Immersions into manifolds without conjugate points

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Immersions Into Manifolds Without Conjugate Points

By

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To :

- My wife Wafaa,
- My daughters Manar and Eeman,
- The memory of my father.

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SYNOPSIS

Many differential geometric concepts such as (isometric) immersion, stability ,...., etc., realized in Euclidean spaces proved to be also realized in manifolds without conjugate points while other concepts are found to be strictly associated with Euclidean spaces. In fact, this thesis may be considered as a trial for finding out to what extent geometric phenomena in Euclidean spaces are still valid in manifolds without conjugate points.

In the introduction, we have quoted the necessary background material for the following chapters. Specially, we have concentrated on the geometry of submanifolds.

The interesting problem of rigidity of submanifolds lies in three different categories : finite rigidity, continuous rigidity and infinitesimal rigidity. These three types of rigidity have been studied in hyperbolic spaces in chapter I, sections 1 and 2.

K. Nomizu, B. Smyth (1969) and S. Braidi, C.C. Hsuing (1970) studied some geometric properties of immersed submanifolds in Euclidean sphere essentially the behaviour of the second fundamental form and the Gauss map. In chapter II (sections 1, 2) we have carried out similar study for immersed submanifolds in hyperbolic spaces which shows some deviations from the corresponding one in Euclidean sphere.

Since B.Y. Chen's paper (1973) which established the geometric concept of stability of submanifolds in Euclidean spaces, other geometers tried to extend this concept to non-Euclidean spaces. In chapter II (section 3) we share this development through studying stability of surfaces in hyperbolic 3-dimensional space.

The most interesting part of our thesis is the last chapter which deals with tight and taut (convex-minimal) immersions in manifolds without conjugate points. Some geometric concepts such as (spherical) two-piece property, h-two-piece property, total (absolute) curvature,... etc., have been introduced. Relations between the above concepts have been adopted. We expect for this part to receive more attention in the future to discover more results and to generalize other Euclidean concepts which we did not touch.

CONTENTS

CHAPTER 0

INTRODUCTION

INTRODUCTION

This chapter reviews briefly the standard concepts and theorems of differential geometry that will be needed in the main part of this work, our aim here being to establish notation and terminology. When M is a C^∞ manifold we use the following notations : $T_x(M) = M_x$ the tangent space of M at x , $\mathcal{F}(M)$ the algebra of (differentiable) C^∞ functions on M and $\mathfrak{X}(M)$ the Lie algebra of vector fields on M. T(M) will denote the tangent bundle of M while S(M) will denote its unit sphere bundle. $\mathbb{R}$ always denotes the real numbers and $\mathbb{R}^n$ denotes the vector space of n-tuples of real numbers $(x^1,\ldots,x^n)$ while E^n denotes the Euclidean n-dimensional space. An n-dimensional manifold will be called n-manifold.

Section 1 : Preliminaries

Our principal references for this section are [2] and [19].

Let M and N be C^∞ manifolds and let $\phi : M \to N$ be a C^∞ mapping from M into N. Let $v\in M_p$ be a tangent vector at $p\in M$. If we set

$$((\phi_*)_p v)f = v(\phi^* f) = v(f\circ\phi)$$

where $f\in\mathcal{F}(N)$, one can see that $(\phi_*)_p v$ is a tangent vector in $N_{\phi(p)}$ and $(\phi_*)_p$, which is the differential of the mapping ϕ, is a linear mapping from M_p into $N_{\phi(p)}$.

The mapping ϕ is called regular at p if $(\phi_*)_p$ is injective. If ϕ is regular at every point of M, then we call ϕ an immersion and M an immersed submanifold of N. When an immersion ϕ is injective it is called an imbedding of M into N. In this case M (or $\phi(M)$) is an imbedded submanifold (or simply a submanifold) of N. If $(\phi_*)_p$ is not injective, then p and $\phi(p)$ are called critical point and critical value of the map ϕ, respectively.

Related to the last concept we mention the "Sard's theorem" as follows:

Let M and N be two C^1 n-manifolds and $\phi : M \to N$ is a C^1 mapping of M into N, then the image $\phi(E)$ of the set E of critical points of ϕ is a set of measure zero of N.

Let X be a C^∞ vector field on a C^∞ manifold M. We associate with X a local one-parameter group of transformations $\{\phi_t\}$ which, for every point $p \in M$ and real number t sufficiently close to zero, assigns the point $\phi(t,p) = \phi_t(p) = \gamma(t)$, where γ is the integral curve of X starting at p. It is known that for every $p \in M$ there is a positive number c and a neighbourhood U of p such that ϕ is defined and C^∞ on $U \times (-c,c)$. For $q \in U$ and $t,s,t+s \in (-c,c)$ we have $\phi_{t+s}(q) = \phi_t(\phi_s(q))$. Conversely, if we are given a C^∞ map having domain of the same type as ϕ and satisfying the above additive property, then we again calling it ϕ get a vector field X having ϕ as its local one-parameter group. In fact X is related to ϕ as follows:

$$X(q)f = \lim_{t \to o} \{ f(\phi_t(q)) - f(q) \} / t$$

for $f \in \mathcal{F}(M)$. The one-parameter group of transformations of M can be defined similarly.

Let M be a C^∞ n-manifold and let $m \in M$. Since M_m is an n-dimensional vector space, the theory of linear algebra can be applied to define tensors and forms. A p-covariant tensor at m ($p>o$) or a p- co tensor at m is a real valued p-linear function on $M_m \times \cdots \times M_m$ (p copies). In a similar way, one can define a V-valued p- co tensor at m, where V is any vector space over $\mathbb{R}$.

The set of real valued 1- co tensors at m is called the dual space of M_m and is denoted by M_m^*. Naturally, M_m^* is a vector space over $\mathbb{R}$ and dim $M_m^* = n$. Similarly, the set of p-co tensors at m, denoted by $T^{o,p}(M_m)$, is a vector space over $\mathbb{R}$. A p-contravariant or p-contra tensor at m ($p > o$) is a real valued p-linear function on $M_m^* \times \dots \times M_m^*$ (p copies) and the natural vector space formed by p-contra tensors at m is denoted by $T^{p,o}(M_m)$. Finally, a p-co and q-contra tensor at m is a (p+q)-linear real valued function on $(M_m)^p \times (M_m^*)^q$ and the vector space of these tensors is denoted by $T^{q,p}(M_m) = T_p^q(M_m)$. In particular, a vector at m is a 1-contra tensor at m. A 1-co tensor at m is called a 1-form at m. A p-form at m ($p > o$) is a skew-symmetric p-co tensor at m and the set of p-forms at m will be denoted by $F^p(M_m)$.

A p-co tensor field on a set $U \subset M$ is a mapping that assigns to each $m \in U$ a p-co tensor at m. A p-co tensor field α on U is C^∞ if and only if U is open and for all sets of C^∞ vector fields $X_1, \dots, X_p$ on U, the function $[\alpha(X_1, \dots, X_p)](m) = \alpha_m(X_1(m), \dots, X_p(m))$ is C^∞ on U. A C^∞ p-form on an open set $U \subseteq M$ is called a differential p-form on U.

If $\alpha \in T^{o,p}(M_m)$ and $\beta \in T^{o,q}(M_m)$, then the tensor product $\alpha \otimes \beta$ of α and β is an element in $T^{o,p+q}(M_m)$ defined by

$$(\alpha \otimes \beta)(X_1, \dots, X_{p+q}) = \alpha(X_1, \dots, X_p)\, \beta(X_{p+1}, \dots, X_{p+q})$$

where $X_1, \dots, X_{p+q}$ are vectors in M_m. The tensor product is bilinear and associative but not commutative.

If α and β are forms of degree p and q, respectively, then their exterior product $\alpha \wedge \beta$ which is a (p+q)-form, is given by

$$(\alpha \wedge \beta)(X_1, \dots, X_{p+q}) = \frac{1}{(p+q)!} \Sigma (-1)^\pi \alpha(X_{\pi(1)}, \dots, X_{\pi(p)})\, \beta(X_{\pi(p+1)}, \dots, X_{\pi(p+q)})$$

where the sum is taken over all permutations π of the set $\{1, 2, \dots, p+q\}$. The exterior product has the properties $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$, $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$ and $\alpha \wedge (\beta_1 + \beta_2) = \alpha \wedge \beta_1 + \alpha \wedge \beta_2$ where β_1 and β_2 are forms of the same degree. In the local coordinate system $(x^1, \dots, x^n)$ on M, a differential p-form ω can be expressed uniquely as

$$\omega = \sum_{i_1<i_2<\ldots<i_p} f_{i_1\ldots i_p} \, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

where all $f_{i_1 \ldots i_p}$ are in $\mathcal{F}(M)$.

A C^∞ symmetric 2-Co tensor field g on a C^∞ manifold M is called a pseudo-Riemannian metric if g_m $(=g|M_m)$ is a non-degenerate bilinear form on M_m at each point $m \in M$ while g is called a Riemannian metric if g_m is positive definite for all m. Clearly, a Riemannian metric g on a C^∞ manifold M induces an inner product on each M_m. A pair (M,g) consisting of a C^∞ manifold M and a (pseudo) Riemannian metric g is called a (pseudo) Riemannian manifold. In local coordinates we write

$$g = \sum_{i,j} g_{ij} \; dx^i \otimes dx^j$$

where $g_{ij} = g\,(\partial/\partial x^i \;,\; \partial/\partial x^j)$. We sometimes write $g = \langle , \rangle$.

For any two points p_1, $p_2 \in$ M in a C^∞ Riemannian n-manifold (M,g) we define the distance $d(p_1 \;, p_2)$ between them to be the greatest lower bound of the lengths of all piecewise differentiable (C^1) curves joining p_1 and p_2. The manifold M together with the metric d turns out to be a metric space. The volume element dv of the Riemannian manifold (M,g) is defined in local coordinates $(x^1,\ldots,x^n)$ to be

$$dv = \sqrt{\det(g_{ij})} \;\; dx^1 \wedge \ldots \wedge dx^n.$$

Let M be a C^∞ manifold and let $A \subseteq M$ be an open set. For $p \geq o$ we define the exterior differentiation map $d: F^p(A) \to F^{p+1}(A)$ through the following properties which it has :

(i) $dF^p(A) \subset F^{p+1}(A)$ for each $p \geq o$.

(ii) If $f \in F^0(A)$ then $df(X) = Xf$ for $X \in \mathfrak{X}(M)$.

(iii) $d^2 = o$

(iv) $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2$ where $\omega_1 \varepsilon F^r(A)$

(v) For $\omega \varepsilon F^p(A)$,

$$d\omega(X_1,\ldots,X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} X_i\,\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{p+1}) + \\ + \sum_{i<j} (-1)^{i+j} \omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{p+1})$$

where $X_1,\ldots,X_{p+1} \varepsilon \mathfrak{X}(A)$ and $\hat{X}_i$ indicates that X_i is omitted as an argument.

Let K be a C^∞ tensor field and X be a C^∞ vector field on M. We define a tensor field $(L_X K)$ and call it the "Lie derivative of K" with respect to X as follows:

$$(L_X K)_p = \lim_{t \to o} \{K_p - (\tilde{\phi}_t K)_p\}/t$$

where $\tilde{\phi}_t$ denotes the induced mapping from the local one-parameter group of transformations $\{\phi_t\}$ around $p \varepsilon M$ generated by X. The operator $L_X : K \to L_X K$ has the following properties :

(i) $L_X(K + K') = L_X K + L_X K'$, $L_X(K \otimes K') = (L_X K) \otimes (L_X K')$ for tensor fields K and K'.

(ii) $L_X f = Xf$, $L_X Y = [X,Y]$ where $X,Y \varepsilon \mathfrak{X}(M)$ and $f \varepsilon \mathcal{F}(M)$.

(iii) $L_{[X,Y]} = L_X L_Y - L_Y L_X$

(iv) K is invariant under $\{\phi_t\}$ if and only if $L_X K = o$

(v) $L_X \omega$, where $\omega \varepsilon F^p(M)$, is in $F^p(M)$ and for $X_1,\ldots,X_p \varepsilon \mathfrak{X}(M)$ we have

$$(L_X\omega)(X_1,\ldots,X_p) = X(\omega(X_1,\ldots,X_p)) - \sum_{i=1}^{p} \omega(X_1,\ldots,[X,X_i],\ldots,X_p).$$

Let (P,G,M) be a C^∞ principal bundle over the C^∞ manifold M where P is the bundle space and G is the structural group [2]. Let $\pi : P \to M$ denote the C^∞ projection. It is known that G acts transitively without fixed point on each fibre. For $a \in G$ and $x \in P$ we write $R_a(x) = xa$ where $R_a : P \to P$ denotes the right action of G on P. The maps $R_{a_*} : T_x(P) \to T_{xa}(P)$ and $\pi_* : T_x(P) \to T_{\pi(x)}(M)$ are the induced ones from R_a and π. The kernel of $(\pi_*)_x$, denoted by $V_x(P)$, is said to be vertical and every element in $V_x(P)$ is tangent to the fibre through x.

We say that a connection Γ is given in P if for each $x \in P$ a subspace H_x of $T_x(P)$ is given such that the following three conditions are satisfied :

(i) $T_x(P) = V_x(P) \oplus H_x(P)$

(ii) $R_{a_*}(H_x) = H_{xa}$

(iii) The map $x \to H_x$ is C^∞.

A vector in H_x is said to be horizontal. Condition (iii) means that if X is in $\mathfrak{X}(P)$ then both horizontal and vertical components of X are C^∞ vector fields on P.

Suppose that a connexion Γ is given on P. If c is a piece-wise differentiable (C^1) curve in the base space M, we can define a mapping ϕ that maps the fibre over the initial point p of c onto the fibre over the end point q as follows:

Take an arbitrary point x on the fibre $\pi^{-1}(p)$, then we have a unique curve c_x^* in P starting at x such that $\pi(c_x^*) = c$ and the velocity vector field of c_x^* is in $H_{c_x^*}$ and we set $y = \phi(x)$ where y is the end point of c_x^*. We call ϕ the parallel displacement along the curve c.

Let $(L(M), Gl(n,\mathbb{R}),M)$ be the principal fibre bundle of tangent n-frames where M is a C^∞ n-manifold and $Gl(n,\mathbb{R})$ is the general linear group acting on $\mathbb{R}^n$. A connexion Γ in this bundle is called an affine connexion. An affine connexion Γ which gives a parallel displacement in L(M), induces in a natural way a parallel displacement in the associated bundle T(M). [see appendix (i)].

Let X and Y be C^∞ vector fields on a C^∞ manifold M with an affine connexion Γ. We define the covariant derivative $\nabla_X Y$ of the vector field Y in the direction of the vector field X as follows:
Let p_o be a point in M and let $c = c(t)$ $(-\varepsilon \leq t \leq \varepsilon)$ be an integral curve of X through p_o. Let $\{\phi_t\}$ be the parallel displacement along c.
We set

$$(\nabla_X Y)_{p_o} = \lim_{t \to o} \{\phi_t^{-1}(Y_{c(t)}) - Y_{p_o}\}/t$$

Hence $\nabla_X Y$ is also a C^∞ vector field on M. The mapping $(X,Y) \to \nabla_X Y$ has the following properties :

(i) $\nabla_X Y$ is linear with respect to X and Y.

(ii) $\nabla_X f = Xf$.

(iii) $\nabla_{fX} T = f\nabla_X Y$, $\nabla_X(fY) = f(\nabla_X Y) + (Xf)Y$.

where $X, Y \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$.

Conversely, if the mapping conditions (i) - (iii) are given, then there exists a unique affine connexion Γ on M whose covariant derivative operator coincides with the given mapping.

