Structure functions and quark models

Buck, Colin

STRUCTURE FUNCTIONS

AND

QUARK MODELS

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I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world .... I have just one chance of escaping a verdict of complete triviality, that I may be judged to have created something worth creating. And that I have created something is undeniable: the question is about its value.

G.H. Hardy (1877-1947)

A Mathematician's Apology

ABSTRACT

The purpose of this research is the utilization of experimental structure function moments to test various models of the quark energy-momentum distribution inside the nucleon and to extend these models to discussion in a wider context: the contemporary interacting field theory of quarks and gluons (Quantum Chromodynamics - QCD); and static properties of the nucleon.

A complete connection between a general quark energy-momentum distribution and the structure function moments is presented and a cure for the pathology of the lack of necessary kinematical restrictions in these models is extensively validated.

PREFACE

The research presented herein was carried out at the Department of Mathematics, University of Durham, between January, 1978, and May, 1980.

My thanks are due to Professor Euan Squires for his advice and enthusiasm, and astute observations on the early manuscript. I would also like to thank Dr. Sheena Bartlett for her excellent typing.

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CONTENTS

CHAPTER 1 INTRODUCTION AND THE QUARK PARTON MODEL

The proliferation of particles that were experimentally discovered in the 1950s prompted Gell-Mann and Zweig [1] to independently suggest that all these particles could be built from more fundamental entities, namely quarks. Moreover the early high-energy inelastic electron-proton scattering experiments [2] demonstrated similar features to lower energy electron-nucleus scattering which revealed the composite nature of the nucleus, and could thus be evidential that the proton was in turn also a composite object, containing more fundamental particles called partons. Subsequent experiments have accurately reaffirmed our notions of the elementary nature of quarks, in the measurement of the magnetic moment ratio μ_p/μ_n, and the pointlike structure of partons in deep-inelastic scattering: there is as yet no evidence to suggest that the properties of quarks are different from those of partons.

This then is the very simple model (naive parton model) which provides our point of departure: the proton is composed of three free quarks, or partons, of types u, u, d (to additively explain the proton's SU(3) properties). Since we will be primarily concerned with inelastic lepton scattering off such a proton we make the further assumption that the photon or vector boson couples minimally to the quarks in the proton.

In Chapter 2 we derive fundamental relationships between the experimental structure functions and a general distribution in energy and three-momentum of the quarks inside the proton. The deep-inelastic structure functions are kinematically restricted to the region $0 < x < 1$ whereas, if our quarks are initially free and massless with varying energy, the range of x is unlimited; the same pathology occurs in the cavity approximation to the bag model [3]. We refer to such a model as the *x model*. The cure is to replace x by $\log_e(1-x)$ to formulate a *$\log_e(1-x)$ model*. The comparison between these two models is extensively investigated.

In Chapter 3 we define and relate the moments of the structure functions with various properties of such a distribution in the context of both the x model and the $\log_e(1-x)$ model; the two subsequent chapters discuss recent experimental analyses of the nucleon structure functions and the possibilities of fitting the data with different modelled distributions.

However experiments have also shown [4] that the quarks carry only 45% of the proton's momentum. The interacting field theory of quarks and gluons (Quantum Chromodynamics - QCD) then pictures the quarks bound by gluon-exchange and it is these gluons which carry the missing momentum. The modifications thus required to our simple model and consequent scaling violations are discussed in Chapter 6.

Finally, with reference to specific fitted momentum distributions of the constituent quarks, we calculate the ratio of the axial-vector and vector coupling constants of the nucleon, G_A/G_V, as an example of a static property of the nucleon, and describe further applications of various aspects of this work.

CHAPTER 2 QED DERIVATION OF THE STRUCTURE FUNCTION

Consider the inelastic scattering of leptons on nucleons via one particle exchange. Our kinematical variables are defined in figure 1, where ℓ may be an electron or neutrino, and writing the energy transfer in the laboratory system as:

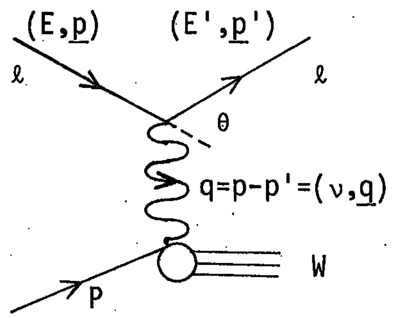


Figure 1. Inelastic lepton-nucleon scattering

$$\nu \equiv E - E' \qquad (2.1)$$

and the invariant four-momentum as:

$$Q^2 \equiv -q^2 = -(p-p')^2$$
$$= -2m_\ell^2 + 2EE' - 2|\underline{p}||\underline{p}'|\cos\theta$$

Thus at high energy where the lepton mass may be neglected:

$$Q^2 \simeq 4EE'\sin^2\frac{\theta}{2} \qquad (2.2)$$

Further the mass of the produced hadronic system is given by:

$$W^2 \equiv (P+q)^2 = M^2 + 2P\cdot q + q^2$$
$$= M^2 + 2M\nu - Q^2 \qquad (2.3)$$

in the laboratory.

The naive parton model assumes that the momentum transfer in figure 1 is sufficiently high to consider the inelastic lepton scattering process as due to incoherent elastic scattering off the point-like constituents, initially considered as free quarks by ignoring any interaction. Thus:

$$k^2 = (k+q)^2$$

i.e. $$Q^2 = 2k\cdot q = 2xP\cdot q$$

where x is the fraction of the proton's momentum carried by the quark. In the laboratory frame $P\cdot q = M\nu$, so we may write the dimensionless variable x as:

$$x \equiv \frac{Q^2}{2M\nu} \qquad (2.4)$$

Figure 1 is now replaced by figure 2 for the case of electron scattering, summing over all quarks. The mass condition on the struck quark gives:

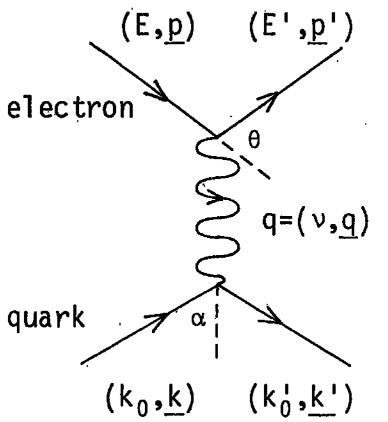


Figure 2. Electron-quark scattering

$$(k_0+\nu)^2 = |\underline{k}+\underline{q}|^2$$

i.e. $$2k_0\nu+q^2-2|\underline{k}||\underline{q}|\cos\alpha = 0$$

or $$Mx = k_0-k\cos\alpha \qquad (2.5)$$

Since the initial quark is also assumed to be free and massless then:

$$Mx = k_0(1-\cos\alpha) \qquad (2.6)$$

The differential cross-section is then just as in QED (5), i.e:

$$d\sigma = \int \frac{d^3p'}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{m_q^2m_e^2}{EE'k_0k_0'}(2\pi)^4\,\delta^4\,(p'+k'-p-k)\,|M|^2\frac{1}{VF} \qquad (2.7)$$

where F is the flux of incident particles and V is the spatial volume of interaction. Integrating over d^3k' and writing $d^3p' = |\underline{p}'|^2dp'd\Omega' = |\underline{p}'|E'dE'd\Omega'$ we get:

$$\frac{d^2\sigma}{dE'd\Omega'} = \frac{2m_q^2m_e^2|\underline{p}'|}{Ek_0}\frac{|M|^2}{(2\pi)^2}\frac{1}{VF}\,\delta(2m_e^2-2EE'+2|\underline{p}||\underline{p}'|\cos\theta+2k\cdot q) \qquad (2.8)$$

Since we are not in the quark's rest frame we must average over angles α between quark and photon for a spherically symmetric quark distribution. From equation (2.6) x then ranges from zero to $2k_0/M$ * and:

$$\frac{d^2\sigma}{dE'd\Omega'} = \frac{2m_q^2m_e^2|\underline{p}'|}{Ek_0}\frac{|M|^2}{(2\pi)^2}\frac{1}{VF}\frac{\int_{-1}^{+1}d\cos\alpha\;\delta(2m_e^2-2EE'+2|\underline{p}||\underline{p}'|\cos\theta-2|\underline{k}||\underline{q}|\cos\alpha+2k_0\nu)}{\int_{-1}^{+1}d\cos\alpha}$$

$$= \frac{m_q^2m_e^2|\underline{p}'|}{2Ek_0^2|\underline{q}|}\frac{|M|^2}{(2\pi)^2}\frac{1}{VF} \qquad (2.9)$$

* A.C Davis and E.J. Squires, *Phys. Lett.* 69B (1977) 249, refer to this as the mass-shell-allowed region.

The invariant amplitude $|M|^2$ is formally written [5]:

$$|M|^2 = \frac{e^4}{2m_e^2m_q^2q^4}\left\{(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') - m_e^2(k'\cdot k) - m_q^2(p'\cdot p) + 2m_e^2m_q^2\right\} \qquad (2.10)$$

after summing over final spins and averaging over initial spins. Since the experiments measure the differential cross-section for electron scattering off a *proton* the factor VF in the proton rest frame is given by [5]:

$$VF = \frac{\sqrt{(p\cdot P)^2 - m_e^2M^2}}{EM} = 1 \qquad (2.11)$$

since m_e is negligible, and thus $|\underline{p}'| \simeq E'$. Substituting equations (2.10) and (2.11) (neglecting terms proportional to m_e^2, m_q^2) into equation (2.9):

$$\frac{d^2\sigma}{dE'd\Omega'} = \frac{e^4}{4\pi^2q^4}\,\frac{E'}{4k_0^2E|\underline{q}|}\left\{(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p')\right\} \qquad (2.12)$$

Define Mandelstam variables for the electron—*quark* interaction:

$$s = (p+k)^2 = (p'+k')^2$$

$$\simeq 2k\cdot p = 2k'\cdot p' \qquad (2.13)$$

$$u = (p'-k)^2 = (p-k')^2$$

$$\simeq -2k\cdot p' = -2k'\cdot p \qquad (2.14)$$

$$t = (p'-p)^2 = (k'-k)^2$$

$$= q^2 \qquad (2.15)$$

Hence:

$$\{(k'\cdot p)(k\cdot p) + (k'\cdot p)(k\cdot p')\} = \tfrac{1}{4}\{s^2 + (s+t)^2\} \qquad (2.16)$$

In the laboratory system define:

$$p = (E, E\cos\theta, E\sin\theta, 0) \qquad (2.17)$$

$$q = (\nu, q_z, 0, 0) \tag{2.18}$$

$$k = (k_0, k_0\cos\alpha, k_0\sin\alpha\cos\beta, k_0\sin\alpha\sin\beta) \tag{2.19}$$

where $\overline{\cos\beta} = 0$, $\overline{\cos^2\beta} = \frac{1}{2}$ for a spherically symmetric quark distribution. The on mass shell conditions for the final electron and quark then give respectively:

$$\cos\theta = \frac{2E\nu - q^2}{2Eq_z} \tag{2.20}$$

$$\cos\alpha = \frac{2k_0\nu + q^2}{2k_0 q_z} \tag{2.21}$$

Therefore:

$$\bar{s} = 2k\cdot p = 2k_0E(1-\cos\alpha\cos\theta-\sin\alpha\overline{\cos\beta}\sin\theta)$$

$$= \frac{-q^2}{q_z^2}\nu(E+Mx-k_0+\frac{2Ek_0}{\nu}) \tag{2.22}$$

and

$$\overline{s^2} = \bar{s}^{\,2} + 2E^2k_0^2\sin^2\alpha\sin^2\theta$$

$$= \bar{s}^{\,2} + \frac{q^4}{8q_z^4}(q^2+4EE')(q^2+4k_0\nu+4k_0^2) \tag{2.23}$$

Further:

$$\overline{(s+t)^2} = \overline{s^2} + 2\bar{s}t + t^2$$

$$= \overline{s^2} + 2\bar{s}q^2 + q^4 \tag{2.24}$$

Therefore:

$$\left\{\overline{s^2} + \overline{(s+t)^2}\right\} = 2\overline{s^2} + 2\bar{s}q^2 + q^4$$

$$= \frac{2q^4}{q_z^4}\nu^2(E+Mx-k_0+\frac{2Ek_0}{\nu})^2 + \frac{q^4}{4q_z^4}(q^2+4EE')(q^2+4k_0\nu+4k_0^2)$$

$$- \frac{2q^4}{q_z^2}\nu(E+Mx-k_0+\frac{2Ek_0}{\nu}) + q^4$$

$$= \frac{q^4}{q_z^4}\Big\{2\nu^2(E+Mx)^2 - 2\nu q_z^2(E+Mx) + q_z^4 + \tfrac{1}{4}q^2(q^2+4EE') + 2\nu q_z^2k_0(1-\frac{2E}{\nu})$$

$$- 4\nu^2k_0(E+Mx)(1-\frac{2E}{\nu}) + k_0(q^2+4EE')(\nu+k_0) + 2\nu^2k_0^2(1-\frac{2E}{\nu})^2\Big\}$$

$$\simeq \frac{q^4}{q_z^4}\left\{2\nu^2(E+Mx)^2 - 2\nu q_z^2(E+Mx) + q_z^4 + \tfrac{1}{4}q^2(q^2+4EE') + \nu k_0(2\nu^2+12EE'+q^2)\right\}$$

where the k_0^2 terms have been neglected. Using equation (2.2) and $\nu(E+Mx) = \nu E - \frac{1}{2}q^2$, this leads to:

$$\{\overline{s^2} + \overline{(s+t)^2}\} = 8MxEE'\sin^2\frac{\theta}{2}\,\frac{\nu^5}{q_z^4}\left\{1 + \frac{2EE'}{\nu^2} + \frac{3q^4}{\nu^4} - \frac{q^2}{\nu^2} + \frac{q^2EE'}{\nu^4} + \frac{k_0}{\nu^3}(2\nu^2+12EE'+q^2)\right\} \tag{2.25}$$

Substituting equation (2.25) into equation (2.16) into equation (2.12) gives:

$$\frac{d^2\sigma}{dE'd\Omega'} = \frac{e^4}{4\pi^2q^4}E'^2\,\frac{Mx\sin^2\frac{\theta}{2}}{2k_0^2}\,\frac{\nu^5}{q_z^5}\left\{1 + \frac{2EE'}{\nu^2} + \frac{3q^4}{\nu^4} - \frac{q^2}{\nu^2} + \frac{q^2EE'}{\nu^4} + \frac{k_0}{\nu^3}(2\nu^2+12EE'+q^2)\right\} \tag{2.26}$$

Using

$$\frac{\nu^5}{q_z^5} = \left(1 - \frac{q^2}{\nu^2}\right)^{-5/2} \simeq \left(1 - \frac{5Mx}{\nu} + \frac{35M^2x^2}{2\nu^2}\right) \tag{2.27}$$

$$\frac{2EE'}{\nu^2} = \frac{Mx}{\nu\sin^2\frac{\theta}{2}} \tag{2.28}$$

$$\frac{3q^4}{\nu^4} = \frac{3M^2x^2}{\nu^2} \tag{2.29}$$

$$-\frac{q^2}{\nu^2} = \frac{2Mx}{\nu} \tag{2.30}$$

$$\frac{q^2EE'}{\nu^4} = -\frac{M^2x^2}{\nu^2\sin^2\frac{\theta}{2}} \tag{2.31}$$

equation (2.26) reduces to (up to terms of order ν^{-2}):

$$\frac{d^2\sigma}{dE'd\Omega'} = \frac{e^4}{4\pi^2q^4}E'^2\,\frac{Mx}{2k_0^2}\left\{\sin^2\frac{\theta}{2} + \frac{1}{\nu}\left(Mx+[2k_0-3Mx]\sin^2\frac{\theta}{2}\right)\right.$$
$$\left. + \frac{1}{\nu^2}\left(6Mx[k_0-Mx]+Mx\left[\frac{21}{2}Mx-12k_0\right]\sin^2\frac{\theta}{2}\right)\right\} \tag{2.32}$$

We may compare this with the standard deep-inelastic electron-proton cross-section formula (6):

$$\frac{d^2\sigma}{dE'd\Omega'} = \frac{e^4}{4\pi^2 q^4} E'^2 \left\{\frac{F_2}{\nu} + \left(\frac{2F_1}{M} - \frac{F_2}{\nu}\right)\sin^2\frac{\theta}{2}\right\} \tag{2.33}$$

and deduce the dimensional relationships between F_1 and F_2 and the Feynman variable x by equating the coefficients of $\sin^2\frac{\theta}{2}$ in equations (2.32) and (2.33):

$$F_1(x,\nu) = \frac{M^2x}{4k_0^2}\left(1 + \frac{2[k_0-Mx]}{\nu} + 0\frac{1}{\nu^2}\right) \tag{2.34}$$

$$F_2(x,\nu) = \frac{M^2x^2}{2k_0^2}\left(1 + \frac{6[k_0-Mx]}{\nu} + 0\frac{1}{\nu^2}\right) \tag{2.35}$$

We have already noted prior to equation (2.9) the range of x for which this on mass shell picture is justifiable, namely:

$$0 < x < \frac{2k_0}{M} \tag{2.36}$$

Noting the relationship (2.4) the above equations can be simply rewritten:

$$F_1(x,Q^2) = \frac{M^2x}{4k_0^2}\left(1 + \frac{4Mx[k_0-Mx]}{Q^2} + 0\frac{1}{Q^4}\right) \tag{2.37}$$

$$F_2(x,Q^2) = \frac{M^2x^2}{2k_0^2}\left(1 + \frac{12Mx[k_0-Mx]}{Q^2} + 0\frac{1}{Q^4}\right) \tag{2.38}$$

In figure 2 we have implicitly assumed a single value $\underline{k}$ for the initial momentum of the free quark and thus ignored confinement. This can be phenomenologically remedied by considering a distribution in energy and three-momentum of the quarks, $P(k_0,k)$, normalised such that:

$$\int_0^\infty\int_0^\infty dk_0 k^2 dk P(k_0,k) = 1 \tag{2.39}$$

Our previous argument then only requires modification in equation (2.8), through:

$$\int_0^\infty \int_0^\infty dk_0 k^2 dk P(k_0,k) \int_{-1}^{+1} d\cos\alpha \; \delta(2m_e^2 - 2EE' + 2|\underline{p}||\underline{p}'|\cos\theta + 2|\underline{k}||\underline{q}|\cos\alpha)$$

$$= \int_0^\infty dk_0 \int_{|Mx-k_0|}^\infty kdkP(k_0,k) . \frac{1}{2|\underline{q}|} \tag{2.40}$$

using equation (2.5). Equation (2.35) then becomes, to leading order:

$$F_2(x) = \frac{M^2x^2}{2} \int_0^\infty \frac{dk_0}{k_0} \int_{|Mx-k_0|}^\infty kdkP(k_0,k) \tag{2.41}$$

This equation expresses the experimental structure function of a single quark in terms of the general distribution in energy and three-momentum of the quark. The reader will note that the range of x is no longer limited through equation (2.6) but can, in principle, take any value (unless a cut-off is imposed on k_0), whereas kinematics restricts x to the region $0 < x < 1$. The calculation of structure functions in the cavity approximation to the bag model (3) encounters the same pathology because of lack of translational invariance, but this may be cured by replacing x by $-\log_e(1-x)$. This has been demonstrated in the two-dimensional bag (7) and we adopt the same prescription in the integral of our fundamental relation, equation (2.41), but leaving the 'coefficient' x^2 which is of kinematic origin. Thus:

$$F_2(x) = \frac{M^2x^2}{2} \int_0^\infty \frac{dk_0}{k_0} \int_{|M\log_e(1-x)+k_0|}^\infty kdkP(k_0,k) \tag{2.42}$$

This then ensures that the structure function goes to zero as x goes to unity.

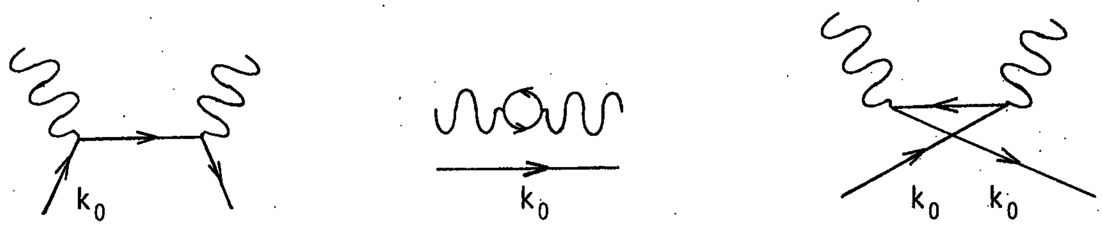
CHAPTER 3 COMPLETE CONNECTION BETWEEN THE EXPERIMENTAL MOMENTS AND THE QUARK DISTRIBUTION IN TWO MODELS

The arguments of the previous chapter are incomplete since we have only considered the 'direct' contribution to the structure function (see figure 2), which can be fully represented as the sum of the three Feynman diagrams in figure 3, namely:

(i) The familiar 'direct' contribution.

(ii) The 'sea' contribution, in which a quark-antiquark pair is created by the incoming photon. We consider only this 'sea' mechanism due to confinement and ignore the perturbative QCD contribution of figure 4.

(iii) The 'Z-graph' contribution which contributes negatively to (ii) since the created quark is excluded from occupying the ground state along with the 'valence' quark.



(i) (ii) (iii)

Figure 3. Graphical representation of the contributions to the structure function $F_2(x)$

Figure 4. Perturbative gluon contribution to the 'sea'

We define the 'direct' contribution in (i) by I(x) so that from equation (2.41) we write:

$$I(x) = \frac{M^2}{2}\int_0^\infty \frac{dk_0}{k_0}\int_{|Mx-k_0|}^\infty kdkP(k_0,k) \tag{3.1}$$

Comparing the kinematics of the 'Z-graph' and 'direct' contributions in the Bjorken limit it is straightforward to demonstrate (see Appendix A) that the 'Z-graph' contribution is obtained by $x \to -x$ in the 'direct' contribution. Hence the 'Z-graph' contribution may be written:

$$I(-x) = \frac{M^2}{2}\int_0^\infty \frac{dk_0}{k_0}\int_{Mx+k_0}^\infty kdkP(k_0,k) \tag{3.2}$$

Temporarily ignoring the color and spin of the quarks the total contribution to the structure function $F_2(x)$ (divided by x^2) * from a single quark is:

$$\frac{f_q(x)}{x} = I(x) + \Delta' \tag{3.3}$$

where Δ' represents the remaining 'sea' contribution after subtraction of the 'Z-graph' contribution §, i.e. when the quark is produced in an energy state above the ground state occupied by the 'valence' quark. Similarly the contribution from an antiquark is given by †:

$$\frac{f_{\bar{q}}(x)}{x} = I(-x) + \Delta' \tag{3.4}$$

* Our notation is defined by writing the Callan-Gross relationship (8) as: $2xF_1(x) = F_2(x) = \sum_i e_i^2 x f_i(x)$

§ J.S. Bell and A.J.G. Hey, *Phys. Lett.* 74B (1978) 77, refer to this latter contribution as the antiparton "valence" term.

† Bell and Hey *(op. cit.)* effectively have -I(-x) here since they neglect color *and* flavor in their equation (28).

where no 'direct' contribution exists and no 'Z-graph' subtraction is required since no Pauli exclusion principle operates to prevent the antiquark from occupying a ground state.

