as

$$K_{1f}(z,z') = \frac{1}{2(z-z')}\left\{\prod'\left(\frac{z-z_r}{z'-z_r}\right)^{\frac{1}{2}}\prod''\left(\frac{z'-z_r}{z-z_r}\right)^{\frac{1}{2}} + \prod'\left(\frac{z'-z_r}{z-z_r}\right)^{\frac{1}{2}}\prod''\left(\frac{z-z_r}{z'-z_r}\right)^{\frac{1}{2}}\right\}$$

$$K_{2b}(z,z') = K_{1b}(z^*,z') \quad , \quad K_{a2}(z,z') = K_{a1}(z,z'^*)$$

Mandelstam[84] using these methods suceeded in calculating the fermion-antifermion scattering amplitude and showed that the amplitude is dual but the s and t channels are not identical being constructed out of the different types of meson either quark-antiquark or zero quark mesons.

LOOP DIAGRAMS.

So far we have not discussed loop diagrams in this approach to the Neveu-Schwarz-Ramond Model, one method these can be calculated in the Neveu-Schwarz Model is in terms of automorphic functions.[85]

It is necessary to start from the operator formalism and then to rewrite the amplitudes in terms of automorphic functions. This necessitates the introduction of a Grassmann algebra[85,86] which consists of anticommuting variables ϕ_i such that

$$\{\phi_i,\phi_j\} = 0 \quad , \quad \phi_i^* \neq \phi_i$$

A variable ϕ_i is introduced to correspond to each operator b_i. From these coherent like states can be constructed which enables the amplitudes to be written in terms of automorphic functions. Integrals over the anticommuting variables are formally defined as $\int d\phi_i = 0 \quad , \quad \int \phi_i d\phi_i = 1$

The automorphic functions are slightly different from the Conventional Dual Model.

THE NEVEU-SCHWARZ MODEL IN A KOBA-NIELSEN FORM.[91]

The Grassmann algebra can be used to formulate the Neveu-Schwarz Model using Koba-Nielsen variables. Each particle has associated with it a variable z_i and a variable ϕ_i, an element of a finite Grassmann algebra. The N point amplitude, the Born term, can be written as

$$A_N = \int\cdots\int \prod_{i=1}^{N} d\phi_i \frac{\prod_{i=1}^{N} dz_i}{dV_{a,b,c}} \prod_{1\le i<j\le N} (z_i - z_j - \phi_i\phi_j)^{-2k_i\cdot k_j}$$

The integral over the ϕ's can be written as a contour integral with measure $\phi^{-2}\, d\phi$ giving

$$A_N = \int \frac{\prod_{i=1}^{N} dz_i}{dV_{a,b,c}} \oint\cdots\oint \prod_{i=1}^{N} \frac{d\phi_i}{\phi_i^2} \prod_{1\le i<j\le N} (z_i - z_j - \phi_i\phi_j)^{-2k_i\cdot k_j}$$

This is evaluated by expanding the integrand and using the property of the algebra that $\phi^2 = 0$.

$$\prod_{1\le i<j\le N} (z_i - z_j - \phi_i\phi_j)^{-2k_i\cdot k_j} = \prod_{1\le i<j\le N} (z_i - z_j)^{-2k_i\cdot k_j} \left(1 + \frac{2k_i\cdot k_j\, \phi_i\phi_j}{(z_i - z_j)}\right)$$

The only terms that contibute to the integral are those with a complete set of the elements of the Grassmann algebra, i.e. $\phi_1\phi_2\ldots\phi_N$ or a permutation of this. If the number of external particles is odd then the amplitude is automatically zero because in any term of the expansion there will be an even number of elements of the algebra, if all elements are not represented the integral over the missing ones gives zero, and if all are represented then there must be two the same which gives zero. To do the ϕ integration the elements have to be ordered as above and this introduces a factor of $(-1)^p$ where p denotes the parity of the permutation.

This then gives the amplitude in terms of Koba-Nielsen variables,[87]

$$\int \frac{\prod_{i=1}^{N} dz_i}{dV_{a,b,c}} \prod_{1\le i<j\le N} (z_i - z_j)^{2k_i \cdot k_j} \sum (-1)^{\rho} \frac{(2k_{i_1}\cdot k_{i_2})(2k_{i_3}\cdot k_{i_4}) \dots (2k_{i_{N-1}}\cdot k_{i_N})}{(z_{i_1}-z_{i_2})(z_{i_3}-z_{i_4}) \dots (z_{i_{N-1}}-z_{i_N})}$$

where the sum is over all possible permutation of the indices, represented by i_j where

$$i_1 < i_2 \,,\; i_3 < i_4 \,,\; \dots \dots \; i_{N-1} < i_N$$

$$i_1 < i_3 < i_5 \dots \; i_{N-1}$$

As the Koba-Nielsen form is Mobius invariant it is natural to look for a way of writing the amplitude in terms of the ϕ's such that Mobius invariance is still true. For the Mobius transformation

$$z_i \to z_i' = \frac{a z_i + b}{c z_i + d} \qquad , \; ad - bc = 1$$

$$dz_i \to dz_i (c z_i + d)^{-2}$$

$$\therefore \frac{\prod_{i=1}^{N} dz_i}{dV_{a,b,c}} \to \frac{\prod_{i=1}^{N} dz_i (c z_i + d)^{-2}}{dV_{a,b,c}}$$

As $(z_i - z_j) \to \dfrac{(z_i - z_j)}{(c z_i + d)(c z_j + d)}$ then to preserve the structure of the integrand it is necessary to have

$$\phi_i \to \pm \phi_i (c z_i + d)^{-1}$$

$$\therefore (z_i - z_j - \phi_i \phi_j) \to \frac{(z_i - z_j - \phi_i \phi_j)}{(c z_i + d)(c z_j + d)}$$

but to preserve the integration $\int \phi_i \, d\phi_i = 1$ it is necessary

to have $d\phi_i \to d\phi_i (cz_i+d)$

Putting all these factors together produces a term $\prod_{i=1}^{N} (cz_i+d)^{2k_i^2-1}$ so for Mobius invariance $k_i^2 = -\frac{1}{2}$ which is just the mass of the "pion" in the Neveu-Schwarz Model. So we see that for Mobius invariance to be possible a necessary condition is that the ground state mass has $m^2 = -\frac{1}{2}$.

The question then arises is it possible to extend the Neveu-Schwarz Model in the same way? In this case Mobius invariance means that the lowest mass particle has zero mass and using this condition it is found that the four point function vanishes. This then appears to be a self-consistency in the model.

Now that it is possible to write the tree graph in terms of the ϕ's the loop graphs can now be considered. Starting with the one loop planar diagram[63] (table three) we want to find a method of obtaining the function $\chi^+(x)$ from the function $\psi(x)$ by using the ϕ's.

$\chi^+(x)$ can be written as[63]

$$\chi^+(x) = \frac{i}{2}\,\theta_2(0|\tau)\theta_4(0|\tau)\frac{\theta_3(v|\tau)}{\theta_1(v|\tau)}\,, \qquad \frac{\ln x}{2\pi i} = v$$

which then can be written as

$$\chi^+(x) = \frac{i}{\pi}\frac{d}{d(\frac{x}{2})}\ln\left(\frac{\theta_1(\frac{x}{2}|\tau)\theta_3(\frac{x}{2}|\tau)}{\theta_2(\frac{x}{2}|\tau)\theta_4(\frac{x}{2}|\tau)}\right)$$

by changing variables, using the Jacobi transformation, the function can be written as an infinite series.

$$\chi^+(x_{ij}) = \frac{1}{\tau}\left(\frac{\sqrt{z}}{(1-z)} - \frac{(1-z)}{\sqrt{z}}\sum_{n=1}^{\infty}\frac{(-r)^n(1+r^{2n})}{(1-zr^{2n})(1-\frac{r^{2n}}{z})}\right), \qquad z = \frac{z_j}{z_i}$$

Now $\psi(x_{ij})$ can be written in terms of the z variables as

$$\psi(x_{ij}) = \pi\gamma \frac{(z_i - z_j)}{\sqrt{z_i z_j}} \prod_{n=1}^{\infty} \frac{(z_i - r^{2n} z_j)(z_j - r^{2n} z_i)}{(z_i - z_i r^{2n})(z_j - z_j r^{2n})}$$

which suggests making the replacement

$$(z_i - z_j) \rightarrow (z_i - z_j - \phi_i \phi_j)$$

and

$$(z_i - z_j r^{2n}) \rightarrow (z_i - z_j r^{2n} - \phi_i \phi_j (-r)^n$$

where the factor $(-r)^n$ comes from the Mobius transformation of ϕ_j the minus sign being chosen. Expanding in terms of the ϕ's gives

$$[\psi(x_{ij})]^{-2k_i \cdot k_j} \left(1 + \frac{2 k_i \cdot k_j}{\sqrt{z_i z_j}} \phi_i \phi_j \left(\frac{-i\pi}{\ln r}\right) \chi^+(x_{ij})\right)$$

This procedure then produces the correct amplitude apart from the partition function where $f(\omega)$ is replaced by $\frac{f(\omega)}{\Phi(\omega)}$ the power of this factor being $2-D=-8$.

Using the z variables makes the prescription for the amplitude more obvious as the term involving the exponential in the function $\psi(x)$ is cancelled by a term arising from the Jacobi transformation on the theta functions.

An explanation for the choice of replacement for the factor $(z_i - z_j r^{2n})$ is given in terms of the Mobius transformation given by $b=c=0$, $a=d^{-1} = \pm r^n$. When this transformation acts on z_j this gives $z_j r^{2n}$ and when it acts on ϕ_j gives $\phi_j (\pm r)^n$; in the planar loop the minus sign is choosen. This can also be considered as the n times iterated Mobius transformation given by

$b=c=0$, $a=d^{-1} = (\pm\dot{r})$.

To obtain the functions for the non-planar orientable loops it is necessary to work in terms of the variables defined on the real axis as the functions can all be written in terms of χ^+ (see table four) so it is relatively easy to extend the above procedure using the ψ's in these variables but not modifying the exponential term. The variables are given by $x_{ij} = \frac{\rho_j}{\rho_i}$ then for N particles on one boundary and M on the other [41]

$$-1 = \rho_1 < \rho_2 < \dots < \rho_N < -w < 0 < w\rho_{N+1} < \rho_{N+M} < \dots < \rho_{N+1} < 1$$

$$w < \rho_{N+M} < 1$$

As the function $\chi^{\pm}(x)$ can be written as

$$\chi^{\pm}(x_{ij}) = \sum_{n=-\infty}^{\infty} \frac{(\mp 1)^n w^{\frac{n}{2}} \sqrt{x_{ij}}}{(1 - w^n x_{ij})} = \sum_{n=-\infty}^{\infty} \frac{(\mp 1)^n w^{\frac{n}{2}} (\rho_i \rho_j)^{\frac{1}{2}}}{(\rho_i - \rho_j w^n)}$$

then this is given by the replacement

$$(\rho_i - \rho_j w^n) \longrightarrow (\rho_i - \rho_j w^n - (\mp 1)^n w^{\frac{n}{2}} \phi_i \phi_j)$$

where the minus sign is taken for χ^+ and the plus sign for χ^- which give the positive and negative G-parity pomerons respectively. The functions $\chi^{\pm}(xe^{i\pi})$ are obtained in the same manner from $\psi(xe^{i\pi})$. So now we can describe the one loop terms using the ϕ's.

This method can easily be extended to multiloop amplitudes by making use of the structure of automorphic functions. As we have seen the Conventional Dual Model can be written in terms of automorphic functions $\Omega(\rho_i, \rho_j)$ which can be written as[72,73]

$$\exp(-2k_i\cdot k_j\,\Omega(\rho_i,\rho_j)) = \left[\frac{(\rho_i-\rho_j)}{\sqrt{\rho_i\rho_j}}\prod_\alpha \frac{(\rho_i-T_\alpha(\rho_j))(\rho_j-T_\alpha(\rho_i))}{(\rho_i-T_\alpha(\rho_i))(\rho_j-T_\alpha(\rho_j))}\right]^{-2k_i\cdot k_j}$$

the form of which resembles the function used at the one loop level, of course. This immediately suggests the replacement of $(\rho_i-T_\alpha(\rho_j)$ by

$$\rho_i - T_\alpha(\rho_j) - \phi_i T_\alpha(\phi_j)\ ,$$

$$T_\alpha(\rho_j) = \frac{a_\alpha\rho_j+b_\alpha}{c_\alpha\rho_j+d_\alpha}\ , \quad T_\alpha(\phi_j) = \frac{(\mp 1)^{n_\alpha}}{(c_\alpha\rho_j+d_\alpha)}\phi_j$$

and where n_α is the order of the transformation T_α . The positive or negative sign in $T_\alpha(\phi_i)$ is chosen according to whether the channel under consideration has negative or positive G-parity. The modification of the amplitude is given by

$$\prod_{pairs}(-1)^p\, k_i\cdot k_j\, F(\rho_i,\rho_j)\sqrt{\rho_i\rho_j}$$

$$F(\rho_i,\rho_j) = \sum_\alpha \frac{(\mp 1)^{n_\alpha}}{(c_\alpha\rho_j+d_\alpha)(\rho_i-T_\alpha(\rho_j))}$$

which agrees with the method of Montonen.[85] So we can see the usefulness of this method in obtaining the multiloop amplitudes in terms of automorphic functions which are obtained from the automorphic functions of the Conventional Dual Model.

CHAPTER FOUR.

NEVEU-SCHWARZ-RAMOND MODEL AND NEUMANN FUNCTIONS.[92]

So far we have only considered the Neveu-Schwarz Model and naturally would like to extend the method of using a finite Grassmann algebra to include the Ramond Model. As a first step we shall rewrite the Neveu-Schwarz amplitudes in terms of Neumann functions and then, at the one loop level, consider both meson and fermion loops but with external mesons only. As the only difference between mesons and fermions are the boundary conditions then the difference manifests itself in the Neumann functions used.

Returning for the moment to the Conventional Dual Model, amplitudes for excited particles can be constructed using the Analogue Model by considering a distribution of momenta entering the interaction region [88] rather than at a point as for ground state particles. The multipole expansion then gives the amplitude for excited particles, and in particular for spin one zero mass particles the integrand for the tree amplitude is given by expanding

$$\exp\left(-\sum_{i\neq j}^{N}\sum_{j=1}^{N}\left(k_i.k_j\ln(z_i-z_j)+\frac{e_i.k_j}{(z_i-z_j)}+\frac{e_i.e_j}{(z_i-z_j)^2}\right.\right.$$

and taking the coefficient of $e_{1\mu}e_{2\nu}\ldots\ldots e_{N\sigma}$,

where the e_i is the polarisation vector for the i'th particle and $k_i.e_i=0$. The three types of terms can be interpreted in terms of the analogue approach as interaction terms between pole-pole, pole-dipole, and dipole-dipole.

This then suggests the following approach for the Neveu-Schwarz Model, that the amplitude is given by the coefficient of $\phi_1\phi_2\ldots\ldots\phi_N$ in the expansion of

$$\exp\left(-2\sum_{i<j}k_i.k_j\left(\ln(z_i-z_j)+\frac{\phi_i\phi_j}{(z_i-z_j)}\right)\right.$$

Remembering that $\phi^2 = 0$, this can easily be seen to be equivalent to the previous procedure for the tree amplitude. The second term can now be interpreted as as the interaction due to spin $\frac{1}{2}$ sources, which have charges ϕ_i which anticommute.