For a (pseudo) Riemannian metric g on a C^∞ manifold M, there exists a unique affine connexion Γ, called the Riemannian connexion, such that

(i) $\nabla g = o$ (ii) $\nabla_X Y - \nabla_Y X = [X,Y]$ for $X, Y \in \mathfrak{X}(M)$.

For an affine connexion Γ on a C^∞ n-manifold M and in local coordinate system $(x^1,\ldots,x^n)$ we define the functions Γ_{ij}^k , called the Christoffel symbols of Γ, as follows :

$$\nabla_{\partial/\partial x^i}(\partial/\partial x^j) = \sum_{k=1}^{n} \Gamma^k_{ij}(\partial/\partial x^k)$$

In particular, for a Riemannian connexion we have

$$\Gamma^k_{ij} = \tfrac{1}{2} \sum_r g^{rk} \left(\frac{\partial g_{ji}}{\partial x^i} + \frac{\partial g_{ir}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^r} \right)$$

where the matrix (g^{ij}) is the inverse of (g_{ij}).

Let M and N be two C^∞ n-manifolds and let $\phi : M \to N$ be a diffeomorphism. Let ∇ and $\bar{\nabla}$ be the covariant differentiation operators of the affine connexions Γ and $\bar{\Gamma}$ on M and N, respectively. The mapping ϕ is called affine (or connexion preserving) if $\phi_*(\nabla_X Y) = \bar{\nabla}_{\phi_* X}(\phi_* Y)$ for all $X,Y \in \mathfrak{X}(M)$. The affine transformation $\phi : M \to M$ can be defined similarly.

If the tangent vector c' of a curve $c = c(t)$ in a C^∞ manifold M with an affine connexion Γ has the property $\nabla_{c'} c' = o$, then c is said to be a geodesic. In terms of local coordinates, the geodesic c is defined by

$$\frac{d^2x^k}{dt^2} + \sum_{ij} \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = o$$

This geodesic c is defined uniquely when knowing its initial point and velocity. If every geodesic of an affine connexion can be extended so it is a geodesic for all $t \in \mathbb{R}$, then the connexion is said to be complete. For an affine connexion Γ on M, the exponential map $\exp_p : M_p \to M$ for fixed $p \in M$ is defined by $\exp_p v = \gamma_v(1)$ for all $v \in M_p$ where 1 is in the domain of γ_v and γ_v is a geodesic such that $\gamma_v(o) = p$ and $\gamma_v'(o) = v$. The exponential map is a local diffeomorphism in a neighbourhood of the origin O in M_p. An important fact concerning the exponential map is the so called "Gauss' lemma" which may be stated as follows:

If $\rho(t) = tv$, $t \in \mathbb{R}$, is a ray through the origin 0 in M_p, where M is a Riemannian manifold, and if $\omega \in (M_p)_{\rho(t)}$ is perpendicular to $\rho'(t)$, then $(\exp_*)_{\rho(t)}\, \omega$ is perpendicular to $(\exp_*)_{\rho(t)} (\rho'(t))$.

The theorem that follows gives useful criteria of a Riemannian manifold to be complete :

Theorem (Hopf-Rinow)

If M is a connected Riemannian manifold, then (a), (b), (c), and (d), stated below, are equivalent statements, and any one of them implies (e).

(a) The exponential map is everywhere defined on T(M).

(b) The manifold is complete with respect to its Riemannian metric.

(c) Bounded closed sets in M are compact.

(d) The closed balls $\bar{B}(m,r)$ are compact for one m in M and all $r > o$.

(e) Any two points in M can be joined by a geodesic segment whose length equals the distance between the two points.

It is also known that if all geodesics starting from a particular point x of a connected C^∞ Riemannian manifold M are infinitely extendable, then M is complete. Every compact Riemannian manifold is complete.

Let M be a C^∞ n-manifold with an affine connexion Γ. The curvature tensor of this affine connexion is a linear transformation valued tensor R that assigns to each pair of vectors X_x and Y_x at $x \in M$ a linear transformation $R(X_x,Y_x)$ of M_x into itself. We define $R(X,Y)Z$ by imbedding X_x,Y_x and Z_x in C^∞ fields about x and setting

$$R(X_x,Y_x)Z_x = (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z)_x$$

The torsion tensor of the connexion Γ on M is a vector valued tensor T that assigns to each pair of vector fields $X,Y \in \mathfrak{X}(M)$, with domain $A \subset M$, a C^∞ vector field T(X,Y), with domain A, by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

If $T \equiv 0$ then we say that Γ is symmetric, or torsion free.

Let (M,g) be a C^∞ Riemannian manifold with Riemannian connexion Γ. The Riemannian curvature tensor of type (0,4) is the 4-co tensor $R(X,Y,Z,W) = g(R(X,Y)Z,W)$ for X,Y,Z,W in M_x, $x \in M$. The following relations are satisfied with R

(a) $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$

(b) $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z) = R(Z,W,X,Y)$

Relation (a) is called the first Bianchi identity and it holds for any symmetric connexion.

Let $u,v \in M_x$ and let

$$A(u,v) = g(u,u)\, g(v,v) - g(u,v)^2$$

For $A(u,v) \neq 0$ we define the sectional curvature (or the Riemannian curvature) for the 2-dimensional subspace P of M_x spanned by u and v by

$$K(P) = K(u,v) = K(u \wedge v) = g(R\ (u,v)\ u,v)/A(u,v)$$

Let $\phi : M \to M'$ be a C^∞ map between C^∞ Riemannian manifolds. If there is a C^∞ real valued positive function F on M such that for all $x \in M$ and all $u,v \in M_x$, $g'((\phi_*)_x u, (\phi_*)_x v) = F(x)\, g(u,v)$ where g and g' are the Riemannian metrics of M and M', respectively, then ϕ is called a conformal mapping and F is called the scale function. If F = 1 then ϕ is called an isometry. If ϕ is an isometry and a diffeomorphism, then we say that M is isometric to M'. If F is constant, then ϕ is called homothetic.

In terms of local coordinates, we have for the C^∞ Riemannian manifold (M,g) ,

$$K(P) = \{ \sum_k g_{ki} \; R^k_{jij} \} / \{ g_{ii} \, g_{jj} - g^2_{ij} \}$$

where

$$R^k_{jij} = \frac{\partial \Gamma^k_{jj}}{\partial x^i} - \frac{\partial \Gamma^k_{ii}}{\partial x^j} + \sum_r (\Gamma^k_{ri} \; \Gamma^r_{jj} - \Gamma^k_{rj} \; \Gamma^r_{ji})$$

If $K(P) = k$, k is constant, for all plane sections P in M_x and for all points $x \in M$, then M is called a space of constant curvature and in this case we have

$$R(X,Y)Z = k\{ g(Z,Y)X - g(Z,X)Y\}$$

Let M be a C^∞ Riemannian n-manifold and let $U \subset M$ be a fixed open set. Let $e_1, \ldots e_n$ be a fixed base field of independent C^∞ vector fields on U and let $\omega^1, \ldots, \omega^n$ be the dual C^∞ 1-forms on U. Define n^2 connexion C^∞ linear 1-forms $\{\omega^i_j\}$ on U which are associated with the Riemannian connexion Γ on M by

$$\nabla_X e_j = \sum_{i=1}^{n} \omega^j_i (X) \, e_i$$

where X is a C^∞ vector field on U. The following equations are called Cartan structural equations

$$d\omega^i = - \sum_{j=1}^{n} \omega_j{}^i \wedge \omega^j$$

$$d\omega_i{}^j = - \sum_{k=1}^{n} \omega_i{}^k \wedge \omega_k{}^j + \tfrac{1}{2} \sum_{k,\ell} R^i_{jk\ell} \; \omega^k \wedge \omega^\ell$$

For $\{e_1, \ldots, e_n\}$ orthonormal basis, we have in addition $\omega_j{}^i + \omega_i{}^j = 0$ and $R^i_{jk\ell} + R^i_{j\ell k} = 0$.

From now and for the rest of the thesis, manifolds, mappings, vector field,..., etc. are sufficiently differentiable for all computations to make sense unless otherwise stated.

Section 2 : On submanifolds :

Our approach to this section is mainly based on chapter VII [19].

Let M be an n-dimensional manifold immersed into a Riemannian manifold N. We denote by $\tilde{\tilde{\nabla}}$ the covariant differentiation operator in N. Since the following discussion is local, we may assume that M is imbedded into N.

Let X and Y be vector fields on M. Since $(\tilde{\nabla}_X Y)_x$ is defined for each $x \in M$, we shall denote by $(\nabla_X Y)_x$ its tangential component and by $\alpha_x(X,Y)$ its normal component so that

$$(\tilde{\nabla}_X Y)_x = (\nabla_X Y)_x + \alpha_x(X,Y) \tag{2.1}$$

where $(\nabla_X Y)_x \in M_x$ and $\alpha_x(X,Y) \in T_x(M)^{\perp}$. In fact ∇ is the covariant differentiation for the induced Riemannian connexion on M. The vector field $\nabla_X Y$ which assigns the vector $(\nabla_X Y)_x$ to each point $x \in M$ is differentiable and $\alpha(X,Y)$ is a differentiable field of normal vectors to M. The mapping $\alpha : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^{\perp}$, where $\mathfrak{X}(M)^{\perp}$ denotes the set of all differentiable fields of normal vectors to M, is symmetric and bilinear over $\mathcal{F}(M)$. Consequently, $\alpha_x(X,Y)$ depends only on X_x and Y_x.

The map $\alpha : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^{\perp}$ is called the second fundamental form of M (for the given immersion in N). In fact, for each $x \in M$, $\alpha_x ; M_x \times M_x \to M_x^{\perp}$ is called the second fundamental form of M at x.

If M has codimension p, then we may locally choose p fields of unit normal vectors $\xi_1, \ldots, \xi_p$ that are orthogonal at each point such that

$$\alpha(X,Y) = \sum_{i=1}^{p} h^i(X,Y)\xi_i$$

Thus we have p second fundamental forms in the classical sense.

Let $X \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}(M)^{\perp}$ and write

$$(\tilde{\nabla}_X \xi)_x = -(A_\xi(X))_x + \tilde{\nabla}_X^{\perp} \xi \qquad (2.2)$$

where $-(A_\xi(X))_x$ and $(\tilde{\nabla}_X^{\perp} \xi)_x$ are the tangential and normal components of $(\tilde{\nabla}_X \xi)_x$, respectively. The vector fields $x \to (A_\xi(X))_x$ and $x \to (\tilde{\nabla}_X^{\perp} \xi)_x$ are differentiable on M. Actually, the mapping $(X,\xi) \in \mathfrak{X}(M) \times \mathfrak{X}(M)^{\perp} \to A_\xi(X) \in \mathfrak{X}(M)$ is bilinear over $\mathcal{F}(M)$ and consequently $(A_\xi(X))_x$ depends only on X_x and ξ_x. The two mappings α and A_ξ are related by

$$g(A_\xi(X),Y) = g(\alpha(X,Y), \xi)$$

for all $X,Y \in M_x$. This shows that $A_{\xi_x} : M_x \to M_x$ is a symmetric linear transformation of M_x with respect to g_x.

On the other hand, the mapping $(X,\xi) \in \mathfrak{X}(M) \times \mathfrak{X}(M)^{\perp} \to (\tilde{\nabla}_X^{\perp} \xi) \in \mathfrak{X}(M)^{\perp}$ coincides with the covariant differentiation of the cross-section ξ of the normal bundle $T(M)^{\perp}$ in the direction of X with respect to the connexion in $T(M)^{\perp}$.

The two formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X,Y) \qquad (2.3)$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \tilde{\nabla}_X^{\perp} \xi \qquad (2.4)$$

are called <u>Gauss' formula</u> and <u>Weingarten's formula</u>, respectively. In case of a hypersurface M, the Weingarten's formula reduces to

$$\tilde{\nabla}_X \xi = -A_\xi (X)$$

for the field ξ of unit normal vectors.

The following is quite useful especially for the last section of the thesis. Let M_1 and M_2 be two n-submanifolds in a Riemannian (n+p)-manifold N. Let $\tau = x(t)$, $0 \leqslant t \leqslant 1$, be a differentiable curve in $M_1 \cap M_2$. We say that M_1 and M_2 are tangent to each other along τ if

$T_{x(t)}(M_1) = T_{x(t)}(M_2)$ for each $t \in [o,1]$. In this case the parallel displacement along τ in M_1 coincides with the parallel displacement along τ in M_2. In particular, if τ is a geodesic in M_1, τ is a geodesic in M_2 as well.

Let M be an n-submanifold of N and let $\tau = x(t)$, $t \in [o,1]$, be a curve in M. Then τ is a geodesic in M if and only if $\tilde{\nabla}_{x'(t)} x'(t)$ is normal to M. In particular, if τ is a geodesic of N contained in M, it is a geodesic in M. (A geodesic in M is not, in general, a geodesic in N).

Let N be a Riemannian (n+k)-manifold ($k \geq 1$) and let M be a connected n-submanifold. Let $p \in M$, the submanifold M is said to be a geodesic submanifold of N at p if each geodesic of N which is tangent to M at p is a curve in M. The submanifold M is called totally geodesic if it is geodesic at each of its points. In other words, a submanifold M of a Riemannian manifold N together with the induced Riemannian structure is called totally geodesic if every geodesic of M is a geodesic of N. Great spheres in the unit n-sphere S^n and m-dimensional planes in E^n are totally geodesic submanifolds.

Using expressions (2.3-2.4) we can easily prove the following:

Theorem:

M is totally geodesic submanifold of the Riemannian manifold N if and only if its second fundamental form vanishes identically. ($\alpha \equiv o$).

For a Riemannian manifold N with constant curvature, it is convenient to mention the following two facts:

(i) Every submanifold which is geodesic at a point is totally geodesic.

(ii) Conversely, if every submanifold which is geodesic at a point is totally geodesic, then N has constant curvature.

Let M be an n-dimensional Riemannian manifold isometrically immersed in an (n+p)-dimensional Riemannian manifold N. Again since the following discussion is local, we can choose p orthonormal fields of normal vectors $\xi_1, \ldots, \xi_p$ to M. Let h^i be the corresponding second fundamental forms and let $A_i = A_{\xi_i}$. Using the formulas of Gauss and Weingarten we have for any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$\tilde{\nabla}_X(\tilde{\nabla}_Y Z) = \nabla_X(\nabla_Y Z) - \sum_i h^i(Y,Z) A_i(X) + \sum_i \{ X.h^i(Y,Z) + h^i(X, \nabla_Y Z) \} \xi_i + \xi h^i(Y,Z) \tilde{\nabla}_X \xi_i$$

A similar expression can be written for $\tilde{\nabla}_Y (\tilde{\nabla}_X Z)$. We have also

$$\tilde{\nabla}_{[X,Y]} Z = \nabla_{[X,Y]} Z + \sum_i \{ h^i(\nabla_X Y, Z) - h^i(\nabla_Y X, Z) \} \xi_i$$

Using these equations, we find that the tangential component of $\tilde{R}(X,Y)z$ has the following form

$$R(X,Y)Z + \sum_i \{ h^i(X,Z) A_i(Y) - h^i(Y,Z) A_i(X) \}$$

where $\tilde{R}$ and R are the curvature tensors of N and M, respectively. If $W \in \mathfrak{X}(M)$, then

$$g(\tilde{R}(X,Y)Z,W) = g(R(X,Y)Z, W) + g(\alpha(X,Z), \alpha(Y,W)) - g(\alpha(Y,Z), \alpha(X,W)) \quad (2.6)$$

This equation is called the <u>equation of Gauss</u>.

