The construction of monopoles in gauge theories

Athorne, Christopher

THE CONSTRUCTION OF MONOPOLES IN GAUGE THEORIES

by

Christopher Athorne, MA.

Thesis presented for the degree of Doctor of Philosophy in the University of Durham.

August, 1982.

ABSTRACT

We discuss the algebraic geometry of certain finite energy, static field configurations of an SU(N) Yang - Mills - Higgs theory in the limit of a massless scalar field and how it may be applied to their construction. We develop a systematic way of locating solutions with spherical or cylindrical symmetries. Consideration of the fields' regularity is extended from SU(2) to SU(3) as also is the description of the general monopole configuration.

Preface

The work contained in this thesis was carried out between October, 1979 and August, 1982 under the supervision of Dr E. Corrigan in the Dept. of Mathematics in the University of Durham. I am grateful for the support of an SERC research studentship during this period.

No claim of originality is made for chapters I and II. The approach and formulation, but only some of the results (as indexed) of chapters III and IV are original. This material is under submission for publication. Sections 5.2-4 are original and to be published in a forthcoming issue of Phys. Lett. B. Sections 3.1 and 5.1 are also due to E. Corrigan, D. B.Fairlie and P.Goddard. The work of chapter VI is new but leans heavily on previous work of E. Corrigan and P. Goddard. The results and conjecture of the appendix are original.

I should like to thank Drs D.B.Fairlie and L.M.Woodward for discussion and encouragement, R.S.Ward for comments on some of this work, but especially E.Corrigan for all these, his many suggestions and for help at crucial points.

Dept. of Mathematics,
University of Durham.

August, 1982.

Contents

Introduction

Magnetic monopoles first entered the world of physics as a means of explaining the quantisation of electric charge. Dirac[1] considered a quantum field theory of electric and magnetic charges whose consistency demanded a relation of the type $eg = 4\pi n\hbar$ where e and g are respectively electric and magnetic charges, n is an integer and $2\hbar\pi$ Plank's constant.

Such a monopole may be described by taking a semi-infinite magnet along the z-axis with one pole at the origin whose vector potential and field strength tensor will be non-singular except on the magnet itself. We can chop off all but the pole by adding to the field strength tensor the " Dirac string ",

$$\epsilon_{ijk}\, 4\pi g\, \delta(x,y)\, n_k \qquad , \quad n_k = (0,0,1) .$$

where g is the pole strength, to leave $F_{\mu\nu}$, so redefined, singular at the origin only. The quantization condition of Dirac is that this extra term in F will be unphysical. An electron will fail to notice it if, on completing a circuit of the string it undergoes no overall change in phase. The Bohm-Aharonov[2] experiment tells us that this phase factor,

$$\exp\left(\frac{ie}{\hbar} \oint A_\mu dx^\mu \right)$$

is the relevant physical quantity and also that it is unity if the magnetic flux in the string, g , obeys the said relation.

With the advent of spontaneously broken gauge theories it was found that electric charge quantization could be described by the breaking of a compact gauge symmetry group down to factors one of which was the $U(1)$ gauge group of electromagnetism. Such a theory however also admits solutions which are extended, non-singular, of finite energy and possess a magnetic charge as was pointed out independently by 'tHooft and Polyakov.[3]

Suppose we have a tube of confined magnetic flux Φ

entering the surface of a finite sphere. We may write the vector potential in the vacuum outside the tube as $A_\mu = \partial_\mu \alpha$ and since the flux is $\Phi = \oint_C A_\mu dx^\mu$ which is to be non-zero, α must be multivalued on C , a loop enclosing the flux tube.

If the gauge group is $U(1)$, α will represent some integral number of windings around the $U(1)$ manifold, a circle. If α is to be a continuous function on the sphere no deformation of C can alter this number of windings and so we cannot contract it elsewhere on the sphere without enclosing another tube of the same flux. Thus in a $U(1)$ gauge theory we are allowed strings of flux but no monopoles.

If however $U(1)$ is embedded in a gauge group with compact covering group, such as $SU(2)$ whose manifold is a 3-sphere (the covering group of $U(1)$ is the real line), such an α which winds around the $U(1)$ subgroup (for $SU(2)$, a circle on the sphere) can be continuously deformed to a constant. Hence the flux does not need to leave the sphere and there must be a monopole inside which can be non-singular because no Dirac string is needed.

For a long time only the simplest explicit solution and some generalizations were known. These were the charge 1 spherical Bogomolnyi'-Prasad-Sommerfield (BPS) monopole in an $SU(2)$ gauge group [4] and its trivial and non-trivial spherical embeddings of charges 1 to $n-1$ in $SU(n)$ [5].

At the same time work had been going on in another area of classical configurations in field theories : the instanton solutions of the pure Yang-Mills equations.[6] These are solutions of the self-dual and antiself-dual equations,

$$F_{\mu\nu} = \pm\tfrac{1}{2}\epsilon_{\mu\nu\sigma\tau} F_{\sigma\tau} = F^*_{\mu\nu}$$

By virtue of the Bianchi identity they are automatically solutions of the full equations of motion,

$$D_\mu F^{\mu\nu} = 0 \quad , \quad D_\mu F^{\mu\nu *} = 0$$

In particular the equivalence of these configurations and certain holomorphic vector bundles over a twistor space had been established.[7] These holomorphic bundles could be studied directly as in the Atiyah - Ward construction[8], or as sub-bundles embedded in a trivial bundle, the Atiyah - Drinfield- Hitchin - Manin construction.[9] In the former approach the bundle is described by a Hermitian patching function, holomorphic on the intersections of coordinate patches on twistor space from which the fields can be recovered via a splitting procedure (see chapter II). In the latter the rank n bundle is embedded in a rank $n+k$ bundle and the connection fields defined by projection back,

$$A_\mu = v^\dagger \partial_\mu v \ .$$

v is an $(n+k)\times n$ matrix of quaternion entries which lies in the null space of an $(n+k)\times(n+k)$ operator, linear in x, $\Delta^\dagger$. For self duality $\Delta^\dagger\Delta$ must be real and invertible. This second approach proved more fruitful for understanding the instanton.

In the limit of a massless, scalar field the first order, static monopole equations are equivalent to the static sector of the instanton equations and so the above constructions were implemented in the search for " multimonopoles ", higher charge solutions in $SU(2)$ and other gauge groups about whose existence there had been some doubt until Taubes[10] gave an existence proof. That multi-monopoles might have static configurations, in the BPS limit, was reasonable as Manton[11] had shown that to low order in perturbation theory the repulsion of widely separated and equal charge monopoles was exactly cancelled by the long range scalar force. O'Raifeartaigh et. al.[12] had also shown that higher charge solutions with cylindrical, but not spherical, symmetry might be expected.

Nahm[13] adapted the ADHM construction to the BPS monopole using differential operators following Manton[14] and Rossi[15] who thought of it as an infinite string of instantons. But generalization of this was to be difficult.

Real progress came from Ward[16] who provided a charge two axially symmetric $SU(2)$ monopole, but with only five degrees of freedom, using the Atiyah-Ward construction which seemed now to be tailored for monopoles rather than instantons. This was followed by axial solutions of arbitrary integer charge.[17,18] None of these had any internal degrees of freedom above the three translations and two rotations of the axis and appeared to be localised. Weinberg[19] had shown that the general solution of charge n should have $4n-1$ free parameters.

Independently a Hungarian group[20] managed to find these solutions as well but using the Bäcklund transformations of the Ernst equation, to which the Bogomolnyi' equations reduce under axial symmetry.[14]

The first " separated " multimonopole, a solution of seven degrees of freedom, was again found by Ward[21] and generalised by Corrigan and Goddard[22] to a charge n family with $4n-1$ degrees of freedom. That this family contains the general solution was then shown by Hitchin.[23] The parameter space is complicated and little understood.[24]

Meanwhile the Hungarian group recovered the same solutions by applying the inverse scattering method to the monopole equations.[25]

Finally Nahm[26] was able to transform the monopole problem for an arbitrary group into a simpler shape along the lines of the ADHM construction involving an ordinary linear differential operator

$$i\frac{d}{dz} + X + i\,\underline{\sigma}\cdot\underline{A}$$

with a potential satisfying the non-linear equation,

$$\frac{dA^1}{dz} = [A^2, A^3] + \sum \alpha_i^1\,\delta(z-z_i) \quad \text{and cyclic}$$

The fields are constructed as before. The description is rather implicit and so, whilst it obviates some of the technical difficulties of the Atiyah-Ward approach, it is by no means trivial to locate even known solutions within it.[27]

Some patching functions have been found for the Atiyah-Ward construction in $SU(3)$.[28,29,30]

For a complete review of the state of the art see ref.[31].

In this thesis we shall concentrate our efforts on the Atiyah - Ward construction in $SU(3)$, but with excursions to $SU(n)$.

We shall start by setting up the problem, discussing the boundary conditions on the Higgs field, deriving the Bogomolnyi' equations in the Prasad-Sommerfield limit and defining the magnetic charge. Then we shall review the twistor idea in two, four and three dimensions, representing the self-dual and Bogomolnyi' equations as integrability conditions and presenting Ward's ansätze and some generalizations for $SU(n)$. Working within these ansätze we go on, by a consideration of the effect on twistor variables of real space rotations, to identify families of cylindrically and spherically symmetric monopoles in $SU(2)$ and $SU(3)$ recovering old solutions and discovering some new ones. This further enables us to find an algorithm for classifying spherical monopoles in $SU(n)$.

Next we consider the problem of proving the regularity of the solutions on $\mathbb{R}^3$. In the absence of any general proof we proceed by example : a charge two cylindrically symmetric family in $SU(3)$.

Finally we consider the general multimonopole in $SU(3)$, discussing some eventualities not present in $SU(2)$. This work and its generalization to $SU(n)$ is still under study and a number of questions are not answered completely.

Within this construction most of the outstanding problems are technical and apparently very complex ; a general proof of the fields' regularity ; a more explicit understanding of the parameter space of the general configuration.

It is important to understand the connection with Nahm's modification of the ADHM construction. This may provide a means of finding criteria on γ that guarantee the regularity of the corresponding fields. In the ADHMN form this follows from the invertibility of a second order elliptic differential operator and in the Atiyah-Ward construction from that of a finite dimensional matrix defined on $\mathbb{R}^3$. The trace of the square of the Higgs field appears to have a similar expression (see appendix) in terms of the determinants of both these objects. One can conjecture that these determinants are equal. But Nahm's operator is of the form

$$\Delta^+\Delta = -\left(\frac{d}{dz}\right)^2 + A^+(z)A(z) + x^+x$$

on a set of compact intervals and so is positive definite.

Chapter I

The Static Monopole Equations

1.1 The Lagrangian

We consider a Lagrangian density[32] of the shape ,

$$\mathcal{L} = \mathrm{Tr}\,\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathrm{Tr}\,\tfrac{1}{2}D_\mu\phi . D^\mu\phi + \lambda V(\phi)$$

where the trace is over the indices of the Lie algebra of a gauge group $\mathcal{G}$. The scalar field ϕ belongs to the adjoint representation of this algebra and F is the curvature 2-form of a Lie algebra valued connection A ,

$$F = F_{\mu\nu}\,dx_\mu \wedge dx_\nu = dA + i\,A \wedge A \qquad , \; A = A_\mu dx_\mu$$

and is Hermitian if A is Hermitian.

D is the covariant derivative with respect to this connection,

$$D_\mu f = \partial_\mu f + i\,[A_\mu , f]$$

F and $D\phi$ both transform covariantly under the action of the gauge group ,

$$\phi \rightarrow g^{-1}\phi\, g$$
$$A \rightarrow g^{-1}A\, g - i\, g^{-1}dg$$

and so the trace operation ensures the gauge invariance of the first two terms of $\mathcal{L}$.

$\lambda V(\phi)$ is also gauge invariant, a potential function of the scalar field which we shall take to have the form,

$$\lambda V(\phi) = \tfrac{1}{4!}\lambda\,(\mathrm{Tr}\,\phi^2 - C)^2$$

where λ and C are constants.

We shall be seeking static solutions to the equations of motion of this Lagrangian and we can make the gauge choice $A_0 = 0$ [33,32] . Hence the energy density is :

$$\mathrm{Tr}\,\tfrac{1}{4}F_{ij}F_{ij} + \mathrm{Tr}\,\tfrac{1}{2}D_i\phi . D_i\phi + \lambda V(\phi)$$

1.2 Asymptotic symmetry breaking.

Note that the potential $V(\phi)$ has minima only for non-vanishing values of the scalar (Higgs) field. Since for a finite energy solution we require that V vanish asymptotically, we must make choices of such non-zero field values on the sphere at infinity and each such choice leaves V with some residual symmetry, the unbroken symmetry group $\mathcal{H}$ [34,32].

Suppose $\phi = \alpha + \varphi$ where $\varphi \to 0$ as $r \to \infty$ and α is a function of angle only such that $\mathrm{Tr}\,\alpha^2 = C$. Then,

$$V(\alpha+\varphi) = \frac{1}{4!}\left(2\,\mathrm{Tr}\,\alpha\varphi + \mathrm{Tr}\,\varphi^2\right)^2$$

The residual symmetry will be the group generated by those $h \in \mathcal{G}$ which leave $V(\alpha+\varphi)$ invariant under the action $\varphi \to h^{-1}\varphi h$, that is, those for which,

$$\mathrm{Tr}\,\alpha\varphi = \mathrm{Tr}\,\alpha h^{-1}\varphi h$$

or

$$[\alpha, h] = 0$$

So $h \in \mathcal{H}_\alpha \subset \mathcal{G}$, a subgroup commuting with α.

Now $\mathrm{Tr}\,\alpha^2 = C$ and so, locally on S^2,

$$\alpha(\theta,\psi) = g^{-1}(\theta,\psi)\,\alpha_0\,g(\theta,\psi)$$

and $\mathcal{H}_\alpha = g^{-1}(\theta,\psi)\mathcal{H}_{\alpha_0}g(\theta,\psi)$. We may choose α_0 to belong to the Cartan subalgebra of $\mathcal{G}$ and hence to be a traceless diagonal matrix,

$$\alpha_0 = \mathrm{diag}(\alpha_1, \alpha_2, \ldots\ldots \alpha_n)$$

where $\sum_1^n \alpha_i = 0$.

The unbroken symmetry will then depend on the groups of equal α_i's. If in general we have,

$$\alpha_0 = \bigoplus_{i=1}^{p} \alpha_i I_{n_i}$$

where $\sum_1^p n_i\alpha_i = 0$ and $\sum_1^p n_i = n$ then locally, as Lie algebras,

$$\mathcal{H}_{\alpha_0} \cong \left(\bigotimes_1^p U(n_i)\right)/U(1)$$

For example, $SU(3)$ may break to $U(1)\times U(1)$ or $U(2)$ and $SU(4)$ to $U(1)\times U(1)\times U(1)$, $U(2)\times U(1)$, $U(2)\times SU(2)$, or $U(3)$.

A further consequence of the non-zero vacuum expectation value of ϕ is the labelling of possible solutions by the elements of a homotopy group. The condition $\mathrm{Tr}\,\alpha^2 = C$ means that α is a map from S^2, the sphere at infinity, into $\mathcal{G}/\mathcal{H}$: that is, it is an element of $\pi_2(\mathcal{G}/\mathcal{H})$. If $\mathcal{G}$ is compact and simply connected then,[33,35]

$$\pi_2(\mathcal{G}/\mathcal{H}) \cong \pi_1(\mathcal{H})$$

If $\mathcal{H}$ has a $U(1)$ factor then $\pi_1(\mathcal{H})$ will have a component $\mathbb{Z}$. The solutions can be labelled by (at least one) integer which can be identified with a charge (or set of charges)[36]. In a physical theory the Higgs field will select a particular $U(1)$ subgroup and the associated integer be identified with the magnetic charge of electromagnetism.[34]

1.3 The Prasad-Sommerfield limit.

Because we are in a static theory the energy of a configuration will be the integral over $\mathbb{R}^3$ of the Lagrangian density, which we write,

$$\mathcal{E} = \int d^3x \left(\lambda V + \tfrac{1}{2}\mathrm{Tr}\left[(D\phi)^2 + (B_i)^2 \right] \right)$$

where $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$ is the magnetic field. Completing the square in the integrand we obtain :

$$\mathcal{E} = \int d^3x \left[\lambda V + \tfrac{1}{2}\mathrm{Tr}\left[D_i\phi \pm B_i \right]^2 \mp \mathrm{Tr}(B_i D_i \phi) \right)$$

Now ,

$$\begin{aligned} \mathrm{Tr}(B_i D_i\phi) &= \mathrm{Tr}\left(\partial_i(B_i\phi) - (\partial_i B_i)\phi + iB_i[A_i,\phi] \right) \\ &= \mathrm{Tr}\left(\partial_i(B_i\phi) - (D_i B_i)\phi \right) \end{aligned}$$

but $D_i B_i = \frac{1}{4}\epsilon_{ijk}[D_i[D_j,D_k]] = 0$, the Bianchi identity, and so,

$$\mathcal{E} = \int d^3x \left(\lambda V + \tfrac{1}{2}\mathrm{Tr}(D_i\phi \pm B_i)^2 \right) \mp \int_{S^2} dS_i\, \mathrm{Tr}(B_i\phi)$$

where the surface integral is taken over the sphere at infinity. This last term depends only upon the boundary conditions and hence on the relevant element of $\pi_1(\mathcal{H})$.

Now let us consider the Prasad-Sommerfield limit[4] $\lambda \to 0$, in which ϕ becomes massless and so mediates an infinite range scalar force, say,

$$\phi \sim \alpha - \frac{m}{r} + o\left(\frac{1}{r^2}\right)$$

Then within a given topological sector the energy will be minimised by either of the Bogomolnyi' equations,[37]

$$D_i \phi = \pm B_i$$

taking on the value,

$$\begin{aligned} \mathcal{E}_m &= \int_{S^2} dS_i \,\mathrm{Tr}\,(\phi D_i \phi) \\ &= \frac{1}{2} \int d^3x \, \nabla^2 \,\mathrm{Tr}\,\phi^2 \\ &> 0 \end{aligned}$$

in either case.

We may also interpret $\mathcal{E}_m = \int_{S^2} dS_i \,\mathrm{Tr}(\phi B_i)$ as the total magnetic flux in the direction specified by the symmetry breaking. For a finite energy monopole B_i is $o\left(\frac{1}{r^2}\right)$ and so $D_i\phi \to 0$. The Higgs field is asymptotically covariantly constant and on a coordinate patch of the sphere at infinity will be a gauge transformation of, say, its value on the z-axis,

$$\phi_0 \sim \alpha_0 - \frac{m_0}{r}$$

We shall assume that $[A_i, \phi] = o\left(\frac{1}{r^2}\right)$, asymptotically, so that,

$$B_i \sim m_0 \frac{r_i}{r^3}$$

Then $[B_i, \phi] = o\left(\frac{1}{r^3}\right)$ and m_0 and α_0 commute. So we choose them both to lie in the Cartan subalgebra,

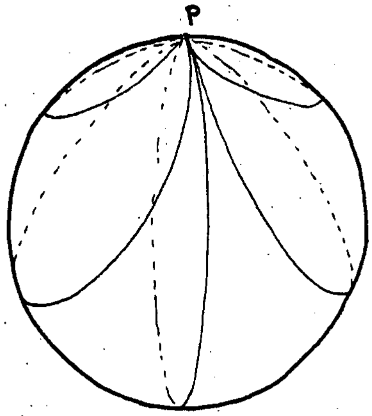
$$m_0 = \underline{m} \cdot \underline{H} \quad , \quad \alpha_0 = \underline{\alpha} \cdot \underline{H}$$

Now the minimised energy, or the flux, is,

$$\begin{aligned} \mathcal{E}_m &= \int_{S^2} dS_i \,\mathrm{Tr}\left(g^{-1} \left(\underline{\alpha} - \frac{\underline{m}}{r}\right) \cdot \underline{H}\, g . g^{-1} \frac{r_i}{r^3} \underline{m} \cdot \underline{H}\, g \right) \\ &= 4\pi \, \alpha_i K_{ij} m_j \end{aligned}$$

where K_{ij} is the Cartan matrix, $\mathrm{Tr}(H_i H_j)$ [38].

From the long range behaviour of the magnetic field a generalisation of the Dirac quantization condition can be deduced. [39,36] The Higgs field is covariantly constant and is asymptotically an element of $\pi_2(G/H)$. Since $\pi_2(G/H) \cong \pi_1(H)$ we seek a map from S^1 to H. To construct it we consider a series of closed loops, with one fixed point, which pass around the sphere.



$h \in H$ is defined by integrating the equation of covariant constancy (which depends only on B_i) along each loop. As the loop encompasses the sphere, so h traces out a path in H and this path is closed and so will be an element of $\pi_1(H)$ provided,

$$\exp(4\pi i\, \underline{m}\cdot\underline{H}) = 1$$

(We have taken $e = 1$; otherwise the covariant derivative is simply $\partial_i + ieA_i$ and a factor e appears in the exponent of the quantization condition.)

The quantization condition can be solved to see that $\underline{m}$ belongs to the weight lattice of a Lie group dual to H [36,40] and so the magnetic flux is restricted to discrete values.

Chapter II

The Atiyah-Ward Construction [7,8,23]

2.1 The Laplacian in two dimensions.

The motivation behind the twistor approach[41] to solving differential equations is to replace a (complicated) differential equation on one manifold by a holomorphic structure on another (complex) manifold in such a way that the holomorphic structure absorbs those equations. A simple instance is the following. We consider the equation $\nabla^2\phi=0$ on $\mathbb{R}^2$ and wish to replace it by an analytic object. We note that any analytic function of a complex variable satisfies $\nabla^2\phi=0$ by virtue of the Cauchy-Riemann equations. So we identify $\mathbb{R}^2$ with a copy of the complex plane $\mathbb{C}$ by choosing unit orthogonal vectors $\underline{e}_x, \underline{e}_y$ in $\mathbb{R}^2$ and making the correspondence,

$$v_x\underline{e}_x + v_y\underline{e}_y \longleftrightarrow v_x + i v_y$$

between vectors on $\mathbb{R}^2$ and complex numbers. The usual real structure is defined by $i \rightarrow -i$ or $\sigma(\underline{e}_x,\underline{e}_y)=(\underline{e}_x,-\underline{e}_y)$. In particular the gradient operator $\underline{\nabla} \rightarrow \partial_x + i\partial_y = 2\partial_{\bar{z}}$ where $\bar{z}=x-iy$. Laplace's equation becomes $\partial\bar{\partial}\phi=0$ and so $\phi=f(z)+g(\bar{z})$, for both f and g arbitrary analytic functions. They will be fixed by constraints of reality and the boundary conditions.

2.2 The Laplacian in four dimensions.

Although we have a choice of complex structure for $\mathbb{R}^2$ this amounts only to a change $z \rightarrow \eta z$ for $\eta\bar{\eta}=1$, or a change in orientation $z \rightarrow \bar{z}$ and these do not alter the form of ϕ. In more than two dimensions however the situation is complicated by the extra variety of complex structures available. Were we to solve the same equation on $\mathbb{R}^4$ there are many ways we can make an identification with $\mathbb{C}^2$ by choices of orthogonal planes.

In fact the general solution is [42,41]

$$\phi = \frac{1}{2\pi i}\oint_C d\zeta\, f(\zeta,\mu,\nu)$$

where $i\mu = z + \bar{X}\zeta$, $i\nu = \bar{z} - \frac{X}{\zeta}$ and $z = t - iz$, $X = y + ix$. ζ is a complex variable and C a contour which passes around an open annulus in the complex plane which remains singularity free for all z and X. We can write μ and $\zeta\nu$ as

$$\mu = \underline{x}\cdot\underline{e}_1 + i\,\underline{x}\cdot\underline{e}_2 \quad , \qquad \zeta\nu = \underline{x}\cdot\underline{e}_3 + i\,\underline{x}\cdot\underline{e}_4$$

where $\underline{e}_1 = (0,-1,-\alpha,\beta)$, $\underline{e}_2 = (-1,0,-\beta,-\alpha)$,
$\underline{e}_3 = (\beta,\alpha,-1,0)$, $\underline{e}_4 = (-\alpha,\beta,0,1)$,

are an orthogonal basis of $\mathbb{R}^4$ depending upon $\zeta = \alpha + i\beta$. μ and $\zeta\nu$ are then the variables on $\mathbb{C}^2$. ζ parametrizes the complex structures and must be eliminated by the contour integration. We note that f is defined only up to the addition of any function analytic within and on the contour.

The relations between μ, ν and ζ can also be written

$$\begin{pmatrix} z & \bar{X} \\ -X & \bar{z} \end{pmatrix}\begin{pmatrix} 1 \\ \zeta \end{pmatrix} = i\begin{pmatrix} \mu \\ \zeta\nu \end{pmatrix} \qquad (*)$$

in which case $q = \begin{pmatrix} z & \bar{X} \\ -X & \bar{z} \end{pmatrix}$ is the quaternion representation of $\underline{x} = (t,x,y,z)$:

$$q = t - i\,\underline{\sigma}\cdot\underline{x}$$
$$= t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - iz\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - ix\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - iy\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Now think of of $\mathbb{R}^4$ as a real slice of $\mathbb{C}^4$ by allowing z, $\bar{z}$, X and $\bar{X}$ to become independent complex variables, defining a real structure by $(\zeta,\mu,\nu) \to (\frac{-1}{\bar{\zeta}}, -\bar{\nu}, -\bar{\mu})$ or $q \to \tilde{q} = \varepsilon^T \bar{q}\,\varepsilon$ for $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\bar{q} = (q^+)^T$. Then $\tilde{q} = q$ describes the slice $\mathbb{R}^4$. For specified values of ζ, μ and ν, $(*)$ is then the equation of a plane in $\mathbb{C}^4$. It is null because $\det(dq) = dz.d\bar{z} + dx.d\bar{x} = 0$ for non-trivial $(1\ \zeta)$. It is, in fact, antiself-dual,

$$z_\mu w_\nu - z_\nu w_\mu = -\tfrac{1}{2}\epsilon_{\mu\nu\sigma\tau}(z_\sigma w_\tau - z_\tau w_\sigma)$$

for any two such null separations z and w. Because they are null their quaternion representations factor into tensor products of two-spinors: $z_{AB'} = z_A\zeta_{B'}$, $w_{AB'} = w_A\zeta_{B'}$, $\zeta_{B'} = (1\ \zeta)$ and so are easily seen to satisfy antiself-duality which, in the same notation, can be written[41]:

$$z_{AA'}w_{BB'} - z_{BB'}w_{AA'} = z_{AB'}w_{BA'} - z_{BA'}w_{AB'}.$$

The pairs $(1,\zeta)$, $(\mu,\zeta\nu)$ are defined by such a plane only up to a complex factor. So the antiself-dual planes in $\mathbb{C}^4$ correspond to the points of $\mathbb{C}^4$ up to such an equivalence, that is, to the points of $\mathbb{P}_3(\mathbb{C})$ - at least up to a copy of $\mathbb{P}_1(\mathbb{C})$ for the planes at infinity. $\mathbb{P}_3(\mathbb{C})$ is then the (complex projective) twistor space, holomorphic functions on which lead, via contour integration, to solutions of the four-dimensional Laplacian.