The color and spin of the quarks are now considered in a three valence quark model and this leads to modifications due to reapplication of the exclusion principle: the quark produced in the 'bubble' diagram of figure 3(ii) *can* be in the ground state with energy k_0 *if* it has a different color or spin to the valence quark. Consequently (see figure 5):

$$\frac{f_u(x)}{x} = 2I(x) + 4\Delta + \Delta' \tag{3.5}$$

where Δ represents the 'sea' contribution when the quark is produced in the ground state (but with different color or spin to the valence quark), i.e.:

$$\Delta = I(-x) \tag{3.6}$$

The factor 4 derives from the exclusion principle: of the six possible states (3 colors, 2 spins) the 'sea' quark is excluded from the two valence quark states. Similarly:

$$\frac{f_d(x)}{x} = I(x) + 5\Delta + \Delta' \tag{3.7}$$

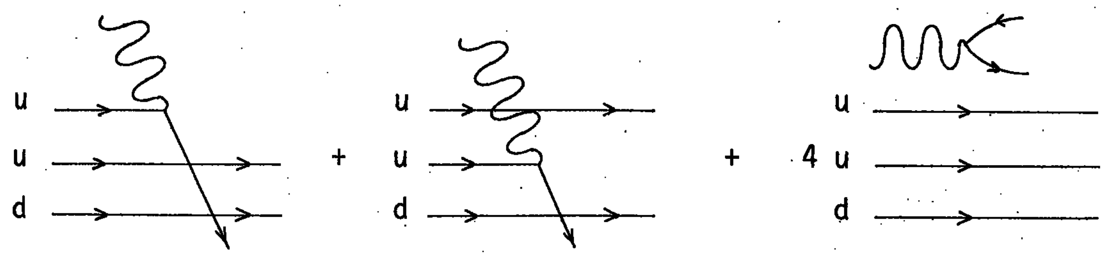


Figure 5. The 'direct' and 'sea' contributions to the u-quark structure function

We have assumed SU(3) symmetry so that no distinction is made between u and d quarks in I(x). We make the same assumption about I(-x), thus neglecting any possible effect the extra valence u quark may have in constraining $\bar{u}$ production in the 'sea' relative to $\bar{d}$ production. Also:

$$\frac{f_{\bar{u}}(x)}{x} = I(-x) + 5\Delta + \Delta' \tag{3.8}$$

$$\frac{f_{\bar{d}}(x)}{x} = I(-x) + 5\Delta + \Delta' \tag{3.9}$$

Rearranging equations (3.5)-(3.9) gives:

$$\begin{aligned} I(x) &= \frac{1}{2}\frac{f_u(x)}{x} - \frac{1}{3}\frac{f_{\bar{u}}(x)}{x} - \frac{1}{6}\Delta' \\ &= \frac{f_d(x)}{x} - \frac{5}{6}\frac{f_{\bar{d}}(x)}{x} - \frac{1}{6}\Delta' \end{aligned} \tag{3.10}$$

$$\begin{aligned} I(-x) &= \frac{1}{6}\frac{f_{\bar{u}}(x)}{x} - \frac{1}{6}\Delta' \\ &= \frac{1}{6}\frac{f_{\bar{d}}(x)}{x} - \frac{1}{6}\Delta' \end{aligned} \tag{3.11}$$

The electromagnetic structure functions are given by the relationships [9]:

$$\begin{aligned} 2xF_1^{ep}(x) = F_2^{ep}(x) = &\ \tfrac{4}{9}x(f_u(x)+f_{\bar{u}}(x)) + \tfrac{1}{9}x(f_d(x)+f_{\bar{d}}(x)) + \tfrac{1}{9}x(f_s(x)+f_{\bar{s}}(x)) \\ &+ \tfrac{4}{9}x(f_c(x)+f_{\bar{c}}(x)) + \cdots\cdots \end{aligned} \tag{3.12}$$

$$\begin{aligned} 2xF_1^{en}(x) = F_2^{en}(x) = &\ \tfrac{4}{9}x(f_d(x)+f_{\bar{d}}(x)) + \tfrac{1}{9}x(f_u(x)+f_{\bar{u}}(x)) + \tfrac{1}{9}x(f_s(x)+f_{\bar{s}}(x)) \\ &+ \tfrac{4}{9}x(f_c(x)+f_{\bar{c}}(x)) + \cdots\cdots \end{aligned} \tag{3.13}$$

where isospin reflection has been used. Therefore:

$$F_2^{ep}(x) - F_2^{en}(x) = \tfrac{1}{3}x(f_u(x)+f_{\bar{u}}(x)) - \tfrac{1}{3}x(f_d(x)+f_{\bar{d}}(x)) \tag{3.14}$$

Using equations (3.5)-(3.9):

$$\frac{F_2^{ep}(x)}{x^2} - \frac{F_2^{en}(x)}{x^2} = \tfrac{1}{3}(I(x) - I(-x)) \tag{3.15}$$

The neutrino structure functions can similarly be written:

$$
\begin{aligned}
F_2^{\nu p}(x) &= 2x(f_d(x) + f_{\bar{u}}(x) + f_s(x) + f_{\bar{c}}(x) + \cdots\cdots) \\
F_2^{\nu n}(x) &= 2x(f_u(x) + f_{\bar{d}}(x) + f_{\bar{s}}(x) + f_c(x) + \cdots\cdots) \\
xF_3^{\nu p}(x) &= 2x(f_d(x) - f_{\bar{u}}(x) + f_s(x) - f_{\bar{c}}(x) + \cdots\cdots) \\
xF_3^{\nu n}(x) &= 2x(f_u(x) - f_{\bar{d}}(x) + f_{\bar{s}}(x) - f_c(x) + \cdots\cdots)
\end{aligned}
\qquad (3.16)
$$

where the factor 2 derives from the coupling of axial and vector currents.

We expect the charm contribution to be small because of the larger mass of the charmed quark and experimental evidence [10] further suggests that the strange contribution is of the order of only 2% of the total quark + antiquark contribution: we thus neglect both charm and strange contributions in equation (3.16). Averaging over a nucleon target composed of an equal number of neutrons and protons:

$$F_2^{\nu N}(x) = x(f_u(x) + f_d(x) + f_{\bar{u}}(x) + f_{\bar{d}}(x)) \qquad (3.17)$$

$$xF_3^{\nu N}(x) = x(f_u(x) + f_d(x) - f_{\bar{u}}(x) - f_{\bar{d}}(x)) \qquad (3.18)$$

and substituting equations (3.5)-(3.9) we get:

$$\frac{F_2^{\nu N}(x)}{x^2} = 3I(x) + 21I(-x) + 4\Delta' \qquad (3.19)$$

$$\frac{xF_3^{\nu N}(x)}{x^2} = 3I(x) - 3I(-x) \qquad (3.20)$$

which can be inverted to give:

$$24I(x) = \frac{F_2^{\nu N}(x)}{x^2} + 7\frac{xF_3^{\nu N}(x)}{x^2} - 4\Delta' \qquad (3.21)$$

$$24I(-x) = \frac{F_2^{\nu N}(x)}{x^2} - \frac{xF_3^{\nu N}(x)}{x^2} - 4\Delta' \qquad (3.22)$$

We define the moments of the 'direct' contribution by:

$$\mu_n = \int_{-\infty}^{\infty} x^n I(x)dx = \int_0^{\infty} x^n\{I(x) + (-1)^n I(-x)\}dx \qquad (3.23)$$

From equation (3.15) this can be written in terms of the experimental structure function moments:

$$\mu_n \text{ (n odd)} = 3\int_0^{\infty} x^{n-2}\{F_2^{ep}(x) - F_2^{en}(x)\}dx \qquad (3.24)$$

Taking n=1 in equations (3.23) and (3.24):

$$\begin{aligned}\mu_1 &= \int_{-\infty}^{\infty} xI(x)dx \\ &= 3\int_0^{\infty} \frac{dx}{x}\{F_2^{ep}(x) - F_2^{en}(x)\} \\ &= 1 \qquad (3.25)\end{aligned}$$

where the last equality follows from imposing the charge constraint in equation (3.14) and assuming an SU(3) symmetric sea. Equation (3.25) expresses the Adler sum rule (11), and the experimental value is 0.81, the small x region being inaccessible (12). This consistency justifies our previous assumption about the SU(3) symmetry of I(-x). The validity of this sum rule is equivalent to normalising the 'direct' contribution such that:

$$\int_{-\infty}^{\infty} xI(x)dx = 1 \qquad (3.26)$$

Using equations (3.21)-(3.23) we can also write:

$$\mu_n = \frac{1}{24}\int_0^{\infty} x^{n-2}\{(F_2^{\nu N}(x)+7xF_3^{\nu N}(x)) + (-1)^n(F_2^{\nu N}(x)-xF_3^{\nu N}(x))\}dx - \frac{1}{6}\Delta_n' \qquad (3.27)$$

where Δ_n' is the nth moment of the remaining 'sea' contribution and vanishes identically in equation (3.27) for odd n. Moreover the 'sea' distribution is expected to be small and concentrated at low x (13), the even moments thus being negligible: consequently Δ_n' may be omitted in the following analysis without undue error. The right-hand side of equation (3.27) is a linear combination of the neutrino structure function moments and is thus well provided with data (see Chapter 4). Again taking n=1:

$$\mu_1 = \frac{1}{3}\int_0^\infty F_3^{\nu N}(x)dx = 1 \tag{3.28}$$

This is the Gross-Llewellyn Smith sum rule [14] which has the experimental value [4] around 1.06 ± 0.16, thereby consistent with the normalisation criteria of equation (3.26).

We now consider the moments of I(x) in terms of quark momentum distribution referred to in equation (2.39). Thus using equation (3.1) to evaluate equation (3.23) for a distribution $P(k_0,k)$:

$$M_n = \frac{M^2}{2}\int_{-\infty}^{\infty} x^n dx \int_0^\infty \frac{dk_0}{k_0}\int_{|Mx-k_0|}^\infty kdkP(k_0,k)$$

$$= \frac{M^2}{2}\int_0^\infty \frac{dk_0}{k_0}\left\{\int_{-\infty}^{\frac{k_0}{M}} x^n dx \int_{k_0-Mx}^\infty kdkP + \int_{\frac{k_0}{M}}^\infty x^n dx \int_{Mx-k_0}^\infty kdkP\right\}$$

$$= \frac{M^2}{2}\int_0^\infty \frac{dk_0}{k_0}\left\{\int_0^\infty kdkP \int_{\frac{k_0-k}{M}}^{\frac{k_0}{M}} x^n dx + \int_0^\infty kdkP \int_{\frac{k_0}{M}}^{\frac{k_0+k}{M}} x^n dx\right\}$$

$$\therefore \quad M_n = \frac{1}{2(n+1)M^{n-1}}\int_0^\infty \frac{dk_0}{k_0}\int_0^\infty kdk\left\{(k_0+k)^{n+1} - (k_0-k)^{n+1}\right\}P(k_0,k) \tag{3.29}$$

So for consecutive n values:

$$M_1 = \int_0^\infty\int_0^\infty dk_0 k^2 dkP(k_0,k) \tag{3.30}$$

$$M_2 = \frac{1}{3M}\left\langle\frac{k^2}{k_0}\right\rangle + \frac{1}{M}\left\langle k_0\right\rangle \tag{3.31}$$

$$M_3 = \frac{1}{M^2}\left\langle k^2\right\rangle + \frac{1}{M^2}\left\langle k_0^2\right\rangle \tag{3.32}$$

$$M_4 = \frac{1}{5M^3}\left\langle\frac{k^4}{k_0}\right\rangle + \frac{2}{M^3}\left\langle k^2k_0\right\rangle + \frac{1}{M^3}\left\langle k_0^3\right\rangle \quad (3.33)$$

$$M_5 = \frac{1}{M^4}\left\langle k^4\right\rangle + \frac{10}{3M^4}\left\langle k^2k_0^2\right\rangle + \frac{1}{M^4}\left\langle k_0^4\right\rangle \quad (3.34)$$

$$M_6 = \frac{1}{7M^5}\left\langle\frac{k^6}{k_0}\right\rangle + \frac{3}{M^5}\left\langle k^4k_0\right\rangle + \frac{5}{M^5}\left\langle k^2k_0^3\right\rangle + \frac{1}{M^5}\left\langle k_0^5\right\rangle \quad (3.35)$$

where we define:

$$\left\langle k^m k_0^n\right\rangle \equiv \int_0^\infty k_0^n dk_0 \int_0^\infty k^{m+2} dk\, P(k_0,k) \quad (3.36)$$

The various combinations of mean properties of the distribution in equations (3.30)-(3.35) can therefore be specified in terms of the experimental moments' values of equation (3.27).

We can do a similar analysis in terms of the $\log_e(1-x)$ model alluded to at the end of Chapter 2, *viz.* by replacing x by $-\log_e(1-x)$ in equation (3.1):

$$I(x) = \frac{M^2}{2}\int_0^\infty \frac{dk_0}{k_0}\int_{|M\log_e(1-x)+k_0|}^\infty kdk\, P(k_0,k) \quad (3.37)$$

The evaluation of equation (3.23) is then modified (we denote the moments in this model by L_n):

$$L_n = \frac{M^2}{2}\int_{-\infty}^\infty x^n dx \int_0^\infty \frac{dk_0}{k_0}\int_{|M\log_e(1-x)+k_0|}^\infty kdk\, P(k_0,k)$$

$$= \frac{M^2}{2}\int_0^\infty \frac{dk_0}{k_0}\left\{\int_{-\infty}^{1-\exp(-\frac{k_0}{M})} x^n dx \int_{M\log_e(1-x)+k_0}^\infty kdk\, P + \int_{1-\exp(-\frac{k_0}{M})}^\infty x^n dx \int_{-M\log_e(1-x)-k_0}^\infty kdk\, P\right\}$$

$$= \frac{M^2}{2}\int_0^\infty \frac{dk_0}{k_0}\left\{\int_0^\infty kdk\, P \int_{1-\exp(-\frac{k_0-k}{M})}^{1-\exp(-\frac{k_0}{M})} x^n dx + \int_0^\infty kdk\, P \int_{1-\exp(-\frac{k_0}{M})}^{1-\exp(-\frac{k_0+k}{M})} x^n dx\right\}$$

$$\therefore L_n = \frac{M^2}{2(n+1)} \int_0^\infty \frac{dk_0}{k_0} \int_0^\infty kdk \left\{ \left[1 - \exp(-\frac{k_0+k}{M})\right]^{n+1} - \left[1 - \exp(-\frac{k_0-k}{M})\right]^{n+1} \right\} P(k_0,k) \tag{3.38}$$

After expanding the brackets for consecutive n (see Appendix B), we can write expressions corresponding to equations (3.30)-(3.35):

$$L_1 = \int_0^\infty\int_0^\infty dk_0 k^2 dk\, P(k_0,k) - \frac{1}{2M}\left\langle\frac{k^2}{k_0}\right\rangle - \frac{3}{2M}\left\langle k_0\right\rangle + \frac{7}{6M^2}\left\langle k^2\right\rangle + \frac{7}{6M^2}\left\langle k_0^2\right\rangle + \cdots\cdots$$

$$= \int_0^\infty\int_0^\infty dk_0 k^2 dk\, P(k_0,k) - \frac{3}{2}\left(\frac{1}{3M}\left\langle\frac{k^2}{k_0}\right\rangle + \frac{1}{M}\left\langle k_0\right\rangle\right) + \frac{7}{6}\left(\frac{1}{M^2}\left\langle k^2\right\rangle + \frac{1}{M^2}\left\langle k_0^2\right\rangle\right)$$

$$- \frac{5}{8}\left(\frac{1}{5M^3}\left\langle\frac{k^4}{k_0}\right\rangle + \frac{2}{M^3}\left\langle k^2 k_0\right\rangle + \frac{1}{M^3}\left\langle k_0^3\right\rangle\right) + \frac{31}{120}\left(\frac{1}{M^4}\left\langle k^4\right\rangle + \frac{10}{3M^4}\left\langle k^2 k_0^2\right\rangle + \frac{1}{M^4}\left\langle k_0^4\right\rangle\right)$$

$$- \frac{7}{80}\left(\frac{1}{7M^5}\left\langle\frac{k^6}{k_0}\right\rangle + \frac{3}{M^5}\left\langle k^4 k_0\right\rangle + \frac{5}{M^5}\left\langle k^2 k_0^3\right\rangle + \frac{1}{M^5}\left\langle k_0^5\right\rangle\right)$$

$$+ \frac{127}{5040}\left(\frac{1}{M^6}\left\langle k^6\right\rangle + \frac{7}{M^6}\left\langle k^4 k_0^2\right\rangle + \frac{7}{M^6}\left\langle k^2 k_0^4\right\rangle + \frac{1}{M^6}\left\langle k_0^6\right\rangle\right) - \cdots\cdots \tag{3.39}$$

$$L_2 = \left(\frac{1}{3M}\left\langle\frac{k^2}{k_0}\right\rangle + \frac{1}{M}\left\langle k_0\right\rangle\right) - 2\left(\frac{1}{M^2}\left\langle k^2\right\rangle + \frac{1}{M^2}\left\langle k_0^2\right\rangle\right)$$

$$+ \frac{25}{12}\left(\frac{1}{5M^3}\left\langle\frac{k^4}{k_0}\right\rangle + \frac{2}{M^3}\left\langle k^2 k_0\right\rangle + \frac{1}{M^3}\left\langle k_0^3\right\rangle\right) - \frac{3}{2}\left(\frac{1}{M^4}\left\langle k^4\right\rangle + \frac{10}{3M^4}\left\langle k^2 k_0^2\right\rangle + \frac{1}{M^4}\left\langle k_0^4\right\rangle\right)$$

$$+ \frac{301}{360}\left(\frac{1}{7M^5}\left\langle\frac{k^6}{k_0}\right\rangle + \frac{3}{M^5}\left\langle k^4 k_0\right\rangle + \frac{5}{M^5}\left\langle k^2 k_0^3\right\rangle + \frac{1}{M^5}\left\langle k_0^5\right\rangle\right)$$

$$- \frac{23}{60}\left(\frac{1}{M^6}\left\langle k^6\right\rangle + \frac{7}{M^6}\left\langle k^4 k_0^2\right\rangle + \frac{7}{M^6}\left\langle k^2 k_0^4\right\rangle + \frac{1}{M^6}\left\langle k_0^6\right\rangle\right) + \cdots\cdots \tag{3.40}$$

$$L_3 = \left(\frac{1}{M^2}\langle k^2\rangle + \frac{1}{M^2}\langle k_0^2\rangle\right) - \frac{5}{2}\left(\frac{1}{5M^3}\left\langle\frac{k^4}{k_0}\right\rangle + \frac{2}{M^3}\langle k^2k_0\rangle + \frac{1}{M^3}\langle k_0^3\rangle\right)$$

$$+ \frac{13}{4}\left(\frac{1}{M^4}\langle k^4\rangle + \frac{10}{3M^4}\langle k^2k_0^2\rangle + \frac{1}{M^4}\langle k_0^4\rangle\right)$$

$$- \frac{35}{12}\left(\frac{1}{7M^5}\left\langle\frac{k^6}{k_0}\right\rangle + \frac{3}{M^5}\langle k^4k_0\rangle + \frac{5}{M^5}\langle k^2k_0^3\rangle + \frac{1}{M^5}\langle k_0^5\rangle\right)$$

$$+ \frac{81}{40}\left(\frac{1}{M^6}\langle k^6\rangle + \frac{7}{M^6}\langle k^4k_0^2\rangle + \frac{7}{M^6}\langle k^2k_0^4\rangle + \frac{1}{M^6}\langle k_0^6\rangle\right) - \cdots\cdots \qquad (3.41)$$

$$L_4 = \left(\frac{1}{5M^3}\left\langle\frac{k^4}{k_0}\right\rangle + \frac{2}{M^3}\langle k^2k_0\rangle + \frac{1}{M^3}\langle k_0^3\rangle\right) - 3\left(\frac{1}{M^4}\langle k^4\rangle + \frac{10}{3M^4}\langle k^2k_0^2\rangle + \frac{1}{M^4}\langle k_0^4\rangle\right)$$

$$+ \frac{14}{3}\left(\frac{1}{7M^5}\left\langle\frac{k_6}{k_0}\right\rangle + \frac{3}{M^5}\langle k^4k_0\rangle + \frac{5}{M^5}\langle k^2k_0^3\rangle + \frac{1}{M^5}\langle k_0^5\rangle\right)$$

$$- 5\left(\frac{1}{M^6}\langle k^6\rangle + \frac{7}{M^6}\langle k^4k_0^2\rangle + \frac{7}{M^6}\langle k^2k_0^4\rangle + \frac{1}{M^6}\langle k_0^6\rangle\right) + \cdots\cdots \qquad (3.42)$$

$$L_5 = \left(\frac{1}{M^4}\langle k^4\rangle + \frac{10}{3M^4}\langle k^2k_0^2\rangle + \frac{1}{M^4}\langle k_0^4\rangle\right) - \frac{7}{2}\left(\frac{1}{7M^5}\left\langle\frac{k^6}{k_0}\right\rangle + \frac{3}{M^5}\langle k^4k_0\rangle + \frac{5}{M^5}\langle k^2k_0^3\rangle + \frac{1}{M^5}\langle k_0^5\rangle\right)$$

$$+ \frac{19}{3}\left(\frac{1}{M^6}\langle k^6\rangle + \frac{7}{M^6}\langle k^4k_0^2\rangle + \frac{7}{M^6}\langle k^2k_0^4\rangle + \frac{1}{M^6}\langle k_0^6\rangle\right) - \cdots\cdots \qquad (3.43)$$

$$L_6 = \left(\frac{1}{7M^5}\left\langle\frac{k^6}{k_0}\right\rangle + \frac{3}{M^5}\langle k^4k_0\rangle + \frac{5}{M^5}\langle k^2k_0^3\rangle + \frac{1}{M^5}\langle k_0^5\rangle\right)$$

$$- 4\left(\frac{1}{M^6}\langle k^6\rangle + \frac{7}{M^6}\langle k^4k_0^2\rangle + \frac{7}{M^6}\langle k^2k_0^4\rangle + \frac{1}{M^6}\langle k_0^6\rangle\right) + \cdots\cdots \qquad (3.44)$$

$$L_7 = \frac{1}{M^6}\langle k^6\rangle + \frac{7}{M^6}\langle k^4k_0^2\rangle + \frac{7}{M^6}\langle k^2k_0^4\rangle + \frac{1}{M^6}\langle k_0^6\rangle + \cdots\cdots \qquad (3.45)$$

In equations (3.39)-(3.45) the moments in the $\log_e(1-x)$ model, L_n, are expressed as linear combinations of the moments in the x model, M_n, and in principle these linear combinations form an infinite series. Normalisation common to both models is defined by equation (3.26) rather than by equations (3.30) and (3.39) separately. In this respect 'renormalisation' in the $\log_e(1-x)$ model is automatic if the first moment, L_1, is set equal to $\mu_1(q^2)$, which will be assumed to equal unity in accordance with the Adler and Gross-Llewellyn Smith sum rules. Jaffe and Ross [15] incorporate the Jacobean factor $(1-x)^{-1}$ to achieve the same satisfaction.

CHAPTER 4 EXPERIMENTAL MOMENTS ANALYSIS

Recent experimental analyses of nucleon structure functions have concentrated on the moments of the structure functions since QCD gives direct predictions for the behaviour of the latter [16]. Kinematical considerations allow the lowest moments to be measured more accurately than the structure functions themselves. This is our motivation for calculating the moments of the quark distribution in equations (3.30)-(3.35) and (3.39)-(3.45), so that accurate experimental data can be utilized through equation (3.27).

We rely on the ABCLOS bubble chamber group analysis [17] which combines neutrino and antineutrino data from low energy interactions in Gargamelle with higher energy data from BEBC: specifically we have values and errors for both the structure functions and their moments at various q^2 [18]. However, since bubble chamber neutrino experiments are of limited statistical precision compared with counter experiments, we further utilize the CDHS counter group results for neutrino and antineutrino interactions in iron [4] to check consistency where possible, although this is limited because only the structure functions and their n=2 and n=3 moments are published from the CDHS experiments. In the following analysis Nachtmann moments are used; the difference between these and Cornwall-Norton moments is unimportant at this level.

In equation (3.27) the moments of the 'direct' contribution, μ_n, are expressed as linear combinations of the experimental neutrino structure function moments and in Table 1 we present the experimental values and errors for μ_n (with Δ_n' set to zero) at various q^2 §. As we have seen in

§ The values of q^2 given in Table 1 correspond to average values according to the binning; these will differ slightly from the published q^2 values for the structure functions (see Table 1 in ref. 17) since the structure functions are interpolated to *fixed* q^2 whereas the moments are binned in different q^2 *intervals*.

equation (3.28), μ_1 should equal unity in agreement with the Gross-Llewellyn Smith sum rule and this is assumed in Table 1. In experiments the measured quantity is xF_3, which is theoretically expected to go to zero as x goes to zero, and the x denominator in the sum rule thus presents problems at small x. The CDHS group incorporate an extrapolation to x=0 and get fair agreement with the expected quark parton model prediction (*cf* their section 6.5 with Table 2 in ref. 17).

Through the expansion of the exponential in equation (3.38) the moments in the $\log_e(1-x)$ model, L_n, were seen in principle to be linear combinations of all M_m ($m \geqslant n$), and, since the range of x in the x model is infinite, the higher moments M_m need not necessarily be negligible. A straightforward approximation from the x model to the $\log_e(1-x)$ model (for example by truncating the series in equations (3.39)-(3.45)) is thus impossible, and to compare the models forms for the distribution $P(k_0,k)$ will be chosen such that the integrals in equations (3.29) and (3.38) can be performed analytically.

It will be sufficient to perform the subsequent analysis by selecting moments at specific values of q^2 from Table 1. However, for completeness, we include the relevant moments at all available q^2 and the subsequent analysis is presented in such a way that the fastidious reader can reproduce the results at any q^2, thus investigating the q^2 dependence of distributions. Of course a more interesting q^2 dependence arises through the second term in equation (2.38) and this will be examined in Chapter 6.

