Harmonic Riemannian manifolds

Carpenter, Paul

HARMONIC RIEMANNIAN MANIFOLDS

by

PAUL CARPENTER

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Contents

Abstract

In this thesis work is described that arose out of a study of harmonic Riemannian manifolds. A definition of harmonicity is given and from this it is shown how the Ledger conditions on the curvature of a harmonic manifold may be derived in principle and the first four are written down. The first three Ledger conditions are put into local co-ordinate form and simpler conditions are derived, the most important being the super-Einstein condition. The idea of the Schur property is also introduced. The mean-value work of Gray and Willmore is described and extended as far as the r^8 term under some simplifying conditions. Finally there is an investigation of the extent to which the compact classical simple Lie groups with bi-invariant metrics can satisfy Ledger's first three conditions.

Preface

The work presented here was carried out between October 1977 and August 1980 in the University of Durham under the supervision of Professor T.J. Willmore.

The material contained in this thesis has not been submitted previously for any degree in this or any other university. Those parts of Chapters 1 and 2 not contained in [B] or [PA], of Chapter 3 not contained in [GW] and of Chapter 4 not contained in [PO] are, as far as I know, original pieces of work.

My thanks are due to Professor Willmore for his broad knowledge of differential geometry, his enthusiasm for the subject and his constant encouragement. Also I would like to thank other members of the differential geometry group at Durham who have made my time here interesting and stimulating. Thanks are also due to Dr. Sheena Bartlett, the excellence of whose typing is self-evident. Finally I would like to express my gratitude to the Science Research Council for making this period of study possible.

Introduction

The work contained in this thesis arose out of a study of Riemannian harmonic manifolds. These were originally considered by Ruse fifty years ago to be manifolds in which there exist solutions of Laplace's equation on a neighbourhood of each point which were functions only of the radial distance from that point. Since then the only examples found of harmonic manifolds are the rank one symmetric spaces. This has led to the so-called fundamental conjecture of the subject: a Riemannian harmonic manifold is a rank one symmetric space. The resolution of this conjecture still seems far off and awaits either a new approach to harmonicity or new examples of Riemannian manifolds which can easily be tested for harmonicity in the hope of finding a counterexample.

In Chapter 1 we define harmonicity using the determinant of the metric tensor in normal co-ordinates. The first four Ledger conditions for harmonicity are derived, as are similar conditions derived by using the trace of the metric tensor and the trace of its inverse. It is noted that these conditions come from matrix differential equations which have similar properties and are worthy of investigation in their own right.

Chapter 2 contains a description of the first three Ledger conditions in local co-ordinates. The second and third are rather complicated, so at the expense of losing some information simpler 2-tensor conditions are derived. This leads in the case of the second condition to the notion of a super-Einstein space. Harmonic spaces are super-Einstein but the reverse need not be true. Also we introduce the idea of the Schur property for symmetric 2-tensors, the guiding example being the Ricci tensor. We give some conjectures on the 2-tensors derived from further Ledger conditions which, on the face of it, cannot be resolved by the methods of the chapter.

In Chapter 3 the work of Gray and Willmore on the power series of the mean-value of a function over a geodesic sphere is described and extended. It is shown, rather disappointingly, that at least as far as r^8 term the mean-value power series gives no more information concerning harmonicity than is already contained in the first three Ledger conditions.

Ledger was the first to show that if a symmetric space is harmonic then it is rank one. We show in Chapter 4 that the classical Lie groups with bi-invariant metrics which are not rank one are quite a long way from being harmonic in the sense that only two can satisfy Ledger's 2nd condition and none satisfy the third. This work has been generalised and extended to symmetric spaces in a joint paper with A. Gray and T.J. Willmore currently in preparation.

In all chapters M is an n-dimensional connected analytic Riemannian manifold with metric tensor g and arc length s . The sign convention for the curvature tensor is that of [GR], [H] and [E]. The Einstein summation convention is assumed throughout, apart from §3 of Chapter 3. The end of a proof will be denoted by $\square$.

Chapter 1 Definition of Harmonicity and Some Necessary Conditions

We first define the very useful tool of normal co-ordinates about a point of a Riemannian manifold, then using these we give one of the classical definitions of harmonicity. (For other definitions and their equivalence, see [RWW] pp. 34 - 43.) The major part of this chapter is concerned with deriving information of a tensorial nature from this definition. The method used is that of [B], Chapter 6, with a slightly different emphasis. As is usual in this subject, the information comes in form of the vanishing of nearly all the terms of a power series, giving rise to an infinite number of necessary conditions for harmonicity which, when taken together, are sufficient. We only consider the first four of these.

§1 Normal co-ordinates, the function Θ_m and the matrix A

There is associated with each point of a Riemannian manifold a geometrically defined class of local co-ordinate systems which, when used in calculations, give information about the geometry of the manifold. These are the normal co-ordinate systems and will be used extensively in what follows.

We define normal co-ordinates about a point by using the exponential map. Let $m \in M$. The exponential map at m, $\exp_m$, is a diffeomorphism of some neighbourhood of 0 in T_mM onto some neighbourhood of m in M. By definition, the injectivity radius i(m) is the supremum (possibly infinite) of positive real numbers ε such that $\exp_m | \mathring{B}(0,\varepsilon)$ is a diffeomorphism onto its image. We shall say that the image of the open ball radius i(m), denoted V_m, is the maximal normal co-ordinate neighbourhood of m, and we co-ordinatise it as follows. Choose an orthonormal basis of T_mM, $(U_1, \ldots, U_n)$, and define normal co-ordinates

$(x^1, \ldots, x^n)$ of a point $\tilde{n} \in V_m$ by

$$(1.1) \qquad x^i(\tilde{n}) = t_i, \qquad i = 1, \ldots, n,$$
$$\text{if} \quad \tilde{n} = \exp_m(t_j U_j).$$

A normal co-ordinate neighbourhood of m is a nhd of m contained in V_m with the obvious parametrisation.

As the image of a straight line through 0 in $T_m M$ is a geodesic through m in M, a geodesic through m restricted to V_m has the equation in normal co-ordinates

$$(1.2) \qquad x^i(s) = sa^i, \qquad i = 1, \ldots, n,$$

where a^i are the components of the tangent vector of the geodesic at m with respect to $(U_1, \ldots, U_n)$.

Note that the definition of normal co-ordinates requires a choice of orthonormal basis of $T_m M$. However it can be easily seen that any two sets of normal co-ordinates about m are expressible in terms of each other by an orthogonal matrix.

We shall often refer to tensorial equations as being "in orthonormal co-ordinates". This means that we have chosen a point m of our manifold, a system of normal co-ordinates about m, and have expressed the components of the tensors evaluated at m in these co-ordinates.

We also define a geodesic ball and sphere, centre m, radius r, for $r < i(m)$. These are the images under $\exp_m$ of a ball and sphere, centre 0, radius r in $T_m M$.

Θ_m is a real-valued function defined on V_m for each $m \in M$ as follows. Let g_{ij} be the components of the metric tensor in a set of normal co-ordinates about m. Then

$$\Theta_m = \sqrt{\det g}. \tag{1.3}$$

Note that this definition is independent of the choice of orthonormal frame.

We are ready to define harmonicity of a manifold.

<u>Definition:</u> A Riemannian manifold M is harmonic if for each pt $m \in M$, the function Θ_m is a function of distance from m only.

We wish to translate this condition into information about the curvature of the manifold. The most direct way of doing so would be to exploit the available power series expansion of Θ_m about m as given for example in [GR]. However we follow the method of [B] which is, in fact, similar to Gray's method of calculation.

The first step is to exploit (1.2) as the equation for geodesics through m.

<u>Lemma 1.1</u> In normal co-ordinates the vector field $s\,c^i\frac{\partial}{\partial x^i}$, where c^i are constants, is a Jacobi vector field along any geodesic through m restricted to V_m.

<u>Proof</u> We demonstrate that $s\,c^i\frac{\partial}{\partial x^i}$ is the variation vector field of a variation through geodesics of any geodesic γ with equation $\gamma^i(s) = sa^i$ for $c^i \neq a^i$. When $c^i = a^i$, $s\,c^i\frac{\partial}{\partial x^i}$ is obviously a Jacobi field along γ.

Consider the two parameter family of vectors $s(a^i + tc^i)U_i \in T_mM$. For small enough s,t these are contained in the ball radius i(m). Their images under $\exp_m$ are then seen to be geodesics through m giving a variation through geodesics of γ. The variation vector field is then seen to be $s\,c^i\frac{\partial}{\partial x^i}$ along γ. $\square$

If we denote this vector field by J(s), then along γ it satisfies the vector differential equation

$$(1.4) \qquad \nabla_{\dot\gamma}\nabla_{\dot\gamma}J + R(\dot\gamma,J)\dot\gamma = 0,$$

with initial conditions

$$J(0) = 0, \quad \nabla_{\dot\gamma}J(0) = c^iU_i$$

(since $\nabla_{\dot\gamma}J(0) = \nabla_{\dot\gamma}(sc^i\frac{\partial}{\partial x^i})(0) = \nabla_{\dot\gamma}(s)c^i\frac{\partial}{\partial x^i}(0) = c^iU_i$).

Let us denote by $E_1(s), \ldots, E_n(s)$ the parallel translation of $U_1, \ldots, U_n$ along γ. We can thus define an $n \times n$ matrix of functions A along γ, restricted to V_m, by

$$(1.5) \qquad s\frac{\partial}{\partial x^i} = A_{ji}E_j, \qquad i = 1, \ldots, n.$$

From Lemma 1.1 $s\frac{\partial}{\partial x^i}$ are Jacobi fields along γ, and A is the matrix of components of these with respect to a parallel frame.

It is easy to determine an ordinary differential equation satisfied by A by substituting the Jacobi field $A_{ji}E_j$ into (1.4) to get

$$\frac{d^2A_{ji}}{ds^2}E_j + A_{ji}R(\dot\gamma,E_j)\dot\gamma = 0, \quad i = 1, \ldots, n.$$

By taking the scalar product with E_ℓ we get

$$\frac{d^2A_{\ell i}}{ds^2} + A_{ji}R_{j\ell} = 0, \qquad i,\ell = 1, \ldots, n,$$

where R is a symmetric matrix of functions along γ defined by

$$R_{ij} = g(R(\dot\gamma,E_i)\dot\gamma,E_j), \qquad i,j = 1, \ldots, n.$$

In matrix form this can be expressed by

$$(1.6) \qquad A'' + RA = 0.$$

The initial conditions for this equation are

(1.7) $$A(0) = 0, \quad A'(0) = I$$

(since $A'_{ji}(0)U_i = \nabla_{\dot\gamma}(A_{ji}E_j)(0) = \nabla_{\dot\gamma}(s\frac{\partial}{\partial x^i})(0) = U_i$).

The significance of A is its relationship with Θ_m. From (1.5)

$$s^2 g_{ij} = g(s\frac{\partial}{\partial x^i}, s\frac{\partial}{\partial x^j}) = g(A_{ki}E_k, A_{\ell j}E_\ell) = A_{ki}A_{kj} = (A^TA)_{ij}.$$

Thus concisely

(1.8) $$s^2 g = A^TA.$$

Taking determinants, we have

(1.9) $$\Theta_m = \frac{1}{s^n}\det A.$$

<u>Remark 1</u> Note that power series expansion of g can be found from that of A by (1.8). However we will give later an easier method of finding the expansions of the determinant and trace of g.

<u>Remark 2</u> This approach in defining Θ_m is essentially that of [B]. However there Θ_m is defined directly from A by (1.9) and has the advantage that it is defined along the whole length of the geodesic.

Before we come to the calculation of the power series of Θ_m, we prove a rather unexpected property of Θ_m in a harmonic manifold. We show that Θ_m is, loosely speaking, the same function of s for each $m \in M$. Precisely:

<u>Proposition 1.2</u> Let $m \in M$, and $r_0 < i(m)$. The injectivity radius is a continuous function on M and thus attains its minimum on $B(m,r_0)$, $i_0(m)$ say. Then if M is harmonic, $\Theta_m(r) = \Theta_{\tilde n}(r)$ for $r < i_0(m)$ and $\tilde n \in \mathring{B}(m,r_0)$.

We first need a lemma, a classical result in the theory of differential equations.

Lemma 1.3 Suppose m, γ and A are as above. Let B be a matrix satisfying the same differential equation (1.6) but with initial conditions

$$B(t) = 0\,, \quad B'(t) = -I\,, \quad \text{for particular } t,\ t \neq 0.$$

Then $B(0) = A^T(t)$.

Proof We know that

$$A'' + RA = 0\,, \quad A(0) = 0\,, \quad A'(0) = I\,,$$

$$B'' + RB = 0\,, \quad B(t) = 0\,, \quad B'(t) = -I.$$

Transpose the first equation and postmultiply by B, and premultiply the second by A^T. On subtracting, the symmetry of R gives

$$A^{T''}B - A^TB'' = 0.$$

We can integrate this to

$$A^{T'}B - A^TB' = C\,, \qquad C \text{ constant},$$

and the result follows from substituting in the prescribed values at 0 and t. $\square$

Proof of Proposition 1.2 Let $\tilde{n} \in \overset{o}{B}(m,r_0)$ and $U \in T_{\tilde{n}}M$. We shall show that $df_r(U) = 0$ where f_r is the real-valued function defined on $\overset{o}{B}(m,r_0)$ by

$$f_r(\tilde{m}) = \Theta_{\tilde{m}}(r), \quad r < i_0(m),\ \tilde{m} \in \overset{o}{B}(m,r_0).$$

As f_r is continuous (when expressed as a power series in r, the coefficients are polynomials in the curvature tensor and its covariant derivatives, so it is, in fact, analytic) this will prove the proposition.

We are assuming $r < i(\tilde{n})$, so $\Theta_{\tilde{n}}(r) \neq 0$. We consider a geodesic γ (parametrised by arc length) through $\tilde{n}$ such that $\gamma(r) = \tilde{n}$ and

$\gamma'(r) \perp U$. Let $\gamma(0) = p$. As $r < i(\tilde{n})$, $\tilde{n}$ and p are not conjugate along γ. Thus there exists a 1-parameter family of geodesics γ_t such that $\gamma_t(0) = p$, $\gamma_0 = \gamma$ and $\frac{\partial}{\partial t}(\gamma_t(r))(0) = U$.
Applying Lemma 1.3 to each γ_t we see that, on taking determinants,

$$\Theta_{\gamma_t(r)}(r) = \Theta_p(r),$$

and so $df(U) = \frac{\partial}{\partial t}(\Theta_{\gamma_t(r)}(r))(0) = 0.$ $\square$

This proof is based upon [B] p. 157, but the proof there has been modified to overcome a few apparent problems.

§2 The power series expansion of Θ_m and other functions

Our method of calculation depends on the following classical result on the derivative of the determinant Δ of a matrix function M.

$$(1.10) \qquad (\log \Delta)' = \operatorname{tr}(M'M^{-1}).$$

We are led to a consideration of the matrix $A'A^{-1}$. Unfortunately for small s

$$(1.11) \qquad A = sI + O(s^2)$$

by (1.7), and so A^{-1} is not defined at 0. However $\frac{1}{s}A$ is invertible in a nhd of 0 with inverse sA^{-1}, and so we consider the matrix $C = sA'A^{-1}$, which is well-defined for small s.

Using (1.9) and (1.10) we find that

$$(1.12) \qquad s(\log \Theta_m)' = \operatorname{tr}C - n.$$

It is as well to point out now that the title of this section is misleading. We will not calculate the power series of Θ_m, but rather that of $\log \Theta_m$ via (1.12). This is enough for our purposes as Θ_m is a function of s alone iff $\log \Theta_m$ is a function of s alone. The power series of Θ_m can be calculated from that of $\log \Theta_m$, e.g. by use of the formula $\Theta' = \Theta(\log\Theta)'$.

An elementary calculation shows that C satisfies

$$sC' = -s^2R - C^2 + C. \tag{1.13}$$

Unfortunately this equation has a singular point at 0, but, in the analytic case, inspection shows that given C(0) and C'(0) all the other derivatives of C at 0 can be found, and hence a unique solution is generated.*

We calculate

$$C(0) = \lim_{s\to 0} sA'A^{-1} = \lim_{s\to 0} A' \lim_{s\to 0} sA^{-1}$$
$$= I.I = I.$$

Also since $A''(0) = 0$, equation (1.11) becomes

$$A = sI + O(s^3).$$

Hence $(sA^{-1})'(0) = 0$ and

$$C'(0) = A''(0)(sA^{-1})(0) + A'(0)(sA^{-1})'(0)$$
$$= 0.$$

We are now in a position to calculate the power series expansion of C as far as we wish from the following recurrence relation for the derivatives of C at 0, derived from (1.13).†

$$(p+1)C^{(p)}(0) = -p(p-1)R^{(p-2)}(0) - \sum_{k=2}^{p-2}\binom{p}{k} C^{(k)}(0)\, C^{(p-k)}(0) \qquad p \geq 2$$

* It also follows that C is symmetric as C^T satisfies the same equation and conditions at 0

† This corrects the formula in [B]

The first nine derivatives are given here:

$$C(0) = I,$$

$$C'(0) = 0,$$

$$C^{(2)}(0) = -\tfrac{2}{3}R(0),$$

$$C^{(3)}(0) = -\tfrac{3}{2}R'(0),$$

$$C^{(4)}(0) = -\tfrac{12}{5}R^{(2)}(0) - \tfrac{8}{15}R(0)R(0),$$

$$C^{(5)}(0) = -\tfrac{10}{3}R^{(3)}(0) - \tfrac{5}{3}(R'(0)R(0) + R(0)R'(0)),$$

$$\begin{aligned} C^{(6)}(0) = &- \tfrac{30}{7}R^{(4)}(0) - \tfrac{45}{7}R'(0)R'(0) - \tfrac{24}{7}(R^{(2)}(0)R(0) + R(0)R^{(2)}(0)) \\ &- \tfrac{32}{21}R(0)R(0)R(0), \end{aligned}$$

$$\begin{aligned} C^{(7)}(0) = &- \tfrac{21}{4}R^{(5)}(0) - \tfrac{35}{6}(R^{(3)}(0)R(0) + R(0)R^{(3)}(0)) \\ &- \tfrac{77}{12}(R'(0)R(0)R(0) + R(0)R(0)R'(0)) - \tfrac{35}{6}R(0)R'(0)R(0) \\ &- \tfrac{63}{4}(R^{(2)}(0)R'(0) + R'(0)R^{(2)}(0)), \end{aligned}$$

$$\begin{aligned} C^{(8)}(0) = &- \tfrac{56}{9}R^{(6)}(0) - \tfrac{80}{9}(R^{(4)}(0)R(0) + R(0)R^{(4)}(0)) \\ &- \tfrac{280}{9}(R^{(3)}(0)R'(0) + R'(0)R^{(3)}(0)) - \tfrac{16.14}{5}R^{(2)}(0)R^{(2)}(0) \\ &- \tfrac{256}{15}(R^{(2)}(0)R(0)R(0) + R(0)R(0)R^{(2)}(0)) - \tfrac{128}{9}R(0)R^{(2)}(0)R(0) \\ &- \tfrac{260}{9}(R'(0)R'(0)R(0) + R(0)R'(0)R'(0)) - \tfrac{280}{9}(R'(0)R(0)R'(0)) \\ &- \tfrac{128}{15}R(0)R(0)R(0)R(0), \end{aligned}$$

$$\begin{aligned} C^{(9)}(0) = &- \tfrac{36}{5}R^{(7)}(0) - \tfrac{63}{5}(R^{(5)}(0)R(0) + R(0)R^{(5)}(0)) \\ &- 54(R^{(4)}(0)R'(0) + R'(0)R^{(4)}(0)) - \tfrac{21.24}{5}(R^{(3)}(0)R^{(2)}(0) \\ &\quad + R^{(2)}(0)R^{(3)}(0)) - \tfrac{26.7}{5}(R^{(3)}(0)R(0)R(0) + R(0)R(0)R^{(3)}(0)) \\ &- 28R(0)R^{(3)}(0)R(0) - 18.9R'(0)R'(0)R'(0) - \tfrac{21.21}{5}R^{(2)}(0)R'(0)R(0) \\ &- \tfrac{39.12}{5}R'(0)R(0)R^{(2)}(0) - 81R(0)R^{(2)}(0)R'(0) \\ &- \tfrac{39.12}{5}R^{(2)}(0)R(0)R'(0) - 81R'(0)R^{(2)}(0)R(0) \\ &- \tfrac{21.21}{5}R(0)R'(0)R^{(2)}(0) - \tfrac{229}{5}(R'(0)R(0)R(0)R(0) + R(0)R(0)R(0)R'(0)) \\ &- \tfrac{29.7}{5}(R(0)R'(0)R(0)R(0) + R(0)R(0)R'(0)R(0)). \end{aligned}$$

We next give the traces of these matrices:

$$
\begin{aligned}
\mathrm{tr}C(0) &= n\,,\\
\mathrm{tr}C'(0) &= 0\,,\\
\mathrm{tr}C^{(2)}(0) &= -\tfrac{2}{3}\mathrm{tr}R(0)\,,\\
\mathrm{tr}C^{(3)}(0) &= -\tfrac{3}{2}\mathrm{tr}R'(0)\,,\\
\mathrm{tr}C^{(4)}(0) &= -\tfrac{12}{5}\mathrm{tr}R^{(2)}(0) - \tfrac{8}{15}\mathrm{tr}R(0)R(0)\,,\\
\mathrm{tr}C^{(5)}(0) &= -\tfrac{10}{3}\mathrm{tr}R^{(3)}(0) - \tfrac{10}{3}\mathrm{tr}R'(0)R(0)\,,\\
\mathrm{tr}C^{(6)}(0) &= -\tfrac{30}{7}\mathrm{tr}R^{(4)}(0) - \tfrac{45}{7}\mathrm{tr}R'(0)R'(0) - \tfrac{48}{7}\mathrm{tr}R^{(2)}(0)R(0)\\
&\quad - \tfrac{32}{21}\mathrm{tr}R(0)R(0)R(0)\,,\\
\mathrm{tr}C^{(7)}(0) &= -\tfrac{21}{4}\mathrm{tr}R^{(5)}(0) - \tfrac{35}{3}\mathrm{tr}R^{(3)}(0)R(0) - \tfrac{56}{3}\mathrm{tr}R'(0)R(0)R(0)\\
&\quad - \tfrac{63}{2}\mathrm{tr}R^{(2)}(0)R'(0)\,,\\
\mathrm{tr}C^{(8)}(0) &= -\tfrac{56}{9}\mathrm{tr}R^{(6)}(0) - \tfrac{160}{9}\mathrm{tr}R^{(4)}(0)R(0) - \tfrac{560}{9}\mathrm{tr}R^{(3)}(0)R'(0)\\
&\quad - \tfrac{16.14}{5}\mathrm{tr}R^{(2)}(0)R^{(2)}(0) - \tfrac{64.34}{45}\mathrm{tr}R^{(2)}(0)R(0)R(0)\\
&\quad - \tfrac{800}{9}\mathrm{tr}R'(0)R'(0)R(0) - \tfrac{128}{15}\mathrm{tr}R(0)R(0)R(0)R(0)\,,\\
\mathrm{tr}C^{(9)}(0) &= -\tfrac{36}{5}\mathrm{tr}R^{(7)}(0) - \tfrac{126}{5}\mathrm{tr}R^{(5)}(0)R(0) - 108\,\mathrm{tr}R^{(4)}(0)R'(0)\\
&\quad - \tfrac{56.18}{5}\mathrm{tr}R^{(3)}(0)R^{(2)}(0) - \tfrac{28.18}{5}\mathrm{tr}R^{(3)}(0)R(0)R(0)\\
&\quad - 9.18\,\mathrm{tr}R'(0)R'(0)R'(0) - \tfrac{73.18}{5}\mathrm{tr}R^{(2)}(0)R'(0)R(0)\\
&\quad - \tfrac{73.18}{5}\mathrm{tr}R^{(2)}(0)R(0)R'(0) - \tfrac{48.18}{5}\mathrm{tr}R'(0)R(0)R(0)R(0)\,.
\end{aligned}
\tag{1.14}
$$

Using (1.12) necessary conditions for harmonicity can be deduced from these equations. This will be done in the next section.

Remark Normal co-ordinates about a point are but a special example of Fermi co-ordinates associated with a submanifold. The construction of the matrices A and C along a geodesic perpendicular to the submanifold is similar; in fact they satisfy the same differential equations but with different initial conditions. An exposition of this generalisation is given in Appendix I, together with a rather surprising property of C.

We now digress from [B] to illustrate a similar method of obtaining the power series expansions of the trace of the metric tensor in normal co-ordinates and the trace of its inverse. This is of interest as Willmore in [W2] has introduced the idea of k-harmonic manifolds which are manifolds where the k^{th} symmetric sum of the eigenvalues of the inverse of g in normal co-ordinates is a function of s alone. Thus n-harmonic is equivalent to harmonic, and 1-harmonic is defined using the trace of the inverse of g.

We have already noted that

$$s^2 g = A^T A$$

and that the expansion for g could be found from that of A. It would be convenient if we could find a differential equation for $A^T A$. However the best we can do is find one for AA^T which, while not the same matrix, has the same eigenvalues. Let us denote the matrix $\frac{1}{s^2}AA^T$ by D and its inverse by E. (By (1.11) D is invertible in a nhd of 0.)