For $\xi_1, \ldots, \xi_p$ orthonormal basis of M_x and X_x, Y_x orthonormal pair of vectors in M_x, Gauss' equation (2.6) gives that

$$K_M(X_x, Y_x) = K_N(X_x, Y_x) + \sum_i \{ h^i(X_x, X_x) h^i(Y_x, Y_x) - (h^i(X_x, Y_x))^2 \} \quad (2.7)$$

where K_M and K_N are the sectional curvatures of M and N, respectively. If M is a hypersurface of N, the last equation (2.7) takes the following simple form

$$K_M(X_x, Y_x) = K_N(X_x, Y_x) + h(X_x, X_x) h(Y_n, Y_n) - (h(X_x, Y_x))^2 \quad (2.8)$$

As Gauss' equation deals with the tangential component of the curvature tensor $\tilde{R}$ of N, Codazzi equation deals with the normal one.

The normal component of $\tilde{R}(X,Y)Z$ for $X,Y,Z\epsilon \mathcal{X}(M)$ is equal to

$$\Sigma \{(\nabla_X h^i)(Y,Z) - (\nabla_Y h^i)(X,Z)\} \xi_i + \Sigma\{h^i(Y,Z)\tilde{\nabla}_X \xi_i - h^i(X,Z)\tilde{\nabla}_Y \xi_i\} \tag{2.9}$$

where $(\nabla_X h^i)(Y,Z) = Xh^i(Y,Z) - h^i(\nabla_X Y,Z) - h^i(Y, \nabla_X Z)$ (2.10)

If we use α instead of h^i's, then the normal component of $\tilde{R}(X,Y)Z$ will take the simple form

$$(\bar{\nabla}_X \alpha)(Y,Z) - (\bar{\nabla}_Y \alpha)(X,Z) \tag{2.11}$$

where

$$(\bar{\nabla}_X \alpha)(Y,Z) = \tilde{\nabla}_X^{\perp} (\alpha(Y,Z)) - \alpha(\nabla_X Y,Z) - \alpha(Y, \nabla_X Z) \tag{2.12}$$

and $\bar{\nabla}$ is the covariant derivative of the connexion in $T(M) + T(M)^{\perp}$ obtained by combining the connexions ∇_X in $T(M)$ and $\tilde{\nabla}_X^{\perp}$ in $T(M)^{\perp}$.

From (2.11 - 2.12) we have

$$(\bar{\nabla}_X \alpha)(Y,Z) - (\bar{\nabla}_Y \alpha)(X,Z) = \sum_i \{(\nabla_X h^i)(Y,Z) - (\nabla_Y h^i)(X,Z)\} \xi_i + \sum_i \{h^i(Y,Z)\tilde{\nabla}_X^{\perp} \xi_i - h^i(X,Z)\tilde{\nabla}_Y^{\perp} \xi_i\} \tag{2.13}$$

which is called <u>Codazzi equation</u>.

In the case N is of constant sectional curvature, Codazzi equation (2.13) takes the simple form

$$(\bar{\nabla}_X \alpha)(Y,Z) = (\bar{\nabla}_Y \alpha)(X,Z) \tag{2.14}$$

Section 3 : Manifolds without conjugate points

All quoted theorems and notions for sections (3) and (4) will be found in [2].

In what follows we demonstrate briefly the most important properties of the so called "manifolds without conjugate points" as they will be very oftenly used in the sequal. Throughout this section N is a complete Riemannian manifold of dimension n and with covariant differentiation operator $\tilde{\nabla}$.

Let M be a submanifold of N and let $T(M)^{\perp}$ denote its normal bundle. The exponential map of N by restriction gives the map

$$\exp : T(M)^{\perp} \to N$$

which is a diffeomorphism in a neighbourhood of the zero cross-section. For $p \in M$ let $T_p(M)^{\perp}$ be the fibre of $T(M)^{\perp}$ over p. We say that $t \in T_p(M)^{\perp}$ is a focal point of M if $\exp_*$ is singular at t. If ρ is the ray from 0 to t in $T_p(M)$, then exp(t) is called a focal point of M along $\exp_p \rho$, which is of course, a geodesic perpendicular to M. If M is a single point, say m, so $T_m(M)^{\perp} = N_m$ and a focal point is called a conjugate point to m. The order(multiplicity) of a focal point is the dimension of the linear space annihilated by $\exp_*$.

A vector field on a maximal geodesic γ of N is a Jacobi vector field if

$$\tilde{\nabla}^2_{\gamma'} Y + \tilde{R}\,(\gamma', Y)\,\gamma' = 0$$

where $\tilde{R}$ is the curvature tensor of N. A Jacobi vector field Y is uniquely determined by the values Y(o) and $Y'(o)$. The Jacobi fields along γ form a linear space of dimension 2n. The Jacobi fields along γ which vanish at $\gamma(o)$ form a linear subspace of dimension n.

Let $M \subset N$ be a Riemannian r-submanifold of N and γ be a maximal unit speed geodesic of N such that $\gamma'(o)$ is perpendicular to $M_{\gamma(o)}$. A Jacobi vector field Y along γ is an M-Jacobi field if

(i) Y is perpendicular to γ.

(ii) $Y(o) \in M_{\gamma(o)}$.

(iii) $A_{\gamma'(o)} Y(o) - \tilde{\nabla}_{\gamma'} Y \,|_{\gamma(o)}$ is perpendicular to $M_{\gamma(o)}$.

The M-Jacobi vector fields form a linear space of dimension dim N-1. Geometrically, a vector field V is an M-Jacobi vector field if and only if it is generated by variation of geodesics starting perpendicular to M and parametrized by arc-length.

An alternative definition of focal points can be given in terms of Jacobi fields as follows:

If γ is a geodesic of N which starts perpendicular to M, then $\gamma(b)$ is a focal point of M along γ if and only if there is a non-trivial M-Jacobi field along γ which vanishes at $\gamma(b)$. Analogously, the order of $\gamma(b)$ (multiplicity) is the dimension of the space of such Jacobi fields. Also $\gamma(b)$ is conjugate to $\gamma(a)$ along γ if and only if there is a non-trivial Jacobi field along γ which vanishes at $\gamma(a)$ and $\gamma(b)$.

A geodesic γ, from a point $p \in N$ does not minimize distance from p beyond the first conjugate point. It is also true that γ does not minimize distance to M beyond the first focal point.

Definitions

(3-1) If m is a point of N such that there exists no point of N that is conjugate to m, then m is called a pole.

(3-2) If every point of N is a pole, then N is called a manifold without conjugate points.

(3-3) The manifold N is said to have no focal points if no maximal geodesic $M = \sigma$ has focal points along any geodesic perpendicular to σ.

The "no focal point" property is equivalent to the following : Let γ be a unit speed geodesic in N, and let Y be not necessarily perpendicular Jacobi vector field on γ such that $Y(o) = o$ and $(\tilde{\nabla}_{\gamma'} Y)(o) \neq o$. Then for any $t > o$, $(\|Y\|^2)'(t) > o$. The "no focal point" property is stronger than the "no conjugate point" one, i.e. a manifold N has no conjugate points if it has no focal points. If N has sectional curvature $K \leqslant o$, then N has no focal points.

Moreover, we have the following important theorems:

Theorem (3.1)

If p is a pole in N, then $\exp_p : N_p \to N$ is a covering map. Thus the simply connected covering of N is diffeomorphic to E^n, and if N is simply connected, then N is diffeomorphic to E^n. ($\dim N = n$).

Theorem (3.2)

If p is a pole in N and N is simply connected, then for any point $q \in N$ there is a unique geodesic through p. If N has no conjugate points, then there is a unique geodesic joining any two of its points.

As an application of Sard's theorem, we conclude that the set of focal points C_f, of an immersion $f : M \to N$ of the manifold M into N, has measure zero, hence $N \setminus C_f$ is dense in N.

Two unit speed geodesic rays α, β in a Riemannian manifold N are said to be asymptotic if there exists a number $c \in \mathbb{R}$, $o < c < \infty$, such that $d(\alpha(t), \beta(t)) \leqslant c$ for all $t \geqslant o$. Related to this concept, Midori S. Goto [16] proved the following:

Theorem (3.3)

Let N be a C^∞ complete simply connected Riemannian manifold without focal points. Then any two distinct geodesic rays starting from any point $p \in N$ cannot be asymptotic to each other.

Section 4 : The Morse index theorem

In this section N and P will be submanifolds of a C^∞ Riemannian manifold M with curvature tensor R and sectional curvature K. If $Q : [a,b] \times [c,r] \to M$ $(a,b,c,r \in \mathbb{R})$ is a piecewise smooth rectangle in M we may define the smooth function $d_Q : [c,r] \to \mathbb{R}$, whose value at a point $t \in [c,r]$ is given by the length of the longitudinal path τ_t of Q.

Theorem (4.1)

Let τ : $[o,b] \to M$ be a piecewise smooth path joining N to P. Then τ is a geodesic in M with $\tau'(o) \in T(N)^{\perp}$ and $\tau'(b) \in T(P)^{\perp}$ if and only if $d_Q'(o) = o$ for all piecewise smooth rectangles in M with base curve τ, initial transversal in N final transversal in P and all transversals normal to τ.

Not let τ : $[o,b] \to$ M be a geodesic in M such that $\tau'(o) \in T(N)^{\perp}$ and $\tau'(b) \in T(P)^{\perp}$. Let $\mathcal{L}(N,P)$ be the space of piecewise smooth vector fields along τ which are orthogonal to τ and have their initial and final vectors tangent to N and P, respectively. Then if $\xi \in \mathcal{L}(N,P)$, we can find a piecewise smooth rectangle Q in M which represents ξ and has initial and final transversals in N and P, respectively. From theorem (4.1) we see that $d_Q'(o) = o$. Further,

$$d_Q''(o) = {}_a\int^b \{\|\nabla_{\tau'}\xi\|^2(u) - K(\xi,\tau')\|\xi\|^2(u)\}\, du + h_N^{\tau'(o)}(\xi,\xi) - h_P^{\tau'(b)}(\xi,\xi)$$

where $h_N^{\tau'(o)}$ and $h_P^{\tau'(b)}$ are the second fundamental forms of N and P in the directions $\tau'(o)$ and $\tau'(b)$, respectively. The above expression for $d_Q''(o)$ is independent of the choice of piecewise smooth rectangle representing ξ, and hence we have a quadratic form defined on $\mathcal{L}(N,P)$. The index form $I_{N,P}$ is the symmetric bilinear form on $\mathcal{L}(N,P)$ associated with this quadratic form and hence is defined as follows : Let $\xi,\eta \in \mathcal{L}(N,P)$, then

$$I_{N,P}(\xi,\eta) = {}_o\int^b \{\langle\nabla_{\tau'}\xi, \nabla_{\tau'}\eta\rangle(u) - \langle R(\tau',\xi)\tau',\eta\rangle(u)\}\, du + h_N^{\tau'(o)}(\xi,\eta) - h_P^{\tau'(b)}(\xi,\eta).$$

Theorem (4.2)

With notations as above, if there exists $\xi \in \mathcal{L}(N,P)$ such that $I_{N,P}(\xi,\xi) < o$ then every neighbourhood of τ in M contains shorter piecewise smooth paths from a neighbourhood of $\tau(o)$ in N to a neighbourhood of $\tau(b)$ in P.

Theorem (4.3)

With notations as above, if there exists a focal point of N along τ, then every neighbourhood of τ in M contains shorter piecewise smooth paths from a neighbourhood of $\tau(o)$ in N to $\tau(b)$.

From theorems (4.2-3) we see that if N has a focal point along τ, then $I_{N,\tau(b)}$ is not positive definite on $\mathcal{L}(N,\tau(b))$. Theorem (4.4), the Morse index theorem, is a refinement of this statement.

Theorem (4.4)

Under the same notations as before, the augmented index of $I_{N,\tau(b)}$ is finite and is equal to the sum of the orders (multiplicities) of the focal points of N along τ. Further, the index of $I_{N,\tau(b)}$ is also finite and is equal to the sum of orders of the focal points of N along $\tau| [o,b)$.

Section 5 : Busemann functions and horospheres

Horospheres have always been a central point of interest in hyperbolic geometry. In modern language, horospheres are defined as enveloping hypersurfaces of all Riemannian spheres having a common normal vector in the hyperbolic space. In fact, using this definition, horospheres can be defined for all simply connected Riemannian manifolds without conjugate points.

Let M be a C^∞ complete, simply connected Riemannian manifold without conjuage points. For every $p,q \in M$ call $d(p,q) = |p,q|$ the distance function between p and q. For each $v \in SM$ and each $s \geqslant o$ define the function

$$b_{vs}(q) = s - |\gamma_v(s),q|$$

where $\gamma_v(s)$ is a unit speed geodesic with the property $\gamma'_v(o) = v$.

Further define the ball $B_{vs} = b_{vs}^{-1}$ $((o,s])$. The functions b_{vs} are all smooth (C^1) except at $\gamma_v(s)$, increasing with s and absolutely bounded by $|\gamma_v(o), q|$. Hence the function $b_v = \lim_{s \to \infty} b_{vs}$ is defined everywhere on M. Call $H_v = b_v^{-1}$ (o) the horosphere and $B_v = b_{vs}^{-1}$ ((o,∞)) the horodisc of v. The function b_v is called the Busemann function of v. J.H. Escheburg [14] proved the following:

Let M be a C^∞ complete, simply connection Riemannian manifold without conjugate points. Then b_v is C^∞ -differentiable with gradient $\nabla b_v = \lim_{s \to \infty} \nabla b_{vs}$ (pointwise convergence) for each unit vector $v \in SM$.

When dealing with tight and taut immersions into manifolds without conjugate points (last chapter) some extra conditions on Busemann functions b_v, such as being C^2 -differentiable, are needed. For this reason we define the so called manifolds with bounded asymptote.

<u>Definition</u> :

A manifold M is called with bounded asymptote if it is C^∞, complete, connected without conjugate points, and if there exists a uniform bound $\rho \geq 1$ for the stable Jacobi tensor D [14] such that $\|D_v(t)\| \leq \rho$ for all $v \in SM$, $t \geq o$.

For example all manifolds without focal points are of bounded asymptote. For more examples and details see [14].

The condition that M has sectional curvatures bounded from below together with the bounded asymptote property is enough to ensure that the following conditions are satisfied:

(a) Each Busemann function is C^2 and has gradient vector field of unit length.

(b) The level hypersurfaces (horospheres) of each Busemann function form an equidistant family whose orthogonal trajectories are geodesics.

(c) If u is a unit vector at $p \varepsilon M$, then $u = \text{grad } b_u(p)$. Moreover, if $v = \text{grad } b_u(q)$ for $q \varepsilon M$ then b_u and b_v differ only by a constant. Hence the horospheres determined by b_u are the same as those determined by b_v.

In a C^∞ , complete, simply connected, Riemannian manifold without conjugate points, each $v \varepsilon SM$ determines a family of horospheres orthogonal to the unit vector field grad b_v. If $u = \text{grad } b_v(q)$, $q \varepsilon M$, we say that u is asymptotic to v. If in addition M is of bounded asymptote and sectional curvatures bounded from below, condition (c) above shows that grad b_v = grad b_u, and asymptotic is an equivalence relation on SM. The equivalence classes form a regular continuous foliation whose leaves are C^1 vector fields on M of the form grad b_v. (see [3]).

It should be known that horospheres in E^n are nothing but hyperplanes while Busemann function b_v, $v \varepsilon SE^n$, is the usual height function in the direction of v.

The following proposition gives some characterization of the horospheres in manifolds without conjugate points. We use S(p,r) to denote the geodesic sphere of center $p \varepsilon M$ and radius r.

