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by

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CONTENTS

ABSTRACT

In chapter I we give an account of the important theorems and results needed in the subsequent work, we also include all the references from which a general point of view can be obtained. Chapter II is a continuation of Chapter I, where we consider only one specific concept Lie groups and homogeneous spaces.

Chapter III deals with Riemannian (locally) symmetric manifolds. The theorems and results in this chapter are included with their proofs, since both are very relevant for the work in the coming chapters.

The main original contributions of this thesis are presented in Chapters IV and V. In Chapter IV, Riemannian s-manifolds, and Riemannian k-symmetric spaces, in the sense of A.J. Ledger, are defined. We also define Riemannian s-regular manifolds, and Riemannian k-regular symmetric spaces. We discuss in detail the case when k is an odd positive integer, and we establish some results concerning this case. The whole of Chapter V is concerned with Riemannian (locally) 5 - (regular) symmetric manifolds. Our treatment of these manifolds is in some way similar to that adopted by Gray [8] for 3 - (regular) symmetric manifolds. We will also show that Riemannian (locally) 5 - (regular) symmetric manifolds diverge from Riemmannian (locally) 3 - (regular) symmetric manifolds.

Finally, the appendix contains calculations needed in Chapter V, section 5.3.

CHAPTER I

Basic Definitions and Fundamental Results

This chapter deals with the basic geometric properties of a manifold. These properties are very important in the next chapers, and they are put in the form required for the subsequent work. However, each section will include references in which generalizations of these properties may be found.

1.1. Manifolds:-

Definition 1.1.1. Let M be a Hausdorff topological space. The pair (U,ϕ) is an open chart, or a co-ordinate neighbourhood of M, if U is an open subset of M, and ϕ is a homeomorphism of U into $\mathbb{R}^n$.

Definition 1.1.2. A differentiable manifold M, with a differentiable structure of class C^r is a Hausdorff space with a collection of open charts (U_α, ϕ_α), $\alpha \in A$, where A is an index set, such that the following properties are satisfied.

(a) U_α covers M.

(b) The mapping $f_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ of $\phi_\alpha(U_\alpha \cap U_\beta)$ onto $\phi_\beta(U_\alpha \cap U_\beta)$ is differentiable of class C^r for all $\alpha, \beta \in A$.

(c) The collection (U_α, ϕ_α), $\alpha \in A$ is a maximal family of open charts such that (a) and (b) hold.

The dimension of M is n, i.e. the same as the dimension of $\mathbb{R}^n$. The mapping $f_{\alpha\beta}$ is a diffeomorphism of $\phi_\alpha(U_\alpha \cap U_\beta)$ onto $\phi_\beta(U_\alpha \cap U_\beta)$. M is said to be analytic if $f_{\alpha\beta}$ is analytic. We shall only consider manifolds of class C^∞, therefore, unless otherwise stated, all manifolds are of class C^∞. If $p \in U_\alpha$, then $\phi_\alpha(p) \in \mathbb{R}^n$, and so it is an n-tuple of real numbers. Let the j^{th} slot be $x^j(p)$, then the n-tuple $(x^1, \ldots, x^n)$ of real-valved functions on U_α is called the local co-ordinate system on (U_α, ϕ_α).

Denote by $C^{\infty}(x,M)$ the algebra of all real-valued functions of class C^{∞} whose domains include a neighbourhood of the point $x \in M$, while we denote by $C^{\infty}(M)$, the algebra of all real-valved functions on M. Consider the real-valued function

$$X_x : C^{\infty}(x,M) \longrightarrow \mathbb{R}$$

satisfying

(1) $X_x(af+bg) = aX_x(f) + bX_x(g)$,

(2) $X_x(fg) = (X_x f)(g(x)) + (f(x))(X_x g)$

for all f, $g \in C^{\infty}(x,M)$, and all a, $b \in \mathbb{R}$. X_x is called a tangent vector at x. At each point $x \in M$, the tangent vectors form a vector space over $\mathbb{R}$, denoted by M_x.

<u>Theorem 1.1.1.</u> Let M be an n-dimensional manifold, and let $\{x^i\}$ $(i = 1, \ldots, n)$ be a local co-ordinate system about a point $x \in M$. Then if $X \in M_x$, $X = (X_x x^i)(\frac{\partial}{\partial x^i})_x$ (We use the Einstien summation convention), and the co-ordinate vectors $(\frac{\partial}{\partial x^i})_x$ form a basis for M_x, which thus has dimension n.

<u>Proof</u>:- See Hicks [12] page 7.

If at each point $p \in M$, we pick a tangent vector $X_p \in M_p$, then the correspondence $X : p \longrightarrow X_p$ is called a <u>vector field</u> on M. X is <u>differentiable</u> if $Xf \in C^{\infty}(M)$ for all $f \in C^{\infty}(M)$, where $(Xf)(p) = X_p f$. Denote by $\mathfrak{X}(M)$ the set of all differentiable vector fields on M, it forms a real Lie algebra with bracket defined by

$$[X,Y](f) = X(Yf) - Y(Xf) \quad ; \quad X,Y \in \mathfrak{X}(M) \text{ and } f \in C^{\infty}(M)$$

A <u>covector</u> at a point $x \in M$, is a vector ω_x which belongs to the dual space M_x^* of M_x. Similar to vector fields on M, a 1-form is an assignment of a covector to each point of M. In local co-ordinates system about x, every one form ω can be uniquely written as

$$\omega = f_i \, dx^i \quad ; \quad i = 1, \ldots, n$$

where f_i are functions defined on a neighbourhood of x, and $\left\{dx^i\right\}$ are the duals of $\left\{\frac{\partial}{\partial x^i}\right\}$ for all $i = 1,\ldots,n$. ω is differentiable if $f_i \in C^{\infty}(M)$ Denote by D(M) the set of differentiable 1-forms . We shall consider only differentiable vector fields, and differentiable 1-forms.

The union of all tangent spaces M_x as x varies on M is called the <u>tangent bundle</u> of M, and is denoted by TM. The map π: TM $\longrightarrow$ M, defined by $\pi X = x$ if and only if $X \in M_x$, defines a projection from TM onto M. Similarly, we define the cotangent bundle TM*.

Let f: M $\longrightarrow$ N, where M and N are m and n - dimensional manifolds respectively. Let $p \in M$, and $p' \in N$ be such that $p' = f(p)$, then f is said to be of class C^{∞} at p, if for any open charts (U, ϕ_α), and (V_β, ψ_β) of p and p′ respectively, we have the map $F = \psi_\beta \circ f \circ \phi_\alpha^{-1} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is C^{∞} at x_o, where $x_o = \phi_\alpha(p)$ · f is said to be C^{∞}, if it is C^{∞} at all points of M. We shall consider only C^{∞} - maps.

For each $p \in M$, f induces a linear transformation of M_p into $N_{f(p)}$, called the <u>derived linear function on M_p</u>, and is denoted by $(df)_p$. If $X \in M_p$, define $(df)_p X$ to be the vector in $N_{f(p)}$ such that if $h \in C^{\infty}(N)$ then $((df)_p(X))(h) = X_p(hof)$, where $hof \in C^{\infty}(M)$. In local co-ordinates system, the action of $(df)_p$ is determined by

$$\left(\frac{\partial}{\partial x^i}\right)_p \longrightarrow \left(\frac{\partial(y^j of)}{\partial x^i}\right)_p \left(\frac{\partial}{\partial y^j}\right)_{f(p)}$$

where $\{x^i\}$, $(i = 1,\ldots,m)$ and $\{y^j\}$, $(j = 1,\ldots,n)$ are co-ordinate systems of U_α and V_β respectively. If $g: M \longrightarrow N$ and $f: N \longrightarrow H$, then we have

$$(d(fog))p = (df)_{g(p)} \, o(dg)_p$$

where $p \in \left\{ \text{domain of fog} \right\}$ (Cf Brickell and Clark [3], Chapter IV page 57).

A map f: $M \to N$, where M and N are m and n-dimensional manifolds, determines a map df: $TM \to TN$, defined by $X \mapsto (df)_p X$, where $\pi X = p$. df is called the differential of f. A vector field $X \in \mathfrak{X}(M)$ is f-related to a vector field $X' \in \mathfrak{X}(N)$ if $(df)_x X_x = X'_{f(x)}$, for all $x \in M$. If X and Y in $\mathfrak{X}(M)$ are f-related to X' and $Y' \in \mathfrak{X}(N)$, then $[X,Y]$ is f-related to $[X',Y']$.

Let $\sigma: E \to M$ be C^∞, where $E \subset \mathbb{R}$ is open containing $[a,b]$, then the restriction of σ to $[a,b]$ is said to be a C^∞ curve. Let $t \in [a,b]$, and consider $(d\sigma)_t\, ({}^d/_{dt})_t = T(t)$, then T(t) is a tangent vector to σ at $\sigma(t)$. σ is an integral curve to a vector field $X \in \mathfrak{X}(M)$, whenever σ is in the domain of X, and X is tanget to σ for all $t \in [a,b]$.

A homoomorphism f: $M \to N$, such that f and f^{-1} are both C^∞, is called a diffeomorphism. If M = N, then f is called a transformation of M. If X and Y are in $\mathfrak{X}(M)$ and f: $M \to N$ is a diffeomorphism, then (df)X and (df)Y are in $\mathfrak{X}(N)$ with $(df)[X,Y] = [(df)X, (df)Y]$.

The mapping $\mathbb{R} \times M \to M$; $(s,x) \mapsto \phi_s(x)$ such that

(1) $\phi_s : M \to M$ is a transformation of M, for all $s \in \mathbb{R}$,

(2) $\phi_{s+t}(x) = \phi_s(\phi_t(x))$ for all s, $t \in \mathbb{R}$, and all $x \in M$, is called a 1-parameter group of transformation. Each 1-parameter group of transformation induces a vector field X on M, where if $p \in M$, the curve $\phi_s(p)$ (called the orbit of p, and $\phi_0(p) = p$) is an integral curve to X.

A local 1-parameter group of local transformations can be defined in the same way, except that $\phi_t(x)$ is defined only for t in a neighbourhood of o, and x in an open set of M. Conversely, let $X \in \mathfrak{X}(M)$, and $x \in M$, there exist a neighbourhood V of x in M and a 1-parameter group of transformation $\phi_s : V \to M$ such that $|s| < \epsilon$, for some positive ϵ, and this 1- parameter group induces X. (Cf. Kobayashi and Nomizu Vol. I [13], page 12).

1.2 Affine Connections:-

Definition 1.2.1. Let M be an n-dimensional manifold, the map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$, defined by $(X,Y) \longmapsto \nabla_X Y$

is called an affine connection, or covariant differentiation, if it satisfies the following properties

(i) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$

(ii) $\nabla_{fX} Y = f\nabla_X Y$

(iii) $\nabla_{(X+W)} Y = \nabla_X Y + \nabla_W Y$

(iv) $\nabla_X(fY) = (Xf)(Y) + f\nabla_X Y$

for all $X, Y, Z, W \in \mathfrak{X}(M)$, and all $f \in C^{\infty}(M)$

The symbol (M,∇) is taken to mean that we are given a manifold M with affine connection ∇ .

A vector field X along a C^{∞} curve σ in M is said to be parallel if $\nabla_T X \equiv 0$, where T is the tangent vector field to σ. If $\nabla_T T \equiv 0$, i.e. T is parallel along σ, then σ is said to be a geodesic.

Given a differentiable 1-form ω, we define $\nabla_X \omega$, $X \in \mathfrak{X}(M)$, to be the 1-form such that $(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y)$, for all $Y \in \mathfrak{X}(M)$

Proposition 1.2.1. Let σ be a curve in M, and suppose that $X_{\sigma(c)} \in M_{\sigma(c)}$ for some $c \in [a,b]$, then there exists a unique parallel vector field X(t) along σ such that $X(c) = X_{\sigma(c)}$. If $d \in [a,b]$, then correspondence $M_{\sigma(c)} \longrightarrow M_{\sigma(d)}$ given by $X_{\sigma(c)} \longrightarrow X_{\sigma(d)}$ is called parallel translation along σ from c to d.

Proof:- See Willmore [23] page 209.

Proposition 1.2.2. Let M be an n-dimensional manifold, and let $p \in M$.

Then for every $X \in M_p$, there exists an $\epsilon > 0$, and a unique geodesic σ_X defined on $[-\epsilon, \epsilon]$ such that $\sigma(0) = p$ and $\dot{\sigma}(0) = X$.

Proof:- See Hicks [12] page 58.

An affine mapping from (M, ∇) to $(\overline{M}, \overline{\nabla})$ is a diffeomorphism $f: M \to M$ such that $(df)(\nabla_X Y) = \overline{\nabla}_{df(X)} (df)Y$, for all $X, Y \in \mathcal{X}(M)$.

If $M = \overline{M}$, then f is called an affine transformation. A parallel vector field X(t) along a curve σ in M is mapped under f to a parallel vector field (df)(X(t)) along the curve $f(\sigma)$ in $\overline{M}$. In particular f maps geodesics to geodesics.

Let M be an n-dimensional manifold, and let $p \in M$, suppose that σ_X is the unique geodesic such that $\dot{\sigma}(0) = X \in M_p$ (proposition 1.2.2.). Define $\exp_p X = \sigma_X(1)$, when $\sigma_X(1)$ is defined. $\exp_p$ is called the exponential map. From the definition we see that at each point $p \in M$, M_p has a subset H_p for which the geodesics $\sigma_Y(1)$ are defined, for all $Y \in H_p$.

Proposition 1.2.3. Let M be an n-dimensional manifold, and let $p \in M$. Then there exists a neighbourhood V of $\underline{o}$ in M_p such that $\exp_p$ maps V diffeomorphically onto a neighbourhood of p in M.

Proof:- See Wolf [24] Chap. I page 22.

Let $x \in M$ and U be a neighbourhood of x in M, then U is called a normal neighbourhood if $\exp_x(V) = U$, where V is an open neighbourhood of $\underline{o}$ in M , and such that $\exp_x : V \to U$ is a diffeomorphism. Proposition 1.2.3. says that every point $x \in M$ has a normal neighbourhood. Let $\{e_i\}$, $i = 1, \dots, n$ be a basis for M_x, and let $\{e^i\}$ be the dual basis, and define $x^1, \dots, x^n$ $C^\infty(x, M)$ by $x^i = e^i \circ \exp^{-1}$ for all i. The functions $x^1, \dots, x^n$ are called a normal co-ordinate system. Let $y \in U$ be an arbitrary point, then y can be joined to x by a unique geodesic, and this geodesic is given by σ_X, where $\exp_x X = y$. An important theorem due to J. Whitehead states that if we are given a (M, ∇), and $p \in M$ is any point, then there exists a neighbourhood U of p in M such that any two points in U can be joined by a unique geodesic, this means that U is normal at each of its points, such a neighbourhood is called convex.

Definition 1.2.2. Let U be a normal neighbourhood of a point $p \in M$, and let $q \in U$, consider the geodesic $\sigma(t)$ within U such that $\sigma(o) = p$ and $\sigma(1) = q$. Put $\sigma(-1) = q'$. The mapping $q \mapsto q'$ of U onto itself is called a geodesic symmetry with respect to p, and it is denoted by s_p.

1.3 Tensors and Tensor Fields:-

Let $p \in M$, then M_p is an n-dimensional vector space over $\mathbb{R}$. Consider the real-valued bilinear maps defined on $M_p^* \times M_p^*$, these maps form a vector space over $\mathbb{R}$ called the tensor product of M_p with itself, and it is denoted by $M_p \otimes M_p$. The dual of any basis $\{e_i\}$, $i = 1, \ldots, n$ of M_p determines a unique basis for $M_p \otimes M_p$.

The set of linear maps $L : M_p^* \longrightarrow M_p$ form a vector space over $\mathbb{R}$, denoted by $L(M_p^*, M_p)$, which is naturally isomorphic (i.e. independent of particular basis) to $M_p \otimes M_p$, therefore each element of $M_p \otimes M_p$ can be identified with a linear map of M_p^* into M_p. Furthermore, the two vector spaces $(M_p \otimes M_p) \otimes M_p$, and $M_p \otimes (M_p \otimes M_p)$ are naturally isomorphic, and there will be no confusion if they are identified with the symbol $M_p \otimes M_p \otimes M_p$.

The set of all real-valued trilinear maps defined of $M_p^* \times M_p^* \times M_p^*$ form a vector space over $\mathbb{R}$ which is naturally isomorphic to $M_p \otimes M_p \otimes M_p$

In this way, the vector space of all real-valued r-linear maps defined over $M_p^* \times \ldots \times M_p^*$ (r-times) is naturally isomorphic to the tensor product

$M_p \otimes \ldots \otimes M_p$ (r-times). Denote by $T_o^{\ r}(M_p)$ the vector space of the tensor product $M_p \otimes \ldots \otimes M_p$ (r-times). An element of $T_o^{\ r}(M_p)$ is said to be of type (r, o). Similarly, elements of $T_s^{\ o}(M\ ) = M_p^* \otimes \ldots \otimes M_p^*$ (s-times) are called tensors of type (o, s) at p. More generally, the elements of $T_o^{\ r}(M_p) \otimes T_s^{\ o}(M_p) \equiv T_s^{\ r}(M_p)$ are called tensors of type (r, s) at p. A tensor of type (r, s) is identified with the multilinear map on $M_p^* \times \ldots \times M_p^* \times M_p \times \ldots \times M_p$ (r- copies of M_p^* and s-copies of M_p). From above we have $T_o^{\ 1}(M_p) = M_p^*$, and $T_1^{\ o}(M_p) = M_p$. We define $T_o^{\ o}(M_p) = \mathbb{R}$. Since $T_s^{\ r}(M_p)$ is a vector space over $\mathbb{R}$, then if A, $B \in T_s^{\ r}(M_p)$, we have $aA + bB\ T_s^{\ r}(M_p)$; $a, b \in \mathbb{R}$. Also if $R \in T_s^{\ r}(M_p)$, and $S \in T_q^{\ t}(M_p)$, we define $R \otimes S \in T_{s+q}^{\ r+t}(M_p)$ (cf. Willmore [23] Chap. V section 3).

Similar to the definition of the tangent bundle, one can define the <u>tensor bundle</u> $T_s^{\ r}M$ as the union of all the vector spaces $T_s^{\ r}(M_p)$ as p vaires over M.

<u>Proposition 1.3.1</u>. $T_r^{\ 1}(M_p)$ is naturally isomorphic to the vector space of all r-linear maps of $M_p \times \ldots \times M_p$ into M_p.

<u>Proof</u>:- See Kobayashi and Nomizu [13] Vol. I. page 23.

It follows that if $A \in T_1^{\ 1}(M_p)$ i.e. if A is a tensor of type (1,1) at p, then A can be regarded as a linear endomorphism of M_p.

Let $\{e_i\}$, $i = 1, \ldots, n$ be a basis for M_p, and let $\{e^i\}$ be their duals, then every tensor $K \in T_s^{\ r}(M_p)$ can be uniquely expressed (using Einstein summation convention) as

$$K = K^{i_1 \ldots i_r}_{j_1 \ldots j_s}\, e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{j_1} \otimes \ldots e^{j_s}$$

where $\{e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{j_1} \otimes \ldots \otimes e^{j_s}\}$ is a basis for $T_s^{\ r}(M_p)$,

$K^{i_1 \ldots i_r}_{j_1 \ldots j_s} \in \mathbb{R}$ are called the components of K with respect to $\{e_i\}$.

An r-form at $p \in M$ is a skew-symmetric element of $T_r^{\,o}(M_p)$. The set of r-forms at $p \in M$ is a subspace of $T_r^{\,o}(M_p)$ of dimension $\binom{n}{r}$, where n is the dimension of M. Denote by $F^r(M_p)$ the space of all r-forms at $p \in M$.

Analogous to the way we defined vector fields on M, we can define tensor fields i.e. at each point $p \in M$, we pick a tensor $A_p \in T_s^{\,r}(M_p)$, then the correspondence $A: p \longrightarrow Ap$ is called a tensor field on M of type (r, s). If $\{x^i\}$, $i = 1, \ldots, n$ is a local co-ordinate system in a neighbourhood of p, put $X_i = \partial/\partial x^i$ as a basis for M_p, and $w^i = dx^i$ are their duals. Then A can be expressed as

$$A = A^{i_1 \ldots i_r}_{j_1 \ldots j_s} \; X_{i_1} \otimes \ldots \otimes X_{i_r} \otimes \omega^{j_1} \otimes \ldots \otimes \omega^{j_s}$$

where $A^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ are real-valued functions on M, called the components of A with respect to $\{x^i\}$, A is said to be differentiable if $A^{i_1 \ldots i_r}_{j_1 \ldots j_s} \in C^\infty(M)$ for all $i_1, \ldots, i_r, j_1, \ldots, j_s$. A differentiable r-form is a tensor field such that at each point of M we have an r-form. Denote by $T_s^{\,r}(M)$ and $F^r(M)$ the vector spaces of differentiable tensor fields and differentiabe r-forms, respectively. We shall consider only elements of $T_s^{\,r}(M)$ and $F^r(M)$.

Proposition 1.3.2. A tensor field K of type (o, r) (respectively of type (1, r)) on a manifold M can be considered as a multilinear map of $\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)$ into $C^\infty(M)$ (respectively $\mathfrak{X}(M)$) such that

$K(f_1X_1, \ldots, f_rX_r) = f_1 \ldots f_r K(X_1, \ldots, X_r)$, for all $f_i \in C^\infty(M)$ and all $X_i \in \mathfrak{X}(M)$. Conversely, any such mapping can be considered as a tensor field of type (o, r) (respectively (1, r)).

Proof:- See Kobayashi and Nomizu [13] Vol. I page 26.

Given a (M, ∇), we define the curvature tensor $R \in T_3^{\,1}(M)$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \qquad ; X, Y, Z \in \mathfrak{X}(M)$$

We also define the <u>torsion tensor</u> $T \in T_2^1(M)$ by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X, Y] \quad ; \qquad X, Y \in \mathfrak{X}(M)$$

Let $T(M) = \bigcup_{r,s \geq 0} T_s^r(M)$, then $T(M)$ is an associative algebra with multiplication $\otimes$, i,e, if $K \in T_s^r(M)$ and $S \in T_l^t(M)$, then $K \otimes S \in T_{s+l}^{r+t}(M)$ is such that if $x \in M$, we have $(K \otimes S)_x = K_x \otimes S_x$

<u>Remark</u>:- Analytic manifolds, analytic maps, analytic vector fields, and analytic tensor fields are defined in a similar way to which differentiable manifolds, differentiable maps, differentiable vector fields, and differentiable tensor fields were defined, e.g. for analytic manifolds, we need the functions $f_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ in definition 1.1.2. to be analytic.

Let $A \in T_s^r(M)$, then $\nabla A \in T_{s+1}^r(M)$ is called the <u>covariant differential of A</u>. It is defined as follows. Let $x \in M$, $A_x \in T_s^r(M_p)$ is considered as a multilinear map of $M_p \times \ldots \times M_p$ (s-times) into $T_o^r(M_p)$. Set

$$(\nabla A)(Y, X_1, \ldots, X_s) = (\nabla_Y A)(X_1, \ldots, X_s) \quad ; \quad Y, X_i \in M_p$$

Now define $\nabla_Y A$ by solving the equation

$$\nabla_Y (A(X_1, \ldots, X_s)) = (\nabla_Y A)(X_1, \ldots, X_s) + \sum_{i=1}^{s} A(X_1, \ldots, \nabla_Y X_i, \ldots, X_s)$$

where $Y, X_i \in \mathfrak{X}(M)$.

A is said to be <u>parallel</u> if and only if $\nabla_X A = o$, for all $X \in \mathfrak{X}(M)$.

<u>Proposition 1.3.3</u>. Given two manifolds (M, ∇), and (M', ∇'). Assume that

$$\nabla_X T = \nabla_X R = \nabla'_{X'} R' = \nabla'_{X'} T' = o, \text{ for all } X \in \mathfrak{X}(M),$$

and all $X' \in \mathfrak{X}(M')$. Let $p \in M$, $p' \in M'$, and suppose that A is a linear 1-1 map of M_p onto $M'_{p'}$. Let $\tilde{A}$ denote the unique type preserving isomorphism of the mixed tensor algebra $T(M_p)$ onto $T(M'_{p'})$, we extend

A such that $\bar{A}$ coincides with $(A^t)^{-1}$ on the dual space M_p^*. Assume that $\bar{A} R_p = R'_{p'}$, and $\bar{A} T_p = T'_{p'}$. Then, there exists an open neighbourhood U of p in M and an affine transformation ϕ of U onto an open neighbourhood U′ of p′ in M′ such that $\phi(p) = p'$ and $(d\phi)_p = A$.

Proof: - See Helgason [10] page 165.

Proposition 1.3.4. Let (M,∇) be an n-dimensional manifold, such that $\nabla T = \nabla R = o$. With respect to the atlas consisting of normal co-ordinate systems, M is an analytic manifold, and the connection is analytic.

Proof: - See Kobayashi and Nomizu [13] Vol I page 263.

1.4 Riemannian Manifolds: -

Definition 1.4.1. Let M be an n-dimensional manifold. Then M is called pseudo-Riemannian if there exists a tensor field $g \in T_2^{\,o}(M)$ on M such that at each point $p \in M$, g_p is a bilinear, non-degenerate, symmetric form of $M_p \times M_p \longrightarrow \mathbb{R}$. If g_p is positive definite, then M is said to be a Riemannian manifold . g is called the metric tensor on M.

Denote by (M, g) a pseudo-Riemannian manifold with a metric g. It is characterised by having a unique affine connection ∇, with two useful properties.

Theorem 1.4.1. (The Fundamental theorem of Riemannian geometry). There exists a unique affine connection on a pseudo-Riemannian manifold with the following properties

(i) $\nabla g = o$, (ii) $T = o$

Proof: - See Wolf [24] Chap I, page 47.

The above mentioned connection is called the Riemannian connection on M, if g is positive definite. When we consider Riemannian manifolds, we will refer to the Riemannian connection (also called Levi-Civita connection).

With the help of the metric tensor, we can define angles between two vectors at each point of M, distance between two points, and length of a curve in M.

The Riemann Christoffel curvature tensor is defined by

$$R(X,Y,Z,W) = g(R(X,Y)Z,W) \quad ; \quad X,Y,Z,W \in \mathfrak{X}(M)$$

It is a tensor of type (o,4), and it has the following properties

$$R(X,Y,Z,W) = R(Z,W,X,Y,) = -R(Y,X,Z,W) = -R(X,Y,W,Z)$$

and $R(X,Y,Z,W) + R(Z,X,Y,W) + R(Y,Z,X,W) = o$

The last property is called the first Bianchi identity. We also have

$$(\nabla_V R)(X,Y,Z,W) = (\nabla_V R)(Z,W,X,Y) = -(\nabla_V R)(Y,X,Z,W)$$

$$= -(\nabla_V R)(X,Y,W,Z) \quad ; \quad V,X,Y,Z,W \in \mathfrak{X}(M)$$

and $(\nabla_V R)(X,Y,Z,W) + (\nabla_W R)(X,Y,V,Z) + (\nabla_Z R)(X,Y,W,V) = o$

where the last property is called the second Bianchi identity.

For all X, $Y \in M_p$, and all $p \in M$, the sectional curvature is defined by

$$K_p = \frac{R_p(X,Y,Y,X,)}{A_p(X,Y)} = \frac{g_p(R_p(X,Y)Y,X)}{A_p(X,Y)}$$

where $o \neq A_p(X,Y) = g_p(X,X)g_p(Y,Y) - (g_p(X,Y))^2$

Definition 1.4.2. Let M and N be two Riemannian manifolds with Riemannian metrics g and h respectively. Let f: $M \longrightarrow N$ be a diffeomorphism of M onto N, then f is called an isometry if for all $X,Y \in M_p$, and $p \in M$, we have

$$g_p(X,Y) = h_{f(p)}((df)_p X,(df)_p Y)$$

f is called a local isometry, if at each point $p \in M$, there exist neighbourhoods U of p, and V of f(p) in M and N respectively, such that f is an isometry of U onto V.

An isometry of (M,g) onto itself is necessarily an affine transformation with respect to the Riemannian connection. It also preserves distances; and the converse is true, i.e. if $f: M \longrightarrow M$ is a distance preserving transformation, then f is an isometry.

Proposition 1.4.2. Let f be an affine transformation of a pseudo-Riemannian manifold M. Suppose that for some point $q \in M$, the map $(df)_q : M_q \longrightarrow M_{f(q)}$ is an isometry. Then f is an isometry of M onto itself.

Proof:- See Helgason [10] Chap. II, page 166.

The following theorem is very useful in subsequent work.

Theorem 1.4.3. Let (M, g) be a Riemannian manifold, and let f and g be two isometrics of M onto itself. Suppose that for some $p \in M$, $f(p) = g(p)$, and $(df)_p = (dg)_p$, then $f = g$ on M.

Proof:- See Helgason [10] Chap. I, page 62.

Definition 1.4.3. Let (M, g) be a Riemannian manifold, and let $p \in M$, an (local) isometry which leaves p as an isolated fixed point, is called a (local) symmetry at p.

For a Riemannian manifold, at each point $p \in M$, there exists a neighbourhood U of p in M, such that if $q \in U$, M_q is spanned by an orthonormal basis, i.e. there exists a basis $X, \ldots, X_n \in M_q$, such that $g_q(X_i, X_j) = \delta_{ij}$ $1 \leq i, j \leq n$, and δ_{ij} is the Kronecker delta.

The Ricci curvature on (M, g) is defined as $s_p(X, Y) = \sum_{i=1}^{n} R_p(X_i, X, X_i, Y)$
$= \sum_{i=1}^{n} g_p(R(X_i, X)X_i, Y)$

where $X, Y \in M_p$, and $\{X_i\}$ $i = 1, \ldots, n$ is an orthonormal basis for M_p.

1.5. Minimal Submanifolds:-

Definition 1.5.1. Let $f: M \longrightarrow N$ be a map, then f is called an immersion if df is injective. f is called an imbedding, if f is an immersion, and f itself is injective.

It follows that for a map to be an immersion is that its rank (dimention of range of $(df)_p$, for all $p \in M$) should be equal to the dimension of M. Locally, we can consider an immersion as an imbedding (away from self intersection).

Definition 1.5.2. Let M be a subset of an n-dimensional manifold N, and let $j: M \longrightarrow N$ be the natural injection. Then M is said to be a submanifold of N, if j is an imbedding.

We must notice that the topology of M is not necessarily the same as the subspace topology, but if they coincide, then M is called a regular submanifold.

Definition 1.5.3. Let M be an n-dimensional manifold. A map D, which assigns to each point $p \in M$ an m-dimensional subspace of M_p, denoted by D_p, in such a way that each $p \in M$, has a neighbourhood U and vector fields $X_1, \ldots, X_m \in \mathcal{X}(U)$, such that D_q is spanned by $\{X_i\}$ $i = 1, \ldots, m$ at q for all $q \in U$, then D is called a smooth m-distribution $(0 \leq m \leq n)$. The vector fields $\{X_i\}$ are called basis for D in U. D is called involutive if $[X, Y] \in D$, whenever $X, Y \in D$.

Definition 1.5.4. Let M be an n-dimensional manifold, and let D be a smooth distribution on M, an integral manifold of D is a submanifold P of M, such that $P_x = D_x$ for all $x \in P$.

Definition 1.5.5. Let M be an n-dimensional manifold, and let D be a distribution on M, then D is said to be integrable, if every point of M is contained in a maximal integral submanifold.

Theorem 1.5.1. Let M be an n-dimensional manifold, and let D be a smooth m-distribution on M. Then D is integrable if and only if it is involutive.

Proof:- See Bishop and Crittenden [2] Chap. I page 22.

Let (G, g) be an m+n-dimensional Riemannian manifold with metric tensor g and Riemannian connection ∇'. Let $f: M \longrightarrow G$ be an immersion, where M is an m-dimensional manifold. f will induce a Riemannian metric on M defined by $h(X_p, Y_p) = g((df)_p X_p, (df)_p Y_p)$, for all $X_p, Y_p \in M_p$, and all $p \in M$.

Discussing local properties, M can considered as imbedded in G. Let $\mathfrak{X}(G,M)$ be the algebra of vector fields of G restricted to M. $\mathfrak{X}(G,M) = \mathfrak{X}(M) \oplus \mathfrak{X}(M)^{\perp}$ (direct sum), where $\mathfrak{X}(M)^{\perp}$ is a subspace of $\mathfrak{X}(G,M)$ of dimension n, and perpendicular to M. For any vector fields $X, Y \in \mathfrak{X}(M)$, $\nabla'_X Y \in \mathfrak{X}(G,M)$, where ∇' is the Riemannian connection on G, and so at any point $p \in M$, we have

$$(\nabla'_X Y)_p = \tan(\nabla'_X Y)_p + V(X,Y)_p$$

where $\tan(\nabla'_X Y)_p$ and $V(X,Y)_p$ are the tangential and the normal components respectively. Let $\tan(\nabla'_X Y)_p = (\nabla_X Y)_p$.

Proposition 1.5.2. (i) $\nabla_X Y$ is the covariant differentiation for the Riemannian connection on M.

(ii) $V: \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)^{\perp}$ is symmetric and bilinear over $C^{\infty}(M)$, called the second fundamental form.

Conversely, $V(X,Y)_p$ depends only on X_p, Y_p, and there is induced a symmetric bilinear map $V_p: M_p \times M_p \longrightarrow M_p^{\perp}$.

Proof:- See Kabayashi and Nomizu [14] Vol. II page 11, 12.

Let $p \in M$, and consider $(\nabla'_X N)_p$, where $X \in \mathfrak{X}(M)$, and $N \in \mathfrak{X}(M)^{\perp}$. Let the tengential and the normal components of $(\nabla'_X N)_p$ be denoted by $-(A_N X)_p$, and $(D_X N)_p$ respectively, i.e. $(\nabla'_X N)_p = -(A_N X)_p + (D_X N)_p$

Proposition 1.5.3. (i) The map $A: \mathfrak{X}(M)^{\perp} \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ given by $(N,X) \mapsto -(A_N X) \in \mathfrak{X}(M)$ is bilinear over $C^{\infty}(M)$. Conversely $-(A_N X)_p$ depends only on N_p and X_p, and a bilinear map is induced on $M_p^{\perp} \times M_p$ into M_p, where p is any arbitrary point of M.

(ii) $h((A_N X), Y) = g(V(X,Y), N)$ for each $N \in M_p^{\perp}$, consequently, A_N is a symmetric - linear transformation of M_p with respect to the metric h at p.

Proof:- See Kabayashi and Nomizu [14] Vol. II, page 15.

Let $p \in M$, for each $N \in M_p^{\perp}$, A_N is a symmetric linear transformation on M_p. Define a real-valued function on $M_p^{\perp}$ by $1/m$ (trace A_N). From linear algebra, there exists a unique element $H \in M_p^{\perp}$ such that $1/m$ (trace A_N) = $g(N,H)$, for every $N \in M_p^{\perp}$. H is called the <u>mean curvature normal at $p \in M$</u>. M is said to be a minimal submanifold if H is identically zero on M, i.e. if trace $A_N = o$, $N \in M_p$.

<u>Definition 1.5.6.</u> M is said to be a totally umbilic at $x \in M$, if A_N, for all $N \in M_x^{\perp}$ is equal to λI, where λ is any scaler, and I is the identity transformation of M_x. M is called a totally umbilic if it is a totally umbilic at each of its points.

<u>1.6 Almost Complex Manifolds</u>:-

Definitions 1.1.1. and 1.1.2. can go over to define a complex manifold by replacing $\mathbb{R}^n$ by $\mathbb{C}^n$, and we assume that the function $f_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ is holomorphic, i.e. its co-ordinate functions can be expanded in convergent power series at each point of its domain.

<u>Definition 1.6.1.</u> Let M be an n-dimensional manifold, then M is said to be an almost complex manifold, if there exists a fixed tensor field of type (1,1), such that if this tensor field, J say, is regarded as a $C^\infty(M)$ - linear map of $\mathfrak{X}(M) \to \mathfrak{X}(M)$, then J satisfies $J^2 = -I$, where I is the identity transformation of $\mathfrak{X}(M)$.

Any complex manifold carries in a natural way an almost complex structure.

<u>Proposition 1.6.1.</u> Let M be an n-dimensional almost complex manifold. Then

(i) n is even.

(ii) M is orientable, i.e. it admits a differentiable n-form, which vanishes nowhere on M.

<u>Proof</u>:- See Kabayashi and Nomizu [14], Vol. II, Chap. IX, page 121.

A smooth map $f: M \to M'$, where M and M' are two almost complex manifolds with almost complex structures J and J' respectively, is said to be almost complex if

$$(df) \circ J' = J' \circ (df)$$

On the other hand if M and M' are complex manifolds, and $(df) \circ J = J' \circ (df)$, then f is called holomorphic.

The Nijeenhuis tensor (or torsion tensor) of an almost complex manifold M, with almost complex structure J, is a tensor of type (1,2) defined by $E(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$, for all $X, Y \in \mathfrak{X}(M)$. J is said to be integrable if E is identically zero on M.

Proposition 1.6.2. An almost complex structure is a complex structure, i.e. the underlying manifold is a complex manifold, if and only if, the almost complex structure is integrable.

Proof:- See Kabayashi and Nomizu [14], Vol II, page 124.

Proposition 1.6.3. Let M be an almost complex manifold with almost complex structure J. Suppose that J is integrable. Then, there exists a unique complex structure on M such that J is the natural almost complex structure.

Proof:- See Helgason [10] page 285.

Definition 1.6.2. Let (M, g) be an almost complex Riemannian manifold with almost complex structure J, and Riemannian connection ∇, if $g(X,Y) = g(JX,JY)$ for all $X, Y \in \mathfrak{X}(M)$. Then M is called an almost Hermitian manifold with almost complex structure J.

For an almost Hermitian manifold, the Kähler 2 - form F is defined by the formula $F(X,Y) = g(JX,Y)$, for all $X, Y \in \mathfrak{X}(M)$. F is a skew symmetric differentiable 2-form.

Extending the Riemannian connection ∇ to act as a derivation on tensors of M, M is called

(i) Kähler if $\nabla_X(J)Y = o$, for all $X, Y \in \mathfrak{X}(M)$.

(ii) Nearly Kähler if $\nabla_X(J)X = o$, for all $X \in \mathfrak{X}(M)$.

(iii) Quasi Kähler if $\nabla_X(J)Y + \nabla_{JX}(J)(JY) = o$, for all $X, Y \in \mathfrak{X}(M)$.

(iv) Hermitian if $E \equiv o$ on M.

In Gray[7] it is proved that the class of Kähler manifolds is a subset of both the classes of nearly Kähler and Hermitian manifolds. On the other hand, the class of nearly Kähler manifolds is a subset of the class of quasi Kähler manifolds.

Definition 1.6.3. Let M and N be two almost Hermitian manifolds, and let $M \subset N$. Then M is said to be an almost Hermitian submanifold of N, if $JX \in \mathfrak{X}(M)$ whenever $X \in \mathfrak{X}(M)$, where J is the almost complex structure on N.

This means that the almost complex structure on M is the restrictions of the almost complex structure of N to M.

Proposition 1.6.3. Any almost Hermitian submanifold of a Kähler, nearly Kähler, quasi Kähler or Hermitian manifold has the same property.

Proof:- See Gray[7].

Proposition 1.6.4. Let N be a quasi Kähler manifold, and let M be an almost Hermitian submanifold of N. Then M is a minimal submanifold. In particular, any almost Hermitian submanifold of a nearly Kähler manifold is a minimal submanifold.