2.3 The Laplacian in three dimensions.

Clearly in three dimensions we cannot apply quite the same procedure as above but we might modify it in one of two ways. Either we factor out one of the $\mathbb{R}^4$ coordinates or we replace $\mathbb{R}^3$ by a four dimensional manifold and put a complex structure on this. In fact the first approach leads to the second in the following way.

If ϕ is to be independent of, say, t we must have,

$$\phi = \frac{1}{2\pi i}\oint_C d\zeta\, f(\gamma,\zeta) \qquad \text{where } \gamma = \mu - \nu \ .$$

ζ belongs to $\mathbb{P}_1(\mathbb{C})$, the Riemann sphere, and $\zeta\gamma$ is a quadratic form. Specific values of ζ and γ give rise to a straight line in $\mathbb{R}^3$ defined by the real and imaginary parts of the equation

$$\zeta\gamma = X\zeta^2 - 2z\zeta - \bar{X}$$

namely : $$\underline{x} = \underline{u}t + \underline{v}$$

where $$\underline{u} = \frac{1}{1+\zeta\bar{\zeta}}\left(\zeta+\bar{\zeta}, i(\zeta-\bar{\zeta}), \zeta\bar{\zeta}-1\right) \quad , \quad \underline{u}\cdot\underline{u} = 1$$

$$\underline{v} = \frac{\zeta\gamma}{(1+\zeta\bar{\zeta})^2}\left(\frac{\bar{\zeta}^2-1}{2}, \frac{\bar{\zeta}^2+1}{2i}, -\bar{\zeta}\right) + \frac{\bar{\zeta}\bar{\gamma}}{(1+\zeta\bar{\zeta})^2}\left(\frac{\zeta^2-1}{2}, -\frac{\zeta^2+1}{2}, -\zeta\right) , \quad \underline{u}\cdot\underline{v} = 0$$

Likewise every such straight line gives rise to a pair (ζ,γ) simply by writing $\underline{u}$ in stereographic projection

$$\underline{u} = \left(\frac{\zeta+\bar{\zeta}}{1+\zeta\bar{\zeta}}, i\frac{\zeta-\bar{\zeta}}{1+\zeta\bar{\zeta}}, \frac{\zeta\bar{\zeta}-1}{1+\zeta\bar{\zeta}}\right)$$

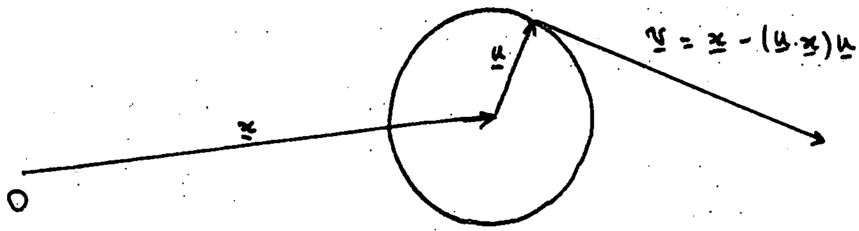
then choosing a complex structure on each tangent space to the sphere and making the correspondence,

$$\underline{v} \to \underline{v} - (\underline{u}\cdot\underline{v})\underline{u} = v_\theta \underline{e}_\theta + v_\varphi \underline{e}_\varphi \longrightarrow v_\theta - i v_\varphi$$

where, for instance, $\underline{e}_\theta = \frac{\partial}{\partial\theta}$, $\underline{e}_\varphi = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}$ are unit tangential vectors to S^2. Then

$$v_\theta - i v_\varphi = [(\underline{e}_\theta + i\underline{e}_\varphi)\cdot\underline{v}](\underline{e}_\theta - i\underline{e}_\varphi)$$
$$= (v_x \zeta^2 - 2 v_z \zeta - v_{\bar{x}})\frac{\partial}{\partial\zeta}$$

using the fact that $\zeta = \cot\frac{\theta}{2}\cdot e^{-i\varphi}$. In particular the point $\underline{x} \in \mathbb{R}^3$ defines such a vector field on S^2 by virtue of the pencil of straight lines passing through it.



At each point $\underline{u}$ of S^2 there is a $\underline{v}$, defined by projection onto the tangent plane at $\underline{u}$, which gives rise to the section

$$(x\zeta^2 - 2z\zeta - \bar{x})\frac{\partial}{\partial\zeta}$$

of the tangent bundle to the sphere looked at as the holomorphic tangent bundle of $\mathbb{P}_1(\mathbb{C})$. The real structure on $\mathbb{P}_3(\mathbb{C})$ then becomes on $T\mathbb{P}_1(\mathbb{C}) : (\zeta,\gamma) \to (-\frac{1}{\bar{\zeta}}, \bar{\gamma})$ or $(\underline{u},\underline{v}) \to (-\underline{u},\underline{v})$. It simply reverses the direction of the straight line. In general a section $(a\zeta^2+b\zeta+c)\frac{\partial}{\partial\zeta}$ will be real if

$$a\zeta^2 + b\zeta + c = -(\bar{a}/\zeta^2 - \bar{b}/\zeta + \bar{c})\zeta^2$$

where the minus sign comes from the derivative of $1/\zeta$. So $a = -\bar{c}$ and $b = \bar{b}$ giving three degrees of freedom. Clearly $\zeta\gamma$ is real and hence the twistor correspondence for $\mathbb{R}^3$ is between straight lines in $\mathbb{R}^3$ and points of $T\mathbb{P}_1(\mathbb{C})$, and between points of $\mathbb{R}^3$ and real sections of $T\mathbb{P}_1(\mathbb{C})$.

TS^2 is the four dimensional Riemannian manifold with which we replace $\mathbb{R}^3$ and which we endow with complex and real structures.

2.4 The Self-dual equations.

These twistor correspondences for $\mathbb{R}^4$ and $\mathbb{R}^3$ also turn out to be useful in solving some of the non-linear equations arising in

Yang-Mills Theory, notably the (anti)self-dual equations,[7]

$$F_{\mu\nu} = \pm\tfrac{1}{2}\epsilon_{\mu\nu\sigma\tau}F_{\sigma\tau}$$

and the Bogomolnyi' equations,[23]

$$D_i\phi = \tfrac{1}{2}\epsilon_{ijk}F_{jk}$$

In the first instance, consider the operators :

$$\partial_1 = \tfrac{1}{2}(\partial_z + \tfrac{1}{\zeta}\partial_{\bar{x}}) \qquad \bar{\partial}_1 = \tfrac{1}{2}(\partial_{\bar{z}} + \zeta\partial_x)$$

$$\partial_2 = \tfrac{1}{2}(\partial_{\bar{z}} - \zeta\partial_x) \qquad \bar{\partial}_2 = \tfrac{1}{2}(\partial_z - \tfrac{1}{\zeta}\partial_{\bar{x}})$$

So $\Box^2 = 2(\partial_1\bar{\partial}_1 + \partial_2\bar{\partial}_2)$ and $\bar{\partial}_1 f(\zeta,\mu,\nu) = \bar{\partial}_2 f(\zeta,\mu,\nu) = 0$. They are the (anti)holomorphic derivatives on $\mathbb{C}^4$. Now we modify them to act on the principle fibre bundle of a Yang-Mills theory by adding a connection term :

$$\nabla_1 = \partial_1 + \tfrac{i}{2}(A_z + \tfrac{1}{\zeta}A_{\bar{x}}) \;,\quad \bar{\nabla}_1 = \bar{\partial}_1 + \tfrac{i}{2}(A_{\bar{z}} + \zeta A_x) \;,$$

$$\nabla_2 = \partial_2 + \tfrac{i}{2}(A_{\bar{z}} - \zeta A_x) \;,\quad \bar{\nabla}_2 = \bar{\partial}_2 + \tfrac{i}{2}(A_z - \tfrac{1}{\zeta}A_{\bar{x}}) \,.$$

The pair of linear equations

$$\bar{\nabla}_1 f = 0, \quad \bar{\nabla}_2 f = 0$$

where f is now an n-component complex vector , are compatible (integrable) provided,

$$[\bar{\nabla}_2, \bar{\nabla}_1] = [\nabla_z - \tfrac{1}{\zeta}\nabla_{\bar{x}}, \nabla_{\bar{z}} + \zeta\nabla_x] = 0$$

that is, provided Yang's equations[43] are satisfied :

$$F_{z\bar{z}} + F_{x\bar{x}} = 0, \qquad F_{zx} = F_{\bar{z}\bar{x}} = 0 \quad .$$

The integrability condition is equivalent to the vanishing of a curvature and since $\bar{\nabla}_1$ and $\bar{\nabla}_2$ are vectors in an antiself-dual plane it is the self dual part of $F_{\mu\nu}$ which vanishes.

Hence the self-dual equations are naturally coded into the integrability of a linear system. These linear equations are the equations of parallel transport over the antiself-dual plane. If we choose a point and a corresponding value for f on each plane, this will specify the vector field on the whole plane. So for each plane, each (ζ,μ,ν) , we have a vector and hence a vector field over $P_3(\mathbb{C})$, a section of a rank n vector bundle. It is then the theorem of Atiyah and Ward[8] that it is the holomorphic vector bundles of rank n over $P_3(\mathbb{C})$, with certain other properties, which correspond to those Yang-Mills connections which have self-dual curvature. Such a bundle will be defined by a transition matrix g such that $\bar{\partial}_1 g = 0$ and $\bar{\partial}_2 g = 0$.

2.5 <u>The Bogomolnyi' equations.</u>

In the second instance, the monopole equations, let us define the operators,

$$D = \underline{u}\cdot\underline{\nabla} - i\phi \qquad , \quad \bar{\partial} = \nabla_\theta + i\nabla_\varphi$$

Then the equation of parallel transport $Ds = 0$ defines on each straight line a vector field and as above we choose a point (holomorphically in the coordinates of $TP_1(\mathbb{C})$) and a value for the field thereat. So at each pair (ζ, γ) we have a vector and hence a section of the rank n vector bundle over $TP_1(\mathbb{C})$. Again every such holomorphic vector bundle, with certain other properties, corresponds to a solution of the Bogomolnyi' equations, without boundary conditions. The equations themselves arise in the following way.

$\bar{\partial}$ takes $(0,p)$-forms to $(0,p+1)$-forms on the bundle defined by $Ds=0$. If $\bar{\partial}s$ is to be a form on this bundle , and not on some larger space, then $D\bar{\partial}s$ must vanish. This will be the case if $[D,\bar{\partial}]=0$. But,

$$[\nabla_t - i\phi, \nabla_\theta + i\nabla_\varphi] = F_{t\theta} - \nabla_\varphi\phi + iF_{t\varphi} + i\nabla_\theta\phi$$

and so, taking real and imaginary parts,

$$F_{t\theta} = \nabla_\varphi\phi \quad , \quad F_{\varphi t} = \nabla_\theta\phi \, .$$

For all directions $\underline{u}$ these two equations give the full set with the correct orientation of $\mathbb{R}^3$.

As we have seen, the two twistor spaces used above are related by factoring out a real (time) translation in $\mathbb{R}^4$. Likewise if we set $\frac{\partial}{\partial t} = 0$ in $\bar{\nabla}_1$ and $\bar{\nabla}_2$ their commutation relation reduces to the three equations :

$$[\nabla_x, -i\nabla_z + A_0] = [i\nabla_z + A_0, \nabla_y] = 0$$
$$2[i\nabla_z, A_0] = [\nabla_x, \nabla_y] = 0$$

For the identification $A_0 \equiv \phi$ these are the Bogomolnyi' equations again. These monopoles " belong " to the static sector of the self-dual equations.

In this way the problem of constructing monopoles is reduced to one of finding bundles, that is , of specifying patching functions, that give rise to fields which satisfy the boundary conditions, are Hermitian and regular. So let us see how we may recover the fields from the bûndle.

2.6 The Higgs and Gauge fields. [8]

Let us take the self-dual construction. We can cover $P_3(\mathbb{C})/P_1(\mathbb{C})$ with a pair of coordinate patches U_0 , on which $|\zeta| < 1+\varepsilon$, and U_∞ , on which $|\zeta| > 1-\varepsilon$. The patching function then specifies how the vector space at each point of $U_0 \cap U_\infty$ transforms under the change of coordinates $\zeta \to \frac{1}{\zeta}$. At each point of U_0 we choose a point in the respective plane, x_0 , a holomorphic function of the coordinates, and specify, again holomorphically, a vector ψ_0 at this point. Similarly we select an x_∞ on the same plane, holomorphically in the U_∞ coordinates and specify a ψ_∞ . Then the patching function g will satisfy, $\psi_\infty = g(x_\infty, x_0)\psi_0$. But to each such plane in $\mathbb{C}^4$ there is only one real point, x , and the equations of parallel transport tell us that, $\psi(x) = g(x, x_0)\psi_0$, $\psi(x) = g(x, x_\infty)\psi_\infty$ where $\bar{\nabla}_{1,2}\, g(x, x_0) = \bar{\nabla}_{1,2}\, g(x, x_\infty) = 0$. Hence the function g " splits " ,

$$g(x_\infty, x_0) = g^{-1}(x, x_\infty)\, g(x, x_0) = h\, k^{-1} \qquad (2.1)$$

into factors holomorphic on each patch. Further, from the covariant constancy of the factors h^{-1}, k^{-1} we can recover the Higgs and gauge fields (remembering that $\frac{\partial}{\partial t} = 0$ and identifying A_0 with ϕ).

$$(-i\partial_{\bar{z}} + \zeta\partial_x + i\phi + A_{\bar{z}} + i\zeta A_x)\, h^{-1} = 0$$

$$(i\partial_z - \tfrac{1}{\zeta}\partial_{\bar{x}} + i\phi - A_z - \tfrac{i}{\zeta} A_{\bar{x}})\, h^{-1} = 0$$

Now h is holomorphic at $\zeta = \infty$. So let $\zeta \to \infty$ in both these equations, then,

$$\partial_x h_\infty^{-1} + i A_x h_\infty^{-1} = 0$$

and,

$$i\partial_z h_\infty^{-1} + (i\phi - A_z) h_\infty^{-1} = 0$$

Similarly, letting $\zeta \to 0$ in the equations for k^{-1} gives,

$$-i\partial_z k_0^{-1} + (i\phi + A_z) k_0^{-1} = 0$$

and,

$$\partial_{\bar{x}} k_0^{-1} + i A_{\bar{x}} k_0^{-1} = 0 .$$

Hence :

$$\begin{aligned} \phi &= \tfrac{1}{2}\left(h_\infty^{-1} \partial_z h_\infty - k_0^{-1} \partial_z k_0 \right) \\ A_z &= -\tfrac{i}{2}\left(h_\infty^{-1} \partial_z h_\infty + k_0^{-1} \partial_z k_0 \right) \\ A_{\bar{x}} &= -i\, k_0^{-1} \partial_{\bar{x}} k_0 \, , \qquad A_x = -i\, h_\infty^{-1} \partial_x h_\infty \, . \end{aligned} \tag{2.2}$$

where $h_\infty = h(\zeta = \infty)$ and $k_0 = k(\zeta = 0)$. From these expressions for the fields we can in principle check their regularity, always provided that we can achieve the splitting of the patching function.

2.7 Ansätze for patching functions.

The fields will be Hermitian if the bundle is, that is, if $g(\zeta, \gamma) = g^{+}(-\frac{1}{\bar{\zeta}}, \gamma)$, according to the real structure on $TP_1(\mathbb{C})$. This will be the case if $k_0^{-1} = h_\infty^{+}$ and we can check the reality of the fields directly from the above expressions :

$$\begin{aligned} \phi^{+} &= \tfrac{1}{2}\left(\partial_z h_\infty^{+} . h_\infty^{+\,-1} - \partial_z k_0^{+} . k_0^{+\,-1} \right) \\ &= \tfrac{1}{2}\left(\partial_z k_0^{-1} . k_0 - \partial_z h_\infty^{-1} . h_\infty \right) \\ &= \phi \, , \qquad\qquad \text{etc.} \end{aligned}$$

The boundary conditions are less easy to satisfy. Originally the Atiyah-Ward construction was intended for use with instanton solutions. They suggested [8] an ansatz for $SU(2)$ instanton patching functions which was explicitly splittable and of the form,

$$g = \begin{pmatrix} \zeta^{l} & \rho \\ 0 & \zeta^{-l} \end{pmatrix}$$

for a positive integer l and where ρ is regular on $U_0 \cap U_\infty$. These correspond to those bundles which can be expressed as extensions of a standard line bundle. This turned out not to be very useful for that problem but the analogous ansatz on $\mathbb{R}^3$ is useful for the monopole problem. Here the suggested bundle [21] is an extension of a tensor product of the previous line bundle with a $U(1)$ line bundle :

$$g = \begin{pmatrix} \zeta^{l} e^{\gamma} & \rho(\zeta,\gamma) \\ 0 & \zeta^{-l} e^{-\gamma} \end{pmatrix}$$

Hitchin [13] has shown that this ansatz contains all those static monopole solutions to the Bogomolnyi' equations which have as boundary conditions,

$$\|\phi\| = 1 - \frac{m}{r} + O(r^{-2})$$
$$\frac{\partial}{\partial r}\|\phi\| = O(r^{-2})$$
$$\|\nabla\phi\| = O(r^{-2})$$

where $\|\cdot\|^2 = \frac{1}{2}\mathrm{Tr}(\cdot)^2$ and $\frac{\partial f}{\partial \Omega} = \left[\left(\frac{\partial f}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial f}{\partial \varphi}\right)^2\right]^{1/2}$.

Clearly such a function does not satisfy the reality condition. But g is unique only up to the equivalence,

$$g \cong a\, g\, A$$

where a is holomorphic on U_∞ , and A on U_0 . These simply represent holomorphic changes in the $\mathbb{C}^n$-fibre of the bundle. So it suffices to write reality in the form : $g^{+} \cong g$.

A suggestion for an ansatz in $SU(n)$ is the following :

$$g = \left[\begin{array}{c|c|c|c} \zeta^{l_1} e^{\alpha_1\gamma} M_1 & \rho_{12} & \rho_{13} \;\cdots\cdots & \rho_{1p} \\ \hline 0 & \zeta^{l_2} e^{\alpha_2\gamma} M_2 & \rho_{23} \;\cdots\cdots & \rho_{2p} \\ \hline 0 & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ \hline 0 & 0 & \cdots\cdots & \zeta^{l_p} e^{\alpha_p\gamma} M_p \end{array}\right] \qquad \sum_{i=1}^{p} n_i l_i = 0 \;, \quad \sum_{i=1}^{p} n_i = n \;, \quad \sum_{i=1}^{p} n_i \alpha_i = 0 \;. \tag{2.3}$$

where ρ_{ij} is a matrix of functions of dimension $n_i \times n_j$ and the diagonal blocks are $n_i \times n_i$. In particular for the case $\alpha_i \neq \alpha_j$ for all i, j , of " maximal symmetry breaking ", this becomes an upper triangular matrix, $n_i = 1$, $n = p$. It is not clear what the M_i should be, but choosing them to be $n_i \times n_i$ identity matrices does give some solutions.

Without loss of generality we may order the λ_i in a decreasing fashion down the diagonal. For using matrices analytic in ζ^{-1} and ζ on the left and the right respectively we can extract the factors $e^{\alpha_i \gamma}$ from the diagonal and then, using upper triangular matrices with unit diagonal entries, chop off the upper and lower reaches of the Laurent expension of g_{ij} in ζ, so that,

$$g_{ij} = \sum_{-\infty}^{\infty} g_{ij}^{(r)} \zeta^r \cong \sum_{\lambda_j}^{\lambda_i} g_{ij}^{(r)} \zeta^r$$

Then if $\lambda_j > \lambda_i$, $g_{ij} \cong 0$ and we can simply rearrange the order of λ_i and λ_j. Strictly speaking these equivalence transformations are not bundle equivalences because they are not functions of γ/ζ and $\zeta\gamma$ respectively. But the second set of factors reduce to unit matrices in the $\zeta \to 0$ and $\zeta \to \infty$ limits, so they do not affect the evaluation of the gauge fields and the first set behave as gauge transformations of the conventional type.

Chapter III

Examples of monopoles in $SU(2)$ and $SU(3)$ with cylindrical and spherical symmetry.

3.1 Real patching functions.

The constraint of reality imposed upon the ansatz for g implies a fairly restricted form for the ρ_{ij} . As it stands g is upper triangular but it suffices that $g^{\dagger}(\frac{-1}{\bar{\zeta}},\bar{\gamma})$ be bundle equivalent to $g(\zeta,\gamma)$. Further, any A which is not upper triangular will put gA into a form in which the constraint can be satisfied,

$$gA = A^{\dagger} g^{\dagger} \qquad (3.1)$$

and multiplying on the left by some a will then amount to no more than a further bundle equivalence.

For $SU(2)^{[11]}$ this condition reads :

$$\begin{bmatrix} A_{11}\zeta^{l}e^{\alpha\gamma}+A_{21}\rho & A_{12}\zeta^{l}e^{\alpha\gamma}+A_{22}\rho \\ A_{21}\zeta^{-l}e^{-\alpha\gamma} & A_{22}\zeta^{-l}e^{-\alpha\gamma} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11}(-\zeta)^{-l}e^{\alpha\gamma}+\bar{A}_{21}\bar{\rho} & \bar{A}_{21}(-\zeta)^{l}e^{-\alpha\gamma} \\ \bar{A}_{12}(-\zeta)^{-l}e^{\alpha\gamma}+\bar{A}_{22}\bar{\rho} & \bar{A}_{22}(-\zeta)^{l}e^{-\alpha\gamma} \end{bmatrix}$$

where $\bar{A}_{ij} = [\overline{A_{ij}(-\frac{1}{\bar{\zeta}})}]$, so effectively $\zeta \to \frac{-1}{\bar{\zeta}}$ and the coefficients of ζ^{n} are complex conjugated. Because A_{22} is analytic in ζ and $\bar{A}_{22}$ in $1/\zeta$ the equation,

$$A_{22}\zeta^{-l} = \bar{A}_{22}(-\zeta)^{l} \qquad (3.2)$$

implies that A_{22} is a polynomial of degree $2l$ in ζ . So,

$$\psi = \zeta^{-l}A_{22} = \sum_{0}^{l} A_{n}\gamma^{n}$$

where $\zeta^{l-n}A_{n}$ is of degree $2(l-n)$ in ζ . Then we can write,

$$\psi = c\prod_{1}^{l}(\gamma-\gamma_i(\zeta)) \qquad (3.3)$$

for C real and where the γ_i , the zeros of ψ , are either real or appear in conjugate pairs.

The off-diagonal equations are equivalent and tell us that,

$$\rho = \frac{Fe^{\alpha\gamma}+fe^{-\alpha\gamma}}{\psi} \qquad (3.4)$$

We shall use upper case letters for functions analytic on U_0 and lower case for those analytic on U_{∞}. F and f can be related up to a bundle equivalence but the above form for ρ will do for now.

The remaining reality condition reduces to $\operatorname{Det} A = 1$, since $A \in SL(2,\mathbb{C})$ and so,

$$A = \begin{bmatrix} \dfrac{1 - f^{+}F}{\zeta^{\ell}\psi} & -F \\ f^{+} & \zeta^{\ell}\psi \end{bmatrix}$$

We demand that $\zeta^{\ell}\psi$ divide $1 - f^{+}F$ to give a function analytic on U_0.

A precisely analogous procedure for $SU(3)$ produces two polynomials ψ_1 and ψ_3 of degrees ℓ_1 and $-\ell_3 \geqslant 0$ in γ from conditions like (3.2) on $(A^{-1})_{11}$ and A_{33} respectively. For the ρ_{ij} we obtain :

$$\rho_{12} = \frac{G e^{\alpha_1 \gamma} + \zeta^{-\ell_3} g\, e^{\alpha_2 \gamma}}{\psi_1}, \quad \psi_1 = \prod_1^{\ell_1} (\gamma - \gamma_{1i})$$

$$\rho_{23} = \frac{f e^{\alpha_3 \gamma} + \zeta^{-\ell_1} F e^{\alpha_2 \gamma}}{\psi_3}, \quad \psi_3 = \prod_1^{-\ell_3} (\gamma - \gamma_{3i})$$

$$\rho_{13} = \frac{H e^{\alpha_1 \gamma}}{\psi_1} + \frac{h e^{\alpha_3 \gamma}}{\psi_3} + \frac{F g\, e^{\alpha_2 \gamma}}{\psi_1 \psi_3} \qquad (3.5)$$

Again g and G, and f and F can be related up to bundle equivalence. Note that the coefficient of the $e^{\alpha_2 \gamma}$ term of ρ_{13} is essentially the product of those in ρ_{12} and ρ_{23}.

We have here chosen the case where the α_i are all distinct corresponding to maximal symmetry breaking, $U(1) \times U(1)$. In the limit $\alpha_1 = \alpha_2$ and for $\ell_1 = \ell_2$, ρ_{12} becomes zero up to equivalence and we recover a g of the form of the ansatz (2.3) with $M_1 = I_2$. In what follows we shall restrict attention to upper triangular patching functions and hope that we might recover at least parts of the other symmetry breaking sectors as such limits.

3.2 Rotational symmetry.

Within this ansatz we can start by looking for monopoles which have some symmetry. This is easier than the problem of finding arbitrary monopole configurations which we shall consider later.

In a gauge theory notions of symmetry have to be slightly modified. Let $x \to r^{-1} x r$ be a point transformation of $\mathbb{R}^3$. A symmetric object S obeys a relation like,

$$S(r^{-1} x r) = R S(x) R^{-1}$$

where R is the representation of the symmetry pertinent to the type of object S. S is covariant, that is, every such change in the underlying manifold can accomodated by a given type of change in S so that it looks the same in the new coordinates. However, in a gauge theory the gauge covariant quantities, like $F_{\mu\nu}$, and the physical, gauge invariant quantities, like $\mathrm{Tr}(\phi F_{\mu\nu})$, are not the canonical ones. So whilst $F_{\mu\nu}$ needs to obey a relation of the above sort, the potentials A_μ do not.

Now under gauge transformations the potentials transform inhomogenously :

$$F(g^{-1} A g + g^{-1} dg) = g^{-1} F(A) g$$

It is then necessary for the symmetry of F only that A transform symmetrically up to such a gauge transformation :

$$A(r^{-1} x r) = g^{-1} R A R^{-1} g + g^{-1} dg$$

In the language of the vector bundle over twistor space however things are simpler. Here the action of r on x induces an action ρ on the points of $TP_1(\mathbb{C})$ and we ask that the bundle remain invariant under this action in the sense that,

$$g(\rho(\zeta, \gamma)) = a\, g(\zeta, \gamma)\, A$$

where a and A are holomorphic on U_∞ and U_0 respectively. The relation of the pair a and A to the real space and gauge transformations is not simple.

