Studies in dual models

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STUDIES IN DUAL MODELS

by

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Preface

The work presented in this thesis was carried out between October 1973 and August 1976 in the Department of Mathematics, University of Durham, under the supervision of Dr. P. Goddard (1st year) and Dr. D. B. Fairlie (2nd and 3rd years).

The material in this thesis has not been submitted previously for any degree in this or any other university. No claim of originality is make for Chapters one and three, and most of Chapter two; the remainder is claimed to be original except where otherwise indicated. Chapter two is based on a paper by the author in collaboration with E. Corrigan and D. Olive; part of the work in Chapter five has been published in a paper by the author in collaboration with D. B. Fairlie and R. G. Yates. Relevant unpublished work by the author is also included.

I should like to thank Peter Goddard and David Fairlie for their guidance and encouragement throughout the course of this work; I should also like to thank Ed Corrigan for many helpful descussions over the past three years. Thanks are also due to Jan Peters for typing the manuscript, and to the Science Research Council for a research studentship.

Abstract

This thesis is concerned with Dual Resonance Models, and in particular with their algebraic structures. Chapter one is an introduction to the subject of dual models, in which the known models are surveyed, and their most important features are indicated.

Chapter two deals with the calculation of the determinants and other functions of infinite dimensional matrices which arise in the calculation of fermion and off-shell dual amplitudes. After explaining how these functions arise, a group-theoretical method of calculating them is given, which is much simpler than previous methods.

In Chapter three recent work on supersymmetry and graded Lie algebras is reviewed, and its relevance to theoretical particle physics in general and dual models in particular is indicated. The algebras underlying the known dual models, including the recently suggested O(N) algebras, are seen to be graded Lie algebras.

In Chapter four it is pointed out that the finite subalgebras of these (infinite) dual model algebras are simple graded Lie algebras. Representations of these subalgebras are constructed using the superfield formalism of supersymmetry. Some of these representations extend to representations of the infinite algebras, and in certain cases they can be used to construct Fock space realizations of the generators of these (graded) algebras.

In Chapter five n-point amplitudes corresponding to the O(N) algebras are constructed using bilinear forms which are invariant under their finite subalgebras (described in Chapter four). The 4-point amplitudes are investigated, and it is found that their mass-spectrum contains ghost states for $N > 2$, and for $N=2$ except in two space-time dimensions.

Contents

Chapter 1. Dual Resonance Models.

Introduction: The concept of duality in strong interaction physics was first introduced by Dolen, Horn and Schmid[1]. They had studied the constraints imposed on S- matrix elements by analyticity and crossing symmetry, using the techniques of finite energy sum rules. They found that the direct-channel resonances and the cross-channel Regge poles provided, in an average sense, equivalent descriptions of the same phenomena.

To obtain a scattering amplitude which satisfies analyticity, crossing symmetry, Regge behaviour and Lorentz invariance is a non-trivial task. The discovery by Veneziano[2] of a four-point crossing-symmetric scattering amplitude with linear Regge trajectories initiated the subject of dual resonance models. This amplitude possesses an infinite number of poles (corresponding to narrow resonances) in the s- or the t- channel, and duality is explicitly satisfied: the sum over the s- channel poles is actually equal to the sum over the t- channel poles.

It was soon discovered that Veneziano's amplitude could be generalized to an n- point amplitude[3,4] which is based on the same principles, and is completely factorizable. A powerful operator formalism was developed[5-8] which enabled a proof to be given that the model is free of indefinite-metric ghosts[9]. Although the n- point amplitude is explicitly non-unitary, constructed as it is from narrow resonances, a procedure for unitarization of the amplitudes is known[10].

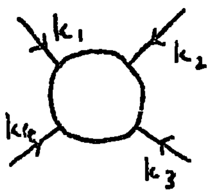
The model developed from Veneziano's original 4-point amplitude possesses remarkable mathematical consistency. There is however very little freedom in the model, and as it stands it is physically unrealistic: the mass spectrum is wrong, there are no fermions, and the model works best in an unphysical number of dimensions of space-time.

To find a way of changing the model in a way which preserves its good qualitative features and its mathematical consistency, and improves the undesirable features Neveu, Schwarz and Ramond[11,12] developed a model which embodies the principles of Veneziano's model, and includes fermions. However the mass spectrum, whilst an improvement on the Veneziano model, is still unrealistic, and the model still works best in an unphysical number of dimensions of space-time.

There are a number of excellent reviews of dual models[13], and in this chapter we intend only to give a brief introduction to the subject, placing an emphasis on those aspects which will be of use in the remainder of this thesis. We shall also briefly describe the new models suggested by Acemollo et al[14], which we shall consider in more detail in later chapters.

1.1. The Veneziano Model

The amplitude first written down by Veneziano was a proposal to describe $\pi\pi \to \pi\omega$ scattering. Adapted for the case of scattering of four identical spinless particles (with no internal quantum numbers)



$$s = (k_1 + k_2)^2$$
$$t = (k_1 + k_4)^2$$
$$u = (k_1 + k_3)^2$$

the proposed amplitude is

$$F(s,t,u) = A(s,t) + A(t,u) + A(u,s)$$

where
$$A(s,t) = \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \tag{1.1}$$

and $\alpha(s) = \alpha(0) + \alpha' s$ is a real linear trajectory function. $A(s,t)$ is an Euler beta- function, and can also be written in integral form as

$$A(s,t) = \int_0^1 dx\; x^{-\alpha(s)} (1-x)^{-\alpha(t)-1} \tag{1.2}$$

The amplitude $A(s,t)$ has a number of important properties. Crossing symmetry is obvious. More important is the property that it can be expressed as a sum of pole terms in either the s or t channel; explicitly we can write

$$A(s,t) = \sum_{n=0}^{\infty} \frac{R_n(t)}{n-\alpha(s)} = \sum_{n=0}^{\infty} \frac{R_n(s)}{n-\alpha(t)}$$

where
$$R_n(x) = \frac{1}{n!}\,\frac{\Gamma(\alpha(x)+n+1)}{\Gamma(\alpha(x)+1)} \tag{1.3}$$

We see that $R_n(x)$ is a polynomial of degree n in x. Hence the residue $R_n(t)$ of the pole in s at $\alpha(s)=n$ is a polynomial of degree n in t; analysing this in terms of angular functions, this residue can be expressed in terms of spins less than or equal to n. This corresponds to the existence of daughter trajectories below the leading trajectory, with particles at the same masses as on the leading trajectory.

We note that if $\alpha(0)=1$, the residues R_n are such that there are no odd daughters. The condition $\alpha(0)=1$ is in fact required for the consistency of the model when generalized to n- point amplitudes; we can take the slope of the trajectory to be $\alpha'=\frac{1}{2}$, so that the leading trajectory is $\alpha(s)=1+\frac{1}{2}s$.

The other most important feature of A(s,t) is that it is Regge-behaved at high energy. Using Sterling's formula we see that

$$A(s,t) \sim (-s)^{\alpha(t)}\,\Gamma(-\alpha(t))$$

as $|s|\to\infty$ (since we are working in the narrow-resonance approximation, we must take this limit away from the real axis). The amplitude F(s,t,u) in this limit is

$$F(s,t,u) \sim \frac{\left(1+e^{i\pi\alpha(t)}\right)}{\Gamma(1+\alpha(t))\,\sin\pi\alpha(t)}\; s^{\alpha(t)} \qquad (1.5)$$

exactly as expected for an even-signatured Regge pole.

The n- point amplitude is a generalization of the integral form of the amplitude (1.3)[3]. It is best written down in terms of Koba-Nielsen variables[4]: associate with each particle of momentum k_i a point z_i on the unit circle, and write the generalization of A(s,t) as

$$A_n(k_1,\dots,k_n) = \int \frac{\prod dz_i}{dV_{abc}} \prod_{1\le i<j\le n} (z_i-z_j)^{-k_i\cdot k_j}\left[\prod_{i=1}^{n}(z_i-z_{i+1})^{\alpha(0)-1}\right]$$

where $$dV_{abc} = \frac{dz_a\,dz_b\,dz_c}{(z_a-z_b)(z_b-z_c)(z_c-z_a)} \qquad (1.6)$$

and the integral is round the unit circle, maintaining the order of the z's. We note that the bracketed term in the integral vanishes when $\alpha(0)=1$. A_n has simultaneous poles in (n-3) channels, lying on Regge trajectories $\alpha_{ij}=1+\frac{1}{2}s_{ij}$, where $s_{ij}=(k_i+..+k_j)^2$.

An important feature of A_n is that it is invariant under the projective or Mobius group SU (1,1): A_n is invariant under a Mobius transformation

$$z_i \rightarrow \frac{az_i+b}{cz_i+d} \qquad ad-bc=1 \tag{1.7}$$

A_n can in fact be written in terms of (Mobius invariant) cross ratios

$$(z_i,z_j;z_{i-1},z_{j+1}) = \frac{(z_i-z_j)(z_{i-1}-z_{j+1})}{(z_i-z_{j+1})(z_{i-1}-z_j)} \tag{1.8}$$

The factor dV_{abc} in (1.6) can be thought of as being due to this projective invariance. If we were to integrate over all z_i in (1.6), without the factor dV_{abc}, we would obtain a constant times A_n(15). This constant is essentially an integral over the group measure, and it is this that we are factoring out by including dV_{abc}.

$A_n(k_1,\ldots\ldots,k_n)$ is invariant under cyclic $(k_i \longrightarrow k_{i+1})$ and anti-cyclic transformations. To obtain the full amplitude we should sum over all inequivalent permutations of $(k_1,\ldots,k_n)$, just as in (1.1) for the 4-point case.

The full amplitude has all the desirable features of the 4-point amplitude. It can be written as the sum of simultaneous poles in (n-3) channels. It exhibits multi-Regge behaviour, that is Regge behaviour in subenergies, when certain ratios of those subenergies are held fixed. It is also fully factorizable with finite degeneracies.(16)

In order to analyse and understand the structure of the amplitude more clearly, it is useful to introduce an operator formalism. In this approach we use an infinite set of harmonic oscillator-type operators a_n^μ, $n > 0$, which satisfy canonical commutation relations

$$[a_n^\mu, a_m^\nu] = [a_n^{\mu\dagger}, a_m^{\nu\dagger}] = 0 \qquad n, m = 1, 2, \ldots \qquad (1.9)$$
$$[a_n^\mu, a_m^{\nu\dagger}] = -g^{\mu\nu}\delta_{nm}$$

The index μ runs over 0, 1, ... , D - 1, denoting one time dimension, and (D - 1) space dimensions. It is often more convenient to use α_n^μ, defined by

$$\alpha_n^\mu = \sqrt{n}\, a_n^\mu \qquad \alpha_{-n}^\mu = \sqrt{n}\, a_n^{\mu\dagger}$$
$$\alpha_0^\mu = p^\mu \qquad (1.10)$$
$$[\alpha_n^\mu, \alpha_m^\nu] = -n\, g^{\mu\nu}\delta_{n+m,0}$$

where p^μ is the momentum operator.

If q^μ is the position operator conjugate to p^μ, then we define generalized position and momentum operators Q^μ and P^μ by

$$Q^\mu(z) = q^\mu - i p^\mu \log z + i \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu z^{-n}$$

$$P^\mu(z) = i z \frac{dQ^\mu}{dz} = \sum_{n=-\infty}^{\infty} \alpha_n^\mu z^{-n} \qquad (1.11)$$

We introduce a vacuum state $|0\rangle$ which satisfies $\alpha_n^\mu|0\rangle = 0$ for $n \geq 0$. Then, defining the Fubini-Veneziano vertex[6,7]

$$V(z,k) = :e^{-ik\cdot Q(z)}: z^{\frac{1}{2}k^2} \qquad (1.12)$$

we can consider a scattering amplitude for n ground-state particles given by

$$A_n = \int \frac{\prod dz_i}{dV_{abc}} \prod_{i=1}^{n} (z_i - z_{i+1})^{\alpha(0)-1} \langle 0| \prod_{i=1}^{n} V(z_i, k_i)|0\rangle \qquad (1.13)$$

where the integration is taken round the unit circle, maintaining the order of the z's, as before. It can easily be shown that this is precisely equal to the Koba-Nielsen form of the amplitude (1.6).

We would now expect the states of the model to be those generated by the creation operators α_{-n}^{μ} acting on the vacuum $|0\rangle$. Because of the sign of the commutation relations satisfied by α_{-n}^{0}, the states created by these operators have a negative norm; they are ghost states which must decouple from the physical states if the model is to be at all realistic.

The ghost states do in fact decouple from the physical states(17), provided that the dimension of space-time $D \leq 26$. This is possible because the states of the model satisfy gauge conditions which restrict the space of physical states. The situation is analogous to that in QED, where the longitudinal components of the photon decouples because of gauge conditions.

To specify the gauge conditions of the Veneziano model, we need to introduce the Virasoro operators(8)

$$L_n = -\oint \frac{dz}{4\pi i z} \, z^n : P^2(z): \qquad n = 0, \pm 1, \ldots \tag{1.14}$$

where the integration is round the unit circle. These satisfy the commutation relations

$$\left[L_n, L_m\right] = (n-m)L_{n+m} + \frac{D}{12} n(n^2-1)\delta_{n+m,0} \tag{1.15}$$

Acting on the vacuum we find that $L_n|0\rangle = 0$ for $n \geq -1$.

An operator is said to have conformal spin J if it satisfies

$$\left[L_n, X(z)\right] = z^n\left(z\frac{d}{dz} - nJ\right)X(z) \tag{1.16}$$

In particular, we note that $Q^{\mu}(z)$, $P^{\mu}(z)$ and $V(z,k)$ have $J=0$, -1, $\frac{1}{2}k^2$ respectively. When $\alpha(0)=1$, the mass of the ground state is $\frac{1}{2}k^2=-1$, so $V(z,k)$ has $J=-1$.

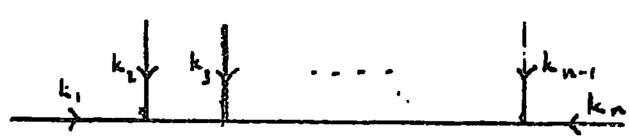
Using (1.14) and (1.16), the n-point amplitude (1.13) can be written in an explicitly factorized form as [6]

$$A_n = \langle 0 | e^{-ik_1\cdot q} V(1,k_2) P V(1,k_3) \dots P V(1,k_{n-1}) e^{-ik_n\cdot q} | 0 \rangle$$

where

$$P = \int_0^1 dx\, x^{L_0-\alpha(0)} (1-x)^{\alpha(0)-1} = \frac{1}{L_0-1} \quad (\alpha(0)=1) \tag{1.17}$$

and $V(1,k)$ is $V(z,k)$ evaluated at $z=1$. Thus the n-point amplitude has been written as a succession of vertices and propagators, corresponding to the diagram



Now consider a tree state $|\psi\rangle$ defined as

$$|\psi\rangle = V(1,k_1)P \dots P V(1,k_{m-1}) e^{-ik_m\cdot q}|0\rangle \tag{1.18}$$

For $\alpha(0)=1$ this satisfies

$$(L_0 - L_n - 1 + n)|\psi\rangle = 0 \qquad n \geq 1 \tag{1.19}$$

It is this gauge condition which allows the decoupling of certain states from the physical states, and in particular decouples all of the ghost states from the physical states for $D \leq 26$.[17]

We note that this only works for $\alpha(0)=1$; in other cases there are ghosts coupling to the physical states. This condition means that the ground state of the model has mass $\frac{1}{2}k^2=-1$: it is a tachyon. This undesirable feature, together with a leading trajectory that passes through $\alpha(0)=1$, means that the spectrum of the model is physically unrealistic. Furthermore, the model prefers to work in 26 dimensions of space-time. The tight constraints placed on the model by requiring the cancellation of ghost states mean that the model cannot simply be modified to rectify these faults; new models must be found.

1.2. Representations of the Projective Group

Before we go on to discuss the construction of further dual models, it is convenient to discuss some aspects of the projective group, since this is an important feature of the Veneziano model and its extensions. In particular we wish to consider the representations of SU(1,1), since we shall use these in later chapters.

The Virasoro algebra (1.15) has a finite subalgebra spanned by $\{L_0, L_{\pm 1}\}$ which is isomorphic to the algebra of SU(1,1). The quadratic Casimir operator of this algebra is

$$C = L_0(L_0+1) - L_1 L_{-1} \tag{1.20}$$

The group SU(1,1) is (isomorphic to) the group of real 2 x 2 matrices with determinant=1. We can think of these as operating on a 2-vector

$$A : \begin{pmatrix} \xi \\ \eta \end{pmatrix} \longrightarrow \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad ad-bc=1 \tag{1.21}$$

The L_0, $L_{\pm 1}$ can be represented as 2 x 2 matrices[18]. Considering terms of the form $(\xi^\alpha \eta^\beta)$ we see that they are mapped into other terms of the same form by the transformation (1.21); hence they are suitable for use as the basis of a representation space for other representations of SU(1,1).

In particular, consider terms $(\xi\eta)^J (\xi/\eta)^{k+m}$ where k is fractional and m is an integer. J and k are left invariant by the transformation (1.21). J is in fact related to the Casimir operator of (1.20); the eigenvalue of C is J(J+1).

The invariants J, k can be used to classify the unitary irreducible representations of SU(1,1)[18-20]. Requiring that the representation be single-valued restricts k to be an integer or half-integer. For $J \geq 0$ we may construct $(2J+1)$ dimensional representations; these are necessarily non-unitary, as we are dealing with a non-compact group.

For $J < 0$ there are various classes of unitary irreducible representations. In particular we have the representations

$$\begin{array}{llll} D_J^{(+)} & J<0,\ k=-J & m=0,1,2\ldots \\ D_J^{(-)} & J<0,\ k=J & m=0,-1,-2\ldots \\ D^{(-\frac{1}{2}\,0)} & J=-\tfrac{1}{2} \quad k=0 & m=0,\pm 1,\pm 2\ldots \end{array} \tag{1.22}$$

As explained above, these representations can be realized on a representation space spanned by a basis

$$|Jkm\rangle = N(J,k,m)\,(\xi\eta)^J \left(\frac{\xi}{\eta}\right)^{k+m} \tag{1.23}$$

where $N(J,k,m)$ is a normalization factor. The representation matrix $D^{(Jk)}_{mn}$ is defined by

$$\begin{aligned} |Jkm\rangle' &\equiv N(J,k,m)\,(\xi'\eta')^J \left(\xi'/\eta'\right)^{k+m} \\ &= \sum_n D^{(Jk)}_{nm}(A)\,|Jkn\rangle \end{aligned} \tag{1.24}$$

where A is the transformation (1.21).

The normalization factors are determined from the condition $D(L_{-1})=D(L_1)^\dagger$, and for the $D_J^{(+)}$, $D_J^{(-)}$ representations can be taken to be

$$N(J,k=\pm J,m) = \sqrt{\frac{\Gamma(m-2J)}{m!}} \tag{1.25}$$

We can obtain (non-unitary) representations of the generators $\{L_0, L_{\pm 1}\}$ for J, k outside the above restrictions. In particular we can consider the representation $D_J^{(+)}$ in the limit $J \longrightarrow 0-$. This can be used to construct a Fock space realization of the generators $\left\{ L_0, L_{\pm 1} \right\}$ [18] which coincides with the Virasoro expression (1.14).

1.3. The Neveu-Schwarz Model

An extension of the Veneziano model was proposed by Neveu and Schwarz[11] who introduced anti-commuting annihalation and creation operators b_r^μ, $b_s^{\mu\dagger}$ satisfying

$$\begin{gathered} \{b_r^\mu, b_s^\nu\} = \{b_r^{\mu\dagger}, b_s^{\nu\dagger}\} = 0 \\ \{b_r^\mu, b_s^{\nu\dagger}\} = -g^{\mu\nu}\delta_{rs} \end{gathered} \tag{1.26}$$

where $r,s = \frac{1}{2}, \frac{3}{2}, \ldots$. The vacuum is now understood to satisfy $b_r^\mu|0\rangle = 0$ as well as $\alpha_n^\mu|0\rangle = 0$ for $n \geqslant 0$. Using the α's and the b's to construct amplitudes gives a model with more structure than the Veneziano model.

Neveu and Schwarz introduced a new field $H^\mu(z)$ defined by

$$H^\mu(z) = \sum_{r=-\infty}^{\infty} b_r^\mu z^{-r} \tag{1.27}$$

(where $b_{-r}^\mu = b_r^{\mu\dagger}$). This field can be used to construct new Virasoro-type guage operators $L_n^{(b)}$:

$$L_n^{(b)} = -\oint \frac{dz}{4\pi i z} z^n :H(z)\cdot \frac{dH(z)}{dz}: \tag{1.28}$$

When added to $L_n^{(a)}$ constructed from α_n^μ as in (1.14), this gives $L_n = L_n^{(a)} + L_n^{(b)}$:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{8} n(n^2-1)\delta_{m+n,0} \tag{1.29}$$

This is of the same form as (1.15), with a different c-number term.

A further set of operators G_r, r half-integral, can be defined[21]:

$$G_r = \oint \frac{dz}{2\pi i z} z^r H(z)\cdot P(z) \tag{1.30}$$

which form a closed algebra with the L_n's

$$\begin{gathered} [L_n, G_r] = \left(\frac{n}{2} - r\right) G_{n+r} \\ \{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \end{gathered} \tag{1.31}$$

If we consider the commutator of L_n with $H^{\mu}(z)$, and compare it with the definition of conformal spin (1.16), we find that it has $J=-\frac{1}{2}$. Under SU(1,1) transformations, $H^{\mu}(z)$ transforms as a ($J=-\frac{1}{2}$, $k=\frac{1}{2}$) representation (it is built from terms of the form z^{k+m} with $k=\frac{1}{2}$).

In order to write an amplitude, a new vertex is defined[11]

$$V(z,k) = k \cdot H(z) : e^{-ik\cdot Q(z)} : z^{\frac{1}{2}k^2-\frac{1}{2}} \tag{1.32}$$

This has conformal spin $J=\frac{1}{2}k^2-\frac{1}{2}$, so that it transforms in the same way as the Fubini-Veneziano vertex under projective transformations, provided that $\frac{1}{2}k^2=-\frac{1}{2}$. As in the Veneziano model, this condition is actually necessary for the consistency of the model, and we assume that it holds in the following. It means in particular that the ground state of the Neveu-Schwarz model is also a tachyon, with mass $m^2=-\frac{1}{2}$.

The n-point Neveu-Schwarz amplitude is

$$A_n = \oint \frac{\prod dz_i}{dV_{abc}} \langle 0 | \prod_{i=1}^{n} V(z_i,k_i) | 0 \rangle \tag{1.33}$$

where dV_{abc} is the projective invariant measure defined before (1.6). A_n is invariant under SU(1,1), since the vertex transforms in the same way as in the Veneziano case. This amplitude has simultaneous poles in (n-3) non-overlapping channels which can be explicitly displayed by writing (1.33) in a factorized form, analagous to (1.17):

$$A_n = \langle 0 | e^{-ik_1\cdot q} k_1\cdot b_{\frac{1}{2}} V(1,k_2) \frac{1}{L_0-1} \dots V(1,k_{n-1}) k_n \cdot b^{\dagger}_{\frac{1}{2}} e^{-ik_n\cdot q} | 0 \rangle \tag{1.34}$$

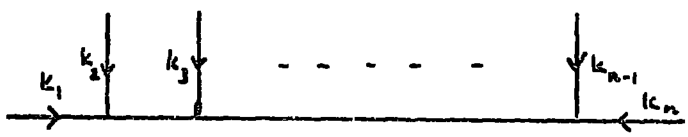
where $V(1,k)=k.H(1)\ V_0(1,k)$

and $V_0(1,k)$ is the Veneziano vertex (1.12) evaluated at z=1.