Proof:- See Gray[7].

CHAPTER II

Lie Groups and Homogeneous Spaces

2.1. Lie Groups:-

Definition 2.1.1. A Lie group G is an analytic manifold, which is also a group, such that group multiplication, and the taking of inverses, are analytic operations, i.e.

$$G \times G \longrightarrow G \quad \text{by} \quad (g,h) \longmapsto goh$$
$$\text{and} \quad G \longrightarrow G \quad \text{by} \quad g \longmapsto g^{-1}$$

are analytic.

Definition 2.1.2. Let G and G' be two Lie groups, and let $f: G \rightarrow G'$ be a map, then f is called an analytic Lie group homomorphism if $f(gh) = f(g)f(h)$, for all $g, h \in G$, and f is an analytic Lie group isomorphism, if f is an isomorphism, and f is analytic.

Definition 2.1.3. Let G be a Lie group, and let $H \subset G$ be a submanifold of G, which is also a Lie group of G using the operations of G. Then H is called a Lie subgroup of G.

Let G be a Lie group, and let $a \in G$. The left transformation $L_a : G \rightarrow G$ of G onto itself is an analytic diffeomorphism given by, $L_a(b) = ab$, $b \in G$.

Theorem 2.1.1. Let G be a Lie group, and let H be a closed subgroup of G. Then H may be given a unique C^{∞} structure in such a way as to make it a Lie subgroup of G, whose topology is the subspace topology.

Proof:- See Hausner and Schwartz[9], page 77.

Let G_o be the largest connected component of a Lie group G, which contains the identity e. G_o is an open, invariant, Lie subgroup of G, and it is generated by a neighbourhood of the identity e in G. (cf Hausner and Schwartz [9], page 37-38.).

Definition 2.1.4. Let M be an n-dimensional manifold, and let G be a Lie group acting on M, i.e. every element of G is a transformation of M, where group multiplications are composition of transformations. Suppose that the map $f : G \times M \longrightarrow M$ defined by $f(g,x) = g(x)$ is C^{∞}, for $x \in M$, and $g \in G$. Then G is called a Lie transformation group of M.

G is said to act effectively if, $gx = x$ for $x \in M$ implies that $g = e$. G is said to act freely on M, if the only element of G which has a fixed point on M is e. G is said to act transitively, if for every $x, y \in M$, there exists a $g \in G$ such that $g(x) = y$.

Definition 2.1.5. A manifold which has a transitive Lie transformation group, is called a homogeneous manifold.

Definition 2.1.6. Let G be a Lie group which acts transitively on a manifold M. Let $x \in M$ be a fixed point, the subgroup H of G whose elements leave x fixed, is called the isotropy group at $x \in M$. $H = \{ g \in G \mid g(x) = x \}$. The orbit of x, denoted by $G(x)$ is the set $\{ g.x \in M \mid g \in G \}$.

Proposition 2.1.2. Let G be the group of isometries acting on a Riemannian manifold M. Let $x \in M$ be any point, then the isotropy subgroup of G at is compact.

Proof:- See Helgason [10], page 169.

Proposition 2.1.3. Let H be a closed subgroup of a Lie group G, denote by G/H the space of left cosets gH with the natural topology. Then the coset space G/H has a unique analytic structure, such that G is a Lie transformation group of $G/_H$. In particular, the projection $\pi : G \to G/_H$ given by $\pi(a) = aH$, $a \in G$, is real analytic.

Proof:- See Chevalley [4] pages 109-111.

Proposition 2.1.4. Let G be a Lie group which acts transitively on a manifold M. Let H be the isotropy subgroup of a fixed point $p \in M$. Then H is closed, and $G/_H$ is diffeomorphic to M under the map

$$f: G/_H \longrightarrow M$$

given by $f(g H) = g.p$, $g \in G$ and $p \in M$.

Proof:- See Helgason [10], page 114

Definition 2.1.7. The group H* of linear transformations $(dh)_{\pi(e)}$, ($h \in H$) of $G/_H$ is called the linear isotropy group.

Definition 2.1.8. Let G be a Lie group, and let $g \in G$.

Suppose that H is the subgroup of G generated by g, i.e.

$$H = \{ h \in G \ / \ h = g^n \text{, n is an integer} \}$$

Then g is a generator of G, if closure H = G. G is monotonic if it has a generator.

Let s^1 be the unit sphere with its standard c^∞-structure (cf Brickell and Clark [3] page 116). Consider the manifold

$$T^n = S^1 \times \ldots \times S^1 \quad \text{(n-times)}$$

T^n is called an n-dimensional torus. T^n is diffeomosphic to $\mathbb{R}^n/\mathbb{Z}^n$.

Proposition 2.1.5. The torus T^n is monotonic, and the generators are dense in T^n .

Proof:- See Adams [1], page 79.

Proposition 2.1.6. A compact, connected, Abelian, Lie group is isomorphic to the n-dimensional torus T^n , where n is the dimension of G.

Proof:- See Adams [1], pages 15-16.

2.2 Lie Algebras:-

Definition 2.2.1. A vector space V over a field K (non characteristic 2) is called a Lie algebra, if there exists a bilinear map

$$[\ ,\ ] : V \times V \longrightarrow V$$

which satisfies the following conditions

(i) $[X,Y] = -[Y,X]$; $X,Y \in V$

(ii) $[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0$

$X,Y,Z \in V$. Condition (ii) is called the Jacobi identity. From (i) it is easy to deduce that $[X,X] = 0$, for all $X \in V$.

Let W be a subset of V, then W is called a subalgebra of V, if $[X,Y] \in W$, whenever $X,Y \in W$, it is called an ideal of V if $[X,Y] \in W$, whenever $X \in W$ and $Y \in V$.

Let $f: V \longrightarrow U$ be a linear transformation of the Lie algebra V into the Lie algebra U, then f is called a homomorphism if $f[X,Y] = [fX,fY]$, for all $X,Y \in V$. If f is 1-1, onto, then it is called an isomorphism. If $V = U$, and f is an isomorphism, then f is called an automorphism.

Let G be a Lie group, and let $L_a : G \longrightarrow G$, be a left transformation. A vector field $Z \in \mathcal{X}(G)$ is said to be a left invariant vector field, if it is invariant under the differential of L_a, for all $a \in G$, i.e. if $(dL_a)Z = Z$; for all $a \in G$. Denote by $\underline{g}$ the set of all left invariant vector fields on G.

$$\underline{g} = \{ X \in \mathcal{X}(G) \ / \ X_{gh} = (dL_g)X_h, \quad h \in G \}$$

If $X,Y \in \underline{g}$, then the Lie bracket $[X,Y]$ is also in $\underline{g}$, i.e. $[X,Y]$ is a left invariant vector field on G. Given a tangent vector $X \in G_e$, where e is the identity of G, then there exists a unique left invariant vector field $\tilde{X}$ on G such that $\tilde{X}_e = X$. $\underline{g}$ is called the Lie algebra of G.

Let H be a Lie subgroup of a Lie group G, let $\underline{g}$ and $\underline{h}$ be the Lie algebras of G and H respectively. Then $\underline{h}$ is a subalgebra of $\underline{g}$. On the other hand, if $\underline{h}$ is a subalgebra of $\underline{g}$, then there corresponds a unique, connected, Lie subgroup of G, whose Lie algebra is $\underline{h}$ (Cf Chevalley[4], pages 107-109).

Given a finite dimensional vector space V over a field K, let GL(V) be the Lie group of all invertable endomorphisms of V. It is well known that the Lie algebra of GL(V), is the set of all endomorphisms of V, denoted by gl(V), with bracket operation

$$[A,B] = AB - BA \quad , \quad A,B \in gl(V).$$

Definition 2.2.2. Let W be a Lie algebra over a field K, and let V be a finite dimensional vector space over the same field K. A homomorphism of W into gl(V) is called a representation of W on V.

Consider the map

$$ad : \underline{g} \longrightarrow gl(\underline{g})$$

defined by the linear transformation

$$(adX)(Y) = [X,Y] \quad , \text{ for all } Y \in \underline{g}$$

$$(ad[X,Y])(Z) = [[X,Y],Z]$$

and

$$([adX, adY])(Z) = (adXadY - adYadX)(Z)$$

$$= [X,[Y,Z]] - [Y,[X,Z]] = -[[Y,Z],X]$$

$$- [[Z,X],Y] = [[X,Y],Z]$$

which shows that ad is a homomorphism of $\underline{g}$ into gl($\underline{g}$), i.e. ad is a representation of $\underline{g}$ into itself. This representation is called the adjoint representation of $\underline{g}$. The center C($\underline{g}$) of $\underline{g}$ is defined to be the kernel of ad (ker ad). C($\underline{g}$) is an ideal of $\underline{g}$. The image of ad (Im $\underline{g}$) is a subalgebra of gl(V). More generally, if f is any homomorphism of a Lie algebra V to a Lie algebra U, then the kernel of f is an ideal of V, and the image of f is a subalgebra of U.

Definition 2.2.3. The adjoint group Int $(\underline{g})$ of $\underline{g}$ is the analytic subgroup of $GL(\underline{g})$, whose Lie algebra is $ad(\underline{g})$.

Definition 2.2.4. Let $\underline{g}$ be a Lie algebra over $\mathbb{R}$. Let $\underline{h}$ be a subalgebra of $\underline{g}$, and let K be the analytic subgroup of Int $(\underline{g})$ which corresponds to the subalgebra $ad(\underline{h})$ of $ad(\underline{g})$. $\underline{h}$ is said to be compactly imbeded subalgebra of $\underline{g}$, if K is compact. $\underline{g}$ is said to be compact if it is compactly imbeded in itself.

Proposition 2.2.1. Let $f : G \longrightarrow G'$ be a homomorphism, let $\underline{g}$ and $\underline{g}'$ be the Lie algebras of G and G' respectively. Then (df) is a homomorphism of $\underline{g}$ into $\underline{g}'$. If f is an isomorphism, then (df) is an isomorphism.

Proof:- See Hausner and Schwartz [9], page 55.

Definition 2.2.5. An affine connection ∇ of a Lie group G is said to be left invariant, if each left invariant transformation of G is an affine transformation with respect to ∇.

Proposition 2.2.2. There is 1-1 correspondence between the set left invariant affine connections ∇ of G, and the set of bilinear functions $d : \underline{g} \times \underline{g} \longrightarrow \underline{g}$, where $\underline{g}$ is the Lie algebra of G, given by

$$d(X,Y) = (\nabla_{\tilde{X}}\tilde{Y})_e \quad , \quad \text{where } \tilde{X}_e = X \, , \tilde{Y}_e = Y$$

Proof:- See Helgason [10], page 92.

Proposition 2.2.3. Let G be a Lie group, and let $\tilde{X} \in \underline{g}$ be such that $\tilde{X}_e = X \in G_e^{\cdot}$. Then there exists a unique analytic homomorphism $f : \mathbb{R} \longrightarrow G$ such that $\dot{f}_{\tilde{X}}(o) = X$, and that $\dot{f}_{\tilde{X}}(t) = \tilde{X}(f_{\tilde{X}}(t))$, for all $t \in \mathbb{R}$, i.e. $f_{\tilde{X}}$ is a maximul integral curve of $\tilde{X}$.

Proof:- See Sagle and Walde [22], page 120.

Definition 2.2.6. Let G be a Lie group with Lie algebra $\underline{g}$, let $\tilde{X} \in \underline{g}$ be such that $\tilde{X}_e = X$. Suppose that $f_{\tilde{X}}(t)$ is the analytic homomorphism of $\mathbb{R}$ into G (of proposition 2.2.3.). Define the exponential map of G by

$$\exp: \underline{g} \longrightarrow G : \tilde{X} \longrightarrow f_{\tilde{X}}(1)$$

Note that for all $s \in \mathbb{R}$

$$\exp s\tilde{X} = f_{s\tilde{X}}(1) = f_{\tilde{X}}(s),$$

and $\exp (s+t)\tilde{X} = \exp s\tilde{X}. \exp t\tilde{X}; \quad s,t \in \mathbb{R}.$

Definition 2.2.7. A one-parameter subgroup of a Lie group G is an analytic homomorphism of $\mathbb{R}$ into G.

From above, we see that the map $\exp t\tilde{X}$, $\tilde{X} \in \underline{g}$ is a one-parameter subgroup of G.

Consider the left invariant affine connection ∇ on G, which corresponds to the bilinear map $d(X,X) = o$, for all $X \in \underline{g}$. With respect to this connection, a one-parameter subgroup of G is a geodesic, and the exponential map of G agrees with the exponential map defined in Chapter I.

Proposition 2.2.4. Let $\theta : G \longrightarrow L$ be a homomorphism of a Lie group G to a Lie group L, let $\underline{g}$ and $\underline{l}$ be the Lie algebras of G and L respectively. Then

$$\exp ((d\theta)_e X) = \theta (\exp X) \quad ; X \in \underline{g}$$

Proof:- See Helgason [10], page 100.

Let $Ad(a) : G \longrightarrow G$ be a map given by $h \longmapsto aha^{-1}$, it is analytic isomorphism of G. By proposition 2.2.4. Ad(a) induces an automorphism of $\underline{g}$, the Lie algebra of G, also denoted by Ad(a). We also have for $X \in \underline{g}$

$$\exp (Ad(a)X) = Ad(a)(\exp X) = a \exp X a^{-1} \quad ; a \in G$$

For every $a \in G$, Ad(a) is an isomorphism of G, hence we have a homomorphism defined by $a \mapsto Ad(a)$ of G into $GL(\underline{g})$, it is called the adjoint representation of G.

2.3 Semisimple Lie Algebras:-

Definition 2.3.1. Let $\underline{g}$ be a Lie algebra over a field of characteristic zero. Consider the following bilinear form

$B(X,Y) = Tr(adX\, adY)$ on $\underline{g} \times \underline{g}$; $X,Y \in \underline{g}$ where Tr means the trace of adXadY. B is called the Killing form of $\underline{g}$.

Let $f : \underline{g} \longrightarrow \underline{g}$ be an automorphism. Then $(ad(fX))(Y) = [fX,Y]$, $X,Y \in \underline{g}$, and $(f \circ adX \circ f^{-1})(Y) = f[X, f^{-1}Y] = [fX,Y]$
Therefore $ad(fX) = f \circ adX \circ f^{-1}$

Definition 2.3.2. A Lie algebra $\underline{g}$ over a field of characteristic zero, is said to be semisimple if the Killing form is nondegenerate i.e. its rank equal to the dimension of $\underline{g}$. $\underline{g}$ is said to be simple if it is semisimple and has no ideals except $\{\underline{o}\}$ and $\underline{g}$. A Lie group is called semisimple (simple) if its Lie algebra is semisimple (simple).

Using that for any automorphism f of the Lie algebra $\underline{g}$, we have $ad(fX) = f \circ adX \circ f^{-1}$, and that for any endomorphism of $\underline{g}$, $T_r(AB) = T_r(BA)$, it is easy to show that

(i) $B(fX, fY) = B(X,Y)$; $X,Y \in \underline{g}$

(ii) $B(X, [Y,Z]) = B(Z, [X,Y]) = B(Y, [Z,X])$; $X,Y,Z \in \underline{g}$

Proposition 2.3.1. Let $\underline{g}$ be a semisimple Lie algebra, and let $\underline{h}$ be an ideal of $\underline{g}$. Let $\underline{b}$ be the set of elements $X \in \underline{g}$ which are orthogonal to $\underline{h}$ with respect to B. Then $\underline{h}$ is semisimple, and $\underline{b}$ is an ideal. Also

$$\underline{g} = \underline{h} \oplus \underline{b} \quad \text{(direct sum)}$$

Proof:- See Helgason [10], page 121.

As a consequence of proposition 2.3.1. we see that the centre of a semisimple Lie algebra is $\{\underline{o}\}$, and that if we continue decomposing $\underline{h}$ and $\underline{b}$ to their constituents ideals,we have

$$\underline{g} = \underline{g}_1 \oplus \cdots \oplus \underline{g}_r$$

where every $\underline{g}_i$ (i = 1,...,r) is a simple ideal of $\underline{g}$.

2.4 Reductive Homogeneous Manifolds:-

Definition 2.4.1. Let G be a connected Lie group, and H a closed subgroup of G. The homogeneous space $G/_H$ is called reductive if the following condition is satisfied. In the Lie algebra $\underline{g}$ of G, there exists a subspace $\underline{m}$ of $\underline{g}$ such that $\underline{g} = \underline{h} \oplus \underline{m}$ (direct sum), where $\underline{h}$ is the Lie algebra of H, and such that Ad(H) $\underline{m} \subset \underline{m}$, for all h $\in$ H.

We can always identify $\underline{m}$ with the tanget space $(G/_H)_o$ $(o \equiv H)$, under the projection $\pi : G \longrightarrow G/_H$.

Definition 2.4.2. A homogeneous space $G/_H$ provided with a G-invariant Riemannian metric g is called a Riemannian homogeneous space. $G/_H$ is said to be naturally reductive if it admits an Ad(H)- invariant decomposition $\underline{g} = \underline{h} \oplus \underline{m}$,satisfying the condition

$$g([X,Y]_{\underline{m}}, Z) = g(X, [Y,Z]_{\underline{m}}); \; X,Y,Z \in \underline{m}$$

From Gray[8], we have the Riemannian connection is given by

$$2\langle\nabla_X Y, Z\rangle_p = -\langle X, [Y,Z]\rangle_p - \langle Y, [X,Z]\rangle_p + \langle Z, [X,Y]\rangle_p$$

$p \in M$; $X,Y,Z \in \underline{m}$.

Hence if $G/_H$ is naturally reductive, we have the Riemannian connection is given by

$$2\langle\nabla_X Y, Z\rangle_p = \langle [X,Y], Z\rangle_p \quad ; \; p \in M, \; X,Y,Z \in \underline{m}$$

Theorem 2.4.1. Let G/H be a reductive homogeneous space with a fixed decomposition of the Lie algebra $\underline{g}$ of G, i.e. $\underline{g} = \underline{h} \oplus \underline{m}$, and $Ad(H)\underline{m} \subset \underline{m}$. Then there exists a 1-1 correspondence between the set of all G-invariant affine connections on G/H and the set of all bilinear functions d on $\underline{m} \times \underline{m}$ with values in $\underline{m}$, which are invariant by Ad(H), i.e. $(Ad(h))\, d(X,Y) = d(Ad(h)X, Ad(h)Y)$, for $X, Y \in \underline{m}$, and $h \in H$. The correspondence is given by

$$d(X,Y) = (\nabla_{\tilde{X}}\tilde{Y})_o \qquad (o \equiv H)$$

$\tilde{X}$, $\tilde{Y}$ are vector fields on G/H such that $\tilde{X}_o = X$, $\tilde{Y}_o = Y$, where we identify the tangent space at o with $\underline{m}$.

Proof:- See Nomizu[21], Chap. II - 8.

Proposition 2.4.2. Let G/H be a reductive homogeneous space, with a decomposition of the Lie algebra $\underline{g}$ of G given by $\underline{g} = \underline{m} \oplus \underline{h}$, where $Ad(H)\,\underline{m} \subset \underline{m}$. Then

(i) There is a natural 1-1 correspondence between the set of all G-invariant almost complex structures J on G/H and the set of linear endomorphisms J_o of $\underline{m}$ satisfying

(1) $J_o^2 = -I$, I is the identity transformation,

(2) $J_o \circ Ad(a) = Ad(a) \circ J_o$, for every $a \in H$;

when H is connected, we have instead of (2)

(2') $J_o \circ Ad(Y) = Ad(Y) \circ J_o$, for every $Y \in \underline{h}$

(ii) An invariant almost complex structure J on G/H is integrable, if and only if, the corresponding linear endomorphism J_o of $\underline{m}$ satisfies

$$[J_oX, J_oY]_{\underline{m}} - [X,Y]_{\underline{m}} - J_o[X, J_oY]_{\underline{m}} - J_o[J_oX, Y]_{\underline{m}} = o$$

for all $X, Y \in \underline{m}$

Proof:- See Kabayashi and Nomizu [14], Vol. II, page 219.

CHAPTER III

Riemannian Symmetric Spaces

In this Chapter we give a brief account of the results on Riemannian (locally) symmetric spaces which are of interest to us. The actual proofs of the results are of importance in later work, and consequently we also include them.

We will not consider affine (locally) symmetric spaces, since they do not play any role in our work.

3.1 Riemannian Locally Symmetric Spaces

Definition 3.1.1. Let (M, g) be a Riemannian manifold, if for each point $p \in M$, there exists a neighbourhood U_p of p in M, and a local geodesic symmetry s_p of p such that s_p is an isometry of U_p, then M is called a Riemannian locally symmetric manifold.

The local geodesic symmetry s_p at each point $p \in U_p$ has the property that $(ds_p)_p = -I_p$, where I_p is the identity transformation.

Theorem 3.1.1. Let (M, g) be a Riemannian manifold. Then M is a Riemannian locally symmetric space, if and only if, the sectional curvature is invariant under all parallel translations.

Proof:- Let $p \in M$, then the sectional curvature of the two deminsional vector space spanned by $X, Y \in M_p$ is given by

$$K_p = \frac{g_p(R_p(X,Y)Y, X)}{A_p(X,Y)}$$

Suppose that M is Riemannian locally symmetric. For all $X, Y, Z, W \in \mathfrak{X}(M)$ we have

$$(\nabla_W R)(X,Y)Z = \nabla_W(R(X,Y)Z) - R(\nabla_W X,Y)Z - R(X,\nabla_W Y)Z - R(X,Y)\nabla_W Z$$

Since s_p is an isometry in a neighbourhood U_p of p, then

$$(ds_p)\left[(\nabla_W R)(X,Y)Z\right] = (\nabla_{ds_p W} R)(ds_p X, ds_p Y)\, ds_p Z$$

But $ds_p = -I_p$, therefore we have

$(\nabla_W R)_p = o$ for all $p \in M$, and this implies that $\nabla_W R = o$ on M.

Assume that X, Y are orthonormal unit vector fields on U, since M is a Riemannian manifold, therefore $\nabla g = o$ on M, and the invariance of K follows.

For the converse, we consider first the following Lemma.

Lemma :- Let A be a ring with identity element e , such that $6a \neq o$ for $a \neq o$ in A. Let E be a module over A. Suppose a mapping $B: E \times E \times E \times E \rightarrow A$ is quadrilinear and satisfies the identities

(a) $B(X,Y,Z,T) = -B(Y,X,Z,T)$

(b) $B(X,Y,Z,T) = -B(X,Y,T,Z)$

(c) $B(X,Y,Z,T) + B(Y,Z,X,T) + B(Z,X,Y,T) = o$

Then

(d) $B(X,Y,Z,T) = B(Z,T,X,Y)$

If in addition to (a), (b), and (c) B saitsfies

(e) $B(X,Y,Y,X,) = o$, for all $X, Y \in E$

then $B = o$

Proof:- See Helgason [10] page 69.

Let γ be a curve joining two points $p, q \in M$, and let τ be the parallel translation along γ . If $X, Y \in M_p$, we have

$$g_p(R_p(X,Y)Y,X) = g_q(R_q(\Upsilon X, \Upsilon Y)\Upsilon Y, \Upsilon X)$$

and $$g_p(R_p(X,Y)Y,X) = g_q(\Upsilon(R_p(X,Y)Y), \Upsilon X)$$

Let B be the quadrilinear form given by

$$B(X,Y,Z,T) = g_q(R_q(\Upsilon X, \Upsilon Y)\Upsilon Z, \Upsilon T) - g_q(\Upsilon(R_p(XY)Z), \Upsilon T)$$

for $X,Y,Z,T \in M_p$, then B satisfies the conditions of the above Lemma

$\therefore$ $B = o$, so

$$\Upsilon(R_p(X,Y)Z) = R_q(\Upsilon X, \Upsilon Y)\Upsilon Z \quad \text{i.e.} \quad \Upsilon R_p = R_q$$

$\therefore$ $\nabla_X R = o$ for each $X \in \mathfrak{X}(M)$.

The diffeomorphism s_p of U_p defines a new connection $\overline{\nabla}$ on U_p by

$$(ds)_p \overline{\nabla}_X Y = \nabla_{ds_p X} ds_p Y \quad , \quad \text{for } X,Y \in \mathfrak{X}(M)$$

Let $\overline{R}$ and $\overline{T}$ be the curvature and torsion tensors with respect to $\overline{\nabla}$. Then

$$(ds_p)(\overline{T}(X,Y)) = T(ds_p X, ds_p Y) = o$$

and

$$(ds_p)((\overline{\nabla}_W R)(X,Y)Z) = (\nabla_{ds_p W} R)((ds_p X, ds_p Y) ds_p Z) = o$$

for all $W,X,Y,Z \in \mathfrak{X}(M)$.

$\therefore$ $\overline{T} = \overline{\nabla}_W \overline{R} = o \qquad W \in \mathfrak{X}(M)$.

Now, $(ds_p)(\overline{R}(X,Y)Z) = R(ds_p X, ds_p Y) ds_p Z$

But $ds_p = -I$, this implies that $R_p = \overline{R}_p$. Hence by proposition 1.3.3. we have s_p an affine transformation. But s_p induces an isometry on M_p. Hence, by proposition 1.4.2. we have s_p an isometry of U_p. This completes the theorem. //

From proposition 1.3.4. we see that every Riemannian locally symmetric space is a real analytic manifold with a real analytic connection with respect to the atlas consisting of normal co-ordinate systems.

Remark:- The definition of an affine locally symmetric space (M, ∇) is similar to definition 3.1.1., where we replace the isometry s_p by an affine transformation. Analogous to theorem 3.1.1., a manifold (M, ∇) is an affine locally symmetric space, if and only if $\nabla R = o$ on M.

3.2. Riemannian Symmetric Spaces:-

Definition 3.2.1. A Riemannian manifold (M, g) is said to be a Riemannian symmetric space, if for each $x \in M$, the symmetry s_x can be extended to a global isometry of M.

Proposition 3.2.1. Every Riemannian symmetric space is complete.

Proof:- Let γ_t , $o \leq t \leq a$, be a geodesic between two points $x, y \in M$. Using the symmetry s_y, we can extend γ_t beyond y as follows. Set

$$\gamma_{a+t} = s_y(\gamma_{a-t}) \qquad o \leq t \leq a \qquad //$$

Theorem 3.2.2. Let M be a Riemannian symmetric manifold. Then,

(i) The set I(M) of isometries on M is a Lie transformation group of M.

(ii) I(M) is transitive on M.

Proof:- See Kobayashi and Nomizu Vol. II [14] pages 223-224.

From proposition 2.1.4. Chapter II, we see that a Riemannian symmetric manifold is diffeomorphic to the homogeneous space G/H, where G is the identity component of the group of isometries, and H is the compact subgroup of G which leaves a fixed point of M fixed. The diffeomorphism is given by $gH \longrightarrow g.p$, where p is the fixed point of M fixed by H; and $g \in G$.

Theorem 3.2.3. Let M be a Riemannian symmetric manifold. Let G be the identity component of I (M), and H the isotropy subgroup of a fixed point $p \in M$. Then,

(i) The map $\sigma : I(M) \to I(M)$ given by $g \mapsto s_p o g o s_p^{-1}$ is an involutive ($\sigma^2 = 1G$, but $\sigma \neq 1G$) automorphism of G, such that H lies between H_σ and $(H_\sigma)o$, where H_σ is the subgroup of G of all fixed points of σ, and $(H_\sigma)_o$ is the identity component of H_σ. Also H contains no normal subgroup of G other than $\{e\}$, where e is the identity of G.

(ii) Let $\underline{g}$ and $\underline{h}$ denote the Lie algebras of G and H, respectively. Then $\underline{h} = \{ X \in \underline{g} \mid (d\sigma)_e X = X \}$, and if we have $\underline{m} = \{X \in \underline{g} \quad (d\sigma)_e X = -X\}$, then $\underline{g} = \underline{h} \oplus \underline{m}$ (direct sum). Let π be the natural map $\pi : G \to M$ given by $g \mapsto g.p$. Then $(d\pi)_e$ maps $\underline{h}$ onto $\{\underline{o}\}$ and $\underline{m}$ isomorphically onto M_p.

Proof:- (i) That σ is an involutive automorphism of I(M) is abvious, and since it maps connected components to connected components, then it maps G to itself. Let $h \in H$, then $s_p h s_p^{-1}$ and h induce the same map of M_p, also $h(p) = p = s_p h s_p^{-1}(p)$= Hence, from theorem 1.4.3. we have $s_p h s_p^{-1} = h$ for all h H. This implies that $H \subset H\sigma$.
Let $s \mapsto g_s$, $s \in \mathbb{R}$, be a 1-parameter subgroup of H_σ. Then $\sigma(g_s) = g_s$. Also $(s_p o g_s)(p) = (g_s o s_p)(p) = g_s(p)$. Hence the orbit $\{g_x(p) \mid s \in \mathbb{R}\}$ is fixed by s_p for all $s \in \mathbb{R}$. But p is an isolated fixed point of S_p, this means that the orbit $\{g_s(p) \mid s \in \mathbb{R}\}$ must reduce to p. Hence $g_s \in H$, but g_s is a 1-parameter subgroup of H_σ, and $(H_\sigma)_o$ is the identity component of H_σ. This implies that $(H_\sigma)_o \subset H$, and we have

$$(H_\sigma)_o \subset H \subset H_\sigma$$

Let T be a normal subgroup of G in H. Let g be any element of G. Then for each $k \in T$, there exists $k' \in T$ such that $k'g = gk$. Hence $k'g(p) = gk(p) = g(p)$ for all $g \in G$, i.e. if $x \in M$, and since G is transitive on M, there exists $g' \in G$ such that $g'(p) = x$, and we have $k'.g'(p) = k'x = g'(p) = x$. But G acts effectively on M, so $k' = e$, and therefore $T = \{e\}$.

(ii) Let $\underline{h}'$ be the Lie algebra of H. Let $X \in \underline{h}'$, then we have from proposition 2.2.4. that

$$\exp (d\sigma)_e X = \sigma(\exp X) = \exp X$$

$$\therefore (d\sigma)_e X = X$$

Hence $\underline{h}' \subset \underline{h}$.

Conversely, let $X \in \underline{h}$, then $\exp tX$; $t \in \mathbb{R}$, is a 1-parameter subgroup of G. From proposition 2.2.4. we have

$$\exp (d\sigma)_e tX = \sigma(\exp tX) = \exp tX$$

or $$s_p \circ \exp tX \circ s_p^{-1}(p) = (\exp t X)(p)$$

$$\therefore s_p((\exp t X)(p)) = (\exp tX)(p)$$

Hence $(\exp tX)(p)$ is a fixed point of s_p, but s_p has p as an isolated fixed point, therefore the orbit $\{(\exp tX)(p) \ / \ t \in \mathbb{R}\}$ must reduce to p. Hence $\exp tX \in H$ and $X \in \underline{h}'$

$$\therefore \underline{h} = \underline{h}'$$

The direct decomposition comes from the identity

$$X = \tfrac{1}{2}(X + (d\sigma)_e X) + \tfrac{1}{2}(X - (d\sigma)_e X)$$

From propositions 2.1.3. and 2.1.4. we see that G acts transitively on $G/_H$, and $G/_H$ is diffeomorphic to M. The projection $\pi : G \longrightarrow M$ maps H onto p, therefore $\underline{h} \subset \text{kernel } (d\pi)_e$.

Now, let $X \in \text{kernel } (d\pi)_e$, then if $g \in C^\infty(M)$ we have
$$0 = ((d\pi)_e X)(g) = X(g \circ \pi) = \left\{ \frac{d}{dt} g(\exp tX.p) \right\}_{t=0}$$

Let $s \in \mathbb{R}$, and consider the function $g^*(q) = g(\exp sX.q)$; $q \in M$. Then

$$0 = \left\{ \frac{d}{dt} g^*(\exp tX.p) \right\}_{t=0} = \left\{ \frac{d}{dt} g(\exp tX.p) \right\}_{t=s}$$

which shows that $g(\exp sX.p)$ is constant in s.g is arbitrary and we have

$(\exp sX)(p) = p$ for all $s \in \mathbb{R}$, and so $X \in \underline{h}$. Hence $(d\pi)_e$ vanishes on $\underline{h}$. So rank π equals (dimension of $\underline{g}$ - dimension of $\underline{h}$), which equals (dimension of G - dimension of H). Hence

$$\text{rank } \pi = \text{dimension } G/H = \text{dimension } M.$$

Hence $(d\pi)_e$ maps $\underline{m}$ isomorphically onto M_p. This completes the theorem. //

Definition 3.2.2. Let G be a connected Lie group and H a closed subgroup. The pair (G, H) is called a symmetric pair if there exists an involutive analytic automorphism σ of G such that $(H_\sigma)_o \subset H \subset H_\sigma$, where H_σ is the set of fixed points of σ, and $(H_\sigma)_o$ is the identity component of H_σ. If in addition the group Ad(H) is compact, (G, H) is called a Riemannian symmetric pair.

Theorem 3.2.4. Let (G, H) be a Riemannian symmetric pair. Let π denote the natural mapping of G onto G/H, and put $o = \pi(e)$. In each G-invariant Riemannian structure Q on G/H, The manifold G/H is a Riemannian symmetric space. The geodesic symmetry s_o satisfies

$$s_o \circ \pi = \pi \circ \sigma$$
$$\tau(\sigma(g)) = s_o \tau(g) s_o, \quad g \in G$$

where $\tau(g)$ is the action of g on G/H.

In particular, s_o is independent of the choice of Q.

Proof:- Let σ be an arbitrary analytic involutive automorphism of G, such that $(H_\sigma)_o \subset H \subset H_\sigma$. Identify the Lie algebra of G with G_e, the tangent space at the identity. The eigenvalues of $(d\sigma)_e$ are ± 1, hence $\underline{g} = \underline{h} \oplus \underline{m}$ (direct sum), where $\underline{h}$ is the Lie algebra of H. For $X \in \underline{m}$ and $k \in H$ we have

$$\sigma(\exp Ad(k)tX) = \sigma(Ad(k)\exp tX) = \sigma(k \exp tX k^{-1})$$
$$= \sigma(k)\sigma(\exp tX)\sigma(k^{-1}) = k \exp(d\sigma)_e tX \; k^{-1}$$
$$= - Ad(k) \exp tX = - \exp Ad(k)tX, \quad t \in \mathbb{R}$$

so that $(d\sigma)_e\ Ad(k)X = -Ad(k)X$; for all $X \in \underline{m}$

Thus $\underline{m}$ is invariant under $Ad(H)$.

$(d\pi)_e$ is an isomorphism of $\underline{m}$ onto $(G/H)_o$, let $h \in H$ and $X \in \underline{m}$. Then

$$\pi(\exp Ad(h)tX) = \pi(Ad(h)\exp tX) = \pi(h \exp tX\, h^{-1})$$
$$= h \exp tX\, h^{-1} H = h \exp tX\, H = \tau(h)\pi(\exp(tX));\ t \in \mathbb{R}$$

Hence we have

$$(d\pi)_e \circ Ad(h)X = d\tau(h) \circ d\pi(X)$$

This means that the isomorphism $(d\pi)_e$ commutes with the action of H.

We have (G, H) a Riemannian symmetric pair, this means that Ad(H) is compact, using Weyl's theorem (Cf Matsushima [20] page 279), there exists a strictly positive definite quadratic form B on $\underline{m}$ invariant by Ad(H). Consider the form $T_o = B \circ (d\pi)_e^{-1}$ on $(G/H)_o$, T_o is invariant under all the maps $d\tau(h)$, $h \in H$, since for $X \in (G/H)_o$ we have

$$T_o(d\tau(h)(X)) = B \circ (d\pi)_e^{-1}(d\tau(h)(X)) = B \circ (d\pi)_e^{-1}((d\pi)_e\, Ad(h)$$
$$(d\pi)_e^{-1}(X)) = B(Ad(h) \circ (d\pi)_e^{-1}(X)) = B \circ (d\pi)_e^{-1}(X) = T_o(X)$$

Let the correspondent symmetric bilinear form on $(G/H)_o \times (G/H)_o$ be Q_o. For each $q \in G/H$, we define $Qq(d\tau(g)(X_o), d\tau(g)(Y_o)) = Q_o(X_o, Y_o)$, $g \in G$, $X_o, Y_o \in (G/H)_o$; on $(G/H)_q \times (G/H)_q$, where $\tau(g).\, o = q$.

Since B is invariant under Ad(H), this guarantees that Qq is well defined. The analyticy of $\tau(g)$, $g \in G$, ensures that Qq is an analytic Riemannian structure on G/H, invariant under G. Define $s_o : G/H \longrightarrow G/H$ by the condition $s_o \circ \pi = \pi \circ \sigma$. Then s_o is an involutive diffeomorphism of G/H onto itself, and we have $(ds_o)_o = -I$.

Now $s_o \circ \pi(gg') = s_o \circ \tau(g)(g'H) = \pi \circ \sigma(gg') = \sigma(g)\sigma(g')H$

$$= \tau(\sigma(g))(\pi(\sigma(g'))) = (\tau(\sigma(g)) \circ s_o)(g'H)$$
$$\therefore\quad s_o \circ \tau(g) = \tau(\sigma(g)) \circ s_o \quad , \quad g, g' \in G$$

Let $g \in G$, $q = \tau(g).o$, and let $X, Y \in ({}^{G}/_{H})_q$, then the vectors $X_o = d\tau(g^{-1})(X)$, $Y_o = d\tau(g^{-1})(Y)$ belong to $({}^{G}/_{H})_o$. Consider the following

$$Q_{s_o(q)}((ds_o)(X),(ds_o)(Y)) = Q_{s_o(q)}(ds_o o d\tau(g)(X_o), ds_o o d\tau(g)(Y_o))$$

$$= Q_{s_o(q)}(d\tau(\sigma(g))ds_o(X_o), d\tau(\sigma(g))\, d\, s_o(Y_o)) = Q_o(ds_o(X_o), ds_o(Y_o))$$

$$= Q_o(-X_o, -Y_o) = Q_o(X_o, Y_o) = Qq\,(d\tau(g)(X_o), d\tau(g)(Y_o))$$

$$= Q_q\,(X,Y)$$

Thus s_o is an isometry. If p is any arbitrary point in ${}^{G}/_{H}$ such that $p = \tau(a)\cdot o$, the symmetry at p is given by $s_p = \tau(a) o s_o o \tau(a^{-1})$ //

Remark:- The proof of the theorem 3.2.3. can go over if we consider an affine symmetric space instead of Riemannian symmetric space, where we replace the isometries by affine transformations. Also, similar to theorem 3.2.4., if we are given a symmetric pair (G, H), then ${}^{G}/_{H}$ is a reductive homogenious space, and has an affine connection such that it beocmes an affine symmetric space.

From theorem 3.2.3. we see that a Riemannian symmetric space induces a pair (g, s), where

(i) g is a real Lie algebra

(ii) $s : g \longrightarrow g$ is an involutive automorphism

(iii) Let h be the set of fixed points of s, then h is a compactly imbedded subalgebra of g.

(iv) Let 3 be the center of g, then $h \cap 3 = \{o\}$

Definition 2.2.3. A pair (g, s) which satisfies the above conditions (i), (ii) and (iii) is called an orthogonal symmetric Lie algebra. It is said to be effective, if it satisfies (iv) also.

Let G be a connected Lie group with Lie algebra g, H is a Lie subgroup of G with Lie algebra h, then the pair (G, H) is said to be associated with the orthogonal Lie algebra (g, s).

