A dual analysis of the $NN\pi\pi\pi$ system

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A DUAL ANALYSIS OF THE $N\bar{N}\pi\pi\pi$ SYSTEM

BY

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A thesis presented for the degree of Doctor of Philosophy
of the University of Durham

Mathematics Department
University of Durham

September 1973

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PREFACE

The work presented in this thesis was commenced in the Department of Mathematics at the University of Durham during the period October 1969 to September 1971 and was subsequently completed, part time, whilst the author was at Heriot-Watt University during the period October 1971 to September 1973. It is a pleasure to thank Professor E J Squires for his supervision of this work and for his guidance, advice and encouragement at all stages of this research. The author also wishes to thank Drs. M Chaichian, R C Johnson and A M S Amatya, colleagues at Durham, for encouragement in this work and for providing many helpful comments. For suggesting the use of the Maximum Likelihood Method we thank Dr A Hawkes, and for a critical reading of section (1.2) on this method thanks are extended to Professor J R Gray of Heriot-Watt University. A considerable proportion of this work was bound up in computing, use being made of the IBM 360/67 Northumbrian Universities' Multiple Access Computer (NUMAC) and the Edinburgh Regional Computing Centre's (ERCC) IBM 360/50. For programming advice and help in using both the Michigan Terminal System (MTS) and the IBM Operating System (OS) we are grateful to Mr A C Heath and other members of the University of Durham Computer Unit. We thank Mr R Broughton (Operations Supervisor) for direct use of the machine at Newcastle and also those members of staff who made this possible by their kind assistance. Similarly, we acknowledge the help of Mr J Leitch and other members of staff at the ERCC who provided assistance in using the computing link at Edinburgh via the land line to the machine at Newcastle, and also in using the IBM machine

in Edinburgh.

For provision of his B_5 programme, the CERN routine ZFACT, and for advice in their use, we thank Dr J F L Hopkinson. For providing the $\bar{p}n \to 3\pi$ (at rest) data we thank Dr A Rothery, and for similarly providing the $\bar{p}n \to 3\pi$ (in flight) data we thank Dr G Rinaudo.

The work in this thesis has not been submitted for any other degree in this or any other University. It is based essentially on two papers by the author.* No claim of originality is made on Chapter One or on the review sections of the other chapters.

The author would like to thank Mrs Phyllis Macdonald for her skilful typing of this thesis, the Science Research Council for a Research Studentship, and lastly, but by no means least, his wife, for her patience and encouragement.

* L. Nicholas, Lett. Nuovo Cimento, 2, 969 (1971);
Phys. Rev. To be published (1974).

ABSTRACT

We describe the application of certain four and five-point Dual Scattering Amplitudes to the $N\bar{N}\pi\pi\pi$ system and compare the results with others in the literature.

In Chapter One we review the basic ideas that led to the construction of the Veneziano model and provide a short introduction to the Maximum Likelihood Method.

In Chapter Two a discussion of five-point dual functions is given, followed by an application to a production process of the above system with an appropriate amplitude.

In Chapter Three a fit by dual four-point functions to some suitable $N\bar{N}$-decay at rest data is presented, together with discussion of related work, and in Chapter Four we use various five-point dual functions to fit the same data and contrast the two sets of results. This is followed by a summary of B_5-phenomenology.

In Chapter Five we apply various four-point function amplitudes to some $N\bar{N}$-decay in flight data and comment on their suitability.

References are provided after each Chapter, and there is some duplication of both references and material between chapters.

CHAPTER 1

Introduction

1.1 An Outline of Duality

1.1.1 Why the word "Duality"?

The words Dual, Duality [1], Self-Dual and their derivatives already enjoyed wide use before High-Energy Theorists employed them.

For example, the Algebraist used Dual Vector Spaces, would take the Dual of the Dual Space and Dual Transformations [2] and might employ Dual Grassmann Coordinates [3] in his Algebraic Geometry. Today he simplifies matters and uses the word "Co" as in Co-homology Group etc. [4]. As early as the 1920's numbers of a form $z + \phi$ were called dual numbers [5]. In Fourier Analysis there are the well-known Duality Theorems [6] concerning functions and their Fourier transforms. Graph Theory also has its Dual-Graphs, Self-Dual Graphs and geometric duals of planar graphs [7] and similarly in Reliability networks there are Dual networks and those that are Dual to themselves [8]. The Mathematical Programmer uses Dual methods in decomposition and has Duality theories in both Linear [9] and non-Linear [10] programming. Even outside of the Mathematical Sciences there is a Dualism both in Theology (in which one description of God's attributes and His nature is said to antagonise another) [11] and in Philosophy (Cartesian dualism, and the traditional dualism of Descartes, in which there is a 'mind-body' dualism) [12], these being the two explanations of the usage of the word 'dualism' in ref. [1], and one speaks of the Social Dualist (who keeps his private life separate from his social life and morals).

In Physics the famous use of these words was for the Wave-Particle behaviour of, for example, light and electrons, spoken of in Quantum Mechanics [13] as the Wave-Particle Duality. There is also, however, a Duality Principle in Continuum Mechanics, the use of the word coming from that in Analysis [14].

For many subjects the word 'duality' expressed a correspondence or correlation of effects between two ideas, things or spaces etc. Thus in about 1968 when High-Energy Theorists wished to express their belief in the correspondence of effects in a scattering process between the 'direct'channel' at 'low' energies and the 'crossed-channel' at 'high' energies they spoke (of some kind, for example Global) of Duality, thus using a word that had the type of connotation they wished to convey.

1.1.2 Phenomenology [15]

In reviewing the work of the Theoretical Study Division at CERN in 1971 [16] M. Jacob said, concerning phenomenology :

> "Lacking a theory for strong interaction processes, models which stress the importance of some specific parameters are tested with variable success against the many experimental results which become available. The aim is thus to ascertain the prominent role of some key parameters around which an actual theory could eventually develop, to test the theoretical pictures thus built up at their predictive value and, by the same token, to help choosing the most significant experiment to do next. Having, however, only theoretical models and not an actual theory, we cannot a priori estimate what is left over by the approximation retained in any specific approach. If in difficulty with experiment, one may often call on this remainder in a particular way in order to help oneself out. As a result, strong interaction models may show to many a somewhat troublesome flexibility. Nevertheless if models may not die, they may well complicate themselves out! A good taste for simplicity is one of the main guiding lines in our search for key parameters".

And:

> "- finding regularity patterns and eventually ascertaining some key parameters for many - particle phenomena is at present one of the most challenging problems in strong interaction phenomenology."

The Veneziano model together with its earlier developments [17] did indeed provide just such simple expressions for strong interaction amplitudes. These models contained the assumption of Regge asymptotic behaviour and Regge-pole-Resonance "duality".

In this thesis the "regularity patterns" exhibited by the $\bar{p}n\to 3\pi$ decay data are studied in the framework of these dual-models. As such the theoretical aspects of our work only concern developments up to about 1971 and no consideration is given to the more formal aspects of the theory which are still being investigated [18].

1.1.3 The Dispersion Relation Approach

Superconvergence

Consider the invariant scattering amplitude as a function of the usual Mandelstam invariants s, t, u and denoted A(s, t).

These invariants are conventionally defined by (see Fig. 1)

$$s = (P_1 + P_2)^2$$
$$t = (P_1 + P_3)^2$$
$$u = (P_1 + P_4)^2$$

with $s + t + u = \sum_{i=1}^{4} m_i^2 = \Sigma$

For fixed t we can write down a dispersion relation, unsubtracted, and not involving kinematic singularities or pole terms, for A(s, t). Consider A(s, t) to be analytic in the s-plane (see Fig. 2) and for fixed t restrict A(s, t) to $A(s, t) \leqslant (s^{-\varepsilon})$, so that:

$$A(s, t) = \frac{1}{\pi} \int_{s_o}^{\infty} \frac{\mathrm{Im}\, A(s',t)\, ds'}{s' - s} + \frac{1}{\pi} \int_{u_o}^{\infty} \frac{\mathrm{Im}\, A(u',t)}{u' - u}\, du'$$

Symbolically written

$$A(s, t) = \frac{1}{\pi} \int \frac{\mathrm{Im}\, A(s', t)\, ds'}{s' - s} \qquad 11.1$$

[Note. Strictly speaking we should use the discontinuity $D_v A \equiv \frac{1}{2i} [A(v + i\varepsilon) - A(v - i\varepsilon)]$ and not Im A. When $t \leqslant t_{th}$ we have $D_v A = \mathrm{Im}\, A$.]

Now suppose we let $s \to \infty$ and make an expansion in terms of $(1 - \frac{s'}{s})$

$$\text{i.e.}\quad \mathrm{Re}\, A(s, t) = -\left\{ \frac{1}{s}\frac{1}{\pi} \int \mathrm{Im}\, A(s',t)\, ds' + \frac{1}{s^2}\frac{1}{\pi} \int \mathrm{Im}\, A(s',t)\, s' + 0\left(\frac{1}{s^3}\right) \text{terms.} \right\}$$

From the first term of the expansion we require that Im A(s',t) vanish faster than $\frac{1}{s'}$, for convergence, and so on.

If $\int \mathrm{Im}\, A(s', t)\, ds' < \infty$ then

$\mathrm{Re}\ A(s,t) \underset{s\to\infty}{\sim} -\frac{1}{s}\left(\frac{1}{\pi}\int \mathrm{Im}\ A(s',t)\ ds'\right)$ (apart from log factors). If in fact $A(s, t) \sim 0\ (\frac{1}{s^2})$ then $\int \mathrm{Im}\ A(s', t)\ ds' = 0$ $\left(\text{if Re } A \not\sim \frac{1}{s}.\right)$
so if $A(s, t) < 0\ (s^{-1-\varepsilon})$, $\varepsilon > 0$ then

$$\int \mathrm{Im}\ A(s, t)\ ds = 0 \qquad 11.2$$

11.2 is called a Superconvergence Relation [19].

This result may be obtained more directly. Take a dispersion relation for A(s, t) multiply by s and subtract a dispersion relation for sA(s, t) viz:-

$$\left.\begin{aligned} sA(s, t) &= s\,\frac{1}{\pi}\int \frac{\mathrm{Im}\ A\ (s', t)\ ds'}{s' - s} \\ sA(s, t) &= \frac{1}{\pi}\int \frac{\mathrm{Im}\ A(s', t) s'\ ds'}{s' - s} \end{aligned}\right\} \Rightarrow$$

$$0 = \int \mathrm{Im}\ A(s', t)\ ds' \quad \text{as above.}$$

In fact if $A(s, t) < 0\ (s^{-n-\varepsilon})$ $\qquad n = 1, 2, 3 \ldots\ldots\ldots$

$$\text{then} \int s^{n-1}\ \mathrm{Im}\ A\ (s, t)\ ds = 0$$

an (n-1)th moment superconvergence relation.

We are more likely to obtain superconvergence relations for processes with spin. [15 (d)]

Take the helicity amplitude

$$< \lambda_1, \ \lambda_2 \mid A \mid \ \lambda_3, \lambda_4 > \sim s^{\mathrm{Max}\left[\,|\lambda_1 - \lambda_3|,\,|\lambda_2 - \lambda_4|\,\right]} \times \text{(Analytic function of s)}$$

$$\sim s^M \times \text{(Analytic function of s).}$$

So that the kinematic singularity free (K.S.F.) amplitude

$$\sim s^{\alpha(t) - M} \qquad [15\ (d)]$$

If $M \geqslant 2$, for example, then since $\alpha(t) < 1$ for $t < 0$ there will always be a superconvergence relation.

Similarly superconvergence relations are more likely when the t - channel has isospin $I = 2$, since there is no known $I = 2$ trajectory with $\alpha(0) > 0$.

1.1.4 Finite Energy Sum Rules (F.E.S.R.)

Finite energy sum rules are a method of exploiting the analytic and asymptotic behaviour (not necessarily Regge) of scattering amplitudes, and as such are little different from the well known dispersion relations. They relate the high-energy asymptotic behaviour of scattering amplitudes to their values at low energies thus providing a method for checking the consistency of asymptotic models such as the Regge model. Historically it was K. Igi in 1962 [20] who first used dispersion relations and asymptotic behaviour to correlate low and high energy properties. The recent work of 1967 from which the term FESR was introduced was carried out by D. Horn and C. Schmid, K. Igi and S. Matsuda, A.A. Logunov et.al., and R. Gatto [21].

Here we follow the derivation given by Dolen et.al. [22] in which the specific application to πN scattering was made.

Consider the relativistic amplitude A (ν, t) where for convenience we take $\nu = \frac{s-u}{4m}$ with m the target mass and s, u and t the usual Mandelstam variables. We assume that A possesses a definite symmetry with respect to $s \leftrightarrow u$ crossing and consider the case where A is antisymmetric in ν the variable at fixed t. i.e. $A(\nu) = - A^*(-\nu)$. Also we assume that A satisfies the fixed t dispersion relation in ν given from

$$A^{\pm}(\nu, t) = \frac{1}{\pi} \int_0^{\infty} d\nu' \; \mathrm{Im}\, A(\nu', t)\left[\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu}\right]$$

by $A(\nu, t) = \frac{1}{\pi} \; 2\nu \int_0^{\infty} d\nu' \; \frac{\mathrm{Im}\, A(\nu')}{\nu'^2 - \nu^2}$ (where the superscript is suppressed for convenience) and where the integration includes pole terms for $0 < \nu' < \nu_{th}$ and continuum distributions for $\nu' > \nu_{th}$.

We now assume that at high energies the amplitude can be written in an expansion of Regge poles i.e. [see e.g. ref. 15(a) or (b)]

For $|\nu| > \nu_N$

$$A^{\mp}(\nu, t) = \sum_i \beta_i(t) \frac{(\pm 1 - e^{-i\pi\alpha_i})}{\alpha_i ! \ \sin \pi\alpha_i} \nu^{\alpha_i(t)}$$

$$= \sum_i \beta_i \nu^{\alpha_i(t)} , \quad \text{say} \qquad 11.3$$

so that if we consider the following result

$$A(\nu) - \sum_{\alpha_i > -1} \beta_i \nu^{\alpha_i(t)} = 0 \, (s^{-1-\varepsilon})$$

this will satisfy the superconvergence relation

$$\int_0^\infty \left[\text{Im } A(\nu) - \sum_{\alpha_i > -1} \beta_i \ \nu'^{\alpha_i} \right] d\nu' = 0$$

Now we cut the integration off at $\alpha_N = N$ and express the high energy behaviour by the Regge terms whose α is below -1

$$\int_0^N \left[\text{Im } A(\nu) - \sum_{\alpha_i > -1} \beta_i \ \nu^{\alpha_i(t)} \right] d\nu + \int_N^\infty \sum_{\alpha_i < -1} \beta_i \ \nu^{\alpha_i} d\nu = 0$$

so that on integrating we obtain the finite energy sum rule

$$\underline{\int_0^N \text{Im } A(\nu) \, d\nu = \sum_{\text{all} \, \alpha_i} \beta_i \frac{N^{\alpha_i + 1}}{\alpha_i + 1}} \qquad 11.4$$

Generalizing to sum rules for higher moments for even integer n we obtain

$$\underline{\int_0^N \nu^n \ \text{Im } A(\nu) \, d\nu = \sum_i \frac{\beta_i \ N^{\alpha_i + n + 1}}{\alpha_i + n + 1}} \qquad 11.5$$

Similarly we can obtain this result for amplitudes even under crossing and for odd integer n. Notice that the relative error made by taking just a fixed number of poles on the R.H.S. is independent of n!

Another way to obtain the relations 11.4 and 11.5 or their equivalents is to apply Cauchy's theorem to the contour of Fig. 3.

Assuming $A(\nu,t) \to \sum_i \frac{\beta_i(t)\left[1 - e^{-i\pi\alpha_i(t)}\right]}{\sin \pi\alpha_i(t)} \nu^{\alpha_i(t)}$

As in 11.3 (Putting $\frac{\beta}{\alpha!} = \beta$) and

taking the countour radius at $|\nu_N| = N$ so that 11.3 holds we have

$$\int_{-N}^{N} A(\nu_+, t)\, d\nu + \int_{\gamma} A(\nu, t)\, d\nu = 0 \text{ with } \nu_+ = \nu + i\varepsilon \quad (\varepsilon > 0)$$

since the function is regular inside the contour.

From the symmetry properties of the amplitude

$$\left.\begin{aligned} \mathrm{Re}\, A(\nu, t) &= -\mathrm{Re}\, A(-\nu, t) \\ \mathrm{Im}\, A(\nu_+, t) &= \mathrm{Im}\, A(-\nu_+, t) \end{aligned}\right\} \quad \text{We obtain}$$

$$2i \int_0^N \mathrm{Im}\, A(\nu, t)\, d\nu + i \sum_j \frac{N^{\alpha_j+1} \beta_j \left[1 - e^{-i\pi\alpha_j}\right]}{\sin \pi\alpha_j} \times$$

$$\int_0^{\pi} e^{i\phi(\alpha_j+1)}\, d\phi = 0$$

where $\nu = N e^{i\phi}$,

and finally

$$\int_0^N \mathrm{Im}\, A(\nu, t)\, d\nu = \sum_j \beta_j \frac{N^{\alpha_j+1}}{\alpha_j+1} \quad \text{as in 11.4}$$

or again using $\nu^n A(\nu, t)$ we could obtain 11.5

$$\int_0^N \nu^n \, \mathrm{Im}\, A\,(\nu,\, t)\, d\nu \;=\; \sum_j \frac{\beta_j \, N^{\alpha_j+n+1}}{\alpha_j+n+1}$$

with n even for antisymmetric amplitudes and

n odd for symmetric amplitudes.

The point of using this method is that one can assume a different asymptotic form for the amplitude, instead of the Regge one in the above, when evaluating the integral over the semicircle.

Continuous moment sum rules CMSR can similarly be obtained.

Recall that $A\,(\nu,\, t) = \frac{1}{\pi} \int_0^\infty d\nu' \; \mathrm{Im}\, A\,(\nu',\, t) \left[\frac{1}{\nu'-\nu} - \frac{1}{\nu'+\nu} \right]$ 11.6

for the odd amplitude A^-, with $\nu_{th} = 0$ then since

$$\mathrm{Im}\,\left([-\,\nu^2]^{\frac{k}{2}} A \right) = |\nu|^{\,k} \left(\cos \frac{k\pi}{2}\; \mathrm{Im}\, A\,(\nu) - \sin \frac{k\pi}{2}\; \mathrm{Re}\, A\,(\nu) \right)$$

(if the phase is chosen as $\exp\,(-\frac{i\pi k}{2})$)

then we require also the real part of A. The imaginary part being taken from the optical theorem [e.g. Ref. 15(a)].

$$\text{Now} \quad \int_0^\infty \mathrm{Im}\,\left([-\nu^2]^{\frac{k}{2}} A\,(\nu) \right) d\nu$$

$$= \int_0^\infty |\nu|^k \cos \frac{k\pi}{2}\; \mathrm{Im}\, A\,(\nu)\, d\nu \;-\; \frac{1}{\pi} \int_0^\infty \sin \frac{k\pi}{2}\, |\nu|^k \; d\nu$$

$$\left[\int_0^\infty d\nu' \left\{ \frac{\mathrm{Im}\, A\,(\nu')}{\nu' - \nu} - \frac{\mathrm{Im}\, A\,(\nu')}{\nu' + \nu} \right\} \right]$$

Changing the order of integration in the last term gives

$$\left.\begin{aligned} \int_0^\infty \frac{\nu^k\, d\nu}{\nu' - \nu} &= +\,\pi \cot \pi k \;\; \nu'^k \\ \int_0^\infty \frac{\nu^k\, d\nu}{\nu + \nu'} &= -\,\pi \operatorname{cosec} \pi k \; \nu'^k \end{aligned}\right\} \qquad \text{From [23]}$$

so that the second term becomes:

$$\int_0^\infty \sin \frac{k\pi}{2} \;\; \nu'^k \operatorname{Im} A(\nu')\, d\nu' \quad x \quad \left(\frac{\cos k\pi + 1}{\sin k\pi}\right)$$

$$= \int_0^\infty \sin \frac{k\pi}{2} \;\; \nu'^k \operatorname{Im} A(\nu')\, d\nu' \quad x \quad \left(\frac{2 \cos^2\left(\frac{k\pi}{2}\right) - 1 + 1}{2 \sin \frac{k\pi}{2} \cos \frac{k\pi}{2}}\right)$$

$$= \cos \frac{k\pi}{2} \int_0^\infty \nu'^k \operatorname{Im} A(\nu')\, d\nu' \qquad \text{and this then}$$

cancels the first term so that the CMSR are identically satisfied if Re A is obtained from the dispersion relations. This type of continuous moment dispersion relation was originally derived and compared with experiment by Y.C. Liu and S. Okubo [24].

The FESR for the odd amplitude (with $\nu_{th} = 0$) is

$$\int_0^N d\nu \;|\nu|^k \left[\cos \frac{k\pi}{2} \operatorname{Im} A(\nu) - \sin \frac{k\pi}{2} \operatorname{Re} A(\nu)\right]$$

$$= \sum_j \frac{\beta_j(t)\; N^{\alpha_j + N + 1}}{\alpha_j + N + 1} \left[\frac{\cos(\alpha_j + N)\, \frac{\pi}{2}}{\cos \alpha_j \,\frac{\pi}{2}}\right] \qquad 11.7$$

See Ref. [15(h)] and [25].

The connection between CMSR and integer moment FESR was given by Ferrari and Violini [26].

The use of FESR for Regge analysis was an accepted and widely used tool, illustration of their more recent use being that by Field and

Jackson [27] in which the effective "pole" parameters of the K* and K** Regge trajectories were obtained using FESR's for the reaction $K^- n \to \pi^- \Lambda$ and $\pi^+ n \to K^+\Lambda$ and a knowledge of the low-lying resonances and their couplings.

A discussion of questions related to the application of FESR in the presence of Regge cuts (which introduce uncertainty of the way to run them) is given by F. Schrempp [28] and such cuts are in the analysis (in which CMSR's were used) of Barger and Phillips in [29] where they are effectively parameterized as secondary Regge poles.

S. Humble [30] has described some of the difficulties associated with writing dispersion relations for production amplitudes and has indicated how these can be overcome to construct FESR's for five point amplitudes.

1.1.5 The Duality Idea

The concept of Duality, first introduced by Dolen, Horn and Schmid [22], states that, in a scattering amplitude A(s, t) for a reaction $A + B \rightarrow A' + B'$, the terms contributed by s-channel resonances and those contributed by t-channel Regge exchanges describe to some extent and in some approximation the same dynamical effects. This duality was expressed in terms of the imaginary parts of the amplitudes with real coupling coefficients in the resonance formulae.

Fig. 4 shows the scattering function for π-N scattering represented both by a Regge fit and by resonances illustrating this concept. The plot is of the difference of π^-p and π^+p total cross sections (which give the imaginary part of the amplitude by the optical theorem) against energy taken from Chiu and Stirling [31]. Curve II is the extrapolation of the contribution of the ρ-trajectory.

In 11.4 we have a sum over all Regge poles significant in the region $\nu > N$ (and neglecting errors due to background terms, lower lying poles etc. in $\nu > N$). On including only poles with $\alpha_j + n > -1$ then the R.H.S. is the Regge pole contribution integrated from threshold to $\nu = N$, so that in this sense the leading Regge pole contribution averages the imaginary part of the amplitude. Or: The prominent resonances at low energies are related to the leading Regge trajectories at high energies. The leading vacuum singularity was excluded from the scheme, for reasons given later, and this type of duality was referred to as global duality.

Dolen, Horn and Schmid [22] applied the F.E.S.R. to πN charge exchange and considered the specific example of $\pi^-p \rightarrow \pi^on$ since for this reaction the t-channel quantum numbers allow only the ρ-meson ($J^P = 1^-$, $I^G = 1^+$, mass M = 765 Mev.) in a single particle intermediate state. So the ρ- Regge pole exchange was assumed for the asymptotic behaviour. See Fig. 5.

This process is described by the invariant amplitudes A' and B (corresponding to t-channel non-flip and helicity flip [32]) which are found from Regge-pole fitting to high-energy data to change sign near t = -0.15 (the "crossover" zero of πN scattering) and t = -0.6 (where a nonsense zero in the ρ-residue at $\alpha_\rho(t) = 0$ is expected), respectively. The sign changes and the approximate magnitudes of the ρ residues in both amplitudes were successfully predicted even though a low cut-off of N = 1.1 Gev was taken. Dolen et. al. suggested two applications of these F.E.S.R.

1. As an aid to determine Regge pole parameters.
2. As a bootstrap (in which Regge poles in "crossed" reactions determine resonances belonging to trajectories in "direct" reactions and the converse).

The following bootstrap ingredients may be noted.

<u>Bootstrap</u>.

<u>F.E.S.R.</u>

1. Analyticity
2. Crossing (Linear Σ = Σ)
3. Regge behaviour
4. $\int^N \mathrm{Im}\, A \, d\nu$ given by Σ Resonances.

'Resonance saturation' assumption.

<u>Conventional</u> (e.g. [15a Ch.6])

1. Analyticity
2. Crossing
3. } Not important.
4. }
5. Unitarity (Non linear process. Force =)
6. Crude approximation (e.g. using nearest singularity)

A typical application which involved a sign problem was that by D. Gross [33] using only scalar mesons (0^+ particles).

This F.E.S.R. duality is incompatible with the old interference model of Barger and Cline [34] and Barger and Olsson [35] because the direct channel resonances and Regge pole approximation are made in different regions so that no question of "double counting" arises. If $\mathrm{Im}\, A = \mathrm{Im}\, A_{Regge}$ for $\nu > N$ and $\mathrm{Im} A = \mathrm{Im}\, A_{Res}$ for $\nu < N$ then

$$\int_0^N \mathrm{Im}\, A_{Res}\, d\nu = \int_0^N \mathrm{Im}\, A_{Regge}\, d\nu$$

Dolen et al. suggest that the amplitude be written $A = A_{Regge} + A_{Res} - \langle A_{Res} \rangle$ where $\langle A_{Res} \rangle$ denotes the locally averaged resonance amplitude, so that for any scattering process where all resonances contribute with the same sign to A one has $A_{Res} \simeq \langle A_{Res} \rangle$ or $A \simeq A_{Regge}$

So that in that case the Interference Form (I.F.) $A = A_{Regge} + A_{Res}$ would imply double counting. If the resonances contributed with different signs so that $\langle A_{Res} \rangle = 0$ one would obtain the I.F. for A.

The Dual Form (D.F.) is defined for intermediate energies by:

$$\mathrm{Im}\, A \simeq \mathrm{Im}\left(\sum_{Res} \overset{R}{\leadsto} \right) \simeq \mathrm{Im}\left(\sum_{Regge} \; R \right) \qquad 11.8$$

F.E.S.R. cannot predict reliably to very high energy since the low energy input may not be sufficiently exact for a large extrapolation. Steiner [36] has given an estimate for the range of t values in which FESR and CMSR can be used and justifies the results of Dolen et.al. concerning the ρ-residue functions at negative t values.

References to the application of the interference model are given in the review article by Hite [15g.] and the connection with FESR is discussed by Kellett [37].

1.1.6 Schmid Loops

Schmid [38] was the first to show that upon analysing the Regge contribution into partial waves, structures may appear in the Argand diagram (in which the Real and Imaginary parts of A_ℓ are plotted) of partial waves which resemble closely resonances. The Regge pole must contain many partial waves so that although these may vary rapidly with energy their total sum is smooth. Schmid took the Regge parameters as determined by fits at high energy to extrapolate to the ρ-exchange amplitude in πN charge exchange scattering down to energies ~ 2 GeV. Then performing a partial wave analysis on this he obtained for each partial wave a loop on the argand diagram very similar to those obtained by phase shift analysis as evidence for nucleon resonances. Moreover, these 'pseudo-resonances' were shown to lie approximately on a linear rising Regge trajectory. Such a behaviour of partial wave phases is an almost exact consequence of the Regge form of the amplitude, for any exchanged trajectory with finite slope [39].

These circles on the Argand diagrams are caused mainly by the changing phase $e^{-i\pi\alpha(t)}$ in the signature factor of the ρ-exchange amplitude. So that in the expression

$$\text{Im } A_\ell(E) = \frac{1}{2} \int_{-1}^{1} dz \, P_\ell(z) \, A(E,z)$$

use is made of the identity

$$i^\ell j_\ell(z) = \frac{1}{2} \int_{-1}^{+1} e^{iz\cos\theta} P_\ell(\cos\theta) \, d(\cos\theta) \quad \left(\equiv i^\ell \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} J_{\ell+\frac{1}{2}}(z)\right)$$

so that for a real linear trajectory (where the equal mass case is taken for simplicity) of slope α', and constant residue, i.e.

$\alpha(t) = \alpha(0) + \alpha' t$, $t = -2q^2(1-\cos\theta)$, we have

$$\frac{1}{2}\int_{-1}^{+1} e^{-i\pi\alpha(t)} P_\ell(\cos\theta)\, d(\cos\theta) = e^{-i\pi(\alpha(0)-2q^2\alpha\ )} \times i^\ell j_\ell(-2q^2\pi\alpha') \qquad 11.9$$

($j_\ell(z)$ is a spherical Bessel function).

For each ℓ the phase of the partial wave amplitude increases with s ($s \sim q^2$) and eventually reaches $\frac{\pi}{2}$ for some $s = s_\ell$. For another partial wave, $\ell' = \ell + \delta\ell$ the phase is reached for $s = s_{\ell'} = s_\ell + \frac{\delta\ell}{\alpha'}$

Collins et. al. [40] give plots of some of the π-N partial wave amplitudes beginning at threshold obtained from a Regge pole fit to high energy data. The agreement of the Regge projection with experiment is not so impressive as it appears to be because the energy dependence is not shown. Some further discussion on the interpretation of these loops is given in refs. [41,42]. (Schmid points out that the authors of ref. [41] obtain unwanted loops in K^+p elastic scattering because in their analysis they failed to include the Y_0^* $(\frac{3}{2}^-, \frac{7}{2}^-, \ldots)$ Regge trajectory).

Incorporating the signature factor into $\beta(t)$ it is seen that the resonance structure is given by the zeroes of $\beta(t)$ which appear as dips in the angular distribution. This correlation is shown to agree experimentally in that channels forbidden by the quark model (called 'exotic') such as pp and K^+p do not show these dips while non-exotic channels like $\bar{p}p$ or K^-p do.

Schmid further claims that the equivalence between t-channel Regge poles and s-channel resonances holds locally at each intermediate energy. This is called <u>local duality</u> and is assumed for the imaginary part of the amplitude only. So that if one considers the difference of two FESR

$$\int_{N_1}^{N_2} d\nu\ \nu^n\ \mathrm{Im}\ A(\nu,t) = \sum_i \beta_i(t)\left[\frac{1 \pm e^{-\pi\alpha_i(t)}}{\sin\pi\alpha_i(t)}\right]\left[\frac{N_2^{\alpha_i+n+1} - N_1^{\alpha_i+n+1}}{\alpha_i + n + 1}\right] \qquad 11.10$$

then for N_2 close to N_1 the Regge formula should be a good approximation to the scattering amplitude in the local sense i.e. point by point.

1.1.7 The Deck Effect

In an effective mass distribution of a resonances' decay products the question arises: What is the background? In the specific reaction $\pi N \rightarrow \pi\rho N$ Deck [43] observed a peak near the A_1 resonance in the final $\pi\rho$ mass spectrum, despite the fact that his model had no pole in this variable. The double peripheral model for three particle final states was used and the substantial low-mass enhancement over phase space was seen in the two-body subchannels.

Further investigation [44,45], using the double Regge model [46], was made into this effect. The Duality explanation [47] was that the "no resonance" situation that gives rise to a "bump" in the cross-section Fig. 6(a) and the "Resonance" situation Fig. 6(b) should not be added as in the Interference model but that these are descriptions of the same phenomena.

The conjecture that the presence of a Deck enhancement could be interpreted as evidence for the existence of the A_1 resonance was however, criticised on two points. First [15h] that Duality was applicable only to the imaginary part and not to the full amplitude and hence not to the cross-section especially if the amplitude were predominantly real. Secondly that the Deck effect is essentially of kinematic origin and should appear for any amplitude with appropriate peripheral properties independently of whether there were resonance poles in the Deck variable or not. Thus one might distinguish a real resonance from a Deck enhancement by a study of the imaginary part or the phase variation in the mass variable.

1.1.8 Pomeranchuk Exchange

Consider collisions of the type $A+B \rightarrow A'+B'$ (see Figs. 7 and 8) which occur without exchange of the internal quantum numbers I.Q.N. (such as baryon number, hypercharge, isospin, or G-parity) i.e.

when $\left.\begin{array}{l}\text{IQN }(A') = \text{IQN }(A)\\ \text{IQN }(B') = \text{IQN }(B)\end{array}\right\}$. In all the measured cases of this type

$\frac{d\sigma}{dt}$ at fixed t shows a weaker s-dependence than for the exchange type collisions.

i.e. when $\left.\begin{array}{l}\text{IQN }(A') \neq \text{IQN }(A)\\ \text{IQN }(B') \neq \text{IQN }(B)\end{array}\right\}$, and is compatible with the approach to a finite limit. The data can be described by an amplitude of the form

$$A \simeq \beta_p(t) \frac{\left[1 + e^{-i\pi\alpha_p(t)}\right]}{\sin \pi\alpha_p(t)} \nu^{\alpha_p(t)} + \Sigma \text{ Reggeized Particle Exchange.} \qquad 11.11$$

where β_p is real and $\alpha_p(t)$, called the Pomeranchuk trajectory (or Pomeron), is subject to

$$0 < \alpha_p'(t=0) < 0.5.$$

The mathematical form for the Pomeron is probably more complicated than the above (Regge cuts for example may be required).

When there are 3 or 4 particles in the final state the Pomeranchuk exchange dominates whenever it is allowed, and this leads to the clustering of the final particles as in Fig. 8(b).

Harari [48] and [17a] (also Freund [49] and Gilman et al [50]) suggested that one takes from the relation

$A = A_{DUAL} + A_{POMERON}$ an identification of the Pomeranchuk term with the non-vanishing and non-resonating background. Thus direct channel resonances are not to be associated with Pomeranchuk exchange as this would have implied isospin degeneracy due to the fact that no non-vacuum trajectories are degenerate with the leading vacuum singularity. This Harari-Freund form of duality assumes that the Pomeron is built exclusively from the background whereas the other Regge poles are built exclusively from the resonances. However, the original dual scheme proposed by Schmid [38] assumed that the resonances built all the Regge poles including the Pomeron whereas the background summed to zero. In the interference model of Barger and Cline [34] on the other hand, the Pomeron and other Regge poles are built from the background while the resonances sum to zero. Each of these schemes is characterized by the fact that the Pomeron on the one hand and the other Regge poles on the other hand are built exclusively from either the background or the resonances. Support for this 'two-component' form of duality was presented by Harari and Zarmi [51] who on analysing πN scattering data found that the Argand diagrams for $I_t=0$ and $I_t=1$ suggested an identification of the large imaginary background (seen in the $I_t=0$ diagram) with the Pomeron. When there are no s-channel resonances the imaginary part of the amplitude in this scheme is entirely given by the Pomeranchuk term, and this vanishes when Pomeron exchange is forbidden. This would apply to reactions like $K^+p \rightarrow K^+p$ for example and implies degeneracy of the ω, ρ, A_2, p' trajectories and allows for the prediction of the SU(3) mixing angles of ω-ϕ, f-f' [52].

Del Guidice and Veneziano [53] have shown, however, that in a crossing symmetric picture, the duality between Pomeron and non-resonant background is not compatible with resonance saturation. If non-resonant background is present in Pomeron channels, crossing puts it also in channels where no Pomeron is possible.

This exceptional role for the Pomeranchuk trajectory is consistent with its apparent flatness and the absence of low mass resonances on the trajectory. The assumption that there is a flat trajectory may however not be correct and Rosner [54] showed that this form of duality leads to an inconsistency in baryon-antibaryon scattering (which imply 6 quark meson states). The role of the Pomeron in the duality picture is thus rather mysterious.

1.1.9 Straight Parallel Trajectories

In the early stages of Regge theory in analogy with potential scattering or from simple S-matrix calculations, which neglected multiparticle intermediate states, a Regge trajectory had a form similar to that in Fig. 9. For example Squires could state [55] in 1963 that the "Re α will probably turn over so that it does not reach very high real values of spin for real s (Mass^2) and: "The approximate agreement of the slopes with $\frac{d\alpha}{ds} \approx 1$ (for the known particles and resonances) - is striking - and better than we have any right to expect! Note that, even when we have two points on the same trajectory, the correct path joining them will not be a straight line but some curve, yet to be determined".

However, it now appears that at least for positive t trajectories are, over several GeV, approximately linear and moreover all the observed trajectories (except possibly for the Pomeron) are approximately parallel with slope $\alpha' \sim 1\ \mathrm{GeV}^{-2}$. Fig. 10 shows a "Chew-Frautshi plot" of spin (J) versus (mass)^2 for the meson trajectories with I=0 and I=1 trajectories coinciding and signature showing no effect so that four trajectories appear to ride on top of each other.

If $\alpha(t)$ increases proportionally with t at large t then we can write a dispersion relation of the form

$$\alpha(t) = a + bt + \frac{1}{\pi}\int \frac{dt'\ \mathrm{Im}\ \alpha(t')}{t' - t}$$

so that if Im $\alpha(t')$ is small then the linearity condition follows. The condition Im α be small amounts to requiring that the resonances be narrow. Should the integral diverge it would require subtraction.

Some plots of the meson trajectories using recent data are given in the review by Collins [15k] for example.

1.1.10 Duality Diagrams

Harari [56] and Rosner [57], following Imachi et al [58], independently suggested that one could represent scattering amplitudes in terms of continuous quark lines and in such a way that exotic resonances were forbidden in both direct (s) and exchanged (t) channels. For the construction of dual models, such graphs were used quite early. In the representation of Gell-Mann [59] Mesons were made of quark - anti-quark pairs ($q\bar{q}$) and baryons of three quarks (qqq) and an "exotic hadron" was defined as any meson whose isospin and hypercharge are such that it cannot be made of a quark (q) and an anti-quark ($\bar{q}$) or any baryon not expressible as (qqq). There appears to be little evidence for the existence of either of these. The quark properties are listed in fig. 11 following the notation p, n, λ of Zweig [60], and various Duality diagrams are shown in fig. 12 including those used more recently for the Regge-Pomeron-Regge cuts of Girardi et al [61] (fig. 12(e)) and also the "illegal" diagrams (fig. 12(c)) which have ($qq\bar{q}\bar{q}$) and ($q\bar{q}qqq$) channels. Certain selection rules were postulated to take into account experimental data on cross-sections (Lipkin's Rule [62]).

Three hadrons can couple to one another only if every pair is connected by at least one quark line [63] and in addition:(dynamical justification given in [64]) No quark line begins and ends in the same hadron.

The quark line from baryon to baryon, that gives the third quark for a baryon, is called a "spectator" and in the non-planar graph of (fig. 12(d)) there is none because each baryon forming quark becomes a meson forming one.

In the notation [64] $M_4 \equiv qq\bar{q}\bar{q}$

$$B_5 \equiv qqqq\bar{q}$$

(M for meson, B for baryon, subscript denoting total number of quarks) the rules forbad the coupling (M_4 M_2 M_2) but not the coupling (M_4 $\bar{B}_3$ B_3) thus allowing the coupling of exotic mesons (M_4) to baryon - anti-baryon pairs (but not to meson-meson pairs). This is known as the $B\bar{B}$ problem [65] and the "illegal" diagram is shown in fig. (12(d)). A strict form of duality would require the existence of such M_4 mesons thus raising some interesting experimental questions. Alternatively some form of "broken duality" is required, [66] in which a complete breakdown of duality in $B\bar{B} \to B\bar{B}$ is used. If the exotic mesons M_4 in $B\bar{B}$ exist they can generate exotic baryons B_5 ($qqqq\bar{q}$) when scattered off baryons ($B_3 \equiv qqq$) [63], and highly exotic states then couple only where they are needed for duality and never destroy earlier sets of constraints. The baryon - anti-baryon elastic channel thus appears to be a place where duality could be crucially tested [67].

Processes which cannot be described by legal diagrams are predicted to have purely real amplitudes (and hence zero polarisation) at small t values as the imaginary part should vanish by duality. A further prediction is that the transitions $\pi^{\pm} \to \phi$ are not allowed by the diagrams so that, for example, $\sigma(\pi N \to \phi N) = 0$ which is in good agreement with experiment. Fig. 13 gives a summary of the well known mesons fitted into the $q\bar{q}$ model.

1.1.11 Exchange Degeneracy (EXD)

We have seen that Duality has given the following two-component prescription:

$$\begin{aligned} \text{Im A (resonances)} &\simeq \text{Im A (Regge poles)} \\ \text{Im A (background)} &\simeq \text{Im A (Pomeron)} \end{aligned} \qquad 11.12$$

where $\simeq$ means approximate equality when averaged over some energy interval at fixed t when t-channel Regge exchange is being considered. From the practical standpoint one of the most striking consequences of this prescription is that if resonance formation $A+B \rightarrow R$ is impossible (i.e. the s-channel is "exotic" e.g. in $K^+n \rightarrow K^o p$ collisions) then Im A(Regge) $\simeq 0$ for both t- and u- channel exchanges. In order not to have the null solution of decoupling all the Regge poles we satisfy Im A (Regge) =0 in the s-channel by imposing that the various cross-channel Regge poles (here ρ, A_2) compensate each other by having opposite signatures but equal couplings and trajectories - EXD.

Consider the specific example of $K^+n \rightarrow K^o p$ which has an exotic s-channel (i.e. no qqq) and ρ and A_2 t-channel Regge exchanges.

For the two exchanges we have the amplitudes:

$$\text{Amp. } (A_2) = \beta^+(t)\left(\frac{-1-e^{-i\pi\alpha_+(t)}}{\sin \pi\alpha_+(t)}\right)\left(\frac{s}{s_o}\right)^{\alpha_+(t)}$$

$$\text{Amp } (\rho) = -\beta^-(t)\left(\frac{+1-e^{-i\pi\alpha_-(t)}}{\sin \pi\alpha_-(t)}\right)\left(\frac{s}{s_o}\right)^{\alpha_-(t)} \qquad 11.13$$

The requirement that on addition this should be purely real leads to the restriction:

$$\alpha_+(t) = \alpha_-(t) = \alpha(t)$$
$$\beta_+(t) = \beta_-(t) = \beta(t) \qquad 11.14$$

and the sum: $\frac{-2\beta}{\sin\pi\alpha}\left(\frac{s}{s_o}\right)^{\alpha}$

The 'poles' here may include cuts i.e. 'Argonne' [69] type cuts which give no effect to the results. (They used the WSZ in contrast to the 'strong' cuts of the 'Michigan' [70] school).

A search for t-channel structure in differential cross-sections for two body reactions which have exotic direct channels (and by this scheme pairs of exchanged poles with opposite signature) shows that the prediction of 'no dip' is widely obeyed and this still holds when making an SU(3) extension to further processes.

The example of the $B\bar{B}$ problem would be:

$$\bar{p}\, p \rightarrow \bar{\Xi}\,\Xi$$

In this case the s-channel is non-exotic but the t- and u-channels are and the requirement

$\mathrm{Im.}\ \Sigma(\text{non exotics}) = 0$ would imply the unreasonable restriction that ρ, f_o, ω, A_2 should decouple from the $\bar{\Xi}\,\Xi$ system.

In meson-meson scattering, $\pi\pi$-scattering implies ρ, f_o EXD, πK-scattering implies ρ, f_o EXD, and KK scattering requires ω, $f_o(I=0)$, ρ, A_2 $(I=1)$ and

$$\beta_{K\bar{K}}^{I=0} = \beta_{K\bar{K}}^{I=1} \quad \text{equalities.}$$

Duality has thus conventionally arranged that an exotic amplitude is made real through EXD: if even and odd signature trajectories having the same quantum numbers are equal, and their residues are equal too ("strong" exchange degeneracy), then the contribution of a pair of trajectories is

$$A_{EXOTIC} \equiv A_E \sim \beta(t) \left[\frac{1+e^{-i\pi\alpha(t)}}{2}\right] s^{\alpha(t)} + \beta(t) \left[\frac{1-e^{-i\pi\alpha(t)}}{2}\right] s^{\alpha(t)} \qquad 11.15$$

$$= \beta(t)\, s^{\alpha(t)}$$

which is purely real. The contribution of the same pair to the corresponding line reversed, or non-exotic amplitude is, however,

$$A_{NON\text{-}EXOTIC} \equiv A_N \sim \beta(t) \left[\frac{1+e^{-i\pi\alpha(t)}}{2}\right] s^{\alpha(t)} - \beta(t) \left[\frac{1-e^{-i\pi\alpha(t)}}{2}\right] s^{\alpha(t)}$$

$$= \beta(t)\; s^{\alpha(t)}\; e^{-i\pi\alpha(t)} \qquad 11.16$$

and is said to have a "rotating phase". "Weak" EXD consists of breaking EXD for the residues ($\beta(t)$) and retaining it for the trajectories ($\alpha(t)$). Experimental support for even strong EXD seems good A. Firestone et al [71] found the "exotic" process $K^+n \to K^0p$ to be overwhelmingly real so that retaining it to the greatest degree possible is desirable. Care in tampering with the residues is required since the roles of A_E and A_N can be interchanged if the residues are altered [72].

The EXD constraints, required by resonance - Regge pole duality and the absence of exotic particles, have many consequences. One of the most interesting predictions is that differential cross-sections should become asymptotically equal for pairs of processes related by line-reversal [73]. Well known examples of this prediction are [71,74].

$$\frac{d\sigma}{dt}\ (K^-p \rightarrow \bar{K}^o n) = \frac{d\sigma}{dt}\ (K^+n \rightarrow K^o p)$$

and

$$\frac{d\sigma}{dt}\ (\pi^+p \rightarrow K^+\ \Sigma^+) = \frac{d\sigma}{dt}\ (K^-p \rightarrow \pi^-\ \Sigma^+)$$

Schmid [75] showed that strong EXD held for the Y^*'s in $\bar{K}N$ which was expected because the FESR are linear relations and refer to one amplitude at a time. Martin and Michael [76] showed that between 3 and 4 GeV/c the differential cross sections for

$K^-\ p \rightarrow \Lambda\ \pi$ (pure I=1 in the s-channel)

$K^-\ p \rightarrow \Lambda\ \eta$ (pure I=0 in the s-channel)

could be related assuming SU(3) and the exchange of exchange degenerate vector (K^*) and tensor (K^{**}) trajectories.

Similar results were presumed to be true for reactions in which resonances are produced, e.g.

$$\frac{d\sigma}{dt}\ (K^+p \rightarrow K^{*+}p) = \frac{d\sigma}{dt}\ (K^-p \rightarrow K^{*-}p)$$

It has been shown [73] that a very general class of dual models predicts that these cross-section equalities are only true for reactions involving stable reactions.

These results depend on "weak" EXD but in general $\frac{d\sigma}{dt}$ (real phase) > $\frac{d\sigma}{dt}$ (rotating phase). Polarization effects (that depend on interference terms) will vanish (when there is no $\mathbb{P}$ exchange) when "strong" EXD holds; a prediction that appears to be violated. In general it was found that the larger the spin 'non-flip' contribution to the amplitude the worse were the results of using EXD whilst the spin 'flip' amplitudes were successful in their predictions and data fits.

Both the straightness of the trajectories, and exchange degeneracy (EXD trajectories occur only in the absence of an exchange (Majorana) force) were completely unexpected, and seem quite at variance with the potential scattering ideas which motivated the introduction of Regge poles into particle physics.

1.1.12 Duality Breaking

If the question asked is: "Given the set of meson trajectories generated by the quark model, [77] what further constraints are imposed by duality?" [78] then the answer is that one requires that the meson trajectories (for no $\lambda(\bar{\lambda})$ quarks in the $q\bar{q}$ state) corresponding to the $(q\bar{q})_L$ model have the form of Fig. 14. The degeneracy of the Fig. is only approximately realized. The main difficulty comes from the $N(= P(-1)^J)=-1$ trajectories which is presumably due to the large deviation from the "ideal" (to give $q\bar{q}$ structure) mixing angle.

Logan and Roy [79] showed that the only solutions of duality and absence of exotic resonances for M-M and M-B scattering, which are consistent with SU(3) symmetric couplings, are the ones in which all the members of the vector and tensor nonets are degenerate with each other. If less stringent degeneracy requirements are assumed and the solution of Harari and Freund taken then they show that it is necessary to have an unreasonable kind of SU(3) breaking of the coupling strengths.

As has been mentioned in (1.1.10) one can eliminate the $\bar{B}B$ problem by abandoning factorization [80] so that a non vanishing polarization then is possible for MB scattering in accord with experiment. We conclude that 'duality' is thus not a perfectly rigorous solution to the strong interaction problem but can be taken as an approximate description of nature.

1.1.13 Successful FESR Bootstrap

Ademollo, Rubinstein, Veneziano and Virasoro [81] applied FESR to a particularly simple case, $\pi\pi\to\pi\omega$, which led Veneziano to his well known formula.

In this process only one amplitude is different from zero, parity being conserved and the ω having no isospin and this amplitude is completely crossing symmetric. The ρ- trajectory will dominate the direct and the crossed channels and in each case one has I=1, G=+1, normal parity and negative signature.