Considering the string picture for the Neveu-Schwarz-Ramond Model[04] and remembering the boundary conditions for the forward arrows are

$$S_1 = -S_2 \,, \quad \frac{\partial S_1}{\partial \sigma} = \frac{\partial S_2}{\partial \sigma}$$

and for the backward arrows are

$$S_1 = S_2 \,, \quad \frac{\partial S_1}{\partial \sigma} = -\frac{\partial S_2}{\partial \sigma}$$

then these imply, that in a source free space, S_α can be written in terms of two real harmonic functions Φ and Ψ such that

$$S_1 = \Phi(\sigma,\tau) + i\,\Psi(\sigma,\tau)$$
$$S_2 = \Phi(\sigma,\tau) - i\,\Psi(\sigma,\tau)$$

The boundary conditions can then be rewritten as $\Phi = 0$, $\frac{\partial \Psi}{\partial \sigma} = 0$ for the forward arrow and $\Psi = 0$, $\frac{\partial \Phi}{\partial \sigma} = 0$ for the backward arrow. The terms in the Neveu-Scharz tree amplitude can be interpreted as the contribution from two spinor sources $k_i \begin{pmatrix} \phi_i \\ \phi_i \end{pmatrix}$ and $k_j \begin{pmatrix} \phi_j \\ \phi_j \end{pmatrix}$ at the points z_i and z_j on the real axis with the source term $k_i \phi_i \, \delta(x - z_i)\,\delta(y)$ which gives $S_1 = S_2^* = \frac{k_j \phi_j}{(x - z_j)}$

Then the contribution to the Lagrangian action in the upper half

plane for the spinor fields is given by

$$\frac{k_i.k_j\,\phi_i\phi_j}{(z_i-z_j)} \quad .$$

This method may readily be extended to one loop amplitudes in the Neveu-Schwarz Model by constructing the potentials for a surface as in the Analogue Model. The interaction region considered is an annulus and the boundary conditions are given by the boundary conditions on the string. For the planar loop the boundary conditions given above mean that the function S_α must be real on one boundary and imaginary on the other. When the radius of the annulus is one and the radius of the hole in the centre is r then the potential can be constructed by an infinite number of poles at $zr^{\pm 2}, zr^{\pm 4}, \ldots$. This then gives

$$S^{\mu}_{1,2} = \sum_{n=-\infty}^{\infty} \sum_j \frac{(-r)^n k_j^{\mu}\phi_j\sqrt{zz_j}}{(z-r^{2n}z_j)}$$

which gives a contribution to the amplitude of

$$k_i.k_j\phi_i\phi_j\chi^+\left(\frac{z_j}{z_i},r^2\right)=\sum_{n=-\infty}^{\infty}\frac{(-r)^n k_i.k_j\phi_i\phi_j(z_iz_j)^{\frac{1}{2}}}{(z_i-z_jr^{2n})}$$

and the function $\chi^+(z_j/z_i, r^2)$ is imaginary when $|z_i| = 1$ and real when $|z_i| = r$. This agrees with the previous calculation but the minus sign is given automatically due to the boundary conditions.

For the one loop non-planar orientable diagrams the even G-parity amplitude is given by $\chi^+(z_i/z_j, r^2)$ when both particles are on the same boundary and when the two particles are on different boundaries this is modified by replacing z_j by $z_j r$ giving $\chi^+(z_j r/z_i, r^2)$. These can be written in the upper half plane variables

using the Jacobi transformation. The function S_α has the property that $S_\alpha(\theta+2\pi) = -S_\alpha(\theta)$ for even G-parity exchange.

For odd G-parity exchange the field S_α must obey the equation $S_\alpha(\theta+2\pi) = S_\alpha(\theta)$ and the solution is

$$S_{1,2} = \sum_{n=-\infty}^{\infty} \sum_j \frac{(z+z_j r^{2n})(-1)^n \phi_j}{(z-z_j r^{2n})}$$

which is imaginary when $|z| = 1$ and real when $|z| = r$. This then gives the contribution for particles on the same boundary as

$$k_i \cdot k_j \phi_i \phi_j \chi_0^+\left(\frac{z_j}{z_i}, r^2\right) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n \frac{(z_i+z_j r^{2n})}{(z_i-z_j r^{2n})} k_i \cdot k_j \phi_i \phi_j$$

The function χ_0^+ can be rewritten as

$$\chi_0^+(z, r^2) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(z^n + (\zeta^2)^n)}{(1+r^{2n})}$$

and by using the Jacobi transformation as

$$\chi_0^+(z, r^2) = \tau\, \chi^+(x, we^{2\pi i})$$

For particles on different boundaries z_j is replaced by $z_j r$ giving

$$\chi_0^+(zr, r^2) = \tau\, \chi^+(xe^{i\pi}, we^{2\pi i})$$

This then covers the case of meson loops but can readily be extended to fermion loops with the external particles being mesons. For the planar loop the annulus must have the same boundary conditions

on each boundary and this is realised by

$$S^{\mu}_{1,2} = \sum_{n=-\infty}^{\infty} \sum_{j} \frac{k^{\mu}_{j} \phi_j \sqrt{z_i z_j}\, r^n}{(z - r^{2n} z_j)}$$

which is imaginary when $|z| = 1, r$, this then gives a contribution to the amplitude of

$$k_i . k_j \phi_i \phi_j \chi^{+}\left(\frac{z_j}{z_i}, r^2 e^{2\pi i}\right) = k_i . k_j \Phi_i \Phi_j \sum_{n=-\infty}^{\infty} \frac{\sqrt{z_i z_j}\, r^n}{(z_i - z_j r^{2n})}$$

which, using the Jacobi transformation, can be written as

$$\chi^{+}(z, r^2 e^{2\pi i}) = \tau\, \chi^{+}_{0}(x, \omega)$$

The even G-parity loops are then given by the same procedure as before, for particles on the same boundary the above function is used and for particles on different boundaries z_j goes to $z_j r$. For odd G-parity loops again $S_\alpha(\theta+2\pi) = S_\alpha(\theta)$ as well as the boundary conditions being the same on each boundary, these conditions give

$$S^{\mu}_{1,2} = \tfrac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{j} \frac{(z + z_j r^{2n})\, k^{\mu}_{j} \phi_j}{(z - z_j r^{2n})} + \tfrac{1}{2} \sum_{j} \frac{\ln\left(\frac{z_j}{z_i}\right)}{\ln r} k^{\mu}_{j} \phi_j$$

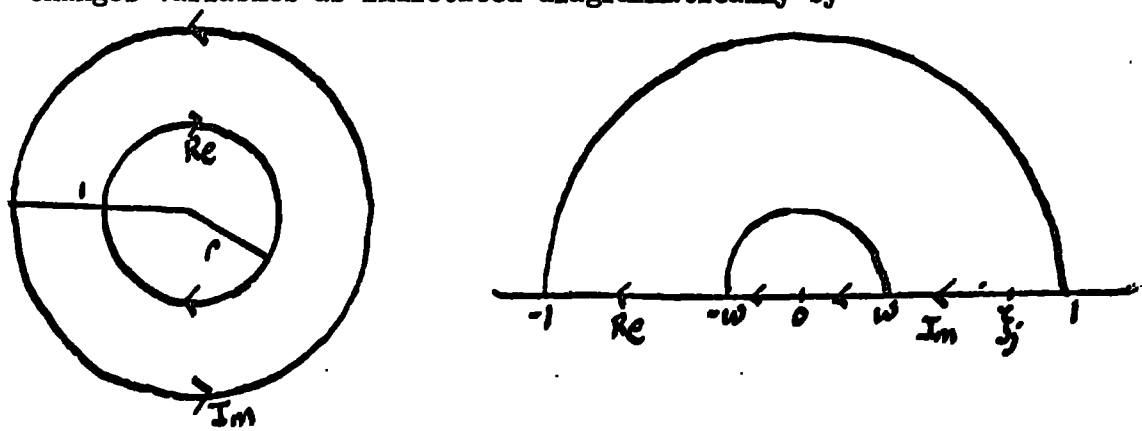
which gives a contribution of

$$k_i . k_j \phi_i \phi_j \left(\chi^{-}_{0}\left(\frac{z_j}{z_i}, r^2\right) + \frac{\ln\left(\frac{z_j}{z_i}\right)}{2 \ln r}\right) = k_i k_j \phi_i \phi_j \left(\sum_{n=-\infty}^{\infty} \frac{(z_i + z_j r^{2n})}{(z_i - z_j r^{2n})} + \frac{\ln\left(\frac{z_j}{z_i}\right)}{2 \ln r}\right)$$

and where

$$\chi_0^-(z,r^2) + \frac{\ln z}{2\ln r} = \tau\, \chi_0^-(x,\omega)$$

These functions can also be derived in the upper half plane variables directly by using the Neumann function given by Mandelstam.[84] For the Neveu-Scharz Model the Jacobi transformation changes variables as indictated diagrammatically by

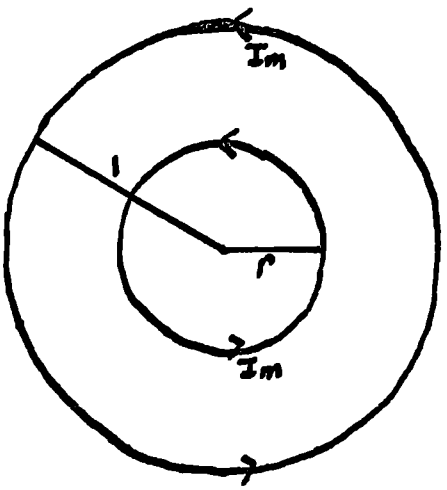


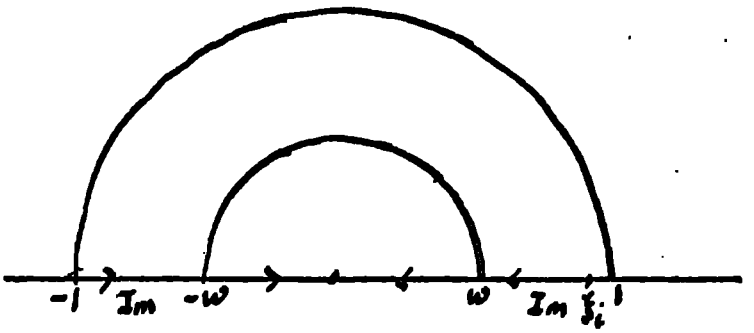
For a variable ξ_j there are image points at $\xi_j\omega^{\pm 1}, \xi_j\omega^{\pm 2}, \ldots\ldots$ and when put in the Neumann function give an infinite sum

$$\sum_{n=-\infty}^{\infty} \frac{(\xi_i\xi_j\omega^n)^{\frac{1}{2}}}{(\xi_i - \xi_j\omega^n)} \qquad , x = \frac{\xi_j}{\xi_i}$$

where a factor $(\xi_i\xi_j\omega^n)^{\frac{1}{2}}$ has been included to give the correct boundary conditions. The positive or negative square root of ω^n give the negative or positive G-parity loops. For particles on different boundaries ξ_j goes to $\xi_j e^{i\pi}$. This then gives the four functions for the Neveu-Schwarz loops.

For the fermion loops the boundary conditions are different giving





and when the image points are taken there appears to be a fermion source at infinity and a antifermion source at zero, so the relevant Neumann function, including the correction factor, is given by

$$\sum_{n=-\infty}^{\infty} \frac{(\xi_i \xi_j)^{\frac{1}{2}} \omega^{n/2} (\mp 1)^n}{2(\xi_i - \xi_j \omega^n)} \left(\left(\frac{\xi_i}{\xi_j \omega^n} \right)^{\frac{1}{2}} + \left(\frac{\xi_j \omega^n}{\xi_i} \right)^{\frac{1}{2}} \right) = \sum_{n=-\infty}^{\infty} \frac{(\mp 1)^n (\xi_i + \xi_j \omega^n)}{2(\xi_i - \xi_j \omega^n)}$$

which gives the correct functions taking the negative sign for positive G-parity and the minus sign for positive G-parity. Again when the particles are on different boundaries ξ_j is replaced by $\xi_j e^{i\pi}$.

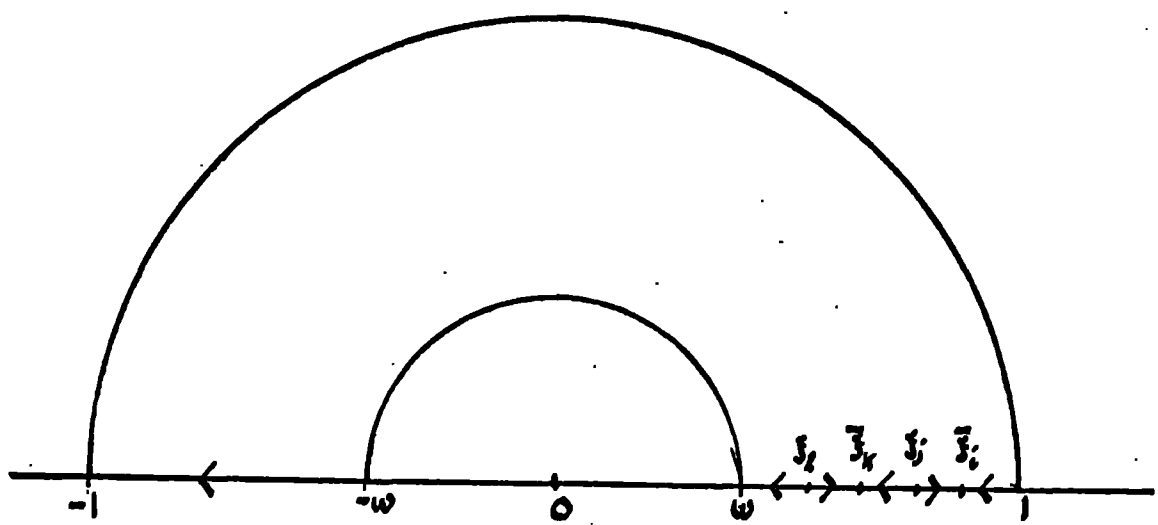
THE PARTITION FUNCTIONS.

Although we have constructed the functions appearing in the integrand of the one loop amplitudes the complete amplitude cannot be written down due to the lack of the partition functions and measure.These have been calculated by Brink and Fairlie[89] using functional techniques. They expanded the spinor fields in terms of anticommuting variables and Fourier series which obeyed the correct boundary conditions. These functions where considered on a rectangle where two opposite edges were identified in such a

manner that the correct conditions were obeyed for either G-parity loop, and the other edges had the correct boundary conditions for the string under consideration. These made it possible to investigate the properties of the pomerons in the Neveu-Schwarz-Ramond Model. They found that the leading even G-parity pomeron with intercept two is present but the leading odd G-parity pomeron with intercept one cancells between the meson and fermion loops. Their results for the partition functions are given in table five.

NEUMANN FUNCTION FOR LOOPS WITH EXTERNAL FERMIONS.