The above form of the amplitude is called the $\mathcal{F}_1$ form. It is sometimes more convenient to work with the $\mathcal{F}_2$ form, given by removing the k.b terms in (1.34) to give

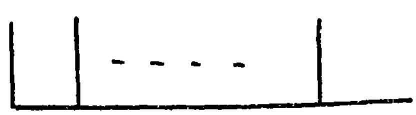
$$A_n = \langle 0|e^{-ik_1\cdot q}V(1,k_2)\frac{1}{L_0-\frac{1}{2}}\cdots\frac{1}{L_0-\frac{1}{2}}V(1,k_{n-1})e^{-ik_n\cdot q}|0\rangle \tag{1.35}$$

Both of these amplitudes represent the scattering process corresponding to the diagram



In the $\mathcal{F}_1$ formalism we consider the 1- particle state to be $k\cdot b^+_{\frac{1}{2}}\, e^{-ik\cdot q}|0\rangle$, in the $\mathcal{F}_2$ formalism we have redefined it to be $e^{-ik\cdot q}|0\rangle$.

As in the Veneziano model, there is the possibility of negative-norm states, created by α^0_{-n} and also by b^0_{-r}. If these are to decouple from the physical states, we require gauge conditions on the physical states; furthermore we expect more gauge conditions than in the Veneziano model as there are more ghost states to be decoupled. For $k^2=-\frac{1}{2}$ the gauge conditions satisfied by a tree state

$|\psi\rangle =$ 

in the $\mathcal{F}_2$ formalism are

$$\begin{aligned} L_n|\psi\rangle &= (L_0+n-\tfrac{1}{2})|\psi\rangle \qquad && n\geq 1 \\ G_r|\psi\rangle &= (L_0+r-\tfrac{1}{2})|\psi_0\rangle \qquad && r\geq \tfrac{1}{2} \end{aligned} \tag{1.36}$$

where $|\psi_0\rangle$ is obtained from $|\psi\rangle$ by replacing the first vertex $V(1,k_1)$ by $V_0(1,k_1)$. These gauge conditions are sufficient to ensure that the ghost states decouple from the physical states, provided that $D\leq 10$[22].

The n-point amplitude can also be written in a form that is very similar to the Koba-Nielsen form of the Veneziano amplitude (1.6)[23]. To do this we introduce variables θ_i associated with each particle k_i, in addition to the variables z_i. The θ_i satisfy

$$\{\theta_i, \theta_j\} = 0 = \theta_i^2 \tag{1.37}$$

An integration over θ_i can be defined [24] by

$$\int d\theta_i = 0 \qquad \int d\theta_i \, \theta_i = 1 \tag{1.38}$$

The n-point amplitude can then be written as $(\frac{1}{2}k_i^2 = -\frac{1}{2})$

$$A_n = \int \frac{\prod dz_i}{dV_{abc}} \prod_{i=1}^{n} d\theta_i \prod_{i<j} (z_i - z_j + \theta_i\theta_j)^{-k_i \cdot k_j} \tag{1.39}$$

We shall see in later chapters that this form of the amplitude is particularly useful for studying the properties of the "supergauges" G_r .

The Neveu-Schwarz model has a built-in 'G-parity': a state has even/odd G-parity as it is created by an even/odd number of $b^\dagger$ operators. The n-point amplitude A_n vanishes for n odd. Because of this it is tempting to identify the ground state of the model as the pion. Isospin factors can be incorporated in A_n by use of the Chan-Paton factors[25].

The trajectories of the Neveu-Schwarz model are spaced at half-unit intervals, rather than at unit intervals as in the Veneziano model. The leading trajectory still passes through $\alpha(0)=1$; however the pole at $\frac{1}{2}k^2=-1$ is cancelled from the spectrum, and the ground state particle is the pole at $\frac{1}{2}k^2=-\frac{1}{2}$. Although the lower-lying states bear some resemblance to the physical mass-spectrum of strangeness-zero mesons, the similarity is far from satisfactory. In particular the pion should not be a tachyon! Also the candidates for the ρ and ω have significantly different masses (the ρ actually having zero mass) and the leading trajectory still has intercept $\alpha(0)=1$. Thus the Neveu-Schwarz model, whilst an improvement on the Veneziano model, is still physically unrealistic.

An important feature of the Neveu-Schwarz model is that it can be extended to include fermions, as shown by Ramond[12]. Anti-commuting operators d^μ_n for $n=0, \pm 1, \pm 2, \ldots$ are introduced satisfying

$$d^\mu_{-n} = d^{\mu\dagger}_n \qquad \{d^\mu_m, d^\nu_n\} = -g^{\mu\nu}\delta_{m+n,0} \tag{1.40}$$

The basic field of the fermion sector is a generalization of the γ matrices

$$\Gamma^\mu(z) = \gamma^\mu + i\sqrt{2}\,\gamma_5 \sum_{n\neq 0} d^\mu_n z^{-n} \tag{1.41}$$

The Virasoro operator $L^{(a)}_n$ can be extended to $L_n = L^{(a)}_n + L^{(d)}_n$ by defining

$$L^{(d)}_n = -\oint \frac{dz}{8\pi i}\, z^n :\Gamma(z)\cdot\frac{d\Gamma(z)}{dz}: \qquad n=0,\pm1,\ldots \tag{1.42}$$

Defining further operators F_n by

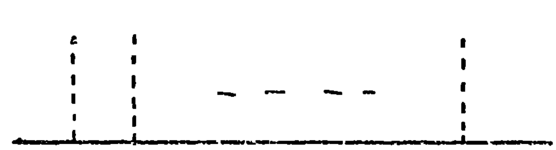
$$F_n = \sqrt{2}\oint \frac{dz}{4\pi i z}\, z^n\, \Gamma(z)\cdot P(z) \qquad n=0,\pm1,\ldots \tag{1.43}$$

we find that the operators L_n, F_m form a closed algebra

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{D}{8}n^3\delta_{n+m,0} \\ [L_n, F_m] &= \left(\frac{n}{2}-m\right)F_{n+m} \\ \{F_n, F_m\} &= -2L_{n+m} + \frac{D}{2}n^2\delta_{n+m,0} \end{aligned} \tag{1.44}$$

Acting on $\Gamma^\mu(z)$ with L_0, $L_{\pm1}$, we find that Γ^μ transforms as a $(J=-\frac{1}{2},\ k=0)$ representation of $SU(1,1)$.

A vertex for the emission of a meson from a fermion line can easily be constructed using the field Γ^μ, and using this an n-point amplitude for the process



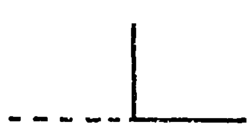
can be written down[21]. In this diagram the solid line is a fermion, the broken lines are mesons. The tree states of the model

$|\psi\rangle$ =

satisfy L and F gauge conditions, provided that the meson momenta k_i satisfy $\frac{1}{2}k_i^2=-\frac{1}{2}$. This suggests that the meson states should be those of the Neveu-Schwarz model, and this is in fact so. These gauge conditions are sufficient to decouple the ghost states from the physical states when D=10 (the critical dimension of the Neveu-Schwarz model) and the mass of the fermion M is zero[13].

As in the meson case, the amplitude may be written in two forms, $\mathcal{F}_1$ and $\mathcal{F}_2$. The fermion propagators in $\mathcal{F}_1$ and $\mathcal{F}_2$ are (omitting spinors) $(L_0)^{-1}$ and $(F_0)^{-1}$ respectively. The $\mathcal{F}_2$ amplitude is actually the correct one; the $\mathcal{F}_1$ amplitude contains an extra factor of $M^2=0$. However the $\mathcal{F}_1$ states satisfy simple gauge conditions; it is often easier to calculate in the $\mathcal{F}_1$ formalism with $M^2\neq 0$, extact the factor of M^2 and then set $M^2=0$ to give the correct ($\mathcal{F}_2$) result.

The other vertex which must be present in the theory is the fermion emission vertex



where the meson line couples to a Neveu-Schwarz state. This must be a more complicated object than the previous vertices considered, as it has to transform the Fock space spanned by $\{\alpha^r_{-n}, d^r_{-m}\}$ to that spanned by $\{\alpha^r_{-n}, b^r_{-r}\}$. It was found by Thorn to be[26] (omitting spinors)

$$V(z,k) = e^{zL_{-1}^{d}} \; {}_b\langle 0 | e^{T} | 0 \rangle_d \tag{1.45}$$

where T is quadratic in the b's and d's. The inclusion of two such vertices in an amplitude leads to expressions involving the determinants of infinite dimensional matrices[27] which are difficult to evaluate. We shall show in Chapter Two how such calculations may be simplified.

1.4. Off-Shell States

In all of the models discussed so far, amplitudes have been written down for the scattering of n ground state particles on their mass-shell. The extension of the Veneziano model to an amplitude containing one or two off-shell states (currents) was achieved by Schwarz and Wu[28]. A more general operator formulation of the off-shell sector of the model was suggested by Corrigan and Fairlie[29]. This requires the introduction of commuting operators c_r^μ , r half-integral, satisfying

$$c_{-r}^\mu = c_r^{\mu\dagger} \qquad [c_r^\mu, c_s^\nu] = -g^{\mu\nu}\delta_{r+s,0} \tag{1.46}$$

Consider the field $R^\mu(z)$ defined by

$$R^\mu(z) = -i\sum_{r=\frac{1}{2}}^{\infty} \frac{1}{\sqrt{r}}\left(c_r^{\mu\dagger} z^r - c_r^\mu z^{-r}\right) \tag{1.47}$$

This can be used to construct Virasoro operators $L_n^{(c)}$ satisfying

$$\left[L_n^{(c)}, L_m^{(c)}\right] = (n-m)L_{n+m}^{(c)} + \frac{D}{12}\, n\left(n^2+\tfrac{1}{2}\right)\delta_{n+m,0} \tag{1.48}$$

and a vertex

$$V_R(z,k) = \,:e^{-ik\cdot R(z)}:\, z^{\frac{1}{2}k^2} \tag{1.49}$$

in the same way that the field $Q^\mu(z)$ is used in the Veneziano model. $R^\mu(z)$ transforms as a (J=0, k=½) representation of SU(1,1). The n-point amplitude for one off-shell particle, (n-1) on-shell (Veneziano) ground state particles can then be written using vertices V_R sandwiched between the c-space vacuum $|0\rangle_c$.

The amplitude referred to above describes the process

where the wavy line is the current. To write down amplitudes with more than one current we need the vertex for the emission of a current from a meson line

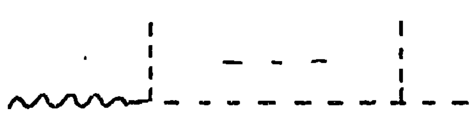
This can be obtained from the one-current amplitude by rearranging the operators to display (n-1) ordinary Veneziano vertices and one other vertex, now considered between a-space vacua $|0\rangle_a$. The one-current vertex is a complicated object which maps from the a-space to the c-space and back to the a-space; it is very similar to two fermion emission vertices (1.45)(29). This complicated form of the vertex leads to the evaluation of determinants of infinite-dimensional matrices, and makes the computation of amplitudes containing two or more currents difficult.

The off-shell amplitudes described above are unsatisfactory for a number of reasons. The ground state form-factor for a current of momentum Q is (29)

$$\int_0^1 \frac{4\,dv}{1-v^2} \, v^{-\frac{1}{2}Q^2-2} \tag{1.50}$$

which diverges. Also the scheme is only consistent when D=16, as opposed to the critical dimension D=26 for the Veneziano model.(30)

A one current amplitude for the Neveu-Schwarz model can also be written down(29), at least for the configuration



This works for D=10, the same as the critical dimension for the Neveu-Schwarz model. Also the form-factor is now finite.

A one-current vertex may be written down[31] but it is difficult to prove that it satisfies appropriate gauge conditions. Multi-current amplitudes will be even more difficult to compute than in the Veneziano case, and generally the off-shell sector is still not completely understood[30].

1.5. String Models

In an attempt to provide an underlying physical picture for the Veneziano and other dual models, the string model was developed. This is a Lagrangian theory based on an action first proposed by Nambu[32] describing the motion of a free (infinitely thin) string moving in D dimensions of space-time. The action is a natural generalization of the action for the motion of a free point particle.

A free (massless) point particle moving along a path $x^\mu(\tau)$ parametrized by τ is described by the action

$$A = \int_{\tau_0}^{\tau_1} \sqrt{\left(\frac{dx}{d\tau}\right)^2}\, d\tau \tag{1.51}$$

A is equal to the length along the path, and the variational principle gives that the particle moves along a path whose length is stationary with respect to small variations.

To generalize this to a free(massless) string, we parametrize the surface traced out in space-time by the parameters σ , τ , and define the action such that the area traced out by the string is stationary with respect to small variations:

$$A = \int d\sigma\, d\tau \sqrt{\left(\frac{\partial x}{\partial\sigma}\cdot\frac{\partial x}{\partial\tau}\right)^2 - \left(\frac{\partial x}{\partial\sigma}\right)^2\left(\frac{\partial x}{\partial\tau}\right)^2} \tag{1.52}$$

The integrand of A can also be written as $\sqrt{-\det g}$, where g is the metric on the surface traced out by the string:

$$g_{ij} = \frac{\partial x}{\partial\sigma_i}\cdot\frac{\partial x}{\partial\sigma_j} \qquad (\sigma_1, \sigma_2) \equiv (\sigma, \tau) \tag{1.53}$$

The classical and quantum dynamics following from (1.52) have been thoroughly investigated[32-34]. We note that A is invariant under local re-parametrization of the surface

$$\sigma, \tau \rightarrow \sigma'(\sigma,\tau), \tau'(\sigma,\tau) \tag{1.54}$$

This invariance means that there is an underlying gauge group in the theory; the generators of this group of transformations correspond exactly to the gauge algebra of the Veneziano model. In fact, after quantization, the spectrum of excitations of the string is precisely that of the Veneziano model. The leading trajectory is described by the excitations of a string of constant length rotating about its centre, with the ends moving at the speed of light(34).

Since there is a gauge invariance in the theory, the action (1.52) has to be quantized in a particular gauge. As in QED this can either be an explicitly Lorentz-covariant gauge, or a non-covariant gauge(34). In the first case it must be checked after quantization that the physical states of the system decouple from the ghost states created by the operators $a_n^{o\dagger}$, as described in Section 1.1. In the second case we can eliminate such operators at the classical level, but then Lorentz invariance must then be checked after quantization; it was found that the theory is only Lorentz invariant if D=26 and the ground state is a tachyon, $m^2=-1$.

In order to prove the complete equivalence of the string model and the Veneziano model (in 26 dimensions at least), a theory of interacting strings is necessary. This was constructed by Mandelstam(35) who showed that the 3-string vertex is formed by allowing one string to split into two others. Further investigation showed that a 4-string vertex is also necessary(36).

The string picture can be extended to the Neveu-Schwarz model(37,38) although not totally satisfactorily as the simple underlying geometrical picture becomes obscured. It is possible that a geometrical picture of the Neveu-Schwarz model might be obtained using anti-commuting variables θ as introduced in (1.37)[39,40]. Zumino[40] has constructed a complicated action which seems to give the Neveu-Schwarz gauge conditions. Other attempts to provide a geometrical picture of the Neveu-Schwarz model involve putting some extra structure onto the Veneziano string[41]. A string picture of the off-shell states has been suggested by Green and Shapiro[30].

A second-quantized field theory of strings has been developed by Kaku and Kikkawa[36].

1.6. O(N) Models

Until recently the models that we have described were, apart from minor modifications, the only consistent dual models (except for the non-planar Shapiro-Virasoro model which we shall mention later). Ademollo et al [42] have recently suggested generalizations of the Virasoro and supergauge algebras on which dual models are based. These generalizations will be described in more detail in Chapters 4 and 5.

They can be regarded as extensions of the (z_i, θ_i) form of the Neveu-Schwarz amplitude described in Section 1.3., and correspond to associating N θ's (θ_i^α with $\alpha=1,\ldots,N$) with each particle k_i. The Veneziano and Neveu-Schwarz models can be thought of as the cases N=0 and 1 respectively. The elements of the algebras generate certain transformations on (z, θ^α); in particular the set of local O(N) transformations on θ^α is included, hence their description as O(N) models.

Fock-space representations have only been found for the O(2) algebra and one other algebra derived from the O(4) algebra, and these have been investigated by Ademollo et al [43]. However amplitudes can be written down for all O(N) models as will be shown in Chapter 5.

1.7. Summary

In this Chapter we have discussed the most important planar dual models. There are other models which can be obtained from the Veneziano or Neveu-Schwarz models by introducing internal symmetries(44) in a way which reduces the number of space-time dimensions of the model. These do not introduce any basically new factors into the theory.

The only other consistent dual model is the non-planar Shapiro-Virasoro model. This is based on a generalization of the Veneziano 4-point function due to Virasoro(45) which exhibits non-planar duality: s,t and u channel poles are all present in one term. The extension to an n-point amplitude was discovered by Shapiro(46) to be

$$A_n = \int \frac{\prod d z_i}{(dV_{abc})^2} \prod_{1\le i<j\le n} |z_i - z_j|^{-k_i \cdot k_j} \tag{1.55}$$

where

$$(dV_{abc})^2 = \frac{d^2 z_a \, d^2 z_b \, d^2 z_c}{(z_a - z_b)^2 (z_b - z_c)^2 (z_c - z_a)^2}$$

The integration is over the whole z-plane, and we must have $\frac{1}{2}k_1^2=-2$; the leading trajectory thus has intercept $\alpha(0)=2$. An operator formalism for this model can be developed using two sets of oscillators a_n^μ and $\bar{a}_n^\mu$, and factorization of the amplitude shown in the same way as for the Veneziano model. It is probable that this model describes the pomeron sector of the Veneziano model.

The pomeron arises in the Veneziano model on consideration of non-planar loop diagrams. It has intercept twice that of the Veneziano model leading trajectory and half the slope, which is in rough agreement with the real world; unfortunately this means that it has intercept $\alpha(0)=2$ which is totally unrealistic.

Consideration of non-planar loops in the Neveu-Schwarz model also leads to the identification of pomeron terms, again corresponding to a leading trajectory of intercept $\alpha(0)=2$ and slope half that of the Neveu-Schwarz model. It might be expected that this is a general feature of dual models.

As we have pointed out, dual models describe many of the qualitative features of strong interactions, but suffer from severe defects. The most important of these are the existence of a tachyon, the incorrect leading trajectory, and the unphysical number of dimensions in which the model works best. In these circumstances it might be imagined that the amplitudes are not very useful phenomenologically. Nonetheless Veneziano-type amplitudes have been used, with a physical leading trajectory, in phenomenological fits to experimental data, often with surprisingly good results.

It may be hoped that one day the 'correct' dual model will be discovered which will incorporate the good qualitative features of the existing dual models, but will not suffer from their defects. Even if this is not the case, their study will have been worthwhile; not just for the discovery of an elegant and complex mathematical structure stemming from a simple 4-point amplitude, but also as a workshop for ideas about strong interactions where concepts such as duality can be explicitly realized. Their study has also led to the development of new areas of research such as supersymmetry and graded Lie algebras (both described in Chapter 3), vortices in gauge field theories(47) and models based on simple geometrical principles, such as the bag model(48).

Chapter 2. Some Calculations Using Group-Theoretical Methods

Introduction: In the introductory Chapter we described briefly the fermion emission and one-current vertices, and pointed out that the evaluation of amplitudes containing them requires the calculation of quantities (such as determinants) built from infinite dimensional matrices. In this Chapter we shall explain in more detail how these quantities arise, and how their evaluation can be simplified by making use of the group representation properties of the underlying fields.

2.1. Determinants in Fermion and Off-Shell Amplitudes

The determinants (and other functions) of infinite dimensional matrices which arise in fermion and off-shell amplitudes do so because the vertices in these models satisfy more complicated gauge conditions than the vertices in the (on-shell) meson sectors of the Veneziano and Neveu-Schwarz models. The fermion emission vertex

is given by [26,49]

$$V(z,k) = e^{zL_{-1}^{d}}\, W(z) :e^{-ik\cdot Q(z)}: \tag{2.1}$$

where $\frac{1}{2}k^2=0$, and $W(z)$ is given by

$$\begin{aligned} W(z) &= {}_b\langle 0|\exp\{A(z)+B(z)\}|0\rangle_d \\ A(z) &= \frac{1}{2}\sum_{r,s=\frac{1}{2}}^{\infty} b_r\cdot A_{rs}(z)\, b_s \\ B(z) &= -i\sum_{n=0}^{\infty}\sum_{r=\frac{1}{2}}^{\infty} d_{-n}\cdot B_{nr}(z)\, b_r\gamma^5 \end{aligned} \tag{2.2}$$

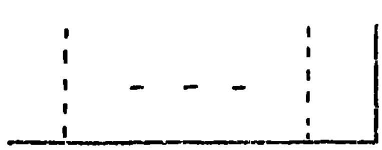
$W(z)$ is constructed so that it satisfies

$$\frac{\Gamma^{\mu}(x-z)}{\sqrt{(x-z)}}\, W(z) = W(z)\,\frac{H^{\mu}(x)}{\sqrt{x}} \tag{2.3}$$

which requires the infinite dimensional matrices A and B to be

$$\begin{aligned} A_{rs}(z) &= \frac{1}{2}\, z^{r+s}\,\frac{r-s}{r+s}\binom{-\frac{1}{2}}{r-\frac{1}{2}}\binom{-\frac{1}{2}}{s-\frac{1}{2}}(-1)^{r+s+1} \\ B_{nr}(z) &= \frac{1}{\sqrt{2}}\, z^{n-r}\binom{n-\frac{1}{2}}{r-\frac{1}{2}}(-1)^{n-r+\frac{1}{2}} \end{aligned} \tag{2.4}$$

Using this vertex, consider the state



which we denote by $\langle F\bar{F}|$. For D=10 and fermion mass =0 this satisfies the gauge conditions[49-51]

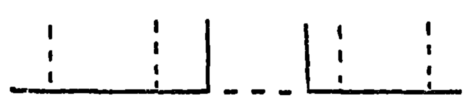
$$\langle F\bar{F}|L_{-n} = \langle F\bar{F}|\left(L_0 + \frac{5(n-1)}{8}\right) \qquad n \geq 1$$

$$\langle F\bar{F}|G_{-r} = \langle F\bar{F}|\sum_{s=\frac{1}{2}}^{\infty} \alpha_{rs} G_s \qquad r \geq \tfrac{1}{2} \tag{2.5}$$

where $\alpha_{rs} = (-1)^{r+s+1} \dfrac{r}{r+s} \begin{pmatrix} -\frac{1}{2} \\ r-\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ s-\frac{1}{2} \end{pmatrix}$

The first gauge condition is similar to that satisfied by the meson states. However the second is significantly different; G_{-r} acting on $\langle F\bar{F}|$ gives rise to an infinite number of 'reflected' G's. If we consider G_{-r} between two such states $\langle F\bar{F}| G_{-r} \ldots\ldots |F\bar{F}\rangle$, G_{-r} acting on $\langle F\bar{F}|$ gives rise to an infinite number of G's, each of which acting on $|F\bar{F}\rangle$ gives an infinite number of G's, and so on; the gauges are reflected back and forth.