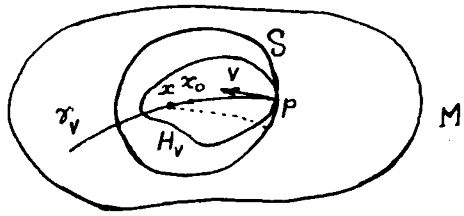
Proposition :

Let M be a complete, simply connected, Riemannian (n+1)-manifold without conjugate points. Then horospheres are complete non-compact hypersurfaces.

Proof :

Suppose that H_v is a compact horosphere of the manifold M, $v \varepsilon SM$. The Horosphere H_v can be contained inside some geodesic sphere with finite radius. Shrink this geodesic sphere radially to the geodesic sphere $S = S(x_o, r)$ which touches H_v. Let $p \varepsilon S \cap H_v$. Draw the unit speed geodesic ray $\gamma_v(t)$ such that $\gamma_v(o) = p$, $\gamma'_v(o) = v$. The geodesic ray γ_v will pass through the center x_0 of S, i.e. $x_0 = \gamma_v(r)$.

Consider the map $g : H_v \to S$ which is defined to be the projection of H_v onto S through geodesic rays from $x = \gamma_v(r + \varepsilon)$ for sufficiently small positive real number ε.



Let L_x be the distance function from x. The point p is a critical point of both $L_x|H_v$ and $L_x|S$. Using the above mapping g, it is easy to see that

$$L_x(q) \leqslant L_x \circ g\,(q)$$

for any $q \in H_v$.

It is now clear that $L_x \circ g$ has index n, so is $L_x|H_v$. Using the Morse index theorem, we conclude that H_v has focal points on the geodesic segment $\gamma_v((o, r + \varepsilon])$. This contradicts the fact that the horosphere H_v has focal points at infinity. Hence H_v is non-compact. As being a level surface of b_v, the horosphere H_v is closed and therefore complete. Since $v \in SM$ is arbitrary, hence the result.

Corollary :

Let M be as in the above proposition, then horodisc B_v and its complement $M \backslash B_v$, for any $v \in SM$, are both unbounded bodies.

Section 6 : Hyperbolic spaces

A complete, simply connected C^∞ Riemannian manifold of constant sectional curvatures is called a space form. A space form is said to be elliptic, hyperbolic or Euclidean according as the sectional curvature is positive, negative or zero, respectively.

For hyperbolic space we have disc model, half-space model and projective model. The geometry of these models can be found in [29]. An important model which we call H-model will be given in the sequel. Consider the real vector space $\mathbb{R}^{n+1}$ equipped with a non-degenerate quadratic form $<,>$ (Lorentz inner product) of signature (n,1). The H-model is either of the two connected components of the hypersurface of $(\mathbb{R}^{n+1}, <,>)$, $\{x \in \mathbb{R}^{n+1} ; <x,x> = -1\}$ on which $<,>$ restricts to a Riemannian metric of constant sectional curvature -1. For geodesics, horospheres ,..., etc, of this model, see [29]. In the following we call $(\mathbb{R}^{n+1}, <,>)$ the Minkowski space.

Geodesics of the H-model can be taken as $x = \rho(c + dt)$ where c is the position vector of the initial point, $d \in H_c$, t is the parameter and $\rho = -(\|d\|^2 t^2 - 1)^{-1}$ is the normalization factor. Using this parametrization we can prove that for any pair of points $p, q \in H$

$$d(p,q) = |\cosh^{-1}(-<p,q>)| \tag{6.1}$$

Taking into account equation (6.1), geodesic sphere S(p,r) of center $p \in H$ and radius r in the H-model is defined as

$$S(p,r) = \{x \in H ; \; r = |\cosh^{-1}(-<x,p>)|\} \tag{6.2}$$

It is clear from this definition that $S(p,r) = L^n \cap H$ where L^n is a hyperplane in $(\mathbb{R}^{n+1}, <,>)$ with p as its normal. As horosphere is a limit of geodesic sphere sequence we can show that a horosphere of the H-model is just $\tilde{L}^n \cap H$ where $\tilde{L}^n$ is a hyperplane in $(\mathbb{R}^{n+1}, <,>)$ which is parallel to a generator of the cone $<x,x> = 0$

Let p be an arbitrary point of H. Then

$$E_p^n = \{q \in \mathbb{R}^{n+1} ; \; <q,p> = 0\}$$

is an n-dimensional subspace of $\mathbb{R}^{n+1}$ on which $<,>$ restricts to a Euclidean metric. Let D_p^n be the unit disc centered at the origin in E_p^n . The diffeomorphism $P : H \to D_p^n$ given by

$$P(x) = \frac{x + \langle x,p \rangle p}{1 - \langle x,p \rangle}$$

is called the stereographic projection with respect to the pole -p. We observe that P is a conformal mapping with scale function $(1-\langle x,p \rangle)^2$. The map P has the property : an r-dimensional submanifold in H is umbilical if and only if its image under P is umbilical in E_p^n .

If D denotes the Riemannian connection on $(\mathbb{R}^{n+1}, \langle\ ,\ \rangle)$ and $\tilde{\nabla}$ denotes the induced Riemannian connexion on H, we have

$$D_X y = \tilde{\nabla}_X y + \langle X,y \rangle\ x \qquad (6.3)$$

where $x \in H$ and $X, Y \in \mathfrak{X}(H)$. The formula (6.3) can be used successfully in showing that the H-model has constant sectional curvature K = -1.

The following proposition gives some characterization of geodesic spheres in hyperbolic space.

Proposition (6.1)

Let M be a C^∞ hypersurface in H, then M is compact umbilical if and only if it is a geodesic sphere.

Proof :

First, assume that M is the geodesic sphere S(p,r) with center $p \in H$ and finite radius r. Clearly M is compact. Let $x \in M$, then from equation (6.1) we have

$$\langle x,p \rangle = -\cosh r$$

Let $X,Y \in \mathfrak{X}(M)$, then

$$X \langle x,p \rangle = \langle D_X x,p \rangle = \langle X,p \rangle = 0 \qquad (6.4)$$

Using equation (6.3) we obtain

$$D_X Y = \nabla_X Y + h(X,Y)\xi \; - <X,Y> \; x \tag{6.5}$$

where ∇ is the induced covariant differentiation operator on M and ξ is the field of unit normal vectors to M as a hypersurface of H.

From equations (6.4 and (6.5) we have

$$h(X,Y) = - \frac{\cosh r}{<n,p>} <X,Y> \tag{6.6}$$

Since $h(X,Y) = <-A_\xi X, Y>$ we have

$$A_\xi = \frac{\cosh r}{<\xi,p>} I \tag{6.7}$$

where I is the identity map. Differentiating (6.1) with respect to we get

$$<\xi,p> = -\sinh r \tag{6.8}$$

From equations (6.7) - (6.8) we obtain

$$A_\xi = -\coth r . \; I \tag{6.9}$$

which shows that M is umbilical.

Conversely, let M be a compact umbilical hypersurface of H, then $A = \lambda I$ where λ is a differentiable function on M. Using Codazzi equation (2.14), direct computations show that λ is constant. Under the above notations we have

$$D_X (\lambda x + \xi) = o \tag{6.10}$$

Hence $\lambda x + \xi$ is a constant vector in $(\mathbb{R}^{n+1}, <,>)$, i.e.

$$\lambda x + \xi = a \tag{6.11}$$

for some constant vector a. Now, equation (6.11) shows that

$$<x,a> = (1 + \lambda^2 - \|a\|^2)/2\lambda = \text{constant}$$

which represents a geodesic sphere in H.

Corollary (6.1)

Applying Gauss equation and using equation (6.7) we get that $S(p,r)$ has positive sectional curvature $1/\sinh^2 r$.

Corollary (6.2)

In hyperbolic space, horospheres are complete flat umbilical hypersurfaces. Consequently, such horospheres are free of conjugate points.

It has been proved that in symmetric spaces of rank 1 and negative curvature, horospheres have curvatures of both signs and even conjugate points (see [17]).

We close off this section by the following [2]:

Theorem (6.1)

Let M and N be C^∞ complete Riemannian n-manifolds and have the same constant sectional curvature k. Then

(1) M and N are locally isometric.

(2) If M and N are connected and simply connected with $k \leqslant o$ then they are isometric.

(3) If $k = a^2 > o$, then the geodesic sphere of radius π/a in M_m, $m \in M$, is mapped to a point by $\exp_m$, and $\exp_m$ is regular within that sphere. If M is simply connected, it is isometric to an n-sphere of radius $1/a$.

Section 7 : On bundles :

In this section we show that the tangent bundle $T(\tilde{M})$ of a complete, Riemannian manifold $(\tilde{M},g)$ and the normal bundle $T(M)^\perp$ of a submanifold M of $(\tilde{M},g)$ are Riemannian manifolds.

It is known (§ 1) that the tangent bundle $T(\tilde{M})$ decomposes naturally under the Riemannian connexion Γ of $(\tilde{M},g)$ into the direct sum $H \oplus V$ of a horizontal subbundle H and a vertical subbundle V. The horizontal subbundle H is the kernel of the so called connexion map $K : T(T\tilde{M}) \to T(\tilde{M})$ which is defined as follows : If $\omega \in T(T\tilde{M})$ is the initial tangent vector to a curve $X(t) \in T(\tilde{M})$ and $\pi_* \omega \neq o$, then $K(\omega) = \nabla_{y'(o)} X(o)$ where $y(t) = \pi(X(t))$.

In view of the mappings π_* and K we can identify the horizontal and vertical subspaces H and V, respectively, with $T_{\pi(x)}(\tilde{M})$. In this way, the Riemannian metric g on $\tilde{M}$ gives rise, via π and K, to a Riemannian metric $\langle\langle \cdot,\cdot \rangle\rangle$ on $T(\tilde{M})$ known as Sasaki metric and is given by

$$\langle\langle v,\omega \rangle\rangle = g(\pi_* v , \pi_* \omega) + g(K(v), K(\omega))$$

for $v,\omega \in T_X(T\tilde{M})$. In a similar way, the unit sphere bundle $S\tilde{M}$ can be shown to be a Riemannian manifold.

For the submanifold M, the normal bundle $T(M)^{\perp}$ is a C^{∞} submanifold of $T(\tilde{M})$. This shows that the Sasaki metric induces, in a natural way, a Riemannian metric on $T(M)^{\perp}$ and hence $T(M)^{\perp}$ is a Riemannian manifold. Same thing is true for the unit normal bundle $S(M)^{\perp}$ of M.

CHAPTER I

THE RIGIDITY PROBLEM

Section 1 : Finite Rigidity

(I.1.0) - Introduction

For this section all manifolds are assumed to be connected. All manifolds and mappings are assumed sufficiently differentiable for all computations to make sense.

When a Riemannian manifold M occurs as a submanifold of another Riemannian manifold $\tilde{M}$, rigidity question naturally arises. The term "finite rigidity" is generally used to refer to the following concept: M is finitely rigid (or simply rigid) as an immersed submanifold of $\tilde{M}$ if whenever $r_0, r_1 : M \to \tilde{M}$ are isometric immersions, there exists an isometry ϕ of $\tilde{M}$ such that $r_1 = \phi o r_0$. Generally speaking, a rigidity theory enumerates the different ways in which M can be isometrically immersed in $\tilde{M}$.

In a paper by M.P. DoCarmo and F.W. Warner [13], utilizing the rigidity studies in Euclidean space carried out by R. Sacksteder [26, 27], the following has been proved:

Theorem (I.1.1)

Let $x : M \to S^{n+1}$ be an isometric immersion of a compact, connected orientable, C^∞ Riemannian n-manifold M into the (n + 1) - sphere S^{n+1} of constant sectional curvature 1 and assume that all sectional curvatures K of M satisfy $K \geqslant 1$.

(a) Then x is an imbedding, M is diffeomorphic with S^n, and x(M) is either totally geodesic or contained in an open hemisphere, in the latter case x(M) is the boundary of a convex body(*) in S^{n+1}.

(*) A set B in a Riemannian manifold $(\tilde{M},g)$ is called convex body if for every pair of points p, $q \in B$, there exists a unique minimal geodesic segment from p to q and this segment is in B. A hypersurface M of $(\tilde{M},g)$ is called convex if it lies on one side of each tangent geodesic hypersurface (see § 1. chapter III).

(b) if $y : M \to S^{n+1}$ is another isometric immersion, then there is an isometry ϕ of S^{n+1} such that $\phi o x = y$.

The main aim of this section is to prove a similar theorem in hyperbolic space. In fact some conditions on the sectional curvatures of M are needed by the following theorem [1] :

Theorem (I.1.2)

Let M be a compact Riemannian manifold of dimension n. If the sectional curvatures of M are non-positive (not necessarily constant) then M can not be isometrically immersed in a hyperbolic space of dimension (n + 1). [For the proof see appendix (ii)].

Now, we state our theorem.

Theorem (I.1.3)

Let $x : M \to H$ be an isometric immersion of a compact, connected, orientable n-dimensional C^∞ Riemannian manifold M into the (n + 1) - dimensional hyperbolic space H of constant sectional curvature -1, and assume that all sectional curvatures K of M satisfy $K \geqslant -1$

(a) Then x is an imbedding, M is diffeomorphic to S^n, and x (M) is the boundary of a convex body in H.

(b) If $y : M \to H$ is another isometric immersion, then there is an isometry ϕ of H such that $\phi o x = y$.

Remarks :

(i) If the sectional curvatures of M are strictly greater than - 1, then the case (b) for n > 2 follows trivially from the classical rigidity theorem [2,p.211].

(ii) Assuming the truth of (a) in the above theorem, assertion (b) for n = 2 follows depending on a theorem which has been proved by

Cohn-Vossen [29] and stated as follows :

Theorem (I.1.4)

If $M \subset E^3$ is a compact convex surface, then M is rigid.

(I.1.1) The Beltrami maps :

The proof of theorem (I.1.3) will require extensive use of the Beltrami maps in transforming problems on the hyperbolic space H to problems in a Euclidean space. We devote this part to defining these maps and deriving their relevant properties.

The Beltrami map (or central projection) $\beta : H \to E^{n+1}$ is defined to be the map which takes $x \in H$ to the intersection of $E^{n+1} = \{(1, a^1, \ldots, a^{n+1}) \subset (\mathbb{R}^{n+2}, <, >)\}$, where H here is considered to be the model described in §6 - chapter 0, with the straight line through x and the origin 0 of $(\mathbb{R}^{n+2}, <, >)$. In this case, $\beta(H)$ is the open $(n + 1)$-ball $B(1)$ of radius 1 in the above E^{n+1} and centered at $(1,o,\ldots,o)$.

In the usual coordinates $(x^0,\ldots,x^{n+1})$ in $(\mathbb{R}^{n+2}, <, >)$, the map β can be expressed as follows :

$$\beta(x) = -x / < x, e_0 > = x/x^0 = (1, x^1/x^0, \ldots, x^{n+1}/x^0)$$

where $e_0 = (1,o,\ldots,o)$. The map β is a geodesic map and we shall use it to transfer hypersurfaces of H with sectional curvatures $K \geq -1$ into hypersurfaces of E^{n+1} with sectional curvatures $K \geq o$, and vice versa. To see that β indeed does have this effect we first give two lemmas (lemmas (I.1.1) & (I.1.2) below) for which we need the following [2] :

Theorem (I.1.5)

Let $\phi : M \to \tilde{M}$ be an immersion of a manifold M into a Riemannian manifold $\tilde{M}$. For the point $p \in M$, let V be the normal coordinate neighbourhood of $\tilde{M}$ around the point $q = \phi(p)$. Let $u = \Sigma a_i u^i$, where $u^i = v^i \circ \phi$ are the normal coordinates pulled back to M. Assume that $z = \Sigma a_i V_i(q) \varepsilon T(M)^{\perp}$ where $V_i(q) = (\partial | \partial v^i)(q)$ are an orthonormal basis of $T(M)^{\perp}|_q$. Then u has a critical point at p and its hessian form is the negative of the second fundamental form h^z.