From Fig. 15 they get:

$$T_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma} \; e_{\mu}^{(\lambda)} \; \varepsilon_{\mu\nu\rho\sigma} \; P_{1\nu} \; P_{2\rho} \; P_{3\sigma} \; A(\nu,t)$$

[Where $\alpha\beta\gamma$ are isospin indices, $\varepsilon_{\alpha\beta\gamma}$, $\varepsilon_{\mu\nu\rho\sigma}$ are Ricci tensors, $e_{\mu}^{(\lambda)}$ is the polarization vector of the ω; $A(\nu,t)$ is an invariant kinematic singularity free amplitude; $\nu = \frac{s-u}{4}$ where s, t and u are the Mandelstam variables]. The one independent Helicity amplitude has asymptotic behaviour

$$A(\nu,t) \underset{\nu\to\infty}{\longrightarrow} \beta(t) \left[\frac{1 - e^{-i\pi\alpha(t)}}{\sin\pi\alpha(t)}\right] \left(\frac{\nu}{\nu_1}\right)^{\alpha(t)-1} \qquad 11.17$$

where $\beta(t)$ was parameterized as $\beta(t) = \frac{\bar{\beta}(t)}{\Gamma(\alpha(t))}$ so that the proper zeros appeared at nonsense points.

The FESR for the nth moment is

$$\int_0^{\bar{\nu}} \nu^n \; \mathrm{Im}\, A(\nu,t)\, d\nu = \frac{\beta(t)}{\alpha(t)+n} \left(\frac{\bar{\nu}}{\nu_1} \right)^{\alpha(t)-1} \bar{\nu}^{n+1} \qquad 11.18$$

Assuming a linear trajectory over the range of interest, a "narrow width" approximation for the resonances and then that $\bar{\beta}(t)$ was a constant $\bar{\beta}$ they firstly took $\bar{\nu}$ in a suitable range and found the cut-off to be midway between the $J^P = 1^-$ and 3^- resonances in $(\text{mass})^2$ units.

For the n=1 case they found that $\text{Im}\, A \sim \frac{\bar{\beta}(m_\rho^{\,2})}{\alpha'} \quad \delta(\nu-\nu_\rho)$

and their sum rule was:

$$4\,\nu_\rho = (2\,m^2 + t - \Sigma) = \frac{\alpha(t)}{\alpha'} \underbrace{\left[\frac{(2\bar{\nu}\alpha')^2}{\Gamma(\alpha+2)} \quad \left(\frac{\bar{\nu}}{\nu_1}\right)^{\alpha-1} \right]}_{\phi} \qquad 11.19$$

where $\Sigma = 3\,m_\pi^{\,2} + m_\omega^{\,2}$.

When $t = \Sigma - 2m_\rho^{\,2} \sim -m_\rho^{\,2}$, $\nu_\rho = 0$ and the equation is satisfied if $\alpha(-m_\rho^{\,2}) = \alpha(-0.53) = 0$.

This zero was confirmed from the experimental analysis of the CEX.πN scattering data where a dip is observed.

On putting the term $\phi = 1$, from $\nu_\rho = \frac{\alpha}{4\alpha'}$, the cut off was then

$$\bar{\nu} = \frac{\alpha(t)+2}{4\alpha'} = \frac{1}{4}\left[2m_\rho^{\,2} + \frac{2}{\alpha'} + t - \Sigma \right]$$

and this choice of cut-off midway between the last resonance included and the first left out turned out to be a general property of the equations used.

In order to enlarge the region of t where the FESR was satisfied other resonances lying on the ρ-trajectory were taken into account, their contribution being evaluated from the crossing symmetry of the amplitude.

They show that in general the cut off is $\bar{\nu} = \frac{\nu_n + \nu_{n+1}}{2}$

and the i-th resonance position is

$$\nu_i = \frac{4(i-1) + \delta + \alpha(t)}{4\alpha'} \quad \text{where } \delta = -\alpha \quad -2m_\rho^{\,2} + \Sigma \qquad 11.20$$

For n=1 as above $\delta = 0$ corresponded to $\alpha(-m_\rho^{\,2})=0$ and a good Regge-Resonance agreement was found.

For n=2 $\delta = -0.05$ corresponding to $\alpha(-0.58)=0$ and again good agreement was found.

However, increasing n led to bad agreement and the Regge pole no longer averaged the resonance contribution because the resonances on the one ρ-trajectory could not keep up with the Regge side.

A solution was suggested which possessed daughter trajectories assumed to be linear, and then found to be parallel, with small residues $\beta(t)$ which also agreed with a theorem due to Khuri on the singularity of $\beta(t)$ at ∞.

1.1.14 The Veneziano Model

It is the aim of the present-day S-matrix theory, which follows the first proposals of Heisenberg [82], to obtain scattering functions such that the following fundamental assumptions are satisfied:

1. Analyticity in the kinematic variables.
2. Crossing symmetry under the interchange of scattering channels.
3. Unitarity. Required in order to preserve probability under the assumption of complete sets of initial and final states.

From the theoretical framework of Regge theory one could require

4. Regge asymptotic behaviour.

From the previous section on duality one could add

5. 'Duality' in the global and local senses.
6. Resonances on linear rising trajectories with the possibility of 'daughters'.

As a result of extensive work on FESR and in particular the success of the application to the process $\pi\pi\to\pi\omega$ with Ademollo et al. [81], G. Veneziano [83] wrote down a neat simple invariant amplitude, in terms of Euler Beta functions, for the process $\pi\pi\to\pi\omega$ which satisfied all but no. 3 of the properties listed above as the resonance poles were actually on the s-axis.

From the definition $T_\mu = \varepsilon_{\mu\nu\rho\sigma} P_{1_\nu} P_{2_\rho} P_{3_\sigma} A(\nu,t)$ given in 1.1.13 and assuming parallel linear trajectories it was found that asymptoticly ($s\to\infty$, fixed t)

$$A \to \frac{\bar{\beta}}{\pi} \Gamma(1 - \alpha(t)) (- \alpha(s))^{\alpha(t)-1} + (s\leftrightarrow u) \qquad 11.21$$

and that this was a good parametrization for the amplitude in the high s-region in the sense that it was able to reproduce itself when introduced in FESR. Veneziano replaced the term $(-\alpha(s))^{\alpha(t)-1}$ by $\Gamma(1-\alpha(s))$ and divided by another Γ function in order to have the correct asymptotic behaviour and was led to the expression:

$$A(s,t,u) = \frac{\bar{\beta}}{\pi}\left[B(1-\alpha(t),\ 1-\alpha(s)) + B(1-\alpha(t),1-\alpha(u)) + B(1-\alpha(s),\ 1-\alpha(u))\right] \qquad 11.22$$

where $\bar{\beta}$ is a constant, and $\bar{\beta} = \Gamma(\alpha(t))\ \beta(t)$ and

$B(x,y) = \dfrac{\Gamma(x)\ \Gamma(y)}{\Gamma(x+y)}$ is the well known Euler Beta Function.

The expression $$B(1-\alpha_t,\ 1-\alpha_s) = \frac{\Gamma(1-\alpha_t)\ \Gamma(1-\alpha_s)}{\Gamma(2-\alpha_s-\alpha_t)} \qquad 11.23$$

(where $\alpha(x) = \alpha_x$) has the properties:

(i) $B(1-\alpha_t,1-\alpha_s) \xrightarrow[s\to\infty]{} (-\ \alpha_s)^{\alpha_t\ -1}\ \Gamma(1-\alpha_t)$ (for fixed t)

thus reproducing the asymptotic relation in (11.21)

(ii) Whenever α_s or α_t take positive integral values the function will have poles, but because of the denominator contribution there is no double pole in the two variables. Lines of poles and lines of zeros in the plane given by $\alpha_s \vee \alpha_t$ are thus of a simple straight line pattern.

(iii) $$B(1-\alpha_s,\ 1-\alpha_t) = \sum_{n=o}^{\infty}\left[\frac{\Gamma(\alpha_t+n)}{\Gamma(n+1)\Gamma(\alpha_t)}\right]\frac{1}{(n+1-\alpha_s)}$$

$$\text{or}\ \sum_{n=o}^{\infty}\left[\frac{\Gamma(\alpha_s+n)}{\Gamma(n+1)\Gamma(\alpha_s)}\right]\frac{1}{(n+1-\ \alpha_t)}$$

The residue of a pole in the s-variable is a polynomial in the t-variable and vice-versa. This function can thus be written either as a sum of s-channel poles or t-channel poles. The coefficient of the form $C_n(t) = \Gamma(n+\alpha_t)\ /\ \Gamma(\alpha_t)$ is an n^{th} degree polynomial with n equally spaced zeros but it does not coincide with the Legendre polynomial $P_n(\cos\theta)$ associated with a resonance at n=J. The pole at some $t=t_n$ in fact corresponds to a multiplet of n-particles with the same mass $m_n = \sqrt{t_n}$ and spins J=0, 1, ..., n.

(iv) $B(1-\alpha_s, 1-\alpha_t) = B(1-\alpha_t, 1-\alpha_s)$ so that crossing symmetry is obeyed for this expression. In fact the amplitude is invariant under cyclic and anti-cyclic permutation of the external lines and the fully crossing-symmetric expression is given by the sum of the three non-equivalent terms.

For the linear trajectory case $\alpha_x = \alpha_o + \alpha'.x$ Regge asymptotism is true in the whole complex s-plane except on the real axis where the narrow resonances lie and if the trajectories are strictly real the absorbtive part is just a sum of δ-functions. (In this narrow-width resonance assumption when $\alpha_s \to J$, $\frac{1}{\alpha_s - J \pm i\varepsilon} = \frac{P}{\alpha_s - J} \pm i\pi\delta(\alpha_s - J)$

so that Im A involves Im $\frac{1}{(\alpha_s - J \pm i\varepsilon)} = \pm i\pi\delta(\alpha_s - J)$.) If α_s were given an imaginary part increasing with energy, however, then unwanted "ancestors" (in which arbitrary high spins are associated with a pole in the s-channel) would appear. Since the amplitude gives both low energy resonances and high energy Regge behaviour 'duality' is obeyed in some sense.

The amplitude is appropriate for the non-diffraction reaction $0^+ + 0^+ \to 0^+ + 0^+$ so that in order to remove poles at even values of α for the reaction being studied Veneziano applied the constraint $\alpha_s + \alpha_t + \alpha_u = 2$.

Finally it was shown that the formula (11.22) was a solution of the superconvergence relations.

This expression for a scattering amplitude in terms of Beta functions raised several important problems:

(i) Unitarity

The narrow resonance approximation of the amplitude with poles along the real axis violates unitarity. Three significant different approaches to this problem were suggested:

(a) A simple and crude solution was to introduce complex trajectories to take the poles off the real axis in the physical region and assure the correct high energy Regge behaviour. This leads to unwanted ancestors (although their residues could be very small) [84] by destroying the polynomial form of the residues and gives equal total widths to all partial waves that resonate at the same mass.

(b) Another approach, due to Martin and subsequent workers [85], was to consider the Beta function as a distribution to be "smeared" out by a suitable convolution integration process that moved the poles out of the physical region but modified the high energy Regge behaviour.

(c) A more ambitious scheme than the phenomenological approaches of (a) and (b) was to treat the Veneziano formula as a Born term in a perturbative approach [86]. Work on this approach is still in hand [18]. Other methods of unitarizing included the K-matrix of Lovelace [87] and several further ingeneous models [88].

(ii) Non Uniqueness and Satellite Terms.

The Beta function amplitude (11.23) could equally well have been written in the form:

$$V_{nmp} = \frac{\Gamma(m-\alpha_s)\,\Gamma(n-\alpha_t)}{\Gamma(m+n+p - \alpha_s-\alpha_t)} \qquad 11.24$$

This has the same basic properties as the expression given by Veneziano. The first poles appear at α_s=m and α_t=n (m and n positive integers and p is required to be an integer and ≤ 0 if the residues on the poles are to be polynomials) instead of zero and the asymptotic behaviours correspond respectively to

$$s^{\alpha_t - n - p} \quad \text{and} \quad t^{\alpha_s - m - p}$$

with n+p and m+p>0, corresponding to daughter behaviour. We may add such "satellite" terms together without modifying any of the desired properties, such as leading high energy behaviour, and in so doing can eliminate unwanted daughter contributions, in particular odd daughters [89], and ghosts (when residues have negative values).

(iii) Extension to Physical Particles

(a) In order to apply the Veneziano formula to physical processes several authors suggested various formulations to include fermions [90], mesons [91] and baryons [92]. Once the Veneziano formula had been extended to the five-point function [93] and then the N-point function [94] attempts were made to include fermions and bosons in a consistent procedure [95]. One solution is to use Veneziano forms for invariant amplitudes that are kinematical singularity free and which have their meaning unchanged under crossing. There are parity doubling problems for this approach as well as that of the relativistic quark models [91] and even departing only slightly from the straight line trajectories modifies the Regge behaviour [84] and no longer gives residues polynomial in the dual variable ("ancestors"). Heimann [96] has discussed some of the questions involved in including fermions in dual amplitudes.

(b) The role of the Pomeron ($\mathbb{P}$) when applying a Veneziano type of amplitude needs to be clarified. (Roberts [97] for example, found that a particular generalization of the Veneziano model gave rise to a trajectory of the form $\alpha_{\mathbb{P}}(t) = 1 + (0.2 \pm 0.4)t$ and concluded that the small slope was consistent with almost complete absence of shrinkage of the diffraction peak in πN scattering so that the $\mathbb{P}$ did not fit a Regge pole scheme. See also section 1.1.8).

(c) The incorporation of isospin into the Veneziano model by a very simple general method was made by Chan and Paton [98]. This method preserved all the desired properties of the model, gave no unwanted states of high isospin and avoided the presence of exotic resonances. The desired isospin factors were given as certain trace terms corresponding to the ordering of the particles (the explicit realization for five particles being given in Chapter 2).

(iv) Duality.

Some difficulty was involved in sorting out exactly which notion of 'duality' was used in the model [99]. However, Sasaki and Sugano [100] demonstrated the Regge poles - Regge poles duality in each channel ignoring Regge cuts, in a Veneziano like amplitude possessing a family of parallel trajectories.

1.1.15 Generalizations of the Veneziano Formula

Many authors have tried to derive the Veneziano representation from general properties of scattering amplitudes [101], in particular using meromorphic approximations. The suggestion of Bassetto that the amplitude be written:

$$A(s,t,\lambda) = \frac{\Gamma(1-\alpha_s)\ \Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} \quad {}_2F_1(1-\alpha_s, 1-\alpha_t;\ 1-\frac{\alpha_s+\alpha_t}{2};\ \lambda) \qquad 11.25$$

produces a Veneziano pattern of straight line zeros for $\lambda=0$, an alternating straight line - wavy line pattern for $\lambda=\frac{1}{2}$ and an 'Odorico' [102] pattern of straight line zeros for $\lambda=1$. Ferrari and Grillo [103] gave a more general form than the Beta function for the integral representation of the amplitude and Virasoro [104] produced an example of how the Veneziano expression could be extended so that when $\alpha_s+\alpha_t+\alpha_u=2$ the Veneziano form for $\pi\pi\to\pi\omega$ is reproduced. Explicitly this expression was:

$$A(s,t,u) = \frac{\beta\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_s)\ \Gamma(\frac{1}{2}-\frac{1}{2}\alpha_t)\ \Gamma(\frac{1}{2}-\frac{1}{2}\alpha_u)}{\Gamma(1-\frac{1}{2}(\alpha_u+\alpha_t))\ \Gamma(1-\frac{1}{2}(\alpha_s+\alpha_u))(\Gamma(1-\frac{1}{2}(\alpha_s+\alpha_t))} \qquad 11.26$$

Additions to the Veneziano expression in order that Regge cuts may be introduced have also been given [105] and in order to incorporate Mandelstan analyticity a new integral formulation for the dual crossing symmetric amplitude was suggested [106].

1.1.16 Applications of the Veneziano formula

(i) The $\pi\pi\to\pi\pi$ scattering process

For $\pi\pi$ scattering [107,108,109] one starts with a linear exchange degenerate ρ - f_o Regge trajectory

$$\alpha_\rho(x) = \alpha_\rho(o) + x.\alpha' = \alpha_x$$

and requires the lowest particle on the trajectory to have spin 1 (since $\alpha(o) > 0$ and the zero point must have no particle). For $\pi^+\pi^-$ elastic scattering the $\pi^+\pi^+$ u-channel is exotic, hence implying exchange degeneracy of these two ρ and f trajectories and resonances in both s- and t- channels should then be spaced by one unit of spin instead of two. Introducing the function

$$V(x,y) = -\lambda \frac{\Gamma(1-\alpha_x)\ \Gamma(1-\alpha_y)}{\Gamma(1-\alpha_x-\alpha_y)} \qquad 11.27$$

where λ is an overall constant, (which can be obtained from $g_\rho\pi\pi$), the $\pi\pi$ amplitudes and isospin amplitudes are :

$$\begin{cases} A(\pi^+\pi^-\to\pi^+\pi^-) = -\lambda\ V(s,t) \\ A(\pi^+\pi^o\to\pi^+\pi^o) = -\frac{\lambda}{2}(V(s,t)+V(t,u) - V(u,s)\ ) \\ A(\pi^o\pi^o\to\pi^o\pi^o) = -\frac{\lambda}{2}(V(s,t)+V(t,u)+V(u,s)\ ) \end{cases}$$

and(neglecting an overall coupling constant):

$$\begin{cases} A^{I_s=0} = \frac{3}{2}\left[V(s,t)+V(s,u)\right] - \frac{1}{2}V(t,u) \\ A^{I_s=1} = V(s,t) - V(s,u) \\ A^{I_s=2} = V(t,u). \end{cases} \qquad 11.28$$

Satellite terms having been disregarded in the amplitude.

The slope of the trajectory is taken from experiment, $\alpha' \sim 0.9\ (\text{GeV})^2$, and this fixes the mass scale. Lovelace [109] showed that the remaining parameters could be determined from current algebra with the off-mass shell continuation being made by considering s, t and u as independent variables in (11.27). The Adler self-consistency condition [110] states that the amplitude should vanish when one of the π's has zero mass, the remaining pions being kept on the mass shell,

i.e. $$s = t = u = m_\pi^2 \quad (s+t+u = 3\, m_\pi^2).$$

Thus either $\lambda=0$ or $V(s,t)=0$ when one of $m_\pi=0$.

i.e. $$1 - \alpha_\rho(m_\pi^2) - \alpha_\rho(m_\pi^2) = 0$$

$$\alpha_\rho\,(m_\pi^2) = \tfrac{1}{2}$$

and taking the ρ-mass as 764 MeV, this gives an intercept of $\alpha_\rho(o)\sim 0.48$ in remarkable agreement with experiment.

The two-soft pion limit [111] $s=u=m_\pi^2, t=0$ fixes λ in terms of the pion decay constant

$f_\pi \sim 95$ MeV (from $\pi\to\mu\nu$) as

$$\lambda = \frac{1}{\pi f_\pi^2\, \alpha'} \;.$$

Thus, by construction, the s-wave scattering lengths agree with current algebra predictions. Kawarabayashi et al [112] constructed such a model for $\pi\pi$, πK, KK and $K\bar{K}$ scattering and normalized at the ρ-pole. In order to compute phase shifts and resonance widths, the amplitude must be unitarized; one such prescription being given by Lovelace [109] is to treat the partial wave projection as a K matrix.

The partial wave projections of (11.28) are

$$A(\ell,s) = R_\ell(s) = \tfrac{1}{2} \int P_\ell(\cos\theta)\, A^1\,(s,t)\; d(\cos\theta)$$

and are purely real with poles at the resonance positions.

The interpretation is to take

$$A_\ell(s) \;=\; \frac{R_\ell(s)}{1 + \rho(s)\, R_\ell(s)}$$

where for the ρ term in a channel with masses M and m Lovelace obtained:

$$\mathrm{Re}\rho(s) = \frac{(M^2-m^2)\ln}{\pi s}\frac{M}{m} - \frac{2(M+m)^2}{\pi s}\left[\frac{s-(M-m)}{s-(M+m)}\right]^{\frac{1}{2}} \times$$

$$\ln\left[\left[\frac{s-(M+m)^2}{4Mm}\right]^{\frac{1}{2}} + \left[\frac{s-(M-m)^2}{4Mm}\right]^{\frac{1}{2}}\right]$$

and

$$\mathrm{Im}\,\rho(s) = -\left[\frac{s-(M-m)^2}{s-(M+m)^2}\right]^{\frac{1}{2}}$$

Using this prescription and the given λ the result

$$\Gamma_\rho \sim 120 \quad \mathrm{MeV}$$

was obtained (a consequence of KSFR [Kawarabayashi, Suzuki, Fayzazudin and Riazzudin] formula [113] known to work well). The I=0, s-wave daughter of the ρ is very broad:

$$\Gamma_\varepsilon \,/\, \Gamma_\rho \sim \frac{9}{2}$$

(a consequence of duality itself [114]) and the f_o parameters are well reproduced: M=1289, Γ=110 MeV. (experimentally M≈1300,Γ≈130).

Roberts and Wagner [115] applied this Lovelace model to experimental data on $\pi\pi$ interactions and also to $\pi^-p\rightarrow \pi^-\pi^+n$ at low energies [116]. Using the same model Wagner [117] predicted the $\pi\pi$ scattering amplitude up to 1 GeV and the off-shell Δ^2 (momentum transfer to the nucleons) dependence from $\pi N\rightarrow\pi\pi N$, as fig. 17.

The K-matrix method was also extended into the inelastic region by Roberts who added an empirical Pomeron and absorptive corrections [118].

Chung and Feldman [119] have presented a formulation of the integral representations of the partial waves of the amplitude (11.27) studied the threshold behaviour in detail, demonstrated how to reduce all partial waves to finite sums of s-waves and reproduced certain power bounds.

(ii) Other Processes.

Although the Veneziano formula was originally devised for meson-meson processes it was soon extended to meson-baryon processes by Igi [120] and by various authors to kaon-nucleon scattering by the use of various simple formulae to give an overall description of the process in agreement with experimental data [121]. It was, however, pointed out that for processes like pion-nucleon scattering there are no reliable principles to construct a concise formula incorporating appropriate signatures and isospin structures of baryon trajectories [122]. Explicit SU(3) symmetric Veneziano models for pseudoscalar meson-baryon scattering have however been constructed [123]. Studies on other processes such as $\pi N \to \eta N$ [124] $\pi N \to K\Lambda$ [125] and $\pi\pi \to \eta$(Boson) [126] were also made, and extensions to five particle processes using pion exchange also given [127].

That is an appropriate point on which to close this section, as the Veneziano formula was itself suitably generalized to five and then N-point processes shortly after its appearance.

1.2 The Maximum Likelihood Method (MLM)

Introduction

In the article "Likelihood" [1] A W F Edwards traces the history of statistical inference through Bernoulli's "Ars Conjectandi" (1713), de Moivre's "Doctrine of Chances" (1718), Bayes (1763) approach of "After-trial evaluation" Lambert (1760) and Daniel Bernoulli (1777) who both maximised likelihoods, Gauss (1809) following Bayes, and Laplace (1820). It was R A Fisher who in 1912 [2] proposed the method of maximum likelihood which he claimed suffered none of the objections of least squares methods, which depended on the measurement scale of the variables, or of the method of moments, which depended on an arbitrary choice of moments to equate in the population and the sample, or of Bayesian estimation methods, which depended on the parametric form adopted. Edwards states as his likelihood axiom that:

> "Within the framework of a statistical model, all the information which the data provide concerning the relative merits of two hypotheses is contained in the likelihood ratio of those hypotheses on the data, and the likelihood ratio is to be interpreted as the degree to which data support the one hypothesis against the other"

or:

> "- the hypothesis which best fits the data is to be preferred, and the relative excellence of the fit is to be measured by the probability of obtaining the data."

We shall not be concerned with general theoretical questions but shall state the widely used practical results of the method.

Method

The method of maximum likelihood [3-10] will be used in order to estimate parameters $a_1, a_2, \ldots, a_n$ of a given function from experimental data. A likelihood function $L(\underline{a})$ being constructed ($\underline{a} = (a_1, a_2, \ldots, a_n)^T$) such that it is maximised for certain values of $\underline{a}$, $\underline{a}^*$ say, called the maximum-likelihood estimator (or solution). Errors ε_j for the parameters a_j can be estimated.

For an experiment consisting of N independent observations (events) at coordinates x_i, $i = 1, \ldots, N$, suppose the expected distribution of observation (probability density) to be given by a function $f(x_i, \underline{a})$ depending on the n parameters $a_1, \ldots, a_n$. $f(x_i, \underline{a})$ is assumed to be normalised to unity, so that if X = range of observation:

$$\int_X f(x, \underline{a})\,dx = 1.$$

The likelihood function for the problem is given by the product:

$$L(\underline{a}) = \prod_{i=1}^{N} f(x_i, \underline{a})$$

being the joint probability density of getting a particular result, $x_1, \ldots, x_n$, and the Log-likelihood function by:

$$\mathcal{L} = \ell n L(\underline{a}) = \sum_i \ell n f(x_i, \underline{a}) \tag{12.1}$$

From
$$\frac{\partial \mathcal{L}}{\partial a_j} = \sum_i \frac{\partial f}{\partial a_j} \Big/ f(x_i, \underline{a}) \tag{12.2}$$

$$-\frac{\partial^2 \mathcal{L}}{\partial a_j \partial a_k} = \sum_i \left(\frac{\partial f}{\partial a_j} \frac{\partial f}{\partial a_k} - \frac{\partial^2 f}{\partial a_j \partial a_k} f(x_i, \underline{a}) \right) \Big/ f^2(x_i, \underline{a}) \tag{12.3}$$

from which $E_{jk} = \left[-\frac{\partial^2 \mathcal{L}}{\partial a_j \partial a_k} \right]^{-1}$ the "error-matrix" or "covariance matrix" gives a measure of the variance (diagonal elements) or the co-variance (off-diagonal elements) and hence an estimate of the confidence intervals on each a_j which are proportional to $\pm\sqrt{E_{jj}}$ (the 'correlation matrix' is $\rho_{jk} = E_{jk} / (E_{jj}.E_{kk})^{\frac{1}{2}}$.) L is then maximised, corresponding to determining a solution to:

$$\frac{\partial L}{\partial a_j} = 0 \quad \text{or} \quad \frac{\partial \mathcal{L}}{\partial a_j} = 0, \qquad j = 1, \ldots, N.$$

In practice, for convenience, one usually uses this latter form since it is easier to work out sums and their derivatives rather than products and since f can often take an exponential form.

Inherent in the MLM there may be a systematic error, or "bias", in $\underline{a}$ [8], but the MLM is said to be unbiased for large N (where N is some number proportional to the "amount of statistics" gathered) because as N increases the bias typically vanishes like N^{-1}.

The MLM therefore enables one to put statistical bounds on the a_j's and to show which if any of them may be neglected, thus giving a "best" set of a_j's for the given data. Given an alternative function, $F(x_i, \underline{b})$ say, then the ratio of the L values (or correspondingly the difference in $\mathcal{L}$ values) for f and F, known as a likelihood ratio test, will indicate which function best'fits'the data. The MLM gives a comparison of fits to the data but in order to indicate the quality of any "best" fit one would use the usual χ^2-test, in which the given data is "binned" (subdivided into suitable groups) and the sum $\sum_{i=1}^{n} (\mu_i - N_i)^2 / N_i$ computed when i refers to the bin, n the number of bins of events, N_i the number of events in each bin and μ_i the predicted number of events in each bin.

In the Method of Least Squares if an experiment consists of N independent measurements y_i, $i = 1, \ldots, N$, of some quantity y at coordinates x_i and if the errors of each y are σ_i (standard deviations) then a minimization is made of the quantity

$$S(\underline{a}) = \sum_{i=1}^{N} (f(x_i, \underline{a}) - y_i)^2 / \sigma_i^2$$

where $f(x_i, \underline{a})$ is the fitting function as before (but is not now a probability distribution). The minimum value of S is called S*, the least squares sum, and is found from $\frac{\partial S}{\partial a_j} = 0$. If the y_i have a probability density function that is normal, mean (expected value) $f(x_i, \underline{a})$ and standard deviation σ_i and if $\phi(y_i, f(x_i, \underline{a})) = \frac{1}{\sigma_i\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma_i^2}\left\{y_i - f(x_i, \underline{a})\right\}^2\right]$ $(i = 1, \ldots, N)$, represents each density function then the joint probability density function for $y_1, y_2, \ldots, y_N$ is

$$L(\underline{a}) = \frac{1}{\sigma_1\sigma_2\ldots\sigma_N}\left(\frac{1}{\sqrt{2\pi}}\right)^N \exp\left[-\frac{1}{2}\sum_{i=1}^{N}\frac{(y_i - f(x_i, \underline{a}))^2}{\sigma_i^2}\right]$$

thus

$$\mathcal{L}(\underline{a}) = \ln L(\underline{a})$$

$$= -N\ln(\sqrt{2\pi}) - \sum_{i=1}^{N}\ln\sigma_i - \frac{1}{2}\sum_{i=1}^{N}\frac{[y_i - f(x_i, \underline{a})]^2}{\sigma_i^2}$$

so that for normally distributed errors with mean 0 and standard deviation σ_i the relation

$$\mathcal{L} = -\frac{1}{2}S(\underline{a}) + \text{const.}$$

holds where $\mathcal{L}$ is the log-likelihood function for the problem. Minimizing the quantity S (the least squares sum) thus corresponds to maximizing $\mathcal{L}$.

In such a case omitting the constant term leads to

$$\mathcal{L} = -\frac{1}{2}\sum_i (f(x_i, \underline{a}) - y_i)^2 / \sigma_i^2 \tag{12.4}$$

$$\frac{\partial \mathcal{L}}{\partial a_j} = \sum_i \frac{\partial f}{\partial a_j}(y_i - f(x_i, \underline{a})) / \sigma_i^2 \tag{12.5}$$

$$-\frac{\partial^2 \mathcal{L}}{\partial a_j \partial a_k} = \sum_i \left[\frac{\partial f}{\partial a_j}\frac{\partial f}{\partial a_k} - \frac{\partial^2 f}{\partial a_j \partial a_k}(y_i - f(x_i, \underline{a}))\right] / \sigma_i^2 \tag{12.6}$$

For Gaussian-distributed y_j's the distribution of $S^* \equiv S(a_j^*)$, the least-squares sum, is the χ^2 distribution of $(p - M)$ degrees of freedom where p is the number of experimental points and M is the number of parameters solved for. If the values are <u>Poisson distributed</u>, e.g, they could represent counting rates (number of events) in a small region of x, then the quantity

$$p(x_i, \underline{a}) = \frac{\left\{f(x_i, \underline{a})\right\}^{y_i}}{y_i!} \exp(-f(x_i, \underline{a}))$$

would give the probability of observing a counting rate y_i in a region x_i expecting a counting rate $f(x_i, \underline{a})$. The likelihood function L is then given by $L = \prod_i p(x_i, \underline{a})$

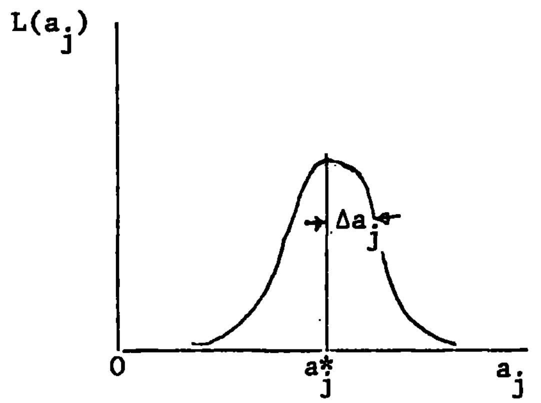
$$\ell n L = \mathcal{L} = \sum_i y_i \ell n f(x_i, \underline{a}) - \sum_i f(x_i, \underline{a}) + \text{const.} \tag{12.7}$$

$$\frac{\partial \mathcal{L}}{\partial a_j} = \sum_i y_i \frac{\partial f}{\partial a_j} / f(x_i, \underline{a}) - \sum_i \frac{\partial f}{\partial a_j} \tag{12.8}$$

$$\frac{-\partial^2 \mathcal{L}}{\partial a_j \partial a_k} = \sum_i y_i \left(\frac{\partial f}{\partial a_j}\frac{\partial f}{\partial a_k} - \frac{\partial^2 f}{\partial a_j \partial a_k} f(x_i, \underline{a})\right) / f^2(x_i, \underline{a}) - \sum_i \frac{\partial^2 f}{\partial a_j \partial a_k} \tag{12.9}$$

J. Orear [9] quotes the following results (for one parameter):

In general, the likelihood function will be close to Gaussian as in the figure, where $\Delta a_j = \left[\frac{\int (a_j - a_j^*)^2 L da_j}{\int L da_j}\right]^{\frac{1}{2}}$



It is a known property of M L estimates (referred to as "The M L Theorem") that in the limit of large N, $a_j^* \rightarrow a_{jo}$ (the true physical parameter value); and furthermore, there is no other method of estimation that is more accurate. Also, the condition for the maximum-likelihood solution is unique and independent of the arbitrariness involved in the choice of physical parameters. B R Martin [4] quotes the following important results (for one parameter). that are valid for common distributions met in practice:

Maximum likelihood estimators (MLE)

a) are consistent (i.e, as the sample size increases the estimate tends to the value of the population parameter),

b) have a distribution which tends to normality for large samples,

c) have minimum variance in the limit of large samples.

If a sufficient estimator (one that contains all the information about the population parameter) for a parameter exists then it is a function of the maximum likelihood estimator.

Blobel [7] gives a programming routine for finding the maximum of $\mathcal{L}$. In fact the routines referred to by Swanson [6] under "Fitierte Literatur" for finding the minimum of a general function of N variables have now been written up as a CERN report MINUIT [10] and this is the routine that has been used. A general review of the main ideas of unconstrained optimization, in which the problem of calculating the greatest value of a given real function $F(x_1, x_2, \ldots, x_N)$, where each variable x_i $(i = 1, 2, \ldots, N)$ can take the value of any real number, is given by M J D Powell [11], and also E Polak [12], in Appendix C of the book by B R Martin [4], and in Chapter 10 of that by S Brandt [5].

A typical M L formulation for a data fitting procedure is given in [13] and for the "Likelihood handling of scattering data with previous experimental information" J Bystricky et al have given a short paper [14].

Fig. 1 Four Line Connected Part

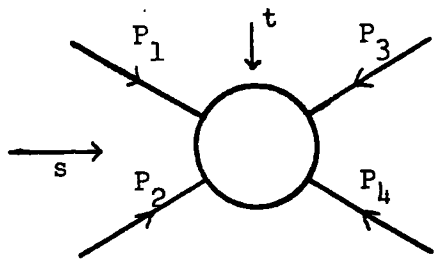


Fig. 2 Complex s-plane

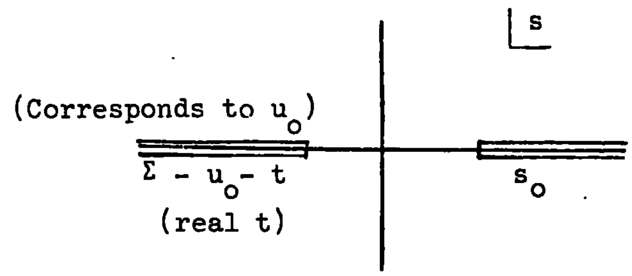
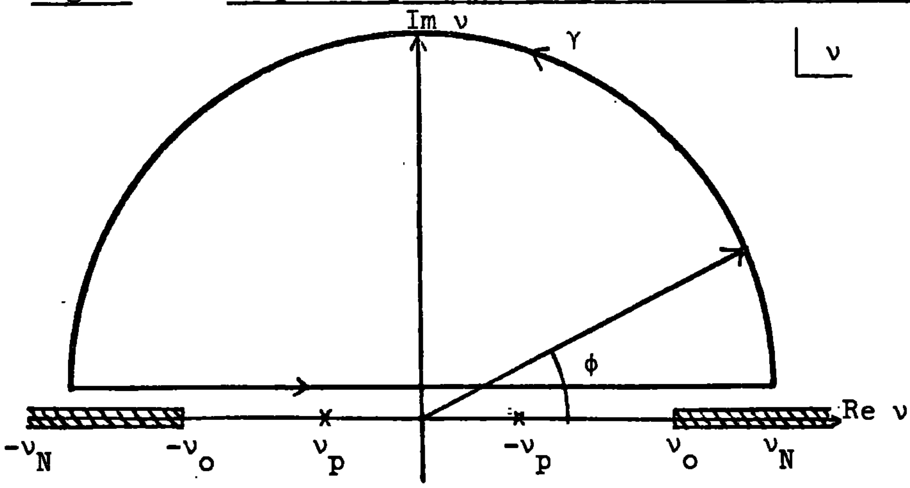


Fig. 3 Complex ν plane, with poles and unitarity cuts



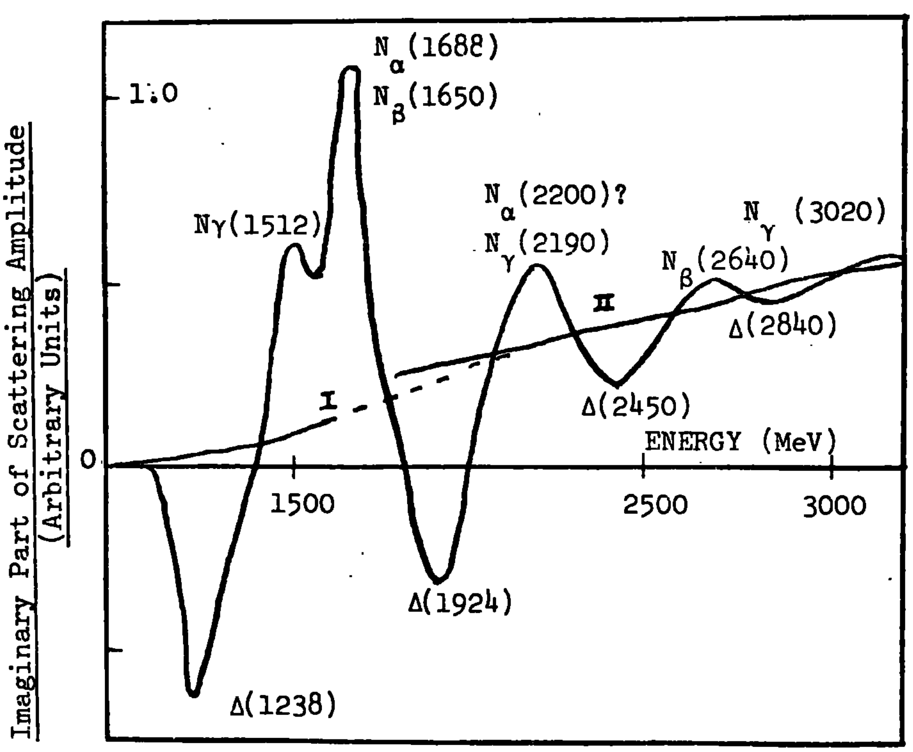


Fig. 4 Scattering Amplitude for π-n scattering represented by two Regge fits I & II and by resonances.

Fig. 5 ρ-exchange in the process $\pi^- p \to \pi^0 n$

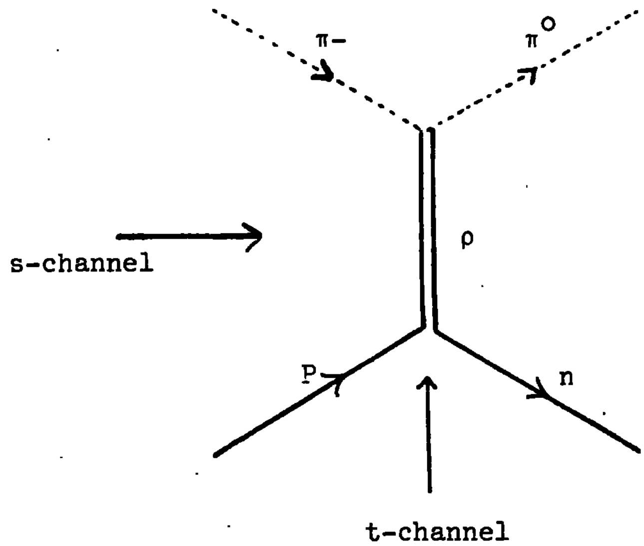
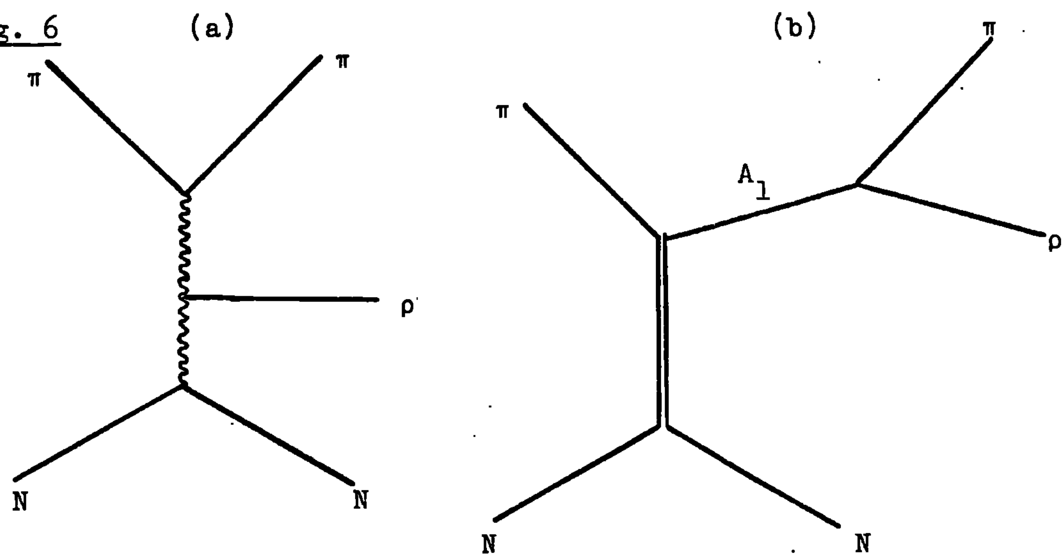


Fig. 6



'Deck Effect' diagrams

Fig. 7 Exchange collision. R = P when IQN permit this

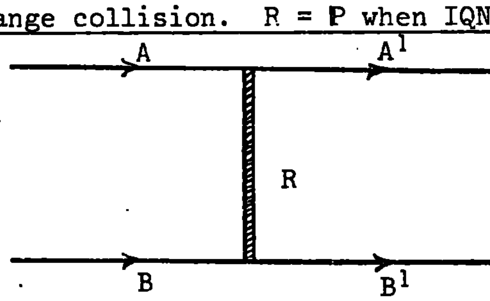
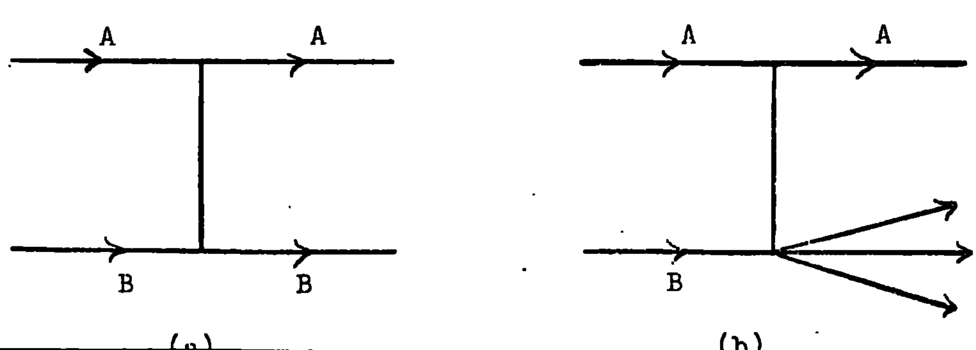


Fig. 8 Two kinematical configurations with Pomeron exchange



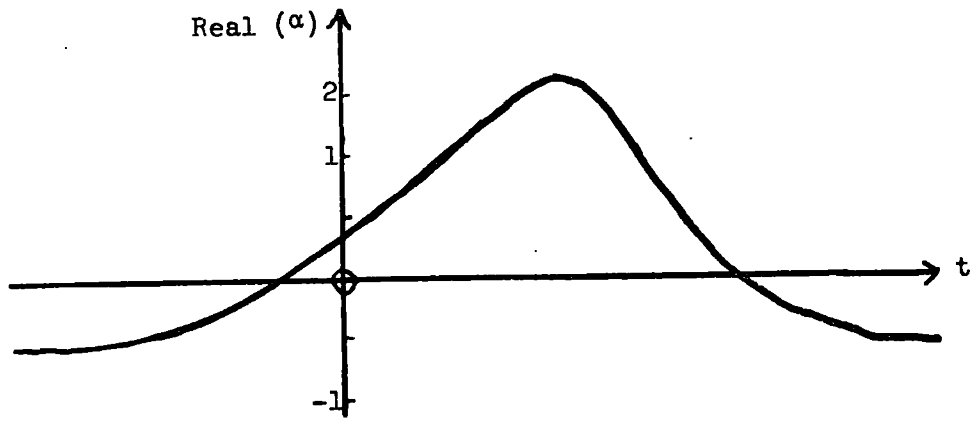


Fig. 9 A Regge trajectory $\alpha(t)$ in potential theory

Fig. 10

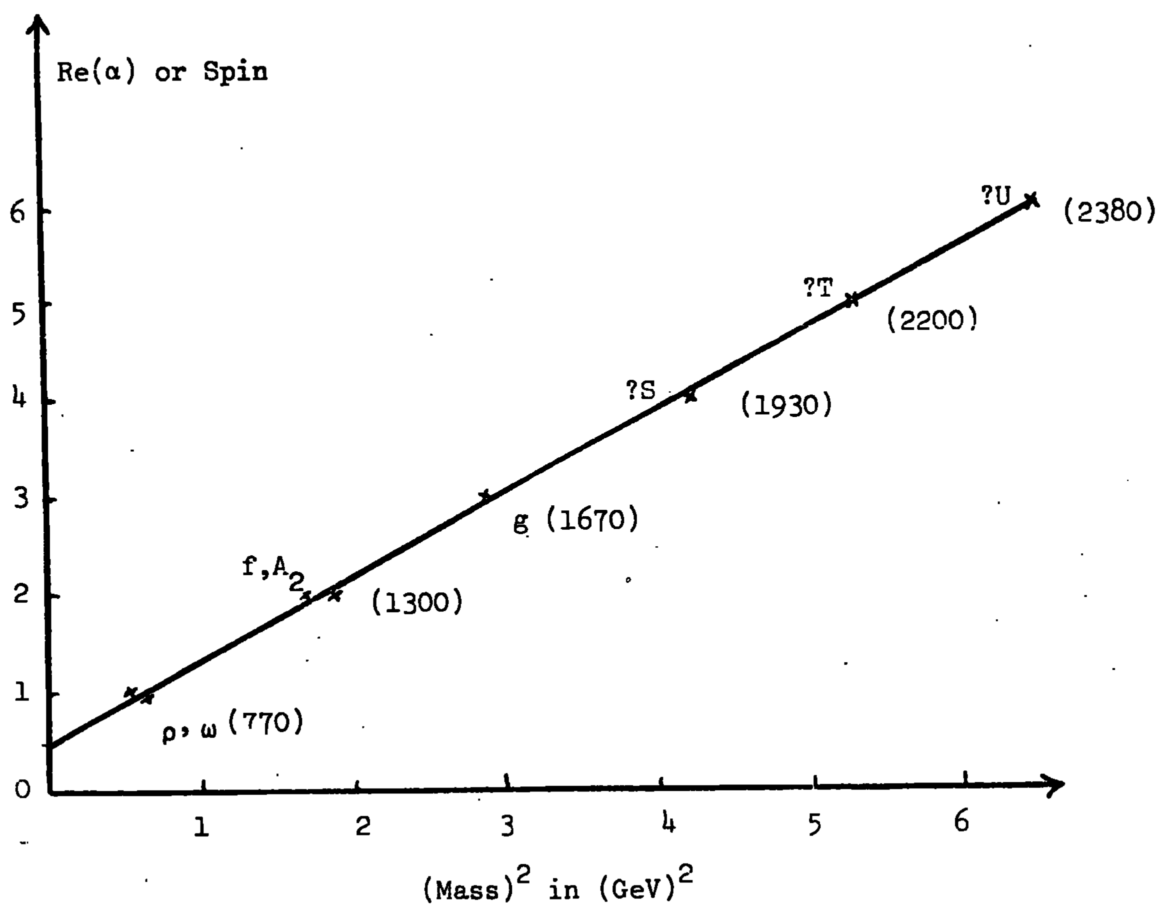


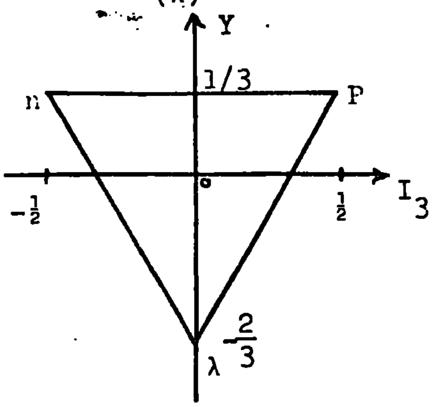
Fig. 11 Quark Properties used in the Duality Diagrams

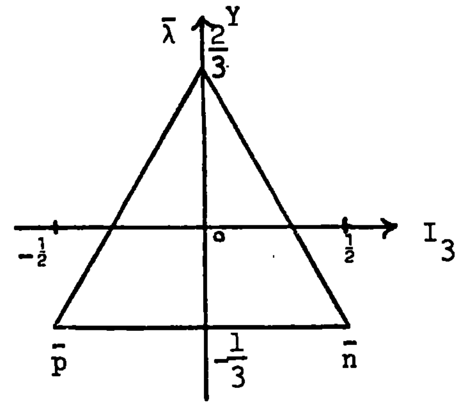
quarks

$$q = \begin{pmatrix} P \\ n \\ \lambda \end{pmatrix}$$

anti-quarks

$$\bar{q} = (\bar{P}, \bar{n}. \bar{\lambda})$$





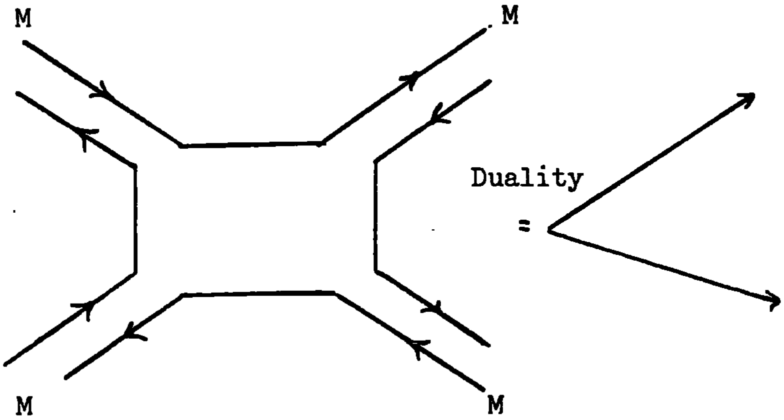
B = Baryon Number
Q = Charge
S = Strangeness
Y = Hypercharge = S + B
I = Isospin
I_3 = Third Conpt. of Isospin = $Q - \frac{1}{2}Y$

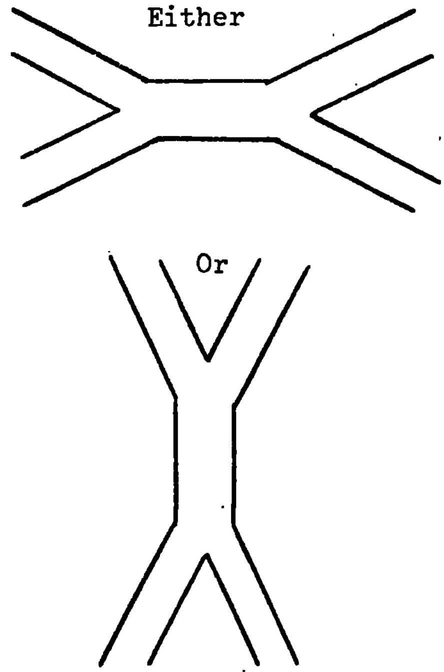
	I	I_3	B	Y	Q	S
P	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0
n	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0
λ	0	0	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	-1

Mesons ($q\bar{q}$) $3 \otimes 3 = 8 \oplus 1$ (8,1)
Baryons (qqq) $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$
Baryon-Meson $8 \otimes 8 = 27 \oplus \overline{10} \oplus 10 \oplus 8_s \oplus 8_a \oplus 1$
and $10 \otimes 8 = 35 \oplus 27 \oplus 10 \oplus 8^s$
The Baryon-Antibaryon problem has ($qq\bar{q}\bar{q}$) (27, $\overline{10}$, 10, 8, 1) exchange dual to ($q\bar{q}$) (8, 1).