<u>Proposition 1.4</u> D and E satisfy the following differential equations and initial conditions:

$$(1.15) \qquad sD' = -2D + CD + DC, \qquad D(0) = I, \; D'(0) = 0,$$

$$(1.16) \qquad sE' = 2E - CE - EC, \qquad E(0) = I, \; E'(0) = 0.$$

<u>Proof</u> This follows easily from

$$sD' = s\left(\frac{1}{s^2}AA^T\right)' = -\frac{2}{s^2}AA^T + s.\frac{1}{s^2}A'(A^{-1}A)A^T + s.\frac{1}{s^2}A(A^T(A^T)^{-1})(A^T)'$$

and

$$D'E + \cancel{ED'}\,DE' = 0.^{\dagger}$$

The initial conditions are easily derived from those of $\frac{1}{s}A$:

† We also need that C is symmetric (page 8)

$$\left(\frac{1}{s}A\right)(0) = I, \quad \left(\frac{1}{s}A\right)'(0) = 0. \qquad \square$$

(1.15) and (1.16) lead to recurrence formulae for the derivatives of D and E at 0:

$$pD^{(p)}(0) = \sum_{k=2}^{p-2} \binom{p}{k} \left(C^{(k)}(0)\, D^{(p-k)}(0) + D^{(p-k)}(0)\, C^{(k)}(0)\right) + 2\,C^{(p)}(0), \quad p \geqslant 2,$$

$$pE^{(p)}(0) = -\sum_{k=2}^{p-2} \binom{p}{k} \left(C^{(k)}(0)\, E^{(p-k)}(0) + E^{(p-k)}(0)\, C^{(k)}(0)\right) - 2\,C^{(p)}(0), \quad p \geqslant 2.$$

With our knowledge of the first nine derivatives of C we can write those of D and E. However, we will give just the traces:

$$\begin{aligned}
\mathrm{tr}D(0) &= n,\\
\mathrm{tr}D'(0) &= 0,\\
\mathrm{tr}D^{(2)}(0) &= -\tfrac{2}{3}\mathrm{tr}R(0),\\
\mathrm{tr}D^{(3)}(0) &= -\mathrm{tr}R'(0),\\
\mathrm{tr}D^{(4)}(0) &= -\tfrac{6}{5}\mathrm{tr}R^{(2)}(0) + \tfrac{16}{15}\mathrm{tr}R(0)R(0),\\
\mathrm{tr}D^{(5)}(0) &= -\tfrac{4}{3}\mathrm{tr}R^{(3)}(0) + \tfrac{16}{3}\mathrm{tr}R'(0)R(0),\\
\mathrm{tr}D^{(6)}(0) &= -\tfrac{10}{7}\mathrm{tr}R^{(4)}(0) + \tfrac{55}{7}\mathrm{tr}R'(0)R'(0) + \tfrac{68}{7}\mathrm{tr}R^{(2)}(0)R(0)\\
&\quad - \tfrac{16}{7}\mathrm{tr}R(0)R(0)R(0),\\
\mathrm{tr}D^{(7)}(0) &= -\tfrac{3}{2}\mathrm{tr}R^{(5)}(0) + \tfrac{46}{3}\mathrm{tr}R^{(3)}(0)R(0) - 24\mathrm{tr}R'(0)R(0)R(0)\\
&\quad + 33\mathrm{tr}R^{(2)}(0)R'(0),\\
\mathrm{tr}D^{(8)}(0) &= -\tfrac{14}{9}\mathrm{tr}R^{(6)}(0) + \tfrac{200}{9}\mathrm{tr}R^{(4)}(0)R(0) + \tfrac{19.28}{9}\mathrm{tr}R^{(3)}(0)R'(0)\\
&\quad + \tfrac{14.14}{5}\mathrm{tr}R^{(2)}(0)R^{(2)}(0) - \tfrac{83.32}{45}\mathrm{tr}R^{(2)}(0)R(0)R(0)\\
&\quad - \tfrac{16.53}{9}\mathrm{tr}R'(0)R'(0)R(0) + \tfrac{256}{45}\mathrm{tr}R(0)R(0)R(0)R(0),\\
\mathrm{tr}D^{(9)}(0) &= -\tfrac{8}{5}\mathrm{tr}R^{(7)}(0) + \tfrac{8.19}{5}\mathrm{tr}R^{(5)}(0)R(0) + 96\mathrm{tr}R^{(4)}(0)R'(0)\\
&\quad + \tfrac{16.49}{5}\mathrm{tr}R^{(3)}(0)R^{(2)}(0) - \tfrac{16.38}{5}\mathrm{tr}R^{(3)}(0)R(0)R(0)\\
&\quad - 8.17\mathrm{tr}R'(0)R'(0)R'(0) - \tfrac{8.161}{5}\mathrm{tr}R^{(2)}(0)R'(0)R(0)\\
&\quad - \tfrac{8.161}{5}\mathrm{tr}R^{(2)}(0)R(0)R'(0) + \tfrac{64.8}{5}\mathrm{tr}R'(0)R(0)R(0)R(0),
\end{aligned} \tag{1.17}$$

$$
\begin{aligned}
\operatorname{tr}E(0) &= n,\\
\operatorname{tr}E'(0) &= 0,\\
\operatorname{tr}E^{(2)}(0) &= \tfrac{2}{3}\operatorname{tr}R(0),\\
\operatorname{tr}E^{(3)}(0) &= \operatorname{tr}R'(0),\\
\operatorname{tr}E^{(4)}(0) &= \tfrac{6}{5}\operatorname{tr}R^{(2)}(0) + \tfrac{8}{5}\operatorname{tr}R(0)R(0),\\
\operatorname{tr}E^{(5)}(0) &= \tfrac{4}{3}\operatorname{tr}R^{(3)}(0) + 8\operatorname{tr}R'(0)R(0),\\
\operatorname{tr}E^{(6)}(0) &= \tfrac{10}{7}\operatorname{tr}R^{(4)}(0) + \tfrac{85}{7}\operatorname{tr}R'(0)R'(0) + \tfrac{100}{7}\operatorname{tr}R^{(2)}(0)R(0)\\
&\quad + \tfrac{160}{21}\operatorname{tr}R(0)R(0)R(0),\\
\operatorname{tr}E^{(7)}(0) &= \tfrac{3}{2}\operatorname{tr}R^{(5)}(0) + 22\operatorname{tr}R^{(3)}(0)R(0) + 51\operatorname{tr}R^{(2)}(0)R'(0)\\
&\quad + 80\operatorname{tr}R'(0)R(0)R(0),\\
\operatorname{tr}E^{(8)}(0) &= \tfrac{14}{9}\operatorname{tr}R^{(6)}(0) + \tfrac{280}{9}\operatorname{tr}R^{(4)}(0)R(0) + \tfrac{28.29}{9}\operatorname{tr}R^{(3)}(0)R'(0)\\
&\quad + \tfrac{28.11}{5}\operatorname{tr}R^{(2)}(0)R^{(2)}(0) + \tfrac{28.304}{45}\operatorname{tr}R^{(2)}(0)R(0)R(0)\\
&\quad + \tfrac{14.208}{9}\operatorname{tr}R'(0)R'(0)R(0) + \tfrac{32.28}{15}\operatorname{tr}R(0)R(0)R(0)R(0),\\
\operatorname{tr}E^{(9)}(0) &= \tfrac{8}{5}\operatorname{tr}R^{(7)}(0) + \tfrac{208}{5}\operatorname{tr}R^{(5)}(0)R(0) + 144\operatorname{tr}R^{(4)}(0)R'(0)\\
&\quad + \tfrac{77.16}{5}\operatorname{tr}R^{(3)}(0)R^{(2)}(0) + \tfrac{32.58}{5}\operatorname{tr}R^{(3)}(0)R(0)R(0)\\
&\quad + 16.31\operatorname{tr}R'(0)R'(0)R'(0) + \tfrac{16.271}{5}\operatorname{tr}R^{(2)}(0)R'(0)R(0)\\
&\quad + \tfrac{16.271}{5}\operatorname{tr}R^{(2)}(0)R(0)R'(0) + \tfrac{64.84}{5}\operatorname{tr}R'(0)R(0)R(0)R(0).
\end{aligned}
\tag{1.18}
$$

§3 The first four harmonic conditions

Using now our equations for the derivatives of trC, (1.14), we can write down some necessary conditions that trC only depends on s. Before we do so, we change our notation slightly (à la Besse) to emphasise the choice of initial vector of the geodesic we have been considering. Thus we shall denote the endomorphism from T_mM to T_mM given by $V \to R(U,V)U$ for $U,V \in T_mM$ by R_U and the endomorphisms given by $V \to \nabla^n_{U \ldots U}R(U,V)U$ by $R^{(n)}_U$.

The definition of harmonicity requires that trC should depend only upon distance along the geodesic and not upon the initial direction of the geodesic so the derivatives of trC at 0 must be independent of the initial direction. From Proposition 1.~~4~~ 2 we see that they must also be locally constant as we allow our initial point to vary, and, as we are assuming M connected, globally constant. We would expect, then, from equations (1.14) to have 8 pieces of information to make use of. This, however, is not the case, and the total information is summed up in the following theorem.

<u>Theorem 1.5</u> (Ledger) Suppose M is harmonic. Then for any $U \in UM$, the following equations hold for constants K,H,L,M:

$$\mathrm{tr}R_U = K, \tag{1.19}$$

$$\mathrm{tr}R_U R_U = H, \tag{1.20}$$

$$32\,\mathrm{tr}R_U R_U R_U - 9\,\mathrm{tr}R'_U R'_U = L, \tag{1.21}$$

$$3\,\mathrm{tr}R_U^{(2)} R_U^{(2)} + 8\,\mathrm{tr}R_U^{(2)} R_U R_U - 50\,\mathrm{tr}R'_U R'_U R_U + 72\,\mathrm{tr}R_U R_U R_U R_U = M. \tag{1.22}$$

<u>Proof</u> (1.19) is obviously derived from the formula for $\mathrm{tr}C^{(2)}$. However assuming (1.19) immediately implies that $\mathrm{tr}C^{(3)} = 0$, so no new information is derived from that condition.

Again, assuming (1.19) we get (1.20) from the formula for $\mathrm{tr}C^{(4)}$. Once more no new information is gained from the formula for $\mathrm{tr}C^{(5)}$ as it is, assuming (1.19), a constant multiple of the derivative of (1.20).

(1.21) and (1.22) are derived similarly. One assumes the previously gained conditions to obtain them, and one finds also that under these conditions the next (odd) condition is a constant multiple of the derivative of the condition preceding it. □

<u>Conjecture</u> Assuming the conditions for harmonicity derived from

the formulae for $\mathrm{tr}C^{(k)}$, $k = 2, \ldots, 2p-1$, then $\mathrm{tr}C^{(2p+1)}$ is a constant multiple of the derivative of the condition derived from $\mathrm{tr}C^{(2p)}$.

Conditions (1.19) – (1.22) are known as the first four Ledger conditions for harmonicity. The k^{th} Ledger condition, which we shall denote by L_k, is derived from the formula for $\mathrm{tr}C^{(2k)}$, assuming the previous k-1 Ledger conditions.

It is of some interest to investigate the conditions derived from equations (1.17) and (1.18). These are set out in the following proposition.

Proposition 1.6 Suppose at each point of M trD is required to depend only on s, then the following equations hold for constants K_1, H_1, L_1, M_1, for any $U \in UM$:

$$
\begin{aligned}
&\mathrm{tr}R_U = K_1, \\
&\mathrm{tr}R_UR_U = H_1, \\
(1.23) \qquad &16\mathrm{tr}R_UR_UR_U + 13\mathrm{tr}R'_UR'_U = L_1, \\
&13.13\mathrm{tr}R_U^{(2)}R_U^{(2)} - 37.8\mathrm{tr}R_U^{(2)}R_UR_U + 23.50\mathrm{tr}R'_UR'_UR_U \\
&\qquad + 32.13\mathrm{tr}R_UR_UR_UR_U = M_1.
\end{aligned}
$$

Similarly, if the same is required of trE, we have the following equations for constants K_2, H_2, L_2, M_2, and any $U \in UM$:

$$
\begin{aligned}
&\mathrm{tr}R_U = K_2, \\
&\mathrm{tr}R_UR_U = H_2, \\
(1.24) \qquad &32\mathrm{tr}R_UR_UR_U - 9\mathrm{tr}R'_UR'_U = L_2, \\
&3\mathrm{tr}R_U^{(2)}R_U^{(2)} + 8\mathrm{tr}R_U^{(2)}R_UR_U - 50\mathrm{tr}R'_UR'_UR_U \\
&\qquad + 72\mathrm{tr}R_UR_UR_UR_U = M_2.
\end{aligned}
$$

Proof We first note that if trD or trE is assumed to be a function of s at each point, then by a similar proof to that of Proposition 1.2, it must be the same function of s at each point. For, according to Lemma 1.3, in that notation,

$$\operatorname{tr} A A^T(t) = \operatorname{tr} B^T B(0) = \operatorname{tr} B B^T(0),$$

and the rest of that proof carries through in the same way.

The proof of the current proposition uses the same method of proof as Theorem 1.5 but see Remark 1 below. $\square$

Remark 1 As is seen when the calculations in Proposition 1.6 are carried out, equations (1.15) and (1.16) have the same property as equation (1.13): the odd derivatives give no new information, at least as far as the ninth derivative. Again we conjecture that this happens for all the odd derivatives.

Remark 2 It will be noted that conditions (1.24) and (1.19) - (1.22) are identical, and this leads to the obvious conjecture:

trC constant function of s over M <=> trE constant function of s over M.

Willmore [W2] has proved the '=>' part, but the converse is as yet unproved. It may be asked why the trace of the inverse of g should be closely related to the determinant of g rather than the trace of g itself. The only light we can shed on this is to observe that the forms of equation for C and E are similar (try substituting C for E in equation (1.16)).

Chapter 2 Local Co-ordinates, Super-Einstein Spaces

In this chapter the conditions for harmonicity are put into local co-ordinate form. This leads to the definition of super-Einstein space which is an Einstein space with an extra condition on the curvature. We prove some curvature formulae in super-Einstein spaces and also introduce the notion of the Schur property which seems to be shared by a number of 2-tensors in this theory.

§1 Harmonic conditions in local co-ordinates

We return to our normal co-ordinate system about $m \in M$ and suppose a geodesic γ through m has initial vector $U = a^i U_i$ and hence equation

$$\gamma^i(s) = a^i s, \qquad i = 1, \ldots, n,$$

then along γ,

$$R_{\dot\gamma} V = R^i{}_{jk\ell} a^j V^k a^\ell \frac{\partial}{\partial x^i} \quad \text{for vector field } V = V^i \frac{\partial}{\partial x^i}.$$

Thus at m

$$R_U U_k = R^i{}_{jk\ell}(m) a^j a^\ell U_i,$$

and

$$\mathrm{tr} R_U = \rho_{j\ell}(m) a^j a^\ell, \qquad \text{where } \rho \text{ is the Ricci tensor.}$$

Ledger's 1st condition, (1.19), states that

$$\mathrm{tr} R_U = K, \qquad \forall U \in \mathrm{UM},$$

and this leads to

$$\rho_{j\ell}(m)a^j a^\ell = K\delta_{j\ell}a^j a^\ell,$$

since U is unit and $g_{j\ell}(m) = \delta_{j\ell}$. As this is true for all choices of the a's we have

$$(2.1) \qquad S(\rho_{j\ell}(m)) = KS(\delta_{j\ell}),$$

where by definition $S(T_{i_1, \ldots, i_k})$ is the sum over all permutations of the free $i_1, \ldots, i_k$ i.e.

$$S(T_{i_1, \ldots, i_k}) = \sum_{\sigma \in S_k} T_{\sigma(i_1) \ldots \sigma(i_k)},$$

where S_k is the symmetric group on k objects. As ρ is symmetric we can rewrite (2.1) as

$$(2.2) \qquad \rho_{j\ell}(m) = K\delta_{j\ell} \qquad \text{in normal co-ordinates}$$

about m and so M must be Einstein.

Considering now $L2$, (1.20), we have at m

$$R_U R_U U_k = R^i{}_{jk\ell}(m)a^j a^\ell R^p{}_{qis}(m)a^q a^s U_p.$$

Thus

$$\operatorname{tr} R_U R_U = (R^i{}_{jk\ell}R^k{}_{qis})(m)\,a^j a^\ell a^q a^s.$$

Ledger's 2nd condition becomes, in normal co-ordinates based at m,

$$(2.3) \qquad S(R_{iajb}R_{ka\ell b})(m) = HS(\delta_{ij}\delta_{k\ell})$$

(Note that because $g_{ij}(m) = \delta_{ij}$ we can write contracted sums with all indices downstairs).

Noting that $R'_{\dot\gamma}V = \nabla_{\dot\gamma}(R^i{}_{jk\ell})a^j V^k a^\ell \frac{\partial}{\partial x^i}$

$$= \nabla_r R^i{}_{jk\ell}a^j V^k a^\ell a^r \frac{\partial}{\partial x^i},$$

and

$$R^{(p)}{}_{\dot{\gamma}}V = \nabla_{r_1} \cdots {}_{r_p} R^{i}{}_{jk\ell} a^j a^\ell a^{r_1} \cdots a^{r_p} V^k \frac{\partial}{\partial x^i},$$

we can write down $L3$ and $L4$ in normal co-ordinates based at m:

$$(2.4) \qquad S(32R_{iajb}R_{kb\ell c}R_{mcha} - 9\nabla_i R_{jakb}\nabla_\ell R_{mahb})(m) = LS(\delta_{ij}\delta_{k\ell}\delta_{mh}),$$

$$(2.5) \qquad \begin{aligned} &S(3\nabla_{ij}R_{ka\ell b}\nabla_{mn}R_{paqb} + 8\nabla_{ij}R_{ka\ell b}R_{mbnc}R_{pcqa} \\ &- 50\nabla_i R_{jakb}\nabla_\ell R_{mbnc}R_{cpaq} + 72R_{iajb}R_{kb\ell c}R_{mcnd}R_{pdqa})\ (m) \\ &= MS(\delta_{ij}\delta_{k\ell}\delta_{mn}\delta_{pq}). \end{aligned}$$

In later sections we shall gain useful information from (2.3) and (2.4) but (2.5) is too unwieldy to be of much use. It was originally hoped that the first three conditions with some manipulation might lead to the proof of the fundamental conjecture, but this has not been the case. The conjecture has been proved by Lichnerowicz and Walker (see [B], p.166) in the case of dimension 4 from the first three conditions, but this makes heavy use of the low dimensionality.

We now discuss a notion which occurs frequently in this context, the easiest example of which is contained in the following well-known theorem of Schur.

Theorem 2.1 Suppose the curvature of M satisfies $\rho = fg$ for some function f. Then if dim M $\neq$ 2, f is a constant.

Proof In orthonormal co-ordinates we are given that

$$(2.6) \qquad \rho_{ij} = f\delta_{ij}.$$

Applying ∇_i to both sides and summing, we find

$$\nabla_i \rho_{ij} = \nabla_j f.$$

Using the 2nd Bianchi identity, this becomes

$$\tfrac{1}{2}\nabla_j \tau = \nabla_j f .$$

On the other hand, taking the trace of (2.6) reveals that

$$\frac{\tau}{n} = f .$$

Comparing the last two equations we see that if $n \neq 2$, then $\nabla_j f = 0$ and f is a constant. $\square$

In the light of this, we shall say that a symmetric covariant 2-tensor T defined on M has the Schur property if, given $T = fg$ for some function f, then f is a constant. Further, we shall say that a symmetric covariant 2-tensor T defined on M has the Schur property of order k if it has the Schur property except when $\dim M = k$. Thus the Ricci tensor has the Schur property of order 2.

§2 Consequences of $L2$. Super-Einstein spaces.

We recall Ledger's 2nd condition in orthonormal co-ordinates:

$$S(R_{iajb}R_{ka\ell b}) = HS(\delta_{ij}\delta_{k\ell}) .$$

We can gain a 2-tensor relation by putting $k = \ell$ and summing from 1 to n. As we carry out many calculations of this kind, we write this one out explicitly.

Proposition 2.2 If the curvature of M satisfies $L1$ and $L2$, then it also satisfies

$$(2.7) \qquad R_{iabc}R_{jabc} = S\delta_{ij} \qquad \text{in orthonormal co-ordinates,}$$

where $\quad S = \tfrac{2}{3}((n+2)H - K^2) .$

<u>Proof</u> It is necessary to consider the permutations on 4 letters, 2 of which are the same, say ijkk. We first write down those with i preceding j:

ijkk, ikjk, ikkj, kijk, kikj, kkij.

The corresponding terms on the LHS of (2.3) are now written down and summed over k, simplifying where possible:

$$R_{iajb}R_{kakb} = K^2\delta_{ij},$$

$$R_{iakb}R_{jakb},$$

$$\left.\begin{aligned} R_{iakb}R_{kajb} &= \tfrac{1}{2}R_{iakb}R_{jakb}, \\ R_{kaib}R_{jakb} &= \tfrac{1}{2}R_{iakb}R_{jakb}, \end{aligned}\right\} \text{see Lemma 2.4 below}$$

$$R_{kaib}R_{kajb} = R_{iakb}R_{jakb},$$

$$R_{kakb}R_{iajb} = K^2\delta_{ij}.$$

And the RHS:

$$\delta_{ij}\delta_{kk} = n\delta_{ij},$$

$$\delta_{ik}\delta_{jk} = \delta_{ij},$$

$$\delta_{ik}\delta_{kj} = \delta_{ij},$$

$$\delta_{ki}\delta_{jk} = \delta_{ij},$$

$$\delta_{ki}\delta_{kj} = \delta_{ij},$$

$$\delta_{kk}\delta_{ij} = n\delta_{ij}.$$

Noting that the LHS and RHS are both symmetric in i and j, and thus there is no need to consider terms with j preceding i, we get by adding,

$$\tfrac{3}{2}R_{iakb}R_{jakb} + K^2\delta_{ij} = H(n+2)\,\delta_{ij}$$

and (2.7) follows easily. $\square$

We shall denote the tensor $R_{iabc}R_{jabc}$ by $\dot{R}_{ij}$.

<u>Proposition 2.3</u> In an Einstein space, $\dot{R}$ has the Schur property of order 4.

<u>Proof</u> We are given that

$$(2.9) \qquad R_{iabc}R_{jabc} = f\,\delta_{ij}, \qquad \text{for some function } f.$$

As in Theorem 2.1, we take ∇_i of each side and sum:

$$\nabla_i R_{iabc}R_{jabc} + R_{iabc}\nabla_i R_{jabc} = \nabla_j f.$$

By the 2nd Bianchi identity,

$$\nabla_i R_{iabc} = \nabla_b \rho_{ac} - \nabla_c \rho_{ab} = 0, \text{ in an Einstein space}.$$

Also

$$\begin{aligned} R_{iabc}\nabla_i R_{jabc} &= \tfrac{1}{2}R_{iabc}\nabla_j R_{iabc} \qquad \text{(Lemma 2.4)} \\ &= \tfrac{1}{4}\nabla_j(R_{iabc}R_{iabc}). \end{aligned}$$

Thus we have shown that

$$\nabla_j(\tfrac{1}{4}R_{iabc}R_{iabc}) = \nabla_j f.$$

On the other hand taking the trace of (2.9) gives

$$\frac{1}{n}R_{iabc}R_{iabc} = f$$

and the proposition follows. $\square$

An Einstein space of dim > 4 which satisfies (2.9) has been defined by Gray and Willmore [GW] to be a super-Einstein manifold. An Einstein space of dimension $\leqslant 4$ automatically satisfies (2.9). This can be shown either by direct calculation using the Singer-Thorpe form of the curvature of a 4-dimensional Einstein manifold [ST] or by the algebraic method of Patterson [PA], a description of which will be given at the end of this chapter. A super-Einstein space of dimension 4 is defined to be an Einstein space with $|R|^2 = R_{iabc}R_{iabc}$ constant.

Examples of super-Einstein spaces are not difficult to find. Any irreducible symmetric space is super-Einstein, since any 2-tensor obtained from the curvature tensor has covariant derivative zero and by a theorem of Walker [WA] in an irreducible space it must be a multiple of the metric tensor. Most known examples of Einstein spaces are also super-Einstein, but Gray and Vanhecke have shown that there exist metrics on spheres of dimension 4n + 3 which are Einstein, but not super-Einstein.

As we have seen, a harmonic manifold must be super-Einstein (including the case of dimension 4. A 4-dimensional harmonic manifold is locally symmetric, so $|R|^2$ must be constant.).

§3 Curvature formulae in a super-Einstein space

In this section we prove some interesting relations between various tensors in a super-Einstein space. First we make some definitions. The following equations in orthonormal co-ordinates define the tensors on the LHS:

$$\overset{\underline{o}}{R}_{ij} = R_{iafb}R_{jcfd}R_{acbd}\,, \qquad \nabla_i R \nabla_j R = \nabla_i R_{abcd} \nabla_j R_{abcd}\,,$$

$$\overset{o}{R}_{ij} = R_{ifab}R_{jfcd}R_{abcd}\,, \qquad \nabla R_i \nabla R_j = \nabla_a R_{ibcd} \nabla_a R_{jbcd}\,,$$

$$T_i = \nabla_a R_{bcde} R_{bcdf} R_{iaef}\,,$$

and the scalars

$$\overset{\underline{o}}{R} = \overset{\underline{o}}{R}_{ii}\,, \qquad |\nabla R|^2 = \nabla_i R \nabla_i R = \nabla R_i \nabla R_i\,,$$

$$\overset{o}{R} = \overset{o}{R}_{ii}\,.$$

The next lemma is a consequence of the Bianchi identities, and was used in the last section.

Lemma 2.4 Suppose A_{ij}, $1 \leqslant i,j \leqslant n$, is a set of numbers anti-symmetric in i and j. Then (i) $A_{ij}R_{iajb} = \frac{1}{2}A_{ij}R_{ijab}$,

(ii) $A_{ij}\nabla_i R_{cjab} = \frac{1}{2}A_{ij}\nabla_c R_{ijab}$.