Consider an amplitude



This consists of a meson propagator between two $|F\bar{F}\rangle$ states. As shown by Olive and Scherk[52] the propagator is not the usual meson propagator of the Neveu-Schwarz model. A correction factor is necessary, because of the reflection property of the G gauges, in order that the spectrum of physical states coupling in the residues of this propagator is just that of the Neveu-Schwarz model. The corrected propagator is

$$\int_0^1 \frac{dx}{x} \frac{x^{L_0 - \frac{1}{2}}}{\Delta(x)} \tag{2.6}$$

where the correction factor is

$$\Delta(x) = \det\left(1 - M(\lambda)^2\right) \tag{2.7}$$

and $$M_{rs}(\lambda) = \lambda^{r+s} \alpha_{rs} \qquad \left(\lambda = \sqrt{x}\right)$$

We note that M_{rs} is related to A_{rs} defined in (2.4) by

$$A_{rs} = \frac{1}{2}\left(M_{rs} - M_{sr}\right) \tag{2.8}$$

The value of $\Delta(x)$ was guessed by Schwarz and Wu[28] and proved by Corrigan et al[53], who evaluated the 4-fermion amplitude

This requires the computation of $\Delta(x)$, as well as the quantities $\det(1-A^2)$ and $v^T(1-A^2)^{-1}v$, where A is $A_{rs}(\lambda)$ and v_r is given by

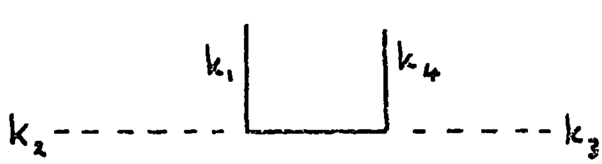
$$v_r = \frac{1}{\sqrt{2}}(-1)^{r-\frac{1}{2}}\lambda^r\binom{-\frac{1}{2}}{r-\frac{1}{2}} \qquad r = \tfrac{1}{2}, \tfrac{3}{2}, \ldots \tag{2.9}$$

The evaluation of these functions was carried out by an indirect method:related quantities were introduced, differential equations for these quantities were set up and solved, and these results were then used to calculate the required functions. In this way it was found[53] that

$$\Delta(x) = (1-\lambda^2)^{1/4} = (1-x)^{1/4}$$
$$v^T(1-A^2)^{-1}v = \frac{1}{\lambda}\left\{1-(1-\lambda^2)^{1/2}\right\} \tag{2.10}$$

A simpler method of calculating these functions will be described in the next section.

Another amplitude in which a corrected propagator must be used is the 2-fermion 2-meson amplitude written in the form



The fermion propagator in this case (omitting spinors) is[54]

$$\int_0^1 \frac{dx}{x} \frac{F_0 x^{L_0}}{K(\sqrt{x})(1-x)^{1/4}} \tag{2.11}$$

where $K(\sqrt{x})$ is the complete elliptic integral of the first kind. We can also calculate this amplitude in the form

in which case the propagator is uncorrected. Hence we expect the two fermion emission vertices to give rise to a factor which cancels the $K(\sqrt{x})$ in the denominator.

In calculating this amplitude we find that the following quantities arise

$$S_{mn}(\lambda) = (-\lambda)^{m+n} \frac{m}{m+n} \binom{-\frac{1}{2}}{m} \binom{-\frac{1}{2}}{n} \qquad m, n = 1, 2, ..$$

$$T_{mn}(\lambda) = \tfrac{1}{2}(S_{mn} - S_{nm})$$

$$w_n(\lambda) = \frac{1}{\sqrt{2}} (-\lambda)^n \binom{-\frac{1}{2}}{n} \tag{2.12}$$

and we obtain a factor

$$\delta = (1-r)(1+r)^{D-1} \det(1-T^2)^{D/2} \tag{2.13}$$

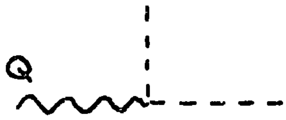
where

$$r = w^T (1-T^2)^{-1} w \quad ,$$

This can be evaluated to give[28]

$$\delta = K(\lambda)(1-\lambda^2)^{\frac{1}{2}-\frac{D}{8}}$$

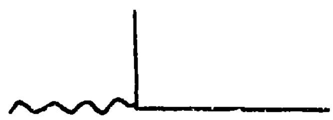
Correction factors such as those described above also occur in the off-shell model, because of the structure of the one-current vertex, which is similar to two fermion emission vertices. For example Corrigan and Fairlie[29] showed that the one-current state coupling to Neveu-Schwarz mesons



can be written as

$$\langle \pi(Q) | \int_0^1 \frac{dx}{x} \; x^{L_0 - \frac{1}{2}} \tag{2.14}$$

$\langle \pi(Q)|$ satisfies gauge conditionsidentical to those satisfied by the fermion $\langle F\bar{F}|$ state. Hence if we calculate the fermion-meson form factor



we find that it is given by

$$\langle \pi(Q) | \int_0^1 \frac{dx}{x} \; \frac{x^{L_0 - \frac{1}{2}}}{\Delta(x)} \, | F \bar{F} \rangle \tag{2.15}$$

where $\Delta(x)$ is the correction factor defined above. Evaluation of this gives a finite answer, in contrast with the divergent form factors of the Veneziano model[29].

2.2. Group-Theoretical Calculation

As pointed out in the first Chapter, the projective group plays an essential role in the theory of dual models. The underlying fields from which the vertices and gauge operators are constructed transform as irreducible representations of SU(1,1). The Virasoro gauge operators L_n constructed from these fields always contains a subalgebra isomorphic to that of SU(1,1). To see this in the cases of the c and d Fock space representations we actually have to redefine the gauge operators $L_0^{c,d}$; instead of the definitions in Chapter one, we take $L_0^{c,d} + \frac{D}{16}$.

The c and d gauge operators differ from their a and b counterparts in another respect: the SU(1,1) subalgebra does not annihilate the vacuum. In fact (with the redefined $L_0^{c,d}$) we have

$$L_1|0\rangle = 0 \qquad L_0|0\rangle = \frac{D}{16}|0\rangle \qquad L_{-1}|0\rangle \neq 0 \tag{2.16}$$

where $|0\rangle$ is the c or d space vacuum as appropriate. Since $L_{-1}|0\rangle \neq 0$, we can consider various expectation values with respect to $e^{\lambda L_{-1}}|0\rangle$, and this proves a useful thing to do.

States such as this arise naturally using the fermion emission vertex:

$$V(z,k)|0\rangle_b = e^{zL_{-1}^d}|0\rangle_d \tag{2.17}$$

and it has already been noted[49] that

$$e^{\lambda L_{-1}^d}|0\rangle_d = \exp\{\gamma^5\gamma w\cdot d^\dagger - \tfrac{1}{2}d^\dagger\cdot T d^\dagger\} \tag{2.18}$$

where $w(\lambda)$ and $T(\lambda)$ are as in (2.12). Similarly we find that[55]

$$e^{\lambda L^c_{-1}}|0\rangle_c = \exp\left(\tfrac{1}{2}c^{\dagger}\cdot P c^{\dagger}\right)|0\rangle_c$$

where
$$P_{rs} = -\left(N^{-1}MN\right)_{rs} \qquad (2.19)$$
$$N_{rs} = \sqrt{r}\,\delta_{rs}$$

Consider now the function

$$F_1 = {}_c\langle 0|e^{\lambda L^c_1}e^{\lambda L^c_{-1}}|0\rangle \qquad (2.20)$$

We can rewrite this using (2.19), and evaluate it using the identity[24,56]

$$\langle 0|e^{\frac{1}{2}\gamma A\gamma}e^{\frac{1}{2}\gamma^{\dagger}B\gamma^{\dagger}}|0\rangle = \left\{\det(1\mp AB)\right\}^{\mp\frac{D}{2}} \qquad (2.21)$$

The - or + sign is used according to whether γ satisfies canonical commutation or anti-commutation relations respectively. Applying this identity to (2.20) gives

$$F_1 = \left\{\det(1-P^2)\right\}^{-\frac{D}{2}} \qquad (2.22)$$

We can evaluate (2.18) in a second way, using the group property (easily checked for the two dimensional representation of the L's)

$$e^{\lambda L_1}e^{\mu L_{-1}} = e^{\frac{\mu}{1-\lambda\mu}L_{-1}}(1-\lambda\mu)^{-2L_0}e^{\frac{\lambda}{1-\lambda\mu}L_1} \qquad (2.23)$$

Using this together with (2.16) we find that

$$F_1 = (1-\lambda^2)^{-D/8} \qquad (2.24)$$

Since $P=-N^{-1}MN$, we have shown that

$$\det(1-M^2) = \det(1-P^2) = (1-\lambda^2)^{1/4} \qquad (2.25)$$

This is the correction factor $\Delta(x)$ for the propagator in the 4-fermion amplitude, as defined earlier.

The other quantities of interest involving the A and M matrices can be evaluated, given (2.25), provided that we also know

$$v^T (1 - M^2)^{-1} v \tag{2.26}$$

We can calculate this by evaluating in two ways the quantity

$$F_2 = {}_c\langle 0 | e^{\lambda L_1^c} \, x \cdot c \; y \cdot c^\dagger e^{\lambda L_{-1}^c} | 0 \rangle_c \tag{2.27}$$

where $\quad x_r = (Nv)_r \qquad y_r = (N^{-1} v)_r \qquad r = \frac{1}{2}, \frac{3}{2}, \ldots$

Evaluating F_2 using (2.19) and (2.21) gives

$$F_2 = - \{ \det (1 - M^2) \}^{-\frac{D}{2}} \; v^T (1 - M^2)^{-1} v \tag{2.28}$$

To calculate F_2 in a different way we use (2.23) to write

$$F_2 = (1 - \lambda^2)^{-D/8} \langle 0 | R(\lambda) \, x \cdot c \; y \cdot c^\dagger R^{-1}(\lambda) | 0 \rangle \tag{2.29}$$

$R(\lambda)$ is given by

$$R(\lambda) = e^{\frac{\lambda}{1-\lambda^2} L_1^c} \, e^{-\lambda L_{-1}^c} \tag{2.30}$$

and it is the representation in the c Fock space of the projective transformation

$$R(\lambda) : z \rightarrow \frac{1}{1 - \lambda^2} \; \frac{z - \lambda}{1 - \lambda z} \tag{2.31}$$

Now we use the fact that the off-shell field $R^\mu(z)$ transforms as a ($J=0, k=\frac{1}{2}$) representation of SU(1,1), as noted in section 1.4. This means that under a transformation

$$A : z \rightarrow \frac{az + b}{cz + d} \qquad ad - bc = 1 \tag{2.32}$$

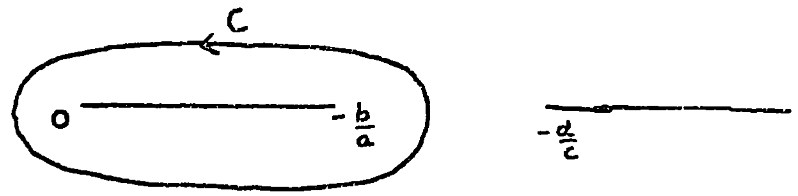
the Fock space representation of A acts on c_r as

$$A \, c_r \, A^{-1} = \sum_{s=-\infty}^{\infty} c_s \, D_{sr}^{(0\frac{1}{2})} (A) \tag{2.33}$$

Clearly $D^{(0,\frac{1}{2})}$ satisfies $D(A)D(B)=D(AB)$. We can obtain an integral expression for $D_{rs}^{(0,\frac{1}{2})}$ from (1.24) which is[57]

$$D_{rs}^{(0\frac{1}{2})}(A) = \sqrt{\frac{s}{r}} \oint_C \frac{dy}{2\pi i y}\, y^{-s}\,(ay+b)^{r}(cy+d)^{-r} \qquad r,s=\pm\tfrac{1}{2},.. \tag{2.34}$$

where the contour C is



Inserting $R^{-1}(\lambda)R(\lambda)$ between $x.c$ and $y.c^{\dagger}$ in (2.29), we can use (2.33) for the case $A=R(\lambda)$ and calculate F_2 to be

$$F_2 = (1-\lambda^2)^{-D/8}\,\frac{\lambda}{2\sqrt{1-\lambda^2}} \tag{2.35}$$

Thus, using (2.28) and (2.25), we have that

$$v^{T}(1-M^2)^{-1}v = \frac{\lambda}{2\sqrt{1-\lambda^2}} \tag{2.36}$$

From the above results we can calculate $v^{T}(1-A^2)^{-1}v$, which is in agreement with the result given in (2.10).

We can calculate the corresponding determinants and functions involving S, T and w by considering expectation values with respect to $e^{\lambda L_{-1}^{d}}|0\rangle_d$. For example consider

$$F_3 = {}_d\langle 0|e^{\lambda L_{+1}^{d}}\,\gamma^{\mu}\,e^{\lambda L_{-1}^{d}}|0\rangle_d \tag{2.37}$$

Using (2.23) we can write this in a form similar to (2.29). The fermion field $\Gamma^{\mu}(z)$ transforms as a $(J=-\frac{1}{2},k=0)$ representation of $SU(1,1)$, and γ^{μ} is the zero mode of Γ^{μ}, so that under the transformation $R(\lambda)$ we have that

$$R(\lambda)\,\gamma^{\mu}R(\lambda)^{-1} = \sum_{n=-\infty}^{\infty}\Gamma_n^{\mu}\,D_{n0}^{(-\frac{1}{2}0)}(R) \tag{2.38}$$

(Here we are writing Γ_n^μ for $i\sqrt{2}\,\gamma_5\, d_n^\mu$). Using this we find that

$$F_3 = D_{00}^{(-\frac{1}{2}0)}(R)\,\gamma^\mu\,(1-\lambda^2)^{-D/8} \tag{2.39}$$

$D_{mn}^{(-\frac{1}{2}0)}$ can be obtained in integral form as

$$D_{mn}^{(-\frac{1}{2}0)}(A) = \oint_C \frac{dy}{2\pi i y}\, y^{m-n}\left(a+\frac{b}{y}\right)^{m-\frac{1}{2}}(cy+d)^{-m-\frac{1}{2}} \tag{2.40}$$

where the contour C is the same as above. In particular we see that

$$D_{00}^{(-\frac{1}{2}0)}(R) = (1-\lambda^2)^{1/2}\,K(\lambda) \tag{2.41}$$

where $K(\lambda)$ is the complete elliptic integral of the first kind.

F_3 can also be calculated using (2.18) and (2.21), and we find that

$$(1+r)(1-r)^{D-1}\{\det(1-T^2)\}^{-\frac{D}{2}} = K(\lambda)\,(1-\lambda^2)^{\frac{1}{2}-\frac{D}{8}} \tag{2.42}$$

where $\quad r = w^T(1-T^2)^{-1}w$

This expression is more complicated than the previous cases because of the zero mode terms in (2.16), which are not present in (2.17).

We can also calculate

$$F_4 = {}_d\langle 0|\, e^{\lambda L_1^d}\, e^{\lambda L_{-1}^d}\, |0\rangle_d \tag{2.43}$$

in two ways, and in doing so we find that

$$(1-r)^D\{\det(1-T^2)\}^{\frac{D}{2}} = (1-\lambda^2)^{-\frac{D}{8}} \tag{2.44}$$

Finally we have the algebraic identity

$$\det(1-S^2) = (1-r^2)\det(1-T^2) \tag{2.45}$$

Using these results we can calculate the remaining quantities which arise in fermion and off-shell amplitudes. For example

$$\det(1-S^2) = K(\lambda)\sqrt{1-\lambda^2} \tag{2.46}$$

as previously calculated[28].

Hence we see that, using the properties of the projective group and its representations, we have been able to give simpler derivations of the functions of infinite dimensional matrices which arise in amplitudes than those previously available.

Chapter 3. Supersymmetry and Graded Lie Algebras

Introduction: In order to understand some of the features of the algebras underlying dual models, in particular the O(N) models, it is useful to discuss some preliminary ideas. The properties of the G gauges in the Neveu-Schwarz model have led to the consideration of theories other than dual models which possess a symmetry under transformations which mix up fermion and boson operators; such theories are called supersymmetric theories. The algebraic structure underlying this supersymmetry is now understood to be that of a graded Lie algebra (GLA).

Graded Lie algebras were the subject of mathematical interest some 15-20 years ago when their role in deformation theory was discussed. Interest in them has recently been revived, stimulated largely by the interest of the physics community, and a number of mathematical results concerning GLA's are now available.

In this Chapter we shall review the concept of supersymmetry and some of the features of GLA's, with a view to applying them to gain further insight into the algebraic structure of dual models.

3.1. Supersymmetry

Supersymmetry first arose in dual models with the extension of the Veneziano model to the Neveu-Schwarz model. The introduction of additional anti-commuting operators b_r leads to the definition of 'supergauge' operators G_r in addition to the Virasoro gauge operators L_n, as explained in Chapter one. These operators form a closed algebra

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{D}{8}\, n(n^2-1)\delta_{n+m,0} \\
[L_n, G_r] &= \left(\frac{n}{2}-r\right)G_{n+r} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{D}{2}\left(r^2-\frac{1}{4}\right)\delta_{r+s,0}
\end{aligned} \tag{3.1}$$

As we shall see later, these relations (without the c-number terms) are typical of GLA's.

The operators G_r have the property that operating on a boson state they give a fermion state, and vice-versa. (In this section only we use boson/fermion state to mean a state created by a commuting/anti-commuting operator.) For example,

$$G_{+\frac{1}{2}}\, \alpha_{-1}^{\mu}\, |0\rangle = -\, b_{-\frac{1}{2}}^{\mu}\, |0\rangle \tag{3.2}$$

Theories possessing an invariance under transformations which map boson states to fermion states and vice-versa are called supersymmetric theories.

Gervais and Sakita[58] realized that the supersymmetry invariance can be expressed as the invariance of an action under transformations in which boson and fermion fields are mixed up. For example, consider the action corresponding to free boson and fermion fields in two dimensions

$$I = \int dx_0\, dx_1 \left\{ -\frac{1}{2}(\partial\phi)^2 - \frac{i}{2}\,\bar{\psi}\gamma\cdot\partial\psi \right\} \tag{3.3}$$

where ψ is an anti-commuting Majorana 2-spinor. This is invariant (subject to suitable boundary conditions) under the infinitessimal conformal transformations

$$\begin{aligned} \phi &\to \phi + \varepsilon \cdot \partial \phi \\ \psi(x) &\to \begin{pmatrix} 1+\frac{1}{2}a' & 0 \\ 0 & 1+\frac{1}{2}b' \end{pmatrix} \psi(x) \end{aligned} \tag{3.4}$$

where

$$\varepsilon^{\mu}(x) = \left(\tfrac{1}{2}(a(x_+) + b(x_-)), \tfrac{1}{2}(a(x_+) - b(x_-)) \right)$$

together with the supergauge transformations

$$\begin{aligned} \phi &\to \phi + i\alpha\psi \\ \psi &\to \psi + \gamma_\mu \alpha \partial^\mu \phi \end{aligned} \tag{3.5}$$

where α is an anti-commuting Majorana spinor that satisfies

$$\gamma^\mu \gamma^\nu \partial_\mu \alpha = 0 \tag{3.6}$$

The Lagrangian itself is not invariant, under supergauge transformations, but the action I is.

Volkov and Akulov[59] extended the idea of supersymmetry to four dimensions. They considered actions which are invariant under the Poincare group together with supergauge transformations which are generalizations of (3.5). The Lagrangians are made up from ordinary fields which depend on the space-time variables x^μ. A better formalism was soon developed[60] which enabled the transformations involved to be expressed more compactly: this involves the introduction of superfields.

Superfields are functions of the space-time variables x^μ together with the further variables θ_i. These θ's have the properties described in (1.37): they anti-commute amongst themselves, and commute with x^μ. In particular $(\theta_i)^2=0$; hence a superfield $\Phi(x^\mu, \theta_i)$ can always be written as a polynomial in θ_i, since if we have n θ_i, the monomial $\theta_{i_1} \dots \theta_{i_{n+1}}$

must vanish. The coefficient of each $\theta_{i_1}\ldots\theta_{i_r}$ term is a function of x^μ only.

The basic supersymmetry transformations are obtained using 4 θ's, chosen so that $\{\theta_i\}$ forms a Majorana spinor. It is often more convenient to use the notation of dotted and undotted spinors(61), where we consider a 4-spinor to be written as a complex 2-spinor $\theta=(\theta_a)$ together with its conjugate spinor $\bar{\theta}=(\bar{\theta}_{\dot{a}})$, $a,\dot{a}=1,2$.

The transformations that we consider are transformations in the 8-dimensional 'superspace' $(x^\mu, \theta_a, \bar{\theta}_{\dot{a}})$, generated by the generators of the Poincare group $\{P^\mu, J^{\mu\nu}\}$, together with four supersymmetry generators $Q_a, \bar{Q}_{\dot{a}}$ which satisfy anti-commutation relations amongst themselves and commutation relations with P^μ, $J^{\mu\nu}$ (60). We can define three functions of $x_\mu, \theta_a, \bar{\theta}_{\dot{a}}$ by

$$\Phi(x,\theta,\bar{\theta}) = \exp(-ix\cdot P + i\theta Q + i\bar{Q}\bar{\theta})$$
$$\Phi_1(x,\theta,\bar{\theta}) = \exp(-ix\cdot P + i\theta Q)\exp(i\bar{Q}\bar{\theta})$$
$$\Phi_2(x,\theta,\bar{\theta}) = \exp(-ix\cdot P + i\bar{Q}\bar{\theta})\exp(i\theta Q) \qquad (3.7)$$

Using the algebra of $\{P^\mu, J^{\mu\nu}, Q_a, \bar{Q}_{\dot{a}}\}$ we can calculate the effects of transformations generated by these operators on the functions in (3.7). For example, a supergauge transformation acting on Φ is given by

$$e^{i\varepsilon Q + i\bar{Q}\bar{\varepsilon}}\,\Phi(x_\mu, \theta_a, \bar{\theta}_{\dot{a}}) = \Phi(x_\mu + i\theta\sigma_\mu\bar{\varepsilon} - i\varepsilon\sigma_\mu\bar{\theta},\ \theta+\varepsilon,\ \bar{\theta}+\bar{\varepsilon}) \qquad (3.8)$$

where

$$(\sigma_\mu)_{a\dot{b}} = (1, \sigma_i) \qquad (\bar{\sigma}_\mu)_{\dot{a}b} = (1, -\sigma_i)$$

and σ_i are the usual Pauli matrices.

The supergauge transformations for Φ, Φ_1, Φ_2 differ slightly; the other transformations have the same effects on all three expressions. These transformation laws, such as (3.8), can be abstracted from the definitions (3.7), and superfields Φ, Φ_1, Φ_2 defined by their transformation properties alone.

Given a superfield Ψ, transforming as one of the types in (3.7), 'covariant' derivatives D can be defined with the property that $D\Psi$ is a superfield of the same type as Ψ. For example, the covariant derivatives acting on type Φ_1 are

$$D = \frac{\partial}{\partial\theta} + 2i\sigma_\mu \bar{\theta}\,\partial^\mu \qquad \bar{D} = \frac{\partial}{\partial\bar{\theta}} \tag{3.9}$$

These covariant derivatives can be used to simplify the form of the superfield under consideration. In the case of Φ_1 we could impose the constraint

$$\bar{D}\,\Phi_1(x,\theta,\bar{\theta}) = 0 \tag{3.10}$$

Since this condition is invariant under all of the transformations considered, Φ_1 subject to (3.10) is still a superfield. However it is independent of $\bar{\theta}$, and may be written as

$$\Phi_1(x,\theta) = \phi(x) + \theta_a \psi_a(x) + \varepsilon_{ab}\theta_a\theta_b F(x) \tag{3.11}$$

Internal symmetrics can be incorporated into the supersymmetry scheme by attaching indices corresponding to a representation of some symmetry group to the θ and $\bar{\theta}$'s[62]. In this case there are more covariant derivatives, and more possibilities for imposing constraints such as (3.10).