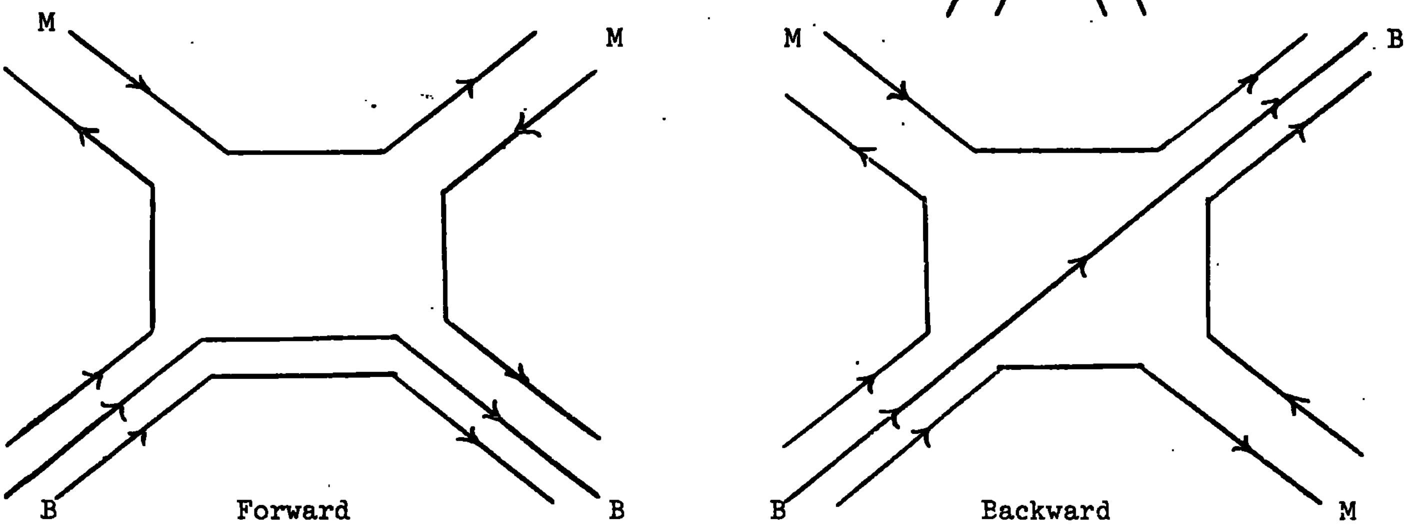
Fig. 12 Duality Diagrams

(a) Meson-meson scattering. ("H-type")

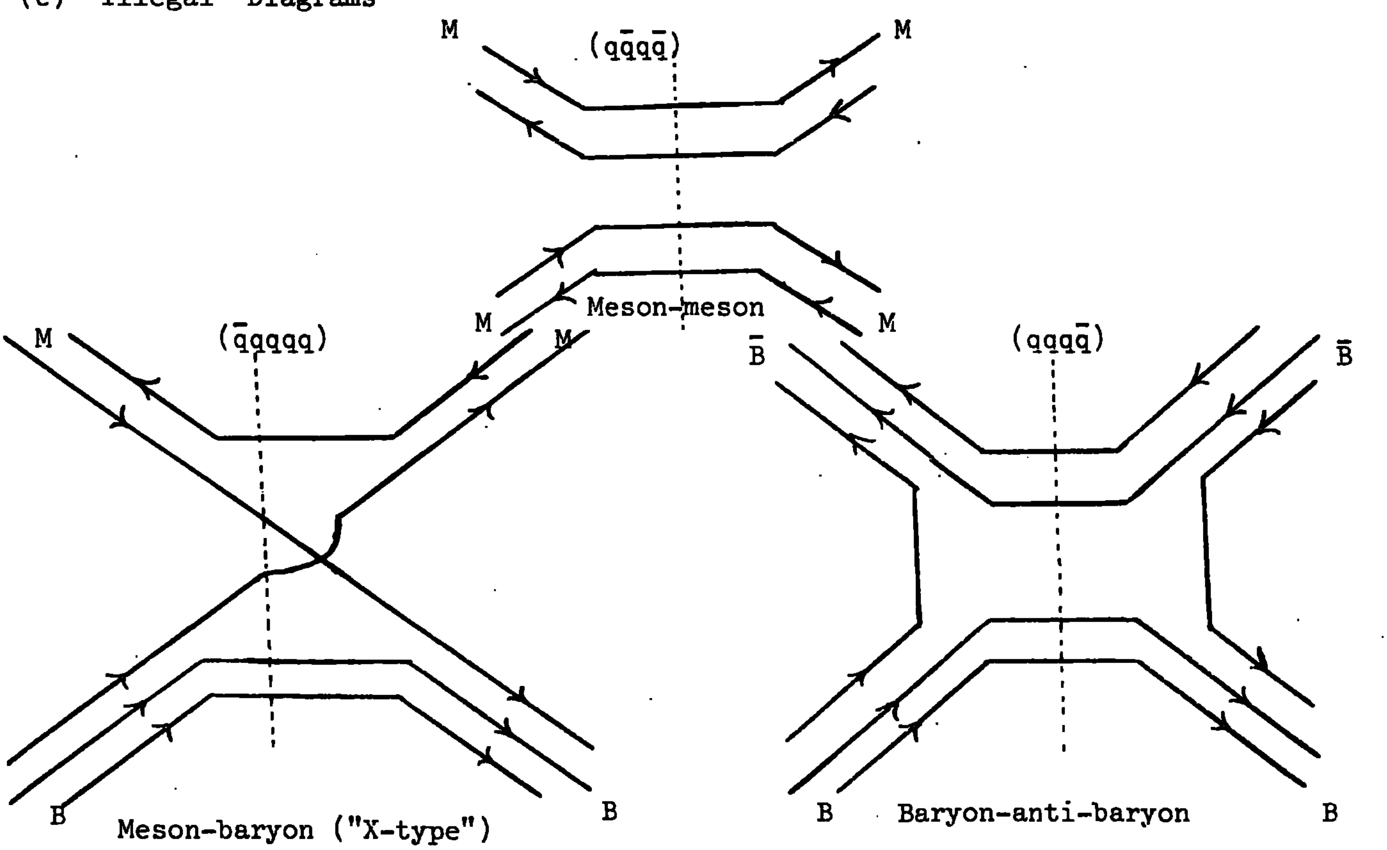




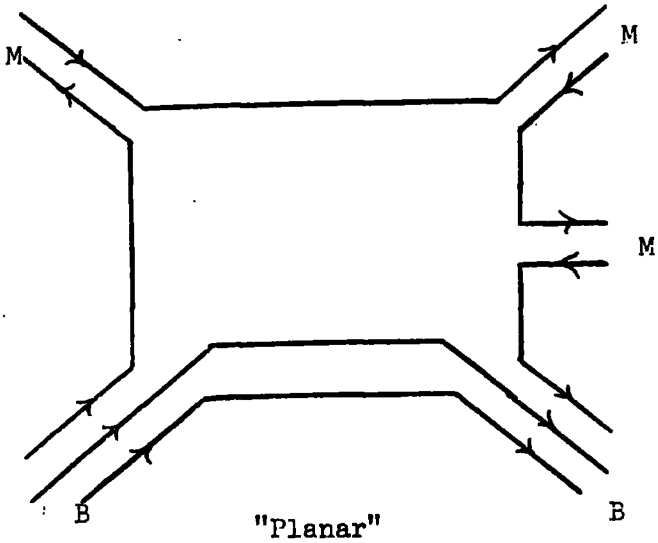
(b) Meson-baryon scattering.

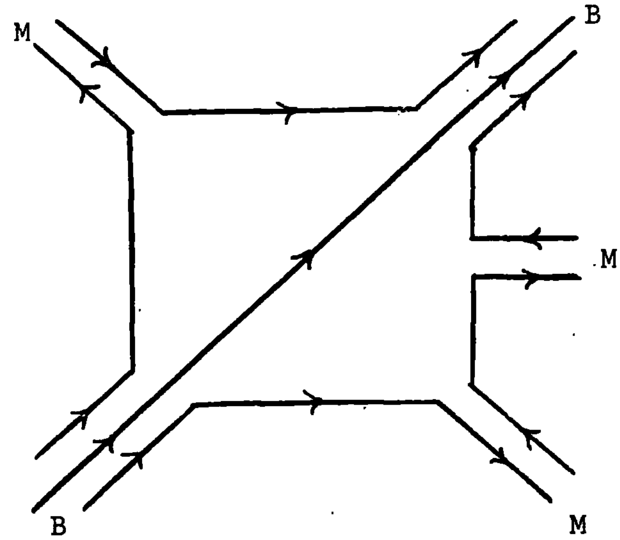


(c) "Illegal" Diagrams

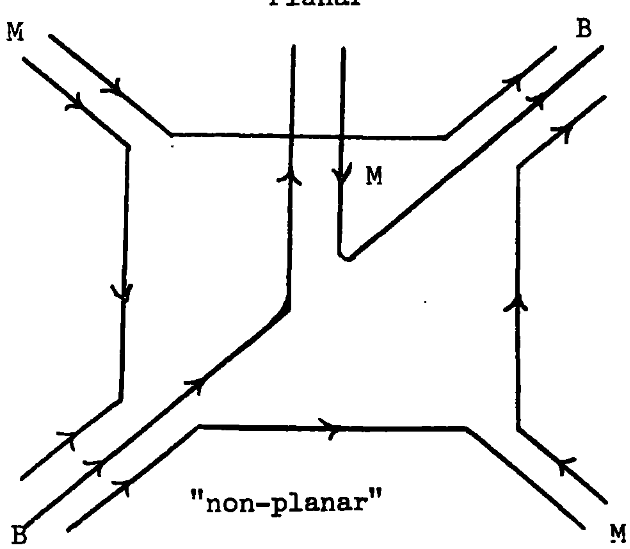


(d) A Production process

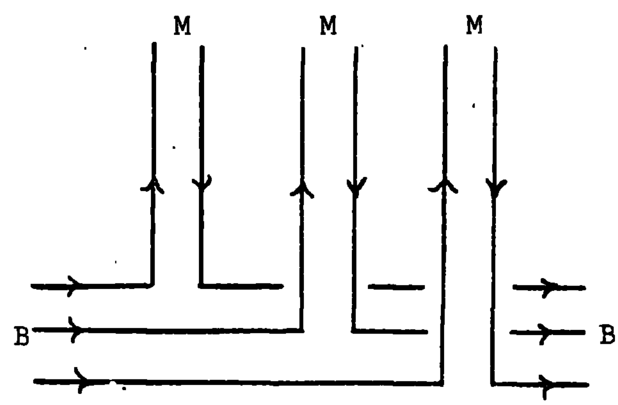




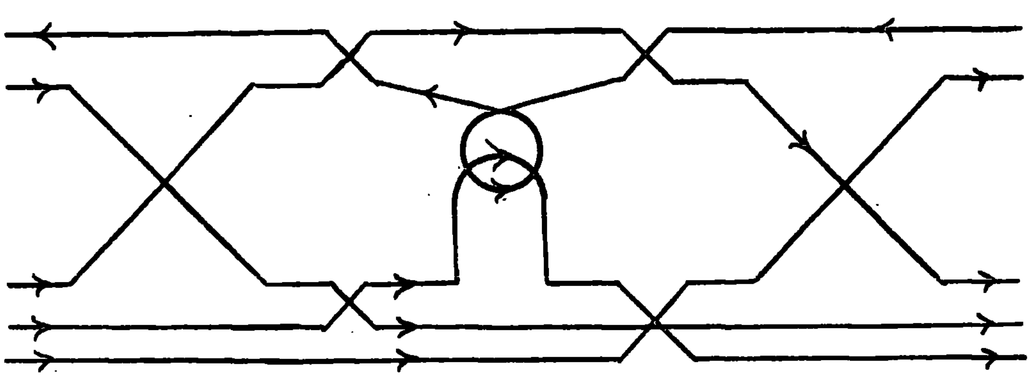
"Planar"

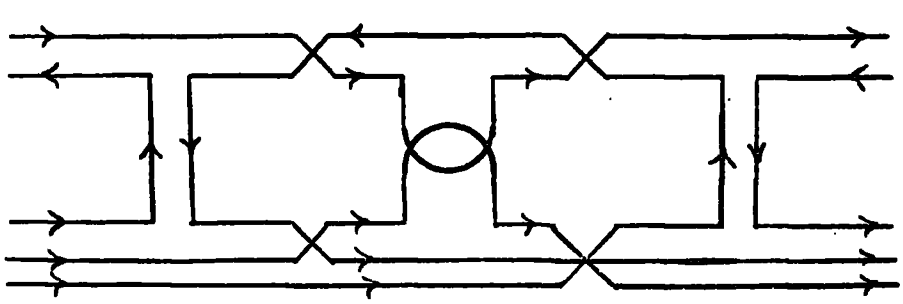


"non-planar"



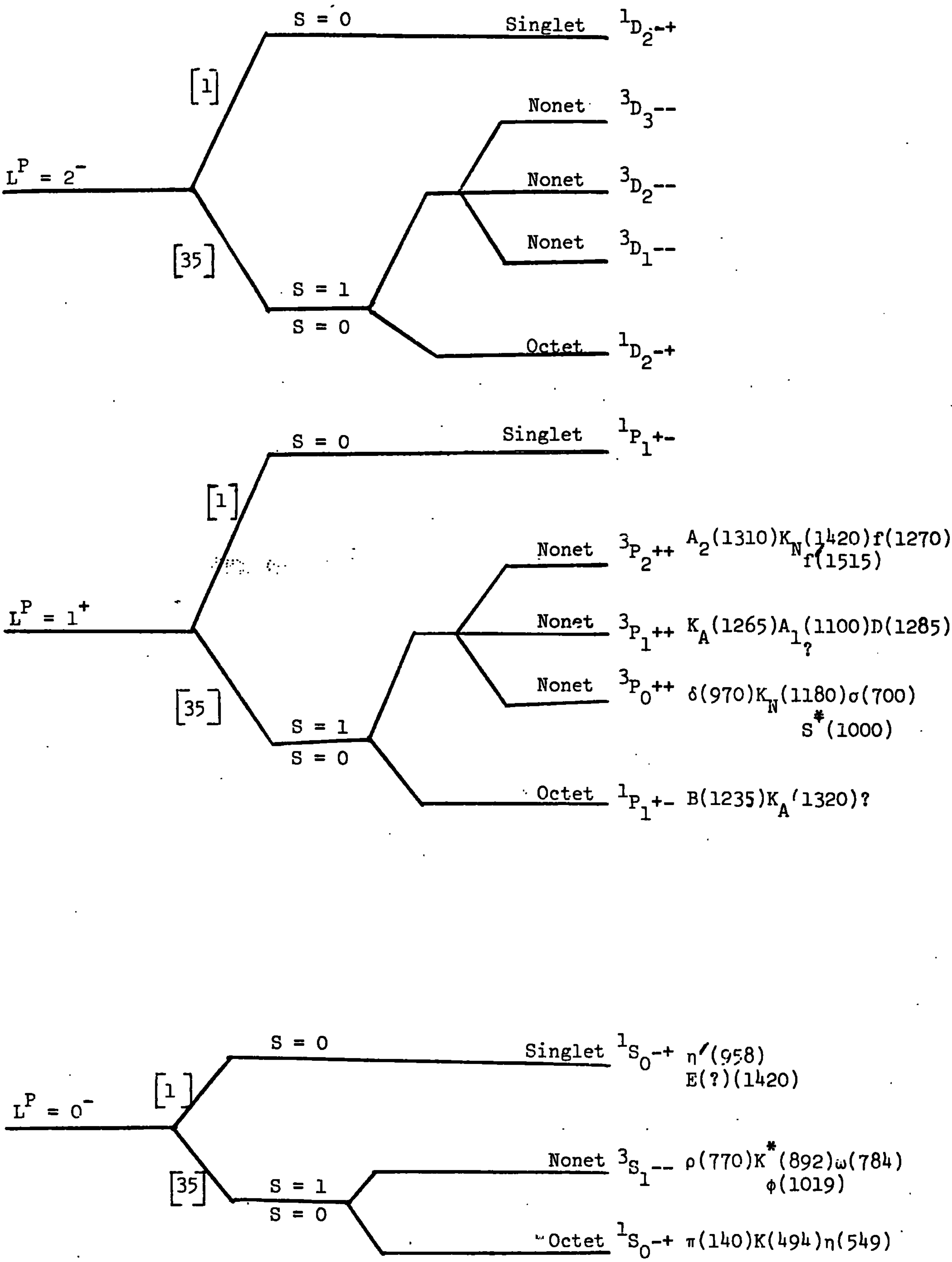
Redrawn as the Burnett and Schwarz five point function in Ref. (68)





(e) Regge-Pomeron-Regge cut diagrams as given in ref. (61).

Fig. 13 Summary for mesons (in the $q\bar{q}$ model)
(masses from "Review of Particle Properties" April 1973).



(Note: Mesons with $J^{PC} = 0^{--}$, $(\text{even})^{+-}$, $(\text{odd})^{-+}$ excluded.)

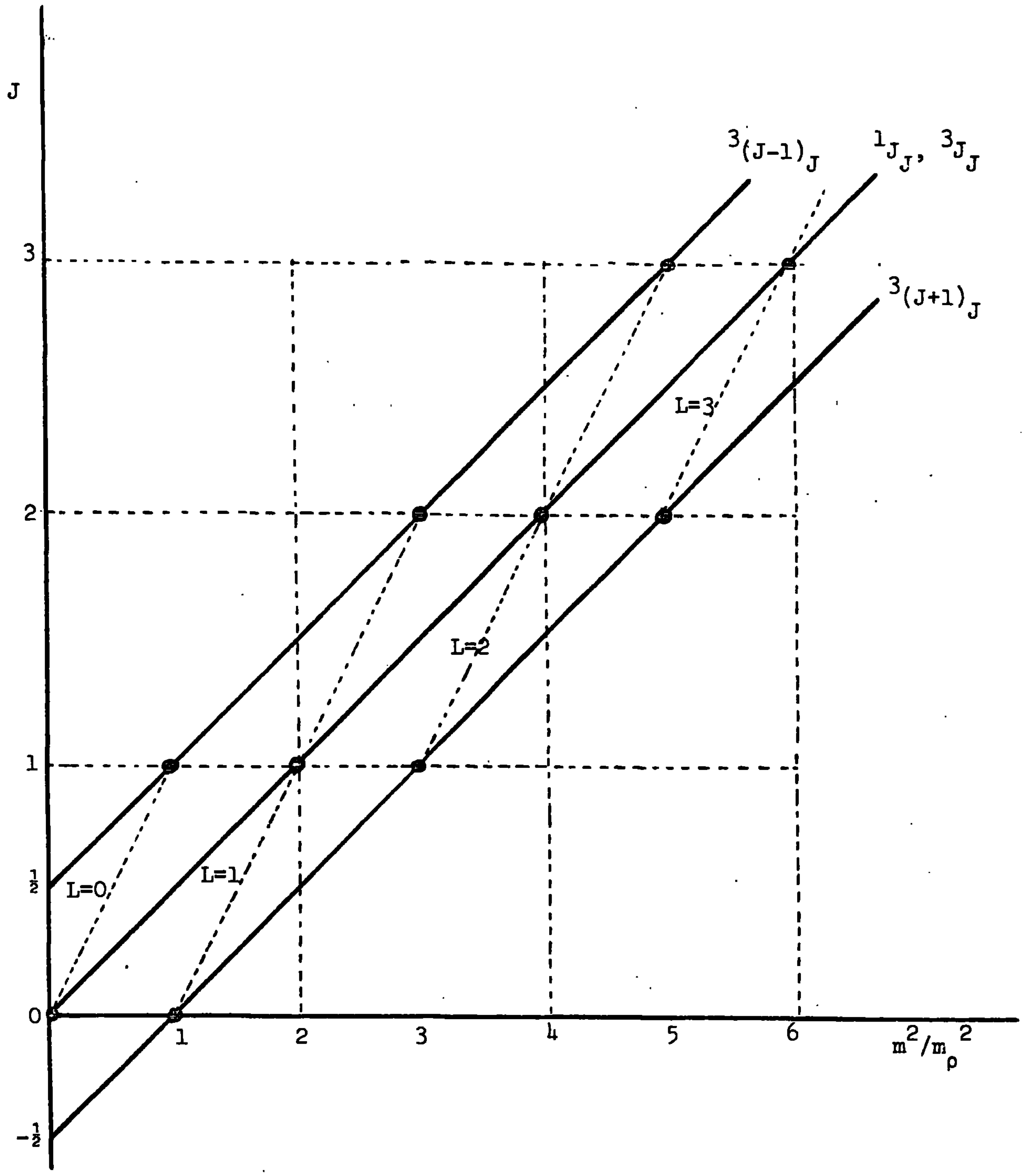


Fig. 14 Chew-Frautschi plot of meson trajectories which are assumed linear and parallel with a common slope $\frac{1}{2m_\rho^2}$ and where $m_\pi^2 = 0$. The notation $^{2s+1}L_J$ (s = total quark spin, J = total angular momentum) is used and the rule $m^2(s,L,J,n) = m^2(s,L,J) + n\delta$ with $n = 0$. Each dot at integer J for the middle line has four states (except J=0 where there are two) the rest have two.

Fig. 15 Kinematics

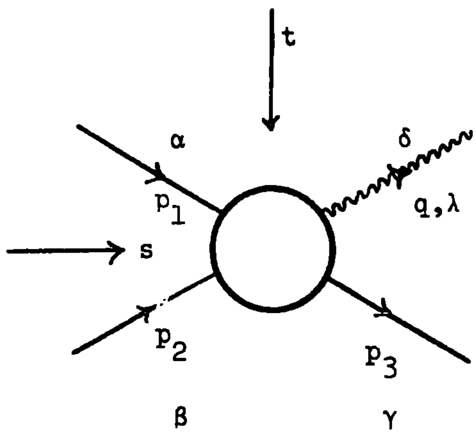
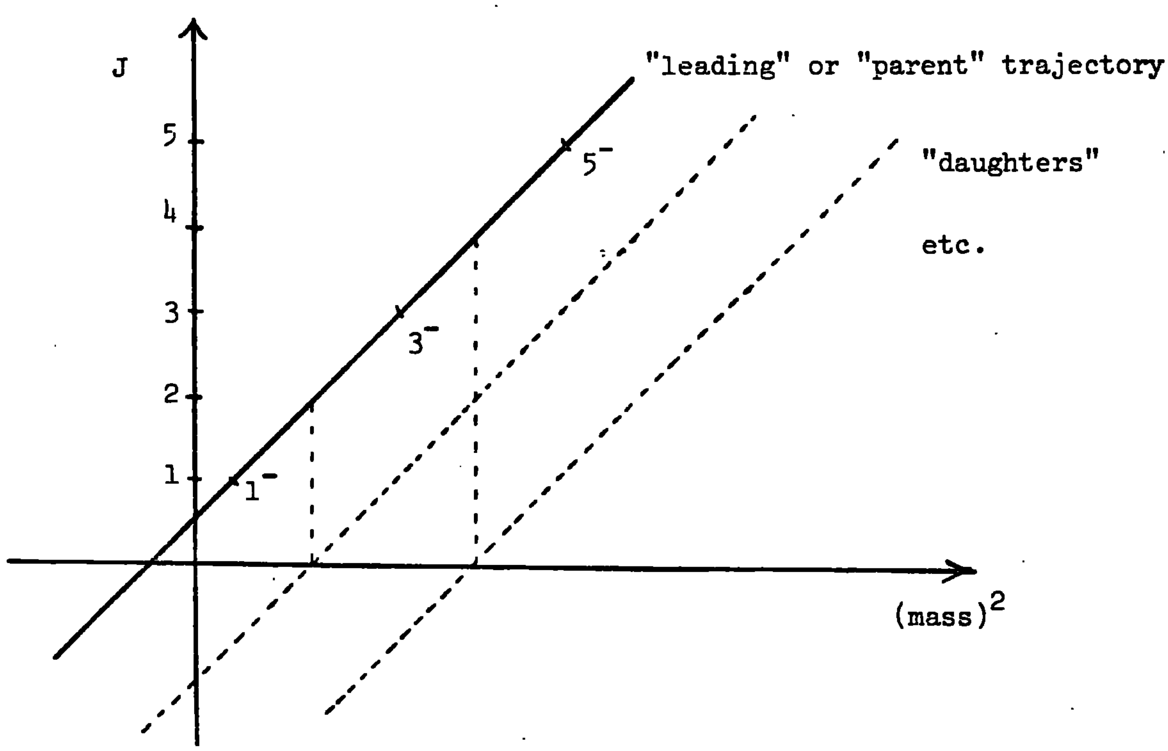


Fig. 16 ρ-trajectory



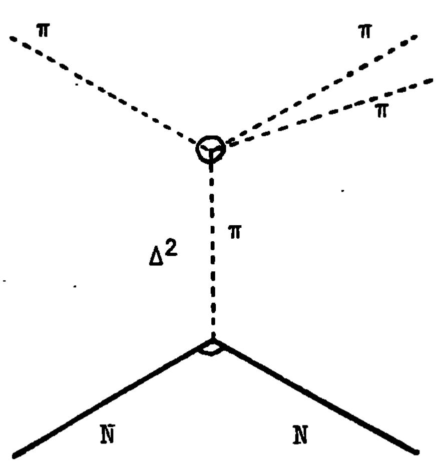


Fig. 17 $\pi N \to \pi\pi N$ process as considered by Wagner

CHAPTER 1 REFERENCES

Part 1.1

Section 1.1.1

Examples of references where the use of
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CHAPTER 2

Application of a Veneziano-type Amplitude to the process $\pi^- p \rightarrow \pi^- \pi^+ n$

2.1 Introduction

The generalization of the Veneziano four-point formula to that suitable for five neutral bosons was first given by Bardakçi-Ruegg [1] and Virasoro [2] .

For a five particle amplitude there are five independent variables [(3N - 10) for N particles] and for this extension the five independent scalar Mandelstam variables $S_{i,\ i+1} = (P_i + P_{i+1})^2$, $P_6 = P_1$, were used where P_i are the incoming particle four-momenta. The linear Regge trajectories given by $\alpha_{i,i+1} = \alpha^o{}_{i,i+1} + \alpha' S_{i,i+1}$ were assumed to hold.

Starting from the four-point function:

$$A_4 = B_4(-\alpha_{12}, -\alpha_{23}) = \int_0^1 du_{12} \int_0^1 du_{23} u_{12}^{-\alpha_{12}-1} u_{23}^{-\alpha_{23}-1} \delta(u_{12}+u_{23}-1)$$

where the(auxiliary) variable u_{23} (called dual to u_{12}) is fixed by forbidding coincident poles, the extension is made to:

$$A_5 = B_5\ (-\alpha_{12}, -\alpha_{23}, -\alpha_{34}, -\alpha_{45}, -\alpha_{51}) =$$

$$\int_0^1 \int_0^1 \frac{du_{i,i+1} du_{j,j+1}}{1-u_{i,i+1}u_{j,j+1}}\ u_{12}^{-\alpha_{12}-1} u_{23}^{-\alpha_{23}-1} u_{34}^{-\alpha_{34}-1} u_{45}^{-\alpha_{45}-1} u_{51}^{-\alpha_{51}-1}$$

where i and j are any two non-consecutive integers and the variables

$u_{i,i+1}$ obey the (duality) constraints:

$$u_{i,i+1} = 1 - u_{i-1,i}\, u_{i+1,i+2}, \quad i = 1, \ldots, 5, u_{01}=u_{51} \text{etc},$$

so that

$$u_{23} = \frac{1 - u_{12}}{1 - u_{12}u_{45}} \quad \text{and} \quad u_{34} = \frac{1 - u_{45}}{1 - u_{12}u_{45}} .$$

The Bardakçi-Ruegg-Virasoro form of B_5 is thus:

$$A_5 = \int_0^1 du_{12} \int_0^1 du_{45}\, u_{12}^{-\alpha_{12}-1} u_{45}^{-\alpha_{45}-1} \left[\frac{1 - u_{12}}{1 - u_{12}u_{45}}\right]^{-\alpha_{23}-1} * \\ * \left[\frac{1 - u_{45}}{1 - u_{12}u_{45}}\right]^{-\alpha_{34}-1} * (1 - u_{12}u_{45})^{-\alpha_{51}-2} \qquad (21.1)$$

This amplitude is cyclicly invariant in the $\alpha_{i,i+1}$ terms; has simple poles for $\alpha_{i,i+1} = 1, 2, 3, \ldots$; has simultaneous poles in $\alpha_{i,j}$ and $\alpha_{m,n}$ if i, j, m, n are all different; gives the correct single and double Regge behaviour in all channels and has no "ancestors" to the leading Regge trajectory.

A compact way of writing this expression is obtained by putting it into the form

$$B_5(x_1,\ldots,x_5) = \int_0^1 \prod_{\kappa=1}^{5} du_\kappa u_\kappa^{x_\kappa-1} \prod_{\bar{\kappa}=1}^{5}{}' \delta(u_{\bar{\kappa}} + u_{\bar{\kappa}-1}u_{\bar{\kappa}+1} - 1)$$

where $u_{i,i+1} = u_i$, $u_0 = u_5$, $u_6 = u_1$ and $x_i = -\alpha_{i,i+1}$. The second (primed) product in the integral is defined to run over all u_κ except the two (called mutually non-dual) chosen as independent variables. The argument of the delta-function is of the form "variable plus product of all dual variables, minus one". This is the Chan form [3] of B_5 which clearly exhibits both the invariances under cyclic and anticyclic permutations as well as the absence of double poles in dual variables. Extension was made to firstly the case $N = 6$ [3], then $N = 7$ [4], then $N = 8$ [5], and finally to that for arbitrary N [6,7] particles. The corresponding Chan form for arbitrary N being given by:

$$B_N(x_1,\ldots,x_{R_N}) = \int_0^1 \prod_{\kappa=1}^{R_N} du_\kappa \, u_\kappa^{x_\kappa - 1} \left\{ \prod_{\bar\kappa=1}^{R_N'} \delta(u_{\bar\kappa} + \prod_{\bar{\bar\kappa}=1}^{R_N''} u_{\bar{\bar\kappa}} - 1) \right\} \tag{21.2}$$

where $R_N = N(N-3)/2$, the (conjugate Mandelstam) variables are denoted u_κ, $\kappa = 1, \ldots, R_N$, the primed product runs over all $\bar\kappa$ except the N-3 (mutually non-dual) independent variables (whichever are chosen), and the doubly primed overall variables dual to $\bar\kappa$. The $u_\kappa^{x_\kappa-1}$ bring in the pole structure, while the product of delta functions enforces the absence of coincident poles in dual variables, thereby determining (N-2)(N-3)/2 auxiliary variables (so that when $N = 5$ the 5 variables are reduced to two independent ones as (5-2)(5-3)/2 are integrated out). The corresponding Bardakci-Ruegg form [8] for N particles can be obtained by defining

$$\omega_{q,r} = u_{1,q} u_{1,q+1} \cdots u_{1,r}, \quad q < r$$

where u_{1j} with $j = 2, \ldots, N-2$ are the (N-3) independent variables and integrating overall delta functions, so that

$$B_N\,(1\,\ldots\,N) =$$

$$\int_0^1 du_{12}du_{13}\ldots du_{1,N-2}\; u_{12}^{x_{12}-1}\ldots u_{1,N-2}^{x_{1,N-2}-1} * (1-u_{12})^{x_{23}-1} *$$

$$(1-u_{13})^{x_{34}-1}\ldots(1-u_{1,N-2})^{x_{N-2,N-1}-1} * (1-\omega_{23})^{-2(p_2\cdot p_4)\alpha'+\alpha_0+\alpha'} * \ldots$$

$$\ldots\,(1-\omega_{N-3,N-2})^{-2(p_{N-3}\cdot p_{N-2})\alpha'+\alpha_0+\alpha'} * (1-\omega_{24})^{-2(p_2\cdot p_5)\alpha'}\ldots$$

$$\ldots(1-\omega_{N-4,N-2})^{-2(p_{N-4}\cdot p_{N-1})\alpha'} * \ldots * (1-\omega_{2,N-2})^{-2(p_2\cdot p_{N-1})\alpha'}$$

This was the form used extensively in the first investigations of level structure [9,10].

The form (21.1) of the amplitude for N = 5 may be obtained directly using graphical rules suitable also for extension to the N-point case. Ordering the momenta as in Fig. 1 we define a partition as a set of at least two momenta with relation to the order of Fig. 1, e.g. (123), (34) etc. Two partitions are said to be dual if they share elements without one being contained in the other, so that, for example, (12) and (23) are dual but not (12) and (34). To every partition is associated the invariant

$$s_{ij} = (p_i + p_{i+1} + \ldots + p_j)^2 .$$

To the graph in Fig. 1 is associated its dual in Fig. 2, where the condition $\sum_i p_i = 0$ is explicitly taken into account by the closed polygon, and where each diagonal of this polygon corresponds to a partition; dual partitions are associated with intersecting diagonals.

There are N-3,2 in this case, non-intersecting, and hence mutually non-dual, variables which correspond to u_{12} and u_{45} in (21.1). These conjugate variables are usually written in this form to correspond to each S_{ij} and although any 2(N-3) non-intersecting variables can be chosen (cf. Figs 3 and 4) the "multiperipheral" form shown in Fig. 4 is easy to visualise and the set S_{ij}, j=2,3,..(N-2) is often used.

From Fig. 2 choose a vertex V of the polygon, associate the independent variables u_{12}, u_{45} to the diagonals concurring there. Then to the diagonal corresponding to u_{23} (i.e, 23) we associate the expression $\dfrac{(1 - u_{12})}{(1 - u_{12}u_{45})}$ since it crosses the diagonal line for u_{12}, i.e (12), and is diagonal to the quadrilateral (123 u_{45}). Similarly the diagonal for u_{34}, i.e (34) crosses that for u_{45}, i.e (45) and (12) makes up the side of the appropriate quadrilateral. The diagonal corresponding to u_{51}, i.e (51), crosses both (12) and (45) and is associated with $(1 - u_{12}u_{45})$ since it must be unity when u_{12} or u_{45} are zero. The rules are now:

i) Integrate from zero to one on all independent variables

ii) Write the factors corresponding to the diagonals of the polygon, each one to a power $(-1 - \alpha_{ij})$, where the ij correspond to the diagonals

iii) Divide by the factor $1 - u_{12}u_{45}$ which guarantees invariance when another set of variables is chosen, i.e.

$$A_5 = \int_0^1\int_0^1 \frac{du_{12}du_{45}}{1 - u_{12}u_{45}} \; u_{12}^{-\alpha_{12}-1} \; u_{45}^{-\alpha_{45}-1} \left(\frac{1 - u_{12}}{1 - u_{12}u_{45}}\right)^{-\alpha_{23}-1} \left(\frac{1 - u_{45}}{1 - u_{12}u_{45}}\right)^{-\alpha_{34}-1} (1 - u_{12}u_{45})^{-\alpha_{15}-1} \tag{21.3}$$

Several equivalent forms of the generalized beta function were suggested and its various important properties established.

By expanding the integral for B_N in a power series in various ways, one can obtain it as an infinite series of beta functions of lower order. Such series expansions were considered in some detail by Hopkinson and Plahte [11] and yield a practical iterative method for the numerical evaluation of these beta functions.

In particular for the Bardakci-Ruegg form of (21.1) by expanding the term in $(1 - u_{12}u_{45})$ in a binomial series, we may obtain after integration:

$$B_5(x_{12},\ldots,x_{51}) = \sum_{k=0}^{\infty} (-1)^k \binom{z}{k} B_4(x_{12}+k, x_{23}) B_4(x_{34}, x_{45}+k) \tag{21.4}$$

where $z = x_{51} - x_{23} - x_{34}$, and $\binom{z}{k} = \dfrac{\Gamma(z+1)}{\Gamma(k+1)\Gamma(z-k+1)}$, $\binom{z}{0} = 1$.

Using the gamma function representation of B_4, this may be rewritten (dropping the x's) as:

$$B_5(1,2,\ldots,5) = B_4(12,51) B_4(34,45)\, {}_3F_2(12,45,-z;12+23,34+45;1) \tag{21.5}$$

where $\quad {}_3F_2\,(a_1,\, a_2,\, a_3;\;\; b_1,\, b_2;\;\; 1) =$

$$\sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)}{\Gamma(a_1)} \; \frac{\Gamma(a_2+n)}{\Gamma(a_2)} \; \frac{\Gamma(a_3+n)}{\Gamma(a_3)} \; \frac{\Gamma(b_1)}{\Gamma(b_1+n)} \; \frac{\Gamma(b_2)}{\Gamma(b_2+n)} \; \frac{z^n}{n!}$$

is a generalized hypergeometric function [12] with unit argument. This series converges when

$$\mathrm{Re}\left[1+(x_{12}+x_{23}) + (x_{34}+x_{45}) - x_{12} + z - x_{45}\right] > 1$$

ie, when $\mathrm{Re}(x_{51})$ is positive. Thus we have found a representation for B_5 which has a much larger region of convergence than the integral. This is because the integral representation of the beta function is only convergent when both arguments have a positive real part, while the function is well-defined, through the gamma function, for all values of its arguments.

The series (21.5) is the starting point for any method of calculating B_5 numerically [13,14] although since it is not convergent in a big enough region to be useful, recursion relations (which increase the range of convergence) are used. The program for this calculation is listed in ref.[13] together with details of the recursion relations and truncation error terms.

Since the beta function B_4 has simple poles in each variable at the non-positive integers, so, using cyclic symmetry, $B_5(x_1,x_2,x_3,x_4,x_5) = B_5(x_2,x_3,x_4,x_5,x_1)$, has B_5 similarly for each argument separately, and these are all the poles of B_5.

From (21.4) taking the limit $x_{12} \to 0$ yields

$$B_5(x_{12},\ldots,x_{51}) = \frac{1}{x_{12}} \left[\sum_{k=0}^{\infty} (-1)^k \binom{z}{k} B_4(x_{34},x_{45}+k) \frac{x_{12}}{x_{12}+k} * \right.$$

$$\left. * \frac{\Gamma(x_{12}+k+1)}{\Gamma(x_{12}+x_{23}+k)}\right]_{x_{12}=0} = \frac{1}{x_{12}} B_4(x_{34}, x_{45}) ,$$

i.e, restriction to the first pole of any variable gives the Veneziano four-point formula - bootstrap consistency [see Fig. 5]. So if the amplitude for N = 5 is known then it is uniquely fixed for N = 4, as residue of the first pole.

In general, if $x_{12} \to -N$

$$B_5(x_{12}, \ldots, x_{51}) = \frac{1}{x_{12}+N} \left\{ \sum_{k=0}^{N} \binom{z}{k} \binom{x_{23}-1}{N-k} (-1)^N B_4(x_{34}, x_{45}+k) \right\}$$

and thus writing

$$B_5(x_{12}, \ldots, x_{51}) = \sum_{k=0}^{\infty} C_k\, (x_{23}, \ldots, x_{51}) \frac{1}{x_{12}+k}$$

the residues $C_k\ (x_{23}, \ldots, x_{51})$ are polynomials of degree k in the "angle" variables x_{51} and x_{23} and the k^{th} pole corresponds to the exchange of a family of particles with spins from k down to zero. (Mother plus daughters).

Various "high energy" limits of B_5 were given by Białas and Pokorski [15] using the limiting properties of the series. For example, if x_{45} and $x_{12} \to \infty$, with x_{23} and $\eta = \frac{x_{12}}{x_{45}}$ fixed, the "single Regge limit" [see Fig. 6], then $B_5(1\ldots5) = B_4(45,51)R(23,34)$ where

$B_4(45,51)$ is the Veneziano amplitude for the reaction

$4 + 5 \to (23) + 1$. [as $x_{45} \to \infty$, $B_4\ (45,51) \to (x_{45})^{x_{51}} * f(x_{51})$]

and $$R(23,24) = \sum_{m=0}^{\infty} \frac{C_m(34,\eta)}{x_{23}+m}$$ is the vertex function

expressed as a sum over resonances in the (23) system.

In a similar fashion, if x_{45}, x_{12} and $x_{23} \to \infty$ with

$\mathcal{H} = x_{12}\, x_{23} \,/\, x_{45}$ fixed (the "double Regge limit") then

$$B_5(1 \ldots 5) = B_4(45,51)B_4(23,34\text{-}51)f(34,51,\mathcal{H})$$

$$\simeq x_{12}^{\;x_{51}}\, x_{23}^{\;x_{34}}\, \tilde{f}\,(34,\, 51,\mathcal{H})$$

Both Regge limits of B_5 introduce a well-defined dependence on the Toller angle $(\sim \eta,\mathcal{H})$.

The Regge limits are taken giving to α an imaginary part (or alternatively avoiding the real axis where the amplitude develops an infinite number of poles in the narrow width approximation). The correct signature factors [16] follow on summing over twelve different orders of external lines and properly considering the Regge limits. Thus B_5 [17] has the essential properties that one would wish to generalize from B_4: dual pole structure, residues polynomial in angle variables and thus correct spin structure, factorization and thus bootstrap consistency, and high energy Regge behaviour. These all remain true for arbitrary N.

A further representation of the N-point amplitude in a compact and manifestly crossing symmetric form was given by Koba and Nielson [18] and also Plahte [19]. This was used in certain aspects of the formal developments of dual theories.

So far we have a formula that violates unitarity (being a narrow resonance approximation amplitude) and has non-physical particles (being built up only of scalars). For phenomenological applications one usually chooses the trajectory $\alpha(x)$ to be complex with $\mathrm{Im}\alpha(x)$ increasing with x, the energy, and determined so as to reproduce correctly all observed resonance widths. The poles are thus taken off the real axis in the physical region and at the same time correct high energy Regge behaviour is assured, although various disadvantages (such as residues, in general, losing polynomial behaviour in the momentum transfers) also follow. We have followed this procedure throughout our work. The introduction of physical trajectories is a non-trivial problem; if we retain the term $u^{-\alpha-1}$ in (21.3) we have a ghost when $\alpha = 0$, if we change the exponent $u^{-\alpha}$ we lose the correct asymptotic behaviour. When the external particles are pseudoscalars, kinematical factors are needed in order to obey parity conservation and they can have just the effect of restoring Regge behaviour in an amplitude with physical trajectories. For reactions involving fermions one can take Veneziano forms for invariant amplitudes free of kinematical singularities.

Although this leads to the desired pole structure, the straight line trajectories in s, t and u give rise to an amplitude invariant under change of sign of the amplitude in $W = \sqrt{S}$ and thus by MacDowell symmetry [20] to parity doubling. Removal of baryon parity doublets in the Veneziano model [21] and discussion is given by Storrow [22] and a re-examination of the arguments using a particular spin formalism by Enflo [23].

One of the further properties that would have to be taken into account in constructing a realistic system is internal symmetry, or the incorporation of isospin. That is, we wish to determine the coefficient $C_5(P)$ multiplying $B_5(P)$ corresponding to permutations P of external particles

$$T_5 = \sum_{\{P\}} C_5(P)\, B_5(P)$$

such that: (i) $C_5(P)\, B_5(P)$ remains invariant under cyclic and anti-cyclic permutations; (ii) factorization is retained; and (iii) no exotics are to occur in any channel. This is dictated by experimental evidence.

For five external isovector particles condition (iii) requires absence of poles with isospin larger than one. The solution takes the simple form given by [24,25]

$$C_5(1,\, 2,\, \ldots\, 5) = \frac{1}{2}\,\mathrm{Tr}\,(\tau_{a_1}\, \tau_{a_2}\, \cdots\, \tau_{a_5})$$

where each τ_{a_i} denotes the 2×2 Pauli matrix representing the isospin state a_i of external particle i. Condition (i) follows from the result TrABC = TrCAB, and (iii) from the closure under multiplication of the 2×2 Pauli matrices (i.e, the product of any number of 2×2 matrices is a 2×2 matrix and hence can represent only a combination of isospin 0 and 1). To see the factorization property, we note the identity:

$$\frac{1}{2}\,\mathrm{Tr}(\tau_{a_1}\cdots\tau_{a_5}) = \left[\frac{1}{2}\,\mathrm{Tr}\,(\tau_{a_1}\cdots\tau_{a_M})\frac{1}{2}\,\mathrm{Tr}(\tau_{a_{M+1}}\cdots a_5)\right]$$

$$+ \sum_{I=1}^{3} \left[\frac{1}{2}\,\mathrm{Tr}(\tau_{a_1}\cdots\tau_{a_M}\tau_{a_I})\right]\left[\frac{1}{2}\,\mathrm{Tr}(\tau_{a_I}\tau_{a_{M+1}}\cdots\tau_{a_5})\right]$$

The first term in the RHS corresponds to a singlet (isospin zero) and the second to a triplet (isospin one) intermediate state; i.e, an isospin degeneracy. Summing over all permutations shows that the two states have different signature (cf. the identity $\tau_a\tau_b = \delta_{ab} + i\varepsilon_{abx}\tau_x$ where the two terms on the RHS have opposite symmetry under the interchange of a and b). In effect this gives the $\rho - f^o$ degeneracy from the isospin factor.

The extension to include kaons as external lines is straightforward using the Gell-Mann λ Matrices. The isospin factor corresponding to the ordering (1, 2, ..., 5) is then simply

$$\mathrm{Tr}\,(\lambda_{a_1}\,\lambda_{a_2}\,\ldots\,\lambda_{a_5}).$$

(The extension to N particles is made by replacing 5 by N). A question of uniqueness has been answered by Törnqvist [26].

The first practical five-point processes to be analysed using B_5 were the $K\bar{K}\pi\pi\pi$ and $K\bar{K}K\bar{K}\pi$ [27,28] systems. For the former process in the form $K\bar{K} \to \pi\pi\pi$ (i.e, 1,2 → 3,4,5) Bardakçi and Ruegg gave the amplitude in the form

$$A_5 = \bar{\beta}\Sigma K_2^{+}\tau_{i1}\tau_{i3}\tau_{i5}K_1\varepsilon_{\mu_1\mu_2\mu_3\mu_4}\,p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_4^{\mu_4}{*}$$

$$B_5(1-\alpha_{12}^{\omega},\ 1-\alpha_{23}^{K*},\ 1-\alpha_{34}^{\rho},\ 1-\alpha_{45}^{\rho},\ 1-\alpha_{51}^{K*})$$

where the sum is performed over all permutations of the three pions. The first factor is due to isospin and the second to parity conservation. They obtained correct poles on the lowest values of the trajectories with correct factorization properties and went on to show that the four- and five-particle Veneziano amplitudes gave consistent results, including the standard mixing angles for ω and ϕ, f and f$'$, a universal relation for 2^+ and 1^- meson decays, and pure F coupling for the decay $2^+ \to 1^- 0^-$. Gunion and Yesian [28] looked at some of the experimental implications for particular process of the two systems. The difficulties that arise when one tries to extend B_5 to include physical bosons are illustrated by considering the amplitude for the $\sigma 4\pi$ system (1,2,...5) [29,30], where σ denotes a $J^P = 0^+$ boson. Taking

$$B_5(-\alpha_\pi^{12}, -\alpha_\rho^{23}, -\alpha_\rho^{34}, -\alpha_\rho^{45}, -\alpha_\pi^{51})$$

introduces a spurious state of negative mass at $\alpha_\rho = 0$, and if the rho trajectory is started at one using

$$B_5(-\alpha_\pi^{12}, 1-\alpha_\rho^{23}, 1-\alpha_\rho^{34}, 1-\alpha_\rho^{45}, -\alpha_\pi^{51})$$

then it has to be multiplied suitably to obtain this, a possible solution being:

$$\alpha_\rho^{34}\, B_5\, (-\alpha_\pi^{12}, 1-\alpha_\rho^{23}, -\alpha_\rho^{34}, 1-\alpha_\rho^{45}, -\alpha_\pi^{51})$$

where the multiplicative factor kills the $\alpha_\rho^{34} = 0$ ghost and provides the correct spin for all the rho poles without altering that of the pion poles.