Proof (i)

$$\begin{aligned} A_{ij}R_{iajb} &= \tfrac{1}{2}(A_{ij} - A_{ji})R_{iajb} \\ &= \tfrac{1}{2}A_{ij}(R_{iajb} - R_{jaib}) \\ &= \tfrac{1}{2}A_{ij}R_{ijab}\,, \end{aligned}$$

by the 1st Bianchi identity.

(ii) Proved similarly, using the 2nd Bianchi identity. $\square$

These results will be made considerable use of in the sequel, as will the Ricci identity for the non-commutativity of covariant differentiation (see e.g. [E]) and the fact that the 3-tensor $\nabla_i R_{jbcd} R_{kbcd}$ is anti-symmetric in j and k in a super-Einstein space.

Generally there is no relation between the four 2-tensors defined above even in an Einstein space. However we have the result:-

Proposition 2.5 In an Einstein space with $\rho = Kg$,

$$\overset{o}{R}_{ij} + 4\overset{\underline{o}}{R}_{ij} - \nabla R_i \nabla R_j = 2K\dot{R}_{ij} - \tfrac{1}{2}\Delta(\dot{R}_{ij})\,. \tag{2.10}$$

Proof By definition

$$\Delta(\dot{R}_{ij}) = \nabla_{kk}(R_{iabc}R_{jabc})$$

$$= \nabla_{kk}R_{iabc}R_{jabc} + 2\nabla_k R_{iabc}\nabla_k R_{jabc} + R_{iabc}\nabla_{kk}R_{jabc}$$

$$(2.11) \qquad = 2\nabla_{kb}R_{iakc}R_{jabc} + 2\nabla R_i\nabla R_j + 2R_{iabc}\nabla_{kb}R_{jakc}$$

(Lemma 2.4(ii)).

By the Ricci identity,

$$\nabla_{kb}R_{iakc} = \nabla_{bk}R_{iakc} + R_{hibk}R_{hakc} + R_{habk}R_{ihkc} + R_{hkbk}R_{iahc} + R_{hcbk}R_{iakh}\,.$$

The Einstein condition implies that $\nabla_{bk}R_{iakc} = 0$ and $R_{hkbk}R_{iahc} = KR_{iabc}$, and so (2.11) becomes

$$\Delta(\dot{R}_{ij}) = 2(KR_{iabc}R_{jabc} + R_{hibk}R_{hakc}R_{jabc} + R_{habk}R_{ihkc}R_{jabc}$$
$$+ R_{hcbk}R_{iakh}R_{jabc}) + 2\nabla R_i\nabla R_j + 2(KR_{iabc}R_{jabc} + R_{iabc}R_{hjbk}R_{hakc}$$
$$+ R_{iabc}R_{habk}R_{jhkc} + R_{iabc}R_{hcbk}R_{jakh})\,.$$

Use of Lemma 2.4(i) gives

$$\Delta(\dot{R}_{ij}) = 4K\dot{R}_{ij} - 2\overset{\circ}{R}_{ij} - 8\overset{\circ}{\bar{R}}_{ij} + 2\nabla R_i\nabla R_j\,. \qquad \square$$

Corollary 2.6 (i) In an Einstein space,

$$\overset{\circ}{R} + 4\overset{\circ}{\bar{R}} - |\nabla R|^2 = 2K|R|^2 - \tfrac{1}{2}\Delta(|R|^2).$$

(ii) In a super-Einstein space, with $\dot{R} = Sg$,

$$\overset{\circ}{R}_{ij} + 4\overset{\circ}{\bar{R}}_{ij} - \nabla R_i\nabla R_j = 2KS\delta_{ij}\,.$$

Thus in a super-Einstein space only three of our defined tensors are independent. We wish to see which combinations of these tensors can have the Schur property on a super-Einstein space. To this end we prove the following proposition (cf. proofs of Theorem 2.1, Proposition 2.3).

<u>Proposition 2.7</u> In a super-Einstein space,

$$\text{(i)}\quad \nabla_i(\overset{\underline{o}}{R}_{ij}) = \tfrac{1}{6}\nabla_j(\overset{\underline{o}}{R}) - \tfrac{1}{2}T_j\,,$$

$$\text{(ii)}\quad \nabla_i(\overset{o}{R}_{ij}) = \tfrac{1}{6}\nabla_j(\overset{o}{R}) - T_j\,,$$

$$\text{(iii)}\quad \nabla_i(\nabla_i R\nabla_j R) = \tfrac{1}{6}\nabla_j(|\nabla R|^2) - 4T_j\,.$$

<u>Proof</u> (i)

$$\nabla_i(\overset{\underline{o}}{R}_{ij}) = \nabla_i(R_{iahb}R_{jchd}R_{acbd})$$

$$= R_{iahb}\nabla_i R_{jchd}R_{acbd} + R_{iahb}R_{jchd}\nabla_i R_{acbd}$$

since the space is Einstein. But, by the 2nd Bianchi identity,

$$R_{iahb}\nabla_i R_{jchd}R_{acbd} = R_{iahb}\nabla_j R_{ichd}R_{acbd} + R_{iahb}\nabla_c R_{jihd}R_{acbd}$$

$$= \tfrac{1}{3}\nabla_j(\overset{\underline{o}}{R}) - R_{iahb}\nabla_i R_{jchd}R_{acbd}\,,$$

where an interchange of dummy indices has been made in the second term. Thus

$$R_{iahb}\nabla_i R_{jchd}R_{acbd} = \tfrac{1}{6}\nabla_j(\overset{\underline{o}}{R}).$$

Also

$$R_{iahb}R_{jchd}\nabla_i R_{acbd} = \tfrac{1}{2}R_{iahb}R_{jchd}\nabla_c R_{aibd} \qquad \text{(Lemma 2.4(ii))}$$

$$= -\tfrac{1}{2}T_j\,.$$

(Note that this formula holds if only the Einstein condition is assumed.)

(ii)

$$\nabla_i(\overset{o}{R}_{ij}) = \nabla_i(R_{ihab}R_{jhcd}R_{abcd})$$

$$= R_{ihab}\nabla_i R_{jhcd}R_{abcd} + R_{ihab}R_{jhcd}\nabla_i R_{abcd}$$

again, as the space is Einstein. Then

$$R_{ihab}\nabla_i R_{jhcd} R_{abcd} = \tfrac{1}{2} R_{ihab}\nabla_j R_{ihcd} R_{abcd} \qquad \text{(Lemma 2.4(ii))}$$

$$= \tfrac{1}{6}\nabla_j(\overset{o}{R}).$$

Also

$$R_{ihab}R_{jhcd}\nabla_i R_{abcd} = 2R_{ihab}R_{jhcd}\nabla_c R_{abid} \qquad \text{(Lemma 2.4(ii))}$$

$$= \nabla_c R_{abid} R_{abih} R_{jchd},$$

using the super-Einstein condition and Lemma 2.4(i). Thus

$$R_{ihab}R_{jhcd}\nabla_i R_{abcd} = -T_j.$$

(iii)
$$\nabla_i(\nabla_i R \nabla_j R) = \nabla_i(\nabla_i R_{abcd}\nabla_j R_{abcd})$$

$$= \nabla_{ii}R_{abcd}\nabla_j R_{abcd} + \nabla_i R_{abcd}\nabla_{ij}R_{abcd}.$$

Considering the first term,

$$\nabla_{ii}R_{abcd}\nabla_j R_{abcd} = 2\nabla_{ia}R_{ibcd}\nabla_j R_{abcd} \qquad \text{(Lemma 2.4(ii))}$$

$$= 2R_{hiai}R_{hbcd}\nabla_j R_{abcd} + 2R_{hbai}R_{ihcd}\nabla_j R_{abcd}$$

$$+ 2R_{hcai}R_{ibhd}\nabla_j R_{abcd} + 2R_{hdai}R_{ibch}\nabla_j R_{abcd},$$

using the Ricci identity and the Einstein condition. Thus

$$\nabla_{ii}R_{abcd}\nabla_j R_{abcd} = 2KR_{abcd}\nabla_j R_{abcd} + R_{abhi}R_{ihcd}\nabla_j R_{abcd} - 2R_{aich}R_{ibhd}\nabla_j R_{abcd}$$

$$- 2R_{iahd}R_{ibch}\nabla_j R_{abcd},$$

since the space is Einstein. Using the super-Einstein condition we have that

$$\nabla_{ii}R_{abcd}\nabla_j R_{abcd} = -\tfrac{1}{3}\nabla_j(\overset{o}{R}) - \tfrac{4}{3}\nabla_j(\overset{\underline{o}}{R})$$

$$= -\tfrac{1}{3}\nabla_j(|\nabla R|^2)$$

by Corollary 2.6(ii). Finally, using the Ricci identity once more we have

$$\nabla_i R_{abcd}\nabla_{ij}R_{abcd} = \nabla_i R_{abcd}\nabla_{ji}R_{abcd} + \nabla_i R_{abcd}R_{haji}R_{hbcd} + \nabla_i R_{abcd}R_{hbji}R_{ahcd}$$

$$+ \nabla_i R_{abcd}R_{hcji}R_{abhd} + \nabla_i R_{abcd}R_{hdjc}R_{abch}$$

$$= \tfrac{1}{2}\nabla_j(|\nabla R|^2) + 4\nabla_i R_{abcd}R_{haji}R_{hbcd}$$

by changing dummy indices. Thus

$$\nabla_i R_{abcd}\nabla_{ij}R_{abcd} = \tfrac{1}{2}\nabla_j(|\nabla R|^2) - 4T_j. \qquad \square$$

<u>Proposition 2.8</u> In a super-Einstein space where $T \neq 0$, a necessary and sufficient condition that a tensor of the form $A\overset{o}{R}_{ij} + B\overset{o}{R}_{ij} + C\nabla_i R\nabla_j R$ has the Schur property of order 6 is that

$$(2.14) \qquad A + 2B + 8C = 0.$$

If $T = 0$ then any tensor of that form has the Schur property of order 6.

<u>Proof</u> As usual we suppose that

$$(2.15) \qquad A\overset{o}{R}_{ij} + B\overset{o}{R}_{ij} + C\nabla_i R\nabla_j R = f\delta_{ij}$$

in orthonormal co-ordinates for some function f. Taking ∇_i of both sides and summing,

$$A\nabla_i(\overset{o}{R}_{ij}) + B\nabla_i(\overset{o}{R}_{ij}) + C\nabla_i(\nabla_i R\nabla_j R) = \nabla_j f.$$

Proposition 2.7 gives

$$(2.16) \qquad \tfrac{1}{6}\nabla_j(A\overset{o}{R} + B\overset{o}{R} + C|\nabla R|^2) - (\tfrac{1}{2}A + B + 4C)T_j = \nabla_j f.$$

On the other hand, taking the trace of (2.15) gives

$$(2.17) \qquad \frac{1}{n}(A\overset{\Omega}{R} + B\overset{o}{R} + C|\nabla R|^2) = f.$$

If $T = 0$ we see that $A\overset{\Omega}{R} + B\overset{o}{R} + C|\nabla R|^2$ has the Schur property of order 6. Suppose $T \neq 0$ and the tensor in question has the Schur property of order 6, then for $n \neq 6$ f is constant and thus (2.14) holds, by comparing (2.16) and (2.17).

Conversely if (2.14) holds the tensor has the Schur property of order 6. $\square$

Corollary 2.9 In a 6-dimensional super-Einstein space $T = 0$.

Proof Take a 2-tensor of the form $A\overset{\Omega}{R}_{ij} + B\overset{o}{R}_{ij} + C\nabla_i R\nabla_j R$ where

$$A + 2B + 8C \neq 0.$$

Substitution of (2.17) into (2.16) when $n = 6$ gives

$$(A + 2B + 8C)T_j = 0. \qquad \square$$

The final proposition in this section gives a list of identities valid in a super-Einstein space, useful in this and the next chapter.

Proposition 2.10 In a super-Einstein space (with $\rho = Kg$, $\dot{R} = Sg$),

(i) if $T_{ijk\ell hm} = S(\nabla_{ij}R_{ka\ell b}R_{hamb})$, then $T_{ijkk\ell\ell} = 8(-5\nabla_i R\nabla_j R - 4\nabla R_i \nabla R_j)$,

(ii) if $U_{ijk\ell hm} = S(\nabla_i R_{jakb}\nabla_\ell R_{hamb})$, then $U_{ijkk\ell\ell} = 8(5\nabla_i R\nabla_j R + 4\nabla R_i \nabla R_j)$,

(iii) if $V_{ijk\ell hm} = S(R_{iajb}R_{kb\ell c}R_{hcma})$, then $V_{ijkk\ell\ell} = 8(\frac{21}{2}\overset{o}{R}_{ij} - 3\overset{\Omega}{R}_{ij} + (3K^3+\frac{27}{2}KS)\delta_{ij})$,

(iv) if $W_{ijk\ell hmn} = S(\nabla_i R_{jakb}R_{\ell bhc}R_{mcna})$, then $W_{ijjkk\ell\ell} = 48(\frac{7}{3}\nabla_i(\overset{o}{R}) - \frac{2}{3}\nabla_i(\overset{\Omega}{R}) - 6T_i)$,

(v) if $X_{ijk\ell hmn} = S(\nabla_{ij}R_{ka\ell b}\nabla_h R_{manb})$, then $X_{ijjkk\ell\ell} = 48(3\nabla_i(|\nabla R|^2) - 32T_i)$,

(vi) if $Y_{ijk\ell hmn} = S(\nabla_{ijk}R_{\ell ahb}R_{manb})$, then $Y_{ijjkk\ell\ell} = 48(-9\nabla_i(|\nabla R|^2) + 96T_i)$.

Proof The method of proof is the same in each case. It is in principle the same as Proposition 2.2, but more laborious. All possible permutations are written down and full use is made of the identities used in previous propositions. □

Remark From Proposition 2.10 we note the following identities in a super-Einstein space:

$$T_{ijkk\ell\ell} + U_{ijkk\ell\ell} = 0, \tag{2.18}$$

$$3X_{ijjkk\ell\ell} + Y_{ijjkk\ell\ell} = 0. \tag{2.19}$$

We also know that if we assume instead Ledger's 2nd condition, by differentiating twice and three times we obtain

$$T_{ijk\ell hm} + U_{ijk\ell hm} = 0,$$

$$3X_{ijk\ell hmn} + Y_{ijk\ell hmn} = 0.$$

Thus while it is clear that (2.18) and (2.19) hold in a space which satisfies $L2$, it is not obvious why they hold when only the super-Einstein condition is assumed. This must have a significance which indicates an easier proof than the hard labour of Proposition 2.10.

§4 Consequences of $L3$

Ledger's 3rd condition which, it will be recalled, is derived assuming the 1st and 2nd conditions is, in orthonormal co-ordinates,

$$S(32R_{iajb}R_{kb\ell c}R_{mcna} - 9\nabla_i R_{jakb}\nabla_\ell R_{manb}) = LS(\delta_{ij}\delta_{k\ell}\delta_{mn}). \tag{2.20}$$

The next proposition is the analogue of Proposition 2.2.

Proposition 2.11 If the curvature of a manifold satisfies $L1$, $L2$ and $L3$, then it also satisfies

$$(2.21) \qquad 16\overset{\underline{o}}{R}_{ij} - 20\overset{o}{R}_{ij} + 3\nabla_i R\nabla_j R = N\delta_{ij} \quad \text{for some constant } N.$$

Proof By setting $k = \ell$, $m = n$, summing and applying Proposition 2.10 we get

$$32\overset{\underline{o}}{R}_{ij} - 112\overset{o}{R}_{ij} + 15\nabla_i R\nabla_j R + 12\nabla R_i \nabla R_j = C\delta_{ij} \quad \text{for some constant } C.$$

Then applying Corollary 2.6(ii) we get (2.21). $\square$

From Corollary 2.6 we have the following relation in a super-Einstein space:

$$\overset{o}{R} + 4\overset{\underline{o}}{R} - |\nabla R|^2 = 2KSn,$$

and from (2.21)

$$16\overset{\underline{o}}{R} - 20\overset{o}{R} + 3|\nabla R|^2 = Nn.$$

These formulae are of interest for if we can find another linearly independent linear combination of $\overset{o}{R}$, $\overset{\underline{o}}{R}$ and $|\nabla R|^2$ which is constant, we could infer that each of these was constant. If the fundamental conjecture is true, then they must be constant; more importantly if $|\nabla R|^2$ could be proved to be the constant 0, then the conjecture is proved. However, as yet, no third condition has been found, and should one be found it would be no easy matter to infer that $|\nabla R|^2 = 0$ (cf. [B] p.174).

Also it is interesting to note that if, instead of simplifying the condition gained from $\mathrm{tr}C^{(6)}$ in Chapter 1 by use of $L1$ and $L2$, we had assumed only that the space was super-Einstein,we would have obtained

$$S(135\nabla_i R_{jakb} \nabla_\ell R_{hamb} + 144\nabla_{ij} R_{ka\ell b} R_{hamb} + 32 R_{iajb} R_{kb\ell c} R_{hcma}) = L' S(\delta_{ij}\delta_{k\ell}\delta_{hm})$$

whence, by use of Proposition 2.10, we would have arrived at (2.21).

Proposition 2.12 In a super-Einstein space, the 2-tensor $16\overset{o}{\underline{R}}_{ij} - 20\overset{o}{R}_{ij} + 3\nabla_i R \nabla_j R$ has the Schur property of order 6.

Proof Its coefficients satisfy the condition of Proposition 2.8. $\square$

We shall denote the 2-tensor $16\overset{o}{\underline{R}}_{ij} - 20\overset{o}{R}_{ij} + 3\nabla_i R\nabla_j R$ by ϕ_{ij}. We have seen that ρ has the Schur property of order 2, in an Einstein space $\dot{R}$ has the Schur property of order 4, and in a super-Einstein space ϕ has the Schur property of order 6. So we are led to a

Conjecture. Suppose $T(k)$ is the symmetric 2-tensor derived from the k^{th} Ledger condition (which is derived assuming the preceding $k-1$ Ledger conditions) then $T(k)$ has the Schur property of order $2k$. However, as we have noted above, there are grounds for a second

Conjecture. Suppose $T(k)$ is the symmetric 2-tensor derived from the $2k^{th}$ derivative of trC. If it is assumed that $T(p) = C_p g$ for constants C_p, $1 \leqslant p \leqslant k-1$, then $T(k)$ has the Schur property of order $2k$.

§5 Results of Patterson

In this section we describe some of the results obtained in a recent paper of Patterson [PA]. In particular we note the discovery of an infinite sequence of symmetric 2-tensors, all of which have the general Schur property. However these are constructed from algebraic combinations of the curvature tensor and do not involve its covariant derivatives. We also mention some results of consequence for low-dimensional harmonic manifolds.

Firstly we define the main tool of the paper, a contravariant tensor of order 2r:

(2.22) $$\alpha^{i_1 i_2 \ldots i_r | j_1 j_2 \ldots j_r} = \sum_{\sigma \in S_r} \varepsilon_\sigma g^{i_1\sigma(j_1)} g^{i_2\sigma(j_2)} \ldots g^{i_r\sigma(j_r)}.$$

It is seen immediately that the i's and j's are anti-symmetric in pairs. We can now define a sequence of contravariant 2-tensors $F(k)$, $k \geqslant 1$, by

$$F^{ij}(k) = \alpha^{ii_1 \ldots i_{2k} | jj_1 \ldots j_{2k}} R_{i_1 i_2 j_1 j_2} \ldots R_{i_{2k-1} i_{2k} j_{2k-1} j_{2k}}.$$

$E(k)$ are defined to be the corresponding covariant tensors obtained by lowering the indices.

Proposition 2.13 $E(k)$ has the Schur property, $k \geqslant 1$.

Proof We work in orthonormal co-ordinates so that

$$E_{ij}(k) = \alpha_{ii_1 \ldots i_{2k} | jj_1 \ldots j_{2k}} R_{i_1 i_2 j_1 j_2} \ldots R_{i_{2k-1} i_{2k} j_{2k-1} j_{2k}},$$

where

$$\alpha_{ii_1 \ldots i_{2k} | jj_1 \ldots j_{2k}} = \sum_{\sigma \in S_{2k+1}} \varepsilon_\sigma \delta_{i\sigma(j)} \delta_{i_1\sigma(j_1)} \ldots \delta_{i_{2k}\sigma(j_{2k})}.$$

We are given that

$$E_{ij}(k) = f\delta_{ij} \qquad \text{for some function } f.$$

As usual we take ∇_i of both sides and sum so that

(2.23) $$\nabla_i E_{ij}(k) = \nabla_j f.$$

But

$$\begin{aligned}\nabla_i E_{ij}(k) &= \nabla_i (\alpha_{ii_1 \ldots i_{2k} | jj_1 \ldots j_{2k}} R_{i_1 i_2 j_1 j_2} \ldots R_{i_{2k-1} i_{2k} j_{2k-1} j_{2k}}) \\ &= \alpha_{ii_1 \ldots i_{2k} | jj_1 \ldots j_{2k}} (\nabla_i R_{i_1 i_2 j_1 j_2} \ldots R_{i_{2k-1} i_{2k} j_{2k-1} j_{2k}} + \ldots \\ &\qquad \ldots + R_{i_1 i_2 j_1 j_2} \ldots \nabla_i R_{i_{2k-1} i_{2k} j_{2k-1} j_{2k}})\end{aligned}$$

(2.24) $$= k\alpha_{ii_1 \ldots i_{2k} | jj_1 \ldots j_{2k}} \nabla_i R_{i_1 i_2 j_1 j_2} \ldots R_{i_{2k-1} i_{2k} j_{2k-1} j_{2k}}$$

(since $\nabla_i R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2k-1} i_{2k} j_{2k-1} j_{2k}}$ can be derived from the other terms by even permutations of the i's and j's.).

$$= \tfrac{1}{2}k\alpha_{ii_1 \ldots i_{2k}|jj_1 \ldots j_{2k}} \nabla_{i_2} R_{i_1 i j_1 j_2} \cdots R_{i_{2k-1} i_{2k} j_{2k-1} j_{2k}}$$

(by Lemma 2.4(ii) using anti-symmetry of i and i_1)

$$(2.25) \qquad = \tfrac{1}{4}k\alpha_{ii_1 \ldots i_{2k}|jj_1 \ldots j_{2k}} \nabla_i R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2k-1} i_{2k} j_{2k-1} j_{2k}}$$

(by Lemma 2.4(ii) using anti-symmetry of i_1 and i_2).

Comparing (2.24) and (2.25) we see that both must be zero. Hence from (2.23) f is constant and E(k) has the Schur property. □

We can now derive from these tensors a sequence D(k) of covariant 2-tensors, $k \geqslant 1$, such that the k^{th} has the Schur property of order 2k, thus inviting comparison with our T's of the last section. This is accomplished by subtracting a certain term from each E(k). We define scalars G(k), $k \geqslant 1$, by

$$G(k) = \alpha_{i_1 \ldots i_{2k}|j_1 \ldots j_{2k}} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2k-1} i_{2k} j_{2k-1} j_{2k}}$$

and covariant 2-tensors D(k) by, in orthonormal co-ordinates,

$$(2.26) \qquad D_{ij}(k) = E_{ij}(k) - G(k)\delta_{ij}.$$

<u>Proposition 2.14</u> D(k) has the Schur property of order 2k, $k \geqslant 1$.

<u>Proof</u> We calculate first the trace of $E_{ij}(k)$. We have

$$E_{ii}(k) = \alpha_{ii_1 \ldots i_{2k}|ij_1 \ldots j_{2k}} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2k-1} i_{2k} j_{2k-1} j_{2k}}.$$

Using the usual expansion of a determinant by the first row, we have

$$\alpha_{ii_1 \ldots i_{2k}|jj_1 \ldots j_{2k}} = \delta_{ij}\alpha_{i_1 \ldots i_{2k}|j_1 \ldots j_{2k}} - \delta_{ij_1}\alpha_{i_1 \ldots i_{2k}|jj_2 \ldots j_{2k}} + \ldots - \delta_{ij_{2k}}\alpha_{i_1 \ldots i_{2k}|jj_1 \ldots j_{2k-1}}.$$

and so

$$\alpha_{ii_1\ldots i_{2k}|ij_1\ldots j_{2k}} = (n - 2k)\alpha_{i_1\ldots i_{2k}|j_1\ldots j_{2k}}.$$

Thus

$$E_{ii}(k) = (n - 2k)G(k). \tag{2.27}$$

We are given that

$$D_{ij}(k) = f\delta_{ij} \qquad \text{for some function } f.$$

Once more we take ∇_i of both sides and sum:

$$\nabla_i D_{ij}(k) = \nabla_j f. \tag{2.28}$$

But, from (2.26),

$$\nabla_i D_{ij}(k) = \nabla_i E_{ij}(k) - \nabla_j G(k)$$

$$= -\nabla_j G(k) \tag{2.29}$$

(see proof of Proposition 2.13).

On the other hand taking the trace of $D_{ij}(k)$,

$$D_{ii}(k) = E_{ii}(k) - nG(k)$$

$$= -2kG(k) \qquad \text{(from (2.27)).}$$

Thus

$$f = \frac{D_{ii}}{n}$$

$$= \frac{-2k}{n} G(k). \tag{2.30}$$

Combining (2.28), (2.29) and (2.30), we see that if $n \neq 2k$, f must be constant. □

Calculation reveals that D(1) is a multiple of the Ricci tensor and D(2) is a multiple of following quadratic polynomial in the components of the curvature tensor:

$$R_{iabc}R_{jabc} + \tau\rho_{ij} + 2R_{iajb}\rho_{ab} + 2\rho_{ia}\rho_{ja}.$$

Note that in an Einstein space this reduces to

$$\dot{R}_{ij} + C\delta_{ij} \qquad \text{for some constant } C.$$

D(3) requires a lot more calculation, but in a super-Einstein space it is a constant multiple of this tensor:

$$2\overset{\Omega}{R}_{ij} - \overset{o}{R}_{ij} + B\delta_{ij} \qquad \text{for some constant } B.$$

Its having the Schur property is verified by Proposition 2.8.