Table 1. The values for the moments μ_n (see equation (3.27)) from experiment at different q^2

q^2 GeV^2	n	μ_n
0.387	1	1.0000[(b)]
	2	0.1756 ± 0.0348
	3	0.0561 ± 0.0210
	4	0.0278 ± 0.0081
	5	0.0116 ± 0.0049
	6	0.0061 ± 0.0019
	7	0.0027 ± 0.0012
	8	0.0014 ± 0.0004
	9	0.0006 ± 0.0003
	10	0.0003 ± 0.0001
	11	(a)
	12	(a)
0.592	1	1.0000[(b)]
	2	0.2339 ± 0.0254
	3	0.0891 ± 0.0158
	4	0.0419 ± 0.0070
	5	0.0220 ± 0.0047
	6	0.0113 ± 0.0021
	7	0.0064 ± 0.0014
	8	0.0034 ± 0.0006
	9	0.0020 ± 0.0004
	10	0.0010 ± 0.0002
	11	0.0006 ± 0.0001
	12	(a)
0.837	1	1.0000[(b)]
	2	0.1930 ± 0.0169
	3	0.0826 ± 0.0097
	4	0.0403 ± 0.0047
	5	0.0239 ± 0.0033
	6	0.0125 ± 0.0017
	7	0.0080 ± 0.0012
	8	0.0043 ± 0.0006
	9	0.0029 ± 0.0004
	10	0.0015 ± 0.0002
	11	0.0010 ± 0.0002
	12	(a)

q^2 GeV^2	n	μ_n
1.225	1	1.0000(b)
	2	0.1787 ± 0.0175
	3	0.0620 ± 0.0055
	4	0.0304 ± 0.0023
	5	0.0171 ± 0.0020
	6	0.0098 ± 0.0011
	7	0.0062 ± 0.0009
	8	0.0038 ± 0.0005
	9	0.0025 ± 0.0004
	10	0.0016 ± 0.0002
	11	0.0011 ± 0.0002
	12	0.0007 ± 0.0001
1.732	1	1.0000(b)
	2	0.1828 ± 0.0218
	3	0.0500 ± 0.0045
	4	0.0221 ± 0.0022
	5	0.0118 ± 0.0016
	6	0.0072 ± 0.0010
	7	0.0046 ± 0.0008
	8	0.0030 ± 0.0005
	9	0.0021 ± 0.0004
	10	0.0014 ± 0.0003
	11	0.0010 ± 0.0002
	12	0.0007 ± 0.0001
2.449	1	1.0000(b)
	2	0.1755 ± 0.0250
	3	0.0479 ± 0.0049
	4	0.0200 ± 0.0016
	5	0.0098 ± 0.0010
	6	0.0059 ± 0.0006
	7	0.0036 ± 0.0005
	8	0.0025 ± 0.0003
	9	0.0017 ± 0.0003
	10	0.0012 ± 0.0002
	11	0.0009 ± 0.0002
	12	0.0007 ± 0.0001
3.873	1	1.0000(b)
	2	0.1455 ± 0.0247
	3	0.0338 ± 0.0063
	4	0.0138 ± 0.0020
	5	0.0061 ± 0.0011
	6	0.0036 ± 0.0007
	7	0.0020 ± 0.0005
	8	0.0013 ± 0.0004
	9	0.0008 ± 0.0003
	10	0.0006 ± 0.0002
	11	(a)
	12	(a)

q^2 GeV^2	n	μ_n
7.071	1	1.0000[(b)]
	2	0.1549 ± 0.0206
	3	0.0414 ± 0.0078
	4	0.0162 ± 0.0034
	5	0.0089 ± 0.0024
	6	0.0045 ± 0.0016
	7	0.0031 ± 0.0013
	8	0.0018 ± 0.0010
	9	0.0013 ± 0.0009
	10	0.0008 ± 0.0007
	11	0.0006 ± 0.0006
	12	(a)
14.100	1	1.0000[(b)]
	2	0.1204 ± 0.0174
	3	0.0278 ± 0.0092
	4	0.0098 ± 0.0049
	5	0.0030 ± 0.0042
	6	0.0013 ± 0.0026
	7	0.0001 ± 0.0024
	8	(a)
	9	(a)
	10	(a)
	11	(a)
	12	(a)
20.000	1	1.0000[(b)]
	2	0.1114 ± 0.0105
	3	0.0254 ± 0.0042
	4	0.0090 ± 0.0022
	5	0.0031 ± 0.0019
	6	0.0015 ± 0.0017
	7	0.0005 ± 0.0012
	8	(a)
	9	(a)
	10	(a)
	11	(a)
	12	(a)
28.284	1	1.0000[(b)]
	2	0.1000 ± 0.0127
	3	0.0244 ± 0.0041
	4	0.0083 ± 0.0022
	5	0.0032 ± 0.0019
	6	0.0015 ± 0.0013
	7	0.0007 ± 0.0013
	8	0.0005 ± 0.0010
	9	(a)
	10	(a)
	11	(a)
	12	(a)

q^2 GeV^2	n	μ_n
48.990	1	1.0000$^{(b)}$
	2	0.0809 ± 0.0105
	3	0.0243 ± 0.0036
	4	0.0085 ± 0.0017
	5	0.0034 ± 0.0012
	6	0.0017 ± 0.0008
	7	0.0008 ± 0.0006
	8	0.0005 ± 0.0004
	9	(a)
	10	(a)
	11	(a)
	12	(a)
63.246	1	1.0000$^{(b)}$
	2	0.0746 ± 0.0099
	3	0.0217 ± 0.0025
	4	0.0076 ± 0.0010
	5	0.0032 ± 0.0007
	6	0.0016 ± 0.0004
	7	0.0009 ± 0.0004
	8	0.0006 ± 0.0002
	9	(a)
	10	(a)
	11	(a)
	12	(a)
77.460	1	1.0000$^{(b)}$
	2	0.0755 ± 0.0904
	3	0.0201 ± 0.0136
	4	0.0069 ± 0.0023
	5	0.0031 ± 0.0008
	6	0.0016 ± 0.0005
	7	0.0010 ± 0.0004
	8	(a)
	9	(a)
	10	(a)
	11	(a)
	12	(a)

(a) The moments are less than 0.0001 for these n values

(b) The Gross-Llewellyn Smith sum rule is assumed correct - see equation (3.28)

CHAPTER 5 THE QUARK DISTRIBUTIONS

In Chapter 2 we introduced a normalised distribution in energy and three-momentum of the quarks, $P(k_0,k)$, to phenomenologically account for confinement and in Chapters 3 and 4 we defined the moments of this distribution and related these to the experimental moments of neutrino structure functions. We now test various models of the quarks' motion by attempting to fit various parameterizations of $P(k_0,k)$ to the data in its explicit form in Table 1.

§5.1 Models with free, massless quarks

In the first instance we ignore confinement and consider the deep-inelastic lepton scattering as due to incoherent elastic scattering off free, massless quarks, the final-state interactions taking place on a much longer time-scale and thus being disregarded. It is evident from Appendix A that such a model with quarks remaining on mass shell will have the 'sea' contributing in the unphysical region $x < 0$ and consequently F_2 and xF_3 will be identical in the region $x > 0$ (see equations (3.19) and (3.20)). This pathology will be rectified in §5.3 when confinement is taken into account.

For the present, with $|\underline{k}| = k_0$, we may rewrite equations (3.30)-(3.35) as:

$$M_1 = 1 \tag{5.1}$$

$$M_2 = \frac{4}{3M}\left\langle k\right\rangle \tag{5.2}$$

$$M_3 = \frac{2}{M^2}\left\langle k^2\right\rangle \tag{5.3}$$

$$M_4 = \frac{16}{5M^3}\left\langle k^3\right\rangle \tag{5.4}$$

$$M_5 = \frac{16}{3M^4}\left\langle k^4\right\rangle \tag{5.5}$$

$$M_6 = \frac{64}{7M^5}\left\langle k^5 \right\rangle \tag{5.6}$$

and these equations may in turn be equated with the experimental numbers (at any q^2) given in Table 1.

With the definition equation (3.36) a simple generalization of the Cauchy-Schwarz inequality in terms of the moments [19] requires the following inequalities to be valid:

$$\left\langle k^a \right\rangle \left\langle k^b \right\rangle > \left\langle k^{\frac{a+b}{2}} \right\rangle^2 \tag{5.7}$$

(provided $a \neq b$), which may be rewritten in terms of M_n:

$$\Delta_{13} \equiv M_1 M_3 - \frac{9}{8} M_2^2 > 0 \tag{5.8}$$

$$\Delta_{15} \equiv M_1 M_5 - \frac{4}{3} M_3^2 > 0 \tag{5.9}$$

$$\Delta_{24} \equiv M_2 M_4 - \frac{16}{15} M_3^2 > 0 \tag{5.10}$$

$$\Delta_{26} \equiv M_2 M_6 - \frac{25}{21} M_4^2 > 0 \tag{5.11}$$

$$\Delta_{35} \equiv M_3 M_5 - \frac{25}{24} M_4^2 > 0 \tag{5.12}$$

$$\Delta_{46} \equiv M_4 M_6 - \frac{36}{35} M_5^2 > 0 \tag{5.13}$$

Therefore initial criteria for the simple model of free, massless quarks are that the quantities Δ must be positive. Taking values for M_n from the experimental moments in Table 1 the values for Δ at various q^2 are calculated in Table 2. We see that the criteria are satisfied except for the larger moments, where the experimental errors are more substantial; the discrepancy is not serious and, for initial orientation, we may proceed with models incorporating free, massless quarks.

Table 2. The quantities Δ defined by equations (5.8)-(5.13), which are required to be positive

$q^2 GeV^2$	Δ_{13}	Δ_{15}	Δ_{24}	Δ_{26}	Δ_{35}	Δ_{46}
0.387	0.0214	0.0074	0.0015	0.0002	-0.0002	+0.0000 [a]
0.592	0.0276	0.0114	0.0013	0.0006	0.0001	-0.0000 [a]
0.837	0.0407	0.0148	0.0005	0.0005	0.0003	-0.0001
1.225	0.0261	0.0120	0.0013	0.0007	0.0001	-0.0000 [a]
1.732	0.0124	0.0085	0.0014	0.0007	0.0001	+0.0000 [a]
2.449	0.0132	0.0067	0.0011	0.0006	0.0001	+0.0000 [a]
3.873	0.0100	0.0046	0.0008	0.0003	+0.0000 [a]	+0.0000 [a]
7.071	0.0144	0.0066	0.0007	0.0004	0.0001	-0.0000 [a]
14.100	0.0115	0.0020	0.0004	+0.0000 [a]	-0.0000 [a]	+0.0000 [a]
20.000	0.0114	0.0022	0.0003	0.0001	-0.0000 [a]	+0.0000 [a]
28.284	0.0132	0.0024	0.0002	0.0001	+0.0000 [a]	+0.0000 [a]
48.990	0.0169	0.0026	0.0001	0.0001	+0.0000 [a]	+0.0000 [a]
63.246	0.0154	0.0026	0.0001	0.0001	+0.0000 [a]	+0.0000 [a]
77.460	0.0137	0.0026	0.0001	0.0001	+0.0000 [a]	+0.0000 [a]

(a) Where the quantity is less than 0.0001 the sign is stated.

§5.1.1 Free, massless quarks with a single fixed energy

We begin with the simplest possibility of a fixed energy k_0 for the quarks which remain on mass shell. The contribution to the structure function from a single quark is then given by the leading term of equation (2.38):

$$I(x) = \frac{M^2}{2k_0^2} \tag{5.14}$$

a uniform distribution which satisfies the normalisation requirement (3.26) over the range $0 < x < \frac{2k_0}{M}$. With $|\underline{k}| = k_0$ (fixed), the first four moments are given by equations (3.30)-(3.33):

$$M_1 = 1 \tag{5.15}$$

$$M_2 = \frac{k^2}{3Mk_0} + \frac{k_0}{M} = \frac{4}{3M}k_0 \tag{5.16}$$

$$M_3 = \frac{k^2}{M^2} + \frac{k_0^2}{M^2} = \frac{2}{M^2}k_0^2 \tag{5.17}$$

$$M_4 = \frac{k^4}{5M^3k_0} + \frac{2k^2k_0}{M^3} + \frac{k_0^3}{M^3} = \frac{16}{5M^3}k_0^3 \tag{5.18}$$

Equations (5.16)-(5.18) thus offer an immediate test of this elementary model when equated with the experimental moments μ_2, μ_3 and μ_4 from Table 1, *viz*:

$$\frac{k_0}{M} = \tfrac{3}{4}\mu_2 = (\tfrac{1}{2}\mu_3)^{\frac{1}{2}} = (\tfrac{5}{16}\mu_4)^{\frac{1}{3}} \tag{5.19}$$

The values of these quantities and their errors are displayed in figure 6. It is not surprising to find that, even within experimental errors, the equality (5.19) cannot hold in general and that a model with free, massless quarks of fixed energy is incompatible with the data. However, in the next section, a modification will be seen to lead to considerable improvement.

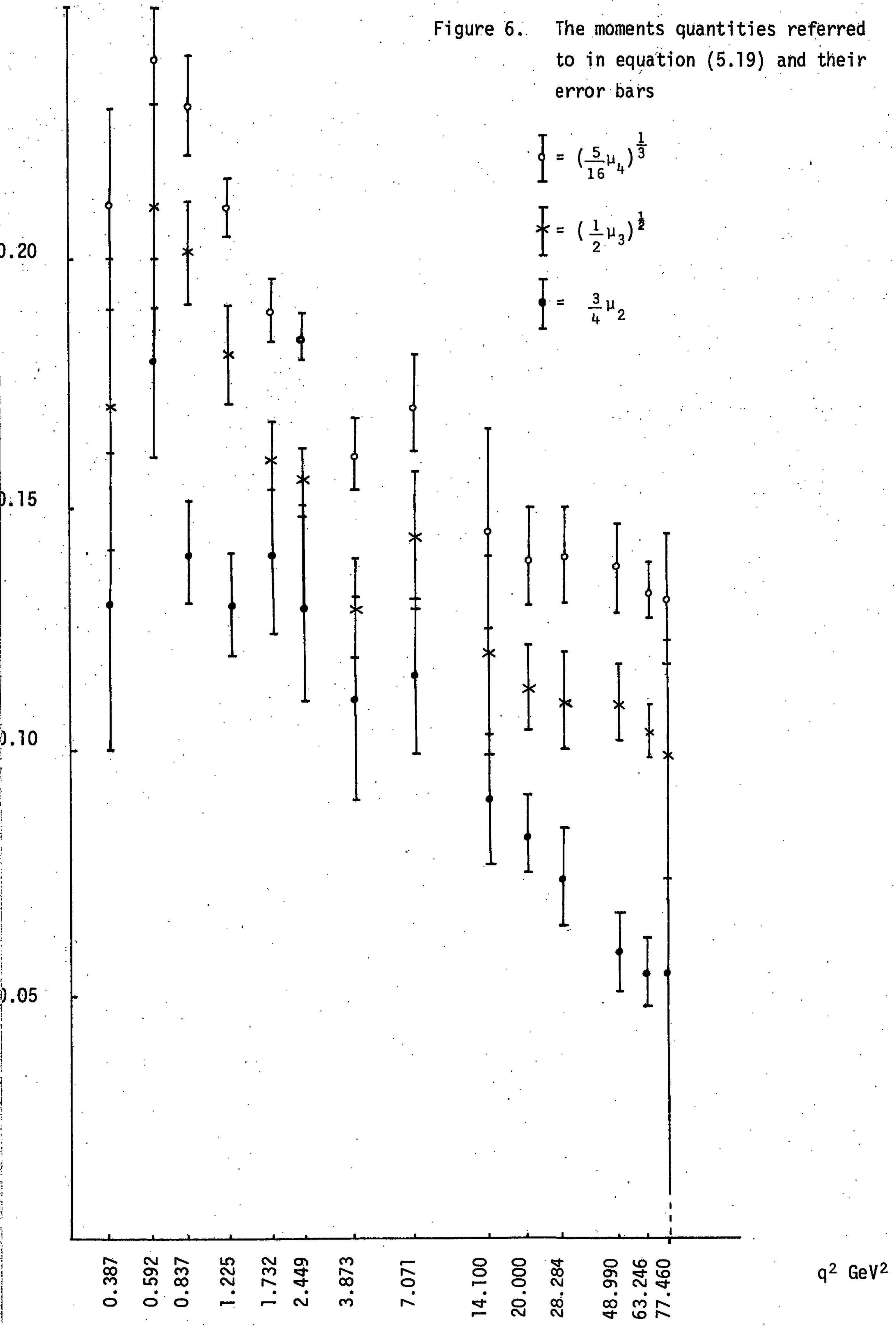


Figure 6. The moments quantities referred to in equation (5.19) and their error bars

§5.1.2 Free, massless quarks with varying energy

In considering a distribution of free, massless quarks with varying energy we seek to include the smallest number of parameters which will give a reasonable fit to the experimental moments, although in theory an infinite number of moments and therefore an infinite number of parameters will be needed to precisely determine the function. For guidance on this point the method of Yndurain [20] has been utilized (see Appendix C and the comments therein) to show that the first three moments (including normalisation) are adequate for our purposes to reconstruct the structure function and we thus include three parameters (including normalisation constant) in our distribution:

$$P(k_0,k) = N\,\frac{e^{-\alpha k_0}}{k_0^2}\,(1 + ak_0)\;\delta(k - k_0) \tag{5.20}$$

The form of this distribution allows us to perform the necessary integrals analytically in both the x model and the $\log_e(1-x)$ model.

When this form for the distribution is substituted into equations (3.30)-(3.32) for the *x model* the integrals may be performed straightforwardly and equated with the experimental moments at various q^2:

$$M_1 = N\left(\frac{1}{\alpha} + \frac{a}{\alpha^2}\right) = \mu_1(q^2) = 1 \tag{5.21}$$

$$M_2 = \frac{4N}{3M}\left(\frac{1}{\alpha^2} + \frac{2a}{\alpha^3}\right) = \mu_2(q^2) \tag{5.22}$$

$$M_3 = \frac{2N}{M^2}\left(\frac{2}{\alpha^3} + \frac{6a}{\alpha^4}\right) = \mu_3(q^2) \tag{5.23}$$

This allows solutions α,a (at various q^2) given by:

$$\alpha M = \frac{3\mu_2 \pm \sqrt{9\mu_2{}^2 - 4\mu_3}}{\mu_3} \tag{5.24}$$

$$a = \left(\frac{4 - 3\alpha M\mu_2}{8 - 3\alpha M\mu_2}\right)\alpha = \left(\frac{4 - \alpha^2M^2\mu_3}{12 - \alpha^2M^2\mu_3}\right)\alpha \tag{5.25}$$

Our normalisation constant, N, may be obtained from equation (5.21):

$$N^{-1} = \frac{1}{\alpha} + \frac{a}{\alpha^2} \tag{5.26}$$

To illustrate the procedure which may be adopted if one wishes to examine the q^2 dependence, we select the experimental moments at the particular $q^2 = 2.449$ GeV^2 from Table 1 and obtain the following two sets of solutions:

$$\alpha M = 4.884 \quad \text{or} \quad 17.100 \tag{5.27}$$

$$aM = -1.285 \quad \text{or} \quad -85.293 \tag{5.28}$$

$$NM = 6.628 \quad \text{or} \quad -4.288 \tag{5.29}$$

The second set of solutions is inadmissible because of a negative normalisation constant which thus prevents the distribution in equation (5.20) from remaining positive definite.

We test the consistency of the first set of solutions by calculating the structure function F_2 (or xF_3) through equations (3.1) and (3.19) (or (3.20), recalling that in this model there is no 'sea' contribution at $x > 0$) and comparing with the combined BEBC and Gargamelle structure functions of ref. 17. With the distribution (5.20) equation (3.19) (or (3.20)) becomes:

$$F_2^{\nu N}(x) \text{ (or } xF_3^{\nu N}) = \frac{3NM^2x^2}{2}\int_{\frac{Mx}{2}}^{\infty} \frac{e^{-\alpha k_0}}{k_0^2}(1 + ak_0)dk_0 \tag{5.30}$$

$$= \frac{3NMx^2}{2}\int_{\frac{x}{2}}^{\infty} \frac{e^{-\alpha My}}{y^2}(1 + aMy)dy \tag{5.31}$$

which may be integrated directly [21] to give:

$$F_2^{\nu N}(x) \text{ (or } xF_3^{\nu N}) = \frac{3NMx^2}{2}\left[\frac{2}{x}e^{-\frac{\alpha Mx}{2}} - (a - \alpha)M\mathrm{Ei}\left(-\frac{\alpha Mx}{2}\right)\right] \tag{5.32}$$

With the admissible values of α, a and N given by equations (5.27)-(5.29) this function is shown in figure 7 along with the experimental $xF_3^{\nu N}$ from ref. 17: it is evident that by fitting only the first three moments more emphasis is attached to $xF_3^{\nu N}$ than $F_2^{\nu N}$ and, for clarity, we show only the former for comparison. Within the naivety of the model and the large experimental errors the agreement is adequate. However, at large x our structure function takes on small negative values due to the negative value of a: we will return to this point later.

We now perform the same procedure in the *$\log_e(1-x)$ model*. With the same distribution (5.20) equation (3.38) becomes:

$$L_n = \frac{NM^2}{2(n+1)} \left(I_1^{n+1} + aI_2^{n+1}\right) \tag{5.33}$$

where:

$$I_1^{n+1} = \int_0^\infty \frac{e^{-\alpha k_0}}{k_0^2}\left(1 - e^{-\frac{2k_0}{M}}\right)^{n+1} dk_0$$

$$= \frac{\alpha}{2}\sum_{m=0}^{n+1} (-1)^m \binom{n+1}{m} (pm+1)^2 \log_e(pm+1) \tag{5.34}$$

$$I_2^{n+1} = \int_0^\infty \frac{e^{-\alpha k_0}}{k_0}\left(1 - e^{-\frac{2k_0}{M}}\right)^{n+1} dk_0$$

$$= (-1)^n \sum_{m=0}^{n+1} (-1)^{n+1-m} \binom{n+1}{n+1-m} \log_e(pm+1) \tag{5.35}$$

and

$$p \equiv \frac{2}{\alpha M} \tag{5.36}$$

Therefore, in the $\log_e(1-x)$ model, the analogous equations to (5.21)-(5.23) are:

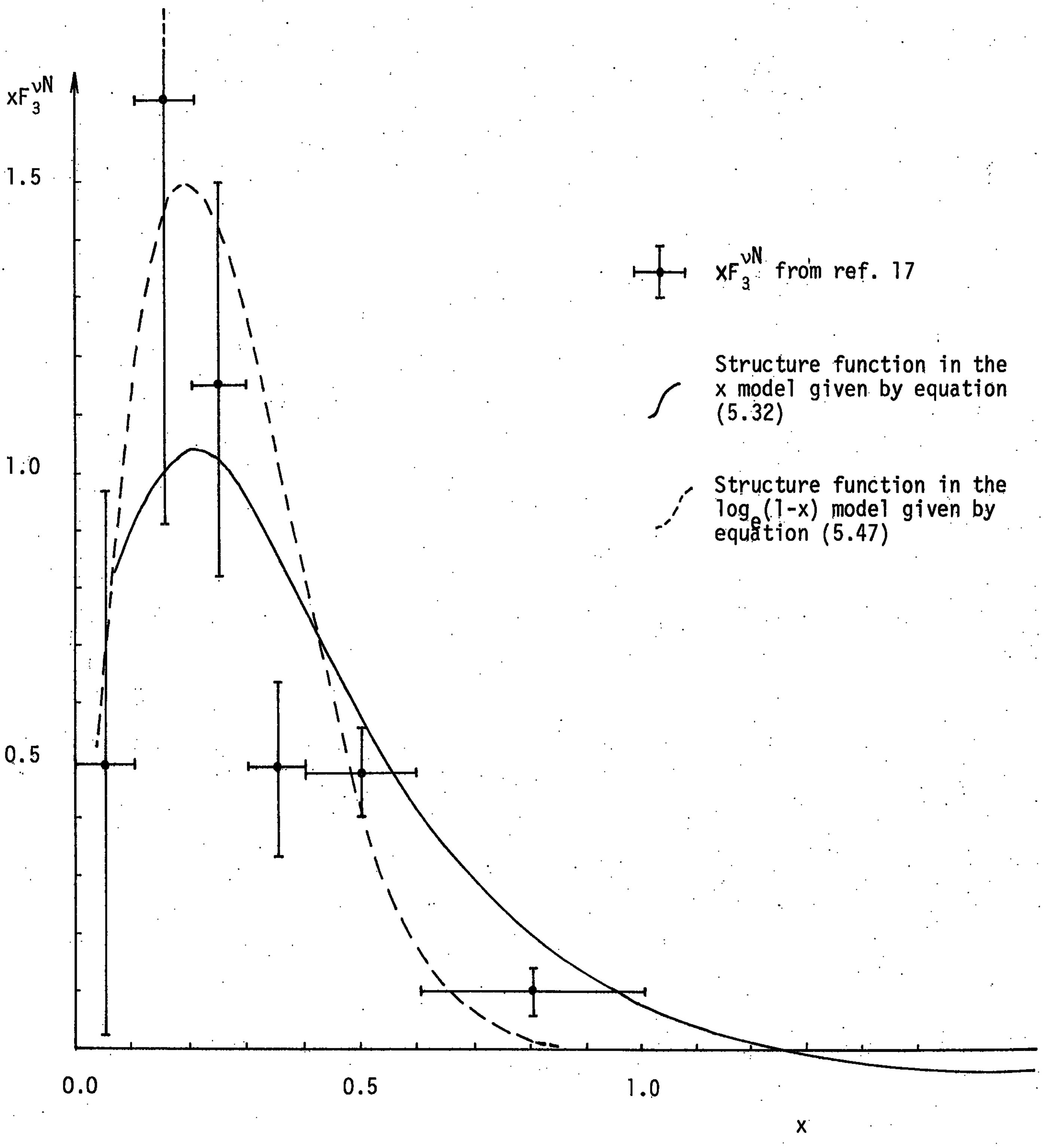


Figure 7. Comparison of the structure functions in different models at q^2 = 2.449 GeV^2 with the combined BEBC and Gargamelle data points (only statistical errors and estimates of smearing and center-of-bin correction are incorporated in error bars).

$$L_1 = \frac{NM}{4}\left(\frac{1}{p}\sum_{m=0}^{2} (-1)^m \binom{2}{m} (pm+1)^2 \log_e(pm+1) - aM\sum_{m=0}^{2} (-1)^{2-m}\binom{2}{2-m}\log_e(pm+1)\right)$$

$$= \mu_1(q^2) = 1 \qquad (5.37)$$

$$L_2 = \frac{NM}{6}\left(\frac{1}{p}\sum_{m=0}^{3} (-1)^m \binom{3}{m} (pm+1)^2 \log_e(pm+1) + aM\sum_{m=0}^{3} (-1)^{3-m}\binom{3}{3-m}\log_e(pm+1)\right)$$

$$= \mu_2(q^2) \qquad (5.38)$$

$$L_3 = \frac{NM}{8}\left(\frac{1}{p}\sum_{m=0}^{4} (-1)^m \binom{4}{m} (pm+1)^2 \log_e(pm+1) - aM\sum_{m=0}^{4} (-1)^{4-m}\binom{4}{4-m}\log_e(pm+1)\right)$$

$$= \mu_3(q^2) \qquad (5.39)$$

Note that in this model our distribution is automatically 'renormalised' by assuming that $\mu_1(q^2) = 1$, so that the Gross-LLewellyn Smith sum rule is satisfied. After expanding the summations and eliminating N, the solution for p (i.e. α) is given by solving the equation:

$$F(p,\mu_2(q^2)) = G(p,\mu_2(q^2),\mu_3(q^2)) \qquad (5.40)$$

where

$$F(p,\mu_2(q^2)) \equiv \frac{6(1-\mu_2)(p+1)^2\log_e(p+1) - 3(2-\mu_2)(2p+1)^2\log_e(2p+1) + 2(3p+1)^2\log_e(3p+1)}{6(1-\mu_2)\log_e(p+1) - 3(2-\mu_2)\log_e(2p+1) + 2\log_e(3p+1)}$$

$$(5.41)$$

$$G(p,\mu_2(q^2),\mu_3(q^2)) \equiv$$

$$\frac{4(1-\frac{\mu_3}{\mu_2})(p+1)^2\log_e(p+1)-2(3-\frac{2\mu_3}{\mu_2})(2p+1)^2\log_e(2p+1)+4(1-\frac{\mu_3}{3\mu_2})(3p+1)^2\log_e(3p+1) - (4p+1)^2\log_e(4p+1)}{4(1-\frac{\mu_3}{\mu_2})\log_e(p+1) - 2(3-\frac{2\mu_3}{\mu_2})\log_e(2p+1) + 4(1-\frac{\mu_3}{3\mu_2})\log_e(3p+1) - \log_e(4p+1)}$$

$$(5.42)$$

The solution for a is then:

$$a = \frac{\alpha F}{2} = \frac{\alpha G}{2} \qquad (5.43)$$

The normalisation constant, N, is then most easily obtained from equation (5.37).