2.2 The Reaction $\pi^- p \to \pi^+\pi^- n$ in the ρ and f^0-mass regions

A specific five-point function amplitude for the process $\bar{p}n \to \pi^-\pi^+\pi^-$ (4,5 → 1,2,3) which ignored spin complications was proposed by Rubinstein, Squires and Chaichian [31]. Using the notation $F(\alpha_{12}, \ldots) = B_5(-\alpha_{12}, \ldots)$ they took their amplitude to be:

$$A = \beta\Big[\alpha_{12}^{\rho}\, F(\alpha_{12}^{\rho}, \alpha_{23}^{\rho} - 1, \alpha_{34}^{B} - \tfrac{1}{2}, \alpha_{45}^{\pi}, \alpha_{15}^{B} - \tfrac{3}{2}) + C(\alpha_{34}^{B} - \tfrac{1}{2})\, F(\alpha_{12}^{\rho} - 1, \alpha_{23}^{\rho} - 1, \alpha_{34}^{B} - \tfrac{1}{2}, \alpha_{45}^{\pi} - 1, \alpha_{15}^{B} - \tfrac{1}{2})\Big] \qquad (22.1)$$

where β is a constant, α_{ij} refers to the Regge trajectory associated with each S_{ij}, as in Fig. 7, and B refers to either the nucleon or delta (1238) baryons. Spinor factors were introduced later [32] in a normalization comparison (see Fig. 8). The constant C was calculated by comparing the amplitude with a fit to the data [33] given by Altarelli and Rubinstein [34]. We show,in fact,in Chapter 3 that this data fit was not satisfactory (although this did not affect the C value), in Chapter 4 that the value of C should be slightly different from the value of −1.25 given, and, in fact, that the second term in the amplitude could be neglected (i.e, C = 0). An explanation of the choice of the terms used by Rubinstein et al in (22.1) is given in Chapter 4 where the amplitude is compared with the data of [33].

We investigated the process $N\bar{N}\pi\pi\pi$ in the production form $\pi^- p \to \pi^-\pi^+ n$ using the amplitude (22.1). The differential cross-section, for small values of the momentum transfer, and the angular distributions are both given with the pole positions $M_{\pi\pi} = M_{\rho}$ and

$M_{\pi\pi} = M_f$ being explicitly taken to simplify the calculations. An interesting point here was to see if the second term in (22.1) would eliminate any, spin 1, ρ' contribution from the, spin 2, f pole. Experimentally there seemed to be no evidence for such a particle which is predicted by the usual Veneziano four-point function formulae that has families of daughter trajectories. The effect of having the second term in (22.1) was therefore to be seen in our results.

Support was lent to applying B_5 to nucleon-antinucleon annihilation reactions by a successful application to the reaction $\bar{p}p \rightarrow 4\pi$ (at rest) by Hopkinson and Roberts [35]. Later applications to the process are considered in Chapter 4.

As regards the production form, Jones and Wylde [36] calculated the differential cross-sections for the quasi two body processes $\sigma\sigma \rightarrow (\sigma\sigma)\sigma$ using the B_5 formula suitable for $\sigma\sigma \rightarrow \sigma\sigma\sigma$. Taking the same trajectory $\alpha_{ij} = \left(S_{ij} - (.138)^2\right) + i\,0.1\sqrt{S_{ij} - 4(.138)^2}$ for each of the five channels and an amplitude of the form $B_5(-\alpha_{12}, -\alpha_{23}, -\alpha_{34}, -\alpha_{45}, -\alpha_{51})$ they were able to correctly reproduce the observed change of slope of $\frac{d\sigma}{dt}$ with $(\sigma\sigma)$ mass that $\pi N \rightarrow \pi\pi\Delta$ results suggest. Using the same amplitude they [37] investigated the Regge residue function with their model and simulated the pion-exchange processes such as $\pi N \rightarrow \rho N$ comparing differential cross-sections and calculating spin-1 and spin-2 density matrix elements. They suggested further investigations with more detailed models.

Waltz [38] applied the production amplitude

$$\varepsilon_{\alpha\beta\gamma\delta}\, p_1{}^{\alpha} p_3{}^{\beta} p_4{}^{\gamma} p_5{}^{\delta}\, B_5\, (1 - \alpha_{12}^{A_2},\ 1 - \alpha_{23},\ 1 - \alpha_{34}^{\rho - f},\ 1 - \alpha_{45}^{\rho - f},\ 1 - \alpha_{51})$$

$$+ \ (3 \leftrightarrow 5)$$

$$+ \ \ldots\ldots$$

to the process $B(0^+)\pi^+ \rightarrow B'(0^+)\ (\pi^-\pi^+)$ (i.e, 2,3 → 1,5,4) and found that the expected shape of the differential cross-section $\frac{d\sigma}{dt}$ $(t = S_{12})$ together with the resonance mass spectrums followed in a straightforward way from the dual amplitude considered. Further, Pokorski, Szeptycka and Zieminski [39] showed that the mass dependence of the slopes in differential cross-sections could be explained using the Bardakçi-Ruegg B_5 function, with finite width resonances, and the kinematics appropriate to $\pi N \rightarrow \pi\pi N$.

a) Phase Space [40]

A reaction with three particles in the final state has five independent Lorentz-invariant variables. Some such suitable variables are indicated in Fig. 7, where, for example,

$$S_{34} = (P_3 + P_4)^2 = P^2 = (P_1 + P_2 + P_5)^2 \tag{22.2}$$

For three particles in the final state the restricted phase space element is given by:

$$\begin{aligned} d\ \text{Lips}\ (S_{34}; P_1, P_2, P_5) = \frac{1}{(2\pi)}\ d\ \text{Lips}\ (S_{34}; P_{12}, P_5)\ * \\ d\ \text{Lips}\ (S_{12}, P_1, P_2)\, dS_{12} \end{aligned} \tag{22.3}$$

where:

$$d\ \mathrm{Lips}(S_{34},P_{12},P_5) = \sqrt{\frac{\lambda(S_{34},m_5^2,S_{12})}{4S_{34}}}\ \frac{d\Omega_5}{16\pi^2 S_{34}^{\frac{1}{2}}} \tag{22.4}$$

where $d\Omega_5$ is the differential solid angle of particle 5's momentum in the centre of mass

and:

$$d\ \mathrm{Lips}(S_{12},P_1,P_2) = \sqrt{\frac{\lambda(S_{12},m_1^2,m_2^2)}{4S_{12}}}\ \frac{d\Omega}{16\pi^2 S_{12}^{\frac{1}{2}}} \tag{22.5}$$

where $d\Omega$ is the corresponding differential solid angle in the rest frame of particles 1 and 2, and $\sqrt{\frac{\lambda(S_{12},m_1^2,m_2^2)}{4S_{12}}} = k$ is the magnitude of particles 1 (or 2) momentum in that rest frame. Hence, (22.3) becomes:

$$d\ \mathrm{Lips}(S_{34};P_1,P_2,P_5) = \sqrt{\frac{\lambda(S_{34},m_5^2,S_{12})}{4S_{34}}}\ \frac{k\ d\Omega d\Omega_5\ dS_{12}}{(16\pi^2)^2 S_{34}^{\frac{1}{2}} S_{12}^{\frac{1}{2}}\quad (2\pi)} \tag{22.6}$$

with

$$\left.\begin{aligned} \lambda(x,y,z) &= x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \\ &= (x - y - z)^2 - 4yz \\ &= (x - (\sqrt{y} + \sqrt{z})^2)\ (x - (\sqrt{y} - \sqrt{z})^2) \end{aligned}\right\} \tag{22.7}$$

a Lorentz-invariant.

b) Cross-Section

The cross-section for this process is given from the general case of N final particles and a and b initial particles of

$$\sigma = \frac{1}{2\sqrt{\lambda(S,m_a^2,m_b^2)}} \int d\ \mathrm{Lips}\ (S;P_1,\ \ldots,\ P_N)\,|T_{if}|^2 \tag{22.8}$$

where T is the connected part of the S matrix.

By neglecting spin effects so that $|T_{if}|^2 = |A|^2$ this becomes:

$$\sigma = \frac{1}{2\sqrt{\lambda(S_{34},m_3^{\,2},m_4^{\,2})}} \int d\ \mathrm{Lips}\ (S_{34};P_1,P_2,P_5)\ |A|^2 \qquad (22.9)$$

c) Kinematics

The details are given in the Appendix, where it is shown that:

$$\frac{d\sigma}{dS_{45}dS_{12}d\Omega} = \left(\frac{k}{2E}\right) |A|^2 \times f(S_{34}) \qquad (22.10)$$

where:

$$f(S_{34}) = \frac{1}{\lambda(S_{34},m^2,M^2)\pi^4 2^9} \qquad (22.11)$$

and

$$\begin{cases} E = \text{Energy of particle 1 in (1,2) CMS} \\ m = \text{pion mass} \\ M = \text{nucleon mass} \end{cases}$$

d) Evaluating the Amplitude for $\pi^-p \to \pi^+\pi^-n$ at the ρ and f^o poles

The details are given in the Appendix, where for the ρ-region ($\alpha_{12} = 1$) we had:

$$A = \beta \left\{ \frac{\Gamma(0-i\mathrm{Im}\rho)\Gamma(\frac{1}{2}-\alpha_{34})\Gamma(-\alpha_{45})}{\Gamma(\frac{1}{2} - \alpha_{34} - \alpha_{45})} \left[\alpha_{23} + \alpha_{45}\ \frac{(\alpha_{15}-\alpha_{23}-\alpha_{34})}{(\alpha_{34}+\alpha_{45}-\frac{1}{2})} + C\ \frac{(\alpha_{34} - \frac{1}{2})\ \ \alpha_{45}}{(\alpha_{34}+\alpha_{45} - \frac{1}{2})} \right] \right\} \qquad (22.12)$$

where Imρ refers to the imaginary part used on the ρ-trajectory at the ρ-pole to keep the expression finite. C is the constant term given as -1.25 in Rubinstein et al [31].

For the f-region ($\alpha_{12} = 2$) we had:

$$A = \beta K \left\{ \alpha_{23}(\alpha_{23}+1) + 2\alpha_{23}\alpha_{45} \frac{(\alpha_{15} - \alpha_{23} - \alpha_{34})}{(\alpha_{34} + \alpha_{45} - \frac{1}{2})} \right.$$
$$- \frac{(\alpha_{15} - \alpha_{23} - \alpha_{34})(\alpha_{15} + 1 - \alpha_{23} - \alpha_{34})(1 - \alpha_{45})\alpha_{45}}{(\alpha_{34} + \alpha_{45} - \frac{3}{2})(\alpha_{34} + \alpha_{45} - \frac{1}{2})}$$
$$+ C \frac{(\alpha_{34} - \frac{1}{2})\,\alpha_{23}\,\alpha_{45}}{(\alpha_{34} + \alpha_{45} - \frac{1}{2})}$$
$$\left. - C \frac{(\alpha_{34} - \frac{1}{2})(\alpha_{15} + 1 - \alpha_{23} - \alpha_{34})(1 - \alpha_{45})\,\alpha_{45}}{(\alpha_{34} + \alpha_{45} - \frac{3}{2})(\alpha_{34} + \alpha_{45} - \frac{1}{2})} \right\} \tag{22.13}$$

where
$$K = \frac{-\Gamma(-\alpha_{12} + 1)\Gamma(-\alpha_{45})\Gamma(-\alpha_{34} + \frac{1}{2})}{\Gamma(-\alpha_{45} - \alpha_{34} + \frac{1}{2})} \tag{22.14}$$

i.e,
$$K = \frac{-\Gamma(-1 - i\mathrm{Im}f)\Gamma(-\alpha_{45})\Gamma(-\alpha_{34} + \frac{1}{2})}{\Gamma(\frac{1}{2} - \alpha_{34} - \alpha_{45})} \tag{22.15}$$;

where Imf refers to the imaginary part given at the f-pole.

e) Integration

In the evaluation of the expression $\frac{d\sigma}{dS_{45}}$ ($\frac{d\sigma}{dt}$ say) it is required to integrate over Ω and S_{12}. To integrate over Ω we integrate over θ and ϕ:

$$\int f \, d\Omega \rightarrow \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\, f \tag{22.16}$$

The integration over the ϕ variable is done directly using the Gauss-Mehler method, viz:

$$\int_0^{\pi} f\ (\cos\phi)\ d\phi \;=\; \frac{\pi}{n}\sum_{j=1}^{n} f\ \left(\cos\frac{(2j-1)\pi}{2n}\right) \tag{22.17}$$

This formula is exact up to the order (2n-1) so that for a 4th order case, as here, we therefore require n=3, i.e.

$$\int_0^{2\pi} f\ (\cos\phi)\ d\phi \;=\; \frac{2\pi}{3}\ \left\{ f\left(\frac{\sqrt{3}}{2}\right) + f(0) + f\left(-\frac{\sqrt{3}}{2}\right) \right\} \tag{22.18}$$

To integrate over θ we use the Gaussian method on

$$\int_{-1}^{1} F\ (x)\ dx.$$

(The previous method is not used because F involves sinθ).

Since the pole positions only were taken for S_{12} to integrate over this variable, it was therefore necessary to assume a particular form for the amplitude at this point. The simple Breit-Wigner form was assumed, so that:

$$\begin{aligned}
\int \frac{d^2\sigma}{dt\ dS_{12}}\,dS_{12} &= \int_{S_{th}}^{\infty} \frac{E_R^{\ 2}\,\Gamma_R^{\ 2}\left(\frac{d^2}{dtdS}\right)_{S=S_R}}{(S-S_R)^2 + E_R^{\ 2}\Gamma_R^2}\ dS \quad \text{where} \quad \begin{matrix} S=S_{12} \\ S=S_R \end{matrix} \\
&= \int_{(S-S_R)=S_{th}-S_R}^{\infty} \frac{d(S-S_R)}{(S-S_R)^2 + E_R^{\ 2}\Gamma_R^{\ 2}} \left(E_R^{\ 2}\Gamma_R^{\ 2}\left[\frac{d^2\sigma}{dtdS}\right]_{S=S_R} \right) \\
&= \frac{1}{E_R\Gamma_R}\tan^{-1}\left[\frac{(S-S_R)}{E_R\Gamma_R}\right]\Bigg|_{(S-S_R)=S_{th}-S_R}^{\infty} \left(E_R^{\ 2}\Gamma_R^{\ 2}\left[\frac{d^2\sigma}{dtdS}\right]_{S_R} \right) \\
&= E_R\Gamma_R\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{S_{th}-S_R}{E_R\ \ \Gamma_R}\right)\right]\ \frac{d^2\sigma}{dtdS}\Bigg|_{S_R}
\end{aligned} \tag{22.19}$$

At the ρ-position:

$$E_R = m_\rho, \quad \Gamma_R = \text{Width } \rho, \quad S_{th} = 4(m)^2, \quad S_R = E_R{}^2$$

At the f-position:

$$E_R = m_f, \quad \Gamma_R = \text{Width f}, \quad S_{th} = 4(m)^2, \quad S_R = E_R{}^2$$

m = pion mass.

f) <u>Spinor Factors</u>

We have mentioned that ref. [32] gives spinor factors to account for the fermion's spin. These are given by:

$$\sum_{f,i} |\bar{u}_f \, T \, u_i|^2 \,, \qquad T = \gamma_5 \tag{22.20}$$

Expanding (22.20) by using the usual projection operators and taking $m_\rho = m_n = M$ we get the factor to be:

$$\left(-\frac{S_{45}}{4M^2}\right) \quad \text{as given in ref. [32].} \tag{22.21}$$

g) <u>The Constant β</u>

The constant β is given by:

$$\beta = \alpha' \times \sqrt{2} \times g_{NN\pi} \, f^2_{\rho\pi\pi}$$

Since we will require $|A|^2$ then we take $B = \beta^2$

$$B = \alpha'^2 \times 2 \times g^2_{NN\pi} \times (f^2_{\rho\pi\pi})^2 \tag{22.22}$$

where:

(i) $\alpha' =$ The Universal trajectory slope

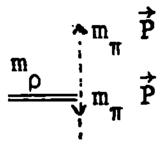
(ii) $f^2_{\rho\pi\pi} \simeq 4\pi \times 2.4$

(iii) $g^2_{NN\pi} \simeq 4\pi \times 14.4$

$$\left.\begin{array}{l} \text{(ii)} \\ \text{(iii)} \end{array}\right\} \text{Coupling constants}$$

f^2 in (ii) may be calculated from:

$$\Gamma_\rho = \frac{f^2}{6\pi}\,\frac{P^3}{m_\rho^{\,2}} \qquad \text{where} \qquad 2\sqrt{P^2 + m_\pi^{\,2}} = m_\rho \quad \text{if}$$



and $P = |\vec{P}|$ (given in, for example [41])

Various values of both (ii) and (iii) are quoted in the literature with $2.1 < \frac{f^2}{4\pi} < 2.8$ and $14 \leq \frac{g^2}{4\pi} \leq 15$. [See for example the values in Ebel et al [42], Sakurai [43] and the Daresbury 4th International Conference (p.94)].

h) Computations

(i) Differential Cross Section v. Momentum Transfer

The following values were used for the masses:

$M_N = 0.94$ GeV, $m_\pi \simeq 0.14$ GeV, $M_{N^*_{33}(1236)} = 1.236$ GeV.

Different widths were tried:

$\Gamma_f \simeq 0.15$ GeV, $\Gamma_\rho \simeq 0.125$ GeV and $\Gamma_\rho = \Gamma_f = .09$ GeV.

Slopes α' were varied from 0.9 to 1.0 and also a slope tried which was given by the ρ-f masses. The conversion factor $1(\text{GeV})^{-2} = 0.38935$ m.b. was used for the ordinates.

The trajectory functions used were:

(22.23)

$$\rho \qquad \alpha_{23} = \alpha_{23}(0) + \alpha' \times S_{23} + i\ (\text{RTERM}) \sqrt{S_{23} - 4m^2}$$

$$B \quad \begin{cases} \alpha_{34} = \alpha_{34}(0) + \alpha' \times S_{34} + i\ 0.14\ \left(S_{34} - (M+m)^2\right) \\ \alpha_{15} = \alpha_{15}(0) + \alpha' \times S_{15} + i\ 0.14\ \left(S_{15} - (M+m)^2\right) \end{cases}$$

$$\pi \qquad \alpha_{45} \simeq \alpha' \times (S_{45} - m^2)$$

$$\text{Poles} \begin{cases} \rho.\ \alpha_{12} \simeq 1 + i\ \underbrace{(\text{RTERM}) \sqrt{S_{12}^{\rho} - 4m^2}} \\ f.\ \alpha_{12} \simeq 2 + i\ \underbrace{(\text{FTERM}) \sqrt{S_{12}^{f} - 4m^2}} \end{cases}$$

where $\quad \text{Im}\alpha \quad = \quad (\alpha' = \text{SLOPE}) \times (\Gamma = \text{WIDTH}) \times (m = \text{MASS})$ (22.24)

This phenomenological fit was first suggested by Peterson and Törnqvist [44] and we follow their prescription for both meson and baryon imaginary parts. The pole positions were used so that:

$$S_{12}^{\rho} = \frac{1.0 - \alpha_{12}(0)}{\alpha'} \quad ; \quad S_{12}^{f} = \frac{2.0 - \alpha_{12}(0)}{\alpha'}$$

$$\alpha_{12}(0) = \alpha_{23}(0) = 2.0 - \alpha' \times S^{f} = 2.0 - \alpha' \times M_f^2$$

$$\alpha_{34}(0) = \alpha_{15}(0) = 1.5 - \alpha' \times M^2_{N^*_{33}(1236)} \qquad \text{(for the } \Delta \text{ as baryon)}$$

The relationship between the laboratory energy and the C M energy is given by:

$$E_{LAB} = \frac{S_{34} - m^2 - M^2}{2M} \quad \text{i.e,} \quad S_{34} = (M^2+m^2) + 2M\sqrt{|P_L|^2 + m^2} \qquad (22.25)$$

(i.e, approx. $|\vec{P}_L| \simeq E_L \simeq \frac{S_{34}}{2}$)

From the appendix to C recall that:

$$S_{23} = \frac{S_{45}}{2} - 2E^2 + \frac{3}{2}m^2 + 2(E^2 - m^2)^{\frac{1}{2}} \lambda_3^{\frac{1}{2}} \cos\theta \tag{22.26}$$

where:

$$E = \frac{1}{2}\sqrt{S_{12}}, \quad \lambda_3 = \lambda\left(S_{45}, m^2, (2E)^2\right) / 16E^2$$

$$\lambda_3 = E_3^2 - m^2, \; E_3 = \frac{4E^2 + m^2 - S_{45}}{4E}$$

$$\therefore \quad \lambda_3 = \frac{\left(S_{45} - (m+2E)^2\right)\left(S_{45} - (m-2E)^2\right)}{16E^2}$$

Similarly,

$$S_{15} = \frac{S_{34}}{2} - 2E^2 + \frac{M^2}{2} + m^2 + 2(E^2 - m^2)^{\frac{1}{2}} \lambda_5^{\frac{1}{2}} \; *$$

$$* \left[-\cos\theta \cos\theta_5 + \sin\theta \sin\theta_5 \cos\phi \right] \tag{22.27}$$

where: $\lambda_5 = E_5^2 - M^2, \quad E_5 = \dfrac{S_{34} - 4E^2 - M^2}{4E}$

$$\therefore \quad \lambda_5 = \frac{\left(S_{34} - (M+2E)^2\right)\left(S_{34} - (M-2E)^2\right)}{16E^2}$$

and $\cos\theta_5 = \dfrac{4E(E_5 - E_3) + 4E^2 - 2E_5E_3 + m^2}{2\lambda_3^{\frac{1}{2}} \lambda_5^{\frac{1}{2}}}$

(E_3, E_5, λ_3, λ_5 and E are given above).

The factor $\left(\dfrac{k}{2E}\right) f(S_{34})$ is found using:

$$E = \frac{1}{2}\sqrt{S_{12}}, \quad k = (E^2 - m^2)^{\frac{1}{2}}, \quad f(S_{34}) = \frac{1}{\lambda(S_{34}, m^2, M^2)\pi^4 \, 2^9} \tag{22.28}$$

where $\lambda(S_{34}, m^2, M^2) = \left(S_{34} - (M-m)^2\right)\left(S_{34} - (M+m)^2\right)$

The amplitude expressions were put into the more compact forms below:

ρ - case (22.12) becomes:

$$A = \beta\left[\frac{\Gamma(0 - i\mathrm{Im}\rho)\Gamma(\frac{1}{2} - \alpha_{34})\Gamma(-\alpha_{45})}{\Gamma(\frac{1}{2} - \alpha_{34} - \alpha_{45})}\right]\left\{\alpha_{23} + \frac{\alpha_{45}}{(\alpha_{34}+\alpha_{45}-\frac{1}{2})} * \right.$$

$$\left. * \left[(\alpha_{15} - \alpha_{23} - \alpha_{34}) + C.(\alpha_{34} - \tfrac{1}{2})\right]\right\} \tag{22.29}$$

For the f-case (22.13) becomes:

$$A = \beta\left[\frac{-\Gamma(-1 - i\mathrm{Im}f)\Gamma(\frac{1}{2} - \alpha_{34})\Gamma(-\alpha_{45})}{\Gamma(\frac{1}{2} - \alpha_{34} - \alpha_{45})}\right]\left\{\alpha_{23}\ (\alpha_{23} + 1)\ + \right.$$

$$+ \frac{1}{(\alpha_{34}+\alpha_{45} - \frac{1}{2})}\left[\alpha_{23}\ \alpha_{45}\left\{2.(\alpha_{15} - \alpha_{23} - \alpha_{34}) + C.(\alpha_{34} - \tfrac{1}{2})\right\} - \right. \tag{22.30}$$

$$\left.\left. - \frac{(1-\alpha_{45})\alpha_{45}(\alpha_{15}-\alpha_{23} - \alpha_{34} + 1)}{(\alpha_{34} + \alpha_{45} - \frac{3}{2})}\left\{(\alpha_{15} - \alpha_{23} - \alpha_{34}) + C.(\alpha_{34} - \tfrac{1}{2})\right\}\right]\right\}$$

(ii) <u>Angular Distribution at f-region $d\sigma / d(\cos\theta)$ v. $\cos\theta$</u>

A similar procedure to that in (i) was carried out except that because we integrate over S_{45} the spinor factor (22.21) and various other terms can not be taken out as factors as was the case in (i). The decay angular distribution (see, for example, ref. [45]) for one pion exchange in the $f^{o}(2^{+})$ resonance region takes the form:

$$\sqrt{W_2}\ (\theta) \sim A \cos^2\theta + B.$$

Since in the ρ (1^{-}) resonance case it takes the form:

$$\sqrt{W_1}\ (\theta) \sim C \cos\theta$$

then if in the region $\alpha_{12} = 2$ there is some spin 1 (ρ') contribution then we would have the form:

$$W_2\ (\theta) \sim \left|\ (D \cos^2\theta + E \cos\theta + F)\right|^2.$$

As this contains terms odd in $\cos\theta$ then a plot of the angular distribution will indicate whether or not there is any spin 1 contribution present.

The Gamma functions were calculated by using ZFACT, a (22.31)
series approximation for $Z! = \Gamma(Z + 1)$ where Z is complex.

A desk-calculator check was made, for one incident energy-momentum transfer case, for those constants not integrated over (e.g, λ_3).

On the first few computer runs various output statement checks were inserted. These were subsequently removed.

The results of the computations compared with experimental data [46,47] are shown in Figs. 11 - 15. Although a fitting procedure was not used there is considerable latitude in any curve presented due to the wide choices of trajectory function, resonance widths and coupling constants. Narrowing the widths increased the magnitude of the differential cross-sections and putting $C = 0$ seemed to produce a slope which corresponded closer to that of the data. In general the agreement is not too good, but is more successful at 8GeV than 16 GeV and for the ρ, rather than the f^o-region. Writing the amplitude in the form $R_1 + CR_2$ then, the value of $C = -1.25$ does not seem to be favoured by the data and in fact a value of $C = 0$ would be not too far out, at least as regards the slope of the curve. These observations were borne out later in direct data fits as reported in Chapters 3 and 4. With $C = 0$, however, the agreement is very poor in the near forward direction of small $|t|$. It would have been, nevertheless, worse still had not the normalisation term [22.21] been included; a fact stressed by Rubinstein et al [32] in their normalization of the amplitude.

The angular distribution for the f^o-region at 8 Gev indicates that this model with C = −1.25 gave rise to ρ' (i.e, f^o-daughter) contributions as seen in the strong asymmetry of the graph in Fig. 15. A symmetric distribution in $\cos\theta$ would require C to be about +2. In Chapter 4 a value of $C \approx \frac{1}{2}$ was found and this would still imply the existence of a small contribution from the ρ' daughter term.

After this work was completed two similar fits to the $N\bar{N}\pi\pi\pi$ complex and the particular process $\pi N \to \pi\pi N$ were given. Bender, Dosch, Müller and Rothe gave a dual resonance model [48] for the complex using B_5 functions suitably multiplied by polynomials for the various invariant amplitudes. They then applied this model to the process $\pi N \to \pi\pi N$ [49] at small momentum transfer (small $|t|$) between the nucleons and found that the differential cross-sections for the ρ and f^o region were in good agreement with the data provided daughter terms were included. Thus, leaving out the I = 1 daughter in the f^o-case led to poor agreement. In the model proposed by Pokorski, Szeptycka and Zieminski [50] in addition to a suitable dual amplitude, with the same vertex factor as used by Rubinstein et al [32], there was used a Pomeron term parametrized according to certain assumptions. Good agreement with the differential cross-section data at 11 and 16 GeV was obtained for the ρ and f-mass regions. These two slightly better fits required, however, some considerable increase in complexity of the amplitude expressions.

Given the relative simplicity of our initial dual amplitude (22.1) and the various assumptions that had to be made we have seen that the crossed, production, process gave moderately successful results for the particular energies chosen and that some ρ'-daughter contributions were present in the given amplitude.

CHAPTER 2 Appendix for (c)

The Kinematics for $\pi^- p \to \pi^+ \pi^- n$

$$\left.\begin{aligned} \text{Let } M &= \text{nucleon mass} \\ m &= \text{pion mass} \end{aligned}\right\}$$

We take, conventionally, all four-momenta as incoming so that in the notation of Fig.7; 3, 4 $\to \bar{1}, \bar{2}, \bar{5}$. Take 3 and 4 to be the incident particles and look for the resonances in the 1, 2 region. Express all variables in terms of S_{34}, S_{12}, S_{45}, and the angles defined in the 1, 2 CMS (i.e, when $\underline{P}_1 + \underline{P}_2 = 0$).

$$\text{In the CMS of 1, 2} \quad E_1 = \frac{S_{12} + m_1^2 - m_2^2}{2\sqrt{S_{12}}}, \quad E_2 = \frac{S_{12} + m_2^2 - m_1^2}{2\sqrt{S_{12}}}$$

so if $m_1 = m_2$ (pion mass) then $E_1 = E_2 = \frac{\sqrt{S_{12}}}{2} = E$ say.

As $\underline{P}_1 + \underline{P}_2 = 0$ let $|\underline{P}_1| = k$ say.

In general we define the four-momenta as follows:

$$\left\{\begin{aligned} & P_1 + P_2 + P_3 + P_4 + P_5 = 0 \\ \text{or } & E_3 + E_4 = E_1 + E_2 + E_5 \\ \text{and } & \underline{P}_3 + \underline{P}_4 = \underline{P}_1 + \underline{P}_2 + \underline{P}_5 \end{aligned}\right. \Longleftarrow \left\{\begin{aligned} &\left.\begin{aligned} -P_1 &= (E_1, \underline{P}_1) \\ -P_2 &= (E_2, \underline{P}_2) \\ -P_5 &= (E_5, \underline{P}_5) \end{aligned}\right\} \text{Outgoing pts.} \\ &\left.\begin{aligned} P_3 &= (E_3, \underline{P}_3) \\ P_4 &= (E_4, \underline{P}_4) \end{aligned}\right\} \text{Incoming pts.} \end{aligned}\right.$$

Note: In Polar Coordinates the three components in (R, θ, ϕ) are:

$(k \cos\theta,\ k \sin\theta\cos\phi,\ k \sin\theta\sin\phi)$.

$$\therefore \quad -P_1 = (E, -k\cos\theta, -k\sin\theta\cos\phi, -k\sin\theta\sin\phi)$$

$$-P_2 = (E, +k\cos\theta, +k\sin\theta\cos\phi, +k\sin\theta\sin\phi).$$

Also $\mathbf{P}_3$ is along the x-axis with $|\mathbf{P}_3| = q$, say, so that

$$P_3 = (E_3, q, 0, 0).$$

The CMS condition that $\mathbf{P}_3 + \mathbf{P}_4 - \mathbf{P}_5 = 0$ implies that these momenta are <u>co-planar</u> so let the 3,4 angle be θ_4 and the 3,5 angle θ_5. See Fig. 16 for these angles.

$$P_4 = (E_4, |\mathbf{P}_4|\cos\theta_4, |\mathbf{P}_4|\sin\theta_4, 0)$$

$$-P_5 = (E_5, -|\mathbf{P}_5|\cos\theta_5, +|\mathbf{P}_5|\sin\theta_5, 0).$$

Using the results

$$P_1 + P_2 + P_3 + P_4 + P_5 = 0 \qquad \text{in 1,2 CMS}$$

$$\Longrightarrow \quad \mathbf{P}_1 + \mathbf{P}_2 = 0 = \mathbf{P}_3 + \mathbf{P}_4 - \mathbf{P}_5$$

and

$$E_3 + E_4 = 2E + E_5$$

gives the following for the invariants:

$$S_{34} = (P_3 + P_4)^2 \quad (\text{in the use of } \mathbf{P}_3 + \mathbf{P}_4 = (+\mathbf{P}_5))$$

$$= (E_3 + E_4)^2 - |\mathbf{P}_5|^2$$

$$= (2E + E_5)^2 - (E_5^2 - M^2) \quad (\text{by CM energy conservation})$$

$$= 4E^2 + 4EE_5 + M^2$$

$$\therefore \quad E_5 = \frac{S_{34} - 4E^2 - M^2}{4E}$$

$$S_{45} = (P_4 + P_5)^2 \qquad \text{(using the CMS condition again)}$$
$$= (E_4 - E_5)^2 - |\mathbf{P}_3|^2$$
$$= (2E - E_3)^2 - (E_3{}^2 - m^2) \qquad \text{(by CMS energy conservation)}$$
$$= 4E^2 - 4EE_3 + m^2$$
$$\therefore \quad E_3 = \frac{4E^2 + m^2 - S_{45}}{4E}$$

$$S_{15} = (P_1 + P_5)^2$$
$$= m^2 + M^2 + 2P_1 . P_5$$
$$= m^2 + M^2 + 2EE_5 - 2\mathbf{P}_1 . \mathbf{P}_5$$
$$= m^2 + M^2 + 2EE_5 - 2k|\mathbf{P}_5| \left[\cos\theta\cos\theta_5 - \sin\theta\sin\theta_5\cos\phi\right]$$
$$S_{15} = m^2 + M^2 + 2EE_5 + 2(E^2 - m^2)^{\frac{1}{2}} (E_5{}^2 - M^2)^{\frac{1}{2}} \left[-\cos\theta\cos\theta_5 + \right.$$
$$\left. + \sin\theta\sin\theta_5\cos\phi\right]$$

$$S_{23} = (P_2 + P_3)^2$$
$$= m^2 + m^2 + 2P_2 . P_3$$
$$= 2m^2 - \left[2E(+E_3) - 2\mathbf{P}_2 . \mathbf{P}_3\right]$$
$$S_{23} = 2m^2 - 2EE_3 + 2\underbrace{(E^2 - m^2)^{\frac{1}{2}}}_{k} \underbrace{(E_3{}^2 - m^2)^{\frac{1}{2}}}_{q} \cos\theta$$

From S_{34}

$$(E_5^2 - M^2) = \frac{(S_{34} - M^2)^2 + 16E^4 - 8E^2(S_{34} - M^2) - 16E^2M^2}{16E^2}$$
$$= \frac{S_{34}^2 + (M^2)^2 + (4E^2)^2 - 2S_{34}M^2 - 2(4E^2)S_{34} - 2(4E^2)M^2}{16E^2}$$
$$E_5{}^2 - M^2 = \frac{\lambda(S_{34}, M^2, (2E)^2)}{16E^2} = \text{(defn.)} \quad \lambda_5 .$$

Where $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$.

Similarly from S_{45}

$$(E_3^{\ 2} - m^2) = \frac{S_{45}^{\ 2} + (4E^2)^2 + (m^2)^2 - 2S_{45}(4E^2) - 2S_{45}m^2 - 2m^2(4E^2)}{16E^2}$$

$$E_3^{\ 2} - m^2 = \frac{\lambda(S_{45}, m^2, (2E)^2)}{16E^2} = \text{(defn.)}\ \lambda_3.$$

N.B.

λ can be put into various different forms (Ref. [40] Pilkuhn, p.6) of products as well as the above:

e.g,

$$\lambda(S_{45}, m^2, (2E)^2) = \left[S_{45} - (m+2E)^2\right]\left[S_{45} - (m-2E)^2\right]$$

$$\lambda(S_{45}, m^2, (2E)^2) = (S_{45} - m^2 - (2E)^2)^2 - 4m^2(2E)^2.$$

Using these results in S_{15}, S_{23} gives:

$$S_{15} = \frac{S_{34}}{2} - 2E^2 + \frac{M^2}{2} + m^2 + 2(E^2-m^2)^{\frac{1}{2}}\ \lambda_5^{\ \frac{1}{2}}\ \left[-\cos\theta\cos\theta_5 + \right.$$

$$\left. + \sin\theta\sin\theta_5\cos\phi\right]$$

$$S_{23} = \frac{S_{45}}{2} - 2E^2 + \frac{3}{2}m^2 + 2(E^2-m^2)^{\frac{1}{2}}\ \lambda_3^{\ \frac{1}{2}}\ \cos\theta$$

From

$$\mathbf{P}_3 + \mathbf{P}_4 - \mathbf{P}_5 = 0$$

$$(\mathbf{P}_3 - \mathbf{P}_5)^2 = (\mathbf{P}_4)^2$$

and

$$\mathbf{P}_3^{\ 2} + \mathbf{P}_5^{\ 2} - 2\mathbf{P}_3.\mathbf{P}_5 = \mathbf{P}_4^{\ 2}$$

we have:

$$\cos\theta_5 = \frac{-(\mathbf{p}_3^2 + \mathbf{p}_5^2 - \mathbf{p}_4^2)}{2|\mathbf{p}_3||\mathbf{p}_5|}$$

$$= \frac{-(\mathbf{p}_3^2 + \mathbf{p}_5^2 - \mathbf{p}_4^2)}{2(E_3^2-m^2)^{\frac{1}{2}}(E_5^2-M^2)^{\frac{1}{2}}}$$

$$= \frac{-(E_3^2-m^2 + E_5^2 - M^2 - (E_4^2-M^2))}{2\lambda_3^{\frac{1}{2}}\lambda_5^{\frac{1}{2}}}$$

But in this CMS, $E_4 = 2E + E_5 - E_3$, so this becomes:

$$\cos\theta_5 = \frac{-\left[\begin{array}{c} E_3^2 - m^2 + E_5^2 - M^2 + M^2 - (4E^2 + E_5^2 + E_3^2 + \\ + 4EE_5 - 2E_5E_3 - 4EE_3) \end{array}\right]}{2\lambda_3^{\frac{1}{2}}\lambda_5^{\frac{1}{2}}}$$

$$\therefore \cos\theta_5 = \frac{-\left[4E(E_3-E_5) - 4E^2 + 2E_5E_3 - m^2\right]}{2\lambda_3^{\frac{1}{2}}\lambda_5^{\frac{1}{2}}}$$

The Differential Cross Section is given from

$$\sigma = \frac{1}{2\sqrt{\lambda(S_{34},m_3^2,m_4^2)}} \int d\,\mathrm{Lips}\,(S_{34},P_1,P_2,P_5)|A|^2$$

and $\mathrm{dLips}\,(S_{34},P_1,P_2,P_5) = \sqrt{\lambda(S_{34},m_5^2,S_r)}\sqrt{\lambda(S_r,m_1^2,m_2^2)} \times$

$$\frac{d\Omega d\Omega_5 dS_r}{(2\pi)4(16\pi^2)^2 S_r S_{34}}$$

where $d\Omega_5$ is the differential solid angle of particle 5's momentum in the CMS and the length of this momentum is $|\mathbf{p}_5| = \sqrt{\lambda(S_{34},m_5^2,S_r)}/2S_{34}^{\frac{1}{2}}$.

$|\mathbf{p}| = \sqrt{\lambda(S_r, m_1^2, m_2^2)}/2S_r^{\frac{1}{2}}$ is the magnitude of the momentum of particle 1 (and 2) in the r or 1,2 rest frame, i.e. = k, and Ω is the corresponding solid angle.

Recall $P_r^2 = S_r$, $P_r = P_1 + P_2$,

$S_r = S_{12} = (P_1 + P_2)^2 = (P_1^o + P_2^o)^2 = (E + E)^2 = 4E^2$.

$S_r = S_{12} = (2E)^2$.

So we have $\dfrac{d\sigma}{d\Omega_5 dS_{12} d\Omega}$ from:

$$\mathrm{dLips}(S_{34},P_1,P_2,P_5) = \frac{d\Omega_5 d\Omega dS_{12} \sqrt{\lambda(S_{34},m_5^2,S_{12})}\,k}{S_{34}\, S_{12}^{\frac{1}{2}}\, 2^{10}\, \pi^5}$$

$$\frac{d\sigma}{d\Omega_5 dS_{12} d\Omega} = \frac{\sqrt{\lambda(S_{34},m_5^2,(2E)^2)}}{2\sqrt{\lambda(S_{34},m_3^2,m_4^2)}} \left(\frac{k}{2E}\right) |A|^2 \left(\frac{1}{2^{10}\pi^5 S_{34}}\right)$$

We now translate $d\Omega_5$ into dS_{45} and to do this we go into the 3,4 CMS (I.e, the 1,2,5 CMS) or the overall CMS of the system. See Fig. 17.

In the 3,4 CMS

$\underline{P}_3 + \underline{P}_4 = 0 = \underline{P}_1 + \underline{P}_2 + \underline{P}_5$. So $\underline{P}_3 = -\underline{P}_4 = -\underline{P}$, say, in this system.

$$S_{45} = (P_4 + P_5)^2$$

$$S_{45} = m_4^2 + m_5^2 + 2P_4P_5 = 2M^2 - 2E_4E_5 + 2|\underline{P}_4||\underline{P}_5|\cos\theta_5'.$$

Recall that $|\underline{P}| = \dfrac{\sqrt{\lambda(S_{34},m^2,M^2)}}{4S_{34}} = |\underline{P}_3| = |\underline{P}_4|$

and that $|\underline{P}_5| = \dfrac{\sqrt{\lambda(S_{34},M^2,(2E)^2)}}{4(S_{34})}$, $S_{12} = 4E^2$,
(in this CMS)

$$\therefore \quad dS_{45} = \frac{2\sqrt{\lambda(S_{34},m^2,M^2)}\ \sqrt{\lambda(S_{34},M^2,S_{12})}}{4S_{34}}\, dz$$

and $d\Omega_5 = 2\pi\ d\cos\theta_5' = 2\pi dz.$

{The int. over $d\phi$ implies no initial spin polarization.}

So we have:

$$\frac{d\sigma}{dS_{45}dS_{12}d\Omega} = \left(\frac{k}{2E}\right) |A|^2 \left[\frac{1}{\lambda(S_{34},m^2,M^2)}\ \pi^4\times 2^9\right]$$

$$\text{i.e,} \quad \frac{d\sigma}{dS_{45}dS_{12}d\Omega} = \left(\frac{k}{2E}\right) |A|^2 \times f(S_{34})$$

$$f(S_{34}) = \frac{1}{\lambda(S_{34},m^2,M^2)\pi^4\ 2^9}$$

CHAPTER 2 Appendix for (d)

Evaluating the amplitude expression for $\pi^-p \to \pi^-\pi^+n$.

Let the amplitude (22.1) be written for convenience

(1) $$A = A_1+A_2 \text{ where } A_1 = \beta\alpha_{12}F(\alpha_{12},\alpha_{23}-1,\alpha_{34}-\tfrac{1}{2},\alpha_{45},\alpha_{15}-\tfrac{3}{2})$$

$$A_2 = \beta C(\alpha_{34}-\tfrac{1}{2})F(\alpha_{12}-1,\alpha_{23}-1,\alpha_{34}-\tfrac{1}{2},\alpha_{45}-1,\alpha_{15}-\tfrac{1}{2})$$

Recall (21.1) and put $x = u_{45}$, $y = u_{12}$ so that

$$F(\alpha_{12},\alpha_{23},\alpha_{34},\alpha_{45},\alpha_{51}) = \int_0^1\int_0^1 dxdyx^{-\alpha_{45}-1}y^{-\alpha_{12}-1}(1-x)^{-\alpha_{34}-1}(1-y)^{-\alpha_{23}-1} *$$

$$* (1-xy)^{\alpha_{23}+\alpha_{34}-\alpha_{15}}$$

Now for the first term A_1 we have:

$$A_1 = \beta\alpha_{12}\int_0^1\int_0^1 dxdy\, x^{-\alpha_{45}-1}y^{-\alpha_{12}-1}(1-x)^{-(\alpha_{34}-\frac{1}{2})-1}(1-y)^{-(\alpha_{23}-1)-1} *$$

$$* (1-xy)^{(\alpha_{23}-1)+(\alpha_{34}-\frac{1}{2})-(\alpha_{15}-\frac{3}{2})}$$

(2) $$A_1 = \beta\alpha_{12}\int_0^1\int_0^1 dxdy\, x^{-\alpha_{45}-1}(1-x)^{-(\alpha_{34}-\frac{1}{2})-1}y^{-\alpha_{12}-1}(1-y)^{-(\alpha_{23}-1)-1} *$$

$$* (1-xy)^{\alpha_{23}+\alpha_{34}-\alpha_{15}}$$

Expanding the last term gives:

(3) $$(1-xy)^{\alpha_{23}+\alpha_{34}-\alpha_{15}} =$$

$$= 1 + (\alpha_{15}-\alpha_{34}-\alpha_{23})xy + \frac{(\alpha_{23}+\alpha_{34}-\alpha_{15})(\alpha_{23}+\alpha_{34}-\alpha_{15}-1)}{2}(xy)^2 + 0(xy)^3$$

Note: Hopkinson and Plahte [11], Hopkinson [13] and Białas and Pokorski [15] have given a general expansion in the form:

$$B_5(x_1,x_2,x_3,x_4,x_5) = \sum_{k=0}^{\infty} (-1)^k \binom{x_5-x_2-x_3}{k} B_4 \quad (x_1+k,x_2)B_4(x_4+k,x_3)$$

as (21.4)

Substitution of the various arguments for the x's and expanding gives a check on the simpler procedure used here to derive (4). and (6).]

Putting (3) into (2) gives:

$$\beta\Big[\alpha_{12}B(-\alpha_{45}, -\alpha_{34} + \tfrac{1}{2})\ B(-\alpha_{12}, -\alpha_{23} + 1)$$

$$+ \alpha_{12}(\alpha_{15} - \alpha_{34} - \alpha_{23})B(-\alpha_{45}+1, -\alpha_{34}+ \tfrac{1}{2})B(-\alpha_{12}+1, -\alpha_{23}+1)$$

$$+ \frac{\alpha_{12}}{2}(\alpha_{23} + \alpha_{34} - \alpha_{15})(\alpha_{23} + \alpha_{34} - \alpha_{15} -1)B(-\alpha_{45}+2, -\alpha_{34}+ \tfrac{1}{2})$$

$$B(-\alpha_{12} + 2, - \alpha_{23} + 1) + \dots\Big]$$

where

$$\left.\begin{aligned} B(m,n) &= \int_0^1 dx\ x^{m-1}(1-x)^{n-1}. \quad m,n > 0 \\ &\qquad\qquad \underline{\text{or}}\ \operatorname{Re} m > 0 \\ &\qquad\qquad \operatorname{Re} n > 0 \end{aligned}\right\} \qquad (5)$$

and

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

(4) So
$$A_1 = \beta\Big[\alpha_{12}\frac{\Gamma(-\alpha_{45})\Gamma(-\alpha_{34}+ \frac{1}{2})}{\Gamma(-\alpha_{45} - \alpha_{34} + \frac{1}{2})} \times \frac{\Gamma(-\alpha_{12})\Gamma(-\alpha_{23} + 1)}{\Gamma(-\alpha_{12} - \alpha_{23} + 1)} +$$

$$\alpha_{12}(\alpha_{15} - \alpha_{23} - \alpha_{34})\frac{\Gamma(-\alpha_{45}+1)\Gamma(-\alpha_{34}+ \frac{1}{2})}{\Gamma(- \alpha_{34} - \alpha_{45} + \frac{3}{2})} \times \frac{\Gamma(-\alpha_{12}+1)\Gamma(-\alpha_{23}+1)}{\Gamma(-\alpha_{12} - \alpha_{23} + 2)} +$$

$$\alpha_{12}\frac{(\alpha_{23} + \alpha_{34} - \alpha_{15})(\alpha_{23} + \alpha_{34} - \alpha_{15} - 1)}{2} \times \frac{\Gamma(-\alpha_{45}+2)\Gamma(-\alpha_{34}+ \frac{1}{2})}{\Gamma(-\alpha_{34} - \alpha_{45} + \frac{5}{2})} *$$

$$* \frac{\Gamma(-\alpha_{12} + 2)\Gamma(-\alpha_{23} + 1)}{\Gamma(-\alpha_{12} - \alpha_{23} + 3)} + \dots\Big]$$

Repeating this procedure for the second term in (1) gives:

$$\text{(6)}\quad A_2 = \beta\Bigg[C(\alpha_{34}-\tfrac{1}{2})\,\frac{\Gamma(-\alpha_{45}+1)\Gamma(-\alpha_{34}+\frac{1}{2})}{\Gamma(-\alpha_{45}-\alpha_{34}+\frac{3}{2})}\times\frac{\Gamma(-\alpha_{12}+1)\Gamma(-\alpha_{23}+1)}{\Gamma(-\alpha_{12}-\alpha_{23}+2)} \; +$$

$$+\; C(\alpha_{34}-\tfrac{1}{2})(\alpha_{15}-\alpha_{23}-\alpha_{34}+1)\,\frac{\Gamma(-\alpha_{45}+2)\Gamma(-\alpha_{34}+\frac{1}{2})}{\Gamma(-\alpha_{34}-\alpha_{45}+\frac{5}{2})}\times\frac{\Gamma(-\alpha_{12}+2)\Gamma(-\alpha_{23}+1)}{\Gamma(-\alpha_{12}-\alpha_{23}+3)}$$

$$+\;\ldots\Bigg]$$

We can now apply the result $\Gamma(z+1) = z\Gamma(z)$, $(z \neq -n)$, to (4) and (6).