Let ($\underline{g}$, s) be an orthogonal Lie algebra, and let (G, H), (G', H') be associated to it. If H and H' are connected, and G' is simply connected. Then H' is closed and (G', H') is a Riemannian Symmetric pair. If H is closed in G, then G/H is Riemannian locally symmetric for each G-invariant metric, and G'/H' is the universal covering manifold of G/H.

3.3. Hermitian Symmetric Spaces

Definition 3.3.1. Let M be a connected Hermitian manifold, with complex structure J, then M is said to be a Hermitian (locally) symmetric space, if each point $p \in M$ is an isolated fixed point of an involutive holomorphic isometry s_p (in a neighbourhood U of p in M) of M.

Denote by H(M), the group of all holomorphic transformations of M, then the group of all holomorphic isometries of M is given by

$$A(M) = H(M) \cap I(M)$$

where I(M) is the Lie group of all isometries of M. A(M) is a closed subgroup of I(M), hence it is a Lie transformation group of M, it contains all the symmetries on M, and this ensures that it is transitive on M. Let $p \in M$, and H the isotropy subgroup at p, then M is diffeomorphic to $A_o(M)/H$, where $A_o(M)$ is the identity component of A(M).

Proposition 3.3.1. Let M be a Hermitian symmetric space, then M is Kähler.

Proof:- Let J be the complex structure of M, and since each element of $A_o(M)$ is a holomorphic isometry, then by definition we have

$$d\Upsilon(g) \circ J = J \circ d\Upsilon(g) \quad , \quad g \in A_o(M).$$

Let $x \in M$, and s_x the symmetry which leaves x as an isolated fixed point. Then

$$(\nabla_X J)(Y) = \nabla_X(JY) - J\nabla_X Y \quad , \qquad X, Y \in \mathfrak{X}(M)$$

$$\therefore \ (ds_x)\left[(\nabla_X J)(Y)\right] = (ds_x)(\nabla_X JY) - (ds_x)(J\nabla_X Y)$$

$$= \nabla_{(ds_x)X} J(ds_x)Y - J\nabla_{(ds_x)X}(ds_x)Y$$

$$= (\nabla_{(ds_x)X}J)(ds_x)Y$$

But $ds_x = -I$

$$(\nabla_X J)Y = 0$$

which is the condition for a Kähler manifold. //

Proposition 3.3.2. Let a Kähler manifold M, with complex structure J, be a locally symmetric space as a Riemannian manifold. Then M is Hermitian locally symmetric.

Proof:- Let $x \in M$, and let U be a normal neighbourhood of x in M. For any $y \in U$, let σ be the geodesic from x to y. The symmetry s_x maps J_x upon itself, and since s_x is an affine transformation, and J is parallel along σ, we have

$$s_x(J_y) = J_{s_x(y)}$$

i.e. s_x maps J_y the same as it is parallel translated along the image of σ under s_x to the point $s_x(y)$.

Thus s_x preserves J and M is a Hermitian symmetric, this completes proof. //

Proposition 3.3.3. Let (G,H) be a Riemannian symmetric pair. Put $o = \pi(e)$, where π is the natural map of G onto G/H. Let Q be any G-invariant Riemannian structure on G/H, and let A be an endomorphism of $(G/H)_o$ such that

(i) $A^2 = -I$

(ii) $Q_o(AX, AY) = Q_o(X,Y)$, $X,Y \in (G/H)_o$

(iii) A commutes with the elements of the linear isotropy group H*.

Then G/H has a unique G-invariant almost complex structure J, such that $J_o = A$, and that G/H is a Hermitian symmetric space.

Proof:- From proposition 2.4.2. (i) Chapter II, such a unique almost complex structure exists. That M is an almost Hermitian manifold follows from the fact that each Q and J is G-invariant.

If σ is any involutive automorphism of G such that $(H_\sigma)_o \subset H \subset H_\sigma$, then we have $s_o \circ \pi = \pi \circ \sigma$ and $s_o \circ \tau(g) = \tau(\sigma(g)) \circ s_o$ (theorem 3.2.4.). Let $p \in M$, $Z \in M_p$, choose $g \in G$, such that $g.o = p$, let $Z_o = d\tau(g^{-1})Z$. Then using the relation $ds_o J_o = J_o ds_o$, we have

$$ds_o(J_p Z) = ds_o\, d\tau(g) J_o Z_o = d\tau(\sigma(g)) \circ J_o(ds_o Z_o)$$

$$= J_{s_o p}(ds_o\, d\tau(g) Z_o) = J_{s_o p}(ds_o Z)$$

Hence J is invariant under s_o.

Let g be the Lie algebra of G, h the Lie algebra of H, then we have $\underline{g} = \underline{m} \oplus \underline{h}$ (direct sum). For any $X,Y \in \underline{m}$, we have $[X,Y] \in \underline{h}$. Hence the integrability condition of J follows from proposition 2.4.2. (ii) Chapter II. The complex structure on M corresponding to J (see proposition 1.6.3. Chapter I) due to its uniqueness is invariant under each s_p, $p \in M$. Hence M is Hermitian symmetric space. //

3.4 Totally Geodesic Submanifolds:-

Definition 3.4.1. Let M be a Riemannian manifold, and let S be a submanifold of M. Suppose that at each point $p \in S$, we have the geodesic

γ_t of M determined by every $X \in S_p$, lies in S for a small value of the parameter t, then S is called a totally geodesic submanifold of M.

Proposition 3.4.1. A submanifold N of a Riemannian manifold M is totally geodesic if and only if its second fundamental form is identically zero.

Proof: - See Hermann [11], page 338.

In the notations of Chapter I, section 5, this means that for any arbitrary point $p \in M$

$$V_p(X,Y) = o \qquad ; X,Y \in M_p$$

and from proposition 1.5.3., we have that

$$A_N(X)\Big|_p = o \qquad , N \in M_p^{\perp} \ , \ X \in M_p$$

Proposition 3.4.2. Let M be a Riemannian locally symmetric space, and let S be a totally geodesic submanifold of M, then S is a locally symmetric space.

Proof: - Let p be any point in S, and suppose that γ_t be a geodesic starting from p. Since S is totally geodesic, γ_t lies in S for small values of the parameter t. Hence, we can obtain a geodesic symmetry of p in S by the restriction of the geodesic symmetry of p in M. p is arbitrary, and so we have S as a locally symmetric space. //

Riemannian symmetric spaces contain plenty of totally geodesic submanifolds. Define a Lie triple System of a real Lie algebra $\underline{g}$, as the subspace $\underline{m}$ of $\underline{g}$ such that $[X, [Y,Z]] \in \underline{m}$, whenever $X,Y,Z \in \underline{m}$. It is proved in Helgason [10] , page 189 that if we have a Riemannian space $M = G/H$, and $\underline{g} = \underline{p} \oplus \underline{h}$ is a fixed decomposition of the Lie algebra $\underline{g}$ of G, then, any Lie triple system of $\underline{g}$ contained in $\underline{p}$, gives rise to a totally geodesic submanifold of M.

CHAPTER IV

Riemannian s-Manifolds and Riemannian k-Symmetric Spaces

Generalized Riemannian symmetric spaces were first introduced in Ledger [16], then in Ledger and Obata [17] s-manifolds were defined. s-Manifolds are divided into two parts, affine s-manifolds and Riemannian s-manifolds, which generalize the affine symmetric spaces and the Riemannian symmetric spaces of E. Cartan. In this thesis we are mainly concerned with Riemannian s-manifolds.

Riemannian s-regular manifolds form a subset of Riemannian s-manifolds. Many results of Riemannian symmetric spaces can be generalized and are valid for Riemannian s-regular manifolds.

4.1. Riemannian s-Manifolds

Definition 4.1.1. Let (M, g) be a connected Riemannian manifold, and consider the map $s: M \longrightarrow I(M)$ such that for each $x \in M$, $s(x) = s_x$ is a symmetry at x, then M is called a Riemannian s-manifold and denoted by the triple (M, g, s).

The family of symmetries $\{s_x \mid x \in M\}$ is said to form a Riemannian s-structure on (M, g).

Definition 4.1.2. Let S be the tensor field of type (1, 1) on M such that $S_x = (ds_x)_x$, $x \in M$, then S is called the symmetry tensor field. A Riemannian s-structure is smooth if S is smooth.

S_x is an orthogonal transformation of M_x, and it does not fix any vector of M_x except the vector $\underline{o}$, therefore S_x does not have 1 as an eigenvalue, and since it is non-singular, it does not have zero as an eigenvlaue.

For a connected Riemannian manifold (M, g), the group $I(M)$ of isometries is a Lie transformation group of M with respect to the compact-open topology (Cf Kabayashi and Nomizu [13] Vo. I, page 239). For a transitivity of $I(M)$, an important theorem due to F. Brickell, whose proof is found in [17], states,

Theorem 4.1.1. Let M be a Riemannian s-manifold. Then $I(M)$ is transitive on M.

This theorem shows that M is diffeomorphic to the homogeneous space $G/_H$ of proposition 2.1.4., Chapter II, where H is the isotropy subgroup of a fixed point $p \in M$.

Remark 1 Affine s-manifolds are defined similarly to Riemannian s-manifolds, each symmetry (affine symmetry) s_x, $x \in M$ is an affine transformation, and $I(M)$ is replaced by $A(M)$, the Lie group of all affine transformations on M. The transitivity of $A(M)$ is proved in [17], where it is assumed that the symmetry tensor field S defined by $S_x = (ds_x)_x$ is smooth, which is not the case for Riemannian s-manifolds.

4.2 Riemannian (Locally) s-Regular Manifolds

Definition 4.2.1. An (M, g, s) is called a Riemannian s-regular manifold, if the symmetry tensor field S is smooth and invariant by each s_x, $x \in M$, i.e. if $X \in \mathfrak{X}(M)$, we have

$$ds_x(SX) = S(ds_x X)$$

In this case the Riemannian s-structure is called regular.

Proposition 4.2.2. Let M be a Riemannian s-regular manifold, then for all $x, y, z \in M$, such that $s_x(y) = z$, we have

$$s_x \circ s_y = s_z \circ s_x$$

Proof:- We have for any $X \in \mathfrak{X}(M)$

$$ds_x(SX)_y = S_{s_x(y)}(ds_x X_y) = S_{\mathfrak{z}}(ds_x X_y)$$

$$\therefore (ds_x \circ ds_y)(X_y) = (ds_{\mathfrak{z}} \circ ds_x)(X_y)$$

or $\quad d(s_x \circ s_y)(X_y) = d(s_{\mathfrak{z}} \circ s_x)(X_y)$

But $\quad s_x \circ s_y(y) = s_x(y) = \mathfrak{z}$, and $s_{\mathfrak{z}} \circ s_x(y) = s_{\mathfrak{z}}(\mathfrak{z}) = \mathfrak{z}$

Hence by theorem 1.4.3., Chapter I, we have

$$s_x \circ s_y = s_{\mathfrak{z}} \circ s_x \quad //$$

Definition 4.2.2. Let (M, g) be a Riemannian manifold which satisfies the following conditions,

(i) At each point $x \in M$ we can assign a local symmetry which has x as an isolated fixed point.

(ii) The tensor field of type (1, 1) defined on M by $S_x = (ds_x)_x$ for all $x \in M$ is smooth and locally invariant by each x. Then M is called a Riemannian locally s-regular manifold.

From the defintion of a Riemannian s-regular manifold, we see that it is also a Riemannian locally s-regular manifold.

Remark 2 Affine (locally) s-regular manifold is defined similarly to Riemannian (locally) s-regular manifold, where the symmetry tensor field S is already smooth, and each symmetry is an affine transformation.

Remark 3 Riemannian symmetric spaces are nothing but Riemannian s-regular manifolds, such that each symmetry is involutive, and where the regularity condition is trivially satisfied. The symmetry tensor field S at any point $x \in M$ is given by $S_x = -I$, where I is the identity transformation of M_x.

Let (M, g) be a Riemannian locally s-regular manifold, with Riemannian convections ∇. Define a new affine convection $\overline{\nabla}$ on M by

$$\overline{\nabla}_X Y = \nabla_X Y - D(X,Y) \quad , \quad X,Y \in \mathfrak{X}(M)$$

where $D(X,Y) = (\nabla_{(I-S)^{-1}X} S)(S^{-1}Y)$

here S - I is non-singular, since S does not have eigenvalue 1, and S - I is invertable.

Graham and Ledger [6] proved that all the symmetrics s_x, $x \in M$ are affine transformations with respect to the affine connection $\overline{\nabla}$. The affine manifold $(M, \overline{\nabla})$ is complete, and admits an analytic atlas in which S is analytic. If $\overline{R}$ and $\overline{T}$ are the curvature and the torsion tensors respectively, then $\overline{\nabla} R = \overline{\nabla} T = \overline{\nabla} g = o$. In [6] it is also shown that the relation between Riemannian (affine) locally s-regular manifolds and Riemannian (affine) s - regular manifolds is similar to the relation between Riemannian (affine) locally symmetric manifolds and Riemannian (affine) symmetric manifolds.

<u>4.3. k-symmetric spaces</u>:-

<u>(I) $k \geq 2$ is any integer</u>

<u>Definition 4.3.1.</u> A Riemannian locally s-manifold M which has at each point p M, a local symmetry s_p such that $s_p^k = id$, where $k \geq 2$ is the least positive integer of that property, is called a Riemannian locally k-symmetric manifold . The local symmetry is called, local k-symmetry and the s-structure $\{ s_p | p \in M \}$ is said to be or order k. M is said to be a Riemannian k-symmetric manifold if each symmetry s_p, p M can be extended such that its domain is the whole of M.

Theorem 4.3.1. (first proved by A.W. Deicke)

Let M be a Riemannian s-manifold, then M admits an s-structure of order k, for some integer $k \geq 2$.

Proof:- Let $p \in M$, and let H be the isotropy subgroup at p, then the symmetry at p, $s_p \in H$. Since M is a Riemannian manifold, then by proposition 2.1.2., Chapter II, H is compact. It is sufficient to prove the theorem at p only, since at any other point $q \in M$, the symmetry is given by $g \circ s_p \circ g^{-1}$, where $g \in I(M)$, the group of all isometries of M, and $g(p) = q$.

Let C be the subgroup of H generated by s_p, the closure $\bar{C}$ of C in H is an Abelian closed subgroup of H, hence it is a compact Abelian Lie group. We have two cases to consider.

(i) If the connected component $\bar{C}_o$ of $\bar{C}$ is trivial, then $\bar{C}$ is finite and the theorem is proved.

(ii) If $\bar{C}_o$ is not trivial, then by proposition 2.1.5., Chapter II it is a torus. By proposition 2.1.4., the elements of finite order are dense in $\bar{C}_o$. //

Let M be a Riemannian locally k-symmetric manifold, $k \geq 2$, suppose that the map $s : M \longrightarrow I(M)$ is differentiable, let $p \in M$ be any point, then in a neighbourhood U of p in M we have s_p^k = identity in U, this implies that $S_p^k = I$, where I is the identity transformation of M_p. The eigenvalues of S_p are k^{th} roots of unity, and since S is continous on M, each root is constant over M. S_p is real, non-singular, and does not have 1 as an eigenvalue, then the possible eigenvalues (we let S act on the complexification M_p^c of M_p) are -1 and pairs of conjugates $w_1, \bar{w}_1, \ldots, w_r, \bar{w}_r$. Since we are dealing with a Riemannian locally k-symmetric manifold, S_p is an orthogonal transformation of M_p. There exists an orthonormal frame $\{e_i\}$, $i = 1, \ldots n$, where n is the dimension of M, of M_p such that the matrix representation of S_p related to the mentioned frame is given by

$$\begin{bmatrix} -1 & & & & & & & & & & \\ & \ddots & & & & & & \bigcirc & & & \\ & & -1 & & & & & & & & \\ & & & \cos\phi_1 & -\sin\phi_1 & & & & & & \\ & & & \cos\phi_1 & \sin\phi_1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & \cos\phi_r & -\sin\phi_r & & & \\ & & \bigcirc & & & & \sin\phi_r & \cos\phi_r & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & x & x \\ & & & & & & & & & x & x \end{bmatrix}$$

where $w_1 = \cos\phi_1 + \sqrt{-1}\,\sin\phi_1$, ..., $\bar{w}_r = \cos\phi_r - \sqrt{-1}\,\sin\phi_r$

Let $T_p = S_p + S_p^{-1} = S_p + S_p^{t}$

$$\therefore\ T_p^{t} = (S_p + S_p^{t})^{t} = S_p^{t} + S_p = T_p$$

Hence T_p is symmetric. Let the eigenvalues of T_p be $\lambda_o, \ldots, \lambda_r$, then from the symmetry of T_p we have $M_p = M_{po} \oplus \ldots \oplus M_{pr}$ (direct sum), where M_{pj} $(j = o, \ldots, r)$ is the eigenspace which corresponds to the eigenvalue λ_j. The M_{pj}'s are orthogonal to each other.

Suppose that $X \in M_{pj}$

$\therefore \quad T_p(X) = \lambda_j X$

$$\therefore \quad T_p(S_p X) = (S_p + S_p^{-1})(S_p X) = S_p(S_p + S_p^{-1})(X) = S_p(T_p X)$$

$$= S_p(\lambda_j X) = \lambda_j(S_p X)$$

That is $S_p X \in M_{pj}$ if $X \in M_{pj}$

Hence we have on M mutually orthogonal differentiable distributions $M_o, \ldots, M_r$. The symmetry tensor field S decomposes into the form

$$S = S_o \oplus \ldots \oplus S_r$$

where S_i acts on M_i $(i = o, \ldots, r)$

(An outline of the above calculations can be found in Ledger and Obata [17]).

M. J. Field [5] made some studies of k-symmetric spaces, where he used the same notations, and followed a similar style of Kobayashi and Nomizu [5], Chapter XI.

He defined a k-symmetric space as a quadruple (G,H,s,k) G is a connected Lie group, H a closed subgroup of G with $(H_s)_o \subseteq H \subseteq H_s$, where H_s is the subgroup of G of fixed points of s and $(H_x)_o$ is the connected component of H_s, and s is an automorphism of order k of G. Field did not show that a k-symmetric manifold in the sense of definition 4.3.1. determines a quadruple (G,H,s,k) as in his definition. He also defined a k-symmetric Lie algebra as a quadruple (g,h,s,k,), consisting of a Lie algebra g, a subalgebra h of g, and an automorphism of order k of g such that h consists of all elements of g which are left fixed by s. Then he used the following Lemma for a natural decomposition of the associated k-symmetric Lie algebra of a k-symmetric space.

Lemma 4.3.2. Let $T: V \longrightarrow V$ be a linear map of a finite dimensional vector space V into itself, and suppose that $f(t) = g(t)h(t)$ are polynomials such that $f(T) = o$, and $g(t)$ and $h(t)$

are relatively prime. Then V is a direct sum of the T- invariant subspaces U and W, where U = kernel g (T) and W = kernel h (T).

Proof: - See Lipschutz [19] Chapter 10, page 232.

Combining the work of Field [5] and the proof of theorem 3.2.4, Chapter III, it can be shown that a quadruple (G, H, s, k) as defined above, with the assumption that Ad (H) is compact determines a Riemannian k -symmetric manifold as in the definition 4.3.1.

Theorem 4.3.3. Let (G, H, s, k) be a quadruple, G is a connected Lie group, H is a closed subgroup of G with $(H_s)_o \subset H \subset H_s$, and s is an automorphism of G of order k. Further assume that Ad(H) is compact. In each G- invariant, s - invariant Riemannian structure Q on G/H, the manifold G/H is a Riemannian k - symmetric manifold. The symmetry $s_o (o \equiv H)$ satisfies

$$s_o \circ \pi = \pi \circ \sigma$$
$$\tau(\sigma(g)) = s_o \tau(g) \circ s_o$$

where $\pi : G \longrightarrow G/H$ is the natural map, and $\tau(g)$ is the action of g on G/H.

Proof: - s induces an automorphism $(ds)_e$ of the Lie algebra g of G, such that $(ds)_e^k = I$. Consider the polynomial $f(t) = t^k - 1 = (t-1)(t^{k-1} + \ldots + 1)$, where g(t) = (t-1) and $h(t) = t^{k-1} + \ldots + 1$ are relatively prime. Moreover f ((ds)e) = O. Hence we have a decomposition.

$$\underline{g} = \underline{h} \oplus \underline{m}$$

where

$$\underline{h} = \{X \in \underline{g} \mid (ds)_e X = X\}$$
and $\underline{m} = \{X \in \underline{g} \mid X + (ds)_e X + \ldots + (ds)_e^{k-1} X = O\}$

Let $X \in \underline{m}$, and $k \in H$. Then for $t \in \mathbb{R}$ we have

$$s(\exp Ad(k)(tX)) = s(Ad(k)\exp(tX)) = s(k\exp(tX)k^{-1})$$
$$= s(k) \circ s(\exp(tX)) \circ s(k^{-1}) = k \circ s(\exp(tX)) \circ k^{-1}$$
$$= Ad(k)\, s(\exp(tX)) = Ad(k)\exp((ds)_e\, tX)$$
$$= \exp(Ad(k)((ds)_e(tX)))$$

i.e. $(ds)_e(Ad(k)(tX)) = Ad(k)((ds)_e(tX))$

Now

$$Ad(k)(X) = -Ad(k)(ds)_e(X) - \dots - Ad(k)(ds)_e^{k-1}(X)$$
$$= -(ds)_e(Ad(k)(X)) - \dots - (ds)_e^{k-1}(Ad(k)(X))$$

i.e. $Ad(H)\underline{m} \subset \underline{m}$, and therefore $G/_H$ is a reductive homogeneous space.

From the proof of theorem 3.2.4., Chapter III, we can construct a Riemannian structure Q, invariant by G. We define $s_o : G/_H \longrightarrow G/_H$ by $s_o \circ \pi = \pi \circ s$, it is easy to verify that s_o^k = identity. We also have by theorem 3.2.4. that $\Upsilon(\sigma(g)) = s_o \circ \Upsilon(g) \circ s_o$, for all $g \in G$.

Our aim now is to prove that s_o is an isometry, having that done, then any symmetry at any point $p \in G/_H$ is given by $s_p = \Upsilon(g) \circ s_o \circ \Upsilon(g^{-1})$, where $p = \Upsilon(g) \cdot o$, $g \in G$.

Let $g \in G$, $q = \Upsilon(g) \cdot o$, let $X, Y \in (G/_H)_q$, then the vectors $X_o = d\Upsilon(g^{-1})(X)$, $Y_o = d\Upsilon(g^{-1})(Y)$ belong to $(G/_H)_o$. Let $X'_o, Y'_o \in \underline{m}$, such that $X_o = (d\pi)_e(X'_o)$, and $Y_o = (d\pi)_e(Y'_o)$. Then

$$Q_{s_o(q)}((ds_o)(X),(ds_o)(Y)) = Q_{s_o(q)}((ds_o) \circ d\Upsilon(g)(X_o), (ds_o) \circ d\Upsilon(g)(Y_o))$$
$$= Q_{s_o(q)}(d\Upsilon(s(g))(ds_o)(X_o), d\Upsilon(s(g))(ds_o)(Y_o)) = Q_o((ds_o)(X_o),(ds_o)(Y_o))$$
$$= Q_o((ds_o) \circ (d\pi)_e(X'_o), (ds_o) \circ (d\pi)_e(Y'_o)) = Q_o((d\pi)_e \circ (ds)_e(X'_o),(d\pi)_e(ds)_e(Y'_o))$$
$$= \overline{B}((ds)_e(X'_o),(ds)_e(Y'_o)) = \overline{B}(X'_o, Y'_o) = Q_o((d\pi)_e(X'_o),(d\pi)_e(Y'_o))$$
$$= Q_o(X_o, Y_o) = Q_q(X,Y)$$

Where $\overline{B}$ is the corresponding symmetric bilinear form on $\underline{m} \times \underline{m}$ induced by the quadratic form B on $\underline{m}$ of theorem 3.2.4. Also $\bar{B}((ds)_e(X'_o), (ds)_e(Y'_o)) = \bar{B}(X'_o, Y'_o)$ since B is s-invariant by assumption. //

(II) k>2 is any odd integer

Proposition 4.3.4. Let M be a Riemannian locally k - symmetric manifold, where k is odd, then there exists an almost complex structure J on M, which makes M into an almost Hermitian manifold.

Proof: - Since k is odd, the symmetry tensor field S does not have -1 as an eigenvalve, and hence the eigenvalues appear as pairs of conjugates. From the calculations in part (I) of this section, we see that at each point $p \in M$, the tangent space $M_p = M_{p1} \oplus \cdots \oplus M_{pr}$ (direct sum), and hence we have mutually orthogonal differentiable distributions on M, $M_1, \ldots, M_r$, where M_j $(j = 1 \ldots, r)$ corresponds to the eigenvalues $\cos\phi_j \pm \sqrt{-1}\,\sin\phi_j$. Also the symmetry tensor field decomposes into the form

$$S = S_1 \oplus \cdots \oplus S_r$$

where S_j acts on M_j. Consider the matrix representation of S at a point $p \in M$ with respect to the orthogonal frame mentioned of part (I). It is in the form.

$$\begin{bmatrix} \cos\phi_1 & -\sin\phi_1 & & & & & & \\ \sin\phi_1 & \cos\phi_1 & & & & & \bigcirc & \\ & & \ddots & & & & & \\ & & & \cos\phi_r & -\sin\phi_r & & & \\ & & & \sin\phi_r & \cos\phi_r & & & \\ & & & & & \ddots & & \\ & \bigcirc & & & & & \times & \times \\ & & & & & & \times & \times \end{bmatrix}$$

If we restrict ourselves to the way S_{pj} acts on M_{pj} we have the matrix representation of S_{pj} in the form

$$\begin{bmatrix} \cos\phi_j & -\sin\phi_j & & & \\ \sin\phi_j & \cos\phi_j & & & \\ & & \ddots & & \\ & & & \cos\phi_j & -\sin\phi_j \\ & & & \sin\phi_j & \mathrm{sos}\,\phi_j \end{bmatrix}$$

$$\therefore\ S_{pj} = (\cos\phi_j)\, I + (\sin\phi_j)\, J_{pj}$$

where I is the identity transformation of M_{pj} and $J_{pj}: M_{pj} \longrightarrow M_{pj}$ such that $J_{pj}^2 = -I$, and this is true for all $j = 1, \ldots, r$.

Let $J_p = J_{p1} \oplus \cdots \oplus J_{pr}$

then $J_p: M_p \longrightarrow M_p$ is a linear transformation of M_p such that $J_p^2 = -I$ and this is true for all $p \in M$, hence we have an almost complex structure J on M.

With respect to the orthogonal frame mentioned, each J_{pj} $(j = 1, \ldots, r)$ can be represented by the matrix

$$\begin{bmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{bmatrix}$$

which is orthogonal. Hence for all $p \in M$, $J_p = J_{p1} \oplus \cdots \oplus J_{pr}$

is orthogonal, and this implies that M is an almost Hermitian manifold with respect to J. //

Proposition 4.3.5. Let M be a Riemannian locally k-symmetric space, where k is odd, let J be the associated almost complex structure on M. Then the following are equivalent.

(i) M is locally regular.

(ii) Each local symmetry is almost complex with respect to J.

Proof; - (i) for simplicity, let X be a vector field on M which belongs to the distribution M_j. Let $p \in M$. Then $(ds_p)(SX)_q = (ds_p)(S_j X)_q$

$$= (ds_p)\left[(\cos\phi_j) X_q + (\sin\phi_j)(J_{qj} X_q)\right]$$

$$= (\cos\phi_j)(ds_p X_q) + (\sin\phi_j)(ds_p o J_{qj})(X_q)$$

Now, if $X_q \in M_{qj}$, then $ds_p X_q \in M_{s_p(q)j}$, because if we assume that $ds_p X_q$ belongs to another subspace, $M_{s_p(q)r}$ say, then we have from the regularity of M that,

$$S_{s_p(q)}(ds_p X_q) = ds_p (S_q X_q)$$

or

$$(\cos\phi_r)(ds_p X_q) + (\sin\phi_r) J_{s_p(q)}(ds_p X_q)$$

$$= (\cos\phi_j)(ds_p X_q) + (\sin\phi_j)\, ds_p\, (J_q X_q)$$

But X_q is perpendicular to $J_q X_q$, and since s_p is an isometry, we also have $ds_p X_q$ is perpendicular to $ds_p(J_q X_q)$ and $ds_p X_q$ is perpendicular to $J_{s_p(q)}(ds_p X_q)$.

$$\therefore \quad (\cos\phi_r - \cos\phi_j)(ds_p X_q) = o \implies \cos\phi_r = \cos\phi_j$$

and we must have $ds_p X_q \in M_{s_p(q)j}$

If $S_{s_p(q)}(ds_p X_q) = \sum_{i=1}^{r} (\cos\phi_i)\, ds_p X_q + (\sin\phi_i) J_{s_p(q)}(ds_p X_q)$

then by similar argument as above we can show that $ds_p X_q \in M_{s_p(q)j}$

$\therefore$ $S(ds_p X_q) = S_{s_p(q)j}(ds_p X_q) = \left[(\cos\phi_j) I + (\sin\phi_j) J_{s_p(q)j}\right](ds_p X_q)$

$$= (\cos\phi_j)(ds_p X_q) + (\sin\phi_j)(J_{s_p(q)j} \circ ds_p)(X_q)$$

Since $\cos\phi_j \neq o$ and $\sin\phi_j \neq o$ for all $j=1, \ldots, r$, we have

$$(ds_p) \circ J_{qj} = J_{s_p(p)} \circ (ds_p)$$

which is true for all $j = 1, \ldots, r$. Hence we have

$$(ds_p) \circ J_q = J_{s_p(q)} \circ (ds_p)$$

and this proves that s_p is an almost complex local isometry.

(ii) Assume that each local symmetry is almost complex with respect to J, i.e. if $p \in M$

$$(ds_p) \circ J_q = J_{s_p(q)} \circ (ds_p)$$

But $J_q = J_{q1} \oplus \ldots \oplus J_{qr}$. Therefore

$$(ds_p) \circ (J_{qj}) = (J_{s_p(q)j}) \circ (ds_p) \qquad j=1, \ldots, r$$

and this gives us that for $X_q \in M_{qj}$, we have

$(\cos\phi_j)(ds_p X_q) + (\sin\phi_j)(ds_p \circ J_{qj})(X_q)$

$$= (\cos\phi_j)(ds_p X_q) + (\sin\phi_j)(J_{s_p(q)j} \circ ds_p)(X_q)$$

or

$(ds_p)(SX)_q = S_{s_p(q)}(ds_p X_q)$, and hence M is regular //

Proposition 4.3.6. Let C (M) be the group of all almost complex isometries on a Riemannian k - regular symmetric manifold, where k is odd. Then C (M) is a transitive Lie transformation group on M.

Proof: - For all $p \in M$, $s_p \in C(M)$, and since in the proof of theorem 4.1.1. in [17] only symmetrics are used, we conclude that C (M) is transitive on M. Let $\{f_n\}$ be a sequence of almost complex isometries which converges to f in I (M). By assumption, the symmetry tensor field is continuous, and hence, the assosiated almost complex structure J is continuous. We have $df_n \circ J = J \circ df_n$, for all n, from the continuity of J we see that $df \circ J = J \circ df$, and $f \in C(M)$. Hence C(M) is closed in I(M). By proposition 2.1.1. Chapter II, we see that C(M) is a Lie transformation group of M. //

Let M be a Riemannian k - regular symmetric manifold, where $k > 2$ is odd integer. Let G be the ~~largest connected~~ identity component of C(M), the group of almost complex isometries on M. Let $x \in M$ be a fixed point, and denote by H the isotropy subgroup of G at x. Finally, let J be the associated almost complex structure on M. Now, we will give a proof analogous to the proof of theorem 3.2.3. Chapter III.

Theorem 4.3.7. Let M be a Riemannian k - regular symmetric manifold, where $k > 2$ is odd integer. Then

(i) The k - symmetry s_x induces a k - automorphism σ of G defined by

$$\sigma(g) = s_x \circ g \circ s_x^{-1} \quad , \text{ for all } g \in G$$

If H_σ is the subgroup of G of fixed points of σ. Then

$$(H_\sigma)_o \subseteq H \subseteq H_\sigma$$

where $(H_\sigma)_o$ is the identity component of H_σ. Also H contains no normal subgroup of G other than e.

(ii) Let $\underline{g}$ and $\underline{h}$ denote the Lie algebras of G and H respectively. Then $\underline{h} = \{X \in \underline{g} \mid (d\sigma)_e X = X\}$, and if we have $\underline{m} = \{X \in \underline{g} \mid X + (d\sigma)_e X + \dots\dots + (d\sigma)_e^{k-1} X = o\}$. Then $\underline{g} = \underline{m} \oplus \underline{h}$ (direct sum).

Let π be the natural map $\pi : G \to M$ given by $g \mapsto g \cdot x$. Then $(d\pi)_e$ maps $\underline{h}$ onto $\{0\}$ and $\underline{m}$ isomorphically onto M_x.

Proof: (i) $\sigma: C(M) \longrightarrow C(M)$ is an automorphism of $C(M)$, and since it maps connected components to connected components, it is also an automorphism of G. Now

$$\sigma^2(g) = \sigma(\sigma(g)) = \sigma(s_x \circ g \circ s_x^{-1}) = s_x^2 \circ g \circ s_x^{-2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\sigma^k(g) = \sigma(\sigma^{k-1}(g)) = s_x^k \circ g \circ s_x^{-k} = g,$$

which proves that σ is a k-automorphism of G.

Let $h \in H$, then at x we have

$$\left[d(\sigma(h))\right]_x = (ds_x \circ dh \circ (ds_x)^{-1})_x$$

$$= \left[((\cos\phi_1)I + (\sin\phi_1)J_1) \oplus \dots \oplus ((\cos\phi_r)I + (\sin\phi_r)J_r)\right]_x \circ dh \circ$$
$$\left[((\cos\phi_1)I + (\sin\phi_r)J_1) \oplus \dots \oplus ((\cos\phi_r)I + (\sin\phi_r)J_r)\right]_x^{-1}$$

$$= \left[((\cos\phi_1)I + (\sin\phi_1)J_1) \oplus \dots \oplus ((\cos\phi_r)I + (\sin\phi_r)J_r)\right]_x \circ dh \circ$$
$$\left[((\cos\phi_1)I + (\sin\phi_1)J_1)^{-1} \oplus \dots \oplus ((\cos\phi_r)I + (\sin\phi_r)J_r)^{-1}\right]_x$$

$$= \left[((\cos\phi_1)I + (\sin\phi_1))J\ ) \oplus \dots \oplus ((\cos\phi_r)I + (\sin\phi_r)J_r)\right]_x \circ dh \circ$$
$$\left[((\cos\phi_r)I + (\sin\phi_r)J_r)^{k-1} \oplus \dots \oplus ((\cos\ _1)I + (\sin\ _1)J_1)^{k-1}\right]_x$$

[where from linear algebra we used that if $T : V \longrightarrow V$ is a linear automorphism of a finite dimensional vector space V, and $T = T_1 \oplus T_2$ with respect to T-invariant direct-sum decomposition, then $T^{-1} = T_1^{-1} \oplus T_2^{-1}$ and $T = T_2 \oplus T_1$]

But h is an almost complex isometry i.e. for all $j = 1, \dots, r$, $(dh) \circ J_{xj} = J_{xj} \circ (dh)$

$$\therefore \quad \left[d(\sigma(h))\right]_x = (dh)_x$$

Also $\sigma(h)(x) = (s_x \circ h \circ s_x^{-1})(x) = x$ and $h(x) = x$

$$\therefore \quad \sigma(h) = h \text{ and } H \subset H_\sigma$$

The rest of (i) is similar as in proof of theorem 3.2.3.

(ii) From proposition 2.2.1. $(d\sigma)_e$ is an automorphism of $\underline{g}$ of order k i.e.

$$(d\sigma)_e^k - I = 0$$

Consider the polynomial $\quad f(t) = t^k - 1 = (t-1)(t^{k-1} + \dots + 1)$

where $g(t) = (t-1)$ and $h(t) = t^{k-1} + \ldots + 1$ are relatively prime.

Moreover $f((d\sigma)_e) = o$.

Hence by Lemma 4.3.2., we have

$$\underline{g} = \underline{h} \oplus \underline{m}$$

where $\underline{h}$ = kernel $g((d\sigma)_e) = \{X \in \underline{g} \mid (d\sigma)_e X = X\}$
and $\underline{m}$ = kernel $h((d\sigma)_e) = \{X \in \underline{g} \mid X + (d\sigma)_e X + \ldots + (d\sigma)_e^{k-1} X = o\}$

The rest of (ii) is as in the proof of theorem 3.2.3. //

Proposition 4.3.8. Let M be a Riemannian k-regular symmetric manifold, where $k > 2$ is an odd integer. Let J be the associated almost complex structure on M. Then every almost Hermitian totally geodesic submanifold of M is k-regular symmetric.

Proof:- Let P be an almost Hermitian totally geodesic submanifold of M. Let $x \in P$ be any point. Then for any $X \in P_x$, $JX \in P_x$ since P is an almost Hermitian submanifold. If S is the symmetry tensor field on M, then

$$S_x X = \left[((\cos\phi_1)I + (\sin\phi_1)J) \oplus \ldots \oplus ((\cos\phi_r)I + (\sin\phi_r)J_r)\right]_x(X)$$

$\therefore$ $S_x X = (ds_x)_x X \in P_x$, and for any positive integer l, such that $l \leq k$

$$S_x^l X \in P_x$$

this implies that $S_x^k X = X$, i.e. S_x^k is the identity on P_x.

Further P is totally geodesic, this implies that s_x is an isometry in a neighbourhood U of x in P, and that s_x^k = identity in U. Hence any symmetry in P is obtained by the restriction of a symmetry in M, and the almost complex structure on P is the restriction of the almost complex structure J on M. This insures that each symmetry is almost complex isometry, and P is a Riemannian k-regular symmetric. //

CHAPTER V

Riemannian 5-Symmetric Manifolds

In this Chapter, we shall consider the case of a Riemannian (locally) 5-(regular) symmetric manifold. Of course, all the results in Chapter IV section (4) part II go over when k = 5. We will generalize some of the results valid for (pseudo) Riemannian 3-regular symmetric manifolds studied by Gray [8].

In section 5.1. we state the important theorems of a (pseudo) Riemannian 3-symmetric manifold. Section 5.3. deals with the curvature relations of a Riemannian locally 5-regular symmetric manifold. Finally, in the last section of this chapter we discuss some properties of a Riemannian 5-regular symmetric manifold, when considered as a reductive homogeneous space.

5.1. Riemannian (locally) 3-Regular Symmetric Manifold

A Riemannian manifold M, which admits at each point $x \in M$, an (a local) isometry of order 3, having x as an isolated fixed point was studied by Gray [8]. He defined such a manifold in a different way from definition 4.3.1., when k = 3, but as we shall see later if the regularity condition is imposed, the two definitions come out to be the same.

Gray [8] first considered an almost Hermitian manifold M with almost complex structure J. Then by putting $S = -\frac{1}{2} I + \sqrt{3}/_2 J$, where I is the identity, he showed that at each point $p \in M$, there exists a neighbourhood U and a diffeomorphism s_p: $V \longrightarrow U$, such that s_p is of order 3, and has p as an isolated fixed point. Further $(ds_p)_p = S_p$. A family of local cubic diffeomorphisms on a manifold M is then defined as a map $p \longmapsto s_p$, which assigns to each point $p \in M$, a diffeomorphism s_p on a neighbourhood U of p in M, and has p as an isolated fixed point. It is then proved that this family give rise to a smooth almost complex structure J on M, called the canonical almost complex structure of the family.