The superfield formalism can easily be adapted to the two dimensional case(43,63,64). Now there are two translation generators P^μ, one Lorentz generator J^{01}, and two supersymmetry generators S_1, S_2 such that $S=\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$ is a Majorana spinor (there

is no need to use dotted spinor notation in this simple case). Using a representation of the γ -matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.12}$$

we can take the charge conjugation matrix $C = \gamma_1$. The Majorana constraint on a spinor α then gives

$$\bar{\alpha} = (-\alpha_2, \alpha_1) \tag{3.13}$$

The superspace (x,θ) now consists of x_μ, $\mu=0,1$, and two θ 's which are the components of a Majorana spinor $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$

The group element can be written in analogy with (3.7) as

$$\begin{aligned} \Phi(x,\theta) &= \exp(-ix\cdot P + i\bar{\theta}S) \\ &= \exp(-ix\cdot P - i\theta_2 S_1 + i\theta_1 S_2) \end{aligned} \tag{3.14}$$

The generators form a closed algebra:

$$\begin{aligned} &[P^\mu, P^\nu] = [P^\mu, S_a] = 0 \qquad \{S_a, \bar{S}_b\} = 2(\gamma\cdot P)_{ab} \\ &[J^{\mu\nu}, P^\lambda] = i g^{\nu\lambda} P^\mu - i g^{\mu\lambda} P^\nu \\ &[J^{\mu\nu}, S_a] = -\tfrac{1}{2}(\sigma^{\mu\nu})_{ab} S_b \end{aligned} \tag{3.15}$$

where $\sigma^{\mu\nu} = \frac{1}{2} i [\gamma^\mu, \gamma^\nu]$. We note in particular that $\{S_1, S_2\} = 0$. This means that the superfields Φ_1, Φ_2, defined in the same way as for the four dimensional case(3.7), are actually equal to Φ: there is only one type of superfield in two dimensions.

The relations (3.15) can be used to express the transformations generated in terms of transformations on (x_μ, θ_a), as in (3.8). These are most conveniently expressed by introducing "light-cone" co-ordinates $x_\pm = x_0 \pm x_1$. Then under a Lorentz transformation $e^{i\omega J^{01}}$

$$\begin{pmatrix} x_+ \\ x_- \end{pmatrix} \longrightarrow \begin{pmatrix} e^{\omega} & 0 \\ 0 & e^{-\omega} \end{pmatrix} \begin{pmatrix} x_+ \\ x_- \end{pmatrix}$$
$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{-\frac{1}{2}\omega} & 0 \\ 0 & e^{\frac{1}{2}\omega} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \tag{3.16}$$

Under a supergauge transformation $e^{i\bar{\alpha}S}$, where α is a 2-spinor, we find that

$$\begin{aligned} x_\mu &\rightarrow x_\mu - i\bar{\alpha}\gamma_\mu\theta \\ \theta &\rightarrow \theta + \alpha \end{aligned} \tag{3.17}$$

i.e.
$$\begin{aligned} x_+ &\longrightarrow x_+ + i\alpha_2\theta_2 & \theta_2 &\rightarrow \theta_2 + \alpha_2 \\ x_- &\longrightarrow x_- - i\alpha_1\theta_1 & \theta_1 &\longrightarrow \theta_1 + \alpha_1 \end{aligned}$$

We see that the two sets of variables (x_+ , θ_2), and (x_-, θ_1) transform amongst themselves; this is because $\{J^{01}, P_-, S_1\}$ and $\{J^{01}, P_+, S_2\}$ are both subalgebras of the algebra (3.15).

The above algebra may be extended to consider conformal transformations on x_μ , rather than just Lorentz transformations. Conformal transformations are defined by

$$(x_0, x_1) \longrightarrow (x_0', x_1')$$

where
$$\frac{\partial x_0'}{\partial x_0} = \frac{\partial x_1'}{\partial x_1} \qquad \frac{\partial x_0'}{\partial x_1} = \frac{\partial x_1'}{\partial x_0} \tag{3.18}$$

In terms of $x_\pm$, these differential equations are satisfied by

$$x_+' = f(x_+) \qquad x_-' = g(x_-) \tag{3.19}$$

where f and g are arbitrary functions. We also require that θ transforms as a conformal spinor; that is, under the transformation (3.19), θ must transform as

$$\theta_1 \to \left(g'(x_-)\right)^{\frac{1}{2}} \theta_1 \qquad \theta_2 \to \left(f'(x_+)\right)^{\frac{1}{2}} \theta_2 \tag{3.20}$$

The infinitessimal parameter α in the supergauge transformations must satisfy (3.6), which gives

$$\alpha(x) = \begin{pmatrix} \alpha_1(x_-) \\ \alpha_2(x_+) \end{pmatrix} \tag{3.21}$$

With this proviso the supergauge transformations for infinitessimal α are just those given in (3.17), and the conformal and supergauge transformations form a closed system.

We see that again each set of variables (x_+, θ_2), (x_-, θ_1) transforms within itself; hence we may consider superfields which depend on (x_+, θ_2) only. If we put $z = x_+$ and $\theta = i\theta_2$, the transformations in infinitessimal form are

conformal:

$$z \to z + u(z)$$
$$\theta \to \left(1 + \tfrac{1}{2} u'(z)\right) \theta$$

supergauge: (3.22)

$$z \to z + \alpha(z)\theta$$
$$\theta \to \theta + \alpha(z)$$

These are just the transformations that are generated by the infinite dimensional algebra $\{L_n, G_r\}$ of the Neveu-Schwarz model[42], as will be explained in Chapter four.

In four dimensions also the conformal algebra may be extended to a supersymmetry algebra[65]. In this case it is necessary to add a further commuting-type generator, corresponding to γ^5 transformations, to the 15 generators of the conformal group, together with 8 supersymmetry generators. The 4-spinor parameter $\alpha(x)$ of a supergauge transformation must be a Majorana spinor satisfying a generalization of (3.6)

$$\left(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu - g_{\mu\nu} \gamma\cdot\partial\right) \alpha(x) = 0 \tag{3.23}$$

This requires α to have the form

$$\alpha(x) = \alpha_0 + \gamma \cdot x\, \alpha_1 \tag{3.24}$$

where α_0 and α_1 are constant 4-spinors, giving 8 parameters corresponding to the 8 supersymmetry generators. The details of this 24 dimensional algebra are given in Corwin, Ne'eman and Sternberg[66].

3.2. Graded Lie Algebras

The algebras described in the previous section are all examples of graded Lie algebras (GLA's - also called pseudo Lie algebras[67] and Lie superalgebras[68]). In the next two sections we shall describe some features of GLA's, following mainly the papers of Nahm, Scheunert and Rittenberg [67,69] and Pais and Rittenberg[70].

A GLA is a graded (non-associative) algebra whose multiplication behaves partly as a commutator and partly as an anti-commutator. They were first studied in connection with work on deformation theory[71] and Hopf algebras[72], and have recently been further studied in view of their occurrence in particle physics.

In order to define a GLA, we recall the definition of graded vector spaces and graded algebras[73]. A vector space V is graded over the integers Z if, for each $n \in Z$, there is a subspace V_n of V, and

$$V = \bigoplus_{i=0}^{n} V_n \qquad (3.25)$$

In fact we are only interested in vector spaces graded over Z_2, the two element group $\{0,1\}$ of the integers mod 2; so that V has the form

$$V = V_0 \oplus V_1 \qquad (3.26)$$

V_0 is called the even subspace, and V_1 the odd subspace. The grading of V is described by a linear map $\gamma_V : V \to V$, the grading automorphism, defined by

$$\gamma_V(x) = (-1)^{\alpha} x \qquad x \in V_\alpha \qquad (3.27)$$

Given two graded vector spaces V and V^1, there is a natural grading that can be defined on $L(V,V^1)$, the space of linear maps from $V \longrightarrow V^1$. Even maps are defined as those which map $V_0 \to V_0{}^1$, $V_1 \to V_1{}^1$, and odd maps $V_0 \to V_1{}^1, V_1 \to V_0{}^1$. In particular we have the graded vector space $L(V)=L(V,V)$ of linear maps of V into itself, and we can write $L(V)=L_0 \oplus L_1$ where

$$L_0(V) = \{g \in L(V) \mid g(V_0) \subset V_0, g(V_1) \subset V_1\}$$

$$L_1(V) = \{g \in L(V) \mid g(V_0) \subset V_1, g(V_1) \subset V_0\} \qquad (3.28)$$

In a similar way we can define a grading on the space $B(V,V^1)$ of bilinear forms on V and V^1.

An algebra A is graded if it is graded as a vector space

$$A = A_0 \oplus A_1 \qquad (3.29a)$$

and its multiplication satisfies

$$A_\alpha A_\beta \subset A_{\alpha+\beta} \qquad \alpha, \beta \in Z_2 \qquad (3.29b)$$

that is, the usual rule for the multiplication of even and odd elements is satisfied.

A graded Lie algebra is a graded (non-associative) algebra $A=A_0 \oplus A_1$, whose multiplication, denoted by a bracket $\langle\ \rangle$, satisfies the identities

$$\langle X,Y\rangle = (-1)^{\alpha\beta}\langle Y,X\rangle$$

$$\langle X,\langle Y,Z\rangle\rangle = \langle\langle X,Y\rangle,Z\rangle + (-1)^{\alpha\beta}\langle Y,\langle X,Z\rangle\rangle \qquad (3.30)$$

where $X \in A_\alpha$, $Y \in A_\beta$, $Z \in A$. The second identity is a generalization of the familiar Jacobi identity of ordinary Lie algebras. We note that A_0 equipped with the rules (3.30) is an ordinary Lie algebra.

As an example of a GLA, take A to be any graded associative algebra, and define

$$\langle X,Y\rangle = [X,Y] \qquad X\in A_\alpha, Y\in A_\beta \quad (\alpha,\beta)\neq(1,1)$$
$$\langle X,Y\rangle = \{X,Y\} \qquad X\in A_1, Y\in A_1 \tag{3.31}$$

This is a GLA. In particular we can consider a graded vector space V, and take the associative algebra L(V) of linear maps of V into itself. Equipped with the rules (3.31), L(V) is then a GLA which we denote by pl(V); it corresponds to gl(V) in the theory of ordinary Lie algebras. If V is n dimensional, then pl(V) is the algebra of n x n matrices, with grading induced from that of V as in (3.28), and multiplication as in (3.31).

Given a GLA A, and a graded vector space V, a (graded) representation of A in V is a homomorphism of A into pl(V) (just as with an ordinary Lie algebra G we would have a homomorphism of G into gl(V)). In particular the map $ad_A : A \longrightarrow pl(A)$ defined by

$$ad_A(X)\,(Y) = \langle X,Y\rangle \qquad X,Y\in A \tag{3.32}$$

is a representation of the algebra A in the vector space A, called the adjoint representation.

We note that, if U is an A_0-invariant subspace of A

$$\langle A_0, U\rangle \subset U \tag{3.33}$$

then the adjoint representation ad_A induces a representation of A_0 in U, called the adjoint representation of A_0 in U, ad_U, which is defined by

$$ad_U(a)\,(u) = \langle a,u\rangle \qquad a\in A_0, u\in U \tag{3.34}$$

In particular, because of the grading condition

$$\langle A_\alpha, A_\beta\rangle \subset A_{\alpha+\beta} \tag{3.35}$$

we see that A_0 and A_1 are A_0 invariant subspaces; hence we have adjoint representations of A_0 in A_0 and A_1. The adjoint representation of A_0 in A_0 is just the usual adjoint representation of A_0 considered as an ordinary Lie algebra.

Given a representation of a GLA A in a representation space V, we can associate with this representation an invariant bilinear form ϕ_V on A defined by

$$\phi_V(a,b) = tr\,(\gamma_V a_V b_V) \qquad a, b \in A \tag{3.36}$$

where a_V, b_V are the representations of a,b in V, and γ_V was defined in (3.27). We denote the "generalized trace" in (3.36) by $Tr(a_V b_V)$. If V is finite dimensional, we choose a basis $\{v_1, \ldots \ldots v_m\}$ of V_0 and $\{v_{m+1}, \ldots, v_{m+n}\}$ of V_1. a_V and b_V are then (m + n) x (m + n) matrices, and if we write

$$a_V b_V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{3.37}$$

where $\alpha, \beta, \gamma, \delta$ are block matrices, α is m x m etc., then the bilinear form (3.36) is given by

$$\phi_V(a,b) = Tr\,(a_V b_V) = tr\,\alpha - tr\,\delta \tag{3.38}$$

If A is finite dimensional, we can consider the adjoint representation and define an even invariant bilinear form on A as above, called the Killing form of A.

In the remainder of this Chapter we consider only finite dimensional GLA's, and we find it convenient to write $G = A_0$, $U = A_1$, so that $A = G \oplus U$. Taking a linearly independent set of generators $\{X_\mu\} = \{Q_m, V_\alpha\}$ where $\{Q_m : m=1,\ldots,d_0\}$ span G and $\{V_\alpha : \alpha = 1,\ldots,d_1\}$ span U, we can write the multiplication $\langle\ \rangle$ of (3.34) as

$$[Q_m, Q_n] = f_{mn}^{\ p} Q_p$$
$$[Q_m, V_\alpha] = F_{m\alpha}^{\beta} V_\beta$$
$$\{V_\alpha, V_\beta\} = A_{\alpha\beta}^{m} Q_m \qquad (3.39)$$

The adjoint representations of G in G and U, ad_G and ad_U respectively, are given by their actions on the generators of G:

$$ad_G : Q_m \longrightarrow (Q_m)_n^{\ p} = f_{mn}^{\ p}$$
$$ad_U : Q_m \longrightarrow (Q_m)_\alpha^{\ \beta} = F_{m\alpha}^{\beta} \qquad (3.40)$$

The adjoint representation of A maps the generators of A into $(d_0 + d_1) \times (d_0 + d_1)$ matrices

$$ad_A : Q_m \longrightarrow \begin{pmatrix} M_m & 0 \\ 0 & N_m \end{pmatrix}$$
$$ad_A : V_\alpha \longrightarrow \begin{pmatrix} 0 & X_\alpha \\ Y_\alpha & 0 \end{pmatrix} \qquad (3.41)$$

where M, N, X, Y are block matrices, M_m is $d_0 \times d_0$ etc.(70).

The Killing form ϕ is given by its action on the generators X_μ of A, and we define the metric

$$g_{\mu\nu} = \phi(X_\mu, X_\nu) = Tr(ad_A X_\mu \cdot ad_A X_\nu) \qquad (3.42)$$

Then, in the notation of (3.39),

$$g_{mn} = g_{nm} = h_{mn} - F_{m\alpha}^{\beta} F_{n\beta}^{\alpha}$$
$$g_{\alpha\beta} = F_{m\alpha}^{\gamma} A_{\beta\gamma}^{m} - F_{m\beta}^{\gamma} A_{\alpha\gamma}^{m} = -g_{\beta\alpha}$$
$$g_{m\alpha} = g_{\alpha m} = 0 \qquad (3.43)$$

where h_{mn} is the usual metric tensor for the ordinary Lie algebra G.

$$h_{mn} = f_{mq}^{\ p} f_{np}^{\ q} \qquad (3.44)$$

A quadratic Casimir operator K may be defined by

$$K = g^{\mu\nu} X_\mu X_\nu \tag{3.45}$$

(where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$) which commutes with all the generators X_μ .

We can now ask whether the types of GLA's can be classified in some way. One way to do so is to classify the simple GLA's: a GLA is simple if it contains no non-trivial (graded) ideals. The simple GLA's have all been classified (68,69,74) and in the next section we shall give the classification.

We note here one important difference between simple GLA's and simple Lie algebras. For an ordinary Lie algebra with metric tensor h_{mn}, G simple => det h $\neq$ 0. For simple GLA's no corresponding statement can be made about $g_{\mu\nu}$. In fact, there exist simple GLA's which have $g_{\mu\nu}$ identically equal to zero.

3.3. Simple Graded Lie Algebras

In order to classify the simple GLA's, it is convenient to consider two cases seperately, as the adjoint representation of G in U is or is not completely reducible.

a) ad_U completely reducible

In listing this class of simple GLA's we follow Scheunert, Nahm and Rittenberg[69]. We do not reproduce their arguments, but list some results which are useful in carrying out their classification.

The adjoint representation ad_U of G in U is completely reducible if and only if G is reductive, that is if

$$G = G_0 \times G' \tag{3.46}$$

where G_0 is the centre of G: $\langle G_0, G\rangle = 0$, and G' is semi-simple; note that $G' = \langle G, G\rangle$.

Given that ad_U is completely reducible, then either ad_U is irreducible or it is the direct sum of two irreducible representations. In this second case, the odd subspace U can be written as the direct sum of two G-invariant subspaces

$$\begin{aligned} &U = U' \oplus U'' \\ \text{where} \quad &\langle U', U'\rangle = \langle U'', U''\rangle = 0 \\ &\langle U', U''\rangle = G \end{aligned} \tag{3.47}$$

If the centre G_0 of G is non-trivial, then it is one dimensional, and the representation ad_U is reducible. Furthermore, there exists a (unique) element $e \in G_0$ such that

$$\begin{aligned} \langle e, u'\rangle &= u' \qquad &\forall\, u' \in U' \\ \langle e, u''\rangle &= -u'' \qquad &\forall\, u'' \in U'' \end{aligned} \tag{3.48}$$

If the ordinary Lie algebra G is semi-simple, then the generators of G can be written in a standard Cartan basis(75)

$$[H_i, H_j] = 0 \qquad i,j = 1,\dots,\ell$$
$$[H_i, E_a] = a_i E_a$$
$$[E_a, E_{-a}] = a^i H_i$$
$$[E_a, E_b] = N_{ab} E_{a+b} \qquad (a+b \neq 0) \qquad (3.49)$$

where l is the rank of the algebra; the a_i and a^i are the covariant and contravariant roots; N_{ab} is a normalization factor. If in addition the metric tensor $g_{\mu\nu}$ of the GLA A is non-degenerate, the basis (3.49a) can be extended to a basis of the algebra A, with additional generators $V_{\pm\alpha}$ (which span the odd subspace U) satisfying

$$[H_i, V_\alpha] = \alpha_i V_\alpha$$
$$[E_a, V_\alpha] = \delta_{a+\alpha} M_{\beta\alpha} V_\beta$$
$$\{V_\alpha, V_{-\alpha}\} = f_{\alpha,-\alpha}\, \alpha^i H_i$$
$$\{V_\alpha, V_\beta\} = -\delta_{\alpha+\beta} f_{\alpha,-\alpha} M_{-\alpha\beta} \quad (\alpha+\beta \neq 0) \qquad (3.49b)$$

The details of δ, f, M are given by Pais and Rittenberg(70). The α_i are the weights of the adjoint representation of G on U; to each weight α there corresponds a root a such that

$$a = 2\alpha \qquad \{V_\alpha, V_\alpha\} \neq 0 \qquad (3.50)$$

Using these results, all the simple GLA's with ad_U completely reducible can be discovered(69). We give a list of the resulting classes of simple GLA's in terms of subalgebras of pl(n,m), the graded algebra of (n + m) x (n + m) matrices, which in block form are

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (3.51)$$

where A is an n x n matrix etc,; the multiplication $\langle\rangle$ is given by

$$\langle X, X' \rangle = \begin{pmatrix} AA'-A'A+BC'+B'C & BD'-B'D+AB'-A'B \\ CA'-C'A+DC'-D'C & DD'-D'D+CB'+C'B \end{pmatrix} \tag{3.52}$$

1) The special linear GLA spl(n,m) is defined by

$$spl(n,m) = \{ X \in pl(n,m) \mid Tr(X) = 0 \} \tag{3.53}$$

Tr(X) is the generalized trace defined in (3.36), so that with X as in (3.51), we have

$$Tr(X) = 0 \Rightarrow tr\, A = tr\, D \tag{3.54}$$

The Lie algebra G of spl(n,m) is sl(n) x sl(n) x U(1).

For n ≠ m, spl(n,m) is simple. When n=m, spl(n,n) has a non-trivial (one dimensional) centre generated by the 2n x 2n unit matrix:

$$Z_n = \left\{ \lambda \begin{pmatrix} 1_n & 0 \\ 0 & 1_n \end{pmatrix} \right\} \tag{3.55}$$

(where λ is a scalar). The quotient algebra

$$spl(n,n)/Z_n \tag{3.56}$$

is simple.

2) The orthosymplectic GLA OSp(2p,m)

Define the 2p x 2p matrix

$$E = \begin{pmatrix} 0 & 1_p \\ -1_p & 0 \end{pmatrix} \tag{3.57}$$

Then the subalgebra of pl(2p,m) consisting of all matrices X of the form (3.51) satisfying

$$\begin{aligned} A^T E + E A &= 0 \\ D^T &= -D \\ C &= B^T E \end{aligned} \tag{3.58}$$

is simple, and has Lie algebra $G=Sp(2p) \times O(m)$. It is called the orthosymplectic GLA $OS(2p,m)$ $(p,m \geq 1)$. We note that $OSp(2,2)$ is isomorphic to $spl(2,1)$.

3) Consider the subalgebra of $pl(n,n)$, $n \geq 3$, consisting of matrices X satisfying

$$A^T = -D \qquad B^T = B \qquad C^T = -C$$
$$tr(A) = 0 \tag{3.59}$$

This is a simple GLA, which we call $b(n)$. Its Lie algebra is $G=sl(n)$.

4) Consider the subalgebra of $pl(n,n)$, $n \geq 3$, called d_n and defined by

$$d_n = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \middle| A \in gl(n), B \in sl(n) \right\} \tag{3.60}$$

The centre of d_n is z_n as in (3.55). The quotient algebra d_n/z_n is simple, and has Lie algebra $G=sl(n)$.

5) There are also some exceptional simple GLA's, analogous to the exceptional Lie algebras. These consist of:

i) A one parameter family of 17 dimensional simple GLA's, whose Lie algebra is $sl(2) \times sl(2) \times sl(2)$

ii) A 31 dimensional simple GLA whose algebra is $sl(2) \times G_2$

iii) A 40 dimensional simple GLA whose Lie algebra is $sl(2) \times O(7)$

These are all of the simple GLA's of this type; they are called the classical simple GLA's, in analogy with ordinary Lie algebras. They can be listed in another way, which reflects the use of the results quoted at the beginning of this section by Scheunert et al[69]. We write $A=G \oplus U$, and ignore the exceptional algebras.

I. G simple

There are 2 cases:

i) Killing form non-degenerate: $A \cong OSp(2p,1)$ $p \geq 1$

ii) Killing form degenerate:

U irreducible: $A \cong d_n/z_n$ $n \geq 3$

U reducible: $A \cong b(n)$ $n \geq 3$

II. G semi-simple, but not simple

There are 2 cases :

i) U irreducible: $A \cong OSp(2p,m)$ $p \geq 1,\ m \geq 3$

ii) U reducible: $G = G_0 \times G$ by (3.46)

G' not simple: $A \cong spl(n,m)$ $n,m \geq 2$

or $A \cong spl(n,n)/z_n$ $n \geq 2$

G' simple: $A \cong OSp(2p,2)$ $p \geq 1$

or $A \cong spl(2,1)$

All of the GLA's which are of interest in physics to date are related to classical simple GLA's. For completeness we mention very briefly the other type of simple GLA.

b) ad_U not completely reducible

The above GLA's are all matrix algebras, and correspond to the classical Lie algebras. There are further simple GLA's defined in terms of the derivations and differential forms on a Grassman algebra[68]. If A is the Grassman algebra on the n variables $x_1,\ldots,x_n$ with the natural grading, the (graded) algebra of derivations on A, W(n), can be written as the set of elements $\sum_i p_i \partial_i$, where $p_i \in A$, and $\partial_i(x_j) = \delta_{ij}$. W(n) is a simple GLA for $n > 1$.