For the ρ-region ($\alpha_{12}=1$) we put $\alpha_{12} = 1 + i\mathrm{Im}\rho$ and for the f-ρ' region ($\alpha_{12}=2$) we can put $\alpha_{12} = 2 + i\mathrm{Im}f$. This will prevent the infinite values of $\Gamma(n)$ for n a negative integer and will be done <u>after</u> the expansion.

<u>ρ-region</u>

<u>For(4)</u>

first term $$\left\{\frac{\alpha_{12}\Gamma(-\alpha_{45})\Gamma(-\alpha_{34}+\frac{1}{2})\Gamma(-\alpha_{12})}{\Gamma(-\alpha_{45}-\alpha_{34}+\frac{1}{2})}\right\}\times\frac{\Gamma(-\alpha_{23}+1)}{\Gamma(-\alpha_{12}-\alpha_{23}+1)}$$

or $$\left\{\frac{-\Gamma(-\alpha_{45})\Gamma(-\alpha_{34}+\frac{1}{2})\Gamma(-\alpha_{12}+1)}{\Gamma(-\alpha_{45}-\alpha_{34}+\frac{1}{2})}\right\}(-\alpha_{23})$$

$$= K_1\,(-\alpha_{23}).$$

second term $$\{K_1\}\times\frac{(\alpha_{15}-\alpha_{23}-\alpha_{34})(-\alpha_{45})(-1)}{(-\alpha_{34}-\alpha_{45}+\frac{1}{2})}$$

$$= \frac{K_1\,\alpha_{45}\,(\alpha_{15}-\alpha_{23}-\alpha_{34})}{(-\alpha_{34}-\alpha_{45}+\frac{1}{2})}$$

For (6)

first term
$$\frac{K_1\ C\ (\alpha_{34} - \frac{1}{2})(-\alpha_{45})(-1)}{(-\alpha_{45} - \alpha_{34} + \frac{1}{2})}$$

$$= \frac{K_1\ C\ (\alpha_{34} - \frac{1}{2})\alpha_{45}}{(-\alpha_{45} - \alpha_{34} + \frac{1}{2})}$$

In this case, therefore,

$$A = \beta \left\{ \frac{\Gamma(0-i\mathrm{Im}\rho)\Gamma(\frac{1}{2} - \alpha_{34})\Gamma(-\alpha_{45})}{\Gamma(\frac{1}{2} - \alpha_{34} - \alpha_{45})} \left[\alpha_{23} + \frac{\alpha_{45}(\alpha_{15} - \alpha_{23} - \alpha_{34})}{(\alpha_{34} + \alpha_{45} - \frac{1}{2})} + \right.\right.$$
$$\left.\left. + \frac{C\ (\alpha_{34} - \frac{1}{2})\alpha_{45}}{(\alpha_{34} + \alpha_{45} - \frac{1}{2})} \right] \right\}$$

Putting in this imaginary part after making the expansion saves having the infinite series and is justified on the basis that we are near a pole with 'small' widths and hence 'small' imaginary parts.

---

f-region

For (4)

first term
$$\left\{ \frac{-\Gamma(-\alpha_{12}+1)\Gamma(-\alpha_{45})\Gamma(-\alpha_{34}+\frac{1}{2})}{\Gamma(-\alpha_{45} - \alpha_{34} + \frac{1}{2})} \right\} \times \frac{\Gamma(-\alpha_{23}+1)}{\Gamma(-2-\alpha_{23}+1)}$$

$$= K\ \frac{(-\alpha_{23})\Gamma(-\alpha_{23})}{\Gamma(-\alpha_{23} - 1)}$$

$$= K\ \alpha_{23}\ (\alpha_{23} + 1)$$

second term $$\frac{K\ (\alpha_{15} - \alpha_{23} - \alpha_{34})(-\alpha_{45})(-\alpha_{12})\Gamma(-\alpha_{23} + 1)}{(-\alpha_{34} - \alpha_{45} + \frac{1}{2})\qquad\qquad \Gamma(-\alpha_{23})}$$

$$= \frac{K\ 2\alpha_{45}\alpha_{23}\ (\alpha_{15} - \alpha_{23} - \alpha_{34})}{(\alpha_{34} + \alpha_{45} - \frac{1}{2})}$$

third term $$\frac{K\ (\alpha_{23} + \alpha_{34} - \alpha_{15})(\alpha_{23} + \alpha_{34} - \alpha_{15} - 1)(-\alpha_{45}+1)(-\alpha_{45})}{2(-\alpha_{34} - \alpha_{45} + \frac{3}{2})(-\alpha_{34} - \alpha_{45} + \frac{1}{2})} *$$

$$* \ (-\alpha_{12})(-\alpha_{12} + 1)\ \frac{\Gamma(-\alpha_{23} + 1)}{\Gamma(-\alpha_{23} + 1)}$$

$$= \frac{-K\ (\alpha_{23} + \alpha_{34} - \alpha_{15})(\alpha_{23} + \alpha_{34} - \alpha_{15} - 1)(1 - \alpha_{45})\ \alpha_{45}}{(-\alpha_{34} - \alpha_{45} + \frac{3}{2})\ (-\alpha_{45} - \alpha_{34} + \frac{1}{2})}$$

<u>For (6)</u>

first term $$\frac{KC(\alpha_{34} - \frac{1}{2})(-\alpha_{45})(-\alpha_{23})(-1)}{(-\alpha_{45} - \alpha_{34} + \frac{1}{2})}$$

$$= \frac{KC\ (\alpha_{34} - \frac{1}{2})\ \alpha_{23}\ \alpha_{45}}{(\alpha_{45} + \alpha_{34} - \frac{1}{2})}$$

second term $$\frac{KC(\alpha_{34} - \frac{1}{2})(\alpha_{15}+1 - \alpha_{23} - \alpha_{34})(-\alpha_{45}+1)(-\alpha_{45})}{(-\alpha_{34} - \alpha_{45} + \frac{3}{2})(-\alpha_{34} - \alpha_{45} + \frac{1}{2})}$$

So that we have ($A = A_1 + A_2$):

$$A = \beta K \left\{\alpha_{23}(\alpha_{23} + 1) + 2\ \alpha_{23}\alpha_{45}\ \frac{(\alpha_{15} - \alpha_{23} - \alpha_{34})}{(\alpha_{34} + \alpha_{45} - \frac{1}{2})}\right.$$

$$- \frac{(\alpha_{15} - \alpha_{23} - \alpha_{34})(\alpha_{15} + 1 - \alpha_{23} - \alpha_{34})(1 - \alpha_{45})\ \alpha_{45}}{(\alpha_{34} + \alpha_{45} - \frac{3}{2})\ (\alpha_{34} + \alpha_{45} - \frac{1}{2})} +$$

$$+ C \frac{(\alpha_{34} - \frac{1}{2})\ \alpha_{23}\ \alpha_{45}}{(\alpha_{34} + \alpha_{45} - \frac{1}{2})}$$

$$\left. - C \frac{(\alpha_{34} - \frac{1}{2})(\alpha_{15} + 1 - \alpha_{23} - \alpha_{34})(1 - \alpha_{45})\ \alpha_{45}}{(\alpha_{34} + \alpha_{45} - \frac{3}{2})\ (\alpha_{34} + \alpha_{45} - \frac{1}{2})} \right\} . \tag{7}$$

$$\text{Where } K = \frac{-\Gamma(-\alpha_{12} + 1)\Gamma(-\alpha_{45})\Gamma(-\alpha_{34} + \frac{1}{2})}{\Gamma(-\alpha_{45} - \alpha_{34} + \frac{1}{2})}$$

$$K = \frac{-\Gamma(-1 - i\mathrm{Im}f)\Gamma(-\alpha_{45})\Gamma(-\alpha_{34} + \frac{1}{2})}{\Gamma(\frac{1}{2} - \alpha_{34} - \alpha_{45})}$$

(Notice that the only difference between the K for the f-region and the K for the ρ-region is that one has $\Gamma(-i\mathrm{Im}f)$ and the other has $\Gamma(-i\mathrm{Im}\rho)$, or simply, the difference is due to the relative widths, and these are almost the same. We could, therefore, equate the K's in magnitude to a good approximation).

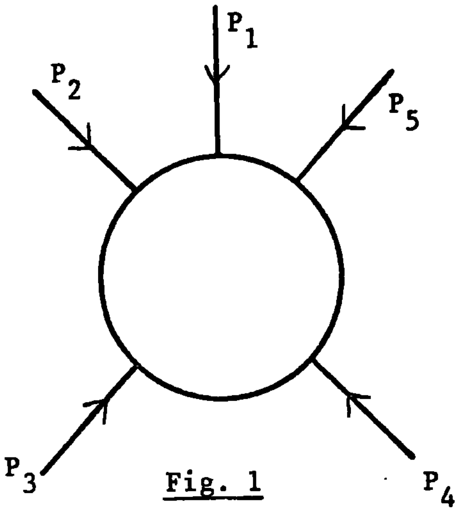


Fig. 1

Diagram representing the five-point amplitude which corresponds to the ordering (1, 2, ... , 5)

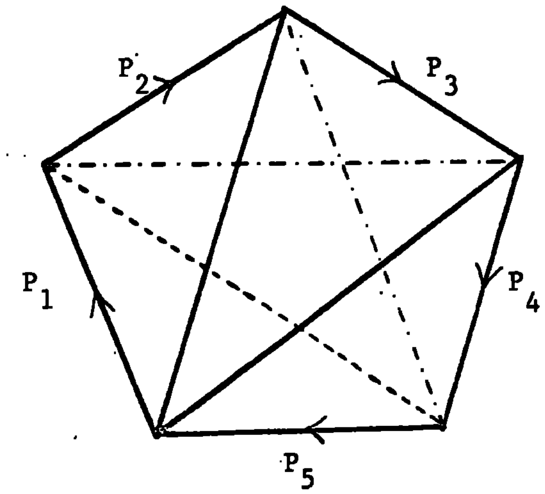


Fig. 2

The dual diagram associated with Fig.1.

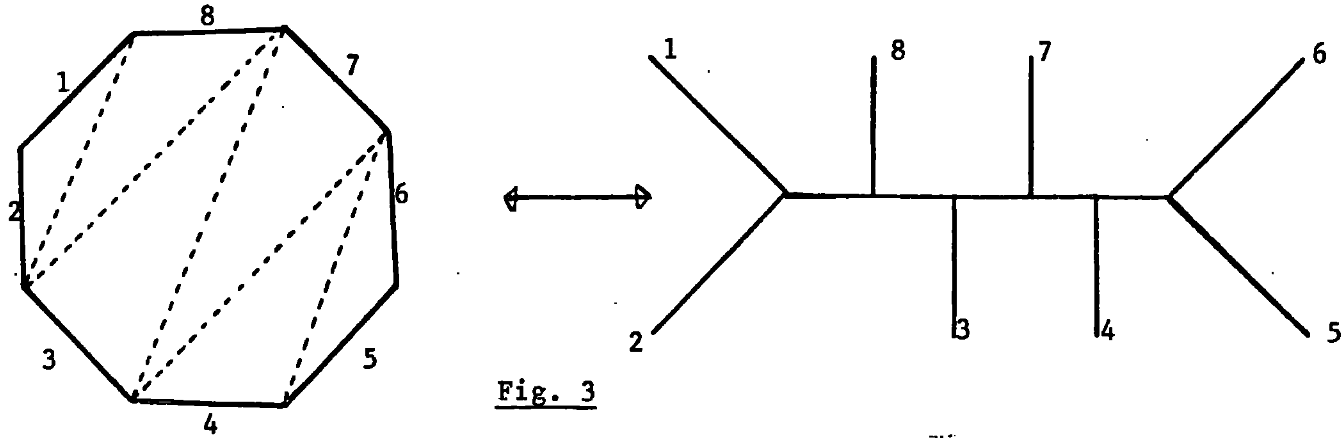


Fig. 3

The set of variables $u_{1,2}u_{2,8}$ etc. on the dual diagram.

Diagram representing the set of independent variables $u_{1,2}$ $u_{2,8}$ etc.

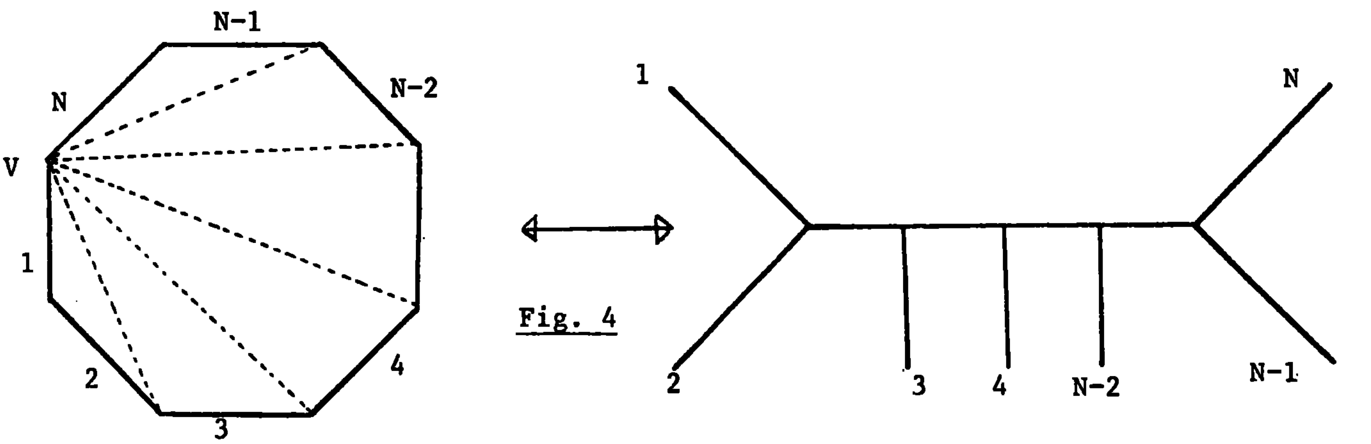


Fig. 4

The set of variables $u_{1,j}$ (j=2,..,N-2) represented on the dual diagram.

Diagram representing the set of independent variables $u_{1,j}$ (j = 2, ..., N-2).

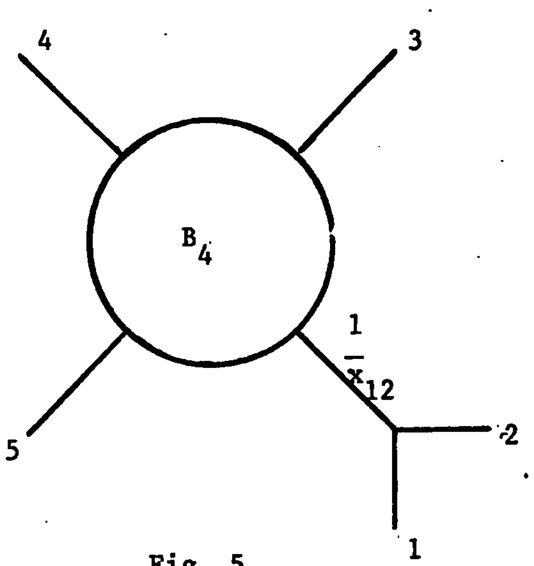


Fig. 5

Diagram illustrating the bootstrap consistency of the five-point amplitude.

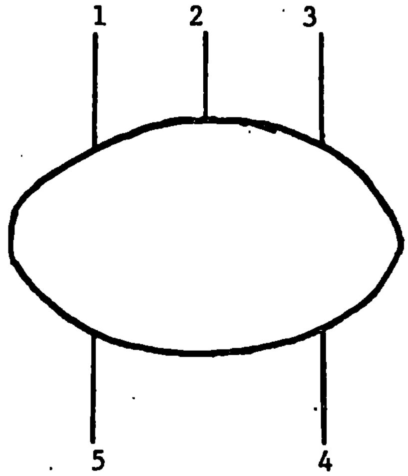


Fig. 6

Diagram illustrating the "single Regge limit" for the five-point amplitude.

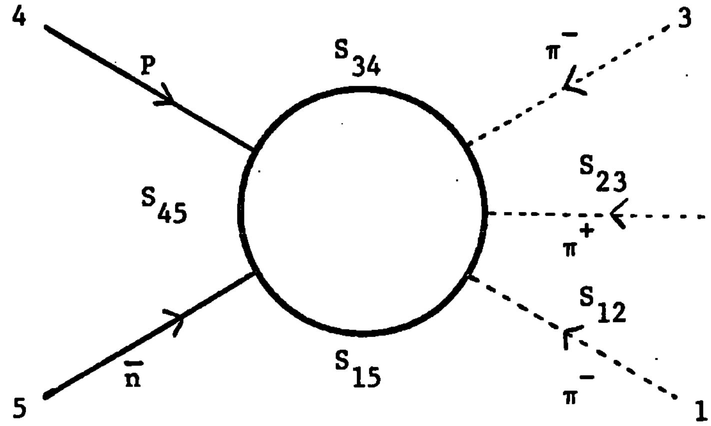


Fig. 7

Notation for the production process $\pi^- p \rightarrow \pi^+ \pi^- n$.

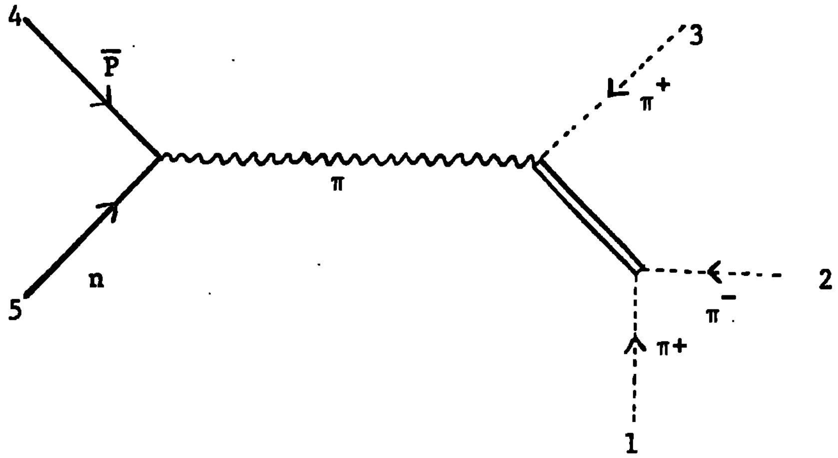
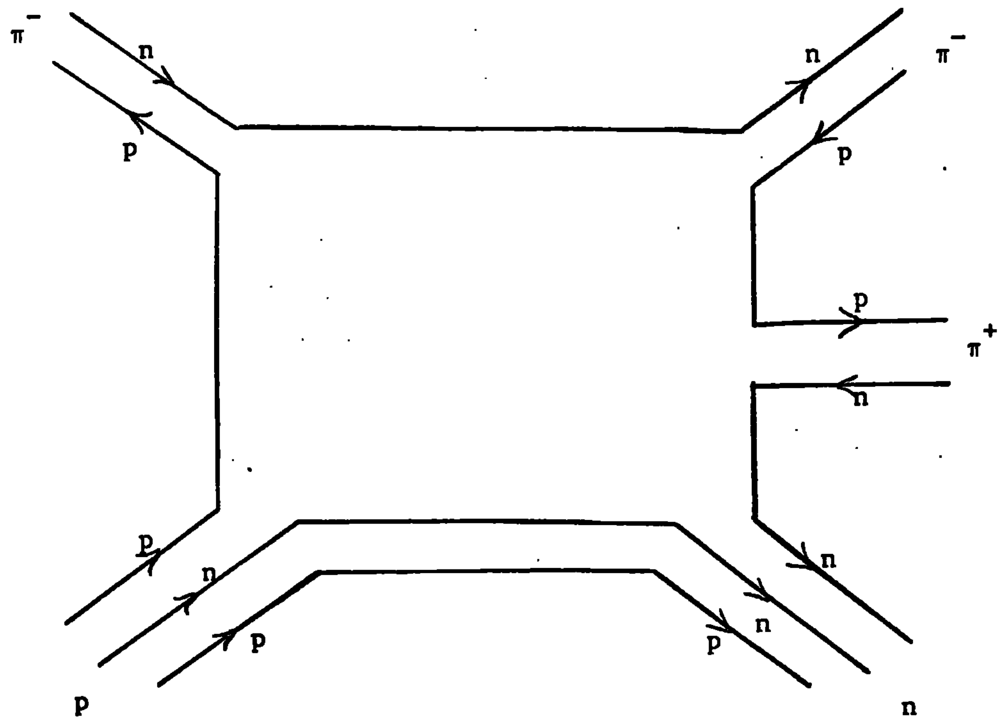


Fig. 8

The π-pole diagram where $S_{45} = m_\pi^2$ used in the normalization comparison with $\bar{p}n$ threshold, where $S_{45} = 4M_N^2$

Fig. 9



A quark duality diagram for the process $\pi^- p \to \pi^- \pi^+ n$.

Fig. 10

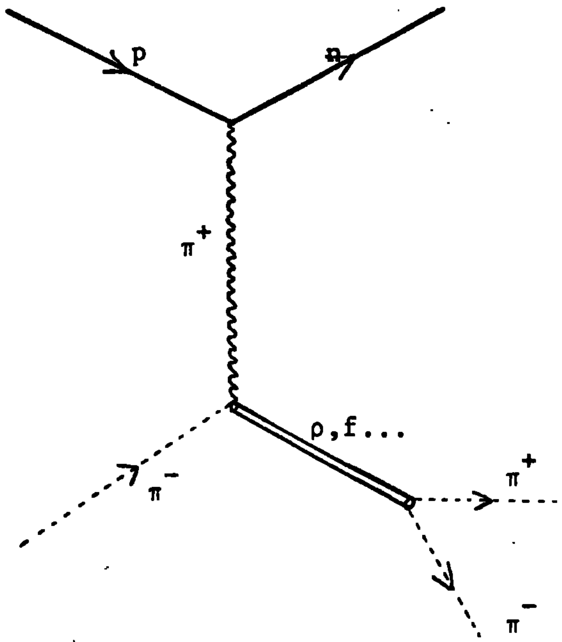


Diagram of the processes considered.

Figs. 11-15

We use the ρ-f-trajectory in the form

$$0.9t + 0.56$$

and the widths are given by

$$\Gamma_\rho = \Gamma_f = 90 \text{ MeV},$$

as used by Bender et al [49].

The dashed lines refer to the given C value and the firm lines refer to the case C = 0, (except for the angular distribution graph of Fig. 15).

Fig. 11

$\pi^- p \to \rho^o n$

8 GeV

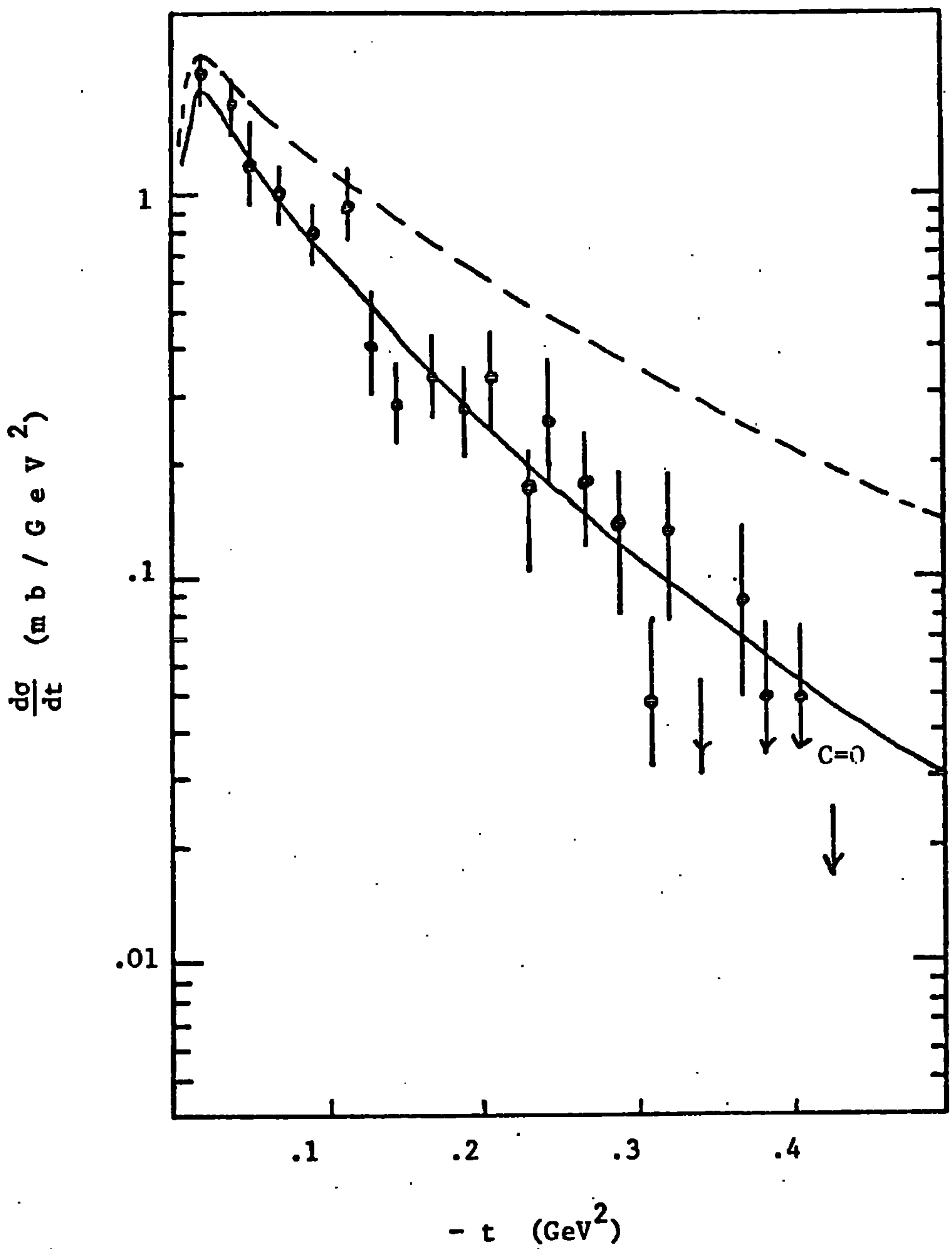
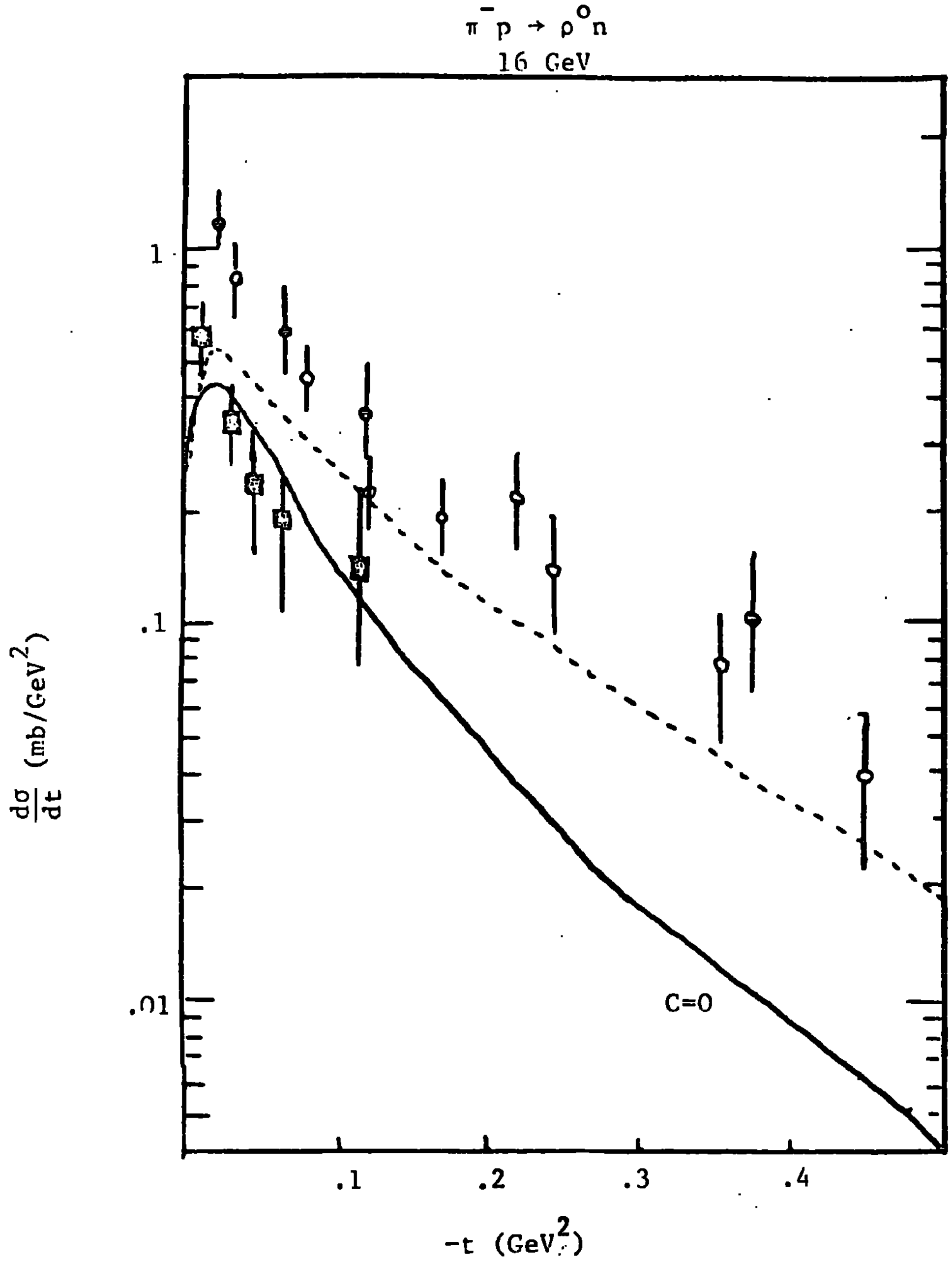


Fig. 12



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G Bellini et al, Nuovo Cimento 53A, 798 (1968)

J Ballam et al, Phys. Letters 31B, 489 (1970)

Fig. 13

$\pi^- p \to f^o n$

8 GeV

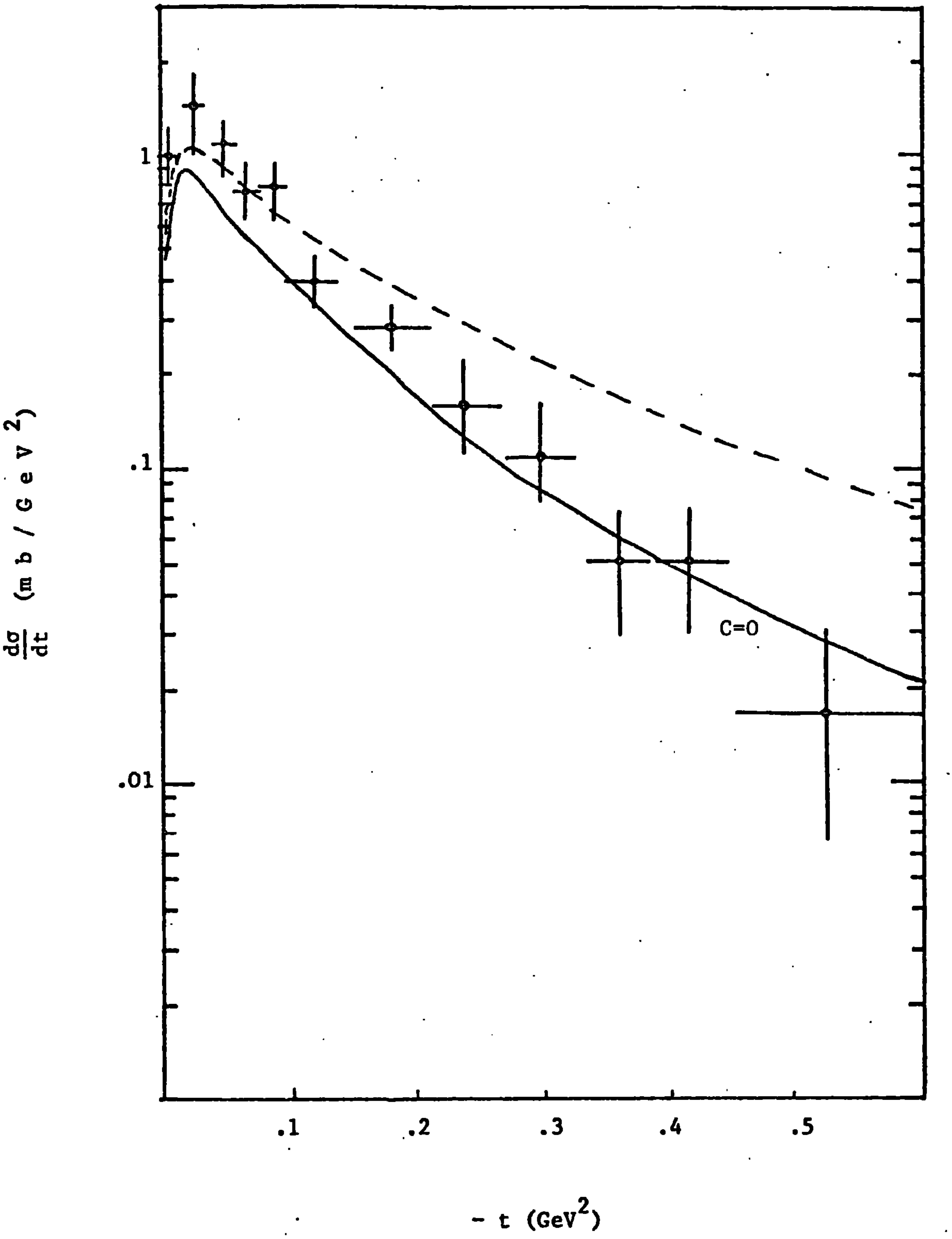
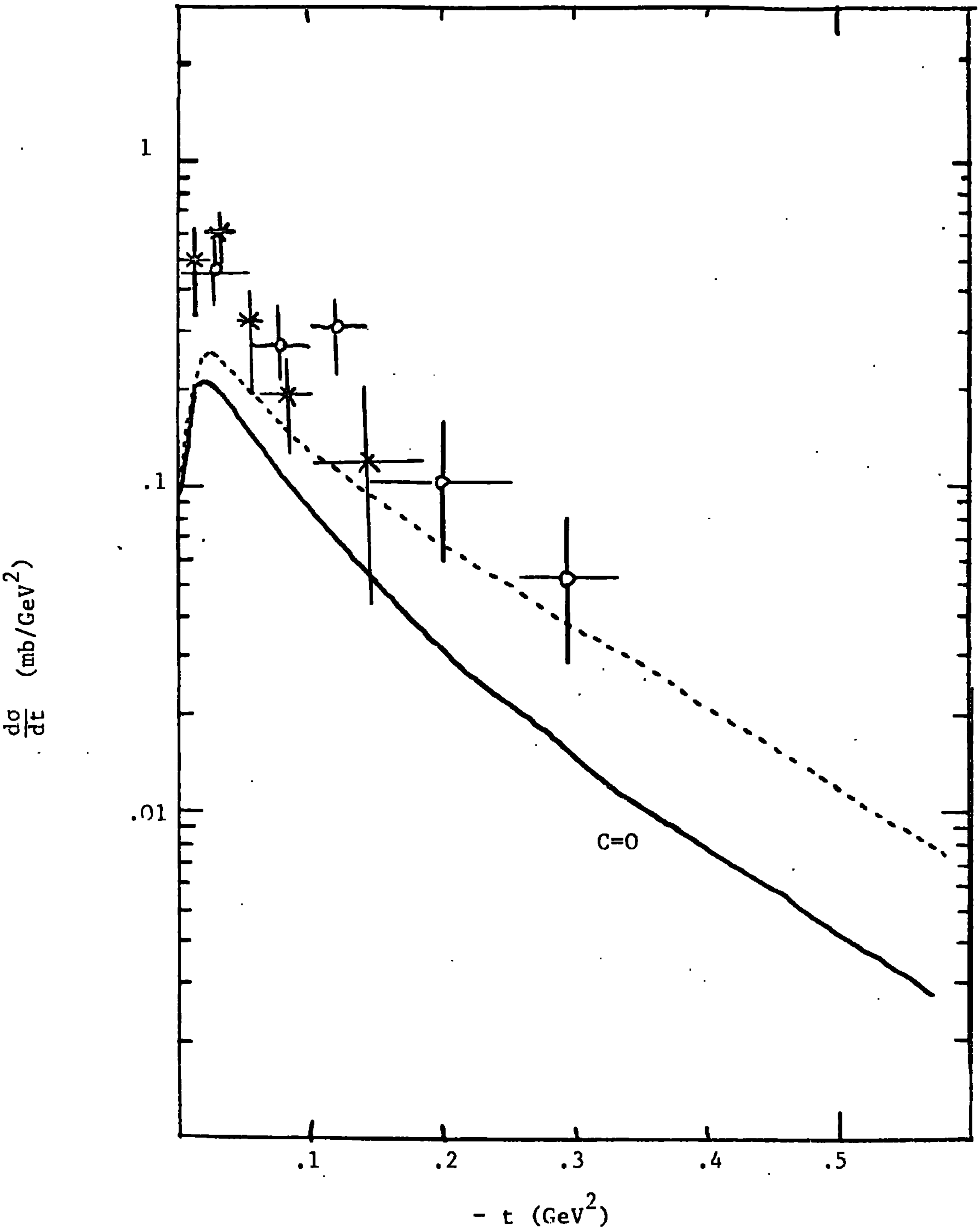


Fig. 14

$\pi^- p \to f^o n$

16 GeV.



G Bellini et al, Nuovo Cimento 53A, 798 (1968)

J Ballam et al, Phys. Letters 31B, 489 (1970)

Fig. 15

Angular Distribution for J = 2, 8 GeV $|t| \leq 0.3$ GeV^2

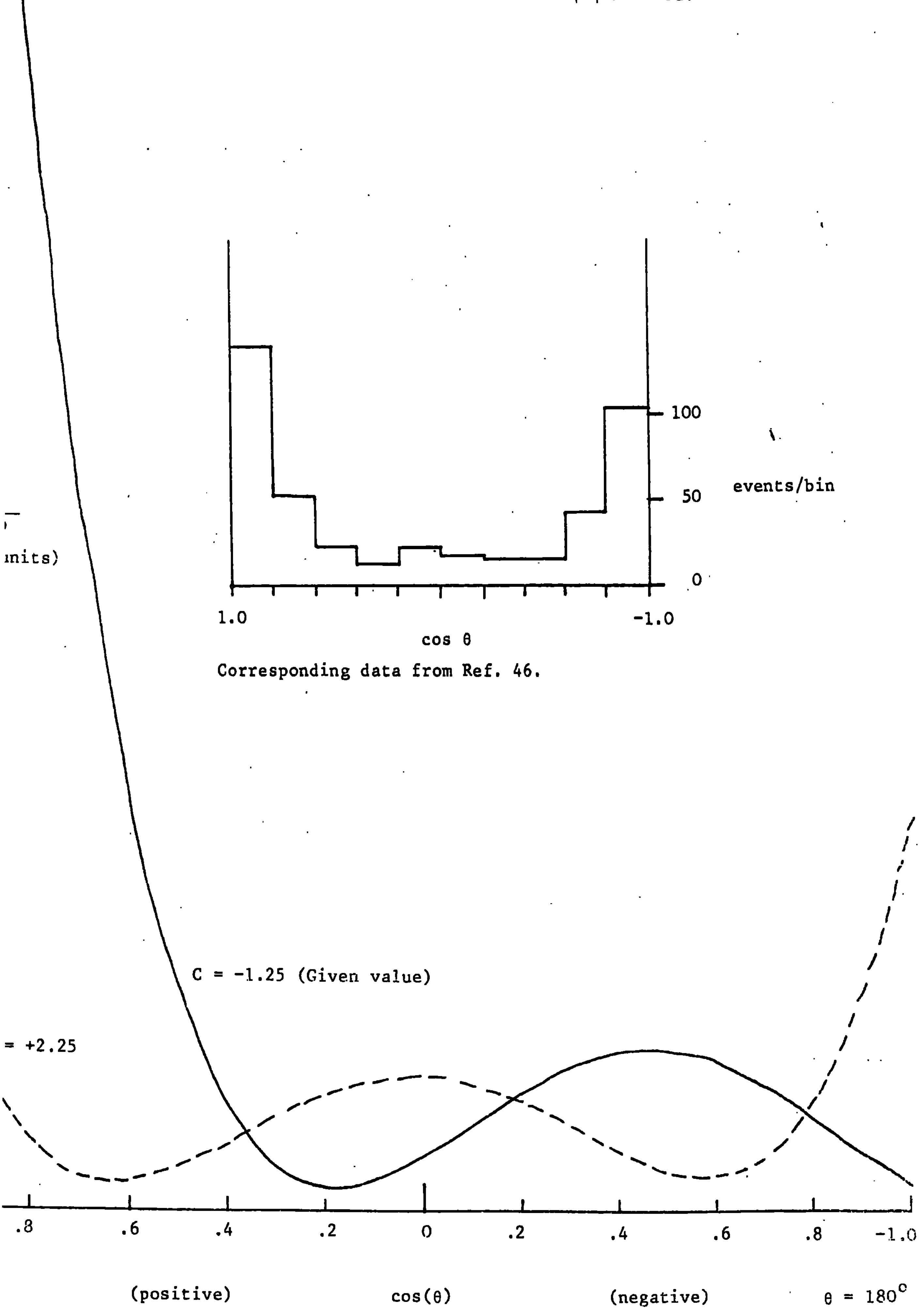
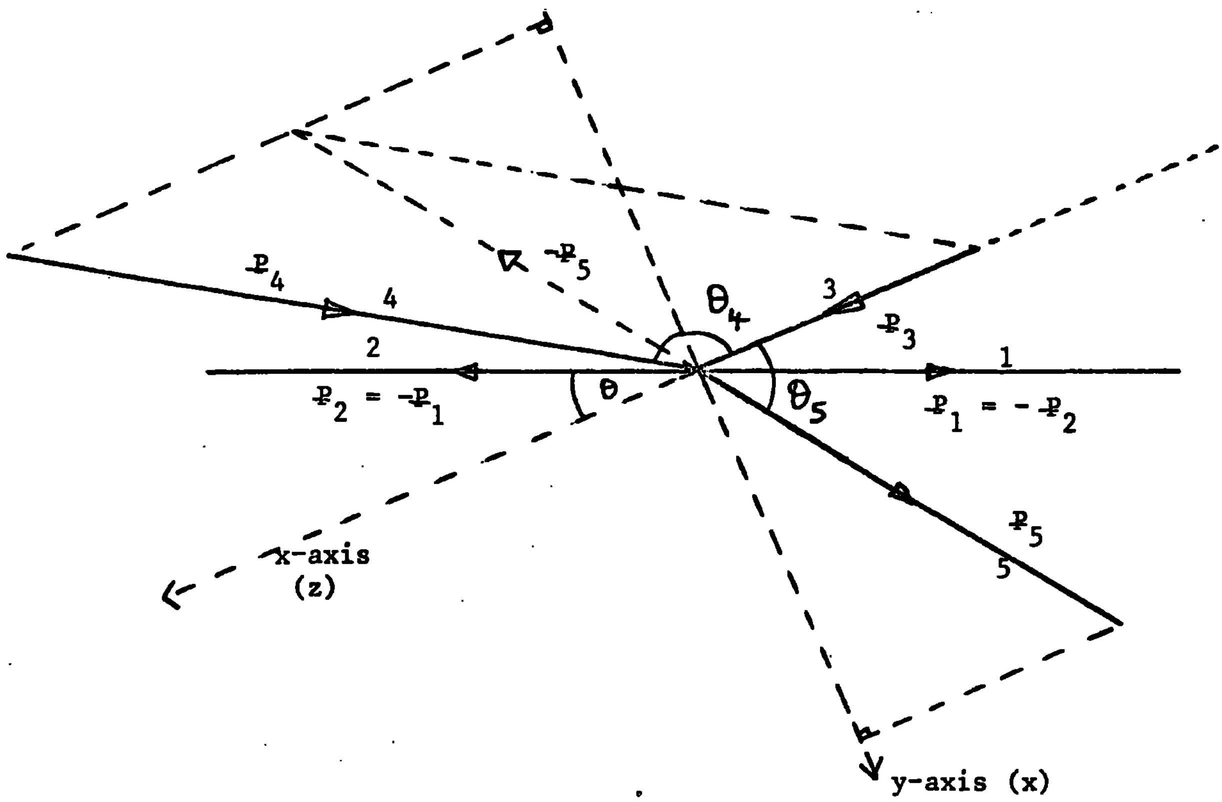


Fig. 16

The 1-2 CMS for 3 4 → 1 2 5

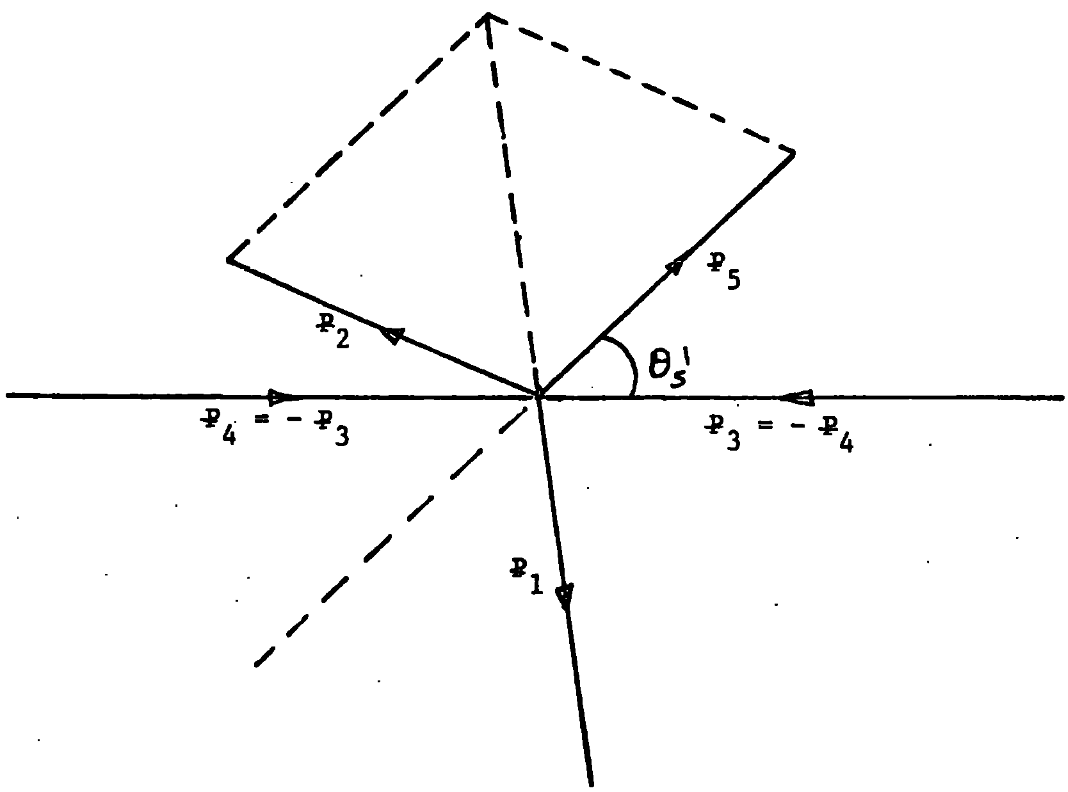
The angles are as defined in this diagram.



In the 1-2 CM system

$$\underline{p}_1 + \underline{p}_2 = 0$$
$$\underline{p}_3 + \underline{p}_4 = \underline{p}_5$$

Fig. 17



CMS system for 3 4 → 1 2 5

In the 3-4 CMS

$$\underline{p}_3 + \underline{p}_4 = 0 = \underline{p}_1 + \underline{p}_2 + \underline{p}_5$$

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CHAPTER 3

A Dalitz Plot Analysis of the Annihilation Process $\bar{p}\, n \rightarrow 3\pi$ at Rest using Veneziano Type 4-point function Amplitudes

3.1 Introduction

A remarkable feature of the reaction $\bar{p}\, n \rightarrow 3\,\pi$ at rest is the very complicated structure of the Dalitz plot for the 3π system. Fig. 1 shows a computer line-printer output of the data (consisting of 2902 events) for this reaction as measured by P Anninos et al [1] in which each event is plotted twice (because of the two π^-) so that the plot is completely symmetric with respect to the diagonal. This group made the following comments on the structure shown:

(i) strong enhancement in the low $M^2(\pi_1^-, \pi_2^-)$ region where $M^2(\pi^+, \pi_1^-) \simeq M^2(\pi^+, \pi_2^-) \simeq 1.64$ GeV^2 (about the f^o mass);

(ii) absence of events in the region $M^2(\pi^+, \pi_1^-) \simeq M^2(\pi^+, \pi_2^-) \simeq 1.08$ GeV^2 (hole near the centre of the Dalitz plot);

(iii) lack of events in the region where one $M^2(\pi^+, \pi^-)$ is small and the other one is large; and

(iv) apparent abundant production of ρ^o and f, as seen in the $M^2(\pi^+, \pi^-)$ distribution.

[N.B.

The line-printer output tends to mask these effects by grouping some of the events for printing purposes.]

They were not able to find a satisfactory fit to the data but could make the conclusions that: ρ production seemed to be very small but f^o production seemed to be very large.

The Veneziano [2] 4-point function formula outlined in Chapter 1 might well never have excited so much interest had it not been seen by C Lovelace [3] to provide a plausible explanation for the complicated dip-bump structure of the $\bar{p}\, n \rightarrow 3\pi$ Dalitz plot.