This raises a number of interesting questions. Do there exist any more naturally occurring sequences of 2-tensors with the same property? Is there an elegant proof of the conjectures in the last section similar to that of the propositions above? The second question would appear to require a different approach to the notion of harmonicity, but one feels that the answer should be 'yes'.

It is easily seen that when $n \leqslant 2k$ then

$$\alpha^{ii_1\ldots i_{2k}|jj_1\ldots j_{2k}} = 0$$

since it is anti-symmetric in the i's and there are $2k + 1$ places to put at most $2k$ different integers. Hence for $n \leqslant 2k$, $E(k) = 0$. This quickly shows that (i) any manifold of dimension $\leqslant 2$ has $\rho = fg$ for some function f,

(ii) any Einstein manifold of dimension $\leqslant 4$ has $\dot{R} = \bar{f}g$ for some function $\bar{f}$,

(iii) any super-Einstein manifold of dimension $\leqslant 6$ has the following property, in orthonormal co-ordinates:

$$2\overset{\underline{o}}{R}_{ij} - \overset{o}{R}_{ij} = \overset{\equiv}{f}\delta_{ij} \qquad \text{and } \overset{\equiv}{f} \text{ is constant for dim } M \leqslant 5.$$

Thus for a harmonic manifold of dimension 5, we have a third relation between the three scalar invariants viz.

$$2\overset{\underline{o}}{R} - \overset{o}{R} = \text{Constant},$$

and we can conclude that $\overset{o}{R}$, $\overset{\underline{o}}{R}$, and $|\nabla R|^2$ are individually constant. However as pointed out earlier, it still seems difficult to prove that the constant in the case of $|\nabla R|^2$ is 0 and hence deduce local symmetry.

Chapter 3 The Mean-Value Theorem

In 1950 another characterisation of harmonic spaces was discovered by Willmore [W1]. Observing the well-known fact that harmonic functions on $\mathbb{R}^n$ are those with the mean-value property, he found that harmonic manifolds are exactly those spaces in which harmonic functions have the mean-value property. This chapter is concerned with exploiting this by calculating the first few terms in the power series expansion of the mean-value of a function over a geodesic sphere and then using it to deduce properties of the manifold. Most of this chapter is modelled on the paper of Gray and Willmore [GW].

§1 Calculation of the mean-value: the preparation

Let us denote by $S_m(r)$ the geodesic sphere centre m, radius r ($r < i(m)$, of course). This is the image under $\exp_m$ of the Euclidean sphere of radius r in T_mM, which we denote by $\$(r)$. The volume elements of these two manifolds will be denoted by $d\omega_S$, $d\omega_\$$ respectively. The mean-value of a function defined on a nhd of m, $M_m(f,r)$, is then given by

(3.1) $$M_m(f,r) = \int_{S_m(r)} f d\omega_S \Big/ \int_{S_m(r)} d\omega_S \qquad \text{for r small enough.}$$

Willmore's result can then be expressed as the following theorem.

Theorem 3.1 A manifold M is harmonic iff for each pt $m \in M$ and each function f harmonic on a nhd of m,

$$M_m(f,r) = f(m) \qquad \text{for r sufficiently}$$

small (see [W1] or [B] p.158ff).

By expanding $M_m(f,r)$ as a power series in r, we can gain necessary conditions for a manifold to be harmonic by demanding that the non-constant terms vanish. We can also obtain results interesting in themselves concerning the vanishing of these terms.

The first step in finding this power series is to transform the integrals over geodesic spheres to integrals over Euclidean spheres, to which we can apply classical results.

<u>Lemma 3.2</u> If $d\omega_S{}^*$ denotes the pull back of $d\omega_S$ under $\exp_m$, then

$$d\omega_S{}^* = \Theta_m d\omega_{\$}\,, \tag{3.2}$$

where Θ_m is as before $\sqrt{\det g}$ in normal co-ordinates.

<u>Proof</u> Let dr_M, dr_E be the radial 1-forms defined on a nhd of m and a nhd of 0 in T_mM by the corresponding radial functions. Then

$$d\omega_E = d\omega_{\$} \wedge dr_E, \quad d\omega_M = d\omega_S \wedge dr_M\,, \tag{3.3}$$

where $d\omega_E$, $d\omega_M$ are the volume elements of T_mM and M. Both equations follow from the Gauss lemma which says that $\frac{\partial}{\partial r_M}$(resp. $\frac{\partial}{\partial r_E}$) is of unit length and is perpendicular to $S_m(r)$ (resp. $\$(r)$).

Also from the definition of Θ_m we have

$$(d\omega_M)^* = \Theta_m d\omega_E\,. \tag{3.4}$$

Hence

$$d\omega_S{}^* \wedge dr_M{}^* = (d\omega_S \wedge dr_M)^* = (d\omega_M)^* = \Theta_m d\omega_E$$
$$= \Theta_m(d\omega_{\$} \wedge dr_E)$$

by (3.3) and (3.4), and as $dr_M{}^* = dr_E$, the result follows. $\square$

We can thus write (3.1) as

$$M_m(f,r) = \int_{\$(r)} f^* d\omega_S^* \Big/ \int_{\$(r)} d\omega_S^*$$

(3.5) $$= \int_{\$(r)} f\, \Theta_m\, d\omega_\$ \Big/ \int_{\$(r)} \Theta_m d\omega_\$$$

by Lemma 3.2, where we abuse the notation slightly and consider f and Θ_m to be defined on a nhd of 0 in T_mM or a nhd of m in M, as suits our needs.

The main tool in calculating the power series is the classical formula of Pizzetti for the power series expansion of the mean-value of a function over a Euclidean sphere:

(3.6) $$\int_{\$(r)} h\, d\omega_\$ \Big/ \int_{\$(r)} d\omega_\$ = \Gamma(\tfrac{1}{2}n) \sum_{k=0}^{\infty} (\tfrac{r}{2})^{2k} \frac{(\tilde{\Delta}^k h)\ (0)}{k!\ \Gamma(\tfrac{1}{2}n+k)},$$

where $\tilde{\Delta}^k$ is the k^{th} iterate of the Euclidean Laplacian i.e.

(3.7) $$\tilde{\Delta}^k f = \frac{\partial^{2k} f}{\partial x^{i_1} \partial x^{i_1} \cdots \partial x^{i_k} \partial x^{i_k}}$$

(see e.g. [CH] p.287ff).

With this in mind we define differential operators $\tilde{\Delta}^k_m$, $k \geq 1$, on a maximal normal co-ordinate nhd of m by

$$\tilde{\Delta}^k_m f = \frac{\partial^{2k} f}{\partial x^{i_1} \partial x^{i_1} \cdots \partial x^{i_k} \partial x^{i_k}},$$

where $x^1 \ldots x^n$ are a set of normal co-ordinates about m. Note that $\tilde{\Delta}^k_m f$ is however independent of the choice of normal co-ordinates, and $(\tilde{\Delta}^k_m)^* = \tilde{\Delta}^k$.

Applying (3.6) to (3.5) we have

$$(3.8) \qquad M_m(f,r) = \frac{\sum\limits_{k=0}^{\infty} (\frac{r}{2})^{2k} \frac{1}{k!\,\Gamma(\frac{1}{2}n+k)} (\tilde{\Delta}_m^k\ (f\Theta_m))(m)}{\sum\limits_{k=0}^{\infty} (\frac{r}{2})^{2k} \frac{1}{k!\,\Gamma(\frac{1}{2}n+k)} (\tilde{\Delta}_m^k\ (\Theta_m))(m)} .$$

To gain information about the geometry of the manifold we need to be able to translate information from partial derivatives of f and Θ_m with respect to normal co-ordinates into information concerning the curvature tensor. Once again the usefulness of normal co-ordinates is demonstrated in the next lemma.

Lemma 3.3 (i) The partial derivatives of Θ_m with respect to normal co-ordinates about m at m can be expressed in terms of the curvature tensor.

(ii) $$\left(\frac{\partial^k f}{\partial x^{i_1} \cdots \partial x^{i_k}}\right)(m) = \frac{1}{k!} S(\nabla_{i_1} \cdots {}_{i_k} f)(m) ,$$

where $(x^1, \ldots, x^n)$ are normal co-ordinates about m.

Proof (i) This was shown in the course of chapter 1.

(ii) In normal co-ordinates about m a geodesic γ through m has the simple equation

$$\gamma^i(s) = a^i s , \qquad i = 1, \ldots, n.$$

The tangent vector along the geodesic has components a^i and as γ is a geodesic

$$(3.9) \qquad \nabla_{\frac{d}{ds}} (a^i) = a^j \nabla_j a^i = 0 , \qquad i = 1, \ldots, n.$$

The formula in (ii) is certainly true when k = 1, by definition of ∇f. Indeed if we take the covariant derivative of f along γ we could write it both as $a^i \nabla_i f$ or $a^i \frac{\partial f}{\partial x^i}$. We can repeat this using these as our scalar functions to get

$$a^j\nabla_j(a^i\nabla_i f) \;=\; a^j \frac{\partial}{\partial x^j}(a^i \frac{\partial f}{\partial x^i}) \text{ along } \gamma,$$

whence, from (3.9) and the fact that the a's are constant, we get

$$a^j a^i \nabla_{ij} f \;=\; a^j a^i \frac{\partial^2 f}{\partial x^i \partial x^j} \quad \text{along } \gamma.$$

In particular this is true at m and as the choice of a's is arbitrary it follows that

$$S(\nabla_{ij} f)\,(m) \;=\; S(\frac{\partial^2 f}{\partial x^i \partial x^j})\,(m) \;=\; 2!\frac{\partial^2 f}{\partial x^i \partial x^j}\,(m).$$

The result for any k follows from applying this a further k - 2 times. □

We shall need the power series expansion of Θ in normal co-ordinates up to the seventh order term in the next section, so we give it here:

$$(3.10)\quad \Theta_m = 1 - \tfrac{1}{6}\rho_{ij}(m)x^i x^j - \tfrac{1}{12}\nabla_i\rho_{jk}(m)x^i x^j x^k$$
$$+ \tfrac{1}{4!}(-\tfrac{3}{5}\nabla_{ij}\rho_{k\ell} - \tfrac{2}{15}R_{iajb}R_{ka\ell b} + \tfrac{1}{3}\rho_{ij}\rho_{k\ell})(m)x^i x^j x^k x^\ell$$
$$+ \tfrac{1}{5!}(-\tfrac{2}{3}\nabla_{ijk}\rho_{\ell h} - \tfrac{2}{3}\nabla_i R_{jakb}R_{\ell ahb} + \tfrac{5}{3}\nabla_i\rho_{jk}\rho_{\ell h})(m)x^i x^j x^k x^\ell x^h$$
$$+ \tfrac{1}{6!}(-\tfrac{5}{7}\nabla_{ijk\ell}\rho_{hg} - \tfrac{15}{14}\nabla_i R_{jakb}\nabla_\ell R_{hagb} - \tfrac{8}{7}\nabla_{ij}R_{ka\ell b}R_{hagb}$$
$$-\tfrac{16}{63}R_{iajb}R_{kb\ell c}R_{hcga} + 3\nabla_{ij}\rho_{k\ell}\rho_{hg} + \tfrac{2}{3}R_{iajb}R_{ka\ell b}\rho_{hg}$$
$$+\tfrac{5}{2}\nabla_i\rho_{jk}\nabla_\ell\rho_{hg} - \tfrac{5}{9}\rho_{ij}\rho_{k\ell}\rho_{hg})(m)x^i x^j x^k x^\ell x^h x^g$$
$$+ \tfrac{1}{7!}(-\tfrac{3}{4}\nabla_{ijk\ell h}\rho_{gf} - \tfrac{5}{3}\nabla_{ijk}R_{\ell ahb}R_{gafb} - \tfrac{8}{3}\nabla_i R_{jakb}R_{\ell bhc}R_{gcfa}$$
$$-\tfrac{9}{2}\nabla_{ij}R_{ka\ell b}\nabla_h R_{gafb} + \tfrac{14}{3}\rho_{ij}\nabla_{k\ell h}\rho_{gf} + \tfrac{14}{3}\rho_{ij}\nabla_k R_{\ell ahb}R_{gafb}$$
$$+\tfrac{21}{2}\nabla_i\rho_{jk}\nabla_{\ell h}\rho_{gf} + \tfrac{7}{3}\nabla_i\rho_{jk}R_{\ell ahb}R_{gafb}$$
$$-\tfrac{35}{12}\nabla_i\rho_{jk}\rho_{\ell h}\rho_{gf})(m)x^i x^j x^k x^\ell x^h x^g x^f.$$

This is easily derived from the results of chapter 1. We now have all we need to begin calculation of the power series of the mean value.

In passing we note the following

Proposition 3.4. In a harmonic manifold formula (3.8) for $M_m(f,r)$ becomes

$$M_m(f,r) = \Gamma(\tfrac{1}{2}n)\sum_{k=0}^{\infty}(\tfrac{r}{2})^{2k}\frac{1}{k!\,\Gamma(\tfrac{1}{2}n+k)}(\tilde{\Delta}_m^k f)(m).$$

Proof Simply note that Θ_m is a function of r alone and

$$M_m(f,r) = \int_{\$(r)} f\, d\omega_{\$} \Big/ \int_{\$(r)} d\omega_{\$},$$

and the result follows by Pizzetti. $\square$

Corollary A sequence of necessary and sufficient conditions that f be harmonic on a nhd of m in a harmonic manifold is

$$\tilde{\Delta}_m^k f(m) = 0 \qquad k = 1,2,\ldots .$$

§2 The Calculation

We write

$$M_m(f,r) = f(m) + A(m)r^2 + B(m)r^4 + C(m)r^6 + D(m)r^8 + O(r^{10})$$

and the object of this section is to calculate A,B,C and D. However B,C, and D will not be found in the most general situation, but under simplifying conditions which will be made clear in each case.

If we multiply out the power series in (3.8) using the binomial theorem, we find that, writing Θ for Θ_m,

(3.11) $$A(m) = \frac{1}{2n}(\tilde{\Delta}_m(f\Theta) - f\tilde{\Delta}_m(\Theta))(m),$$

(3.12) $$B(m) = \frac{1}{8n(n+2)}(\tilde{\Delta}_m^2(f\Theta) - f\tilde{\Delta}_m^2(\Theta))(m) - \frac{1}{4n^2}(\tilde{\Delta}_m(f\Theta)-f\tilde{\Delta}_m(\Theta))(m)\tilde{\Delta}_m(\Theta)(m),$$

$$(3.13)\qquad C(m) = \frac{1}{48n(n+2)(n+4)}(\tilde{\Delta}_m^3(f\Theta) - f\tilde{\Delta}_m^3(\Theta))(m)$$

$$-\frac{1}{16n^2(n+2)}(\tilde{\Delta}_m(f\Theta) - f\tilde{\Delta}_m(\Theta))\tilde{\Delta}_m^2(\Theta)(m)$$

$$-\frac{1}{16n^2(n+2)}(\tilde{\Delta}_m^2(f\Theta) - f\tilde{\Delta}_m^2(\Theta))(m)\tilde{\Delta}_m(\Theta)(m)$$

$$+\frac{1}{8n^3}(\tilde{\Delta}_m(f\Theta) - f\tilde{\Delta}_m(\Theta))(m)[\tilde{\Delta}_m(\Theta)]^2(m),$$

$$(3.14)\qquad D(m) = \frac{1}{48.8n(n+2)(n+4)(n+6)}(\tilde{\Delta}_m^4(f\Theta) - f\tilde{\Delta}_m^4(\Theta))(m)$$

$$-\frac{1}{16.6n^2(n+2)(n+4)}(\tilde{\Delta}_m^3(f\Theta) - f\tilde{\Delta}_m^3(\Theta))(m)\tilde{\Delta}_m(\Theta)(m)$$

$$-\frac{1}{64n^2(n+2)^2}(\tilde{\Delta}_m^2(f\Theta) - f\tilde{\Delta}_m^2(\Theta))(m)(\tilde{\Delta}_m^2(\Theta))(m)$$

$$-\frac{1}{16.6n^2(n+2)(n+4)}(\tilde{\Delta}_m(f\Theta) - f\tilde{\Delta}_m(\Theta))(m)\tilde{\Delta}_m^3(\Theta)(m)$$

$$+\frac{1}{32n^3(n+2)}(\tilde{\Delta}_m^2(f\Theta) - f\tilde{\Delta}_m^2(\Theta))(m)[\tilde{\Delta}_m(\Theta)]^2(m)$$

$$+\frac{1}{16n^4}(\tilde{\Delta}_m(f\Theta) - f\tilde{\Delta}_m(\Theta))(m)[(\tilde{\Delta}_m(\Theta))]^3(m)$$

$$+\frac{1}{32n^3(n+2)}(\tilde{\Delta}_m(f\Theta) - f\tilde{\Delta}_m(\Theta))(m)\tilde{\Delta}_m(\Theta)(m)\tilde{\Delta}_m^2(\Theta)(m).$$

In the ensuing calculations we shall denote partial differentiation of a function h with respect to x^i by h_i.

<u>Calculation of A</u> From (3.11)

$$A(m) = \frac{1}{2n}(\tilde{\Delta}_m(f\Theta) - f\tilde{\Delta}_m(\Theta))(m).$$

Now $$\tilde{\Delta}_m(f\Theta) = (f\Theta)_{ii}$$

$$= f_{ii}\Theta + 2f_i\Theta_i + f\Theta_{ii}$$

so $$\tilde{\Delta}_m(f\Theta) - f\tilde{\Delta}_m(\Theta) = f_{ii}\Theta + 2f_i\Theta_i.$$

Evaluating the RHS at m we have, by Lemma 3.3 and (3.10),

$$f_{ii}(m)\Theta(m) = \Delta_{ii}f(m),$$

$$f_i(m)\Theta_i(m) = 0.$$

So $$A = \frac{1}{2n}\Delta f.$$

Thus we easily get

Proposition 3.5. Suppose f is defined on an open set U of M. A necessary and sufficient condition that for each $m \in U$

$$M_m(f,r) = f(m) + O(r^4) \quad \text{for sufficiently small } r$$

is $\Delta f = 0$.

Recalling now Theorem 3.1, from the point of view of harmonicity we are interested in the mean-value of harmonic functions, so we shall assume from now on that f is harmonic on a nhd of m. However if we had started with a general function and asked that it had the mean-value property where defined, it would have been forced to be harmonic by Proposition 3.5.

Calculation of B assuming $\Delta f = 0$ throughout a nhd of m. (For B in the general case see [GW].) From (3.12),

$$B(m) = \frac{1}{8n(n+2)}(\tilde{\Delta}_m^2(f\Theta) - f\tilde{\Delta}_m^2(\Theta))(m) - \frac{1}{4n^2}(\tilde{\Delta}_m(f\Theta) - f\tilde{\Delta}_m(\Theta))(m)\tilde{\Delta}_m(\Theta)(m).$$

From the calculation of A, we see that the second term vanishes under our assumption. Now

$$\tilde{\Delta}^2_m(f\Theta) - f\tilde{\Delta}^2_m(\Theta) = (f\Theta)_{iijj} - f\Theta_{iijj}$$

$$= f_{iijj}\Theta + 4f_{iij}\Theta_j + 2f_{ii}\Theta_{jj} + 4f_{ij}\Theta_{ij} + 4f_i\Theta_{ijj}.$$

Evaluating the RHS at m, we have by Lemma 3.3 and (3.10)

$$(3.15) \qquad f_{iijj}(m)\Theta(m) = \tfrac{1}{3}(\nabla_{iijj}f + \nabla_{ijij}f + \nabla_{ijji}f)(m),$$

$$f_{iij}(m)\Theta_j(m) = 0,$$

$$f_{ij}(m)\Theta_{ij}(m) = \tfrac{1}{3}\nabla_{ij}f(m)\rho_{ij}(m),$$

$$f_{ii}(m)\Theta_{jj}(m) = 0, \text{ in view of our assumption},$$

$$f_i(m)\Theta_{ijj}(m) = -\tfrac{1}{3}\nabla_i f(m)\nabla_i\tau(m).$$

We define two scalar functions in which B will be expressed:

$$\left\langle \rho, \nabla^2 f \right\rangle = \rho_{ij}\nabla_{ij}f,$$

$$\left\langle \nabla\tau, \nabla f \right\rangle = \nabla_i\tau\nabla_i f.$$

Thus $\quad B(m) = \dfrac{1}{8n(n+2)}\left(f_{iijj} - \tfrac{4}{3}\left\langle \rho, \nabla^2 f\right\rangle - \tfrac{4}{3}\left\langle \nabla\tau, \nabla f\right\rangle\right)(m).$

All that remains to be calculated is the expression for $f_{iijj}(m)$.

We need the next lemma for this and calculations to follow.

<u>Lemma 3.6</u> We have

(i) $\nabla_{jiik}f = \nabla_{jk}(\Delta f) + \nabla_j\rho_{k\ell}\nabla_\ell f + \rho_{k\ell}\nabla_{j\ell}f,$

(ii) $\nabla_{ijik}f = \nabla_{jk}(\Delta f) + \nabla_j\rho_{k\ell}\nabla_\ell f + \rho_{k\ell}\nabla_{j\ell}f + \rho_{j\ell}\nabla_{k\ell}f + R_{ijk\ell}\nabla_{i\ell}f,$

(iii) $\nabla_{iijk}f = \nabla_{jk}(\Delta f) + (\nabla_j\rho_{k\ell} + \nabla_k\rho_{j\ell} - \nabla_\ell\rho_{jk})\nabla_\ell f + \rho_{k\ell}\nabla_{j\ell}f$
$\qquad + \rho_{j\ell}\nabla_{k\ell}f + 2R_{ijk\ell}\nabla_{i\ell}f.$

Proof From the Ricci identity we have

$$\nabla_{iki}f - \nabla_{kii}f = \rho_{k\ell}\nabla_{\ell}f.$$

(i) results from taking ∇_j of this equation. (ii) and (iii) follow from further applications of the Ricci identity. □

We see from this lemma that

$$\nabla_{ijij}f = \nabla_{ijji}f = \Delta^2 f + \tfrac{1}{2}\nabla_{\ell}\tau\nabla_{\ell}f + \rho_{k\ell}\nabla_{k\ell}f,$$

and so from (3.15)

$$f_{iijj}(m) = \tfrac{1}{3}\left(\left\langle \nabla\tau, \nabla f \right\rangle + 2 \left\langle \rho, \nabla^2 f \right\rangle \right)(m),$$

since $\Delta f = 0$ by assumption.

Hence, assuming $\Delta f = 0$,

$$B = -\frac{1}{24n(n+2)}\left(2 \left\langle \rho, \nabla^2 f \right\rangle + 3 \left\langle \nabla\tau, \nabla f \right\rangle\right).$$

Note that a sufficient condition for B to vanish, given $\Delta f = 0$, is that the manifold is Einstein. Later we shall prove the necessity in the following proposition.

Proposition 3.7 A necessary and sufficient condition that for every $m \in M$ and every f harmonic on a nhd of m

$$M_m(f,r) = f(m) + O(r^6) \quad \text{for sufficiently small } r$$

is that M is Einstein.

Thus we recover $L1$ from this term. We carry the conditions $\Delta f = 0$ and M is Einstein over into the calculation of the next term, as we are interested in the circumstances where all the terms up to a certain point vanish.