Again specifically at $q^2 = 2.449$ GeV^2, we obtain the following two sets of solutions:

$$\alpha M = 9.703 \quad \text{or} \quad 124.942 \tag{5.44}$$

$$aM = 811.883 \quad \text{or} \quad -53.916 \tag{5.45}$$

$$NM = 0.162 \quad \text{or} \quad -72.802 \tag{5.46}$$

With the distribution (5.20) the structure function is given by the modification of equation (5.32):

$$F_2^{\nu N}(x) \text{ (or } xF_3^{\nu N}) = \frac{3NMx^2}{2}\left[-\frac{2(1-x)^{\frac{\alpha M}{2}}}{\log_e(1-x)} - (a-\alpha)MEi\left(\frac{\alpha M}{2}\log_e(1-x)\right)\right] \tag{5.47}$$

With the admissible values of α, a and N given by equations (5.44)-(5.46) this function is compared in figure 7 with the structure function in the x model and the experimental values at $q^2 = 2.449$ GeV^2 from ref. 17. The $\log_e(1-x)$ model maps the region $0 < x < \infty$ onto $0 < x < 1$ and this gives better agreement with the data than the x model, in which the contribution from $x > 1$, although small, is significant in fitting the n=3 moment when weighted by a factor x.

§5.2 A model with free, massive quarks with varying energy

We now investigate the effect of giving the quarks a small, constant mass whilst retaining them on mass shell. Our approach will be along the same lines as in the massless case, i.e. §5.1.

With $k^2 = k_0^2 - m^2$, where m is the quark mass, we may write the moments equations (3.30)-(3.35) as:

$$M_1 = 1 \tag{5.48}$$

$$M_2 = \frac{4}{3M}\langle k_0\rangle - \frac{m^2}{3M}\left\langle\frac{1}{k_0}\right\rangle \tag{5.49}$$

$$M_3 = \frac{2}{M^2}\langle k_0^2\rangle - \frac{m^2}{M^2} \tag{5.50}$$

$$M_4 = \frac{16}{5M^3}\langle k_0^3\rangle - \frac{12m^2}{5M^3}\langle k_0\rangle + \frac{m^4}{5M^3}\left\langle\frac{1}{k_0}\right\rangle \tag{5.51}$$

$$M_5 = \frac{16}{3M^4}\langle k_0^4\rangle - \frac{16m^2}{3M^4}\langle k_0^2\rangle + \frac{m^4}{M^4} \tag{5.52}$$

$$M_6 = \frac{64}{7M^5}\langle k_0^5\rangle - \frac{80m^2}{7M^5}\langle k_0^3\rangle + \frac{24m^4}{7M^5}\langle k_0\rangle - \frac{m^6}{7M^5}\left\langle\frac{1}{k_0}\right\rangle \tag{5.53}$$

where the 'natural' definition of $\langle k_0^n\rangle$ follows from equation (3.36):

$$\langle k_0^n\rangle \equiv \int_m^\infty k_0^n dk_0 (k_0^2 - m^2)\, P(k_0,k) \tag{5.54}$$

With this definition the revised moments inequalities of equation (5.7), in terms of k_0, can be tested and (in some cases) used to determine bounds on the quark mass imposed by the values of experiment. Analogous to equations (5.8)-(5.13) we have:

$$\Delta_{13} \equiv M_1(M_3 + \frac{m^2}{M^2}) - \frac{9}{8}(M_2 + \frac{m^2}{3M}\left\langle\frac{1}{k_0}\right\rangle)^2 > 0 \tag{5.55}$$

$$\Delta_{15} \equiv M_1(M_5 + \frac{8m^2}{3M^2}M_3 + \frac{5m^4}{3M^4}) - \frac{4}{3}(M_3 + \frac{m^2}{M^2})^2 > 0 \tag{5.56}$$

$$\Delta_{24} \equiv (M_2 + \frac{m^2}{3M}\left\langle\frac{1}{k_0}\right\rangle)(M_4 + \frac{9m^2}{5M^2}M_2 + \frac{2m^4}{5M^3}\left\langle\frac{1}{k_0}\right\rangle) - \frac{16}{15}(M_3 + \frac{m^2}{M^2})^2 > 0 \tag{5.57}$$

$$\Delta_{26} \equiv (M_2 + \frac{m^2}{3M}\left\langle\frac{1}{k_0}\right\rangle)(M_6 + \frac{25m^2}{7M^2}M_4 - \frac{27m^4}{7M^4}M_2 + \frac{5m^6}{M^5}\left\langle\frac{1}{k_0}\right\rangle)$$

$$- \frac{25}{21}(M_4 + \frac{9m^2}{5M^2}M_2 + \frac{4m^4}{5M^3}\left\langle\frac{1}{k_0}\right\rangle)^2 > 0 \tag{5.58}$$

$$\Delta_{35} \equiv (M_3 + \frac{m^2}{M^2})(M_5 + \frac{8m^2}{3M^2}M_3 + \frac{5m^4}{3M^4}) - \frac{25}{24}(M_4 + \frac{9m^2}{5M^2}M_2 + \frac{2m^4}{5M^3}\left\langle\frac{1}{k_0}\right\rangle)^2 > 0 \tag{5.59}$$

$$\Delta_{46} \equiv (M_4 + \frac{9m^2}{5M^2}M_2 + \frac{2m^4}{5M^3}\left\langle\frac{1}{k_0}\right\rangle)(M_6 + \frac{25m^2}{7M^2}M_4 + \frac{27m^4}{7M^4}M_2 + \frac{5m^6}{7M^5}\left\langle\frac{1}{k_0}\right\rangle)$$

$$- \frac{36}{35}(M_5 + \frac{8m^2}{3M^2}M_3 + \frac{5m^4}{3M^4})^2 > 0 \tag{5.60}$$

Equations (5.8)-(5.13) can be retrieved by letting $m \to 0$. Moreover, by substituting the relevant experimental values μ_n from Table 1 the quantities Δ reduce to quadratic expressions in $\left\langle\frac{1}{k_0}\right\rangle$ or, in the case of Δ_{15}, a direct inequality in m which is satisfied at all q^2 for all $m \geqslant 0$. However, the requirement of positiveness of the quantity $\left\langle\frac{1}{k_0}\right\rangle$ does allow a determination of lower limits on m; for example, at $q^2 = 2.449$ GeV^2, inequalities (5.58) and (5.60) respectively lead to the bounds :

$$m \geqslant 0.47 \text{ GeV} \tag{5.61}$$

$$m \geqslant 0.20 \text{ GeV} \tag{5.62}$$

However the relevant quantities Δ_{26} and Δ_{46} both involve several moments, thus compounding their accompanying errors, and little significance should be attached to (5.61) and (5.62). The validity of the other inequalities for all $m \geqslant 0$ at $q^2 = 2.449\ \text{GeV}^2$ allows us to continue with this model with free, massive quarks.

We adopt the following distribution with free, massive quarks:

$$P(k_0,k) = N(k_0 - m)e^{-\alpha k^2}(1 + ak_0 + bk_0^2)\delta(k - \sqrt{k_0^2-m^2}) \qquad (5.63)$$

The form of this distribution is chosen to allow an analytical evaluation of the relevant integrals in the x model but precludes the $\log_e(1-x)$ model. For various α and m we treat N, a and b as parameters to be determined by the first three moments equations. It seems reasonable in this respect to restrict our attention to the ranges:

$$1 < \alpha M^2 < 300 \qquad (5.64)$$

$$0.01\ \text{GeV} < m < 0.50\ \text{GeV} \qquad (5.65)$$

Using Appendix D, equations (3.30)-(3.32) become:

$$M_1 = Ne^{\alpha m^2}\Big\{a\Big[\sqrt{\pi}\ \text{erfc}\ m\sqrt{\alpha}\Big(\frac{3}{8\sqrt{\alpha^5}} - \frac{m^2}{4\sqrt{\alpha^3}}\Big) + \frac{e^{-\alpha m^2}}{4\alpha^2}m\Big]$$

$$+ b\Big[\sqrt{\pi}\ \text{erfc}\ m\sqrt{\alpha}\ \frac{m}{8\sqrt{\alpha^5}}(2\alpha m^2 - 3) - \frac{e^{-\alpha m^2}}{4\alpha^3}(\alpha m^2 - 4)\Big] + \sqrt{\pi}\ \text{erfc}\ m\sqrt{\alpha}\ \frac{m}{4\sqrt{\alpha^3}}(2\alpha m^2 - 1)$$

$$- \frac{e^{-\alpha m^2}}{2\alpha^2}(\alpha m^2 - 1)\Big\}$$

$$= \mu_1(q^2) = 1 \qquad (5.66)$$

$$M_2 = \frac{Ne^{\alpha m^2}}{3M}\Big\{a\Big[\sqrt{\pi}\ \text{erfc}\ m\sqrt{\alpha}\ \frac{m}{4\sqrt{\alpha^5}}(-2\alpha^2m^4+5\alpha m^2-6) + \frac{e^{-\alpha m^2}}{2\alpha^3}(\alpha^2m^4-3\alpha m^2+8)\Big]$$

$$+ b\Big[\sqrt{\pi}\ \text{erfc}\ m\sqrt{\alpha}\ \frac{1}{8\sqrt{\alpha^7}}(2\alpha^2m^4-15\alpha m^2+30) - \frac{e^{-\alpha m^2}}{4\alpha^3}m(\alpha m^2-14)\Big]$$

$$+ \sqrt{\pi}\ \text{erfc}\ m\sqrt{\alpha}\ \frac{1}{4\sqrt{\alpha^5}}(2\alpha^2m^4-5\alpha m^2+6) + \frac{e^{-\alpha m^2}}{\alpha^2}m - \frac{m^5}{2}\text{Ei}(-\alpha m^2)\Big\}$$

$$= \mu_2(q^2) \qquad (5.67)$$

$$M_3 = \frac{2Ne^{\alpha m^2}}{M^2}\left\{a\left[\sqrt{\pi}\text{ erfc } m\sqrt{\alpha}\frac{1}{16\sqrt{\alpha^7}}(-6\alpha m^2+15) + \frac{e^{-\alpha m^2}}{8\alpha^3}7m\right]\right.$$

$$+ b\left[\sqrt{\pi}\text{ erfc } m\sqrt{\alpha}\frac{m}{16\sqrt{\alpha^7}}(6\alpha m^2-15) + \frac{e^{-\alpha m^2}}{8\alpha^4}(\alpha m^2+24)\right]$$

$$\left.+ \sqrt{\pi}\text{ erfc } m\sqrt{\alpha}\frac{m}{8\sqrt{\alpha^5}}(2\alpha m^2-3) - \frac{e^{-\alpha m^2}}{4\alpha^3}(\alpha m^2-4)\right\}$$

$$= \mu_3(q^2) + m^2 \tag{5.68}$$

One can then patiently eliminate the normalisation constant, N, from equations (5.66) and (5.67) and from equations (5.66) and (5.68) to obtain two simultaneous equations in a and b:

$$f_{11}(\alpha,m,\mu_2(q^2))a + f_{12}(\alpha,m,\mu_2(q^2))b = g_1(\alpha,m,\mu_2(q^2)) \tag{5.69}$$

$$f_{21}(\alpha,m,\mu_3(q^2))a + f_{22}(\alpha,m,\mu_3(q^2))b = g_2(\alpha,m,\mu_3(q^2)) \tag{5.70}$$

where:

$$f_{11}(\alpha,m,\mu_2(q^2)) \equiv \sqrt{\pi}\text{ erfc } m\sqrt{\alpha}\frac{1}{16\sqrt{\alpha^5}}(-8\alpha^2m^5+20\alpha m^3+12\mu_2M\alpha m^2-24m-18\mu_2M)$$

$$+ \frac{e^{-\alpha m^2}}{8\alpha^3}(4\alpha^2m^4-12\alpha m^2-6\mu_2M\alpha m+32) \tag{5.71}$$

$$f_{12}(\alpha,m,\mu_2(q^2)) \equiv \sqrt{\pi}\text{ erfc } m\sqrt{\alpha}\frac{1}{8\sqrt{\alpha^7}}(2\alpha^2m^4-6\mu_2M\alpha^2m^3-15\alpha m^2+9\mu_2M\alpha m+30)$$

$$- \frac{e^{-\alpha m^2}}{4\alpha^3}(\alpha m^3-3\mu_2M\alpha m^2+12\mu_2M-14m) \tag{5.72}$$

$$f_{21}(\alpha,m\ \mu_3(q^2)) \equiv \sqrt{\pi}\text{ erfc } m\sqrt{\alpha}\frac{1}{16\sqrt{\alpha^7}}(2\alpha^2m^4+2\mu_3M^2\alpha^2m^2-9\alpha m^2-3\mu_3M^2\alpha+15)$$

$$- \frac{me^{-\alpha m^2}}{8\alpha^3}(\alpha m^2+\mu_3M^2\alpha-7) \tag{5.73}$$

$$f_{22}(\alpha,m,\mu_3(q^2)) \equiv \sqrt{\pi}\,\mathrm{erfc}\,m\sqrt{\alpha}\frac{1}{16\sqrt{\alpha^7}}(-2[\mu_3M^2+m^2]\alpha^2m^3+9\alpha m^3+3\mu_3M^2\alpha m-15m)$$

$$+\frac{e^{-\alpha m^2}}{8\alpha^4}([\mu_3M^2+m^2]\alpha^2m^2-3\alpha m^2-4\mu_3M^2\alpha+24) \qquad (5.74)$$

$$g_1(\alpha,m,\mu_2(q^2)) \equiv \sqrt{\pi}\,\mathrm{erfc}\,m\sqrt{\alpha}\frac{1}{8\sqrt{\alpha^5}}(-4\alpha^2m^4+12\mu_2M\alpha^2m^3+10\alpha m^2-6\mu_2M\alpha m-12)$$

$$-\frac{e^{-\alpha m^2}}{4\alpha^2}(6\mu_2M\alpha m^2+4m-6\mu_2M) \qquad (5.75)$$

$$g_2(\alpha,m,\mu_3(q^2)) \equiv \sqrt{\pi}\,\mathrm{erfc}\,m\sqrt{\alpha}\frac{1}{8\sqrt{\alpha^5}}(2\alpha^2m^5+2\mu_3M^2\alpha^2m^3-3\alpha m^3-\mu_3M^3\alpha m+3m)$$

$$-\frac{e^{-\alpha m^2}}{4\alpha^3}(\alpha^2m^4+\mu_3M^2\alpha^2m^2-2\alpha m^2-\mu_3M^2\alpha+4) \qquad (5.76)$$

The distribution (5.63) will remain positive definite for all $k_0 > m$ provided:

$$b > 0 \qquad (5.77)$$

$$a^2 < 4b \qquad (5.78)$$

These two conditions impose severe restrictions on the admissibility of solutions for a,b, which are computed and presented in Table 3 for the above ranges of α and m at the specific $q^2 = 2.449\ \mathrm{GeV}^2$; the normalisation constant, N, is computed from equation (5.66). It is evident that an average value can be taken without undue error:

$$m = 0.01\ \mathrm{GeV} \qquad (5.79)$$

$$\alpha M^2 = 79.5 \qquad (5.80)$$

$$aM = -11.48 \qquad (5.81)$$

$$bM^2 = 32.95 \qquad (5.82)$$

$$NM^4 = 1.20\times10^5 \qquad (5.83)$$

Table 3. The sets of solutions to the moments equations at $q^2 = 2.449$ Gev^2 with the parameterized distribution (5.63)

m GeV	αM^2	aM	bM^2	NM^4
0.01	77	-11.21	31.41	1.09×10^5
0.01	78	-11.31	32.02	1.13×10^5
0.01	79	-11.42	32.63	1.18×10^5
0.01	80	-11.53	33.26	1.23×10^5
0.01	81	-11.64	33.88	1.27×10^5
0.01	82	-11.75	34.51	1.32×10^5

Following the procedure of §5.1.2, the structure function F_2 (or xF_3 since our quarks are on mass shell) may be calculated with the distribution (5.63):

$$F_2^{\nu N}(x) \text{ (or } xF_3^{\nu N}) = \frac{3NM^2x^2}{2}\int_{h(x)}^{\infty}\frac{\sqrt{k_0^2-m^2}}{k_0}(k_0-m)e^{-\alpha(k_0^2-m^2)}(1 + ak_0 + bk_0^2)dk_0 \tag{5.84}$$

where:

$$h(x) = \frac{M^2x^2 + m^2}{2Mx} \tag{5.85}$$

Thus, with $z^2 = \alpha(k_0^2 - m^2)$, this becomes:

$$F_2^{\nu N}(x) \text{ (or } xF_3^{\nu N}) = \frac{3NM^2x^2}{2\alpha^2}\left[\sqrt{\alpha}(a - bm)I_0 + \alpha(1 - am)I_1 - m\sqrt{\alpha^3}I_2 + bI_3\right] \tag{5.86}$$

where:

$$I_0 \equiv \int_{\bar{h}(x)}^{\infty} z^2e^{-z^2}dz = \tfrac{1}{2}\Gamma(\tfrac{3}{2}, (\bar{h}(x))^2) \tag{5.87}$$

$$I_1 \equiv \int_{\bar{h}(x)}^{\infty} \frac{z^2}{\sqrt{z^2 + \alpha m^2}} e^{-z^2} dz \tag{5.88}$$

$$I_2 \equiv \int_{\bar{h}(x)}^{\infty} \frac{z^2}{(z^2 + \alpha m^2)} e^{-z^2} dz \tag{5.89}$$

$$I_3 \equiv \int_{\bar{h}(x)}^{\infty} z^2 \sqrt{z^2 + \alpha m^2}\, e^{-z^2}\, dz \tag{5.90}$$

and:

$$\bar{h}(x) \equiv \sqrt{\alpha}\left(\frac{M^2 x^2 - m^2}{2Mx}\right) \tag{5.91}$$

The structure function is computed with the set of parameters given by equations (5.79)-(5.83) and displayed in figure 8. Notwithstanding the double peak (which is caused by the dominance of the 'k_0^2' term with coefficient b at high x and could, in principle, be eliminated by imposing a further restriction), the agreement of this massive x model with experiment is good and should be compared with the analogous massless example of an x model (§5.1.2 and figure 7).

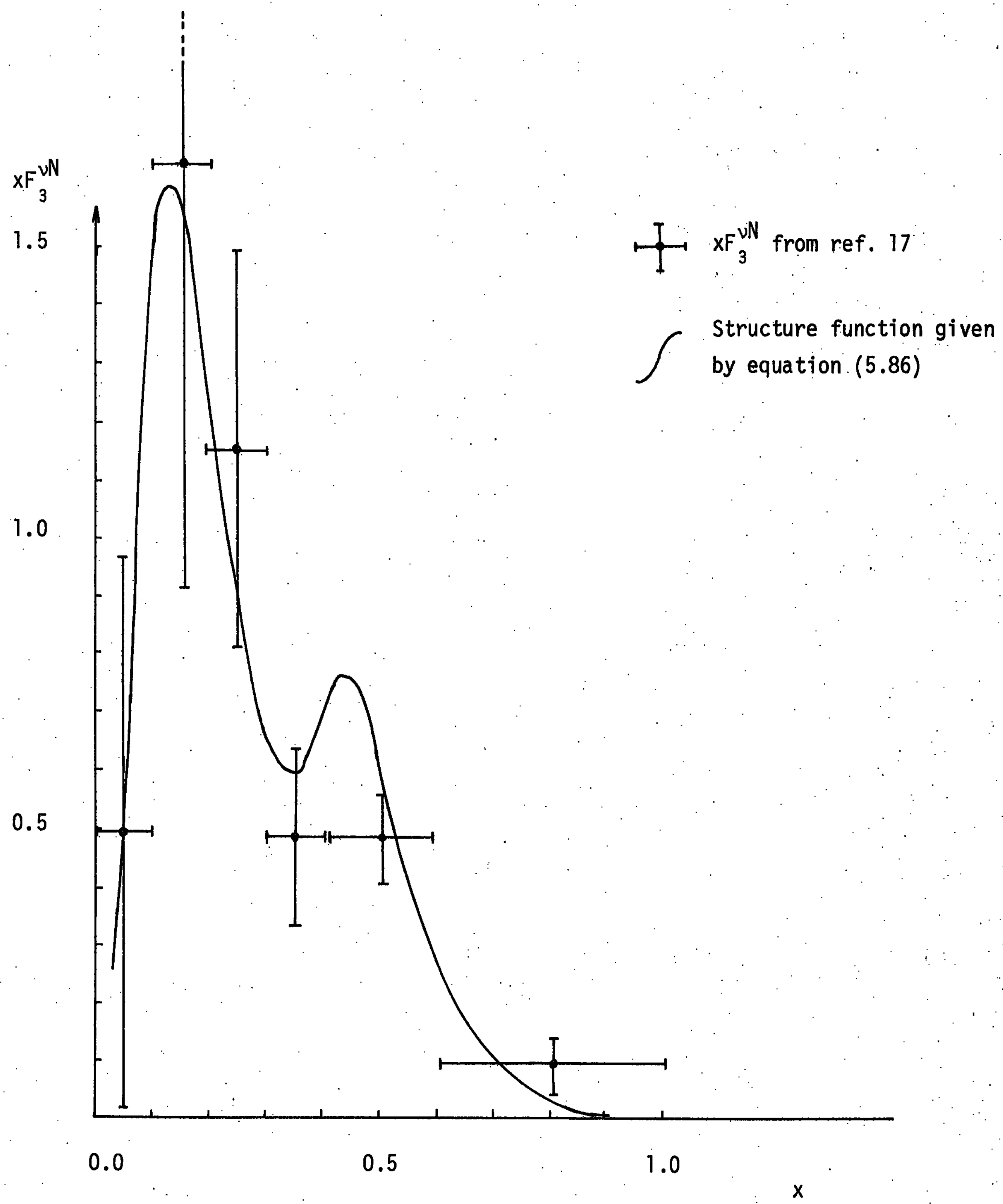


Figure 8. Comparison of the structure function in a model with free, massive quarks, at $q^2 = 2.449$ GeV^2, with the combined BEBC and Gargamelle data points

§5.3 Models with off mass shell, massless quarks

In this section we take account of confinement by treating massless quarks in various off mass shell models with a 'sea' contribution in the physical region $x > 0$. A particularly useful analogue is the MIT bag model [23][24], in which the quarks lie in eigenmodes of energy. Further possibilities of a varying energy as well as a varying three-momentum will be considered in §5.3.2 and §5.3.3.