If r represents a rotation in $\mathbb{R}^3$ about an axis $\underline{n}$ through an angle θ then a vector $\underline{u}$ changes its coordinates in the following manner :

$$\underline{u} \to \cos\theta\, \underline{u} + (1-\cos\theta)(\underline{u}\cdot\underline{n})\,\underline{n} + \sin\theta\, \underline{n} \wedge \underline{u}$$

In terms of the quantity $1+\underline{\sigma}\cdot\underline{u}$ this can be represented by :

$$1+\underline{\sigma}\cdot\underline{u} \longrightarrow \exp\left(-\tfrac{i}{2}\theta\,\underline{\sigma}\cdot\underline{n}\right)\left(1+\underline{\sigma}\cdot\underline{u}\right)\exp\left(\tfrac{i}{2}\theta\,\underline{\sigma}\cdot\underline{n}\right)$$

But $1+\underline{\sigma}\cdot\underline{u}$ is real and of determinant zero for $\underline{u}^2=1$. Hence it is a tensor product of a spinor and its conjugate :

$$1+\underline{\sigma}\cdot\underline{u} = \bar{\eta}\eta = \begin{bmatrix} (1+u_z)^{1/2} \\ \dfrac{u_{\bar{x}}}{(1+u_z)^{1/2}} \end{bmatrix} \otimes \begin{bmatrix} (1+u_z)^{1/2} & \dfrac{u_x}{(1+u_z)^{1/2}} \end{bmatrix}$$

Then under a rotation η transforms thus :

$$\eta \longrightarrow \eta\exp\left(\tfrac{i\theta}{2}\,\underline{\sigma}\cdot\underline{n}\right) = \eta\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

If $\underline{u}$ is the coordinate on S^2 of the twistor space then ζ is the ratio of the two components of η and hence undergoes a Möbius transformation ,

$$\zeta = \frac{1+u_z}{u_x} = \frac{\eta_1}{\eta_2} \longrightarrow \frac{\alpha\eta_1-\bar{\beta}\eta_2}{\beta\eta_1+\bar{\alpha}\eta_2} = \frac{\alpha\zeta-\bar{\beta}}{\beta\zeta+\bar{\alpha}}$$

where $\alpha\bar{\alpha}+\beta\bar{\beta}=1$.

A vector field $\zeta\gamma(v)$ can be written :

$$\begin{aligned}\zeta\gamma &= v_x\zeta^2 - 2v_z\zeta - v_{\bar{x}} \\ &= \begin{pmatrix}\zeta & 1\end{pmatrix}\begin{pmatrix} 1+v_z & v_x \\ v_{\bar{x}} & 1-v_z \end{pmatrix}\begin{pmatrix} -1 \\ \zeta \end{pmatrix}\end{aligned}$$

It will transform to :

$$(\beta\zeta+\bar{\alpha})^{-2}\left[(\alpha\zeta-\bar{\beta},\beta\zeta+\bar{\alpha})\begin{pmatrix}\bar{\alpha} & -\beta \\ \bar{\beta} & \alpha\end{pmatrix}\begin{pmatrix}1+v_z & v_x \\ v_{\bar{x}} & 1-v_z\end{pmatrix}\begin{pmatrix}\alpha & \beta \\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}\begin{pmatrix}-\beta\zeta-\bar{\alpha} \\ \alpha\zeta-\bar{\beta}\end{pmatrix}\right]$$

$$= (\beta\zeta+\bar{\alpha})^{-2}\left[\begin{pmatrix}\zeta & 1\end{pmatrix}\begin{pmatrix}1+v_z & v_x \\ v_{\bar{x}} & 1-v_z\end{pmatrix}\begin{pmatrix}-1 \\ \zeta\end{pmatrix}\right]$$

and so :

$$\gamma \longrightarrow (\beta\zeta+\bar{\alpha})^{-1}\left(\alpha-\bar{\beta}/\zeta\right)^{-1}\gamma$$

Under an infinitesimal rotation,

$$\alpha = 1 + i\varepsilon \quad , \quad \beta = \eta + i\theta$$

for ε and $|\beta|$ small,

we have :
$$\zeta \to \zeta + \zeta\left[2i\varepsilon + (\eta - i\theta)\zeta + (\eta + i\theta)/\zeta\right]$$
$$\gamma \to \gamma + \gamma\left[(\eta - i\theta)\zeta - (\eta + i\theta)/\zeta\right]$$

and so :
$$g \to g + 2i\varepsilon\zeta\frac{\partial g}{\partial \zeta} + (\eta - i\theta)\zeta\left(\zeta\frac{\partial}{\partial\zeta} + \gamma\frac{\partial}{\partial\gamma}\right)g + (\eta + i\theta)\frac{1}{\zeta}\left(\zeta\frac{\partial}{\partial\zeta} - \gamma\frac{\partial}{\partial\gamma}\right)g$$

For symmetry we demand that the transformed g be bundle equivalent to the original,

$$(1 - a)g(1 + A) = g + gA - ag + O(\varepsilon^2, |\beta|^2)$$

where a and A are now traceless and $O(\varepsilon, \eta, \theta)$. Thus for each degree of freedom we obtain a differential equation :

$$2i\zeta\frac{\partial g}{\partial \zeta} = gA - ag$$

$$\left(\zeta^2\frac{\partial}{\partial\zeta} + \gamma\zeta\frac{\partial}{\partial\gamma}\right)g = gB - bg \qquad (3.6)$$

$$\left(\frac{\partial}{\partial\zeta} - \frac{\gamma}{\zeta}\frac{\partial}{\partial\gamma}\right)g = gC - cg$$

The holomorphic constraints on g, A and a etc, as well as the reality of g are enough to allow us to solve these equations without difficulty.

3.3 Cylindrical symmetry in $SU(2)$.

For $\beta = 0$ these equations become :

$$2\zeta\begin{bmatrix} l\zeta^{l-1}e^{\alpha\gamma} & \partial\rho/\partial\zeta \\ 0 & -l\zeta^{-l-1}e^{-\alpha\gamma}\end{bmatrix} = \begin{bmatrix} \zeta^l e^{\alpha\gamma} & \rho \\ 0 & \zeta^{-l}e^{-\alpha\gamma}\end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} \zeta^l e^{\alpha\gamma} & \rho \\ 0 & \zeta^{-l}e^{-\alpha\gamma}\end{bmatrix}$$

The simplest component is :

$$\zeta^{-l}e^{-\alpha\gamma}A_{21} = a_{21}\zeta^l e^{\alpha\gamma}$$

Since A_{21}, a_{21} are to be functions of ζ, $\zeta\gamma$ and $\frac{1}{\zeta}$, γ/ζ respectively they cannot cancel the exponentials and, unless $\alpha = 0$, we must have $A_{21}=0$, $a_{21}=0$. But α sets the scale of the Higgs field and cannot be zero.

The two diagonal entries give

$$A_{11} = c + l \qquad a_{11} = c - l$$

for constant c.

Using the same argument as above we may disconnect the $e^{\alpha\gamma}$ and $e^{-\alpha\gamma}$ parts of the remaining equation to give, using the real form (3.4) of ρ,

$$2\zeta \frac{\partial}{\partial\zeta}\left(\frac{F}{\psi}\right) = \zeta^{l} A_{12} - 2c\frac{F}{\psi}$$

$$2\zeta \frac{\partial}{\partial\zeta}\left(\frac{f}{\psi}\right) = -2c\frac{f}{\psi} - a_{12}\zeta^{-l}$$

which we rewrite as :

$$\begin{aligned} 2\zeta^{1-c}\frac{\partial}{\partial\zeta}\left(\frac{F\zeta^{c}}{\psi}\right) &= \zeta^{l} A_{12} \\ 2\zeta^{1-c}\frac{\partial}{\partial\zeta}\left(\frac{f\zeta^{c}}{\psi}\right) &= -\zeta^{-l} a_{12} \end{aligned} \qquad (3.7)$$

Now F is a function of ζ and $\gamma\zeta$. Therefore if ψ divides it, being a polynomial of order l in γ, it will push out a factor ζ^{l} from F. But if ψ divides F it must also divide f to prevent ρ having singularities in $U_0 \cap U_\infty$. Since f is a function of ζ^{-1} and $\gamma\zeta^{-1}$ this pushes out a factor ζ^{-l}. Then ρ has the form,

$$\rho = \zeta^{l} F'(\zeta,\gamma\zeta)e^{\alpha\gamma} + \zeta^{-l} f'\left(\tfrac{1}{\zeta}, \gamma/\zeta\right)e^{-\alpha\gamma}$$

which is precisely that removable by a bundle equivalence :

$$\begin{bmatrix} \zeta^{l}e^{\alpha\gamma} & \rho \\ 0 & \zeta^{-l}e^{-\alpha\gamma} \end{bmatrix} = \begin{bmatrix} 1 & f' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta^{l}e^{\alpha\gamma} & 0 \\ 0 & \zeta^{-l}e^{-\alpha\gamma} \end{bmatrix} \begin{bmatrix} 1 & F' \\ 0 & 1 \end{bmatrix}$$

So we may assume some part of ψ does not divide F. Since f/ψ must have the same x-dependent poles, the same part does not divide f.

However, A_{12} may only have poles outside U_0 . Therefore (3.7) implies $A_{n}=0$. Similarly $a_{n}=0$. Hence the arguments of the derivatives in (3.7) are functions of γ alone :

$$\frac{F\zeta^c}{\psi} = \tilde{F}(\gamma) \qquad \frac{f\zeta^c}{\psi} = \tilde{f}(\gamma) \tag{3.8}$$

F and f must both have suitable factors of ζ to cancel the power ζ^c which is only possible if $c \leqslant 0$ and $c \geqslant 0$. So $c=0$. But further, since F and f can only be functions of γ if they are also functions of ζ , independence of the latter implies that of the former and F and f are constant, F_0 and f_0 .

Hence the general form for cylindrical symmetry is[16,17,18] :

$$\rho_c = \frac{F_0 e^{\alpha\gamma} + f_0 e^{-\alpha\gamma}}{\prod_1^l (\gamma - c_i)} \tag{3.9}$$

where the c_i are constant, any more general ζ-dependence being incompatible with (3.8) . The constants F_0 and f_0 must be chosen to prevent ρ_c from having χ-dependent singularities. Choosing $\alpha = 1$ we need $F_0/f_0 = (-1)^l$ and the c_i to be l distinct choices of the form $\frac{1}{2}ni\pi$. Regularity of the fields then requires that they take the values[17] :

$$-\frac{l-1}{2}i\pi\ , -\frac{l-3}{2}i\pi, \ \ldots\ldots \ \frac{l-3}{2}i\pi\ , \frac{l-1}{2}i\pi\ .$$

The means of proving regularity will be discussed in chapter V .

3.4 <u>Spherical symmetry in $SU(2)$</u>.

To unearth spherically symmetric monopoles we impose the extra two equations of (3.6) upon ρ_c . Taking the first of the pair, again the upper triangularity of g implies that of B and b . The diagonal entries give :

$$B_{11} = c \qquad b_{11} = c - \frac{1}{3}(l - \alpha\gamma)$$

for some new constant c .

The remaining equation splits into $e^{\alpha\gamma}$ and $e^{-\alpha\gamma}$ parts :

$$-\frac{\gamma F_0}{\zeta}\cdot\frac{\partial}{\partial\gamma}\left(\frac{1}{\psi}\right) = \zeta^{\ell} B_{12} + \left(\frac{\ell}{\zeta} - 2c\right)\frac{F_0}{\psi}$$

$$\frac{\alpha f_0 \gamma}{\zeta\psi} - \frac{\gamma f_0}{\zeta}\cdot\frac{\partial}{\partial\gamma}\left(\frac{1}{\psi}\right) = -\zeta^{-\ell} b_{12} + \left(\frac{1}{\zeta}(\ell-\alpha\gamma) - 2c\right)\frac{f_0}{\psi}$$

Since $\ell > 0$ we may extract the $\frac{1}{\zeta}$ terms from the first of these obtaining :

$$-\gamma\frac{\partial}{\partial\gamma}\left(\frac{1}{\psi}\right) = \frac{\ell}{\psi}$$

and so, up to a multiplicative constant, $\psi = \gamma^{\ell}$. The remainder is then,

$$\zeta^{\ell} B_{12} = \frac{2cF_0}{\gamma^{\ell}}$$

But B_{12} is allowed no such singularities, so $B_{12} = 0$ and $c = 0$.

The other equation simplifies now to :

$$2\alpha f_0\left(\frac{\gamma}{\zeta}\right) = -b_{12}\left(\frac{\gamma}{\zeta}\right)^{\ell}$$

which implies $\ell = 1$ and b_{12} is constant.

The last member of (3.6) leads to the same result. Hence there is a unique spherically symmetric monopole in $SU(2)$ given by,[14]

$$\rho_s = F_0\,\frac{e^{\alpha\gamma} - e^{-\alpha\gamma}}{\gamma} \qquad (3.10)$$

where the choice $f_0 = -F_0$ renders the singularities acceptable. Its regularity will be discussed in a later chapter.

3.5 Cylindrical symmetry in $SU(3)$.

The above arguments extend with some added complexity to the $SU(3)$ case. We shall restrict attention to the instance where the α_i are all distinct. The limits in which a pair become equal do give examples of $U(2)$ breaking.

Looking first at small rotations about the z-axis, A and a are again upper triangular and the diagonal terms in the first of equations (3.6) give,

$$2\ell_j = A_{jj} - a_{jj} \quad , \quad j = 1,2,3 .$$

and so we write : $$A_{jj} = \lambda_j + c_j$$
$$a_{jj} = -\lambda_j + c_j$$

where c_1 , c_2 , c_3 are constant and have zero sum.

Disconnecting the terms in $e^{\alpha_i \gamma}$ from the upper off-diagonal entries and using the real forms (3.5) for the ρ_{ij} leads to three sets of equations :

$$\left.\begin{aligned} 2\zeta \frac{\partial}{\partial \zeta}\left(\frac{G\zeta^{n_{12}}}{\psi_1}\right)\zeta^{-n_{12}} &= A_{12}\zeta^{\lambda_1} \\ 2\zeta \frac{\partial}{\partial \zeta}\left(\frac{g\zeta^{n_{12}-\lambda_3}}{\psi_1}\right)\zeta^{-n_{12}} &= -a_{12}\zeta^{\lambda_2} \end{aligned}\right\} \qquad (3.11)$$

$$\left.\begin{aligned} 2\zeta \frac{\partial}{\partial \zeta}\left(\frac{f\zeta^{n_{23}}}{\psi_3}\right)\zeta^{-n_{23}} &= -a_{23}\zeta^{\lambda_3} \\ 2\zeta \frac{\partial}{\partial \zeta}\left(\frac{F\zeta^{-\lambda_1+n_{23}}}{\psi_3}\right)\zeta^{-n_{23}} &= A_{23}\zeta^{\lambda_2} \end{aligned}\right\} \qquad (3.12)$$

$$\left.\begin{aligned} 2\zeta \frac{\partial}{\partial \zeta}\left(\frac{H\zeta^{n_{13}}}{\psi_1}\right)\zeta^{-n_{13}} &= -a_{13}\zeta^{\lambda_3} + A_{23}\frac{G}{\psi_1} \\ 2\zeta \frac{\partial}{\partial \zeta}\left(\frac{h\zeta^{n_{13}}}{\psi_3}\right)\zeta^{-n_{13}} &= A_{13}\zeta^{\lambda_1} - a_{12}\frac{f}{\psi_3} \\ 2\zeta \frac{\partial}{\partial \zeta}\left(\frac{gF\zeta^{n_{13}}}{\psi_1\psi_3}\right)\zeta^{-n_{13}} &= A_{23}\frac{\zeta^{-\lambda_3}g}{\psi_1} - a_{12}\frac{\zeta^{-\lambda_1}F}{\psi_3} \end{aligned}\right\} \qquad (3.13)$$

where $n_{ij} = \frac{1}{2}(c_i - c_j - \lambda_i - \lambda_j)$, so that,

$$n_{13} - n_{12} - n_{23} = \lambda_2$$

By a bundle equivalence argument similar to that in the $SU(2)$ case we can remove any of the arbitrary functions which divide by ψ_1 or ψ_3 .

Then, either because we can replace G and g by zero or by the argument from poles in G/ψ_1 and g/ψ_1, we have $A_{12}=0$ and $a_{12}=0$. Similarly $a_{23}=0$ and $A_{23}=0$. From these it follows in (3.13) that $a_{13}=0$ and $A_{13}=0$. So all the arguments of the derivatives in (3.11-13) are either zero or ζ-independent. Hence either $\rho_{ij}=0$ or $\rho_{ij}=\zeta^{-n_{ij}}\tilde{\rho}_{ij}(\gamma)$ and the γ_{1i} and γ_{3i} of ψ_1 and ψ_3 are constants, c_{1i} and c_{3i}.

Suppose now that $\rho_{12}\neq 0$. Then,

$$\frac{G}{\psi_1} = \zeta^{-n_{12}}\,\frac{\tilde{G}(\gamma,M_{12})}{\phi_1(\gamma)}$$

$$\frac{g}{\psi_1}\zeta^{-\lambda_3} = \zeta^{-n_{12}}\,\frac{\tilde{g}(\gamma,m_{12})}{\phi_1(\gamma)}$$

where $\phi_1(\gamma)$ is of order $\rho_1\leq\lambda_1$ and gives the common poles which do not divide G and g. $\tilde{G}$ and $\tilde{g}$ are polynomials in γ alone of degrees M_{12} and m_{12} respectively. Then the ζ- dependence of G implies :

$$\frac{G(\gamma\zeta,\zeta)}{\psi_1} = \zeta^{\lambda_1-\rho_1}\,\frac{G'(\gamma\zeta,\zeta)}{\phi_1(\gamma)} = \zeta^{-n_{12}}\,\frac{\tilde{G}(\gamma,M_{12})}{\phi_1(\gamma)}$$

Therefore $\quad 0\leqslant M_{12}\leqslant -n_{12}+\rho_1-\lambda_1$.

Also,

$$\frac{g(\gamma/\zeta,1/\zeta)}{\psi_1} = \zeta^{-\lambda_1+\rho_1}\,\frac{g'(\gamma/\zeta,1/\zeta)}{\phi_1} = \zeta^{\lambda_3-n_{12}}\,\frac{\tilde{g}(\gamma,m_{12})}{\phi_1}$$

and $\quad 0\leqslant m_{12}\leqslant n_{12}+\rho_1+\lambda_2$. These inequalities imply,

$$\lambda_3\leqslant n_{12}\leqslant 0 \quad , \quad \lambda_1\geqslant\rho_1\geqslant\tfrac{1}{2}(\lambda_1-\lambda_2).$$

Similarly, if $\rho_{23}\neq 0$ we obtain :

$$\frac{f}{\psi_3} = \zeta^{-n_{23}}\,\frac{\tilde{f}(\gamma,m_{23})}{\phi_3(\gamma)}$$

$$\frac{F\zeta^{-\lambda_1}}{\psi_3} = \zeta^{-n_{23}}\,\frac{\tilde{F}(\gamma,M_{23})}{\phi_3(\gamma)}$$

where ϕ_3 has degree $-\rho_3\leqslant-\lambda_3$ and,

$$0\leqslant m_{23}\leqslant n_{23}+\lambda_3-\rho_3$$

$$0\leqslant M_{23}\leqslant -n_{23}-\lambda_2-\rho_3$$

so that : $\quad 0 \leqslant n_{23} \leqslant \lambda_1 \quad , \quad \lambda_3 \leqslant p_3 \leqslant \frac{1}{2}(\lambda_3 - \lambda_2)$

We can identify the following cases :-

Case (i). ψ_1 divides G , ψ_3 divides f .

Then $\rho_{12} \cong 0$, $\rho_{23} \cong 0$ and ρ_{13} loses its $e^{\alpha_2 \gamma}$ term :

$$\rho_{13} = \frac{H e^{\alpha_1 \gamma}}{\psi_1} + \frac{h e^{\alpha_3 \gamma}}{\psi_3} = \zeta^{-n_{13}} \tilde{\rho}(\gamma)$$

by (3.13) . The analyticities of $H(\gamma\zeta, \zeta)$ and $h(\gamma/\zeta, 1/\zeta)$ then require $0 \leqslant n_{13} \leqslant 0$ since we can only extract positive powers from the former and negative from the latter. This means that H and h are constant since any γ-dependence must be accompanied by ζ-dependence. ψ_1^{-1} and ψ_3^{-1} must have the same (constant) poles in γ to ensure ρ_{13} is analytic in $U_0 \cap U_\infty$. Therefore $\lambda_1 = -\lambda_3$, $\lambda_2 = 0$ and ρ_{13} looks exactly like the ρ_c of $SU(2)$, but with α and $-\alpha$ replaced by α_1 and α_3 . Regularity will be ensured by the same choices of constants C_i , up to a factor of $\alpha_1 - \alpha_3$, as in the earlier case and we have embeddings in $SU(3)$ of $SU(2)$ cylindrical monopoles.

Case (ii). ψ_3 divides f .

$\rho_{23} \cong 0$ and,

$$\rho_{12} = \zeta^{-n_{12}} \frac{\tilde{G}(\gamma, M_{12}) e^{\alpha_1 \gamma} + \tilde{g}(\gamma, m_{12}) e^{\alpha_2 \gamma}}{\phi_1(\gamma)}$$

ρ_{13} will vanish or not according to whether ψ_1 divides H and ψ_3 divides h or not, since $F=0$ by its divisibility by ψ_3 . If it does not vanish it will have the same form as in case (i) and for the same reasons. $\lambda_2 = 0$ and ϕ_1 is a selection of p_1 factors from ψ_1 .

Case (iii). ψ_1 divides G .

This is similar to the previous case.

Case (iv). ψ_1 does not divide G , ψ_3 does not divide F .

Since $n_{13} = \lambda_2 + n_{12} + n_{23}$, ρ_{13} has the general form :

$$\rho_{13} = \zeta^{-n_{13}} \left(\frac{H e^{\alpha_1 \gamma}}{\psi_1} + \frac{h e^{\alpha_3 \gamma}}{\psi_3} + \frac{\tilde{g}(\gamma, m_{12}) \tilde{F}(\gamma, M_{23}) e^{\alpha_2 \gamma}}{\phi_1 \phi_3} \right)$$

(a) If ψ_1 divides H and ψ_3 divides h then ϕ_1 divides $\tilde{F}$, ϕ_3 divides $\tilde{g}$ and,

$$0 \leq p_1 \leq M_{23} \leq -n_{23} - l_2 - p_3$$

$$0 \leq -p_3 \leq m_{12} \leq n_{12} + p_1 + l_2$$

These imply $n_{12} \geqslant n_{23}$. But $n_{12} \leqslant 0$, $n_{23} \geqslant 0$ so both vanish and $l_1 = p_1$, $l_3 = p_3$. So,

$$\rho_{12} = \frac{\tilde{G}_0 e^{\alpha_1 \gamma} + \tilde{g}(\gamma, -l_3) e^{\alpha_2 \gamma}}{\psi_1}$$

$$\rho_{23} = \frac{\tilde{f}_0 e^{\alpha_3 \gamma} + \tilde{F}(\gamma, l_1) e^{\alpha_2 \gamma}}{\psi_3}$$

$$\rho_{13} = 0$$

(b) ψ_1 does not divide H, ψ_3 does not divide h. As in case (i), $n_{13} = 0$ and H and h are constants. All we can say in general, since the $e^{\alpha_2 \gamma}$ term in ρ_{13} is determined from those in ρ_{12} and ρ_{23}, is that enough cancellation must occur between ϕ_1 and $\tilde{F}$, and ϕ_3 and $\tilde{g}$ to ensure that the poles in $\tilde{g}\tilde{F}/\phi_1\phi_3$ in $U_0 \cap U_\infty$ balance the distinct poles in ψ_1^{-1} and ψ_3^{-1}.

(c) ψ_1 divides H or ψ_3 divides h. In this case n_{13} is no longer zero and the usual remarks apply as regards singularities.

It is not easy to analyse these cases further without choosing particular values for the l_i. The greatest freedom seems to belong to case (iv)(b). The simplest choice is $l_1 = 1$, $l_2 = 0$, $l_3 = -1$ in which case $\psi_1 = \gamma - c_1$, $\psi_3 = \gamma - c_3$. Then $n_{12} = -n$, $n_{23} = n$ and,

$$0 \leq M_{12} \leq n \qquad 0 \leq m_{12} \leq 1 - n$$

$$0 \leq m_{23} \leq n \qquad 0 \leq M_{23} \leq 1 - n$$

So $n = 1$ or 0.

In the first case :

$$\rho_{12} = \zeta \, \frac{(\tilde{G}_0 + \gamma \tilde{G}_1) e^{\alpha_1 \gamma} + \tilde{g}_0 e^{\alpha_2 \gamma}}{\gamma - c_1}$$

$$\rho_{23} = \frac{1}{\zeta} \cdot \frac{(\tilde{f}_0 + \gamma \tilde{f}_1) e^{\alpha_3 \gamma} + \tilde{F}_0 e^{\alpha_2 \gamma}}{\gamma - c_3}$$

$$\rho_{13} = \frac{H_0 e^{\alpha_1 \gamma}}{\gamma - c_1} + \frac{h_0 e^{\alpha_3 \gamma}}{\gamma - c_3} + \frac{\tilde{F}_0 \tilde{g}_0 e^{\alpha_2 \gamma}}{(\gamma - c_1)(\gamma - c_3)}$$

By a bundle equivalence argument we can choose $\tilde{G}_1 = \tilde{f}_1 = 0$. It is then easy to choose the remaining constants in such a way as to remove the apparent singularities :

$$\rho_{12} = \zeta \cdot \frac{e^{\alpha_1(\gamma - c_1)} - e^{\alpha_2(\gamma - c_1)}}{\gamma - c_1}$$

$$\rho_{23} = \frac{1}{\zeta} \cdot \frac{e^{\alpha_3(\gamma - c_3)} - e^{\alpha_2(\gamma - c_3)}}{\gamma - c_3} \qquad (3.15)$$

$$\rho_{13} = \frac{(\gamma - c_3) H_0 e^{\alpha_1 \gamma} + e^{\alpha_2(\gamma - c_1 - c_3)} + (\gamma - c_1) h_0 e^{\alpha_3 \gamma}}{(\gamma - c_1)(\gamma - c_3)}$$

where $H_0 = \dfrac{e^{-\alpha_1 c_1 - \alpha_2 c_3}}{c_3 - c_1}$ and $h_0 = \dfrac{e^{-\alpha_3 c_3 - \alpha_2 c_1}}{c_1 - c_3}$.

This appears to be an " elongated " embedding of the charge one $SU(2)$ monopole.[29] It has been shown to be regular and the limit $c_1 = c_3$ yields a spherically symmetric, simple embedding in the ρ_{13} position.