Lemma (I.1.1)

Let m be a point on an oriented hypersurface in the Euclidean space E^{n+1}, and suppose that in a neighbourhood of m on the hypersurface the eigenvalues of the second fundamental forms do not have different signs. Then there is a neighbourhood of m on the hypersurface which lies on one side of the tangent hyperplane at m.

The proof of this lemma which can be found in [13] makes use of theorem (I.1.5) by taking u to be the height function of the hypersurface above its tangential hyperplane in E^{n+1}.

Lemma (I.1.2)

Let m be an arbitrary point on an oriented hypersurface in hyperbolic space and suppose that in a neighbourhood of m on the hypersurface the eigenvalues of the second fundamental forms do not have different signs. Then there is a neighbourhood of m on the hypersurface which lies on one side of the tangent totally geodesic hypersurface at m.

Proof :

The proof is carried out through contradiction. Assume that the hypersurface cuts its tangent totally geodesic hypersurface Π at m. Consider U to be a neighbourhood of m on the hypersurface on which the second fundamental forms are everywhere, say, negative semi-definite. This choice is possible through considering a convenient orientation if necessary. By theorem (I.1.5), the hessian forms of the height functions are all positive semi-definite.

Without loss of generality, we can consider the H-model to be the ambient hyperbolic space as it is more suitable than any other model. Now use the central projection β to transfer the hypersurface together with its orientation into the open unit ball B(1) in the Euclidean space $E^{n+1} = H_{e_0}$. Clearly, β(Π) will be a hyperplane in E^{n+1}. Since the hypersurface β(U) cuts its tangent hyperplane β(Π) at β(m), according to the preceding lemma there must be a point $\beta(p) \epsilon \beta(U)$ at which the hessian of the height function has a negative eigenvalue, and therefore in the direction of the corresponding eigenvector the hypersurface β(U) locally lies on the side of its tangent hyperplane at β(p) opposite from the oriented normal direction. Accordingly, the hypersurface β(U) and its tangent hyperplane at β(p) have contact of order exactly 1 in the corresponding eigen-direction. As contact order is always preserved under diffeomorphism, U and its tangent totally geodesic hypersurface at p have contact of order exactly 1 in the corresponding direction. Therefore in this direction the height function at p must have a non-zero second derivative, which is necessarily negative since in this direction U lies for a while on the side of its tangent totally geodesic hypersurface at p opposite from the oriented normal direction. Thus the hessian of the height function at p is not positive semi-definite, which is a contradiction.

The following proposition describes the effect of the central projection mapping β on sectional curvatures. In the following $K_X(P)$ will denote the sectional curvature of the Riemannian manifold X with respect to the 2-plane section P.

Proposition (I.1.1)

Let X be an n-dimensional hypersurface in the hyperbolic space (H-model) and let $\tilde{X} = \beta(X)$. Then $K_X \geqslant -1$ everywhere if and only if $K_{\tilde{X}} \geqslant o$ everywhere. Moreover, if $K_X \geqslant -1$, and if the rank of the second fundamental form of X at $p \in X$ is r, $o \leqslant r \leqslant n$, then the rank of the second fundamental form of $\tilde{X}$ at $\beta(p)$ is also r.

Proof:

Let $p \in X$ and let $K_X \geqslant -1$. Let P be a 2-plane in X_p. Applying Gauss' equation ((2.8) chapter 0) we have

$$K_X(P) = -1 + h_X(u,u)\, h_X(v,v) - (h_X(u,v))^2$$

where h_X denotes the second fundamental form of X and $\{u,v\}$ is an orthornomal basis of P.

Now suppose that not all eigenvalues of the second fundamental forms are zero. Since $K_X \geqslant -1$, then all the non-zero eigenvalues of h_X will have the same sign, and nearby p all the non-zero eigenvalues of h_X will have the same fixed sign. Using lemma (I.1.2) we see that X locally lies on one side of the tangent totally geodesic hypersurface at p. Hence $\beta(X)$ locally lies on one side of its tangent hyperplane at $\beta(p)$. Applying Gauss' equation again in Euclidean space we get that all sectional curvatures $K_{\tilde{X}}$ at $\beta(p)$ are $\geqslant o$.

If all eigen-values of h_X are identically zero on an entire neighbourhood of p, then X is totally geodesic near p, so is $\tilde{X}$ near $\beta(p)$. If each neighbourhood of p contains points at which there are non-zero eigenvalues of h_X, then there is a sequence of points $\{p_i\}$ in X converging to p, for which we already know that all $K_{\tilde{X}} \geqslant o$ at $\beta(p_i)$. Hence by continuity, all $K_{\tilde{X}} \geqslant o$ at $\beta(p)$.

A similar argument can be stated for the reverse direction, namely, if $K_{\tilde{X}} \geqslant o$ then $K_X \geqslant -1$.

Now assume that the rank of h_X at p is r. Equivalently, the hessian of the height function at p has rank r. If in addition $K_X \geqslant -1$, then there is an r-dimensional subspace A of X_p, on which the hessian is either positive or negative definite. It follows that in each direction in A the hypersurface X has contact of order exactly 1 with the tangent totally geodesic hypersurface through p. Since contact is preserved by the diffeomorphism β, there is an r-dimensional subspace of $\tilde{X}_{\beta(p)}$ along which $\tilde{X}$ has contact of order exactly 1 with the tangent hyperplane through $\beta(p)$. Consequently, the rank of the hessian of the height function for $\tilde{X}$ at $\beta(p)$ must be at least r, so that rank $h_{\tilde{X}}(\beta(p)) \geqslant$ rank $h_X(p)$. Reversing the argument we obtain that rank $h_{\tilde{X}}(\beta(p) \leqslant$ rank $h_X(p)$. Hence rank $h_{\tilde{X}}(\beta(p)) =$ rank $h_X(p)$.

In particular, proposition (I.1.1) gives that a point $p \in X$ has all $K_X > -1$ if and only if all $K_{\tilde{X}} > o$ for $\beta(p)$.

(I.1.2) Proof of theorem (I.1.3)(a) :

It is known that for a compact, n-hypersurface M in E^{n+1}, $n > 1$, there exists at least one point $p \in M$ at which all sectional curvatures $K_M > o$. Using this fact together with the last proposition, it is easy to prove the following:

Proposition (I.1.2)

Let M be a compact, Riemannian n-manifold with sectional curvature $K_M \geqslant -1$. Let $x : M \to H$ be an isometric immersion into the (n + 1) - hyperbolic space H. Then there exists at least one point $p \in M$ at which all sectional curvatures $K_M > -1$. (A stronger result can be obtained by theorem (I.1.2)).

For being important, we mention the following theorem without proof and for more details see [25].

Theorem (I.1.6)

Let M be a complete, Riemannian n-manifold ($n \geqslant 2$) and let $x : M \to E^{n+1}$ is a C^{n+1} isometric immersion. Suppose that every sectional curvature of M is non-negative, and at least one is positive. Then the image x(M) is the boundary of a convex body in E^{n+1}.

Now assume that M - as a Riemannian manifold - has g_M as its own Riemannian structure under which $x : M \to H$ is an isometric immersion. The manifold M will have another Riemannian structure, $\bar{g}_M$ say, as an induced one from the mapping $\bar{x} = \beta \circ x$ such that

$$\bar{g}_M = \bar{x}^* g_E = (\beta \circ x)^* g_E$$

where g_E is the usual Euclidean metric on E^{n+1}. Accordingly, $\bar{g}_M$ makes $\bar{x} : M \to E^{n+1}$ to be an isometric immersion. As mentioned before $\bar{x}(M)$ has $K \geqslant 0$, so under $\bar{g}_M$, M has $K \geqslant 0$, call $(M,\bar{g}_M) = \bar{M}$. Applying theorem (I.1.6) we get that $\bar{x}(M)$ is a boundary of a convex body in E^{n+1}. Applying β^{-1} we immediately obtain the conclusions of part (a) of theorem (I.1.3).

(I.1.3) Proof of theorem (I.1.3)(b) :

The following material is needed to complete the proof of the theorem. Let $f_1, f_2 : M \to H$ be two maps of a Riemannian manifold M into the hyperbolic space (H-model). Define $\tilde{f}_1 : M \to E^{n+1}$ by

$$\tilde{f}_1 = - \frac{f_1 + \langle f_1, e_0 \rangle e_0}{\langle f_1 + f_2, e_0 \rangle} \tag{1.1}$$

and define $\tilde{f}_2$ similarly. (This formula of $\tilde{f}_1$ makes sense as $f_1 + f_2$ is never perpendicular to e_0).

Proposition (I.1.3)

The two maps $f_1, f_2 : M \to H$ induce the same metric on M if and only if the two maps $\tilde{f}_1, \tilde{f}_2 : M \to E^{n+1}$ induce the same metric on M.

Proof :

From the formula (1.1) above, we have

$$-\langle f_1 + f_2, e_0 \rangle^2 \tilde{f}_{1*} = \langle f_1 + f_2, e_0 \rangle [f_{1*} + \langle f_{1*}, e_0 \rangle e_0] - \langle f_{1*} + f_{2*}, e_0 \rangle \cdot [f_1 + \langle f_1, e_0 \rangle e_0] \tag{1.2}$$

Taking into account that $<f_1,f_1> = <f_2,f_2> = -1$ and $<f_{1*},f_1> = <f_{2*},f_2> = o$, we have by direct computations that :

$$< f_1 + f_2 ,e_0>^2 [<\tilde{f}_{2*},\tilde{f}_{2*}> - <\tilde{f}_{1*},\tilde{f}_{1*}>] = <f_{2*},f_{2*}> - <f_{1*},f_{1*}> \quad (1.3)$$

which completes the proof.

Let γ be an arc-length parametrized curve in H. Then the Frenet equations for H give [29]

$$\gamma''(s) = k(s)\, n(s) + \gamma(s) \quad (1.4)$$

where n(s) is the principal unit normal vector and k(s) is the curvature of γ. Hence

$$\gamma(s + h) = (1 + \frac{h^2}{2})\gamma(s) + ht(s) + \frac{kh^2}{2} n(s) + O(h^3) \quad (1.5)$$

where $t(s) = \gamma'(s)$ and h is a small real number.

In the following we say that a hypersurface M in H is star-shaped with respect to some point $p \in H$ if each geodesic ray starting from p intersects M exactly once.

Proposition (I.1.4)

Let M be an oriented n-manifold, and let $f_1,f_2 : M \to H$ be two imbeddings such that $f_1(M)$ and $f_2(M)$ are convex and star-shaped with respect to $e_0 \varepsilon$ H, and such that f_1 and f_2 induce the same metric on M and the natural orientations on $f_1(M)$ and $f_2(M)$. Suppose moreover that the second fundamental forms of $f_1(M)$ and $f_2(M)$ are positive semi-definite. Then the same is true for the second fundamental forms of $\tilde{f}_1(M)$ and $\tilde{f}_2(M)$ in E^{n+1}.

Proof :

Let c be a curve parametrized by arc-length s in M (with the metric induced by f_1 and f_2). Applying formula (1.4) to the arc-length parametrized curve $\gamma_i = f_i \circ c$ we have

$$f_i(c(s+h)) = (1+\frac{h^2}{2})\, x_i + ht_i + \frac{h^2}{2} k_i n_i + O(h^3) \qquad (1.6)$$

where $x_i = f_i(c(s))$ and t_i, k_i, n_i are just t, k, n associated with the curve γ_i, i = 1,2. We also have

$$-<\tilde{f}_1(c(s+h)) - \tilde{f}_1(c(s)), \tilde{n}_1> = \frac{k_1 h^2}{2<x_1+x_2,e_0>} <-<n_1,e_0>\tilde{f}_1(c(s)) +$$

$$+ n_1, \tilde{n}_1> - \frac{k_2 h^2}{2<x_1+x_2, e_0>} <n_2,e_0><\tilde{f}_1(c(s)), \tilde{n}_1> + O(h^3) \qquad (1.7)$$

where $\tilde{n}_1$ is the unit normal of $\tilde{f}_1(M)$ at $\tilde{f}_1(c(s))$. Using the definition of the second derivative α'' of a function α given by

$$\alpha''(x) = \lim_{h\to o} \frac{\alpha(x+h) + \alpha(x-h) - 2\alpha(x)}{h^2}$$

we get

$$<(\tilde{f}_1 \circ c)''(s), \tilde{n}_1> = \frac{-k_1}{<x_1+x_2,e_0>} <-<n_1,e_0>\tilde{f}_1(c(s)) + n_1, \tilde{n}_1> +$$

$$+ \frac{k_2}{<x_1+x_2,e_0>} <n_2, e_0><\tilde{f}_1(c(s)), \tilde{n}_1> \qquad (1.8)$$

The term on the left hand is the second fundamental form of $\tilde{f}_1(M)$ applied to $((\tilde{f}_1 \circ c)'(s), (\tilde{f}_1 \circ c)'(s))$. So it suffices to show that it is always $\geqslant$ o. Since

$$\frac{-k_1}{<x_1+x_2,e_0>} \geqslant o \qquad \text{and} \qquad \frac{k_2}{<x_1+x_2,e_0>} \geqslant o$$

it suffices to prove that

$$<-<n_1,e_0>\tilde{f}_1(c(s)) + n_1, \tilde{n}_1> > o \quad \text{and} \quad -<n_2, e_0><\tilde{f}_1(c(s)), \tilde{n}_1> > o$$

The following lemma completes the proof

Lemma (I.1.3)

Let P and Q be the tangent hyperplanes of $f_1(M)$ and $f_2(M)$ at the point $a_0 = f_1(p)$ and $b_0 = f_2(p)$, and let $c_0 = \tilde{f}_1(p)$. Let a_{n+1}

and b_{n+1} be the unit normals to P and Q at a_0 and b_0 and let c_{n+1} be the unit normal to the tangent hyperplane of $\tilde{f}_1(M)$ at c_0. Then

$A = \langle -\langle a_0, e_0\rangle c_0 + a_{n+1}, c_{n+1}\rangle > o$ and $B = -\langle b_{n+1}, e_0\rangle\langle c_0, c_{n+1}\rangle > o$

Proof:

Choose positively oriented orthonormal vectors $a_1,\ldots,a_n$ at the point a_0 in P, let $b_1,\ldots,b_n$ be the corresponding vectors at b_0 in Q, and let $c_1,\ldots,c_n$ be the corresponding vectors at c_0 in the tangent hyperplane of $\tilde{f}_1(M)$ at c_0. Then for some $c > o$

$$a_{n+1} = a_0 x \ldots x\, a_n, \quad b_{n+1} = b_0\, x \ldots x b_n, \quad c_{n+1} = c e_0 x c_1 x \ldots x c_n$$

Applying formula (1.2) to the tangent vectors X_i in M_p such that $f_{1*}(X_i) = a_i$, we have

$$c_i = \frac{-1}{\lambda_0^2}(-\lambda_0 a_i + \lambda_i a_0) = (\ldots)\, e_0 \qquad i = 1, 2, \ldots, n$$

where

$$\lambda_i = -\langle a_i + b_i, e_0\rangle, \lambda_0 > o, i = o, 1, \ldots, n$$

Notice also that

$$c_0 = \frac{a_0}{\lambda_0} + (\ldots.)e_0$$

Hence

$$c_{n+1} = \frac{c}{\lambda_0^n}\{\lambda_0(e_0 \times a_1 x \ldots x\, a_n) - \lambda_1(e_0 x\, a_0 \times a_2\, x \ldots x\, a_n) + \ldots\}$$

$$\langle e_0, b_{n+1}\rangle = \det\begin{bmatrix} e_0 \\ b_0 \\ \vdots \\ b_n \end{bmatrix} \quad (1.9\text{-a}), \quad \langle c_0, c_{n+1}\rangle = -\frac{c}{\lambda^{n+1}}\det\begin{bmatrix} e_0 \\ a_0 \\ \vdots \\ a_n \end{bmatrix} \quad (1.9\text{-b})$$

Since f_1 and f_2 induce the natural orientation on $f_1(M)$ and $f_2(M)$, the determinants in equations (.19) are both positive, hence

$$-\langle b_{n+1}, e_0\rangle\langle c_0, c_{n+1}\rangle = \frac{c}{\lambda_0^{n+1}} > o$$

Similar computations on B show that

$$B \geqslant \frac{c}{2\lambda_0^{n+1}}\left\{ \left\{\det \begin{bmatrix} e_0 \\ b_0 \\ \vdots \\ b_n \end{bmatrix}\right\}^2 + \left\{\det \begin{bmatrix} e_0 \\ a_n \\ \vdots \\ a_n \end{bmatrix}\right\}^2 \right\} > 0$$

and the proof now is complete.