Two distinct algebras, Ω and S, of differential forms on A can be defined, generated by the anti-commuting (commuting) defferentials dx_i (δx_i):

$$\Omega : dx_1, \dots, dx_n \qquad S : \delta x_1, \dots, \delta x_n \tag{3.61}$$

The grading on A is extended to Ω and S by putting dx_i odd, δx_i even. Every derivation on A induces derivations on Ω and S; three further sets of GLA's can be defined as subalgebras of the algebras of the derivations on Ω and S. Further details are given by Kats[68].

3.4. Graded Lie Algebras and Physics

As we have stated in section 3.1., supersymmetry is an example of a GLA in physics, and we shall see below which are the relevant GLA's. However the earliest appearance of GLA's in physics was the f,d algebra which arises in the consideration of representations of SU(3)[76].

The generators of SU(3) can be represented by 3 x 3 traceless matrices $\frac{1}{2}\lambda_i$, i=1,...,8. which satisfy

$$\begin{aligned} \left[\tfrac{1}{2}\lambda_i, \tfrac{1}{2}\lambda_j\right] &= i f_{ijk}\left(\tfrac{1}{2}\lambda_k\right) \\ \{\lambda_i, \lambda_j\} &= \tfrac{4}{3}\delta_{ij}\mathbb{1} + 2 d_{ijk}\lambda_k \end{aligned} \qquad (3.62)$$

The f_{ijk} are the structure constants of SU(3). If we write matrices f_i, d_i as

$$(f_i)_{jk} = f_{ijk} \qquad (d_i)_{jk} = d_{ijk} \qquad (3.63)$$

then the adjoint representation F_i of SU(3) is given by $F_i = -if_i$, which satisfies

$$[F_i, F_j] = i f_{ijk} F_k \qquad (3.64)$$

We can also define $D_i = d_i$, and these satisfy

$$[D_i, F_j] = i f_{ijk} D_k \qquad (3.65)$$

As they stand the D's and F's do not form a closed algebra. However, if we redefine F_i and D_i to be the 16 x 16 matrices

$$F_i = \begin{pmatrix} -if_i & 0 \\ 0 & -if_i \end{pmatrix} \qquad D_i = \begin{pmatrix} 0 & d_i \\ \underset{if_i}{\cancel{d_i}} & 0 \end{pmatrix} \qquad (3.66)$$

then we do have a closed algebra:

$$\begin{aligned} [F_i, F_j] &= i f_{ijk} F_k \\ [D_i, F_j] &= i f_{ijk} D_k \\ \{D_i, D_j\} &= -d_{ijk} F_k \end{aligned} \qquad (3.67)$$

We recognize (3.66) as the adjoint representation of the GLA (3.67). The algebra (3.67) is the simple GLA d_3/z_3, as in section 3.3.

The f,d algebra can be generalized to SU(n), and such algebras are known as the Gell-Mann-Michel-Radicati algebras(77). They are precisely the simple GLA's d_n/z_n, $n \geq 3$.

Now consider the four (space-time) dimensional supersymmetry algebras. The 24 dimensionsl algebra of Wess and Zumino(65) based on an extension of the conformal transformations is(after complexification) precisely the simple GLA spl(4,1)(74). The 14 dimensional algebra based on an extension of the Poincare group(59) is not a simple GLA; it may however be obtained from the simple 14 dimensional GLA OSp(4,1) by an Inönu-Wigner type contraction.

The ordinary Lie algebra of OS(4,1) is (isomorphic to) the de Sitter algebra SO(3,2). The 10 generators of this algebra can be represented by(78)

$$M_{ab} = \gamma_a \gamma_b = -M_{ba} \qquad (a \neq b;\ a,b = 1,\dots,5) \tag{3.68}$$

(where $\gamma_4 = i\gamma_0$). If we define new generators(74)

$$\begin{aligned} \bar{M}_{ab} &= M_{ab} \qquad a,b = 1,\dots,4 \\ \bar{M}_{a5} &= \lambda M_{a5} \end{aligned} \tag{3.69}$$

and perform a contraction by letting $\lambda \to 0$, we obtain the Poincare algebra, with $M_{a5} \to P^\mu$, and the other generators $\to$ Lorentz generators.

If we now take the 4 odd generators of OSp(4,1), F_α say, and define new generators(74)

$$\bar{F}_\alpha = \sqrt{\lambda}\ F_\alpha \tag{3.70}$$

then in the limit $\lambda \to 0$, (3.69) and (3.70) together give the 14 dimensional supersymmetry algebra.

The two (space-time) dimensional supersymmetry of dual models is infinite dimensional, because of the special properties of the conformal group in two dimensions; however, it has a finite subalgebra spanned by $\{L_0, L_{\pm 1}, G_{\pm\frac{1}{2}}\}$. This algebra is the simple GLA OSp(2,1). We shall see that the O(N) algebras of Ademollo et al[42] are also related to simple GLA's.

Finally, in the graded Riemannian geometry of supergauge theories[79], one considers manifolds that can be locally mapped onto a flat (graded) space with orthosymplectic metric.

Hence we see that all of the GLA's that have arisen in physics to date are either classical simple GLA'S, or are closely related to them.

Chapter 4. Representations of Orthosymplectic Algebras

Introduction: In this Chapter we are going to explore some aspects of the algebras underlying the known dual models. The gauge algebra of the Veneziano model is (up to c-number terms) the infinite algebra corresponding to the conformal transformations in two dimensions. It has a finite subalgebra which generates the Mobius transformations of one complex variable; this group of transformations is isomorphic to the symplectic group Sp(2).

The supergauge algebra of the Neveu-Schwarz model (without c-numbers) generates conformal and supergauge transformations on the variables (z, θ) where θ is an anti-commuting variable, as in (3.22). This algebra has a finite subalgebra $\{L_0, L_{\pm 1}, G_{\pm\frac{1}{2}}\}$ which is isomorphic to the orthosymplectic algebra OSp(2,1);this will be explained in section 2.

Further extensions of the conformal algebra which have been proposed correspond to the generators of certain transformations on the set of variables (z, θ^{α}), where $\alpha=1,\ldots, N$ labels the vector representation of an O(N) symmetry. This O(N) algebra contains the conformal algebra as a subalgebra, and has a finite subalgebra which is isomorphic to OSp(2,N).

We wish to study in particular some representations of these algebras, as they are useful in constructing amplitudes and Fock-space operators for the corresponding dual models.

Some of the details of the representations have been relegated to an appendix.

4.1. The Virasoro Algebra

As explained in Chapter 1, the Virasoro algebra is spanned by an infinite set of gauge operators L_n, which satisfy the commutation relations (1.15). If we omit the c-number terms, these relations are

$$[L_n, L_m] = (n-m) L_{n+m} \tag{4.1}$$

The algebra (4.1) can be realized by putting[80]

$$L_{-n} = z^{n+1} \frac{d}{dz} - nJz^n \tag{4.2}$$

where J is any number. This realization is appropriate for showing that the algebra(4.1) corresponds to the generators of conformal transformations. If we consider an infinitessimal transformation generated by the L s ($1 + \sum_n u_{n+1} L_{-n}$), we find that (with J=0)

$$\left(1 + \sum_n u_{n+1} L_{-n}\right) z = z + u(z) \tag{4.3}$$

where $\qquad u(z) = \sum u_n z^n$

This is just the form of an infinitessimal conformal transformation, as in (3.22).

If we consider the action of (4.2) on z^{k+m}, where m is an integer, we obtain a linear combination of terms of the form $z^{k+m'}$, where m' is an integer. Hence, the vector space with $\{z^{k+m} : m \in \mathbb{Z}\}$ as a basis is mapped into itself by L_n. We can represent L_n as an infinite dimensional matrix with respect to this space.

We write the basis of this vector space as[18]

$$V_m = N(J,k,m) z^{k+m} \tag{4.4}$$

where $N(J,k,m)$ is a normalization factor; different choices of $N(J,k,m)$ give equivalent representations. Operating with L_n as in (4.2), we find that

$$L_n : v_m \rightarrow v_p D^{(J\,k)}_{pm}(L_n) \tag{4.5}$$

gives a matrix representation $D^{(J,k)}$ of L_n, satisfying (4.1).

Instead of considering L_n as in (4.2) acting on elements of the form (4.4), we could consider a vector space spanned by basis elements $z'^{J} z^{k+m}$, and write L_{-n} as

$$L_{-n} = z^n\left(z\frac{\partial}{\partial z} - nz'\frac{\partial}{\partial z'}\right) \tag{4.6}$$

J is invariant under L_{-n}, so that we can consider $z'^{J} z^{k+m}$ for fixed J, in which case L_{-n} takes the form (4.2). It is sometimes convenient to write[80]

$$z' = \xi\eta \qquad z = \xi/\eta \tag{4.7}$$

L_{-n} may then be written in terms of ξ and η derivatives (see appendix).

As explained in section 1.2., the vector spaces described above can be used to derive the representations of the $SU(1,1)$ subalgebra of (4.1) spanned by $\{L_0, L_{\pm 1}\}$, and the group generated by this algebra. There are different classes of representation of $SU(1,1)$, some of which which were given in (1.22). The $D^{(+)}_J$ representation is obtained when $J < 0$ and $k=-J$. In this case the subspace spanned by $\{z'^{J} z^{-J+m} : m=0,1,\ldots\}$ is invariant under $\{L_0, L_{\pm 1}\}$, and has no invariant subspaces; hence the representation $D^{(+)}_J$ of $\{L_0, L_{\pm 1}\}$ obtained as in (4.5) is irreducible. Similarly, for $J < 0$ and $k = J$, the subspace spanned by $\{z'^{J} z^{J+m} : m=0,-1,\ldots\}$ is invariant, giving the irreducible representation $D^{(-)}_J$. The normalization factors for these representations may be fixed by requiring $D(L_1)=D(L_{-1})^{\dagger}$, which gives $N(J,k=\pm J,m)$ as in (1.25).

The representation $D^{(+)}_J$ can be used to obtain a Fock-space realization of $\{L_0, L_{\pm 1}\}$ (18). We put

$$L_i = -\sum_{nm\geq 0} a_m^{\mu +} \cdot D_{mn}^{(+)}(L_i)\, a_{n\mu} \qquad (i=0,\pm 1) \tag{4.8}$$

where $a_n^{\mu +}$ are canonical creation operators. If we take $a_n^{\mu +}$ to be the usual Veneziano operators, introduced in Chapter one, and consider (4.8) in the limit $J \to 0$ (from below), then we obtain the usual operator expression for the Virasoro generators $\{L_0, L_{\pm 1}\}$, as defined in (1.14), provided that we take the zero modes(18)

$$\sqrt{-2J}\, a_0^{\mu}\, ,\ \sqrt{-2J}\, a_0^{\mu +} \longrightarrow p^{\mu} \tag{4.9}$$

A Fock space realization of all the Virasoro generators L_n can actually be obtained using the representations of (4.1). Take J=0, k≠0, and consider the vector space spanned by

$$V_m = \frac{1}{\sqrt{k+m}}\, z^{k+m} \tag{4.10}$$

The derivative expressions (4.2) for L_n induce a (non-unitary) representation $D^{(0k)}(L_n)$ of (4.1) on this space by (4.5). We can now use this representation in the limit $k \to 0$ to obtain the usual expression for Virasoro generators (1.14); we put

$$L_n = \lim_{k\to 0} -\tfrac{1}{2} : \sum_{pm} a_m'^{+} \cdot D_{mp}(L_n)\, a_p' :$$

where

$$\begin{aligned} a_m'^{+} &= a_m^{+} \ (m\geq 0) & a_p' &= a_p \ (p>0) \\ a_m'^{+} &= -i a_m \ (m<0) & a_p' &= -i a_p^{+} \ (p<0) \\ & \sqrt{k}\, a_0'\, ,\ \sqrt{k}\, a_0'^{+} \longrightarrow p \end{aligned} \tag{4.11}$$

The factors of i associated with the a's in (4.11) just cancel similar factors in $D^{(0k)}$ which arise because of the normalization factors in (4.10). We note that, although we started with a representation of the conformal algebra (4.1), the L_n defined in (4.11) satisfy the Virasoro algebra(1.15).

For $J \geqslant 0$, J half-integral or integral, there is an irreducible representation of $SU(1,1)$ of dimension $(2J + 1)$ [19]; since $SU(1,1)$ is a non-compact group, this representation is necessarily non-unitary. In the present formalism, these representations can be realized on the vector space $\{ z'^{J} z^{a} : a=J,J-1,\ldots,-J \}$, which is invariant under $\{L_0, L_{\pm 1}\}$. The case $J=0$ (distinct from the case $J < 0$, $J \rightarrow 0$) gives the trivial representation $L_i=0$. We shall look at the cases $J=\frac{1}{2}$ and $J=1$ in a little detail; it is convenient to use the ξ, η notation of (4.7).

In the $J=\frac{1}{2}$ case, we consider the 2 dimensional space of vectors (ξ, η). Using the derivative expressions in the appendix for L_i, we find that

$$(\xi, \eta) \rightarrow (\xi', \eta') = (\xi, \eta) D(L_i) \tag{4.12}$$

gives the usual 2 dimensional representation for $L_0, L_{\pm 1}$ [80].

$$L_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad L_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \qquad L_{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \tag{4.13}$$

In the $J=1$ case we consider the 3 dimensional space of vectors $(\eta^2, \xi\eta, \xi^2)$, and we find the representation

$$L_1 = \begin{pmatrix} & 1 & \\ & & 2 \\ & & \end{pmatrix} \qquad L_0 = \begin{pmatrix} -1 & & \\ & & \\ & & 1 \end{pmatrix} \qquad L_{-1} = \begin{pmatrix} & & \\ -2 & & \\ & -1 & \end{pmatrix} \tag{4.14}$$

If we compare this with the adjoint representation of $SU(1,1)$, given by the structure constants

$$(\mathrm{ad}\, X_\mu)^\lambda_{\ \nu} = c_{\mu\nu}^{\ \ \lambda} \tag{4.15}$$

where $X_\mu = (L_1, L_0, L_{-1})$ and

$$[X_\mu, X_\nu] = c_{\mu\nu}^{\ \ \lambda} X_\lambda \tag{4.16}$$

we find that this is precisely the J=1 representation(4.14).

The metric tensor

$$g_{\mu\nu} = \mathrm{tr}\left(\mathrm{ad}\, X_\mu \cdot \mathrm{ad}\, X_\nu\right) \tag{4.17}$$

and its inverse can be used to define a quadratic Casimir operator(18)

$$C = \tfrac{1}{2}\left\{L_0(L_0+1) - L_1 L_{-1}\right\} \tag{4,18}$$

For a given J this has eigenvalue $C=\frac{1}{2}J(J+1)$; the representations (4.13), (4.14) each give $\frac{1}{2}J(J+1)\mathbb{1}$ as they must by Schur's lemma.

Consider now the transformations on ξ and η realized by the group generated by the algebra $\{L_0, L_{\pm 1}\}$. As in (1.21) these are

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} \to \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = A\begin{pmatrix}\xi\\ \eta\end{pmatrix} \tag{4.19}$$

where ad-bc=1; $z=\xi/\eta$ undergoes a Mobius transformation(1.7). The matrices A satisfy

$$A^T\begin{pmatrix} & 1\\ -1 & \end{pmatrix}A = \begin{pmatrix} & 1\\ -1 & \end{pmatrix} \tag{4.20}$$

which is just the condition for A to belong to the symplectic group Sp(2). In particular, the transformation A leaves invariant the bilinear form

$$(\xi_1, \eta_1)\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\begin{pmatrix}\xi_2\\ \eta_2\end{pmatrix} = \eta_1\eta_2(z_1 - z_2) \tag{4.21}$$

This bilinear form is intimately connected with the Killing form, the invariant bilinear form obtained using the metric tensor $g_{\mu\nu}$. In fact we find that the Killing form is

$$(\eta_1^2, \xi_1\eta_1, \xi_1^2)\begin{pmatrix} & & -1\\ & 2 & \\ -1 & & \end{pmatrix}\begin{pmatrix}\eta_2^2\\ \xi_2\eta_2\\ \xi_2^2\end{pmatrix} = -\eta_1^2\eta_2^2(z_1 - z_2)^2 \tag{4.22}$$

This is just the square of the bilinear form (4.21). If we wish we can write Mobius transformations as 3 x 3 matrices A acting on $(\eta^2, \zeta\eta, \zeta^2)$ and satisfying det A=1 and $A^T gA=g$.

We have developed the details of the representations in this section so that we can see easily how they extend to the graded cases. Representations of the Virasoro algebra have been studied in a more rigorous fashion by Goddard and Horsley.(81)

4.2. The Neveu-Schwarz Algebra

In this section we extend the ideas of the last section to the Neveu-Schwarz case by considering transformations on a superspace (z', z, θ). As noted before, the gauge algebra of the Neveu-Schwarz model is (without c-number terms)

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} \\ [L_n, G_r] &= \left(\tfrac{n}{2} - r\right) G_{n+r} \\ \{G_r, G_s\} &= 2L_{r+s} \end{aligned} \tag{4.23}$$

This algebra can be realized in terms of derivatives of z', z and θ, where θ is an anti-commuting variable as defined in (1.37):

$$\begin{aligned} L_{-n} &= z^n \left\{ z\frac{\partial}{\partial z} - nz'\frac{\partial}{\partial z'} + \frac{1}{2}(n+1)\theta\frac{\partial}{\partial\theta} \right\} \\ G_{-r} &= z^{r-\frac{1}{2}} \left\{ \theta z\frac{\partial}{\partial z} - 2r\theta z'\frac{\partial}{\partial z'} + z\frac{\partial}{\partial\theta} \right\} \end{aligned} \tag{4.24}$$

Here $\frac{\partial}{\partial\theta}$ is a left-hand derivative : $\frac{\partial}{\partial\theta}\theta = 1$. As in the Virasoro case, it is sometimes useful to write L_{-n}, G_{-r} in terms of derivatives of ξ, η and θ (see appendix.

We see that L_n and G_r acting on $z'^J z^{k+m}\theta^\nu$, where $\nu = 0$ or 1, leave J invariant, and for fixed J $z'\frac{\partial}{\partial z'}$ may be replaced in (4.24) by J.

We can consider the infinitessimal transformations on z and θ (J=0) generated by $\{L_n, G_r\}$. In the Virasoro case, we found that the L_n generated conformal transformations on z. Now we find that the transformations

$$1 + \sum u_{n+1} L_{-n}$$

and

$$1 + \sum \alpha_{r+\frac{1}{2}} G_{-r} \tag{4.25}$$

generate the conformal and supergauge transformations(3.22) on z and θ (where $u(z)=\sum u_n z^n$ and $\alpha(z)=\sum \alpha_{r+\frac{1}{2}} z^{r+\frac{1}{2}}$ are the parameters of the transformation(3.22)).

Consider the vector space spanned by the basis

$$V = \{ v_m^\nu = N(Jkm\nu) z'^J z^{k+m} \theta^\nu : m \in Z; \nu = 0,1 \} \tag{4.26}$$

for fixed J and k, where $N(J,k,m,\nu)$ is a normalization factor. A natural grading can be defined on V by defining v_m^ν to be even or odd as $\nu=0$ or 1 respectively. This space is mapped into itself by L_n and G_r; in analogy with (4.5) we can represent $X=L_n$ or G_r on this space by the infinite dimensional matrix $D_{nm}^{\nu'\nu}(X)$ defined by

$$X : v_m^\nu \rightarrow v_n^{\nu'} D_{nm}^{\nu'\nu}(X) \tag{4.27}$$

where sums over n and ν' are understood. Actually, to write this as a matrix representation we should incorporate (m,ν) into one index; however we prefer to write D(X) as in (4.27) and understand that matrix multiplication implies summation of ν indices as well.

Because the derivative expressions (4.24) satisfy (4.23), the matrices defined by (4.27) satisfy (4.23):

$$\langle D(X_1), D(X_2) \rangle = D(\langle X_1, X_2 \rangle) \tag{4.28}$$

where $\langle\,\rangle$ denotes commutator or anti-commutator, as defined in Chapter 3.

The finite subalgebra of (4.23), spanned by $\{L_0, L_{\pm1}, G_{\pm\frac{1}{2}}\}$, is isomorphic to the simple GLA OSp(2,1). (We note that these generators are actually in the form of the standard Cartan basis(3.49), and we see that the root-weight theorem (3.50) is satisfied). We can now consider k=-J. The subspace

(4.26) for specific values of J and k, and ask whether there are irreducible representations of this subalgebra similar to those which exist in the Sp(2) case. The finite dimensional representations have also been considered by Pais and Rittenberg[70].

For $J < 0$ we can consider $k=-J$. The subspace spanned by $\{z'^{J} z^{-J+m}\theta^{\nu}: m=0,1,\ldots;\ \nu=0,1\}$ of (4.26) is invariant under OSp(2,1) and has no invariant subspaces; hence we obtain an irreducible representation from (4.27) which we call $D_J^{(+)}$ in analogy with (1.22). Similarly, for $k=J < 0$, the invariant subspace spanned by $\{ z'^{J} z^{J+m}\theta^{\nu}: m=0,1,\ldots: \nu=0,1$ acts as a representation space for an irreducible representation $D_J^{(-)}$.

The normalization factors $N(J,k,m,\nu)$ can be fixed by requiring $D(L_{-1})=D(L_1)^{\dagger}$. For $D_J^{(+)}, D_J^{(-)}$ we have

$$N(J,k=\pm J,m,\nu) = \sqrt{\frac{\Gamma(m-2J+\nu)}{m!}} \qquad (4.29)$$

These factors also ensure that $D(G_{-\frac{1}{2}})=D(G_{\frac{1}{2}})^{\dagger}$, provided that we define

$$D^{\nu'\nu\,\dagger}_{nm} = \overline{D^{\nu\nu'}_{mn}} \qquad (4.30)$$

that is the order of ν,ν' must also be interchanged; this is just what we would expect, since, as pointed out previously, (m,ν) should really be considered as a single matrix label.

The representation $D_J^{(+)}$ may be used to obtain a Fock-space realization of OSp(2,1). We consider a_n^{ν} for $\nu=0,1$, where

$$a_n^{\mu_0} = a_n^{\mu} \qquad a_n^{\mu_1} = b^{\mu}_{n+\frac{1}{2}} \qquad (4.31)$$

and a_n, b_r are the usual Neveu-Schwarz operators. Then if we write

$$X = -\sum_{\substack{m n \geq 0 \\ \nu\nu' = 0,1}} a_n^{\nu' \dagger} D_{nm}^{\nu'\nu}(X)\, a_m^{\nu} \tag{4.32}$$

for $X = \{L_0, L_{\pm 1}, G_{\pm\frac{1}{2}}\}$, where D is the representation $D_J^{(+)}$ in the limit $J \to 0$, we obtain the correct operator forms of these generators as in Chapter 1 (using (4.9) for the zero modes).

The representations $D_{nm}^{\nu'\nu}(L_i)$ are diagonal in ν, ν'; hence we can obtain two representations of $\{L_0, L_{\pm 1}\}$ by considering D_{nm}^{00} and D_{nm}^{11}. These are just the $D^{(+)}$ representations defined in section 4.1., for $J \to 0$ and $J = -\frac{1}{2}$ respectively.