For $n = 0$ we get the same functions but without their powers of ζ .

If we choose similar functions for ρ_{12} and ρ_{23} in the $l_2 = 0$, $l_1 = 2$ ansatz we can identify the following family[30] :

$$\rho_{12} = \zeta \cdot \frac{e^{\alpha_2 \gamma} - e^{\alpha_1 \gamma + \alpha_{21} c_1}}{\alpha_{21}(\gamma - c_1)}$$

$$\rho_{23} = \zeta^{-1} \cdot \frac{e^{\alpha_2 \gamma} - e^{\alpha_3 \gamma + \alpha_{23} c_3}}{\alpha_{23}(\gamma - c_3)} \qquad (3.16)$$

for $c_1 = \bar{c}_3$, $\phi_1 = \gamma - c_1$, $\phi_3 = \gamma - c_3$, $\psi_1 = \psi_3 = \phi_1 \phi_3$

$$\rho_{13} = \frac{e^{\alpha_2\gamma} + H_0 e^{\alpha_1\gamma} + h_0 e^{\alpha_3\gamma}}{\alpha_{21}\alpha_{23}(\gamma - c_1)(\gamma - c_3)} \tag{3.16}$$

and where

$$H_0 = e^{(c_1+c_3)\alpha_2} \cdot \frac{e^{\alpha_{32}c_1} - e^{\alpha_{32}c_3}}{e^{\alpha_3 c_3 + \alpha_1 c_1} - e^{\alpha_1 c_3 + \alpha_3 c_1}}$$

$$h_0 = e^{(c_1+c_3)\alpha_2} \cdot \frac{e^{\alpha_{12}c_3} - e^{\alpha_{12}c_1}}{e^{\alpha_3 c_3 + \alpha_1 c_1} - e^{\alpha_1 c_3 + \alpha_3 c_1}} , \quad \alpha_{ij} = \alpha_i - \alpha_j$$

This will be shown to be regular and may perhaps be a superposition of two differently embedded $SU(2)$ monopoles of " separation " $c_1 - c_3$. In the limit $\alpha_1 = \alpha_2$ which must be very carefully taken, we obtain a patching function equivalent to one for $U(2)$ breaking.[28] When $c_1 = c_3$ we recover the known spherical solution of charge two to be discussed in the next section.

3.6 Spherical symmetry in $SU(3)$.

Not all the above cases possess spherically symmetric examples.

In the lower two equations of (3.6) B , b , C and c will again be upper triangular.

The diagonal terms yield :

$$B_{ii} = k_i \quad , \quad b_{ii} = k_i - \frac{1}{\zeta}(l_i - \gamma\alpha_i)$$

$$C_{ii} = k_i' + \zeta(l_i + \gamma\alpha_i) \, , \qquad c_{ii} = k_i'$$

for k_i and k_i' constant.

In any of the cases where $\rho_{13} = 0$ we have the equation ,

$$0 = \left(\frac{\partial}{\partial\zeta} - \frac{\gamma}{\zeta}\frac{\partial}{\partial\gamma}\right)\rho_{13} = B_{13}\zeta^{l_1}e^{\alpha_1\gamma} + B_{23}\rho_{12} - b_{12}\rho_{23} - b_{13}\zeta^{l_3}e^{\alpha_3\gamma}$$

and similarly for C and c . If ρ_{12} and ρ_{23} are non-vanishing this gives $B_{23} = \zeta^{l_1}\psi_1 B_{23}'$ and $b_{12} = \zeta^{l_3}\psi_3 b_{12}'$ and so also, from the $e^{\alpha_2\gamma}$ term,

$$B_{23}'\,\tilde{g}\,\zeta^{l_1 - n_{12}} = -b_{12}'\,\tilde{F}\,\zeta^{l_3 - n_{23}} = 0 \quad \text{since the}$$

left hand side has positive powers and the right hand side negative powers of ζ. Thus only B_{12} and b_{23} fail to vanish. The ρ_{12} equation becomes :

$$\left(\frac{\partial}{\partial\zeta} - \frac{\gamma}{\zeta}\cdot\frac{\partial}{\partial\gamma}\right)\rho_{12} = B_{12}\zeta^{\lambda_1}e^{\alpha_1\gamma} + \left((k_2-k_1) + \frac{1}{\zeta}(\lambda_1 - \alpha_1\gamma)\right)\rho_{12}$$

in which the $e^{\alpha_2\gamma}$ term is :

$$\frac{\partial}{\partial\gamma}\left(\frac{\tilde{g}}{\phi_1}\right) = \alpha_{21}\,\tilde{g}/\phi_1 + \frac{\lambda_1 - n_{12}}{\gamma}\cdot\tilde{g}/\phi_1 + \frac{k_2-k_1}{\gamma}\cdot\zeta$$

The ζ-dependence implies $k_2 = k_1$. But this leaves us with an equation which for $\alpha_1 \neq \alpha_2$ has no finite polynomial solution and we have no spherical monopoles here.

So suppose $\rho_{23} = 0$ as well. In this case B_{12} and b_{12} only remain. The equations we have for ρ_{12} are :

$$\left(\frac{\partial}{\partial\zeta} - \frac{\gamma}{\zeta}\cdot\frac{\partial}{\partial\gamma}\right)\rho_{12} = B_{12}\zeta^{\lambda_1}e^{\alpha_1\gamma} + \left((k_2-k_1) + \frac{1}{\zeta}(\lambda_1 - \alpha_1\gamma)\right)\rho_{12} - b_{12}\zeta^{\lambda_2}e^{\alpha_2\gamma}$$

$$\left(\zeta^2\frac{\partial}{\partial\zeta} + \zeta\gamma\frac{\partial}{\partial\gamma}\right)\rho_{12} = C_{12}\zeta^{\lambda_1}e^{\alpha_1\gamma} + \left((k_2'-k_1') + \zeta(\lambda_2 + \alpha_2\gamma)\right)\rho_{12} - c_{12}\zeta^{\lambda_2}e^{\alpha_2\gamma}$$

$$(3.17)$$

The $e^{\alpha_1\gamma}$ term in the first gives :

$$-\gamma\zeta^{-n_{12}-1}\frac{\partial}{\partial\gamma}\left(\frac{\tilde{G}}{\phi_1}\gamma^{\lambda_1+n_{12}}\right)\gamma^{-\lambda_1-n_{12}} = B_{12}\zeta^{\lambda_1} + (k_2-k_1)\zeta^{-n_{12}}\frac{\tilde{G}}{\phi_1}$$

Now if ϕ_1^{-1} had a pole in γ other than at $\gamma = 0$ it would be made second order by differentiation and could not be compensated on the right hand side. So, up to a multiplicative constant, $\phi_1 = \gamma^{p_1}$. Similarly $\phi_3 = \gamma^{-p_3}$. Also the ζ-dependence requires that $k_2 = k_1$. As a result,

$$B_{12} = -\gamma\zeta^{-n_{12}-\lambda_1-1}\frac{\partial}{\partial\gamma}\left(\tilde{G}/\gamma^{p_1-\lambda_1-n_{12}}\right)\gamma^{-\lambda_1-n_{12}}$$

Since $\tilde{G}$ can have no factors of γ we can only avoid poles in B_{12} if $p_1 - \lambda_1 - n_{12} = 0$ and $\tilde{G}$ is constant. Likewise $\tilde{g}$ is constant and $p_1 + n_{12} + \lambda_2 = 0$, from the $e^{\alpha_2\gamma}$ term in the other of equations (3.17).

The remaining equations are :

$$-\gamma \frac{\partial}{\partial\gamma}\tilde{g} = \alpha_{21}\gamma\tilde{g} - b_{12}\zeta^{n_{12}+l_2+1}\gamma^{p_1}$$

$$\gamma\frac{\partial}{\partial\gamma}\tilde{G} = \alpha_{21}\gamma\tilde{G} + C_{12}\zeta^{\lambda_1+n_{12}-1}\gamma^{p_1}$$

The ζ-dependence requires that $n_{12}+\lambda_2+1 \geqslant 0$ and $\lambda_1+n_{12}-1 \leqslant 0$ which can both be written $1-p_1 \geqslant 0$. So $p_1 = 1$ and we have for the patching function :

$$g = \begin{bmatrix} \zeta^{l+2}e^{\alpha_1\gamma} & \zeta^{l+1}\dfrac{\tilde{g}_0 e^{\alpha_2\gamma} + \tilde{G}_0 e^{\alpha_1\gamma}}{\gamma} & 0 \\ 0 & \zeta^{l}e^{\alpha_2\gamma} & 0 \\ 0 & 0 & \zeta^{-2l-2}e^{\alpha_3\gamma} \end{bmatrix} \tag{3.18}$$

Similarly in the case $\rho_{12} = 0$,

$$g = \begin{bmatrix} \zeta^{2l+2}e^{\alpha_1\gamma} & 0 & 0 \\ 0 & \zeta^{-l}e^{\alpha_2\gamma} & \zeta^{-l-1}\dfrac{\tilde{f}_0 e^{\alpha_3\gamma} + \tilde{F}_0 e^{\alpha_2\gamma}}{\gamma} \\ 0 & 0 & \zeta^{-l-2}e^{\alpha_3\gamma} \end{bmatrix}$$

Neither of these can be split unless $l = 0$. They are then simply embeddings of the charge one , $SU(2)$ monopole.

The more interesting case occurs when $\rho_{13} \neq 0$. This time we obtain $\phi_1 = \gamma^{p_1}$ and $\phi_3 = \gamma^{-p_3}$ from the ρ_{13} equations. The others reduce to the following shape, for example :

$$\zeta^{-n_{12}-1}\left((-n_{12}-\lambda_1)\frac{\tilde{G}}{\phi_1} - \gamma\frac{\partial}{\partial\gamma}\left(\frac{\tilde{G}}{\phi_1}\right)\right) = B_{12}\zeta^{\lambda_1} + (k_2-k_1)\zeta^{-n_{12}}\frac{\tilde{G}}{\phi_1}$$

The non-singularity of B_{12} requires $k_1 = k_2$ and that

$$\left((p_1-\lambda_1-n_{12})\tilde{G} - \gamma\frac{\partial}{\partial\gamma}\tilde{G}\right)\gamma^{-p_1}$$

have no poles at $\gamma = 0$. This means that $p_1-\lambda_1-n_{12} = 0$ and either $\frac{\partial}{\partial\gamma}\tilde{G} = 0$ or $p_1 = 1$. But the companion equation to the above is ,

$$\zeta^{-n_{12}-1}\left((\alpha_1-\alpha_2)\tilde{g} - \frac{\partial}{\partial\gamma}\tilde{g}\right)\gamma^{1-p_1} = -b_{12}\zeta^{\lambda_2}$$

and this clearly requires that $p_1 = 1$. Together with similar results from the other three sets of equations we find,

$$\begin{aligned} n_{12} - \lambda_1 &= -1 \\ n_{12} - \lambda_2 &= 1 \\ -n_{23} - \lambda_2 &= -1 \\ -n_{23} - \lambda_3 &= 1 \end{aligned}$$

Since $\lambda_1 + \lambda_2 + \lambda_3 = 0$ we have $\lambda_1 = 2$, $\lambda_2 = 0$ and $\lambda_3 = -2$.

Finally we obtain for the patching function :

$$g = \begin{bmatrix} \zeta^2 e^{\alpha_1\gamma} & \zeta\tilde{\rho}_{12} & \tilde{\rho}_{13} \\ 0 & e^{\alpha_2\gamma} & \zeta^{-1}\tilde{\rho}_{23} \\ 0 & 0 & \zeta^{-2}e^{\alpha_3\gamma} \end{bmatrix}$$

where, having chosen $g_0 = \frac{1}{\alpha_{21}}$, $F_0 = \frac{1}{\alpha_{23}}$,

$$\begin{aligned} \tilde{\rho}_{12} &= \frac{-1}{\alpha_{21}\gamma}\left(e^{\alpha_2\gamma} - e^{\alpha_1\gamma}\right) \\ \tilde{\rho}_{23} &= \frac{1}{\alpha_{23}\gamma}\left(e^{\alpha_2\gamma} - e^{\alpha_3\gamma}\right) \\ \tilde{\rho}_{13} &= \frac{1}{\gamma^2}\left(\frac{e^{\alpha_1\gamma}}{\alpha_{12}\alpha_{13}} + \frac{e^{\alpha_2\gamma}}{\alpha_{21}\alpha_{23}} + \frac{e^{\alpha_3\gamma}}{\alpha_{31}\alpha_{32}}\right) \end{aligned} \qquad (3\cdot19)$$

and $\alpha_{ij} = \alpha_i - \alpha_j$. We need all three exponential terms in ρ_{13} to satisfy the regularity condition.

It is known[5] that the so-called maximal embedding of $SU(2)$ in $SU(3)$ has a spherically symmetric monopole whose behaviour is determined by the functions :

$$\frac{1}{r^2} Q_1(r) = \frac{1}{r^2}\left(\frac{e^{\alpha_1 r}}{\alpha_{12}\alpha_{13}} + \frac{e^{\alpha_2 r}}{\alpha_{21}\alpha_{23}} + \frac{e^{\alpha_3 r}}{\alpha_{31}\alpha_{32}} \right)$$

$$\frac{1}{r^2} Q_2(r) = \frac{1}{r^2}\left(\frac{e^{-\alpha_1 r}}{\alpha_{12}\alpha_{13}} + \frac{e^{-\alpha_2 r}}{\alpha_{21}\alpha_{23}} + \frac{e^{-\alpha_3 r}}{\alpha_{31}\alpha_{32}} \right)$$

where the α_i are associated with the direction of asymptotic symmetry breaking. Then the Higgs field on the γ-axis is given by :

$$2\phi(0,0,\gamma) \longrightarrow H_1 \frac{\partial}{\partial r}\left(\ln \frac{Q_1}{r^2}\right)\Big|_{r=\gamma} + H_2 \frac{\partial}{\partial r}\left(\ln \frac{Q_2}{r^2}\right)\Big|_{r=\gamma}$$

where $H_1 = \mathrm{diag}(1,-1,0)$ and $H_2 = \mathrm{diag}(0,1,-1)$ span the Cartan subalgebra of $SU(3)$. If we assume $\alpha_1 > \alpha_2 > \alpha_3$, then the asymptotic behaviour of ϕ is :

$$2\phi \longrightarrow \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} - \frac{2}{r}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

That the patching function (3.19) gives this monopole is strongly suggested by the facts that,

$$g_{13} = \frac{1}{\gamma^2} Q_1(\gamma)$$

and

$$g_{12}g_{23} - g_{22}g_{13} = \frac{1}{\gamma^2} Q_2(\gamma)$$

This generalises the pattern in $SU(2)$ where

$$2\phi(0,0,\gamma) \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial r}\left(\ln \frac{\mathrm{sh}\, r}{r}\right)\Big|_{r=\gamma}$$

and $g_{12} = \dfrac{\mathrm{sh}\,\gamma}{\gamma}$.

It also suggests a generalisation to $SU(n)$ for the charge $n-1$ spherical monopole :

$$\lambda_1 = n-1\ ,\quad \lambda_2 = n-3\ ,\quad \ldots\ldots \quad \lambda_{n-1} = 3-n\ ,\ \lambda_n = 1-n$$

$$g_{ij} = 0 \qquad , \qquad i > j$$

$$g_{ij} = \frac{y^{\frac{1}{2}(\ell_i+\ell_j)}}{\gamma^{j-i}} \sum_{k=i}^{j} e^{\alpha_k \gamma} \Big[\prod_{\substack{m=i \\ m \neq k}}^{j} \alpha_{mk} \Big]^{-1} \quad , \quad i \leq j \tag{3.20}$$

That these assertions are true will be seen in the following chapters when we discuss spherical symmetry for $SU(n)$ and do some explicit splitting calculations.

Chapter IV

The classification of spherically symmetric monopoles for an $SU(n)$ gauge group.

Using the experience gained in the last chapter we can go on to solve the equations of spherical symmetry in $SU(n)$, using methods developed in this context but from a different point of view by other authors.[44,45] First we establish some notation by briefly reviewing the root theory of $SU(n)$.

4.1 The root decompostion of $SU(n)$.

In general the Lie algebra of a Lie group can be decomposed in the following manner,[38]

$$\mathcal{L} = H \oplus E = H \oplus E^{(+)} \oplus E^{(-)}$$

where H is a set of r commuting generators (r being the rank of the group), the Cartan subalgebra, and E the set of generators remaining. E is spanned by $d-r$ eigenvectors (d the dimension of $\mathcal{L}$) under the adjoint action of H, E_i, whose eigenvalues $\underline{\alpha}_i$ are real vectors in an r dimensional Euclidean space. The basis of this vector space corresponds to a choice of basis for the Cartan subalgebra :

$$[\underline{H}, E_j] = \underline{\alpha}_j E_j$$

$$\underline{H} = (H_1, H_2, \ldots H_r)$$

The vector eigenvalues are called roots. If $\underline{\alpha}_i$ is such a root then so is $-\underline{\alpha}_i$.

Further, we can find a (non-unique) basis of r simple roots which span the set of all roots in such a way that each root is a linear sum of the simple roots with coefficients 1, 0 or -1 and such that the coefficients are either all non-negative or all non-positive. These are the positive and negative roots respectively which give the further decomposition :

$$E = E^{(+)} \oplus E^{(-)}$$

The $E_i^{(+)}$ and $E_i^{(-)}$ are the raising and lowering operators of the algebra. For instance, in $SU(3)$ we have ,

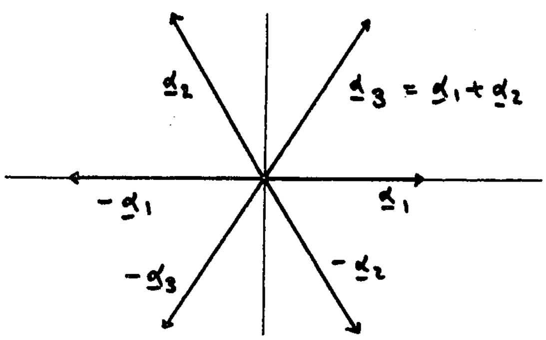
$$H_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad , \quad H_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

$$E_1^{(+)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{\alpha}_1 = (2, 0)$$

$$E_2^{(+)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{\alpha}_2 = (-1, \sqrt{3})$$

$$E_3^{(+)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{\alpha}_3 = (1, \sqrt{3})$$

where $\underline{\alpha}_1, \underline{\alpha}_2$ is a choice of simple roots. We may draw the root diagram :



Any pair of roots subtending an angle of $2\pi/3$ constitute a simple set.

$E_i^{(-)} = E_i^{T(+)}$ and has root $-\underline{\alpha}_i$.

We note that the $E_i^{(+)}$ corresponding to simple roots multiply by addition of roots in the sense that,

$$E_1^{(+)} E_2^{(+)} = E_3^{(+)} \qquad \text{where} \quad \underline{\alpha}_1 + \underline{\alpha}_2 = \underline{\alpha}_3 \ ,$$

but $\quad E_2^{(+)} E_1^{(+)} = 0$

These properties generalise easily to $SU(n)$. We can choose the raising operators of the simple roots to be upper, off-diagonal matrices with a single unit entry, of which there are

$n-1$, the rank of $SU(n)$:

$$E_i^{(+)} = \delta_{i,i+1}$$

The roots are $n(n-1)$ in number, of equal length and form a symmetric $n(n-1)$ -simplex in $\mathbb{R}^{n-1}$.

If $E_i^{(+)}$ and $E_j^{(+)}$ are raising operators of roots $\underline{\alpha}_i$ and $\underline{\alpha}_j$, then one of $E_i^{(+)}E_j^{(+)}$ and $E_j^{(+)}E_i^{(+)}$ will be the raising operator of $\underline{\alpha}_i+\underline{\alpha}_j$, provided this sum is also a root. The other product will be zero. If the sum is not a root both will vanish.

The basis we have chosen above for H is the Chevalley basis. In $SU(n)$ it consists in choosing for H_m $(1\leq m\leq n-1)$ the diagonal matrix ,

$$H_m = N_m \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & -m & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\ \ddots\\1\end{array}}\right\} m \\ \left.\vphantom{\begin{array}{c}-m\\0\\ \ddots\\0\end{array}}\right\} n-m \end{array}$$

and choosing the normalisation N_m so that the roots have length 2 . Considering the raising operator of a simple root :

$$[\underline{H} , E_m] = (0, \ldots N_{m-1}(-m+1), N_m(m+1), \ldots 0) E_m$$

so that , $$(m-1)^2 N_{m-1}^2 + (m+1)^2 N_m^2 = 4$$

and : $N_m = \sqrt{2}/\sqrt{m(m+1)}$.

In this basis the Cartan matrix is ,

$$K_{ij} = \mathrm{Tr}(H_i H_j) = 2\,\delta_{ij}$$

It is useful at this point to rephrase the discussion of symmetry breaking in this language.[19]

We want to solve the equation,

$$[\alpha , h] = 0$$

where $\alpha = \underline{a}\cdot\underline{H}$, where $\underline{a}$ is an r -vector. Let h be a general element of the Lie algebra of $\mathcal{G}$,

$$h = \sum_1^r h_i H_i + \sum_1^{\frac{1}{2}(d-r)} \left(h_i^{(+)} E_i^{(+)} + h_i^{(-)} E_i^{(-)} \right)$$

Then the commutation condition is, by the definition of root,

$$\sum_j \left(\underline{a}\cdot\underline{\alpha}_j\, h_j^{(+)} E_j^{(+)} - \underline{a}\cdot\underline{\alpha}_j\, h_j^{(-)} E_j^{(-)} \right) = 0$$

and by the linear independence of the E_j this vanishes only for those roots which are orthogonal to $\underline{a}$. Clearly too the h_i can be non-zero. Then the unbroken symmetry group will be generated by the Cartan subalgebra and the raising and lowering operators which correspond to these roots. The roots so selected with the requisite elements of H will generate subgroups and products of subgroups of $\mathcal{G}$ and $\mathcal{H}$ will also contain what remains of the Cartan subalgebra as a handful of $U(1)$ factors.

In $SU(4)$ for example, the simple roots can be chosen to be :

$$\underline{\alpha}_1 = (2,0,0)$$
$$\underline{\alpha}_2 = (-1,\sqrt{3},0)$$
$$\underline{\alpha}_3 = \left(0,-\frac{2}{\sqrt{3}},\frac{2\sqrt{2}}{\sqrt{3}}\right)$$

and there are four generic possibilities :

$$\alpha = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix}, \quad \underline{a} = (0,0,\sqrt{6}) \; : \; \left.\begin{array}{l} \underline{a}\cdot\underline{\alpha}_1 = 0 \\ \underline{a}\cdot\underline{\alpha}_2 = 0 \\ \underline{a}\cdot(\underline{\alpha}_1+\underline{\alpha}_2) = 0 \end{array}\right\} \; \mathcal{H} = U(1)\times SU(3)$$

$$\alpha = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \underline{a} = \left(0,\frac{2}{\sqrt{3}},\frac{\sqrt{2}}{\sqrt{3}}\right) \; : \; \left.\begin{array}{l} \underline{a}\cdot\underline{\alpha}_1 = 0 \\ \underline{a}\cdot\underline{\alpha}_3 = 0 \end{array}\right\} \; \mathcal{H} = U(1)\times SU(2) \times SU(2)$$

$$(\underline{\alpha}_1+\underline{\alpha}_3 \neq \text{root})$$

$$\alpha = \begin{pmatrix} -2x-y & & & \\ & y & & \\ & & x & \\ & & & x \end{pmatrix}, \quad \underline{a} = \left(-1,\frac{-2}{\sqrt{3}},-\frac{\sqrt{2}}{\sqrt{3}}\right)x + (-1,0,0)\,y \; : \; \underline{a}\cdot\underline{\alpha}_3 = 0 \Big\} \; \mathcal{H} = U(1)^2\times SU(2)$$

Otherwise $\underline{a}\cdot\underline{\alpha}_i \neq 0$, $\forall i$. $\mathcal{H} = H = U(1)^3$.

4.2 Spherical symmetry.

Now we shall attempt to find a way of locating the spherically symmetric monopoles in any given ansatz, that is, given values for the l_i . We shall be guided by what we know of the simpler cases.

The full equations of spherical symmetry are :

$$2\gamma \frac{\partial}{\partial \gamma} g = gA - ag \tag{4.1}$$

$$\left(\frac{\partial}{\partial \zeta} - \frac{\gamma}{\zeta}\cdot\frac{\partial}{\partial \gamma}\right) g = -bg \tag{4.2}$$

$$\left(\zeta^2 \frac{\partial}{\partial \zeta} + \zeta\gamma \frac{\partial}{\partial \gamma}\right) g = gB \tag{4.3}$$

The fact that the differential operator in $(4.2(3))$ is holomorphic away from $\zeta = 0(\infty)$ allows us to use just the bundle transformation of similar analyticity and to assume that b and B are no more than degree 2 in $\frac{1}{\zeta}$ and ζ respectively.

The operator $\zeta\frac{\partial}{\partial\zeta}$ can introduce no poles at $\zeta = 0,\infty$ and so A and a must be constant. We choose them, after the precedent of $SU(3)$, to lie in the Cartan subalgebra :

$$A = \underline{A}\cdot\underline{H} \quad , \quad a = \underline{a}\cdot\underline{H}$$

In the upper triangular ansatz g has a diagonal of the form,

$$\mathrm{diag}(g) = \zeta^{\underline{l}\cdot\underline{H}} \exp(\underline{d}\cdot\underline{H}\gamma)$$

where $\underline{l}\cdot\underline{H} = \mathrm{diag}(l_1, l_2, \ldots l_n)$ and $\underline{d}\cdot\underline{H} = \mathrm{diag}(d_1, \ldots d_n)$. Then these diagonal components of (4.1) give :

$$2\underline{l}\cdot\underline{H} = \underline{A}\cdot\underline{H} - \underline{a}\cdot\underline{H}$$

Then,

$$\underline{A} = \underline{l} + \underline{c} \quad , \quad \underline{a} = -\underline{l} + \underline{c}$$

and the form,

$$g(\gamma,\zeta) = \left(\frac{\gamma}{\zeta}\right)^{-\frac{1}{2}\underline{l}\cdot\underline{H}} \zeta^{-\frac{1}{2}\underline{c}\cdot\underline{H}}\, \tilde{g}(\gamma)\, (\gamma\zeta)^{\frac{1}{2}\underline{l}\cdot\underline{H}}\, \zeta^{\frac{1}{2}\underline{c}\cdot\underline{H}}$$

where $\tilde{g}$ is upper triangular with diagonal $e^{\underline{\lambda}\cdot\underline{H}\gamma}$, satisfies both the ansatz and equation (4.1) quite generally. A factor of $\gamma^{\frac{1}{2}\underline{\lambda}\cdot\underline{H}}$ has been pulled out from each side to simplify the other two equations. They become :

$$-bg = \frac{1}{2\zeta}[g, \underline{c}\cdot\underline{H}] + \frac{1}{\zeta}\underline{\lambda}\cdot\underline{H}\, g - \frac{\gamma}{\zeta}\left(\frac{\gamma}{\zeta}\right)^{-\frac{1}{2}\underline{\lambda}\cdot\underline{H}} \zeta^{-\frac{1}{2}\underline{c}\cdot\underline{H}} \frac{\partial\tilde{g}}{\partial\gamma} (\gamma\zeta)^{\frac{1}{2}\underline{\lambda}\cdot\underline{H}} \zeta^{\frac{1}{2}\underline{c}\cdot\underline{H}}$$

$$g\mathcal{B} = \frac{\zeta}{2}[g, \underline{c}\cdot\underline{H}] + \zeta g\, \underline{\lambda}\cdot\underline{H} + \gamma\zeta\left(\frac{\gamma}{\zeta}\right)^{-\frac{1}{2}\underline{\lambda}\cdot\underline{H}} \zeta^{-\frac{1}{2}\underline{c}\cdot\underline{H}} \frac{\partial\tilde{g}}{\partial\gamma} (\gamma\zeta)^{\frac{1}{2}\underline{\lambda}\cdot\underline{H}} \zeta^{\frac{1}{2}\underline{c}\cdot\underline{H}}$$

We could remove the $[g, \underline{c}\cdot\underline{H}]$ term by introducing suitable powers of γ into the form of g. But this would have to be done in reciprocal ways for each equation. So it is convenient to set $\underline{c}=0$ and we do not appear to lose any generality this way. Certainly, $\underline{c}=0$ for spherical symmetry in $SU(3)$.