Calculation of C assuming $\Delta f = 0$ throughout a nhd of m and M is

Einstein ($\rho = Kg$). (For computation of C in the general case see [GW].) Under our assumptions (3.13) gives

$$C(m) = \frac{1}{48n(n+2)(n+4)}(\tilde{\Delta}^3_m(f\Theta) - f\tilde{\Delta}^3_m(\Theta))(m).$$

We have that

$$(3.16)\quad \tilde{\Delta}^3_m(f\Theta) - f\tilde{\Delta}^3_m(\Theta) = f_{iijjkk}\Theta + 6f_{iijjk}\Theta_k + 3f_{iijj}\Theta_{kk} + 12f_{iijk}\Theta_{jk} + 12f_{iij}\Theta_{jkk} + 8f_{ijk}\Theta_{ijk} + 3f_{ii}\Theta_{jjkk} + 12f_{jk}\Theta_{iijk} + 6f_k\Theta_{iijjk}.$$

We write down two scalar functions in which C will be expressed:

$$\left\langle \dot{R}, \nabla^2 f\right\rangle = \dot{R}_{ij}\nabla_{ij}f,$$

$$\left\langle \nabla(|R|^2), \nabla f\right\rangle = \nabla_i(|R|^2)\nabla_i f.$$

From (3.10) the required derivatives of Θ are, under our assumptions,

$$\Theta_{jk}(m) = -\frac{1}{3}\rho_{jk}(m) = -\frac{1}{3}K\delta_{jk},$$

$$\Theta_{ijk}(m) = 0,$$

$$\Theta_{ijk\ell}(m) = \frac{1}{4!}S(-\tfrac{2}{15}R_{iajb}R_{ka\ell b} + \tfrac{1}{3}K^2\delta_{ij}\delta_{k\ell})(m),$$

$$\Theta_{ijk\ell h}(m) = \frac{1}{5!}S(-\tfrac{2}{3}\nabla_i R_{jakb}R_{\ell ahb})(m).$$

Summing the last two in the familiar way of Chapter 2 gives

$$\Theta_{iijk}(m) = \frac{1}{4!}(-\tfrac{8}{5}\dot{R}_{jk} + \tfrac{8}{15}K^2(5n+4)\delta_{jk})(m).$$

$$\Theta_{iijjk}(m) = -\frac{1}{15}\nabla_k(|R|^2)(m).$$

From the calculations of A and B, we see that our assumptions imply

$$f_{ii}(m) = 0, \quad f_{iijj}(m) = 0,$$

and hence (3.16) reduces to

$$\tilde{\Delta}_m^3(f\Theta) - f\tilde{\Delta}_m^3(\Theta) = f_{iijjkk}(m) - \frac{4}{5}\left\langle \dot{R}, \nabla^2 f\right\rangle - \frac{2}{5}\left\langle \nabla(|R|^2), \nabla f\right\rangle ;$$

so it remains to find $f_{iijjkk}(m)$. By Lemma 3.3

$$\begin{aligned}(3.17)\quad f_{iijjkk}(m) &= \tfrac{1}{15}\{\nabla_{iijjkk}f + \nabla_{iijkjk}f + \nabla_{iijkkj}f + \nabla_{ijijkk}f \\ &\quad + \nabla_{ijikjk}f + \nabla_{ijikkj}f + \nabla_{ijjikk}f + \nabla_{ijkijk}f \\ &\quad + \nabla_{ijkikj}f + \nabla_{ijjkik}f + \nabla_{ijkjik}f + \nabla_{ijkkij}f \\ &\quad + \nabla_{ijjkki}f + \nabla_{ijkjki}f + \nabla_{ijkkji}f\}(m) \\ &= \tfrac{2}{15}\{\nabla_{ijikjk}f + \nabla_{ijkijk}f + \nabla_{ijjkik}f + \nabla_{ijkjik}f \\ &\quad + \nabla_{ijkkij}f\}(m)\end{aligned}$$

by the commutativity of the last two indices and the assumptions we have made. Under our assumptions formulae (i) - (iii) of Lemma 3.6 become

$$(3.17a)\quad \begin{cases} \text{(i)}\ \nabla_{jiik}f = K\nabla_{jk}f, \\ \text{(ii)}\ \nabla_{ijik}f = 2K\nabla_{jk}f + R_{ijk\ell}\nabla_{i\ell}f, \\ \text{(iii)}\ \nabla_{iijk}f = 2K\nabla_{jk}f + 2R_{ijk\ell}\nabla_{i\ell}f. \end{cases}$$

Thus
$$\begin{aligned}\nabla_{ijikjk}f &= K\nabla_{ijij}f = 0, \\ \nabla_{ijkijk}f &= 2K\nabla_{ijij}f + \nabla_{ij}(R_{kij\ell}\nabla_{k\ell}f) = R_{kij\ell}\nabla_{ijk\ell}f \quad \text{(Einstein)}, \\ \nabla_{ijjkik}f &= K\nabla_{ijij}f = 0, \\ \nabla_{ijkjik}f &= 2K\nabla_{ijij}f + \nabla_{ij}(R_{kij\ell}\nabla_{k\ell}f) = R_{kij\ell}\nabla_{ijk\ell}f, \\ \nabla_{ijkkij}f &= 2K\nabla_{ijij}f + 2\nabla_{ij}(R_{kij\ell}\nabla_{k\ell}f) = 2R_{kij\ell}\nabla_{ijk\ell}f.\end{aligned}$$

And $R_{kij\ell}\nabla_{ijk\ell}f = \frac{1}{2}R_{kij\ell}\nabla_i(R_{hk\ell j}\nabla_h f)$ (Ricci identity)

$$= \tfrac{1}{2}R_{kij\ell}R_{hk\ell j}\nabla_{ih}f + \tfrac{1}{2}R_{kij\ell}\nabla_i R_{hk\ell j}\nabla_h f$$

$$= \tfrac{1}{2}\left\langle \dot{R},\nabla^2 f\right\rangle + \tfrac{1}{8}\left\langle \nabla(|R|^2),\nabla f\right\rangle \qquad \text{(by Lemma 2.4(ii))}$$

Thus from (3.17)

$$f_{iijjkk}(m) = \tfrac{2}{15}\left(2\left\langle R,\nabla^2 f\right\rangle + \tfrac{1}{2}\left\langle \nabla(|R|^2),\nabla f\right\rangle\right)(m),$$

and

$$(\tilde{\Delta}^3_m(f\Theta) - f\tilde{\Delta}^3_m(\Theta))(m) = \left(-\tfrac{8}{15}\left\langle \dot{R},\nabla^2 f\right\rangle - \tfrac{1}{3}\left\langle \nabla(|R|^2),\nabla f\right\rangle\right)(m).$$

Finally

$$C = \frac{1}{720n(n+2)(n+4)}\left(-8\left\langle \dot{R},\nabla^2 f\right\rangle - 5\left\langle \nabla(|R|^2),\nabla f\right\rangle\right).$$

Note that a sufficient condition for C to vanish, given $\Delta f = 0$, is that the manifold is super-Einstein. Later we will prove the necessity in the following proposition.

<u>Proposition 3.8</u> A necessary and sufficient condition that for every $m \in M$ and every f harmonic on a nhd of m

$$M_m(f,r) = f(m) + O(r^8) \quad \text{for sufficiently small } r$$

is that M is super-Einstein.

Thus we recover the 2-tensor condition derived from $L2$ from this term. We carry this condition over into our calculation of D.

<u>Calculation of D</u> assuming $\Delta f = 0$ and M is super-Einstein, with $\rho = Kg$, $\dot{R} = Sg$. From (3.14) under these assumptions we see that

$$D(m) = \frac{1}{48.8n(n+2)(n+4)(n+6)}(\tilde{\Delta}^4_m(f\Theta) - f\tilde{\Delta}^4_m(\Theta))(m).$$

Now $\tilde{\Delta}^4_m(f\Theta) - f\tilde{\Delta}^4_m(\Theta) = f_{iijjkk\ell\ell}\Theta + 8f_{iijjkk\ell}\Theta_\ell + 24f_{iijjk\ell}\Theta_{k\ell}$

$$+ 4f_{iijjkk}\Theta_{\ell\ell} + 32f_{iijk\ell}\Theta_{jk\ell} + 24f_{iijjk}\Theta_{k\ell\ell}$$

$$+ 16f_{ijk\ell}\Theta_{ijk\ell} + 48f_{iik\ell}\Theta_{jjk\ell} + 6f_{iijj}\Theta_{kk\ell\ell}$$

$$+ 32f_{jk\ell}\Theta_{iijk\ell} + 24f_{iij}\Theta_{jkk\ell\ell} + 24f_{ij}\Theta_{ijkk\ell\ell}$$

$$+ 4f_{ii}\Theta_{jjkk\ell\ell} + 8f_i\Theta_{ijjkk\ell\ell}.$$

We write down the scalar functions in which D will be expressed:

$$\left\langle R \circ R, \nabla^4 f\right\rangle = R_{iajb}R_{ka\ell b}\nabla_{ijk\ell}f, \qquad \left\langle \nabla(\overset{\Omega}{R}), \nabla f\right\rangle = \nabla_i(\overset{\Omega}{R})\nabla_i f,$$

$$\left\langle \overset{\Omega}{R}, \nabla^2 f\right\rangle = \overset{\Omega}{R}_{ij}\nabla_{ij}f, \qquad \left\langle \nabla(\mathring{R}), \nabla f\right\rangle = \nabla_i(\mathring{R})\nabla_i f,$$

$$\left\langle \mathring{R}, \nabla^2 f\right\rangle = \mathring{R}_{ij}\nabla_{ij}f, \qquad \left\langle \nabla(|\nabla R|^2), \nabla f\right\rangle = \nabla_i(|\nabla R|^2)\nabla_i f,$$

$$\left\langle \nabla R \otimes \nabla R, \nabla^2 f\right\rangle = \nabla_i R\nabla_j R\nabla_{ij}f, \qquad \left\langle T, \nabla f\right\rangle = T_i\nabla_i f.$$

From (3.10) we can write down the required derivatives of Θ under our assumptions:

$$\Theta_{k\ell}(m) = -\tfrac{1}{3}K\delta_{k\ell},$$

$$\Theta_{jk\ell}(m) = 0,$$

$$\Theta_{ijk\ell}(m) = \frac{1}{4!}S(-\tfrac{2}{15}R_{iajb}R_{ka\ell b} + \tfrac{1}{3}K^2\delta_{ij}\delta_{k\ell})(m),$$

$$\Theta_{ijk\ell h}(m) = \frac{1}{5!}S(-\tfrac{2}{3}\nabla_i R_{jakb}R_{\ell ahb})(m),$$

$$\Theta_{ijk\ell hg}(m) = \frac{1}{6!}S(-\tfrac{15}{14}\nabla_i R_{jakb}\nabla_\ell R_{hagb} - \tfrac{8}{7}\nabla_{ij}R_{ka\ell b}R_{hagb} - \tfrac{16}{63}R_{iajb}R_{kb\ell c}R_{hcga}$$

$$- \tfrac{2}{3}KR_{iajb}R_{ka\ell b}\delta_{hg} - \tfrac{5}{9}K^3\delta_{ij}\delta_{k\ell}\delta_{hg})(m),$$

$$\Theta_{ijk\ell hgf}(m) = \frac{1}{7!}S(-\tfrac{5}{3}\nabla_{ijk}R_{\ell ahb}R_{gafb} - \tfrac{8}{3}\nabla_i R_{jakb}R_{\ell bhc}R_{gcfa}$$

$$- \tfrac{9}{2}\nabla_{ij}R_{ka\ell b}\nabla_h R_{gafb} + \tfrac{14}{3}K\delta_{ij}\nabla_k R_{\ell ahb}R_{gafb})(m).$$

As before we can calculate appropriate sums:

$$\Theta_{jjk\ell}(m) = P\delta_{k\ell} \quad \text{for some number } P \quad \text{(see Proposition 2.2)},$$

$$\Theta_{iijk\ell}(m) = 0$$

(since it is the symmetric sum of terms of the form $\nabla_i R_{jabc} R_{kabc}$ and this is anti-symmetric in j and k in a super-Einstein space),

$$\Theta_{ijkk\ell\ell}(m) = \frac{40}{6!42}\left(3\nabla_i R \nabla_j R - 20\overset{\circ}{R}_{ij} + 16\overset{\Omega}{R}_{ij}\right) + Q\delta_{ij} \quad \text{for some number } Q,$$

$$\Theta_{ijjkk\ell\ell}(m) = \frac{24}{9.7!}\left(-85\nabla_i(\overset{\circ}{R}) + 140\nabla_i(\overset{\Omega}{R})\right).$$

The last two sums use Proposition 2.10. Our assumptions imply that

$$f_{ii}(m) = 0, \qquad f_{iijj}(m) = 0, \qquad f_{iijjkk}(m) = 0.$$

Thus

$$(3.18) \quad (\tilde{\Delta}^4_m(f\Theta) - f\tilde{\Delta}^4_m(\Theta))(m) = \Big(f_{iijjkk\ell\ell} + 16 f_{ijk\ell}\Theta_{ijk\ell}$$
$$+\frac{160}{7!}\left(3\nabla_i R\nabla_j R - 20\overset{\circ}{R}_{ij} + 16\overset{\Omega}{R}_{ij}\right)f_{ij}$$
$$+\frac{24.8}{9.7!}\left(140\nabla_i(\overset{\Omega}{R}) - 85\nabla_i(\overset{\circ}{R})\right)f_i\Big)(m).$$

To express $f_{ijk\ell}\Theta_{ijk\ell}$ in invariant terms we need the next lemma.

<u>Lemma 3.9</u> In normal co-ordinates based at m

$$R_{iajb}R_{ka\ell b}(m)\frac{\partial^4 f}{\partial x^i\partial x^j\partial x^k\partial x^\ell}(m) = \Big(\left\langle R\circ R, \nabla^4 f\right\rangle + \tfrac{2}{3}\left\langle \overset{\Omega}{R}, \nabla^2 f\right\rangle$$
$$- \tfrac{1}{12}\left\langle \overset{\circ}{R}, \nabla^2 f\right\rangle + \tfrac{1}{9}\left\langle \nabla(\overset{\Omega}{R}), \nabla f\right\rangle$$
$$- \tfrac{1}{72}\left\langle \nabla(\overset{\circ}{R}), \nabla f\right\rangle\Big)(m).$$

<u>Proof</u> We have from Lemma 3.3(ii) that

$$\frac{\partial^4 f}{\partial x^i\partial x^j\partial x^k\partial x^\ell}(m) = \tfrac{1}{12}\big(\nabla_{ijk\ell}f + \nabla_{i\ell jk}f + \nabla_{ik\ell j}f + \nabla_{jik\ell}f + \nabla_{j\ell ik}f$$
$$+ \nabla_{jk\ell i}f + \nabla_{kij\ell}f + \nabla_{k\ell ji}f + \nabla_{kji\ell}f + \nabla_{\ell ijk}f$$
$$+ \nabla_{\ell kij}f + \nabla_{\ell jki}f\big)(m).$$

The lemma follows from repeated use of the Ricci identity to express the permuted covariant derivatives in terms of the first, and then use of the curvature identities to further simplify. $\square$

Now

$$16 f_{ijk\ell}\Theta_{ijk\ell}(m) = -\tfrac{2}{3}\cdot\tfrac{2}{15} f_{ijk\ell}(m)\, S(R_{iajb}R_{ka\ell b})(m)$$

$$= -\tfrac{4}{45}\cdot 24 f_{ijk\ell}(m) R_{iajb}R_{ka\ell b}(m)$$

$$(3.19)\qquad = -\tfrac{32}{15}\Big(\Big\langle R \circ R, \nabla^4 f\Big\rangle + \tfrac{2}{3}\Big\langle \overset{\circ}{\overline{R}}, \nabla^2 f\Big\rangle - \tfrac{1}{12}\Big\langle \overset{\circ}{R}, \nabla^2 f\Big\rangle + \tfrac{1}{9}\Big\langle \nabla(\overset{\circ}{\overline{R}}), \nabla f\Big\rangle - \tfrac{1}{72}\Big\langle \nabla(\overset{\circ}{R}), \nabla f\Big\rangle\Big)(m)$$

by Lemma 3.9. The only term left to calculate is $(\tilde{\Delta}^4_m f)(m)$ and the lengthy calculation involved is given in detailed outline in Appendix II. We quote the result:

$$(3.20)\qquad (\tilde{\Delta}^4_m f)(m) = \tfrac{2}{105}\Big(56 \Big\langle R \circ R, \nabla^4 f\Big\rangle + 24\Big\langle \overset{\circ}{\overline{R}}, \nabla^2 f\Big\rangle + 12\Big\langle \overset{\circ}{R}, \nabla^2 f\Big\rangle - \tfrac{5}{2}\Big\langle \nabla R \otimes \nabla R, \nabla^2 f\Big\rangle + \tfrac{7}{3}\Big\langle \nabla(\overset{\circ}{\overline{R}}), \nabla f\Big\rangle + \tfrac{19}{12}\Big\langle \nabla(\overset{\circ}{R}), \nabla f\Big\rangle\Big)(m).$$

Combining (3.18), (3.19) and (3.20), we have this formula for D, assuming $\Delta f = 0$ and the space is super-Einstein:

$$(3.21)\qquad D = \chi(n)\Big(-112\Big\langle R \circ R, \nabla^4 f\Big\rangle - 48\Big\langle \overset{\circ}{\overline{R}}, \nabla^2 f\Big\rangle - 24\Big\langle \overset{\circ}{R}, \nabla^2 f\Big\rangle + 5\Big\langle \nabla R \otimes \nabla R, \nabla^2 f\Big\rangle + 42\Big\langle \nabla(\overset{\circ}{\overline{R}}), \nabla f\Big\rangle - \tfrac{63}{2}\Big\langle \nabla(\overset{\circ}{R}), \nabla f\Big\rangle\Big)$$

where $\chi(n) = \dfrac{1}{48.8.105n(n+2)(n+4)(n+6)}$.

We will also require D in a form involving partial derivatives, so using Lemma 3.9 we rewrite (3.21) as

$$(3.22)\qquad D(m) = \chi(n)\Big(-112 R_{iajb}R_{ka\ell b}f_{ijk\ell} + \tfrac{80}{3}\overset{\circ}{\overline{R}}_{ij} f_{ij} - \tfrac{100}{3}\overset{\circ}{R}_{ij} f_{ij} + 5\nabla_i R \nabla_j R f_{ij} + \tfrac{490}{9}\nabla_i(\overset{\circ}{\overline{R}}) f_i - \tfrac{595}{18}\nabla_i(\overset{\circ}{R}) f_i\Big)(m).$$

§3 The use of the Cauchy-Kowalewski theorem

We now proceed to proving the necessary parts of the Propositions 3.6 and 3.7 and a similar proposition for the next term. This is by means of a well-known theorem in partial differential equation theory, the Cauchy—Kowalewski initial value theorem, which enables us to choose suitable test harmonic functions.

N.B. The Einstein summation convention is not assumed to hold throughout this section. All summations will be written explicity.

Our first proposition is easily seen to finish the proofs of Propositions 3.7 and 3.8.

Proposition 3.10 Suppose that for every f harmonic in a nhd of m, there exist sets of real numbers T_{ij} $(T_{ij} = T_{ji})$ and U_i such that

$$(3.23) \qquad \sum_{i,j=1}^{n} T_{ij}\frac{\partial^2 f}{\partial x^i \partial x^j}(m) + \sum_{i=1}^{n} U_i \frac{\partial f}{\partial x^i}(m) = 0,$$

$(x^1 \ldots x^n)$ being normal co-ordinates about m. Then

$$T_{ij} = \lambda\delta_{ij} \quad i,j = 1, \ldots, n \text{ for some real number } \lambda \text{ and}$$

$$U_i = 0 \qquad i = 1, \ldots, n.$$

Proof The Cauchy-Kowalewski theorem asserts that there is a solution of the elliptic equation $\Delta f = 0$ in a nhd of m such that on the hypersurface $x^1 = 0$ both f and $\frac{\partial f}{\partial x^1}$ take arbitrarily prescribed values, say

$$f = \phi_0(0, x^2, \ldots, x^n),$$

$$\frac{\partial f}{\partial x^1} = \phi_1(0, x^2, \ldots, x^n).$$

We express these functions as power series in the variables $x^2, \ldots, x^n$:

$$\phi_0(0, x^2, \ldots, x^n) = \beta_0 + \sum_{i=2}^{n} \beta_i x^i + \tfrac{1}{2} \sum_{i,j=2}^{n} \beta_{ij} x^i x^j + \cdots,$$

$$\phi_1(0, x^2, \ldots, x^n) = \gamma_0 + \sum_{i=2}^{n} \gamma_i x^i + \cdots$$

where we assume that $\beta_{ij} = \beta_{ji}$. Writing down the first and second partial derivatives of f at m:

$$\frac{\partial f}{\partial x^k}(m) = \beta_k, \qquad 2 \leqslant k \leqslant n,$$

$$\frac{\partial f}{\partial x^1}(m) = \gamma_0,$$

$$\frac{\partial^2 f}{\partial x^i \partial x^j}(m) = \beta_{ij}, \qquad 2 \leqslant i,j \leqslant n,$$

$$\frac{\partial^2 f}{\partial x^1 \partial x^k}(m) = \gamma_k, \qquad 2 \leqslant k \leqslant n,$$

$$\frac{\partial^2 f}{\partial x^1 \partial x^1}(m) = -\sum_{i=2}^{n} \frac{\partial^2 f}{\partial x^i \partial x^i} = -\sum_{i=2}^{n} \beta_{ii} .$$

The last equation follows from $\Delta f(m) = (\tilde{\Delta}_m f)(m) = 0$.

We can make all second derivatives of f at m zero by the choice

$$\beta_{ij} = 0 \quad 2 \leqslant i,j \leqslant n, \qquad \gamma_k = 0 \quad 2 \leqslant k \leqslant n .$$

For an f of this kind (3.23) reduces to

$$\sum_{i=1}^{n} U_i \frac{\partial f}{\partial x^i}(m) = 0 .$$

Then by a further choice of one of $\{\beta_k, k = 2, \ldots, n\} \cup \{\gamma_0\}$ to be 1 and the rest 0 we see that

$$U_i = 0 \quad 1 \leqslant i \leqslant n .$$

Hence (3.23) reduces to, for any harmonic f,

(3.24) $$\sum_{i,j=1}^{n} T_{ij} \frac{\partial^2 f}{\partial x^i \partial x^j}(m) = 0 .$$

Fix integers p,q, $2 \leqslant p,q \leqslant n$, $p \neq q$, set all the β's equal to zero except for $\beta_{pq} = \beta_{qp} = 1$ and set $\gamma_k = 0$, $2 \leqslant k \leqslant n$. Using an f of this kind in (3.24) we find that

$$T_{pq} = 0, \qquad 2 \leqslant p,q \leqslant n,\ p \neq q,$$

as we assumed T to be symmetric.

Similarly fixing p, $2 \leqslant p \leqslant n$, and setting all the γ's equal to zero except $\gamma_p = 1$, and all the β's equal to zero we find that for an f of this kind,

$$T_{ip} = 0, \qquad 2 \leqslant p \leqslant n.$$

Finally fixing p and setting $\beta_{pp} = 1$ and the other β's and γ's equal to zero we obtain

$$-T_{11} + T_{pp} = 0, \qquad 2 \leqslant p \leqslant n,$$

whence $$T_{ij} = \lambda\delta_{ij}. \qquad \square$$

We now turn our attention to the fourth term, and try to find necessary and sufficient conditions for the first four non-constant terms to vanish. From the previous section we recall that

(3.25) $$\begin{aligned} D(m) = \chi(n)\Big(&-112 R_{iajb} R_{ka\ell b} \frac{\partial^4 f}{\partial x^i \partial x^j \partial x^k \partial x^\ell} \\ &+ \big(\tfrac{80}{3}\overset{o}{R}_{ij} - \tfrac{100}{3}\overset{o}{R}_{ij} + 5\nabla_i R \nabla_j R\big) \frac{\partial^2 f}{\partial x^i \partial x^j} \\ &+ \big(\tfrac{490}{9}\nabla_i(\overset{o}{R}) - \tfrac{595}{18}\nabla_i(\overset{o}{R})\big) \frac{\partial f}{\partial x^i}\Big)(m) \end{aligned}$$

for $\Delta f = 0$ in a super-Einstein space. We see that sufficient conditions for

$$M_m(f,r) = f(m) + O(r^{10})$$

are $\Delta f = 0$, M satisfies $L1$ and $L2$ (and hence is super-Einstein) and that the following identities hold in orthonormal co-ordinates:

$$(3.26) \qquad 16\overset{\underline{o}}{R}_{ij} - 20\overset{o}{R}_{ij} + 3\nabla_i R \nabla_j R = N\delta_{ij},$$

$$(3.27) \qquad 28\overset{\underline{o}}{R} - 17\overset{o}{R} = \text{constant}.$$

Note that (3.26) is exactly the 2-tensor condition derived from $L3$. Although it would be sufficient for N to be a function, the tensor has the Schur property of order 6, and so N must be a constant for $n \neq 6$. For $n = 6$ N is a constant because of (3.27) (using corollary 2.6). Also the constant in $L2$ need only be a function, but the super-Einstein condition derived from earlier terms ensures it is a constant.

The content of the next theorem is that these conditions are necessary.

<u>Theorem 3.11</u> Necessary and sufficient conditions that for every $m \in M$ and every f harmonic on a nhd of m

$$M_m(f,r) = f(m) + O(r^{10}) \qquad \text{for sufficiently small } r$$

are that M satisfies the first two Ledger conditions and

$$16\overset{\underline{o}}{R}_{ij} - 20\overset{o}{R}_{ij} + 3\nabla_i R \nabla_j R = N\delta_{ij},$$

$$28\overset{\underline{o}}{R} - 17\overset{o}{R} = C \qquad \text{for } N, C \text{ constants}.$$

Proof Sufficiency we have seen. We need to extend the earlier use of the Cauchy-Kowalewski theorem to cope with fourth derivatives, so we write

$$\phi_0(0, x^2, \ldots, x^n) = \beta_0 + \sum_{i=2}^{n} \beta_i x^i + \frac{1}{2}\sum_{i,j=2}^{n} \beta_{ij} x^i x^j$$

$$+ \frac{1}{3!}\sum_{i,j,k=2}^{n} \beta_{ijk} x^i x^j x^k + \frac{1}{4!}\sum_{i,j,k,\ell=2}^{n} \beta_{ijk\ell} x^i x^j x^k x^\ell$$

$$+ \ldots ,$$

$$\phi_1(0, x^2, \ldots, x^n) = \gamma_0 + \sum_{i=2}^{n} \gamma_i x^i + \frac{1}{2}\sum_{i,j=2}^{n} \gamma_{ij} x^i x^j$$

$$+ \frac{1}{3!}\sum_{i,j,k=2}^{n} \gamma_{ijk} x^i x^j x^k + \ldots$$

where the γ's and β's are assumed symmetric in every index.