As in the Virasoro case, we can in fact derive the Fock space realizations of all the generators L_n, G_r by considering representations of the full algebra(4.23). In particular we consider the representation induced by the derivative forms (4.24) on the vector space(4.26) for J=0 and $k \neq 0$. We take the normalization factors to be

$$N(0,k,m,0) = \frac{1}{\sqrt{k+m}} \qquad N(0,k,m,1) = 1 \tag{4.33}$$

Then, for $X = L_n, G_r$, we can write

$$X = -\frac{1}{2} \lim_{k \to 0} : \sum_{\substack{m n \\ \nu'\nu}} a_n'^{\nu'\dagger} \cdot D_{nm}^{\nu'\nu}(X)\, a_m'^{\nu}$$

where

$$\begin{aligned} a_n'^{0\dagger} &= a_n'^{\dagger} & a_n'^{0} &= a_n' \\ a_n'^{1\dagger} &= b_{(n+\frac{1}{2})}^{\dagger} & a_n'^{1} &= b_{(n+\frac{1}{2})} \end{aligned} \tag{4.34}$$

and $a_n'^{\dagger}, a_n'$ are as given in (4.11). Again, although D is a representation of the algebra (4.23), the operators (4.34) satisfy the correct algebra with c-number terms.

Consider now the case $J \geq 0$. For J integral or half-integral, the (4J + 1) dimensional representation of OSp(2,1).

As in the Sp(2) case, we shall look at the $J=\frac{1}{2}$ and $J=1$ cases in some detail.

In the $J=\frac{1}{2}$ case we again use the ξ, η notation, and consider the space of vectors $(\xi, \eta, \eta\theta)$. Using (4.27) gives the 3 dimensional representation

$$L_1 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \qquad L_0 = \left(\begin{array}{cc|c} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \qquad L_{-1} = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

$$G_{\frac{1}{2}} = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \end{array}\right) . \qquad G_{-\frac{1}{2}} = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & -1 & 0 \end{array}\right) \tag{4.36}$$

When $J=1$ we consider the 5 dimensional space of vectors $(\eta^2, \xi\eta, \xi^2, 2\eta^2\theta, 2\xi\eta\theta)$. The resulting 5 dimensional representation is precisely the adjoint representation, obtained from the structure constants of the algebra as explained in section 3.2.

A metric tensor $g_{\mu\nu}$ for the generators X_μ of the GLA OSp(2,1) may be constructed, as in (3.42), and its inverse used to construct the quadratic Casimir operator, as in (3.45):

$$C = \frac{2}{3}\left\{L_0(L_0+\tfrac{1}{2}) - L_1 L_{-1} + \tfrac{1}{2} G_{\frac{1}{2}} G_{-\frac{1}{2}}\right\} \tag{4.37}$$

For a given J this has eigenvalue $\frac{2}{3}J(J+1)$. The representations for $J=\frac{1}{2},1$ each give $\frac{2}{3}J(J+1)\mathbb{1}$ so that Schur's lemma generalizes to this case(70).

We can consider the transformations realized on z and θ by the group generated by the algebra $\{L_0, L_{\pm 1}, G_{\pm\frac{1}{2}}\}$. The infinitessimal transformations generated by $(1+\alpha G_{-\frac{1}{2}} + \beta G_{\frac{1}{2}})$, where α and β are anti-commuting parameters, are

$$\begin{aligned} z &\rightarrow z + (\alpha z + \beta)\theta \\ \theta &\rightarrow \theta + (\alpha z + \beta) \end{aligned} \tag{4.38a}$$

Finite supergauge transformations, generated by successive transformations of the type above, are

$$z \to z + \alpha z + \beta$$
$$\theta \to (1+\tfrac{1}{2}\alpha\beta)\theta + (\alpha z + \beta) \tag{4.38b}$$

The (finite) projective transformations are

$$z \to \frac{az+b}{cz+d} \qquad \theta \to \frac{1}{cz+d}\,\theta \tag{4.39}$$

when ad-bc=1.

A general finite transformation generated by $\{L_0, L_{\pm1}, G_{\pm\frac{1}{2}}\}$ can be written as

$$\begin{pmatrix} \zeta \\ \eta \\ \eta\theta \end{pmatrix} \to \left(\begin{array}{cc:c} a & b & a\beta - b\alpha \\ c & d & c\beta - d\alpha \\ \hdashline \alpha & \beta & 1+\alpha\beta \end{array}\right) \begin{pmatrix} \zeta \\ \eta \\ \eta\theta \end{pmatrix} = A \begin{pmatrix} \zeta \\ \eta \\ \eta\theta \end{pmatrix} \tag{4.40}$$

subject to a 'generalized determinant' condition

$$\det A = ad - bc + \alpha\beta = 1 \tag{4.41}$$

This definition of the determinant of A coincides with that given by Nath and Arnowitt(79) for matrices in which some of the elements are of anti-commuting type; this definition will be explaned further in Chapter 5.

The supergauge transformations (4.38) are obtained from (4.40) by putting b=c=0, a=d=1-$\frac{1}{2}\alpha\beta$; the conformal transformations (4.39) correspond to $\alpha = \beta = 0$, subject to ad-bc=1. In both cases we are taking z= ζ/η.

The matrices A satisfy the condition

$$A^T \left(\begin{array}{cc:c} & 1 & \\ -1 & & \\ \hdashline & & 1 \end{array}\right) A = \left(\begin{array}{cc:c} & 1 & \\ -1 & & \\ \hdashline & & 1 \end{array}\right) \tag{4.42}$$

where A^τ is defined as

$$A^\tau = \left(\begin{array}{cc:c} a & c & \alpha \\ b & d & \beta \\ \hdashline -(a\beta - b\alpha) & -(c\beta - d\alpha) & (1+\alpha\beta) \end{array}\right) \tag{4.43}$$

Note the change of sign of 2 elements in the bottom row compared to the usual transpose of A. This ensures that the transformation A leaves invariant the bilinear form

$$(\xi_1, \eta_1, \eta_1\theta_1) \left(\begin{array}{cc:c} & 1 & \\ -1 & & \\ \hdashline & & 1 \end{array}\right) \left(\begin{array}{c} \xi_2 \\ \eta_2 \\ \eta_2\theta_2 \end{array}\right) \tag{4.44}$$

We see that (4.44) is anti-symmetric under interchange of labels 1 and 2, since the θ's anti-commute. The orthosymplectic transformations correspond to the natural generalization of the symplectic group, the group which leaves an anti-symmetric bilinear form invariant, to the case where we allow anti-commuting as well as commuting variables.

We note that, in the Koba-Nielsen formulation of the Neveu-Schwarz amplitude, using anti-commuting θ's[23], expressions similar to (4.44) appear as the natural generalizations of the forms similar to (4.21) which appear in the Veneziano model. We shall comment further on this in the next Chapter.

Given four (ξ_i , η_i , $\eta_i\theta_i$), we can define a generalized 'cross-ratio'

$$x = \frac{(z_1 - z_2 + \theta_1\theta_2)(z_3 - z_4 + \theta_3\theta_4)}{(z_1 - z_3 + \theta_1\theta_3)(z_2 - z_4 + \theta_2\theta_4)} \tag{4.45}$$

Since this is constructed from invariant bilinear forms (4.44), it is invariant under OSp(2,1). However it no longer has the simple property that (1-x) is also a cross-ratio.

If we also consider the invariant bilinear form constructed using the metric tensor $g^{\mu\nu}$ and the 5 dimensional vector ($\eta^2, \xi\eta, \xi^2, 2\eta^2\theta, 2\xi\eta\theta$), we obtain the form

$$- \eta_1^2 \eta_2^2 (z_1 - z_2 + \theta_1\theta_2)^2 \tag{4.46}$$

which is just (minus) the square of the form (4.45), as in the Veneziano case.

4.3. The Fermion and Off-Shell Algebras

In discussing the Neveu-Schwarz algebra in the previous section, we only considered that part of the algebra corresponding to the (on-shell) meson sector of the Neveu-Schwarz model, the gauge operators built from a_n and b_r operators. We might ask whether the formalism can also be applied to the fermion and off-shell sectors of the model.

Brink and Winnberg(64) have shown that the fermion amplitudes can be written in terms of the z,θ superfield formalism, so we would expect to be able to represent the fermion operators in a manner analgous to that used for the meson sector. The gauge operators in the fermion sector are L_n and F_m, where $n,m = 0, \pm 1, \ldots$, and they satisfy the relations (omitting the c-number terms)

$$
\begin{aligned}
[L_n, L_m] &= (n-m) L_{n+m} \\
[L_n, F_m] &= \left(\tfrac{n}{2} - m\right) F_{n+m} \\
\{F_n, F_m\} &= 2 L_{n+m}
\end{aligned}
\qquad (4.47)
$$

The F's satisfy almost exactly the same relations as the G's, the only difference being that the F's are labelled by integers whereas the G's are labelled by half-integers. Hence we can immediately write down a realization of L_n, F_m in terms of derivatives of z', z and θ; we take L_{-n} to be as in (4.24) and F_m to be the expression for G_r with r replaced by m.

These realizations can again be used to construct representations of L_n and F_m on the vector space spanned by $\{ z'^J z^{k+m} \theta^{\nu} \}$. The only finite subalgebra of (4.47) is the Mobius subalgebra $\{L_0, L_{\pm 1}\}$, so that we cannot look for finite dimensional representations as before. We can however find representations of the whole algebra (4.47), using (4.27).

Although J is again invariant under the transformations (4.47), k is not: F_n changes k by $\frac{1}{2}$. The vector space spanned by $\{ z^{\prime J}\, z^{k+m}, z^{\prime J}\, z^{k+m-\frac{1}{2}}\theta \ : \ m=0,\pm 1,\ldots \}$ for fixed J and k is invariant under (4.47), and can be used to define a representation of the algebra (4.47).

In particular we can consider the representation D obtained by putting J=0, k$\neq$0, realized on the representation space

$$\left\{ \frac{1}{\sqrt{k+m}}\, z^{k+m} \, ; \, z^{k+m-\frac{1}{2}}\,\theta : m=0,\pm 1,\ldots \right\} \qquad (4.48)$$

As in the previous cases, this representation gives the usual Fock space representation of the full algebra (1.44) (with c-number terms). We write $X=L_n$ or F_m as

$$X = \lim_{k\to 0} : \sum a_n^{\prime\, \nu' \dagger}\, D_{nm}^{\nu'\nu}(X)\, a_m^{\prime\,\nu} :$$

where (4.49)

$$a_n^{\prime 0 \dagger} = a_n^{\prime} \qquad a_n^{\prime 0} = a_n^{\prime}$$
$$a_n^{\prime 1 \dagger} = d_n^{\dagger} \qquad a_n^{\prime 1} = d_n$$
$$a_0^{\prime 1 \dagger} = a_0^{\prime \dagger} = (i\sqrt{2})^{-1}\gamma_5\gamma_0$$

and $a_n^{\prime\dagger}$, $a_n^{\prime}$ are as given in (4.11).

A similar procedure can be carried out for the off-shell sectors of the model, the off-shell meson sector consisting of c_r and b_s, and the off-shell fermion sector consisting of c_r and d_n. In each case we can define a representation D on the invariant subspaces

$$c_r, b_s : \left\{ N_m\, z^{k+m+\frac{1}{2}} \, , \, z^{k+m}\,\theta : m=0,\pm 1,\ldots \right\}$$
$$c_r, d_n : \left\{ N_m\, z^{k+m+\frac{1}{2}} \, , \, z^{k+m+\frac{1}{2}}\,\theta : m=0,\pm 1,\ldots \right\} \qquad (4.50)$$

where $N_m = (k + m + \frac{1}{2})^{-\frac{1}{2}}$. This representation can be used to construct the appropriate gauge operators X as in (4.49), where now we take

$$\begin{aligned} a_n^{\prime 0 \dagger} &= c_{n+\frac{1}{2}}^{\dagger} & a_n^{\prime 0} &= c_{n+\frac{1}{2}} & (n \geq 0) \\ a_n^{\prime 0 \dagger} &= -i c_{n+\frac{1}{2}} & a_n^{\prime 0} &= -i c_{n+\frac{1}{2}} & (n < 0) \end{aligned} \tag{4.51}$$

and $a_n^{\prime 1} = b_{n+\frac{1}{2}}$ or d_n, as in (4.34) or (4.49), as appropriate.

In all of the representations descrobed above, $D_{nm}^{\nu'\nu}(L_n)$ is diagonal in ν and ν'; hence these matrices provide a representation of the conformal algebra (4.1). In particular, if we consider the D s defined aboce in the limit $k \to 0$, then in the fermion case D^{00} gives the $(J=0, k \to 0)$ representation of the L_n, and D^{11} gives the $(J=-\frac{1}{2}, k=0)$ representation (where now J,k are those appropriate to the L_n algebra, as in section 4.1.; in the off-shell case, D^{00} gives the $(J=0, k=\frac{1}{2})$ representation, and D^{11} gives the $(J=-\frac{1}{2}, k=\frac{1}{2})$ or $(J=-\frac{1}{2}, k=0)$ representation as $a^{\prime 1}$ =b or d respectively.

4.4. O(N) Algebras

The supergauge algebra of the Neveu-Schwarz model can be generalized by considering transformations on z and θ^α, for $\alpha=1,\ldots,N$[42]. The infinitessimal transformations on (z, θ^α) are given by

$$\delta z = (2-n)\,\varepsilon^{\alpha_1\ldots\alpha_n}(z)\,\theta^{\alpha_1}\ldots\theta^{\alpha_n}$$
$$\delta\theta^\alpha = n\,\varepsilon^{\alpha_1\ldots\alpha_{n-1}\alpha}(z)\,\theta^{\alpha_1}\ldots\theta^{\alpha_{n-1}} + \varepsilon^{\alpha_1\ldots\alpha_n}(z)'\,\theta^{\alpha_1}\ldots\theta^{\alpha_n}\theta^\alpha \tag{4.52}$$

where $\alpha=1,\ldots,N$ and $n=0,\ldots,N$ form a closed system; the parameters $\varepsilon^{\alpha_1\ldots\alpha_n}$ are taken as commuting/anti-commuting as n is even/odd. The conformal and supergauge transformations correspond to n=0 and 1 respectively. The n=2 transformations leave z unchanged, but perform a (local) O(N) transformation on the θ's.

The generators of these transformations consist of one set of L_m (n=0 transformations), N sets of G's, G_r(n=1 transformations), $\frac{1}{2}N(N-1)$ sets of T's corresponding to local O(N) generators $T_m^{\alpha\beta}$ (n=2), together with generators corresponding to higher ($n > 2$) transformations. The generators corresponding to n_1 and n_2 satisfy mutual commutation/anti-commutation relations as $(n_1 + n_2)$ is even/odd. Further details are given by Ademollo et al[42].

The simplest algebra after the Veneziano (N=0) and Neveu-Schwarz (N=1) cases is the O(2) algebra which has one set of O(2) generators T_m and two sets of G's, G_r^1 and G_r^2. The algebra is[42]:

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} \\
[L_n, G_r^\alpha] &= \left(\tfrac{n}{2}-r\right) G_{n+r}^\alpha \\
\{G_r^\alpha, G_s^\beta\} &= 2\delta^{\alpha\beta} L_{r+s} + \varepsilon^{\alpha\beta}(r-s)T_{r+s} \\
[T_n, T_m] &= 0 \\
[T_n, L_m] &= n\,T_{n+m} \\
[T_n, G_r^\alpha] &= -\varepsilon^{\alpha\beta} G_{n+r}^\beta
\end{aligned} \tag{4.53}$$

As in the previous cases, this algebra can be realized in terms of derivatives of z', z, θ^α:

$$L_{-n} = z^n\left(z\frac{\partial}{\partial z} - nz'\frac{\partial}{\partial z'} + \frac{1}{2}(n+1)\,\theta^\alpha\frac{\partial}{\partial\theta^\alpha}\right)$$

$$G^\alpha_{-r} = z^{r-\frac{1}{2}}\left\{\theta^\alpha z\frac{\partial}{\partial z} - 2r\,\theta^\alpha z'\frac{\partial}{\partial z'} + z\frac{\partial}{\partial\theta^\alpha} + (r+\tfrac{1}{2})\theta^\alpha\theta^\beta\frac{\partial}{\partial\theta^\beta}\right\} \quad (4.54)$$

$$T_{-n} = z^n\left[\theta^1\frac{\partial}{\partial\theta^2} - \theta^2\frac{\partial}{\partial\theta^1}\right] - 2nz^{n-1}\theta^1\theta^2 z'\frac{\partial}{\partial z'}$$

Clearly these can also be written in terms of $(\xi, \eta, \eta\theta^\alpha)$ (see appendix). The transformations generated on z,θ^α by (4.54) are just those of (4.25) (with N=2).

We see that (4.54) acting on $z'^J\ z^{k+m}\theta^{1\nu_1}\theta^{2\nu_2}$; ν_i=0 or 1, leave J invariant. This extends to the O(N) case, and we can look for representations on the vector space spanned by

$$\left\{ v^\nu_m = N(J\,k\,m\,\nu)\,z'^J z^{k+m}\theta^{1\nu_1}\theta^{2\nu_2} : m\in Z,\ \nu_i = 0,1 \right\} \quad (4.55)$$

for fixed J and k. Representations are induced on this space as in (4.27) by

$$X:\ v^\nu_m \to v^{\nu'}_n\, D^{\nu'\nu}_{nm}(X) \quad (4.56)$$

provided that the representation matrix is labelled by the set $\nu = \{\nu_i\}$, and in matrix multiplication we understand that ν_i are to be summed over.

The finite subalgebra in the O(N) case is spanned by $\{L_0, L_{\pm1}, G^\alpha_{\pm\frac{1}{2}}, T^{\alpha\beta}_0\}$. and is isomorphic to the simple GLA OSp(2,N). The ordinary Lie algebra formed by $\{L_0, L_{\pm1}, T^{\alpha\beta}_0\}$ is Sp(2) x O(N). As pointed out at the end of section 3.3., the adjoint representation of Sp(2) x O(N) on the subspace of OSp(2,N) is irreducible for all N except N=2; for N=2, $\{G^\alpha_{\pm\frac{1}{2}}\}$ splits into two subspaces, each invariant under $\{L_0, L_{\pm1}, T_0\}$:

$$U' = \{G^1_{\frac{1}{2}} + i\,G^2_{\frac{1}{2}},\ G^1_{-\frac{1}{2}} + i\,G^2_{-\frac{1}{2}}\}$$

$$U'' = \{G^1_{\frac{1}{2}} - i\,G^2_{\frac{1}{2}},\ G^1_{-\frac{1}{2}} - i\,G^2_{-\frac{1}{2}}\} \quad (4.57)$$

which satisfy
$$\{U', U'\} = \{U'', U''\} = 0$$
$$\{U', U''\} = \{L'_0, L_{\pm 1}, T_0\}$$

We can define representations of OSp(2,N) using (4.55) and (4.56). In particular we can define irreducible representations $D_J^{(+)}$ for J=-k< 0, as in previous cases, using the invariant subspace

$$\left\{ N(J, k=-J, m, \nu) z'^J z^{k+m} \theta^{1\nu_1} \dots \theta^{N\nu_N} : m \in \mathbb{Z}, \nu_i = 0, 1 \right\} \qquad (4.58)$$

The normalization factors, fixed by $D(L_1)=D(L_{-1})^\dagger$, are given by

$$N(J, k=-J, m, \nu) \sqrt{\frac{\Gamma(m-2J+\Sigma\nu_i)}{m!}} \qquad (4.59)$$

These give $D(G_{\frac{1}{2}})=D(G_{-\frac{1}{2}})^\dagger$ and $D(T_0^{\alpha\beta})=-D(T_0^{\alpha\beta})^\dagger$, provided that we define $D^\dagger$ as in (4.30).

The representation D_J in the limit $J \to 0$ may be used to obtain a Fock space of OSp(2,N) by writing

$$X = -\sum_{\substack{n m \geq 0 \\ \nu, \nu'}} a_n^{\nu' \dagger} D_{nm}^{\nu'\nu}(X) a_n^\nu \qquad (4.60)$$

where a_n^ν is a commuting or anti-commuting operator as $(\nu_1 + \dots + \nu_n)$ is even or odd respectively.

In the O(2) case, there is a Fock space representation of the full algebra (4.52)[42,43]. We can obtain this as in previous cases by considering the J=0,k≠0 representation D, induced by the derivative expressions (4.54), on the vector space spanned by

$$\left\{ \frac{1}{\sqrt{k+m}} z^{k+m},\ z^{k+m}\theta^\alpha,\ \sqrt{k+m+1}\, z^{k+m}\theta^1\theta^2 : m \in \mathbb{Z} \right\} \qquad (4.61)$$

In this way we obtain the Fock space representation given by Ademollo et al[43] if we write

$$X = \lim_{k\to 0} -\tfrac{1}{2} : \sum_{\substack{nm \\ \nu\nu'}} a_n'^{\nu' +} \cdot D_{nm}^{\nu'\nu}(X)\, a_n'^{\nu} : \qquad (4.62)$$

where $\quad a_n'^{00} \sim a_n^1 \qquad a_n'^{11} \sim a_{n+1}^2$

$\qquad\quad a_n'^{10} \sim b_{n+\frac{1}{2}}^1 \qquad a_n'^{01} \sim b_{n+\frac{1}{2}}^2$

Here $a_n^1, a_n^2/b_r^1, b_r^2$ are canonical commuting/anti-commuting operators, as in reference (43). Factors of (-i) must be associated with a_n^i for $n < 0$, as in (4.11).

The operators (4.62) satisfy the algebra (4.53) with additional c-number terms.(43) This algebra can also be obtained from the Neveu-Schwarz algebra(43) by writing the a_m, b_r operators of the Neveu-Schwarz model as $a_m = a_m^1 + i a_m^2$, $b_r = b_r^1 + i b_r^2$, and putting $G_r = G_r^1 + i G_r^2$: T_n are then the generators which close the algebra. We shall come back to this point in the next section.

For $N > 2$, there is in general no suitable Fock space realization of the full algebra of the transformations (4.52) (42,43,82) (the $N=4$ case is an exception, as will be explained in the next section). Actually Fock space realizations of the algebra can always be written down. For any representation D of the algebra of (4.52) on the vector space (4.58), the generators X of the algebra can be realized as

$$X = -\sum_{\substack{nm \\ \nu\nu'}} a_n^{\nu' +} \cdot D_{nm}^{\nu'\nu}(X)\, a_m^{\nu} \qquad (4.63)$$

This gives the algebra without any c-number terms. If we wish to obtain a realization where we identify a_{-n} and a_n^{+}, and obtain the algebra with c-numbers, we have to consider an expression such as (4.62); this will only work if the representation D has suitable symmetry properties, and for general N this is not the case.

We can obtain finite dimensional representations of OSp(2,N) for J half-integral or integral, $J \geq 0$. For $J=\frac{1}{2}$ we have a (2 + N) dimensional representation on the vector space $(\zeta, \eta, \eta\theta^{\alpha})$, induced by the derivative realizations of the generators. For J=1 we have a $(3 + \frac{1}{2}N(N + 3))$ dimensional representation on the vectors

$$(\eta^2, \zeta\eta, \zeta^2, -2\eta^2\theta^1\theta^2, -2\eta^2\theta^1\theta^3, \ldots, 2\eta^2\theta^1, 2\zeta\eta\theta^1, 2\eta^2\theta^2 \ldots) \tag{4.64}$$

which is precisely the adjoint representation of OSp(2,N), constructed as in section 3.2.