We write b and $\mathcal{B}$ in the shapes,

$$b = -\frac{1}{\zeta}\underline{\lambda}\cdot\underline{H} + \beta$$

$$\mathcal{B} = \zeta\,\underline{\lambda}\cdot\underline{H} + \mathfrak{B}$$

and the above equations reduce to :

$$\gamma\frac{\partial\tilde{g}}{\partial\gamma}\tilde{g}^{-1} = \zeta\left(\frac{\gamma}{\zeta}\right)^{\frac{1}{2}\underline{\lambda}\cdot\underline{H}} \beta \left(\frac{\gamma}{\zeta}\right)^{-\frac{1}{2}\underline{\lambda}\cdot\underline{H}}$$

$$\gamma\,\tilde{g}^{-1}\frac{\partial\tilde{g}}{\partial\gamma} = \zeta^{-1}(\gamma\zeta)^{\frac{1}{2}\underline{\lambda}\cdot\underline{H}}\, \mathfrak{B}\, (\gamma\zeta)^{-\frac{1}{2}\underline{\lambda}\cdot\underline{H}}$$

Now β is upper triangular and can be written ,

$$\beta = \underline{p}\cdot\underline{H} + \sum_{\Delta^{(+)}} \beta_i E_i^{(+)}$$

where $\Delta^{(+)}$ is the set of positive roots. Using the formula, which is easily proved,

$$e^{\underline{p}\cdot\underline{H}} E_i^{(+)} e^{-\underline{p}\cdot\underline{H}} = e^{\underline{p}\cdot\underline{\alpha}_i} E_i^{(+)}$$

we obtain,

$$\gamma \frac{\partial \tilde{g}}{\partial \gamma} \tilde{g}^{-1} = \zeta \underline{\beta} \cdot \underline{H} + \sum_{\Delta^{(+)}} \zeta \beta_i \left(\frac{\gamma}{\zeta}\right)^{\frac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_i} E_i^{(+)} \qquad (4.4)$$

The diagonal elements of $\tilde{g}$ then require that $\underline{\beta} = \frac{\gamma}{\zeta} \underline{\alpha}$. Otherwise the right hand side must be independent of ζ , so that,

$$\beta_i = \zeta^{\frac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_i - 1} \tilde{\beta}_i(\gamma)$$

If β_i is to be at most quadratic in ζ^{-1} , $\frac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_i$ has to take on the values -1 , 0 or 1 , in which case $\tilde{\beta}_i$ must be of degree at most 0 , 1 or 0 respectively in γ . If for a particular root $\frac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_i$ is some other value then the corresponding β_i vanishes. In fact the case $\frac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_i = -1$ is also unacceptable because it introduces a γ^{-2} term into $\frac{\partial \tilde{g}}{\partial \gamma}\tilde{g}^{-1}$ which could only come from something like $\exp \frac{1}{\gamma}$ and this is disallowed on grounds of reality and singularity.

(4.4) can easily be solved in the case where for those $\frac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_i$ which vanish we choose $\beta_i = \tilde{\beta}_{i1} \frac{\gamma}{\zeta}$ so that no γ^{-1} term appears. In that case,

$$\frac{\partial \tilde{g}}{\partial \gamma} \tilde{g}^{-1} = \underline{\alpha}\cdot\underline{H} + \sum_{\Delta^{(+)}(\underline{\lambda})} \tilde{\beta}_i E_i^{(+)} \text{ , constant}$$

and so :

$$g = \left(\frac{\gamma}{\zeta}\right)^{-\frac{1}{2}\underline{\lambda}\cdot\underline{H}} \exp\left(\underline{\alpha}\cdot\underline{H}\gamma + \sum_{\Delta^{(+)}(\underline{\lambda})} \tilde{\beta}_i E_i^{(+)} \gamma\right)(\gamma\zeta)^{\frac{1}{2}\underline{\lambda}\cdot\underline{H}} \qquad (4.5)$$

where $\Delta^{(+)}(\underline{\lambda})$ consists of those positive roots which have $\frac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_i = 0, 1$. The same form can be deduced from the equation in β .

Such a g has the correct singularities, that is, it has no poles at $\gamma = 0$ but only at $\zeta = 0$, $\zeta = \infty$. To see this consider a typical term in the expansion of the exponential, neglecting powers of ζ :

$$\frac{1}{m!} \gamma^{-\frac{1}{2}\underline{\lambda}\cdot\underline{H}} \left(\underline{\alpha}\cdot\underline{H} + \sum_{\Delta^{(+)}(\underline{\lambda})} \tilde{\beta}_i E_i^{(+)}\right) \gamma^m \gamma^{\frac{1}{2}\underline{\lambda}\cdot\underline{H}}$$

The effects of the matricial exponents of γ on a general $E_i^{(+)}$ is to multiply it by a power $\gamma^{m-\frac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_i}$. But the worst behaviour in the expansion of the bracketed term must come from a concatenation of m raising operators for which $\frac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_i = 0,1$ provided the sum of their roots is again a root. But then the scalar product of $\underline{\lambda}$ with this root must be less than or equal to m and so $m-\frac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_i \geqslant 0$. Thus all the powers of γ appearing are positive or zero and $\mathcal{g}$ is holomorphic except at $\gamma=0$, ∞.

Reality is less easily proved. Suffice it to say that all the known spherical monopoles are of this form for α_i and $\tilde{\beta}_i$ real.

The constraint that there be at least one root for which $\frac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_i = 0,1$ is clearly very restrictive on $\underline{\lambda}$. It is always sufficient to look at the values for the simple roots to see which, if any, other roots have either of these values. Hence, given an $\underline{\lambda}$ (that is, a choice of asymptotic behaviour for the magnetic field) there may or may not be corresponding spherically symmetric monopoles. But for small enough suitable values of $\underline{\lambda}$ they will exist and in this case $\underline{\lambda}$ selects roots of the Lie algebra and the monopoles are constructed from these in a simple way, (4.5) .

The orthogonality of $\underline{\lambda}$ with some set of roots has been discussed already under the guise of symmetry breaking. It will happen when sets of the λ_i are equal. If in such a case we choose the corresponding α_i to be equal also then the relevant $\tilde{\beta}_i$ can be gauged to zero.

4.3 Results for $SU(2)$ and $SU(3)$.

To see that we do indeed recover the results from chapter III consider first $SU(2)$ and $SU(3)$. The former has only one positive root, length 2, and so $\frac{1}{2}\underline{l}\cdot\underline{\alpha} = 1$ only for $l = 1$. We have a single monopole with patching function :

$$g = \left(\frac{\gamma}{\zeta}\right)^{-\frac{1}{2}H_1} \exp\left[(\alpha H_1 + \beta_1 E_1^{(+)})\gamma\right] (\gamma\zeta)^{\frac{1}{2}H_1}$$

where $H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $E_1^{(+)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It can be rewritten in the familiar form,

$$g = \begin{pmatrix} (\gamma/\zeta)^{-1/2} & 0 \\ 0 & (\gamma/\zeta)^{1/2} \end{pmatrix} \exp\begin{pmatrix} \alpha\gamma & \beta_1\gamma \\ 0 & -\alpha\gamma \end{pmatrix} \begin{pmatrix} (\gamma\zeta)^{1/2} & 0 \\ 0 & (\gamma\zeta)^{-1/2} \end{pmatrix}$$

$$= \begin{pmatrix} \zeta e^{\alpha\gamma} & \beta_1 \dfrac{\operatorname{sh}\alpha\gamma}{\alpha\gamma} \\ 0 & \zeta^{-1} e^{-\alpha\gamma} \end{pmatrix}$$

$SU(3)$ has three positive roots,

$$\underline{\alpha}_1 = (2,0) \quad , \quad \underline{\alpha}_2 = (-1,\sqrt{3}) \quad , \quad \underline{\alpha}_3 = \underline{\alpha}_1 + \underline{\alpha}_2 .$$

There are two possible $\underline{l}$, in view of the ordering of the l_i :

$$\underline{l} = (1,\sqrt{3}) \qquad \tfrac{1}{2}\underline{l}\cdot\underline{\alpha}_1 = \tfrac{1}{2}\underline{l}\cdot\underline{\alpha}_2 = 1 \qquad \underline{l}\cdot\underline{H} = \operatorname{diag}(2,0,-2)$$

$$\underline{l} = \left(\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\right) \qquad \tfrac{1}{2}\underline{l}\cdot\underline{\alpha}_1 = \tfrac{1}{2}\underline{l}\cdot\underline{\alpha}_2 = \tfrac{1}{2} \qquad \underline{l}\cdot\underline{H} = \operatorname{diag}(1,0,-1)$$

Note that we do not have $\underline{l}\cdot\underline{H} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$. Although there is a spherically symmetric monopole in this ansatz, it occurs for $\alpha_1 = \alpha_2$ (breaking to $U(2)$) in which case it is bundle equivalent to the first of the above cases.

In the second case only $\underline{\alpha}_3$ contributes to $\tilde{g}$ and we simply have an $E_3^{(+)}$ embedding of the $SU(2)$ monopole.

In the first case there are three possibilities.

If only one of $\tilde{\beta}_1$ and $\tilde{\beta}_2$ is non-zero, we again have embeddings of the $SU(2)$ monopole :

$$\begin{pmatrix} \gamma^2 e^{\alpha_1\gamma} & \rho & 0 \\ 0 & e^{\alpha_2\gamma} & 0 \\ 0 & 0 & \gamma^{-2}e^{\alpha_3\gamma} \end{pmatrix} , \quad \begin{pmatrix} \gamma^2 e^{\alpha_1\gamma} & 0 & 0 \\ 0 & e^{\alpha_2\gamma} & \rho \\ 0 & 0 & \gamma^{-2}e^{\alpha_3\gamma} \end{pmatrix}$$

If neither vanish,

$$g = \begin{pmatrix} \gamma/\gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma/\gamma \end{pmatrix} \exp\left[\begin{pmatrix} \alpha_1 & 1 & 0 \\ 0 & \alpha_2 & 1 \\ 0 & 0 & \alpha_3 \end{pmatrix}\gamma\right]\begin{pmatrix} \gamma\gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\gamma\gamma} \end{pmatrix}$$

where we choose $\tilde{\beta}_1 = \tilde{\beta}_2 = 1$. But this is easily verified to be the maximal embedding of charge 2 that we recovered earlier, (3.19) .

4.4 __$SU(4)$ and maximal embeddings.__

The Chevalley basis is :

$$H_1 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} , \quad H_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -2 & \\ & & & 0 \end{pmatrix} , \quad H_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{pmatrix}$$

and the simple roots :

$$\underline{\alpha}_1 = (2,0,0) \qquad \underline{\alpha}_2 = (-1,\sqrt{3},0) \qquad \underline{\alpha}_3 = \left(0, \frac{-2}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}\right)$$

We identify the following cases.

(i) $$\underline{\lambda}\cdot\underline{H} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix} , \quad \underline{\lambda} = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right) , \quad \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_1 = \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_3 = \tfrac{1}{2} , \ \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_2 = 0$$

We have non-vanishing β_i for the $\underline{\alpha}_2$ and $\underline{\alpha}_1 + \underline{\alpha}_2 + \underline{\alpha}_3$ positions. The second gives a simple $SU(2)$ embedding.

(ii) $$\underline{\lambda}\cdot\underline{H} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} , \quad \underline{\lambda} = \left(0, \frac{2}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right) , \quad \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_1 = \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_3 = 0, \ \frac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_2 = 1 .$$

All roots contribute. This must include charge one simple embeddings and their superpositions as well as case (i).

(iii)

$$\underline{\lambda}\cdot\underline{H} = \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & -2 \end{pmatrix}, \quad \underline{\lambda} = \left(1, \frac{1}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}\right), \quad \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_1 = 1, \ \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_2 = 0, \ \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_3 = 1 .$$

Only $\underline{\alpha}_1 + \underline{\alpha}_2 + \underline{\alpha}_3$ does not contribute. It will have a pair of simple embeddings for $\underline{\alpha}_1$ and $\underline{\alpha}_3$ and two embeddings of the $SU(3)$ charge two monopole, $(\underline{\alpha}_1, \underline{\alpha}_2 + \underline{\alpha}_3)$ and $(\underline{\alpha}_1 + \underline{\alpha}_2, \underline{\alpha}_3)$.

(iv)

$$\underline{\lambda}\cdot\underline{H} = \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & -1 & \\ & & & -2 \end{pmatrix}, \quad \underline{\lambda} = \left(\frac{1}{2}, \frac{5}{2\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}\right), \quad \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_1 = \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_3 = \tfrac{1}{2}, \quad \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_2 = 1 .$$

Here is only one simple embedding at $\underline{\alpha}_2$.

(v)

$$\underline{\lambda}\cdot\underline{H} = \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{pmatrix}, \quad \underline{\lambda} = (1, \sqrt{3}, \sqrt{6}), \quad \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_1 = \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_2 = \tfrac{1}{2}\underline{\lambda}\cdot\underline{\alpha}_3 = 1$$

Only the simple roots contribute. It is the maximal embedding in $SU(4)$ of charge three when all of $\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_2$ are non-zero. Otherwise we have two embeddings of the charge two, $SU(3)$ monopole at $(\underline{\alpha}_1, \underline{\alpha}_2)$ and $(\underline{\alpha}_2, \underline{\alpha}_3)$, a charge two double embedding of $SU(2)$ monopoles at $(\underline{\alpha}_1, \underline{\alpha}_3)$ and three simple embeddings at each of the simple roots.

The pattern for maximal embeddings means that for each $SU(n)$ we get a new charge $n-1$ monopole in the ansatz where $\underline{\lambda}$ is $(N_1, N_2, \ldots N_n)$, so $\underline{\lambda}\cdot\underline{H} = \mathrm{diag}(n-1, n-3, \ldots -n+1)$ and the patching function is,

$$g = \left(\frac{\gamma}{\zeta}\right)^{-\frac{1}{2}\underline{\lambda}\cdot\underline{H}} \exp\left[\begin{pmatrix} \alpha_1 & 1 & 0 & 0 & \cdots & 0 \\ & \alpha_2 & 1 & 0 & & \vdots \\ & & \alpha_3 & 1 & & 0 \\ & 0 & & & \ddots & 1 \\ & & & & & \alpha_n \end{pmatrix}\gamma\right] (\gamma\zeta)^{\frac{1}{2}\underline{\lambda}\cdot\underline{H}} \tag{4.6}$$

which is precisely (3.20) .

For the other possible $\underline{l}$ there is generally room for maximal embeddings from lower groups and for their superpositions .

It remains to ennumerate all the possibilities and to check their reality and regularity. When we have completed the splitting we shall see that (4.6) is the charge $n-1$ solution of earlier work.[5]

Chapter V

Regularity of the fields

The real bugbear of the Atitah-Ward construction is the post hoc necessity of showing that a given patching matrix can be split in a regular fashion, that is, in such a way that the Higgs and Gauge fields do not have singularities.

5.1 The $SU(2)$ splitting procedure.[8]

We start by looking at the simplest case : g is,

$$\begin{pmatrix} \zeta^{l} e^{\alpha\gamma} & \rho(\gamma,\zeta) \\ 0 & \zeta^{-l} e^{-\alpha\gamma} \end{pmatrix}$$

where $\gamma = \zeta x - 2z - \frac{1}{\zeta}\bar{x}$ and we can write :

$$g = \begin{pmatrix} e^{-\alpha(z+\bar{x}/\zeta)} & 0 \\ 0 & e^{\alpha(z+\bar{x}/\zeta)} \end{pmatrix} \begin{pmatrix} \zeta^{l} & \tilde{\rho} \\ 0 & \zeta^{-l} \end{pmatrix} \begin{pmatrix} e^{\alpha(\zeta x - z)} & 0 \\ 0 & e^{-\alpha(\zeta x - z)} \end{pmatrix}$$

where $\tilde{\rho} = \rho \exp(\alpha(\frac{\bar{x}}{\zeta} + \zeta x))$. Now we can attempt the splitting in two stages. Let,

$$\begin{pmatrix} \zeta^{l} & \tilde{\rho} \\ 0 & \zeta^{-l} \end{pmatrix} = \begin{pmatrix} 1 & h_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta^{l} & \hat{\rho} \\ 0 & \zeta^{-l} \end{pmatrix} \begin{pmatrix} 1 & H_{12} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \zeta^{l} & h_{12}\zeta^{-l} + \hat{\rho} + H_{12}\zeta^{l} \\ 0 & \zeta^{-l} \end{pmatrix}$$

If we expand $\tilde{\rho}$ as a Laurent series on an open annulus in $U_0 \cap U_\infty$, in the ζ-plane,

$$\tilde{\rho} = \sum_{-\infty}^{\infty} \zeta^{r} \tilde{\rho}_r \qquad \text{where} \qquad \tilde{\rho}_r = \frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta}\, \zeta^{-r}\tilde{\rho}$$

we see that H_{12} and h_{12} can be chosen so that $\hat{\rho}$ has only the coefficients between $-\ell+1$ and $\ell-1$,

$$\hat{\rho} = \sum_{-\ell+1}^{\ell-1} \zeta^r \tilde{\rho}_r$$

and the splitting of g is now equivalent to that of $\hat{g}$.

To calculate the fields proper it is more circumspect to choose H_{12}, h_{12} to have no constant terms, in which case $\hat{\rho}$ will consist in the central $2\ell+1$ (instead of $2\ell-1$) coeffi-cients of $\tilde{\rho}$. Then $H(0) = h(\infty) = 1$, and the splitting of $\hat{g}$ gives the correct factors to calculate ϕ and A_i, (2.2) modulo the exponential factors extracted at the start.

For proof of regularity however the first choice of $\hat{\rho}$ serves us more simply. Accordingly we seek to solve the four equations :

$$\begin{pmatrix} \zeta^{\ell} & \sum_{-\ell+1}^{\ell-1} \zeta^r \tilde{\rho}_r \\ 0 & \zeta^{-\ell} \end{pmatrix} \begin{pmatrix} \hat{k}_{1i} \\ \hat{k}_{2i} \end{pmatrix} = \begin{pmatrix} \hat{h}_{1i} \\ \hat{h}_{2i} \end{pmatrix}, \quad i = 1,2. \tag{5.1}$$

Using the Taylor and Laurent expansions of $\hat{k}$ and $\hat{h}$ we obtain :

$$\hat{h}_{2i,0} = \hat{k}_{2i,\ell} \tag{5.2}$$

$$\hat{h}_{1i,0} = \hat{k}_{1i,\ell-1} \tilde{\rho}_{-\ell+1} + \hat{k}_{1i,\ell-2} \tilde{\rho}_{-\ell+2} + \cdots + \hat{k}_{1i,0} \tilde{\rho}_0$$

and, from the coefficients of positive powers of ζ, which vanish on the right hand side :

$$\begin{aligned}
\hat{k}_{2i,\ell} \tilde{\rho}_{-\ell+1} + \hat{k}_{2i,\ell-1} \tilde{\rho}_{-\ell+2} + \cdots &+ \hat{k}_{2i,0} \tilde{\rho}_1 = 0 \\
\hat{k}_{2i,\ell} \tilde{\rho}_{-\ell+2} + \hat{k}_{2i,\ell-1} \tilde{\rho}_{-\ell+3} + \cdots &+ \hat{k}_{2i,0} \tilde{\rho}_2 = 0 \\
\vdots \qquad\qquad \vdots \qquad\qquad &\qquad \vdots \qquad \vdots \\
\hat{k}_{2i,\ell} \tilde{\rho}_0 + \hat{k}_{2i,\ell-1} \tilde{\rho}_1 + \cdots &+ \hat{k}_{2i,0} \tilde{\rho}_{\ell-1} = -\hat{k}_{1i,0}
\end{aligned}$$

We can solve this system for $(\hat{k}_{2i,1}, \hat{k}_{2i,2}, \ldots, \hat{k}_{2i,\ell})$ in terms of $\hat{k}_{ij,0}$, then use (5.2) to relate $\hat{h}_{ij,0}$ to $\hat{k}_{rj,0}$. Then we make a choice of gauge, by fixing $\hat{k}_{ij,0}$ say, and so obtain $\hat{h}(\infty) = \hat{h}_{ij,0}$ and thus the fields from (2.2).

But the system's solubility depends on inverting the $\ell \times \ell$ matrix,

$$S = \begin{bmatrix} \tilde{\rho}_{-\ell+1} & \tilde{\rho}_{-\ell+2} & \tilde{\rho}_{-\ell+3} & \cdots & \tilde{\rho}_0 \\ \tilde{\rho}_{-\ell+2} & \tilde{\rho}_{-\ell+3} & & \iddots & \\ \tilde{\rho}_{-\ell+3} & & \iddots & & \\ \vdots & \iddots & & & \\ \tilde{\rho}_0 & & & & \tilde{\rho}_{\ell-1} \end{bmatrix}, \quad S_{ij} = \tilde{\rho}_{-\ell-1+i+j}$$

So for a given ρ we need to calculate $\mathrm{Det}\, S$ and show that it is non-vanishing throughout $\mathbb{R}^3$. In this event we can calculate explicit forms for the fields, although this is in general extremely messy and unenlightening. The form of $\mathrm{Det}\, S$ however is often quite interesting. It also allows us to check the asymptotic behaviour of the Higgs field via the equation :

$$\mathrm{Tr}\, \phi^2 = \mathrm{Const} - \tfrac{1}{2} \nabla^2 \ln \mathrm{Det}\, S$$

which has been shown to be generally true for $SU(2)$ [18]. An analogous formula has been demonstrated for some cases in $SU(3)$ [28,29].

5.2 The $SU(3)$ splitting procedure.

In $SU(3)$ the splitting procedure is complicated by the greater variety of diagonal entries. Let us take the form :

$$g = \begin{pmatrix} \zeta^n e^{\alpha_1 \gamma} & \rho_{12} & \rho_{13} \\ 0 & \zeta^m e^{\alpha_2 \gamma} & \rho_{23} \\ 0 & 0 & \zeta^{-n-m} e^{\alpha_3 \gamma} \end{pmatrix}, \quad \sum_1^3 \alpha_i = 0.$$

for $n \geq m \geq 0$ which includes the case we shall want to consider in detail. The cases $m < 0$ can be similarly treated.

As before we may remove the exponential factors and then truncate ρ_{ij}. However, because of the way the upper triangular matrices H and h affect the entries of $\tilde{g}$,

$$\begin{pmatrix} 1 & h_{12} & h_{13} \\ 0 & 1 & h_{23} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} z^{n} & \tilde{\rho}_{12} & \tilde{\rho}_{13} \\ 0 & z^{m} & \tilde{\rho}_{23} \\ 0 & 0 & z^{-n-m} \end{pmatrix}\begin{pmatrix} 1 & H_{12} & H_{13} \\ 0 & 1 & H_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} z^{n} & z^{n}H_{12} + \tilde{\rho}_{12} + z^{m}h_{12} & z^{n}H_{13} + \tilde{\rho}_{13} + z^{-n-m}h_{13} + h_{12}\tilde{\rho}_{23} + H_{23}\tilde{\rho}_{12} \\ & & + z^{m}H_{23}h_{12} \\ 0 & z^{m} & z^{m}H_{23} + \tilde{\rho}_{23} + z^{-n-m}h_{23} \\ 0 & 0 & z^{-n-m} \end{pmatrix}$$

one each of the truncated portions of $\tilde{\rho}_{12}$ and $\tilde{\rho}_{23}$ reappear in $\tilde{\rho}_{13}$:

$$\hat{\rho}_{12} = \sum_{m+1}^{n-1} \tilde{\rho}_{12,r} z^{r}$$

$$\hat{\rho}_{23} = \sum_{-n-m+1}^{m-1} \tilde{\rho}_{23,r} z^{r}$$

but,

$$\hat{\rho}_{13} = \sum_{-n-m+1}^{n-1} \tilde{P}_{13,r} z^{r}$$

where

$$\tilde{P}_{13} = \tilde{\rho}_{13} - z^{m}\left(\tilde{\rho}_{12}\tilde{\rho}_{23} - \left(\sum_{m+1}^{\infty} \tilde{\rho}_{12,r} z^{r} \right)\left(\sum_{-\infty}^{m-1} \tilde{\rho}_{23,s} z^{s} \right)\right)$$

Examination of the real form for ρ_{13}, (3.5), shows that this apparent awkwardness is a blessing in disguise, for it removes the term in $\tilde{\rho}_{13}$,

$$\frac{Fg}{\psi_1\psi_3}\, e^{\alpha_{23}(\zeta X-\zeta)+\alpha_{12}(\zeta+\bar{X}/\zeta)}$$

which would otherwise present difficulties to contour integration.