Now, we return back to complete the proof of theorem (I.1.3)(b). We assume that M is a compact, connected, orientable, C^∞ Riemannian n-manifold with $K_M \geqslant -1$, and that f_1 and f_2 are two isometric immersions of M into the (n+1) - dimensional model H. From theorem (I.1.3)(a) we have that both f_1 and f_2 are imbeddings and both $f_1(M)$ and $f_2(M)$ are boundaries of convex bodies. Without loss of generality we can assume that both $f_1(M)$ and $f_2(M)$ are star-shaped with respect to the point $e_0 \varepsilon$ H. From the last proposition it is clear that $\tilde{f}_1(M)$ and $\tilde{f}_2(M)$ are locally convex hypersurfaces in E^{n+1} with sectional curvatures greater than or equal to zero. Since $\tilde{f}_1(M)$ is a compact hypersurface of E^{n+1}, then there exists a point of $\tilde{f}_1(M)$ at which all sectional curvatures are strictly positive. M.P.DoCarmo and E. Lima jointly proved the following [12].

<u>Theorem (I.2.7)</u>

Assume that all second quadratic forms of the immersion $x : M \to E^{n+N}$ of a compact, connected, orientable, n-dimensional Riemannian manifold M into the Euclidean space E^{n+N}, $N \geqslant 1$, to be semi-definite, and definite at one point $(p, v_0(p)) \varepsilon S(M)^\perp$ {this condition is relevant by the last paragraph} . Then x(M) belongs to a linear subvariety of E^{n+N} and $x : M \to E^{n+1}$ imbeds M as the boundary of a convex body, in particular M is homeomorphic to a sphere.

Applying this theorem to our case we conclude that $\tilde{f}_1(M)$ and $\tilde{f}_2(M)$ bound convex bodies in E^{n+1}. In addition, R. Sacksteder [27] proved the following :

Theorem (I.1.8)$^{(*)}$

Let M be a complete, Riemannian n-manifold and let $x, y : M \to E^{n+1}$ be two C^2 isometric imbeddings such that $x(M)$ and $y(M)$ bound convex bodies. Then if $r \geqslant 3$ is the maximum rank of the second fundamental forms, $x(M)$ and $y(M)$ are congruent. (i.e. there is an Euclidean motion (isometry) T such that $T(x(M)) = y(M)$.).

From all the above arguments, the maximum rank of the second fundamental forms is n and this happens at the points of $\tilde{f}_1(M)$ and $\tilde{f}_2(M)$ of positive sectional curvatures. Hence we conclude that there exists an isometry $\tilde{\alpha}$ of E^{n+1} such that $\tilde{\alpha} \circ \tilde{f}_1 = \tilde{f}_2$.

We define the mappings $\rho_1, \rho_2 : E^{n+1} \to H$ by

$$\rho_1(p) = \frac{2p + e_0\,(-1 + <\tilde{\alpha}(p), \tilde{\alpha}(p)> - < p,p >)}{\| \text{numerator} \|}$$

and

$$\rho_2(p) = \frac{2p + e_0\,(-1 + <\tilde{\alpha}^{-1}(p), \tilde{\alpha}^{-1}(p)> - < p,p >)}{\| \text{numerator} \|}$$

Direct computations show that ρ_1 and ρ_2 are C^∞ mappings, moreover, $\rho_1(\tilde{f}_1(p)) = f_1(p)$ and $\rho_2(\tilde{f}_2(p)) = f_2(p)$ for all $p \in M$.

Proposition (I.1.5)

The mappings ρ_1 and ρ_2 are both injective.

Proof :

Suppose that p, q are two points in E^{n+1} such that $p \neq q$ but

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(*) Theorem (I.1.8) applies to the case when dim M > 2 but if dim M = 2 we use theorem (I.1.4).

$\rho_1(p) = \rho_1(q)$. From the nature of ρ_1 and ρ_2 we have that $\vec{p}$ and $\vec{q}$ must be parallel vectors in E^{n+1}, so there exists a unit vector $v \in E^{n+1}$ such that $p = \lambda v$ and $q = \mu v$. Substituting these two expressions in the equality $\rho_1(p) = \rho_1(q)$ we have

$$[-1 + < \tilde{\alpha}(p), \tilde{\alpha}(p) > - \lambda^2]/\lambda = [-1 + < \tilde{\alpha}(q), \tilde{\alpha}(q) > - \mu^2]/\mu$$

From the nature of the group of isometries of E^{n+1} we can assume that $\tilde{\alpha}$ is a rotation $\tilde{\alpha}^*$ followed by a translation by the vector c. Hence the last equation becomes

$$\frac{-1+c^2}{\lambda} + 2 < \tilde{\alpha}^*(v), c > = \frac{-1+c^2}{\mu} + 2 < \tilde{\alpha}^*(v), c >$$

which shows that $\lambda = \mu$ and hence $p = q$ which is a contradiction leading to the fact that ρ_1 is an injective map and so is ρ_2.

It follows from the invariance of domain theorem that ρ_1 and ρ_2 are both open maps. Hence $\alpha = \rho_2 \circ \tilde{\alpha} \circ \rho_1^{-1}$ defines an injection on some connected open neighbourhood of $f_1(M)$ in H and $\alpha(f_1(p)) = f_2(p)$. It remains now to show that α extends to an isometry of H and for this it is sufficient to show that α preserves the Lorentz inner product $< , >$. In terms of ρ_1 and ρ_2 we show that

$$< \rho_1(p) - \rho_1(q), \rho_1(p) - \rho_1(q) > = < \rho_2(\tilde{\alpha}(p)) - \rho_2(\tilde{\alpha}(q)), \rho_2(\tilde{\alpha}(p)) - \rho_2(\tilde{\alpha}(q)) > \quad (1.10)$$

From relation ((6.1) chapter 0) it can be proved that the Lorentz distance $<p-q, p-q>$ between any pair of points p,q in H uniquely determines their H-distance (the length of the geodesic segment joining p and q in H) which will complete the proof.

Instead of proving relation (1.10) we show that

$$< \rho_1(p), \rho_1(q) > = < \rho_2(\tilde{\alpha}(p)), \rho_2(\tilde{\alpha}(q)) > \quad (1.11)$$

Let

$$a(p) = 2p + e_0(-1 + < \tilde{\alpha}(p), \tilde{\alpha}(p) > - < p,p >)$$

$$b(p) = 2\tilde{\alpha}(p) + e_0(-1 + < p,p > - < \tilde{\alpha}(p), \tilde{\alpha}(p) >)$$

Then

$$\rho_1(p) = a(p)/\sqrt{<a(p),\ a(p)>} \ \& \ \rho_2(p) = b(p)/\sqrt{<b(p),b(p)>}$$

Again to show that (1.11) holds, it suffices to prove that

$$<a(p),\ a(q)> \ = \ <b(p),\ b(q)> \tag{1.12}$$

Now

$$<a(p),a(q)> = 4<p,q> -(-1+<\tilde{\alpha}(p),\tilde{\alpha}(p)> - <p,p>)\cdot$$
$$\cdot(-1+<\tilde{\alpha}(q),\tilde{\alpha}(q)> - <q,q>) \tag{1.13-a}$$

$$<b(p),b(q)> = 4<\tilde{\alpha}(p),\tilde{\alpha}(q)> - (-1+<p,p> - <\tilde{\alpha}(p),\tilde{\alpha}(p)>)\cdot$$
$$\cdot(-1+<q,q> - <\tilde{\alpha}(q),\tilde{\alpha}(q)>) \tag{1.13-b}$$

Writing $\alpha = \tilde{\alpha}^* + c$ as before we get

$$\begin{aligned} <\tilde{\alpha}(p),\tilde{\alpha}(p)> &= <p,p> + 2<\tilde{\alpha}^*(p),c> + c^2 \\ <\tilde{\alpha}(q),\tilde{\alpha}(q)> &= <q,q> + 2<\tilde{\alpha}^*(q),c> + c^2 \\ <\tilde{\alpha}(q),\tilde{\alpha}(q)> &= <p,q> + <c,\tilde{\alpha}^*(p) + \tilde{\alpha}^*(q)> + c^2 \end{aligned} \tag{1.14}$$

Equations (1.14) together with (1.13) give the required result.

This completes the proof of theorem (I.1.3)

Section 2 : Infinitesimal and Continuous Rigidity

(I.2.0) Introduction

We have seen in the previous section that a Riemannian manifold M is rigid as a submanifold of another Riemannian manifold $\tilde{M}$ if whenever r_1 and r_2 are isometric immersions of M into $\tilde{M}$, there exists an isometry ϕ of $\tilde{M}$ such that $r_2 = \phi \circ r_1$. A second theory which is the subject of this section is called "infinitesimal rigidity".

As a prototype we have the classical Liebmann problem which can be stated as follows [29]:

A closed convex surface in Euclidean three-space is given. It is to be shown that the only small deformations of it which preserves the line element within terms of second order in the deformation parameter are small rigid motions.

In this section we try to extend the concept of infinitesimal rigidity to submanifolds of hyperbolic spaces using the original ideas formulated in Elliptic and Euclidean spaces by A.V. Pogorelov [25] and latterly by R.A. Goldstein and P.J. Ryan [15]. The contrast between (finite) rigidity and infinitesimal rigidity will be clarified through the present work. We conclude this section with mentioning some notes about the theory of continuous rigidity as a third theory of rigidity.

One of the aims of this section is to establish a one-to-one mutual correspondence between submanifolds in the Euclidean and hyperbolic spaces and their respective infinitesimal deformations. In this way, the questions regarding infinitesimal rigidity of submanifolds in hyperbolic spaces will reduce to those regarding infinitesimal rigidity of submanifolds in Euclideanspaces.

In this section, all manifolds and maps are assumed sufficiently differentiable for all computations to make sense. All manifolds are assumed connected.

The following notations will be used throughout. A submanifold $S = (M,r)$ of a Riemannian manifold $(\tilde{M},g)$ consists of a manifold M and an immersion r of M into $\tilde{M}$. The group of isometries of a manifold $\tilde{M}$ is denoted by $I(\tilde{M})$. Some familiarity with [15] is required for reading this section.

(I.2.1) Deformations of submanifolds

Let $S = (M,r)$ be a submanifold of a Riemannian manifold $(\tilde{M},g)$. Let $I = [-\delta, \delta]$ for some $\delta > o$. A map

$$\gamma : I \times M \to M$$

is said to be a deformation of S if $\gamma_0 = r$ and γ_t is an immersion for each $t \in I$. (We have written $\gamma_t(x)$ for $\gamma(t,x)$). Each immersion γ_t induces a Riemannian metric g_t on M. Each closed curve on M has a length $L(t)$ measured by the metric g_t.

Definition

Let γ be a deformation of S. We say that γ is an isometric deformation (ID) of S if $g_t = g_0$ for each $t \in I$. We say that γ is an infinitesimal isometric deformation (IID) of S if $g'(o) = o$.

Notice that when we write $g'(o)$, we regard g_t as a curve in the finite dimensional vector space of tensors of type $(o,2)$ at a point of M. It is easy to show that γ is an ID if and only if $L(t)$ is independent of t for each closed curve in M. Furthermore, γ is an IID if and only if $L'(o) = o$ for each such curve. Actually, the definition of the IID given above can be written in a clearer way as follows :

A deformation $\gamma : I \times M \to \tilde{M}$ is said to be an IID if and only if the relation

$$g_t(u,v) = g_0(u,v) + O(t^2)$$

is true for each $p \varepsilon M$ and each pair $u,v \varepsilon M_p$.

In [15], R.A. Goldstein and P.J. Ryan gave an example of infinitesimal isometric deformation in Euclidean space E^3. In what follows we give an example for the same kind of deformation in hyperbolic space.

Example (I.2.1)

Consider, for this example, the 3-dimensional hyperbolic space represented by the half-space model

$$IR^{3+} = \{ x \varepsilon IR^3 \ : \ x = (x^1, x^2, x^3),\ x^3 > o,\ g = (\sum_i dx^i \otimes dx^i)/(x^3)^2 \}$$

Let M be the hypersurface of IR^{3+} given by

$$M = \{ x \varepsilon IR^{3+} \ : \ x = (o, x^2, x^3) \text{ where } x^3 > o,\ -\infty < x^2 < \infty \}$$

and $r : M \to IR^{3+}$ to be the inclusion map in IR^{3+}. Consider the following deformation of $S = (M,r)$ defined for $t \varepsilon [-1,1]$

$$\gamma(t,x) = \gamma_t(x) = (t\Psi(x), x^2, x^3)$$

where $\Psi(x)$ is a smooth function with compact support on M. For simplicity we write $\gamma_t(M) = M_t$.

It is clear that under the above deformation γ, the basis of the tangent space $(M_t)_{\gamma_t(x)}$ are

$$\partial/\partial x^2 = (t\partial\Psi/\partial x^2, 1, o) \quad \text{and} \quad \partial/\partial x^3 = (t\partial\Psi/\partial x^3, o, 1)$$

It is easy to see that for $t = o$, $\partial/\partial x^2 = (o,1,o)$ and $\partial/\partial x^3 = (o,o,1)$. Consider $U = (U_1, U_2)$ and $V = (V_1, V_2)$ to be two tangent vectors to M_o at x. Then direct computations show that

$$g(\gamma_{t*}(U), \gamma_{t*}(V)) = g(U,V) + (t/x^3)^2 \{U_1 V_1 \Psi_2^2 + U_2 V_2 \Psi_3^2 + (U_1 V_2 + U_2 V_1)\Psi_2 \Psi_3\}$$

where $\Psi_i = \partial\Psi/\partial x^i$. Clearly, the last expression of g_t shows that the deformation γ is an IID.

The above example can be generalized for the (n+1) - dimensional half-space model $IR^{(n+1)+}$ and M to be the hyperplane defined by $x^1 = o$, say, Moreover, g_t will have the following form :

$$g(\gamma_{t*}(U), \gamma_{t*}(V)) = g(U,V) + (t/x^{n+1})^2 (U\Psi)(V\Psi)$$

(I.2.2) Vector fields associated with a deformation:

Let S = (M,r) be a submanifold of the Riemannian manifold $(\tilde{M},g)$, and let $\gamma : IxM \to \tilde{M}$ be a deformation of S. For each $x \varepsilon M$, let Z_x be the tangent vector to the curve $t \to \gamma(t,x)$ at t = o. Thus Z is a vector field along the immersion r (or simply along $\gamma_0(M)$) whose value at x is the initial velocity of the motion of x under the deformation γ. We call Z the deformation field of γ. It is, in fact, Z which determines the infinitesimal properties of γ.