Definition 5.1.1. (Due to A. Gray) A Riemannian locally 3-symmetric space M is an analytic Riemannian manifold M together with a family of cubic diffeomorphisms $p \mapsto s_p$, such that s_p is a holomorphic (almost complex) isometry, in a neighbourhood U of p in M, with respect to the canonical almost complex structure of the family. If the domain of each local cubic isometry is all of M, then M is called Riemannian 3-symmetric space.

Graham and Ledger [6] showed that a Riemannian 3-symmetric manifold M defined as in definition 4.3.1., when k = 3, always admits an analytic atlas, where the symmetry tensor field S is analytic. If M is regular, then by proposition 4.3.5. each symmetry is an almost complex isometry (holomorphic in definition 5.1.1.) with respect to the almost complex structure J, (called the canonical almost complex structure in definition 5.1.1.) and finally, from proposition 4.3.4., the symmetry tensor field is given by

$$S = -\tfrac{1}{2} I + \sqrt{3}/_2 J$$

from this, we see that a Riemannian 3-regular symmetric manifold is in fact the same as Riemannian 3-symmetric space defined by Gray [8].

Proposition 5.1.1. Let M be a Riemannian locally 3-regular symmetric manifold. Then

(i) $R(X,Y,Z,W) = R(JX,JY,Z,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW)$
$X,Y,Z,W \in \mathcal{X}(M)$

(ii) $\nabla_V(R)(X,Y,Z,W) + \nabla_V(R)(JX,JY,JZ,JW) = o\ ,\quad V,X,Y,Z,W \in \mathcal{X}(M)$

Proof:- See Gray [8] page 24.

Theorem 5.1.2. Let (G,H,t,3) be a symmetric quadruple, where G is a connected Lie group, H is a closed subgroup of G with $(H_t)_o \subset H \subset H_t,$

and t is an automorphism of order 3 of G. Further, assume that Ad(H) is compact.

(i) In each G-invariant, t-invariant Riemannian structure Q on $G/_H$, the manifold $G/_H$ is a Riemannian 3-regular symmetric manifold.

(ii) If we write $(dt)_e \mid \underline{m} = -\frac{1}{2}I + \sqrt{3}/_2 J$, then J induces the almost complex structure on $G/_H$, and Q is almost Hermitian with respect to J.

Proof:- (i) Exactly the same proof of theorem 4.3.3., where in this case k = 3, and t = s.

(ii) See Gray [8], page 35.

Proposition 5.1.3. Let M be a Riemannian 3-regular symmetric manifold, with almost complex J. Then the following conditions are equivalent.

(i) M is naturally reductive.

(ii) The almost complex structure J is nearly Kählerian.

Proof:- See Gray [8], page 36.

Using proposition 5.1.3. we have the following

Proposition 5.1.4. A totally umbilic almost Hermitian submanifold M of a naturally reductive Riemannian 3-regular symmetric manifold N is a naturally reductive Riemannian 3-regular symmetric manifold.

Proof:- From proposition 5.1.3. N is nearly Kähler. By proposition 1.6.4., M is a minimal submanifold, but M is totally umbilic, i.e. if $p \in M$ is any point, then using the notations of section 5, Chapter I, we have

$$A_N \big|_p = \lambda I$$

where λ is a scaler and I is the identity transformation of M_p.

$$\text{trace } A_N = n\lambda = o \iff \lambda = o$$

$$\therefore A_N \big|_p = o \ , \text{ for all } p \in M.$$

∴ M is a totally geodesic almost Hermitian submanifold of N. By proposition 4.3.8. M is a Riemannian 3-regular symmetric manifold.

From proposition 1.6.3. M is a nearly Kähler, hence by proposition 5.1.3. M is naturally reductive. //

Gray [8] gave a classification of (pseudo) Riemannian locally 3-regular symmetric manifolds, his arguments depended on a joint work done by him and Wolf [25].

Proposition 5.1.4. Let M be a regular Riemannian s-manifold of order k, such that the only eigenvalues of the symmetry tensor field S of M are θ and $\bar{\theta}$ (θ is not real). Then either M is locally symmetric, or k = 3.

Proof:- See Ledger and Obata [17].

Riemannian 3-symmetric manifolds appear in the recent work of Ledger and Pettitt [18]. They consider a metrizable s-regular manifold (M, s) (i.e. for some metric g, M is a Riemannian s-regular manifold) for which the symmetry tensor field S has a quadratic minimal polynomial, in this case (M, s) is called a quadratic s-manifold. Such manifolds are found to admit an almost complex structure J. It is also proved that either all the symmetries are of order 3 or J is integrable, and there exists a metric g such that (M, g) is Hermitian symmetric with respect to J. A classification up to equivalence of all compact quadratic manifolds (M, s) is given.

One may ask about a classification of a metrizable s-regular manifolds for which the symmetry tensor field S has a minimal polynomial $S^4 + \alpha S^3 + \beta S^2 + \gamma S + \nu I = o$, and in general a classification of a metrizable s-regular manifolds, for which the symmetry tensor field S has minimal polynomial $S^m + \ldots + \nu I = o$, where m is an even integr $> o$.

5.2 Riemannian (locally) 5-Symmetric Manifolds:-

If we put k = 5 in definition 4.3.1., we get the definition of a Riemannian (locally) 5-regular symmetric manifold.

In the following proposition we will prove a statement similar to the one given in proposition 5.1.4., where we assume that the symmetry tensor field has four distinct eigenvalues.

Proposition 5.2.1. Let M be a Riemannian s-manifold of order k, such that the eigenvalues of the symmetry tensor field S are θ_1, $\bar{\theta}_1$, θ_2 and $\bar{\theta}_2$, where all the θ's are distinct (θ_1, θ_2 are not real). Further assume that $\theta_1^2 \neq \bar{\theta}_1$ and $\theta_2^2 \neq \bar{\theta}_2$. Then either

(i) k = 5 or (ii) M is locally symmetric.

Proof:- At each point $x \in$ M, denote the θ_1-eigenspace and θ_2-eigenspace of S_x on the complex tangent space M_x^c by N_{1x} and N_{2x}. Then their complex conjugates $\bar{N}_{1x}$ and $\bar{N}_{2x}$ are the $\bar{\theta}_1$-eigenspace and $\bar{\theta}_2$-eigenspace. Let D_1, D_2, $\bar{D}_1$, and $\bar{D}_2$ be the complex distributions which assign N_{1x}, N_{2x}, $\bar{N}_{1x}$ and $\bar{N}_{2x}$ at x. If X and Y are complex-valued vector fields, then

$$S_x[X,Y]_x = (ds_x)_x [X,Y]_x = [ds_xX, ds_xY]_x = [SX,SY]_x$$

Consider the following cases

(1) $(X,Y \in D_1)$; $[\theta_1 X, \theta_1 Y]_x = \theta_1^2 [X,Y]_x$, then either $[X,Y]_x = 0$

or one of the following is valid

(i) $\theta_1^2 = \theta_2$, (ii) $\theta_1^2 = \bar{\theta}_2$

(2) $(X,Y \in D_2)$; $[\theta_2 X, \theta_2 Y]_x = \theta_2^2 [X,Y]_x$, then either $[X,Y]_x = 0$

or one of the following is valid

(i) $\theta_2^2 = \bar{\theta}_1$, (ii) $\theta_2^2 = \theta_1$

(3) $(X,Y \in \bar{D}_1)$; $[\bar{\theta}_1 X, \bar{\theta}_1 Y]_x = \bar{\theta}_1^2 [X,Y]_x$, then either $[X,Y]_x = o$

or one of the following is valid

(i) $\bar{\theta}_1^2 = \theta_2$, (ii) $\bar{\theta}_1^2 = \bar{\theta}_2$, and this implies that either

$[X,Y]_x = o$, or one of the following is valid

(i) $\theta_1^2 = \bar{\theta}_2$, (ii) $\theta_1^2 = \theta_2$ (the same as (1))

(4) $(X,Y \in \bar{D}_2)$; $[\bar{\theta}_2 X, \bar{\theta}_2 Y]_x = \bar{\theta}_2^2 [X,Y]_x$, then either $[X,Y]_x = o$

or one of the following is valid

(i) $\bar{\theta}_2^2 = \theta_1$, (ii) $\bar{\theta}_2^2 = \theta_1$, and this implies that either

$[X,Y]_x = o$, or one of the following is valid

(i) $\theta_2^2 = \bar{\theta}_1$, (ii) $\theta_2^2 = \theta_1$ (the same as (2))

(5) $(X \in D_1, Y \in D_2)$; $[\theta_1 X, \theta_2 Y]_x = \theta_1 \theta_2 [X,Y]_x$. One of the

following cases is valid

(i) $\theta_1 \theta_2 = \theta_1$, (ii) $\theta_1 \theta_2 = \theta_2$, (iii) $\theta_1 \theta_2 = \bar{\theta}_1$,

(iv) $\theta_1 \theta_2 = \bar{\theta}_2$ (v) $[X,Y]_x = o$

(i) and (ii) are rejected. For (iii) $\theta_1 \theta_2 = \bar{\theta}_1 \Leftrightarrow \theta_1 \theta_2 \bar{\theta}_2 \theta_1 = \bar{\theta}_1 \bar{\theta}_2 \theta_1$

$\Leftrightarrow \theta_1^2 = \bar{\theta}_2$. For (iv) $\theta_1 \theta_2 = \bar{\theta}_2 \Leftrightarrow \theta_2^2 = \bar{\theta}_1$. Hence either

$[X,Y]_x = o$, or one of the following is valid

(i) $\theta_1^2 = \bar{\theta}_2$, (ii) $\theta_2^2 = \bar{\theta}_1$

(6) $(X \in D_1, Y \in \bar{D}_1)$; $[\theta_1 X, \theta_1 Y]_x = \theta_1 \bar{\theta}_1 [X,Y]_x = [X,Y]_x \Rightarrow [X,Y]_x = o$

(7) $(X \in D_1, Y \in \bar{D}2)$; $[\theta_1 X, \bar{\theta}_2 Y]_x = \theta_1 \bar{\theta}_2 [X,Y]_x$. One of the

following is valid.

(i) $\theta_1 \bar{\theta}_2 = \theta_1$, (ii) $\theta_1 \bar{\theta}_2 = \bar{\theta}_2$, (iii) $\theta_1 \bar{\theta}_2 = \theta_2$, (iv) $\theta_1 \bar{\theta}_2 = \bar{\theta}_1$

(v) $[X,Y]_x = o$

(i) and (ii) are rejected. For (iii) $\theta_1 \bar{\theta}_2 = \theta_2 \Leftrightarrow \theta_2^2 = \theta_1$, and for (iv) $\theta_1 \bar{\theta}_2 = \bar{\theta}_1 \Leftrightarrow \theta_1^2 = \theta_2$. Hence either $[X,Y]_x = o$, or one of the following is valid

(i) $\theta_2^2 = \theta_1$, (ii) $\theta_1^2 = \theta_2$.

(8) $(X \in D_2, Y \in D_1)$; $[\theta_2 X, \bar{\theta}_1 Y]_x = \theta_2 \bar{\theta}_1 [X,Y]_x$. One of the following cases is valid:-

(i) $\bar{\theta}_1 \theta_2 = \bar{\theta}_1$, (ii) $\bar{\theta}_1 \theta_2 = \theta_2$, (iii) $\bar{\theta}_1 \theta_2 = \theta_1$,

(iv) $\bar{\theta}_1 \theta_2 = \bar{\theta}_2$, (v) $[X,Y]_x = o$

(i) and (ii) are rejected. For (iii) $\bar{\theta}_1 \theta_2 = \theta_1 \Leftrightarrow \theta_1^2 = \theta_2$, and for (iv) $\bar{\theta}_1 \theta_2 = \bar{\theta}_2 \Leftrightarrow \theta_2^2 = \theta_1$. Hence, either $[X,Y]_x = o$, or one of the following is valid

(i) $\theta_1^2 = \theta_2$, (ii) $\theta_2^2 = \theta_1$ (the same as (7))

(9) $(X \in D_2, Y \in \bar{D}_2)$; $[\theta_2 X, \bar{\theta}_2 Y]_x = \theta_2 \bar{\theta}_2 [X,Y]_x = [X,Y]_x \Rightarrow [X,Y]_x = o$

(10) $(X \in \bar{D}_1, Y \in \bar{D}_2)$; $[\bar{\theta}_1 X, \bar{\theta}_2 Y]_x = \bar{\theta}_1 \bar{\theta}_2 [X,Y]_x$. One of the following cases is valid

(i) $\bar{\theta}_1 \bar{\theta}_2 = \bar{\theta}_1$, (ii) $\bar{\theta}_1 \bar{\theta}_2 = \bar{\theta}_2$, (iii) $\bar{\theta}_1 \bar{\theta}_2 = \theta_1$, (iv) $\bar{\theta}_1 \bar{\theta}_2 = \theta_2$

(v) $[X,Y]_x = o$

(i) and (ii) are rejected. For (iii) $\bar{\theta}_1 \bar{\theta}_2 = \theta_1 \Leftrightarrow \theta_1^2 = \bar{\theta}_2$, and for

$\bar{\theta}_1 \bar{\theta}_2 = \theta_2 \Longleftrightarrow \theta_2^2 = \bar{\theta}_1$. Hence, either $[X,Y]_x = o$, or one of the following is valid.

(i) $\theta_1^2 = \bar{\theta}_2$, (ii) $\theta_2^2 = \bar{\theta}_1$ (the same as (5))

All the cases from (1) to (10) are reduced to

(I) (i) $\theta_1^2 = \theta_2$ or (ii) $\theta_1^2 = \bar{\theta}_2$ or (iii) $[X,Y]_x = o$

and (II) (i) $\theta_2^2 = \theta_1$ or (ii) $\theta_2^2 = \bar{\theta}_1$ or (iii) $[X,Y]_x = o$

and (III) (i) $\theta_1^2 = \bar{\theta}_2$ or (ii) $\theta_2^2 = \bar{\theta}_1$ or (iii) $[X,Y]_x = o$

and (IV) (i) $\theta_2^2 = \theta_1$ or (ii) $\theta_1^2 = \theta_2$ or (iii) $[X,Y]_x = o$

Assume that (I) (i) is valid i.e. $\theta_1^2 = \theta_2$. Square both sides we have $\theta_1^4 = \theta_2^2$. This with (II) (i) gives $\theta_1^4 = \theta_1 \Longleftrightarrow \theta_1^3 = 1$ which is rejected, while $\theta_1^4 = \theta_2^2$ with (II) (ii) gives $\theta_1^4 = \bar{\theta}_1 \Longleftrightarrow \theta_1^5 = 1$. (III) and (IV) do not give any new information. Similarly, if we assume that (I) (ii) is valid, then this with (II) (i) give $\theta_2^5 = 1$, while with (II) (ii) give $\theta_2^3 = 1$, which is rejected. (III) and (IV) do not give any new information.

Finally, assume that $[X,Y]_x = o$. From theorem 2.1.4., $M = G/H$, where G is the largest connected component of the Lie group I(M) of all isometries of M, and H is the isotropy subgroup of G at some fixed point $x \in M$. G/H is a reductive homogeneous space, with fixed decomposition of the Lie algebra $\underline{g}$ of G, i.e. $\underline{g} = \underline{h} \oplus \underline{m}$ (direct sum), where $\underline{h}$ is the Lie algebra of H, and $\underline{m}$ is a subspace of $\underline{g}$ such that $Ad(H)(\underline{m}) \subset \underline{m}$. Denote by $\underline{m}^c$ the complexification of $\underline{m}$, and since we have $[X,Y] = o$, for all complex vector fields X and Y, we have $[\underline{m}^c , \underline{m}^c] \subseteq \underline{h}^c$, where $\underline{h}^c$ is the complexification of $\underline{h}$. We also have $\underline{m} = \underline{m}^c \cap \underline{g}$

$$\therefore [\underline{m}, \underline{m}] = [\underline{m}^c \cap \underline{g}, \underline{m}^c \cap \underline{g}] = [\underline{m}^c, \underline{m}^c] \cap \underline{g} \subseteq \underline{h}^c \cap \underline{g} = \underline{h}$$

i.e. $[\underline{m}, \underline{m}] \subseteq \underline{h}$, and this proves that M is locally symmetric space. //

Remark:- In proposition 5.2.1., the conditions that $\theta_1^2 \neq \bar{\theta}_1$ and $\theta_2^2 \neq \bar{\theta}_2$ are clearly necessary, as it is shown in the following example due to Dr. R. B. Pettitt, to whom I am very grateful.

Example:- Let (M_1, g_1) be s^6 (the six sphere) with the usual metric and let s_1 be the 3-symmetric structure defined via the representation $G_2/SU(3)$. Let ∇_1 be the levi-Civita connection of g_1, and S_1 the symmetry tensor field. The eigenvalues of S_1 are $\cos 2\pi/3 \pm i \sin 2\pi/3$, and $\nabla_1(S_1) \neq o$. Let (M_2, g_2) be $\mathbb{R}^2$ with the usual flat metric, and let s_2 be the k-symmetric structure defined by $(s_2)_p$ = rotation about p by $2\pi/k$. Let ∇_2 be the Levi-Civita connection of g_2, and S_2 the symmetry tensor field. The eigenvalues of S_2 are $\cos 2\pi/k \pm i \sin 2\pi/k$. Define the product s-manifold

$(M, g, s) = (M_1 \times M_2, g_1 \times g_2, s_1 \times s_2)$, $S = S_1 \oplus S_2$ is the symmetry tensor field of M and $\nabla(S) = \nabla_1(S_1) \oplus \nabla_2(S_2)$. Since $\nabla_1(S_1) \neq o$, then $\nabla(S) \neq o$. The eigenvalues of S are $\cos 2\pi/3 \pm i \sin 2\pi/3$ and $\cos 2\pi/k \pm i \sin 2\pi/k$. Thus if k is any integer > 3, $(M, g, s,)$ is a Riemannian regular s-manifold for which S has eigenvalues $e^{\pm i\theta_1}$, $e^{\pm i\theta_2}$, $o < \theta_1 = 2\pi/k < \theta_2 = 2\pi/3 < \pi$, but neither locally symmetric nor 5-symmetric. //

Let M be a regular Riemannian locally 5-symmetric manifold. Let S be the symmetry tensor field on M. The eigenvlaues of S are the 5th roots afunity, and since they appear in pairs we have three cases to consider

(i) $\cos 2\pi/5 \pm i \sin 2\pi/5$ are the only eigenvalues of S.

(ii) $\cos 4\pi/5 \pm i \sin 4\pi/5$ are the only eigenvalues of S.

(iii) All $\cos 2\pi/5 \pm i \sin 2\pi/5$ and $\cos 4\pi/5 \pm i \sin 4\pi/5$ are eigenvalues of S. $(i = \sqrt{-1})$

In cases (i) or (ii) the symmetry tensor field S has only two eigenvalues conjugate to each other, but the square of one eigenvalue does not equal the conjugate of this eigenvalue, hence by proposition 5.1.4., M is locally symmetric. For case (iii), and using proposition 5.2.1., we see that M is not locally symmetric. From now on we will only consider the case when the symmetry tensor field S has four distinct eignevalues.

At each point $p \in M$, we have $M_p = M_{p1} \oplus M_{p1}$, and this gives rise to two differentiable distributions M_1 and M_2 on M , $S = S_1 \oplus S_2$, where $S_1 = (\cos 2\pi/5)I + (\sin 2\pi/5)J_1$ and $S_2 = (\cos 4\pi/5)I + (\sin 4\pi/5)J_2$, and $J = J_1 \oplus J_2$ is an almost complex structure on M. We will always refer to the almost complex structure mentioned above.

5.3. Curvature Relations in Riemannian Locally 5-Regular Symmetric Manifolds

Proposition 5.3.1. Let $S : V \longrightarrow V$ be an isomorphism of an even dimensional vector space V. Let $V = V_1 \oplus V_2$ (direct sum) , $S = S_1 \oplus S_2$, such that $S_j(V_j) \subset V_j$ $(j = 1,2)$. Suppose that $S_1 = (\cos 2\pi/5)I_1 + (\sin 2\pi/5)J_1$ and $S_2 = (\cos 4\pi/5)I_2 + (\sin 4\pi/5)J_2$, where $I_j = V_j \longrightarrow V_j$ is the identity transformation, and $J_j = V_j \longrightarrow V_j$ $(j = 1,2)$ is an almost complex structure on V_j. Let α and β be two tensors on V of type (4,0) and (5,0) respectively, such that they satisfy the following conditions

$$\alpha(X,Y,Z,W) = -\alpha(Y,X,Z,W) = -\alpha(X,Y,W,Z) = \alpha(Z,W,X,Y)$$
$$\beta(V,X,Y,Z,W) = -\beta(V,Y,X,Z,W) = -\beta(V,X,Y,W,Z) = \beta(V,Z,W,X,Y)$$

for all $V,X,Y,Z,W \in V$. Suppose that S preserves α and β

(1)(i) If X,Y,Z,W belong either to V_1 or V_2 or $X,Y \in V_1$ and $Z,W \in V_2$ or $X,Z \in V_1$ and $Y,W \in V_2$. Then

$$\alpha(X,Y,Z,W) = \alpha(JX,JY,Z,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW)$$
and $\alpha(X,Y,Z,W) = \alpha(JX,JY,JZ,JW)$

(ii) If $X, Y, Z \in V$, and $W \in V_2$. Then

$$-3\alpha(X,Y,Z,W,) = \alpha(JX,JY,Z,W,) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW)$$

and $\alpha(X,Y,Z,W) = \alpha(JX,JY,JZ,JW)$

(iii) If $X \in V$, and $Y, Z, W \in V_2$. Then

$$3\alpha(X,Y,Z,W) = \alpha(JX,JY,Z,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW)$$

and $\alpha(X,Y,Z,W) = -\alpha(JX,JYJZ,JW)$

(2)(i) If V, X, Y, Z, W belong either to V_1 or V_2. Then

$$\begin{aligned} -10\beta(V,X,Y,Z,W) &= \beta(JV,JX,Y,Z,W) + \beta(JV,X,JY,Z,W) \\ &+ \beta(JV,X,Y,JZ,W) + \beta(JV,X,Y,Z,JW) + \beta(V,JX,JY,Z,W) \\ &+ \beta(V,JX,Y,JZ,W) + \beta(V,JX,Y,Z,JW) + \beta(V,X,JY,JZ,W) \\ &+ \beta(V,X,JY,Z,JW) + \beta(V,X,Y,JZ,JW) \end{aligned}$$

(ii) If $V, X, Y, Z \in V_1$ and $W \in V_2$ or $V, X \in V_1$ and $Y, Z, W \in V_2$ or $V \in V_2$ and $X, Y, Z, W \in V_1$ or $V, Z, W \in V_2$ and $X, Y \in V_1$ or $V, Y, W \in V_2$ and $X, Z \in V_1$. Then

$$2\beta(V,X,Y,Z,W) = \beta(JV,JX,Y,Z,W) + \ldots + \beta(V,X,Y,JZ,JW)$$

(iii) If $V, X, Y \in V_1$ and $Z, W \in V_2$ or $V, X, Z \in V_1$, $Y, W \in V_2$ or $V \in V_1$ and $X, Y, Z, W \in V_2$ or $V, W \in V_2$ and $X, Y, Z \in V_1$ or $V, Y, Z, W \in V_2$ and $X \in V_1$. Then

$$-2\beta(V,X,Y,Z,W) = \beta(JY,JX,Y,Z,W) + \ldots + \beta(V,X,Y,JZ,JW)$$

<u>Proof</u>:- Let $Z \in V$ be any vector, then $Z = X + Y$, when $X \in V_1$ and $Y \in V_2$.

$$\begin{aligned} S^2 Z &= S_1^2 X \oplus S_2^2 Y = S_1(S_1 X) \oplus S_2(S_2 Y) \\ &= ((\cos 2\pi/5) I_1 + (\sin 2\pi/5) J_1)((\cos 2\pi/5) X + (\sin 2\pi/5) J_1 X) \\ &\oplus ((\cos 4\pi/5) I_2 + (\sin 4\pi/5) J_2)((\cos 4\pi/5) Y + (\sin 4\pi/5) J_2 Y) \\ &= \left[(\cos^2 2\pi/5) X + 2(\sin 2\pi/5)(\cos 2\pi/5) J_1 X - (\sin^2 2\pi/5) X\right] \\ &\oplus \left[(\cos^4 4\pi/5) Y + 2(\sin 4\pi/5)(\cos 4\pi/5) J_2 Y - (\sin^2 4\pi/5) Y\right] \end{aligned}$$

$$= ((\cos 4\pi/5)X + (\sin 4\pi/5)J_1X) \oplus ((\cos 2\pi/5)Y - (\sin 2\pi/5)J_2Y)$$

$$= (\left[(\cos 4\pi/5) I_1 + (\sin 4\pi/5) J_1\right] \oplus \left[(\cos 2\pi/5) I_2 - (\sin 2\pi/5) J_2\right])(Z)$$

Similarly, we have

$$S^3Z = (\left[(\cos 4\pi/5) I_1 - (\sin 4\pi/5) J_1\right] \oplus \left[(\cos 2\pi/5) I_2 + (\sin 2\pi/5) J_2\right])(Z)$$

$$Z^4 Z = (\left[(\cos 2\pi/5) I_1 - (\sin 2\pi/5) J_1\right] \oplus \left[(\cos 4\pi/5) I_2 - (\sin 4\pi/5) J_2\right])(Z)$$

and $S^5Z = Z$

We have S preserves α, ie. for all $X,Y,Z,W \in V$, we have

$$\alpha(X,Y,Z,W) = \alpha(SX,SY,SZ,SW) = \alpha(S^2X,S^2Y,S^2Z,S^2W)$$

$$= \alpha(S^3X,S^3Y,S^3Z,S^3W) = \alpha(S^4X,S^4Y,S^4Z,S^4W) \ldots ①$$

Let $X = X_1 + X_2$, $Y = Y_1 + Y_2$, $Z = Z_1 + Z_2$ and $W = W_1 + W_2$, where X_1, Y_1

Z_1, $W_1 \in V_1$ and X_2, Y_2, Z_2, $W_2 \in V_2$

Hence $\alpha(X,Y,Z,W) = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \sum_{m=1}^{2} \alpha(X_i,Y_j,Z_k,W_m)$

This gives 16 terms, to prove (1) of the proposition, we make use of the condition α satisfies, and this requires us to consider only 6 cases which are given in (i), (ii) and (iii) of part (1).

(1)(i)(a) $X,Y,Z,W \in V_1$. From ① we have

$$\alpha((\cos 2\pi/5)X \pm (\sin 2\pi/5)JX, (\cos 2\pi/5)Y \pm (\sin 2\pi/5)JY, (\cos 2\pi/5)Z \pm (\sin 2\pi/5)JZ), (\cos 2\pi/5)W \pm (\sin 2\pi/5)JW) = \alpha(X,Y,Z,W) \cdots\cdots (i)$$

$$\alpha((\cos 4\pi/5)X \pm (\sin 4\pi/5)JX, (\cos 4\pi/5)Y \pm (\sin 4\pi/5)JY, (\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ, (\cos 4\pi/5)W, \pm (\sin 4\pi/5)JW) = \alpha(X,Y,Z,W) \cdots\cdots (ii)$$

Using linearity of α, and add (i) to (ii) (in each equation 8 terms out of 16 are cancelled), we have

$2(\cos^4 2\pi/5 + \cos^4 4\pi/5)\,\alpha(X,Y,Z,W) + 2(\cos^2 2\pi/5 \sin^2 2\pi/5 + \cos^2 4\pi/5$

$\sin^2 4\pi/5)$

$\times[\alpha(JX,JY,Z,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW) + \alpha(X,JY,JZ,W)$

$+ \alpha(X,JY,Z,JW) + \alpha(X,Y,JZ,JW)] \quad 2(\sin^4 2\pi/5 + \sin^4 4\pi/5)$

$\alpha(JX,JY,JZ,JW) = 4\alpha(X,Y,Z,W)$ - - - - - - - (iii)

From the appendix (5) (i), (ii) and (iii) we have

$7/16\ \alpha(X,Y,Z,W) + 5/16\,[\alpha(JX,JY,Z,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW)$

$+ \alpha(X,JY,JZ,W) + \alpha(X,JY,Z,JW) + \alpha(X,Y,JZ,JW)] + 15/16$

$\alpha(JX,JY,JZ,JW) = 2\alpha(X,Y,Z,W)$, or

$\alpha(JX,JY,Z,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW) + \alpha(X,JY,JZ,W)$

$+ \alpha(X,JY,Z,JW) + \alpha(X,Y,JZ,JW) + 3\alpha(JX,JY,JZ,JW) = 5\alpha(X,Y,Z,W)$ - - - - (iv)

Replace X, Y, Z, W by JX, JY, JZ, JW in (iv) we have

$\alpha(JX,JY,Z,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW) + \alpha(X,JY,JZ,W)$

$+ \alpha(X,JY,Z,JW) + \alpha(X,Y,JZ,JW) + 3\alpha(X,Y,Z,W) = 5\alpha(JX,JY,JZ,JW)$ - - - - (v)

From (iv) and (v) we get

$5\alpha(X,Y,Z,W) - 3\alpha(JX,JY,JZ,JW) = 5\alpha(JX,JY,JZ,JW) - 3\alpha(X,Y,Z,W)$

or $\alpha(X,Y,Z,W) = \alpha(JX,JY,JZ,JW)$ - - - - - - (vi)

Using (vi) we have

$\alpha(X,Y,Z,W) = \alpha(JX,JY,Z,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW)$

(1) (i) (b) $X,Y,Z,W \in V_2$

From above we see that the coefficient of S_2 are exactly the same as the coefficients of S_1^2, and the coefficients of S_2^2, S_2^3, and S_2^4 are exactly the same as the coefficients of S_1^4, S_1, and S_1^3 respectively. Hence from equation ① we are going to get exactly the same equations as in (1) (i)(a) but this time we have $X,Y,Z,W \in V_2$, and this gives us that

$$\alpha(X,Y,Z,W) = \alpha(JX,JY,JZ,JW)$$

and $\alpha(X,Y,Z,W) = \alpha(JX,JY,Z,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW)$

(1)(i)(c) $X,Y \in V_1$ and $Z,W \in V_2$. From equation ① , we have

$$\alpha((\cos 2\pi/5)X \pm (\sin 2\pi/5)JX,\ (\cos 2\pi/5)Y \pm (\sin 2\pi/5)JY,\ (\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ,\ (\cos 4\pi/5)W \pm (\sin 4\pi/5)JW) = 2\alpha(X,Y,Z,W) \text{ - - - - - - - - (i)}$$

$$\alpha((\cos 4\pi/5)X \pm (\sin 4\pi/5)JX,\ (\cos 4\pi/5)Y \pm (\sin 4\pi/5)JY,\ (\cos 2\pi/5)Z \mp (\cos 2\pi/5)JZ,\ (\cos 2\pi/5)W \mp (\sin 2\pi/5)JW) = 2\alpha(X,Y,Z,W) \text{ - - - - - (ii)}$$

Using linearity of α and add (i) to (ii), we have

$$2(\cos^2 2\pi/5\ \cos^2 4\pi/5 + \cos^2 4\pi/5\ \cos^2 2\pi/5)\alpha(X,Y,Z,W) + 2(\cos^2 2\pi/5\ \sin^2 4\pi/5 + \cos^2 4\pi/5\ \sin^2 2\pi/5)\left[\alpha(X,Y,JZ,JW) + \alpha(JX,JY,Z,W)\right] + 2(\sin^2 2\pi/5\ \sin^2 4\pi/5 + \sin^2 4\pi/5\ \sin^2 2\pi/5)\alpha(JX,JY,JZ,JW) = 4\alpha(X,Y,Z,W)$$

From the appendix (6) (i), (ii) and (iii) we have

$$\alpha(JX,JY,Z,W) + \alpha(X,Y,JX,JW) + \alpha(JX,JY,JZ,JW) = 3\alpha(X,Y,Z,W) \text{ ---(iii)}$$

Replace X,Y,Z,W by JX,JY,JZ,JW we have

$$\alpha(JX,JY,Z,W) + \alpha(X,Y,JZ,JW) + \alpha(X,Y,Z,W) = 3\alpha(JX,JY,JZ,JW) \text{ - - - (iv)}$$

From (iii) and (iv) we have

$$\alpha(X,Y,Z,W) = \alpha(JX,JY,JZ,JW) \text{ - - - - - . . . . . . - . (v)}$$

Hence from (iii) we get

$$\alpha(JX,JY,Z,W) = \alpha(X,Y,Z,W) \text{ - . - . . - . . . . . (vi)}$$

Replace Y, Z by JY, JZ in (vi) we get

$-\alpha(JX,Y,JZ,W) = \alpha(X,JY,JZW)$

But from (v) we have $\alpha(X,JY,JZ,W) = \alpha(JX,Y,Z,JW)$

$\therefore \quad \alpha(JX,Y,JZ,W) + (JX,Y,Z,JW) = 0$ - - - - - - -(vii)

Hence from (vi) and (vii) we have

$\alpha(X,Y,Z,W) = \alpha(JX,JY,Z,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW)$

(1)(i)(d) $X,Z \in V_1$ and $Y,W \in V_2$. From equation ① we have

$$\alpha((\cos 2\pi/5)X \pm (\sin 2\pi/5)JX, (\cos 4\pi/5)Y \pm (\sin 4\pi/5)JY, (\cos 2\pi/5)Z \pm (\sin 2\pi/5)JZ,$$
$$(\cos 4\pi/5)W \pm (\sin 4\pi/5)JW) = 2\alpha(X,Y,Z,W) \text{ - - - - - -(i)}$$

$$\alpha((\cos 4\pi/5)X \pm (\sin 4\pi/5)JX, (\cos 2\pi/5)Y \mp (\sin 2\pi/5)JY, (\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ,$$
$$(\cos 2\pi/5)W \mp (\sin 2\pi/5)JW) = 2\alpha(X,Y,Z,W) \text{ - - - - - - - (ii)}$$

Using linearity of α, and add (i) to (ii), we have

$$2(\cos^2 2\pi/5 \cos^2 4\pi/5 + \cos^2 4\pi/5 \cos^2 2\pi/5)\alpha(X,Y,Z,W) + 2(\cos^2 2\pi/5 \sin^2 4\pi/5 +$$
$$\cos^2 4\pi/5 \sin^2 2\pi/5)\left[\alpha(JX,Y,JZ,W) + \alpha(X,JY,Z,JW)\right] +$$
$$2(\sin^2 2\pi/5 \sin^2 4\pi/5 + \sin^2 4\pi/5 \sin^2 2\pi/5)\alpha(JX,JY,JZ,JW)$$
$$= 4\alpha(X,Y,Z,W) \text{ - - - - - - - - - - - (iii)}$$

From the appendix (6)(i),(ii) and (iii), we have

$$\alpha(JX,Y,JZ,W) + \alpha(X,JY,Z,JW) + \alpha(JX,JY,JZ,JW) = 3\alpha(X,Y,Z,W) \text{ - - - (iv)}$$

Replace X,Y,Z,W by JX,JY,JZ,JW in (iv) we get

$$\alpha(JX,Y,JZ,W) + \alpha(X,JY,Z,JW) + \alpha(X,Y,Z,W) = 3\alpha(JX,JY,JZ,JW) \text{ - - - (v)}$$

From (iv) and (v) we get

$3\alpha(X,Y,Z,W) - \alpha(JX,JY,JZ,JW) = 3\alpha(JX,JY,JZ,JW) - \alpha(X,Y,Z,W)$

or $\alpha(X,Y,Z,W) = \alpha(JX,JY,JZ,JW)$ - - - - - - - - - - (vi)

In (vi) replace Y, Z by JY, JZ we get

$\alpha(X,JY,JZ,W) = \alpha(JX,Y,Z,JW)$ - - - - - - - - - - (vii)

Using (vi) in (iv) we have

$\alpha(JX,Y,JZ,W) = \alpha(X,Y,Z,W)$ - - - - - - - - - - (viii)

In (viii) replace Y, Z by JY, JZ we get

$-\alpha(JX,JY,Z,W) = \alpha(X,JY,JZ,W) = \alpha(JX,Y,Z,JW)$

or $\alpha(JX,JY,Z,W) + \alpha(JX,Y,Z,JW) = 0$

$\therefore \alpha(X,Y,Z,W) = \alpha(JX,JYZ,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW)$

(1)(ii) $X,Y,Z \in V_1$ and $W \in V_2$. From equation ① we have

$\alpha((\cos 2\pi/5)X \pm (\sin 2\pi/5)JX, (\cos 2\pi/5)Y \pm (\sin 2\pi/5)JY, (\cos 2\pi/5)Z \pm (\sin 2\pi/5)JZ,$

$(\cos 4\pi/5)W \pm (\sin 4\pi/5)JW) = 2\alpha(X,Y,Z,W)$ - - - - - - - - (i)

$\alpha((\cos 4\pi/5)X \pm (\sin 4\pi/5)JX, (\cos 4\pi/5)Y \pm (\sin 4\pi/5)JY, (\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ,$

$(\cos 2\pi/5)W \mp (\sin 2\pi/5)JW) = 2\alpha(X,Y,Z,W)$ - - - - - - - - (ii)

Using linearity of α, and add (i) to (ii) we get

$2(\cos^3 2\pi/5 \cos 4\pi/5 + \cos^3 4\pi/5 \cos 2\pi/5)\alpha(X,Y,Z,W) +$

$2(\cos^2 2\pi/5 \sin 2\pi/5 \sin 4\pi/5 - \cos^2 4\pi/5 \sin 2\pi/5 \sin 4\pi/5)\left[\alpha(X,Y,JZ,JW) +\right.$

$\left.\alpha(X,JY,Z,JW) + \alpha(JX,Y,Z,JW)\right] + 2(\cos 2\pi/5 \sin^2 2\pi/5 \cos 4\pi/5 +$

$\cos 4\pi/5 \sin^2 4\pi/5 \cos 2\pi/5)\left[\alpha(X,JY,JZ,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,JY,Z,W)\right]$

$+ 2(\sin^3 2\pi/5 \sin 4\pi/5 - \sin^3 4\pi/5 \sin 2\pi/5)\alpha(JX,JY,JZ,JW) = 4\alpha(X,Y,Z,W)$

From the appendix (7)(i),(ii),(iii) and (iv) we have

$$-\left[\alpha(JX,JY,Z,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW) + \alpha(X,JY,JZ,W) + \alpha(X,JY,Z,JW) + \alpha(X,Y,JZ,JW)\right] + \alpha(JX,JY,JZ,JW) = 7\alpha(X,Y,Z,W) \quad \text{---(iii)}$$

In (iii) replace X,Y,Z,W by JX,JY,JZ,JW we have

$$-\left[\alpha(JX,JY,Z,W,) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW) + \alpha(X,JY,JZ,W) + \alpha(X,JY,Z,JW) + \alpha(X,Y,JZ,JW)\right] + \alpha(X,Y,Z,W) = 7\alpha(JX,JY,JZ,JW) \quad \text{...(iv)}$$

From (iii) and (iv) we get

$$7\alpha(X,Y,Z,W) - \alpha(JX,JY,JZ,JW) = 7\alpha(JX,JY,JZ,JW) - \alpha(X,Y,Z,W)$$

$$\therefore \alpha(X,Y,Z,W) = \alpha(JX,JY,JZ,JW) \quad \text{--------(v)}$$

Using (v) in (iii) we have

$$-3\alpha(X,Y,Z,W) = \alpha(JX,JY,Z,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW)$$