§5.3.1 Off mass shell, massless quarks with a single fixed energy

With quarks of zero mass and fixed energy, k_0, the familiar moments equations (3.30)-(3.32) in the x model are:

$$M_1 = 1 \tag{5.92}$$

$$M_2 = \frac{\langle k^2 \rangle}{3Mk_0} + \frac{k_0}{M} \tag{5.93}$$

$$M_3 = \frac{\langle k^2 \rangle}{M^2} + \frac{k_0^2}{M^2} \tag{5.94}$$

Equating with the relevant experimental moments from Table 1 it is straightforward to deduce the following solution for k_0 at any particular q^2 :

$$\frac{k_0}{M} = \frac{3}{4}\mu_2(q^2) \pm \frac{1}{4}\sqrt{9\mu_2^2\,(q^2) - 8\mu_3(q^2)} \tag{5.95}$$

From figure 6 it is evident that real roots for k_0 are only permitted by the experimental uncertainties in μ_2 and μ_3; then a range of possible k_0 can be deduced within these experimental limits at various q^2, and this is shown in Table 4. The moments at high q^2 incorporate large experimental errors as the structure functions creep towards $x = 0$, so that any conclusions on the solutions to equation(5.95) must be

Table 4. The range of k_0 allowed within the bounds of experimental errors at various q^2, in an off mass shell, massless quark model with a single fixed energy

q^2 GeV^2	Minimum Value of k_0/M	Maximum Value of k_0/M
0.387	0.083	0.233
0.592	0.172	0.217
0.837	(a)	(a)
1.225	(a)	(a)
1.732	0.126	0.181
2.449	0.119	0.182
3.873	0.080	0.176
7.071	0.118	0.145
14.100	0.076	0.131
20.000	(a)	(a)
28.284	(a)	(a)
48.990	(a)	(a)
63.246	(a)	(a)
77.460	0.022	0.227

(a) No real roots to equation (5.95) exist for these q^2 values considered unreliable in this region. On the other hand a fairly consistent range of k_0 is obtained as q^2 goes from 1.732 GeV^2 to 14.100 GeV^2, and the average value, $\frac{k_0}{M} = 0.141$, is taken as typical within the bounds of experimental errors. Although we are unable to similarly deduce a value for k_0 in the $\log_e(1-x)$ model, it is not unreasonable to assume that this 'typical' value is applicable to facilitate the following comparison of the models for different bag radii.

The massless quark distribution in three-momentum in the ground state of the cavity approximation to the MIT bag model is given by [25]:

$$P(k) = NR^6\left(\frac{\sin(k - k_0)R}{kk_0(k - k_0)R^3} - \frac{\sin k_0 R \sin kR}{k^2k_0^2R^4}\right)^2 \tag{5.96}$$

where R is the bag radius and normalisation may be fixed by the first moment in the x model:

$$\int_0^\infty k^2 dk\, P(k) = 1 \tag{5.97}$$

The shape of the probability distribution, $k^2P(k)$, is shown in figure 9, and in general the most probable value of k will be greater than the on mass shell value because of confinement. The oscillatory nature arises from the rigidity of the bag boundary, and the first minimum in kR is given by the second solution ($k \neq k_0$) to the equation:

$$\cot kR - \frac{1}{kR} = \cot k_0R - \frac{1}{k_0R} \tag{5.98}$$

Normalisation could then be fixed by imposing a cut-off in k in equation (5.97) corresponding to this minimum: this is found to be unsatisfactory because of difficulties in adaption to the $\log_e(1-x)$ model, so we will return to the question of normalisation shortly.

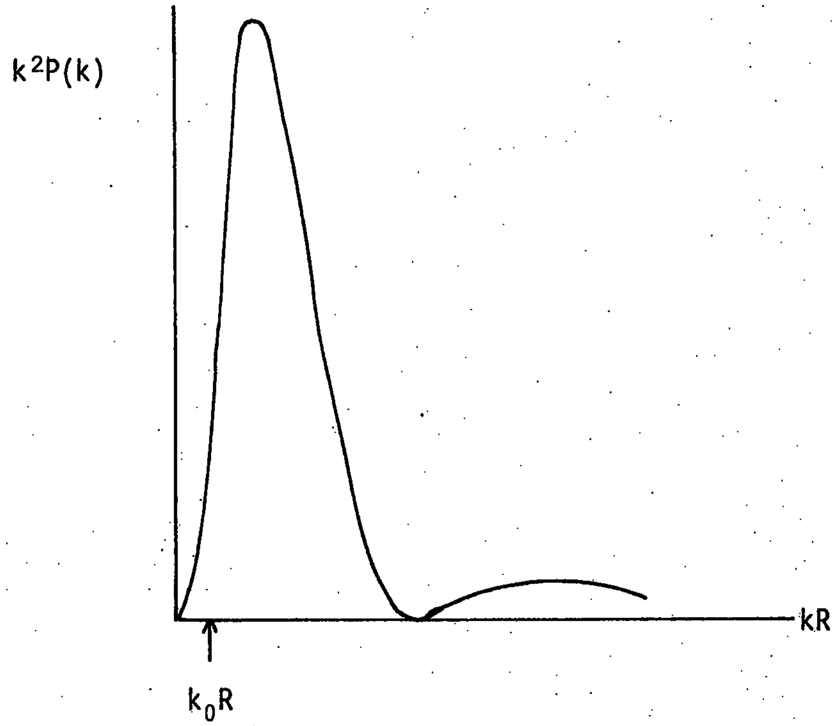


Figure 9. The probability distribution of massless quarks in the ground state of the cavity approximation to the MIT bag model

The structure functions may be calculated for any k_0 in the bag in both the x model and the $\log_e(1-x)$ model, using equations (5.96), (3.1), (3.2), (3.37), (3.19) and (3.20). Moreover it is straightforward to investigate the effect of changing the bag radius R with k_0 fixed. The structure functions in the x model and the $\log_e(1-x)$ model are compared in figures 10 and 11 for R = 9.0 GeV^{-1} and 15.0 GeV^{-1} respectively and $k_0/M = 0.141$, the 'typical' value referred to above. To compare directly with experiment we must fix the normalisation: equation (5.97) is equivalent to the Gross-Llewellyn Smith sum rule (see equations (3.28) and (3.30)) and the validity of the latter is taken to prescribe the normalisation in the following manner. As referred to in Chapter 4 difficulties are encountered at small x when evaluating the area under the calculated structure function $F_3^{\nu N}(x)$. We thus assume that the experimental value,

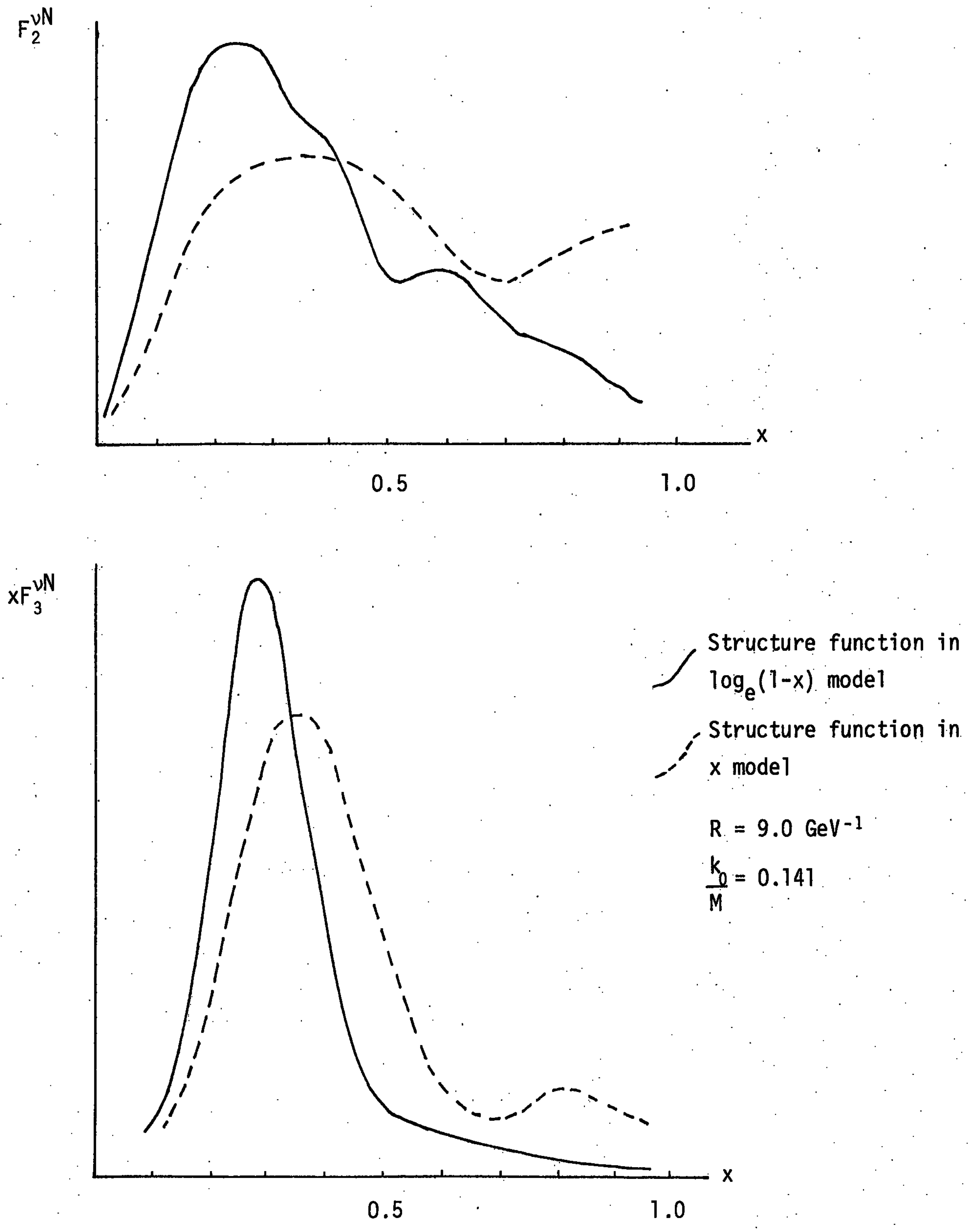


Figure 10. Comparison of structure functions in different models within a bag framework for $k_0/M = 0.141$ and $R = 9.0$ GeV^{-1}

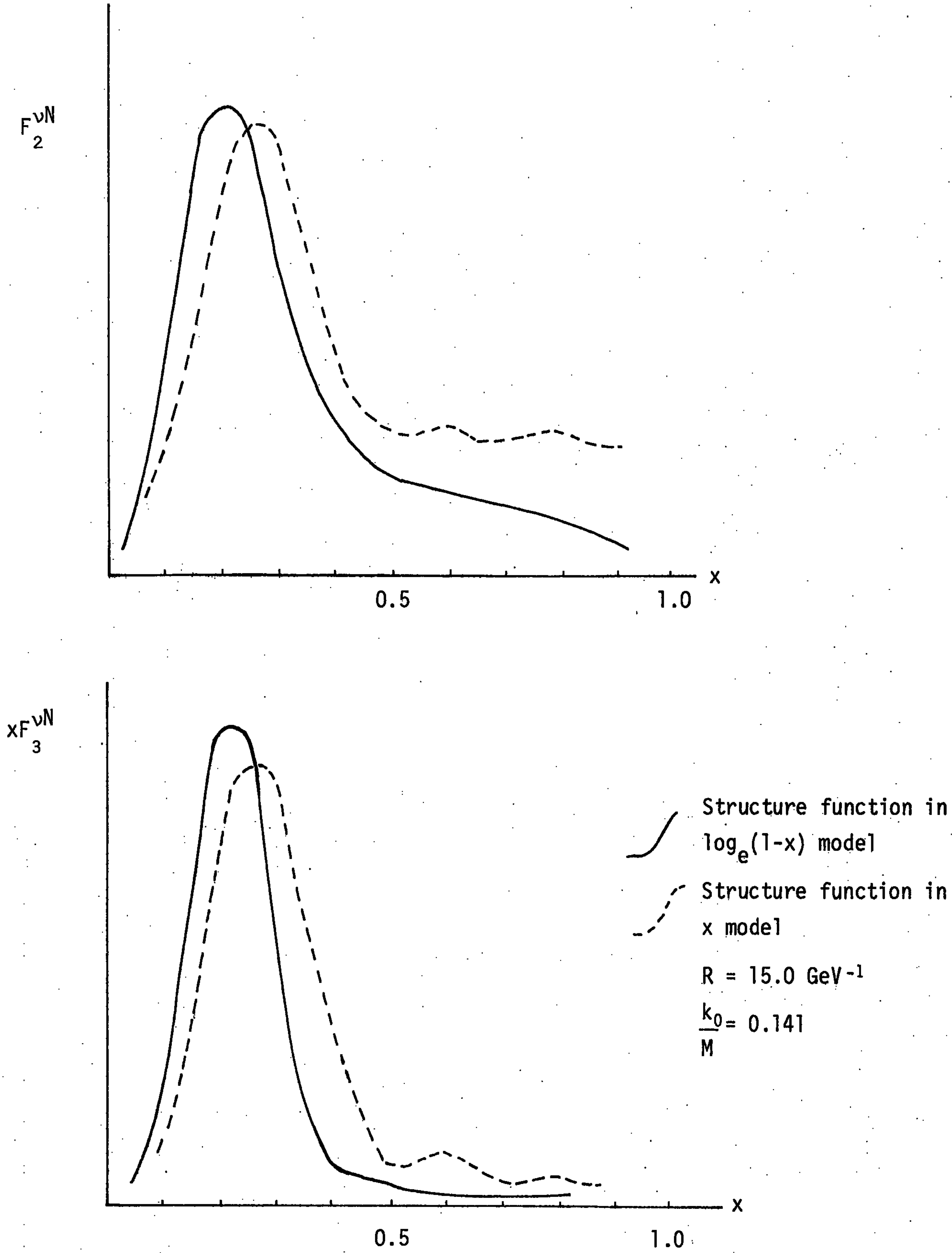


Figure 11. Comparison of structure functions in different models within a bag framework for $k_0/M = 0.141$ and $R = 15.0\ \mathrm{GeV}^{-1}$

$$\int_{0.1}^{1} F_3^{\nu N}(x)dx = 1.09 \tag{5.99}$$

as given in figure 21 of ref. 4, is consistent with the Gross-Llewellyn Smith sum rule. This normalisation prescription is then applicable to the $\log_e(1-x)$ model calculations of the structure functions, and the direct comparison with experiment is shown in figures 12 and 13 for R = 9.0 GeV^{-1} and 15.0 GeV^{-1} and $k_0/M = 0.141$. This latter value was an average over the ABCLOS group's q^2 range from 1.732 GeV^2 to 14.100 GeV^2 and the CDHS group's data provides a very convenient and more accurate structure function analysis over this range for neutrino energies 10 to 20 GeV (see Table 3 of ref. 4) and it is this data which is utilized in figures 12 and 13. The good agreement with $F_2^{\nu N}(x)$ for a bag radius of 15.0 GeV^{-1} is not conclusively duplicated with $xF_3^{\nu N}(x)$ but it should be pointed out that the CDHS data of figure 13 is not entirely consistent with the BEBC data at low x (where q^2 is comparable) of figure 7. Notwithstanding this it would appear that a radius much larger than the bag value of 5 GeV^{-1} [24] is required from this analysis. As clearly demonstrated for the x model in ref. 25 the position of the structure function maximum is determined by k_0/M and a larger bag radius produces a narrower structure function. These features are carried over into the $\log_e(1-x)$ model. It is interesting to note that, with $k_0/M = 0.141$, good agreement is obtained when R = 15.0 GeV^{-1}, thus well satisfying the linear bag boundary condition.

The momentum sum rule [26] can be evaluated in each case:

$$\int_{0}^{1} F_2^{\nu N}(x)dx = 1.02 \; ; \quad R = 9.0 \text{ GeV}^{-1} \tag{5.100}$$

$$\int_{0}^{1} F_2^{\nu N}(x)dx = 0.47 \; ; \quad R = 15.0 \text{ GeV}^{-1} \tag{5.101}$$

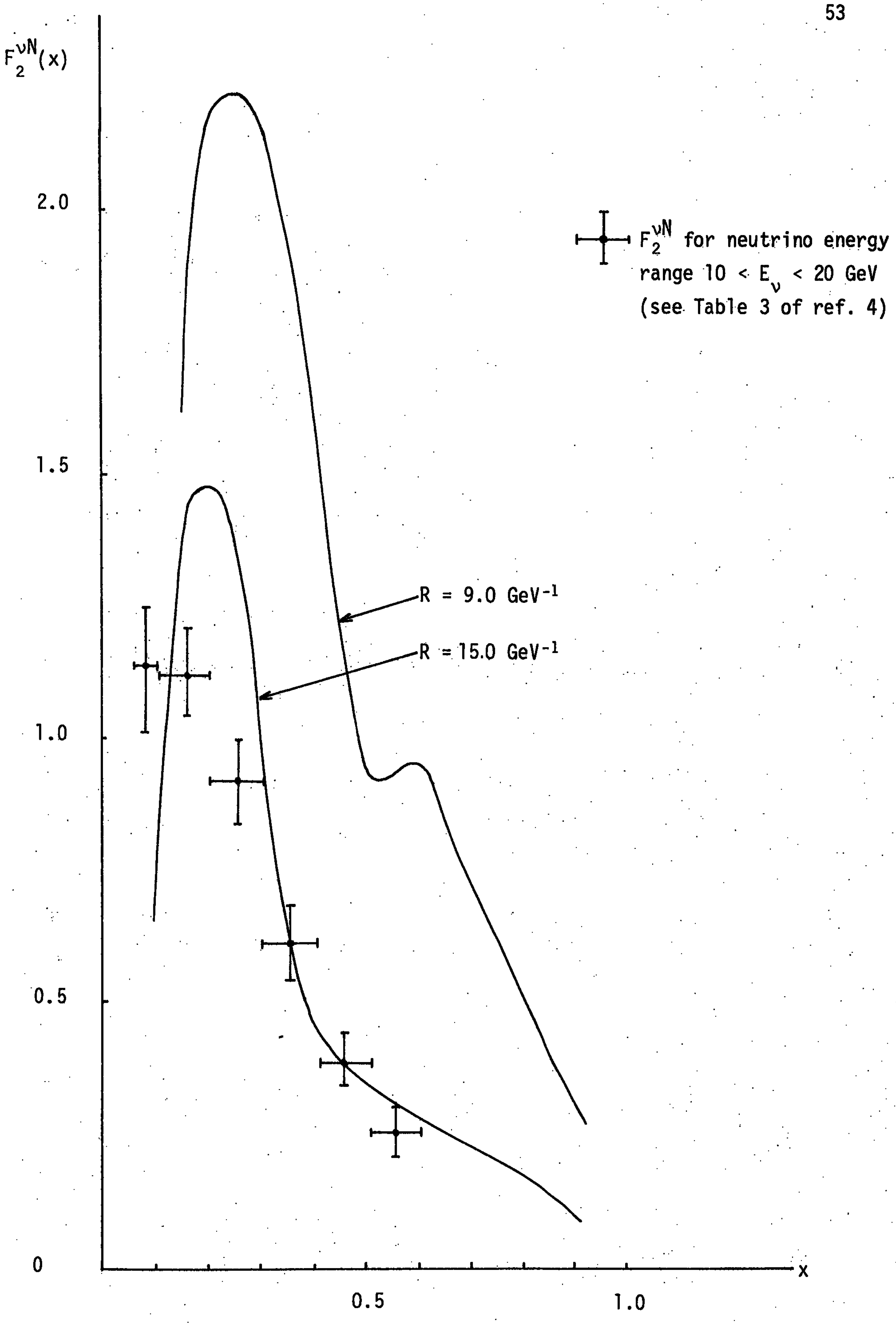


Figure 12. Comparison of structure function in $\log_e(1-x)$ model within a bag framework for $k_0/M = 0.141$, with CDHS data

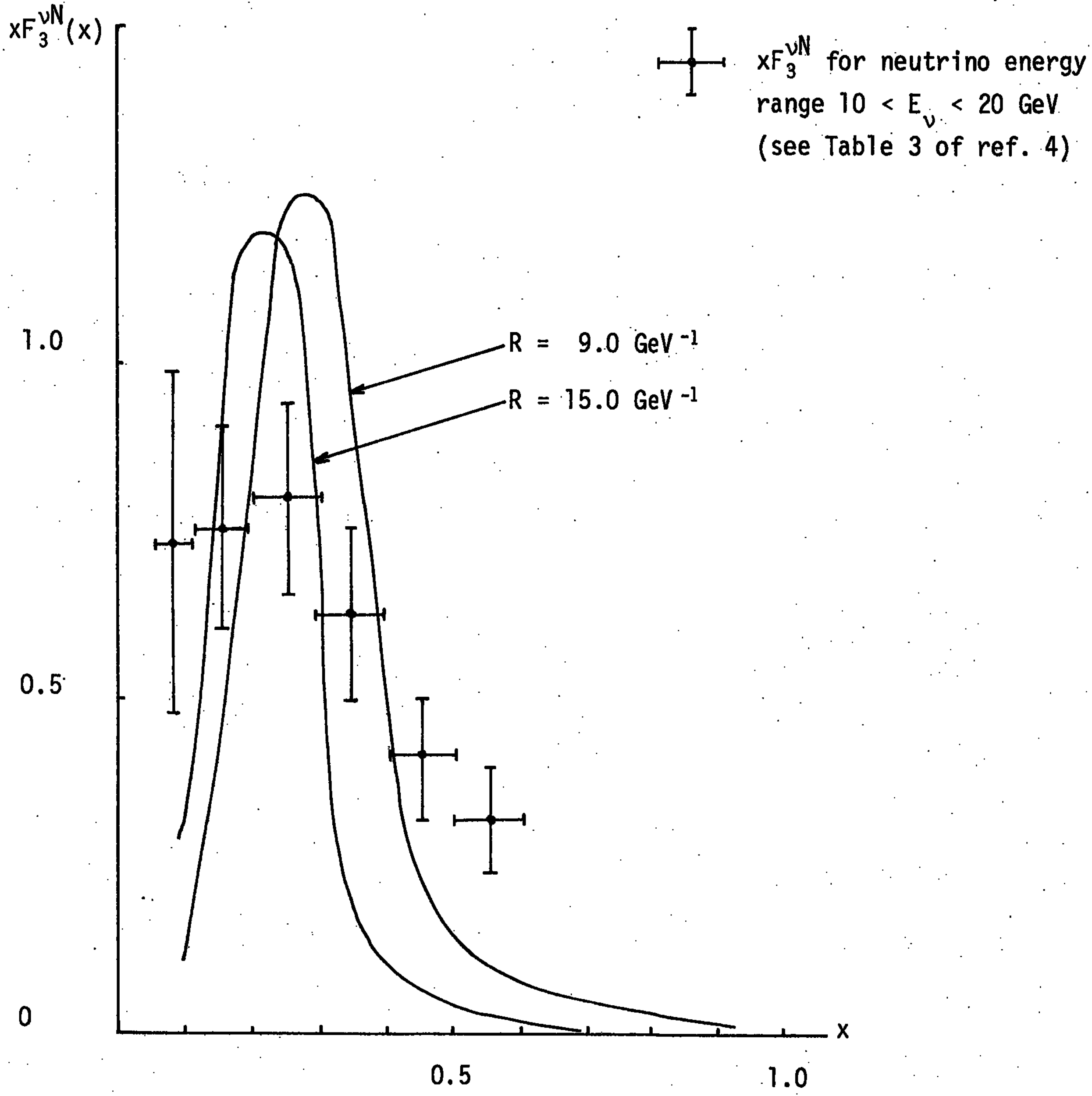


Figure 13. Comparison of structure function in $\log_e(1-x)$ model within a bag framework for $k_0/M = 0.141$, with CDHS data

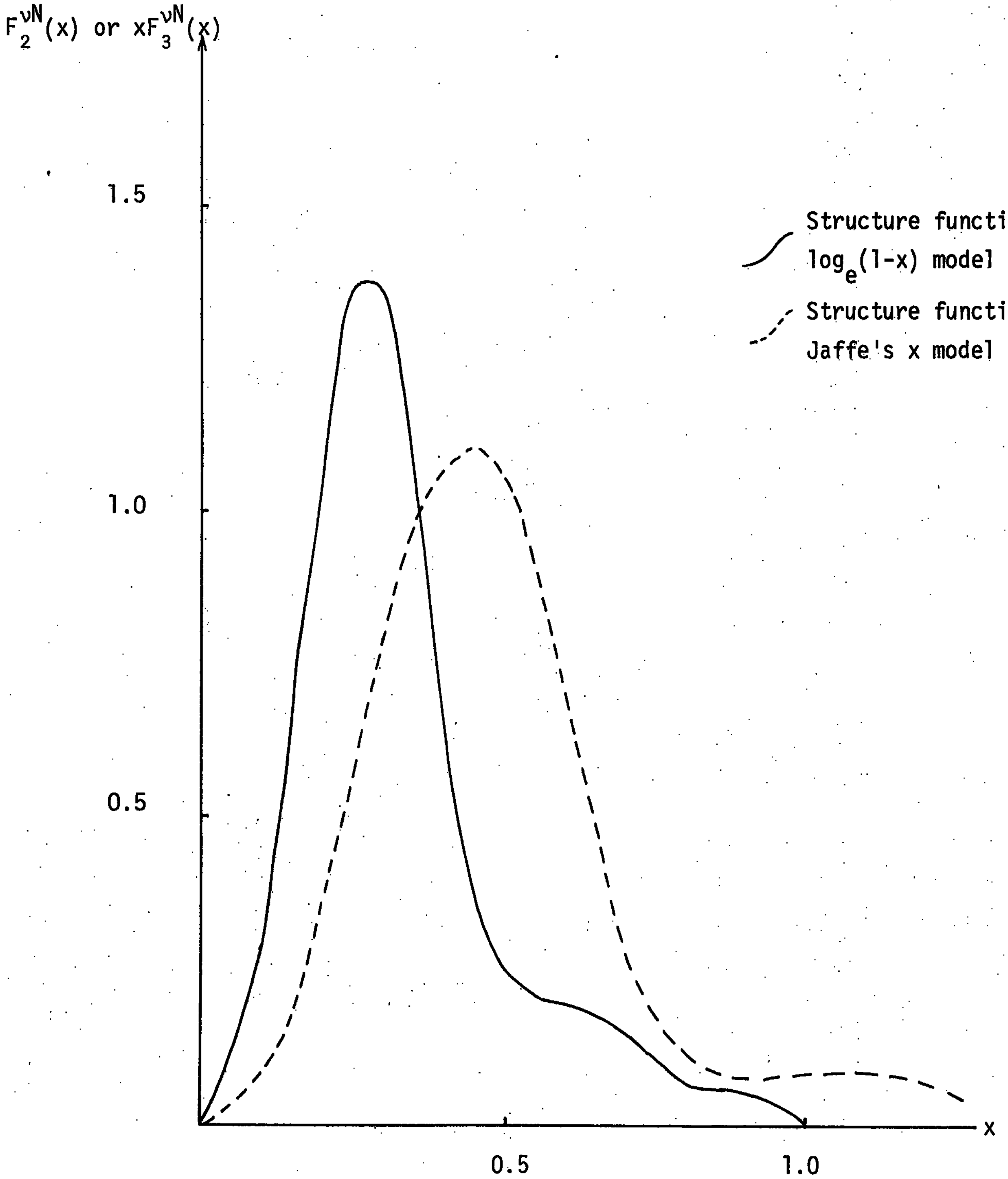
The CDHS group find a value 0.44 ± 0.02 over the relevant neutrino energy range, thus confirming our agreement for a bag radius of 15.0 GeV^{-1}.