Taking Cauchy coefficients of the nine equations,

$$\begin{pmatrix} \zeta^{n} & \hat{\rho}_{12} & \hat{\rho}_{13} \\ 0 & \zeta^{m} & \hat{\rho}_{23} \\ 0 & 0 & \zeta^{-n-m} \end{pmatrix} \begin{pmatrix} \hat{k}_{1i} \\ \hat{k}_{2i} \\ \hat{k}_{3i} \end{pmatrix} = \begin{pmatrix} \hat{h}_{1i} \\ \hat{h}_{2i} \\ \hat{h}_{3i} \end{pmatrix}$$

then gives a linear system whose solubility depends on inverting the $(n+m)\times(n+m)$ matrix S , shown on the following page. For a few simple cases :

$$\left.\begin{array}{l} n=1 \\ m=0 \end{array}\right\} \qquad S = \tilde{P}_{13,0}$$

$$\left.\begin{array}{l} n=1 \\ m=1 \end{array}\right\} \qquad S = \begin{bmatrix} \tilde{P}_{13,0} & \tilde{P}_{13,-1} \\ \tilde{\rho}_{23,0} & \tilde{\rho}_{23,-1} \end{bmatrix}$$

$$\left.\begin{array}{l} n=2 \\ m=0 \end{array}\right\} \qquad S = \begin{bmatrix} \tilde{P}_{13,0} & \tilde{P}_{13,-1} \\ \tilde{P}_{13,1} & \tilde{P}_{13,0}-\tilde{\rho}_{12,1}\tilde{\rho}_{23,-1} \end{bmatrix}$$

The matrix S, for an $SU(3)$ patching function.

$$
\begin{array}{c}
\begin{array}{r}
n\left\{\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right. \\
m\left\{\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right.
\end{array}
\underbrace{\left[\begin{array}{ccc}
\tilde{P}_{13,0} & \cdots & \tilde{P}_{13,-n-m+1} \\
\vdots & & \vdots \\
\tilde{P}_{13,n-1} & \cdots & \tilde{P}_{13,-m} \\
\hline
\tilde{\rho}_{23,0} & \cdots & \tilde{\rho}_{23,-n-m+1} \\
\vdots & & \vdots \\
\tilde{\rho}_{23,m-1} & \cdots & \tilde{\rho}_{23,-n}
\end{array}\right]}_{n+m}
-
\overbrace{\left[\begin{array}{ccccc}
0 & & & & \\
\vdots & & O & & \\
\tilde{\rho}_{12,m+1} & \ddots & & & \\
\vdots & & \ddots & & \\
\tilde{\rho}_{12,n-1} & \cdots & \tilde{\rho}_{12,m+1} & 0 \cdots & 0 \\
\hline
 & & O & &
\end{array}\right]}^{n}
\times
\underbrace{\left[\begin{array}{ccccc}
0 & \tilde{\rho}_{23,m-1} & \cdots & \cdots & \tilde{\rho}_{23,-n+1} \\
\vdots & & \ddots & & \vdots \\
 & O & & \ddots & \vdots \\
 & & & \tilde{\rho}_{23,m-1} \cdots & \tilde{\rho}_{23,0}
\end{array}\right]}_{m+n}
\left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\}n
\end{array}
$$

$$S = A - BC \quad :$$

$$
A_{ij} = \begin{cases} \tilde{P}_{13,i-j} & \text{for } 1 \leq i \leq n,\ 1 \leq j \leq m+n \\ \tilde{\rho}_{23,i-j-n} & \text{for } n+1 \leq i \leq n+m,\ 1 \leq j \leq m+n \end{cases}
$$

$$
B_{ij} = \begin{cases} \tilde{\rho}_{12,i-j} & \text{for } 1 \leq i \leq n,\ j \leq i-m-1 \\ 0 & \text{otherwise} \end{cases}
$$

$$
C_{ij} = \begin{cases} \tilde{\rho}_{23,i-j+m} & \text{for } j > i \\ 0 & \text{otherwise} \end{cases}
$$

5.3 Regularity of maximal embeddings

Before we do an explicit example of this complexity, let us consider the simpler case of the maximally embedded, charge $n-1$ spherically symmetric monopole in $SU(n)$. Because of its symmetry it is necessary to show that it splits satisfactorily in one radial direction. If we choose the z-direction then $\gamma\big|_{x=0} = -2r$ and the only ζ-dependence of g is that of the powers of ζ in the ρ_{ij}. So we must split :

$$\bar{g} = \begin{bmatrix} \zeta^{n-1} e^{-2\alpha_1 r} & \zeta^{n-2}\bar{\rho}_{12} & \cdots & & \bar{\rho}_{1n} \\ 0 & \zeta^{n-3} e^{-2\alpha_2 r} & & & \\ \vdots & 0 & \ddots & & \\ \vdots & \vdots & & \zeta^{-n+3} e^{-2\alpha_{n-1} r} & \zeta^{-n+2}\bar{\rho}_{n-1,n} \\ 0 & 0 & & & \zeta^{-n+1} e^{-2\alpha_n r} \end{bmatrix}$$

where

$$\bar{\rho}_{ij} = \frac{1}{(-2z)^{i+j}} \sum_{i}^{j} e^{-2\alpha_k r} \Big(\prod_{\substack{m\neq k \\ i}}^{j} (\alpha_m - \alpha_k) \Big)^{-1}$$

Examination of $SU(2)$, $SU(3)$ and $SU(4)$ is enough to convince us that (choosing an anti-diagonal form for k_0) :

$$h_0 = \begin{bmatrix} D_1/D_0 & & & & \\ & D_2/D_1 & & 0 & \\ & & \ddots & & \\ & 0 & & D_{n-1}/D_{n-2} & \\ & & & & D_n/D_{n-1} \end{bmatrix}$$

where

$$D_r = \mathrm{Det} \begin{bmatrix} \bar{g}_{1,n+1-r} & \cdots & \bar{g}_{1,n} \\ \vdots & & \vdots \\ \bar{g}_{r,n+1-r} & \cdots & \bar{g}_{r,n} \end{bmatrix} , \quad D_0 = D_n = 1$$

and
$$\mathrm{Det}\, S = \prod_{0}^{n} D_r$$

Further, the Higgs field is,

$$\phi\big|_{x=0} = \frac{1}{2} h_0^{-1} \partial_z h_0 = \frac{1}{2} \sum_{1}^{n-1} H_m \frac{\partial}{\partial r} (\ln D_m)$$

where we have used a basis for the Cartan subalgebra different to that used in Chapter IV, namely,

$$H_m = \mathrm{diag}\,(0, \ldots .\, 0, \overset{m}{1}, \overset{m+1}{-1}, 0 \ldots \; 0)$$

From $\bar{\rho}_{ij}$ we see that the D_m have the shape :

$$D_m = \left(\frac{1}{2r}\right)^{m(n-m)} Q_m(-2r)$$

The Q_m are infact (see Appendix) functions familiar from earlier work,[5] from which we know that ϕ has precisely the above form. They are also non-vanishing except at the origin where they have zeros of order $m(n-m)$, exactly cancelled by the factors of r in D_m to give a non-zero limit. Then $\mathrm{Det}\, S$ is nowhere vanishing on the z-axis.

Spherical symmetry means that under a small rotation,

$$g \rightarrow a\, g\, A$$

and so,
$$h(\infty) \rightarrow a(\infty) h(\infty) \qquad , \qquad k(0) \rightarrow A^{-1}(0)\, k(0)$$

and since no singularities can be introduced this way we are able both to calculate the A_i and to argue that $\mathrm{Det}\, S$ is nowhere vanishing throughout $\mathbb{R}^3$.

5.4 Regularity of a charge 2 , one parameter family.

In general we must avail ourselves of the full splitting procedure.

Those patching functions which have been shown to factorize to give smooth fields are the axially symmetric[16,17,18] and some separated[21,46] ones in $SU(2)$, the "deformed " embedding of the single $SU(2)$

monopole in $SU(3)$ [29] and a family of charge 2 cylindrical monopoles in $SU(3)$ which includes both breakings to $U(2)$ [28] and $U(1)\times U(1)$ [30] and contains the charge 2 spherical one. We shall give the calculation for the $U(1)\times U(1)$ breaking.

It belongs to the ansatz $\underline{\lambda}\cdot\underline{H}=(2,0,-2)$ and the $\tilde{\rho}_{ij}$, once the $\exp(\underline{\alpha}\cdot\underline{H}\gamma)$ has been factored out, are given by :

$$\tilde{\rho}_{12} = \zeta\,\frac{e^{\alpha_{12}\nu}-e^{\alpha_{12}(\mu-c_1)}}{\alpha_{21}(\mu-\nu-c_1)}$$

$$\tilde{\rho}_{23} = \zeta^{-1}\,\frac{e^{\alpha_{23}\mu}-e^{\alpha_{23}(\nu+c_2)}}{\alpha_{23}(\mu-\nu-c_2)}$$

$$\tilde{\rho}_{13} = \frac{e^{\alpha_{12}\nu+\alpha_{23}\mu}+H_0e^{\alpha_{13}\mu}+h_0e^{\alpha_{13}\nu}}{\alpha_{21}\alpha_{23}(\mu-\nu-c_1)(\mu-\nu-c_2)}$$

where $H_0 = e^{\alpha_3 c}\cdot\dfrac{\sin\alpha_{32}\delta}{\sin\alpha_{13}\delta}$, $h_0=e^{\alpha_1 c}\cdot\dfrac{\sin\alpha_{21}\delta}{\sin\alpha_{13}\delta}$, $c_1=c+i\delta$, $c_2=c-i\delta$ and $\mu=-\gamma+X\zeta$, $\nu=\gamma+\bar{X}/\zeta$. c_1 and c_2 are complex conjugate to ensure the reality of γ.

To construct the function $\tilde{P}_{13}$ we need the half expansions $\sum_1^\infty \tilde{\rho}_{12,r}\zeta^r$ and $\sum_{-\infty}^{-1}\tilde{\rho}_{23,r}\zeta^r$. The forms of $\tilde{\rho}_{12}$ and $\tilde{\rho}_{23}$ allow us to find these simply by first separating off the terms in the numerator, then dealing with the residual poles from the denominator. Thus,

$$\tilde{\rho}_{23}\Big|_{-\infty}^{-1} = \frac{1}{\alpha_{23}X}\cdot\frac{e^{\alpha_{23}\mu}-e^{\alpha_{23}\nu+\alpha_{23}c_2}}{(\zeta-\zeta_2^+)(\zeta-\zeta_2^-)}\Big|_{-\infty}^{-1}$$

where $X\zeta_2^\pm=\gamma+\tfrac{1}{2}c_2\pm R_2$ and $R_2^2=(\gamma+\tfrac{1}{2}c_2)^2+\bar{X}X$,

$$\tilde{\rho}_{23}\Big|_{-\infty}^{-1} = \frac{1}{\alpha_{23}X}\cdot\frac{e^{\alpha_{23}(\gamma+c_2)}-e^{\alpha_{23}(\nu+c_2)}}{(\zeta-\zeta_2^+)(\zeta-\zeta_2^-)}$$
$$-\frac{1}{\alpha_{23}X}\cdot\frac{e^{\alpha_{23}(\gamma+c_2)}-e^{\alpha_{23}(\nu_2^++c_2)}}{(\zeta_2^+-\zeta_2^-)(\zeta-\zeta_2^+)}-\frac{1}{\alpha_{23}X}\cdot\frac{e^{\alpha_{23}(\gamma+c_2)}-e^{\alpha_{23}(\nu_2^-+c_2)}}{(\zeta-\zeta_2^-)(\zeta_2^--\zeta_2^+)}$$

where $\nu_2^\pm = \gamma+\bar{X}/\zeta_2^\pm$.

$\tilde{\rho}_{12}$ is a little more involved because of the higher powers of ζ in the numerator which mean that more of the terms from $e^{\alpha_{12}\nu}$ have to be included. But we obtain the results :

$$\tilde{\rho}_{23}\Big|_{-\infty}^{-1} = \frac{\zeta^{-1} e^{\alpha_{23}c_2/2}}{\alpha_{23}(\gamma - c_2)} \cdot \left[\operatorname{ch}\alpha_{23}R_2 + (\mu - \tfrac{c_2}{2}) \frac{\operatorname{sh}\alpha_{23}R_2}{R_2} - e^{\alpha_{23}\nu + \frac{1}{2}\alpha_{23}c_2} \right]$$

$$\tilde{\rho}_{12}\Big|_{1}^{\infty} = \frac{\zeta e^{\frac{1}{2}\alpha_{21}c_1}}{\alpha_{21}(\gamma - c_1)} \cdot \left[\operatorname{ch}\alpha_{12}R_1 + (\nu + \tfrac{c_1}{2}) \frac{\operatorname{sh}\alpha_{12}R_1}{R_1} - e^{\alpha_{12}\mu + \frac{1}{2}\alpha_{21}c_1} \right]$$

and so have,

$$\begin{aligned} \tilde{P}_{13} = \frac{1}{\alpha_{21}\alpha_{23}} \cdot \frac{1}{(\gamma - c_1)(\gamma - c_2)} \Big\{ & e^{\alpha_{13}\mu}(H_0 + e^{\alpha_{21}c_1}) + e^{\alpha_{13}\nu}(h_0 + e^{\alpha_{23}c_2}) \\ & - e^{\alpha_{23}(\nu + c_2) + \frac{1}{2}\alpha_{21}c_1} F_1 - e^{\alpha_{12}(\mu - c_1) + \frac{1}{2}\alpha_{23}c_2} F_2 \\ & + e^{\frac{1}{2}\alpha_{23}c_2 + \frac{1}{2}\alpha_{21}c_1} F_1 F_2 \Big\} \end{aligned}$$

where
$$F_1 = \operatorname{ch}\alpha_{12}R_1 + (\nu + \tfrac{c_1}{2}) \frac{\operatorname{sh}\alpha_{12}R_1}{R_1}$$

$$F_2 = \operatorname{ch}\alpha_{23}R_2 + (\mu - \tfrac{c_2}{2}) \frac{\operatorname{sh}\alpha_{23}R_2}{R_2}$$

Now all the functions $\tilde{\rho}_{12}$, $\tilde{\rho}_{23}$ and $\tilde{P}_{13}$ have singularities at $\zeta = 0$ and $\zeta = \infty$ only. We calculate their Laurent coefficients by integrating around a contour C enclosing the origin :

$$\begin{aligned} \tilde{\rho}_{12,1} &= \frac{1}{2\pi i} \oint_C \frac{d\zeta}{\zeta} \cdot \zeta^{-1} \tilde{\rho}_{12} \\ &= \frac{1}{2\pi i} \oint_C d\zeta \cdot \frac{e^{\alpha_{12}\nu} - e^{\alpha_{12}(\mu - c_1)}}{\alpha_{21} X (\zeta - \zeta_1^+)(\zeta - \zeta_1^-)} \end{aligned}$$

Let C expand towards a circle at infinity. Then we lose the first term, whose behaviour is like $\exp \zeta^{-1}$. If we then shrink the contour to zero we shall lose the other term, which has behaviour $\exp \zeta$, but in so doing pick up the poles at ζ_1^+, ζ_1^- to get :

$$\tilde{\rho}_{12,1} = \frac{1}{\alpha_{21}X} \left\{ \frac{e^{\alpha_{12}(\mu_1^+ - c_1)}}{\zeta_1^+ - \zeta_1^-} + \frac{e^{\alpha_{12}(\mu_1^- - c_1)}}{\zeta_1^- - \zeta_1^+} \right\} = e^{-\frac{1}{2}\alpha_{12}c_1} \frac{\operatorname{sh}\alpha_{12}R_1}{\alpha_{21}R_1}$$

Similar arguments lead to complicated expressions for the $\tilde{P}_{13,r}$:

$$\tilde{P}_{13,0} = \frac{-1}{\alpha_{21}\alpha_{23}} \cdot e^{\frac{3}{2}\alpha_2 c - \frac{i}{2}\alpha_{13}\delta} \cdot \frac{1}{2i\delta} \cdot \left\{ \frac{\sin\alpha_{21}\delta}{\sin\alpha_{13}\delta} \cdot \frac{sh\,\alpha_{13}R_1}{R_1} - \frac{\sin\alpha_{23}\delta}{\sin\alpha_{13}\delta} \cdot \frac{sh\,\alpha_{13}R_2}{R_2} \right.$$

$$\left. + ch\,\alpha_{23}R_2 \frac{sh\,\alpha_{12}R_1}{R_1} + ch\,\alpha_{12}R_1 \frac{sh\,\alpha_{23}R_2}{R_2} + i\delta \frac{sh\,\alpha_{12}R_1}{R_1} \cdot \frac{sh\,\alpha_{23}R_2}{R_2} \right\}$$

$$\tilde{P}_{13,-1} = \frac{-1}{\alpha_{21}\alpha_{23}} \cdot e^{\frac{3}{2}\alpha_2 c - \frac{i}{2}\alpha_{13}\delta} \cdot \frac{1}{2iX\delta} \cdot$$

$$\left\{ (\gamma + \tfrac{1}{2}c_1)\left(\frac{\sin\alpha_{21}\delta}{\sin\alpha_{13}\delta} \cdot \frac{sh\,\alpha_{13}R_1}{R_1} \right) \right.$$

$$-(\gamma + \tfrac{1}{2}c_2)\left(\frac{\sin\alpha_{23}\delta}{\sin\alpha_{13}\delta} \cdot \frac{sh\,\alpha_{13}R_2}{R_2} - \frac{sh\,\alpha_{23}R_2}{R_2} \cdot \left(ch\,\alpha_{12}R_1 + i\delta \frac{sh\,\alpha_{12}R_1}{R_1} \right) \right.$$

$$\left. - \frac{sh\,\alpha_{12}R_1}{R_1} \cdot ch\,\alpha_{23}R_2 \right)$$

$$\left. + \frac{\sin\alpha_{21}\delta}{\sin\alpha_{13}\delta} \cdot ch\,\alpha_{13}R_1 - \frac{\sin\alpha_{23}\delta}{\sin\alpha_{13}\delta} \cdot ch\,\alpha_{13}R_2 + ch\,\alpha_{12}R_1 \cdot ch\,\alpha_{23}R_2 + \frac{R_2}{R_1} sh\,\alpha_{23}R_2 \cdot sh\,\alpha_{12}R_1 \right\}$$

$$\tilde{P}_{13,1} = \frac{1}{\alpha_{21}\alpha_{23}} \cdot e^{\frac{3}{2}\alpha_2 c - \frac{i}{2}\alpha_{13}\delta} \cdot \frac{1}{2i\bar{\chi}\delta}$$

$$\Bigg\{ (\gamma + \tfrac{1}{2}c_1)\left(\frac{\sin\alpha_{21}\delta}{\sin\alpha_{13}\delta} \cdot \frac{\operatorname{sh}\alpha_{13}R_1}{R_1} \right)$$

$$- (\gamma + \tfrac{1}{2}c_2)\left(\frac{\sin\alpha_{23}\delta}{\sin\alpha_{13}\delta} \cdot \frac{\operatorname{sh}\alpha_{13}R_2}{R_2} - \frac{\operatorname{sh}\alpha_{23}R_2}{R_2}\left(\operatorname{ch}\alpha_{12}R_1 - i\delta\,\frac{\operatorname{sh}\alpha_{12}R_1}{R_1}\right) \right.$$

$$\left. - \frac{\operatorname{sh}\alpha_{12}R_1}{R_1} \cdot \operatorname{ch}\alpha_{23}R_2 \right)$$

$$- \frac{\sin\alpha_{21}\delta}{\sin\alpha_{13}\delta}\operatorname{ch}\alpha_{13}R_1 + \frac{\sin\alpha_{23}\delta}{\sin\alpha_{13}\delta}\operatorname{ch}\alpha_{13}R_2 - \frac{R_2}{R_1}\operatorname{sh}\alpha_{23}R_2 \cdot \operatorname{sh}\alpha_{12}R_1$$

$$- \operatorname{ch}\alpha_{23}R_2\left(\operatorname{ch}\alpha_{12}R_1 - i\delta\,\frac{\operatorname{sh}\alpha_{12}R_1}{R_1}\right) \Bigg\}$$

$$\tilde{P}_{13,0} - \tilde{\rho}_{12,1}\,\tilde{\rho}_{23,-1} = \frac{-1}{\alpha_{21}\alpha_{23}} \cdot \frac{e^{\frac{3}{2}\alpha_2 c - \frac{i}{2}\alpha_{13}\delta}}{2i\delta} \cdot$$

$$\Bigg\{ \frac{\sin\alpha_{21}\delta}{\sin\alpha_{13}\delta} \cdot \frac{\operatorname{sh}\alpha_{13}R_1}{R_1} - \frac{\sin\alpha_{23}\delta}{\sin\alpha_{13}\delta} \cdot \frac{\operatorname{sh}\alpha_{13}R_2}{R_2}$$

$$+ \operatorname{ch}\alpha_{23}R_2 \cdot \frac{\operatorname{sh}\alpha_{12}R_1}{R_1} + \operatorname{ch}\alpha_{12}R_1 \cdot \frac{\operatorname{sh}\alpha_{23}R_2}{R_2} - i\delta \cdot \frac{\operatorname{sh}\alpha_{12}R_1}{R_1} \cdot \frac{\operatorname{sh}\alpha_{23}R_2}{R_2} \Bigg\}$$

As a check on the progress of the calculation we can consider the limit $\delta \to 0$ in which we expect to recover the spherically symmetric monopole for which we have already seen the splitting work. So in $\tilde{P}_{13,0}$ put $R_1^2 = z^2 + i\delta z$, $R_2^2 = z^2 - i\delta z$, $c = 0$ and $X = 0$, so we are looking on the z-axis :

$$\tilde{P}_{13,0}(X=0,\delta\to 0) = \frac{1}{2i\delta z\,\alpha_{12}\alpha_{23}}\cdot\left(-\mathrm{sh}\,\alpha_{13}z + \mathrm{ch}\,\alpha_{13}z . \mathrm{sh}\,\alpha_{12}z + \mathrm{ch}\,\alpha_{12}z . \mathrm{sh}\,\alpha_{23}z\right)$$

$$+ \frac{1}{2iz\,\alpha_{12}\alpha_{23}}\cdot\left(\frac{3}{2}\alpha_2\,\mathrm{ch}\,\alpha_{13}z - \frac{3}{2}\alpha_2\,\mathrm{sh}\,\alpha_{23}z . \mathrm{sh}\,\alpha_{12}z - \frac{3}{2}\alpha_2\,\mathrm{ch}\,\alpha_{23}z\,\mathrm{ch}\,\alpha_{12}z\right)$$

$$+ \frac{1}{4iz^2\,\alpha_{12}\alpha_{23}}\cdot\left(-\frac{\alpha_{21}}{\alpha_{13}}\mathrm{sh}\,\alpha_{13}z - \frac{\alpha_{23}}{\alpha_{13}}\mathrm{sh}\,\alpha_{13}z + \mathrm{sh}\,3\alpha_2 z + 2\,\mathrm{sh}\,\alpha_{12}z . \mathrm{sh}\,\alpha_{23}z\right)$$

$$= \frac{e^{-\alpha_2 z}}{4i\,z^2}\left(\frac{e^{-2\alpha_1 z}}{\alpha_{12}\alpha_{13}} + \frac{e^{-2\alpha_2 z}}{\alpha_{21}\alpha_{23}} + \frac{e^{-2\alpha_3 z}}{\alpha_{31}\alpha_{32}}\right)$$

So
$$\check{P}_{13,0}(X=0,\delta=0) = \frac{e^{-\alpha_2 z}}{4iz^2}\,Q_2(2z) \quad \text{(see Appendix)}$$

Likewise,
$$(\check{P}_{13,0} - \check{\rho}_{12,1}\check{\rho}_{23,-1})(X=0,\delta=0) = \frac{e^{\alpha_2 z}}{4iz^2}\,Q_1(2z)$$

and $\tilde{P}_{13,-1}$, $\tilde{P}_{13,1}$ both vanish on the z-axis. Hence we recover the expected result :

$$\mathrm{Det}\, S = -\frac{1}{16z^4}\,Q_1(2z)\,Q_2(2z)$$

In general however we get the following expression :

$$\mathrm{Det}\, S = \check{P}_{13,0}\left(\check{P}_{13,0} - \check{\rho}_{12,1}\check{\rho}_{23,-1}\right) - \check{P}_{13,1}\check{P}_{13,-1}$$

$$= \frac{e^{3\alpha_2 c + i\alpha_{31}\delta}}{4\delta^2\,\bar{X}X}\cdot\left(\frac{\sin\alpha_{21}\delta}{\alpha_{21}}\cdot\frac{\sin\alpha_{23}\delta}{\alpha_{23}}\right)^2$$

$$\left\{\frac{1}{2}(1-\underline{n}_1\cdot\underline{n}_2)\,Q_1^c(R_+)\,Q_2^c(R_+) + \frac{1}{2}(1+\underline{n}_1\cdot\underline{n}_2)\,Q_1^c(R_-)\,Q_2^c(R_-)\right\}$$

where $\underline{R}_1 = \bar{\underline{R}}_2 = (x,y,z+\frac{1}{2}(c+i\delta))$, $\underline{n}_i = \frac{\underline{R}_i}{|\underline{R}_i|}$, $|\underline{R}_i|^2 = \underline{R}_i\cdot\underline{R}_i$, $R_\pm = R_1 \pm R_2$

The Q_i^c functions are simple generalizations of those governing the spherical behaviour and are given by the formulae :

$$Q_1^c(R) = \frac{e^{\alpha_1 R}}{\sin\alpha_{12}\delta . \sin\alpha_{13}\delta} + \frac{e^{\alpha_2 R}}{\sin\alpha_{21}\delta . \sin\alpha_{23}\delta} + \frac{e^{\alpha_3 R}}{\sin\alpha_{31}\delta . \sin\alpha_{32}\delta}$$

$$Q_2^c(R) = \frac{e^{-\alpha_1 R}}{\sin\alpha_{12}\delta . \sin\alpha_{13}\delta} + \frac{e^{-\alpha_2 R}}{\sin\alpha_{21}\delta . \sin\alpha_{23}\delta} + \frac{e^{-\alpha_3 R}}{\sin\alpha_{31}\delta . \sin\alpha_{32}\delta}$$

Finally we must show that $\mathrm{Det}\, S$ is non-vanishing throughout $\mathbb{R}^3$ for some range of values of the parameter δ .

By a translation of the origin along the z-axis we may choose $c = 0$. Putting $R_1 = \bar{R}_2 = a + ib$, $\mathrm{Det}\, S$ can be written,

$$-\frac{e^{i\alpha_{31}\delta}}{2(a^2+b^2)} \left(\frac{\sin\alpha_{21}\delta}{\alpha_{21}} . \frac{\sin\alpha_{23}\delta}{\alpha_{23}} \right)^2 . \left\{ \frac{Q_1^c(2a)\, Q_2^c(2a)}{4a^2+\delta^2} - \frac{Q_1^c(2ib)\, Q_2^c(2ib)}{\delta^2 - 4b^2} \right\}$$

for $a \in \mathbb{R}$, $b \in [-\frac{\delta}{2}, \frac{\delta}{2}]$. The two bracketted terms are dependent only upon a and b respectively. We wish to show that the first is bounded below and the second bounded above in such a way that the second bound does not exceed the first and so the difference is always non-negative. In fact the bounds are equal and attained at a unique point. But in approaching this point as a limit we can show that $\mathrm{Det}\, S$ is not zero there.