(1)(iii) $X \in V_1$ and $Y,Z,W \in V_2$

$$\alpha((\cos 2\pi/5)X \pm (\sin 2\pi/5)JX, (\cos 4\pi/5)Y \pm (\sin 4\pi/5)JY, (\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ, (\cos 4\pi/5)W \pm (\sin 4\pi/5)JW) = 2\alpha(X,Y,Z,W) \quad \text{---(i)}$$

$$\alpha((\cos 4\pi/5)X \pm (\sin 4\pi/5)JX, (\cos 2\pi/5)Y \mp (\sin 2\pi/5)JY, (\cos 2\pi/5)Z \mp (\sin 2\pi/5)JZ, (\cos 2\pi/5)W \mp (\sin 2\pi/5)JW) = 2\alpha(X,Y,Z,W) \quad \text{---(ii)}$$

Using linearity of α, and add (i) to (ii) we get

$$2(\cos 2\pi/5 \cos^3 4\pi/5 + \cos 4\pi/5 \cos^3 2\pi/5)\alpha(X,Y,Z,W) + 2(\cos 2\pi/5 \cos 4\pi/5 \sin^2 4\pi/5 + \cos 4\pi/5 \cos 2\pi/5 \sin^2 2\pi/5)\left[\alpha(X,Y,JZ,JW) + \alpha(X,JY,Z,JW) + \alpha(X,JY,JZ,W)\right]$$

$$+ 2(\sin 2\pi/5 \sin 4\pi/5 \cos^2 4\pi/5 - \sin 4\pi/5 \sin 2\pi/5 \cos^2 2\pi/5)\left[\alpha(JX,Y,Z,JW) + \alpha(JX,Y,JZ,W) + \alpha(JX,JY,Z,W)\right] + 2(\sin 2\pi/5 \sin^3 4\pi/5 - \sin 4\pi/5 \sin^3 2\pi/5)$$

$$\alpha(JX,JY,JZ,JW) = 4\alpha(X,Y,Z,W)$$

From the appendix (7)(i),(ii),(iii) and (iv) we have

$$-\left[\alpha(X,Y,JZ,JW) + \alpha(X,JY,Z,JW) + \alpha(X,JY,JZ,W)\right] + \left[\alpha(JX,Y,Z,JW) + \alpha(JX,Y,JZ,W) + \alpha(JX,JY,Z,W)\right] - \alpha(JX,JY,JZ,JW) = 7\alpha(X,Y,Z,W) \quad \ldots\ldots(iii)$$

In (iii) replace X,Y,Z,W by JX,JY,JZ,JW we have

$$-\left[\alpha(JX,JY,Z,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW)\right] + \left[\alpha(X,JY,JZ,W) + \alpha(X,JY,Z,JW) + \alpha(X,Y,JZ,JW)\right] - \alpha(X,Y,Z,W) = 7\alpha(JX,JY,JZ,JW) \quad \ldots\ldots(iv)$$

Add (iii) to (iv) we have

$$-\left[\alpha(X,Y,Z,W) + \alpha(JX,JY,JZ,JW)\right] = 7\left[\alpha(X,Y,Z,W) + \alpha(JX,JY,JZ,JW)\right]$$

or $\alpha(X,Y,Z,W) = -\alpha(JX,JY,JZ,JW)$ ....(v)

Using (v) in (iii) we get

$$3\alpha(X,Y,Z,W) = \alpha(JX,JY,Z,W) + \alpha(JX,Y,JZ,W) + \alpha(JX,Y,Z,JW)$$

(2) We have S preserves β, i.e. for all V,X,Y,Z,W V, we have

$$\beta(V,X,Y,Z,W) = \beta(SV,SX,SY,SZ,SW) = \beta(S^2V,S^2X,S^2Y,S^2Z,S^2W)$$
$$= \beta(S^3V,S^3X,S^3Y,S^3Z,S^3W) = \beta(S^4V,S^4X,S^4Y,S^4Z,S^4W) \quad \text{--------} \; (2)$$

Let $V = V_1 + V_2$, $X = X_1 + X_2$, $Y = Y_1 + Y_2$, $W = W_1 + W_2$, Z_1+Z_2, *where*

$V_1, X_1, Y_1, Z_1, W_1 \in V_1$ and $V_2, X_2, Y_2, Z_2 W_2 \in V_2$

Hence $\beta(V,X,Y,Z,W) = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{m=1}^{2}\sum_{n=1}^{2} \beta(V_i,X_j,Y_k,Z_m,W_n)$

This gives 32 terms, to prove (2) of the proposition, we make use of the condition β satisfies, and this requires us to consider only 12 cases, which are given in (i),(ii) and (iii) of part (2) of the proposition

2) (i) (a) $V, X, Y, Z, W \in V_1$. Then from equation ② we have

$$\beta((\cos 2\pi/5)V \pm (\sin 2\pi/5)JV, (\cos 2\pi/5)X \pm (\sin 2\pi/5)JX, (\cos 2\pi/5)Y \pm (\sin 2\pi/5)JY,$$
$$(\cos 2\pi/5)Z \pm (\sin 2\pi/5)JZ, (\cos 2\pi/5)W \pm (\sin 2\pi/5)JW) = 2\beta(V,X,Y,Z,W) \cdots\cdots \text{(i)}$$

$$\beta((\cos 4\pi/5)V \pm (\sin 4\pi/5)JV, (\cos 4\pi/5)X \pm (\sin 4\pi/5)JX, (\cos 4\pi/5)Y \pm (\sin 4\pi/5)Jy,$$
$$(\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ, (\cos 4\pi/5)W \pm (\sin 4\pi/5)JW) = 2\beta(V,X,Y,Z,W) \cdots\cdots \text{(ii)}$$

Using linearity of β, and add (i) to (ii) (in each equation 16 terms out of 32 are cancelled), we have

$$2(\cos^5 2\pi/5 + \cos^5 4\pi/5)\beta(V,X,Y,Z,W) + 2(\cos^3 2\pi/5 \sin^2 2\pi/5 + \cos^3 4\pi/5 \sin^2 4\pi/5)$$
$$\big[\beta(JV,JX,Y,Z,W) + \beta(JV,X,JY,Z,W) + \beta(JV,X,Y,JZ,W) + \beta(JV,X,Y,Z,JW)$$
$$+ \beta(V,JX,JY,Z,W) + \beta(V,JX,Y,JZ,W) + \beta(V,JX,YZ,JW) + \beta(V,X,JY,JZ,W)$$
$$+ \beta(V,X,JY,Z,JW) + \beta(V,X,Y,JZ,JW)\big] + 2(\cos 2\pi/5 \sin^4 2\pi/5 + \cos 4\pi/5 \sin^4 4\pi/5)$$
$$\big[\beta(JV,JX,JY,JZ,W) + \beta(JV,JX,JY,Z,JW) + \beta(JV,JX,Y,JZ,JW)$$
$$+ \beta(JV,X,JY,JZ,JW) + \beta(V,JX,JY,JZ,JW)\big] = 4\beta(V,X,Y,Z,W)$$

From the appendix (8)(i), (ii) and (iii) we have

$$-\big[\beta(JV,JX,Y,Z,W) + \beta(JV,X,JY,Z,W) + \beta(JV,X,Y,JZ,W) + \beta(JV,X,Y,Z,JW)$$
$$+ \beta(V,JX,JY,Z,W) + \beta(V,JX,Y,JZ,W) + \beta(V,JX,Y,Z,JW) + \beta(V,X,JY,JZ,W)$$
$$+ \beta(V,X,JY,Z,JW) + \beta(V,X,Y,JZ,JW) - \beta(JV,JX,JY,JZ,W)$$
$$- \beta(JV,JX,JY,Z,JW) - \beta(JV,JX,Y,JZ,JW) - \beta(JV,X,JY,JZ,JW)$$
$$- \beta(V,JX,JY,JZ,JW)\big] = 15\beta(V,X,Y,Z,W) \cdots\cdots \text{(iii)}$$

In (iii) replace V, X, Y, Z by JV, JX, JY, JZ, we get

$-[\beta(JV,JX,Y,Z,W) + \beta(JV,X,JY,Z,W) + \beta(JV,X,Y,JZ,W) + \beta(JV,X,Y,Z,JW)$

$+ \beta(V,JX,JY,Z,W) + \beta(V,JX,Y,JZ,W) + \beta(V,JX,Y,Z,JW) + \beta(V,X,JY,JZ,W)$

$+ \beta(V,X,JY,Z,JW) + \beta(V,X,Y,JZ,JW) - \beta(V,JX,JY,JZ,JW) - \beta(JV,X,JY,JZ,JW)$

$- \beta(JV,JX,Y,JZ,JW) - \beta(JV,JX,JY,Z,JW) - \beta(V,X,Y,Z,W)]$

$= 15\beta(JV,JX,JY,JZ,W)$ . . . . (iv)

Subtract (iv) from (iii) we have

$-\beta(JV,JX,JY,JZ,W) + \beta(V,X,Y,Z,W) = 15[\beta(V,X,Y,Z,W) - \beta(JV,JX,JY,JZ,W)]$

Hence $\beta(V,X,Y,Z,W) = \beta(JV,JX,JY,JZ,W)$ . . . . .(v)

Similarly, if we replace V,X,Y,W or V,X,Z,W or V,Y,Z,W or X,Y,Z,W by JV,JX,JY,JW or JV,JX,JZ,JW or JV,JY,JZ,JW or JX,JY,JZ,JW respectively in (iii) and each time we sub tract the result from (iii), we get

$\beta(V,X,Y,Z,W) = \beta(JV,JX,JY,Z,JW)$ or $\beta(V,X,Y,Z,W) = \beta(JV,JX,Y,JZ,JW)$ or $\beta(V,X,Y,Z,W) = \beta(JV,X,JY,JZ,JW)$ or $\beta(V,X,Y,Z,W) = (V,JX,JY,JZ,JW)$

respectively. Hence (iii) is reduced to

$-10\beta(V,X,Y,Z,W) = \beta(JV,JX,Y,Z,W) + \ldots + \beta(V,X,Y,JZ,JW)$

2) (i) (b) $V,X,Y,Z,W \in V_2$

Since the coefficients of S_2, S_2^2, S_2^3, and S_2^4 are exactly the same as the coefficients of S_1^2, S_1^4, S_1 and S_1^3 respectively, hence from equation (2), we are going to have exactly the same equation as (iii) in (2)(i)(a), and we use the same calculations done there to get

$-10\beta(V,X,Y,Z,W) = \beta(JV,JX,Y,Z,W) + \ldots + \beta(V,X,Y,JZ,JW)$

2)(ii)(a) $V, X, Y, Z \in V_1$ and $W \in V_2$. From equation ② we have

$$\beta((\cos 2\pi/5)X \pm (\sin 2\pi/5)JV, (\cos 2\pi/5)X \pm (\sin 2\pi/5)JX, (\cos 2\pi/5)Y \pm (\sin 2\pi/5)JY, (\cos 2\pi/5)Z \pm (\sin 2\pi/5)JZ, (\cos 4\pi/5)W \pm (\sin 4\pi/5)JW) = 2\beta(V,X,Y,Z,W) \cdots\cdots (i)$$

$$\beta((\cos 4\pi/5)V \pm (\sin 4\pi/5)JV, (\cos 4\pi/5)X \pm (\sin 4\pi/5)JX, (\cos 4\pi/5)Y \pm (\sin 4\pi/5)JY, (\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ, (\cos 2\pi/5)W \mp (\sin 2\pi/5)JW) = 2\beta(V,X,Y,Z,W) \cdots\cdots (ii)$$

Using linearity of β, and add (i) to (ii), we have

$$2(\cos 2\pi/5 \cos^4 4\pi/5 + \cos 4\pi/5 \cos^4 2\pi/5)\beta(V,X,Y,Z,W) + 2(\cos^3 2\pi/5 \sin 2\pi/5 \sin 4\pi/5 - \cos^3 4\pi/5 \sin 4\pi/5 \sin 2\pi/5)\Big[\beta(V,X,Y,JZ,JW) + \beta(V,X,JY,Z,JW) + \beta(V,JX,Y,Z,JW) + \beta(JV,X,Y,Z,JW)\Big] + 2(\cos 2\pi/5 \sin^3 2\pi/5 \sin 4\pi/5 - \cos 4\pi/5 \sin^3 4\pi/5 \sin 2\pi/5)$$

$$\times\Big[\beta(V,JX,JY,JZ,JW) + \beta(JV,X,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,Z,JW)\Big] + 2(\cos 2\pi/5 \sin^4 4\pi/5 + \cos 4\pi/5 \sin^4 2\pi/5)\beta(JV,JX,JY,JZ,W)$$

$$= 4\beta(V,X,Y,Z,W)$$

From the appendix (9)(i),(ii)(iii) and (iv), we have

$$\beta(V,X,Y,JZ,JW) + \beta(V,X,JY,Z,JW) + \beta(V,JX,Y,Z,JW) + \beta(JV,X,Y,Z,JW) + \beta(V,JX,JY,JZ,JW) + \beta(JV,X,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,Z,JW) - 2\beta(JV,JX,JY,JZ,W) = 6\beta(V,X,Y,Z,W) \cdots\cdots (iii)$$

In (iii) replace V,X,Y,Z by JV,JX,JY,JZ, we have

$$-\Big[\beta(JV,JX,JY,Z,JW) + \beta(JV,JX,Y,JZ,JW) + \beta(JV,X,JY,JZ,JW) + \beta(V,JX,JY,JZ,JW) + \beta(JV,X,Y,Z,JW) + \beta(V,JX,Y,Z,JW) + \beta(V,X,JY,Z,JW) + \beta(V,X,Y,JZ,JW)\Big] - 2\beta(V,X,Y,Z,W) = 6\beta(JV,JX,JY,JZ,W) \cdots\cdots (iv)$$

Add (iii) to (iv), we get

$$-2\left[\beta(JV,JX,JY,JZ,W)+\beta(V,X,Y,Z,W)\right] = 6\left[\beta(JV,JX,JY,JZ,W)+\beta(V,X,Y,Z,W)\right]$$

or $\beta(V,X,Y,Z,W) + \beta(JV,JX,JY,JZ,W) = 0$ - - - - - - - (v)

Use (v) in (iii) we get

$$2\left[\beta(V,X,Y,JZ,JW)+\beta(V,X,JY,Z,JW)+\beta(V,JX,Y,X,JW)+\beta(JV,X,Y,Z,JW)\right]$$

$= 4\beta(V,X,Y,Z,W)$ - (vi)

In (V) replace V,X by JV,JX, then replace V,Y by JV,JY and finally replace V,Z by JV,JZ, and add the three results, we get

$$\beta(JV,JX,Y,Z,W)+\beta(V,X,JY,JZ,W)+\beta(JV,X,JY,Z,W)+\beta(V,JX,Y,JZ,W)$$

$+\beta(JV,X,Y,JZ,W) + \beta(V,JX,JY,Z,W) = 0$ . . . . (vii)

From (vi) and (vii) we get

$$2\beta(V,X,Y,Z,W)=\beta(JV,JX,Y,Z,W)+\beta(JV,X,JY,Z,W)+\beta(JV,X,Y,JZ,W)$$

$$+\beta(JV,X,Y,Z,JW)+\beta(V,JX,JY,Z,W)+\beta(V,JX,Y,JZ,W)+\beta(V,JX,Y,Z,JW)$$

$$+\beta(V,X,JY,JZ,W)+\beta(V,X,JY,Z,JW)+\beta(V,X,Y,JX,JW)$$

2)(ii)(b) $V,X\in V_1$ and $Y,Z,W\in V_2$. From equation (2) we have

$\beta((\cos 2\pi/5)V \pm (\sin 2\pi/5)JV, (\cos 2\pi/5)X \pm (\sin 2\pi/5)JX, (\cos 4\pi/5)Y \pm (\sin 4\pi/5)JY,$

$(\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ, (\cos 4\pi/5)W \pm (\sin 4\pi/5)JW) = 2\beta(V,X,Y,Z,W)$ . . . . (i)

$((\cos 4\pi/5)V \pm (\sin 4\pi/5)JV, (\cos 4\pi/5)X \pm (\sin 4\pi/5)JX, (\cos 2\pi/5)Y \mp (\sin 2\pi/5)JY,$

$(\cos 2\pi/5)Z \mp (\sin 2\pi/5)JZ, (\cos 2\pi/5)W \mp (\sin 2\pi/5)JW) = 2\beta(V,X,Y,Z,W)$ . . . . (ii)

Using linearity of β and add (i) to (ii), we have

$2(\cos^2 2\pi/5)\cos^3 4\pi/5 + \cos^2 4\pi/5 \cos^3 2\pi/5)\beta(V,X,Y,Z,W) +$

$$+ 2(\cos^2 2\pi/5 \cos 4\pi/5 \sin^2 4\pi/5 + \cos^2 4\pi/5 \cos 2\pi/5 \sin^2 2\pi/5)\left[\beta(V,X,Y,JZ,JW) + \beta(V,X,JY,Z,JW) + \beta(V,X,JY,JZ,W)\right]$$

$$+ 2(\cos 2\pi/5 \sin 2\pi/5 \cos^2 4\pi/5 \sin 4\pi/5 - \cos 4\pi/5 \sin 4\pi/5 \cos^2 2\pi/5 \sin 2\pi/5)$$

$$\left[\beta(V,JX,Y,Z,JW) + \beta(V,JX,Y,JZ,W) + \beta(V,JX,JY,Z,W) + \beta(JV,X,Y,Z,JW) + \beta(JV,X,Y,JZ,W) + \beta(JV,X,JY,Z,W)\right] + 2(\cos 2\pi/5 \sin 2\pi/5 \sin^3 4\pi/5 - \cos 4\pi/5$$

$$\sin 4\pi/5 \sin^3 2\pi/5)\left[\beta(V,JX,JY,JZ,JW) + \beta(JV,X,JY,JZ,JW)\right] + 2(\sin^2 4\pi/5 \cos^3 2\pi/5 + \sin^2 2\pi/5 \cos^3 4\pi/5)$$

$$\times \beta(JV,JX,Y,Z,W) + 2(\sin^2 4\pi/5 \cos 2\pi/5 \sin^2 2\pi/5 + \sin^2 2\pi/5 \cos 4\pi/5 \sin^2 4\pi/5)$$

$$\left[\beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,Z,JW) + \beta(JV,JX,JY,JZ,W)\right] = 4\beta(V,X,Y,Z,W)$$

From the appendix (10) (i), (ii), (iii), (iv), (v), and (vi) we have

$$\beta(V,X,Y,JZ,JW) + \beta(V,X,JY,Z,JW) + \beta(V,X,JY,JZ,W) + \beta(V,JX,Y,Z,JW)$$
$$+ \beta(V,JX,Y,JZ,W) + \beta(V,JX,JY,Z,W) + \beta(JV,X,Y,Z,JW) + \beta(JV,X,Y,JZ,W)$$
$$+ \beta(JV,X,JY,Z,W) + 3\left[\beta(V,JX,JY,JZ,JW) + \beta(JV,X,JY,JZ,JW)\right]$$
$$- 3\beta(JV,JX,Y,Z,W) - \left[\beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,Z,JW)\right.$$
$$\left. + \beta(JV,JX,JY,JZ,W)\right] = 13\beta(V,X,Y,Z,W) \quad \text{-----------} \text{(iii)}$$

In (iii) replace X, Y, Z, W by JX, JY, JZ, JW, we have

$$\beta(V,JX,JY,Z,W) + \beta(V,JX,Y,JZ,W) + \beta(V,JX,Y,Z,JW) + \beta(V,X,JY,JZ,W)$$
$$+ \beta(V,X,JY,Z,JW) + \beta(V,X,Y,JZ,JW) - \beta(JV,JX,JY,JZ,W) - \beta(JV,JX,JY,Z,JW)$$
$$- \beta(JV,JX,Y,JZ,JW) + 3\left[-\beta(JV,X,Y,Z,JW) - \beta(V,JX,Y,Z,JW)\right]$$
$$- 3\beta(V,X,JY,JZ,W) - \left[-\beta(JV,X,JY,Z,W) - \beta(JV,X,Y,JZ,W)\right.$$

$$-\beta(JV,X,Y,Z,JW)\Big] = 13\beta(V,JX,JY,JZ,JW) \quad \text{- - - - - - - - - - - -} \quad (iv)$$

From (iii) and (iv) we get

$$13\beta(V,X,Y,Z,W) + 3\beta(JV,JX,Y,Z,W) - 3\Big[\beta(V,JX,JY,JZ,JW) + \quad (JV,X,JY,JZ,JW)\Big]$$

$$= 13\beta(V,JX,JY,JZ,JW) - 3\beta(JV,X,JY,JZ,JW) - 3\Big[\beta(V,X,Y,Z,W) - \beta(JV,JX,Y,Z,W)\Big]$$

$$\text{or } \beta(V,X,Y,Z,W) = \beta(V,JX,JY,JZ,JW) \quad \text{- - - - - - - - - - - - - - -} \quad (v)$$

Similarly, in (iii) if we replace V,Y,Z,W by JV,JY,JZ,JW and compare the result with (iii) we get

$$13\beta(V,X,Y,Z,W) + 3\beta(JV,JX,Y,Z,W) - 3\Big[\beta(V,JX,JY,JZ,JW) + \beta(JV,X,JY,JZ,JW)\Big]$$

$$= 13\beta(JV,X,JY,JZ,JW) - 3\beta(V,JX,JY,JZ,JW) - 3\Big[\beta(V,X,Y,Z,W) - \beta(JV,JX,Y,Z,W)\Big]$$

$$\text{or } \beta(V,X,Y,Z,W) = \beta(JV,X,JY,JZ,JW) \quad \text{- - - - - - - - - - - - -} \quad (vi)$$

In (vi) replace V,X by JV,JX and add the result to (v), we get

$$\beta(V,X,Y,Z,W) + \beta(JV,JX,Y,Z,W) = 0 \quad \text{- - - - - - - - - - - -} \quad (vii)$$

Using (v), (vi) and (vii) we have

$$\beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,Z,JW) + \beta(JV,JX,JY,JZ,JW)$$

$$= -\Big[\beta(JV,X,JY,Z,W) + \beta(JV,X,Y,JZ,W) + \beta(JV,X,Y,Z,JW)\Big]$$
$$= -\Big[\beta(V,JX,JY,Z,W) + \beta(V,JX,Y,JZ,W) + \beta(V,JX,Y,Z,JW)\Big]$$
$$= -\Big[\beta(V,X,Y,JZ,JW) + \beta(V,X,JY,Z,JW) + \beta(V,X,JY,JZ,W)\Big]$$

This with (v), (vi) and (vii) in (iii) gives

$$-\Big[\beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,Z,JW,) + \beta(JV,JX,JY,JZ,W)\Big]$$

$$= \beta(V,X,Y,Z,W)$$

Hence (iii) is reduced to

$$2\beta(V,X,Y,Z,W) = \beta(JV,JX,Y,Z,W) + \ldots + \beta(V,X,Y,JZ,JW)$$

(2)(ii)(c) $V \in V_2$ and $X, Y, Z, W \in V_1$. From equation (2) we have

$$\beta((\cos 4\pi/5)V \pm (\sin 4\pi/5)JV, (\cos 2\pi/5)X \pm (\sin 2\pi/5)JX, (\cos 2\pi/5)Y \pm (\sin 2\pi/5)JY,$$
$$(\cos 2\pi/5)Z \pm (\sin 2\pi/5)JZ, (\cos 2\pi/5)W \pm (\sin 2\pi/5)JW) = 2\beta(V,X,Y,Z,W) \ldots\ldots (i)$$

$$\beta((\cos 2\pi/5)V \mp (\sin 2\pi/5)JY, (\cos 4\pi/5)X \pm (\sin 4\pi/5)JX, (\cos 4\pi/5)Y \pm (\sin 4\pi/5)JY,$$
$$(\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ, (\cos 4\pi/5)W \pm (\sin 4\pi/5)JW) = 2\beta(V,X,Y,Z,W) \ldots\ldots (ii)$$

Using Linearity of β, and add (i) to (ii), we have

$$2(\cos 2\pi/5 \cos^4 4\pi/5 + \cos 4\pi/5 \cos^4 2\pi/5)\beta(V,X,Y,Z,W) + (\cos 2\pi/5 \cos^2 4\pi/5 \sin^2 4\pi/5$$
$$+ \cos 4\pi/5 \cos^2 2\pi/5 \sin^2 2\pi/5)\left[\beta(V,X,Y,JZ,JW) + \beta(V,X,JY,Z,JW) + \beta(V,X,JY,JZ,W)\right.$$
$$\left.+\beta(V,JX,Y,Z,JW) + \beta(V,JX,Y,JZ,W) + \beta(V,JX,JY,Z,W)\right] + 2(\cos 2\pi/5 \sin^4 4\pi/5$$
$$+ \cos 4\pi/5 \sin^4 2\pi/5)\beta(V,JX,JY,JZ,JW) + 2(\sin 4\pi/5 \sin 2\pi/5 \cos^3 2\pi/5$$
$$- \sin 4\pi/5 \sin 2\pi/5 \cos^3 4\pi/5)\left[\beta(JV,X,Y,Z,JW) + \beta(JV,X,Y,JZ,W)\right.$$
$$\left.+\beta(JV,X,JY,Z,W) + \beta(JV,JX,Y,Z,W)\right] + 2(\sin 4\pi/5 \cos 2\pi/5 \sin^3 2\pi/5 -$$
$$\sin 2\pi/5 \cos 4\pi/5 \sin^3 4\pi/5)\left[\beta(JV,X,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW)\right.$$
$$\left.+\beta(JV,JX,JY,Z,JW) + \beta(JV,JX,JY,JZ,W)\right] = 4\beta(V,X,Y,Z,W)$$

From the appendix (9)(i),(ii),(iii),(iv) and (v), we have

$$-2\beta(V,JX,JY,JZ,JW) + \left[\beta(JV,X,Y,Z,JW) + \beta(JV,X,Y,JZ,W)\right.$$
$$\left.+\beta(JV,X,JY,Z,W) + \beta(JV,JX,Y,Z,W)\right] + \left[\beta(JV,X,JY,JZ,JW)\right.$$
$$\left.+\beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,Z,JW) + \beta(JV,JX,JY,JZ,W)\right]$$
$$= 6\beta(V,X,Y,Z,W) \text{ - - - - - - - - - (iii)}$$

In (iii) replace X,Y,Z,W by JX,JY,JZ,JW, we have

$$-2\beta(V,X,Y,Z,W)+\left[-\beta(JV,JX,JY,JZ,W)-\beta(JV,JX,JY,Z,JW)\right.$$
$$\left.-\beta(JV,JX,Y,JZ,JW)-\beta(JV,X,JY,JZ,JW)\right]+\left[-\beta(JV,JX,Y,Z,W)\right.$$
$$\left.-\beta(JV,X,JY,Z,W)-\beta(JV,X,Y,JZ,W)-\beta(JV,X,Y,Z,JW)\right]$$
$$=6\beta(V,JX,JY,JZ,JW)\text{ - - - - - - (iv)}$$

Add (iii) to (iv) we have

$$-2\left[\beta(V,X,Y,Z,W)+\beta(V,JX,JY,JZ,JW)\right]$$
$$=6\left[\beta(V,X,Y,Z,W)+\beta(V,JX,JY,JZ,JW)\right]$$

or $\beta(V,X,Y,Z,W)+\beta(V,JX,JY,JZ,JW)=0$ . . . . . . (v)

Use (v) in (iii) we have

$$2\beta(V,X,Y,Z,W)=\beta(JV,JX,Y,Z,W)+\beta(JV,X,JY,Z,W)+\beta(JV,X,Y,JZ,W)$$
$$+\beta(JV,X,Y,Z,JW)\text{ . . . . (vi)}$$

In (v) first replace X,Y by JX,JY, then replace X,Z by JX,JZ, and finally replace X,W by JX,JW, and add the three results we get

$$\beta(V,JX,JY,Z,W)+\beta(V,X,Y,JZ,JW)+\beta(V,JX,Y,JZ,W)$$
$$+\beta(V,X,JY,Z,JW)+\beta(V,JX,Y,Z,JW)+\beta(V,X,JY,JZ,W)=0\text{ - - - - (vii)}$$

Using (vii), (vi) can be written as

$$2\beta(V,X,Y,Z,W)=\beta(JV,JX,Y,Z,W)+\ldots+\beta(V,X,Y,JZ,JW)$$

2)(ii)(d) $V,Z,W\in V_2$ and $X,Y\in V_1$. Then from equation (2) we have

$$\beta((\cos 4\pi/5)V\pm(\sin 4\pi/5)JV,(\cos 2\pi/5)X\pm(\sin 2\pi/5)JX,(\cos 2\pi/5)Y\pm(\sin 2\pi/5)JY,$$
$$(\cos 4\pi/5)Z\pm(\sin 4\pi/5)JZ,(\cos 4\pi/5)W\pm(\sin 4\pi/5)JW)=2\beta(V,X,Y,Z,W)\cdots\text{(i)}$$

$$((\cos 2\pi/5)V \mp (\sin 2\pi/5)JV, (\cos 4\pi/5)X \pm (\sin 4\pi/5)JX, (\cos 4\pi/5)Y \pm (\sin 4\pi/5)JY, (\cos 2\pi/5)Z \mp (\sin 2\pi/5)JZ, (\cos 2\pi/5)W \mp (\sin 2\pi/5)JW)) = 2\beta(V,X,Y,Z,W) \cdots \text{(ii)}$$

Using linearity of β, and add (i) to (ii), we have

$$2(\cos^3 4\pi/5 \cos^2 2\pi/5 + \cos^3 2\pi/5 \cos^2 4\pi/5)\,\beta(V,X,Y,Z,W) + 2(\cos 4\pi/5 \cos^2 2\pi/5 \sin^2 4\pi/5 + \cos 2\pi/5 \cos^2 4\pi/5 \sin^2 2\pi/5)\big[\beta(V,X,Y,JZ,JW) + \beta(JV,X,Y,Z,JW) + \beta(JV,X,Y,JZ,W)\big] + 2(\cos^2 4\pi/5 \cos 2\pi/5 \sin 2\pi/5 \sin 4\pi/5 - \cos^2 2\pi/5 \cos 4\pi/5 \sin 4\pi/5 \sin 2\pi/5)$$
$$\times\big[\beta(JV,X,JY,Z,W) + \beta(V,X,JY,Z,JW) + \beta(V,X,JY,JZ,W) + \beta(V,JX,Y,Z,JW) + \beta(V,JX,Y,JZ,W) + \beta(JV,JX,Y,Z,W)\big] + 2(\cos^3 4\pi/5 \sin^2 2\pi/5 + \cos^3 2\pi/5 \sin^2 4\pi/5)\,\beta(V,JX,JY,Z,W) + 2(\cos 4\pi/5 \sin^2 2\pi/5 \sin^2 4\pi/5 + \cos 2\pi/5 \sin^2 4\pi/5 \sin^2 2\pi/5)\big[\beta(V,JX,JY,JZ,JW) + \beta(JV,JX,JY,Z,JW) + \beta(JV,JX,JY,JZ,W)\big] + 2(\sin 2\pi/5 \cos 2\pi/5 \sin^3 4\pi/5 - \sin 4\pi/5 \cos 4\pi/5 \sin^3 2\pi/5)\big[\beta(JV,X,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW)\big]$$
$$= 4\beta(V,X,Y,Z,W)$$

From the appendix (10)(i),(ii),(iii),(iv),(v) and (vi), we have

$$\beta(V,X,Y,JZ,JW) + \beta(JV,X,Y,Z,JW) + \beta(JV,X,Y,JZ,W) + \beta(V,X,JY,Z,JW) + \beta(V,X,JY,JZ,W) + \beta(V,JX,Y,Z,JW) + \beta(V,JX,Y,JZ,W) + \beta(JV,X,JY,Z,W) + \beta(JV,JX,Y,Z,W) - 3\beta(V,JX,JY,Z,W) - \big[\beta(V,JX,JY,JZ,JW) + \beta(JV,JX,JY,Z,JW) + \beta(JV,JX,JY,JZ,W)\big] + 3\big[\beta(JV,X,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW)\big] = 13\beta(V,X,Y,Z,W) \cdots\cdots \text{(iii)}$$

In (iii) replace V,Y,Z,W by JV,JY,JZ,JW, we have

$$\beta(JV,X,JY,Z,W) + \beta(V,X,JY,JZ,W) + \beta(V,X,JY,Z,JW) + \beta(JV,X,Y,JZ,W) + \beta(JV,X,Y,Z,JW) - \beta(JV,JX,JY,JZ,W) - \beta(JV,JX,JY,Z,JW) + \beta(V,X,Y,JZ,JW) - \beta(V,JX,JY,JZ,JW) + 3\beta(JV,JX,Y,JZ,JW) - \big[-\beta(JV,JX,Y,Z,W) - \beta(V,JX,Y,JZ,W) - \beta(V,JX,Y,Z,JW)\big] + 3\big[\beta(V,X,Y,Z,W) - \beta(V,JX,JY,Z,W)\big] = 13\beta(JV,X,JY,JZ,JW) \cdots \text{(iv)}$$

From (iii) and (iv) we have

$$13\beta(V,X,Y,Z,W) + 3\beta(V,JX,JY,Z,W) - 3\beta(JV,X,JY,JZ,JW) - 3\beta(JV,JX,Y,JZ,JW)$$

$$= 13\beta(JV,X,JY,JX,JW) - 3\beta(JV,JX,Y,JZ,JW) - 3\beta(V,X,Y,Z,W) + 3\beta(V,JX,JY,Z,W)$$

or $\beta(V,X,Y,Z,W) = \beta(JV,X,JY,JZ,JW)$ . . . . (v)

Similarly, if we replace V,X,Z,W by JV,JX,JZ,JW in (iii) and compare the result with (iii) we have

$$13\beta(V,X,Y,Z,W) + 3\beta(V,JX,JY,Z,W) - 3\beta(JV,X,JY,JZ,JW) - 3\beta(JV,JX,Y,JZ,JW)$$

$$= 13\beta(JV,JX,Y,JZ,JW) - 3\beta(JV,X,JY,JZ,JW) + 3\beta(V,JX,JY,Z,W) - 3\beta(V,X,Y,Z,W)$$

or $\beta(V,X,Y,Z,W) = (JV,JX,Y,JZ,JW)$ . . . . . . (vi)

From (v) and (vi) we have by replacing X,Y by JX,JY in (v)

$$\beta(V,X,Y,Z,W) + \beta(V,JX,JY,Z,W) = o \qquad \text{(vii)}$$

Using (v) , (vi) and (vii) we have

$$\beta(V,JX,JY,JZ,JW) + \beta(JV,JX,JY,Z,JW) + \beta(JV,JX,JY,JZ,W)$$

$$= -\left[\beta(JV,JX,Y,Z,W) + \beta(V,JX,Y,JZ,W) + \beta(V,JX,Y,Z,JW)\right]$$

$$= -\left[\beta(JV,X,JY,Z,W) + \beta(V,X,JY,JZ,W) + \beta(V,X,JY,Z,JW)\right]$$

$$= -\left[\beta(V,X,Y,JZ,JW) + \beta(JV,X,Y,Z,JW) + \beta(JV,X,Y,JZ,JW)\right]$$

This with (v) , (vi) and (vii) in (iii), we have

$$-\left[\beta(V,JX,JY,JZ,JW) + \beta(JV,JX,JY,Z,JW) + \beta(JV,JX,JY,JZ,W)\right]$$

$$= \beta(V,X,Y,Z,W)$$

Hence (iii) is reduced to

$2\beta(V,X,Y,Z,W) = \beta(JV, JX,Y,Z,W) + \ldots\ldots + \beta(V,X,Y,JZ,JW)$

2)(ii)(e) $V,Y,W \in V_2$ and $X,Z \in V_1$. From equation ② we have

$\beta((\cos 4\pi/5)V \pm (\sin 4\pi/5)JV, (\cos 2\pi/5)X\ (\sin 2\ /5)JX, (\cos 4\pi/5)Y \pm (\sin 4\pi/5)JY, (\cos 2\pi/5)Z \pm (\sin 2\pi/5)JZ, (\cos 4\pi/5)W \pm (\sin 4\pi/5)JW =$

$2\beta(V,X,Y,Z,W)$ .....(i)

$\beta((\cos 2\pi/5)V \mp (\sin 2\pi/5)JV, (\cos 4\pi/5)X \pm (\sin 4\pi/5)JX, (\cos 2\pi/5)Y \mp (\sin 2\pi/5)JY, (\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ, (\cos 2\pi/5)W \mp (\sin 2\pi/5)JW) =$

$2\beta(V,X,Y,Z,W)$ .....(ii)

Using linearity of β, and add (i) to (ii) we have

$2(\cos^3 4\pi/5 \cos^2 2\pi/5 + \cos^3 2\pi/5 \cos^2 4\pi/5)\, \beta(V,X,Y,Z,W) + 2(\cos^2 4\pi/5 \cos 2\pi/5 \sin 2\pi/5 \sin 4\pi/5 - \cos^2 2\pi/5 \cos 4\pi/5 \sin 4\pi/5 \sin 2\pi/5)$
$[\beta(V,X,Y,JZ,JW) + \beta(V,X,JY,JZ,W) + \beta(V,JX,Y,Z,JW) + \beta(V,JX,JY,Z,W) + \beta(JV,X,Y,JZ,W) + \beta(JV,JX,Y,Z,W)] + 2(\cos 4\pi/5 \cos^2 2\pi/5 \sin^2 4\pi/5 + \cos 2\pi/5 \cos^2 4\pi/5 \sin^2 2\pi/5)[\beta(V,X,JY,Z,JW) + \beta(JV,X,Y,Z,JW) + \beta(JV,X,JY,Z,W)] + 2(\cos^3 4\pi/5 \sin^2 2\pi/5 + \cos^3 2\pi/5 \sin^2 4\pi/5)$
$\beta(V,JX,Y,JZ,W) + 2(\cos 4\pi/5 \sin^2 2\pi/5 \sin^2 4\pi/5 + \sin^2 2\pi/5 \cos 2\pi/5 \sin^2 4\pi/5)[\beta(V,JX,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,JZ,W)]$
$+ 2(\sin^2 \pi/5 \sin^3 4\pi/5 \cos 2\pi/5 - \sin^3 2\pi/5 \sin 4\pi/5 \cos 4\pi/5)$

$[\beta(JV,X,JY,JZ,JW) + B(JV,JX,JY,Z,JW)] = 4\beta(V,X,Y,Z,W)$

From the appendix (10) (i), (ii), (iii), (iv), (v) and (vi), we have

$B(V,Y,JZ,JW) + \beta(V,X,JY,JZ)W) + \beta(V,JX,Y,Z,JW) + \beta(V,JX,JY,Z,W) + \beta(JV,X,Y,JZ,W) + \beta(JV,JX,Y,Z,W)] + [\beta(V,X,JY,Z,JW) + \beta(JV,X,Y,Z,JW) + \beta(JV,X,JY,Z,W)] - 3\beta(V,JX,Y,JZ,W) - [\beta(V,JX,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,JZ,W)] + 3[\beta(JV,X,JY,JZ,JW) + \beta(JV,JX,JY,Z,JW)] = 13\beta(V,X,Y,Z,W)$ ....(iii)

In (iii) replace V,Y,Z,W by JV,JY,JZ,JW, we have $\beta(JV,X,JY,Z,W) + \beta(JV,X,Y,Z,JW) - \beta(JV,JX,JY,JZ,W) - \beta(JV,JX,Y,JZ,JW) + \beta(V,X,JY,Z,JW) + \beta(V,JX,JY,JZ,JW)] + [\beta(JV,X,Y,JZ,W) + \beta(V,X,JY,JZ,W) + \beta(V,X,Y,JZ,JW)] + 3\beta(JV,JX,JY,Z,JW) - [-\beta(JV,JX,Y,Z,W)$

$\dot{-}\ \beta(V,JX,JY,Z,W) - \beta(V,JX,Y,Z,JW)) + 3[\beta(V,X,Y,Z,W)$

$- \beta(V,JX,Y,JZ,W)] = 13\,\beta(JV,X,JY,JZ,JW)$ ....(iv)

From (iii) and (iv) we have.