We first ensure that we are working with functions whose 1st and 2nd derivatives at m are zero by choosing all β_i's, γ_i's, β_{ij}'s and γ_0 to be zero (see proof of Proposition 3.10). Our assumption implies that the space is super-Einstein (Proposition 3.8), so we see from (3.17a) that for an f of this kind

$$\sum_{i=1}^{n} \nabla_{jiik} f(m) = \sum_{i=1}^{n} \nabla_{ijik} f(m) = \sum_{i=1}^{n} \nabla_{iijk} f(m) = 0 .$$

Thus it is clear from Lemma 3.3 that

$$\sum_{i=1}^{n} \frac{\partial^4 f}{\partial x^i \partial x^i \partial x^j \partial x^k}(m) = 0, \qquad 1 \leqslant j,k \leqslant n, \tag{3.28}$$

Thus we can, assuming that the above choice has been made, write down the fourth derivatives of such an f in terms of β's and γ's:

$$\frac{\partial^4 f}{\partial x^i \partial x^j \partial x^k \partial x^\ell}(m) = \beta_{ijk\ell}, \qquad 2 \leqslant i,j,k,\ell \leqslant n,$$

$$\frac{\partial^4 f}{\partial x^1 \partial x^j \partial x^k \partial x^\ell}(m) = \gamma_{jk\ell}, \qquad 2 \leqslant j,k,\ell \leqslant n,$$

$$\frac{\partial^4 f}{\partial x^1 \partial x^1 \partial x^k \partial x^\ell}(m) = -\sum_{i=2}^{n} \beta_{iik\ell}, \qquad 2 \leqslant k,\ell \leqslant n \quad (\text{from } (3.28)),$$

$$\frac{\partial^4 f}{\partial x^1 \partial x^1 \partial x^1 \partial x^\ell}(m) = -\sum_{i=2}^{n} \gamma_{ii\ell}, \qquad 2 \leqslant \ell \leqslant n \quad (\text{from } (3.28)),$$

$$\frac{\partial^4 f}{\partial x^1 \partial x^1 \partial x^1 \partial x^1}(m) = \sum_{i,j=2}^{n} \beta_{iijj} \qquad (\text{from } (3.28) \text{ twice}).$$

The condition that D(m) vanishes reduces, with such an f, to

$$\sum_{a,b,i,j,k,\ell=1}^{n} \left(R_{iajb} R_{ka\ell b} \frac{\partial^4 f}{\partial x^i \partial x^j \partial x^k \partial x^\ell}\right)(m) = 0,$$

and this is equivalent to

$$\sum_{a,b,i,j,k,\ell=1}^{n} \left(S(R_{iajb} R_{ka\ell b}) \frac{\partial^4 f}{\partial x^i \partial x^j \partial x^k \partial x^\ell}\right)(m) = 0.$$

For ease of notation we define

$$C_{ijk\ell} = S\left(\sum_{a,b=1}^{n} R_{iajb} R_{ka\ell b}\right)(m),$$

and the condition can be written as

$$\sum_{i,j,k,\ell=1}^{n} C_{ijk\ell} \frac{\partial^4 f}{\partial x^i \partial x^j \partial x^k \partial x^\ell}(m) = 0 \qquad \text{or}$$

$$(3.29) \quad C_{1111} \frac{\partial^4 f}{\partial x^1 \partial x^1 \partial x^1 \partial x^1}(m) + 4 \sum_{k=2}^{n} C_{111k} \frac{\partial^4 f}{\partial x^1 \partial x^1 \partial x^1 \partial x^k}(m)$$

$$+ 6 \sum_{k,\ell=2}^{n} C_{11k\ell} \frac{\partial^4 f}{\partial x^1 \partial x^1 \partial x^k \partial x^\ell}(m) + 4 \sum_{j,k,\ell=2}^{n} C_{1jk\ell} \frac{\partial^4 f}{\partial x^1 \partial x^j \partial x^k \partial x^\ell}(m)$$

$$+ \sum_{i,j,k,\ell=2}^{n} C_{ijk\ell} \frac{\partial^4 f}{\partial x^i \partial x^j \partial x^k \partial x^\ell}(m) = 0.$$

As in the last proposition we make choices of the γ's and β's and use (3.29) to gain information about the C's. We make the convention that '$\beta_{pqrs} = 1$, rest zero' means 'β_{pqrs} and the β's obtained by permuting p,q,r,s are all equal to 1, the other β's are zero'.

First set $\gamma_{ijk} = 0$, $2 \leqslant i,j,k \leqslant n$. (3.29) becomes

$$C_{1111}\Big(\sum_{i,j=2}^{n} \beta_{iijj}\Big) - 6\sum_{k,\ell=2}^{n} C_{11k\ell}\Big(\sum_{i=2}^{n}\beta_{iik\ell}\Big) + \sum_{i,j,k,\ell=2}^{n} C_{ijk\ell}\beta_{ijk\ell} = 0.$$

Choose $\beta_{pqrs} = 1$, for distinct p,q,r,s, rest zero. Then

(3.30) $\quad C_{pqrs} = 0, \qquad 2 \leqslant p,q,r,s \leqslant n$, p,q,r,s distinct.

Choose $\beta_{pprs} = 1 \;\; r \neq s$, rest zero. Then

(3.31) $\quad C_{11rs} = C_{pprs}, \qquad 2 \leqslant p,r,s \leqslant n \quad r \neq s.$

Choose $\beta_{pppp} = 1$, rest zero. Then

(3.32) $\quad C_{1111} - 6C_{11pp} + C_{pppp} = 0, \; 2 \leqslant p \leqslant n.$

Now we set $\beta_{ijk\ell} = 0$, $2 \leqslant i,j,k,\ell \leqslant n$. (3.29) becomes

$$-\sum_{k=2}^{n} C_{111k}\Big(\sum_{i=2}^{n}\gamma_{iik}\Big) + \sum_{j,k,\ell=2}^{n} C_{1jk\ell}\gamma_{jk\ell} = 0.$$

Choose $\gamma_{pqr} = 1$, for distinct p,q,r, rest zero. Then

(3.33) $\quad C_{1pqr} = 0, \qquad 2 \leqslant p,q,r \leqslant n$, p,q,r distinct.

Choose $\gamma_{ppr} = 1$ for distinct p,r, rest zero. Then

(3.34) $\quad C_{111r} = 3C_{1ppr}, \qquad 2 \leqslant p,r \leqslant n \quad p \neq r.$

Choose $\gamma_{ppp} = 1$, rest zero. Then

(3.35) $\quad C_{111p} = C_{1ppp}, \qquad 2 \leqslant p \leqslant n.$

It should also be remembered that the choice of $x^1 = 0$ as the hypersurface for initial conditions was arbitrary and so there are similar equations obtained by using $x^k = 0$, $2 \leqslant k \leqslant n$.

We wish to show that the space satisfies $L2$ i.e. that

$$C_{ijk\ell} = HS(\delta_{ij}\delta_{k\ell}), \qquad 1 \leqslant i,j,k,\ell \leqslant n.$$

This is equivalent to

$$(3.36) \qquad \left.\begin{cases} C_{ijk\ell} = 0, & i,j,k,\ell \text{ distinct}, \\ C_{iik\ell} = 0, & i,k,\ell \text{ distinct}, \\ C_{iikk} = 8H, & i \neq k, \\ C_{iiik} = 0, & i \neq k, \\ C_{iiii} = 24H, & \end{cases}\right\} 1 \leqslant i,j,k,\ell \leqslant n.$$

Proposition 2.2 involved the calculation of $\sum_{i=1}^{n} C_{iik\ell}$. From that proposition we see that, in a super-Einstein space,

$$(3.37) \qquad \sum_{i=1}^{n} C_{iik\ell} = M\delta_{k\ell} \qquad \text{for some constant } M.$$

Let us take equation (3.32) and sum from $p = 2$ to n to obtain

$$(n-1)C_{1111} - 6\sum_{p=2}^{n} C_{11pp} + \sum_{p=2}^{n} C_{pppp} = 0,$$

and so $$(n+4)C_{1111} - 6\sum_{p=1}^{n} C_{11pp} + \sum_{p=1}^{n} C_{pppp} = 0.$$

Using (3.37) we see that

$$(n+4)C_{1111} = 6M - \sum_{p=1}^{n} C_{pppp}.$$

But, as remarked earlier, we could have obtained a similar equation if we had chosen $x^i = 0$, $2 \leqslant i \leqslant n$, as our initial hypersurface. We conclude that

$$(3.38) \qquad C_{iiii} = H', \qquad 1 \leqslant i \leqslant n.$$

Reconsidering (3.32) in the light of this we see that

$$C_{11ii} = \tfrac{1}{3}H', \qquad 2 \leqslant i \leqslant n,$$

and again by the arbitrariness of choice of index 1 we conclude that

$$(3.39) \qquad C_{iikk} = \tfrac{1}{3}H', \qquad i \neq k.$$

Taking now equation (3.34)

$$C_{111k} = 3C_{ii1k}, \qquad 2 \leqslant i,k \leqslant n, \ i \neq k,$$

and equation (3.35)

$$3C_{111k} = 3C_{kk1k},$$

we find that, summing,

$$(n + 1)C_{111k} = 3\sum_{i=2}^{n} C_{ii1k},$$

and hence that

$$(n + 4)C_{111k} = 3\sum_{i=1}^{n} C_{ii1k}.$$

However from (3.37), the R.H.S $= 0$ as $k \neq 1$, and we obtain by the arbitrary choice of index 1

$$(3.40) \qquad C_{iiik} = 0, \qquad i \neq k.$$

Taking equation (3.31)

$$C_{11k\ell} = C_{iik\ell}, \qquad 2 \leq i,k,\ell \leq n,\ k \neq \ell,$$

and summing over i,

$$(n-1)C_{11k\ell} = \sum_{i=2}^{n} C_{iik\ell},$$

and thus

$$nC_{11k\ell} = \sum_{i=1}^{n} C_{iik\ell}.$$

Again by (3.37) the R.H.S = 0, as $k \neq \ell$, and as the choice of index 1 was arbitrary

$$C_{iik\ell} = 0 \qquad i,k,\ell \text{ distinct}. \tag{3.41}$$

Finally from (3.30) and (3.33)

$$C_{ijk\ell} = 0 \qquad i,j,k,\ell \text{ distinct}. \tag{3.42}$$

We have thus satisfied each of (3.36) in (3.38) - (3.42) and the manifold satisfies $L2$.

This removes the fourth derivative term from (3.25), and we can then conclude from Proposition 3.10 that

$$16\overset{\Omega}{R}_{ij} - 20\overset{o}{R}_{ij} + 3\nabla_i R \nabla_j R = N\delta_{ij},$$

$$28\overset{\Omega}{R} - 17\overset{o}{R} = \text{constant}. \qquad \square$$

It is interesting to consider the relationship between the Ledger conditions and the conditions on the curvature derived from successive terms of the mean value power series in the manner above. The first in either case is the same namely that the manifold be Einstein. The second mean value conditions are obtained from $L2$ by summing over one and two pairs of indices. The third mean value conditions are $L2$ plus conditions obtained from $L3$ by summing over two and three pairs of indices. An obvious conjecture is that the next set of mean-value conditions will include $L3$ and the conditions derived from $L4$ by summing over three and four indices, although verification of this by direct calculation seems out of the question. We know that both sets of infinite conditions are equivalent to the manifold being harmonic and it is of interest to note that the mean-value conditions seem to occur more slowly inasmuch as one has to go further along the sequence to find the corresponding Ledger condition.

We also note the possibility of gaining necessary conditions on M by exploiting Proposition 3.4 and its corollary by use of the Cauchy-Kowalewski theorem. However inspection of our formulae for $\tilde{\Delta}_m^k f(m)$ for $k = 2,3,4$ shows that the conditions found are exactly those gained above.

Chapter 4 Classical Compact Simple Lie Groups and the Ledger Conditions

This chapter contains an investigation of the extent to which the classical compact simple Lie groups can satisfy the first three Ledger conditions for harmonicity. It is well-known that a group of this kind, when endowed with a bi-invariant metric becomes a globally symmetric space ([H] Ch. IV Section 6). By a theorem of Ledger [L] a symmetric space is harmonic if and only if it is a rank one symmetric space. Thus the classical compact simple Lie group of rank one must satisfy all the Ledger conditions and it is an interesting question to ask whether any other classical compact simple Lie group can satisfy $L2$ or $L3$ (they are all Einstein). We answer the question negatively for almost all of these groups by first finding necessary conditions for them to satisfy $L2$ and $L3$ and showing that these cannot be satisfied. The main reference for this chapter is Pontryagin [PO].

§1 Classical compact simple Lie groups, their Lie algebras and root systems

A compact simple Lie group gives rise to a compact simple Lie algebra over $\mathbb{R}$ and the classification of the former is carried out via the classification of the latter. This is achieved by means of the root systems. We now proceed to give a brief description of the root system of a compact semi-simple Lie algebra. (For full details see [PO], Section 62.)

Let R be a Lie algebra over $\mathbb{R}$. Then we can for each $s \in R$ define an endomorphism of R, p_s, by

$$p_s(x) = [s,x], \qquad x \in R.$$

R is then compact semi-simple if the Killing-Cartan form $\langle\ ,\ \rangle$ given by

$$\langle s,t\rangle = \operatorname{tr} p_s p_t, \qquad s,t \in R,$$

is negative definite. For the rest of this section we suppose R to be a compact semi-simple Lie algebra.

For $a \in R$ let S_a be the subalgebra of elements commuting with a. The dimension of S_a varies with a; if it is a minimum for R then a is, by definition, a regular element and S_a is a regular (or Cartan) subalgebra. The dimension of S_a is the rank of R. A non-trivial fact is that a regular subalgebra is commutative ([PO], p.460). Because the Killing-Cartan form is invariant under the adjoint group i.e.

$$\langle [u,v],w\rangle + \langle v,[u,w]\rangle = 0, \qquad u,v,w \in R,$$

each of the commutative set of endomorphisms $\{p_s : s \in S_a\}$ is skew-symmetric with respect to the Killing-Cartan form. Hence they have purely imaginary eigenvalues and common eigenvectors belonging to the complexification of R, $R^{\mathbb{C}}$. The eigenvalues are pure imaginary linear forms on S_a i.e. we can write, if α is the eigenvalue corresponding to the eigenvector r_α,

$$p_s(r_\alpha) = \alpha(s)r_\alpha, \qquad s \in S_a.$$

We have the natural inner product (,) on S_a given by the restriction of the negative of the Killing-Cartan form. Abusing our notation we define the vector $\alpha \in S_a$, the rootvector of r_α, by

$$(4.1) \qquad p_s(r_\alpha) = i(\alpha,s)r_\alpha.$$

The set of α's obtained in this way together with their pairwise inner products form the root system of R.

The complexification of R resolves into the direct sum of the complexification of S_a and $R_{\alpha_1}, \ldots, R_{\alpha_k}$ where R_{α_i} is the eigenspace corresponding to the rootvector α_i. It can be shown that each R_{α_i} is of complex dimension 1, and if $j\alpha$ is a multiple of α and is also a rootvector then $j = \pm 1$. The eigenspace corresponding to $-\alpha$ is $\overline{R}_\alpha$ ([PO] p.463).

A result by no means straightforward to prove is that if an alternative regular subalgebra is chosen, the root system obtained is isometric to the first ([PO] Section 64). Finally the classification theory depends on the fact that a compact semi-simple Lie algebra is determined by its root system; a long constructive proof is required ([PO] Section 63).

We give now the list of classical compact simple Lie groups in its usual form:

$\mathcal{A}_r$: Group of unitary matrices of order r+1 with determinant 1, $r \geqslant 1$,
$\mathcal{B}_r$: Group of orthogonal matrices of order 2r+1 with determinant 1, $r \geqslant 2$,
$\mathcal{C}_r$: Group of symplectic matrices of order 2r, $r \geqslant 3$,
$\mathcal{D}_r$: Group of orthogonal matrices of order 2r with determinant 1, $r \geqslant 4$.

These groups are in fact defined for all values of $r \geqslant 1$, but $\mathcal{A}_1 \cong \mathcal{B}_1 \cong \mathcal{C}_1$, $\mathcal{B}_2 \cong \mathcal{C}_2$, $\mathcal{A}_3 \cong \mathcal{D}_3$, $\mathcal{D}_2 \cong \mathcal{A}_1 \times \mathcal{A}_1$ and $\mathcal{D}_1 \cong S^1$.

The corresponding Lie algebras are:

A_r: Skew-Hermitian matrices of order r+1 with trace zero, $r \geqslant 1$,
B_r: Real skew-symmetric matrices of order 2r+1, $r \geqslant 2$,
C_r: Skew-Hermitian matrices of order 2r such that $M \in C_r$ if $JM = M\dot{J}$ for $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, $r \geqslant 3$,
D_r: Real skew-symmetric matrices of order 2r, $r \geqslant 4$.

The subscript gives the rank of each Lie algebra. The Killing-Cartan forms are found to be:

$$A_r: \quad \langle A,B \rangle = 2(r+1)\,\mathrm{tr}(AB),$$

$$B_r: \quad \langle A,B \rangle = (2r-1)\,\mathrm{tr}(AB),$$

$$C_r: \quad \langle A,B \rangle = 2(r+1)\,\mathrm{tr}(AB),$$

$$D_r: \quad \langle A,B \rangle = 2(r-1)\,\mathrm{tr}(AB).$$

A classical compact simple Lie group is given a metric obtained by translating a negative multiple of the Killing-Cartan form of the associated Lie algebra by the group action. In fact, in each case, we choose the metric such that a matrix of the form

$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}$$

belonging to the Lie algebra has unit length.

Next we give the root systems of these algebras with the metrics just mentioned. Let E^s be an s-dimensional inner product space with orthonormal basis $E_1, \ldots, E_s$ and let $\sum(R)$ denote the root system of R.

A_r: A regular subalgebra can be considered to be the r-dimensional subspace of E^{r+1} given by vectors $s^i E_i$ such that $s^1 + \ldots + s^{r+1} = 0$. Then

$$\sum(A_r) = \{E_j - E_k,\ j \neq k,\ j,k = 1, \ldots, r+1\}.$$

(E_j can be taken to be the diagonal matrix with $\sqrt{2}i$ in (j,j) place, zeroes elsewhere.).

B_r: A regular subalgebra can be considered to be E^r and

$$\sum(B_r) = \{\pm E_j,\ j = 1, \ldots, r;\ \pm E_j \pm E_k\ \ j < k,\ j,k = 1, \ldots, r\}.$$

(E_j can be taken to be the matrix with 1 in $(2j-1, 2j)$ place and -1 in $(2j, 2j-1)$ place, zeroes elsewhere.)

C_r: A regular subalgebra can be considered to be E^r and

$$\sum(C_r) = \{\pm 2E_j,\ j = 1, \ldots, r;\ \pm E_j \pm E_k\ \ j < k,\ j,k = 1, \ldots, r\}.$$

(E_j can be taken to be the matrix with i in $(2j-1, 2j-1)$ place and $-i$ in the $(2j,2j)$ place, zeroes elsewhere.)

D_r: A regular subalgebra can be considered to be E^r and

$$\sum(D_r) = \{\pm E_j \pm E_k,\ j < k,\ j,k = 1, \ldots, r\}.$$

(E_j can be taken as for B_r.)

The proof of this is given in [PO] p.495ff. The dimensions of the Lie algebras can be found by adding the rank to the number of root vectors. Thus

$$\dim(A_r) = r^2 + 2r,\ \dim(B_r) = 2r^2 + r,\ \dim(C_r) = 2r^2 + r,$$

$$\dim(D_r) = 2r^2 - r.$$

§2 A particular co-ordinate system

We wish to choose an orthonormal basis of a compact semi-simple Lie algebra which will make our calculations easier. This basis will then give rise, in the usual way, to normal co-ordinates on a neighbourhood of the identity of the corresponding classical compact simple Lie group.

Suppose we have chosen a regular subalgebra S of our Lie algebra R of rank r. We take an orthonormal basis of S, $e_1, \ldots, e_r$, say.

Let us consider an eigenvector r_α of p_s, $s \in S$, $\alpha \in \sum(R)$, and writing $r_\alpha = x_\alpha + iy_\alpha$ we have from (4.1)

$$[s,x_\alpha] = -(\alpha,s)y_\alpha, \tag{4.2}$$

$$[s,y_\alpha] = (\alpha,s)x_\alpha. \tag{4.3}$$

Since the complexification of R is the direct sum of the complexification of S and $\{R_\alpha,\ \alpha \in \sum(R)\}$, and $r_{-\alpha} = \overline{r}_\alpha$, we have that $\{e_1, \ldots, e_n, x_\alpha, y_\alpha,\ \alpha \in \Sigma'\}$ span R where Σ' is a subset of Σ obtained by choosing just one of each $\pm\alpha$. (Here we use the fact that each R_α has complex dimension 1.) We note that the choice of r_α is not unique; any scalar multiple is also an eigenvector.

<u>Proposition 4.1</u> An orthonormal (with respect to a negative multiple of the Killing-Cartan form) basis of R of the form $\{e_1, \ldots e_n, x_\alpha, y_\alpha,\ \alpha \in \Sigma'\}$ can be chosen.

<u>Proof</u> The proof is based on the fact that the Killing-Cartan form is adjoint invariant ([PO] p.452) i.e, denoting the given metric by (,),

$$([u,v],w) + (v,[u,w]) = 0, \tag{4.4}$$

and so would be true for any other adjoint invariant metric on R.

(i) For any choice of $r_\alpha = x_\alpha + iy_\alpha$, $(x_\alpha,x_\alpha) = (y_\alpha,y_\alpha)$, since for $s \in S$

$$\begin{aligned} -(\alpha,s)(y_\alpha,y_\alpha) &= ([s,x_\alpha],y_\alpha) && (4.2) \\ &= -(x_\alpha,[s,y_\alpha]) && (4.4) \\ &= -(\alpha,s)(x_\alpha,x_\alpha) && (4.3), \end{aligned}$$

and we can choose $s = \alpha \neq 0$. Thus by choosing $r_\alpha' = \dfrac{r_\alpha}{\|x_\alpha\|} = \dfrac{r_\alpha}{\|y_\alpha\|}$

we can ensure that the real and imaginary parts have unit length.

(ii) $(x_\alpha, y_\alpha) = 0$, since for $s \in S$

$$-(\alpha,s)(y_\alpha,x_\alpha) = ([s,x_\alpha],x_\alpha) \qquad (4.2)$$
$$= 0 \qquad (4.4).$$

(iii) $(x_\alpha,x_\beta) = (y_\alpha,y_\beta) = 0$, $\alpha \neq \beta$, $\alpha,\beta \in \Sigma'$, since for $s \in S$

$$(\alpha,s)(x_\alpha,x_\beta) = ([s,y_\alpha],x_\beta) \qquad (4.3)$$
$$= -(y_\alpha,[s,x_\beta]) \qquad (4.4)$$
$$= (\beta,s)(y_\alpha,y_\beta) \qquad (4.2).$$

But α cannot be a multiple of β so $(x_\alpha,x_\beta) = (y_\alpha,y_\beta) = 0$.

(iv) $(x_\alpha,y_\beta) = 0$, $\alpha \neq \beta,\alpha,\beta \in \Sigma'$, since for $s \in S$

$$(\alpha,s)(x_\alpha,y_\beta) = ([s,y_\alpha],y_\beta) \qquad (4.3)$$
$$= -(y_\alpha,[s,y_\beta]) \qquad (4.4)$$
$$= -(\beta,s)(y_\alpha,x_\beta) \qquad (4.3).$$

Thus, as in (iii) $(x_\alpha,y_\beta) = 0$.

(v) Finally $(x_\alpha,s') = (y_\alpha,s') = 0$, $s' \in S$, since for $s \in S$

$$(\alpha,s)(x_\alpha,s') = ([s,y_\alpha],s') \qquad (4.3)$$
$$= 0 \qquad (4.4).$$

Similarly for y_α.

(i) - (v) demonstrate that an orthonormal basis can be chosen of the required form. $\square$

We denote the vectors of Σ' by $\alpha_1, \ldots, \alpha_k$. The corresponding basis of R $\{e_1, \ldots, e_r, x_{\alpha_1}, y_{\alpha_1}, \ldots, x_{\alpha_k}, y_{\alpha_k}\}$ will be renamed

$\{e_1,\ldots, e_r, e_{\alpha_1}, e_{\alpha_1'} \ldots e_{\alpha_k}, e_{\alpha_k'}\}$. Some of the constants of structure take a particularly easy form with respect to this basis viz.

$$\text{(4.5)} \qquad \left\{\begin{aligned} C^{\ell}_{im} &= 0, \\ C^{\ell}_{i\alpha_p} &= -(\alpha_p, e_i)\delta_{\ell\alpha_p'}, \\ C^{\ell}_{i\alpha_p'} &= (\alpha_p, e_i)\delta_{\ell\alpha_p}, \end{aligned}\right\} \quad \text{where } 1 \leqslant i,m \leqslant r,\ 1 \leqslant p \leqslant k,\ 1 \leqslant \ell \leqslant r+2k.$$

Note that in general, when referred to an orthonormal basis the constants of structure satisfy

$$\text{(4.6)} \qquad C^{k}_{ij} = -C^{i}_{kj} = C^{j}_{ki}$$

due to (4.4) and the anti-symmetry in the lower indices.

§3 A necessary condition for $L2$ and the main theorem

We now make some observations about the curvature of a compact Lie group G equipped with a bi-invariant metric. We use the convention that the curvature tensor of G is given by

$$R_e(X,Y)Z = [[X,Y],Z], \qquad X,Y,Z \in T_eG.$$

(This is in fact equivalent to considering G to be the symmetric space $G \times G/\{(g,g)\,|\,g \in G\}$. See e.g. [CGW] Section 4. We do this to avoid awkward factors of $\frac{1}{4}$ in our calculations.) In a co-ordinate system this becomes

$$\text{(4.7)} \qquad R^{i}_{jk\ell} = C^{i}_{jp}C^{p}_{k\ell}.$$

Proposition 4.2 Suppose a compact Lie group G equipped with a bi-invariant metric is Einstein with $\rho = Kg$, then it is super-Einstein with $\dot{R} = K^2g$.