Jaffe's treatment of structure functions in the cavity approximation of the bag [3] suffers from the pathology of 'x models', namely their extension outside the physical region $0 < x < 1$ because of lack of translational invariance. Further the momentum sum rule was well saturated in the physical region:

$$\int_0^1 F_2^{\nu N}(x)dx \approx 0.93 \qquad (5.102)$$

A more direct illustration of the improvement which can be achieved in the $\log_e(1-x)$ model is thus obtained by comparing Jaffe's structure functions (measured from the parton distributions in his figure 7) with our calculation for R = 9.0 GeV^{-1} when the momentum sum rule is also approximately saturated. Two caveats should first be entered: Jaffe takes R = 7.0 GeV^{-1} from the bag fit to the average mass of the N-Δ system [23] and, more importantly, the 'sea' contribution of figure 3(ii) is neglected in his calculations, with the consequence that $F_2^{\nu N}$ and $xF_3^{\nu N}$ differ only by the small 'Z-graph' contribution. A more accurate illustration can thus be obtained by comparing with our structure function $F_2^{\nu N}$ or $xF_3^{\nu N}$ with the 'sea' subtracted, i.e. the quantity $3x^2I(x)$ of equation (3.19) or (3.20). This is shown in figure 14 where the average of Jaffe's $F_2^{\nu N}$ and $xF_3^{\nu N}$ is taken (the difference is negligible above x = 0.5). Even with these caveats the improved features due to the $\log_e(1-x)$ model are evident, namely its forced restriction to the physical region and its more reasonable width and peak.

Figure 14. Comparison of bag structure functions:
(i) in the $\log_e(1-x)$ model with $R = 9.0$ GeV^{-1}
(ii) à la Jaffe with $R = 7.0$ GeV^{-1}



§5.3.2 Off mass shell, massless quarks with independent energy and three-momentum

In an attempt to consider a general off mass shell model without fixed energy a five-parameter distribution separable in energy and three-momentum is hypothesized:

$$P(k_0,k) = C\, k_0^{\,p}\, e^{-\alpha k_0}\, k^{2q+1}\, e^{-\beta k^2} \tag{5.103}$$

where we expect α, $\beta > 0$ and p, $q \geqslant 0$. This form for the distribution permits a straightforward determination of the necessary integrals in equations (3.30)-(3.34) (22), leading to:

$$M_1 = \frac{C(q+1)!\; p!}{2\alpha^{p+1}\,\beta^{q+2}} = \mu_1(q^2) = 1 \tag{5.104}$$

$$M_2 = \frac{1}{3M}\left(\frac{\alpha(q+2)}{\beta p} + \frac{3(p+1)}{\alpha}\right) = \mu_2(q^2) \tag{5.105}$$

$$M_3 = \frac{1}{M^2}\left(\frac{q+2}{\beta} + \frac{(p+1)(p+2)}{\alpha^2}\right) = \mu_3(q^2) \tag{5.106}$$

$$M_4 = \frac{1}{5M^3}\left(\frac{\alpha(q+3)(q+2)}{\beta^2 p} + \frac{10(q+2)(p+1)}{\beta\alpha} + \frac{5(p+3)(p+2)(p+1)}{\alpha^3}\right)$$

$$= \mu_4(q^2) \tag{5.107}$$

$$M_5 = \frac{1}{M^4}\left(\frac{(q+3)(q+2)}{\beta^2} + \frac{10(q+2)(p+2)(p+1)}{3\beta\alpha^2} + \frac{(p+4)(p+3)(p+2)(p+1)}{\alpha^4}\right) = \mu_5(q^2) \tag{5.108}$$

It is straightforward to demonstrate that these equations do not permit realistic solutions for α, β, p and q in the sense of the provisos mentioned after equation (5.103). Eliminating the quantity $(q + 2)/\beta$ from equations (5.105) and (5.106), α may be expressed as the following function of p:

$$\alpha M = \frac{3\mu_2 p \pm \sqrt{9\mu_2{}^2 p^2 - 8\mu_3(p^2 - 1)}}{2\mu_3} \tag{5.109}$$

A further relation between α and p may be obtained by eliminating the quantity $(q + 3)(q + 2)/\beta^2$ from equations (5.107) and (5.108) and using equations (5.105) and (5.106):

$$\alpha M = \left(\frac{-2b(p^2 - 1) \pm \sqrt{4b^2(p^2 - 1)^2 - \frac{32}{3}a(p^2 - 1)(p^2 - 4)}}{2a}\right)^{\frac{1}{2}} \tag{5.110}$$

where

$$a \equiv \frac{1}{3\mu_2}(5\mu_3\mu_4 - 3\mu_2\mu_5) \tag{5.111}$$

$$b \equiv \frac{5}{3\mu_2}(\mu_4 - 2\mu_2\mu_3) \tag{5.112}$$

Through equations (5.105) and (5.106) the provisos $q \geqslant 0$, $\beta > 0$ demand:

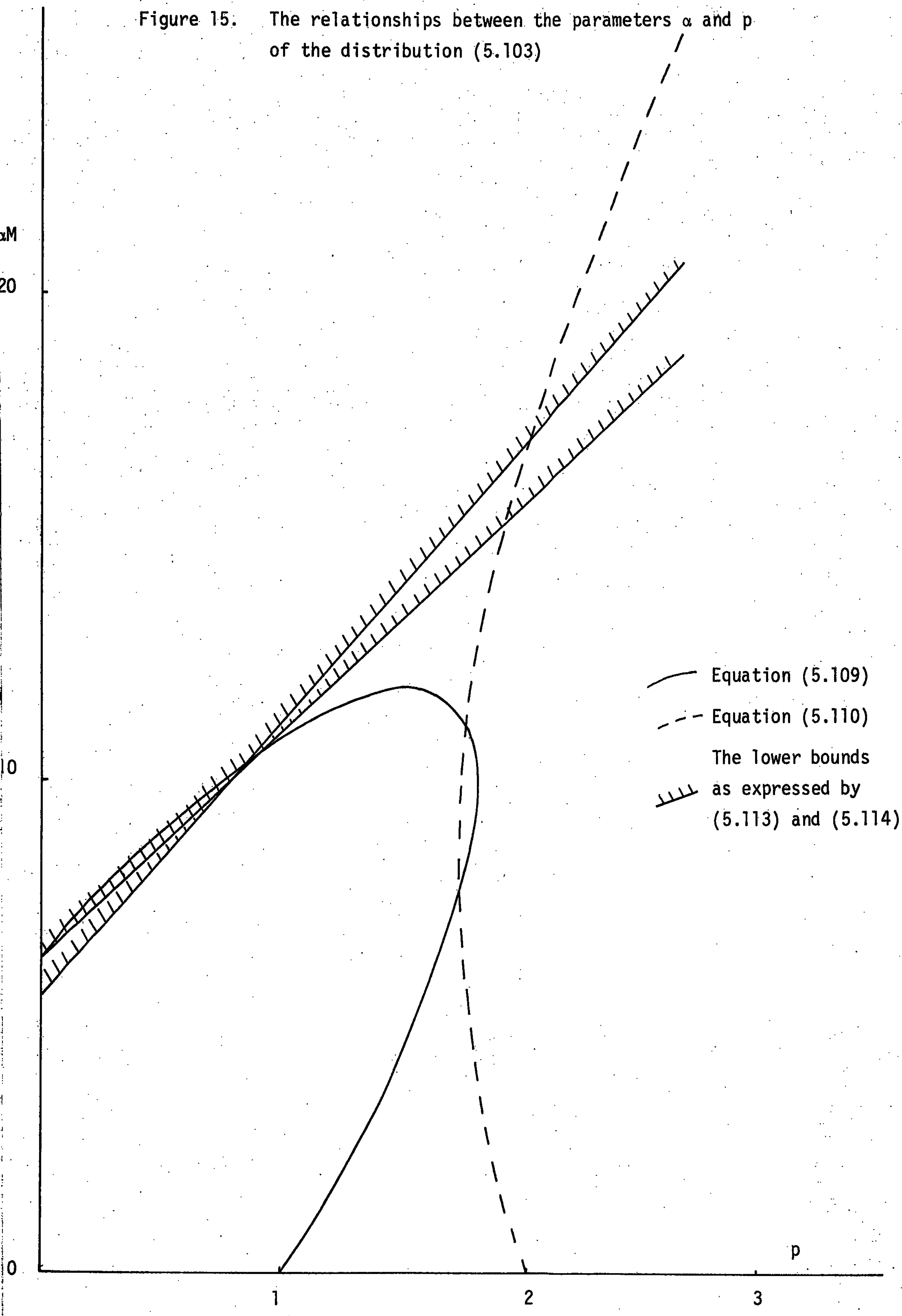
$$\alpha M > \frac{p + 1}{\mu_2} \tag{5.113}$$

$$\alpha M > \left(\frac{(p + 1)(p + 2)}{\mu_3}\right)^{\frac{1}{2}} \tag{5.114}$$

Equations (5.109) and (5.110) and the lower bounds of (5.113) and (5.114) are shown in figure 15 where the moments $\mu_n(q^2)$ have been taken from Table 1 at the specific $q^2 = 2.449$ GeV^2.

The positive solutions for α and p do not satisfy the requirements of (5.113) and (5.114) and this is found to be generally so at all q^2. In the light of the relative successes of the distributions in §5.1.2, §5.2 and §5.3.1 the consequences of the distribution (5.103) will not be pursued further.

Figure 15. The relationships between the parameters α and p of the distribution (5.103)



§5.3.3 Off mass shell, massless quarks with independent energy and deviation from mass shell

As an extension of the previous sub-section we seek a four-parameter distribution separable in energy, k_0, and a quantity $k - k_0$, which is a measure of the deviation of the (massless) quark from its mass shell:

$$P(k_0,k) = C\, k_0\, e^{-\alpha|k_0 - E_0|}\, e^{-\mu|k - k_0|} \qquad (5.115)$$

The parameters C, α and μ can be determined for various E_0 by equating the first three moments, *viz.* equations (3.30)-(3.32), with the experimental quantities μ_1, μ_2 and μ_3. Using the required integrals evaluated in Appendix E, these equations lead to:

$$M_1 = \frac{2C}{\mu^3}\left\{\frac{2\mu^2E_0}{\alpha^3}(\alpha^2E_0^2+6) + \frac{4E_0}{\alpha} + e^{-\alpha E_0}\left(\frac{2}{\alpha^2} + \frac{6\mu^2}{\alpha^4} - \frac{1}{(\alpha-\mu)^2}\right)\right.$$

$$\left. - \frac{2\alpha e^{-\mu E_0}}{(\alpha+\mu)^2(\alpha-\mu)^2}(E_0(\alpha+\mu)(\alpha-\mu) - 2\mu)\right\} = \mu_1(q^2) = 1 \qquad (5.116)$$

$$M_2 = \frac{2C}{3M\mu^5}\left\{\frac{2}{\alpha^5}(\alpha^4\mu^4E_0^4 + 12\alpha^2\mu^2E_0^2(\alpha^2+\mu^2) + 24\alpha^4 + 24\alpha^2\mu^2 + 24\mu^4)\right.$$

$$\left. - \frac{24e^{-\alpha E_0}}{\alpha^5}(\alpha^4+\alpha^2\mu^2+\mu^4) + \frac{12}{(\alpha-\mu)(\alpha+\mu)}\left((\alpha+\mu)e^{-\alpha E_0} - 2\alpha e^{-\mu E_0}\right)\right\}$$

$$+ \frac{2C}{M\mu^3}\left\{\frac{2}{\alpha^5}(\alpha^4\mu^2E_0^4 + (\alpha^2+6\mu^2)(2\alpha^2E_0^2+4)) - \frac{4e^{-\alpha E_0}}{\alpha^5}(\alpha^2+6\mu^2)\right.$$

$$\left. - \frac{2\alpha E_0 e^{-\mu E_0}}{(\alpha+\mu)^2(\alpha-\mu)^2}((\alpha^2-\mu^2)E_0 - 4\mu) - \frac{2e^{-\mu E_0}}{(\alpha+\mu)^3} + \frac{2}{(\alpha-\mu)^3}(e^{-\alpha E_0} - e^{-\mu E_0})\right\}$$

$$= \mu_2(q^2) \qquad (5.117)$$

$$M_3 = \frac{2C}{M^2\mu^5}\left\{\frac{2E_0}{\alpha^5}(\alpha^4\mu^4E_0^4 + 4\alpha^2\mu^2E_0^2(3\alpha^2+5\mu^2) + 24\alpha^4 + 6\alpha^2\mu^2 + 120\mu^4)\right.$$

$$+ \frac{24e^{-\alpha E_0}}{\alpha^6}(\alpha^4+3\alpha^2\mu^2+5\mu^4) - \frac{12e^{-\mu E_0}}{(\alpha-\mu)(\alpha+\mu)^2}(2\alpha E_0(\alpha+\mu) + \alpha - \mu)$$

$$\left. + \frac{12}{(\alpha-\mu)^2}(e^{-\mu E_0} - e^{-\alpha E_0})\right\}$$

$$+ \frac{2C}{M^2\mu^3}\left\{\frac{2E_0}{\alpha^5}(\alpha^4\mu^2E_0^4 + 2\alpha^2E_0^2(\alpha^2+10\mu^2) + 12\alpha^2 + 120\mu^2) + \frac{12e^{-\alpha E_0}}{\alpha^6}(\alpha^2+10\mu^2)\right.$$

$$- 2e^{-\mu E_0}\left(\frac{\alpha E_0^3}{(\alpha+\mu)(\alpha-\mu)} - \frac{6\alpha\mu E_0^2}{(\alpha+\mu)^2(\alpha-\mu)^2} - \frac{6\mu E_0(3\alpha^2+\mu^2)}{(\alpha+\mu)^3(\alpha-\mu)^3} + \frac{3}{(\alpha+\mu)^4}\right)$$

$$\left. + \frac{6}{(\alpha-\mu)^4}(e^{-\mu E_0} - e^{-\alpha E_0})\right\} = \mu_3(q^2) \qquad (5.118)$$

Eliminating the normalisation constant C from equations (5.116) and (5.117) and from equations (5.116) and (5.118) allows solutions for α and μ by solving the two equations:

$$\frac{1}{\alpha^5}\left[2\alpha^4\mu^2E_0^3(4E_0-3M\mu_2(q^2)) + 12\alpha^4E_0(3E_0-M\mu_2(q^2)) + 12\alpha^2\mu^2E_0(8E_0-3M\mu_2(q^2))\right.$$

$$\left. + 72\,\alpha^2 + 192\mu^2 + 48\frac{\alpha^4}{\mu^2}\right]$$

$$- e^{-\alpha E_0}\left[\frac{36}{\alpha^3} + \frac{96\mu^2}{\alpha^5} + \frac{24}{\alpha\mu^2} + \frac{6M\mu_2(q^2)}{\alpha^2} + \frac{18M\mu_2(q^2)\mu^2}{\alpha^4} - \frac{12}{\mu^2(\alpha-\mu)} - \frac{6}{(\alpha-\mu)^3} - \frac{3M\mu_2(q^2)}{(\alpha-\mu)^2}\right]$$

$$- e^{-\mu E_0}\left[\frac{6\alpha E_0(E_0-M\mu_2(q^2))}{(\alpha+\mu)(\alpha-\mu)} - \frac{12\alpha\mu(2E_0-M\mu_2(q^2))}{(\alpha+\mu)^2(\alpha-\mu)^2} + \frac{24\alpha}{\mu^2(\alpha-\mu)(\alpha+\mu)}\right.$$

$$\left. + \frac{6}{(\alpha+\mu)^3} + \frac{6}{(\alpha-\mu)^3}\right] = 0 \qquad (5.119)$$

$$\frac{2E_0}{\alpha^5}\Big[\alpha^4\mu^2E_0^2(2E_0^2-M^2\mu_3(q^2)) + 2\alpha^4(7E_0^2-M^2\mu_3(q^2)) + 2\alpha^2\mu^2(20E_0^2-3M^2\mu_3(q^2))$$

$$+ 18\alpha^2 + 240\mu^2 + 24\frac{\alpha^4}{\mu^2}\Big]$$

$$+ e^{-\alpha E_0}\left[\frac{6}{\alpha^4}(14-M^2\mu_3(q^2)) + \frac{240\mu^2}{\alpha^6} + \frac{24}{\alpha^2\mu^2} - \frac{2M^2\mu_3(q^2)}{\alpha^2} - \frac{6}{(\alpha-\mu)^4} - \frac{12}{\mu^2(\alpha-\mu)^2} + \frac{M^2\mu_3(q^2)}{(\alpha-\mu)^2}\right]$$

$$+ e^{-\mu E_0}\Big[\frac{4\alpha\mu(3E_0^2-M^2\mu_3(q^2))}{(\alpha+\mu)^2(\alpha-\mu)^2} - \frac{2\alpha E_0(E_0^2-M^2\mu_3(q^2))}{(\alpha+\mu)(\alpha-\mu)} - \frac{24\alpha E_0}{\mu^2(\alpha+\mu)(\alpha-\mu)}$$

$$- \frac{12\mu E_0(3\alpha^2+\mu^2)}{(\alpha+\mu)^3(\alpha-\mu)^3} - \frac{12}{\mu^2}\left(\frac{1}{(\alpha+\mu)^2} - \frac{1}{(\alpha-\mu)^2}\right) - \frac{6}{(\alpha+\mu)^4} + \frac{6}{(\alpha-\mu)^4}\Big] = 0 \tag{5.120}$$

We also consider a slight modification to the distribution (5.115) by imposing a minimum 'ground state' energy E_0, i.e.:

$$P(k_0,k) = C\, k_0\, e^{-\alpha(k_0-E_0)}\, e^{-\mu|k-k_0|}\, \theta(k_0-E_0) \tag{5.121}$$

Of course the integrals over k in our previous analysis remain identical and the only modification required is to the lower limit in integrals over k_0, which is now E_0 instead of zero because of the θ function. Again using Appendix E, the first three moments with the distribution (5.121) become:

$$M_1 = \frac{2C}{\mu^3}\left\{\frac{\mu^2}{\alpha^4}(\alpha^3E_0^3+3\alpha^2E_0^2+6\alpha E_0+6) + \frac{2}{\alpha^2}(\alpha E_0+1) - \frac{e^{-\mu E_0}}{(\alpha+\mu)^2}((\alpha+\mu)E_0+1)\right\} = \mu_1(q^2) = 1 \tag{5.122}$$

$$M_2 = \frac{2C}{3M\mu^5}\Big\{\frac{1}{\alpha^5}(\alpha^4\mu^4E_0^4+4\alpha^3\mu^4E_0^3+12\alpha^2\mu^2E_0^2(\alpha^2+\mu^2) + 24\alpha\mu^2E_0(\alpha^2+\mu^2) + 24\alpha^4$$

$$+ 24\alpha^2\mu^2 + 24\mu^4) - \frac{12e^{-\mu E_0}}{(\alpha+\mu)}\Big\}$$

$$+ \frac{2C}{M\mu^3}\Big\{\frac{1}{\alpha^5}(\alpha^4\mu^2E_0^4+4\alpha^3\mu^2E_0^3+2(\alpha^2+6\mu^2)(\alpha^2E_0^2+2\alpha E_0+2))$$

$$- \frac{e^{-\mu E_0}}{(\alpha+\mu)^3}((\alpha+\mu)^2E_0^2 + 2(\alpha+\mu)E_0 + 2)\Big\} = \mu_2(q^2) \tag{5.123}$$

$$M_3 = \frac{2C}{M^2\mu^5}\left\{\frac{1}{\alpha^5}(\alpha^4\mu^4E_0^5+5\alpha^3\mu^4E_0^4+4\alpha\mu^2E_0^2(\alpha E_0+3)(3\alpha^2+5\mu^2)+24E_0(\alpha^4+3\alpha^2\mu^2+5\mu^4)\right.$$

$$\left.+72\alpha\mu^2+24\alpha^3+120\frac{\mu^4}{\alpha}) - \frac{12e^{-\mu E_0}}{(\alpha+\mu)^2}((\alpha+\mu)E_0 + 1)\right\}$$

$$+ \frac{2C}{M^2\mu^3}\left\{\frac{1}{\alpha^5}(\alpha^4\mu^2E_0^5+5\alpha^3\mu^2E_0^4+2\alpha E_0^2(\alpha^2+10\mu^2)(\alpha E_0+3)+12E_0(\alpha^2+10\mu^2)+12\alpha+\frac{120\mu^2}{\alpha})\right.$$

$$\left.- \frac{e^{-\mu E_0}}{(\alpha+\mu)^4}((\alpha+\mu)^3E_0^3+3(\alpha+\mu)^2E_0^2+6(\alpha+\mu)E_0+6)\right\} = \mu_3(q^2) \qquad (5.124)$$

The two simultaneous equations for α and μ are then obtained in an identical fashion:

$$\frac{1}{\alpha^5}\left[\alpha^4\mu^2E_0^3(4E_0-3M\mu_2(q^2)) + \alpha^3\mu^2E_0^2(16E_0-9M\mu_2(q^2)) + 6\alpha^4E_0(3E_0-M\mu_2(q^2))\right.$$

$$+ 6\alpha^2\mu^2E_0(8E_0-3M\mu_2(q^2)) + 6\alpha^3(6E_0-M\mu_2(q^2)) + 6\alpha\mu^2(16E_0-3M\mu_2(q^2)) + 36\alpha^2$$

$$\left.+ 96\mu^2 + \frac{24\alpha^4}{\mu^2}\right] + \frac{e^{-\mu E_0}}{\mu^2(\alpha+\mu)^2}\left[3M\mu_2(q^2)\mu^2((\alpha+\mu)E_0 + 1) - 12(\alpha+\mu)\right]$$

$$- \frac{3e^{-\mu E_0}}{(\alpha+\mu)^3}\left[(\alpha+\mu)^2E_0^2 + 2(\alpha+\mu)E_0 + 2\right] = 0 \qquad (5.125)$$

$$\frac{1}{\alpha^6}\left[\alpha^5\mu^2E_0^3(2E_0^2-M^2\mu_3(q^2)) + \alpha^4\mu^2E_0^2(10E_0^2-3M^2\mu_3(q^2)) + 2\alpha^5E_0(7E_0^2 -M^2\mu_3(q^2))\right.$$

$$+ 2\alpha^3\mu^2E_0(20E_0^2-3M^2\mu_3(q^2)) + 2\alpha^4(21E_0^2-M^2\mu_3(q^2)) + 6\alpha^2\mu^2(20E_0^2-M^2\mu_3(q^2))$$

$$\left.+ 84\alpha^3E_0 + \frac{24\alpha^5E_0}{\mu^2} + 240\alpha\mu^2E_0 + 84\alpha^2 + \frac{24\alpha^4}{\mu^2} + 240\mu^2\right]$$

$$+ \frac{e^{-\mu E_0}}{\mu^2(\alpha+\mu)^2}\left[(M^2\mu_3(q^2)\mu^2-12)((\alpha+\mu)E_0 + 1)\right]$$

$$- \frac{e^{-\mu E_0}}{(\alpha+\mu)^4}\left[(\alpha+\mu)^3E_0^3 + 3(\alpha+\mu)^2E_0^2 + 6(\alpha+\mu)E_0 + 6\right] = 0 \qquad (5.126)$$

Using a computer we find that solutions to equations (5.119) and (5.120) and equations (5.125) and (5.126) may be obtained with very large α and μ. For example, with E_0 = 0.01 GeV and moments taken at q^2 = 14.100 GeV^2,

$$\alpha M \sim 200 \qquad (5.127)$$

$$\mu M \sim 50 \qquad (5.128)$$

This leads us to conclude that a very narrow distribution around E_0 is demanded by the data, reverting us to §5.3.1 where a fixed energy distribution (i.e. δ-function in k_0) was employed. In the light of the satisfactory nature of the analysis in that sub-section, the consequences of equations (5.127) and (5.128) will not be pursued further.