$13\beta(V,X,Y,Z,W) + 3\beta(V,JX,Y,JZ,W) - 3[\beta(JV,X,JY,JZ,JW)$

$+ \beta(JV,JX,JY,Z,JW)] = 13\beta(JV,X,JY,JZ,JW) - 3\beta(JV,JX,JY,Z,JW)$

$- 3[\beta(V,X,Y,Z,W) - \beta(V,JX,Y,JZ,W)]$ or $\beta(V,X,Y,Z,W) = \beta(JV,X,JY,JZ,JW)$

.........(v)

Similarly, if we replace V, X, Y, W by JV, JX, JY, JW in (iii), we have

$13\beta(V,X,Y,Z,W) + 3\beta(V,JX,Y,JZ,W) - 3[\beta(JV,X,JY,JZ,JW)$

$+ \beta(JV,JX,JY,Z,JW)] = 13\beta(JV,JX,JY,Z,JW) - 3\beta(JV,X,JY,JZ,JW)$

$- 3[-\beta(V,JX,Y,JZ,W) + \beta(V,X,Y,Z,W)]$ or $\beta(Y,X,Y,Z,W) =$

$\beta(JV,JX,JY,Z,JW)$ ....(vi)

In (v) replace X, Z by JX, JZ and add the result to (vi), we get

$\beta(V,X,Y,Z,W) + \beta(V,JX,Y,JZ,W) = 0$ ....(vii)

Using (v), (vi) and (vii) we have, $\beta(V,JX,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW)$

$+ \beta(JV,JX,JY,JZ,W) = - [B(JV,JX,Y,Z,W) + \beta(V,JX,JY,Z,W) +$

$\beta(V,JX,Y,Z,JW)] = - [\beta(JV,X,Y,JZ,W) + \beta(V,X,JY,JZ,W) + \beta(V,X,Y,JZ,JW)]$

$= - [\beta(V,X,JY,Z,JW) + \beta(JV,X,Y,Z,JW) + \beta(JV,X,JY,Z,W)]$

This with (v), (vi) and (vii) in (iii) gives

$-[\beta(V,JX,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,JZ,W)] = \beta(V,X,Y,Z,W)$

Hence (iii) is reduced to

$2\beta(V,X,Y,Z,W) = \beta(JV,JX,Y,Z,W) + \ldots\ldots + \beta(V,X,Y,Z,JW)$

2)(iii)(a) $V,X,Y \in V_1$, and $Z,W\ V_2$. Then from equation ② we have

$\beta((\cos 2\pi/5)V \pm (\sin 2\pi/5)JV, (\cos 2\pi/5)X \pm (\sin 2\pi/5)JX, (\cos 2\pi/5)Y$
$\pm (\sin 2\pi/5)JY, (\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ, (\cos 4\pi/5)W \pm (\sin 4\pi/5)JW)$
$= 2\beta(V,X,Y,Z,W)$ ....(i)

$\beta(\cos 4\pi/5)V \pm (\sin 4\pi/5)JV, (\cos 4\pi/5)X \pm (\sin 4\pi/5)JX, (\cos 4\pi/5)Y$
$\pm (\sin 4\pi/5)JY, (\cos 2\pi/5)Z \mp (\sin 2\pi/5)JZ, (\cos 2\pi/5)W \mp (\sin 2\pi/5)JW)$

$= 2\beta(V,X,Y,Z,W)$ ....(ii)

Using lieasity of β, and add (i) to (ii) we have

$2(\cos^3 2\pi/5 \cos^2 4\pi/5 + \cos^3 4\pi/5 \cos^2 2\pi/5)\ \beta(V,X,Y,Z,W)$

$+ 2(\cos^3 2\pi/5 \sin^2 4\pi/5 + \cos^3 4\pi/5 \sin^2 2\pi/5)\ \beta(V,X,Y,JZ,JW)$

$+ 2(\cos^2 2\pi/5 \cos 4\pi/5 \sin \frac{4\pi}{5} \sin\frac{2\pi}{5} - \cos^2 4\pi/5 \cos 2\pi/5\ \sin 4\pi/5 \sin 2\pi/5)$

$\left[\beta(V,X,JY,Z,JW) + \beta(V,X,JY,JZ,W) + \beta(V,JX,Y,Z,JW) + \beta(V,JX,Y,JZ,W) + \beta(JV,X,Y,Z,JW) + \beta(JV,X,Y,JZ,W)\right] + 2(\cos{}^2 \pi/5 \sin{}^2 2\pi/5 \cos{}^2 4\pi/5 + \cos{}^4 \pi/5 \sin{}^2 4\pi/5 \cos{}^2 2\pi/5)\left[\beta(V,JX,JY,Z,W) + \beta(JV,X,JY,Z,W) + \beta(JV,JX,Y,Z,W)\right] + 2(\cos{}^2 \pi/5 \sin{}^2 2\pi/5 \sin{}^2 4\pi/5 + \cos{}^4 \pi/5 \sin{}^2 4\pi/5 \sin{}^2 2\pi/5)\left[\beta(V,J,X,JY,JZ,JW) + \beta(JV,X,JY,JZ,JW) + \beta(JV,J X,Y,JZ,JW)\right] + 2(\sin{}^3 2\pi/5 \cos{}^4 \pi/5 \sin 4\pi/5 - \cos{}^2 \pi/5 \sin{}^2 \pi/5 \sin{}^3 \frac{4\pi}{5}) \left[\beta(JV,JXJY,Z,JW) + \beta(JV,JX,JY,JZ,W)\right] = 4\beta(V,X,Y,Z,W)$

From the appendix (10) (i), (ii), (iii), (iv), (v) and (vi), we have

$- 3\beta(V,X,Y,JZ,JW) - \left[\beta(V,X,JY,Z,JW) + \beta(V,X,JY,JZ,W) + \beta(V,JX,Y,Z,JW) + \beta(V,JX,Y,JZ,W) + \beta(JV,X,Y,Z,JW) + \beta(JV,X,Y,JZ,W)\right] + \left[\beta(V,JX,JY,Z,W) + \beta(JV,X,JY,Z,W) + \beta(JV,JX,Y,Z,W)\right] - \left[\beta(V,JX,JY,JZ,JW) + \beta(JV,X,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW)\right] - 3\left[\beta(JV,JX,JY,Z,JW) + \beta(JV,JX,JY,JZ,W)\right] = 13\beta(V,X,Y,Z,W)$ ....(iii)

In (iii) replace V,X,Y,Z by JV,JX,JY,JZ, we have, $3\beta(JV,JX,JY,Z,JW) - \left[-\beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,Y,Z,W) - \beta(JV,X,JY,JZ,JW) + \beta(JV,X,JY,Z,W) - \beta(V,JX,JY,JZ,JW) + \beta(V,JX,JY,Z,W)\right] + \left[\beta(JV,X,Y,JZ,W) + \beta(V,J X,Y,JZ,W) + \beta(V,X,JY,JZ,W)\right] + \left[\beta(JV,X,Y,Z,JW) + \beta(V,JX,Y,Z,JW) + \beta(V,X,JY,Z,JW)\right] - 3\left[-\beta(V,X,Y,JZ,JW) + \beta(V,X,Y,Z,W)\right] = 13\,(JV,JX,JY,JZ,JW)$ - . . . . . - - - - - - - - - (iv)

From (iii) and (iv) we have

$- 3\left[\beta(JV,JX,JY,JZ,W) + \beta(V,X,Y,Z,W)\right] = 13\left[\beta(V,X,Y,Z,W) + \beta(JV,JX,JY,JZ,W)\right]$ or $\beta(V,X,Y,Z,W) + \beta(JV,JX,JY,JZ,W) = 0$ ....(v)

In (iii) replace V,X,Y,W by JV,JX,JY,JW and compare the result with (iii) we have

$- 3\left[\beta(JV,JX,JY,Z,JW) + \beta(V,X,Y,Z,W)\right] = 13\left[\beta(V,X,Y,Z,W) + \beta(JV,JX,JY,Z,JW)\right]$ or $\beta(V,X,Y,Z,W) + \beta(JV,JX,JY,Z,JW)$ ....(vi)

In (v) replace Z,W by JZ,JW and add the result to (vi) we get

$\beta(V,X,Y,Z,W) + \beta(V,X,Y,JZ,JW) = 0$ ....(vii)

Using (v), (vi) and (vii), consider the following

$\beta(V,JX,JY,JZ,JW) + \beta(JV,X,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW)$

$$\left.\begin{aligned} &= \beta(JV,X,Y,Z,JW) + \beta(V,JX,Y,Z,JW) + \beta(Y,X,JY,Z,JW) \\ &= \beta(JV,X,Y,JZ,W) + \beta(V,JX,Y,JZ,W) + \beta(V,X,JY,JZ,W) \\ &= -\left[\beta(V,JX,JY,Z,W) + \beta(JV,X,JY,Z,W) + \beta(JV,JX,Y,Z,W)\right] \end{aligned}\right\} \ldots\text{(viii)}$$

This and (v), (vi), (vii) in (iii) gives

$$-\left[\beta(V,JX,JY,JZ,JW) + \beta(JV,X,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW)\right] = \beta(V,X,Y,Z,W)$$

Hence (iii) is reduced to

$$-2\beta(V,X,Y,Z,W) = \beta(V,X,JY,Z,JW) + \beta(V,X,JY,JZ,W) + \beta(V,JX,V,Z,JW) + \beta(V,JX,Y,JZ,W) + \beta(JV,X,Y,JZ,W) + \beta(JV,X,Y,Z,JW)$$

But from (viii) we have

$$\begin{aligned} \beta(V,JX,JY,Z,W) + \beta(JV,X,JY,Z,W) + \beta(JV,JX,Y,Z,W) &= \beta(V,X,Y,Z,W) \\ &= -\beta(V,X,Y,JZ,JW) \end{aligned}$$

Hence we finally have

$$\begin{aligned} -2\beta(V,X,Y,Z,W) = {} & \beta(JV,JX,Y,Z,W) + \beta(JV,X,JY,Z,W) + \beta(JV,X,Y,JZ,W) \\ & + \beta(JV,X,Y,Z,JW) + \beta(V,JX,JY,Z,W) + \beta(V,JX,Y,JZ,W) \\ & + \beta(V,JX,Y,Z,JW) + \beta(V,X,JY,JZ,W) + \beta(V,X,JY,Z,JW) \\ & + \beta(V,X,Y,JZ,JW) \end{aligned}$$

2) (iii) (b) $V,X,Z \in V_1$, and $Y,W \in V_2$. Then from equation (2) we have

$$\beta\big((\cos 2\pi/5)V \pm (\sin 2\pi/5)JV, (\cos 2\pi/5)X \pm (\sin 2\pi/5)JX, (\cos 4\pi/5)Y \pm (\sin 4\pi/5)JY, (\cos 2\pi/5)Z \pm (\sin 2\pi/5)JZ, (\cos 4\pi/5)W \pm (\sin 4\pi/5)JW\big)$$

$$= 4\beta(V,X,Y,Z,W) \ldots\ldots\text{(i)}$$

$$\beta\big((\cos 4\pi/5)V \pm (\sin 4\pi/5)JV, (\cos 4\pi/5)X \pm (\sin 4\pi/5)JX, (\cos 2\pi/5)Y \pm (\sin 2\pi/5)JY, (\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ, (\cos 2\pi/5)W \mp (\sin 2\pi/5)JW\big) =$$

$$4\beta(V,X,Y,Z,W) \ldots\ldots\text{(ii)}$$

Using linearity of β, and add (i) to (ii) we have

$$2(\cos^3 2\pi/5 \cos^2 4\pi/5 + \cos^3 4\pi/5 \cos^2 2\pi/5)\,\beta(V,X,Y,Z,W)$$
$$+ 2(\cos^2 2\pi/5 \cos 4\pi/5 \sin 2\pi/5 \sin 4\pi/5 - \cos^2 4\pi/5 \cos 2\pi/5 \sin 2\pi/5 \sin 4\pi/5)\big[\beta(V,X,Y,JZ,JW) + \beta(V,X,JY,JZ,W) + \beta(V,JX,Y,Z,JW)$$
$$+ \beta(V,JX,JY,Z,W) + \beta(JV,X,Y,Z,JW) + \beta(JV,X,JY,Z,W)\big] + 2(\cos^3 2\pi/5$$

$\sin^2 4\pi/5 + \cos^3 4\pi/5 \sin^2 2\pi/5)\ \beta(V,X,JY,Z,JW) + 2(\cos 2\pi/5 \sin^2 2\pi/5$
$\cos^2 4\pi/5 + \cos 4\pi/5 \sin^2 4\pi/5 \cos^2 2\pi/5)\left[\beta(V,JX,Y,JZ,W) + \beta(JV,X,Y,JZ,W)\right.$
$\left. + \beta(JV,JX,Y,Z,W)\right] + 2(\cos 2\pi/5 \sin^2 4\pi/5 \sin^2 2\pi/5 + \cos 4\pi/5 \sin^2 2\pi/5$
$\sin^2 4\pi/5)\left[\beta(JV,X,JY,JZ,JW) + \beta(V,JX,JY,JZ,JW) + \beta(JV,JX,JY,Z,JW)\right]$
$+ 2(\sin 4\pi/5 \cos 4\pi/5 \sin^3 2\pi/5 - \sin^3 4\pi/5 \sin 2\pi/5 \cos 2\pi/5)\left[\beta(JV,JX,Y,JZ,JW)\right.$
$\left. + \beta(JV,JX,JY,JZ,W)\right] = 4\beta(V,X,Y,Z,W)$

From the appendix (10) (i), (ii), (iii), (iv), (v) and (vi), we have

$$-\left[\beta(V,X,Y,JZ,JW) + \beta(V,X,JY,JZ,W) + \beta(V,JX,Y,Z,JW) + \beta(V,JX,JY,Z,W)\right.$$
$$\left.\beta(JV,X,Y,Z,JW) + \beta(JV,X,JY,Z,W)\right] - 3\beta(V,X,JY,Z,JW) + \left[\beta(V,JX,Y,JZ,W)\right.$$
$$\left. + \beta(JV,X,Y,JZ,W) + \beta(JV,JX,Y,Z,W)\right] - \left[\beta(V,JX,JY,JZ,JW) + \beta(JV,X,JY,JZ,JW)\right.$$
$$\left. + \beta(JV,JX,JY,Z,JW)\right] - 3\left[\beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,JZ,W)\right]$$
$$= 13\beta(V,X,Y,Z,W) \quad - - - - - - - \text{(iii)}$$

In (iii) replace V, X, Z, W by JV, JX, JZ, JW, we have

$$-\left[\beta(JY,JX,Y,Z,W) - \beta(JV,JX,JY,Z,JW) + \beta(JV,X,Y,JZ,W) - \beta(JV,X,JY,JZ,JW)\right.$$
$$\left. + \beta(V,JX,Y,JZ,W) - \beta(V,JX,JY,JZ,JW)\right] + 3\beta(JV,JX,JY,JZ,W) +$$
$$\left[\beta(JV,X,Y,Z,JW) + \beta(V,JX,Y,Z,JW) + \beta(V,X,Y,JZ,JW)\right] - \left[-\beta(JV,X,JY,Z,W)\right.$$
$$\left. - \beta(V,JX,JY,Z,W) - \beta(V,X,JY,JZ,W)\right] - 3\left[\beta(V,X,Y,Z,W) - \beta(V,X,JY,Z,JW)\right]$$
$$= 13\beta(JV,JX,Y,JZ,JW) \quad - - - - - - - \text{(iv)}$$

From (iii) and (iv) we have

$$-3\left[\beta(JV,JX,Y,JZ,JW) + \beta(V,X,Y,Z,W)\right] = 13\left[\beta(V,X,Y,Z,W) + \beta(JY,JX,Y,JZ,JW)\right]$$

or $\beta(V,X,Y,Z,W) + \beta(JV,JX,Y,JZ,JW) = 0 \quad - - - - \text{(v)}$

Similarly, if we replace Y, X, Y, Z by JV, JX, JY, JZ, and we compare the result with (iii) we have

$$-3\left[\beta(V,X,Y,Z,W) + \beta(JV,JX,JY,JZ,W) = 13\left[\beta(V,X,Y,Z,W) + \right.\right.$$
$$\left.\beta(JV,JX,JY,JZ,W)\right]$$

or β (V,X,Y,Z,W) + β (JV,JX,JY,JZ,W) = o - - - -(vi)

In (v) replace Y,W by JY,JW and add the result to (vi) we get

β (V,X,Y,Z,W) + β (V,X,JY,Z,JW) = o - - - - (vii)

Using (v), (vi) and (vii), consider the following

β(V,JX,JY,JZ,JW) + β(JV,X,JY,JZ,JW) + β(JV,JX,JY,Z,JW)

= β(JV,X,JY,Z,W) + β(V,JX,JY,Z,W) + β(V,X,JY,JZ,W)

= β(JV,X,Y,Z,JW) + β(V,JX,Y,Z,JW) + β(V,X,Y,JZ,JW)

= - [β(V,JX,Y,JZ,W) + β(JV,X,Y,JZ,W) + β(JV,JX,Y,Z,W)]

This and (v), (vi) (vii) in (iii) gives

$$-2\beta(V,X,Y,Z,W) = \Big[\beta(V,X,Y,JZ,JW) + \beta(V,X,JY,JZ,W) + \beta(V,JX,Y,Z,JW)$$
$$+\beta(V,JX,JY,Z,W) + \beta(JV,X,Y,Z,JW) + \beta(JV,X,JY,Z,W)\Big]$$

But from (iii) we have

$$\beta(V,JX,Y,JZ,W) + \beta(JV,X,Y,JZ,W) + \beta(JV,JX,Y,Z,W) = \beta(V,X,Y,Z,W)$$
$$= -\beta(V,X,JY,Z,JW)$$

Hence we finally have

$$-2\beta(V,X,Y,Z,W) = \beta(JV,JX,Y,Z,W) + \cdots + \beta(V,X,Y,JZ,JW)$$

(2)(iii)(c) :- $V \in V_1$ and $X,Y,Z,W \in V_2$. From equation ② we have $\beta\Big[(\cos 2\pi/5)V \pm (\sin 2\pi/5)JV,$

$(\cos 4\pi/5)X \pm (\sin 4\pi/5)JX, (\cos 4\pi/5)Y \pm (\sin 4\pi/5)JY, (\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ,$

$(\cos 4\pi/5)W \pm (\sin 4\pi/5)JW\Big] = 2\beta(V,X,Y,Z,W) \cdots$ (i)

$\beta\Big[(\cos 4\pi/5)V \pm (\sin 4\pi/5)JV, (\cos 2\pi/5)X \mp (\sin 2\pi/5)JX, (\cos 2\pi/5)Y \mp (\sin 2\pi/5)JY,$

$(\cos 2\pi/5)Z \mp (\sin 2\pi/5)JZ, (\cos 2\pi/5)W \mp (\sin 2\pi/5)JW\Big] = 2\beta(V,X,Y,Z,W) \cdots\cdots$ (ii)

Using linearity of β, and add (i) to (ii) we have

$$2(\cos 2\pi/5 \cos^4 4\pi/5 + \cos 4\pi/5 \cos^4 2\pi/5)\,\beta(V,X,Y,Z,W) + 2(\cos 2\pi/5 \cos^2 4\pi/5$$
$$\sin^2 4\pi/5 + \cos 4\pi/5 \cos^2 2\pi/5 \sin^2 2\pi/5)\Big[\beta(V,JX,JY,Z,W) + \beta(V,JX,Y,JZ,W)$$
$$+\beta(V,JX,Y,Z,JW) + \beta(V,X,JY,JZ,W) + \beta(V,X,JY,Z,JW) + \beta(V,X,Y,JZ,JW)\Big]$$
$$+2(\cos 2\pi/5 \sin^4 4\pi/5 + \cos 4\pi/5 \sin^4 2\pi/5)\,\beta(V,JX,JY,JZ,JW) + 2(\sin 2\pi/5$$
$$\cos^3 4\pi/5 \sin 4\pi/5 - \sin 4\pi/5 \sin 2\pi/5 \cos^3 2\pi/5)\Big[\beta(JV,X,Y,Z,JW) + \beta(JV,X,Y,JZ,W)$$
$$+\beta(JV,X,JY,Z,W) + \beta(JV,JX,Y,Z,W)\Big] + 2(\sin 2\pi/5 \cos 4\pi/5 \sin^3 4\pi/5 -$$
$$\sin 4\pi/5 \sin^3 2\pi/5 \cos 2\pi/5)\Big[\beta(JV,X,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW) +$$
$$\beta(JV,JX,JY,Z,JW) + \beta(JV,JX,JY,JZ,W)\Big] = 4\beta(V,X,Y,Z,W)$$

From the appendix (9) (i), (ii), (iii), (iv) and (v) we have

$$-2\beta(V,JX,JY,JZ,JW) - [\beta(JV,X,Y,Z,JW) + \beta(JV,X,Y,JZ,W) + \beta(JV,X,JY,Z,W) + \beta(JV,JX,Y,Z,W)] - [\beta(JV,X,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,Z,JW) + \beta(JV,JX,JY,JZ,W)] = 6\beta(V,X,Y,Z,W) \quad \text{——— (iii)}$$

Replace X,Y,Z,W by JX,JY,JZ,JW in (iii) we have

$$-2\beta(V,X,Y,Z,W) - [-\beta(JV,JX,JY,JZ,W) - \beta(JV,JX,JY,Z,JW) - \beta(JV,JX,Y,JZ,JW) - \beta(JV,X,JY,JZ,JW)] - [-\beta(JV,JX,Y,Z,W) - \beta(JV,X,JY,Z,W) - \beta(JV,X,Y,JZ,W) - \beta(JV,X,Y,Z,JW)]$$
$$= 6\beta(V,JX,JY,JZ,JW) \quad \text{——— (iv)}$$

From (iii) and (iv) we have

$$-2[\beta(V,JX,JY,JZ,JW) + \beta(V,X,Y,Z,W)] = 6[\beta(V,X,Y,Z,W) + \beta(V,JX,JY,JZ,JW)] \text{ or, } \beta(V,X,Y,Z,W) + \beta(V,JX,JY,JZ,JW) = 0 \quad \text{——— (v)}$$

Using (v) in (iii) we have

$$-2\beta(V,X,Y,Z,W) = \beta(JV,JX,Y,Z,W) + \beta(JV,X,JY,Z,W) + \beta(JV,X,Y,JZ,W) + \beta(JV,X,Y,Z,JW) \quad \text{——— (vi)}$$

Also if in (v) we first replace X,Y by JX,JY, then we replace X, Z by JX,JZ and finally we replace X,W by JX,JW, and we add the three results we have

$$\beta(V,JX,JY,Z,W) + \beta(V,X,Y,JZ,JW) + \beta(V,JX,Y,JZ,W) + \beta(V,X,JY,Z,JW) + \beta(V,JX,Y,Z,JW) + \beta(V,X,JY,JZ,W) = 0 \quad \text{——— (vii)}$$

Equations (vi) and (vii) give

$$-2\beta(V,X,Y,Z,W) = \beta(JV,JX,Y,Z,W) + \dots + \beta(V,X,Y,JZ,JW)$$

2) (iii) (d) $V,W \in V_2$ and $X,Y,Z \in V_1$. From equation ② we have

$$\beta\Big[(\cos 4\pi/5)V \pm (\sin 4\pi/5)JV,\ (\cos 2\pi/5)X \pm (\sin 2\pi/5)JX,\ (\cos 2\pi/5)Y \pm (\sin 2\pi/5)JY,$$

$$(\cos 2\pi/5)Z \pm (\sin 2\pi/5)JZ,\ (\cos 4\pi/5)W \pm (\sin 4\pi/5)JW\Big] = 2\beta(V,X,Y,Z,W) \text{ ——— (i)}$$

$$\beta\Big[(\cos 2\pi/5)V \mp (\sin 2\ /5)JV,\ (\cos 4\pi/5)X \pm (\sin 4\pi/5)JX,\ (\cos 4\pi/5)Y \pm$$

$$(\sin 4\pi/5)JY,\ (\cos 4\pi/5)Z \pm (\sin 4\pi/5)JZ,\ (\cos 2\pi/5)W \mp (\sin 2\pi/5)JW\Big] =$$

$$2\beta(V,X,Y,Z,W) \text{ ——— (ii)}$$

Using linearity of β , and add (i) to (ii) , we have

$$2(\cos^2 4\pi/5 \cos^3 2\pi/5 + \cos^2 2\pi/5 \cos^3 4\pi/5)\ \beta(V,X,Y,Z,W) +$$

$$2(\cos 4\pi/5 \cos^2 2\pi/5 \sin 2\pi/5 \sin 4\pi/5 - \cos 2\pi/5 \cos^2 4\pi/5 \sin 4\pi/5 \sin 2\pi/5)$$

$$\Big[\beta(V,X,Y,JZ,JW), \beta(V,X,JY,Z,JW) + \beta(V,JX,Y,Z,JW) + \beta(JV,X,Y,JZ,W)$$

$$+ \beta(JV,X,JY,Z,W) + \beta(JV,JX,Y,Z,W)\Big] + 2(\cos^2 4\pi/5 \sin^2 2\pi/5 \cos 2\pi/5$$

$$+ \cos 4\pi/5 \sin^2 4\pi/5 \cos^2 2\pi/5)\Big[\beta(V,X,JY,JZ,W) + \beta(V,JX,Y,JZ,W)$$

$$+ \beta(V,JX,JY,Z,W)\Big] + 2(\cos^3 2\pi/5 \sin^2 4\pi/5 + \cos^3 4\pi/5 \sin^2 2\pi/5)$$

$$\beta(JV,X,Y,Z,JW) + 2(\cos 4\pi/5 \sin^3 2\pi/5 \sin 4\pi/5 - \cos 2\pi/5 \sin^3 4\pi/5 \sin 2\pi/5)$$

$$\Big[\beta(V,JX,JY,JZ,JW) + \beta(JV,JX,JY,JZ,W)\Big] + 2(\cos 2\pi/5 \sin^2 2\pi/5 \sin^2 4\pi/5$$

$$+ \cos 4\pi/5 \sin^2 4\pi/5 \sin^2 2\pi/5)\Big[\beta(JV,X,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW)$$

$$+ \beta(JV,JX,JY,Z,JW)\Big] = 4\beta(V,X,Y,Z,W)$$

From the appendix (10) (i), (ii), (iii), (iv), (v) and (vi) we have

$$-\Big[\beta(V,X,Y,JZ,JW) + \beta(V,X,JY,Z,JW) + \beta(V,JX,Y,Z,JW)$$

$$+\beta(JV,X,Y,JZ,W) + \beta(JV,X,JY,Z,W) + \beta(JV,JX,Y,Z,W)\Big] + \Big[\beta(V,X,JY,JZ,W)$$

$$+ \beta(V,JX,Y,JZ,W) + \beta(V,JX,JY,Z,W)\Big] - 3\Big[\beta(V,JX,JY,JZ,JW)$$

$$+ \beta(JV,X,Y,Z,JW) + \beta(JV,JX,JY,JZ,W)\Big] - \Big[\beta(JV,X,JY,JZ,JW)$$

$$+ \beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,Z,JW)\Big] = 13\beta(V,X,Y,Z,W) \quad \text{———— (iii)}$$

In (iii) replace X,Y,Z,W by JX,JY,JZ,JW, we have

$$-\Big[\beta(V,JX,JY,Z,W) + \beta(V,JX,Y,JZ,W) + \beta(V,X,JY,JZ,W)$$

$$- \beta(JV,JX,JY,Z,JW) - \beta(JV,JX,Y,JZ,JW) - \beta(JV,X,JY,JZ,JW)\Big]$$

$$+\Big[\beta(V,JX,Y,Z,JW) + \beta(V,X,JY,Z,JW) + \beta(V,X,Y,JZ,JW)\Big]$$

$$-3\Big[\beta(V,X,Y,Z,W) - \beta(JV,JX,JY,JZ,W) - \beta(JV,X,Y,Z,JW)\Big]$$

$$-\Big[-\beta(JV,JX,Y,Z,W) - \beta(JV,X,JY,Z,W) - \beta(JV,X,Y,JZ,W)\Big]$$

$$=13\beta(V,JX,JY,JZ,JW) \quad \text{———— (iv)}$$

From (iii) and (iv) we have

$$3\Big[\beta(V,JX,JY,JZ,JW) + \beta(JV,X,Y,Z,JW) + \beta(JV,JX,JY,Z,JW)\Big]$$

$$+13\beta(V,X,Y,Z,W) = -3\Big[\beta(V,X,Y,Z,W) - \beta(JV,JX,JY,JZ,W)$$

$$+\beta(JV,X,Y,Z,JW)\Big] - 13\beta(V,JX,JY,JZ,JW) \quad \text{or} \quad \beta(V,X,Y,Z,W)$$

$$+ \beta(V,JX,JY,JZ,JW) = 0 \quad \text{———— (v)}$$

Similarly if in (iii) we replace V,X,Y,Z, by JV,JX,JY,JZ and compare

the result by (iii) we have

$$\beta(V,X,Y,Z,W) + \beta(JV,JX,JY,JZ,W) = 0 \quad \text{———— (vi)}$$

In (v) replace V,W by JV, JW and add the result to (vi), we get

$$\beta(V,X,Y,Z,W) + \beta(JV,X,Y,Z,JW) = 0 \quad \text{———— (vii)}$$

Using (v), (vi) and (vii), consider the following

$$\left.\begin{aligned} &\beta(JV,X,JY,JZ,JW) + \beta(JV,JX,Y,JZ,JW) + \beta(JV,JX,JY,JZ,W) \\ &= \beta(JV,JX,Y,Z,W) + \beta(JV,X,JY,Z,W) + \beta(JV,X,Y,Z,JW) \\ &= \beta(V,JX,Y,Z,JW) + \beta(V,X,JY,Z,JW) + \beta(V,X,Y,JZ,JW) \\ &= -\Big[\beta(V,X,JY,JZ,W) + \beta(V,JX,Y,JZ,W) + \beta(V,JX,JY,Z,W)\Big] \end{aligned}\right\} \text{———— (viii)}$$

This and (v), (vi), (vii) in (iii) gives

$$-\left[\beta(JV,X,JY,JZ,JW)+\beta(JV,JX,Y,JZ,JW)+\beta(JV,JX,JY,JZ,W)\right]$$

$$=\beta(V,X,Y,Z,W)$$

Hence (iii) is reduced to

$$-2\beta(V,X,Y,Z,W)=\beta(V,X,Y,JZ,JW)+\beta(V,X,JY,Z,JW)+\beta(V,JX,Y,Z,JW)$$
$$+\beta(JV,X,Y,JZ,W)+\beta(JV,X,JY,Z,W)+\beta(JV,JX,Y,Z,W)$$

But from (viii) we have

$$\beta(V,X,JY,JZ,W)+\beta(V,JX,Y,JZ,W)+\beta(V,JX,JY,Z,W)=\beta(V,X,Y,Z,W)$$

$$=-\beta(JV,X,Y,Z,JW)$$

Hence we can write

$$-2\beta(V,X,Y,Z,W)=\beta(JV,JX,Y,Z,W)+\ldots+\beta(V,X,Y,JZ,JW)$$

2) (iii) (e) $X\in V_1$, and $V,Y,Z,W\in V_2$. From equation ② we have

$$\beta\left[(\cos 4\pi/5)V\pm(\sin 4\pi/5)JV,\ (\cos 2\pi/5)X\pm(\sin 2\pi/5)JX,(\cos 4\pi/5)Y\pm\right.$$
$$\left.(\sin 4\pi/5)JY,\ (\cos 4\pi/5)Z\pm(\sin 4\pi/5)JZ,(\cos 4\pi/5)W\pm(\sin 4\pi/5)JW\right]$$

$$=2\beta(V,X,Y,Z,W)\ \text{———}\ (i)$$

$$\beta\left[(\cos 2\pi/5)V\mp(\sin 2\pi/5)JV,\ (\cos 4\pi/5)X\pm(\sin 4\pi/5)JX,\ (\cos 2\pi/5)Y\right.$$
$$\left.\mp(\sin 2\pi/5)JY,(\cos 2\pi/5)Z\mp(\sin 2\pi/5)JZ,\ (\cos 2\pi/5)W\mp(\sin 2\pi/5)JW\right]$$

$$=2\beta(V,X,Y,Z,W)\ \text{———}\ (ii)$$

Using linearity of β, and add (i) to (ii), we have

$$2(\cos^4 4\pi/5\cos^2 \pi/5+\cos^4 2\pi/5\cos 4\pi/5)\ \beta(V,X,Y,Z,W)+2(\cos^2 4\pi/5\cos 2\pi/5$$
$$\sin^2 4\pi/5+\cos^2 2\pi/5\cos 4\pi/5\sin^2 2\pi/5)\left[\beta(JV,X,JY,Z,W)+\beta(JV,X,Y,JZ,W)\right.$$
$$\left.+\beta(JV,X,Y,Z,JW)+\beta(V,X,JY,JZ,W)+\beta(V,X,JY,Z,JW)+\beta(V,X,Y,JZ,JW)\right]$$

$$+2(\cos^3 4\pi/_5 \sin^2 \pi/_5 \sin^4 \pi/_5 - \cos^3 2\pi/_5 \sin^4 \pi/_5 \sin^2 \pi/_5)[\beta(V,JX,Y,Z,JW)$$

$$+\beta(V,JX,Y,JZ,W)+\beta(V,JX,JY,Z,W)+\beta(JV,JX,Y,Z,W)]$$

$$+2(\cos^4 \pi/_5 \sin^2 \pi/_5 \sin^3 4\pi/_5 - \cos^2 \pi/_5 \sin^4 \pi/_5 \sin^3 2\pi/_5)[\beta(V,JX,JY,JZ,JW)$$

$$+\beta(JV,JX,Y,JZ,JW)+\beta(JV,JX,JY,Z,JW)+\beta(JV,JX,JY,JZ,W)]$$

$$+2(\cos^2 \pi/_5 \sin^4 4\pi/_5 + \cos^4 \pi/_5 \sin^4 2\pi/_5)\beta(JV,X,JY,JZ,JW) = 4\beta(V,X,Y,Z,W)$$

From the appendix (9) (i), (ii), (iii), (iv) and (v), we have

$$-[\beta(V,JX,Y,Z,JW)+\beta(V,JX,Y,JZ,W)+\beta(V,JX,JY,Z,W)+\beta(JV,JX,Y,Z,W)]$$

$$-[\beta(V,JX,JY,JZ,JW)+\beta(JV,JX,Y,JZ,JW)+\beta(JV,JX,JY,Z,JW)$$

$$+\beta(JV,JX,JY,JZ,W)] - 2\beta(JV,X,JY,JZ,JW) = 6\beta(V,X,Y,Z,W) \quad \text{——— (iii)}$$

Replace V, Y, Z, W by JV, JY, JZ, JW in (iii), we have

$$-[-\beta(JV,JX,JY,JZ,W)-\beta(JV,JX,JY,Z,JW)-\beta(JV,JX,Y,JZ,JW)$$

$$-\beta(V,JX,JY,JZ,JW)] - [-\beta(JV,JX,Y,Z,W)-\beta(V,JX,JY,Z,W)$$

$$-\beta(V,JX,Y,JZ,W)-\beta(V,JX,Y,Z,JW)]$$

$$-2\beta(V,X,Y,Z,W) = 6\beta(JV,X,JY,JZ,JW) \quad \text{——— (iv)}$$

Add (iii) to (iv), we have

$$-2[\beta(JV,X,JY,JZ,JW)+\beta(V,X,Y,Z,W)] = 6[\beta(V,X,Y,Z,W)$$

$$+\beta(JV,X,JY,JZ,JW)]$$

or $\beta(V,X,Y,Z,W)+\beta(JV,X,JY,JZ,JW) = 0$ ——— (v)

Use (v) in (iii) we have

$$-2\beta(V,X,Y,Z,W) = \beta(V,JX,Y,Z,JW)+\beta(V,JX,Y,JZ,W)$$

$$+\beta(V,JX,JY,Z,W)+\beta(JV,JX,Y,Z,W) \quad \text{——— (vi)}$$

In (v) first replace V, Y by JV, JY, then V, Z by JV, JZ, and finally

V, W, by JV, JW and add the results, we have

$$\beta(JV,X,JY,Z,W)+\beta(V,X,Y,JZ,JW)+\beta(JV,X,Y,JZ,W)$$

$$+\beta(V,X,JY,Z,JW)$$

$+\beta(JV,X,Y,Z,JW) + \beta(V,X,JY,JZ,W) = 0$, hence we can write $-2\beta(V,X,Y,Z,W) = \beta(JV,JX,Y,Z,W) + \ldots + \beta(V,X,Y,JZ,JW)$. This complete the proof of proposition 5.3.1. //

Proposition 5.3.2. Let M be a Riemanenan locally 5 regular symmetric manifold. Let J be the almost complex structure on M. Let M_1 and M_2 be the two differentiable distributions on M, such that at each point

$p \in M$, we have $M_p = M_{p1} \oplus M_{p2}$

(1) (i) If X, Y, Z, W, are vector fields on M belonging either to M_1 or M_2 or $X, Y \in M_1$ and $Z, W \in M_2$ or $X, Z \in M_1$ and $Y, W \in M_2$ Then $R(X,Y,Z,W) = R(JX,JY,Z,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW)$ and $R(X,Y,Z,W) = R(JX,JY,JZ,JW)$

(ii) If $X, Y, Z \in M_1$ and $W \in M_2$. Then, $-3\ R(X,Y,Z,W) = R(JX,JY,Z,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW)$ and $R(X,Y,Z,W) = R(JX,JY,JZ,JW)$

(iii) If $X \in M_1$ and $Y, Z, W \in M_2$. Then $3R(X,Y,Z,W) = R(JX,JY,Z,W) + R(JX,Y,JZ,W) + (RJX,Y,Z,JW)$ and $R(X,Y,Z,W) = -R(JX,JY,JZ,JW)$

(2) (i) If V, X, Y, Z, W are vector fields on M belonging either to M_1 or M_2. Then $-10(\nabla_V R)(X,Y,Z,W) = (\nabla_{JV} R)(JX,Y,Z,W) + (\nabla_{JV} R)(X,JY,Z,W) + (\nabla_{JV} R)(X,Y,JZ,W) + (\nabla_{JV} R)(X,Y,Z,JW) + (\nabla_V R)(JX,JY,Z,W) + (\nabla_V R)(JX,Y,JZ,W) + (\nabla_V R)(JX,Y,Z,JW) + (\nabla_V R)(X,JY,JZ,W)$

$(\nabla_V R)(X,JY,Z,JW) + (\nabla_V R)(X,Y,JZ,JW)$

(ii) If $V,X,Y,Z \in M_1$ and $W \in M_2$ or $V,X \in M_1$ and $Y,Z,W \in M_2$ or $V \in M_2$ and $X,Y,Z,W \in M_1$ or $V,Z,W \in M_2$ and $X,Y \in M_1$ or $V,Y,W \in M_2$ and $X,Z \in M_1$. Then $2(\nabla_V R)(X,Y,Z,W) = (\nabla_{JV} R)(JX,Y,Z,W) + \ldots + (\nabla_V R)(X,Y,JZ,JW)$

(iii) If $V,X,Y \in M_1$ and $Z,W \in M_2$ or $V,X,Z \in M_1$ and $Y,W \in M_2$ or $V \in M_1$ and $X,Y,Z,W \in M_2$ or $V,W, \in M_2$ and $X,Y,Z \in M_1$ or $V,Y,Z,W \in M_2$ and $X \in M_1$. Then $-2(\nabla_V R)(X,Y,Z,W) = (\nabla_{JV} R)(JX,Y,Z,W) + \ldots + (\nabla_V R)(X,Y,JZ,JW)$

Proof This follows from proposition 5.3.1 and that the curvature tensor R is determined by its value at a fixed point, say $0 \in M$. //

Remark In part (1) of the above proposition, we only considered 6 combinations of vector fields belonging to M_1 and M_2, where in fact we have 16 combinations, but since the curvature tensor field R satisfies $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z) = R(Z,W,X,Y)$ $X,Y,Z,W \in \mathfrak{X}(M)$ any case which is not considered above, can be obtained from part (1). Part (2) in the above proposition may be treated in the same way.