<u>Proof</u> We choose an orthonormal basis of the Lie algebra of G (not necessarily that of §2). Then in these co-ordinates

$$\rho_{ij}(e) = K\delta_{ij}. \tag{4.8}$$

As G is globally symmetric we need only prove that in these co-ordinates

$$\dot{R}_{ij}(e) = K^2\delta_{ij}.$$

From (4.7) we have

$$\rho_{ij}(e) = R^k{}_{ikj}(e) = C^k{}_{ip}C^p{}_{kj}.$$

Thus from (4.8) we deduce

$$C^k{}_{ip}C^p{}_{kj} = K\delta_{ij}. \tag{4.9}$$

On the other hand

$$\begin{aligned}
R_{iabc}R_{jabc}(e) &= C^i{}_{ap}C^p{}_{bc}C^j{}_{aq}C^q{}_{bc} && (4.7)\\
&= C^i{}_{ap}C^j{}_{aq}C^p{}_{bc}C^q{}_{bc} \\
&= C^i{}_{ap}C^j{}_{aq}C^b{}_{pc}C^c{}_{pq} && (4.6)\\
&= KC^i{}_{ap}C^j{}_{ap} && (4.9)\\
&= KC^a{}_{ip}C^p{}_{aj} && (4.6)\\
&= K^2\delta_{ij} && (4.9)
\end{aligned}$$

and the proposition is proved. (We can use (4.6) since the group metric is bi-invariant and thus is adjoint invariant on the Lie algebra). □

<u>Proposition 4.3</u> Suppose a compact Lie group of dimension n equipped with a bi-invariant metric satisfies $L1$ and $L2$ with constants K and H. Then

$$(4.10) \qquad K^2 = \frac{2(n+2)H}{5}.$$

<u>Proof</u> From Proposition 2.2 we see that if an Einstein space satisfies $L2$ then

$$|R|^2 = \frac{2n}{3}((n+2)H - K^2).$$

However from Proposition 4.2

$$|R|^2 = K^2 n$$

and the proposition follows. $\square$

We return now to the cases of interest, the classical compact simple Lie groups, and the co-ordinate system of §2. If in this co-ordinate system we denote $R_{1a1b}R_{1a1b}(e)$ by H, then a group cannot satisfy $L2$ unless (4.10) holds, where K can be calculated from $R_{1a1a}(e)$ for example. We now establish our main theorem by calculating K and H from the root systems and seeing if they satisfy (4.10). It is quickly seen from (4.5) that in these co-ordinates

$$\begin{aligned} R_{1a1b} &= C^h{}_{1a}C^h{}_{1b} = (\alpha_\ell, e_1)^2, \text{ if } a = b = \alpha_\ell \text{ or } \alpha_\ell', \\ &= 0 \text{ otherwise.} \end{aligned}$$

Hence

$$(4.11) \qquad K = R_{1a1a} = 2\sum_{\ell=1}^{k}(\alpha_\ell, e_1)^2,$$

$$(4.12) \qquad H = R_{1a1b}R_{1a1b} = 2\sum_{\ell=1}^{k}(\alpha_\ell, e_1)^4.$$

Theorem 4.4 Of the classical compact simple Lie groups only A_1, A_2 and $\mathcal{D}_4$ can satisfy Ledger's 2nd condition for harmonicity.

Proof We calculate K and H in each case from the root systems given in §1 using (4.11) and (4.12) and show that only the cases cited above can satisfy (4.10).

A_r. We choose an orthonormal basis $(e_1, \ldots, e_r)$ of S such that $e_1 = \frac{E_1 - E_2}{\sqrt{2}}$. We take $\Sigma' = (E_i - E_j,\ i < j,\ i,j = 1 \ldots r+1)$. Thus

$$K = 2\sum_{\alpha\in\Sigma'} (\alpha,e_1)^2 = 2(r+1), \quad H = 2\sum_{2\in\Sigma'} (\alpha,e_1)^4 = r+7.$$

Substituting these with $n = r^2 + 2r$ into (4.10) we find that the condition is satisfied if

$$4(r+1)^2 = \tfrac{2}{5}(r^2 + 2r + 2)(r + 7)$$

$$\text{or} \qquad 0 = (r - 1)(r - 2)(r + 2).$$

Thus only A_1, A_2 can satisfy $L2$.

B_r. We take the orthonormal basis of S to be $E_1, \ldots, E_r$ and $\Sigma' = (E_i,\ i = 1,\ldots,r,\ E_j \pm E_k,\ j < k\ j,k = 1,\ldots,r)$. Thus

$$K = 2\sum_{\alpha\in\Sigma'} (\alpha,E_1)^2 = 2(2r-1), \quad H = 2\sum_{\alpha\in\Sigma'} (\alpha,E_1)^4 = 2(2r-1).$$

Together with $\dim (B_r) = 2r^2 + r$ we find that (4.10) is satisfied if

$$4(2r - 1)^2 = \tfrac{4}{5}(2r - 1)(2r^2 + r + 2)$$

$$\text{or} \qquad 0 = (2r - 1)(2r - 7)(r - 1).$$

Thus no B_r, $r \geqslant 2$ can satisfy $L2$. (Note that $B_1 = A_1$ can.)

C_r. We take the orthonormal basis of S given by $E_1, \ldots, E_r$ and $\Sigma' = (2E_i,\ i = 1,\ldots,r,\ E_i \pm E_j\ i < j,\ i,j = 1,\ldots,r)$. Hence

$$K = 2\sum_{\alpha\in\Sigma'} (\alpha,E_1)^2 = 4(r+1), \quad H = 2\sum_{\alpha\in\Sigma'} (\alpha,E_1)^4 = 4(r+7).$$

Together with dim $C_r = 2r^2 + r$ we find that (4.10) is satisfied if

$$16(r+1)^2 = \tfrac{8}{5}(2r^2 + r + 2)(r + 7)$$

$$\text{or} \qquad 0 = (r-1)(2r-1)(r+4).$$

Thus no C_r, $r \geqslant 3$ can satisfy $L2$. (Note that $C_1 = A_1$ can.)

D_r. Once more we take the orthonormal basis of S given by $E_1, \ldots, E_r$ and $\Sigma' = (E_i \pm E_j,\ i < j,\ i,j = 1,\ldots,r)$. We find

$$K = 2\sum_{\alpha\in\Sigma'} (\alpha,E_1)^2 = 4(r-1), \quad H = 2\sum_{\alpha\in\Sigma'} (\alpha,E_1)^4 = 4(r-1).$$

Together with dim $D_r = 2r^2 - r$ we find that (4.10) is satisfied if

$$16(r-1)^2 = \tfrac{8}{5}(2r^2 - r + 2)(r - 1)$$

$$\text{or} \qquad 0 = (r-1)(2r-3)(r-4).$$

Thus only $\mathcal{D}_4$ can satisfy $L2$ and the theorem is proved. $\square$

Corollary 4.5 There exist super-Einstein manifolds which do not satisfy $L2$.

Proof Clear from Theorem 4.4 as all the classical compact simple Lie groups are super-Einstein (Proposition 4.2). $\square$

As to whether A_2 and $\mathcal{D}_4$ do satisfy $L2$, see §5. A_1 being harmonic satisfies all the Ledger conditions.

§4 The third condition

We can follow a similar procedure for $L3$. It has been remarked before that in using this condition one should be aware that it has been derived assuming the preceding Ledger conditions. However, in the case of a symmetric space, they can be considered as being independent, since the derivative of any condition is zero and hence cannot be used to simplify later conditions.

For a symmetric space $L3$ becomes

$$S(R_{iajb}R_{kb\ell c}R_{hcma}) = LS(\delta_{ij}\delta_{k\ell}\delta_{hm}) \qquad \text{in orthonormal co-ordinates.}$$

Summing over two pairs of indices, and assuming the space is super-Einstein with $\rho = Kg$, $\dot{R} = Sg$, we see that a necessary condition for $L3$ is

$$(4.13) \quad \tfrac{7}{2}\overset{o}{R}_{ij} - \underset{\text{-}}{\overset{o}{R}}_{ij} + (K^3 + \tfrac{9KS}{2})\delta_{ij} = (n^2 + 6n + 8)L\,\delta_{ij}$$

where use has been made of Proposition 2.10(iii).

We now prove a similar proposition to Proposition 4.2.

<u>Proposition 4.6</u> Suppose a compact Lie group equipped with a bi-invariant metric is Einstein with $\rho = Kg$, then the 2-tensors $\underset{\text{-}}{\overset{o}{R}}$, $\overset{o}{R}$ satisfy $\underset{\text{-}}{\overset{o}{R}} = \frac{K^3}{4}g$ and $\overset{o}{R} = K^3g$.

<u>Proof</u> Working in orthonormal co-ordinates:

$$\begin{aligned}
\underset{\text{-}}{\overset{o}{R}}_{ij}(e) &= R_{iakb}R_{jckd}R_{acbd}(e) \\
&= C^i{}_{a\ell}C^\ell{}_{kb}C^j{}_{cm}C^m{}_{kd}C^a{}_{cn}C^n{}_{bd} && (4.7) \\
&= C^i{}_{a\ell}C^a{}_{cn}C^j{}_{cm}C^m{}_{kd}C^\ell{}_{kb}C^n{}_{bd} \\
&= C^i{}_{\ell a}C^a{}_{cn}C^j{}_{cm}C^m{}_{kd}C^\ell{}_{kb}C^b{}_{nd} && (4.6) \\
&= R_{i\ell cn}R_{jckd}R_{\ell knd}(e) && (4.7) \\
&= \tfrac{1}{4}R_{ic\ell n}R_{jckd}R_{\ell nkd}(e) && \text{(Lemma 2.4(i))}
\end{aligned}$$

i.e. $$\underset{\text{-}}{\overset{o}{R}}_{ij} = \tfrac{1}{4}\overset{o}{R}_{ij}.$$

Further
$$\overset{o}{R}_{ij}(e) = R_{ikab}R_{jkcd}R_{abcd}(e) \tag{4.7}$$
$$= C^i{}_{k\ell}C^\ell{}_{ab}C^j{}_{km}C^m{}_{cd}C^a{}_{bn}C^n{}_{cd}$$
$$= C^i{}_{k\ell}C^\ell{}_{ab}C^j{}_{km}C^a{}_{bn}C^m{}_{cd}C^n{}_{cd}$$
$$= KC^i{}_{k\ell}C^\ell{}_{ab}C^j{}_{km}C^a{}_{bm} \qquad (4.6),(4.9)$$
$$= KC^i{}_{k\ell}C^j{}_{km}C^\ell{}_{ab}C^m{}_{ab}$$
$$= K^2C^i{}_{k\ell}C^j{}_{k\ell}$$
$$= K^3\delta_{ij}. \qquad \square$$

Proposition 4.7 Suppose a compact Lie group of dimension n equipped with a bi-invariant metric satisfies $L1$ and $L3$ with constants K and L. Then

$$35K^3 = 4(n^2 + 6n + 8)L. \tag{4.14}$$

Proof Combine Proposition 4.6 with equation (4.13), using Proposition 4.2 to assert that the group is super-Einstein and $S = K^2$. $\square$

If we are given a compact simple group and the co-ordinate system of §2 we can set $L = R_{1a1b}R_{1b1c}R_{1c1a}(e)$ then $L3$ cannot be satisfied unless (4.14) holds. Using (4.5) we see that

$$L = 2\sum_{\alpha\in\Sigma'}(\alpha,e_1)^6.$$

Theorem 4.8 The only classical compact simple Lie group which can satisfy $L3$ is A_1.

Proof All that is required is a calculation of K and L and to see whether they satisfy (4.14).

A_r. $K = 2(r+1)$, $L = \dfrac{r+31}{2}$, $n = r^2 + 2r$.

Hence (4.14) is satisfied if

$$35.8(r+1)^3 = 2((r^2+2r)^2 + 6(r^2+2r) + 8)(r+31)$$

$$\text{or} \qquad 0 = (r-1)(r^4 + 36r^3 + 30r^2 - 68r - 108),$$

and it is easily verified that there are no positive integral solutions apart from $r = 1$.

B_r. $K = 2(2r-1)$, $L = 2(2r-1)$, $n = 2r^2 + r$.

Hence (4.14) is satisfied if

$$35.8(2r-1)^3 = 8(2r-1)((2r^2+r)^2 + 6(2r^2+r) + 8)$$

$$\text{or} \qquad 0 = (2r-1)(r-1)(4r^3 + 8r^2 - 119r + 27),$$

and there are no positive integral solutions apart from $r = 1$.

C_r. $K = 4(r+1)$, $L = 4(r+31)$, $n = 2r^2 + r$.

Hence (4.14) is satisfied if

$$35.64(r+1)^3 = 16((2r^2+r)^2 + 6(2r^2+r) + 8)(r+31)$$

$$\text{or} \qquad 0 = (r-1)(4r^4 + 132r^3 + 129r^2 + 118r - 108),$$

and there are no positive integral solutions apart from $r = 1$.

D_r. $K = 4(r-1)$, $L = 4(r-1)$, $n = 2r^2 - r$.

Hence (4.14) is satisfied if

$$35.64(r-1)^3 = 16(r-1)((2r-r)^2 + 6(2r^2-r) + 81)$$

$$\text{or} \qquad 0 = (r-1)(4r^4 - 4r^3 - 127r^2 + 274r - 132),$$

and there are no positive integral solutions apart from $r = 1$. $\square$

§5 Postscript

The contents of the previous sections have, at the time of writing, been overtaken by events. An improved method of proving Theorems 4.4 and 4.8 has been found which, as well as giving necessary and sufficient conditions for a compact simple Lie group to satisfy any of Ledger's conditions, is easily generalised to the case of symmetric spaces. For a complete exposition of this method and the results obtained, the reader is referred to [CGW], which is still in preparation. Below we give a brief description of this approach together with some of the results.

Let G be a compact semi-simple Lie group and g its Lie algebra, as usual. Let S be a regular subalgebra and $x \in S$. Suppose we have a co-ordinate system as in §2 and $x = x^i e_i$. Then

$$\begin{aligned}\rho(x,x) &= \rho_{ij}x^i x^j \\ &= C^i{}_{k\ell}C^j{}_{k\ell}x^i x^j \\ &= 2\sum_{\alpha\in\Sigma'}(\alpha,e_i)(\alpha,e_j)x^i x^j, \qquad \text{using (4.5)}, \\ &= 2\sum_{\alpha\in\Sigma'}(\alpha,x)^2.\end{aligned}$$

Thus using the Einstein condition we have

$$(4.15) \qquad 2\sum_{\alpha\in\Sigma'}(\alpha,x)^2 = K(x,x)^2, \qquad x \in S.$$

$L2$ can be written as

$$(4.16) \qquad R_{iajb}R_{ka\ell b}x^i x^j x^k x^\ell = H(x,x)^4.$$

Again, using (4.5) we find that

$$(4.17) \qquad R_{iajb}R_{ka\ell b}x^i x^j x^k x^\ell = 2\sum_{\alpha\in\Sigma'}(\alpha,x)^4, \qquad x \in S.$$

Combining (4.15), (4.16) and (4.17) we see that a necessary condition for the group to satisfy $L2$ is that

$$(4.18) \qquad \sum_{\alpha \in \Sigma'} (\alpha,x)^4 = A\Big(\sum_{\alpha \in \Sigma'} (\alpha,x)^2\Big)^2, \qquad A \text{ constant, } x \in S.$$

On the other hand given that (4.18) holds for every x belonging to a regular subalgebra then certainly (4.16) holds for all regular elements of g. However the regular elements are dense in g (see [H] p. 297 where it is shown that the set of non-regular elements has dimension dim g − 3) and thus extending by continuity we see that M satisfies $L2$.

If we revert to describing roots as 1-forms on S, and $w_1, \ldots, w_\ell$ are the positive roots then a Lie group satisfies $L2$ iff

$$\sum_{i=1}^{\ell} w_i^4 = A\Big(\sum_{i=1}^{\ell} w_i^2\Big)^2, \quad A \text{ constant,}$$

where by w^k we mean the symmetrised k-fold tensor product of w with itself. The extension to Lk is obvious and the necessary and sufficient condition is that

$$\sum_{i=1}^{\ell} (w_i)^{2k} = A_k\Big(\sum_{i=1}^{\ell} w_i^2\Big)^k, \quad A_k \text{ constant.}$$

It is a straightforward computation to check Theorems 4.4 and 4.8 and also to show that A_2 and D_4 do satisfy $L2$. One can also show that no classical compact simple Lie group can satisfy Lk, $k \geqslant 3$, apart from A_1.

When applying this method to the exceptional Lie groups an important fact emerges. All of these groups satisfy $L2$, but E_8 is the first known example of a manifold which satisfies $L1$, $L2$, and $L3$ but which is not harmonic. Thus the first three Ledger conditions do not characterise harmonicity, but they still might imply local symmetry and so unfortunately the solution to the fundamental conjecture seems no nearer.

Appendix I A,C and Θ in Fermi Co-ordinates

Suppose we are given a q-dimensional submanifold P of M and let $p \in P$. Then given a co-ordinate system in a co-ordinate neighbourhood W of p in P, $(y_1, \ldots, y_q)$, and a set of orthonormal vector fields on W perpendicular to P, $(U_{q+1}, \ldots U_n)$, we can define Fermi co-ordinates $(x_1, \ldots, x_n)$ on a small enough neighbourhood of p in M as follows:

$$x_a(\exp_{\tilde{n}}(\sum_{j=q+1}^{n} t_j U_j(\tilde{n}))) = y_a(\tilde{n}), \qquad a = 1, \ldots, q,$$

$$x_i(\exp_{\tilde{n}}(\sum_{j=q+1}^{n} t_j U_j(\tilde{n}))) = t_i, \qquad i = q+1, \ldots, n.$$

Informally: to find the Fermi co-ordinates of a point m close to P, follow the shortest geodesic γ from m to P and let $\tilde{n}$ be the point where it intersects P. The first q Fermi co-ordinates of m are those of $\tilde{n}$ in the co-ordinates $(y_1, \ldots, y_q)$ and the last $n-q$ are those of the negative of the tangent vector of γ at $\tilde{n}$ referred to the basis $U_{q+1}(\tilde{n}), \ldots, U_n(\tilde{n})$.

Normal co-ordinates are the special case of when P is a single point.

We concentrate on a geodesic γ perpendicular to P emanating from $p \in P$. It is easy to see that the vector fields $s \sum_{i=q+1}^{n} c^i \frac{\partial}{\partial x^i}$ and $\sum_{i=1}^{q} d^i \frac{\partial}{\partial x^i}$, c^i, d^i constant are Jacobi vector fields along γ, the former for the same reason as the normal co-ordinate case and the latter because it is easily seen to give rise to a variation in geodesics. (Indeed, if we choose $p = \gamma(0)$, then γ has co-ordinates $(0, \ldots, 0, se^{q+1}, \ldots, se^n)$ for constant e's, and $(s,t) \to (td^1, \ldots, td^q; se^{q+1}, \ldots, se^n)$ is the required variation.)

As in the normal co-ordinate case we extend $\frac{\partial}{\partial y^1}(p), \ldots \frac{\partial}{\partial y^q}(p)$, $U_{q+1}(p), \ldots, U_n(p)$ along γ by parallel translation and denote these vector fields by $E_1(s), \ldots, E_n(s)$. Then we can define an $n \times n$ matrix A along γ by

$$
(I.1) \qquad \begin{cases} \dfrac{\partial}{\partial x^i} = A_{ji}E_j\,, & i = 1, \ldots, q\,, \\[2ex] s\dfrac{\partial}{\partial x^i} = A_{ji}E_j\,, & i = q+1, \ldots, n\,. \end{cases}
$$

Since the vector fields on the LHS are Jacobi along γ, A satisfies the same differential equation as in the normal co-ordinate case viz.

$$
(I.2) \qquad A'' + RA = 0,
$$

but with different initial conditions. Before writing these down we establish the convention of exhibiting $n \times n$ matrices as blocks of $q \times q$, $(n-q) \times q$, $q \times (n-q)$, $(n-q) \times (n-q)$ submatrices with the $q \times q$ submatrix in the top left corner.

It is clear from (I.1) that

$$
(I.3) \qquad A(0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.
$$

To find $A'(0)$ we note that

$$
\nabla_{\dot\gamma}\left(\frac{\partial}{\partial x^i}\right)(0) = A'_{ji}(0)E_j(0), \qquad i = 1, \ldots, q,
$$

$$
U_i(0) = \frac{\partial}{\partial x^i}(0) = \nabla_{\dot\gamma}\left(s\frac{\partial}{\partial x^i}\right)(0) = A'_{ji}(0)E_j(0),
$$

$$
i = q+1, \ldots, n.
$$

Thus, if $$\nabla_{\dot\gamma}\left(\frac{\partial}{\partial x^i}\right)(0) = \sum_{j=1}^{q} M_{ji}\frac{\partial}{\partial x^j}(0) + \sum_{j=q+1}^{n} N_{ji}U_j(0),$$

$$
i = 1, \ldots, q,
$$

we can write

$$\text{(I.4)} \qquad A'(0) = \begin{pmatrix} M & 0 \\ N & I \end{pmatrix}.$$

By taking inner products of (I.1) we see that in these co-ordinates the metric tensor has the form

$$\begin{aligned} g_{ij} &= (A^T A)_{ij}, && 1 \leqslant i,j \leqslant q, \\ g_{ij} &= s(A^T A)_{ij}, && 1 \leqslant i \leqslant q,\ q+1 \leqslant j \leqslant n, \\ g_{ij} &= s^2(A^T A)_{ij}, && q+1 \leqslant i,j \leqslant n. \end{aligned}$$

On taking determinants we see that

$$\text{(I.5)} \qquad \tilde{\theta}_p = s^{n-q}\det A,$$

where $\tilde{\theta}_p = \sqrt{\det g}$.

We wish to calculate the first few terms in the power series expansion of $\tilde{\theta}_p$ and so, as in Chapter 1, we consider the matrix $C = sA'A^{-1}$ along γ.

<u>Proposition I.1</u> The matrix $C = sA'A^{-1}$ exists for s small enough, and is independent of the matrix N occurring in (I.4).

<u>Proof</u> For small s we have, using (I.3) and (I.4)

$$A(s) = \begin{pmatrix} I + sM & 0 \\ sN & sI \end{pmatrix} + O(s^2),$$

so A is invertible for small $s > 0$, but not at $s = 0$. However $\lim_{s \to 0} sA^{-1}$ exists since

$$\det A(s) = s^{n-q} + O(s^{n-q+1}),$$

and the cofactor of any element of A is of the form $K s^{n-q-1} + O(s^{n-q})$.

Thus sA^{-1} is an analytic function of s, and C exists in some nhd of 0 and is analytic.

As in the normal co-ordinate case C satisfies the differential equation

$$sC' = -s^2R - C^2 + C$$

and is determined (in the analytic case) by the values of C(0) and C'(0). Hence if we show that C(0) and C'(0) are independent of N, the proposition is proved.

In order to find C(0) and C'(0) we calculate the first two terms of the power series expansion of sA^{-1}. Suppose

$$\text{(I.6)} \quad sA^{-1} = \begin{pmatrix} P_0 & Q_0 \\ S_0 & T_0 \end{pmatrix} + s\begin{pmatrix} P_1 & Q_1 \\ S_1 & T_1 \end{pmatrix} + s^2\begin{pmatrix} P_2 & Q_2 \\ S_2 & T_2 \end{pmatrix} + 0(s^3).$$

We have that $A'' = -RA$ along γ so writing

$$R(0) = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}$$

(the symmetry of R implies that $R_2 = R_3^T$ and R_1, R_4 are symmetric), we have that

$$A''(0) = -\begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -R_1 & 0 \\ -R_3 & 0 \end{pmatrix}.$$

Thus

$$\frac{1}{s}A(s) = \frac{1}{s}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} M & 0 \\ N & I \end{pmatrix} + \frac{s}{2}\begin{pmatrix} -R_1 & 0 \\ -R_3 & 0 \end{pmatrix} + 0(s^2).$$

Multiplying (I.6) by (I.7) we find that

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \frac{1}{s}\begin{pmatrix} P_0 & 0 \\ S_0 & 0 \end{pmatrix} + \begin{pmatrix} P_0M + Q_0N + P_1 & Q_0 \\ S_0M + T_0N + S_1 & T_0 \end{pmatrix}$$

$$+ s\begin{pmatrix} -\frac{P_0R_1}{2} - \frac{Q_0R_3}{2} + P_1M + Q_1N + P_2 & Q_1 \\ -\frac{S_0R_1}{2} - \frac{T_0R_3}{2} + S_1M + T_1N + S_2 & T_1 \end{pmatrix} + O(s^2).$$

From the first term we deduce that

$$P_0 = 0, \quad S_0 = 0.$$

From the constant term we have that

$$Q_0 = 0, \quad T_0 = I, \quad P_1 = I, \quad S_1 = -N.$$

The coefficient of s must be zero, so

$$Q_1 = 0, \quad T_1 = 0.$$

Hence

$$sA^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} + s\begin{pmatrix} I & 0 \\ -N & 0 \end{pmatrix} + O(s^2).$$

Finally

$$C(0) = A'(0)sA^{-1}(0) = \begin{pmatrix} M & 0 \\ N & I \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

$$\begin{aligned} C'(0) &= A''(0)sA^{-1}(0) + A'(0)(sA^{-1})'(0) \\ &= \begin{pmatrix} -R_1 & 0 \\ -R_3 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} M & 0 \\ N & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -N & 0 \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}. \quad \square \end{aligned}$$

<u>Remark</u> As $s(\log\tilde{\theta}_p)' = \mathrm{tr}C - (n-q)$,
we see that $\tilde{\theta}_p$ is independent of N.