CHAPTER 6 SCALING VIOLATIONS

In the previous chapters, and specifically in the definition of I(x) through equation (3.1), the second term of order Q^{-2} in equation (2.38) has been ignored and the subsequent structure function analysis in Chapter 5 has been independent of Q^2 at fixed x and thus independent of any mass scale. This is the phenomenon of scaling in the naive parton model, deriving from the assumption that the inelastic lepton scattering occurs through incoherent elastic scattering off point-like quarks. However, both the ABCLOS and CDHS data clearly demonstrate a violation of scaling (27), the structure function $F_2(x,Q^2)$ falling at large x and rising at smaller x as Q^2 increases.

Three possible sources of such effects are:

(i) Kinematic corrections due to non-negligible quark and target masses, which may be accomodated by a modification of the x variable (28):

$$\xi = \frac{(Q^2 + m_f^2 - m_i^2) + \sqrt{(Q^2 + m_f^2 - m_i^2)^2 + 4m_i^2Q^2}}{2M(\nu + \sqrt{\nu^2+Q^2})} \tag{6.1}$$

where m_i, m_f are the initial and final quark masses and M is the target mass. This variable then reduces to the x of equation (2.4) at large Q^2.

(ii) Other neglected properties like the transverse momentum and off mass shell effects of the quarks, which arise in twist-four operators in the operator product expansion and are of order M_0^2/Q^2, where M_0 is the fundamental scale of the strong interactions, phenomenologically estimated at about 400 MeV (29).

(iii) Logarithmic corrections due to the quark-gluon interactions of quantum chromodynamics (30). The agreement of such QCD predictions with experiment is extensively claimed (31) but it is worth pointing out that some authors (32) have commented that the higher-twist terms of (ii), besides becoming significant below 4 GeV^2, may account for the high experimental value of R, the ratio of the photoabsorption total cross section for photons of helicity zero or ±1. However it is agreed that the Q^2/ν^2 effects in (i) cannot be the sole source of scaling violation, and it is this point on which we wish to focus in this chapter.

We adopt the following method. It has been shown [33] that the inclusion of the first order term ($\sim Q^{-2}$) in the structure function of equation (2.38) is equivalent to the light-cone and operator product expansion techniques of Barbieri *et. al.* [28] which yield the scaling variable ξ. Maintaining massless quarks on mass shell, the moments of I(x) in the x model now include corrections (*cf.* equation (3.29)):

$$M_n = \frac{2^n}{(n+1)M^{n-1}} \int_0^\infty dk_0 k_0^{n+1} P(k_0) + \left(\frac{1}{n+2} - \frac{2}{n+3}\right) \frac{6.2^{n+2}}{Q^2 M^{n-1}} \int_0^\infty dk_0 k_0^{n+3} P(k_0)$$

$$= \frac{2^n}{(n+1)M^{n-1}} \int_0^\infty dk_0 k_0^{n+1} \left(1 - \frac{24(n+1)^2}{(n+2)(n+3)} \frac{k_0^2}{Q^2}\right) P(k_0) \tag{6.2}$$

Hence:

$$M_1 = \int_0^\infty k_0^2 dk_0 P(k_0) - \frac{8}{Q^2} \left\langle k_0^2 \right\rangle = \mu_1(q^2) = 1 \tag{6.3}$$

$$M_2 = \frac{4}{3M} \left\langle k_0 \right\rangle - \frac{72}{5MQ^2} \left\langle k_0^3 \right\rangle = \mu_2(q^2) \tag{6.4}$$

$$M_3 = \frac{2}{M^2} \left\langle k_0^2 \right\rangle - \frac{128}{5MQ^2} \left\langle k_0^4 \right\rangle = \mu_3(q^2) \tag{6.5}$$

Repeating the analysis of §5.1.2. (in the x model at $q^2 = 2.449$ GeV^2) the corrected solutions for α, a, N are found to be:

$$\alpha M = 6.392 \tag{6.6}$$

$$aM = -1.265 \tag{6.7}$$

$$NM = 7.439 \tag{6.8}$$

The structure function (*cf.* equation (5.30)) is then given by:

$$F_2^{\nu N}(x,Q^2)(\text{or } xF_3^{\nu N}) = \frac{3NM^2x^2}{2}\int_{\frac{Mx}{2}}^{\infty} \frac{e^{-\alpha k_0}}{k_0^2}(1+ak_0)\left(1 + \frac{12Mx[k_0 - Mx]}{Q^2}\right)dk_0$$

$$= \frac{3NMx^2}{2}\left(1 - \frac{12M^2x^2}{Q^2}\right)\left[\frac{2}{x}e^{-\frac{\alpha Mx}{2}} - (a-\alpha)MEi\left(\frac{-\alpha Mx}{2}\right)\right]$$

$$+ \frac{18NM^3x^3}{Q^2}\left[\frac{a}{\alpha}e^{-\frac{\alpha Mx}{2}} - Ei\left(\frac{-\alpha Mx}{2}\right)\right] \qquad (6.9)$$

This first order corrected structure function is compared with the zeroth order parameterization of equation (5.32) in figure 16. The corrections have a small effect and we assume that the set of parameters (5.44)-(5.46) adequately accomodate the inclusion of the first order correction in the $\log_e(1-x)$ model at $q^2 = 2.449$ GeV^2, so that the q^2 dependence of the structure function in this model may be directly investigated through the modification of equations (5.47) and (6.9):

$$F_2^{\nu N}(x,Q^2)(\text{or } xF_3^{\nu N}) = \frac{3NM^2x^2}{2}\left(1 - \frac{12M^2x^2}{Q^2}\right)\left[-\frac{2(1-x)^{\frac{\alpha M}{2}}}{\log_e(1-x)} - (a-\alpha)MEi\left(\frac{\alpha M}{2}\log_e(1-x)\right)\right]$$

$$+ \frac{18NM^3x^3}{Q^2}\left[\frac{a}{\alpha}(1-x)^{\frac{\alpha M}{2}} - Ei\left(\frac{\alpha M}{2}\log_e(1-x)\right)\right] \qquad (6.10)$$

To make a direct comparison with the ABCLOS data we compute this function for varying q^2 at center-bin values x = 0.05, 0.15, 0.25, 0.35 and 0.50 (x = 0.80 is not included since there are experimental discrepancies between the Gargamelle and BEBC data at this value). The results are displayed in figure 17. Little significance should be attached to the absolute magnitude of the structure function since, as figure 7 demonstrates, at specific center-bin values the uncorrected, parameterized function is sometimes higher, sometimes lower than the experimental value at that point, and only at

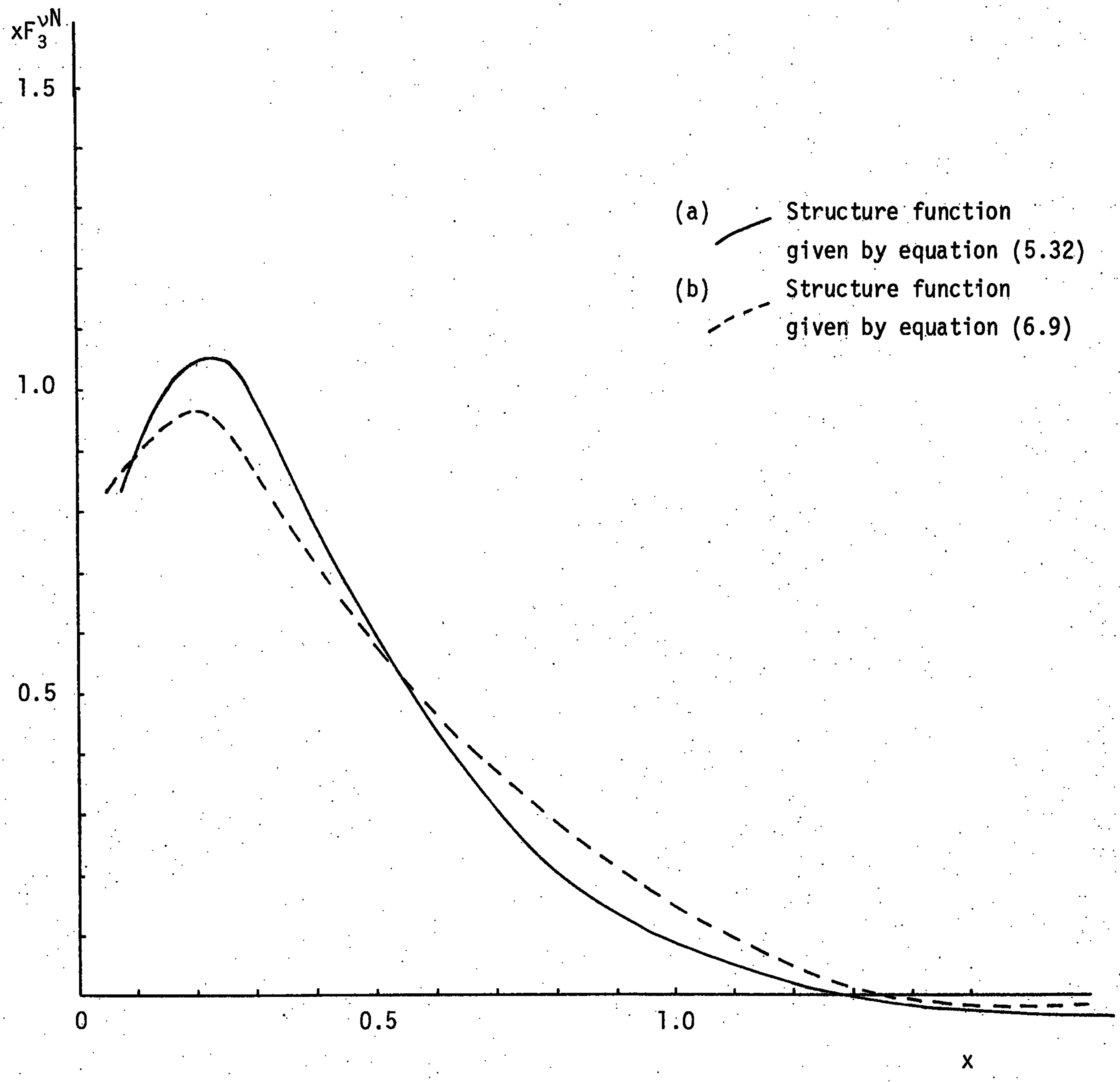


Figure 16. Comparison of the structure function fitted to the moments in the x model,
(a) to zeroth order
(b) with first order correction,
at q^2 = 2.449 GeV^2.

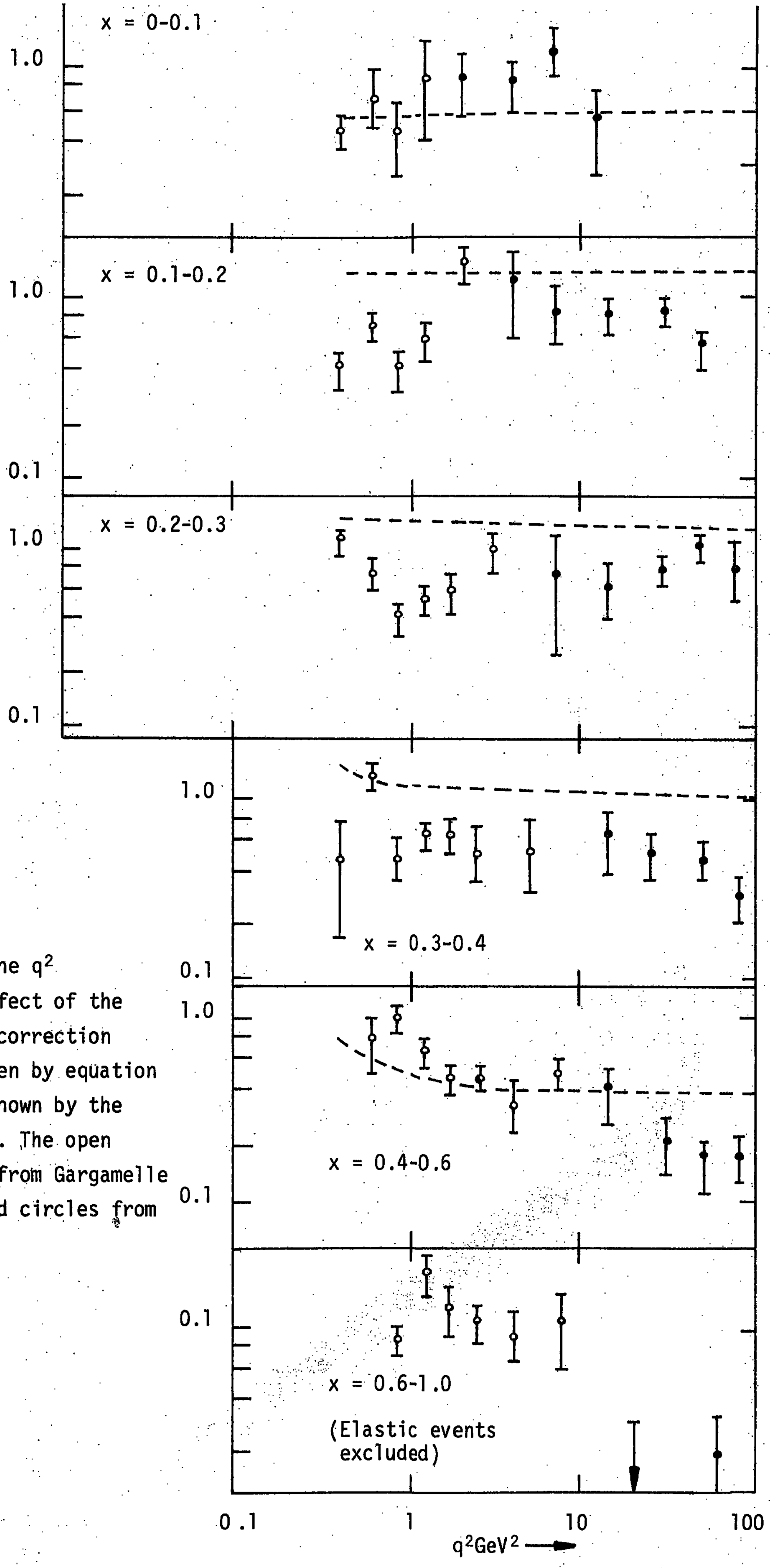


Figure 17. The q^2 dependent effect of the first order correction to $xF_3^{\nu N}$, given by equation (6.10), is shown by the dashed lines. The open circles are from Gargamelle and the solid circles from BEBC.

$x = 0.50$ is good agreement obtained. The first order correction, and correspondingly the target mass effects of (i), is then unable to sufficiently account for the high q^2 decrease in the structure function at $x > 0.30$, requiring some further scale violating mechanism, namely the QCD effects referred to in (iii).

CHAPTER 7 THE RATIO G_A/G_V

In §5.2 the parameters of a distribution with free, massive quarks with varying energy were deduced from the experimental structure function moments. We now apply this distribution to calculate the vector and the axial-vector coupling constants of the nucleon, G_V and G_A. These operators are defined by:

$$G_V \equiv \bar{\Psi}\gamma_0\Psi \tag{7.1}$$

$$G_A \equiv \bar{\Psi}\gamma_3\gamma_5\Psi \tag{7.2}$$

where Ψ is the plane wave solution to the Dirac equation for a particle of mass m. The detailed calculation is performed in Appendix F, including the quark energy-momentum distribution and the SU(6) factor, to give:

$$\frac{G_A}{G_V} = \frac{5}{9}\int_m^\infty\int_0^\infty \left(1 + \frac{2m}{k_0}\right) dk_0 k^2 dk\, P(k_0,k) \tag{7.3}$$

With the normalised distribution (5.63) and the definition (5.54), therefore:

$$\frac{G_A}{G_V} = \frac{5}{9} + \frac{10m}{9}\left\langle\frac{1}{k_0}\right\rangle$$

$$= \frac{5}{9} + \frac{10}{9m}\left(4\left\langle k_0\right\rangle - 3\mu_2(q^2)M\right) \tag{7.4}$$

where equation (5.49) has been used. Using Appendix D, or deciphering equation (5.67), the relevant quantity $\langle k_0\rangle$ is given by:

$$\left\langle k_0 \right\rangle = N\left\{\frac{1}{2\sqrt{\alpha^7}}\left[\pi e^{\alpha m^2}\,\mathrm{erfc}\; m\sqrt{\alpha}\left(\tfrac{1}{2}\alpha^2 m^2(am-1) + \tfrac{3}{4}\alpha(1-am-bm^2) + \tfrac{15}{8}b\right)\right.\right.$$

$$\left. + m\sqrt{\alpha}\left(\alpha^2 m^2(am-1) + \alpha(\alpha m^2+\tfrac{3}{2})(1-am-bm^2) + b(\alpha^2 m^4+\tfrac{5}{2}\alpha m^2+\tfrac{15}{4})\right)\right]$$

$$\left. + \frac{1}{2\alpha^3}\left[(\alpha^2 m^4+2\alpha m^2+2)(a-bm) - \alpha m(\alpha m^2+1)(1+am-bm^2) + \alpha^2 m^3\right]\right\} \qquad (7.5)$$

Assuming that the dependence of G_A/G_V on the momentum transfer in the decay is negligible, μ_2 is taken at $q^2 = 2.449\ \mathrm{GeV}^2$ and each set of solutions in Table 3, conveniently labelled by the value of α, may then be used to give a value for the ratio: these are displayed in figure 18. The non-relativistic quark model prediction of $\frac{5}{3}$ (34) is also indicated, as is the experimental value of 1.253 ± 0.007 (35).

It should be emphasized that the value given by equation (7.4) is for *on mass shell* quarks with a very small, but *non-zero*, mass. A direct comparison with the off mass shell model of §5.3.1 requires modifications to the bag distribution of equation (5.96) to include massive quarks:

$$P(k) = NR^6\left(\frac{\sin(k-k_0)R}{2kk_0(k-k_0)R^3}\left[\left(\frac{\omega+m}{\omega}\right)^{\frac{1}{2}} + \left(\frac{\omega-m}{\omega}\right)^{\frac{1}{2}}\right] - \frac{\sin(k+k_0)R}{kk_0(k+k_0)R^3}\left[\left(\frac{\omega+m}{\omega}\right)^{\frac{1}{2}} - \left(\frac{\omega-m}{\omega}\right)^{\frac{1}{2}}\right]\right.$$

$$\left. - \frac{\sin k_0 R \sin kR}{k^2 k_0^2 R^4}\left[\left(\frac{\omega-m}{\omega}\right)^{\frac{1}{2}}\right]\right)^2 \qquad (7.6)$$

where

$$\omega^2 = k_0^2 + m^2 \qquad (7.7)$$

It must be recalled that the normalisation is prescribed to satisfy the Gross-Llewellyn Smith sum rule (see the arguments on pages 48 and 52). Using the parameters which gave good agreement with the structure functions in §5.3.1 ($k_0/M = 0.141$, $R = 15.0\ \mathrm{GeV}^{-1}$), equation (7.3) then reproduces the bag model result since our quark mass is very small and the linear

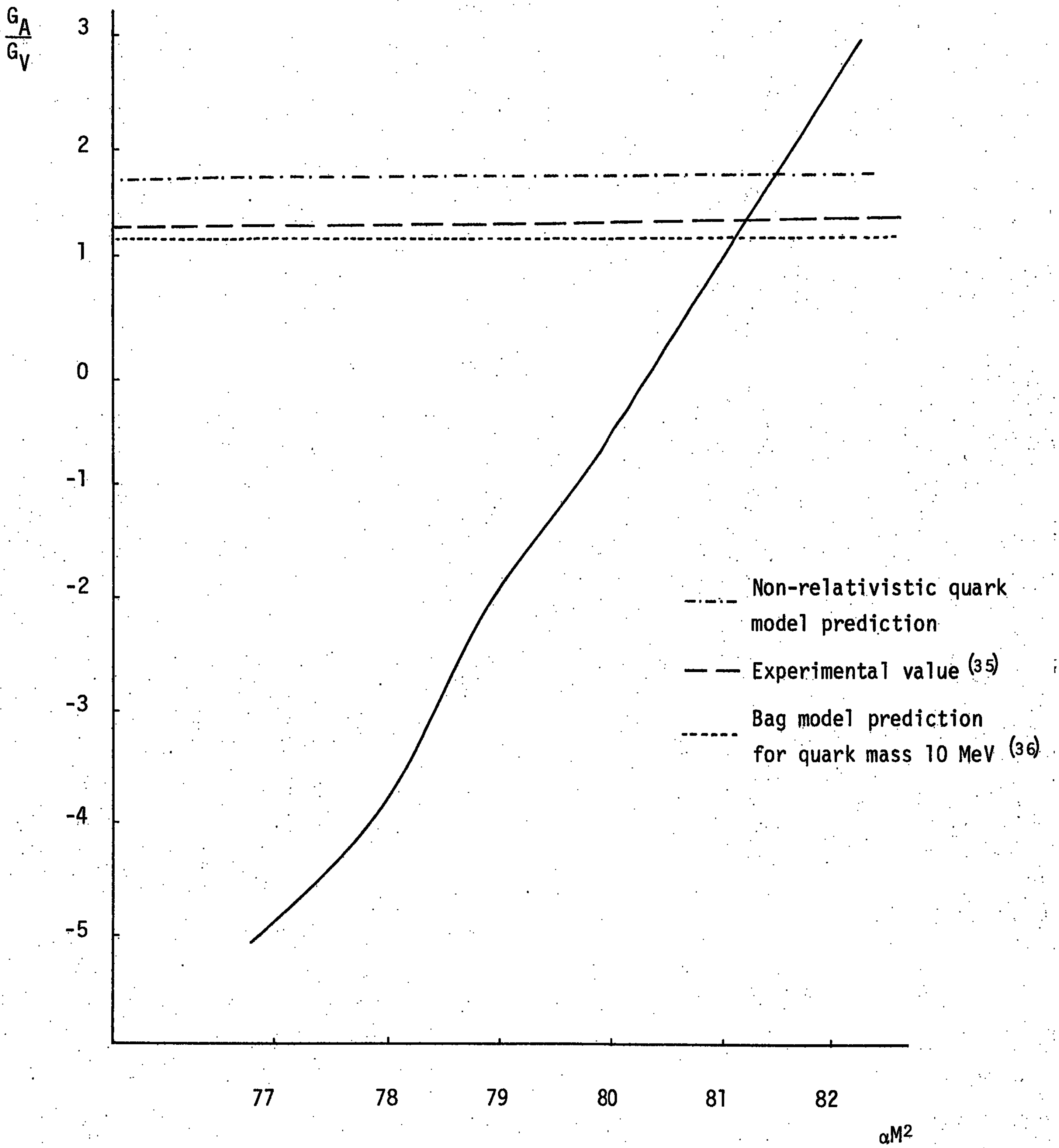


Figure 18. The ratio G_A/G_V for the model §5.2 and for each set of solutions in Table 3, labelled by the value of αM^2

bag boundary condition is well satisfied with our parameters: the value for G_A in the bag depends essentially on the product k_0R and only weakly on the quark mass (23)(36). In figure 18 we indicate the value 1.10 as an extrapolation to mass 10 MeV from Table I of ref. 36.

To obtain an approximate comparison with the *massless* case we surmise the following generalization of equation (7.3):

$$\frac{G_A}{G_V} = \frac{5}{9}\int_0^\infty\int_0^\infty\left(1 + \frac{2\sqrt{|k^2 - k_0^2|}}{k_0}\right)dk_0k^2dk\,P(k_0,k) \tag{7.8}$$

which, with fixed quark energy, becomes:

$$\frac{G_A}{G_V} = \frac{5}{9} + \frac{10}{9k_0}\int_0^\infty \sqrt{|k^2 - k_0^2|}\;k^2dk\,P(k) \tag{7.9}$$

Employing the bag distribution with massless quarks, equation (5.96), and again with $k_0/M = 0.141$, $R = 15.0\ \text{GeV}^{-1}$, we compute:

$$\frac{G_A}{G_V} = 0.93 \tag{7.10}$$

Notwithstanding the approximate nature of equation (7.8), this value and the values of G_A/G_V given in figure 18, obtained using the parameters deduced from the experimental structure functions, allow us to conclude that these two phenomena are not incompatible.