For simplicity we are considering the case when $\alpha_2 = 0$, $\alpha_1 = -\alpha_3 = \alpha$. Otherwise it may be the extrema are non-unique and the result is harder to show. So we wish to consider the function,

$$\mathrm{Det}\, S = K \left\{ \frac{(\cos\alpha\delta - \cosh 2\alpha a)^2}{4a^2+\delta^2} - \frac{(\cos\alpha\delta - \cos 2\alpha b)^2}{\delta^2 - 4b^2} \right\}$$

where $K(\delta)$ has factors of $(\sin\alpha\delta)^2$ and so is non-vanishing for $0 < |\delta| < \frac{\pi}{\alpha}$ (and we already know that the $\delta = 0$ limit is regular).

This is an even function of a and b , and

$$\frac{1}{8a}\cdot\frac{\partial}{\partial a}\left[\frac{(\cos\alpha\delta-\operatorname{ch}2\alpha a)^2}{4a^2+\delta^2}\right]$$

$$= \frac{\operatorname{ch}2\alpha a-\cos\alpha\delta}{(4a^2+\delta^2)^2}\left[\cos\alpha\delta+\frac{1}{2}e^{2\alpha a}\left(\frac{4a^2+\delta^2}{2a}\alpha-1\right)-\frac{1}{2}e^{-2\alpha a}\left(\frac{4a^2+\delta^2}{2a}\alpha+1\right)\right]$$

$$\geqslant \frac{\operatorname{ch}2\alpha a-\cos\alpha\delta}{(4a^2+\delta^2)^2}\left[\cos\alpha\delta+\frac{(1+2\alpha a)}{2}\left(\frac{4a^2+\delta^2}{2a}\alpha-1\right)-\frac{1}{2(1+2\alpha a)}\left(\frac{4a^2+\delta^2}{2a}\alpha+1\right)\right]$$

if $a \geqslant 0$,

with equality for $a=0$,

$$\geqslant \frac{\operatorname{ch}2\alpha a-\cos\alpha\delta}{(4a^2+\delta^2)^2}\left[\cos\alpha\delta-1+2a^2\alpha^2+\delta^2\alpha^2\cdot\frac{1+\alpha a}{1+2\alpha a}\right]$$

$$\geqslant \frac{\operatorname{ch}2\alpha a-\cos\alpha\delta}{(4a^2+\delta^2)^2}\left[2a^2\alpha^2+\delta^2\alpha^2\left(\frac{1+\alpha a}{1+2\alpha a}-\frac{1}{2}\right)\right]$$

since $\cos\alpha\delta \geqslant 1-\frac{1}{2}\alpha^2\delta^2$ for $|\delta|<\frac{\pi}{\alpha}$,

$$\geqslant \frac{\operatorname{ch}2\alpha a-\cos\alpha\delta}{(4a^2+\delta^2)^2}\left[2a^2\alpha^2+\frac{1}{2}\,\frac{\delta^2\alpha^2}{1+2\alpha a}\right]$$

$$> 0$$

for $a>0$.

Hence the first term has a unique minimum at $a=0$.

Likewise, consider,

$$\frac{1}{8b}\cdot\frac{\partial}{\partial b}\left[\frac{(\cos\alpha\delta-\cos2\alpha b)^2}{\delta^2-4b^2}\right]$$

$$= \frac{\cos2\alpha b-\cos\alpha\delta}{(\delta^2-4b^2)^2}\left[\cos2\alpha b-\cos\alpha\delta-\frac{\delta^2-4b^2}{2b}\cdot\alpha\sin2\alpha b\right]$$

$$= \alpha^2 \cdot \frac{\cos 2\alpha b - \cos\alpha\delta}{\delta^2 - 4b^2} \cdot \left(\frac{1}{2} \cdot \frac{\sin\alpha(b+\frac{\delta}{2})}{\alpha(b+\frac{\delta}{2})} \cdot \frac{\sin\alpha(\frac{\delta}{2}-b)}{\alpha(\frac{\delta}{2}-b)} - \frac{\sin 2\alpha b}{2\alpha b} \right)$$

$$\leqslant \alpha^2 \cdot \frac{\cos 2\alpha b - \cos\alpha\delta}{\delta^2 - 4b^2} \left(\frac{1}{2} - \frac{\sin\alpha\delta}{\alpha\delta} \right)$$

since $-\frac{\delta}{2} \leqslant b \leqslant \frac{\delta}{2}$ and $\frac{\sin x}{x}$ is monotone decreasing for $0 < x < \pi$. This is negative for $\frac{\sin\alpha\delta}{\alpha\delta} > \frac{1}{2}$ which is certainly the case for $\delta < \frac{\pi}{2\alpha}$.

Hence the second term has a unique maximum at $b = 0$.

But further,

$$\left.\frac{(\cos\alpha\delta - \mathrm{ch}\,2\alpha a)^2}{4a^2+\delta^2}\right|_{a=0} = \left.\frac{(\cos\alpha\delta - \cos 2\alpha b)^2}{\delta^2 - 4b^2}\right|_{b=0}$$

and so $\mathrm{Det}\, S$ is non-vanishing except possibly at $a = b = 0$.

For a and b small,

$$\mathrm{Det}\, S \sim \frac{1}{a^2+b^2} \left[\frac{\cos\alpha\delta - 1}{\delta^2} \cdot (a^2+b^2) \cdot \left(\alpha^2 + \frac{\cos\alpha\delta - 1}{\delta^2}\right) \right]$$

which is non-zero in the range $|\delta| < \frac{\pi}{2\alpha}$. This bound might be pushed a little higher by a more accurate analysis.

It has also been shown that in the $U(2)$ breaking case, $\alpha_1 = \alpha_2 = 1$, $\lambda_1 = \lambda_2 = 1$ the fields are regular for $|\delta| < \frac{\pi}{3}$ [28].

In $SU(2)$ the imaginary displacements of the constituents of cylindrically symmetric configurations are restricted to certain discrete values. In $SU(3)$ it appears that it can vary continuously but it is not at all clear how this parameter might be interpreted.

Chapter VI

The general monopole solution in $SU(3)$.

Now we have a number of explicit patching functions we might attempt to find the fully general classes of monopole in an $SU(3)$ gauge theory. This has been done for $SU(2)$ [22] and the solution shown to be complete. [23] Similar methods can be used to analyse the next case.

6.1 Multimonopoles in $SU(3)$ broken to $U(1) \times U(1)$.

In $SU(3)$ we have the following gauge freedom :

$$a g' = g A$$

for $a, A \in SL(3,\mathbb{C})$ which leads to,

$$\begin{aligned} \rho_{12}' &= \frac{A_{12}}{k_{11}} \zeta^{l_1} e^{\alpha_1 \gamma} + \frac{k_{22}}{k_{11}} \rho_{12} - \frac{a_{12}}{k_{11}} \zeta^{l_2} e^{\alpha_2 \gamma} \\ \rho_{23}' &= \frac{A_{23}}{k_{22}} \zeta^{l_2} e^{\alpha_2 \gamma} + \frac{k_{33}}{k_{22}} \rho_{23} - \frac{a_{23}}{k_{22}} \zeta^{l_3} e^{\alpha_3 \gamma} \\ \rho_{13}' &= \frac{A_{13}}{k_{11}} \zeta^{l_1} e^{\alpha_1 \gamma} + \frac{k_{33}}{k_{11}} \rho_{13} + k_{33}(a_{12}a_{23} - k_{22}a_{13}) \zeta^{l_3} e^{\alpha_3 \gamma} \\ &\quad + \frac{A_{23}}{k_{22}} \rho_{12} - a_{12} k_{33}^2 \rho_{23} - a_{12} A_{23} k_{33} \zeta^{l_2} e^{\alpha_2 \gamma} \end{aligned} \tag{6.1}$$

where k_{11}, k_{22} and k_{33} are non-zero constants with $k_{11}k_{22}k_{33} = 1$.

The reality conditions,

$$g A = A^\dagger g^\dagger$$

give :

$$A_{33} = \zeta^{-l_3} \psi_3 \quad , \qquad \psi_3 = \psi_3^\dagger = \sum_{1}^{-l_3} \gamma^s \psi_{3s}(\zeta)$$

$$A_{11}^{-1} = A_{22}A_{33} - A_{23}A_{32} = \zeta^{l_1} \psi_1 \, , \qquad \psi_1 = \psi_1^\dagger = \sum_{1}^{l_1} \gamma^s \psi_{1s}(\zeta)$$

and, for instance,

$$\rho_{23} = \frac{1}{A_{33}} \left((-\zeta)^{-l_3} e^{\alpha_3 \gamma} A_{32}^\dagger - \zeta^{l_2} e^{\alpha_2 \gamma} A_{23} \right)$$

But we can relate the coefficients A_{23} and A^{+}_{32} via the polynomial ψ_1, so that,

$$\rho_{23} = -\frac{\zeta^{-\lambda_1}}{\psi_3}\left(e^{\alpha_2\gamma}A_{23} + (-1)^{\lambda_2}\frac{\psi_1 e^{\alpha_3\gamma}}{A^{+}_{23}}\right) + \zeta^{\lambda_3}e^{\alpha_3\gamma}\frac{A^{+}_{22}}{A^{+}_{23}}$$

and the extra term is simply a gauge transformation if we assume that A_{23} is non-vanishing in its region of analyticity,

$$\rho_{23} \cong -\frac{\zeta^{-\lambda_1}}{\psi_3}\left(e^{\alpha_2\gamma}A_{23} + (-1)^{\lambda_2}e^{\alpha_3\gamma}\psi_1/A^{+}_{23}\right) \tag{6.2}$$

This assumption is only reasonable if at the same time we require ψ_1 and ψ_3 to have no common zeros, in view of the equation :

$$A_{22}\zeta^{-\lambda_3}\psi_3 - \zeta^{\lambda_1}\psi_1 = A_{23}A_{32}$$

We shall examine the coincidence of zeros later.

Making similar assumptions about A^{-1}_{21}, we find :

$$\rho_{12} \cong -\frac{\zeta^{-\lambda_3}}{\psi_1}\left((-1)^{\lambda_1}(A^{-1}_{21})^{+}e^{\alpha_2\gamma} + e^{\alpha_1\gamma}\psi_3/A^{-1}_{21}\right) \tag{6.3}$$

Neither of these bundle equivalences affect the third function,

$$\rho_{13} = \frac{e^{\alpha_1\gamma}A^{-1}_{13}}{\psi_1} + (-1)^{\lambda_1}\frac{e^{\alpha_2\gamma}A_{23}A^{-1+}_{21}}{\psi_1\psi_3} + (-1)^{\lambda_3}\frac{e^{\alpha_3\gamma}A^{+}_{31}}{\psi_3} \tag{6.4}$$

The forms of ρ_{12} and ρ_{23} are the same as that which can be deduced for ρ in the $SU(2)$ case, except for the presence of the polynomials upstairs. Nevertheless we can use the same argument as in that case [23] and reproduce it here.

We have assumed that $A^{-1}_{21} \neq 0$ in its region of analyticity and so write it as an exponential,

$$A^{-1}_{21} = \exp(\chi_1 + \zeta^{\lambda_1}\psi_1\delta_1) \quad .$$

where χ_1 is of order $\lambda_1 - 1$ in $\zeta\gamma$, and δ_1 is analytic on U_0. Since then,

$$(A^{-1}_{21})^{-1} = e^{-\chi_1}(1 - \zeta^{\lambda_1}\psi_1 B_1)$$

where B_1 is analytic on U_0, and,

$$(A^{-1}_{21})^{+} = e^{\chi_1^{+}}(1 + (-\zeta)^{-\lambda_1}\psi_1 B_2)$$

where B_2 is analytic on U_∞, we can use further bundle equivalence transformations to write :

$$\rho_{12} \cong -\frac{\zeta^{-\lambda_3}}{\psi_1}\left((-1)^{\lambda_1} e^{\alpha_2\gamma + \chi_1^{\dagger}} + \psi_3 e^{\alpha_1\gamma - \chi_1}\right)$$

Such a transformation will affect the terms of ρ_{13} but not its canonical form which is a consequence of reality.

If ρ_{12} is not to have poles at the zeros of ψ_1 we shall require ,

$$\exp\left(\alpha_{21}\gamma + \chi_1^{\dagger} + \chi_1\right)\Big|_{\psi_1=0} = -(-1)^{\lambda_1}\psi_3\Big|_{\psi_1=0}$$

and since ψ_1 and ψ_3 have no common zeros the right hand side cannot vanish and these constraints are soluble.

We have that,

$$\chi_1 = \sum_0^{\lambda_1-1} \chi_{1s}(\zeta)(\gamma\zeta)^s \quad \text{and} \quad \psi_1 = \prod_1^{\lambda_1}(\gamma - \gamma_{1s}(\zeta))$$

up to a real constant as a polynomial of degree λ_1 in γ . ψ_1 can also be written,

$$\psi_1 = \sum_0^{\lambda_1} \gamma^s \psi_s(\zeta)$$

where ψ_s has powers of ζ from $-\lambda_1+s$ to λ_1-s . Since $\psi_1 = \psi_1^{\dagger}$ implies $\psi_s(\zeta) = \psi_s^{\dagger}(-\frac{1}{\zeta})$ there are $2(\lambda_1-s)+1$ real degrees of freedom in ψ_s and so in ψ_1 there are,

$$\sum_{s=0}^{\lambda_1}\left(2(\lambda_1-s)+1\right) = (\lambda_1+1)^2$$

One overall constant factor can be removed by a bundle transformation, (6·2) , leaving $\lambda_1^2+2\lambda_1$ degrees of freedom.

The constraint on χ_1 is,

$$\Theta_1\Big|_{\gamma=\gamma_{1s}} = i\pi(\lambda_1+1) + \ln\psi_3(\gamma_{1s}) = 2\pi i\, \nu_{1s}\,, \quad 1 \leqslant s \leqslant \lambda_1 .$$

where $\Theta_1(\gamma,\zeta) = \alpha_{21}\gamma + \chi_1^{\dagger} + \chi_1$ a real polynomial of degree λ_1-1 in γ . Then the form of χ_1 implies,

$$\Theta_1(\gamma,\zeta) = \sum_0^{\lambda_1-1} \gamma^s\, \Theta_{1s}(\zeta)$$

$$\Theta_{1r} = \alpha_{21}\delta_{r1} + (-\zeta)^{-r}\chi_{1r}^{\dagger}\left(\frac{-1}{\zeta}\right) + \zeta^r \chi_{1r}(\zeta)$$

The analyticity of χ_{1r} means that Θ_{1r} for $r>1$ has no powers of ζ in the range $-r+1$ to $r-1$, that is :

$$\oint \frac{d\zeta}{\zeta}\, \zeta^s\, \Theta_{1,r} = 0 \qquad -r+1 \leqslant s \leqslant r-1$$

and that for $r = 1$, $$\oint \frac{d\zeta}{\zeta} \Theta_{1,1} = \alpha_{21} \tag{6.6}$$

About $\Theta_{1,0}$ we can say nothing. From the λ_1 values $2\pi i \nu_{1s}$ of this polynomial of degree $\lambda_1 - 1$ we may use the Lagrange interpolation formula,

$$\Theta_1(\gamma,\zeta) = 2\pi i \sum_{s=1}^{\lambda_1} \nu_{1s} \prod_{r \neq s} \left(\frac{\gamma - \gamma_{1r}}{\gamma_{1s} - \gamma_{1r}} \right)$$

where the γ_{1r} , the zeros of ψ_1 , are in general functions of ζ . Then the above conditions on $\Theta_{1,r}$ represent,

$$1 + \sum_{2}^{\lambda_1 - 1} (2s-1) = (\lambda_1 - 1)^2$$

constraints on the $\gamma_{1r}(\zeta)$.

Hence we are left with,

$$\lambda_1(\lambda_1 + 2) - (\lambda_1 - 1)^2 = 4\lambda_1 - 1$$

degrees of freedom.

The same argument in the case of ρ_{23} leads to a further (independent) $-4\lambda_3 - 1$ degrees of freedom, and the form ;

$$\rho_{23} \cong - \frac{\zeta^{-\lambda_1}}{\psi_3} \left(e^{\alpha_2 \gamma + \chi_3} + (-1)^{\lambda_2} \psi_1 e^{\alpha_3 \gamma - \chi_3^\dagger} \right)$$

where $\chi_3(\gamma\zeta,\zeta)$ is of degree $-\lambda_3 - 1$ in γ .

It remains to consider what extra constraints might be imposed by the analyticity of ρ_{13} . It takes the form ,

$$\rho_{13} = \frac{e^{\alpha_1 \gamma} A_{13}^{-1}}{\psi_1} + (-1)^{\lambda_1} \frac{e^{\alpha_2 \gamma + \chi_3 + \chi_1^\dagger}}{\psi_1 \psi_3} + (-1)^{\lambda_3} \frac{e^{\alpha_3 \gamma} A_{31}^\dagger}{\psi_3}$$

For $\lambda_2 \geqslant 0$, $\lambda_1 \leqslant -\lambda_3$, put,

$$A_{13}^{-1} = \exp\left(\chi_3 - \chi_1 - \zeta^{\lambda_1} \psi_1 F \right)$$
$$A_{31}^\dagger = \exp\left(\chi_1^\dagger - \chi_3^\dagger \right)$$

where F is the polynomial part of $\dfrac{\chi_3 - \chi_1}{\zeta^{\lambda_1} \psi_1}$ and a function of $\gamma\zeta$ and ζ . Such a ρ_{13} satisfies the regularity conditions by virtue of the fact that both ρ_{12} and ρ_{23} do. Thus at $\psi_1 = 0$,

$$(-1)^{\lambda_3} \frac{e^{\alpha_3 \gamma} A_{31}^\dagger}{\psi_3} + \frac{e^{\chi_3}}{\psi_3} \cdot \frac{e^{\alpha_1 \gamma - \chi_1 - \zeta^{\lambda_1} \psi_1 F} \psi_3 + (-1)^{\lambda_1} e^{\alpha_2 \gamma + \chi_1^\dagger}}{\psi_1}$$ is non-singular,

and at $\psi_3 = 0$,

$$\frac{e^{\alpha_1 \gamma} A_{13}^{-1}}{\psi_1} + (-1)^{\lambda_1} \frac{e^{\chi_1^\dagger}}{\psi_1} \cdot \frac{e^{\alpha_2 \gamma + \chi_3} + (-1)^{\lambda_2} \psi_1 e^{\alpha_3 \gamma - \chi_3^\dagger}}{\psi_3} \quad \text{is non-singular.}$$

The matrix A is :

$$A = \begin{bmatrix} \frac{1 - e^{-\zeta^{\ell_1} \psi_1 F}}{\zeta^{\ell_1} \psi_1} & e^{-\chi_1 - \zeta^{\ell_1} \psi_1 F} & \frac{\zeta^{\lambda_2} \psi_3}{\psi_1} e^{\chi_3 - \chi_1} \left(1 - e^{-\zeta^{\ell_1} \psi_1 F}\right) \\ 0 & 0 & e^{\chi_3} \\ e^{\chi_1 - \chi_3} & -\zeta^{\lambda_1} \psi_1 e^{-\chi_3} & \zeta^{-\ell_3} \psi_3 \end{bmatrix}$$

which has determinant unity and is holomorphic in V_0 , and the Hermitian form of g is shown on the following page.

The extraction of the factor $e^{-\zeta^{\ell_1} \psi_1 F}$ in A_{13}^{-1} is really just a bundle transformation to remove the excess powers of γ in the exponentiated polynomial which are over and above the degree of ψ_1. Clearly for $\lambda_2 \leqslant 0$ an analogous procedure involving $A_{31}^\dagger$ should be undertaken.

This form of ρ_{13} satisfies all the requirements but involves no more constraints on the free parameters. Thus we have a family of $4(\lambda_1 - \ell_3) - 2$ degrees of freedom. This agrees with the result for $U(1) \times U(1)$ breaking predicted by index theorem methods.[19]

The SU(3) multimonopole patching function in Hermitian form.

$$\begin{bmatrix} \dfrac{e^{\alpha_1\gamma}}{\psi_1} + (-1)^{l_1}\dfrac{e^{\alpha_2\gamma+\chi_1+\chi_1^\dagger}}{\psi_1\psi_3} + (-1)^{l_3}\dfrac{e^{\alpha_3\gamma+\chi_1+\chi_1^\dagger-\chi_3-\chi_3^\dagger}}{\psi_3} & -(-\zeta)^{l_1}\dfrac{e^{\chi_1^\dagger}}{\psi_3}\left(e^{\alpha_2\gamma}+\psi_1(-1)^{l_2}e^{\alpha_3\gamma-\chi_3-\chi_3^\dagger}\right) & (-\zeta)^{-l_3}e^{\alpha_3\gamma+\chi_1^\dagger-\chi_3^\dagger} \\ -\zeta^{-l_1}\dfrac{e^{\chi_1}}{\psi_3}\left(e^{\alpha_2\gamma}+(-1)^{l_2}\psi_1 e^{\alpha_3\gamma-\chi_3^\dagger-\chi_3}\right) & \dfrac{\psi_1}{\psi_3}\left(e^{\alpha_2\gamma}+(-1)^{l_2}\psi_1 e^{\alpha_3\gamma-\chi_3-\chi_3^\dagger}\right) & -(-\zeta)^{l_2}\psi_1 e^{\alpha_3\gamma-\chi_3^\dagger} \\ \zeta^{l_3}\, e^{\alpha_3\gamma+\chi_1-\chi_3} & -\zeta^{-l_2}\psi_1 e^{\alpha_3\gamma-\chi_3} & \psi_3 e^{\alpha_3\gamma} \end{bmatrix}$$

6.2 Coincidence of zeros

In $SU(2)$ a pair of coincident monopoles in $\mathbb{R}^3$, the axially symmetric charge 2 configuration, does not have coincident parameters in twistor space. The polynomial ψ is not allowed double zeros. In $SU(3)$ however we have already seen that between ψ_1 and ψ_3 we can have coincident zeros, the cylindrically symmetric family (3.16), and even zeros of order two in each polynomial, the spherically symmetric monopole (3.19). For the general maximal embedding we have even higher order zeros. With a better understanding of the parameter space it should be possible to recover these examples as limits using the results of the previous section, where we had to assume that ψ_1 and ψ_3 had no common zeros. Without that understanding we can still attempt to see how the arguments should be modified in respect of the matrix A.

So let us consider what happens when zeros from ψ_1 and ψ_3 coalesce so that $\psi_1 = \phi_1\phi$ and $\psi_3 = \phi_3\phi$, and ϕ_1 and ϕ_3 have no common zeros. We shall use the symbols for the polynomials interchangeably with their sets of zeros.

Now $\phi = \phi^+$. For suppose not. Then there is a zero in ϕ whose conjugate zero is not in ϕ. Since $\psi_1 = \psi_1^+$ it must be in ϕ_1. Likewise it must be in ϕ_3 because $\phi_3 = \phi_3^+$. But this contradicts $\phi_1 \cap \phi_3 = 0$. Then we can write,

$$\phi = \phi_r \phi_c^+ \phi_c$$

where ϕ_r consists of distinct real zeros and ϕ_c of a choice of one each of the pairs of conjugate zeros.

We still have the two relations which follow from reality :

$$A_{23}A_{32} = (A_{22}\zeta^{-\lambda_3}\phi_3 - \zeta^{\lambda_1}\phi_1)\phi_r\phi_c^+\phi_c \qquad (*)$$

and, for instance :

$$\rho_{23} = \frac{\zeta^{\lambda_3}}{\phi_3\phi_r\phi_c^+\phi_c}\left((-\zeta)^{-\lambda_3} e^{\alpha_3\gamma} A_{32}^+ - \zeta^{\lambda_2} e^{\alpha_2\gamma} A_{23}\right)$$

Let $\gamma - \gamma_0$ be a simple factor of ϕ_r. If $\gamma-\gamma_0 \nmid A_{23}$ then $\gamma-\gamma_0 \nmid A_{32}^+$, for otherwise ρ_{23} would have an arbitrary pole, and so $\gamma-\gamma_0 \nmid A_{32}$. But this contradicts (*). Hence every factor of ϕ_r divides both A_{23} and A_{32} and dis-appears out of ρ_{23}. For simplicity let us assume $\phi_r = 1$.

Let ϕ_m be the largest part of ϕ dividing A^{+}_{32} and ϕ_M that dividing A_{23}. Then $\phi_m | A_{23}$ and $\phi^{+}_M | A_{32}$ also. But the maximality of ϕ_m and ϕ_M then implies $\phi_M | \phi_m$ and $\phi_m | \phi_M$. So, up to a constant, $\phi_m = \phi_M$ and,

$$A^{+}_{32} = (-\zeta)^{-m} \phi_m \tilde{A}^{+}_{32} \, , \quad A_{23} = \zeta^{m} \phi_m \tilde{A}_{23}$$

where the tilda'd functions have no zeros in common with ϕ.

(*) now implies that $\phi \mid \phi^{+}_m \phi_m$ and so $\phi_c | \phi_m$ without loss of generality because we can reallocate conjugate zeros in ϕ_c and ϕ^{+}_c. Thus $\phi_m = \phi_c \psi$, $\psi | \phi^{+}_c$ and,

$$\tilde{A}_{23} \tilde{A}_{32} = (A_{22} \zeta^{-\lambda_3} \phi_3 - \zeta^{\lambda_1} \phi_1) \frac{\zeta^{-2m}}{\psi^{+}\psi}$$

We can apply the previous bundle equivalence argument, replacing A^{+}_{32} by $\phi_1 (A^{+}_{23})^{-1}$, only if $\psi^{+}\psi = 1$, that is, $\phi_m = \phi_c$. In this case :

$$\rho_{23} = - \frac{\zeta^{-\lambda_1+m}}{\phi_3 \phi^{+}_c} \left(e^{\alpha_2 \gamma + \chi_3} + (-1)^{\lambda_2+m} \phi_1 e^{\alpha_3\gamma - \chi^{+}_3} \right)$$

Similarly,

$$\rho_{12} = - \frac{\zeta^{-\lambda_3-m}}{\phi_1 \phi_c} \left(\phi_3 e^{\alpha_1\gamma - \chi_1} + (-1)^{\lambda_1-m} e^{\alpha_2\gamma + \chi^{+}_1} \right)$$

and,

$$\rho_{13} = \frac{e^{\alpha_1\gamma} A^{-1}_{13}}{\phi_1 \phi^{+}_c \phi_c} + (-1)^{\lambda_1-m} \frac{e^{\alpha_2\gamma + \chi_3 + \chi^{+}_1}}{\phi_1 \phi_3 \phi^{+}_c \phi_c} + (-1)^{\lambda_3} \frac{e^{\alpha_3\gamma} A^{+}_{31}}{\phi_3 \phi^{+}_c \phi_c}$$

Regularity of ρ_{13} at $\phi_1 = 0$ and $\phi_3 = 0$ is guaranteed by that of ρ_{12} and ρ_{23} provided $A^{-1}_{13}(\phi_1 = 0) = e^{\chi_3 - \chi_1}$ and $A^{+}_{31}(\phi_3 = 0) = e^{\chi^{+}_1 - \chi^{+}_3}$. It is not easy to see what the general choice of the functions A^{-1}_{13} and A^{-1}_{31} must be from the analyticity conditions :

$$\left. \phi_3 e^{\alpha_1\gamma} A^{-1}_{13} + (-1)^{\lambda_1-m} e^{\alpha_2\gamma + \chi_3 + \chi^{+}_1} + (-1)^{\lambda_3} e^{\alpha_3\gamma} A^{+}_{31} \right|_{\phi^{+}_c \phi_c} = 0$$

but we have seen already in Chapter III an example for which both

ϕ_1 and ϕ_2 are constant and $\phi_c = \gamma - c_1$, $\phi_c^+ = \gamma - c_1^+$. In this case none of the χ's or A_{13}^{-1} and A_{31}^+ are γ-dependent and regularity is easily satisfied.