To extend the model further necessitates the inclusion of fermions. The Born term for fermion-antifermion scattering has been given by Mandelstam.[84] So the next step would be the calculation of loop diagrams involving external fermions. The first step in calculating the loop diagrams is the construction of the Neumann function for mesons in the presence of fermions. We shall restrict ourselves to two incoming fermions and two incoming antifermions but the Neumann function can easily be extended to include any number of fermion-antifermion pairs. The Neumann function can be calculated by using the Neumann function for the Born term given by Mandelstam and the procedure used before. On the upper half plane this can be represented by



For a meson at ξ' there are image points at $\xi'\omega^{\pm n}$, $n=1,2,\ldots\infty$ and for the fermions image points at $\bar{\xi}_i\omega^{\pm n}$, $\xi_j\omega^{\pm n}$, $\bar{\xi}_k\omega^{\pm n}$, $\xi_l\omega^{\pm n}$.

Again modifying the Neumann function by the square root factor, gives

$$K(\xi,\xi') = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(\xi\xi')^{\frac{1}{2}}\omega^{\frac{n}{2}}}{2(\xi-\xi'\omega^n)}\left[\left[\prod_{m=-\infty}^{\infty}\left(\frac{(\xi-\bar{\xi}_i\omega^m)(\xi-\bar{\xi}_k\omega^m)(\xi'\omega^n-\xi_j\omega^m)(\xi'\omega^n-\xi_l\omega^m)}{(\xi'\omega^n-\bar{\xi}_i\omega^m)(\xi'\omega^n-\bar{\xi}_k\omega^m)(\xi-\xi_j\omega^m)(\xi-\xi_l\omega^m)}\right)\right]^{\frac{1}{2}}\right.$$

$$\left. + \left[\prod_{m=-\infty}^{\infty}\left(\frac{(\xi-\xi_j\omega^m)(\xi-\xi_l\omega^m)(\xi'\omega^n-\bar{\xi}_i\omega^m)(\xi'\omega^n-\bar{\xi}_k\omega^m)}{(\xi'\omega^n-\xi_j\omega^m)(\xi'\omega^n-\xi_l\omega^m)(\xi-\bar{\xi}_i\omega^m)(\xi-\bar{\xi}_k\omega^m)}\right)\right]^{\frac{1}{2}}\right] .$$

The infinite products can be written in terms of theta functions giving

$$K(\xi,\xi') = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(\xi\xi')^{\frac{1}{2}}\omega^{\frac{n}{2}}}{2(\xi-\xi'\omega^n)}\left[\left[\frac{\theta_1\left(\frac{\xi}{\bar{\xi}_i},\tau\right)\theta_1\left(\frac{\xi}{\bar{\xi}_k},\tau\right)\theta_1\left(\frac{\xi'}{\xi_j},\tau\right)\theta_1\left(\frac{\xi'}{\xi_l},\tau\right)}{\theta_1\left(\frac{\xi'}{\bar{\xi}_i},\tau\right)\theta_1\left(\frac{\xi'}{\bar{\xi}_k},\tau\right)\theta_1\left(\frac{\xi}{\xi_j},\tau\right)\theta_1\left(\frac{\xi}{\xi_l},\tau\right)}\times\right.\right.$$

$$\left.\left.\times\left(\frac{\xi_j\xi_l}{\bar{\xi}_i\bar{\xi}_k}\right)^n\right]^{\frac{1}{2}} + \left[\frac{\theta_1\left(\frac{\xi'}{\xi_i},\tau\right)\theta_1\left(\frac{\xi'}{\bar{\xi}_k},\tau\right)\theta_1\left(\frac{\xi}{\xi_j},\tau\right)\theta_1\left(\frac{\xi}{\xi_l},\tau\right)}{\theta_1\left(\frac{\xi}{\xi_i},\tau\right)\theta_1\left(\frac{\xi}{\bar{\xi}_k},\tau\right)\theta_1\left(\frac{\xi'}{\xi_j},\tau\right)\theta_1\left(\frac{\xi'}{\xi_l},\tau\right)}\left(\frac{\xi_i\bar{\xi}_k}{\xi_j\xi_l}\right)^n\right]^{\frac{1}{2}}\right]$$

(For details of the calculation see appendix .)

As it is easier to work with the theta functions, especially when when using the Jacobi imaginary transformation to consider the behaviour of the pomeron, it would be useful to write the remaining terms as theta functions. This is achieved by noting that [90]

$$\frac{\theta_1(u+v|\tau)\theta_1'(0|\tau)}{\theta_1(u|\tau)\theta_1(v|\tau)} = \cot\pi u\cot\pi v + 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\omega^{nm}\sin 2\pi(nu+mv)$$

which, by changing the variable v to $v+\frac{\tau}{2}+\frac{1}{2}$ and rewriting both sides gives the left hand side in terms of θ_3 functions and the right hand side in terms of a infinite sum.

$$\frac{i}{4}\frac{\theta_3(u+v|\tau)\theta_1'(0|\tau)}{\theta_3(v|\tau)\theta_1(u|\tau)} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n(\zeta\zeta')^{\frac{1}{2}}\omega^{\frac{n}{2}}}{2(\zeta-\zeta'\omega^n)}\left(\frac{\zeta_j\zeta_\ell}{\zeta_i\zeta_k}\right)^{\frac{n}{2}}$$

where $u=\dfrac{\ln(\zeta/\zeta')}{2\pi i}$ and $v=\dfrac{\ln\left(\frac{\zeta_j\zeta_\ell}{\zeta_i\zeta_k}\right)^{\frac{1}{2}}}{2\pi i}$

The Neumann function can now be written in terms of theta functions as

$$K(\zeta,\zeta') = \frac{i}{4}\frac{\theta_3\left(\frac{1}{2}\ln\left(\frac{\zeta}{\zeta'}\right)+\frac{1}{2}\ln\left(\frac{\zeta_j\zeta_\ell}{\zeta_i\zeta_k}\right)^{\frac{1}{2}},\tau\right)\theta_1'(0|\tau)}{\theta_3\left(\frac{1}{2}\ln\left(\frac{\zeta_j\zeta_\ell}{\zeta_i\zeta_k}\right)^{\frac{1}{2}},\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta}{\zeta'}\right),\tau\right)}\left[\frac{\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta}{\zeta_i}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta}{\zeta_k}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta'}{\zeta_j}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta'}{\zeta_\ell}\right),\tau\right)}{\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta'}{\zeta_i}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta'}{\zeta_k}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta}{\zeta_j}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta}{\zeta_\ell}\right),\tau\right)}\right]^{\frac{1}{2}}$$

$$+\frac{i}{4}\frac{\theta_3\left(\frac{1}{2}\ln\left(\frac{\zeta'}{\zeta}\right)+\frac{1}{2}\ln\left(\frac{\zeta_i\zeta_k}{\zeta_j\zeta_\ell}\right)^{\frac{1}{2}},\tau\right)\theta_1'(0|\tau)}{\theta_3\left(\frac{1}{2}\ln\left(\frac{\zeta_i\zeta_k}{\zeta_j\zeta_\ell}\right)^{\frac{1}{2}},\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta'}{\zeta}\right),\tau\right)}\left[\frac{\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta}{\zeta_j}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta}{\zeta_\ell}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta'}{\zeta_i}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta'}{\zeta_k}\right),\tau\right)}{\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta'}{\zeta_j}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta'}{\zeta_\ell}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta}{\zeta_i}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta}{\zeta_k}\right),\tau\right)}\right]^{\frac{1}{2}}$$

It is now obvious how to include more fermion-antifermion pairs, for a fermion at ζ_f and an antifermion at ζ_a the first term is modified by the factors in the bracket being multiplied by

$$\frac{\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta'}{\zeta_f}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta}{\zeta_a}\right),\tau\right)}{\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta}{\zeta_f}\right),\tau\right)\theta_1\left(\frac{1}{2}\ln\left(\frac{\zeta'}{\zeta_a}\right),\tau\right)}$$

and the variables of the θ_3 functions being modified to

$$\left(\frac{\zeta_i\zeta_k\zeta_f}{\zeta_j\zeta_\ell\zeta_a}\right)^{\frac{1}{2}}$$

the second term is similarly altered.

The Neumann function for a fermion-antifermion pair on the other boundary is given by changing, say, $\bar{\xi}_k$ and ξ_1 to $\bar{\xi}_k e^{i\pi}$ and $\xi_1 e^{i\pi}$, noting that if a fermion is on one boundary then there also must be an antifermion on that boundary, and vice versa. These conditions are readily seen by considering the boundary conditions on an annulus and noting that it is impossible to match the boundary conditions otherwise.

If we are just considering mesons then the orientable non-planar loop Neumann functions are given as before, for an even number of mesons on each boundary the function is as given and for an odd number on each boundary ω goes to $\omega e^{i\pi}$. For mesons on different boundaries ξ' goes to $\xi' e^{i\pi}$.

If we consider the case where $\bar{\xi}_i = \xi_j$ and $\bar{\xi}_k = \xi_l$ then the diagram indicates that the function should reduce to the function for a meson loop. All the square root factors vanish giving

$$\frac{i}{2}\,\frac{\theta_2\left(\frac{\xi'}{\xi},\tau\right)\theta_1'(0|\tau)}{\theta_3(1,\tau)\,\theta_1\left(\frac{\xi'}{\xi},\tau\right)}$$

which by using $\theta_3(1,\tau) = \theta_3(0|\tau)$ and $\theta_1'(0|\tau) = \theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau)$

gives

$$\frac{i}{2}\,\theta_2(0|\tau)\theta_4(0|\tau)\,\frac{\theta_3\left(\frac{\xi'}{\xi},\tau\right)}{\theta_1\left(\frac{\xi'}{\xi},\tau\right)}$$

which is just the function for a meson loop.

A similar procedure can be used to odtain the function for a fermion loop. In this case $\xi_j = \bar{\xi}_k$ and $\xi_i = \bar{\xi}_i \omega$, making use of the

infinite number of image points. This gives

$$\frac{i}{4}\frac{\theta_3(\frac{\xi'}{\xi}\omega^{\frac{1}{2}},\tau)\theta_1'(0|\tau)}{\theta_3(\omega^{\frac{1}{2}},\tau)\theta_1(\frac{\xi'}{\xi},\tau)}\left[\frac{\theta_1(\frac{\xi}{\xi_i},\tau)\theta_1(\frac{\xi'}{\xi_i}\omega^{-1},\tau)}{\theta_1(\frac{\xi'}{\xi_i},\tau)\theta_1(\frac{\xi}{\xi_i}\omega^{-1},\tau)}\right]^{\frac{1}{2}}+$$

$$+\frac{i}{4}\frac{\theta_3(\frac{\xi'}{\xi}\omega^{-\frac{1}{2}},\tau)\theta_1'(0|\tau)}{\theta_3(\omega^{-\frac{1}{2}},\tau)\theta_1(\frac{\xi'}{\xi},\tau)}\left[\frac{\theta_1(\frac{\xi}{\xi_i}\omega^{-1},\tau)\theta_1(\frac{\xi'}{\xi_i},\tau)}{\theta_1(\frac{\xi}{\xi_i},\tau)\theta_1(\frac{\xi'}{\xi_i}\omega^{-1},\tau)}\right]^{\frac{1}{2}}$$

$$=\frac{i}{2}\frac{\theta_2(\frac{\xi'}{\xi},\tau)\theta_1'(0|\tau)}{\theta_2(0|\tau)\theta_1(\frac{\xi'}{\xi},\tau)}=\frac{i}{2}\,\theta_3(0|\tau)\theta_4(0|\tau)\frac{\theta_2(\frac{\xi'}{\xi},\tau)}{\theta_1(\frac{\xi'}{\xi},\tau)}$$

just the function for the fermion loop.

The Neumann function can easily be written down in terms of the variables on the annulus by using the Jacobi transformation on the theta functions, and most of the additional factors given cancel with each other giving

$$\frac{i}{4\tau}\frac{\theta_3(\frac{z'}{z}(\frac{z_j z_\ell}{z_i z_k})^{\frac{1}{2}},\tau')\theta_1'(0|\tau')}{\theta_3((\frac{z_j z_\ell}{z_i z_k})^{\frac{1}{2}},\tau')\theta_1(\frac{z'}{z},\tau')}\left[\frac{\theta_1(\frac{z}{z_i},\tau')\theta_1(\frac{z}{z_k},\tau')\theta_1(\frac{z'}{z_j},\tau')\theta_1(\frac{z'}{z_\ell},\tau')}{\theta_1(\frac{z'}{z_i},\tau')\theta_1(\frac{z'}{z_k},\tau')\theta_1(\frac{z}{z_j},\tau')\theta_1(\frac{z}{z_\ell},\tau')}\right]^{\frac{1}{2}}+$$

$$+\frac{i}{4\tau}\frac{\theta_3(\frac{z'}{z}(\frac{z_i z_k}{z_j z_\ell})^{\frac{1}{2}},\tau')\theta_1'(0|\tau')}{\theta_3((\frac{z_i z_k}{z_j z_\ell})^{\frac{1}{2}},\tau')\theta_1(\frac{z'}{z},\tau')}\left[\frac{\theta_1(\frac{z}{z_i},\tau')\theta_1(\frac{z}{z_k},\tau')\theta_1(\frac{z'}{z_j},\tau')\theta_1(\frac{z'}{z_\ell},\tau')}{\theta_1(\frac{z'}{z_i},\tau')\theta_1(\frac{z'}{z_k},\tau')\theta_1(\frac{z}{z_j},\tau')\theta_1(\frac{z}{z_\ell},\tau')}\right]^{\frac{1}{2}},$$

$$u'=\frac{\ln(\frac{z'}{z})}{2\pi i},\quad u'=\frac{u}{\tau},\quad u=\frac{\ln(\frac{\xi'}{\xi})}{2\pi i}$$

$$v'=\frac{\ln(\frac{z_j z_k}{z_i z_\ell})^{\frac{1}{2}}}{2\pi i},\quad v'=\frac{v}{\tau},\quad v=\frac{\ln(\frac{\xi_j \xi_k}{\xi_i \xi_\ell})^{\frac{1}{2}}}{2\pi i}$$

THE PARTITION FUNCTION.

In order to write down the amplitudes using the Neumann function involving external fermions it is necessary to know the partition functions used. The methods used by Brink and Fairlie[87] for meson and fermion loops cannot be extened to cover this case due to the presence of fermion and antifermion sources on the boundaries, which means that the function S_α cannot be written in terms of a Fourier series.