Proposition 5.3.3. Let M be a Riewannian locally 5 - regular symmetric manifold with almost complex structure J

Then

(i) If $X \in M_1$, and $Y, Z, W \in M_2$, we have $\frac{1}{2}(m^2 - 4m - 44)$ $[(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW)] = 24\, R(J\nabla_{JV}(J)X, JY, JZ, JW) + 4\,[R(X, \nabla_{JV}(J)Y,Z,W) + R(X,Y,\nabla_{JV}(J)Z,W) + R(X,Y,Z,\nabla_{JV}(J)W) - R(JX, J\nabla_{JV}(J)Y,Z,W) - R(JX,Y,J\nabla_{JV}(J)Z,W) - R(JX,Y,Z,\nabla_{JV}(J)W)] - (6m-48)\, R(\nabla_V(J)X,JY,JZ,JW) + (2m-16)\,[R(JX,\nabla_V(J)Y,Z,W) + R(JX,Y,\nabla_V(J)Z,W) + R(JX,Y,Z,\nabla_V(J)W)]$, where $m = -10$ or 2 or -2; $V \in \mathfrak{X}(M)$

(ii) If X,Y,Z,W belong to either M_1 or M_2 or $X, Y \in M_1$ and $Z, W \in M_2$ or $X, Z \in M_1$ and $Y, W \in M_2$ or $X,Y,Z \in M_1$ and $W \in M_2$, we have $(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW) = 0$, or $(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW) = (\nabla_V R)(JX,JY,Z,W) + \dots + (\nabla_V R)(X,Y,JZ,JW)$

<u>Proof</u>:- (i) From proposition 5.3.2 we have $R(X,Y,Z,W) + R(JX,JY,JY,JW) = 0$ ------- (1) and $3\,R(X,Y,Z,W) = R(JX,JY,Z,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW)$ ------- (2) . Take the covariant derivative of (2) with respect to $V \in \mathfrak{X}(M)$ we get $3[(\nabla_V R)(X,Y,Z,W) + R(\nabla_V X,Y,Z,W) + R(X,\nabla_V Y,Z,W) + R(X,Y,\nabla_V Z,W) + R(X,Y,Z,\nabla_V W)] - [(\nabla_V R)(JX,JY,Z,W) + R(\nabla_V(J)X,JY,Z,W) + R(J\nabla_V X,JY,Z,W) + R(JX,\nabla_V(J)Y,Z,W) + R(JX,J\nabla_V Y,Z,W)$

$+ R(JX,JY,\nabla_V Z,W) + R(JX,JY,Z,\nabla_V W) + (\nabla_V R)(JX,YJZ,W)$
$+ R((\nabla_V J)X,Y,JZ,W) + R(J\nabla_V X,Y,JZ,W) + R(JX,\nabla_V Y,JZ,W)$
$+ R(JX,Y,\nabla_V(J)Z,W) + R(JX,Y,J\nabla_V Z,W) + R(JX,Y,JZ,\nabla_V W)$
$+ (\nabla_V R)(JX,Y,Z,JW) + R(\nabla_V(J)X,Y,Z,JW) + R(J\nabla_V X,Y,Z,JW)$
$+ R(JX,\nabla_V Y,Z,JW) + R(JX,Y,\nabla_V Z,JW) + R(JX,Y,Z,\nabla_V(J)W)$
$+ R(JX,Y,Z,J\nabla_V W) = 0$ . . . . (3)

In (2) if we replace X by $\nabla_V X$, we have

$3R(\nabla_V X,Y,Z,W) = R(J\nabla_V X,JY,Z,W) + R(J\nabla_V X,Y,JZ,W) +$
$R(J\nabla_V X,Y,Z,JW)$ and we also have similar idendities if in (2) we replace Y, Z, W by $\nabla_V Y$, $\nabla_V Z$ and $\nabla_V W$ respectively. Hence (3) is reduced to $3(\nabla_V R)(X,Y,Z,W) - (\nabla_V R)(JX,JY,Z,W) - (\nabla_V R)(JX, Y,JZ,W) - (\nabla_V R)(JX,Y,Z,JW) = R(\nabla_V(J)X,JY,Z,W) + R(JX,$
$\nabla_V(J)Y,Z,W) + R(\nabla_V(J)X,Y,JZ,W) + R(JX,Y,\nabla_V(J)Z,W)$
$+ R(\nabla_V(J)X,Y,Z,JW)$

$+ R(JX,Y,Z,\nabla_V(J)W)$ . . . . (4)

From (1) and (2) we have $3R(\nabla_V(J)X,JY,JZ,JW) = -R(\nabla_V(J)X, JY,Z,W) - R(\nabla_V(J)X,Y,JZ,W) - R(\nabla_V(J)X, Y,Z,JW)$ Hence (4) can be written as $3(\nabla_V R)(X,Y,Z,W) - (\nabla_V R)(JX,JY,Z,W) - (\nabla_V R)(JX,Y,JZ,W) - (\nabla_V R)(JX,Y,Z,JW) = -3R(\nabla_V(J)X,JY,JZ,JW) + R(JX,\nabla_V(J)Y,Z,W) + R(JX,Y,\nabla_V(J)Z,W) + R(JX,Y,Z,\nabla_V(J)W)$ . . . . (5)

In (5) replace X,Y,Z,W by JX,JY,JZ,JW, using that $(\nabla_V J)(JX) = -J(\nabla_V J)(X)$, we have $3(\nabla_V R)(JX,JY,JZ,JW) - (\nabla_V R)(X,Y,JZ,JW) - (\nabla_V R)(X,JY,Z,JW) - (\nabla_V R)(X,JY,JZ,W) - 3R(J\nabla_V(J)X,Y,Z,W) + R(X,J\nabla_V(J)Y,JZ,JW) + R(X,JY,J\nabla_V(J)Z,JW) + R(X,JY,JZ,J\nabla_V(J)W)$ . . . . (6). Add (5) to (6), we have $3[(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW)] - [(\nabla_{JV} R)(JX,Y,Z,W)$

$+ (\nabla_V R)(JX, Y, JZ, W) + (\nabla_V R)(JX, Y, Z, JW) + (\nabla_V R)(X, JY, JZ, W)$
$+ (\nabla_V R)(X, JY, Z, JW) + (\nabla_V R)(X, Y, JZ, JW) = -6R(\nabla_V (J) X, JY, JZ, JW) + 2\left[R(JX, \nabla_V (J) Y, Z, W) + R(JX, Y, \nabla_V (J) Z, W) + R(JX, Y, Z, \nabla_V (J) W)\right]$ - - - - - — (7)

From proposition 5.3.2. we have

$$\left.\begin{aligned} & m(\nabla_V R)(X, Y, Z, W) - \left[(\nabla_{JV} R)(JX, Y, Z, W) + (\nabla_{JV} R)(X, JY, Z, W) + (\nabla_{JV} R)(X, Y, JZ, W) + (\nabla_{JV} R)(X, Y, Z, JW)\right] \\ & = (\nabla_V R)(JX, JY, Z, W) + (\nabla_V R)(JX, Y, JZ, W) + (\nabla_V R) \end{aligned}\right\} \quad \ldots\ldots (8)$$

$(X, JY, JZ, W) + (\nabla_V R)(X, JY, Z, JW) + (\nabla_V R)(X, Y, JZ, JW) + (\nabla_V R)(JX, Y, Z, JW)$

where $m = 10$ or 2 or - 2. In (8) replace X, Y, Z, W by JX, JY, JZ, JW and add the result to 8, we have $m\left[(\nabla_V R)(X, Y, Z, W) + (\nabla_V R)(JX, JY, JZ, JW)\right] - \left[(\nabla_{JV} R)(JX, Y, Z, W) + (\nabla_{JV} R)(X, JY, Z, W) + (\nabla_{JV} R)(X, Y, JZ, W) + (\nabla_{JV} R)(X, Y, Z, JW)\right] + (\nabla_{JV} R)(X, JY, JZ, JW) + (\nabla_{JV} R)(JX, Y, JZ, JW) + (\nabla_{JV} R)(JX, JY, Z, JW) + (\nabla_{JV} R)(JX, JY, JZ, W)] = 2\,((\nabla_V R)(JX, JY, Z, W) + (\nabla_V R)(JX, Y, JZ, W)$

$+ (\nabla_V R)(JX, Y, Z, JW) + (\nabla_V R)(X, JY, JZ, W) + (\nabla_V R)(X, JY, Z, JW)$

$+ (\nabla_V R)(X, Y, JZ, JW)\Big]$ - - - - - - - - (9)

Use (7) in (9) we have $(3 - \frac{1}{2}m)\Big[(\nabla_V R)(X, Y, Z, W) + (\nabla_V R)(JX, JY, JZ, JW)\Big] - \frac{1}{2}\Big\{ -\Big[(\nabla_{JV} R)(JX, Y, Z, W) + (\nabla_{JV} R)(X, JY, Z, W)$

$+ (\nabla_{JV} R)(X, Y, JZ, W) + (\nabla_{JV} R)(X, Y, Z, JW)\Big] + \Big[(\nabla_{JV} R)(X, JY, JZ, JW)$

$+ (\nabla_{JV} R)(JX, Y, JZ, JW) + (\nabla_{JV} R)(JX, JY, Z, JW) + (\nabla_{JV} R)(JX, JY, JZ, W)\Big]$

$= - 6R(\nabla_V (J)X, JY, JZ, JW) + 2\Big[R(JX, \nabla_V (J)Y, Z, W) + R(JX, Y, \nabla_V (J)Z, W)$

$+ R(JX, Y, Z, \nabla_V (J)W)\Big]$ - - - - - - - - (10)

In (10) replace first V, X by JV, JX then V, Y by JV, JY then V, Z by JV, JZ and finally V, W by JV, JW, we have the following four equations.

$(3 - \frac{1}{2}m)\Big[(\nabla_{JV} R)(JX, Y, Z, W) - (\nabla_{JV} R)(X, JY, JZ, JW)\Big] - \frac{1}{2}\Big\{ -$

$\Big[(\nabla_V R)(X, Y, Z, W) - (\nabla_V R)(JX, JY, Z, W) - (\nabla_V R)(JX, Y, JZ, W)$

$- (\nabla_V R)(JX, Y, Z, JW)\Big] + \Big[- (\nabla_V R)(JX, JY, JZ, JW) + (\nabla_V R)(X, Y, JZ, JW)$

$+ (\nabla_V R)(X, JY, Z, JW) + (\nabla_V R)(X, JY, JZ, W)] \} = 6R(J\nabla_{JV}(J)$
$X, JY, JZ, JW) + 2[- R(X, \nabla_{JV}(J)Y, Z, W) - R(X, Y, \nabla_{JV}(J)Z, W)$
$- R(X, Y, Z, \nabla_{JV}(J)W)]$ ------ (11) $\therefore$ $(3 - \frac{1}{2}m)$
$(\nabla_{JV} R)(X, JY, Z, W) - (\nabla_{JV} R)(JX, Y, JZ, JW) - \frac{1}{2}\{ -$
$[(\nabla_V R)(JX, JY, Z, W) + (\nabla_V R)(X, Y, Z, W) - (\nabla_V R)(X, JY, JZ, W)$
$- (\nabla_V R)(X, JY, Z, JW)] + (\nabla_V R)(X, Y, JZ, JW) - (\nabla_V R)$
$(JX, JY, JZ, JW) + (\nabla_V R)(JX, Y, Z, JW) + (\nabla_V R)(JX, Y, JZ, W)]\}$
$= 6R(\nabla_{JV}(J)X, Y, JZ, JW) + 2[- R(JX, J\nabla_{JV}(J)Y, Z, W) + R(JX, JY, \nabla_{JV}$
$(J)Z, W) + R(JX, JY, Z, \nabla_{JV}(J)W)$ ------ (12)
$(3 - \frac{1}{2}m)[(\nabla_{JV} R)(X, Y, JZ, W) - (\nabla_{JV} R)(JX, JY, Z, JW)] - \frac{1}{2}\{ -$
$[- (\nabla_V R)(JX, Y, JZ, W) - (\nabla_V R)(X, JY, JZ, W) + (\nabla_V R)(X, Y, Z, W)$
$- (\nabla_V R)(X, Y, JZ, JW)] + [(\nabla_V R)(X, JY, Z, JW) + (\nabla_V R)$
$(JX, Y, Z, JW) - (\nabla_V R)(JX, JY, JZ, JW) + (\nabla_V R)(JX, JY, Z, W) =$
$6R(\nabla_{JV}(J)X, JY, Z, JW) + 2[R(JX, \nabla_{JV}(J)Y, JZ, W) - R(JX, Y, J\nabla_{JV}(J)Z, W)$
$+ R(JX, Y, JZ, \nabla_{JV}(J)W)]$ .... (13)

$(3 - \frac{1}{2}m)[(\nabla_{JV} R)(X, Y, Z, JW) - (\nabla_{JV} R)(JX, JY, JZ, W)] - \frac{1}{2}\{ -$

$[-(\nabla_V R)(JX,Y,Z,JW) - (\nabla_V R)(X,JY,Z,JW) - (\nabla_V R)(X,Y,JZ,JW)$
$+ (\nabla_V R)(X,Y,Z,W)] + [(\nabla_V R)(X,JY,JZ,W) + (\nabla_V R)(JX,Y,JZ,W)$
$+ (\nabla_V R)(JX,JY,Z,W) - (\nabla_V R)(JX,JY,JZ,JW) = 6R(\nabla_{jV}(J)X,JY,JZ,W)$
$+ 2[R(JX,\nabla_{jV}(J)Y,Z,JW) + R(JX,Y,\nabla_{jV}(J)X,JW) - R(JX,Y,Z,J\nabla_{jV}(J)W)$ ...(14)

Add (11), (12), (13) and (14), we have L.H.S. equal to $(3 - \frac{1}{2}m)$
$(\nabla_{jV}R)(JX,Y,Z,W) + (\nabla_{jV}R)(X,JY,Z,W) + (\nabla_{jV}R)(X,Y,JZ,W)$
$+ (\nabla_{jV}R)(X,Y,Z,JW) - (\nabla_{jV}R)(X,JY,JZ,JW) - (\nabla_{jV}R)(JX,Y,JZ,JW)$
$- (\nabla_{jV}R)(JX,JY,Z,JW) + (\nabla_{jV}R)(JX,JY,JZ,W) - \frac{1}{2}\{-4(\nabla_V R)$
$(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW)] + 4[(\nabla_V R)(JX,JY,Z,W) +$
$(\nabla_V R)(JX,Y,JZ,W) + (\nabla_V R)(JX,Y,Z,JW) + (\nabla_V R)(X,JY,JZ,W)$
$+ (\nabla_V R)(X,JY,Z,JW) + (\nabla_V R)(X,Y,JZ,JW)]\}$

Use quations (10) and (7) we have the L.H.S. is reduced to
$(3 - \frac{1}{2}m)\{-12R(\nabla_V(J)X,JY,JZ,JW) + 4[R(JX,\nabla_V(J)Y,Z,W) +$
$R(JX,Y,\nabla_V(J)Z,W) + R(JX,Y,Z,\nabla_V(J)W) - 2(3-\frac{1}{2}m)[(\nabla_V R)$
$(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW)]\}$

$$-\tfrac{1}{2}\Big\{-4\big[(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW)\big] + 12\big[(\nabla_V R)(X,Y,Z,W)$$

$$+ (\nabla_V R)(JX,JY,JZ,JW)\big] + 24R(\nabla_V(J)X,JY,JZ,JW) - 8\big[R(JX,\nabla_V(J)Y,Z,W)$$

$$+R(JX,Y,\nabla_V(J)Z,W) + R(JX,Y,Z,\nabla_V(J)W)\big]\Big\} = \big[-2(3-\tfrac{1}{2}m)^2 - 4\big]$$

$$\big[(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW)\big] + \big[-12(3-\tfrac{1}{2}m) - 12\big]$$

$$R(\nabla_V(J)X,JY,JZ,JW) + \big[4(3-\tfrac{1}{2}m) + 4\big]\,R(JX,\nabla_V(J)Y,Z,W)$$

$$+ R(JX,Y,\nabla_V(J)Z,W) + R(JX,Y,Z,\nabla_V(J)W)\big]$$

The R.H.S. is equal to

$$6\big[R(J\nabla_{JV}(J)X,JY,JZ,JW) + R(\nabla_{JV}(J)X,Y,JZ,JW) + R(\nabla_{JV}(J)X,JY,Z,W)$$

$$+ R(\nabla_{JV}(J)X,JY,JZ,W) + 2\big[-R(X,\nabla_{JV}(J)Y,Z,W) - R(X,Y,\nabla_{JV}(J)Z,W)$$

$$- R(X,Y,Z,\nabla_{JV}(J)W) - R(JX,J\nabla_{JV}(J)Y,Z,W) + R(JX,JY,\nabla_{JV}(J)Z,W)$$

$$+ R(JX,JYZ,\nabla_{JV}(J)W) + R(JX,\nabla_{JV}(J)Y,JZ,W) - R(JX,Y,J\nabla_{JV}(J)Z,W)$$

$$+ R(JX,Y,JZ,\nabla_{JV}(J)W) + R(JX,\nabla_{JV}(J)Y,Z,JW) + R(JX,Y,\nabla_{JV}(J)Z,JW)$$

$$- R(JX,Y,Z,J\nabla_{JV}(J)W)\big].$$

Using equation (2) we have the R.H.S. equal to

$$24R(J\nabla_{JV}(J)X,JY,JZ,JW) + 4\big[R(X,\nabla_{JV}(J)Y,Z,W) + R(X,Y,\nabla_{JV}(J)Z,W)$$

$$+ R(X,Y,Z,\nabla_{JV}(J)W) - R(JX,J\nabla_{JV}(J)Y,Z,W) - R(JX,Y,J\nabla_{JV}(J)Z,W)$$

$$- R(JX,Y,Z,J\nabla_{JV}(J)W)\big]$$

Hence from all this we have

$$\tfrac{1}{2}(m^2 - 4m - 44)\big[(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW)$$

$$= 24R(J\nabla_{JV}(J)X,JY,JZ,JW) + 4\big[R(X,\nabla_{JV}(J)Y,Z,W) + R(X,Y,\nabla_{JV}(J)Z,W)$$

$$+ R(X,Y,Z,\nabla_{JV}(J)JW) - R(JX,J\nabla_{JV}(J)Y,Z,W) - R(JX,Y,J\nabla_{JV}(J)Z,W)$$

$$- R(JX,Y,Z,J\nabla_{JV}(J)W)\Big] - 6(m-8)\ R(\nabla_V(J)X,JY,JZ,JW)$$

$$+ 2(m-8)\Big[R(JX,\nabla_V(J)Y,Z,W) + R(JX,Y\nabla_V(J)Z,W) + R(JX,Y,Z,\nabla_V(J)W)$$

(ii) From proposition 5.3.2. we have

$$R(X,Y,Z,W) = R(JX,JY,JZ,JW) \quad \text{——— (1)}$$

and $kR(X,Y,Z,W) = R(JX,JY,Z,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,W)$ ——— (2)

where $k = 1$ or -3. As in part (i) of this proposition, if we take the covariant derivative with respect to a $V \in \mathcal{X}(M)$ of (2) and we use the following equation deduced from (2) by replacing X by $\nabla_V X$

$$kR(\nabla_V X,Y,Z,W) = R(J\nabla_V X,JY,Z,W) + R(J\nabla_V X,Y,JZ,W) + R(J\nabla_V X,Y,Z,JW)$$

and similar equations deduced from (2) by replacing Y, Z, and W by $\nabla_V Y$, $\nabla_V Z$ and $\nabla_V W$. We have

$$k(\nabla_V R)(X,Y,Z,W) - (\nabla_V R)(JX,JY,Z,W) - (\nabla_V R)(JX,Y,JZ,W)$$

$$- (\nabla_V R)(JX,Y,Z,JW) = R(\nabla_V(J)X,JY,Z,W) + R(\nabla_V(J)X,Y,JZ,W)$$

$$+ R(\nabla_V(J)X,Y,Z,JW) + R(JX,\nabla_V(J)Y,Z,W) + R(JX,Y,\nabla_V(J)Z,W)$$

$$+ R(JX,Y,Z,\nabla_V(J)W) \quad \text{——— (3)}$$

From equations (1) and (2) we have

$$kR(\nabla_V(J)X,JY,JZ,JW) = R(\nabla_V(J)X,JY,Z,W) + R(\nabla_V(J)X,Y,JZ,W)$$
$$+ R(\nabla_V(J)X,Y,Z,JW)$$

Therefore (3) can be written as

$$k(\nabla_V R)(X,Y,Z,W) - (\nabla_V R)(JX,JY,Z,W) - (\nabla_V R)(JX,Y,JZ,W)$$

$$-(\nabla_V R)(JX,Y,Z,JW) = kR(\nabla_V(J)X,JY,JZ,JW) + R(JX,\nabla_V(J)Y,Z,W)$$

$$+ R(JX,Y,\nabla_V(J)Z,W) + R(JX,Y,Z,\nabla_V(J)W) \text{——————} (4)$$

In (4) replace X, Y, Z, W by JX, JY, JZ, JW and add the result to (4) we have

$$k\left[(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW)\right] = (\nabla_V R)(JX,JY,Z,W)$$

$$+ (\nabla_V R)(JX,Y,JZ,W) + (\nabla_V R)(JX,Y,Z,JW) + (\nabla_V R)(X,JY,JZ,W)$$

$$+ (\nabla_V R)(X,JY,Z,JW) + (\nabla_V R)(X,Y,JZ,W) \text{——————} (5)$$

where $R(J\nabla_V(J)X,Y,Z,W) + R(\nabla_V(J)X,JY,JZ,JW) = o$

and $R(X,J\nabla_V(J)Y,JZ,JW) + R(JX,\nabla_V(J)Y,Z,W) = o$

and $R(X,JY,J\nabla_V(J)Z,JW) + R(JZ,Y,\nabla_V(J)Z,W) = o$

where we used equation (1) [Because of these three equations part (ii) of the proposition is different from part (i), since in part (i) such equations give twice each term instead of identically zero]. Use equation (5) in equation (9) part (i) we have

$$(m-2k)\left[(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW)\right]$$

$$= (\nabla_{JV} R)(JX,Y,Z,W) + (\nabla_{JV} R)(X,JY,Z,W) + (\nabla_{JV} R)(X,Y,JZW)$$

$$+ (\nabla_{JV} R)(X,Y,Z,JW) - \left[(\nabla_{JV} R)(X,JY,JZ,JW) + (\nabla_{JV} R)(JX,Y,JZ,JW)\right.$$

$$\left.+ (\nabla_{JV} R)(JX,JY,Z,JW) + (\nabla_{JV} R)(JX,JY,JZ,W)\right] \text{——————} (6)$$

In (6) we first replace V, X by JV, JX, and then we replace V, Y by JV, JY, and then we replace V, Z by JV, JZ, finally, we replace, V, W by JV, JW and we add all the four resulting equations we have

$$(m-2k)\left\{\left[(\nabla_{JV} R)(JX,Y,Z,W) + (\nabla_{JV} R)(X,JY,Z,W) + (\nabla_{JV} R)(X,Y,JZ,W)\right.\right.$$

$$\left.+ (\nabla_{JV} R)(X,Y,Z,JW)\right] - \left[(\nabla_{JV} R)(X,JY,JZ,JW) + (\nabla_{JV} R)(JX,Y,JZ,JW)\right.$$

$$\left.\left.+ (\nabla_{JV} R)(JX,JY,Z,JW) + (\nabla_{JV} R)(JX,JY,JZ,W)\right]\right\} = 4\left\{\left[(\nabla_V R)(X,Y,Z,W) + \right.\right.$$

$(\nabla_V R)(JX,JY,JZ,JW)\Big] - \Big[(\nabla_V R)(JX,JY,Z,W) + (\nabla_V R)(JX,Y,JZ,W)$

$+ (\nabla_V R)(JX,Y,Z,JW) + (\nabla_V R)(X,JY,JZ,W) + (\nabla_V R)(X,JY,Z,JW)$

$+ (\nabla_V R)(X,Y,JZ,JW)$ ——— (7)

Use equations (5) and (6) in (7) we have

$(m-2k)\Big\{(m-2k)\Big[(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW)\Big]\Big\}$

$= 4\Big\{\Big[(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW)\Big]$

$-k\Big[(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW)\Big]$

or $\Big[(m-2k)^2 - 4(k-1)\Big]\Big[(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW)\Big] = o$

Hence, if $k \neq 1$ and $m \neq 2$, we have

$(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW) = o$ ——— (8)

If $k = 1$ and $m = 2$, we have from equation (6) that

$(\nabla_{JV} R)(JX,Y,Z,W) + (\nabla_{JV} R)(X,JY,Z,W) + (\nabla_{JV} R)(X,Y,JZ,W)$

$+ (\nabla_{JV} R)(X,Y,Z,JW) - \Big[(\nabla_{JV} R)(X,JY,JZ,JW) + (\nabla_{JV} R)(JX,Y,JZ,JW)$

$+ (\nabla_{JV} R)(JX,JY,Z,JW) + (\nabla_{JV} R)(JX,JY,JZ,W)\Big] = o$ ——— (9)

In (9) replace V, X, by JV, JX, we have

$(\nabla_V R)(X,Y,Z,W) - (\nabla_V R)(JX,JY,Z,W) - (\nabla_V R)(JX,Y,JZ,W)$

$- (\nabla_V R)(JX,Y,Z,JW) - \Big[-(\nabla_V R)(JX,JY,JZ,JW) + (\nabla_V R)(X,Y,JZ,JW)$

$+ (\nabla_V R)(X,JY,Z,JW) + (\nabla_V R)(X,JY,JZ,JW)\Big] = o$

or $(\nabla_V R)(X,Y,Z,W) + (\nabla_V R)(JX,JY,JZ,JW) = (\nabla_V R)(JX,JY,Z,W)$

$+ (\nabla_V R)(JX,Y,JZ,W) + (\nabla_V R)(JX,Y,Z,JW) + (\nabla_V R)(X,JY,JZ,W)$

$+ (\nabla_V R)(X, JY, Z, JW) + (\nabla_V R)(X, Y, JZ, JW)$

This completes the proof of proposition 5.3.3. //

Remark:- In fact, when k = 1 and m = 2, the two equations (ii)(5) and (i) (9) are not independent of each other, since if in (i)(9) we put m = 2, replace V, X by JV, JX, and compare the result with (i) (9) we have (ii)(5) again.

5.4 Riemannian 5-Regular Symmetric Manifolds As Coset Manifolds:-

Let M be a Riemannian 5-regular symmetric manifold, with associated almost complex structure J, let G be the largest connected component of C(M), where C(M) is the transitive Lie transformation group of almost complex isometries. If $x \in M$ is any point, denote by H the isotropy subgroup of G at x. Theorem 4.3.7. goes over when k = 5, and we have the homogeneous space $G/_H$ is isomorphic to M. On the other hand, if G is any connected Lie group, H is a closed subgroup of G, s is an automorphism of G of order 5 such that $(H_s)_o \subseteq H \subseteq H_s$, where H_s is the subgroup of G of fixed points of s, $(H_s)_o$ is the identity component of H_s, and finally, if we assume that Ad(H) is compact, then theorem 4.3.3. is valid when k = 5, and we have the coset space $G/_H$ is a Riemannian 5-symmetric manifold.

Proposition 5.4.1. Let M be a Riemannian 5-regular manifold and let M_1 and M_2 be the two differentiable distributions on M.

(i) If $X, Y \in M_1$. Then $[X, Y]$, $\nabla_X Y \in M_2$

(ii) If $X, Y \in M_2$ Then $[X, Y]$, $\nabla_X Y \in M_1$

Proof:- Let $p \in M$ be any point, we have $M_p = M_{p_1} \oplus M_{p_2}$. S_p is a linear transformation of M_p, and it can be extended to act of M_p^c, the complexification of M_p, denote this extension by S_p also. From proposition 5.4.1. we have four complex distributions D_1, $\bar{D}_1$, D_2, $\bar{D}_2$ on M, corresponding to the four eignevalues θ_1, $\bar{\theta}_1$, θ_2, and $\bar{\theta}_2$.

(i) Extend X, Y to be complex-valved vector fields on M, denoted also by X, Y, then X, Y $D_1 \oplus \bar{D}_1$. Consider the following four cases

(1) $(X_1, Y_1 \in D_1)$: $S_p[X_1, Y_1] = [SX_1, SY_1]_p = [\theta_1 X_1, \theta_1 Y_1]_p = \theta_1^2 [X_1, Y_1]$

and $S_p \nabla_{X_1} Y_1 = \nabla_{S_p X_1} S_p Y_1 \big|_p = \nabla_{\theta_1 X_1} \theta_1 Y_1 \big|_p = \theta_1^2 \nabla_{X_1} Y_1 \big|_p$

but $\theta_1^2 = \theta_2$ or $\theta_1^2 = \bar{\theta}_2$, and in both cases $[X_1, Y_1]$, $\nabla_{X_1} Y_1 \underset{p}{\in} M_{p2}$

(2) $(X_2, Y_2 \in \bar{D}_1)$: $S_p[X_2, Y_2] = [SX_2, SY_2]_p = [\bar{\theta}_1 X_2, \bar{\theta}_1 Y_2] = \bar{\theta}_1^2 [X_2, Y_2]_p$

and $S_p \nabla_{X_2} Y_2 = \nabla_{SX_2} SY_2 \big|_p = \nabla_{\bar{\theta}_1 X_2} \theta_1 Y_2 \big|_p = \theta_1^2 \nabla_{X_2} Y_2 \big|_p$

but $\bar{\theta}_1^2 = \theta_2$ or $\theta_1^2 = \bar{\theta}_2$, and in both cases we have $[X_2, Y_2]_p$, $\nabla_{X_2} Y_2 \underset{p}{\in} M_{p2}$

(3) $(X_1 \in D_1, Y_2 \in \bar{D}_1)$: $S_p[X_1, Y_2] = [SX_1, SY_2]_p = [\theta_1 X_1, \bar{\theta}_1 Y_2]_p$

$= \theta_1 \bar{\theta}_1 [X, Y]_p = [X, Y]_p = \underline{o} \in M_{p2}$

and $S_p \nabla_{X_1} Y_2 = \nabla_{SX_1} SY_2 \big|_p = \nabla_{\theta_1 X_1} \bar{\theta}_1 Y_2 \big|_p = \theta_1 \bar{\theta}_1 \nabla_{X_1} Y_2 \big|_p = \nabla_{X_1} Y_2 \big|_p$

$= \underline{o} \in M_{p2}$

(4) $(X_2 \in \bar{D}_1, Y_1 \in D_1)$: $S_p[X_2, Y_1] = [SX_2, SY_1]_p = [\bar{\theta}_1 X_2, \theta_1 Y_1]_p =$

$\bar{\theta}_1 \theta_1 [X_2, Y_1]_p = [X_2, Y_1]_p = \underline{o} \in M_{p2}$

and $S_p \nabla_{X_2} Y_1 = \nabla_{SX_2} SY_2 \big|_p = \nabla_{\bar{\theta}_1 X_2} \theta_1 Y_1 = \bar{\theta}_1 \theta_1 \nabla_{X_2} Y_1 = \nabla_{X_2} Y_1 = \underline{o} \in M_{p2}$

Hence if $X = X_1 + X_2$ and $Y = Y_1 + Y_2$, we have

$[X, Y]_p = [X_1 + X_2, Y_1 + Y_2]_p = [X_1, Y_1]_p + [X_1, Y_2]_p + [X_2, Y_1]_p +$

$[X_2, Y_2]_p \in M_{p2}$

and

$$\nabla_X Y_p = \nabla^{Y_1 + Y_2}_{X_1 + X_2}\Big|_p = \nabla_{X_1} Y_1\Big|_p + \nabla_{X_1} Y_2\Big|_p + \nabla_{X_2} Y_1\Big|_p + \nabla_{X_2} Y_2\Big|_p \in M_{p2}.$$

and this is true for all $p \in M$. Hence (i) is proved

(ii) Here we also extend X, and Y to be complex-valued vector fields on M, denoted also by X, Y, then $X, Y \in D_2 \oplus D_2$.

A similar proof is given as in part (i), where we have θ_2, θ_2 instead of θ_1, $\bar{\theta}_1$. Hence we have $[X,Y]_p$, $\nabla_X Y\Big|_p \in M_{p1}$, and this is true for all $p \in M$.

Proposition 5.4.2. Let M be a Riemannian 5-regular symmetric manifold

(i) If $X, Y \in M_1$. Then

$[JX, Y] = J[X,Y]$

(ii) If X, Y M_2. Then

$[JX, Y] = -J[X,Y]$

Proof:- Let $p \in M$ be any point. We have

$$S_p = \left[(\cos 2\pi/5)I + (\sin 2\pi/5)J\right]_p \oplus \left[(\cos 4\pi/5)I + (\sin 4\pi/5)J\right]_p$$

(i) $X, Y \in M_{p1}$, then by proposition 5.4.1. $[X,Y]_p \in M_{p2}$. We also have

$$S_p[X,Y] = [S\,X, S\,Y]_p$$

$$S_p[X,Y] = \left[(\cos 4\pi/5)I + (\sin 4\pi/5)J\right][X,Y]_p$$

$$= (\cos 4\pi/5)[X,Y]_p + (\sin 4\pi/5)J\,[X,Y]_p \quad \text{-------- (i)}$$

$$[S\,X, S\,Y]_p = \left[(\cos 2\pi/5)X + (\sin 2\pi/5)JX,\ (\cos 2\pi/5)Y + (\sin 2\pi/5)JY\right]_p$$

$$= (\cos^2 2\pi/5)[X,Y]_p + \cos 2\pi/5 \sin 2\pi/5([X,JY]_p + [JX,Y]_p)$$

$$+ (\sin^2 2\pi/5)[JX,JY]_p \qquad \text{(ii)}$$

Subtract (ii) from (i) we have

$$\left[(\cos 4\pi/5) - (\cos^2 2\pi/5)\right] \left[X,Y\right]_p + (\sin 4\pi/5)J\left[X,Y\right]_p - (\cos 2\pi/5 \sin 2\pi/5)$$

$$(\left[X,JY\right]_p + \left[JX,Y\right]_p) - (\sin^2 2\pi/5)\left[JX,JY\right]_p = o \quad \ldots\ldots \quad (iii)$$

In (iii) replace Y by JX, we have

$$\left[(\cos 4\pi/5) - (\cos^2 2\pi/5) - (\sin^2 2\pi/5)\right]\left[X,JX\right]_p + (\sin 4\pi/5)J\left[X,JX\right]_p = o$$

$$\text{or } \left[X,JX\right]_p = o \quad \ldots\ldots \quad (iv)$$

We also have

$$\left[X+Y, JX+JY\right]_p = o = \left[X,JX\right]_p + \left[X,JY\right]_p + \left[Y,JX\right]_p + \left[Y,JY\right]_p$$

$$\therefore \left[X,JY\right]_p + \left[Y,JX\right]_p = o \quad \ldots\ldots \quad (v)$$

In (v) replace X by JX, we have

$$\left[JX,JY\right]_p - \left[Y,X\right]_p = o$$

$$\text{or } \left[X,Y\right]_p + \left[JX,JY\right]_p = o \quad \ldots\ldots \quad (vi)$$

Consider the identities

$$\cos 4\pi/5 = \cos^2 2\pi/5 - \sin^2 2\pi/5 \text{ and } 2\cos 2\pi/5 \sin 2\pi/5 = \sin 4\pi/5$$

Using (v) and (vi) in (iii), we have

$$\left[(\cos^2 2\pi/5) - (\sin^2 2\pi/5) - (\cos^2 2\pi/5)\right]\left[X,Y\right]_p + (\sin 4\pi/5)J\left[X,Y\right]_p$$

$$- (\sin 4\pi/5)\left[JX,Y\right]_p - (\sin^2 2\pi/5)\left[JX,JY\right]_p = o$$

$$\therefore (\sin 4\pi/5)(J\left[X,Y\right]_p - \left[JX,Y\right]_p) = o$$

$$\text{or } \left[JX,Y\right] = J\left[X,Y\right]$$

(ii) If $X, Y \in M_{p2}$, then $\left[X,Y\right] \in M_{p1}$. Also we have

$$S_p\left[X,Y\right]_p = \left[S\ X, S\ Y\right]_p$$

$$S_p[X,Y] = ((\cos 2\pi/5)I + (\sin 2\pi/5)J)([X,Y]_p)$$

$$= (\cos 2\pi/5)[X,Y]_p + (\sin 2\pi/5)J[X,Y]_p \qquad \text{(i)}$$

$$[S X, S Y]_p = [(\cos 4\pi/5)X + (\sin 4\pi/5)JX, (\cos 4\pi/5)Y + (\sin 4\pi/5)JY]_p$$

$$= (\cos^2 4\pi/5)[X,Y]_p + (\cos 4\pi/5 \sin 4\pi/5)([X,JY]_p + [JX,Y]_p)$$

$$+ (\sin^2 4\pi/5)[JX,JY]_p \qquad \text{(ii)}$$

Subtract (ii) from (i) we have

$$((\cos 2\pi/5) - (\cos^2 4\pi/5))[X,Y]_p + (\sin 2\pi/5)J[X,Y]_p - (\cos 4\pi/5 \sin 4\pi/5)$$

$$([X,Jy]_p + [JX,Y]_p) - (\sin^2 4\pi/5)[JX,Jy]_p = 0 \quad \text{------ (iii)}$$

In (iii) replace Y by JX, we have

$$(\cos 2\pi/5 - \cos^2 4\pi/5 - \sin^2 4\pi/5)[X,JX]_p + (\sin 2\pi/5)J[X,JX]_p = 0$$

or $[X,JX]_p = 0$

From part (i) we have

$$[X,JY]_p + [Y,JX]_p = 0 \quad \text{------------ (iv)}$$

and $$[X,Y]_p + [JX,JY]_p = 0 \quad \text{------------ (v)}$$

Consider the identities

$$\cos 2\pi/5 = \cos(2\pi - 2\pi/5) = \cos 8\pi/5 = \cos^2 4\pi/5 - \sin^2 4\pi/5$$

and $$2\cos 4\pi/5 \sin 4\pi/5 = \sin 8\pi/5 = -\sin 2\pi/5$$

Using (iv) and (v) in (iii) we have

$$[(\cos^2 4\pi/5) - (\sin^2 4\pi/5) - (\cos^2 4\pi/5)][X,Y]_p + (\sin 2\pi/5)J[X,Y]_p$$

$$+ (\sin 2\pi/5)[JX,Y]_p - (\sin^2 4\pi/5)[JX,JY]_p = 0$$

or $(\sin 2\pi/5)(J[X,Y]_p + [JX,Y]_p) = 0$

or $[JX,Y]_p = -J[X,Y]_p$

This completes the proof. //

Let M be a Riemannian 5- regular symmetric manifold, we will denote by $\langle X,Y\rangle$, the metric $g(X,Y)$; $X,Y\in \mathfrak{X}(M)$. We recall the definition of a reductive homogeneous space G/H to be naturally reductive if

$$\langle [X,Y]_{\underline{m}}, Z\rangle = \langle X, [Y,Z]_{\underline{m}}\rangle \quad , X,Y,Z\in \underline{m}$$

where if $\underline{g}$ is the lie algebra of the Lie group G, then $\underline{g} = \underline{h} \oplus \underline{m}$.