The main theorem of this section, which is given below, has been proved in [15] through adapting the theorem of Nash [19] which deals with the imbedding problem of Riemannian manifolds. We give here another proof which does not need such a background material. Our proof is also much easier in computations.

Theorem (I.2.1)

A deformation γ is an IID if and only if for $X,Y \varepsilon \mathfrak{X}(M)$

$$g(\tilde{\nabla}_X Z,Y) + g(X,\tilde{\nabla}_Y Z) = o \qquad (2.2.1)$$

where Z is the deformation vector field of γ and $\tilde{\nabla}$ is the covariant differentiation operator of the Riemannian manifold $(\tilde{M},g)$.

Proof:

Under the same notations and writing, for simplicity, $\gamma_{t_*}(X) = X_t$ for $X \varepsilon \mathfrak{X}(M)$ we have

$$Z_t g(X_t,Y_t) - g([Z_t,X_t],Y_t) - g(X_t,[Z_t,Y_t]) = g(\tilde{\nabla}_{X_t} Z_t,Y_t) + g(X_t,\tilde{\nabla}_{Y_t} Z_t)$$

where Z_t is the velocity field of the curve $t \to \gamma(t,x)$.

In terms of the Lie derivative, the last relation may be written as

$$(L_{Z_t}\, g)(X_t,Y_t) = g(\tilde{\nabla}_{X_t}Z_t,Y_t) + g(X_t, \tilde{\nabla}_{Y_t}Z_t) \qquad (2.2.2)$$

Since Z is an IID vector field, then the original definition of the Lie derivative gives

$$(L_{Z_t}g)(X_t,Y_t)\mid_{t=o} = \lim_{t\to o} \frac{1}{t}\left[g(X_t,Y_t) - g(X_o,Y_o)\right] = \lim_{t\to o} \frac{O(t^2)}{t} = o \qquad (2.2.3)$$

Accordingly, equation (2.2.2) when computed at t = o using equation (2.2.3) we get

$$g(\tilde{\nabla}_X Z,Y) + g(X,\tilde{\nabla}_Y Z) = o$$

The converse can be proved if we assume that

$$g(\tilde{\nabla}_{X_t}Z_t,Y_t) + g(X_t,\tilde{\nabla}_{Y_t}Z_t)$$

is of order O(t). The integration gives that g_t is itself of order $O(t^2)$ and this completes the proof of the theorem.

In fact, the above theorem is a successful tool for dealing with the problems of infinitesimal isometric deformations as it will be clear through the following work.

Proposition (I.2.1-a)

Let M be a hypersurface of the Riemannian manifold $(\tilde{M},g)$ and let Z be any IID vector field along M which is everywhere normal to M. Then at every point $p \in M$ where $Z_p \neq o$, the second fundamental form h(p) of M at p vanishes.

Proof:

Let Z be as in the proposition, hence it satisfies equation (2.2.1)

$$< \tilde{\nabla}_X Z, Y > + < X, \tilde{\nabla}_Y Z > = o$$

where $X,Y \in \mathfrak{X}(M)$. Let $p \in M$ be a point of M at which $Z_p \neq o$. Using the Weingarten's formula (§ 2-chapter 0) we have

$$\tilde{\nabla}_X Z = - AX + \tilde{\nabla}_X^{\perp} Z$$

and substituting in the above equation we obtain

$$< AX, Y >_p = h(p)(X,Y) = o$$

Since X and Y are arbitrary vector fields in $\mathfrak{X}(M)$ we get the result that $h(p) = o$.

Corollary :

If $Z_p \neq o$ for all p in an open set $U \subset M$, then U lies in a totally geodesic hypersurface of $(\tilde{M},g)$. If Z does not vanish globally along M then M will be a totally geodesic hypersurface of $(\tilde{M},g)$.

Moreover, we can prove the following :

Proposition (I.2.1-b)

Every normal deformation of a totally geodesic hypersurface is an IID.

Proof :

Let M be a totally geodesic hypersurface of the Riemannian manifold $(\tilde{M}, <,>)$ and let γ be a normal deformation of M. Let Z be the associated vector field of γ. For arbitrary $X,Y \in \mathfrak{X}(M)$ we have

$$h(X,Y) = o$$

or equivalently

$$< AX,Y > + < X,AY > = o$$

Using Weingarten's formula together with this equation we get

$$< \tilde{\nabla}_X Z,Y > + < X, \tilde{\nabla}_Y Z > = o$$

which shows, by virtue of theorem (I.2.1), that Υ is an infinitesimal isometric deformation of M as a submanifold of $(\tilde{M}, < , >)$.

Concerning the infinitesimal isometric deformations of totally geodesic hypersurfaces we have the following

Proposition (I.2.2)

Let M be a totally geodesic hypersurface of the Riemannian manifold $(\tilde{M},g)$. Then a vector field Z along M is an IID vector field of M if and only if its tangential component is an IID vector field of M as well.

Proof:

Let M be a totally geodesic hypersurface of $(\tilde{M}, < , >)$ with unit normal field ν. Consider a vector field Z along M, then we can write

$$Z = T + \phi\nu$$

where T denotes the tangential component of Z and ϕ the length of its normal component. For $X \in M_p$ we have

$$\tilde{\nabla}_X Z = \tilde{\nabla}_X T + h(p)(X,T)\nu + X(\phi)\nu - \phi(p)\tilde{\nabla}_X \nu \tag{2.2.9}$$

Now suppose that Z is an IID vector field, then for $X, Y \in M_p$ we have

$$< \tilde{\nabla}_X Z, Y > + < X, \tilde{\nabla}_Y Z > = o \tag{2.2.10}$$

Substituting (2.2.9) in (2.2.10) we have

$$< \tilde{\nabla}_X T, Y > + < X, \tilde{\nabla}_Y T > = 2\phi(p)\ h(p)\ (X,Y) \tag{2.2.11}$$

Since M is a totally geodesic hypersurface then $h(p)(X,Y) = o$ for every point $p \in M$. Hence (2.2.11) becomes

$$< \tilde{\nabla}_X T, Y > + < X, \tilde{\nabla}_Y T > = o$$

which means that the tangential component T of Z is also an IID field of M in $(\tilde{M}, < , >)$. The converse is direct.

A.V. Pogorelov [25] proved that for two close isometric surfaces F_1, F_2, in 3-dimensional elliptic space, which are defined by $x = x_1(u,v)$ and $x = x_2(u,v)$, respectively, the surface F defined by $x = \rho(x_1 + x_2)$ has the vector field $\xi = \rho(x_1 - x_2)$ as an IID field (ρ is defined below). For hyperbolic space and more generally we have:

Proposition (I.2.3)

Let M be a Riemannian m-manifold and let $y_1, y_2 : M \to H$ be two isometric immersions of M into the n-dimensional hyperbolic space model H. Let y_1 and y_2 have the property that $y = \rho(y_1 + y_2) : M \to H$ is an immersion. Then for the submanifold $S = (M, \rho(y_1 + y_2))$, the vector field $Z = \rho(y_1 - y_2)$ is an infinitesimal isometric deformation field of S in H.

{ ρ = is a normalization factor making $\rho(y_1 + y_2) : M \to H$ to be an immersion into H, i.e. $\rho^2 (-2 + 2 < y_1, y_2 >) = -1$ }

Proof :

At first, we show that Z is a vector field along S and is tangent to H. This can be carried out by showing that $< y,Z > = 0$ everywhere along S. In fact

$$< y,Z > = \rho^2 < y_1 + y_2, y_1 - y_2 > = \rho^2 \{< y_1, y_1 > - < y_2, y_2 >\} \qquad (2.2.12)$$

Since $M_1 = (M, y_1)$ and $M_2 = (M, y_2)$ are submanifolds of H, then

$$< y_1, y_1 > = < y_2, y_2 > = -1 \qquad (2.2.13)$$

Equation (2.2.12) together with (2.2.13) show that $< y,Z > = 0$ everywhere along S.

To complete the proof consider two arbitrary vector fields $X, Y \in \mathfrak{X}(S)$. Let $X_i \in \mathfrak{X}(M_i)$, i = 1,2, be the natural projection of X on the appropriate submanifold M_i. Since X is tangental to S, then we can write

$$X = D_X y = X(\rho)(y_1 + y_2) + \rho (D_X y_1 + D_{X_2} y_2)$$
$$= X(\rho)(y_1 + y_2) + \rho(X_1 + X_2) \qquad (2.2.14)$$

A similar expression can be written for the vector field Y. Notice that D denotes the covariant differentation in the Minkowski space $(\mathbb{R}^{n+1}, <,>)$. We also have

$$D_X Z = X(\rho)\ (y_1 - y_2) + \rho(X_1 - X_2) \qquad (2.2.15)$$

Using equations (2.2.14 and 2.2.15) we obtain

$$< D_X Z, Y> + < X, D_Y Z > = 2\{< X_1, Y_1> - < X_2, Y_2> \}$$

Taking into account that y_1 is isometric to y_2, we get

$$< D_X Z, Y> + < X, D_Y Z> = o \qquad (2.2.16)$$

Since X and Z are tangential to H, then by formula ((6.3) - chapter 0) we have

$$\left.\begin{array}{l} D_X Z = \tilde{\nabla}_X Z + < X,Z > x \\ < X,x> = < Y,x> = 0 \end{array}\right\} \qquad (2.2.17)$$

From (2.2.16) and 2.2.17) we have that

$$< \tilde{\nabla}_X Z, Y> + < X, \tilde{\nabla}_Y Z> = o$$

which shows that Z is an IID vector field of S in H.

If ϕ is a curve in $I(\tilde{M})$ with $\phi(o) = \imath$, then the deformation $\gamma : IxM \to \tilde{M}$ defined by

$$\gamma(t,x) = \phi(t)\ r(x)$$

where $r = \gamma_0$, gives an isometric deformation of the submanifold $S = (M,r)$ in $\tilde{M}$ since

$$(\gamma_t)_* X = (\phi(t))_*\ (r_* X)$$

for each $X \in \mathfrak{X}(M)$.

Definition :

An IID $\gamma : I \times M \to \tilde{M}$, whose deformation vector field Z coincides with that of a deformation induced by a curve $\phi(t)$ in $I(\tilde{M})$, is said to be trivial. (and the vector field Z is called trivial as well).

Before being involved in any other details, we give the following notes concerning the trivial deformations of submanifolds in the Minkowski space $(\mathbb{R}^{n+2}, <\ ,\ >)$.

Definition :

An $n \times n$ matrix B is called S-skew-symmetric if $(SB)^* = -(SB)$ where $S = \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix}$.

It is known that for a submanifold M in a Euclidean space E^{n+2}, the deformation vector field Z, associated with some deformation of M, is trivial if and only if Z can be expressed in the form

$$Z_x = a\ r(x) + b$$

for all $x \in M$ where "a" is a skew-symmetric matrix and "b" is a constant vector in E^{n+2}. In the following proposition, a similar result has been proved.

Proposition (I.2.4)

An IID $\gamma : I \times M \to (\mathbb{R}^{n+2}, <\ ,\ >)$ is trivial if and only if, for some S-skew-symmetric matrix "a" and some constant vector "b", the deformation vector field Z can be written as

$$Z_x = a\ r(x) + b$$

for all $x \in M$.

Proof :

It is known that each curve $\phi(t)$ in the group of isometrics $I(\mathbb{R}^{n+2}, <\ ,\ >)$ of the Minkowski space has the following form

$$\phi(t)\ r(x) = \alpha(t)\ r(x) + \beta(t)$$

where $\alpha(t)$ is an element of the Lorentz group $O^1\ (n+2)$ [29] and $\beta(t)$ is a vector, and $\alpha(t)$ acts on $r(x)$ by matrix multiplication. We also have that $\alpha(o) = 1$ and $\beta(o) = o$.

The deformation vector field of the above deformation can be written in the following form

$$Z_X = \alpha'(o)\ r(x) + \beta'(o)$$

The proof of the necessity part of the proposition will be complete when showing that $\alpha'(o)$ is an S-skew-symmetric matrix.

Since $\alpha(t) \varepsilon O^1\ (n+2)$, then it satisfies the equation

$$\alpha^*(t).S = S.\alpha^{-1}(t)$$

Differentiating this relation with respect to t we have

$$\alpha^{*\prime}(t).S = -S.\ \alpha^{-1}(t).\alpha'(t).\ \alpha^{-1}(t)$$

Computing at $t = o$ and taking into account that $\alpha(o) = 1$, we obtain

$$\alpha^{*\prime}(o).S = (S.\alpha'(o))^* = -\ S.\alpha'(o)$$

Hence $\alpha'(o)$ is an S-skew-symmetric matrix.

Conversely, if a,b are given, put

$$\gamma\ (t,x) = \exp(ta).\ r(x) + tb$$

It is easy to check that exp(ta) is in $O^1(n+1)$, hence γ is an isometric deformation with deformation field

$$Z_X = ar(x) + b$$

and the proof now is complete.

<u>Proposition (I.2.5)</u>

Let Z be an IID vector field of an immersion $f : M \to H$ where M is an r-dimensional manifold. Define the deformation $\gamma : I \times M \to H$ by

$$\gamma(t,x) = \gamma_t(x) = \rho(f(x)+tZ_x)$$

Then in a neighbourhood of any point $x \in M$, the map γ_t is an immersion for sufficiently small t, and the induced metric $g_t = \gamma_t^* <\ ,\ >$ on M is related to the metric $f^*<\ ,\ > = g_o$

$$g_t(X,Y) = g_o(X,Y) + t^2 \{ <X,Y> Z^2 + < \underset{X}{D}Z.\underset{Y}{D}Z > \} + O(t^4)$$

In particular, the metric $\gamma_t^* <\ ,\ >$ and $\gamma_{-t}^* <\ ,\ >$ on M are the same.

Proof:

If X is a tangent vector on M, with $X = c'(o)$ for some curve c in M, then

$$\gamma_{t*}(X) = \frac{d}{ds}\Big|_{s=o} \gamma_t(c(s)) = \frac{d}{ds}\Big|_{s=o} \rho\{c(s) + tZ_{c(s)}\} =$$

$$\rho\{c'(o) + tD_{c'(o)} Z_{c(o)}\} + \frac{d\rho}{ds}\Big|_{s=o} \{c(s) + tZ_{c(s)}\}_{s=o} =$$

$$\rho\{X + t\underset{X}{D}Z\} + X(\rho)\{f(x) + tZ_x\}$$

In a similar way we can write the expression of $\gamma_{t*}(Y)$ for another tangent vector Y to M at x as follows

$$\gamma_{t*}(Y) = \rho\{Y + tD_y Z\} + Y(\rho)\{f(x) + tZ_x\}$$

For the map γ_t to be an immersion for small values of t is clear from the last formulas for γ_{t*}.

The aim now is to compute $g_t = \gamma_t^* <\ ,\ >$ which can be done as follows:

$$g_t(X,Y) = (\gamma_t^* <\ ,\ >)(X,Y) = \rho^2 g_o(X,Y) + \rho t\{<X,D_YZ> + <D_XZ,Y>\} +$$

$$+\rho^2 t^2 <D_XZ, D_YZ> + X(\rho)\, Y(\rho) < f(X) + tZ_X, f(x) + tZ_X> +$$

$$+\rho\, Y(\rho) < X + tD_XZ, f(x) + tZ_X> + \rho\, X(\rho) < Y + tD_YZ, f(x) + tZ_X> \quad (2.2.18)$$

Taking into account that Z is an IID vector field of M in H, we have

$$< X , D_Y Z > + < D_X Z, Y > = o \qquad (2.2.19)$$

Since

$$\rho^{-2} = 1 - t^2 \| Z_X \|^2 , < x, Z_x > = o$$

we have

$$\rho^{-3} X(\rho) = t^2 < Z, D_X Z > \qquad (2.2.20)$$

Substituting (2.2.19) and (2.2.20) into (2.2.18) and expressing ρ as a power series in t we obtain the required result.