In the annihilation process $\bar{p}\, n \rightarrow \pi^+ \pi^- \pi^-$ at rest the initial state has orbital angular momentum L = 0 and the total angular momentum J equals S, where either S = 0 (singlet) or S = 1 (triplet). The initial state is charged so that T = I = 1 and since L = 0 it has $P = (-1)^{L+1} = -1$. Thus, the initial state is either $J^P = 0^-$ or $J^P = 1^-$, but since $G = (-\ )^{L+S+I}$ the latter state has $G = (-\ )^{0+1+1} = +$ and cannot decay into three pions. The initial state $\bar{p}\, n$ is thus uniquely an isovector pseudoscalar $I^G J^P = 1^- 0^-$ state, or has exactly the same quantum numbers as the π meson, but of mass $m_p + m_n$, also written as ${}^1S_o$ and illustrated in Fig. 2. Fuller discussion on the evidence for S-state capture of the antiproton at rest is given by Gray et al [4], based on the original study of Day, Snow and Sucher [5]. This assumption about the initial state may, however, not be justified since recently there has been evidence against complete S-state capture of the $\bar{p}$ reported by Devons et al [6] in the process $\bar{p}\, p \rightarrow 2\pi^o$ at rest and discussed further by R Bizzarri [7] and T E Kalogeropoulos [8] at the Chexbres Symposium on Nucleon - Antinucleon Annihilations.

Lovelace took the π-π Veneziano type 4-point function to describe this process making an "off mass-shell" continuation on the grounds that the exchanged trajectories were not modified thereby. Since Veneziano forms

$$V_{nm} = \frac{\Gamma(n - \alpha_s)\,\Gamma(n - \alpha_t)}{\Gamma(m+n - \alpha_s - \alpha_t)}$$

depend explicitly on linear trajectories an extrapolation from the mass of the pion to that of the two nucleons is performed by changing the connection $s + t + u = 4m_\pi^2$ to $s + t + u = \sum = (m_p + m_n)^2 + 3m_\pi^2 = m_b^2 + 3m_\pi^2$ where s, t and u are the Mandelstan variables for the decay of the dinucleon system into the three pions. Coefficients for terms like V_{nm} depend on the external masses and could be allowed to change representing, for each term, just a scale change.

Lovelace suggested the two term formula:

$$A(s,t) = -\beta\,\frac{\Gamma(1 - \alpha_s)\,\Gamma(1 - \alpha_t)}{\Gamma(1 - \alpha_s - \alpha_t)} + \gamma\,\frac{\Gamma(1 - \alpha_s)\,\Gamma(1 - \alpha_t)}{\Gamma(2 - \alpha_s - \alpha_t)}$$

$$= \{\gamma - \beta\,(1 - \alpha_s - \alpha_t)\}\,B\,(1 - \alpha_s,\ 1 - \alpha_t) \tag{31.1}$$

for the amplitude A(s,t) to describe this process and took a phenomenological Regge trajectory:

$$\alpha_x = 0.483 + 0.885x + i0.28\sqrt{x - 4m_\pi^2}\quad \theta(x - 4m_\pi^2) \tag{31.2}$$

For comparison with experimental data he gave α_s and α_t imaginary parts in order to remove the poles away from the real axis. The residues are then no longer polynomials in the crossed channel invariant but the "ancestor" problems are not too serious for this particular application. Lovelace ended up by setting $\beta = 0$ in his comparison with experiment, so that the standard $\pi\pi$ Veneziano amplitude [9] was eliminated leaving the satellite term which does not have the leading $(\rho, f \ldots)$ trajectory. This was done because of the apparent absence of an appreciable ρ- signal in the data of Anninos as mentioned above.

The distribution of events on the $s = M^2_{\pi^+ \pi^-_1}$ versus $t = M^2_{\pi^+ \pi^-_2}$ Dalitz plot is given by

$$\frac{\partial^2\sigma}{\partial s \partial t} \alpha \quad |A(s,t)|^2$$

(The phase space distribution on the Dalitz plot is constant).

Lovelace claimed that his version of the Veneziano type amplitude given by (31.1) predicted the marked depletion of events, corresponding to $\alpha_s + \alpha_t = 3$, and the strong accumulation at the edge of the plot given at $\alpha_s = 1$ (ρ band) or $\alpha_s = 2$ (f band) and at $\alpha_t = 1$ or $\alpha_t = 2$. In fact the hole is so deep and the depletion of events on the lines $\alpha_s + \alpha_t = 2$ or 4 is so much weaker that a fit to the data will require an additional line of zeros at $\alpha_s + \alpha_t = 3$, which could be obtained, for example, by setting $\beta = -\gamma/2$ in (31.1) so that $A(s,t) = [\alpha_s + \alpha_t - 3]B(1 - \alpha_s, 1 - \alpha_t)$, explaining qualitatively why satellite terms are needed [10]. Although the amplitude (31.1) with $\beta = 0$ could not be said to "fit" the data the idea of applying the Veneziano formula to this particular reaction was an important one. If, for example, the reaction $\bar{p}\,p \rightarrow \pi^+ \pi^- \pi^0$ had been chosen then the initial state at rest having the quantum numbers of either π, which couples to ${}^1S_0$, or, say, ω, which couples to the ${}^3S_1$ states of $\bar{p}\,p$, would have required a more complicated analysis and also as the final state has no exotic $\pi\pi$ channels it is less interesting anyway. Jengo and Remiddi in fact looked at this problem [11] and gave an adequate fit to the data.

Berger [12] pointed out that Lovelace's fit did not match the angular distribution in the ρ and f-region and he was not convinced of the theoretical justification for equation (31.1) since it was not clear how the details of the model had entered, beyond the fact that the $\pi\pi$ system contains a ρ, f and a large S-wave phase shift. Using the Lovelace ansatz for the trajectory function (31.2) Berger allowed β and γ in (31.1) to be free parameters and found a best fit to the invariant mass distributions with $\beta = -1.0$ and $\gamma = 1.95$. The experimental data for the decay angular distribution of the π^+ in the mass region of the f show a sharp forward peak that needs an $L = 2$ contribution present in Berger's model, but absent in Lovelace's. In the ρ-region neither model fitted the decay angular distribution very well.

The Lovelace results were also compared by Boldrighini and Pugliese [13] with the phenomenological consequences of using amplitudes of the form:

$$F(s,t) = \beta\,\frac{\Gamma(1-\alpha_s/\gamma)\Gamma(1-\alpha_t/\gamma)}{\Gamma(1-(\alpha_s+\alpha_t)/\gamma)} \qquad \text{and}$$

$$F(s,t) = \beta\,\frac{\Gamma(1-\alpha_s/\gamma)\Gamma(1-\alpha_t/\gamma)}{\Gamma(2-(\alpha_s+\alpha_t)/\gamma)}$$

assuming

$$\alpha_s = b - \left(\frac{S_0 - S}{c}\right)^{\gamma}$$

and taking $b = 0.52$, $c = 1.29$ and $\gamma = 0.93$.

They obtained only a fair agreement with the data and concluded that this was a reflection of the lack of higher thresholds in their amplitudes.

By expanding $A(s,t)$ simultaneously in poles in α_s and α_t Boguta [14] was able to make the structure of the Dalitz plot appear very obvious. For $\pi\pi$ scattering he took the convergent expansion for the amplitude V_{10} as:

$$\begin{aligned} A(s,t) &= \sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{\Gamma(n+1-\alpha_s-\alpha_t)}{\Gamma(n)\Gamma(1-\alpha_s-\alpha_t)}\left\{\frac{1}{n-\alpha_s}+\frac{1}{n-\alpha_t}\right\} \\ &= (\alpha_s+\alpha_t-1)\left\{\frac{1}{\alpha_t-1}+\frac{1}{\alpha_s-1}\right. \\ &\quad \left. + \frac{\alpha_t+\alpha_s-2}{\alpha_t-2} + \frac{\alpha_t+\alpha_s-2}{\alpha_s-2} + \text{etc.}\right\} \end{aligned}$$

This converges for $\mathrm{Re}(1 - \alpha_s - \alpha_t) < 0$ or $s + t > (1 - 2\alpha_0)\,\alpha'$ (where α_0 refers to the intercept and α' the slope of the Regge trajectory) $\simeq 0$ (as $\alpha_0 \simeq \frac{1}{2}$). The residues of the poles in this expansion grow when s and t increase so that constructive interference gets stronger when s and t increase - a specific prediction. By taking the first few terms of such an expansion one has a specifically non-dual isobar model.

Boguta was able to reproduce Lovelace's original results by using such a method, illustrating that within a limited kinematical range a dual model can always be approximated arbitrarily well by a non-dual model [10].

A similar type of fit but using non linear daughters was the rising phase shift model of Gleeson, Meggs and Parkinson [15]. Their flexible parameterization allowed mass shifts between resonances in each tower of daughters and different widths. By letting the masses of the resonances vary they were testing one of the assumptions of the Veneziano approach, that resonances occur in degenerate towers.

Moen and Moffat [16] instead of taking a product of Gamma functions for each term took a sum to ensure that there were no 'ancestors' as occurs in the Veneziano type of approach. They claimed a fit to the data at least as good as Berger. Other models for this process will be considered later on in the chapter.

Lovelace had used a one term Veneziano type of fit and Berger a two term fit, neither of which amplitude is unique. Altarelli and Rubinstein [17] suggested, therefore, using the decay amplitude for this process in the form:

$$A(s,t) \; \alpha \; \sum_{n=1} \; \sum_{m=0}^{n} C_{nm} V_{nm} \tag{31.3}$$

where the C_{nm} are coefficients to be determined by the fit, and V_{nm} was as above.

In order to restrict the possible values of n and m they utilised the experimental feature of the Dalitz plot that there is a "hole" at values of s and t such that $\alpha_s \simeq \alpha_t \simeq 1.5$. This led them to a five term fit using $n + m \leqslant 3$, so keeping only those terms that vanished at $\alpha_s + \alpha_t = 3$, and they obtained for the coefficients:

$$C_{10} = 1 \qquad \text{(normalization)}$$

$$C_{11} = 1.89$$

$$C_{30} = 0.57$$

$$C_{20} = C_{21} = 0$$

They claimed a good fit to the data and ruled out the possibility of a one term four point function, such as Lovelace's, fitting the data. (In fact they possibly slipped up in computing their coefficients and their decay rate ratios given in equation (9) are miscalculated). Using Bizzarri's (1968) estimate of the conversion factor $(\bar{p}\,p \to \text{all})/(\bar{p}\,n \to \text{all})$ they reached rough agreement between the experimental and theoretical values of the decay rates for $(\bar{p}\,p \underset{(T=1)}{\to} \pi^+\,\pi^-\,\pi^0)$ vs $\bar{p}\,n$.

Jengo and Remiddi computed this ratio for Lovelace's amplitude but found their result to be (a factor of 10) different from the (rough) estimate of Altarelli and Rubinstein.

Boguta [18] was again able to generalise this model to reproduce the invariant mass-distribution results of Altarelli and Rubinstein and also showed that ancestors played an important role in their fits. He took a finite number of terms and a suitable ansatz to give identical ancestor, parent daughter structure. Similar agreement with the experimental data using these simple isobaric amplitudes was also made for other related reactions [19].

The somewhat arbitrary nature of taking sums of terms such as (31.3) was pointed out by Rubinstein, Squires and Chaichian [20]. Instead of using the Veneziano formula appropriate for two body scattering processes they took the generalized forms suitable for processes as given by Bardakçi-Ruegg and Virasoro [21]. We have discussed this amplitude in Chapter 2 and further comment will be given in Chapter 4. In order to cast their amplitude in the form of (31.3) they evaluated the amplitude at the threshold, i.e, $S_{45} = 4M_N^2$ (their diagram for the definition of the kinematical variables is given in Fig. 4), took the approximation $\alpha^{\pi}_{45} = 3$, and obtained the results:

$$C_{10} = -3C\,(A^2 + 8A + 15) - (2A^3 + 21A^2 + 70A + 75)$$

$$C_{11} = 3C(2A^3 + 17A^2 + 38A + 15) - (3A^2 + 24A + 45)$$

$$C_{20} = 3C + 6A + 21$$

$$C_{21} = 9 - 3C\,(2A + 3)$$

$$C_{22} = 3C\,(2A + 3) - 9$$

$$C_{nm} = 0, \qquad \text{otherwise}$$

(NB - The C_{10}, C_{11} coefficients given in their paper were in error and a publication of the corrected terms was made)

where $A = - 2\alpha' M_N^2 + 2\alpha^B(0) - \alpha^\rho(0) - 1$, α' being the universal trajectory slope, $\alpha(0)$ the trajectory intercept, M_N the nuclear mass and $C = \frac{3}{2A+3}$ if C_{22} is required to be zero.

Using the Δ trajectory for α^B, they gave (corrected entries):

$$C = -1.25$$

$$C_{10} = 1 \qquad \text{(normalization)}$$

$$C_{11} = 3.2$$

$$C_{20} = 0.39$$

$$C_{21} = 0$$

Rubinstein et al claimed excellent agreement with the Altarelli and Rubinstein results. This was the case, and their signs were consistent, but the Altarelli and Rubinstein fit was itself not very good in the form they gave. Boguta [22], not aware of the corrected coefficients, pointed out the deficiencies of the fit to the data.

Since none of these previously cited adaptations of the Veneziano model to $\bar{p}\,n$ annihilation made direct fits to the full two-dimensional Dalitz plot representation the accuracy of the predicted patterns was not fully tested. Gopal, Migneron and Rothery [23] made such a fit to the data using the same Veneziano like terms as Altarelli and Rubinstein with $m + n \leqslant 3$ but found that their coefficients C_{mn} were different. Their procedure was to divide up the $M^2_{\pi^-\pi^-}$ (= u) vs $M^2_{\pi^-\pi^+}$ (= s) Dalitz plot into a 30 x 30 grid and obtain the predicted probability distribution p of each square by integrating the expression

$$\frac{d^2p}{dsdu} = C. \left| A(s,u) \right|^2$$

over the area of the square. A method of obtaining an indication of goodness of fit was presented. They found that:

(1) For the restriction $m + n \leqslant 3$ the secondary terms with $n \geqslant 2$ were essential.

(2) The overall fit of Altarelli and Rubinstein was worse than Lovelace's.

(3) The best fit trajectory of the form

$$\alpha_x = \alpha_0 + \alpha' x + iA(x - 4m^2)^B \, \theta(x - 4m^2)$$

had α_0, α' and $B = \frac{1}{2}$ as for Lovelace but that $A = 0.33$ ($A = 0.28$ gave only slightly inferior results).

(4) Using this trajectory (3) their coefficients were:

$$C_{10} = 1.00 \qquad \text{(normalization)}$$

$$C_{11} = 2.90$$

$$C_{20} = 2.14$$

$$C_{21} = 7.31$$

$$C_{30} = -3.74$$

We agree that a direct fit to the Dalitz plot is essential for determining the quality of any parametrization. However, the use of a grid over the plot (without a good criterion for its size) and of the Poisson distribution seem unnecessary (except for a χ^2-type test of a fit). Further, exactly which of the C_{nm}'s are important and what other ones might be required should be investigated.

3.2 The Model

We wish to fit the $\bar{p}\, n \to 3\,\pi$ at rest annihilation process with a sum of four-point functions of the form (31.3), to give the statistical errors on the coefficients found and to see if additional terms other than those previously used are required.

Events for this study were those given by Anninos et al [1] in which 2902 points were recorded on the $s = M^2\,(\pi+\ \pi_1-)$ vs. $t = M^2(\pi+\ \pi_2-)$ plot. (As $s + t + u = \sum$ we did not follow Gopal et al [23] who use s vs. u). Using the amplitude:

$$A(s,t) = \sum_{\substack{n,m \\ n \geq 1 \\ m \leq n}} C_{nm} \frac{\Gamma(n - \alpha_s)\Gamma(n - \alpha_t)}{\Gamma(n+m - \alpha_s - \alpha_t)}$$

$$= \sum_{\substack{n,m \\ n \geq 1 \\ m \leq n}} C_{nm}\ V_{nm} = \sum_I C_I\ V_I$$

where $\alpha_x = 0.483 + 0.885x + iA\sqrt{x - 4m^2}\ \theta(x - 4m^2)$

with $s + t + u = (2M_N)^2 + 3m^2$

(M_N = Nucleon mass, m = Pion mass)

we performed a maximum likelihood [ML] fit [see Chapter 1] to the (s,t) data using for the likelihood function:

$$L = \prod_{i=1}^{N=2902} F(x_i, \underline{c}) \quad \text{with} \quad \int_{\underline{x}} F(\underline{x}, \underline{c})d\underline{x} = 1$$

where

$$F(x_i, \underline{c}) = \frac{|A((s_i,t_i), C_{nm}\text{'s})|^2}{\int_\sigma\int |A((s,t), C_{nm}\text{'s})|^2 ds dt},$$

(s_i,t_i) are the data points of the plot and the integration is taken over the Dalitz plot. Maximum values of $\mathcal{L}$ = Ln L were evaluated using the CERN library routines MINUITS on $-\mathcal{L}$, ZFACT for the Gamma function with complex arguments, and the Dalitz plot integration was performed using Simpson's rule (72 intervals) and Gaussian Quadrature (32 points).

With $n + m \leq 3$ the value of A was varied between 0.12 and 0.35 to find an optimum and $\mathcal{L}$ was evaluated for the Lovelace [3], Altarelli and Rubinstein [17], Rubinstein, Squires and Chaichian [20], and Gopal, Migneron and Rothery [23] parametrizations. The results of the $\mathcal{L}$ values for these cases are given in Tables 1 and 2 where it is seen that the four-point function fit of Rubinstein et al does not give $\mathcal{L}$ values anywhere near so good as those of Altarelli and Rubinstein. The C_{nm} coefficients of these two cases were then optimised and their $\mathcal{L}$ values consequently improved. However, even with these improvements they come nowhere near the $A = 0.33$ results of Gopal et al.

The analysis was then extended to include the C_{22} and C_{31} coefficients and it was found that $C_{31} \simeq 0$ and so terms in the series of (31.3) after V_{30} were neglected. C_{22} was also found to be small and including the V_{22} term only changed the value of $\mathcal{L}$ by approximately one so that this term too could be left out of the series.

An estimate of the errors (statistical) on the C_I's was obtained from the error-matrix.

$$\left(- \frac{\partial^2 \mathcal{L}}{\partial C_I \, \partial C_J} \right)^{-1} = E_{IJ} \quad \text{where} \quad \pm\lambda \sqrt{E_{II}}$$

gives the confidence interval for C_I, λ being given for both 95% (1.96) and 99% (2.576) levels. These in turn implied changes of the order of $\frac{\lambda^2}{2}$ in $\mathcal{L}$ for $C_I \pm \lambda \sqrt{E_{II}}$, (for a 'normal distribution' the variation is exactly $\frac{\lambda^2}{2}$) and the effect on $\mathcal{L}$ of changing some of the coefficients was also calculated. These results are presented in Table 3.

We also show the ratios of the decay rates for each case, following Altarelli and Rubinstein, given by:

$$R(\bar{p}p \to 3\pi^o) : R(\bar{p}n \to \pi^+\pi^-\pi^-) : R(\underset{T=1}{\bar{p}p \to \pi^+\pi^-\pi^o}).$$

If $$P = \iint_\sigma |A(s,t)|^2 \, ds \, dt$$ and

$$Q = \iint_\sigma 2 \, \mathrm{Re}\left\{ A(s,t) \, A^*(t,u) \right\} ds \, dt ,$$

and the integration is over the Dalitz plot, then these are given by

$$1 : \frac{4P}{P+Q} : \frac{6P - 2Q}{P+Q}$$

These are compared with the approximate experimental results for

$$\frac{R(\underset{T=1}{\bar{p}p \to \pi^+\pi^-\pi^o})}{R(\bar{p}n \to \pi^+\pi^-\pi^-)}$$

and are given in Table 4.

The kinematical and computational details are given in the Appendix.

The results of Gopal et al are seen to agree remarkably well with ours and the requirement of Altarelli and Rubinstein that there should be just the five terms with $m + n \leqslant 3$ is here established statistically. The decay rate ratios indicate that for our fit

$$R(\bar{p}n \to \pi^+\pi^-\pi^-) \simeq R(\underset{T=1}{\bar{p}p \to \pi^+\pi^-\pi^o}) \simeq 2R(\bar{p}p \to 3\pi^o)$$

otherwise they are inconclusive due to the uncertainty in the experimental value used. Those of Altarelli and Rubinstein are almost reproduced by the Rubinstein et al N-trajectory ones.

From a theoretical point of view the justification for Eq. (31.3) is perhaps rather slight. The initial 0^- state of mass $2M_N$ is a very low-lying object on the Chew-Frantschi plot, perhaps a particle on the third daughter trajectory of the pion, as in Fig. 5. However, very little is known about $\alpha_\pi(t)$ and nothing about its daughters. The physical interpretation of these Veneziano model fits to $\bar{p}n \to 3\pi$ is that at rest the reaction is dominated by the five $\pi\,\pi$ resonances:

ρ, ε, f, ρ', and ε' shown in Fig. 6.

These fits have no role for any $I = 2$ $\pi\,\pi$ interaction.

Summary

Veneziano four-point function fits have been applied with some success to five particle processes following the original idea cf Lovelace [3]. In an analysis of the specific annihilation process $\bar{p}n \to 3\pi$ (at rest) we employed the ML method to give an indication of the relative importance of various Veneziano satellite terms. We were able to give support to the idea [17] that this process could be fitted with just a few such terms given from certain observations of the structure of the data. For such a fit we presented the corresponding decay rate ratios. In the next chapter we describe efforts to fit five point functions to the same annihilation data.

3.3 Other Methods

A more careful treatment of the problems inherent in the ad hoc "unitarization" procedure of adding an imaginary part to the ρ trajectory is given by Pokorski, Raitio and Thomas [24]. The difficulty with the rather crude treatment of the unitarization problem is that the Imα prescription forces the total widths of all resonances within a given tower to be the same, even though partial wodths of parents and daughters are very different. Pokorski et al followed the method of Boguta [14] and decomposed the Veneziano amplitude into a convergent sum of resonance terms which enabled them to "unitarize" each resonance term separately. Total widths were given to the ε, ρ' and ε' while the partial widths were determined by the coefficients of the Veneziano functions. Mass and angular distributions which showed the qualitative features of the model were first presented by Pokorski and Thomas and then a fit to the data was made by dividing it into 120 bins and applying an ML type of fit. A χ^2 test was used to compare their data fits and some significant qualitative differences in overall fits was noted. An important feature of their dual model was the presence of the non-resonant background determined by the resonance coupling strengths. The table of relative contributions of resonance terms for the various models as given in Pokorski et al [24] is reproduced as Table 5.

In a similar analysis to determine the resonance structure for this process Gopal et al [25] found that the dominant contributions to the $(\pi^+\pi^-)$ system in $\bar{p}n \to 3\pi$ came from ε, ε', ε'' and ρ' daughter resonances and they confirmed the "decoupling" effect of the parent trajectory suggested by Lovelace [3]. Rothery [26] suggested a possible explanation for the small ρ- signal by using a simple model of ρ/photon analogy.

Barnes, Sarkar and Wells [27] used a scheme in which essentially the Lovelace form was multiplied by a $\left[\tilde{U}(12) \text{ trace terms}\right]$ polynomial factor. Their fit proved inferior to Lovelace's and they concluded that this was due to their use of $\tilde{U}(12)$ rather than chiral symmetry.

In a similar manner Franzen and Römer [28] constructed a dual quark model with Regge-behaviour in all channels and absence of both exotics and parity doublets. They claimed reasonable fits to the data not only for the $\bar{p}n \to 3\pi$ process but also for $\bar{p}p \to 3\pi$, $\bar{p}p \to \eta\pi^+\pi^-$ and $\bar{p}p \to \omega\pi^+\pi^-$. Their resonance spectrum, however, contains the well-established resonances ρ, ρ', f, A_2, B but no $\varepsilon(0^+0^+)$.

The problem of unwanted ancestors is eliminated by Gaskell [29] who uses a model in which complex trajectories appear naturally. He does not make a direct data fit for $\bar{p}n \to 3\pi$ but claims that the qualitative features of his model agree with the data for a suitable choice of his parameters.

More ambitious still was the model of Cohen-Tannoudji et al [30] in which a definite model suitable for $\pi\pi$ scattering was constructed incorporating analyticity, crossing, Regge behaviour, "duality" and partial unitarity via requiring second sheet resonance poles at low energies and absorption effects at high energies. Analytically continuing in s, t, u the $\pi^+\pi^-$ elastic amplitude to the $\bar{p}n \to 3\pi$ decay region they produced Dalitz plots and mass distributions that gave qualitatively good results without making any further kind of parameter adjustment. However, this success required a fairly complicated amplitude.

Hicks, Shukre and Winternitz [31] took a two-variable expansion of decay amplitudes, based on the representation theory of the group O(4) and applied their formalism, applicable to four particle cases where the masses and spins are arbitrary, to the $\bar{p}n \to 3\pi$ annihilation at rest data. The numerical fit was made to some more recent data from T. Kalogeropoulos and account of final state Coulomb interactions was also made, slightly improving their fits which were claimed to be reasonably good. No assumptions were made about the initial or final states or the annihilation dynamics, giving a purely kinematical fit.

A Moments Analysis has been given by G Rinaudo [32], and Bjørneboe [33] has presented a Grand Angular Momentum Analysis, both being given for the $\bar{p}n \rightarrow 3\pi$ annihilation reaction.

3.4 Application to Other Final States

Veneziano four-point function fits have also been applied to other $N\bar{N}$ decay processes with varying success.

The $\bar{p}n \to \pi K\bar{K}$ and $\bar{p}p \to \pi K\bar{K}$ annihilations were fitted by a Rome group [34] using a least squares fit to the experimental data. The fits reproduced the qualitative features of the data and the main difficulty of the model was stated to be that all the resonances of the same mass were given the same widths. Satellite terms were disregarded.

A good fit to both the Dalitz plot and the ω - decay angular distributions was obtained using Veneziano-type amplitudes for the process $\bar{p}p \to \pi^+\pi^-\omega$ by Chung, Montanet and Reucroft [35]. A χ^2 fit was made to the Dalitz plot which had a large amount of data. Franzen and Römer [28] pointed out that in the Chung et al model parity doublets appear. A similar model for this process was given by Hussain, Rahman and Razmi [36] in which a fit was made to the mass distributions but not the Dalitz plot. Both found a small singlet to triplet decay ratio.

The Chung et al group [37] have made a similar sort of analysis of the process $\bar{p}p \to \eta\pi^+\pi^-$ at rest and by an ML fit together with a χ^2 test they obtain good fits to the data using

Veneziano-type amplitudes with satellite terms. In both of these cases the Chung et al group use the Veneziano-type amplitudes in the form

$$V_{\ell mn} = \frac{\Gamma(\ell - \alpha_1)\Gamma(m - \alpha_2)}{\Gamma(n - \alpha_1 - \alpha_2)}$$

rather than the symmetric form as in (31.3). Similar results for a final state interaction model for this process using an ML fit were claimed by P Espigat et al [38].

Białas, Turnau and Zalewski [39] have shown that in the Veneziano model the resonances observed in the decay and production channels in general cannot have the same properties enjoyed by resonances formed in the direct scattering channel. A modified Veneziano form is presented by Goebel, Blackman and Wali [40] for dealing with the reaction $\pi\pi \to \pi S$ where S is a particle of arbitrary spin S and parity $\pm(-1)^S$.

APPENDIX

Dalitz Plot Boundary

In the figure of the Dalitz Plot boundary the points A-F are given in the accompanying Table 6. The limiting curve of the Dalitz plot corresponds to events which are collinear, so that in the notation of Fig. 4 if $s_{12} = s$ and $s_{23} = t$, this becomes:

$$\sqrt{E_1^{\,2} - m_1^{\,2}} \pm \sqrt{E_2^{\,2} - m_2^{\,2}} \pm \sqrt{E_3^{\,2} - m_3^{\,2}} = 0$$

where $m_1 = m_2 = m_3 = m$.

By conservation of energy:

$$E_1 + E_2 + E_3 = 2M.$$

The energies are given by:

$$E_3 = \frac{(2M)^2 + m^2 - s}{2(2M)}, \quad E_1 = \frac{(2M)^2 + m^2 - t}{2(2M)}$$

so that the limiting curve is given by

$$(2M)^2 - 2(2M)\,(E_1 + E_3) + 2E_1E_2 + m^2$$
$$= \pm\, 2\sqrt{(E_1^{\,2} - m^2)\,(E_3^{\,2} - m^2)}$$

Substituting for the values of E, multiplying by $2(2M)^2$, squaring and collecting terms gives this equation in the form:

$$s^2 t + t^2 s - st(\sum) + m^2\left[(2M)^2 - m^2\right]^2 = 0 \tag{A.1}$$

where $s + t + u = \sum = (2M)^2 + 3m^2$.

This equation is symmetric under interchange of s, t and u and is a quadratic in each of the variables so that for each value of s there corresponds two values of t and vice-versa. A check on (A.1) is that at s (or t) $= 4m^2$ and at s (or t) $= (2M - m)^2$ the equation should give equal roots corresponding to the one value of t (or s) at the minimum and maximum of s (or t) respectively, given by the points B and E (or A and D). The points C and F are given by putting $s = t$.

An alternative set of axes for giving the boundary of the plot are given in Fig. 7, with

$$s = \frac{1}{\sqrt{2}}(x - y), \qquad t = \frac{1}{\sqrt{2}}(x + y)$$

$$x_C = \sqrt{2}\,(m^2 + 2Mm), \qquad x_F = \frac{1}{\sqrt{2}}\left((2M)^2 - m^2\right)$$

The boundary curve now becomes:

$$(y^2 - x^2)\left[\frac{\sum}{2} - \frac{x}{\sqrt{2}}\right] + m^2\left[(2M)^2 - m^2\right]^2 = 0 \tag{A.2}$$

giving two y values for each x value.

Integration over the Dalitz plot

When optimizing the C_{nm} coefficients this was performed by taking

$$|\sum_I C_I V_I|^2 = \sum_I C_I^2 |V_I|^2 + \sum_{IJ} C_I C_J 2\mathrm{Re}(V_I V_J^*)$$

and integrating each term separately.

The evaluation of P and Q for the Decay-Rate ratios was made using both (s,t) and(x,y) axes and as

$$P = \int |A(s,t)|^2 d\sigma = \int |A(t,u)|^2 d\sigma = \int |A(s,u)|^2 d\sigma$$

and

$$\begin{aligned} Q &= \int 2\mathrm{Re}\ [A(s,t)A^*(t,u)]\ d\sigma \\ &= \int 2\mathrm{Re}\ [A(s,t)A^*(s,u)]\ d\sigma \\ &= \int 2\mathrm{Re}\ [A(t,u)A^*(s,u)]\ d\sigma \end{aligned}$$

(where the integration is over the Dalitz plot) a check on the precision of these results was made by evaluating each possibility.

The various decay rates (4a, 4b and 4c in Altarelli and Rubinstein) are found as follows using the well known $\pi\pi$ isospin relations for the s-channel:

$$\begin{cases} A_s^o = \frac{3}{2}\left[A(s,t) + A(s,u)\right] - \frac{1}{2} A(t,u) \\ A_s^1 = A(s,t) - A(s,u) \\ A_s^2 = A(t,u) \end{cases}$$

For $1 \to 2, 3, 4$ consider $1, 2 \to 3, 4$ and use the Clebsch-Gordon coefficients for $(T = 1)x(T = 1)$

(a) $\bar{p}n \to \pi^+\pi^-\pi^-$ is considered to go via

$\pi^- \to \pi^+\pi^-\pi^-$

i.e. $\pi^-\pi^- \to \pi^-\pi^-$, $T = 2$ amplitude A^2

or $A(s_{23}, s_{34}) = A(t,s)$ (u and s interchange)

(b) $\underset{T=1}{\bar{p}p} \to \pi^+\pi^-\pi^o$ is considered to go via

$\pi^o \to \pi^+\pi^-\pi^o$

i.e. $\pi^o\pi^- \to \pi^-\pi^o$, amplitude $\frac{1}{2}(A^2 - A^1)$

or $\frac{1}{2}\left(-A(s_{12}, s_{23}) + A(s_{12}, s_{34}) + A(s_{23}, s_{34})\right)$

$= \frac{1}{2}\left(-A(u,t) + A(s,u) + A(t,s)\right)$ (u and s interchange)

(c) $\bar{p}p \to 3\pi^o$ is considered to go via

$\pi^o \to \pi^o\pi^o\pi^o$

i.e. $\pi^o\pi^o \to \pi^o\pi^o$ Using the isospin relations

(no interchange needed) on $\frac{1}{3} A^o + \frac{2}{3} A^2$ gives

$\frac{1}{2}\left[A(s,t) + A(s,u)\right] - \frac{1}{6} A(t,u) + \frac{2}{3} A(t,u)$

$= \frac{1}{2}\left[A(s,t) + A(s,u) + A(t,u)\right]$

The decay rates are then proportional to the modulus squared of these results. So that if

$$\left\{\begin{array}{lll} A_1 & = & A(s,t) \\ A_2 & = & A(s,u) \\ A_3 & = & A(t,u) \end{array}\right. \quad \text{using Bose statistics gives:}$$

$$R_a = \frac{1}{2!}\,|A_1|^2$$

$$R_b = \frac{1}{4}\,|A_3 - A_1 - A_2|^2$$

$$R_c = \frac{1}{4} \times \frac{1}{3!}\,|A_1 + A_2 + A_3|^2$$

Integrating using the relations for P and Q gives

$$\left\{\begin{array}{lll} R_a & = & \frac{1}{2} P \\ R_b & = & \frac{3}{4} P - \frac{1}{4} Q \\ R_c & = & \frac{1}{8} (P + Q) \end{array}\right.$$

so that $\quad R_c : R_a : R_b$

or $\quad 1 : \dfrac{R_a}{R_c} : \dfrac{R_b}{R_c}$

becomes $\quad 1 : \dfrac{4P}{P+Q} : \dfrac{6P-2Q}{P+Q}$

Also $\quad 2(R_a - R_c) = R_b.$

From $$\frac{R_b}{R_a} = \frac{3}{2} - \frac{1}{2}\frac{Q}{P} .$$

and $$-1 \leqslant \frac{Q}{P} \leqslant 2$$

we obtain the relationship:

$$\frac{1}{2} \leqslant \frac{R_b}{R_a} \leqslant 2,$$

i.e. $$\frac{1}{2} \leqslant \frac{R(\bar{p}p_{T=1} \rightarrow \pi^+\pi^-\pi^o)}{R(\bar{p}n \rightarrow \pi^+\pi^-\pi^-)} \leqslant 2.$$

<u>Amplitudes</u>

The amplitudes used were of the form:

$$\sum_{n,m} C_{nm} \frac{\Gamma(n - \alpha_s)\Gamma(n - \alpha_t)}{\Gamma(m + n - \alpha_s - \alpha_t)}$$

where α_x was the Lovelace type Regge trajectory

$$\alpha_x = 0.483 + 0.885x + i\, A \sqrt{x - 4m^2} \;\; \theta(x - 4m^2).$$

Taking out the $\pi\pi$ amplitude as a factor we have

<u>Lovelace</u>

$C_{11} = 1.0, \quad C_{nm} = 0 \quad$ otherwise

$$A(s,t) = \frac{\Gamma(1 - \alpha_s)\Gamma(1 - \alpha_t)}{\Gamma(1 - \alpha_s - \alpha_t)} \quad \frac{1}{(1 - \alpha_s - \alpha_t)}$$

Altarelli and Rubinstein

$C_{10} = 1.0, \quad C_{11} = 1.89, \quad C_{30} = 0.57, \quad \text{other } C_{nm} = 0$

$$A(s,t) = \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} \left[C_{10} + \frac{1}{(1-\alpha_s-\alpha_t)} \left\{ C_{11} + C_{30} \left[\frac{(1-\alpha_s)(1-\alpha_t)(2-\alpha_s)(2-\alpha_t)}{(2-\alpha_s-\alpha_t)} \right] \right\} \right]$$

Gopal, Migneron and Rothery

A=0.28: $C_{10} = 1.0$, $C_{11} = 2.55$, $C_{20} = 2.96$, $C_{21} = 7.80$, $C_{30} = -4.52$

A=0.33: $C_{10} = 1.0$, $C_{11} = 2.90$, $C_{20} = 2.14$, $C_{21} = 7.31$, $C_{30} = -3.74$

$$A(s,t) = \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} \left[C_{10} + \frac{1}{(1-\alpha_s-\alpha_t)} \left\{ C_{11} + (1-\alpha_s)(1-\alpha_t) \left[C_{20} + \frac{\{C_{21} + C_{30}(2-\alpha_s)(2-\alpha_t)\}}{(2-\alpha_s-\alpha_t)} \right] \right\} \right]$$

C_{22} and C_{31} terms were included by replacing C_{21} by $C_{21} + \dfrac{C_{22}}{(3-\alpha_s-\alpha_t)}$

and C_{30} by $C_{30} + \dfrac{C_{31}}{(3-\alpha_s-\alpha_t)}$ respectively.

Rubinstein, Squires and Chaichian

In the C_{nm} terms below

$$A = -2\alpha' M^2 + 2\alpha^B(0) - \alpha^\rho(0) - 1,$$

$$C = \frac{3}{2A-3}$$

where $\alpha' = 0.885$ and $\alpha^{\rho}(0) = 0.483$

and for $N^{*} = \Delta, \quad \alpha^{B}(0) = 1.5 - \alpha' \; (1.236)^{2}$

$N \quad , \quad \alpha^{B}(0) = 0.5 - \alpha' \; M^{2}.$

Putting $C_{10} = 1$ for normalization

$$C_{101} = -3C(A^{2} + 8A + 15) - (2A^{3} + 21A^{2} + 70A + 75)$$

$$C_{11} = \left[3C(2A^{3} + 17A^{2} + 38A + 15) - (3A^{2} + 24A + 45)\right] \Big/ C_{101}$$

$$C_{20} = \left[3C + 6A + 21\right] \Big/ C_{101}$$

$$A(s,t) = \frac{\Gamma(1 - \alpha_{s})\Gamma(1 - \alpha_{t})}{\Gamma(1 - \alpha_{s} - \alpha_{t})} \left[C_{10} + \frac{\left[C_{11} + C_{20}(1-\alpha_{s})(1-\alpha_{t})\right]}{(1 - \alpha_{s} - \alpha_{t})} \right]$$

For the case $C = 0$,

$$C_{21} = 9 \Big/ C_{101} \quad , \quad C_{22} = -C_{21}$$

$$A(s,t) = \frac{\Gamma(1 - \alpha_{s})\Gamma(1 - \alpha_{t})}{\Gamma(1 - \alpha_{s} - \alpha_{t})} \left[C_{10} + \frac{1}{(1 - \alpha_{s} - \alpha_{t})} \left[C_{11} + (1 - \alpha_{s})(1 - \alpha_{t}) \left(C_{20} + \frac{C_{21}}{(3 - \alpha_{s} - \alpha_{t})} \right) \right] \right]$$

Error Matrix

The amplitude $A = \sum_{n,m} C_{nm} V_{nm}$ was written

$$A = \sum_I X_I V_I$$

so that $A A^* = (\sum_K X_K V_K) (\sum_L X_L V_L^*)$.

From $L = \prod_i \frac{|A|_i^2}{\int |A|^2 d\sigma}$ taking logs gives

$$-\mathcal{L} = n \ln \int |A|^2 d\sigma - \sum_i \ln |A|_i^2$$

$$= n \mathcal{L}_1 - \sum_i \mathcal{L}_2 .$$

The error-matrix is given by

$$\left[- \frac{\partial^2 \mathcal{L}}{\partial X_i \partial X_j} \right]^{-1} \quad \text{where } X_1 = 1 \text{ for normalization}$$

$$\frac{\partial \mathcal{L}_2}{\partial X_I} = \frac{\sum_{J=1}^{6} 2 \text{ Real } (V_I V_J^*) X_J}{| A |^2} , \quad I = 2, \ldots , 6$$

$$\therefore \frac{\partial^2 \mathcal{L}_2}{\partial X_I \partial X_K} = \frac{2 \text{ Real } (V_I V_J^*)}{| A |^2} -$$

$$- \sum_{J=1}^{6} \frac{2 \text{ Real } (V_I V_J^*) X_J}{| A |^2} \sum_{L=1}^{6} \frac{2 \text{ Real } (V_K V_L^*) X_L}{| A |^2}$$

This was then summed to include each i value and held in store to be subtracted from the corresponding value of $\frac{\partial^2 \mathcal{L}_1}{\partial X_I \partial X_K}$, (each term $\frac{-\partial^2 \mathcal{L}}{\partial X_I \partial X_J}$ being calculated and noted separately due to the large computing time of completing the whole operation in one process). The values of the error matrix were found by inverting the resultant matrix and the square-root of each diagonal element taken and then multiplied by λ for the required parameter confidence interval.

Fig. 1

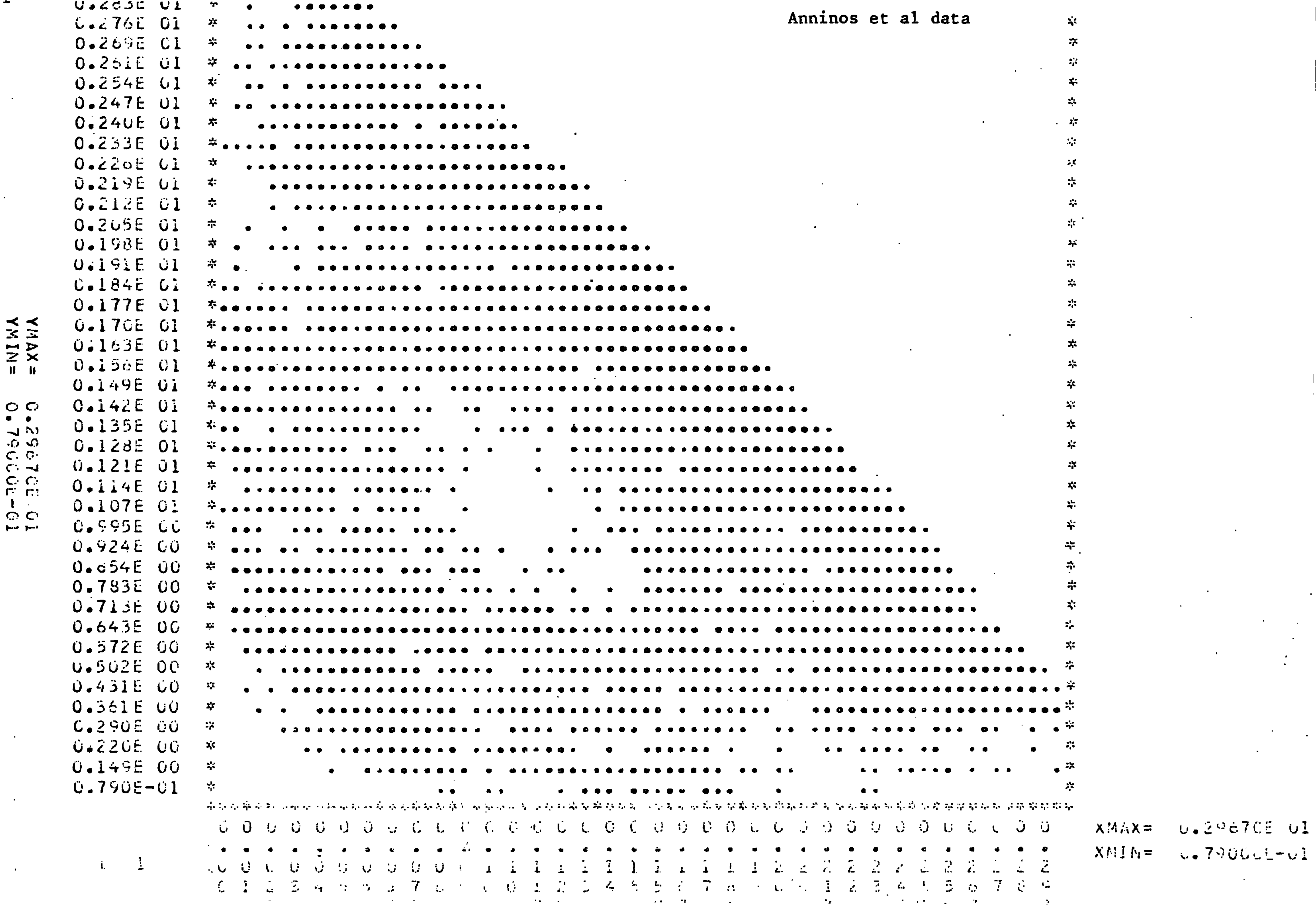

Anninos et al data
YMAX= 0.29670E 01
YMIN= 0.79000E-01
0.283E 01
0.276E 01
0.269E 01
0.261E 01
0.254E 01
0.247E 01
0.240E 01
0.233E 01
0.226E 01
0.219E 01
0.212E 01
0.205E 01
0.198E 01
0.191E 01
0.184E 01
0.177E 01
0.170E 01
0.163E 01
0.156E 01
0.149E 01
0.142E 01
0.135E 01
0.128E 01
0.121E 01
0.114E 01
0.107E 01
0.995E 00
0.924E 00
0.854E 00
0.783E 00
0.713E 00
0.643E 00
0.572E 00
0.502E 00
0.431E 00
0.361E 00
0.290E 00
0.220E 00
0.149E 00
0.790E-01
XMAX= 0.29670E 01
XMIN= 0.79000E-01

Fig. 2 $\bar{p}\,n \rightarrow 3\pi$ Annihilation at Rest

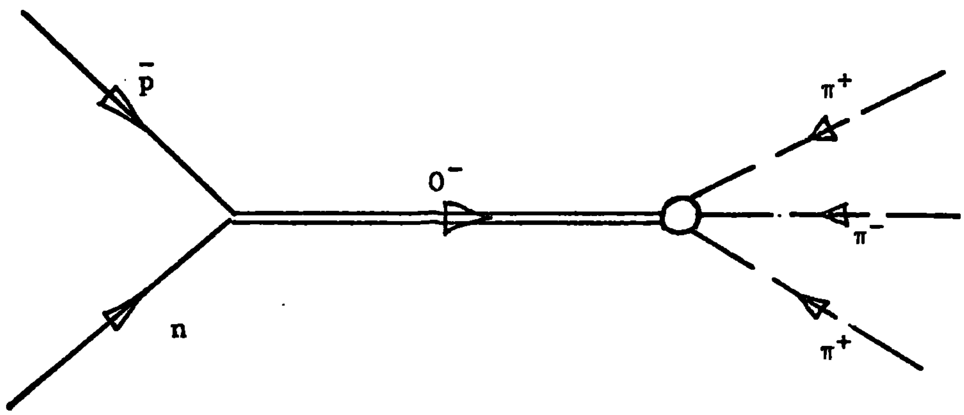
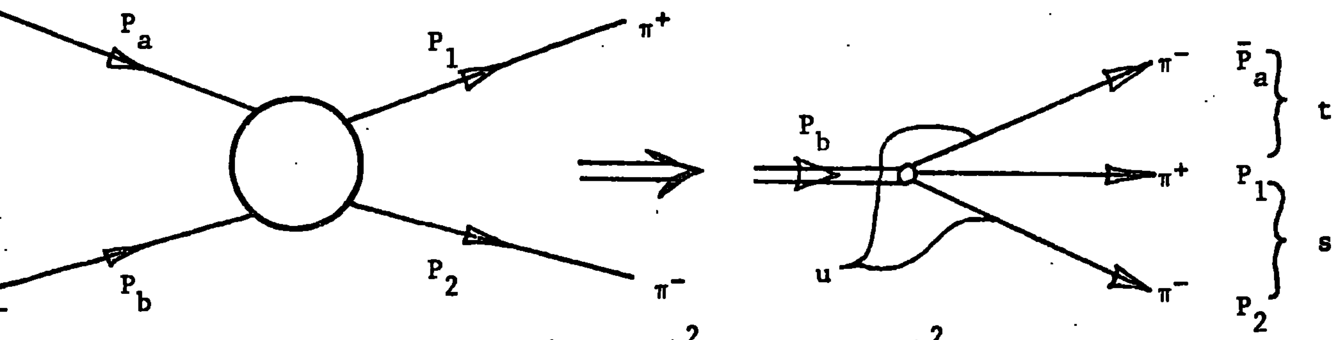


Fig. 3 Mandelstam Variables for $\bar{p}n \rightarrow 3\pi$



$$s = (P_1 + P_2)^2 = (P_a + P_b)^2$$
$$t = (\bar{P}_a + P_1)^2 = (P_1 - P_a)^2$$
$$u = (\bar{P}_a + P_2)^2 = (P_2 - P_a)^2$$

$$P_a + P_b = P_1 + P_2 \qquad s + t + u = m_b^2 + 3m_\pi^2 = \sum$$

Fig. 4 Mandelstam Variables for ref. [20]

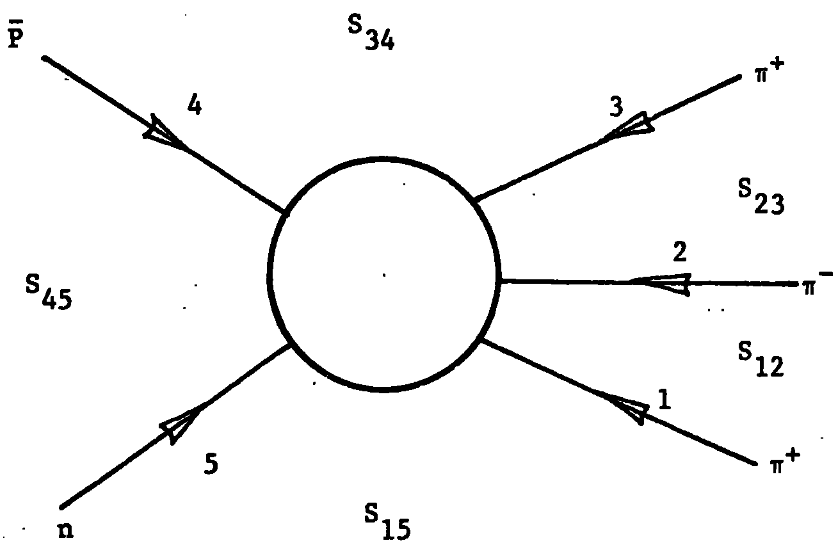


Table 1

VALUES OF THE LOG LIKELIHOOD FUNCTION $\mathcal{L}$.