As in (3.42) we can construct a metric tensor $g_{\mu\nu}$ for the generators X_μ of OSp(2,N), and obtain a Casimir operator $C=g^{\mu\nu}X_\mu X_\nu$. Acting on $z^{1\zeta} z^{k+m} \theta^{1\nu_1} \cdots \theta^{N\nu_N}$ in the O(N) case, we find that the Casimir operator is (up to an overall constant)

$$C \sim J\left(J+1-\frac{N}{2}\right) \tag{4.65}$$

Considering transformations of $(\zeta, \eta, \eta\theta^{\alpha})$ under the group generated by the OSp(2,N) algebra

$$\begin{pmatrix} \zeta \\ \eta \\ \eta\theta^{\alpha} \end{pmatrix} \longrightarrow A \begin{pmatrix} \zeta \\ \eta \\ \eta\theta^{\alpha} \end{pmatrix} \tag{4.66}$$

we find that A satisfies the condition

$$A^T \left(\begin{array}{cc|c} & 1 & \\ -1 & & \\ \hline & & 1_N \end{array}\right) A = \left(\begin{array}{cc|c} & 1 & \\ -1 & & \\ \hline & & 1_N \end{array}\right) \tag{4.67}$$

where 1_N is the N x N unit matrix, and A^T is an obvious generalization of (4.43). In particular A leaves invariant the bilinear form

$$(\zeta_1, \eta_1, \eta_1\theta_1^{\alpha}) \left(\begin{array}{cc|c} & 1 & \\ -1 & & \\ \hline & & 1_N \end{array}\right) \begin{pmatrix} \zeta_2 \\ \eta_2 \\ \eta_2\theta_2 \end{pmatrix} = \eta_1\eta_2\left(z_1 - z_2 + \theta_1^{\alpha}\theta_2^{\alpha}\right) \tag{4.68}$$

The bilinear form in the J=1 case is just (minus) the square of this, as in the Veneziano and Neveu-Schwarz cases. This bilinear form occurs in the n-point amplitude for the O(N) model, as will be explained in the next Chapter.

There is one exception to the above generalizations. The algebra OSp(2,4) has a metric tensor $g_{\mu\nu}$ which vanishes identically. However there is an invariant bilinear form, and there is a 17 x 17 matrix which has the properties of non-zero $g_{\mu\nu}$ and which can be extrapolated, either from the cases N=4 or from the square of the bilinear form (4.68).

4.5. O(2) and Quaternion Algebras

In the previous section we gave some irreducible representations of the OSp(2,N). There may however be further irreducible representations which are useful. For example, consider the OSp(2,2) algebra. From Chapter 3 we know that this is isomorphic to the spl(2,1) algebra; hence there must be a representation of osp(2,2) in terms of matrices of spl(2,1) type; that is 3 x 3 matrices with (generalized) trace zero. This representation is

$$L_1 = \left(\begin{array}{cc|c} & & \\ 1 & & \\ \hline & & \end{array}\right) \qquad L_0 = \left(\begin{array}{cc|c} \frac{1}{2} & & \\ & -\frac{1}{2} & \\ \hline & & \end{array}\right) \qquad L_{-1} = \left(\begin{array}{cc|c} & -1 & \\ & & \\ \hline & & \end{array}\right)$$

$$G^1_{\frac{1}{2}} = \left(\begin{array}{cc|c} & & \\ & & 1 \\ \hline 1 & & \end{array}\right) \qquad G^1_{-\frac{1}{2}} = \left(\begin{array}{cc|c} & & 1 \\ & & \\ \hline & -1 & \end{array}\right) \qquad G^2_{\frac{1}{2}} = \left(\begin{array}{cc|c} & & \\ & & i \\ \hline -i & & \end{array}\right)$$

$$G^2_{-\frac{1}{2}} = \left(\begin{array}{cc|c} & & i \\ & & \\ \hline & i & \end{array}\right) \qquad T_0 = \left(\begin{array}{cc|c} i & & \\ & i & \\ \hline & & 2i \end{array}\right) \tag{4.69}$$

It has been noted by Ademollo et al[43] that the O(2) algebra can be realized by transformations on (x, θ), where x and θ are given by

$$x = z + i\,\theta^1\theta^2 \qquad \theta = \theta^1 + i\,\theta^2 \tag{4.70}$$

Guided by this, we suppose that the representation (4.69) acts on $(\xi, \eta, \eta\theta)$; we can translate the representation (4.69) into a realization of the algebra in terms of derivatives (see appendix), which generalizes to the full algebra (4.53).

If we express the finite transformations on $(\xi, \eta, \eta\theta)$ as a 3 x 3 matrix A, then A satisfies

$$A^{\dagger}\left(\begin{array}{cc|c} & 1 & \\ -1 & & \\ \hline & & 1 \end{array}\right)A = \left(\begin{array}{cc|c} & 1 & \\ -1 & & \\ \hline & & 1 \end{array}\right) \tag{4.71}$$

where $A^{\dagger} = \overline{A^{T}}$, and A^{T} is defined as in (4.43). We see that the matrix $A = e^{i\omega}\mathbb{1}$ satisfies (4.71); hence the (graded) group of matrices which satisfy (4.71) is actually spl(2,1) x U(1).

The bilinear form which is left invariant by spl(2,1) x U(1) is

$$(\bar{\xi}_1, \bar{\eta}_1, \bar{\eta}_1\bar{\theta}_1)\left(\begin{array}{cc|c} & 1 & \\ -1 & & \\ \hline & & 1 \end{array}\right)\begin{pmatrix} \xi_2 \\ \eta_2 \\ \eta_2\theta_2 \end{pmatrix} = \bar{\eta}_1\eta_2(\bar{x}_1 - x_2 + \bar{\theta}_1\theta_2) \tag{4.72}$$

where $x = \xi/\eta$. Using (4.70) this is related to the form (4.68) left invariant by spl(2,1) by

$$\begin{aligned} &\bar{\eta}_1\eta_2(\bar{x}_1 - x_2 + \bar{\theta}_1\theta_2)\bar{\eta}_2\eta_1(\bar{x}_2 - x_1 + \bar{\theta}_2\theta_1) \\ &\qquad = -|\eta_1|^2|\eta_2|(z_1 - z_2 + \theta_1^{\alpha}\theta_2^{\alpha})^2 \end{aligned} \tag{4.73}$$

This is just (minus) the square of the spl(2.1) form, except that $\eta_1^2\eta_2^2$ has been replaced by $|\eta_1|^2|\eta_2|^2$, reflecting the additional U(1) invariance.

We can also obtain infinite dimensional representations of OSp(2,2), and the full O(2) algebra, on subspaces of the vector space spanned by $\{x'^{J}x^{k+m}\theta^{\nu}\}$, where $x' = \xi/\eta$. In particular we can consider the J=0,k≠0 representation of the full algebra, and use it to obtain a Fock space realization of the generators X:

$$X = \lim_{k\to 0} -\frac{1}{2} : \sum_{\substack{nm \\ \nu\nu'}} a_n'^{\nu'\dagger} D_{nm}^{\nu'\nu}(X)\, a_n'^{\nu} : \tag{4.74}$$

where

$$\begin{aligned} a_n'^{0} &\sim a_n^1 + i a_n^2 \\ a_n'^{1} &\sim b^1_{n+\frac12} + i b^2_{n+\frac12} \end{aligned}$$

and we must associate factors of (-i) with $a_n^{\prime 0}, a_n^{\prime 0\dagger}$ for $n < 0$, as in (4.11). This gives exactly the Fock space realization described before. In this way we see why the O(2) Fock-space realization is obtained from the Neveu-Schwarz model by 'complexifying' the operators a_m, b_r: the representation space on which it is based is similar to that for the Neveu-Schwarz model, with real variables replaced by complex ones(4.70).

As pointed out in the previous section, the O(N) algebras do not in general have a (suitable) Fock space realization. However there is another algebra which has a Fock space realization; this algebra is obtained from the Neveu-Schwarz algebra by expanding the canonical operators a_n, b_r as quaternions [42,43], rather than complex operators as described above for the O(2) case. This quaternion algebra does not correspond to any of the O(N) algebras; it is however a subalgebra of the O(4) algebra[42]. It is made up of 1 set of L_n operators, 4 sets of G's, G_r, and 3 sets of T's, $T_n^{\alpha\beta}$, satisfying the relations (omitting c-number terms)

$$
\begin{aligned}
\left[L_n, L_m\right] &= (n-m) L_{n+m} \\
\left[L_n, G_r^\alpha\right] &= \left(\tfrac{n}{2} - r\right) G_r^\alpha \\
\left\{G_r^\alpha, G_s^\beta\right\} &= 2\delta^{\alpha\beta} L_{r+s} + 2(r-s) T_{r+s}^{\alpha\beta} \\
\left[2T_n^{\alpha\beta}, G_r^\gamma\right] &= -\delta^{\alpha\gamma} G_{n+r}^\beta + \delta^{\beta\gamma} G_{n+r}^\alpha + \varepsilon^{\alpha\beta\gamma\delta} G_{n+r}^\delta \\
\left[T_n^{\alpha\beta}, T_m^{\gamma\delta}\right] &= -\delta^{\alpha\gamma} T_{n+m}^{\beta\delta} - \delta^{\beta\delta} T_{n+m}^{\alpha\gamma} + \delta^{\alpha\delta} T_{n+m}^{\beta\gamma} + \delta^{\beta\gamma} T_{n+m}^{\alpha\delta} \\
\left[T_n^{\alpha\beta}, L_m\right] &= n\, T_{n+m}^{\alpha\beta}
\end{aligned}
\tag{4.75}
$$

where $\alpha = 1, \ldots, 4$, and $T_0^{\alpha\beta} = -\varepsilon^{\alpha\beta\gamma\delta} T_0^{\gamma\delta}$

There is a finite subalgebra of this algebra, the 14 dimensional algebra spanned by $\{L_0, L_{\pm 1}, G_{\pm\frac{1}{2}}, T_0^{\alpha\beta}\}$. This is isomorphic to the simple GLA $spl(2,2)/z_2$, where z_2 is the 1 dimensional algebra generated by the 4 x 4 identity matrix, as explained in section 3.3.; spl(2,2) is the algebra of 4 x 4 matrices with (generalized) trace equal to zero. We can thus obtain a projective representation of the finite subalgebra by matrices of spl(2,2). A projective representation of a GLA is (in analogy with the ordinary Lie algebra case) defined to be a representation $D(X_\mu)$ of the generators X_μ which satisfies

$$\langle D(X_\mu), D(X_\nu)\rangle = D(\langle X_\mu, X_\nu\rangle) + \lambda_{\mu\nu} \mathbb{1} \qquad (4.76)$$

where $\lambda_{\mu\nu}$ is some number; thus a projective representation is a representation up to multiples of the unit matrix. Since the finite subalgebra is isomorphic to $spl(2,2)/\{\lambda \mathbb{1}\}$, it follows that we can find spl(2,2) matrices which satisfy (4.76). This projective representation is given in the appendix; it is very similar to the O(2) representation (4.69). We see that it satisfies all of the $spl(2,2)/z_2$ relations, except that

$$\{G^1_{\frac{1}{2}}, G^2_{-\frac{1}{2}}\} = 2T_0^{12} + i\mathbb{1}$$
$$\{G^3_{\frac{1}{2}}, G^4_{-\frac{1}{2}}\} = 2T_0^{34} + i\mathbb{1} = -2T_0^{12} + i\mathbb{1} \qquad (4.77)$$

Guided by the O(2) case, we suppose that the (projective) representation acts on $(\xi, \eta, \eta\omega', \eta\omega^2)$, where the variables are complex. We can translate the 4 x 4 matrices into expressions for the generators in terms of the derivatives of $x = (\xi/\eta)$, $x' = \xi\eta$, ω', ω^2. The unit matrix is expressed as

$$i\mathbb{1} \longrightarrow 2i\, x' \frac{\partial}{\partial x'} \qquad (4.78)$$

Hence if we operate on (x, ω^α) alone, the derivative expressions for the generators form a closed algebra, and we can use them to obtain genuine representations, not merely projective ones.

Furthermore, the derivative realizations can be extended to the full algebra (4.75) (see appendix).

In particular, we can consider the representation of the full algebra induced on the space spanned by

$$\left\{ \frac{1}{\sqrt{k+m}}\, x^{k+m},\ x^{k+m}\,\omega^{\alpha},\ \sqrt{k+m+1}\; x^{k+m}\,\omega^{1}\omega^{2} \right\} \qquad (4.79)$$

and use this in the limit $k \to 0$ to obtain a Fock space realization of the generators of the full algebra; we use (4.74), where now $\nu = \{\nu_1, \nu_2\}$, and

$$\begin{aligned} a_n'^{\,00} &\sim a_n^1 + i\, a_n^2 \\ a_n'^{\,11} &\sim a_{n+1}^3 + i\, a_{n+1}^4 \\ a_n'^{\,10} &\sim b_{n+\frac12}^1 + i\, b_{n+\frac12}^2 \\ a_n'^{\,01} &\sim b_{n+\frac12}^3 + i\, b_{n+\frac12}^4 \end{aligned} \qquad (4.80)$$

where a_n^i / b_r^i are canonical commuting/anti-commuting operators, and factors of $(-i)$ must be associated with $a_n^i, a_n^{i\dagger}$ for $n < 0$, as before (4.11). This is the same as the realization obtained by writing the Neveu-Schwarz operators as quaternions; we are thinking of the quaternion as made up of two complex parts.

The finite transformations A generated by the 4 x 4 (projective) representation of $spl(2.2)/z_2$ satisfy

$$A^{\dagger} \left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right) A = \left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \qquad (4.81)$$

where again $A^{\dagger} = \overline{A^{T}}$, and A^{T} is the obvious generalization of (4.43). We note that (4.81) is also satisfied by

$$A = e^{i\omega}\,\mathbb{1} \qquad\qquad A = e^{i\omega} \left(\begin{array}{c|c} \mathbb{1}_2 & \\ \hline & -\mathbb{1}_2 \end{array}\right) \qquad (4.82)$$

so that the group of A satisfying (4.81) is actually $spl(2,2) \times U(1) \times U(1)$.

The bilinear form left invariant by this group is

$$\bar{\eta}_1 \eta_2 \left(\bar{x}_1 - x_2 + \bar{\omega}_1^{\alpha}\, \omega_2^{\alpha} \right) \qquad (4.83)$$

We see that one of the U(1) invariances has factored out, expressed as $\eta_i \to e^{i\omega}\eta_i$; the other is $\omega^\alpha_i \to e^{i\omega}\omega^\alpha_i$.

If we define a metric tensor $g_{\mu\nu}$ for spl(2,2)/z_2 using the adjoint representation, we find that it vanishes identically. However we can use the projective representation to define a non-zero $g_{\mu\nu}$ (74). We put

$$g_{\mu\nu} = \mathrm{Tr}\left(D(X_\mu)D(X_\nu)\right) \tag{4.84}$$

where $D(x_\mu)$ is the projective representation, and Tr is the generalized trace. There is no ambiguity in (4.84), since

$$\begin{aligned}\mathrm{Tr}\left\{\left(D(X_\mu) + a\mathbb{1}\right)\left(D(X_\nu) + b\mathbb{1}\right)\right\} \\ = \mathrm{Tr}\left(D(X_\mu)D(X_\nu)\right)\end{aligned} \tag{4.85}$$

using $\mathrm{Tr}\, D(X_\mu) = \mathrm{Tr}\,\mathbb{1} = 0$. This is entrirely analagous to the construction of a metric tensor for the ordinary Lie algebra of 2n x 2n matrices with trace zero.

Chapter 5. O(N) Amplitudes

Introduction: In the last Chapter we considered the O(N) algebras discovered by Ademollo et al[42]. In this Chapter we shall consider the dual amplitudes corresponding to these algebras; they are constructed from an obvious extension of the (z, θ) form of the Neveu-Schwarz n-point amplitude. This sort of extension had been considered previously, but rejected because it apparently gave a vanishing four-point amplitude. We shall see that this is due to the zero mass of the ground state of the O(2) model, and the fact that the amplitude had been evaluated in the $\mathcal{F}_1$ formalism; evaluation of the amplitude in the $\mathcal{F}_2$ gauge gives a non-zero result.

Given the n-point amplitude for the O(N) model, we may ask whether there are any ghost states in its spectrum. We analyse the O(N) 4-point amplitude, with disappointing results.

Finally we point out how our techniques would apply in the quaternion model.

5.1. n-point Amplitudes

As explained in section 1.3., the n-point amplitude of the Neveu-Schwarz model can be written in Koba-Nielsen form as a straightforward generalization of the Veneziano n-point amplitude, where we have an anti-commuting variable θ_i, as well as a z_i variable, associated with each particle

$$A_n = \int \frac{\prod_i dz_i\, d\theta_i}{dV_{abc}} \prod_{i<j} \left(z_i - z_j + \theta_i\theta_j\right)^{-k_i \cdot k_j} \tag{5.1}$$

dV_{abc} is the usual Veneziano measure factor, and the θ integrations are defined as in (1.38) by

$$\int d\theta_i = 0 \qquad \int d\theta_i\ \theta_i = 1 \tag{5.2}$$

We note that if θ_i changes by a factor, $\theta_i \rightarrow a\theta_i$, then (5.2) requires that $d\theta_i \rightarrow \frac{1}{a} d\theta_i$. The integrand of (5.1) (including the measure factors) is invariant under a Mobius transformation (4.39) on all (z_i, θ_i) provided that $\frac{1}{2}k_i^2 = -\frac{1}{2}$.

The obvious generalization of (5.1) to the case when we have θ_i^α, $\alpha=1,,,,.N$, associated with each particle k_i is the amplitude

$$A_n = \int \frac{\prod_{\alpha,i} dz_i\, d\theta_i^\alpha}{dV_{abc}} \prod_{i<j} \left(z_i - z_j + \theta_i^\alpha \theta_j^\alpha\right)^{-k_i \cdot k_j} \tag{5.3}$$

The integrand of this is invariant under a Mobius transformation (4.39), provided that $\frac{1}{2}k_i^2 = (\frac{1}{2}N-1)$. However, if we consider the 4-point amplitude A_4, we find that it is identically equal to zero for $N \geq 2$, as was pointed out by Fairlie and Martin(83).

If we look carefully at the evaluation of the 4-point amplitude, we find that the cause of its vanishing for $N \geq 2$

is a factor of the O(2) ground state mass (=zero) in the answer. The amplitude can be evaluated (as was done for the O(2) case by Ademollo et al[43]) to give[84]

$$A_4 = C_N \frac{\Gamma(N-\alpha(s))\,\Gamma(N-\alpha(t))}{\Gamma(N-\alpha(s)-\alpha(t))} \tag{5.4}$$

where $\alpha(s) = 1 + \frac{1}{2}s$ is the leading trajectory. This result may easily be established by induction, since we may integrate over all θ_i^N in (5.3) to give an expression of the same form with N-1 θ's. By doing this, we find that

$$C_N = \frac{1}{2} k_N^2 . C_{N-1} \tag{5.5}$$

where $\frac{1}{2}k_N^2 = \frac{1}{2}N-1$ is the O(N) ground state mass. Thus, for $N \geqslant 2$, C_N contains a factor $\frac{1}{2}k_2^2=0$.

This is analagous to the fermion case of the Neveu-Schwarz model, where certain amplitudes are not well-defined in the gauge because of the vanishing of the fermion mass. Brink and Winnberg[64] showed that (5.1) is in fact the $\mathcal{F}_1$ expression, and that the (correct) $\mathcal{F}_2$ amplitude is obtained by changing the measure factor

$$\frac{1}{dV_{abc}} \longrightarrow \frac{1}{dV_{abc}} \; \frac{\theta_d\,\theta_e}{z_d - z_e} \tag{5.6}$$

We should like to show how this can be related to the invariance of (5.1) under supergauge transformations.

The Veneziano measure factor dV_{abc} can be thought of as arising from consideration of

$$\int \prod dz_i \prod_{i<j} (z_i - z_j)^{-k_i \cdot k_j} \tag{5.7}$$

We can rewrite this in terms of an integration over $(n-3)z_i$, with the other 3 z_i fixed, together with an integration over the 3 parameters a,b,c of the Mobius transformation which takes the fixed z_i's into their values in (5.7). The integral then separates into an integration over the group variables (infinite since it is a non-compact group) multiplied by the usual Veneziano n-point function[15]. The factor $(dV_{abc})^{-1}$ is then the Jacobian of the transformation from the 3 z_i's to be fixed to the 3 group variables a,b.c.

In the Neveu-Schwarz case, the integrand is built from terms of the form $(z_i - z_j + \theta_i\theta_j)$. We note that this is just $(\eta_i\eta_j)^{-1}$ times the bilinear form (4.44), which is invariant under the extension of the Mobius algebra $\{L_0, L_{\pm 1}, G_{\pm\frac{1}{2}}\}$. The terms $\prod_i dz_i d\theta_i$ are just sufficient to compensate for the transformations of $\prod_{i<j}(\eta_i\eta_j)^{-k_i\cdot k_j}$ under this algebra, provided that $\frac{1}{2}k_i^2 = -\frac{1}{2}$, so that

$$\prod dz_i\, d\theta_i \prod_{i<j} (z_i - z_j + \theta_i\theta_j)^{-k_i\cdot k_j} \tag{5.8}$$

is invariant under the (graded) group of transformations (4.40), not just the Mobius transformations. Hence we would expect to obtain the Neveu-Schwarz amplitude by factoring out an integration over the group, leaving the Jacobian of the transformation to the group parameters, as in the Veneziano case.

Suppose that we fix $(z_1, z_2, z_3, \theta_1, \theta_2)$, and transform these integrations to integrations over group variables a,b,c,α,β. An infinitessimal transformation generated by the OSp(2,1) algebra is

$$\left(\mathbb{1} + aL_1 + bL_0 + cL_{-1} + \alpha G_{\frac{1}{2}} + \beta G_{-\frac{1}{2}} \right) \tag{5.9}$$

The action of this on z_i, θ_i is given by the derivative expressions (4.24). The Jacobian that we require is

$$\frac{\partial(z_1, z_2, z_3, \theta_1, \theta_2)}{\partial(a, b, c, \alpha, \beta)} \tag{5.10}$$

To define this, we need a definition of the determinant of a matrix in which some of the elements are commuting and some are anti-commuting. Such a definition has been given by Nath and Arnowitt(79); it corresponds to generalizing the identity log (det A)=tr(log A) by replacing the trace by the generalized trace defined in section 3.2. For a matrix written in block form as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (5.11)$$

Where A and D have commuting elements, and B and C have anti-commuting elements, Nath and Arnowitt's definition gives

$$\det M = \frac{\det A}{\det(D - CA^{-1}B)} \qquad (5.12)$$

Using (5.12) to evaluate (5.10), we find

$$\frac{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)}{(z_1 - z_2)} \qquad (5.13)$$

Thus we would expect the n-point Neveu-Schwarz amplitude to be

$$\int \frac{\prod dz_i}{dV_{abc}} \frac{d\theta_3 \ldots d\theta_n}{(z_1 - z_2)} \prod_{i<j} (z_i - z_j + \theta_i \theta_j)^{-k_i \cdot k_j} \qquad (5.14)$$

We know that there can be no θ_1, θ_2 dependence in this, since we started with an expression in which all of the θ's were integrated over, just as in the Veneziano case there is no dependence on z_a, z_b, z_c; it can easily be checked explicitly for the 4-point function that the θ_1, θ_2 terms do vanish. Using the definition of the θ-integration, (5.14) can be rewritten as

$$\int \frac{\prod_{i=1}^{n} dz_i \, d\theta_i}{dV_{abc}} \frac{\theta_1 \theta_2}{(z_1 - z_2)} \prod_{i<j} (z_i - z_j + \theta_i \theta_j)^{-k_i \cdot k_j} \qquad (5.15)$$

which is just the J_2 form of the amplitude[64].