One way it might be hoped to solve for the partition function is by taking a net on a rectangle and defining a function at the intersections of the net and then solving this with the appropriate boundary conditions and then taking the limit when the net becomes infinitly fine. This method is set up by writing $S_1 = \phi + i\psi$ and $S_2 = \phi - i\psi$, when these are put in the Lagrangian

$$S_1\left(\frac{\overleftrightarrow{\partial}}{\partial x} + i\frac{\overleftrightarrow{\partial}}{\partial y}\right)S_1 - \overleftrightarrow{S_2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\overleftrightarrow{S_2}$$

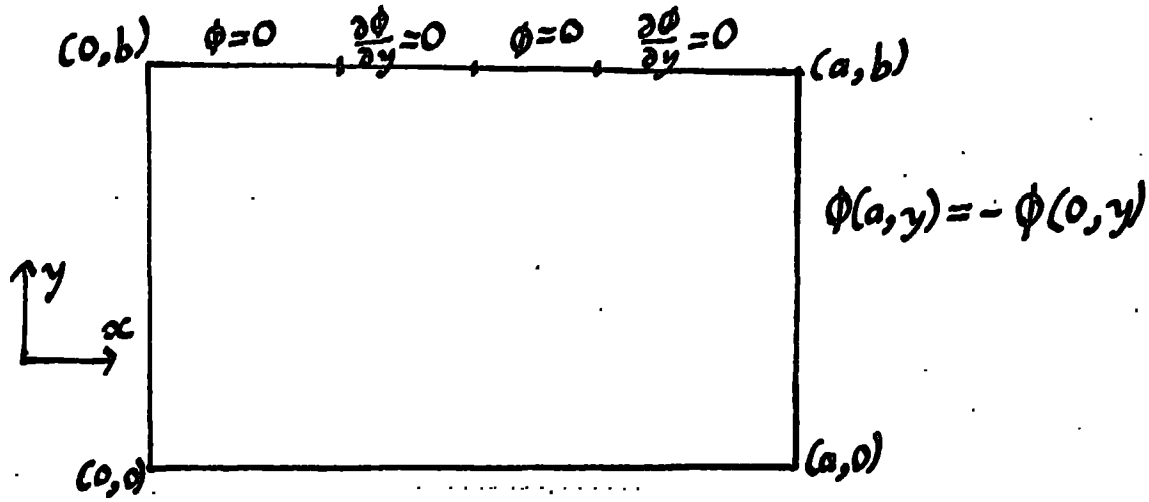
the following conditions are given

$$\frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} \quad , \quad \frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y}$$

which means that

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad , \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \; .$$

Considering the first equation and the boundary conditions necessary for fermion and antifermion source terms this gives the boundary conditions for the rectangle as



This, in terms of the annulus, means that the fermion antifermion pairs are on the same boundary. The condition $\phi(a,y) = -\phi(0,y)$ is just what is needed to give the functions on an annulus and is related to the relation $S_\alpha(\theta+2\pi) = -S_\alpha(\theta)$, for odd G-parity exchanges this would be replaced by $\phi(a,y) = \phi(0,y)$. The partition function is given in terms of the eigenvalues of the function $\phi(x,y)$ so the equation that has to be solved on the net is the matrix equation

$$-\phi_{i,j-1} - \phi_{i,j+1} - \phi_{i-1,j} - \phi_{i+1,j} + 4\phi_{ij} = \lambda\phi_{ij}$$

where ϕ_{ij} is defined at the point $x = i\Delta x$ and $y = j\,\Delta y$ where Δx and Δy give the size of the mesh. The above equation is obtained from

$$-\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} = -\lambda\phi$$

The partition function is given by the square root of the product of eigenvalues which is just the determinant of a diagonal matrix with enteries which are the eigenvalues. In fact the partition function is given in terms of the determinant of any matrix which is similar to this

$$D_A = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}, \quad |D_A| = |B^{-1}AB| = |ABB^{-1}| = |A|$$

Unfortunately it is not possible , so far, to solve this problem in a closed form, which is necessary to be able to obtain the partition function, due to the necessity of the change in the boundary conditions at fermion and antifermion sources.

Although we cannot calculate the partition function we can try and guess what it might be. Remembering that for the Neumann function when $\xi_i = \xi_j$ and $\xi_k = \xi_l$ the Neumann function for the meson loop is given and when $\xi_j = \xi_k$ and $\xi_l = \xi_i \omega$ the Neumann function for the fermion loop is given then it seems reasonable to assume that under the same conditions the partition functions for the meson and fermion loops are given. Calling the partition function required $\phi_F^+(r^2)$ then a reasonable guess seems to be

$$\phi_F^+(r^2) f(r^2) = 2^{\frac{1}{2}} \left[\frac{\theta_3\left(\left(\frac{z_j z_l}{z_i z_k} \right)^{\frac{1}{2}}, \tau' \right)}{\theta_1'(0|\tau')} \right]^{\frac{1}{2}}$$

For the meson loop $\left(\frac{\xi_j \xi_l}{\xi_i \xi_k} \right)^{\frac{1}{2}} = 1$ which when transformed into the z variables gives $\left(\frac{z_j z_l}{z_i z_k} \right)^{\frac{1}{2}} = 1$, and for the fermion loop $\left(\frac{\xi_j \xi_l}{\xi_i \xi_k} \right)^{\frac{1}{2}} = \omega^{\frac{1}{2}}$ which gives $\left(\frac{z_j z_l}{z_i z_k} \right)^{\frac{1}{2}} = (-1)$.

For the meson loop this gives

$$2^{\frac{1}{2}} \left[\frac{\theta_3(0|\tau')}{\theta_1'(0|\tau')} \right]^{\frac{1}{2}} = 2^{\frac{1}{2}} \left[\frac{\theta_3(0|\tau)}{\theta_1'(0|\tau)} \right]^{\frac{1}{2}} \left(\frac{-2\pi}{\ln \omega} \right)^{\frac{1}{2}}$$

and for the fermion loop

$$2^{\frac{1}{2}}\left[\frac{\theta_3(-1,\tau')}{\theta_1'(0|\tau')}\right]^{\frac{1}{2}} = 2^{\frac{1}{2}}\left[\frac{\theta_4(0|\tau')}{\theta_1'(0|\tau')}\right]^{\frac{1}{2}}$$

$$= 2^{\frac{1}{2}}\left(\frac{-2\pi}{\ln w}\right)^{\frac{1}{2}}\left[\frac{\theta_3(w^{\frac{1}{2}},\tau')}{\theta_1'(0|\tau')}\right]^{\frac{1}{2}} \exp\left(\frac{\ln w}{16}\right) = 2^{\frac{1}{2}}\left(\frac{-\pi 2}{\ln w}\right) w^{\frac{1}{16}}\left[\frac{\theta_3(\frac{\tau}{2}|\tau)}{\theta_1'(0|\tau)}\right]^{\frac{1}{2}}$$

$$= 2^{\frac{1}{2}}\left(\frac{-2\pi}{\ln w}\right)\left[\frac{\theta_2(0|\tau)}{\theta_1'(0|\tau)}\right]^{\frac{1}{2}}$$

These results then agree with the partition functions for the meson and fermion loops (see table five) and hence the guess seems reasonable. This then gives for the partition function

$$\Phi_F^+(r^2) = \frac{2^{\frac{1}{6}}\left[\theta_3\left(\left(\frac{z_j z_l}{\bar{z}_i \bar{z}_k}\right)^{\frac{1}{2}},\tau\right)\right]^{\frac{1}{2}}}{\left[\theta_1'(0|\tau')\right]^{\frac{1}{6}}}$$

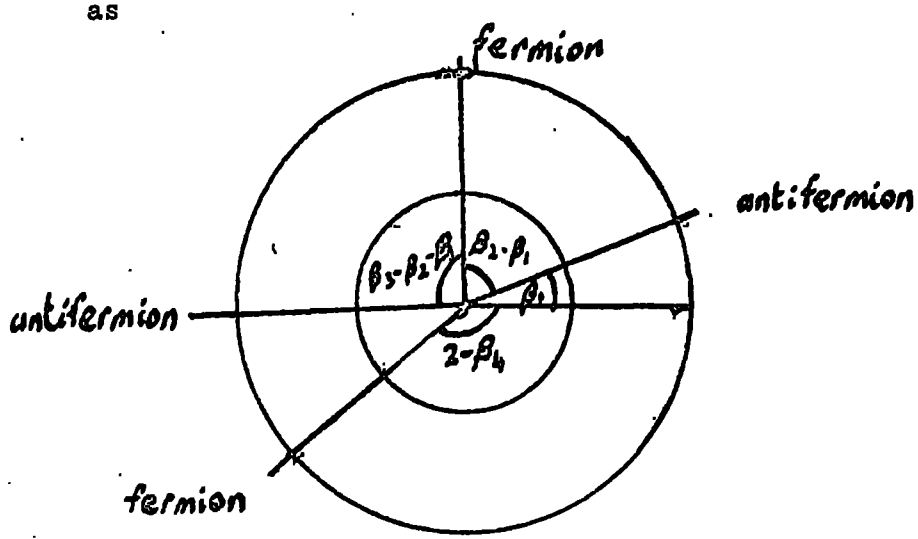
$$= r^{-\frac{1}{24}}\prod_{n=1}^{\infty}\left(1+r^{2n-1}\left(\frac{z_j z_l}{\bar{z}_i \bar{z}_k}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\left(1+r^{2n-1}\left(\frac{\bar{z}_i \bar{z}_k}{z_j z_l}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$= 2^{\frac{1}{6}}\frac{\left[\theta_3\left(\left(\frac{\xi_j \xi_l}{\bar{\xi}_i \bar{\xi}_k}\right)^{\frac{1}{2}},\tau\right)\right]^{\frac{1}{2}}}{\left[\theta_1'(0|\tau)\right]^{\frac{1}{6}}} \exp\left(\frac{i\pi v^2}{2\tau}\right)$$

$$v = \frac{\ln\left(\frac{\xi_j \xi_0}{\bar{\xi}_i \bar{\xi}_k}\right)^{\frac{1}{2}}}{2\pi i}$$

ONE LOOP AMPLITUDE WITH EXTERNAL FERMIONS.

To calculate the one loop amplitudes with four external fermions on the same boundary we follow the method of Mandelstam[84] which he used to calculate the Born term. The Neumann functions used are those on the annulus and the variables are taken to be the angles the sources make on the annulus. Diagrammatically this is represented as



The Neumann functions are given by

$$K^{+-}(\beta_i,\beta_j)=\frac{i}{2}\,\frac{\theta_3(\beta_j-\beta_i+\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4)|\tau')}{\theta_3(\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4)|\tau')}\,\frac{\theta_1'(0|\tau')}{\theta_1(\beta_j-\beta_i|\tau')}\times$$

$$\times\left[\frac{\theta_1(\beta_j-\beta_1|\tau')\theta_1(\beta_j-\beta_3|\tau')\theta_1(\beta_i-\beta_2|\tau')\theta_1(\beta_i-\beta_4|\tau')}{\theta_1(\beta_i-\beta_1|\tau')\theta_1(\beta_i-\beta_3|\tau')\theta_1(\beta_j-\beta_2|\tau')\theta_1(\beta_j-\beta_4|\tau')}\right]^{\frac{1}{2}}$$

$$K^{-+}(\beta_j,\beta_i)=-K^{+-}(\beta_i,\beta_j)$$

These are obtained from the Neumann functions on the upper half plane by the Jacobi transformation and multiplying by τ to cancel the factor given by the transformation. To do the calculation the annulus has to be mapped onto a strip on the upper half plane.

On this strip the lengths of two of the strings are taken to be infinitesimally small, if the length of string i is α_i then

$$\alpha = \alpha_1 = -\alpha_3 \ , \quad \alpha_2 = -\alpha_3 = \delta\alpha$$

where δ is small. The points where the strings join together ρ_α and ρ_β on the strip correspond to the points on the annulus β_α and β_β which are taken to be, to the accuracy required ,

$$\beta_\alpha = \beta_2 + \delta A \ , \quad \beta_\beta = \beta_3 + \delta B$$

To be able to do the calculation it is necessary to know the form of the conformal map from the annulus to the strip. The conformal map[93]

$$k^{\frac{1}{2}} sn\left(\frac{2iK}{\pi} \ln\frac{\nu}{r} + K ; r^2\right) = \frac{\theta_2\left(\frac{i}{\pi}\ln\frac{\nu}{r} \middle| \tau''\right)}{\theta_3\left(\frac{i}{\pi}\ln\frac{\nu}{r} \middle| \tau''\right)}$$

where $\nu = c e^{i\pi\theta}$ and where $\tau'' = \frac{2\ln r}{\pi i}$

maps an annulus onto a disc with a slit in it and the map

$$\rho = \ln\left(\left(1 - \frac{\theta_2\left(\frac{i}{\pi}\ln\frac{\nu}{r} \middle| \tau''\right)}{\theta_3\left(\frac{i}{\pi}\ln\frac{\nu}{r} \middle| \tau''\right)}\right) \Big/ \left(1 + \frac{\theta_2\left(\frac{i}{\pi}\ln\frac{\nu}{r} \middle| \tau''\right)}{\theta_3\left(\frac{i}{\pi}\ln\frac{\nu}{r} \middle| \tau''\right)}\right)\right)$$

maps the annulus onto a strip. So the mapping required is given by

$$\rho = \sum_{s=1}^{4} \alpha_s \ln\left(\frac{\left(1 - \frac{\theta_2\left(\frac{i}{\pi}\ln\left(\frac{c}{r}\right) + (\beta - \beta_s) \middle| \tau''\right)}{\theta_3\left(\frac{i}{\pi}\ln\left(\frac{c}{r}\right) + (\beta - \beta_s) \middle| \tau''\right)}\right)}{\left(1 + \frac{\theta_2\left(\frac{i}{\pi}\ln\left(\frac{c}{r}\right) + (\beta - \beta_s) \middle| \tau''\right)}{\theta_3\left(\frac{i}{\pi}\ln\left(\frac{c}{r}\right) + (\beta - \beta_s) \middle| \tau''\right)}\right)}\right)$$

when the variable is on the outer boundary of the annulus this reduces to

$$\rho = \sum_{s=1}^{4} a_s \ln \tan \pi\left(\frac{\beta-\beta_s}{2}\right)$$

As two of the strings are infinitesimally small, which means that the points where the strings join are close to β_2 and β_3, the above transformation is taken as being sufficiently accurate for the purposes of this calculation. The points where the strings join are then given by

$$\frac{\partial \rho}{\partial \beta} = 0$$

$$\frac{\partial \rho}{\partial \beta} = \frac{1}{\sin \pi(\beta-\beta_1)} + \frac{\delta}{\sin \pi(\beta-\beta_2)} - \frac{\delta}{\sin \pi(\beta-\beta_3)} - \frac{1}{\sin \pi(\beta-\beta_4)}$$

this then gives the solutions

$$A = \frac{\sin \pi(\beta_1-\beta_2)\sin \pi(\beta_2-\beta_4)}{\sin \pi(\beta_1-\beta_2)+\sin \pi(\beta_2-\beta_4)}, \quad B = -\frac{\sin \pi(\beta_1-\beta_3)\sin \pi(\beta_3-\beta_4)}{\sin \pi(\beta_1-\beta_3)+\sin \pi(\beta_3-\beta_4)}$$

This then gives sufficient accuracy to calculate, in Mandelstam's notation, ϕ_i where

$$\phi_1 = \langle ++|T|++\rangle \qquad \phi_2 = \langle --|T|++\rangle$$

$$\phi_3 = \langle +-|T|+-\rangle \qquad \phi_4 = \langle +-|T|-+\rangle$$

where the plus and minus signs stand for positive and negative helicity states of the particles $\langle 34|T|21\rangle$

The Neumann functions are calculated in terms of the variables on the annulus but have to be multiplied by a factor involved in the transformation from the strip to the annulus, so the Neumann functions are of the form

$$\left(\frac{\partial \rho_i}{\partial \beta_i}\right)^{-\frac{1}{2}}\left(\frac{\partial \rho_j}{\partial \beta_j}\right)^{-\frac{1}{2}} K^{+-}(\beta_i,\beta_j)$$

If the Neumann function involves a factor of $\left(\frac{\partial \rho_\alpha}{\partial \beta_\alpha}\right)^{-\frac{1}{2}}$ or $\left(\frac{\partial \rho_\beta}{\partial \beta_\beta}\right)^{-\frac{1}{2}}$ then these cancel with factors from the remainding terms and, therefore do not have to be calculated explicitly. The other factors give rise to a term which either cancels with a similar factor from the Neumann function or give a zero, as in the Born term.