Proposition 5.4.3. Let M be a Riemannian 5- regular symmetric manifold. Assume that M is naturally reductive. Then we have

$$(\nabla_X J)X = (\nabla_{X_1} J)X_2 + (\nabla_{X_2} J)X_1$$

and $(\nabla_X J)X + (\nabla_{JX} J)(JX) = 2(\nabla_{X_2} J)X_1$

Proof:- Consider the following cases

(1) $\langle (\nabla_{X_1} J)X_1, Z\rangle = \langle (\nabla_{X_1} J)X_1, Z_1\rangle + \langle (\nabla_{X_1} J)X_1, Z_2\rangle$

$Z\in \mathfrak{X}(M)$

(i) $\langle (\nabla_{X_1} J)X_1, Z_1\rangle = \langle \nabla_{X_1}(JX_1), Z_1\rangle - \langle J\nabla_{X_1} X_1, Z_1\rangle$,

but $\langle \nabla_{X_1}(JX_1), Z_1\rangle = \frac{1}{2}\langle [X_1, JX_1], Z_1\rangle = 0$

where from proposition 5.4.2. we have $[X_1, JX_1] = 0$

and $\langle J\nabla_{X_1} X_1, Z_1\rangle = -\langle \nabla_{X_1} X_1, JZ_1\rangle = -\frac{1}{2}\langle [X_1, X_1], JZ_1\rangle = 0$

$\therefore \langle (\nabla_{X_1} J)X_1, Z_1\rangle = 0$

(ii) $\langle (\nabla_{X_1} J)X_1, Z_2\rangle = \langle \nabla_{X_1}(JX_1), Z_2\rangle - \langle J\nabla_{X_1} X_1, Z_2\rangle$

and for the same reason in (i) we have

$$\langle (\nabla_{X_1} J)X_1, Z_2\rangle = 0$$

$\therefore \langle (\nabla_{X_1} J)X_1, Z \rangle = 0$, for all $Z \in \mathcal{H}(M)$

or $(\nabla_{X_1} J)X_1 = 0$

(2) $\langle (\nabla_{X_1} J)X_2, Z \rangle = \langle (\nabla_{X_1} J)X_2, Z_1 \rangle + \langle (\nabla_{X_1} J)X_2, Z_2 \rangle$

(i) $\langle (\nabla_{X_1} J)X_2, Z_1 \rangle = \langle \nabla_{X_1}(JX_2), Z_1 \rangle - \langle J \nabla_{X_1} X_2, Z_1 \rangle$

$\langle \nabla_{X_1}(JX_2), Z_1 \rangle = \frac{1}{2} \langle [X_1, JX_2], Z_1 \rangle = -\frac{1}{2} \langle [JX_2, X_1], Z_1 \rangle$

$= -\frac{1}{2} \langle JX_2, [X_1, Z_1] \rangle = \frac{1}{2} \langle X_2, J[X_1, Z_1] \rangle$

$= -\frac{1}{2} \langle X_2, J[Z_1, X_1] \rangle = -\frac{1}{2} \langle X_2, [JZ_1, X_1] \rangle$

$= \frac{1}{2} \langle X_2, [X_1, JZ_1] \rangle = \frac{1}{2} \langle [X_1, JZ_1], X_2 \rangle = \langle \nabla_{X_1}(JZ_1), X_2 \rangle$

where from proposition 5.4.2., we have $J[Z_1, X_1] = [JZ_1, X_1]$

$\langle J \nabla_{X_1} X_2, Z_1 \rangle = -\langle \nabla_{X_1} X_2, JZ_1 \rangle = -\frac{1}{2} \langle [X_1, X_2], JZ_1 \rangle$

$= \frac{1}{2} \langle [X_2, X_1], JZ_1 \rangle = \frac{1}{2} \langle X_2, [X_1, JZ_1] \rangle$

$= \frac{1}{2} \langle [X_1, JZ_1], X_2 \rangle = \langle \nabla_{X_1}(JZ_1), X_2 \rangle$

$\therefore \langle (\nabla_{X_1} J)(X_2), Z_1 \rangle = \langle \nabla_{X_1}(JX_2), Z_1 \rangle - \langle J \nabla_{X_1} X_2, Z_1 \rangle = 0$

(ii) $\langle (\nabla_{X_1} J)X_2, Z_2 \rangle = \langle \nabla_{X_1}(JX_2), Z_2 \rangle - \langle J \nabla_{X_1} X_2, Z_2 \rangle$

$\langle \nabla_{X_1}(JX_2), Z_2 \rangle = \frac{1}{2} \langle [X_1, JX_2], Z_2 \rangle = \frac{1}{2} \langle X_1, [JX_2, Z_2] \rangle$

$= -\frac{1}{2} \langle X_1, J[X_2, Z_2] \rangle = \frac{1}{2} \langle X_1, J[Z_2, X_2] \rangle$

$= -\frac{1}{2} \langle X_1, [JZ_2, X_2] \rangle = \frac{1}{2} \langle X_1, [X_2, JZ_2] \rangle$

$= \frac{1}{2} \langle [X_2, JZ_2], X_1 \rangle = \langle \nabla_{X_2}(JZ_2), X_1 \rangle$

$\langle J \nabla_{X_1} X_2, Z_2 \rangle = -\langle \nabla_{X_1} X_2, JZ_2 \rangle = -\frac{1}{2} \langle [X_1, X_2], JZ_2 \rangle$

$= -\frac{1}{2} \langle X_1, [X_2, JZ_2] \rangle = -\langle \nabla_{X_1}(JZ_2), X_1 \rangle$

$\therefore \langle (\nabla_{X_1} J)X_2, Z_2 \rangle = 2 \langle \nabla_{X_2}(JZ_2), X_1 \rangle$

Replace X_1, X_2 by JX_1, JX_2, we have

$$\langle (\nabla_{JX_1} J)(JX_2), Z_2 \rangle = 2 \langle \nabla_{JX_2}(JZ_2), JX_1 \rangle$$

$$= \langle [JX_2, JZ_2], JX_1 \rangle = -\langle J[X_2, JZ_2], JX_1 \rangle$$

$$= -\langle [X_2, JZ_2], X_1 \rangle = -2 \langle \nabla_{X_2}(JZ_2), X_1 \rangle$$

where from proposition 5.4.2. $J[X_2, Z_2] = -[JX_2, Z_2]$

$$\therefore \langle (\nabla_{X_1} J)X_2 + (\nabla_{JX_1} J)JX_2, Z_2 \rangle = 0$$

But from (i) we have

$$\langle (\nabla_{X_1} J)(X_2) + (\nabla_{JX_1} J)(JX_2), Z_1 \rangle = 0$$

$$\langle (\nabla_{X_1} J)X_2 + (\nabla_{JX_1} J)JX_2, Z \rangle = 0 \quad , \text{for all } Z \in \mathcal{X}(M)$$

or $(\nabla_{X_1} J)X_2 + (\nabla_{JX_1} J)JX_2 = 0$

(3) $\langle (\nabla_{X_2} J)X_1, Z \rangle = \langle (\nabla_{X_2} J)X_1, Z_1 \rangle + \langle (\nabla_{X_2} J)X_1, Z_2 \rangle$

(i) $\langle (\nabla_{X_2} J)X_1, Z_1 \rangle = \langle \nabla_{X_2}(JX_1), Z_1 \rangle - \langle J \nabla_{X_2} X_1, Z_1 \rangle$

$$\langle \nabla_{X_2}(JX_1), Z_1 \rangle = \tfrac{1}{2} \langle [X_2, JX_1], Z_1 \rangle = \tfrac{1}{2} \langle X_2, [JX_1, Z_1] \rangle$$

$$= \tfrac{1}{2} \langle X_2, J[X_1, Z_1] \rangle = -\tfrac{1}{2} \langle X_2, J[Z_1, X_1] \rangle$$

$$= -\tfrac{1}{2} \langle X_2, [JZ_1, X_1] \rangle = \tfrac{1}{2} \langle X_2, [X_1, JZ_1] \rangle$$

$$= \tfrac{1}{2} \langle [X_1, JZ_1], X_2 \rangle = \langle \nabla_{X_1}(JZ_1), X_2 \rangle$$

$$\langle J \nabla_{X_2} X_1, Z_1 \rangle = -\langle \nabla_{X_2} X_1, JZ_1 \rangle = -\tfrac{1}{2} \langle [X_2, X_1], JZ_1 \rangle$$

$$= -\tfrac{1}{2} \langle X_2, [X_1, JZ_1] \rangle = -\tfrac{1}{2} \langle [X_1, JZ_1], X_2 \rangle$$

$$= -\langle \nabla_{X_1}(JZ_1), X_2 \rangle$$

$$\therefore \langle (\nabla_{X_2} J)X_1, Z_1 \rangle = 2 \langle \nabla_{X_1}(JZ_1), X_2 \rangle$$

Replace X_1, X_2 by JX_1, JX_2 we have

$$\langle (\nabla_{JX_2} J)(JX_1), Z_1 \rangle = 2 \langle \nabla_{JX_1}(JZ_1), JX_2 \rangle$$

$$= \langle [JX_1 JZ_1], JX_2 \rangle = \langle J[X_1 JZ_1], JX_2 \rangle$$

$$= \langle [X_1, JZ_1], X_2 \rangle = 2 \langle \nabla_{X_1}(JZ_1), X_2 \rangle$$

$$\langle (\nabla_{X_2} J)X_1 - (\nabla_{JX_2} J)JX_1, Z_1 \rangle = 0$$

(ii) $\langle (\nabla_{X_2} J)X_1, Z_2 \rangle = \langle \nabla_{X_2}(JX_1), Z_2 \rangle - \langle J \nabla_{X_2} X_1, Z_2 \rangle$

$$\langle \nabla_{X_2}(JX_1), Z_2 \rangle = \tfrac{1}{2} \langle [X_2, JX_1], Z_2 \rangle = -\tfrac{1}{2} \langle [JX_1, X_2], Z_2 \rangle$$

$$= -\tfrac{1}{2} \langle JX_1, [X_2, Z_2] \rangle = \tfrac{1}{2} \langle X_1, J[X_2, Z_2] \rangle$$

$$= -\tfrac{1}{2} \langle X_1, J[Z_2, X_2] \rangle = \tfrac{1}{2} \langle X_1, [JZ_2, X_2] \rangle$$

$$= -\tfrac{1}{2} \langle X_1, [X_2, JZ_2] \rangle = \tfrac{1}{2} \langle [X_2, JZ_2], X_1 \rangle$$

$$= -\langle \nabla_{X_2}(JZ_2), X_1 \rangle$$

$$\langle J \nabla_{X_2} X_1, Z_2 \rangle = -\langle \nabla_{X_2} X_1, JZ_2 \rangle = -\tfrac{1}{2} \langle [X_2, X_1], JZ_2 \rangle$$

$$= \tfrac{1}{2} \langle [X_1, X_2], JZ_2 \rangle = \tfrac{1}{2} \langle X_1, [X_2, JZ_2] \rangle$$

$$= \tfrac{1}{2} \langle [X_2, JZ_2], X_1 \rangle = \langle \nabla_{X_2}(JZ_2), X_1 \rangle$$

$$\therefore \quad \langle (\nabla_{X_2} J)X_1, Z_2 \rangle = 0$$

or $\langle (\nabla_{X_2} J)X_1 - (\nabla_{JX_2} J)JX_1, Z_2 \rangle = 0$

This with part (i) we have

$$\langle (\nabla_{X_2} J)X_1 - (\nabla_{JX_2} J)JX_1, Z \rangle = 0 \quad , \text{ for all } Z \in \mathfrak{X}(M)$$

$$\therefore \quad (\nabla_{X_2} J)X_1 - (\nabla_{JX_2} J)JX_1 = 0$$

(4) $\langle (\nabla_{X_2} J)X_2, Z \rangle = \langle (\nabla_{X_2} J)X_2, Z_1 \rangle + \langle (\nabla_{X_2} J)X_2, Z_2 \rangle$

(i) $\langle(\nabla_{X_2}J)X_2, Z_1\rangle = \langle\nabla_{X_2}(JX_2), Z_1\rangle - \langle J\nabla_{X_2}X_2, Z_1\rangle$

$\langle\nabla_{X_2}(JX_2), Z_1\rangle = \frac{1}{2}\langle[X_2, JX_2], Z_1\rangle = 0$

Since by proposition 5.4.2. we have $[X_2, JX_2] = 0$

$\langle J\nabla_{X_2}X_2, Z_1\rangle = -\langle\nabla_{X_2}X_2, JZ_1\rangle = -\frac{1}{2}\langle[X_2, X_2], JZ_1\rangle = 0$

$\therefore \langle(\nabla_{X_2}J)X_1, Z_1\rangle = 0$

(ii) $\langle(\nabla_{X_2}J)X_2, Z_2\rangle = \langle\nabla_{X_2}(JX_2), Z_2\rangle - \langle J\nabla_{X_2}X_2, Z_2\rangle$

By the same reason in (i) we have

$\langle(\nabla_{X_2}J)X_2, Z_2\rangle = 0$

$\langle(\nabla_{X_2}J)X_2, Z\rangle = 0$, for all $Z \in \mathfrak{X}(M)$

$\therefore (\nabla_{X_2}J)X_2 = 0$

For any $X \in \mathfrak{X}(M)$ $(X = X_1 + X_2)$, we have

$$(\nabla_X J)X = (\nabla_{X_1 + X_2}J)(X_1 + X_2) = (\nabla_{X_1}J)X_1 + (\nabla_{X_1}J)X_2$$
$$+ (\nabla_{X_2}J)X_1 + (\nabla_{X_2}J)X_2$$
$$= (\nabla_{X_1}J)X_2 + (\nabla_{X_2}J)X_1 \quad \text{———— (1)}$$

and this is the first relation required. In (1) replace X by JX we have

$$(\nabla_X J)X + (\nabla_{JX}J)(JX) = (\nabla_{X_1}J)X_2 + (\nabla_{X_2}J)X_1 + (\nabla_{JX_1}J)JX_2 + (\nabla_{JX_2}J)JX_1$$

but we have

$$(\nabla_{X_1}J)X_2 + (\nabla_{JX_1}J)JX_2 = 0 = (\nabla_{X_2}J)X_1 - (\nabla_{JX_2}J)JX_1$$

$$(\nabla_X J)X + (\nabla_{JX}J)(JX) = 2(\nabla_{JX_2}J)JX_1 = 2(\nabla_{X_2}J)X_1 \quad //$$

APPENDIX

) The 5th roots of unity which do not equal one are

$$w_1 = e^{i\,2\pi/5} = \cos 2\pi/5 + i \sin 2\pi/5 \;,\; w_2 = w_1^2 = e^{i\,4\pi/5} = \cos 4\pi/5 + i \sin 4\pi/5$$

$$w_3 = w_1^3 = e^{i6\pi/5} = \cos 6\pi/5 + i \sin 6\pi/5 = \cos 4\pi/5 - i \sin 4\pi/5 ,(i = \sqrt{-1})$$

$$w_4 = w_1^4 = e^{i8\pi/5} = \cos 8\pi/5 + i \sin 8\pi/5 = \cos 2\pi/5 - i \sin 2\pi/5$$

Consider the polynomial $Z^5 = 1$, its roots are the 5th roots of unity including 1. $Z = W_1$, also satisfies it. Hence

$$(w_1 - 1)(w_1^4 + w_1^3 + w_1^2 + w_1 + 1) = 0$$

Since w_1 1 , we have $w_1^4 + w_1^3 + w_1^2 + w_1 = -1$,

from the above we have

$$2(\cos 2\pi/5 + \cos 4\pi/5) = -1 \Leftrightarrow \cos 2\pi/5 + \cos 4\pi/5 = -1/2 \qquad \text{\_\_\_\_\_\_\_\_\_\_ (1)}$$

(i) $\sin 4\pi/5 \cos 4\pi/5 = 1/2 \sin 8\pi/5 = -1/2 \sin 2\pi/5$

(ii) $\sin 2\pi/5 \cos 2\pi/5 = 1/2 \sin 4\pi/5$

(iii) $\sin^2 2\pi/5 = 1/2(1 - \cos 4\pi/5)$

(iv) $\sin^2 4\pi/5 = 1/2(1 - \cos 8\pi/5) = 1/2(1 - \cos 2\pi/5)$

(v) $\cos^2 2\pi/5 = 1/2(1 + \cos 4\pi/5)$

(vi) $\cos^2 4\pi/5 = 1/2(1 + \cos 8\pi/5) = 1/2(1 + \cos 2\pi/5)$

(i) $\sin (4\pi/5 + 2\pi/5) = \sin 4\pi/5 \cos 2\pi/5 + \cos 4\pi/5 \sin 2\pi/5$ __________ (a)

$\sin (4\pi/5 - 2\pi/5) = \sin 4\pi/5 \cos 2\pi/5 - \cos 4\pi/5 \sin 2\pi/5$ __________ (b)

Add (a) to (b) we get

$2 \sin 4\pi/5 \cos 2\pi/5 = \sin 6\pi/5 + \sin 2\pi/5 = - \sin 4\pi/5 = \sin 2\pi/5$ __________ (c)

Subtract (b) from (a) we get

$$2\cos\frac{4\pi}{5}\sin\frac{2\pi}{5} = \sin\frac{6\pi}{5} - \sin\frac{2\pi}{5} = -\left(\sin\frac{4\pi}{5} + \sin\frac{2\pi}{5}\right) \quad \text{———— (d)}$$

(ii) $\cos\left(\frac{4\pi}{5} + \frac{2\pi}{5}\right) = \cos\frac{4\pi}{5}\cos\frac{2\pi}{5} - \sin\frac{4\pi}{5}\sin\frac{2\pi}{5}$ ———— (a)

$\cos\left(\frac{4\pi}{5} - \frac{2\pi}{5}\right) = \cos\frac{4\pi}{5}\cos\frac{2\pi}{5} + \sin\frac{4\pi}{5}\sin\frac{2\pi}{5}$ ———— (b)

Add (a) to (b) we get

$$2\cos\frac{4\pi}{5}\cos\frac{2\pi}{5} = \cos\frac{6\pi}{5} + \cos\frac{2\pi}{5} = \cos\frac{4\pi}{5} + \cos\frac{2\pi}{5}$$

But from (1) we have $\cos\frac{4\pi}{5} + \cos\frac{2\pi}{5} = -\frac{1}{2}$

$\therefore \cos\frac{4\pi}{5}\cos\frac{2\pi}{5} = -\frac{1}{4}$ ———————————— (c)

Subtract (a) from (b) we get

$2\sin\frac{4\pi}{5}\sin\frac{2\pi}{5} = \cos\frac{2\pi}{5} - \cos\frac{4\pi}{5}$ ———————————— (d)

4) (i) $\cos^2\frac{4\pi}{5} + \cos^2\frac{2\pi}{5} = \frac{1}{2}\left(1 + \cos\frac{2\pi}{5}\right) + \left(1 + \cos\frac{4\pi}{5}\right)$

$$= 1 + \frac{1}{2}\left(\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}\right)$$

But from (1), $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

$\therefore \cos^2\frac{4\pi}{5} + \cos^2\frac{2\pi}{5} = 1 + \frac{1}{2}\left(-\frac{1}{2}\right) = \frac{3}{4}$

(ii) $\sin^2\frac{4\pi}{5} + \sin^2\frac{2\pi}{5} = 1 - \cos^2\frac{4\pi}{5} + 1 - \cos^2\frac{2\pi}{5}$

Using (4)(i) we have

$\sin^2\frac{4\pi}{5} + \sin^2\frac{2\pi}{5} = 2 - \frac{3}{4} = \frac{5}{4}$

(iii) $\left(\cos\frac{2\pi}{5} - \cos\frac{4\pi}{5}\right)^2 = \cos^2\frac{2\pi}{5} + \cos^2\frac{4\pi}{5} - 2\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}$

Using (4)(i) and (3)(ii)(c) we get

$\left(\cos\frac{2\pi}{5} - \cos\frac{4\pi}{5}\right)^2 = \frac{3}{4} - 2\left(-\frac{1}{4}\right) = \frac{5}{4}$

(iv) $\cos^3\frac{4\pi}{5} + \cos^3\frac{2\pi}{5} = \left(\cos\frac{4\pi}{5} + \cos\frac{2\pi}{5}\right)\left(\cos^2\frac{4\pi}{5} + \cos^2\frac{2\pi}{5} - \cos\frac{4\pi}{5}\cos\frac{2\pi}{5}\right)$

From 1 , (4) (i) and 3 (ii) (C) we have

$$\cos^3\frac{4\pi}{5} + \cos^3\frac{2\pi}{5} = \left(-\tfrac{1}{2}\right)\left(\tfrac{3}{4} - \left(-\tfrac{1}{4}\right)\right) = -\tfrac{1}{2} \times 1 = -\tfrac{1}{2}$$

5) (i) Let $T = \cos^4 2\pi/5 + \cos^4 4\pi/5$

$\therefore \quad T = \left[\frac{1}{2}(1 + \cos 4\pi/5)\right]^2 + \left[\frac{1}{2}(1 + \cos 2\pi/5)\right]^2$

$= \frac{1}{4}\left[(1 + 2\cos 4\pi/5 + \cos^2 4\pi/5) + (1 + 2\cos 2\pi/5 + \cos^2 2\pi/5)\right]$

$= \frac{1}{4}\left[2 + 2(\cos 4\pi/5 + \cos 2\pi/5) + (\cos^2 4\pi/5 + \cos^2 2\pi/5)\right]$

Using ① and (4)(i) we have

$T = \frac{1}{4}\left[2 + 2(-\frac{1}{2}) + \frac{3}{4}\right] = \frac{1}{4}(1 + \frac{3}{4}) = 7/16$

(ii) Let $T = \cos^2 2\pi/5 \sin^2 2\pi/5 + \cos^2 4\pi/5 \sin^2 4\pi/5$

Using (2)(i) and (2)(ii) we have

$T = \frac{1}{4}(\sin^2 4\pi/5 + \sin^2 2\pi/5)$, using (4)(ii) we have

$T = \frac{1}{4} \cdot 5/4 = 5/16$

(iii) Let $T = \sin^4 2\pi/5 + \sin^4 4\pi/5$

Using (2)(iii) and (2)(iv) we have

$T = \left[\frac{1}{2}(1 - \cos 4\pi/5)\right]^2 + \left[\frac{1}{2}(1 - \cos 2\pi/5)\right]^2$

$= \frac{1}{4}\left[(1 - 2\cos 4\pi/5 + \cos^2 4\pi/5) + (1 - 2\cos 2\pi/5 + \cos^2 2\pi/5)\right]$

$= \frac{1}{4}\left[2 - 2(\cos 4\pi/5 + \cos 2\pi/5) + (\cos^2 4\pi/5 + \cos^2 2\pi/5)\right]$

Using ① and (4)(i) we have

$T = \frac{1}{2}(2 - 2(-\frac{1}{2}) + \frac{3}{4}) = 15/16$

6) (i) Let $T = 2\cos^2 2\pi/5 \cos^2 4\pi/5$

Using (3)(ii)(c) we have

$T = 2(1/16) = \frac{1}{8}$

(ii) Let $T = \sin^2 2\pi/5 \cos^2 4\pi/5 + \sin^2 4\pi/5 \cos^2 2\pi/5$

$= \cos^2 4\pi/5 (1 - \cos^2 2\pi/5) + \cos^2 2\pi/5 (1 - \cos^2 4\pi/5)$

$= \cos^2 4\pi/5 + \cos^2 2\pi/5 - 2\cos^2 2\pi/5 \cos^2 4\pi/5$

Using (4)(i) and (6)(i) we have

$T = \frac{3}{4} - \frac{1}{8} = 5/16$

(iii) Let $T = 2\sin^2 2\pi/5 \sin^2 4\pi/5$

Using (3)(ii)(d) and (4)(iii) we have

$T = 2\left[\frac{1}{2}(\cos 2\pi/5 - \cos 4\pi/5)\right]^2 = 2\ \frac{1}{4}(\cos 2\pi/5 - \cos 4\pi/5)^2 = \frac{1}{2}\ 5/4 = \frac{5}{8}$

7) (i) Let $T = \cos^3 2\pi/5 \cos 4\pi/5 + \cos^3 4\pi/5 \cos 2\pi/5$

$= \cos 2\pi/5 \cos 4\pi/5 (\cos^2 2\pi/5 + \cos^2 4\pi/5)$

Using (3)(ii)(c) and (4)(i) we have

$T = (-\frac{1}{4})(\frac{3}{4}) = -3/16$

(ii) Let $T = \cos^2 2\pi/5 \sin 2\pi/5 \sin 4\pi/5 - \cos^2 4\pi/5 \sin 2\pi/5 \sin 4\pi/5$

$= \sin 2\pi/5 \sin 4\pi/5 (\cos^2 2\pi/5 - \cos^2 4\pi/5)$

Using (3)(ii)(d) , (4)(iii) and ① we have

$T = \frac{1}{2}(\cos 2\pi/5 - \cos 4\pi/5)^2 (\cos 2\pi/5 + \cos 4\pi/5) = (\frac{1}{2})(5/4)(-\frac{1}{2}) = -5/16$

(iii) Let $T = \cos 2\pi/5 \sin^2 2\pi/5 \cos 4\pi/5 + \cos 4\pi/5 \sin^2 4\pi/5 \cos 2\pi/5$

$= \cos 2\pi/5 \cos 4\pi/5 (\sin^2 2\pi/5 + \sin^2 4\pi/5)$

Using (3)(ii)(c) and (4)(ii) we have

$T = (-\frac{1}{2})(5/4) = -5/16$

(iv) Let $T = \sin^3 2\pi/5 \sin 4\pi/5 - \sin^3 4\pi/5 \sin 2\pi/5$

$= \sin 2\pi/5 \sin 4\pi/5 (\sin^2 2\pi/5 - \sin^2 4\pi/5)$

Using (3)(ii)(d) we get

$T = \frac{1}{2}(\cos 2\pi/5 - \cos 4\pi/5)(1 - \cos^2 2\pi/5 - 1 + \cos^2 4\pi/5)$
$= -\frac{1}{2}(\cos 2\pi/5 - \cos 4\pi/5)^2 (\cos 2\pi/5 + \cos 4\pi/5)$

Using (4)(iii) and 1 we have

$T = (-\frac{1}{2})(-\frac{1}{2})(5/4) = 5/16$

(8)(i) Let $T = \cos^5 2\pi/5 + \cos^5 4\pi/5 = \cos^3 2\pi/5(1 - \sin^2 2\pi/5) + \cos^3 4\pi/5(1 - \sin^2 4\pi/5)$

$$= \cos^3 2\pi/5 \left[1 - \tfrac{1}{2}(1 - \cos 4\pi/5)\right] + \cos^3 4\pi/5 \left[1 - \tfrac{1}{2}(1 - \cos 2\pi/5)\right]$$

$$= \tfrac{1}{2}\cos^3 2\pi/5 (1 + \cos 4\pi/5) + \tfrac{1}{2}\cos^3 4\pi/5 (1 + \cos 2\pi/5)$$

$$= \tfrac{1}{2}\left[\cos^3 2\pi/5 + \cos^3 4\pi/5 + \cos 2\pi/5 \cos 4\pi/5 (\cos^2 2\pi/5 + \cos^2 4\pi/5)\right]$$

Using (4)(iv), (4)(i) and (3)(ii)(c) we have

$$T = \tfrac{1}{2}\left[-\tfrac{1}{2} + (-\tfrac{1}{4})(\tfrac{3}{4})\right] = -\tfrac{1}{2}(\tfrac{1}{2} + 3/16) = -11/32$$

(ii) Let $T = \cos^3 2\pi/5 \sin^2 2\pi/5 + \cos^3 4\pi/5 \sin^2 4\pi/5$

$$= \cos^3 2\pi/5 (1 - \cos^2 2\pi/5) + \cos^3 4\pi/5 (1 - \cos^2 4\pi/5)$$

$$= \cos^3 2\pi/5 + \cos^3 4\pi/5 - (\cos^5 2\pi/5 + \cos^5 4\pi/5)$$

Using (4)(iv) and (8)(i) above, we have

$$T = -\tfrac{1}{2} - (-11/32) = \frac{-16}{32} + \frac{11}{32} = -5/32$$

(iii) Let $T = \cos 2\pi/5 \sin^4 2\pi/5 + \cos 4\pi/5 \sin^4 4\pi/5$

$$= \cos 2\pi/5 \left[\tfrac{1}{2}(1 - \cos 4\pi/5)\right]^2 + \cos 4\pi/5 \left[\tfrac{1}{2}(1 - \cos 2\pi/5)\right]^2$$

$$= \tfrac{1}{4}\left[\cos 2\pi/5 (1 - 2\cos \tfrac{4\pi}{5} + \cos^2 \tfrac{4\pi}{5} + \cos 4\pi/5 [1 - 2\cos 2\pi/5 + \cos^2 2\pi/5)\right]$$

$$= \tfrac{1}{4}\left[(\cos 2\pi/5 + \cos 4\pi/5) - 4\cos 2\pi/5 \cos 4\pi/5 + \cos 2\pi/5 \cos 4\pi/5 (\cos 2\pi/5 + \cos 4\pi/5)\right]$$

Using (3) (ii) (c) and (1) we have

$$T = \tfrac{1}{4} \; -\tfrac{1}{2} - 4(-\tfrac{1}{4}) + (-\tfrac{1}{4})(-\tfrac{1}{2}) = \tfrac{1}{4}(\tfrac{1}{2} + \tfrac{1}{8}) = 5/32$$

(9)(i) Let $T = \cos^4 4\pi/5 \cos 2\pi/5 + \cos^4 2\pi/5 \cos 4\pi/5$

$$= \cos {}^{4\pi}/_5 \cos {}^{2\pi}/_5 (\cos^3 {}^{4\pi}/_5 + \cos^3 {}^{2\pi}/_5)$$

Using (3)(ii)(c) and (4)(iv) , we have

$$T = (-\tfrac{1}{4})(-\tfrac{1}{2}) = \tfrac{1}{8}$$

(ii) Let $T = \cos^3 {}^{2\pi}/_5 \sin {}^{2\pi}/_5 \sin {}^{4\pi}/_5 - \cos^3 {}^{4\pi}/_5 \sin {}^{4\pi}/_5 \sin {}^{2\pi}/_5$

$$= \sin {}^{2\pi}/_5 \sin {}^{4\pi}/_5 (\cos^3 {}^{2\pi}/_5 - \cos^3 {}^{4\pi}/_5)$$

Using (3)(ii)(d) we have

$$T = \tfrac{1}{2} (\cos {}^{2\pi}/_5 - \cos {}^{4\pi}/_5)^2 (\cos^2 {}^{2\pi}/_5 + \cos^2 {}^{4\pi}/_5 + \cos {}^{2\pi}/_5 \cos {}^{4\pi}/_5)$$

Using (4) (iii) and (3)(ii)(C), we have

$$T = (\tfrac{1}{2})({}^5/_4)(\tfrac{1}{2}) = {}^5/_{16}$$

(iii) Let $T = \cos^2 {}^{4\pi}/_5 \sin^2 {}^{4\pi}/_5 \cos {}^{2\pi}/_5 + \cos^2 {}^{2\pi}/_5 \sin^2 {}^{2\pi}/_5 \cos {}^{4\pi}/_5$

$$= \cos {}^{4\pi}/_5 \cos {}^{2\pi}/_5 (\cos {}^{4\pi}/_5 \sin^2 {}^{4\pi}/_5 + \cos {}^{2\pi}/_5 \sin^2 {}^{2\pi}/_5)$$

$$= \cos {}^{4\pi}/_5 \cos {}^{2\pi}/_5 (-\tfrac{1}{2} \sin {}^{2\pi}/_5 \sin {}^{4\pi}/_5 + \tfrac{1}{2} \sin {}^{4\pi}/_5 \sin {}^{2\pi}/_5 = o$$

Using (2)(i) and (2)(ii)

(iv) Let $T = \cos {}^{2\pi}/_5 \sin^3 {}^{2\pi}/_5 \sin {}^{4\pi}/_5 - \cos {}^{4\pi}/_5 \sin^3 {}^{4\pi}/_5 \sin {}^{2\pi}/_5$

$$= \tfrac{1}{2}(\sin^2 {}^{2\pi}/_5 \sin^2 {}^{4\pi}/_5 + \sin^2 {}^{2\pi}/_5 \sin^2 {}^{4\pi}/_5)$$

$$= \sin^2 {}^{4\pi}/_5 \sin^2 {}^{2\pi}/_5$$

Using (3)(ii)(d) and (4) (iii) , we have

$$T = \tfrac{1}{4} (\cos {}^{2\pi}/_5 - \cos {}^{4\pi}/_5)^2 = \tfrac{1}{4} {}^5/_4 = {}^5/_{16}$$

(v) Let $T = \cos {}^{2\pi}/_5 \sin^4 {}^{4\pi}/_5 + \cos {}^{4\pi}/_5 \sin^4 {}^{2\pi}/_5$

$$= \cos {}^{2\pi}/_5 \left[\tfrac{1}{2}(1 - \cos {}^{2\pi}/_5)\right]^2 + \cos {}^{4\pi}/_5 \left[\tfrac{1}{2} (1 - \cos {}^{4\pi}/_5)\right]^2$$

$$= \frac{1}{4}\left[\cos \frac{2\pi}{5}\left(1 - 2\cos \frac{2\pi}{5} + \cos^2 \frac{2\pi}{5}\right) + \cos \frac{4\pi}{5}\left(1 - 2\cos \frac{4\pi}{5} + \cos^2 \frac{4\pi}{5}\right)\right]$$

$$= \frac{1}{4}\left[\left(\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5}\right) - 2\left(\cos^2 \frac{2\pi}{5} + \cos^2 \frac{4\pi}{5}\right) + \left(\cos^3 \frac{2\pi}{5} + \cos^3 \frac{4\pi}{5}\right)\right]$$

Using (1), (4)(i) and (4)(iv), we have

$$T = \frac{1}{4}\left[\left(-\frac{1}{2}\right) - 2\left(\frac{3}{4}\right) - \left(\frac{1}{2}\right)\right] = -\frac{1}{4}\cdot\frac{5}{2} = -\frac{5}{8}$$

(10)(i) Let $T = \cos^2 \frac{2\pi}{5} \cos^3 \frac{4\pi}{5} + \cos^2 \frac{4\pi}{5} \cos^3 \frac{2\pi}{5}$

$$= \cos^2 \frac{2\pi}{5} \cos^2 \frac{4\pi}{5}\left(\cos \frac{4\pi}{5} + \cos \frac{2\pi}{5}\right)$$

Using (1) and (3)(ii)(c), we have

$$T = \frac{1}{16}\left(-\frac{1}{2}\right) = -\frac{1}{32}$$

(ii) Let $T = \cos^2 \frac{2\pi}{5} \cos \frac{4\pi}{5} \sin^2 \frac{4\pi}{5} + \cos^2 \frac{4\pi}{5} \cos \frac{2\pi}{5} \sin^2 \frac{2\pi}{5}$

$$= \cos \frac{2\pi}{5} \cos \frac{4\pi}{5}\left[\cos \frac{2\pi}{5}\left(1 - \cos^2 \frac{4\pi}{5}\right) + \cos \frac{4\pi}{5}\left(1 - \cos^2 \frac{2\pi}{5}\right)\right]$$

$$= \cos \frac{2\pi}{5} \cos \frac{4\pi}{5}\left[\left(\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5}\right) - \cos \frac{2\pi}{5} \cos \frac{4\pi}{5}\left(\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5}\right)\right]$$

Using 1 and (3)(ii)(c), we have

$$T = \left(-\frac{1}{4}\right)\left[-\frac{1}{2} + \frac{1}{4}\left(-\frac{1}{2}\right)\right] = \frac{5}{32}$$

(iii) Let $T = \cos \frac{2\pi}{5} \sin \frac{2\pi}{5} \cos^2 \frac{4\pi}{5} \sin \frac{4\pi}{5} - \cos \frac{4\pi}{5} \sin \frac{4\pi}{5} \cos^2 \frac{2\pi}{5} \sin \frac{2\pi}{5}$

$$= \cos \frac{2\pi}{5} \cos \frac{4\pi}{5} \sin \frac{2\pi}{5} \sin \frac{4\pi}{5}\left(\cos \frac{4\pi}{5} - \cos \frac{2\pi}{5}\right)$$

Using (1) and (3)(ii)(d), we have

$$T = (-\tfrac{1}{4})(-\tfrac{1}{2})(\cos 4\pi/5 - \cos 2\pi/5)^2 = 1/8 \;\; 5/4 = 5/32$$

where we used (4)(iii)

(iv) Let $T = \cos 2\pi/5 \sin 2\pi/5 \sin^3 4\pi/5 - \cos 4\pi/5 \sin 4\pi/5 \sin^3 2\pi/5$

$$= \sin 2\pi/5 \sin 4\pi/5 \left[\cos 2\pi/5(1 - \cos^2 4\pi/5) - \cos 4\pi/5(1 - \cos^2 2\pi/5)\right]$$

$$= \sin 2\pi/5 \sin 4\pi/5 \left[(\cos 2\pi/5 - \cos 4\pi/5) - \cos 2\pi/5 \cos 4\pi/5 \cdot (\cos 4\pi/5 - \cos 2\pi/5)\right]$$

Using (3)(ii)(d), (1), and 4 (iii), we have

$$T = (\tfrac{1}{2})(\cos 2\pi/5 - \cos 4\pi/5)^2 \left[1 + (-\tfrac{1}{4})\right] = \tfrac{1}{2} \cdot 5/4 \cdot \tfrac{3}{4} = 15/32$$

(v) Let $T = \sin^2 4\pi/5 \cos^3 2\pi/5 + \sin^2 2\pi/5 \cos^3 4\pi/5$

$$= \cos^3 2\pi/5(1 - \cos^2 4\pi/5) + \cos^3 4\pi/5 (1 - \cos^2 2\pi/5)$$

$$= \cos^3 2\pi/5 + \cos^3 4\pi/5 - \cos^2 2\pi/5 \cos^2 4\pi/5 (\cos 2\pi/5 + \cos 4\pi/5)$$

Using (4)(iv), (1), and (3)(ii)(d), we have

$$T = -\tfrac{1}{2} - 1/16 (-\tfrac{1}{2}) = -15/32$$

(vi) Let $T = \sin^2 4\pi/5 \cos 2\pi/5 \sin^2 2\pi/5 + \sin^2 2\pi/5 \cos 4\pi/5 \sin^2 4\pi/5$

$$= \sin^2 4\pi/5 \sin^2 2\pi/5 (\cos 2\pi/5 + \cos 4\pi/5)$$

Using (3)(ii)(d), and (1), we have

$$T = (\tfrac{1}{4})(\cos 2\pi/5 - \cos 4\pi/5)^2 (-\tfrac{1}{2}) = \tfrac{1}{4} \cdot 5/4 \cdot (-\tfrac{1}{2}) = -5/32$$

where we used (4)(iii)

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