CHAPTER 8 CONCLUSIONS AND ONGOING SITUATIONS

Within the framework of the quark parton model a fundamental relationship between the structure function $F_2(x)$ and a general distribution in energy and three-momentum of the quarks, $P(k_0,k)$, was derived in Chapter 2, and reformulated in terms of the moments in Chapter 3. However it was noted that the range of the Feynman variable, x, was unlimited, a characteristic of so-called *x models*, and, in order to restrict x to its kinematically allowed region, $0 < x < 1$, we invoked the replacement of x by $-\log_e(1-x)$, thus constructing *$\log_e(1-x)$ models* for comparison.

Our main conclusions originate from Chapter 5, in which different modelled distributions were tested against the data. They may be summarized thus:

(i) With free, massless quarks (thus ignoring confinement) it is necessary to have a varying energy to obtain reasonable agreement with experiment; moreover, the expected improvement in agreement in the $\log_e(1-x)$ model is well demonstrated.

(ii) With free, massive quarks agreement could be achieved with a parameterized distribution in which the quark mass was 10 MeV. An analytical comparison with the $\log_e(1-x)$ model was precluded in this case, and this part of the work remains to be completed numerically.

(iii) Using the MIT bag model as an analogue for the confinement mechanism, the improvement in the structure function behaviour in the $\log_e(1-x)$ model was again explicitly demonstrated. Moreover, experimental agreement necessitated a fixed quark energy ≈ 0.13 GeV and a bag radius ≈ 15 GeV^{-1}. These values differ considerably from fits to the hadron spectrum but the linear bag boundary condition remains well satisfied [24].

(iv) It was shown that two examples of models with off mass shell, massless quarks with independent energy and three-momentum, and with independent energy and deviation from mass shell, were unable to fit the data.

The possible sources of corrections to the scaling phenomenon of the naive parton model were considered in Chapter 6, in particular the incorporation of kinematic corrections in our analysis through the inclusion of the first order Q^{-2} term. Within the context of the free, massless quark model of §5.1.2 it was demonstrated that a scale violating mechanism beyond target mass effects is required to explain the decrease in the structure function with increasing q^2 at $x \geqslant 0.3$.

Much recent attention [37] has been devoted to the relative size of the higher-twist terms and the logarithmic scaling violations in QCD referred to in Chapter 6. Having treated target mass effects equivalently through the Q^{-2} term in the expression for the structure function (equation (2.38)), the inclusion of other effects which are departures from the naive parton model picture (specifically quark transverse momentum, which has a twist-four form) and the consequences of the second order Q^{-4} term (a higher-twist contribution) must be considered. The overall x dependence of the latter term may be deduced quite straightforwardly by extending equation (6.9) to order Q^{-4} and then compared with the x dependence hypothesized from quark-counting arguments [38]; a more physical insight may then be developed.

In Chapter 7 we endeavoured to make the connection between the deep-inelastic structure function analysis of the previous chapters and a static property of the nucleon, namely the ratio G_A/G_V. Although this requires some refinement the compatibility suggests further applications of our parameterized distributions: the calculation of nucleon magnetic moments; and nucleon charge radii, where inconsistencies between structure function behaviour and the sign of the neutron charge radius may be resolved within the framework of the MIT bag by including admixtures of higher bag states [39]. However this contradicts our conclusion that a *lower* quark energy than the bag ground state is required to fit the structure functions. These calculations can be performed with free, massless quark distributions as well as bag distributions, as instanced for G_A/G_V in Chapter 7.

Our analysis may also be extended to polarized structure functions with a view to the sensitivity of the Bjorken sum rule [40] to light quark masses [41]: in this respect the use of the free, massive quark distribution with different u and d quark masses would be interesting *.

These then are some of the conclusions which may be proffered and the further applications which are at various stages of development.

* It has come to the author's attention that the effects of an anomaly in the divergences of the axial-vector current, ignored in ref. 41, cancel the isospin violating corrections to the Bjorken sum rule due to different u and d quark masses - see D.J. Gross, S.B. Treiman and F. Wilczek, *Phys. Rev.* D19 (1979) 2188.

Appendix A. Kinematic relationship between 'direct' and 'Z-graph' contributions to the structure function

Compare the 'direct' and 'Z-graph' contributions (see figures A1 and A2) and apply the mass condition to the final quark in each case. Then:

in the 'direct' case, $(k_0+\nu)^2 = |\underline{k}+\underline{q}|^2$

$\therefore$ $2k_0\nu+q^2-2|\underline{k}||\underline{q}|\cos\alpha = 0$

so, in the Bjorken limit, $2\nu(k_0-k\cos\alpha) \approx -q^2$ (A.1)

In the 'Z-graph' case, $(k_0-\nu)^2 = |\underline{k}-\underline{q}|^2$

$\therefore$ $2k_0\nu-q^2-2|\underline{k}||\underline{q}|\cos\alpha = 0$

so, in the Bjorken limit, $2\nu(k_0-k\cos\alpha) \approx q^2$ (A.2)

Therefore, defining x through equation (2.4), we see that the 'Z-graph' contribution can be simply obtained from the 'direct' contribution by replacing x with -x.

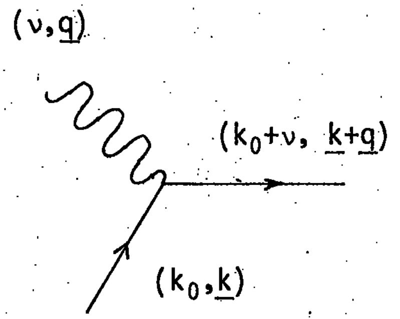


Figure A1. 'Direct' graph kinematics

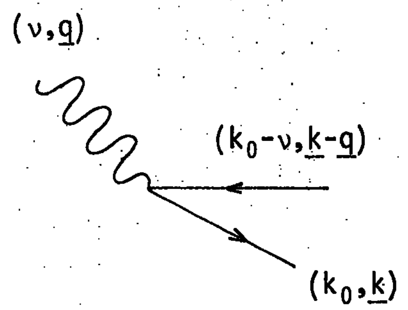


Figure A2. 'Z-graph' kinematics

Appendix B. Expansions of the integrand in the moments in the $\log_e(1-x)$ model

We can expand the bracket in equation (3.38) quite generally:

$$E_n \equiv \left[1 - \exp\left(-\frac{k_0+k}{M}\right)\right]^{n+1} - \left[1 - \exp\left(-\frac{k_0-k}{M}\right)\right]^{n+1}$$

$$= (n+1)e^{-\frac{k_0}{M}}\left(e^{\frac{k}{M}} - e^{-\frac{k}{M}}\right) - \frac{(n+1)n}{2!}e^{-\frac{2k_0}{M}}\left(e^{\frac{2k}{M}} - e^{-\frac{2k}{M}}\right)$$

$$+ \frac{(n+1)n(n-1)}{3!}e^{-\frac{3k_0}{M}}\left(e^{\frac{3k}{M}} - e^{-\frac{3k}{M}}\right) - \cdots\cdots \qquad (B.1)$$

The exponentials can then be expanded individually, so that for consecutive n:

$$E_1 = \frac{4}{M^2}k_0k - \frac{6}{M^3}k_0^2k - \frac{2}{M^3}k^3 + \frac{14}{3M^4}k_0^3k + \frac{14}{3M^4}k_0k^3 - \frac{5}{2M^5}k_0^4k - \frac{5}{M^5}k_0^2k^3 - \frac{1}{2M^5}k^5$$

$$+ \frac{31}{30M^6}k_0^5k + \frac{31}{9M^6}k_0^3k^3 + \frac{31}{30M^6}k_0k^5 - \frac{7}{20M^7}k_0^6k - \frac{7}{4M^7}k_0^4k^3 - \frac{21}{20M^7}k_0^2k^5$$

$$- \frac{1}{20M^7}k^7 + \frac{127}{1260M^8}k_0^7k + \frac{127}{180M^8}k_0^5k^3 + \frac{127}{180M^8}k_0^3k^5 + \frac{127}{1260M^8}k_0k^7 + \cdots\cdots \qquad (B.2)$$

$$E_2 = \frac{6}{M^3}k_0^2k + \frac{2}{M^3}k^3 - \frac{12}{M^4}k_0^3k - \frac{12}{M^4}k_0k^3 + \frac{25}{2M^5}k_0^4k + \frac{25}{M^5}k_0^2k^3 + \frac{5}{2M^5}k^5 - \frac{9}{M^6}k_0^5k$$

$$- \frac{30}{M^6}k_0^3k^3 - \frac{9}{M^6}k_0k^5 + \frac{301}{60M^7}k_0^6k + \frac{301}{12M^7}k_0^4k^3 + \frac{301}{20M^7}k_0^2k^5 + \frac{301}{420M^7}k^7 - \frac{23}{20M^8}k_0^7k$$

$$- \frac{161}{10M^8}k_0^5k^3 - \frac{161}{10M^8}k_0^3k^5 - \frac{23}{10M^8}k_0k^7 + \cdots\cdots \qquad (B.3)$$

$$E_3 = \frac{8}{M^4}k_0^3k + \frac{8}{M^4}k_0k^3 - \frac{20}{M^5}k_0^4k - \frac{40}{M^5}k_0^2k^3 - \frac{4}{M^5}k^5 + \frac{26}{M^6}k_0^5k + \frac{260}{3M^6}k_0^3k^3 + \frac{26}{M^6}k_0k^5$$

$$- \frac{70}{3M^7}k_0^6k - \frac{350}{3M^7}k_0^4k^3 - \frac{70}{M^7}k_0^2k^5 - \frac{10}{3M^7}k^7 + \frac{81}{5M^8}k_0^7k + \frac{567}{5M^8}k_0^5k^3 + \frac{567}{5M^8}k_0^3k^5$$

$$+ \frac{81}{5M^8}k_0k^7 + \cdots\cdots \qquad (B.4)$$

$$E_4 = \frac{10}{M^5}k_0^4k + \frac{20}{M^5}k_0^2k^3 + \frac{2}{M^5}k^5 - \frac{30}{M^6}k_0^5k - \frac{100}{M^6}k_0^3k^3 - \frac{30}{M^6}k_0k^5 + \frac{140}{3M^7}k_0^6k + \frac{700}{3M^7}k_0^4k^3$$

$$+ \frac{140}{M^7}k_0^2k^5 + \frac{20}{3M^7}k^7 - \frac{50}{M^8}k_0^7k - \frac{350}{M^8}k_0^5k^3 - \frac{350}{M^8}k_0^3k^5 - \frac{50}{M^8}k_0k^7 + \cdots\cdots \qquad (B.5)$$

$$E_5 = \frac{12}{M^6}k_0^5k + \frac{40}{M^6}k_0^3k^3 + \frac{12}{M^6}k_0k^5 - \frac{42}{M^7}k_0^6k - \frac{210}{M^7}k_0^4k^3 - \frac{126}{M^7}k_0^2k^5 - \frac{6}{M^7}k^7 + \frac{76}{M^8}k_0^7k$$

$$+ \frac{532}{M^8}k_0^5k^3 + \frac{532}{M^8}k_0^3k^5 + \frac{76}{M^8}k_0k^7 + \cdots\cdots \qquad (B.6)$$

$$E_6 = \frac{14}{M^7}k_0^6k + \frac{70}{M^7}k_0^4k^3 + \frac{42}{M^7}k_0^2k^5 + \frac{2}{M^7}k^7 - \frac{56}{M^8}k_0^7k - \frac{392}{M^8}k_0^5k^3 - \frac{392}{M^8}k_0^3k^5 - \frac{56}{M^8}k_0k^7 + \cdots\cdots \qquad (B.7)$$

$$E_7 = \frac{16}{M^8}k_0^7k + \frac{112}{M^8}k_0^5k^3 + \frac{112}{M^8}k_0^3k^5 + \frac{16}{M^8}k_0k^7 + \cdots\cdots \qquad (B.8)$$

Appendix C Reconstruction of Structure Functions from their Moments

Although a general, point-like reconstruction of a structure function, F, from its moments is impossible, the use of normalised Bernstein polynomials allows an *average* reconstruction of F given by [20] *:

$$\tilde{F}(x_{N,k}) = \frac{(N+1)!}{k!} \sum_{\ell=0}^{N-k} \frac{(-1)^{\ell}}{\ell!(N-k-\ell)!} \mu_{k+\ell+2} \tag{C.1}$$

where $\tilde{F}(x_{N,k})$ is the average of F around the point $x_{N,k}$:

$$x_{N,k} = \frac{k+1}{N+2} \quad ; \quad k = 0,1 \ldots N \tag{C.2}$$

and μ_n is the nth moment and (N + 1) moments are considered. Under the reasonable assumption that F is twice differentiable (except at its endpoints) we can proceed further and reconstruct a *pointwise* function:

$$F(x_{N,k}) = \tilde{F}(x_{N,k}) + \delta(x_{N,k}) \tag{C.3}$$

where

$$\delta(x_{N,k}) = \frac{(N+1)!(N-k+1)}{2k!(N+2)^2(N+3)} \sum_{\ell=0}^{N-k} \frac{(-1)^{\ell}(k+\ell)(k+\ell-1)}{\ell!(N-k-\ell)!} \mu_{k+\ell} \tag{C.4}$$

These formulae are used in an attempt to reconstruct the structure functions F_2 and xF_3 from just the n=2, 3 and 4 moments at 7.071 GeV^2 and to compare with the averaged experimental structure functions. The results are shown in Table 5; within the experimental errors the agreement is satisfactory. However it must be pointed out that the moments used in this reconstruction are n=2, 3 and 4 and do not include the 'normalisation moment' (n=1) incorporated in Chapter 5. Hence, in fitting the moments therein, we

* Our notation differs from that of Yndurái̇n in ref. 20 by relabelling the μ_n to correspond with the notation of Chapter 3 - note, however, that in this appendix the μ_n refer specifically to either F_2 or xF_3 and not the linear combinations of Chapter 3.

must not be surprised if the reproduced structure function does not agree too well with the data. The use of more parameters and more moments would obviously give improvement (in theory an infinite number are required), but we only wish to suggest the smallest number of parameters which may be adequate.

Table 5. Comparison of reconstructed pointwise structure functions with experimental averages

k	x	$\tilde{F}_2$	δ	F_2	Experimental Bin	Experimental value of F_2
0	0.25	0.980	0.057	1.037	$0.2 < x < 0.3$	1.141 ± 0.146
1	0.50	0.423	-0.076	0.348	$0.4 < x < 0.6$	0.258 ± 0.051
2	0.75	0.112	0.019	0.131	$0.6 < x < 1.0$	0.049 ± 0.049

k	x	$x\tilde{F}_3$	δ	xF_3	Experimental Bin	Experimental value of xF_3
0	0.25	0.765	0.051	0.816	$0.2 < x < 0.3$	0.792 ± 0.455
1	0.50	0.431	-0.068	0.363	$0.4 < x < 0.6$	0.513 ± 0.117
2	0.75	0.158	0.017	0.175	$0.6 < x < 1.0$	0.107 ± 0.080

Appendix D Evaluation of the definite integrals required in §5.2

With the distribution (5.63) substituted in equation (3.29) the moments M_n (n=1,2,3) are:

$$M_1 = Ne^{\alpha m^2} \int_m^\infty (k_0^2-m^2)(k_0-m)e^{-\alpha k_0^2}(1+ak_0+bk_0^2)dk_0 = 1 \qquad (D.1)$$

$$M_2 = \frac{4Ne^{\alpha m^2}}{3M} \int_m^\infty k_0(k_0^2-m^2)(k_0-m)e^{-\alpha k_0^2}(1+ak_0+bk_0^2)dk_0$$

$$- \frac{Nm^2e^{\alpha m^2}}{3M} \int_m^\infty \frac{(k_0^2-m^2)(k_0-m)}{k_0}\, e^{-\alpha k_0^2}(1+ak_0+bk_0^2)dk_0$$

$$= \mu_2(q^2) \qquad (D.2)$$

$$M_3 = \frac{2Ne^{\alpha m^2}}{M^2} \int_m^\infty k_0^2(k_0^2-m^2)(k_0-m)e^{-\alpha k_0^2}(1+ak_0+bk_0^2)dk_0$$

$$= \mu_3(q^2) + \frac{m^2}{M^2} \qquad (D.3)$$

Expanding the integrands then demands the following integrals [21][22]:

$$\int_m^\infty \frac{1}{k_0}\, e^{-\alpha k_0^2}\, dk_0 = -\frac{1}{2}\,\mathrm{Ei}(-\alpha m^2) \qquad (D.4)$$

$$\int_m^\infty e^{-\alpha k_0^2}\, dk_0 = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\,\mathrm{erfc}\; m\sqrt{\alpha} \qquad (D.5)$$

where erfc is the complementary error function, and:

$$\int_m^\infty k_0^n\, e^{-\alpha k_0^2}\, dk_0 = \frac{1}{2}\,\alpha^{-\left(\frac{n+1}{2}\right)}\,\Gamma\left(\frac{n+1}{2},\, \alpha m^2\right) \qquad (D.6)$$

where Γ is the incomplete gamma function [42]. After substituting and manoeuvring these expressions into equations (D.1)-(D.3), equations (5.66)-(5.68) will ensue.

Appendix E Evaluation of the definite integrals required in §5.3.3

With the distribution (5.115) substituted in equation (3.29) the moments M_n (n=1,2,3) are:

$$M_1 = C\int_0^\infty k_0 e^{-\alpha|k_0-E_0|}dk_0 \int_0^\infty k^2 e^{-\mu|k-k_0|}dk = 1 \qquad \text{(E.1)}$$

$$M_2 = \frac{C}{3M}\int_0^\infty e^{-\alpha|k_0-E_0|}dk_0 \int_0^\infty k^4 e^{-\mu|k-k_0|}dk$$

$$+\frac{C}{M}\int_0^\infty k_0^2 e^{-\alpha|k_0-E_0|}dk_0 \int_0^\infty k^2 e^{-\mu|k-k_0|}dk = \mu_2(q^2) \qquad \text{(E.2)}$$

$$M_3 = \frac{C}{M^2}\int_0^\infty k_0 e^{-\alpha|k_0-E_0|}dk_0 \int_0^\infty k^4 e^{-\mu|k-k_0|}dk$$

$$+\frac{C}{M^2}\int_0^\infty k_0^3 e^{-\alpha|k_0-E_0|}dk_0 \int_0^\infty k^2 e^{-\mu|k-k_0|}dk = \mu_3(q^2) \qquad \text{(E.3)}$$

We thus require the following integrals:

$$\int_0^\infty k^n e^{-\mu|k-k_0|}dk = e^{-\mu k_0}\int_0^{k_0} k^n e^{\mu k}dk + e^{\mu k_0}\int_{k_0}^\infty k^n e^{-\mu k}dk$$

$$= e^{-\mu k_0}\left[e^{\mu k}\left(\frac{k^n}{\mu} - \frac{nk^{n-1}}{\mu^2} + \cdots + (-1)^n \frac{n!}{\mu^{n+1}}\right)\right]_0^{k_0}$$

$$+ e^{\mu k_0}\left[-e^{-\mu k}\left(\frac{k^n}{\mu} + \frac{nk^{n-1}}{\mu^2} + \cdots + \frac{n!}{\mu^{n+1}}\right)\right]_{k_0}^\infty$$

$$= \frac{2k_0{}^n}{\mu} + \frac{2n(n-1)k_0{}^{n-1}}{\mu^3} + \cdots + (1+(-1)^n)\frac{n!}{\mu^{n+1}} + e^{-\mu k_0}(-1)^{n+1}\frac{n!}{\mu^{n+1}}$$

(E.4)

Making the replacements $k_0 \to E_0$, $\mu \to \alpha$:

$$\int_0^\infty k_0{}^n e^{-\alpha|k_0-E_0|}dk_0 = \frac{2E_0{}^n}{\alpha} + \frac{2n(n-1)E_0{}^{n-1}}{\alpha^3} + \cdots + (1 + (-1)^n)\frac{n!}{\alpha^{n+1}}$$

$$+ e^{-\alpha E_0}(-1)^{n+1}\frac{n!}{\alpha^{n+1}} \tag{E.5}$$

And:

$$\int_0^\infty k_0{}^n e^{-\alpha|k_0-E_0|}e^{-\mu k_0}dk_0 = e^{-\alpha E_0}\int_0^{E_0} k_0{}^n e^{(\alpha-\mu)k_0}dk_0 + e^{\alpha E_0}\int_{E_0}^\infty k_0{}^n e^{-(\alpha+\mu)k_0}dk_0$$

$$= e^{-\alpha E_0}\left[e^{(\alpha-\mu)k_0}\left(\frac{k_0{}^n}{(\alpha-\mu)} - \frac{nk_0{}^{n-1}}{(\alpha-\mu)^2} + \cdots + (-1)^n\frac{n!}{(\alpha-\mu)^{n-1}}\right)\right]_0^{E_0}$$

$$- e^{\alpha E_0}\left[e^{-(\alpha+\mu)k_0}\left(\frac{k_0{}^n}{(\alpha+\mu)} + \frac{nk_0{}^{n-1}}{(\alpha+\mu)^2} + \cdots + (+1)^n\frac{n!}{(\alpha+\mu)^{n+1}}\right)\right]_{E_0}^\infty$$

$$= e^{-\mu E_0}\left(\frac{E_0{}^n}{(\alpha+\mu)} + \frac{E_0{}^n}{(\alpha-\mu)} + \frac{nE_0{}^{n-1}}{(\alpha+\mu)^2} - \frac{nE_0{}^{n-1}}{(\alpha-\mu)^2} + \cdots + (+1)^n\frac{n!}{(\alpha+\mu)^{n+1}}\right.$$

$$\left. + (-1)^n\frac{n!}{(\alpha-\mu)^{n+1}}\right) + e^{-\alpha E_0}(-1)^{n+1}\frac{n!}{(\alpha-\mu)^{n+1}} \tag{E.6}$$

Armed with these expressions, and after some rearrangement, equations (5.116)-(5.118) may be extorted from equations (E.1)-(E.3).

With the modified distribution (5.121) the moments M_n are identical except for the lower limit of the k_0 integral, which is now E_0 instead of zero. The integrals over k will remain the same but the integrals over k_0 will now involve:

$$\int_{E_0}^\infty k_0{}^n e^{-\alpha(k_0-E_0)}dk_0 = \frac{E_0{}^n}{\alpha} + \frac{nE_0{}^{n-1}}{\alpha^2} + \cdots + \frac{n!}{\alpha^{n+1}} \tag{E.7}$$

and

$$\int_{E_0}^\infty k_0{}^n e^{-\alpha(k_0-E_0)}e^{-\mu k_0}dk_0 = e^{-\mu E_0}\left(\frac{E_0{}^n}{(\alpha+\mu)} + \cdots + \frac{n!}{(\alpha+\mu)^{n+1}}\right) \tag{E.8}$$

These two expressions can be simply read off from the second parts of equations (E.5) and (E.6) respectively. Equations (5.122)-(5.124) then follow.

Appendix F Calculation of the ratio G_A/G_V

The vector coupling constant and the axial-vector coupling constant of β decay are defined by:

$$G_V \equiv \bar{\Psi} \gamma_0 \Psi \tag{F.1}$$

$$G_A \equiv \bar{\Psi} \gamma_3 \gamma_5 \Psi \tag{F.2}$$

Our intention is to apply the model of §5.2 with free, massive quarks with varying energy, hence Ψ is the plane wave solution to the Dirac equation for a particle of mass m:

$$\Psi = \frac{1}{\sqrt{2k_0}} \sum_s u_s e^{-ikx} \tag{F.3}$$

and u is a suitably normalised four-component spinor:

$$u_s = \begin{pmatrix} \sqrt{k_0+m} & w_s \\ \sqrt{k_0-m} & \underline{\sigma}\cdot\hat{\underline{k}}\, w_s \end{pmatrix} \tag{F.4}$$

We refer to the standard representation of Berestetskii *et al.* (43) to obtain:

$$G_V = 1 \tag{F.5}$$

$$G_A = \bar{\Psi} \begin{pmatrix} \underline{\sigma}\cdot\hat{\underline{z}} & 0 \\ 0 & -\underline{\sigma}\cdot\hat{\underline{z}} \end{pmatrix} \Psi \tag{F.6}$$

Substituting equations (F.3) and (F.4) into equation (F.6) and using

$$(\underline{\sigma}\cdot\hat{\underline{z}})(\underline{\sigma}\cdot\hat{\underline{k}}) = (\underline{\sigma}\cdot\hat{\underline{k}})(\underline{\sigma}\cdot\hat{\underline{z}}) + 2i\underline{\sigma}\cdot(\hat{\underline{z}}\wedge\hat{\underline{k}}) \tag{F.7}$$

gives:

$$G_A = \left(\frac{1}{3} + \frac{2m}{3k_0}\right) \sum_s w_s^{\dagger}\, \underline{\sigma}\cdot\hat{\underline{z}}\, w_s \tag{F.8}$$

Sandwiching this between nucleon states gives the SU(6) factor $\frac{5}{3}$ [34] and including the spherically symmetric quark energy-momentum distribution gives the result of equation (7.3):

$$\frac{G_A}{G_V} = \frac{5}{9}\int_m^\infty\int_0^\infty \left(1 + \frac{2m}{k_0}\right) dk_0 k^2 dk\, P(k_0, k) \qquad \text{(F.9)}$$

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