For example

$$\left(\frac{\partial \rho_2}{\partial \beta_2}\right)^{-\frac{1}{2}}\left(\frac{\partial \rho_1}{\partial \beta_1}\right)^{-\frac{1}{2}} K^{-+}(\beta_2,\beta_1) = \left(\frac{\partial \rho_2}{\partial \beta_\beta}\right)^{-\frac{1}{2}} \left(\sin\pi(\beta_1-\beta_1)\right)^{\frac{1}{2}} \times \left(\sin\pi(\beta_1-\beta_1)\right)^{-\frac{1}{2}}$$

$$\times \frac{i}{2}\frac{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2-\beta_3+\beta_4)|\tau'\right)\theta_1'(0|\tau')}{\theta_3\left(\frac{1}{2}(\beta_2-\beta_1-\beta_3+\beta_4)|\tau'\right)\theta_1(\beta_1-\beta_2|\tau')}\left[\frac{\theta_1(\beta_2-\beta_1|\tau')\theta_1(\beta_2-\beta_3|\tau')\theta_1(\beta_1-\beta_2|\tau')\theta_1(\beta_1-\beta_4|\tau')}{\theta_1(\delta A|\tau')\theta_2(\beta_2-\beta_4|\tau')\theta_1(\beta_1-\beta_3|\tau')\theta_1'(0|\tau')}\right]^{\frac{1}{2}}$$

$$= \frac{i}{2}\left(\frac{\partial \rho_2}{\partial \beta_2}\right)^{-\frac{1}{2}}\delta^{-\frac{1}{2}}A^{-\frac{1}{2}}\frac{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2-\beta_3+\beta_4)|\tau'\right)}{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4)|\tau'\right)}\left[\frac{\theta_1(\beta_2-\beta_3|\tau')\theta_1(\beta_1-\beta_4|\tau')}{\theta_1(\beta_4-\beta_2|\tau')\theta_1(\beta_1-\beta_3|\tau')}\right]^{\frac{1}{2}}$$

This is expanded to the necessary order of δ to give the amplitudes ϕ_1, ϕ_2, and ϕ_3, the other Neumann functions involved with these amplitudes have to be expanded to similar order. For the amplitude ϕ_4 the Neumann functions have to be expanded to a further order of δ as when the Neumann functions are written in the amplitude the

lowest order cancels. Although it. is necessary to expand the points where the strings join to second order in δ for the separate Neumann functions it is not necessary to calculate them explicitly as the terms in the amplitude involving this second order correction cancel. The Neumann functions used are given, to appropriate order, in table six.

The term in the functional integration which involves the points where the strings join together is

$$G_1(\rho) = S_1^i(\rho)\left(\frac{\partial}{\partial\tau} + i\frac{\partial}{\partial\sigma}\right)x^i(\rho)$$

where $S_1^i(\rho)$ is the field due to the spin and $x^i(\rho)$ is the usual variable of the relativistic string. At the ends of the string the term $\frac{\partial x^i}{\partial\sigma}$ vanishes. The terms $\frac{\partial x^i}{\partial\tau}$ give rise to a term

$$\frac{\partial}{\partial\tau}\sum_r p_r^i N(\rho,\rho_r)$$

where p_r is the momentum of particle r and $N(\rho,\rho_r)$ is the Neumann function for the orbital modes , that is the Neumann function for the ordinary relativistic string. In this case

$$N(\beta_i,\beta_j) = \ln V(\beta_i,\beta_j) = \ln\frac{\theta_1(\beta_j-\beta_i|\tau')}{\theta_1'(0|\tau')}$$

The terms $\frac{\partial x^i}{\partial\tau}$ can be contracted in pairs to give $(d-2)\frac{\partial^2}{\partial\tau\partial\tau'}N(\rho,\rho')$ but such terms do not contribute in this particular calculation. The terms $S_1^i(\rho)$ are contracted in pairs with the similar terms arising from the Lagrangian giving the Neumann functions all readily discussed. The terms involved are $S_1^i(\rho_2), S_1^i(\rho_\beta)$ and the terms where the helicity states of the particles are full. Positive helicity states for fermions are taken to be empty and negative helicity states are taken to be full, for the antifermion

negative helicity states are empty and positive helicity states full. As we are following Mandelstam we are working in four dimensions. The amplitudes also involves terms from the partition function, from the orbital terms, and from a term Γ which is used to restore projective invariance, we consider these later.

The parts of the amplitude which are given from the Neumann functions are denoted by F_i. For the amplitude Φ_3 all the helicity states are empty so the only contribution comes from

$$G(\rho_\alpha)G(\rho_\beta) = S_1^i(\rho_\beta)\, S_1^j(\rho_\alpha)\frac{\partial x^i(\rho_\beta)}{\partial \tau}\frac{\partial x^j(\rho_\alpha)}{\partial \tau}$$

and rewritten in terms of helicity combinations the contraction of

$$S_1^+(\rho_\beta)\, S_1^-(\rho_\alpha) \qquad \text{gives} \quad -\tfrac{1}{2}\left(\frac{\partial \rho_\alpha}{\partial \beta_\alpha}\right)^{-\frac{1}{2}}\left(\frac{\partial \rho_\beta}{\partial \beta_\beta}\right)^{-\frac{1}{2}} K^{+-}(\beta_\beta, \beta_\alpha)$$

The remaining factors $\left(\frac{\partial x^{(-)}(\rho_\alpha)}{\partial \tau}\right)\left(\frac{\partial x^{(+)}(\rho_\beta)}{\partial \tau}\right)$ give a term

$$\left\{\frac{\partial}{\partial \tau_\beta}\sum_{r=1}^{4} \rho_r^{(-)} N(\beta_\beta, \beta_r)\right\}\left\{\frac{\partial}{\partial \tau_\alpha}\sum_{r=1}^{4} \rho_r^{(+)} N(\beta_\alpha, \beta_r)\right\}$$

where $\rho_r^{(\pm)} = \frac{1}{\sqrt{2}}(\rho_r^x \pm \rho_r^y)$ (see table seven)

This gives to dominant order in δ

$$-\frac{t}{2}\left(\frac{\partial \rho_\alpha}{\partial \beta_\alpha}\right)^{-1}\left(\frac{\partial \rho_\beta}{\partial \beta_\beta}\right)^{-1}\delta^{-2}(1-A)(1-B)$$

The same term also appears in Φ_3 and Φ_4. For Φ_2 the terms to be contracted which correspond to these are $\frac{\partial x^{(-)}(\rho_\alpha)}{\partial \tau}$ and $\frac{\partial x^{(-)}(\rho_\beta)}{\partial \tau}$

which give

$$\left\{\frac{\partial}{\partial\tau_\rho}\sum_{r=1}^{4}\rho_r^{(-)}N(\beta_\rho,\beta_r)\right\}\left\{\frac{\partial}{\partial\tau_\alpha}\sum_{r=1}^{4}\rho_r^{(-)}N(\beta_\beta,\beta_r)\right\}=$$

$$=\delta^{-1}\left(\frac{\partial\rho_\alpha}{\partial\beta_\alpha}\right)^{-1}\left(\frac{\partial\rho_\beta}{\partial\beta_\beta}\right)^{-1}(1-B)\left(-\frac{t}{4}\frac{\theta_1'(\beta_1-\beta_1|\tau')}{\theta_1(\beta_2-\beta_1|\tau')}-\frac{(2s+t)}{4}(1-A)+\right.$$

$$\left.+\frac{t}{2}\frac{\theta_1'(\beta_2-\beta_3|\tau')}{\theta_1(\beta_2-\beta_3|\tau')}-\frac{t}{4}\frac{\theta_1'(\beta_2-\beta_4|\tau')}{\theta_1(\beta_2-\beta_4|\tau')}\right)$$

where $\theta_i'(u)=\frac{\partial}{\partial(\pi u)}\theta_i(u|\tau')$

For Φ_1 the expression to be contracted is

$$2\alpha\, S_1^-(\rho_4)\,G(\rho_\beta)\,G(\rho_\alpha)\,S^+(\rho_1)$$

as the helicity states of particles one and four are full, this then gives for the spin field terms

$$\frac{\alpha}{2}\left(\frac{\partial\rho_\alpha}{\partial\beta_\alpha}\right)^{\frac{1}{2}}\left(\frac{\partial\rho_\beta}{\partial\beta_\beta}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_1}{\partial\beta_1}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_4}{\partial\beta_4}\right)^{-\frac{1}{2}}\left(K^{-+}(\beta_\alpha,\beta_\beta)K^{+-}(\beta_1,\beta_4)-\right.$$

$$\left.K^{-+}(\beta_\alpha,\beta_1)K^{+-}(\beta_\beta,\beta_4)+K^{+-}(\beta_\alpha,\beta_4)K^{-+}(\beta_\beta,\beta_1)\right)$$

The last term in this expression does not contribute. Similarly for Φ_2 and Φ_4 the terms which are non zero are given by, for

Φ_2

$$-\frac{i}{2}\alpha\,\delta^{\frac{1}{2}}\left(\frac{\partial\rho_\alpha}{\partial\beta_\alpha}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_\beta}{\partial\beta_\beta}\right)^{\frac{1}{2}}\left(\frac{\partial\rho_2}{\partial\beta_2}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_4}{\partial\beta_4}\right)^{-\frac{1}{2}}K^{+-}(\beta_\alpha,\beta_2)K^{+-}(\beta_\beta,\beta_4)$$

and for ϕ_4 by

$$\frac{i}{2}\alpha\delta^{\frac{1}{2}}\left(\frac{\partial\rho_\alpha}{\partial\beta_\alpha}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_\beta}{\partial\beta_\beta}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_4}{\partial\beta_4}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_3}{\partial\beta_3}\right)^{-\frac{1}{2}}\Big\{K^{-+}(\beta_\alpha,\beta_\beta)K^{+-}(\beta_3,\beta_4)+$$

$$+K^{-+}(\beta_\alpha,\beta_3)K^{+-}(\beta_\beta,\beta_4)\Big\}$$

When we multiply by the Lorentz transformation factors

$$h_1=-\frac{\delta s}{t}\ ,\ h_2=-1\ ,\ h_3=-\delta\frac{(s+t)}{t}\ ,\ h_4=1$$

these give

$$h_1F_1=\frac{s}{2^4}\left(\frac{\partial\rho_\alpha}{\partial\beta_\alpha}\right)^{-\frac{3}{2}}\left(\frac{\partial\rho_\beta}{\partial\beta_\beta}\right)^{-\frac{3}{2}}\frac{\delta^{-2}(1-A)(1-B)}{(AB)^{\frac{1}{2}}}\times$$

$$\times\left[\left[\frac{\theta_2\left(\frac{1}{2}(\beta_1+\beta_2-\beta_3-\beta_4)\middle|\tau'\right)}{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4)\middle|\tau'\right)}\right]^2\left[\frac{\theta_1(\beta_1-\beta_2|\tau')\,\theta_1(\beta_3-\beta_4|\tau')}{\theta_1(\beta_1-\beta_3|\tau')\,\theta_1(\beta_2-\beta_4|\tau')}\right]+\right.$$

$$\left.+\left[\frac{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2-\beta_3+\beta_4)\middle|\tau'\right)}{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4)\middle|\tau'\right)}\right]^2\left[\frac{\theta_1(\beta_1-\beta_4|\tau')\,\theta_1(\beta_2-\beta_3|\tau')}{\theta_1(\beta_2-\beta_4|\tau')\,\theta_1(\beta_1-\beta_3|\tau')}\right]\right]$$

$$h_2 F_2 = \frac{\delta^{-2}}{2^3}\left(\frac{\partial \rho_\alpha}{\partial \beta_\alpha}\right)^{-\frac{3}{2}}\left(\frac{\partial \rho_\beta}{\partial \beta_\beta}\right)^{-\frac{3}{2}}\frac{(1-A)}{(AB)^{\frac{1}{2}}}\frac{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2+\beta_4-\beta_3)|\tau'\right)}{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4)|\tau'\right)} \times$$

$$\times\left[\frac{\theta_1(\beta_1-\beta_4|\tau')\theta_1(\beta_3-\beta_2|\tau')}{\theta_1(\beta_1-\beta_3|\tau')\theta_1(\beta_2-\beta_4|\tau')}\right]^{\frac{1}{2}}\left(-\frac{t}{4}\frac{\theta_1'(\beta_2-\beta_1|\tau')}{\theta_1(\beta_2-\beta_1|\tau')}-\frac{(2s+t)}{4}(1-A)+\right.$$

$$\left.+\frac{t}{2}\frac{\theta_1'(\beta_2-\beta_3|\tau')}{\theta_1(\beta_2-\beta_3|\tau')}-\frac{t}{4}\frac{\theta_1'(\beta_2-\beta_4|\tau')}{\theta_1(\beta_2-\beta_4|\tau')}\right)$$

$$h_3 F_3 = \delta^{-2}\frac{(s+t)}{2^2}\left(\frac{\partial \rho_\alpha}{\partial \beta_\alpha}\right)^{-\frac{3}{2}}\left(\frac{\partial \rho_\beta}{\partial \beta_\beta}\right)^{-\frac{3}{2}}\frac{(1-A)(1-B)}{(AB)^{\frac{1}{2}}}\frac{\theta_3\left(\frac{1}{2}(\beta_1+\beta_2-\beta_3-\beta_4)|\tau'\right)}{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4|\tau'\right)} \times$$

$$\times\left[\frac{\theta_1(\beta_1-\beta_2|\tau')\theta_1(\beta_3-\beta_4|\tau')}{\theta_1(\beta_1-\beta_3|\tau')\theta_1(\beta_2-\beta_4|\tau')}\right]^{\frac{1}{2}}$$

$$h_4 F_4 = \delta^{-2}\frac{(-t)}{2^3}\left(\frac{\partial \rho_\alpha}{\partial \beta_\alpha}\right)^{-\frac{3}{2}}\left(\frac{\partial \rho_\beta}{\partial \beta_\beta}\right)^{-\frac{3}{2}}\left(\frac{B}{A}\right)^{\frac{1}{2}}(1-A)(1-B)\frac{\theta_3\left(\frac{1}{2}(\beta_1+\beta_2-\beta_3-\beta_4)|\tau'\right)}{\left[\theta_3\left(\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4)|\tau'\right)\right]^2} \times$$

$$\times\frac{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2-\beta_3+\beta_4)|\tau'\right)}{\theta_1(\beta_2-\beta_4|\tau')\,\theta_1(\beta_1-\beta_3|\tau')}\left[\theta_1(\beta_1-\beta_2|\tau')\theta_1(\beta_3-\beta_4|\tau')\theta_1(\beta_1-\beta_4|\tau')\theta_1(\beta_3-\beta_2|\tau')\right]^{\frac{1}{2}}$$

$$\times\left[\frac{\theta_1'(\beta_4-\beta_3|\tau')}{\theta_1(\beta_4-\beta_3|\tau')}+\frac{\theta_1'(\beta_3-\beta_2|\tau')}{\theta_1(\beta_3-\beta_2|\tau')}-\frac{1}{2}\frac{\theta_3'\left(\frac{1}{2}(\beta_1+\beta_2-\beta_3-\beta_4)|\tau'\right)}{\theta_3\left(\frac{1}{2}(\beta_1+\beta_2-\beta_3-\beta_4)|\tau'\right)}-\frac{1}{2}\frac{\theta_3'\left(\frac{1}{2}(\beta_1-\beta_2-\beta_3+\beta_4)|\tau'\right)}{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2-\beta_3+\beta_4)|\tau'\right)}\right]$$