Taking $M_\pi = m = 140$ MeV. $M = M_p = M_n = 940$ MeV.

$m + n \leq 3$ FIVE TERM FIT

A	$-\mathcal{L}$
0.12	4780
0.28	4220
0.31	4214
0.32	4214
0.33	4214
0.35	4217

Table 2

VALUES OF THE LOG LIKELIHOOD FUNCTION $\mathcal{L}$.

Taking $m = M_\pi = 140$ MeV. $M = M_p = M_n = 940$ MeV.

		A	$-\mathcal{L}$.
LOVELACE	$C_{11} = 1 \quad C_{nm} = 0$	0.28	4531
If A = .33		0.33	4512
ALTARELLI and RUBINSTEIN	$C_{10} = 1,\ C_{11} = 1.89,\ C_{30} = 0.57$	0.28	4603
Best Fit with $C_{20}=C_{21}=0$	$C_{10} = 1,\ C_{11} = 1.67,\ C_{30} = 2.98$	0.28	4409
	$C_{10} = 1,\ C_{11} = 1.67,\ C_{30} = 2.98$	0.33	4356
RUBINSTEIN et al	$C_{10} = 1,\ C_{11} = 3.22,\ C_{20} = 0.39$	0.28	5776
Best Fit with $C_{21}=C_{30}=0$	$C_{10} = 1,\ C_{11} = 4 \times 10^5,\ C_{20} = 0.39$	0.28	4531
GOPAL et al	$C_{10} = 1,\ C_{11} = 2.55,\ C_{20} = 2.96,$ $C_{21} = 7.80,\ C_{30} = -4.52$	0.28	4220
	$C_{10} = 1,\ C_{11} = 2.90,\ C_{20} = 2.14,$ $C_{21} = 7.31,\ C_{30} = -3.74$	0.33	4214

Table 3

VALUES OF THE LOG LIKELIHOOD FUNCTION $\mathcal{L}$

TAKING MASSES FROM 'PARTICLE PROPERTIES' TABLE

		Confidence Levels 99% (λ=2.576)	95% (λ=1.96)	A	$-\mathcal{L}$
Best Fit including C_{22}	$C_{10} = 1$				
	$C_{11} = 2.89$	± 0.47	± 0.35		
	$C_{20} = 2.16$	± 1.04	± 0.79		
	$C_{21} = 7.35$	± 1.80	± 1.37	0.33	4212
	$C_{30} = -3.65$	± 0.94	± 0.72		
	$C_{22} = -0.10$	± 0.45	± 0.34		
Best Fit with $C_{22} = 0$	$C_{10} = 1$				
	$C_{11} = 2.86$	± 0.45	± 0.34		
	$C_{20} = 2.14$	± 0.99	± 0.76	0.33	4213
	$C_{21} = 7.31$	± 1.75	± 1.33		
	$C_{30} = -3.65$	± 0.86	± 0.65		
Best Fit including C_{22} for A = .28	$C_{10} = 1$				
	$C_{11} = 2.55$	± 0.25	± 0.19		
	$C_{20} = 2.96$	± 0.96	± 0.73		
	$C_{21} = 7.74$	± 1.62	± 1.23	0.28	4219
	$C_{30} = -4.52$	± 0.91	± 0.69		
	$C_{22} = -0.10$	± 0.33	± 0.25		
Best Fit with $C_{22} = 0$ for A = .28	$C_{10} = 1$				
	$C_{11} = 2.52$	± 0.25	± 0.19		
	$C_{20} = 2.96$	± 0.93	± 0.71		
	$C_{21} = 7.74$	± 1.63	± 1.24	0.28	4219
	$C_{30} = -4.52$	± 0.88	± 0.67		

Table 4

DECAY RATE RATIOS

		$R(\bar{p}p \to 3\pi^0)$:	$R(\bar{p}n \to \pi^+\pi^-\pi^-)$:	$R(\bar{p}p \to \pi^+\pi^-\pi^0)_{T=1}$	$\dfrac{R(\bar{p}p \to \pi^+\pi^-\pi^0)_{T=1}}{R(\bar{p}n \to \pi^+\pi^-\pi^-)}$
LOVELACE		1	1.53	1.05	0.69
ALTARELLI and RUBINSTEIN		1	2.47	2.95	1.19
RUBINSTEIN et al	Δ	1	1.60	1.20	0.75
	N	1	2.54	3.09	1.21
RUBINSTEIN et al with C = 0	Δ	1	1.63	1.27	0.78
	N	1	1.66	1.33	0.80
GOPAL et al					
A = 0.28		1	2.30	2.60	1.13
A = 0.33		1	2.01	2.02	1.00
Best Fit					
A = 0.28					
$C_{22} \neq 0$		1	2.32	2.64	1.14
$C_{22} = 0$		1	2.31	2.62	1.13
A = 0.33					
$C_{22} \neq 0$		1	2.03	2.05	1.01
$C_{22} = 0$		1	2.02	2.04	1.01
EXPERIMENTAL VALUE FROM ALTARELLI [17]					$1.6^{+1.1}_{-0.8}$

Fig. 5 Chew-Frantschi Plot of the $\alpha_\pi(t)$ parent trajectory

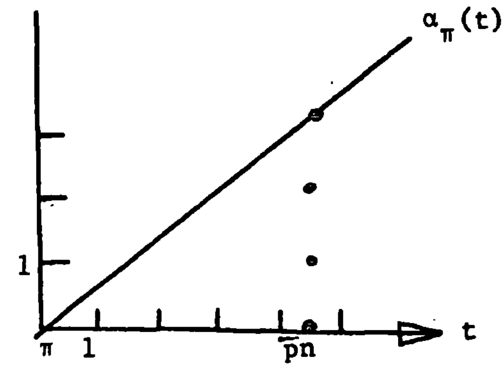


Fig. 6 The low-spin parent and daughter states included in the model amplitude (31.3). Resonances are labelled by their common names (ρ, ε etc.) and J designates their spin.

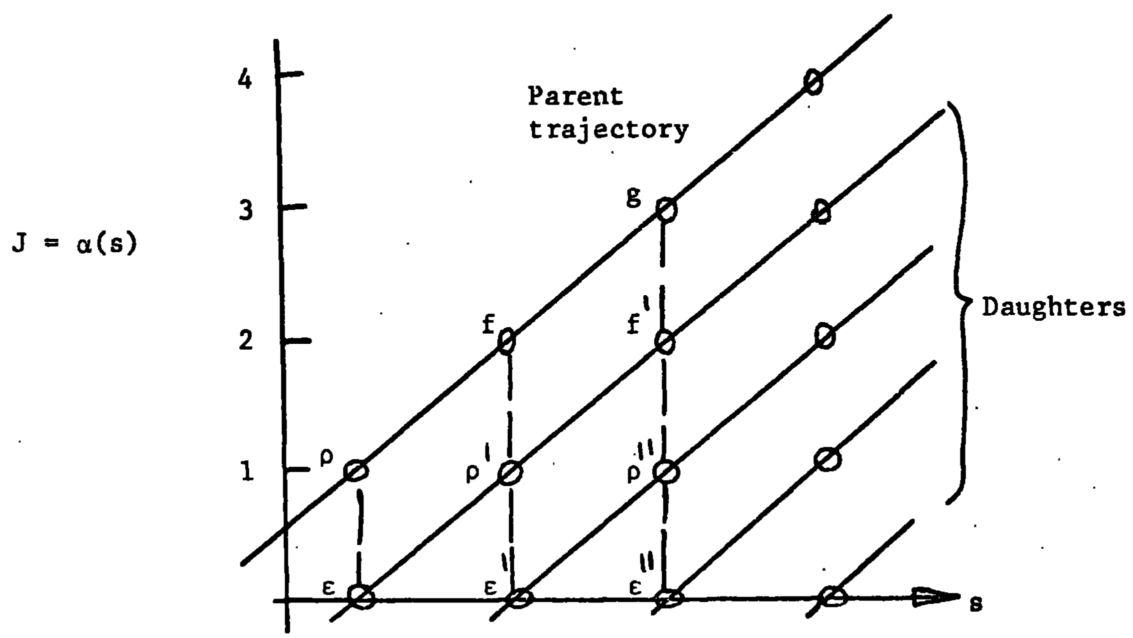


Table 5

The relative magnitudes $C_R(M_R^2)$ as defined by

$$A(s,t) = \sum_R \frac{(2L+1)C_R(s)P_L(\cos\theta)}{s - M_R^2} + (s \leftrightarrow t) + \text{(non-resonant background)}$$

where the relative contribution of each resonance, neglecting interference effects, is

$$(2\ell + 1) \mid C_R(M_R^2) \mid^2 / M_R \Gamma_R$$

when a finite width is given to the resonance.

	C_ε	C_ρ	$C_{\varepsilon'}$	$C_{\rho'}$	C_f
LOVELACE	1	0	1	-0.2	0
ALTARELLI and RUBINSTEIN	0.2	1	-2.1	1.2	-0.22
GOPAL et al. A=0.28 case	1.8	1	-1.7	2.6	0.42
POKORSKI et al	1	-0.052	1.0	-0.56	0.072
$\pi\pi$ AMPLITUDE (LOVELACE)	1	-0.2	2.0	-0.5	0.04

Fig. 7

THE DALITZ PLOT BOUNDARY

$\bar{p}n \to 3\pi$ (At rest)

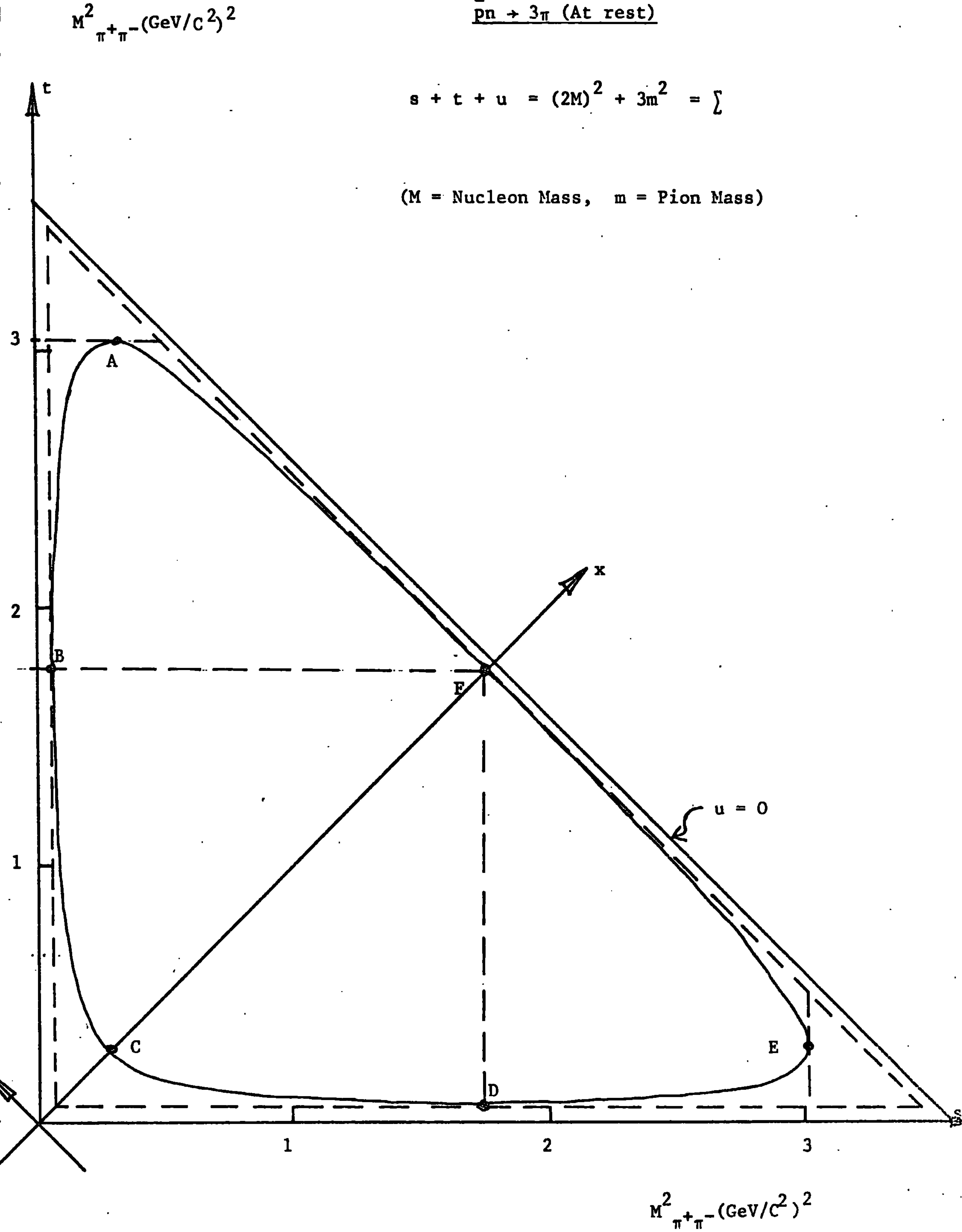


Table 6

POINTS ON THE DALITZ PLOT BOUNDARY

	s	t
A	$m^2 + 2Mm$	$(2M - m)^2$
B	$(2m)^2$	$\frac{(2M)^2 - m^2}{2}$
C	$m^2 + 2Mm$	$m^2 + 2Mm$
D	$\frac{(2M)^2 - m^2}{2}$	$(2m)^2$
E	$(2M - m)^2$	$m^2 + 2Mm$
F	$\frac{(2M)^2 - m^2}{2}$	$\frac{(2M)^2 - m^2}{2}$

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See also/...

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CHAPTER 4

Five-point function fit to the $\bar{p}n \to 3\pi$ at rest Dalitz plot data, and B_5 Phenomenology

4.1 Introduction

In Chapter 3 various four-point function fits to the $\bar{p}n \to 3\pi$ at rest data of Anninos et al [1] were discussed. Berger [2] recommended the use of five-point B_5 function fits to Dalitz plot data for $2 \to 3$ body processes and in particular for the $\bar{p}n \to 3\pi$ process in which one might have expected some contribution from baryon exchange graphs. If $t_{N\pi}$ denotes the four momentum transfer for an initial nucleon and final pion then the allowed kinematic range of $t_{N\pi}$ is given by:

$$- M_N^2 + 2.5\ M_\pi^2 \leqslant t_{N\pi} \leqslant (M_N - M_\pi)^2 \simeq 0.64\ \text{GeV}^2 .$$

At the upper limit, $t_{N\pi}$ is not far from the ($M_N^2 = 0.88\ \text{GeV}^2$) nucleon pole position, so that Berger expected large contributions to any amplitude from nucleon exchange. Further weight to this argument was lent by observing that $t_{N\pi}$ being near its maximum implied that $M^2_{\pi^+\pi^-}$ was also and that therefore the baryon exchange should be largest in the two corners of the Dalitz plot where indeed maxima of the density distributions are observed. Sivers [3] has pointed out that there is a limitation to the use of the four-point function models used as a convenience for reproducing a general final state $\pi\pi$ interaction with a reasonable spectrum of resonances, in that there is a level at which they can do no more in describing the data. The suggestion is that the t-channel (the cross-channel if $N\bar{N}$ is in the direct-channel) exchange picture and the final state interaction picture (based on direct-channel resonances) may be combined in a consistent way by a five-point function approach. Such an approach considers the process to be a $2 \rightarrow 3$ reaction in which two duality diagrams correspond to functions with poles in the $\bar{p}n$ channel. Some duality diagrams for $\bar{p}n \rightarrow \pi^+\pi^-\pi^-$ are shown in Fig. 1, and indicate that they each have an exchanged nucleon pole. Sivers suggests that looking at the exchange picture is more appropriate and more complete than that of the final state interaction picture and therefore one should factorize at the exchanged nucleon pole rather than in the $N\bar{N}$ channel and one should also look at the structure of the Dalitz plot in this light. However, Sivers points out that if B_5 functions are used then these should not be used to make detailed fits but rather to give a qualitative guide to the data.

Reference has already been made in Chapters 2 and 3 to the five-point function given by Rubinstein, Squires and Chaichian[4] for this ($\bar{p}n$) threshold annihilation into three pions process. These authors started from the assumption that, when the external particles lie on leading trajectories, a good approximation to the amplitude is provided by the leading Veneziano terms. It was then necessary to construct physically acceptable five-point functions and the following conditions were required: all desired poles, leading Regge behaviour in all channels, no spin-zero ghosts when trajectories have positive intercepts. The demand was then made that the relevant piece of the five-point function, i.e. the invariant non-flip amplitude, reduces to the leading term in each channel when we go to a pole on a leading trajectory. In particular this gives the important restriction that the amplitude does not have the nucleon pole in both baryon channels simultaneously, since otherwise we would obtain an incorrect $\pi N \to \pi N$ non-flip amplitude.

They take for that part of the amplitude which has poles in the $N\bar{N}$-channel, corresponding to the configuration of figure 2

$$
\begin{aligned}
A = \alpha^{\rho}_{12}\, F\,(\alpha^{\rho}_{12},\ \alpha^{\rho}_{23} - 1,\ \alpha^{B}_{34} - \tfrac{1}{2},\ \alpha^{\pi}_{45},\ \alpha^{B}_{15} - \tfrac{3}{2}) \\
+ C\,(\alpha^{B}_{34} - \tfrac{1}{2}) F(\alpha^{\rho}_{12} - 1,\ \alpha^{\rho}_{23} - 1,\ \alpha^{B}_{34} - \tfrac{1}{2},\ \alpha^{\pi}_{45} - 1,\ \alpha^{B}_{15} - \tfrac{1}{2}) \\
+ \ldots\ldots
\end{aligned}
\tag{41.1}
$$

where C is a constant and the terms not written come from non-cyclic reordering of the external particles of figure 2, and

$$F(x_1, x_2, x_3, x_4, x_5) = B_5(-x_1, -x_2, -x_3, -x_4, -x_5)$$

where B_5 is the Bardakçi-Ruegg-Virasoro form [5] given in Chapter 2.

α^B refers to either the N or the Δ trajectory, with the notation $\alpha_{ij} = \alpha(S_{ij})$ where each α_{ij} refers to the appropriate S_{ij} of Fig. 2. The factor $\alpha^B_{34} - \frac{1}{2}$ was chosen to eliminate the double nucleon pole and the term $\alpha^\rho_{23} - 1$ kills the ghost. They point out that Bose - statistics demands the addition of an identical term with 1 and 3 (referring to the two π^-'s) interchanged, and also that instead of their second term they could have added a term like the first but symmetrised in 4 and 5 that would also give spin $\frac{1}{2}$ poles in the 15 channel. We have seen in Chapter 2 that differential cross sections are not fitted well with their given C value and are improved if we put C = 0. Similarly in Chapter 3 the four-point function fit derived from this amplitude with the same C value was not very successful. An additional reason for having such a second term was the hope that it might have enabled an uncoupling of the ρ' daughter of the f as at the time (and also the present [2]) the status of this ρ' was not established. This would have been a means of eliminating a particle that appeared in the resonance towers of the usual Veneziano four-point function expressions. The relative contribution of each resonance as evaluated by Pokorski et al [6] is presented in Table 5 of Chapter 3, showing that the four-point function fits each have a ρ' contribution. This second term does not have leading behaviour in all channels as, for example, it behaves like $S_{15}^{\alpha_\rho - 1}$ when S_{15} and S_{23} are large and their ratio is constant. Schematically then the Rubinstein et al amplitude is:

$$A \begin{matrix}\text{(appropriate to}\\ \text{non-flip)}\end{matrix} = \beta \left\{ A_1 \begin{matrix}\text{(with}\\ \text{pion-pole)}\end{matrix} + CA_2 \begin{matrix}\text{(a satellite}\\ \text{term)}\end{matrix} \right\} .$$

Pokorski et al [6] point out that when evaluating the Rubinstein et al B_5 model at a pole in the $\bar{p}n$ channel the resulting B_4 four-point function fit does not give a reasonable description of the data. They note also that $\varepsilon(1234)$. B_5 models seem to work well only for peripheral collisions [7] and suggest that the five-point function model for $\bar{p}n$ annihilation still has unsolved problems. We have seen in Chapter 3 that the criticism of the four-point function fit (with $C = -1.25$) is indeed justified.

Boguta [8] also criticised the Rubinstein et al amplitude and computing exactly the model predictions via the standard B_5 program of Hopkinson and Plahte [9] showed that the results did not fit the data at all. However, Boguta does not make it clear how he performed the fits and does not produce any goodness of fit criterion. Repeating the Rubinstein arguments he wrote down amplitude (41.1) plus the same thing with 1 and 3 interchanged, as demanded by Bose statistics. Using the result for $\bar{p}n$ at rest that

$$\alpha^B_{34} = \alpha^B_{35}$$

$$\alpha^B_{51} = \alpha^B_{14}$$

he singled out the terms involving the factor C which were

$$C(\alpha^B_{34} - \tfrac{1}{2})F(\alpha^\rho_{12} - 1,\ \alpha^\rho_{23} - 1,\ \alpha^B_{34} - \tfrac{1}{2},\ \alpha^\pi_{45} - 1,\ \alpha^B_{51} - \tfrac{1}{2})$$

$$+\ C(\alpha^B_{14} - \tfrac{1}{2})F(\alpha^\rho_{23} - 1,\ \alpha^\rho_{12} - 1,\ \alpha^B_{14} - \tfrac{1}{2},\ \alpha^\pi_{45} - 1,\ \alpha^B_{35} - \tfrac{1}{2})$$

and then applied the permutation $12345 \to 32154$ together with these

α^B equalities to give the sum as:

$$C(\alpha_{34}^B + \alpha_{14}^B - 1)B(\alpha_{12}^\rho-1,\ \alpha_{23}^\rho-1,\ \alpha_{34}^B-\tfrac{1}{2},\ \alpha_{45}^\pi-1,\ \ \alpha_{15}^B-\tfrac{1}{2})$$

at $\bar{p}n$ threshold.

Quoting Rubinstein et al that:

$$\alpha^\pi(4M_N{}^2) \simeq 3$$

and that the residuum of B_5 can be used in an approximation for A where $B_5 \simeq \frac{1}{\alpha^\pi-3}\ \text{Res}\ B_5 \Big|_{\alpha^\pi=3}$ to get the decomposition

$$A = \sum_{nm} C_{nm} \frac{\Gamma(n - \alpha_s)\Gamma(n - \alpha_t)}{\Gamma(m + n - \alpha_s - \alpha_t)}$$

where the coefficients are given in the Rubinstein et al paper, Boguta then pointed out that if all the terms of C_{nm} are collected having the factor C, it must be divisible by $\alpha_{34}^B + \alpha_{14}^B - 1$ and since this was not the case for the Rubinstein et al coefficients then one of them must be wrong. In fact, Rubinstein et al had simply forgotten to symmatrise in the 1 and 3 variables and they later published corrections as pointed out in Chapter 3. A further comment of Boguta's was on the approximation of taking the residuum of B_5 in the α_{45} variable. He stated that this destroys the pole structure in the dual α_{15} and α_{34} (i.e, baryon exchange) variables and that for the approximation to make sense one should be certain that no characteristic features of baryon exchange are present when computing with the total B_5. The approximation is claimed unfounded for the delta exchange which was not at all negligible and the nucleon exchange was no better. In the actual numerical computations no width was assigned to the π-trajectory and a poor data fit was obtained even when C was allowed to vary. No best C value was given.

Pokorski, Szeptycka and Zieminski [10] applied the generalised Veneziano model to the related process $\pi^- p \to \pi^- \pi^+ n$ in the laboratory momentum range from 5 to 16 GeV/C. They considered the following four processes, each of which was thought to be dominated by π exchange:

$$\pi^- p \to \pi^- \pi^+ n$$

$$\pi^+ p \to \pi^+ \pi^- N^{*++}$$

$$K^- p \to K^- \pi^+ n$$

$$K^+ p \to K^+ \pi^- N^{*++}$$

They claimed that their approach had the following nice properties as compared to previous versions of one pion exchange (OPE) models:

(a) nucleon-nucleon four-momentum transfer dependence is fully predicted by our amplitude without any additional phenomenological form factors,

(b) factorization of the amplitude into the $\pi\pi \to \pi\pi$ part and the πNN vertex is not assumed,

(c) the effects of πN resonance production in the (15) system (see Figs. 1 and 2) are taken into account in a natural way.

Their calculations were an attempt to include π exchange into the duality frame, even though the dual nature of the pion is not clear, but they do not test the model's crossing properties.

The form of amplitude used for the given reactions was $|A|^2 = C|A_D|^2 + |A_P|^2$ where A_P represents a Pomeron term which they claim has to be taken into account. The dual amplitude A_D for the particular process $\pi^- p \to \pi^+\pi^- n$ had the form

$$A_D = \bar{u}(p)\gamma_5 u(q) \times [1 - \alpha^\rho_{23} - \alpha^\rho_{12}] \times \left[B_5\ (1-\alpha^\rho_{23},\ 1-\alpha^\rho_{12},\ \tfrac{3}{2} - \alpha^B_{15},\ -\alpha^\pi_{45},\ \tfrac{3}{2} - \alpha^B_{34}) + (4\leftrightarrow 5)\right] \quad (41.2)$$

using the labelling of Fig. 2.

The presence of fermions in the calculations was taken into account by the πNN vertex factor $\bar{u}(p)\gamma_5 u(q)$ which when averaged over initial and summed over final nucleon spins gave the factor S_{45}. The kinematic factor which multiplies the B_5 functions was introduced in order to get the Lovelace amplitude for $\pi\pi$ elastic scattering as a residue at the pion pole. This type of model with the Adler condition built in is criticised by Pokorski et al in [6] since the resulting sum of B_4 amplitudes does not seem to give a reasonable description of the annihilation data. Thomas [11] points out that the approximations used for fermion spin, isospin and unitarity for vector exchange reactions may not apply to pion exchange reactions, and that the approximation of making α complex as followed by Pokorski et al [10] is a very poor one. He claims that in such cases instead of the B_5 method an entirely equivalent description to the data may be made by using a B_4 amplitude times the pion pole:

$$\beta \bar{u}\, \gamma_5 u\, \frac{s^{\alpha_\pi}}{\alpha_\pi} (1 - \alpha_\rho - \alpha_\rho) B_4(1 - \alpha_\rho,\, 1 - \alpha_\rho)$$

Then he suggested that no new insight is obtained by including the pion pole in a dual model because, for example, the pion amplitude at small t_{pn} is mostly real, and hence plays no part in building up the imaginary part of the dual resonance contribution. Thus there are important differences between vector exchange and pion exchange. Thomas also pointed out the further difficulty of finding a reliable model in which to include the Pomeron. These points are amplified later.

Pokorski et al [10] attempted a detailed comparison of their model with experimental data giving different widths to resonances on parent and daughter trajectories by adding suitable terms to (41.2). They found that the dual B_5 model describes the details of the experimental data very well and was better than a given reggeized π exchange amplitude. They stated that the dual nature of the pion was not strongly tested by the application of the B_5 model to their given reactions.

This $\pi\pi\pi N\bar{N}$ process has also been examined in detail in terms of these five-point functions by a Heidelberg-Karlsruhe group of workers. Starting from the covariant decomposition of the $\pi\pi\pi N\bar{N}$- five-point function chosen by Dosch and Müller [12] Bender, Dosch, Müller and Rothe [13] made a particular ansatz for the invariant functions in terms of B_5-functions multiplied by polynomials of the invariant variables. The construction of a dual model for this process was based on these invariant functions because of their known and simple crossing properties. Each invariant function was expressed as a sum of twelve terms, each of the above form, and it was demanded that the invariant functions should factorize correctly at the nucleon and pion poles so that the πN and $\pi\pi$-amplitudes appearing in the residues were supposed to have the Igi [14] and Lovelace [15] structures respectively. This forced a minimal set of Regge trajectories to be those of the π, ρ, p', ω and N. Using certain asymptotic requirements and that the spin-averaged cross section shall behave in all single Regge limits like a corresponding scalar five-point function their rather unwieldy expressions were somewhat simplified, but despite the complicated formulae various simplifications in the physics remained. Application was made to the process $\pi^+p \rightarrow \rho^+p$ with apparently good agreement with the differential cross section data, all parameters now being fixed. Further applications, to show the prediction for $\pi\pi$-resonance production at high energies, were made [16] and the resulting differential cross section for the $\pi^-p \rightarrow \pi^+\pi^-n$ process at various energies and for the ρ and f-mass regions were found to be quite good in comparison with the experimental data. Of particular relevance to this section was the application of the foregoing five-point function dual model to the special case of $n\bar{p}$ annihilation at rest made by Bender and Rothe [17]. Using the requirement of absence of exotics in the isospin 2 $\pi\pi$-channel and that at the $n\bar{p}$ threshold $\mathrm{Re}\ \alpha_\pi(4M^2) \simeq 3$ they reduced the T-matrix element for the annihilation process to

$$T = -\sqrt{2}\ \bar{v}\ \gamma_5\ u\ H\ (S_{12},\ S_{23})$$

where $$H \simeq g^3\ \alpha'^2\ H_N - 2g\ \alpha'\ f^2_{\rho\pi\pi}\ H_\pi$$

and $$H_N(S_{23},S_{12}) = \left(S_{23} + S_{12} - (2M)^2 - m^2\right)\ B_5(1-\alpha^\rho_{23},\ 1-\alpha^\rho_{12},\ \tfrac{1}{2} -\alpha^B_{15},\ 2-\alpha^\pi_{45},\ \tfrac{1}{2} -\alpha^B_{34}) \tag{41.3}$$

$$H_\pi(S_{23},S_{12}) = (1 - \alpha^\rho_{23} - \alpha^\rho_{12})B_5(1-\alpha^\rho_{23},\ 1-\alpha^\rho_{12},\ \tfrac{3}{2} - \alpha^B_{15},\ -\alpha^\pi_{45},\ \tfrac{3}{2} -\alpha^B_{34})$$

g, f and α' being constants, and the notation being as previously. Use of the result $B_5(x_5,x_4,x_3,x_2,x_1) = B_5(x_1,x_2,x_3,x_4,x_5)$ together with cyclic symmetry will restore the expressions for H_N and H_π to the form used in (41.2). A free parameter λ was then introduced in front of the first term to compensate for the deficiency of the model regarding those terms containing the nucleon poles. Proceeding "à la Rubinstein et al" [4] they further reduced the expression to sums of Veneziano four-point functions with suitable coefficients but to obtain a good data fit some liberal variation of the widths and slope was required. The authors suggested the direction for improvement in the $\pi\pi\pi N\bar{N}$-five-point function model but have not followed it.

4.2 Comparison of five-point function fits

In this section we investigate the quality of the fits to the $\bar{p}n \to 3\pi$ at rest data made by the five-point function amplitudes of the last section as given by Rubinstein et al [4], Pokorski et al [10] and Bender and Rothe [17]. Taking these amplitudes to be respectively A_1, A_2 and A_3 they are given by

$$A_1 = R_1 + C\,R_2.$$

$$R_1 = \alpha^{\rho}_{12} B_5(-\alpha^{\rho}_{12},\ 1-\alpha^{\rho}_{23},\ \tfrac{1}{2}-\alpha^{B}_{34},\ -\alpha^{\pi}_{45},\ \tfrac{3}{2}-\alpha^{B}_{15}) + (1\leftrightarrow 3)$$

$$R_2 = (\alpha^{B}_{34} - \tfrac{1}{2}) B_5(1-\alpha^{\rho}_{12},\ 1-\alpha^{\rho}_{23},\ \tfrac{1}{2}-\alpha^{B}_{34},\ 1-\alpha^{\pi}_{45},\ \tfrac{1}{2}-\alpha^{B}_{15}) + (1\leftrightarrow 3)$$

$$A_2 = (1 - \alpha^{\rho}_{23} - \alpha^{\rho}_{12}) B_5(1-\alpha^{\rho}_{23},\ 1-\alpha^{\rho}_{12},\ \tfrac{3}{2}-\alpha^{B}_{15},\ -\alpha^{\pi}_{45},\ \tfrac{3}{2}-\alpha^{B}_{34}) + (1\leftrightarrow 3)$$

$$A_3 = A_2 + \lambda B_1.$$

$$B_1 = \left(S_{23} + S_{12} - (2M)^2 - m^2\right) B_5(1-\alpha^{\rho}_{23},\ 1-\alpha^{\rho}_{12},\ \tfrac{1}{2}-\alpha^{B}_{15},\ 2-\alpha^{\pi}_{45},\ \tfrac{1}{2}-\alpha^{B}_{34}) + (1\leftrightarrow 3)$$

where C and λ were taken to be free parameters, M = nucleon mass, m = pion mass and each α_{ij} corresponds to an S_{ij} of Fig. 2.

For the case of decay from rest if $S_{12} = s$, $S_{23} = t$ and $S_{45} = (2M)^2$ then $S_{14} = S_{15} = \frac{t + m^2 - 2M^2}{2}$, $S_{34} = S_{35} = \frac{s + m^2 - 2M^2}{2}$ and $s + t + u = (2M)^2 + 3m^2 \doteq \Sigma$. (See Appendix).

The ρ Regge trajectory was taken in the form

$$\alpha_x^\rho = 0.483 + 0.885x + i\,0.33\sqrt{x - 4m^2}$$

as given in [18] and α^B refers to the nucleon (B=N) or $\Delta(1238)$(B=Δ) trajectory with the same slope of 0.885.

$$\alpha^\pi = \alpha'(S_{45} - m^2) + iI_1 \quad \text{or} \quad \alpha'(S_{45} - (3m)^2) + iI_2$$

refers to the pion trajectory or a '3π' daughter trajectory, again with the same slope, α'. Boguta [8] in his fitting procedure for A_1 did not put any imaginary part to the pion trajectory and it is not necessary in fact although improvements to all fits can be obtained with its use.

It will be noticed that each of these amplitudes are appropriate for the diagrams A and B of Fig. 1 and no terms for the diagrams C and D are included. We do not present suitable amplitudes for these latter configurations.

Since Hicks et al [19] could comment on their fitting procedure that the χ^2-method is not too applicable in regions where the χ^2 contributions vary greatly from bin to bin (large statistical errors) and that it might be preferable to consider, for example, the "likelihood of observation", we continue to use the same procedure as in Chapter 3.

We perform a Maximum Likelihood [ML] fit to the Anninos et al [1] (s,t) data using for the likelihood function:

$$L = \prod_{i=1}^{N} F(s_i, t_i), \quad \mathcal{L} = \ln L, \quad N = 2902 \quad \text{and}$$

$$F(s_i, t_i) = \frac{|A(s_i,t_i)|^2}{\int_\sigma |A(s,t)|^2 ds dt}$$

where (s_i, t_i) are the data points of the Dalitz plot and the integration is taken over this plot. In this case the numerical integration was performed using the Gauss-Legendre quadrature scheme (with double precision arithmetic). $\mathcal{L}$ was maximised for various cases by applying the CERN routine MINUIT [20] to $-\mathcal{L}$ and the imaginary parts I_1 and I_2 were optimised (regardless of signs) to give minimum $-\mathcal{L}$ for each particular case. The B_5 terms were evaluated using an adaptation of the computer subroutine written by Hopkinson [21] and the gamma functions were evaluated using the CERN routine ZFACT. The four B_5 terms were optimised first with both (B=N) and (B=Δ) being used for A_2, the best value being obtained for A_2 with (B=Δ) and the '3π' trajectory with an imaginary part of $0.071 \sqrt{s_{45} - (3m)^2}$. Combinations of amplitudes were next optimised using firstly pairs, then triplets and then all four (with B=N).

Some of these results are summarised in Table 1 where, for comparison, the values of $\mathcal{L}$ for the one, three and five term four-point function fits [18] are also given. It was found that a slight improvement in the values of $\mathcal{L}$ was to be obtained by using the '3π' daughter trajectory instead of the pion trajectory for the initial state as prescribed by Rubinstein et al [22], i.e, $I_2 = Im_{\pi}(x) = \lambda(x-9m^2)$ (a straight line $Im\alpha$). However I_1 and I_2 both changed signs when going from (B=Δ) to (B=N). In fact, I_1 and I_2 could both have been previously fixed prior to making a fit if some theoretical restraint were required such as, for example, that given by Rubinstein et al in ref. [22] equation (9). Another computing difficulty was that I_1 and I_2 had to be varied both for sums of amplitudes as well as for individual ones as they changed from case to case. The value of C obtained in the fit for A_1 was about 0.5 but in any case R_2 only had a small effect on the amplitude A_1. C varied according to the value of I_2 but was not near the value -1.25 as given by Rubinstein et al [4] which seems to confirm the results of Chapters 2 and 3 that this R_2 term could be neglected. Most of the likelihood values come nowhere near those of the four-point function fits [i.e, they are not within about 3.3 at the 99% level] but, however, a change in the argument of the B_5 function for B_1 of $\alpha_{34} \leftrightarrow \alpha_{15}$ produced much more encouraging results. [This would correspond to the suggestion of Rubinstein et al in [4] that one could, and in general should, add terms similar to the amplitude but with 4 and 5 interchanged and multiplied by an arbitrary coefficient.] Therefore, although the amplitudes given by these three groups did not produce good fits to the Dalitz plot data it should be possible to give an amplitude in terms of, say, two B_5 functions that does do so, at least to the order of the four-point function fits given in [18].

Summary

In chapter three it was shown that a suitable sum of B_4 terms gave a reasonable and economical parameterisation of the $\bar{p}n \to 3\pi$ (at rest) data. In this chapter we have tested some B_5 functions using the same data with the hope that with just a few parameters we may have been able to give a comparable fit to the data and thus predict the various coefficients in the former fit. This would then have been a real test of the whole dual model idea, giving a check on the large mass - small spin region (see Fig. 5 of Chapter 3) which is more significant than tests which lie on the leading trajectory. From our results we have shown that this project was not successful so that real support for the model was not provided. However, since the results were not too absurd we can attribute the failure in detail to the fact that we did not know how to write down dual amplitudes with fermions. There are diagrams without resonance in the $N\bar{N}$ channel which we did not include in the annihilation fit. Sivers [3] argued that all four diagrams A, B, C, D of Fig. 1 should be taken into account in a data fit since all have the exchanged nucleon pole and including functions appropriate for diagrams C and D would have been an obvious next step to test the $N\bar{N}$ channel "factorization" assumed in the four-point function approach. In that case the crossing predictions of the model assumed in chapter two would have required the use of extra terms to give the amplitude for $\pi N \to \pi\pi N$ from that of $N\bar{N} \to \pi\pi\pi$.

4.3 B_5 Phenomenology

There are several excellent reviews of the application of the B_5 formulae to five-point function processes and the development and progress of this work can be traced through those in, for example,

Chan (1969)
Lovelace (1969)
Satz (1970)
Berger (1971)
Thomas (1971) [23]

and in the Introductions to some of the original papers. N point Veneziano formulae give a new approach to multi-Regge phenomenology. They include resonances and Regge exchange in a dual manner, they should be valid for all values of the subenergies, and they have well determined and theoretically plausible Regge residues. All the main drawbacks of the multi-Regge model are thereby removed and a unified description of mechanisms previously considered separate, such as "resonance production", "background" and "double-peripheralism" is provided. The attractive properties of correct Regge asymptotic behaviour, crossing symmetry and duality possessed by such models lead to the expectation that the same amplitude describes different amplitudes related by crossing and that it should describe also two body reactions related by "bootstrap consistency". A further attractive feature is that, due to the theoretical constraints imposed in constructing the model, in applications there are relatively few unknown parameters. The shortcomings of the B_5 model as applied to data are that spin and isospin are not correctly included in the model and unitarity is violated, being simulated usually by adding in an imaginary part to the trajectory function as required. Further, the model requires a method of dealing with the Pomeron, since the Pomeranchuk singularity has no place in a dual model of this type without unitarity.

We have mentioned some of the early applications of the uses of B_5 in Chapter 2 where the particles were mesons (π, κ, σ). However, the B_5 amplitude was first applied in the analysis of a production experiment by Petersson and Törnqvist [24] who studied the reaction

$$\kappa^- p \rightarrow \Lambda \pi^+ \pi^-$$

over the energy range 3 - 10 GeV/C. This reaction was suitable for such an analysis because of the absence both of Pomeron and pion exchange, the restriction by quantum number of permissible graphs, and the dominance of normal parity exchange. The baryons were put in with spin zero. The specific amplitude chosen for this process was of the form

$$A = C\, \varepsilon_{\alpha\beta\gamma\delta}\, P_\alpha^{\ 1} P_\beta^{\ 2} P_\gamma^{\ 3} P_\delta^{\ 4} \times$$
$$\Big[B_5 \left(1-\alpha_{\kappa^*}, 1-\alpha_\rho, \tfrac{3}{2} - \alpha_{Y_1^*}, 1-\alpha_{\kappa^*}, \tfrac{3}{2} - \alpha_{Y_1^*}\right) +$$
$$B_5 \left(1-\alpha_{\kappa^*}, \tfrac{3}{2} - \alpha_{Y_1^*}, \tfrac{3}{2} - \alpha_{Y_1^*}, 1-\alpha_N, \tfrac{3}{2} - \alpha_{Y_1^*}\right) \Big]$$

where C was a normalizing parameter and the trajectories were given the universal slope of 0.9 $(\mathrm{GeV})^2$ and had imaginary parts, inserted above thresholds, of the form $A\sqrt{S-S_0}$ for the ρ and $B(S-S_0)$ for the Y_1^* resonances respectively. A further difficulty in interpretation, however, was that the graphs chosen for the above amplitude were not those that the Harari-Rosner [25] quark duality graphic rules would suggest in that the "heretical" model with four quarks and an antiquark in the $\bar{K}N$ channel was chosen. This is illustrated in Fig. 3 where the two sets of duality diagrams considered are shown. The reasonable agreement obtained with the large range of experimental data treated was most impressive and certainly encouraged further applications. Törnqvist [26] then crossed to the process $\pi^+p \rightarrow K^+\pi^+\Lambda$ and found that the normalization was too large by a factor of two for the process and its related quasi-two-body process $\pi^+p \rightarrow Y_1^*(1385)\,K^+$ the dominant sub-channel, which is Pomeron free. This was nevertheless considered by Lovelace [23] to rank among the very best existing checks of crossing symmetry since a well-known backward π^-p Regge fit when extrapolated to the Δ pole was out by a factor of 2000! This example of crossing illustrates the novel feature of these types of models in which the legs of the B_5 formulae can be permuted by crossing symmetry to predict ten different $2 \rightarrow 3$ reaction channels, several of which are often observable. If the five external particle lines are permuted then there are $(N-1)!/2 = 12$ in-equivalent such diagrams. Further, in each reaction a considerable number of charge combinations are also possible so that several processes could be fitted simultaneously and for a range of energies.

Further applications were made by Hoyer et al [27] but most of the work on production processes was, however, concerned with the $K\bar{K}N\bar{N}\pi$ system first investigated by Chan, Raitio, Thomas and Törnqvist [28]. The four channels $KN \to K\pi N$, $\bar{K}N \to \bar{K}\pi N$, $\pi N \to K\bar{K}N$, $\bar{N}N \to K\bar{K}\pi$ were considered and 21 charged states that had enough data for study were classified into those that were considered to require (i) a vector exchange model, (ii) a vector + a Pomeron exchange model, (iii) a pion + a Pomeron exchange model. They then considered the three reactions of type (i) $K^+p \to K^0\pi^+p$, $K^-p \to \bar{K}^0\pi^-p$ and $\pi^-p \to K^0\bar{K}^-p$ and used the orthodox Harari-Rosner diagrams(and absence of exotics) to obtain three B_5 terms of the Petersson and Törnqvist form for their amplitude. For each channel the dominant trajectory was inserted and the imaginary part of α above threshold was found using the formula $\mathrm{Im}\alpha = \alpha' M_{res} \Gamma_{res}$.

A large wide ranging quantity of data was fitted by this one parameter fit although once again the cross-section normalization was predicted badly from reaction to reaction. This apparently significant work which tested global duality was then continued in several directions.

By taking the Chan et al [28] amplitude at the nucleon or Λ, pole predictions for the two-body reactions of the kind $K^-p \to \bar{K}^0n$ and $\pi^-p \to K^0\Lambda$ were made by Peterson and Thomas [29]. Bartsch et al [30] made a study of the reaction $K^-p \to \bar{K}^0\pi^-p$ similar to the one above and Raitio [31] subsequently studied the reactions $K^+n \to K^0\pi^+n$ and $K^-n \to \bar{K}^0\pi^-n$ related to those considered by the Chan group by isospin invariance. These global successes with so few adjustable parameters at first seemed impressive, especially when compared to other models that have much more inherent freedom but fail to do better. However, a closer look at the above works showed that to some degree the quality of the fits reflected a judicious input into the model, so that it became evident that the claims of one parameter fits were somewhat misleading. (Discussed in the Review by Berger [2]). The CERN group also looked at the complex where π-exchange is thought to be dominant and considered the group of reactions derived from $\kappa^- p \to \kappa^- \pi^+ n$ [32]. An overall crossing symmetric description was attempted and the main features of the data were found to be determined by pion exchange, and the daughter structure and relative coupling constants which follow from the zero width model were supported by the data. The dominant baryon resonances were concluded to be dual to the ρ and no experimental evidence was found for pion duality to known baryon resonances. For a B_5 model describing these reactions, the ε kinematic factor, spin, isospin and unitarity solution used for vector exchange reactions may not be appropriate. In contrast to the situation with vector exchange, for pion exchange daughter states give appreciable contributions in all but the $\bar{p}n$ channel, even for the lowest position on the trajectory, thus making the approximation of α to be complex a very poor one, as remarked earlier. An entirely equivalent description to using the sort of $\beta(1 - \alpha_\rho - \alpha_{\kappa^*})\, B_5\, \bar{u}\, \gamma_5\, u$ form for the amplitude was found to be a B_4 amplitude times the pion pole:

$$\beta \bar{u} \gamma_5 u \; \frac{s^{\alpha_\pi}}{\alpha_\pi} (1 - \alpha_\rho - \alpha_{\kappa^*})\; B_4\; (1 - \alpha_\rho, 1 - \alpha_{\kappa^*})$$

Some simplicity in understanding might thus be obtained by excluding the pion from the dual framework, a conclusion that the Pokorski paper [10] did not thoroughly test, as remarked in the earlier section. The fears expressed by Lovelace [23], that no B_5 phenomenology existed outside the CERN group and that spin would therefore never be put in to the amplitudes properly, were no doubt overcome by the work that gradually appeared from America, Europe and the USSR.

A detailed test of the Bardakci-Ruegg model applied to the data of $K^+p \to \pi^+pK^o$ was made by Waluch et al [33] in order to determine what portions of the success of the model were truly independent of the input. It was shown that even without *ad hoc* modifications of trajectory functions a good fit could be obtained, but at the expense of using several kinematic factors and five adjustable parameters. A report of the experiment at 12 GeV/C and an extension of their study is given by Waluch [34]. Several authors have also extended the study of the complex to other energies, e.g. [35].

An Imperial College group considered a number of different reactions using the procedure of these previous authors and in the $K\bar{K}N\bar{N}\pi$ system considered the process $\pi^-p \to K^o\bar{K}^on$ at 12 GeV/C [36], and found only a limited success with their model when assuming only vector exchange. Other dual resonance models for the pion-dominant reactions $K^-p \to K^-\pi^+n$, $K^+p \to K^+\pi^+n$, $K^+n \to K^+\pi^-p$, and $K^-n \to K^-\pi^+p$ [37-39] have been presented. Shafee [40] analysed the effect of the first daughter of K^*_{1400} in the reaction $K^+p \to K^0\pi^+p$ using a B_5 formularism, and others [41-43] have fitted $KN \to K^+\pi N$ reactions with $\varepsilon.B_5$ models.

Other Complexes

Besides the $N\bar{N}K\bar{K}\pi$ complex other types of reactions were investigated: $KN\bar{N}N\Lambda$ by Dunwoodie and Tuominiemi [44] and others [45-46], $NK\bar{K}K\Lambda$ by a UCLA-Oxford group [47], $K^+p \to K^+p\omega$ by Jerome and Simmons [48], $NK\bar{K}^*\pi\Delta^{++}$ by Baier et al [49], $K^-p \to \Xi\pi K$ by Ross and Lyons [50], and recently Chu [51] constructed a dual resonance model to describe the reaction $\pi^+p \to \pi^+\pi^0p$.

Spin and Isospin

Spin and Isospin have been incorporated in various ways. Benfatto et al [52] proposed a model for the $N\bar{N}K\bar{K}\pi$ process (having the correct asymptotic behaviour and spin structure, the right isospin and signature on the parent trajectories and the appropriate factorization properties on the lowest poles) using invariant amplitudes. A similar Veneziano type ansatz for invariant amplitudes to suit this complex was given by Schmidt [53]. The most general spinless dual amplitude describing the set of reactions $KN \to K\pi N$, by imposing isospin, charge conjugation and crossing symmetry, as well as absence of exotic states, was given by a group at Technion [54]. The group at Imperial College presented a method for including both spin and unitary spin by combining the U(6,6) supermultiplet formalism with the Veneziano spinless amplitude, application being made both to $K^-p \to \pi^-\pi^+\Lambda$ [55] and $K^-p \to \bar{K}^o\pi^-p$ [56], with greater success in the latter, and also to $K^-p \to K^{*-}\pi^+n$ [57]. Hirshfeld and Schmidt [58] also looked at the dual $\bar{K}N\bar{\Lambda}\ \pi\pi$ system with spin.