The extension of this to the O(N) case is now obvious. The bilinear form invariant under the finite subalgebra $\{L_0, L_{\pm 1}, G^{\alpha}_{\pm\frac{1}{2}}, T^{\alpha\beta}_0\}$ of the O(N) algebra is $\eta_1\eta_2(z_1 - z_2 + \theta^{\alpha}_1\theta^{\beta}_2)$, as in (4.68). If we consider

$$\prod_{i,\alpha} dz_i \, d\theta^{\alpha}_i \prod_{i<j}\left(z_i - z_j + \theta^{\alpha}_i\theta^{\alpha}_j\right)^{-k_i\cdot k_j} \tag{5.16}$$

this is invariant under the transformations generated by this OSp(2,N) algebra, provided that $\frac{1}{2}k_i^2 = (\frac{1}{2}N-1)$. Factoring out this invariance gives a Jacobian factor, as above; the θ-integrations which have beenremoved can be re-instated as in (5.15), to give the J_2 form of the amplitude

$$A_n = C_n \int \frac{\prod_{i\alpha} dz_i \, d\theta^{\alpha}_i}{dV_{abc}} \, \frac{\prod_{\alpha} \theta^{\alpha}_d \theta^{\alpha}_e}{(z_d - z_e)^N} \prod_{i<j}\left(z_i - z_j + \theta^{\alpha}_i\theta^{\alpha}_j\right)^{-k_i\cdot k_j} \tag{5.17}$$

The constant C_n is now non-zero for all N. In particular, the 4-point function is as given in (5.4), except that the overall constant is now non-zero for all N.

For N=2 the n-point amplitude (5.17) agrees with the expression obtained from the operator formulation of the model[43]. For $N \geq 2$ there is no Fock space realization of the O(N) algebra, so there is no operator expression to check (5.17) against. If we had a simple geometrical picture of the z_2 form of the Neveu-Schwarz model, we could perhaps understand $\log\left|z_i - z_j + \theta^{\alpha}_i\theta^{\alpha}_j\right|$ as a Green's function, just as $\log\left|z_i - z_j\right|$ is a Green's function in the Veneziano case.

5.2. Ghosts

Now that we have an n-point amplitude for the O(N) model, we can ask whether it is ghost free. In the Veneziano and Neveu-Schwarz models, we know that this is the case only when the number of space-time dimensions is $D \leq 26$ or 10 respectively, and that the upper limit is the critical dimension in which the model is most consistent.

We consider the 4-point amplitude(84), given (up to a positive constant) by

$$A_4 = \frac{\Gamma(N-\alpha(s))\,\Gamma(N-\alpha(t))}{\Gamma(N-\alpha(s)-\alpha(t))}$$

$$= \frac{\Gamma(1-k_1\cdot k_2)\,\Gamma(1-k_1\cdot k_4)}{\Gamma(k_1\cdot k_3)} \qquad (5.18)$$

We note that, in the second way of writing this amplitude, the value of N appears only in the restriction on the ground state mass: $\frac{1}{2}k_1^2=(\frac{1}{2}N-1)$.

Consider the residues of the leading trajectory poles of A_4 at $\alpha(s)=M$ for $M > (\frac{1}{2}N-1)$. Each residue can be decomposed into a sum of terms, each of which corresponds to definite angular momentum; to do this we write the residue as a linear combination of the angular functions appropriate to D dimensions. For D=4 these functions would be the Legendre polynomials; for general D the relevant functions are the Gegenbauer polynomials $C_n^{\frac{1}{2}(D-3)}(z)$. The condition that the amplitude be ghost-free is that the coefficient of each $C_n^{\frac{1}{2}(D-3)}$ for $n=0,1,\ldots,M$ should be positive.

The residue of A_4 at the pole $\alpha(s)=M$ is given by (5.19)

$$R(z) = c_M \prod_{k=0}^{\frac{1}{2}M-1} \{(2N-M-3)^2 z^2 - (2k+1)^2\} \qquad M \text{ even}$$

$$R(z) = c_M (2N-M-3) \prod_{k=0}^{\frac{1}{2}M-1} \{(2N-M-3)^2 z^2 - (2k+2)^2\} \qquad M \text{ odd}$$

where c_M is a positive constant, and z is the cosine of the centre of mass scattering angle. We note that only even/odd powers of z appear in R(z) as M is even/odd respectively, so that there are no odd daughters in this amplitude.

Consider the case when M is even. R(z) may be expanded as an even series of Gegenbauer polynomials, given (up to an unimportant positive normalization factor) in terms of hypergeometric functions by[85]

$$C_n^{\frac{1}{2}(D-3)}(z) = (-1)^n F_{21}\left(-n, n+\tfrac{1}{2}(D-3); \tfrac{1}{2}; z^2\right) \tag{5.20}$$

The first four polynomials are given by

$$\begin{aligned}
C_0^{\frac{1}{2}(D-3)}(z) &= 1 \\
C_2^{\frac{1}{2}(D-3)}(z) &= (D-1)z^2 - 1 \\
C_4^{\frac{1}{2}(D-3)}(z) &= \tfrac{1}{3}(D+1)(D+3)z^4 - 2(D+1)z^2 - 1 \\
C_6^{\frac{1}{2}(D-3)}(z) &= \tfrac{1}{15}(D+3)(D+5)(D+7)z^6 - (D+3)(D+5)z^4 \\
&\quad + 3(D+3)z^2 - 1
\end{aligned} \tag{5.21}$$

We see that D=4 gives the usual Legendre polynomials.

When N=0 or 1, the Veneziano and Neveu-Schwarz cases, the coefficient of $C_0^{\frac{1}{2}(D-3)}$ in the residue at $\alpha(s)=2$ vanishes for the critical dimension D=26 or 10 respectively. In the N=2 case, it vanishes when D=2, and Ademollo et al[43] have shown that the critical dimension of the O(2) model is in fact D=2.

For N $\geqslant$ 2 it is easy to see that the amplitude possesses ghosts. Consider the residue at $\alpha(s)=2N-2$:

$$R(z) = c \prod_{k=0}^{N-2} \left\{z^2 - (2k+1)^2\right\} \tag{5.22}$$

$$= \sum_{k=0}^{N-1} a_k \, C_{2k}^{\frac{1}{2}(D-3)}(z)$$

In the second expression we have written the residue as a sum of Gegenbauer polynomials. We find that the coefficient a_{N-2} is

$$a_{N-2} = (\text{const})\left\{\frac{1}{D-1+4(N-2)} - \frac{2(N-2)+3}{3}\right\} \qquad (5.23)$$

For N=2,D=2 this gives $a_0=0$, as noted above. For $N > 2$, the overall constant is positive, and we see that $a_{N-2} < 0$ for all $D \geq 2$. Hence the spectrum of A_4 as in (5.18) possesses ghosts for all $N > 2$.

5.3. The Quaternion Model

We have shown that there are ghosts in all of the O(N) models, except for the O(2) model in two dimensions. The only other model which has arisen from the O(N) algebras is the quaternion model. Can we construct the n-point amplitude for this model using the invariance arguments of before? We can construct an amplitude, but it does not appear to be dual.

As pointed out in section 4.5., we can realize the quaternion algebra in terms of derivatives of x, ω^1, ω^2. The corresponding bilinear form is

$$\bar{\eta}_1 \eta_2 (\bar{x}_1 - x_2 + \bar{\omega}_1 \omega_2) \qquad (5.24)$$

We can also realize the algebra in terms of z, z', θ^α (the finite subalgebra is given in the appendix). If we put

$$\begin{aligned} x &= z + i(\theta^1\theta^2 + \theta^3\theta^4) - \tfrac{1}{2}\theta^1\theta^2\theta^3\theta^4 \\ \omega^1 &= \left(\theta^1 - \tfrac{1}{2z}\theta^2\theta^3\theta^4\right) + i\left(\theta^2 + \tfrac{1}{2z}\theta^1\theta^3\theta^4\right) \\ \omega^2 &= \left(\theta^3 - \tfrac{1}{2z}\theta^1\theta^2\theta^4\right) + i\left(\theta^4 + \tfrac{1}{2z}\theta^1\theta^2\theta^3\right) \end{aligned}$$

and act on x, ω^1, ω^2 only, then the two realizations are the same.

We would expect the n-point amplitude to be obtained from

$$\int \prod_{i\alpha} dz_i \, d\theta_i^\alpha \prod_{i \neq j} (\bar{x}_i - x_j + \bar{\omega}_i^m \omega_j) \qquad (5.25)$$

by factoring out the integration over the group variables, giving a Jacobian factor, as before. However, if we do this and evaluate the resulting 4-point amplitude it is no longer dual.

Ademollo at al (43) have shown that there are in fact ghosts in the quaternion model.

5.4. Conclusion

In this thesis we have emphasized some aspects of the role played in dual models by the conformal algebra and its extension to supergauge algebras. Most of this work was stimulated by the discovery of Ademollo et al [42] of further extensions of the conformal algebra in what had previously thought to be a tightly constrained situation.

We have largely been concerned with the representations of these algebras, and in Chapter 2 we showed how representations of the projective group can be used to simplify the calculation of functions of infinite dimensional matrices, which are otherwise tedious to calculate.

In order to explain how such representations extend to the supergauge algebras it was found convenient to explain the formalism, and some recent results, of graded Lie algebras. We listed the classification of simple GLA's, and noted that the important (finite) graded algebras which have occurred in physics to date are either (classical) simple GLA's, or derived from them by contraction.

We have shown how representations of the (graded) Neveu-Schwarz and O(N) algebras can be constructed in a manner completely analagous to the conformal case, provided that the conformal variable z is extended to a superspace of variables (z, θ^{α}). These representations can be used in certain cases to construct Fock space realizations of the algebras; however the O(N) algebras do not in general admit a (suitable) Fock space realization.

The finite subalgebras of the O(N) supergauge algebras are the simple GLA's OSp(2,N), and it has been shown that the invariant bilinear forms of OSp(2,N) are important for building n-point amplitudes. Using them we were able to construct an n-point amplitude for the O(N) algebras, even where no Fock space realization exists. Unfortunately we found that all of these amplitudes have ghosts in their spectrum, except for the O(2) model which has critical space-time dimension two, and in which the only physical particle is in fact the massless ground state(43).

Disappointing though the conclusion of the study of the O(N) algebras is, we believe that it has been worthwhile, not least for the examination of the graded Lie algebra structure of the Neveu-Schwarz model. We have seen that the representations of the meson and fermion sectors, on and off shell, of the Neveu-Schwarz algebra are intimately connected. The (z,θ) formalism seems to be such a natural one for describing the extension of the Veneziano model to the Neveu-Schwarz model, that it may be hoped that some simple geometrical picture of the Neveu-Schwarz model in superspace may soon be found(39,40).

Appendix

In this appendix we list some details of the representations of the orthosymplectic algebras and their infinite dimensional generalizations which are referred to in Chapters 4 and 5.

ξ - η Derivative Realizations

The derivative realizations of the O(N) algebras for N=0,1 and 2 are given by

$$L_{-n} = \left(\frac{\xi}{\eta}\right)^{n}\left(\frac{(1-n)}{2}\,\xi\frac{\partial}{\partial\xi} - \frac{(1+n)}{2}\,\eta\frac{\partial}{\partial\eta}\right) \qquad n = 0, \pm 1, \ldots$$

$$G^{\alpha}_{-r} = \left(\frac{\xi}{\eta}\right)^{r-\frac{1}{2}}\left\{\xi\frac{\partial}{\partial(\eta\theta^{\alpha})} - \left(r-\tfrac{1}{2}\right)\theta^{\alpha}\xi\frac{\partial}{\partial\xi} - \left(r+\tfrac{1}{2}\right)\theta^{\alpha}\eta\frac{\partial}{\partial\eta}\right\} \qquad r = \pm\tfrac{1}{2}, \ldots$$

$$T_{-n} = \left(\frac{\xi}{\eta}\right)^{n}\varepsilon^{\alpha\beta}\eta\theta^{\alpha}\frac{\partial}{\partial(\eta\theta^{\beta})} - n\left(\frac{\xi}{\eta}\right)^{n-1}\left(\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta}\right) \qquad n = 0, \pm 1, \ldots \tag{A.1}$$

The Virasoro algebra (N=0) consists of $\{L_n\}$; the Neveu-Schwarz algebra (N=1) consists of $\{L_n, G_r\}$; with one $\theta = \theta^1$; the O(2) algebra consists of the full algebra (A.1) with θ^1 , θ^2 . In (A.1) the ξ and η derivatives are understood to be taken keeping $\eta\theta^{\alpha}$ fixed.

Metric Tensors

The metric tensors for some of the finite subalgebras considered are (up to a constant):

$$Sp(2) \qquad g^{\mu\nu} = \begin{pmatrix} & & -1 \\ & 2 & \\ -1 & & \end{pmatrix}$$

$$OSp(2,1) \qquad g^{\mu\nu} = \left(\begin{array}{ccc|cc} & & -1 & & \\ & 2 & & & \\ -1 & & & & \\ \hline & & & & \tfrac{1}{2} \\ & & & -\tfrac{1}{2} & \end{array}\right)$$

$$OSp(2,2) \qquad g^{\mu\nu} = \left(\begin{array}{cccc|cccc} & & -1 & & & & & \\ & 2 & & & & & & \\ -1 & & & & & & & \\ & & & 1 & & & & \\ \hline & & & & & \tfrac{1}{2} & & \\ & & & & -\tfrac{1}{2} & & & \\ & & & & & & & \tfrac{1}{2} \\ & & & & & & -\tfrac{1}{2} & \end{array}\right)$$

$$spl((2,2)/Z_2 \qquad g^{\mu\nu} = \left(\begin{array}{cccccc|cccccccc} & & -1 & & & & & & & & & & & \\ & 2 & & & & & & & & & & & & \\ -1 & & & & & & & & & & & & & \\ & & & 2 & & & & & & & & & & \\ & & & & 2 & & & & & & & & & \\ & & & & & 2 & & & & & & & & \\ \hline & & & & & & & \tfrac{1}{2} & & & & & & \\ & & & & & & -\tfrac{1}{2} & & & & & & & \\ & & & & & & & & & \tfrac{1}{2} & & & & \\ & & & & & & & & -\tfrac{1}{2} & & & & & \\ & & & & & & & & & & & \tfrac{1}{2} & & \\ & & & & & & & & & & -\tfrac{1}{2} & & & \\ & & & & & & & & & & & & & \tfrac{1}{2} \\ & & & & & & & & & & & & -\tfrac{1}{2} & \end{array}\right)$$

OSp(2,2)=spl(2,1)

The realization of the generators of OSp(2,2) in terms of derivatives of (complex) x, x', θ is

$$L_{-n} = x^n\left(x\frac{\partial}{\partial x} - n x'\frac{\partial}{\partial x'} + \frac{1}{2}(n+1)\,\theta\frac{\partial}{\partial \theta}\right) \qquad n=0,\pm 1$$

$$G^1_{-r} = x^{r-\frac{1}{2}}\left(\theta x\frac{\partial}{\partial x} - 2r\theta x'\frac{\partial}{\partial x'} + x\frac{\partial}{\partial \theta}\right) \qquad (A.3)$$

$$G^2_{-r} = i x^{r-\frac{1}{2}}\left(-\theta x\frac{\partial}{\partial x} + 2r\theta x'\frac{\partial}{\partial x'} + x\frac{\partial}{\partial \theta}\right) \qquad r=\pm\frac{1}{2},\ldots$$

$$T_{-n} = i x^n\left(2x'\frac{\partial}{\partial x'} + \theta\frac{\partial}{\partial \theta}\right) \qquad n=0,\pm 1,\ldots$$

<u>$spl(2,2)/z_2$</u>

The projective representation of the finite subalgebra of the quaternion algebra by matrices of spl(2,2) is given by

$$L_1 = \left(\begin{array}{cc|cc} & & & \\ 1 & & & \\ \hline & & & \\ & & & \end{array}\right) \qquad L_0 = \left(\begin{array}{cc|cc} \frac{1}{2} & & & \\ & -\frac{1}{2} & & \\ \hline & & & \\ & & & \end{array}\right) \qquad L_{-1} = \left(\begin{array}{cc|cc} & -1 & & \\ & & & \\ \hline & & & \\ & & & \end{array}\right)$$

$$G^1_{\frac{1}{2}} = \left(\begin{array}{cc|cc} & & & \\ & & 1 & \\ \hline 1 & & & \\ & & & \end{array}\right) \qquad G^1_{-\frac{1}{2}} = \left(\begin{array}{cc|cc} & & 1 & \\ & & & \\ \hline & -1 & & \\ & & & \end{array}\right) \qquad G^2_{\frac{1}{2}} = \left(\begin{array}{cc|cc} & & & \\ & & i & \\ \hline -i & & & \\ & & & \end{array}\right)$$

$$G^2_{-\frac{1}{2}} = \left(\begin{array}{cc|cc} & & i & \\ & & & \\ \hline & i & & \\ & & & \end{array}\right) \qquad G^3_{\frac{1}{2}} = \left(\begin{array}{cc|cc} & & & \\ & & & 1 \\ \hline & & & \\ 1 & & & \end{array}\right) \qquad G^3_{-\frac{1}{2}} = \left(\begin{array}{cc|cc} & & & 1 \\ & & & \\ \hline & & & \\ & -1 & & \end{array}\right)$$

$$G^4_{\frac{1}{2}} = \left(\begin{array}{cc|cc} & & & \\ & & & i \\ \hline & & & \\ -i & & & \end{array}\right) \qquad G^4_{-\frac{1}{2}} = \left(\begin{array}{cc|cc} & & & i \\ & & & \\ \hline & & & \\ & i & & \end{array}\right) \qquad \text{(A.4)}$$

$$2T_0^{12} = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & i & \\ & & & -i \end{array}\right) \qquad 2T_0^{13} = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right) \qquad 2T_0^{14} = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & & i \\ & & i & \end{array}\right)$$

$$T_0^{\alpha\beta} = -\varepsilon^{\alpha\beta\gamma\delta}\, T_0^{\gamma\delta}$$

These satisfy the (anti-) commutation relations of $spl(2,2)/z_2$, except for

$$\{ G^1_{\frac{1}{2}}, G^2_{-\frac{1}{2}} \} = 2T_0^{12} + i\mathbb{1}$$

$$\{ G^3_{\frac{1}{2}}, G^4_{-\frac{1}{2}} \} = 2T_0^{34} + i\mathbb{1} \qquad \text{(A.5)}$$

$$= -2T_0^{12} + i\mathbb{1}$$

By taking (A.4) to act on $(\xi, \eta, \eta\omega^1, \eta\omega^2)$ we can obtain a realization of the above matrices in terms of derivatives of $x = \xi/\eta$, $x' = \xi\eta$, ω^1, ω^2. These do not form a closed algebra because of (A.5). However we note that $i\,1 \rightarrow 2ix'\frac{\partial}{\partial x'}$ in derivative form. Hence if we operate on (x, ω^α) only, so that $x'\frac{\partial}{\partial x'}=0$, we will have a closed algebra. Operating in this superspace, we can extend the derivative realization to the full quaternion algebra:

$$
\begin{aligned}
L_{-n} &= x^n\left\{x\frac{\partial}{\partial x} + \frac{1}{2}(n+1)\,\omega^\alpha\frac{\partial}{\partial\omega^\alpha}\right\}\\
G^1_{-r} &= x^{r-\frac{1}{2}}\left\{\omega^1 x\frac{\partial}{\partial x} + x\frac{\partial}{\partial\omega^1} + (r+\tfrac{1}{2})\,\omega^1\omega^2\frac{\partial}{\partial\omega^2}\right\}\\
G^2_{-r} &= i\,x^{r-\frac{1}{2}}\left\{-\omega^1 x\frac{\partial}{\partial x} + x\frac{\partial}{\partial\omega^1} - (r+\tfrac{1}{2})\,\omega^1\omega^2\frac{\partial}{\partial\omega^2}\right\}\\
G^3_{-r} &= x^{r-\frac{1}{2}}\left\{\omega^2 x\frac{\partial}{\partial x} + x\frac{\partial}{\partial\omega^2} + (r+\tfrac{1}{2})\,\omega^2\omega^1\frac{\partial}{\partial\omega^1}\right\}\\
G^4_{-r} &= i\,x^{r-\frac{1}{2}}\left\{-\omega^2 x\frac{\partial}{\partial x} + x\frac{\partial}{\partial\omega^2} - (r+\tfrac{1}{2})\,\omega^2\omega^1\frac{\partial}{\partial\omega^1}\right\}\\
2T^{12}_{-n} &= i\,x^n\left\{\omega^1\frac{\partial}{\partial\omega^1} - \omega^2\frac{\partial}{\partial\omega^2}\right\} = -2T^{34}_{-n}\\
2T^{13}_{-n} &= x^n\left\{\omega^1\frac{\partial}{\partial\omega^2} - \omega^2\frac{\partial}{\partial\omega^1}\right\} = 2T^{24}_{-n}\\
2T^{14}_{-n} &= i\,x^n\left\{\omega^1\frac{\partial}{\partial\omega^2} + \omega^2\frac{\partial}{\partial\omega^1}\right\} = -2T^{23}_{-n}
\end{aligned}
\qquad (A.6)
$$

The realization of $spl(2,2)/z_2$ in terms of $z, z', \theta^\alpha, \alpha=1,\ldots,4$, is given by

$$L_1 = \frac{1}{z}\left(1-\frac{1}{z^2}\theta^1\theta^2\theta^3\theta^4\right)\left(z\frac{\partial}{\partial z}+z'\frac{\partial}{\partial z'}\right)+\frac{1}{12z^2}\varepsilon^{\alpha\beta\gamma\delta}\theta^\alpha\theta^\beta\theta^\gamma\frac{\partial}{\partial\theta^\delta}$$

$$L_0 = z\frac{\partial}{\partial z}+\frac{1}{2}\theta^\alpha\frac{\partial}{\partial\theta^\alpha}$$

$$L_{-1} = z\left(1-\frac{1}{z^2}\theta^1\theta^2\theta^3\theta^4\right)\left(z\frac{\partial}{\partial z}-z'\frac{\partial}{\partial z'}\right)+z\theta^\alpha\frac{\partial}{\partial\theta^\alpha}+\frac{1}{12}\varepsilon^{\alpha\beta\gamma\delta}\theta^\alpha\theta^\beta\theta^\gamma\frac{\partial}{\partial\theta^\delta}$$

$$G^\alpha_{\frac{1}{2}} = \frac{1}{z}\left(\theta^\alpha+\frac{1}{12z}\varepsilon^{\alpha\beta\gamma\delta}\theta^\beta\theta^\gamma\theta^\delta\right)\left(z\frac{\partial}{\partial z}+z'\frac{\partial}{\partial z'}\right) \qquad (A.7)$$

$$+\left(1-\frac{1}{2z^2}\theta^1\theta^2\theta^3\theta^4\right)\frac{\partial}{\partial\theta^\alpha}-\frac{1}{4z}\varepsilon^{\alpha\beta\gamma\delta}\theta^\beta\theta^\gamma\frac{\partial}{\partial\theta^\delta}$$

$$G^\alpha_{-\frac{1}{2}} = \left(\theta^\alpha-\frac{1}{12z}\varepsilon^{\alpha\beta\gamma\delta}\theta^\beta\theta^\gamma\theta^\delta\right)\left(z\frac{\partial}{\partial z}-z'\frac{\partial}{\partial z'}\right)+\frac{\partial}{\partial\theta^\alpha}$$

$$+\theta^\alpha\theta^\beta\frac{\partial}{\partial\theta^\beta}+\frac{1}{4}\varepsilon^{\alpha\beta\gamma\delta}\theta^\beta\theta^\gamma\frac{\partial}{\partial\theta^\delta}$$

$$2T_0^{\alpha\beta} = \theta^\alpha\frac{\partial}{\partial\theta^\beta}-\theta^\beta\frac{\partial}{\partial\theta^\alpha}-\varepsilon^{\alpha\beta\gamma\delta}\theta^\gamma\frac{\partial}{\partial\theta^\delta}$$

If we change variables to $x, x', \omega^1, \omega^2, \bar\omega^1, \bar\omega^2$ using (5.24), and set $\frac{\partial}{\partial x'} = \frac{\partial}{\partial\bar\omega^1} = \frac{\partial}{\partial\bar\omega^2} = 0$, we obtain the subalgebra of (A.6).

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