The term Γ, which ensures projective invariance, contains factors of $\left(\frac{\partial \rho_\alpha}{\partial \beta_\alpha}\right)^{3/2}$ and $\left(\frac{\partial \rho_\beta}{\partial \beta_\beta}\right)^{3/2}$ which cancel with similar terms from $h_i F_i$. Γ must also produce a term of δ^2 to cancel with the term δ^{-2} in the functions $h_i F_i$. In the Born term Γ is constructed from terms of the form $(Z_i - Z_j)$ which correspond in this case to terms of the form $\tan \pi (\beta_i - \beta_j)/2$. So Γ is given by

$$\Gamma = \prod_\alpha \left(\frac{\partial \rho_\alpha}{\partial \beta_\alpha}\right)^{3/2} \prod_{\alpha<\beta} \left(\tan \pi \frac{(\beta_\alpha - \beta_\beta)}{2}\right)^{-1} \prod_{r<s} \left(\tan \pi \frac{(\beta_r - \beta_s)}{2}\right)^{-2} \prod_{\alpha,r} \tan \pi \frac{(\beta_\alpha - \beta_r)}{2} \cdot$$

$$\times \prod_r \tan \pi \frac{(\beta_r - \beta_{r+1})}{2}$$

where the product $\prod_{r<s}$ is for β_r a fermion source and β_s an antifermion source or vice versa. This reduces to

$$\Gamma = \left(\frac{\partial \rho_\alpha}{\partial \beta_\alpha}\right)^{3/2} \left(\frac{\partial \rho_\beta}{\partial \beta_\beta}\right)^{3/2} \delta^2 \frac{AB}{4} \frac{\tan \pi \frac{(\beta_2 - \beta_4)}{2} \tan \pi \frac{(\beta_1 - \beta_3)}{2}}{\tan \pi \frac{(\beta_1 - \beta_4)}{2}}$$

The partition function is given by

$$\left(2^{\frac{1}{2}} \left[\frac{\theta_3\left(\frac{1}{2}(\beta_1 - \beta_2 + \beta_3 - \beta_4)|\tau'\right)}{\theta_1'(0|\tau')}\right]^{\frac{1}{2}}\right)^{d-2} = 2\left[\frac{\theta_3\left(\frac{1}{2}(\beta_1 - \beta_2 + \beta_3 - \beta_4)|\tau'\right)}{\theta_1'(0|\tau')}\right]$$

and the orbital mode term is given by

$$\prod_{i<j} \Psi(\beta_i, \beta_j)^{-2 \rho_i \cdot \rho_j}$$

where

$$\Psi(\beta_i, \beta_j) = \frac{\theta_1(\beta_i - \beta_j|\tau')}{\theta_1'(0|\tau')}$$

The variables β_i are integrated over from zero to two, and the variable r is integrated over zero to one with the measure $1/r$ where $\tau' = \frac{\ln r}{\pi i}$. The amplitudes are then given by

$$\phi_i = \int_0^1 \frac{dr}{r} \int_0^2 \prod_{i=1}^{4} d\beta_i \prod_{i<j} \Psi(\beta_i, \beta_j)^{-2\rho_i \cdot \rho_j} 2\left[\frac{\theta_3(\frac{1}{2}(\beta_1 - \beta_2 + \beta_3 - \beta_4)|\tau')}{\theta_1'(0|\tau')}\right] \times$$

$$\times \Gamma h_i F_i$$

CONCLUSION.

The formulation of the Neveu-Schwarz and Ramond Models in terms of functional integration is an extremely useful method and once the Neumann functions have been calculated it is possible to calculate one loop amplitudes of both fermion and meson loops. For the one loop amplitude with external fermions there is an uncertainty over the partition function but apart from this the amplitude can be calculatedand we have presented this amplitude above in the same manner as Mandelstam calculated the Born term for fermion scattering.

TABLE ONE.

CONVENTIONAL DUAL MODEL.

PLANAR LOOPS.

$$A_N = \int \prod_{i=1}^{N} dx_i \left[\frac{2\pi}{-\ln w}\right]^{\frac{D}{2}} f(w)^{2-D} \prod_{1\le i<j\le N} \Psi(x_{ij}, w)^{-2k_i.k_j}$$

$$w = x_1 x_2 \ldots x_N \quad , \quad x_{ij} = x_i x_{i+1} \ldots x_{j-1}$$

$$\Psi(x,w) = -2\pi i \exp\left[\frac{\ln^2 x}{2\ln w}\right] \frac{\theta_1(v|\tau)}{\theta_1'(0|\tau)}$$

$$= \exp\left[\frac{\ln^2 x}{2\ln w}\right] x^{-\frac{1}{2}} \tilde{\Psi}(x,w)$$

$$\tilde{\Psi}(x,w) = -2\pi i x^{\frac{1}{2}} \frac{\theta_1(v|\tau)}{\theta_1'(0|\tau)}$$

$$f(w) = \prod_{n=1}^{\infty}(1-w^n) \quad , \quad v = \frac{\ln x}{2\pi i} \quad , \tau = \frac{\ln w}{2\pi i}$$

ORIENTABLE NON-PLANAR LOOPS.

$$A_N = \int \prod_{i=1}^{N} dx_i \left[\frac{2\pi}{-\ln w}\right]^{\frac{D}{2}} f(w)^{2-D} \prod_{1\le i<j\le N} \Psi_{(T)}(x_{ij}, w)^{-2k_i.k_j}$$

where $\Psi_{(T)}(x,w) = \Psi(x,w)$ if even no. of twists between x_i and x_j

$= \Psi_T(x,w)$ if odd no. of twists between x_i and x_j

$$\Psi_T(x,w) = 2\pi \exp\left[\frac{\ln^2 x}{2\ln w}\right] \frac{\theta_2(v|\tau)}{\theta_1'(0|\tau)}$$

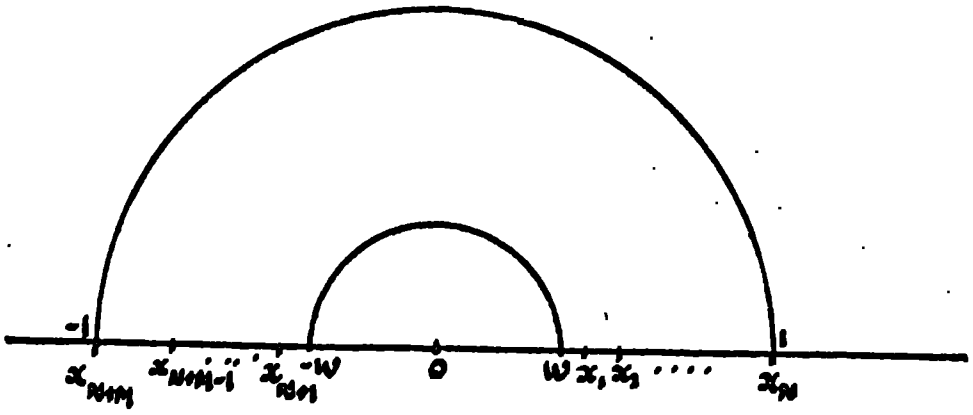
$$= \exp\left[\frac{\ln^2 x}{2\ln w}\right] x^{-\frac{1}{2}} \tilde{\Psi}(xe^{i\pi}, w)$$

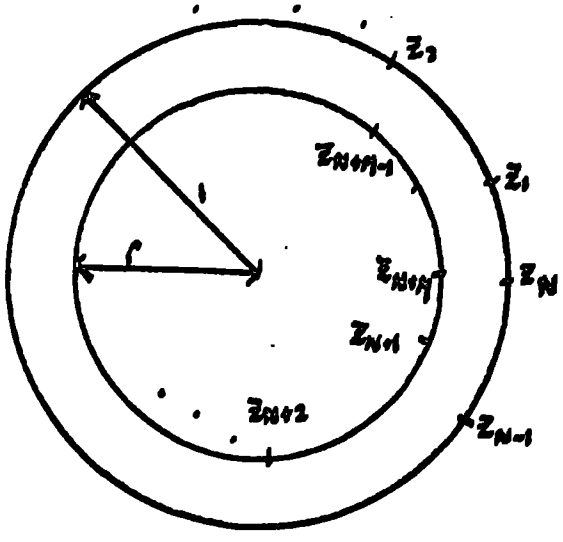
TABLE TWO.

JACOBI TRANSFORMATION AND THETA FUNCTIONS.

$$v=\frac{\ln x}{2\pi i}\ ,\quad \tau=\frac{\ln w}{2\pi i}\ ,\quad v'=\frac{v}{\tau}\ ,\quad \tau'=\frac{-1}{\tau}$$

$$v'=\frac{\ln z}{2\pi i}\ ,\quad \tau'=\frac{\ln r}{\pi i}\ ,\quad \ln r \ln w = 2\pi^2$$





$$\theta_1(v'|\tau') = -i(-i\tau)^{\frac{1}{2}} \exp\left(\frac{i\pi v^2}{\tau}\right) \theta_1(v|\tau)$$

$$\theta_2(v'|\tau') = (-i\tau)^{\frac{1}{2}} \exp\left(\frac{i\pi v^2}{\tau}\right) \theta_4(v|\tau)$$

$$\theta_3(v'|\tau') = (-i\tau)^{\frac{1}{2}} \exp\left(\frac{i\pi v^2}{\tau}\right) \theta_3(v|\tau)$$

$$\theta_4(v'|\tau') = (-i\tau)^{\frac{1}{2}} \exp\left(\frac{i\pi v^2}{\tau}\right) \theta_2(v|\tau)$$

$$\theta_1'(0|\tau') = (-i\tau)^{\frac{3}{2}} \theta_1'(0|\tau)$$

$$\theta_1'(0|\tau) = \frac{\partial}{\partial(\pi v)} \theta_1(v|\tau)\Big|_{v=0}$$

$$\theta_1(v|\tau) = 2\omega^{\frac{1}{8}} \sum_{n=0}^{\infty} (-1)^n \omega^{\frac{n(n+1)}{2}} \sin((2n+1)\pi v)$$

$$= 2\omega^{\frac{1}{8}} \sin\pi v \prod_{n=1}^{\infty}(1-\omega^n) \prod_{n=1}^{\infty}(1-2\omega^n \cos 2\pi v + \omega^{2n})$$

$$\theta_2(v|\tau) = 2\omega^{\frac{1}{8}} \sum_{n=0}^{\infty} \omega^{\frac{n(n+1)}{2}} \cos((2n+1)\pi v)$$

$$= 2\omega^{\frac{1}{8}} \cos\pi v \prod_{n=1}^{\infty}(1-\omega^n) \prod_{n=1}^{\infty}(1+2\omega^n \cos 2\pi v + \omega^{2n})$$

$$\theta_3(v|\tau) = 1 + 2\sum_{n=1}^{\infty} \omega^{\frac{n^2}{2}} \cos(2n\pi v)$$

$$= \prod_{n=1}^{\infty}(1-\omega^n) \prod_{n=1}^{\infty}(1+2\omega^{n-\frac{1}{2}} \cos 2\pi v + \omega^{2n-1})$$

$$\theta_4(v|\tau) = 1 + 2\sum_{n=1}^{\infty} (-1)^n \omega^{\frac{n^2}{2}} \cos(2n\pi v)$$

$$= \prod_{n=1}^{\infty}(1-\omega^n) \prod_{n=1}^{\infty}(1-2\omega^{n-\frac{1}{2}} \cos 2\pi v + \omega^{2n-1})$$

$$\theta_1'(0|\tau) = 2\omega^{\frac{1}{8}} \prod_{n=1}^{\infty}(1-\omega^n)^3$$

TABLE THREE.

NEVEU-SCHWARZ MODEL.

PLANAR LOOPS.

$$A_N = \int \prod_{i=1}^{N} dx_i \, \frac{\omega^{-\frac{3}{2}}}{\ln^3 \omega} \left[\frac{\phi^+(\omega)}{f(\omega)}\right]^{D-2} \psi(x_{ij},\omega)^{-2k_i\cdot k_j} \times$$

$$\times \sum (-1)^p \prod_{pairs} k_{i_\ell}\cdot k_{i_m} \chi^+(x_{i_\ell i_m},\omega)$$

$$\chi^{\pm}(x,\omega) = \sum_{n=1}^{\infty} \frac{x^{n-\frac{1}{2}} \pm \left(\frac{\omega}{x}\right)^{n-\frac{1}{2}}}{1 \pm \omega^{n-\frac{1}{2}}}$$

$$\phi^{\pm}(\omega) = \prod_{n=1}^{\infty} (1 \pm \omega^{n-\frac{1}{2}})$$

ORIENTABLE NON-PLANAR LOOPS.

$$A_N = \int \prod_{i=1}^{N} dx_i \, \frac{\omega^{-\frac{3}{2}}}{\ln^2 \omega} \left[\frac{\phi^{\pm}(\omega)}{f(\omega)}\right]^{D-2} \psi_{(T)}(x_{ij},\omega) \times$$

$$\times \sum (-1)^p \prod_{pairs} k_{i_\ell}\cdot k_{i_m} \chi^{\pm}_{(T)}(x_{i_\ell i_m},\omega)$$

$$\chi^{\pm}_{(T)}(x_{ij},\omega) = \chi^{\pm}(x_{ij},\omega) \quad \text{if } x_i \text{ and } x_j \text{ are on the same boundary}$$

$$= \chi^{\pm}_{T}(x_{ij},\omega) = \chi^{\pm}(x_{ij}\, e^{iT},\omega) \quad \text{if } x_i \text{ and } x_j \text{ are on different boundaries}$$

The plus sign is taken for an even number of particles on each boundary and the minus sign for an odd number on each boundary.

TABLE FOUR.

FUNCTIONS OF THE NEVEU-SCHWARZ-RAMOND MODEL.