Pomeron Exchanges

Pokorski and Satz [59] attempted to describe diffraction dissociation reactions by splitting a five-point function for AB → ACD up into

$$f(t_{AA})\bar{s}\cdot V(4)_{\mathbb{P}B-CD}$$

where $f(t_{AA})$ denoted a form factor for the hadron-hadron-Pomeron vertex, $\bar{s} = (P_A + P_B)^2$ is a factor to account for the Pomeron propagator and V(4) denoted the "amplitude" for the "reaction" $\mathbb{P} + B \to C + D$. Berger [2] doubted the reliability of such a model, but Kajantie and Papageorgiou [60] in their Dual + Pomeron analysis of $K^{\pm}p \to K^{\pm}\pi^o p$ made a good analysis using it,and applications by other authors were made [61,62].

Summary of B_N Phenomenology

Attractive Features:

1) Offers a unified approach to resonance production and multi-Reggeism.

2) Crossing symmetry.

3) Bootstrap consistency (some ambiguity in practice).

4) "Fits" a large amount of data with few parameters.

5) Some complexes have only a few allowed graphs.

Limitations:

1) Unitarity simulated by imaginary part of trajectories.

2) Fermions treated as Bosons

3) One trajectory in each channel-inserted 'dominant' one.

4) Complexity for more bodies in final states (i.e, if $N > 5$)

5) Has the problems of simple Regge theory (which probably need cuts for their resolution).

APPENDIX

In the symmetrization procedure it is necessary to find S_{14} and S_{35}. Using the notation of Fig. 2 with a four-momentum vector P_i associated with each particle i then:

At threshold $P_4 = P_5 = (M,0)$ and $E_1 + E_2 + E_3 = 2M$

So
$$S_{14} = (P_1+P_4)^2 = \left[(P_1+P_5)+(P_4-P_5)\right]^2 = (P_1+P_5)^2 = S_{15}$$
$$S_{34} = (P_3+P_4) = \left[(P_3+P_5)+(P_4-P_5)\right]^2 = (P_3+P_5)^2 = S_{35}.$$

Also

$$\begin{aligned} S_{14} &= (P_1+P_4)^2 = (P_2+P_3+P_5)^2 = S_{23} + M^2 + 2P_5(P_2 + P_3) \\ &= S_{23} + M^2 - 2M\,(E_2 + E_3) \\ &= S_{23} + M^2 - 2M\,(2M - E_1) \end{aligned}$$

But

$$\begin{aligned} S_{23} &= (P_2+P_3)^2 = \left(P_1 + (P_4+P_5)\right)^2 = \left(P_1 + (2M,0)\right)^2 \\ &= \left(m^2 + (2M)^2 + 2(-2ME_1)\right) = -4ME_1 + (2M)^2 + m^2 \end{aligned}$$

Therefore
$$E_1 = \frac{(2M)^2 + m^2 - S_{23}}{4M}$$

Hence
$$S_{14} = S_{23} + M^2 - \frac{1}{2}\,(4M^2 - m^2 + S_{23})$$

i.e.
$$S_{14} = S_{15} = \frac{S_{23} + m^2 - 2M^2}{2}$$

Interchanging indices 1 and 3 gives:

$$S_{34} = S_{35} = \frac{S_{12} + m^2 - 2M^2}{2}$$

In a similar manner

$$S_{24} = (P_2 + P_4)^2$$

$$= M^2 + m^2 - 2ME_2$$

But

$$E_2 = 2M - E_1 - E_3$$

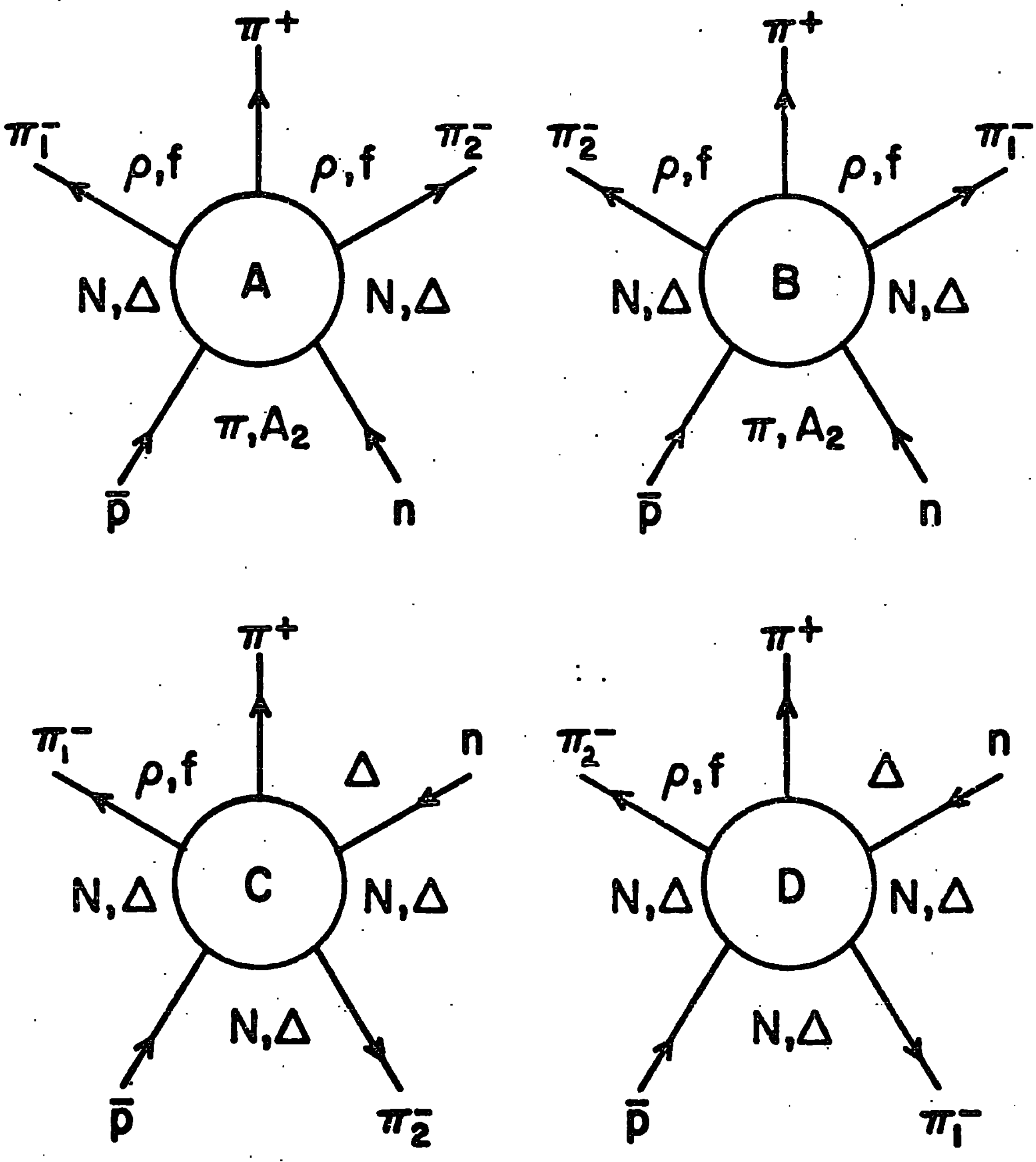
So

$$S_{24} = M^2 + m^2 - 2M \left(\frac{S_{12} + S_{23} - 2m^2}{4M} \right)$$

Therefore

$$S_{24} = S_{25} = \frac{1}{2} \left[4m^2 + 2M^2 - S_{12} - S_{23} \right].$$

Fig. 1



Diagrams giving the singularity structure of a dual model for $\bar{p}n \rightarrow \pi^+\pi_1^-\pi_2^-$.

Poles occur in channels defined by adjacent particles .

FIG. 2

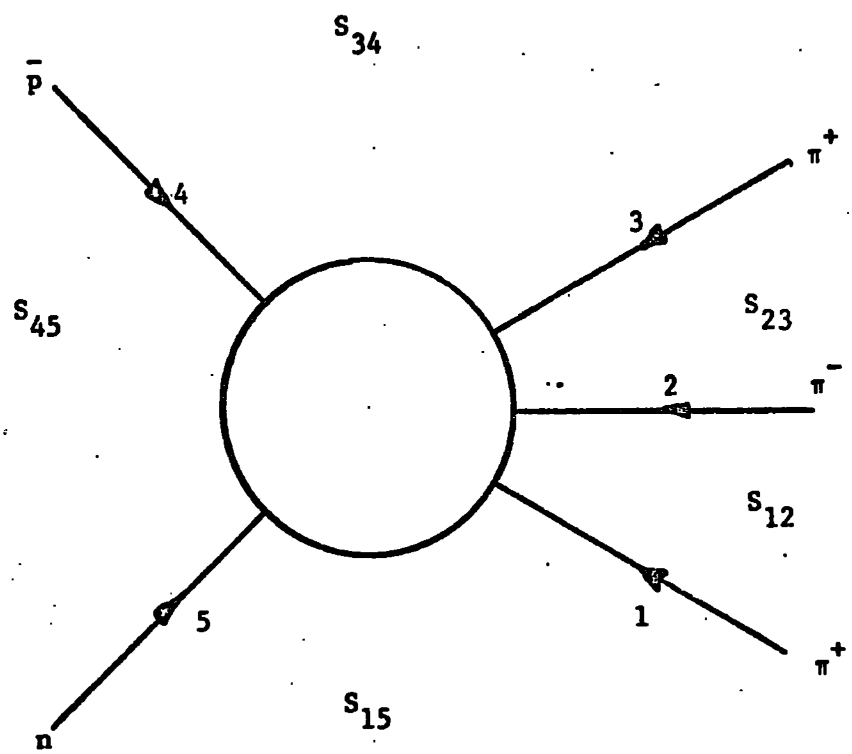
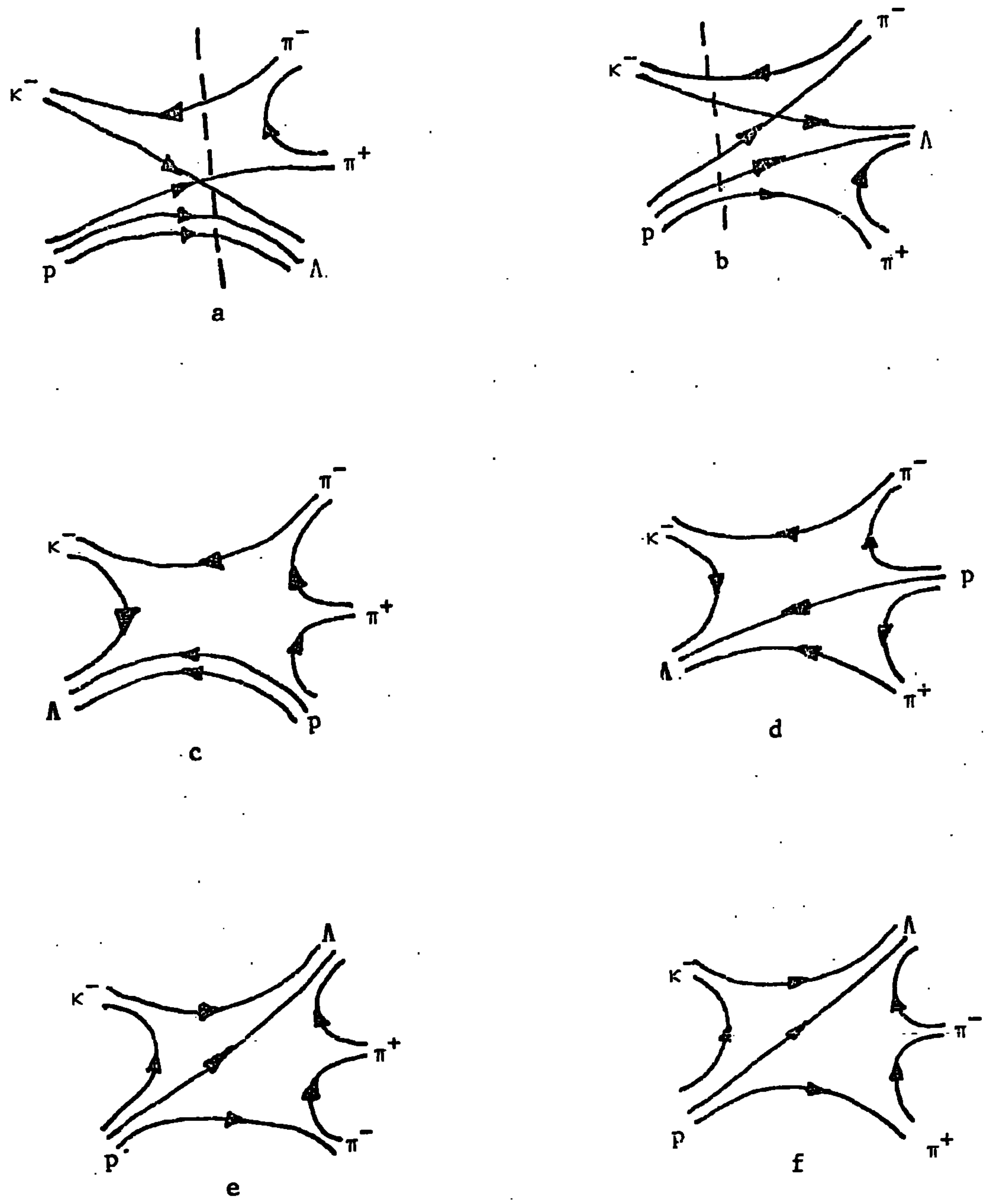


TABLE 1

Values of the log likelihood function $\mathcal{L}$

Amplitude	$-\mathcal{L}$
A_2 $(B = N)$	5006
R_1 $(B = N)$	5002
A_1 $(B = N)$	4855
A_2 $(B = \Delta)$	4679
A_3 $(B = N)$	4619
A_3 $(B = \Delta)$	4576
Lovelace	4531
A_2 $(B = \Delta)$ + B_1 $(B = N)$	4568
R_1 $(B = N)$ + B_1 $(B = N)$	4485
$\alpha_{34} \leftrightarrow \alpha_{15}$ in B_1	
A_3 $(B = \Delta)$	4548
A_2 $(B = \Delta)$ + B_1 $(B = N)$	4470
A_3 $(B = N)$	4415
Altarelli and Rubinstein	4409
R_1 $(B = N)$ + B_1 $(B = N)$	4355
Nicholas	4213

FIG. 3



Quark duality diagram for the reaction $K^-p \rightarrow \pi^+\pi^-\Lambda$ showing the "illegal" diagrams (a and b) used by Petersson and Törnqvist.

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CHAPTER 5

Four-Point Function Fits to the $\bar{p}n \to 3\pi$ 1.2 GeV in flight Dalitz Plot data

5.1 Introduction

The data of Bettini et al [1] for the process $\bar{p}n \to \pi^+\pi^-\pi^-$ at 1.2 GeV/C is shown as a Dalitz plot of the 818 events in Fig. 1. Each event, as in the at rest case, is plotted twice giving a symmetric plot, although use of the line printer has resulted in some bunching of events. This data also shows a striking pattern of zeros in the experimental plot although the features are not quite the same as those for annihilation at rest. Fig. 2 shows the equal density contours on the plot, as given by Bettini et al, from which it is seen that there are:

(i) two symmetrical zeros, at $M^2_{\pi^+\pi^-_1} = M^2_{\pi^+\pi^-_2} \simeq 1$,
$M^2_{\pi^+\pi^-_1} = M^2_{\pi^+\pi^-_2} \simeq 2$.

other zeros, at $M^2_{\pi^+\pi^-_1} \simeq 1$, $M^2_{\pi^+\pi^-_2} \simeq 3$,
$M^2_{\pi^+\pi^-_1} \simeq 3$, $M^2_{\pi^+\pi^-_2} \simeq 1$

(ii) absence of zeros, at $M^2_{\pi^+\pi^-_1} \simeq 1$, $M^2_{\pi^+\pi^-_2} \simeq 2$,
$M^2_{\pi^+\pi^-_1} \simeq 2$, $M^2_{\pi^+\pi^-_2} \simeq 1$

(iii) symmetric maximum, at $M^2_{\pi^+\pi^-_1} = M^2_{\pi^+\pi^-_2} \simeq \frac{3}{2}$

other maxima, at $M^2_{\pi^+\pi^-_1} \simeq \frac{1}{2}$, $M^2_{\pi^+\pi^-_2} \simeq \frac{7}{2}$

$M^2_{\pi^+\pi^-_1} \simeq \frac{7}{2}$, $M^2_{\pi^+\pi^-_2} \simeq \frac{1}{2}$

Bettini et al [1] attempted to fit their data with two types of Veneziano type amplitudes. Firstly a four point function fit, assuming that the decay was from a $J^P = 2^+$ state, where the amplitude was of the form:

$$A = (\text{factors})\ \frac{\Gamma(2 - \alpha_s)\Gamma(1 - \alpha_t)}{\Gamma(3 - \alpha_s - \alpha_t)} + (s \leftrightarrow t)$$

The trajectory used was found from fitting a straight line along the diagonal on the dip-bump-dip structure of the Dalitz plot to give

$$\alpha_s = 0.65 + 0.84S + 0.26\ i\sqrt{S - 4m^2}\ .$$

Their resultant fit to the data did not give the bumps at the ends of the ρ-bands and gave only a rough qualitative agreement over the Dalitz plot.

Secondly a five point function amplitude was suggested, which included only normal parity states in the $\bar{p}n$ channel, ruled out various external line permutations and neglected the nucleon spins. Labelling the particles $\bar{p}_5 n_4 \rightarrow \pi_3^- \pi_2^+ \pi_1^-$ they took

$$A = (\text{factors})\ B_5\left(2 - \alpha_\rho^{45},\ \frac{3}{2} - \alpha_{15}^{N},\ 1 - \alpha_\rho^{12},\ 1 - \alpha_\rho^{23},\ \frac{3}{2} - \alpha_N^{34}\right) - (1 \leftrightarrow 3)$$

and found results similar to those obtained by the four-point function fit.

Odorico [2] noted the failure of the four-point and five-point functions suggested by Bettini et al to explain fully the Dalitz plot data and in particular that they failed to give the prominent hills present at the corners of the plot. Also they failed to explain the fact that when the holes are present they are present alternately only. Pointing to the fact that to fit the data at rest one required several terms of four-point functions in order to reproduce the "hole" at $\alpha_s = \alpha_t = \frac{3}{2}$ he suggested that for the in flight case many such terms might be required for the more complicated Dalitz plot. Specifically Odorico proposed an amplitude of the form

$$A(s,t) = \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)\Gamma\left(\frac{\alpha_s+\alpha_t}{2}\right)}{\Gamma\left(\frac{\alpha_s-\alpha_t}{2}\right)\Gamma\left(\frac{\alpha_t-\alpha_s}{2}+1\right)\Gamma\left(\frac{3-\alpha_s-\alpha_t}{2}\right)} \qquad (51.1)$$

with $\alpha_x = \frac{1}{2} + x$.

This amplitude explicitly gives zeros at $\alpha_s - \alpha_t = 2m$ and removes them for $\alpha_s + \alpha_t = 2n$ (m and n arbitrary integers) so that an alternate presence of holes is automatically incorporated into the expression for the amplitude. The amplitude is Regge behaved, crossing symmetric and has straight line behaving zeros. However it implies the existence of exotic meson resonances with I = 2 (in the u-channel), alternating signs of the residues of successive towers of poles and that increasing $m^2(\bar{p}n)$ increases the mass of the u-channel first resonance position. What Odorico had observed was that, near a pole in s and a pole in t, the amplitude could be written

$$\frac{a}{s-m_1^{\,2}} + \frac{b}{t-m_2^{\,2}} = \frac{1}{(s - m_1^{\,2})(t - m_2^{\,2})} \times$$

$$\times \left[\frac{1}{2}\,(a+b)(t+s-m_1^{\,2}-m_2^{\,2}) + \frac{1}{2}\,(a-b)(t-s+m_1^{\,2}-m_2^{\,2})\right].$$

So that if a = b, the square bracket would generate a line of zeros at constant u, while if a = -b, the line of zeros would be at fixed (s-t). The former occurred with a simple Veneziano type model whereas the latter appeared to agree better with the in flight annihilation data. Writing Odorico's amplitude in the form

$$A(s,t) = \frac{2^{1-\alpha_s-\alpha_t}}{\sqrt{\pi}} \cdot \frac{\sin\frac{\pi}{2}(\alpha_s-\alpha_t)}{\sin\frac{\pi}{2}(\alpha_s+\alpha_t)} \cdot \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(2-\alpha_s-\alpha_t)}$$

(by using the 'duplication' formula) shows that the modification to the Veneziano amplitude used to obtain zeros at fixed (s-t) was just to multiply it by a suitable factor of s and t. As a phenomenological realization it is not clear that this type of amplitude is required to fit the data but the suggestion of having lines of zeros at fixed (s-t), rather than fixed u, was certainly interesting. (Odorico has also looked for fixed u structure in other reactions). Fig. 3 shows the pattern of zeros and poles in both the Veneziano and the Odorico formulae, and Fig. 4 shows the pattern on the Dalitz plot.

Bugrij, Jenkovski and Kobylinski [3] suggested that the most economic amplitude of the Veneziano type giving an absence of zeros at the required points corresponding to $\alpha_s + \alpha_t = 4$ was the form

$$A(s,t) = (3 - \alpha_s - \alpha_t)V_{11} + C(3 - \alpha_s - \alpha_t)^2 V_{32} \quad .$$

with $$V_{nm} = \frac{\Gamma(n - \alpha_s)\Gamma(n - \alpha_t)}{\Gamma(m + n - \alpha_s - \alpha_t)}$$

or

$$A(s,t) = V_{10} + 2V_{11} + C(V_{30} - V_{31} + V_{32}) \tag{51.2}$$

where $\alpha_x = 0.483 + 0.885x + 0.28i\sqrt{x - 4m^2}$ from Lovelace [4]. Like the Odorico amplitude of (51.1) this was equivalent to multiplying the Veneziano-type amplitude by a rational function of α_s and α_t. This form of the amplitude follows that of (31.3) as given by Altarelli and Rubinstein [5] for the at rest data. Bugrij et al made a fit to the experimental distribution and found C to be -1.44. They did not make it clear how such a fit was made and by changing C their fit could in fact be improved. Both the mass distribution and the Dalitz plot are not fitted well with their amplitude. Even when they attempted using a dual amplitude with Mandelstam analyticity (DAMA) the resultant fit to the Dalitz plot was wildly out.

One might say that what is really needed is a fully dual five-point function amplitude (with spin and isospin taken into account) that would fit the in flight $\bar{p}n \rightarrow \pi^-\pi^-\pi^+$ data, would suitably extrapolate to the data at rest, also reproducing the four-point function amplitude results of Chapter 3, and would describe by crossing, $\pi^-p \rightarrow \pi^+\pi^- n$ and the other $\pi N \rightarrow \pi\pi N$ processes. However, the lack of quantitative agreement by the existing five-point functions to fit the at rest data suggests that this would not be a simple task. There is the possibility that the differences in the two Dalitz plots are indicative of important dynamical effects in the initial $N\bar{N}$ state which might mitigate against such a treatment [6].

The fact that the Dalitz plots for both the at rest and in-flight cases have pronounced minima and maxima suggests that one might extend the Lovelace method to the in-flight data. Since the $\bar{p}n$ is no longer at rest it can no longer be asserted that a 'heavy pion' adequately represents the initial state quantum numbers, nor that the pion-trajectory dominates the direct channel [7].

5.2 Four-point function fit

In making a four point function fit one could follow the method of Bettini et al [1] and use sums of terms each of which were appropriate for a particular $\pi\pi \to \pi S$ process where S has arbitrary spin and parity. Alternatively a sum of four-point functions could be used with individual terms of the form

$$\frac{\Gamma(\ell - \alpha_s)\Gamma(m - \alpha_t)}{\Gamma(n - \alpha_s - \alpha_t)}$$

In our fit, however, it was decided to use the same form for the amplitude as had been used in the at rest case. This allowed a comparison with the at rest fit and also with that of Bugrij et al [3].

The amplitude expression was taken to be

$$A(s,t) = \sum_{\substack{n=1 \\ m \leqslant n}} C_{nm} V_{nm} \tag{52.1}$$

with

$$V_{nm} = \frac{\Gamma(n - \alpha_s)\Gamma(n - \alpha_t)}{\Gamma(m + n - \alpha_s - \alpha_t)}$$

and

$$\alpha_x = 0.483 + 0.885x + 0.33\, i \sqrt{x - 4m^2}$$

(the trajectory used for the at rest case).

The fit to the data was performed by maximizing $\mathcal{L}$ in the expression

$$L = \prod_{i=1}^{N} |F(s_i,t_i)|^2 \quad \text{where } \mathcal{L} = \ln L, \quad N = 818$$

and

$$F(s_i, t_i) = \frac{|A\ s_i,t_i)|^2}{\int_\sigma |A(s,t)|^2 ds dt}$$

and where the data points (s_i, t_i) refer to the $M^2_{\pi^+\pi^-_1}$, $M^2_{\pi^+\pi^-_2}$ Dalitz plot events given by Bettini et al for 1.2 GeV/C incident momenta and the integration is taken over this new plot. $\mathcal{L}$ was then maximised, as for the previous cases, by applying the CERN routines MINUETS for $-\mathcal{L}$ and ZFACT for the Gamma functions. The 95% and 99% confidence intervals on the coefficients C_{nm} imply changes of the order of $\frac{\lambda^2}{2}$ in $\mathcal{L}$ where λ is 1.96 and 2.576 respectively and this allowed terms that did not change $\mathcal{L}$ by more than these amounts (about 2 or 3.3 for the two cases) to be discarded from the series. Proceeding in this manner the fit seemed to have approximately the simple form of:

$$A(s,t) \simeq V_{11} - V_{20} + 2(V_{22} - V_{30}) + V_{32}.$$

A practical difficulty with the minimization routine was that as further terms were added in it tended to neglect these in preference to the earlier ones, particularly if the former were only having a small effect on $\mathcal{L}$. In both this and the at rest case therefore there could be higher terms to the series giving a small effect on $\mathcal{L}$. The actual coefficients are given in Table 1 together with the $\mathcal{L}$ values of Odorico and Bugrij et al. These results indicate that the Odorico amplitude does not fit the data as well as the sum of terms but that, like the Lovelace amplitude for the data at rest, it could be one of several similar terms which when combined could do so. The Bugrij et al suggestion of only one free parameter was unduly restrictive and even when this was fitted the $\mathcal{L}$ value, although better than Odorico's, showed that extra terms were required.

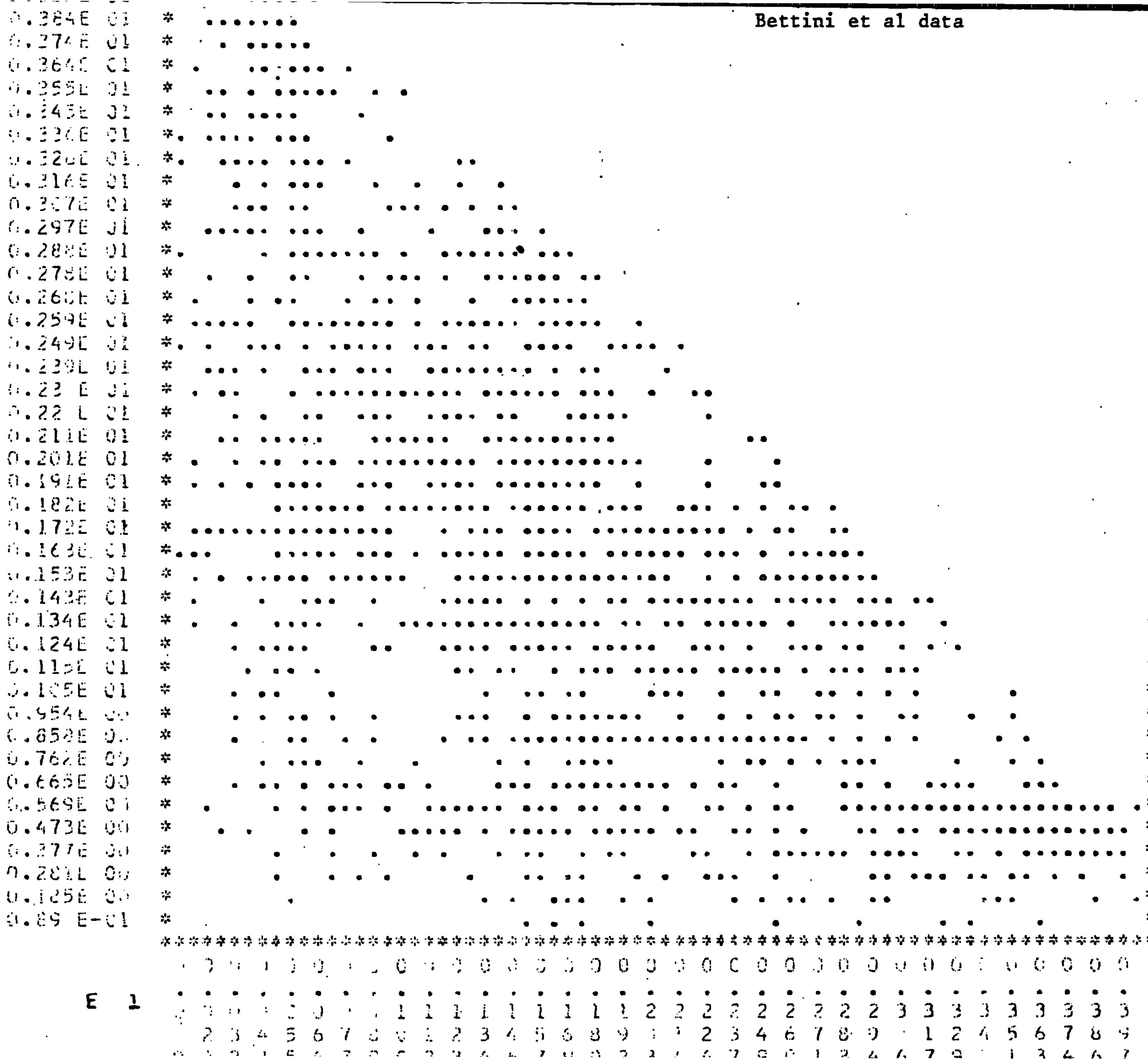
Summary.

The amplitude expression (52.1) gives rise to straight line zeros(assuming α_s and α_t are real) for $\alpha_s + \alpha_t > \max (m+n)$ so that if we wish to preserve this property over the Dalitz plot then we should impose the condition $m+n \leq 5$ in the same spirit as the restriction $m+n \leq 3$ noted in chapter three that was used for the decay (at rest) case. Although the data appears to suggest an Odorico-type pattern of zeros we have nevertheless fitted it with a simple pattern of Veneziano-type amplitudes. We have not performed a fit using a combination of Veneziano and Odorico-type terms although this may have indicated which pattern of zeros the data dictated. The addition of the imaginary part to the trajectory function meant that the lines of zeros were not simply extractable unless for example we neglected these imaginary parts in such considerations.

It might be thought that arguments based on simple four point functions should not be relevant here but quite surprisingly the pattern of zeros seems to exhibit the striking form suggested by Veneziano (or Odorico)-type amplitudes.

We conclude by reiterating that perhaps a suitable five point dual function fit should be made to the data such that the at-rest case is fitted as a particular example of the initial energy.

YMAX= 0.4128 E 01
YMIN= 0.8900 E-01



XMAX= 0.4028 E 01
XMIN= 0.8900 E-01

for $\bar{p}n \rightarrow \pi^-\pi^-\pi^+$, 1.2 GeV/C

Bettini et al (1970)

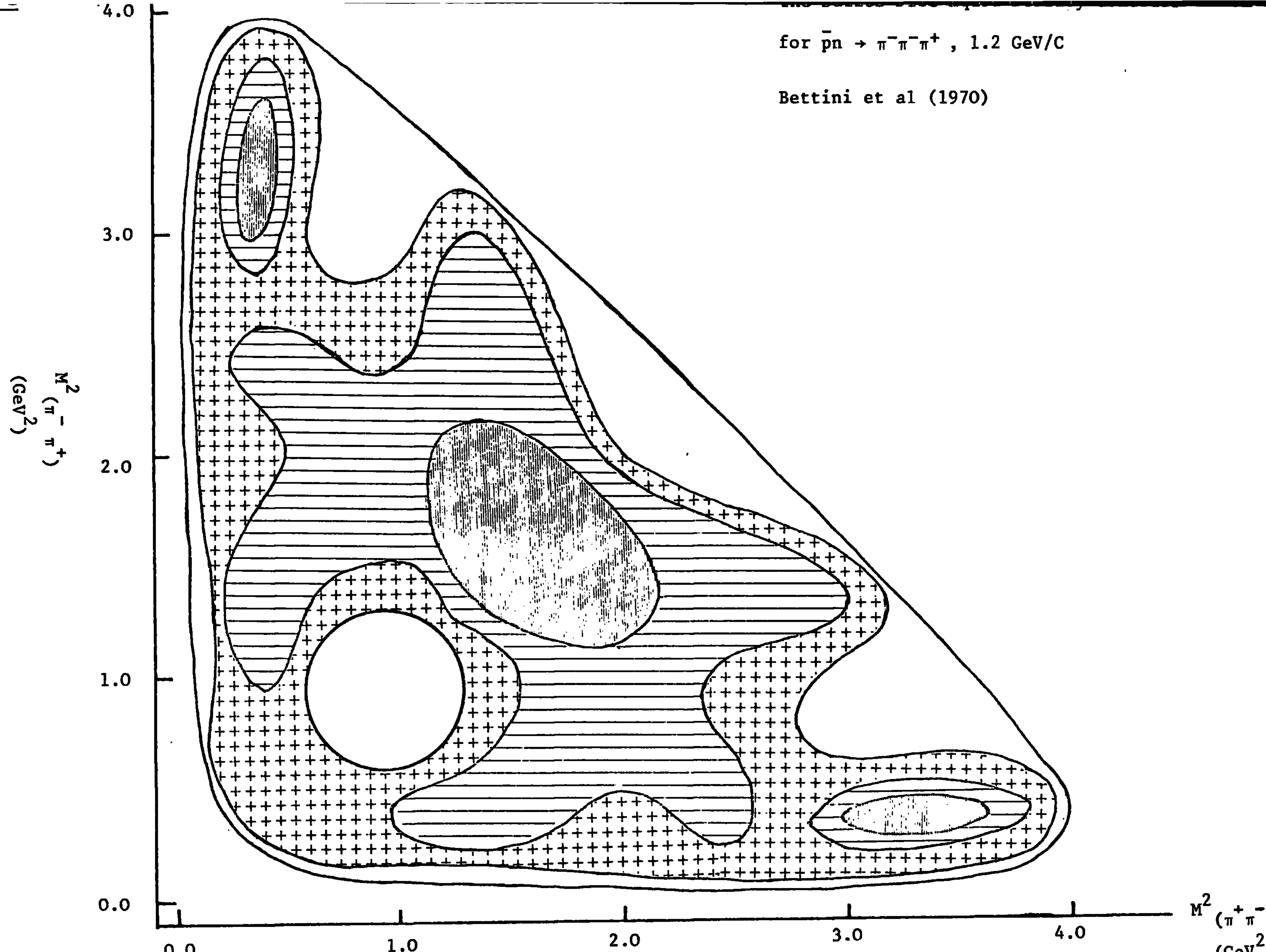
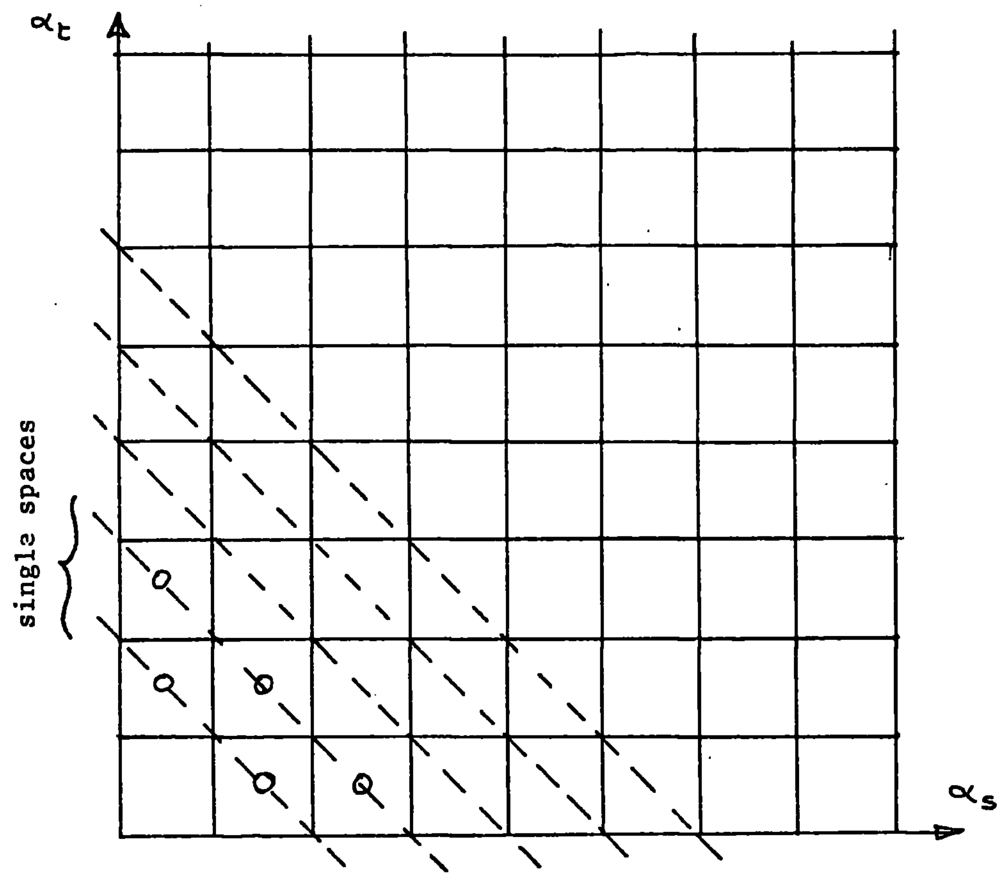


Fig. 3

Veneziano zeros and poles pattern



Veneziano zeros are due to the denominator of $\dfrac{\Gamma(1 - \alpha_s)\Gamma(1 - \alpha_t)}{\Gamma(2 - \alpha_s - \alpha_t)}$

which also removes double poles. Odorico zeros are given at $\alpha_s - \alpha_t = 2m$, and removed at $\alpha_s + \alpha_t = 2n$.

Odorico zeros and poles pattern

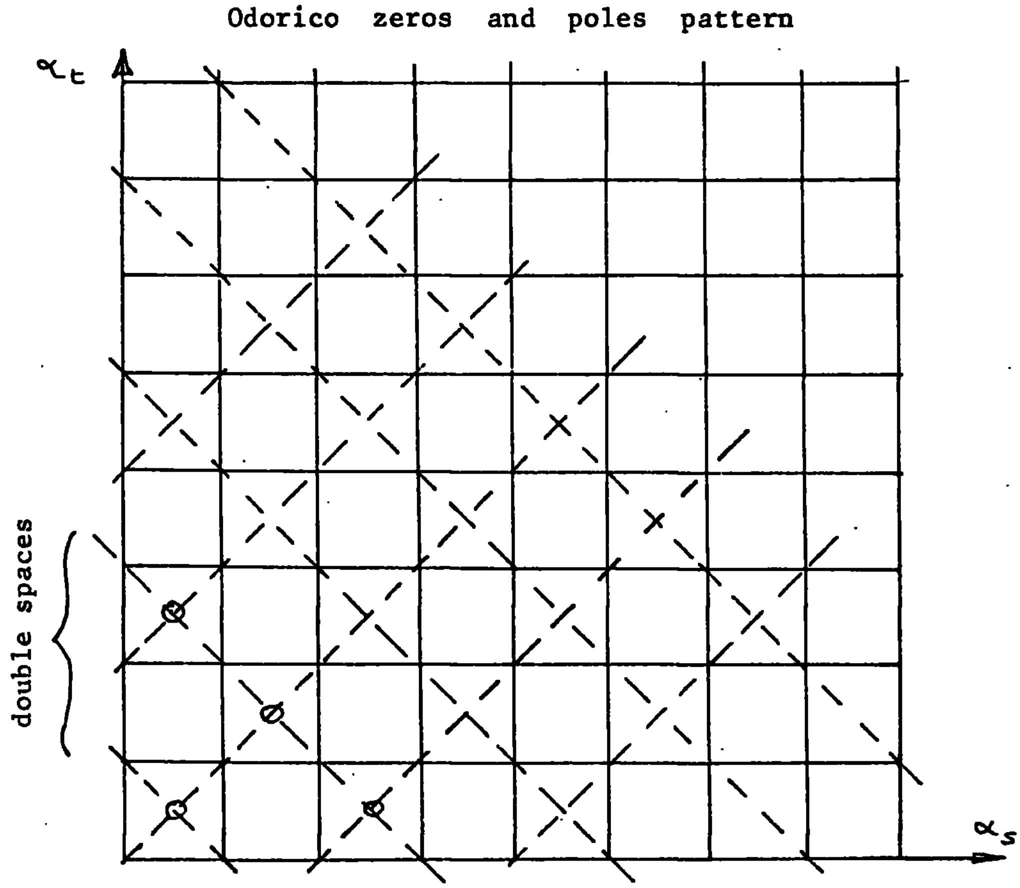
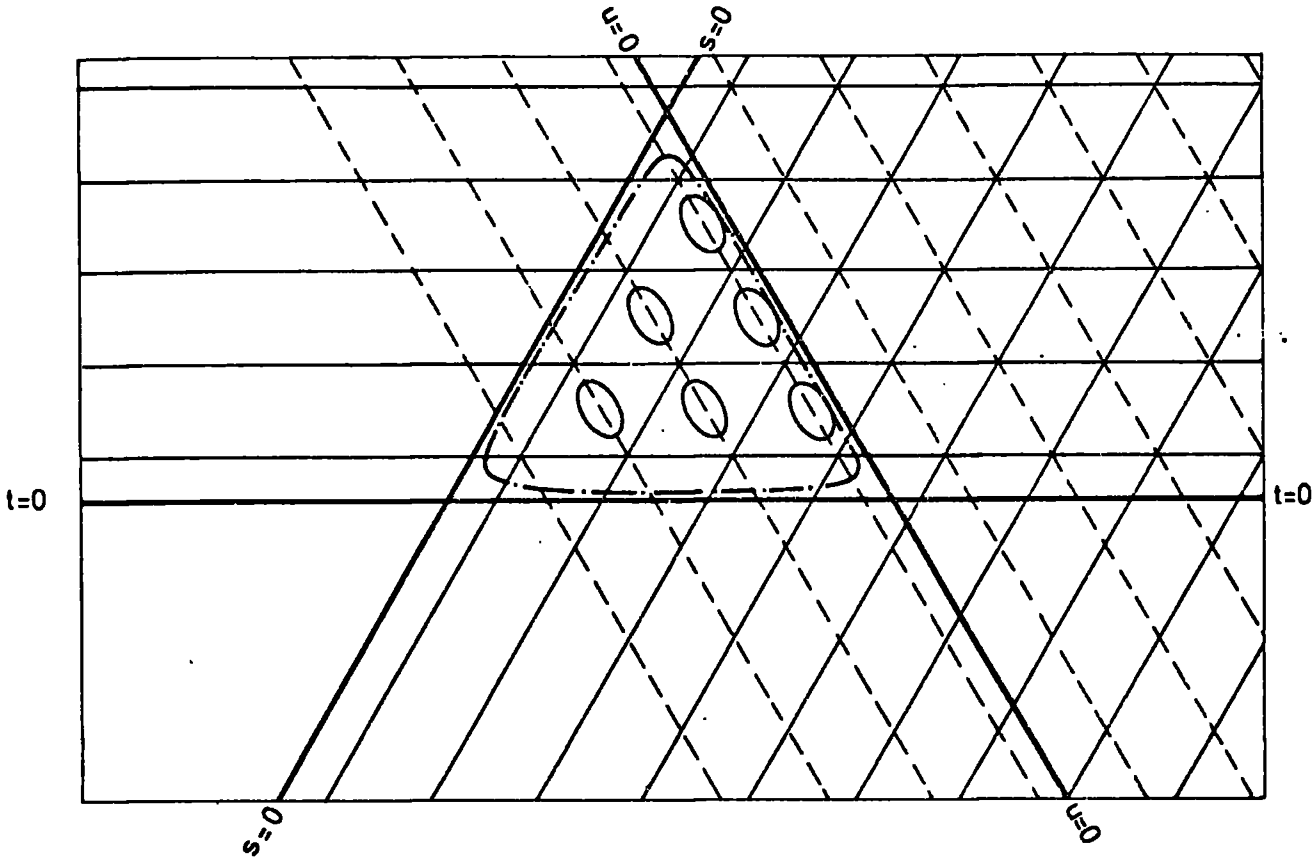


Fig. 4 The zeros and poles pattern in the Mandelstam plane for: a) The Veneziano formula



b) The Odorico formula

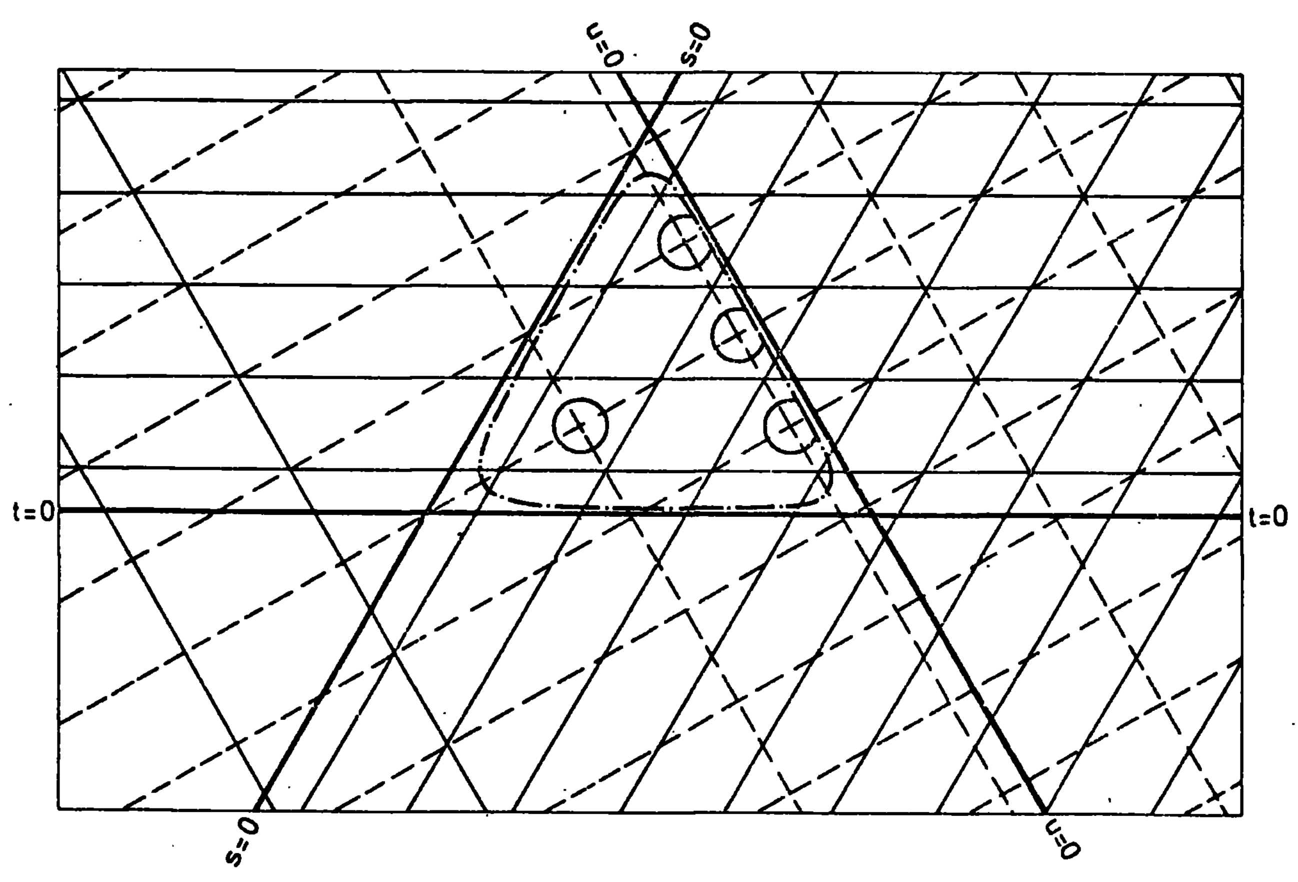


Table 1

Values of the log likelihood function $\mathcal{L}$

Cnm values used	$-\mathcal{L}$
$C_{10} = -0.131$ $C_{11} = 1$ $C_{20} = -0.915$ $C_{21} = -0.280$ $C_{22} = 1.826$ $C_{30} = -2.005$ $C_{31} = 0.979$ $C_{32} = 0.870$ Putting $C_{40} = 0$	1740
$C_{10} = -0.129$ $C_{11} = 1$ $C_{20} = -0.787$ $C_{22} = 1.859$ $C_{30} = -2.238$ $C_{32} = 0.878$ Putting $C_{21} = C_{31} = C_{40} = 0$.	1741
$C_{10} = -0.130$ $C_{11} = 1$ $C_{20} = -0.847$ $C_{22} = 1.862$ $C_{30} = -2.084$ Putting $C_{21} = C_{31} = C_{32} = C_{40} = 0$	1743
Odorico Odorico (Lovelace trajectory)	2766 2685
Bugrij et. al. $C_{10} = 1$ $C_{11} = 2$ $C_{20} = C_{21} = C_{22} = 0$ $C_{30} = -C_{31} = C_{32} = -19.19$	2200

CHAPTER 5 REFERENCES

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