Calculation then gives the following formulae for the first four derivatives of trC at 0:

$$\mathrm{tr}C'(0) = \mathrm{tr}M,$$

$$\mathrm{tr}C^{(2)}(0) = -\tfrac{2}{3}\mathrm{tr}R(0) - \tfrac{4}{3}\mathrm{tr}R_1(0) - 2\mathrm{tr}M^2,$$

$$\mathrm{tr}C^{(3)}(0) = -\tfrac{3}{2}\mathrm{tr}R'(0) - \tfrac{3}{2}\mathrm{tr}R_1'(0) + 6\mathrm{tr}MR_1(0) + 6\mathrm{tr}M^3,$$

$$\begin{aligned}\mathrm{tr}C^{(4)}(0) = {} & -\tfrac{12}{5}\mathrm{tr}R^{(2)}(0) - \tfrac{8}{5}\mathrm{tr}R_1{}^{(2)}(0) - \tfrac{8}{15}\mathrm{tr}R(0)R(0) - \tfrac{112}{15}\mathrm{tr}R_1(0)R_1(0) \\ & -\tfrac{22}{15}\mathrm{tr}R_2(0)R_3(0) - \tfrac{2}{3}\mathrm{tr}R_3(0)R_2(0) + 8\mathrm{tr}MR_1'(0) \\ & -32\mathrm{tr}M^2R_1(0) - 24\mathrm{tr}M^4.\end{aligned}$$

Appendix II Calculation of $(\tilde{\Delta}^4_m f)(m)$ in a super-Einstein space assuming f is harmonic on a nhd of m

We wish to calculate

$$(\tilde{\Delta}^4_m f)(m) = \frac{1}{8!}\sum_{\sigma\in S_8}(\nabla_{\sigma(i)\sigma(i)\sigma(j)\sigma(j)\sigma(k)\sigma(k)\sigma(\ell)\sigma(\ell)}f)(m).$$

Because dummy indices can be interchanged, there are only 105 distinct terms on the RHS. These fall into two classes, 15 begin with ∇_{ii}, the other 90 with two different indices. The first class can be ignored under our assumptions, for its sum is

$$\sum_{\rho\in S_6}(\nabla_{ii\rho(j)\rho(j)\rho(k)\rho(k)\rho(\ell)\rho(\ell)}f)(m),$$

which, from our calculation of C, we see to be zero (in fact each term is zero).

Our strategy will be to calculate the sum of the other 90 terms by first considering all sixth covariant derivatives of f with two free indices, say i and j, and then applying ∇_{ij}.

<u>Lemma II.1</u> Suppose T is a covariant 2-tensor then $\nabla_{ij}T_{ij} = \nabla_{ji}T_{ij}$.

<u>Proof</u> $\nabla_{ij}T_{ij} = \nabla_{ji}T_{ij} + R_{hiji}T_{hj} + R_{hjji}T_{ih}$ (Ricci identity)

$$= \nabla_{ji}T_{ij} + \rho_{jh}T_{hj} - \rho_{ih}T_{ih}$$

$$= \nabla_{ji}T_{ij}. \qquad \square$$

Using the lemma we see that we only need consider those sixth derivatives with i preceding j before applying ∇_{ij}. We write these 45 terms down in three classes according to the permutation of the dummy indices:

$$
\begin{array}{lllll}
A^{(1)}_{ij} = \nabla_{ijk\ell k\ell}f & & & & \\
A^{(2)}_{ij} = \nabla_{ikj\ell k\ell}f & A^{(6)}_{ij} = \nabla_{kij\ell k\ell}f & & & \\
A^{(3)}_{ij} = \nabla_{ik\ell jk\ell}f & A^{(7)}_{ij} = \nabla_{ki\ell jk\ell}f & A^{(10)}_{ij} = \nabla_{k\ell ijk\ell}f & & \\
A^{(4)}_{ij} = \nabla_{ik\ell kj\ell}f & A^{(8)}_{ij} = \nabla_{ki\ell kj\ell}f & A^{(11)}_{ij} = \nabla_{k\ell ikj\ell}f & A^{(13)}_{ij} = \nabla_{k\ell kij\ell}f & \\
A^{(5)}_{ij} = \nabla_{ik\ell k\ell j}f & A^{(9)}_{ij} = \nabla_{ki\ell k\ell j}f & A^{(12)}_{ij} = \nabla_{k\ell ik\ell j}f & A^{(14)}_{ij} = \nabla_{k\ell ki\ell j}f & A^{(15)}_{ij} = \nabla_{k\ell k\ell ij}f\,, \\
B^{(1)}_{ij} = \nabla_{ijk\ell\ell k}f & & & & \\
B^{(2)}_{ij} = \nabla_{ikj\ell\ell k}f & B^{(6)}_{ij} = \nabla_{kij\ell\ell k}f & & & \\
B^{(3)}_{ij} = \nabla_{ik\ell j\ell k}f & B^{(7)}_{ij} = \nabla_{ki\ell j\ell k}f & B^{(10)}_{ij} = \nabla_{k\ell ij\ell k}f & & \\
B^{(4)}_{ij} = \nabla_{ik\ell\ell jk}f & B^{(8)}_{ij} = \nabla_{ki\ell\ell jk}f & B^{(11)}_{ij} = \nabla_{k\ell i\ell jk}f & B^{(13)}_{ij} = \nabla_{k\ell\ell ijk}f & \\
B^{(5)}_{ij} = \nabla_{ik\ell\ell kj}f & B^{(9)}_{ij} = \nabla_{ki\ell\ell kj}f & B^{(12)}_{ij} = \nabla_{k\ell i\ell kj}f & B^{(14)}_{ij} = \nabla_{k\ell\ell ikj}f & B^{(15)}_{ij} = \nabla_{k\ell\ell kij}f\,, \\
C^{(1)}_{ij} = \nabla_{ijkk\ell\ell}f & & & & \\
C^{(2)}_{ij} = \nabla_{ikjk\ell\ell}f & C^{(6)}_{ij} = \nabla_{kijk\ell\ell}f & & & \\
C^{(3)}_{ij} = \nabla_{ikkj\ell\ell}f & C^{(7)}_{ij} = \nabla_{kikj\ell\ell}f & C^{(10)}_{ij} = \nabla_{kkij\ell\ell}f & & \\
C^{(4)}_{ij} = \nabla_{ikk\ell j\ell}f & C^{(8)}_{ij} = \nabla_{kik\ell j\ell}f & C^{(11)}_{ij} = \nabla_{kki\ell j\ell}f & C^{(13)}_{ij} = \nabla_{kk\ell ij\ell}f & \\
C^{(5)}_{ij} = \nabla_{ikk\ell\ell j}f & C^{(9)}_{ij} = \nabla_{kik\ell\ell j}f & C^{(12)}_{ij} = \nabla_{kki\ell\ell j}f & C^{(14)}_{ij} = \nabla_{kk\ell i\ell j}f & C^{(15)}_{ij} = \nabla_{kk\ell\ell ij}f\,.
\end{array}
$$

Thus under our assumptions

$$(\tilde{\Delta}^4_m f)(m) = \frac{2}{105}\Big(\nabla_{ij}\Big(\sum_{k=1}^{15} A^{(k)}_{ij} + B^{(k)}_{ij} + C^{(k)}_{ij}\Big)\Big)(m).$$

Note that

$$X^{(4)}_{ij} = X^{(5)}_{ij},\ X^{(8)}_{ij} = X^{(9)}_{ij},\ X^{(11)}_{ij} = X^{(12)}_{ij},\ X^{(13)}_{ij} = X^{(14)}_{ij},$$

where $X = A, B, C$, since the last two indices commute.

Extensive use is now made of the Ricci identity to express each term in terms of the one to the left of it (except, of course, those in the first column). The initial sums without the use of our assumptions are:

$$\begin{aligned}\sum_{k=1}^{15} A^{(k)}_{ij} = {} & A^{(1)}_{ij} + A^{(2)}_{ij} + 2A^{(3)}_{ij} + 7A^{(4)}_{ij} + 7(\rho_{ih}\nabla_{\ell h\ell j}f + R_{hjik}\nabla_{\ell k\ell h}f) \\ & + 5\nabla_k(R_{hki\ell}\nabla_{h\ell j}f + \rho_{ih}\nabla_{khj}f + R_{hji\ell}\nabla_{k\ell h}f) \\ & + 3\nabla_{k\ell}(R_{h\ell ik}\nabla_{hj}f + R_{hjik}\nabla_{\ell h}f) + \nabla_{k\ell k}(R_{hji\ell}\nabla_h f) \\ & + 2(R_{hjik}\nabla_{\ell hk\ell}f + \rho_{ih}\nabla_{\ell jh\ell}f) + \nabla_k(R_{hji\ell}\nabla_{hk\ell}f + R_{hki\ell}\nabla_{jh\ell}f \\ & \quad + \rho_{ih}\nabla_{jkh}f) + R_{hjik}\nabla_{h\ell k\ell}f + \rho_{ih}\nabla_{j\ell h\ell}f,\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{15} B^{(k)}_{ij} = {} & B^{(1)}_{ij} + B^{(2)}_{ij} + 2B^{(3)}_{ij} + 7B^{(4)}_{ij} + 7(\rho_{ih}\nabla_{\ell\ell hj}f + R_{hjik}\nabla_{\ell\ell kh}f) \\ & + 5\nabla_k(\rho_{ih}\nabla_{hkj}f + R_{hki\ell}\nabla_{\ell hj}f + R_{hji\ell}\nabla_{\ell kh}f) \\ & + 3\nabla_{k\ell}(R_{hki\ell}\nabla_{hj}f + R_{hji\ell}\nabla_{kh}f) + \nabla_{k\ell\ell}(R_{hjik}\nabla_h f) \\ & + 2(R_{hjik}\nabla_{\ell hk\ell}f + \rho_{ih}\nabla_{\ell jh\ell}f) + \nabla_k(R_{hji\ell}\nabla_{hk\ell}f + R_{hki\ell}\nabla_{jh\ell}f \\ & \quad + \rho_{ih}\nabla_{jkh}f) + R_{hjik}\nabla_{h\ell k\ell}f + \rho_{ih}\nabla_{j\ell h\ell}f,\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{15} C^{(k)}_{ij} = {} & C^{(1)}_{ij} + C^{(2)}_{ij} + 2C^{(3)}_{ij} + 7C^{(4)}_{ij} + 7(\rho_{ih}\nabla_{h\ell\ell j}f + R_{hjik}\nabla_{k\ell\ell h}f) \\ & + 5\nabla_k(R_{hjik}\nabla_{\ell\ell h}f) + 3\nabla_{kk}(\rho_{ih}\nabla_{hj}f + R_{hji\ell}\nabla_{\ell h}f) \\ & + \nabla_{kk\ell}(R_{hji\ell}\nabla_h f) + 2(\rho_{ih}\nabla_{hj\ell\ell}f + R_{hjik}\nabla_{kh\ell\ell}f) \\ & + \nabla_k(R_{hjik}\nabla_{h\ell\ell}f) + R_{hjik}\nabla_{hk\ell\ell}f + \rho_{ih}\nabla_{jh\ell\ell}f.\end{aligned}$$

We now make the assumptions of Einstein and $\Delta f = 0$ and recall formulae (3.17a):

$$\nabla_{jiik}f = K\nabla_{jk}f,$$
$$\nabla_{ijik}f = 2K\nabla_{jk}f + R_{ijk\ell}\nabla_{i\ell}f,$$
$$\nabla_{iijk}f = 2K\nabla_{jk}f + 2R_{ijk\ell}\nabla_{i\ell}f.$$

We have these formulae for the remaining A's, B's and C's:

$$A^{(1)}_{ij} = B^{(1)}_{ij} = \nabla_{ij}(\nabla_{k\ell k\ell}f) = 0,$$
$$A^{(2)}_{ij} = B^{(2)}_{ij} = \nabla_{ik}(\nabla_{j\ell k\ell}f) = K^2\nabla_{ij}f,$$
$$A^{(3)}_{ij} = B^{(3)}_{ij} = \nabla_{ik}(\nabla_{\ell jk\ell}f) = 2K^2\nabla_{ij}f + \nabla_i(R_{skj\ell}\nabla_{ks\ell}f),$$
$$A^{(4)}_{ij} = \nabla_{ik}(\nabla_{\ell k\ell j}f) = 2K^2\nabla_{ij}f + \nabla_i(R_{skj\ell}\nabla_{ks\ell}f),$$
$$B^{(4)}_{ij} = \nabla_{ik}(\nabla_{\ell\ell kj}f) = 2K^2\nabla_{ij}f + 2\nabla_i(R_{skj\ell}\nabla_{ks\ell}f),$$
$$C^{(1)}_{ij} = C^{(2)}_{ij} = C^{(3)}_{ij} = 0,$$
$$C^{(4)}_{ij} = \nabla_{ik}(\nabla_{k\ell\ell j}f) = K^2\nabla_{ij}f.$$

Our sums now simplify to

$$\begin{aligned}\sum_{k=1}^{15} A^{(k)}_{ij} = {} & 12\nabla_i(R_{skj\ell}\nabla_{ks\ell}f) + 5R_{hki\ell}\nabla_{kh\ell j}f + R_{hki\ell}\nabla_{kjh\ell}f \\ & + 3R_{hki\ell}\nabla_{\ell khj}f + 5\nabla_k R_{hji\ell}\nabla_{k\ell h}f + \nabla_k R_{hji\ell}\nabla_{hk\ell}f \\ & + 3\nabla_{k\ell}(R_{hjik}\nabla_{\ell h}f) + \nabla_{k\ell k}(R_{hji\ell}\nabla_h f) + 20R_{hjik}R_{\ell hks}\nabla_{\ell s}f \\ & + 51KR_{kij\ell}\nabla_{k\ell}f + 57K^2\nabla_{ij}f,\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{15} B^{(k)}_{ij} = {} & 21\nabla_i(R_{skj\ell}\nabla_{ks\ell}f) + 5R_{hki\ell}\nabla_{k\ell hj}f + 3R_{hki\ell}\nabla_{k\ell hj}f \\ & + R_{hki\ell}\nabla_{kjh\ell}f + 5\nabla_k R_{hji\ell}\nabla_{\ell kh}f + \nabla_k R_{hji\ell}\nabla_{hk\ell}f \\ & + 3\nabla_{k\ell}(R_{hjik}\nabla_{\ell h}f) + \nabla_{k\ell\ell}(R_{hjik}\nabla_h f) + 22R_{hjik}R_{\ell hks}\nabla_{\ell s}f \\ & + 53KR_{kij\ell}\nabla_{k\ell}f + 57K^2\nabla_{ij}f,\end{aligned}$$

$$\sum_{k=1}^{15} C_{ij}^{(k)} = 4\nabla_{kk}(R_{hji\ell}\nabla_{\ell h}f) + 22K^2\nabla_{ij}f + 18KR_{kij\ell}\nabla_{k\ell}f.$$

Summing,

$$\begin{aligned}\sum_{k=1}^{15} (A_{ij}^{(k)} + B_{ij}^{(k)} + C_{ij}^{(k)}) = {} & 33\nabla_i(R_{skj\ell}\nabla_{ks\ell}f) + 8R_{hki\ell}\nabla_{k\ell hj}f \\ & + 2R_{hki\ell}\nabla_{kjh\ell}f + 5R_{hki\ell}\nabla_{kh\ell j}f + 3R_{hki\ell}\nabla_{\ell khj}f \\ & + 5\nabla_k R_{hji\ell}\nabla_{k\ell h}f + 5\nabla_k R_{hji\ell}\nabla_{\ell kh}f \\ & + 2\nabla_k R_{hji\ell}\nabla_{hk\ell}f + 3\nabla_{k\ell}(R_{hji\ell}\nabla_{kh}f) \\ & + 3\nabla_{k\ell}(R_{hjik}\nabla_{\ell h}f) + 4\nabla_{kk}(R_{hji\ell}\nabla_{\ell h}f) \\ & + \nabla_{k\ell\ell}(R_{hjik}\nabla_h f) + \nabla_{k\ell k}(R_{hji\ell}\nabla_h f) \\ & + 42R_{hjik}R_{\ell hks}\nabla_{\ell s}f + 122KR_{kij\ell}\nabla_{k\ell}f \\ & + 126K^2\nabla_{ij}f.\end{aligned}$$

We simplify further by means of the various curvature identities, introducing the super-Einstein condition $\dot{R} = Sg$,

$$33\nabla_i(R_{skj\ell}\nabla_{ks\ell}f) = \tfrac{33}{2}S\nabla_{ij}f,$$

$$\begin{aligned}& 8R_{hki\ell}\nabla_{k\ell hj}f + 2R_{hki\ell}\nabla_{kjh\ell}f + 5R_{hki\ell}\nabla_{kh\ell j}f + 3R_{hki\ell}\nabla_{\ell khj}f \\ & \quad = 13R_{ikh\ell}R_{sjh\ell}\nabla_{ks}f + 2R_{hk\ell i}R_{hj\ell s}\nabla_{sk}f + \tfrac{15}{2}S\nabla_{ij}f \\ & \qquad + R_{hki\ell}\nabla_j R_{s\ell hk}\nabla_s f + \tfrac{11}{2}R_{hki\ell}\nabla_\ell R_{sjhk}\nabla_s f,\end{aligned}$$

$$\begin{aligned}& 5\nabla_k R_{hji\ell}\nabla_{\ell kh}f + 5\nabla_k R_{hji\ell}\nabla_{k\ell h}f + 2\nabla_k R_{hji\ell}\nabla_{hk\ell}f \\ & \quad = 12\nabla_k R_{hji\ell}\nabla_{hk\ell}f + \tfrac{5}{2}\nabla_j R_{hki\ell}R_{hks\ell}\nabla_s f + \tfrac{5}{2}\nabla_k R_{ijh\ell}R_{skh\ell}\nabla_s f,\end{aligned}$$

$$\begin{aligned}& 3\nabla_{k\ell}(R_{hjik}\nabla_{\ell h}f) + 3\nabla_{k\ell}(R_{hji\ell}\nabla_{kh}f) \\ & \quad = 6\nabla_k R_{hji\ell}\nabla_{hk\ell}f + 3\nabla_k R_{ijh\ell}R_{skh\ell}\nabla_s f - 3R_{sj\ell k}R_{sh\ell i}\nabla_{kh}f \\ & \qquad + \tfrac{3}{2}R_{kis\ell}R_{hjs\ell}\nabla_{kh}f + 6R_{hji\ell}R_{sh\ell k}\nabla_{ks}f + 12KR_{hji\ell}\nabla_{h\ell}f,\end{aligned}$$

$$\nabla_{\ell kk}(R_{hji\ell}\nabla_h f) + \nabla_{k\ell k}(R_{hji\ell}\nabla_h f) + 4\nabla_{kk}(R_{hji\ell}\nabla_{\ell h} f)$$

$$= 6\nabla_{kk}(R_{hji\ell}\nabla_{h\ell} f) - 3R_{sj\ell k}R_{sh\ell i}\nabla_{kh} f + \tfrac{3}{2}R_{kis\ell}R_{hjs\ell}\nabla_{kh} f$$
$$+ \tfrac{3}{2}R_{sjk\ell}\nabla_i R_{hsk\ell}\nabla_h f + \tfrac{3}{2}R_{sik\ell}\nabla_s R_{hjk\ell}\nabla_h f$$
$$+ 2KR_{hjik}\nabla_{kh} f.$$

Combining these and simplifying a little more,

$$\sum_{k=1}^{15}(A_{ij}^{(k)} + B_{ij}^{(k)} + C_{ij}^{(k)}) = -10R_{ikh\ell}R_{jsh\ell}\nabla_{ks} f - 4R_{hk\ell i}R_{hj\ell s}\nabla_{sk} f$$
$$- \tfrac{5}{2}\nabla_j R_{hki\ell}R_{hks\ell}\nabla_s f - \tfrac{3}{2}\nabla_i R_{hsk\ell}R_{jsk\ell}\nabla_h f$$
$$+ 4R_{hki\ell}\nabla_s R_{\ell jhk}\nabla_s f + \tfrac{11}{2}\nabla_k R_{ijh\ell}R_{skh\ell}\nabla_s f$$
$$+ 18\nabla_k R_{hji\ell}\nabla_{hk\ell} f + 6\nabla_{kk}(R_{hji\ell}\nabla_{h\ell} f)$$
$$+ 48R_{hji\ell}R_{sh\ell k}\nabla_{ks} f + 24S\nabla_{ij} f$$
$$+ 136KR_{hji\ell}\nabla_{h\ell} f + 126K^2\nabla_{ij} f.$$

It is now that we take ∇_{ij} of both sides. We recall that if a covariant 2-tensor T is anti-symmetric then $\nabla_{ij}T_{ij} = 0$ by the Lemma. Also $\nabla_{ij}(\nabla_{ij} f) = 0$ as the space is Einstein and $\nabla_{ij}(R_{hji\ell}\nabla_{h\ell} f) = 0$ as the space is super-Einstein. We have

$$\nabla_{ij}\Big(\sum_{k=1}^{15}(A_{ij}^{(k)} + B_{ij}^{(k)} + C_{ij}^{(k)})\Big) = -10\nabla_{ij}(R_{ikh\ell}R_{jsh\ell}\nabla_{ks} f)$$
$$- 4\nabla_{ij}(R_{hk\ell i}R_{hj\ell s}\nabla_{sk} f)$$
$$- \nabla_{ij}(\nabla_j R_{hki\ell}R_{hks\ell}\nabla_s f)$$
$$+ 18\nabla_{ij}(\nabla_k R_{hji\ell}\nabla_{hk\ell} f)$$
$$+ 6\nabla_{ijkk}(R_{hji\ell}\nabla_{h\ell} f)$$
$$+ 48\nabla_{ij}(R_{hji\ell}R_{sh\ell k}\nabla_{ks} f).$$

After much manipulation of the kind met in Chapter 2 we can write each term on the RHS in terms of known scalar functions.

$$\nabla_{ij}(R_{ikh\ell}R_{jsh\ell}\nabla_{ks}f) = 2\left\langle \overset{\underline{\circ}}{R},\nabla^2 f\right\rangle - \tfrac{1}{2}\left\langle \nabla R\otimes\nabla R,\nabla^2 f\right\rangle - \tfrac{1}{12}\left\langle \nabla(\overset{\circ}{R}),\nabla f\right\rangle$$
$$+ \left\langle T,\nabla f\right\rangle,$$

$$\nabla_{ij}(R_{hk\ell i}R_{hj\ell s}\nabla_{sk}f) = \left\langle R\circ R,\nabla^4 f\right\rangle - \tfrac{1}{4}\left\langle \overset{\circ}{R},\nabla^2 f\right\rangle + \tfrac{1}{4}\left\langle \nabla R\otimes\nabla R,\nabla f\right\rangle$$
$$+ \tfrac{1}{6}\left\langle \nabla(\overset{\underline{\circ}}{R}),\nabla f\right\rangle - \tfrac{1}{2}\left\langle T,\nabla f\right\rangle,$$

$$\nabla_{ij}(R_{hji\ell}R_{sh\ell k}\nabla_{ks}f) = \left\langle R\circ R,\nabla^4 f\right\rangle + \left\langle \overset{\underline{\circ}}{R},\nabla^2 f\right\rangle + \tfrac{1}{4}\left\langle \overset{\circ}{R},\nabla^2 f\right\rangle$$
$$- \tfrac{1}{4}\left\langle \nabla R\otimes\nabla R,\nabla^2 f\right\rangle + \tfrac{1}{2}\left\langle T,\nabla f\right\rangle,$$

$$\nabla_{ij}(\nabla_{kk}(R_{hji\ell}\nabla_{h\ell}f)) = 2\left\langle R\circ R,\nabla^4 f\right\rangle + 2\left\langle \overset{\underline{\circ}}{R},\nabla^2 f\right\rangle + \tfrac{1}{2}\left\langle \overset{\circ}{R},\nabla^2 f\right\rangle$$
$$- \tfrac{1}{2}\left\langle \nabla R\otimes\nabla R,\nabla^2 f\right\rangle + \left\langle T,\nabla f\right\rangle,$$

$$\nabla_{ij}(\nabla_j R_{hki\ell}R_{hks\ell}\nabla_s f) = -2\left\langle \overset{\underline{\circ}}{R},\nabla^2 f\right\rangle - \tfrac{1}{2}\left\langle \overset{\circ}{R},\nabla^2 f\right\rangle + \tfrac{1}{2}\left\langle \nabla R\otimes\nabla R,\nabla^2 f\right\rangle - \tfrac{1}{2}\left\langle T,\nabla f\right\rangle,$$

$$\nabla_{ij}(\nabla_k R_{hji\ell}\nabla_{hk\ell}f) = -\left\langle \overset{\underline{\circ}}{R},\nabla^2 f\right\rangle - \tfrac{1}{4}\left\langle \overset{\circ}{R},\nabla^2 f\right\rangle + \tfrac{1}{2}\left\langle \nabla R\otimes\nabla R,\nabla^2 f\right\rangle$$
$$+ \tfrac{1}{6}\left\langle \nabla(\overset{\underline{\circ}}{R}),\nabla f\right\rangle + \tfrac{1}{24}\left\langle \nabla(\overset{\circ}{R}),\nabla f\right\rangle - \tfrac{5}{4}\left\langle T,\nabla f\right\rangle.$$

Summing we obtain the result:

$$(\tilde{\Delta}^4_m f)(m) = \tfrac{2}{105}\Big(56\left\langle R\circ R,\nabla^4 f\right\rangle + 24\left\langle \overset{\underline{\circ}}{R},\nabla^2 f\right\rangle + 12\left\langle \overset{\circ}{R},\nabla^2 f\right\rangle$$
$$- \tfrac{5}{2}\left\langle \nabla R\otimes\nabla R,\nabla^2 f\right\rangle + \tfrac{19}{12}\left\langle \nabla(\overset{\circ}{R}),\nabla f\right\rangle + \tfrac{7}{3}\left\langle \nabla(\overset{\underline{\circ}}{R}),\nabla f\right\rangle\Big)(m).$$

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