MESON LOOPS.

$$\chi^+(x,w) = \frac{i}{2}\,\theta_2(0|\tau)\theta_4(0|\tau)\frac{\theta_3(v|\tau)}{\theta_1(v|\tau)} = \frac{1}{\tau}\chi^+(z,r^2)$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+\frac{1}{2}} + \left(\frac{w}{x}\right)^{n+\frac{1}{2}}}{1+w^{n+\frac{1}{2}}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n w^{\frac{n}{2}} (x)^{\frac{1}{2}}}{(1-xw^n)}$$

$$\chi_T^+(x,w) = \chi^+(xe^{i\pi},w) = \frac{i}{2}\,\theta_2(0|\tau)\theta_4(0|\tau)\frac{\theta_4(v|\tau)}{\theta_2(v|\tau)} = \frac{1}{\tau}\chi^+(zr,r^2)$$

$$\chi^-(x,w) = \chi^+(x,we^{2\pi i}) = \frac{i}{2}\,\theta_2(0|\tau)\theta_3(0|\tau)\frac{\theta_4(v|\tau)}{\theta_1(v|\tau)}$$

$$= \frac{1}{\tau}\chi_0^+(z,r^2) = \sum_{n=-\infty}^{\infty} \frac{w^{\frac{n}{2}} x^{\frac{1}{2}}}{(1-xw^n)}$$

$$\chi_T^-(x,w) = \chi^-(xe^{i\pi}, we^{2\pi i}) = \frac{i}{2}\,\theta_2(0|\tau)\theta_3(0|\tau)\frac{\theta_3(v|\tau)}{\theta_2(v|\tau)}$$

$$= \frac{1}{\tau}\chi_0^+(zr,r^2)$$

FERMION LOOPS.

$$\chi_0^+(x,w) = \frac{i}{2}\,\theta_3(0|\tau)\theta_4(0|\tau)\frac{\theta_2(v|\tau)}{\theta_1(v|\tau)} = \frac{1}{\tau}\chi^+(z,r^2e^{2\pi i})$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{x^n + \left(\frac{w}{x}\right)^n}{(1+w^n)} = \frac{1}{2}\sum_{n=-\infty}^{\infty} (-1)^n \frac{(1+xw^n)}{(1-xw^n)}$$

$$\chi_{0T}^+(x,w) = \chi_0^+(xe^{i\pi},w) = -\frac{i}{2}\,\theta_3(0|\tau)\theta_4(0|\tau)\frac{\theta_1(v|\tau)}{\theta_2(v|\tau)}$$

$$= \frac{1}{\tau}\chi^+(zr,r^2e^{2\pi i})$$

$$\chi_0^-(x,\omega) = -\frac{i}{2}\frac{\theta_1'(v|\tau)}{\theta_1(v|\tau)} = -\frac{i}{2}\frac{d}{dv}\ln\theta_1(v|\tau) = \frac{1}{\tau}\left(\chi_0^-(z,r^2)+\frac{\ln z}{2\ln r}\right)$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{(1+\omega^n x)}{(1-\omega^n x)}$$

$$\chi_{0\tau}^-(x,\omega) = \chi_0^-(xe^{i\pi},\omega) =$$

TABLE FIVE.

PARTITION FUNCTIONS IN NEVEU-SCHWARZ-RAMOND MODEL.[89]

MESON EVEN G-PARITY EXCHANGE.

$$\frac{\phi^+(r^2)}{f(r^2)} = r^{\frac{1}{8}} \prod_{n=1}^{\infty} \frac{(1+r^{2n-1})}{(1-r^{2n})} = 2^{\frac{1}{2}} \left[\frac{\theta_3(0|\tau')}{\theta_1'(0|\tau')}\right]^{\frac{1}{2}} = 2^{\frac{1}{2}} \left[\frac{\theta_3(0|\tau)}{\theta_1'(0|\tau)}\right]^{\frac{1}{2}} (-i\tau)^{-\frac{1}{2}}$$

$$= 2^{\frac{1}{2}} \left(\frac{\pi}{-\ln w}\right)^{\frac{1}{2}} w^{-\frac{1}{16}} \prod_{n=1}^{\infty} \frac{(1+w^{n-\frac{1}{2}})}{(1-w^n)} = 2^{\frac{1}{2}} \left(\frac{\pi}{-\ln w}\right)^{\frac{1}{2}} w^{-\frac{1}{16}} \frac{\phi^+(w)}{f(w)}$$

MESON ODD G-PARITY EXCHANGE.

$$\frac{\phi^-(r^2)}{f(r^2)} = \prod_{n=1}^{\infty} \frac{(1+r^{2n})}{(1-r^{2n})} = \left[\frac{\theta_2(0|\tau')}{\theta_1'(0|\tau')}\right]^{\frac{1}{2}} = \left[\frac{\theta_4(0|\tau)}{\theta_1'(0|\tau)}\right]^{\frac{1}{2}} (-i\tau)^{-\frac{1}{2}}$$

$$= \left(\frac{\pi}{-\ln w}\right)^{\frac{1}{2}} w^{-\frac{1}{16}} \prod_{n=1}^{\infty} \frac{(1-w^{n-\frac{1}{2}})}{(1-w^n)} = \left(\frac{\pi}{-\ln w}\right)^{\frac{1}{2}} w^{-\frac{1}{16}} \frac{\phi^-(w)}{f(w)}$$

FERMION EVEN G-PARITY EXCHANGE.

$$\frac{\phi_0^+(r^2)}{f(r^2)} = r^{-\frac{1}{8}} \prod_{n=1}^{\infty} \frac{(1-r^{2n-1})}{(1-r^{2n})} = 2^{\frac{1}{2}} \left[\frac{\theta_4(0|\tau')}{\theta_1'(0|\tau')}\right]^{\frac{1}{2}} = 2^{\frac{1}{2}} \left[\frac{\theta_2(0|\tau)}{\theta_1'(0|\tau)}\right]^{\frac{1}{2}} (-i\tau)^{-\frac{1}{2}}$$

$$= 2^{\frac{1}{2}} \left(\frac{\pi}{-\ln w}\right)^{\frac{1}{2}} \prod_{n=1}^{\infty} \frac{(1+w^n)}{(1-w^n)} = 2^{\frac{1}{2}} \left(\frac{\pi}{-\ln w}\right)^{\frac{1}{2}} \frac{\phi_0^+(w)}{f(w)}$$

FERMION ODD G-PARITY EXCHANGE.

$$\frac{\phi_0^-(r^2)}{f(r^2)} = 1 = \frac{\phi_0^-(w)}{f(w)}$$

ORBITAL CONTRIBUTION.

$$f(r^2) = r^{\frac{1}{12}} \prod_{n=1}^{\infty} (1-r^{2n}) = 2^{-\frac{1}{3}} [\theta_1'(0|\tau')]^{\frac{1}{3}} = 2^{-\frac{1}{3}} (-i\tau)^{\frac{1}{2}} [\theta_1'(0|\tau)]^{\frac{1}{3}}$$

$$= 2^{-\frac{1}{2}} w^{\frac{1}{24}} \left(\frac{\pi}{-\ln w}\right)^{\frac{1}{2}} \prod_{n=1}^{\infty} (1-w^n) = 2^{-\frac{1}{2}} w^{\frac{1}{24}} \left(\frac{\pi}{-\ln w}\right)^{\frac{1}{2}} f(w)$$

APPENDIX.

NEUMANN LOOP FUNCTION WITH EXTERNAL FERMIONS.

$$K(\xi,\xi') = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (\xi\xi')^{\frac{1}{2}} \omega^{\frac{n}{2}}}{2(\xi-\xi'\omega^n)} \left[\left[\prod_{m=-\infty}^{\infty} \frac{(\xi-\bar{\xi}_i\omega^m)(\xi-\bar{\xi}_k\omega^m)(\xi'\omega^n-\xi_j\omega^m)(\xi'\omega^n-\xi_\ell\omega^m)}{(\xi'\omega^n-\bar{\xi}_j\omega^m)(\xi'\omega^n-\bar{\xi}_k\omega^m)(\xi-\xi_j\omega^m)(\xi-\xi_\ell\omega^m)} \right]^{\frac{1}{2}} + \right.$$

$$\left. + \left[\prod_{m=-\infty}^{\infty} \frac{(\xi-\xi_j\omega^m)(\xi-\xi_\ell\omega^m)(\xi'\omega^n-\bar{\xi}_i\omega^m)(\xi'\omega^n-\bar{\xi}_k\omega^m)}{(\xi'\omega^n-\xi_j\omega^m)(\xi'\omega^n-\xi_\ell\omega^m)(\xi-\bar{\xi}_i\omega^m)(\xi-\bar{\xi}_k\omega^m)} \right]^{\frac{1}{2}} \right]$$

To write the Neumann function in terms of theta functions the product is dealt with first.

$$\prod_{m=-\infty}^{\infty} \frac{(\xi-\bar{\xi}_i\omega^m)(\xi-\bar{\xi}_k\omega^m)(\xi'\omega^n-\xi_j\omega^m)(\xi'\omega^n-\xi_\ell\omega^m)}{(\xi-\xi_j\omega^m)(\xi-\xi_\ell\omega^m)(\xi'\omega^n-\bar{\xi}_i\omega^m)(\xi'\omega^n-\bar{\xi}_k\omega^m)}$$

$$= \frac{(\xi-\bar{\xi}_i)(\xi-\bar{\xi}_k)(\xi'\omega^n-\xi_j)(\xi'\omega^n-\xi_\ell)}{(\xi-\xi_j)(\xi-\xi_\ell)(\xi'\omega^n-\bar{\xi}_i)(\xi'\omega^n-\bar{\xi}_k)} \prod_{m=1}^{\infty} \frac{(\xi-\bar{\xi}_i\omega^m)(\bar{\xi}_i-\xi\omega^m)(\xi-\bar{\xi}_k\omega^m)(\bar{\xi}_k-\xi\omega^m)}{(\xi-\xi_j\omega^m)(\xi_j-\xi\omega^m)(\xi-\xi_\ell\omega^m)(\xi_\ell-\xi\omega^m)} \times$$

$$\times \frac{(\xi'\omega^n-\xi_j\omega^m)(\xi_j-\xi'\omega^{n+m})(\xi'\omega^n-\xi_\ell\omega^m)(\xi_\ell-\xi'\omega^{n+m})}{(\xi'\omega^n-\bar{\xi}_i\omega^m)(\bar{\xi}_i-\xi'\omega^{n+m})(\xi'\omega^n-\bar{\xi}_k\omega^m)(\bar{\xi}_k-\xi'\omega^{n+m})}$$

$$= \frac{(\xi-\bar{\xi}_i)(\xi-\bar{\xi}_k)(\xi'\omega^n-\xi_j)(\xi'\omega^n-\xi_\ell)}{(\xi'\omega^n-\bar{\xi}_i)(\xi'\omega^n-\bar{\xi}_\ell)(\xi-\xi_j)(\xi-\xi_\ell)} \prod_{m=1}^{\infty} \frac{(1-\frac{\bar{\xi}_i}{\xi}\omega^m)(1-\frac{\xi}{\bar{\xi}_i}\omega^m)(1-\frac{\bar{\xi}_k}{\xi}\omega^m)(1-\frac{\xi}{\bar{\xi}_k}\omega^m)}{(1-\frac{\xi_j}{\xi}\omega^m)(1-\frac{\xi}{\xi_j}\omega^m)(1-\frac{\xi_\ell}{\xi}\omega^m)(1-\frac{\xi}{\xi_\ell}\omega^m)} \times$$

$$\times \frac{(1-\frac{\xi_j}{\xi'}\omega^{m-n})(1-\frac{\xi'}{\xi_j}\omega^{m+n})(1-\frac{\xi_\ell}{\xi'}\omega^{m-n})(1-\frac{\xi'}{\xi_\ell}\omega^{m+n})}{(1-\frac{\bar{\xi}_i}{\xi'}\omega^{m-n})(1-\frac{\xi'}{\bar{\xi}_i}\omega^{m+n})(1-\frac{\bar{\xi}_k}{\xi'}\omega^{m-n})(1-\frac{\xi'}{\bar{\xi}_k}\omega^{m+n})}$$

but $$\theta_1(\tfrac{x}{y},\tau) = \omega^{\frac{1}{8}}(-i)\frac{(x-y)}{(xy)^{\frac{1}{2}}} \prod_{n=1}^{\infty}(1-\omega^n)\prod_{n=1}^{\infty}(1-\omega^n\tfrac{x}{y})(1-\omega^n\tfrac{y}{x})$$

so the product becomes

$$\frac{\theta_1\left(\frac{\xi}{\xi_i},\tau\right)\theta_1\left(\frac{\xi}{\xi_k},\tau\right)\theta_1\left(\frac{\xi'\omega^n}{\xi_l},\tau\right)\theta_1\left(\frac{\xi'\omega^n}{\xi_j},\tau\right)}{\theta_1\left(\frac{\xi}{\xi_j},\tau\right)\theta_1\left(\frac{\xi}{\xi_l},\tau\right)\theta_1\left(\frac{\xi'\omega^n}{\xi_k},\tau\right)\theta_1\left(\frac{\xi'\omega^n}{\xi_i},\tau\right)}$$

Now $\theta_1(x\omega^r,\tau)=\theta_1(v+p\tau|\tau)=\theta_1(x,\tau)(-1)^p\omega^{-\frac{p^2}{2}}x^{-p}$

so the product of theta functions gives

$$\frac{\theta_1\left(\frac{\xi}{\xi_i},\tau\right)\theta_1\left(\frac{\xi}{\xi_k},\tau\right)\theta_1\left(\frac{\xi'}{\xi_j},\tau\right)\theta_1\left(\frac{\xi'}{\xi_l},\tau\right)}{\theta_1\left(\frac{\xi}{\xi_j},\tau\right)\theta_1\left(\frac{\xi}{\xi_l},\tau\right)\theta_1\left(\frac{\xi'}{\xi_i},\tau\right)\theta_1\left(\frac{\xi'}{\xi_k},\tau\right)}\cdot\left(\frac{\xi_j\xi_l}{\xi_i\xi_k}\right)^n$$

This then leaves terms of the form

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n\omega^{\frac{n}{2}}(\xi\xi')^{\frac{1}{2}}}{(\xi-\xi'\omega^n)}\left(\frac{\xi_j\xi_l}{\xi_i\xi_k}\right)^{\frac{n}{2}}$$

writing $\left(\frac{\xi'}{\xi}\right)=x$ and $\left(\frac{\xi_j\xi_l}{\xi_i\xi_k}\right)=y$

then $u=\frac{\ln x}{2\pi i}$, $v=\frac{\ln y^{\frac{1}{2}}}{2\pi i}$

then the sum can be written as

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n\omega^{\frac{n}{2}}x^{\frac{1}{2}}y^{\frac{n}{2}}}{(1-x)\omega^n)}=\frac{x^{\frac{1}{2}}}{(1-x)}+\sum_{n=1}^{\infty}\frac{(-1)^n x^{\frac{1}{2}}\omega^{\frac{n}{2}}y^{\frac{n}{2}}}{(1-x\omega^n)}+$$

$$+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{-\frac{1}{2}}\omega^{\frac{n}{2}}y^{-\frac{n}{2}}}{(1-\omega^n x^{-1})}$$

which by expanding the denominator and rewriting gives

$$\frac{x^{\frac{1}{2}}}{(1-x)} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^n w^{n(m-\frac{1}{2})}\left(x^{m-\frac{1}{2}}y^{\frac{n}{2}} - x^{-(m-\frac{1}{2})}y^{-\frac{n}{2}}\right)$$