6.3 Multimonopoles for $SU(3)$ broken to $U(2)$.

In the case of $U(2)$ breaking we may apply a similar argument to a patching function of the form :

$$g = \begin{bmatrix} \zeta^{l} e^{\alpha\gamma} & 0 & \rho_{13} \\ 0 & \zeta^{l} e^{\alpha\gamma} & \rho_{23} \\ 0 & 0 & \zeta^{-2l} e^{-2\alpha\gamma} \end{bmatrix}$$

to obtain the two similar functions :

$$\rho_{13} = \frac{1}{\psi}\left(e^{-2\alpha\gamma} A_{31}^+ - \zeta^{-l} e^{\alpha\gamma} A_{13}\right)$$

$$\rho_{23} = \frac{1}{\psi}\left(e^{-2\alpha\gamma} A_{32}^+ - \zeta^{-l} e^{\alpha\gamma} A_{23}\right)$$

where ψ is of degree $2l$ in ζ . Reapplying the arguments of section 6.1 we are able to reduce these to ,

$$\rho_{13} = \frac{1}{\psi}\left(e^{-2\alpha\gamma + \chi_1^+} + \phi_1 e^{\alpha\gamma - \chi_1}\right)$$

$$\rho_{23} = -\frac{\zeta^{-l}}{\psi}\left((-1)^l \phi_2 e^{-2\alpha\gamma - \chi_2^+} + e^{\alpha\gamma + \chi_2}\right)$$

and have for the matrix A ,

$$\begin{bmatrix} 0 & \dfrac{\sigma + \zeta^{l}\phi_1\phi_2 e^{-\chi_2 - \chi_1}}{\zeta^{l}\psi} & -e^{-\chi_1}\zeta^{l}\phi_1 \\ \dfrac{\zeta^{l}\sigma^+ + e^{\chi_2 + \chi_1}}{\zeta^{2l}\psi} & 0 & e^{\chi_2} \\ e^{\chi_1} & -\zeta^{l}\phi_2 e^{-\chi_2} & \zeta^{2l}\psi \end{bmatrix}$$

where $\phi_1 = \phi_1^+$, $\phi_2 = \phi_2^+$ and σ all have powers of ζ from $-\ell$ to ℓ , and satisfy,

$$\sigma^+\sigma = \phi_1\phi_2 - \psi$$

ensuring that $\mathrm{Det}\, A = 1$.

The real form of g is shown on the following page.

If A is to be non-singular at $\psi = 0$ we must have that ψ divide $\zeta^\ell\sigma^+ + e^{\chi_2+\chi_1}$ and hence also $(-\zeta)^{-\ell}\sigma + e^{\chi_2^+ + \chi_1^+}$, since ψ is real. This and the non-singularity of ρ_{23} at $\psi = 0$ is enough to guarantee that of ρ_{13} . For from ρ_{23} at $\psi = 0$ we have,

$$e^{3\alpha\gamma}\Big|_{\psi=0} = -(-1)^\ell \phi_2 e^{-\chi_2^+ - \chi_2}\Big|_{\psi=0}$$

and hence ,

$$\rho_{13}\Big|_{\psi=0} = \frac{e^{-2\alpha\gamma + \chi_1^+}}{\psi}\left(1 - (-1)^\ell \phi_1\phi_2 e^{-\chi_1 - \chi_2 - \chi_1^+ - \chi_2^+}\right)\Big|_{\psi=0}$$

or, equivalently, since $\phi_1\phi_2 = \sigma^+\sigma + \psi$,

$$\rho_{13}\Big|_{\psi=0} = \frac{e^{-2\alpha\gamma + \chi_1^+}}{\psi}\left(1 - (-1)^\ell \sigma^+\sigma e^{-\chi_1 - \chi_2 - \chi_1^+ - \chi_2^+}\right) + \text{non-singular}$$

But we can rewrite the term in brackets as,

$$\tfrac{1}{2}\left(1 + \sigma(-\zeta)^{-\ell} e^{-\chi_1^+ - \chi_2^+}\right)\left(1 - \sigma^+\zeta^\ell e^{-\chi_1 - \chi_2}\right) + \tfrac{1}{2}\left(1 - \sigma(-\zeta)^{-\ell} e^{-\chi_1^+ - \chi_2^+}\right)\left(1 + \sigma^+\zeta^\ell e^{-\chi_1 - \chi_2}\right)$$

which vanishes at $\psi = 0$.

As an example consider the patching function,[28]

$$\begin{bmatrix} \zeta e^\gamma & 0 & \dfrac{e^{-2\gamma} - (\cos\Psi + A\gamma)e^\gamma}{\gamma^2 + (\Psi/3)^2} \\[2ex] 0 & \zeta e^\gamma & \dfrac{(A^{-1}\cos\Psi - \gamma)e^{-2\gamma} - A^{-1}e^\gamma}{\zeta(\gamma^2 + (\Psi/3)^2)} \\[2ex] 0 & 0 & \zeta^{-2}e^{-2\gamma} \end{bmatrix}, \quad A = \frac{-3\sin\Psi}{\Psi}$$

Hermitian form of the patching function for $SU(3)$ broken to $U(2)$.

$$
\begin{bmatrix}
\dfrac{e^{-2\alpha\gamma+\chi_1^\dagger+\chi_1}+\phi_1 e^{\alpha\gamma}}{\psi} & \dfrac{\sigma e^{\alpha\gamma}-\zeta^{\ell}\phi_2 e^{-2\alpha\gamma-\chi_2+\chi_1^\dagger}}{\psi} & \zeta^{2\ell}e^{-2\alpha\gamma+\chi_1^\dagger} \\[2ex]
\dfrac{\sigma^\dagger e^{\alpha\gamma}-(-\zeta)^{-\ell}\phi_2 e^{-2\alpha\gamma-\chi_2^\dagger+\chi_1}}{\psi} & \dfrac{\phi_2}{\psi}\left((-1)^{\ell}\phi_2 e^{-2\alpha\gamma-\chi_2^\dagger-\chi_2}+e^{\alpha\gamma}\right) & -(-\zeta)^{\ell}\phi_2 e^{-2\alpha\gamma-\chi_2^\dagger} \\[2ex]
\zeta^{-2\ell}e^{-2\alpha\gamma+\chi_1} & -\zeta^{-\ell}\phi_2 e^{-2\alpha\gamma-\chi_2} & \psi e^{-2\alpha\gamma}
\end{bmatrix}
$$

We make the following identifications, $\psi = \gamma^2 + (\Psi/3)^2$,

$$\chi_1 = 0 \quad , \quad \phi_1 = -(\cos\Psi + A\gamma)$$

$$e^{\chi_2} = A^{-1} \quad , \quad \phi_2 = (\cos\Psi - A\gamma)/A^2$$

and then,

$$\sigma^+\sigma = -\gamma^2 - (\tfrac{\Psi}{3})^2 - A^{-2}\cos^2\Psi + \gamma^2$$

$$= -\frac{\Psi^2}{9} \cdot \frac{1}{\sin^2\Psi}$$

So $\sigma = \zeta A^{-1}$ and $\zeta\sigma^+ + e^{\chi_2+\chi_1}$ vanishes.

We can count parameters as before. We have $4\ell(\ell+1)$ in ψ and $\ell(\ell+1)$ in each of ϕ_1 and ϕ_2 . σ is then uniquely defined up to (perhaps) reallocation of conjugate factors in σ^+ . The regularity of ρ_{23} imposes $(2\ell-1)^2$ constraints. Also at $\psi=0$ we need $e^{\chi_1} = -e^{\chi_2}\zeta^\ell\sigma^+$. As before this defines χ_1 by Lagrange interpolation but now the fact that,

$$\chi_1(\gamma\zeta,\zeta) = \sum_0^{2\ell-1}\gamma^r\chi_{1r}(\zeta)$$

gives the constraints, $\oint\frac{d\zeta}{\zeta}\zeta^s\chi_{1,r} = 0$ for $s=0,1,\ldots r-1$, and these are only $\sum_1^{2\ell-1} r = \ell(2\ell-1)$ in number. Thus we are left with a family of

$$6\ell(\ell+1) - (2\ell-1)^2 - \ell(2\ell-1) = 11\ell - 1$$

degrees of freedom. Note that this is less than $12\ell-2$, the number which would be expected from taking the limit of $U(1)\times U(1)$ breaking. In any case the ansatz is probably not wholly general and ought to involve a 2×2 block on the diagonal in place of which we have used $\zeta^\ell e^{\alpha\gamma} I_2$.

Finally, it is necessary to show that the sorts of patching functions used above yield the expected charges and also that they give non-singular Higgs and gauge fields.

In respect of the former, if we conjecture that the formula,[28,29]

$$\mathrm{Tr}\,\phi^2 = \mathrm{Const} - \tfrac{1}{2}\nabla^2\ln\mathrm{Det}\,S$$

holds in $SU(3)$ for the matrix defined in section (5.2), then the charge can be shown to be correct. For the ρ_{ij} satisfy the Laplace equation and the $\tilde{\rho}_{ij}$, that is, when we pull out the diagonal exponential factors in g, then satisfy,

$$\nabla^2 \tilde{\rho}_{ij} = \alpha_{ij}^2 \tilde{\rho}_{ij}$$

In consequence, so do all their Laurent coefficients. Hence they behave as $\frac{1}{r} \exp \alpha_{ij} r$. Further the leading behaviour of $\tilde{P}_{13}$ is the same as that of $\tilde{\rho}_{13}$, as r tends to infinity, because the modifying terms in $\tilde{P}_{13}$ are products of coefficients from $\tilde{\rho}_{12}$ and $\tilde{\rho}_{23}$. The same goes for the matrix S itself, and we need to retain only the first, $(l_1+l_2)\times(l_1+l_2)$ matrix in the definition. But then the large distance behaviour of its determinant, which comes from two blocks of dimensions $l_1 \times (l_1+l_2)$ and $l_2 \times (l_1+l_2)$, will be,

$$\mathrm{Det}\, S \sim \left[\frac{1}{r} \exp \alpha_{13} r\right]^{l_1} \left[\frac{1}{r} \exp \alpha_{23} r\right]^{l_2}$$

and so, for the trace of the Higgs field squared,

$$\mathrm{Tr}\, \phi^2 \sim \mathrm{Const} - \frac{1}{r}\left(\alpha_{13} l_1 + \alpha_{23} l_2\right) + o\left(\frac{1}{r^2}\right)$$

$$\sim \mathrm{Const} - \frac{1}{r}\left(\alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3\right)$$

which agrees with the expression for the flux in Chapter I.

As regards the regularity of the fields all we can say at present is that this is alright for small perturbations about those patching functions in $SU(3)$ which have been shown to give regular fields.

Appendix

$Tr\phi^2$ for spherically symmetric monopoles in $SU(n)$.

From the work of previous authors[5] we know that on the z-axis the Higgs field can be written ,

$$\varphi = \frac{1}{2}\sum_{1}^{n-1} \varphi_i H_i$$

where $n-1$ is the rank of $SU(n)$ and the H_i are a choice of basis of the Cartan subalgebra,

$$H_i = \mathrm{diag}(0, \dots 0, \overset{i}{1}, \overset{i+1}{-1}, 0, \dots 0)$$

satisfying,

$$Tr\, H_i H_j = K_{ij}$$

where K_{ij} ,the Cartan matrix is :

$$\begin{bmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot \\ 0 & -1 & 2 & -1 & 0 & \cdot & \cdot \\ \cdot & 0 & -1 & 2 & -1 & 0 & \cdot \\ \cdot & & & \ddots & \ddots & \ddots & \cdot \\ \cdot & & 0 & -1 & 2 & -1 & 0 \\ \cdot & & & 0 & -1 & 2 & -1 \\ \cdot & \cdot & \cdot & \cdot & 0 & -1 & 2 \end{bmatrix}$$

The φ_i are given by,

$$\varphi_i = \frac{n_i}{r} - \frac{1}{Q_i}\frac{\partial}{\partial r} Q_i$$

where the integers n_i are such that,

$$\frac{1}{2}\sum_{1}^{n-1} K_{ij} n_i = 1$$

and define the maximal embedding[47] of $SU(2)$ in $SU(n)$. The Q_i satisfy the Toda lattice equations[48] in the form,

$$\frac{\partial}{\partial r}\left(Q_i^{-1}\frac{\partial}{\partial r}Q_i\right) = -n_i \prod_j Q_j^{-K_{ij}} \qquad \text{(no summation)}$$

The substitution,

$$Q_i = \exp[-q_i - (K^{-1})_{ik} \ln n_k]$$

recasts them in the familiar shape ,

$$\frac{\partial^2}{\partial r^2} q_i = \exp[\sum_j K_{ij} q_j]$$

The topological characteristic of the self-dual solutions of a pure Yang-Mills theory is,

$$\chi_I = \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$$

In the case of the static sector and the gauge choice $A_0 \equiv 0$, the monopole equations, this becomes,

$$\chi_M = \mathrm{Tr}\, F_{ij}F^{ij}$$
$$= 2\nabla^2\, \mathrm{Tr}\,\varphi^2$$

(see section 2.2)

From our expression for φ above, we have :

$$2\,\mathrm{Tr}\,\varphi^2 = \frac{1}{2}\sum_{i,j}\varphi_i K_{ij}\varphi_j$$

$$= \frac{1}{2r^2}\sum_{i,j}\left(n_i + r\frac{\partial q_i}{\partial r}\right)K_{ij}\left(n_j + r\frac{\partial q_j}{\partial r}\right)$$

$$= \frac{1}{r^2}\left(\sum_i n_i + 2r\sum_i\frac{\partial q_i}{\partial r} + \frac{1}{2}r^2\sum_{i,j}K_{ij}\frac{\partial q_i}{\partial r}\cdot\frac{\partial q_j}{\partial r}\right)$$

But consider,

$$\frac{\partial}{\partial r}\left(2\sum_i\frac{\partial^2 q_i}{\partial r^2} - \sum_{i,j}K_{ij}\frac{\partial q_i}{\partial r}\cdot\frac{\partial q_j}{\partial r}\right)$$

$$= \frac{\partial}{\partial r}\left(2\sum_i \exp\left(\sum_j K_{ij}q_j\right) - \sum_{i,j}K_{ij}\frac{\partial q_i}{\partial r}\cdot\frac{\partial q_j}{\partial r}\right)$$

$$= 2\sum_{i,j}K_{ij}\frac{\partial q_j}{\partial r}\exp\left(\sum_l K_{il}q_l\right) - 2\sum_{i,j}K_{ij}\frac{\partial q_i}{\partial r}\exp\left(\sum_l K_{jl}q_l\right)$$

where the last term follows from the symmetry of the Cartan matrix. So the derivative vanishes and,

$$\sum_{i,j}K_{ij}\frac{\partial q_i}{\partial r}\cdot\frac{\partial q_j}{\partial r} = 2\frac{\partial}{\partial r}\sum_i\frac{\partial q_i}{\partial r} - \mathrm{Const}$$

Thus,

$$2\,\mathrm{Tr}\,\varphi^2 = \frac{1}{r^2}\left(\sum_i n_i + 2r\sum_i\frac{\partial q_i}{\partial r} + r^2\frac{\partial}{\partial r}\sum_i\frac{\partial q_i}{\partial r} - \frac{r^2}{2}\mathrm{Const}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\sum_i n_i \ln r + \sum_i q_i + Ar^2 + B/r + C\right)\right)$$

for A , B and C constants.

This can be written in the form :

$$2\,\mathrm{Tr}\,\varphi^2 = \nabla^2 \ln f$$

where,

$$f = \prod_i \left(r^{n_i}/Q_i \right) \exp\left[Ar^2 + B/r + C + \sum_{j,k} (K)^{-1}_{jk} \ln n_k \right]$$

The exponential terms make no conribution to χ_M ,

$$\chi_M = -\nabla^2 \nabla^2 \ln \left[\prod_i \left(Q_i / r^{n_i} \right) \right]$$

and provide the constant in $\mathrm{Tr}\,\varphi^2$,

$$\mathrm{Tr}\,\varphi^2 = \mathrm{Const} - \tfrac{1}{2} \nabla^2 \ln \left[\prod_i \left(Q_i / r^{n_i} \right) \right]$$

We have written χ_M like this because of its similarity to a formula due to Osborn [49] which follows from the ADHM construction :

$$\chi_I = \Box^2 \Box^2 \ln \left(\det (\Delta^{+}\Delta)^{-1} \right)$$

where $\Box^2$ is the four-dimensional Euclidean Laplacian and $\Delta^{+}\Delta$ a real, invertible matrix. In Nahm's modification $\Delta^{+}\Delta$ is a second order elliptic differential operator whose reality corresponds to the satisfaction of the field equations and whose invertibility corresponds to the regularity of the fields.

The Q_i are very interesting functions. In $SU(n)$,

$$_nQ_1 = \sum_{i=1}^{n} e^{\alpha_i r} \left(\prod_{j\neq i} \alpha_{ij} \right)^{-1} , \quad \sum_1^n \alpha_i = 0 , \quad \alpha_{ij} = \alpha_i - \alpha_j .$$

and the Toda lattice equations can be used to express $_nQ_m$ in terms of $_nQ_1$ and its first $2(m-1)$ derivatives, with respect to r ,

$$
{}_nQ_m = \mathrm{Det}\begin{bmatrix} {}_nQ_1 & {}_nQ_1^{(1)} & {}_nQ_1^{(2)} & \cdots & {}_nQ_1^{(m-1)} \\ {}_nQ_1^{(1)} & {}_nQ_1^{(2)} & \cdots & & \vdots \\ {}_nQ_1^{(2)} & \vdots & & & \vdots \\ \vdots & & & & \vdots \\ {}_nQ_1^{(m-1)} & \cdots & & \cdots & {}_nQ_1^{2(m-1)} \end{bmatrix}
$$

The functions ${}_nQ_m/r^{n_m}$ for $n_m = m(n-m)$ are regular at $r = 0$.

In attempting to follow up the link with the ADHMN construction, one may remark that the functions ${}_nQ_m/r^{n_m}$ are the determinants of a very simple set of differential operators.

Consider the n-th order operator ,

$$
{}_mD_n = \prod_{i=1}^{n}\left(\frac{\partial}{\partial\theta} - \alpha_i r\right)
$$

on the compact interval $[0,1]$, with associated boundary conditions :

$$
g(0) = g^{(1)}(0) = \cdots\cdots = g^{(m-1)}(0) = 0 ,
$$

$$
g(1) = g^{(1)}(1) = \cdots\cdots = g^{(n-m-1)}(1) = 0,
$$

for $1 \leqslant m \leqslant n-1$.

It is tedious, but not difficult, to show that,

$$
\mathrm{Det}({}_mD_n) = {}_nQ_m/r^{n_m}
$$

using as a definition of determinant,

$$
\frac{\partial}{\partial r}\ln(\mathrm{Det}\,{}_mD_n) = \mathrm{Tr}\left(\frac{\partial\,{}_mD_n}{\partial r}\cdot{}_mG_n\right)
$$

where ${}_mG_n$ is the Green function,

$$
{}_mD_n(\theta)\,{}_mG_n(\theta,\theta') = \delta(\theta-\theta')
$$

under the relevant boundary conditions.

Such a set of $n-1$ n-th order operators might be related to an elliptic 2-nd order operator of matrix dimension $\frac{1}{2}n(n-1)$ to recover Nahm's $\Delta^{+}\Delta$ operator but this is not evident.

References

(1) P.A.M.Dirac : Proc. Roy. Soc. London. A, vol 133. (1931)

(2) Y.Aharanov ; D.Bohm : Phys. Rev. 115, 485 (1959)

(3) G.'tHooft : Nucl. Phys B79,276 (1974)

A.M.Polyakov : JETP Lett. 20, 194 (1974)

(4) M.K.Prasad ; C.M.Sommerfield : Phys. Rev. Lett. 35, 760 (1975)

E.B.Bogomolnyi' : Sov. J. Nucl. Phys. 24, 861 (1976)

(5) F.A.Bais ; D.Wilkinson : Phys. Rev. D19, 2410 (1979)

F.A.Bais ; H.A.Weldon : Phys. Rev. Lett. 41, 601 (1978)

A.N.Leznov ; M.V.Saveliev : Lett. Math. Phys. 3, 48a (1980)

(6) A.A.Belavin ; A.M.Polyakov ; A.Schwartz ; Y.Tyupkin : Phys. Lett. 59B, 85 (1975)

(7) R.S.Ward : Phys. Lett. 61A (1977), 81

(8) M.F.Atiyah ; R.S.Ward : Comm. Math. Phys. 55, 117 (1977)

E.Corrigan ; D.B.Fairlie ; P.Goddard ; R.G.Yates : Comm. Math. Phys. 58, 223 (1978)

(9) M.F.Atiyah ; V.A.Drinfield ; N.J.Hitchin ; Yu.I.Manin : Phys. Lett. 65A, 185 (1978).

(10) A.Jaffe ; C.H.Taubes : " Vortices and Monopoles ", Birkhäuser (Boston, 1980)

See also, C.Rebbi ; P.Rossi : Phys. Rev. 22D, 2010 (1980)

(11) N.S.Manton : Nucl. Phys. B126, 525 (1977)

See also, W.Nahm : Phys. Lett. B 85, 373 (1979)

(12) L.O'Raifeartaigh ; P.Houston : Phys.Lett 93B, 151 (1980)

(13) W.Nahm : Phys. Lett. 90B, 412 ; 93B, 42, (1980)

(14) N.S.Manton : Nucl. Phys. B135, 319 (1978)

(15) P.Rossi : Nucl. Phys. B149, 170 (1979)

(16) R.S.Ward : Comm. Math. Phys. 79, 317 (1981)

(17) M.K.Prasad ; P.Rossi : Phys. Rev. Lett. 46, 806 (1981)

(18) M.K.Prasad ; Comm. Math. Phys. 80, 137 (1981)

(19) E.Weinberg : Phys. Rev. D20, 936 (1979) and Nucl. Phys. B167, 500 (1980).

(20) P.Forgacs ; Z.Horvath ; L.Palla : Phys. Lett 99B, 232 and 102B, 131 (1981), and Nucl. Phys. 192B, 141 (1981)

See also, F.A.Bais ; R.Sasaki : Nucl. Phys. B195, 522 (1982)

(21) R.S.Ward :Phys. Lett. 102B, 136 (1981)

(22) E. Corrigan ; P.Goddard : Comm. Math. Phys. 80, 575 (1981)

(23) N.J.Hitchin : Comm. Math. Phys. 83, 579 (1982)

(24) L.O'Raifeartaigh ; S.Rouhani ; L.P.Singh : Phys. Lett. 112B, 369 (1982).

A.Soper :" Multimonopoles close together", DAMTP preprint 81/85

(25) P.Forgacs ; Z.Horvath ; L.Palla : Phys. Rev. D23, 1876 (1981) and Phys.Lett. 109B, 200 (1982)

(26) W.Nahm : CERN preprint Th-3172 (1981) and ICTP, Trieste, preprint IC/81/238

(27) S.A.Brown ; H.Panagopoulos ; M.K.Prasad : MIT preprint, " Two separated SU(2) Yang-Mills-Higgs monopoles in the ADHMN construction."

(28) R.S.Ward : Phys. Lett. 107B, 281 (1981)

(29) R.S.Ward : Stony Brook preprint, ITP-SB-82-13," Deformations of the imbedding of the SU(2) monopole in SU(3)."

(30) C.Athorne : " Cylindrically and spherically symmetric monopoles in SU(3) gauge theory ", Durham preprint. submitted to Comm. Math. Phys.

" Factorization of an SU(3) monopole patching function ", Phys. Lett. B, to appear.

(31) " Monopoles in Quantum Field Theory ", Proceedings of the monopole meeting, Trieste, Italy, December 1981. Edited by N.S.Craigie, P.Goddard and W.Nahm, (World Scientific)

(32) P.Goddard ; D.Olive : Rep. Prog. Phys. 41, 1357 (1978)

(33) S. Coleman : " Classical lumps and their Quantum descendants ", Proceedings of the 1975 International School of Subnuclear Physics, " Ettore Majorana ".

(34) E.Corrigan ; D.B.Fairlie ; J.Nuyts ; D.Olive : Nucl. Phys. B106, 475 (1976).

(35) V.A.Fateev ; A.S.Schwartz ; Y.S.Tyupkin : JETP Lett. 21, 42 (1975).

M.I.Monastyrsky ; A.M.Perelomov : JETP Lett. 21, 43 (1975)

(36) P.Goddard ; J.Nuyts ; D.Olive : Nucl. Phys. B125, 1 (1977)

(37) E.B.Bogomolnyi' : Sov. J. Nucl. Phys. 24, 449 (1976)

(38) " Introduction to Lie algebras and Representation Theory," J.Humphreys, Springer (Berlin,1972)

(39) E.Corrigan ; D.Olive : Nucl. Phys. B110, 237 (1976)

(40) P.Goddard ; D.Olive : Nucl. Phys. B191, 511 (1981)

(41) R.Penrose ; M.A.H.Macallum : Phys. Rep. 6C, 241 (1972)

" Advances in Twistor theory ", Edited by P.Hughston and R.S.Ward, (Pitman, 1981)

(42) E.T.Whittacker ; G.N.Watson : " Modern Analysis ", (CUP, 1902)

(43) C.N.Yang : Phys. Rev.Lett. 38, 1377 (1977).

(44) A.N.Leznov ; M.V.Saveliev ; Comm. Math. Phys. 74, 111 (1980)

(45) N.Ganoulis ; P.Goddard ; D.Olive : Imperial College preprint, ICTP/80, 81-1

(46) L.O'Raifeartaigh ; S.Rouhani ; L.P.Singh : " On the finitely separated two monopole solution ", DIAS preprint, (1982)

(47) E.B.Dynkin : Amer. Math. Soc. Trans. Series 2, 6, 111 (1965)

(48) M.Toda : Prog. of Theor. Phys. (Suppl.) 45, 174 (1970)

" Theory of Non-linear lattices ", Springer series in Solid State Physics, 20. (Berlin, 1981)

(49) H.Osborn : Nucl. Phys. B159, 281 (1979)