Now from Watson and Whittaker[80]

$$\frac{\theta_1(u+v|\tau)\theta_1'(0|\tau)}{\theta_1(u|\tau)\theta_1(v|\tau)} = \cot\pi u + \cot\pi v + 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} w^{nm}\sin 2\pi(nu+mv)$$

by changing variables $v \to v + \frac{\tau}{2} + \frac{1}{2}$

$$\theta_1(v+\tfrac{\tau}{2}+\tfrac{1}{2}|\tau) = x^{-\frac{1}{2}}w^{-\frac{1}{8}}\theta_3(v|\tau)$$

$$\therefore \frac{\theta_1(u+v+\frac{\tau}{2}+\frac{1}{2}|\tau)\theta_1'(0|\tau)}{\theta_1(v+\frac{\tau}{2}+\frac{1}{2}|\tau)\theta_1(u|\tau)} = \frac{\theta_3(u+v|\tau)\theta_1'(0|\tau)}{\theta_3(v|\tau)\theta_1(u|\tau)}x^{-\frac{1}{2}}$$

$$= \cot\pi u + \cot\pi(v+\tfrac{\tau}{2}+\tfrac{1}{2}) + 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} w^{nm}\sin 2\pi(nu+mv+m\tfrac{\tau}{2}+\tfrac{m}{2})$$

$$= -i\frac{(1+x)}{(1-x)} - i\frac{(1-y^{\frac{1}{2}}w^{\frac{1}{2}})}{(1+y^{\frac{1}{2}}w^{\frac{1}{2}})} - 2i\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m w^{nm}\left(x^n y^{\frac{m}{2}}w^{\frac{m}{2}} - x^{-n}y^{-\frac{m}{2}}w^{-\frac{m}{2}}\right)$$

$$= -i\frac{(1+x)}{(1-x)} - i + 2i\sum_{n=0}^{\infty}\left(\frac{w^{n+\frac{1}{2}}x^n y^{\frac{1}{2}}}{(1+y^{\frac{1}{2}}w^{n+\frac{1}{2}})} - \frac{x^{-(n+1)}w^{n+\frac{1}{2}}y^{-\frac{1}{2}}}{(1+y^{-\frac{1}{2}}w^{n+\frac{1}{2}})}\right)$$

$$\therefore \frac{\theta_3(u+v|\tau)\theta_1'(0|\tau)}{\theta_3(v|\tau)\theta_1(u|\tau)} = \frac{-2ix^{\frac{1}{2}}}{(1-x)} + 2i\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m w^{(n+\frac{1}{2})(m+1)}\left(x^{n+\frac{1}{2}}y^{\frac{(m+1)}{2}} - x^{-(n+\frac{1}{2})}y^{-\frac{(m+1)}{2}}\right)$$

$$= -i2\left(\frac{x^{\frac{1}{2}}}{(1-x)} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m w^{m(n-\frac{1}{2})}\left(x^{n-\frac{1}{2}}y^{\frac{m}{2}} - x^{-(n-\frac{1}{2})}y^{-\frac{m}{2}}\right)\right.$$

$$\therefore \frac{i}{2}\frac{\theta_3(u+v|\tau)\theta_1'(0|\tau)}{\theta_3(v|\tau)\theta_1(u|\tau)} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(\xi\xi')^{\frac{1}{2}}\omega^{\frac{n}{2}}}{(\xi-\xi'\omega^n)}\left(\frac{\xi_j\xi_l}{\xi_i\xi_k}\right)^{\frac{n}{2}}$$

Putting all the terms together gives, for the Neumann function

$$K(\xi,\xi') = \frac{i}{4}\frac{\theta_3\left(\frac{\xi}{\xi'}\left(\frac{\xi_j\xi_l}{\xi_i\xi_k}\right)^{\frac{1}{2}},\tau\right)\theta_1'(0|\tau)}{\theta_3\left(\left(\frac{\xi_j\xi_l}{\xi_i\xi_k}\right)^{\frac{1}{2}},\tau\right)\theta_1\left(\frac{\xi}{\xi'},\tau\right)}\left[\frac{\theta_1\left(\frac{\xi}{\xi_i},\tau\right)\theta_1\left(\frac{\xi}{\xi_k},\tau\right)\theta_1\left(\frac{\xi'}{\xi_j},\tau\right)\theta_1\left(\frac{\xi'}{\xi_l},\tau\right)}{\theta_1\left(\frac{\xi}{\xi_j},\tau\right)\theta_1\left(\frac{\xi}{\xi_l},\tau\right)\theta_1\left(\frac{\xi'}{\xi_i},\tau\right)\theta_1\left(\frac{\xi'}{\xi_k},\tau\right)}\right]^{\frac{1}{2}}$$

$$+\frac{i}{4}\frac{\theta_3\left(\frac{\xi}{\xi'}\left(\frac{\xi_i\xi_k}{\xi_j\xi_l}\right)^{\frac{1}{2}},\tau\right)\theta_1'(0|\tau)}{\theta_3\left(\left(\frac{\xi_i\xi_k}{\xi_j\xi_l}\right)^{\frac{1}{2}},\tau\right)\theta_1\left(\frac{\xi}{\xi'},\tau\right)}\left[\frac{\theta_1\left(\frac{\xi}{\xi_j},\tau\right)\theta_1\left(\frac{\xi}{\xi_l},\tau\right)\theta_1\left(\frac{\xi'}{\xi_i},\tau\right)\theta_1\left(\frac{\xi'}{\xi_k},\tau\right)}{\theta_1\left(\frac{\xi}{\xi_i},\tau\right)\theta_1\left(\frac{\xi}{\xi_k},\tau\right)\theta_1\left(\frac{\xi'}{\xi_j},\tau\right)\theta_1\left(\frac{\xi'}{\xi_l},\tau\right)}\right]^{\frac{1}{2}}$$

TABLE SIX

NEUMANN FUNCTIONS FOR ONE LOOP AMPLITUDE WITH EXTERNAL FERMIONS.

$$\left(\frac{\partial\rho_\alpha}{\partial\beta_\alpha}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_\beta}{\partial\beta_\beta}\right)^{-\frac{1}{2}} K^{-+}(\beta_\alpha,\beta_\beta)=\frac{i}{2}\left(\frac{\partial\rho_2}{\partial\beta_\alpha}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_\beta}{\partial\beta_\beta}\right)^{-\frac{1}{2}}\delta^{-1}(AB)^{-\frac{1}{2}}\times$$

$$\times\frac{\vartheta_3(\frac{1}{2}(\beta_1+\beta_2-\beta_3-\beta_4)|\tau')}{\vartheta_3(\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4)|\tau')}\left[\frac{\vartheta_1(\beta_1-\beta_2|\tau')\vartheta_1(\beta_3-\beta_4|\tau')}{\vartheta_1(\beta_2-\beta_4|\tau')\vartheta_1(\beta_3-\beta_1|\tau')}\right]^{\frac{1}{2}}\times$$

$$\times\Bigg(1+\delta\Bigg((A-B)\frac{\vartheta_1'(\beta_3-\beta_2|\tau')}{\vartheta_1(\beta_3-\beta_2|\tau')}-\frac{C}{2A}-\frac{D}{2B}+\frac{(B-A)}{2}\frac{\vartheta_3'(\frac{1}{2}(\beta_1+\beta_2-\beta_3-\beta_4)|\tau')}{\vartheta_3(\frac{1}{2}(\beta_1+\beta_2-\beta_3-\beta_4)|\tau')}+$$

$$+\frac{A}{2}\left(\frac{\vartheta_1'(\beta_2-\beta_1|\tau')}{\vartheta_1(\beta_2-\beta_1|\tau')}+\frac{\vartheta_1'(\beta_2-\beta_1|\tau')}{\vartheta_1(\beta_2-\beta_1|\tau')}-\frac{\vartheta_1'(\beta_2-\beta_4|\tau')}{\vartheta_1(\beta_2-\beta_4|\tau')}\right)+$$

$$+\frac{B}{2}\left(\frac{\vartheta_1'(\beta_3-\beta_4|\tau')}{\vartheta_1(\beta_3-\beta_4|\tau')}+\frac{\vartheta_1'(\beta_3-\beta_4|\tau')}{\vartheta_1(\beta_3-\beta_4|\tau')}-\frac{\vartheta_1'(\beta_3-\beta_1|\tau')}{\vartheta_1(\beta_3-\beta_1|\tau')}\right)\Bigg)$$

$$\beta_\alpha=\beta_2+\delta A+\delta^2 C\ ,\ \beta_\beta=\beta_3+\delta B+\delta^2 L$$

$$\left(\frac{\partial\rho_\beta}{\partial\beta_\beta}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_4}{\partial\beta_4}\right)^{-\frac{1}{2}} K^{+-}(\beta_\beta,\beta_4)=(\delta B)^{-\frac{1}{2}}\frac{\alpha^{-1}}{2}\frac{\vartheta_3(\frac{1}{2}(\beta_1-\beta_2-\beta_3+\beta_4)|\tau')}{\vartheta_3(\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4)|\tau')}\left(\frac{\partial\rho_\beta}{\partial\beta_\beta}\right)^{-\frac{1}{2}}\times$$

$$\times\left[\frac{\vartheta_1(\beta_1-\beta_4|\tau')\vartheta_1(\beta_3-\beta_2|\tau')}{\vartheta_1(\beta_1-\beta_3|\tau')\vartheta_1(\beta_2-\beta_4|\tau')}\right]^{\frac{1}{2}}\Bigg(1+\delta\Bigg(-\frac{\beta}{2}\frac{\vartheta_3'(\frac{1}{2}(\beta_1-\beta_2-\beta_3+\beta_4)|\tau')}{\vartheta_3(\frac{1}{2}(\beta_1-\beta_2-\beta_3+\beta_4)|\tau')}+$$

$$+\frac{B\vartheta_1'(\beta_4-\beta_3|\tau')}{\vartheta_1(\beta_4-\beta_3|\tau')}-\frac{D}{2B}+\frac{\beta}{2}\left(\frac{\vartheta_1'(\beta_3-\beta_2|\tau')}{\vartheta_1(\beta_3-\beta_2|\tau')}+\frac{\vartheta_1'(\beta_3-\beta_4|\tau')}{\vartheta_1(\beta_3-\beta_4|\tau')}\right.$$

$$\left.-\frac{\vartheta_1'(\beta_3-\beta_1|\tau')}{\vartheta_1(\beta_3-\beta_1|\tau')}\right)\Bigg)$$

$$\left(\frac{\partial\rho_4}{\partial\beta_4}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_3}{\partial\beta_3}\right)^{-\frac{1}{2}}K^{-+}(\beta_2,\beta_3)=\left(\frac{\partial\rho_4}{\partial\beta_4}\right)^{-\frac{1}{2}}\frac{(2\delta A)^{-\frac{1}{2}}\theta_3\left(\frac{1}{2}(\beta_1+\beta_2-\beta_3-\beta_4)|\tau'\right)}{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4)\tau'\right)}\times$$

$$\times\left[\frac{\theta_1(\beta_1-\beta_2|\tau')\theta_1(\beta_3-\beta_4|\tau')}{\theta_1(\beta_2-\beta_4|\tau')\theta_1(\beta_3-\beta_1|\tau')}\right]^{\frac{1}{2}}\left(1+\delta\left(A\frac{\theta_1'(\beta_3-\beta_2|\tau')}{\theta_1(\beta_3-\beta_2|\tau')}-\frac{C}{2A}\right.\right.$$

$$-\frac{A}{2}\frac{\theta_3'\left(\frac{1}{2}(\beta_3-\beta_2-\beta_1+\beta_4)|\tau'\right)}{\theta_3\left(\frac{1}{2}(\beta_3-\beta_2-\beta_1+\beta_4)|\tau'\right)}+\frac{A}{2}\left(\frac{\theta_1'(\beta_2-\beta_1|\tau')}{\theta_1(\beta_2-\beta_1|\tau')}+\frac{\theta_1'(\beta_2-\beta_3|\tau')}{\theta_1(\beta_2-\beta_3|\tau')}\right.$$

$$\left.\left.-\frac{\theta_1'(\beta_2-\beta_4|\tau')}{\theta_1(\beta_2-\beta_4|\tau')}\right)\right)$$

$$\left(\frac{\partial\rho_1}{\partial\beta_1}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_4}{\partial\beta_4}\right)^{-\frac{1}{2}}K^{+-}(\beta_1,\beta_4)=\frac{\alpha^{-1}}{2}\frac{\theta_3\left(\frac{1}{2}(\beta_1+\beta_2-\beta_3-\beta_4)|\tau'\right)}{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4)|\tau'\right)}\times$$

$$\times\left[\frac{\theta_1(\beta_3-\beta_4|\tau')\theta_1(\beta_1-\beta_2|\tau')}{\theta_1(\beta_1-\beta_3|\tau')\theta_1(\beta_4-\beta_2|\tau')}\right]^{\frac{1}{2}}$$

$$\left(\frac{\partial\rho_2}{\partial\beta_2}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_1}{\partial\beta_1}\right)^{-\frac{1}{2}}K^{+-}(\beta_2,\beta_1)=+\frac{\alpha^{-\frac{1}{2}}}{2}\left(\frac{\partial\rho_3}{\partial\beta_3}\right)^{-\frac{1}{2}}\delta^{-\frac{1}{2}}B^{-\frac{1}{2}}\frac{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2-\beta_3+\beta_4)|\tau'\right)}{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4)|\tau'\right)}\times$$

$$\times\left[\frac{\theta_1(\beta_2-\beta_3|\tau')\theta_1(\beta_1-\beta_4|\tau')}{\theta_1(\beta_4-\beta_2|\tau')\theta_1(\beta_1-\beta_3|\tau')}\right]^{\frac{1}{2}}$$

$$\left(\frac{\partial\rho_2}{\partial\beta_2}\right)^{-\frac{1}{2}}\left(\frac{\partial\rho_4}{\partial\beta_4}\right)^{-\frac{1}{2}}K^{+-}(\beta_4,\beta_2)=-i\frac{\alpha^{-\frac{1}{2}}}{2}\left(\frac{\partial\rho_3}{\partial\beta_3}\right)^{-\frac{1}{2}}\delta^{-1}A^{-\frac{1}{2}}$$

$$\left(\frac{\partial \rho_3}{\partial \beta_3}\right)^{-\frac{1}{2}}\left(\frac{\partial \rho_4}{\partial \beta_4}\right)^{-\frac{1}{2}} K^{+-}(\beta_3,\beta_4) = \frac{i}{2}\alpha^{-1}\delta^{-\frac{1}{2}}\frac{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2-\beta_3+\beta_4)|\tau'\right)}{\theta_3\left(\frac{1}{2}(\beta_1-\beta_2+\beta_3-\beta_4)|\tau'\right)} \times$$

$$\times\left[\frac{\theta_1(\beta_1-\beta_4|\tau')\theta_1(\beta_3-\beta_2|\tau')}{\theta_1(\beta_1-\beta_3|\tau')\theta_1(\beta_2-\beta_4|\tau')}\right]^{\frac{1}{2}}$$

TABLE SEVEN

MOMENTA OF EXTERNAL FERMIONS IN ONE LOOP AMPLITUDE.[84]

$$p_1^{(+)} = -p_1^{(-)} = \frac{i(-t)^{\frac{1}{2}}}{2\sqrt{2}} \quad , \quad p_2^{(+)} = \frac{i(-t)^{\frac{1}{2}}}{\sqrt{2}}$$

$$p_2^{(-)} = \frac{i\delta(2s+t)}{(-t)^{\frac{1}{2}}\,2\sqrt{2}} \quad , \quad p_3^{(+)} = -\frac{i\delta(2s+t)}{(-t)^{\frac{1}{2}}\,2\sqrt{2}}$$

$$p_3^{(-)} = \frac{i(-t)^{\frac{1}{2}}}{2} \quad , \quad p_4^{(+)} = -p_4^{(-)} = \frac{i(-t)^{\frac{1}{2}}}{2\sqrt{2}}$$

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NOTE:

There is an error in pagination in this thesis. The number 54 has been omitted.