Actually, the term $O(t^4)$ is an infinite series of the even powers of t and hence $(\gamma_t^* < , >)(X,Y)$ is an even function of t, hence

$$(\gamma_t^* < , >) (X,Y) = (\gamma_{-t}^* < , >)(X,Y)$$

and the proof is now complete.

It turns out that the map $\rho (f(x) + tZ_X) \to \rho (f(x) - tZ_X)$, which is an isometry by the last proposition, is some sort of reflection which can not be realized, in general, by a trivial motion. Equivalently, the map $\rho(f(x) + tZ_X) \to \rho(f(x) - tZ_X)$ is not a restriction of any Lorentzian motion.

Proposition (I.2.6)

The deformation given in example (I.2.1) is a non-trivial IID.

Proof:

The deformation field which is an IID field in example (I.3.1) may be written as

$$Z = (\Psi , o, \ldots, o)$$

We remark that $Z \neq o$ on the support $\mathcal{O}$ of ψ and hence $Z \equiv o$ on an open set in M.

Suppose that the deformation γ (in the example) is trivial contrary to the claim in the proposition, then there exists an isometric deformation $\bar{\gamma}_t = \phi(t)$, where $\phi(t)$ is a continuous curve in $I(\mathbb{R}^{(n+1)+})$, for which the deformation field $\bar{Z}$ associated with $\bar{\gamma}_t$ coincides with Z at t = o. However, any trivial $\bar{Z}$ (affine map) which is zero on an open set is identically zero. We now conclude that Z is non-trivial unles $\Psi = o$.

In fact the last proposition can be restated in a more general form as follows:

<u>Proposition (I.2.7)</u>

Any hypersurface in hyperbolic space, some open subset of which lies in a totally geodesic hypersurface admits a non-trivial infinitesimal isometric deformation.

In [15], R.A. Goldstein and P.J.Ryan proved that the standard sphere of radius R in E^{n+1} is infinitesimally rigid. They also proved that small spheres on $S^{n+1}(R)$ are infinitesimally rigid. In the next part, similar results have been proved in Minkowski space and in hyperbolic spaces.

<u>(I.2.3) Rigidity of the H-model</u>

We start by defining the concept of infinitesimal rigidity. Let E denote the restriction of the tangent bundle $T(\tilde{M})$ to M where M is an immersed submanifold $(r : M \to \tilde{M})$ of the (pseudo) Riemannian manifold $\tilde{M}$.

<u>Definition:</u>

A submanifold $S = (M,r)$ of $\tilde{M}$ is infinitesimally rigid (IR) if the only sections of E which satisfy (2.2.1) are trivial.

Theorem (I.2.2) :

The H-model of (n+1)-dimensional hyperbolic space in the Minkowski space $(\mathbb{R}^{n+2}, <,>)$ is infinitesimally rigid.

Proof:

Suppose that Z is an IID vector field of H in $(\mathbb{R}^{n+2}, <,>)$, then Z can be written as

$$Z_x = \tau_x + \frac{1}{2}\phi x \qquad (2.3.1)$$

for $x \in H$, τ is the tangential component to H and ϕ is a smooth function on H. Now, we have by using formula ((6.3) - chapter 0)

$$D_X Z = \tilde{\nabla}_X \tau + <X,\tau> x + \tfrac{1}{2}(X\phi)x + \tfrac{1}{2}\phi . X \qquad (2.3.2)$$

for $X \in \mathfrak{X}(H)$ and $\tilde{\nabla}$ the induced Riemannian connexion on H. Using theorem (I.2.1) together with equation (2.3.2) we have

$$<\tilde{\nabla}_X \tau, Y> + <X, \tilde{\nabla}_Y \tau> + \phi <X,Y> = 0$$

for $X,Y \in \mathfrak{X}(H)$. This last relation is equivalent to

$$(L_\tau < , >)(X,Y) + \phi . <X,Y> = 0$$

or simply

$$L_\tau < , > = -\phi < , > \qquad (2.3.3)$$

To complete the proof of the theorem we need the following materials. For more details see [30].

Definition

A vector field X on a Riemannian manifold (M,g) is conformal if it generates a one-parameter group $\{\phi_t\}$, $t \in \mathbb{R}$, of conformal transformations on (M,g).

The following proposition has been proved in [30].

Proposition (I.2.8)

Let X be a complete vector field on a Riemannian manifold (M,g). Then X is a conformal vector field on (M,g) if and only if there exists a real-valued function λ on M, called the characteristic function of X, such that

$$(L_X g)\,(X,Y) = 2\lambda\,(m)\;g(X,Y)$$

for each $m \in M$ and for each pair $X,Y \in \mathfrak{X}(M)$.

The following proposition gives some characterizations of the conformal vector fields.

Proposition (I.2.9)

Each conformal non-Killing vector field on H can be obtained from a non-trivial constant vector field c on $(\mathbb{R}^{n+2}, <,>)$ by an orthogonal projection. The converse is also true.

Proof:

As c is a constant vector field on $(\mathbb{R}^{n+2}, <,>)$, then $D_Y c = o$ for any $Y \in \mathfrak{X}(\mathbb{R}^{n+2}, <,>)$. For $X,Y \in \mathfrak{X}(H)$ we have already

$$D_Y x = Y \qquad \& \quad D_Y X = \tilde{\nabla}_Y X + <X,Y> x$$

for each $x \in H$. Let $\bar{c} = c + <c,x> x$ denote the orthogonal projection on H, hence $\bar{c} \in \mathfrak{X}(H)$ and for $Y \in \mathfrak{X}(H)$ we have by direct computations that

$$\tilde{\nabla}_Y \bar{c} = <c,x> Y$$

Using this relation we find that the Lie derivative of the induced metric on H satisfies

$$(L_{\bar{c}}\, g) = 2 <c,x> g \qquad (2.3.4)$$

which means (by proposition (I.2.8)) that $\bar{c}$ is a conformal vector field on H with characteristic function $2 <c,x>$. The converse is direct.

Now, as the Lie derivative L_X is linear in X, any conformal vector field τ on H can be written as a linear combination of a Killing vector field V and a non-Killing one, c say, i.e.

$$\tau = V + c + \langle c,x \rangle x \qquad (2.3.5)$$

where c is a constant vector field on $(\mathbb{R}^{n+2}, \langle , \rangle)$.

We return back to complete the proof of the theorem. If we write the tangential component τ of Z in the form (2.3.5) and taking $V = ax$ where a is an $(n+2) \times (n+2)$ S-skew-symmetric matrix, we have

$$\tau = ax + c + \langle c,x \rangle x \qquad (2.3.6)$$

Comparing (2.3.3) and (2.3.4) we get that $\phi = -2\langle c,x \rangle$, hence from equation (2.3.1) and (2.3.6) we have

$$Z_x = ax + c$$

which means that Z is a trivial vector field, and so the model H is infinitesimally rigid as a hypersurface in the pseudo-Riemannian manifold $(\mathbb{R}^{n+2}, \langle , \rangle)$.

(I.2.4) Rigidity of geodesic spheres in hyperbolic space

The main result of this part is to prove that :

Theorem (I.2.3) :

Geodesic spheres in hyperbolic space are infinitesimally rigid.

Proof :

The next two lemmas are helpful in carrying out the proof. We will not mention their proofs as they depend on direct and easy computations.

Lemma (I.2.1)

Let S(c,r) be a geodesic sphere in the model H with centre c.

and radius r. The unit normal vector field ξ of S(c,r) as a hypersurface of H is

$$\xi_x = (-c + x.\cosh r)/\sinh r \quad , x \in S(c,r)$$

Lemma (I.2.2)

The unit normal vector field N of S(c,r) as a hypersurface of the hyperplane $\langle x,c \rangle = -\cosh r$ is

$$N_x = (x - c\cosh r)/\sinh r \quad , x \in S(c,r)$$

It is known (§ 6 - chapter 0) that the second fundamental tensor A of S(c,r) as a submanifold of H is given by

$$A = \coth r.\, I$$

Now, consider Z to be an IID vector field of S(c,r) in H. This vector field Z can be written as

$$Z = \tau + \tfrac{1}{2}\phi\xi \qquad (2.4.1)$$

where τ is tangent to S(c,r) and ϕ is a smooth function on S(c,r). It is known from theorem (I.2.1) that

$$\langle \tilde{\nabla}_X Z , y\rangle + \langle X, \tilde{\nabla}_Y Z\rangle = o \qquad (2.4.2)$$

for $X,Y \in \mathfrak{X}(S(c,r))$. Substituting (2.4.1) in (2.4.2) we get

$$\langle \nabla_X \tau , Y\rangle + \langle X, \nabla_Y \tau\rangle + \phi.\coth r.\langle X,Y\rangle = o \qquad (2.4.3)$$

where ∇ is the induced covariant differentiation operator on S(c,r) and $X,\gamma \in \mathfrak{X}(S(c,r))$. Using the same notations as before, we have

$$(L_\tau\langle\ ,\ \rangle) = \phi\,.\coth r.\langle\ ,\ \rangle \qquad (2.4.4)$$

which shows that $\tau \in \mathfrak{X}(S(c,r))$ is a conformal vector field on S(c,r). Taking into account that S(c,r) is a Euclidean hypersphere in the hyperplane $\langle x,c\rangle = -\cosh r$, we can write

$$\tau = V + b - \langle b,N\rangle N \qquad (2.4.5)$$

where V is a Killing vector field on S(c,r) and b is a constant vector in $(\mathbb{R}^{n+2}, <,>)$ which satisfies $<b,c> = 0$.

Similar computations to that of [30] (p.85) show that

$$L_\tau <,>) = \frac{-2}{\sinh r} <b,N>. <,> \tag{2.4.6}$$

From (2.4.3) and (2.4.6) we have

$$\Phi = \frac{2}{\cosh r} <b,N>$$

and from (2.4.1) we obtain

$$Z = V + b + <b,N> \tanh r . c \tag{2.4.7}$$

But since $<b,N> = <x,b> / \sinh r$, we have

$$Z_x = V_x + b + \frac{<x,b>}{\cosh r} c \tag{2.4.8}$$

In a similar way to [30] we can show that there exist two S-skew-symmetric matrices a_0 and a_1 such that

$$Z_x = (a_0 + a_1) x \tag{2.4.9}$$

Choosing $V_x = a_0 x$ and writing x as a linear combination of c,b and some other vector v, we have

$$a_1 v = 0, \quad a_1 c = -b/\cosh r \quad \text{and} \quad a_1 b = (\|b\|^2/\cosh r)\, c$$

Since linear combination of two IID vector fields is again an IID one, this together with (2.4.9) complete the proof.

Although horosphere in hyperbolic space is a limit of sequence of geodesic spheres which are infinitesimally rigid, horosphere itself is not infinitesimally rigid. The following example indicates this fact.

Example (I.2.2)

Consider $\mathbb{R}^{3+}$ to be the 3-dimensional half-space model of hyperbolic 3-spaces and let H_{x^3} be the horosphere given by $x^3 = a$

where a is a positive real number. Consider the deformation γ of H_{x^3} in IR^{3+} which is defined by

$$\gamma_t(x) = (x^1, x^2, a + t^2 \psi) \quad , t \varepsilon [-\delta , \delta]$$

where Ψ is a smooth function with compact support $\mathcal{O}$ on H_{x^3}. The basis $\{ \partial / \partial x^1, \partial / \partial x^2 \}$ of the tangent space $\gamma_t(H_{x^3})_{\gamma_t(x)}$ are given

$$\partial/\partial x^1 = (1,o,t^2 \Psi_1) \quad \text{and} \quad \partial/\partial x^2 = (o,1,t^2 \Psi_2)$$

where $\Psi_i = \partial \Psi / \partial x^i \quad , \quad i = 1,2$

Let $U = (U_1,U_2)$ and $V = (V_1,V_2)$ be two tangent vectors to $\gamma_o(H_{x_3})$ at x, then

$$U_t = \gamma_{t*} (U) = U_1 \; \partial / \partial x^1 + U_2 \; \partial / \partial x^2$$

$$V_t = \gamma_{t*} (V) = V_1 \; \partial / \partial x^1 + V_2 \; \partial / \partial x^2$$

Now, direct computations show that

$$< U_t,V_t> = U_1 V_1 \frac{1+t^4 \Psi_1^2}{(a+t^2 \psi)^2} + U_2 V_2 \frac{1+t^4 \Psi_2^2}{(a+t^2 \psi)^2} + (U_1 V_2 + U_2 V_1) \frac{t^4 \Psi_1 \Psi_2}{(a+t^2 \psi)^2}$$

Expanding $(a+t^2 \Psi)^{-2}$ in a power series of t and substituting, we obtain

$$<U_t,V_t> = <U,V> + O(t^2)$$

which shows that the deformation γ given above is an IID of H_{x^3} in IR^{3+}. In a similar way to that of proposition (I.2.6) we can show that Υ is a non-trivial IID and the proof is now complete.

(I.2.5) Transformation of submanifolds and their infinitesimal isometric deformations :

In this part we establish a mutual correspondence between submanifolds and their infinitesimal isometric deformations in the hyperbolic and the Euclidean spaces. For the rest of this part let H denote - as before - the (n+1) - hyperbolic space model in (IR^{n+2} , < , >). Let, in addition, that D and $\tilde{\nabla}$ denote the covarient differentiation operators on (IR^{n+2} , < , >) and H, respectively.

Before giving the proof of the following two theorems, we state without proof the following proposition which is just a restriction of proposition (I.2.4) to the model H.

Proposition (I.2.10) :

A deformation of a submanifold S = (M,r) in H is trivial if and only if for some S-skew-symmetric matrix "a" , the associated deformation vector field Z is expressed as

$$Z_x = ar(x)$$

for all $x \in M$.

Now, we are in a stage to prove the following :

Theorem (I.2.4)

If ξ is an IID vector field of the submanifold S = (M,r) in H, then the vector field defined by

$$Z_y = \frac{\xi_x + <\xi_x, e_o> e_o}{<e_o, r(x)>}$$

is the field of an IID of the submanifold

$$\phi : y = \frac{r(x) + <r(x), e_o> e_o}{<r(x), e_o>}$$

in E^{n+1}. The field Z is trivial if and only if the field ξ is trivial.

Proof :

Consider a curve in M with velocity field X. Let the corresponding curve in ϕ have Y as its velocity field. From the relations given in the proposition, we have

$$D_Y Z = \frac{D_X\xi + < D_X\xi, e_0 > e_0}{< r(x), e_0 >} - \left[\frac{\xi + < \xi, e_0 > e_0}{< r(x), e_0 >^2} \right] < e_0, X >$$

$$Y = \frac{X + < X, e_0 > e_0}{< r(x), e_0 >} - \left[\frac{r(x) + < r(x), e_0 > e_0}{< r(x), e_0 >^2} \right] < e_0, X >$$

For Y_1 , $Y_2 \varepsilon \mathfrak{X}(\phi)$ and using the last two expressions we get, by direct computations taking into account that ξ is an IID field, that

$$< D_{Y_1} Z, Y_2 > + < D_{Y_2} Z, Y_1 > = 0$$

where D in this equation denotes the induced Riemannian connexion on E^{n+1} and hence Z is an IID field of ϕ in E^{n+1} .

For the second part of the theorem, let ξ be trivial, i.e. ξ has the form

$$\xi_X = ar(x)$$

for some S-skew-symmetric matrix a. Substituting this expression of ξ in the Z expression, we have

$$Z = \alpha . y(r(x)) + \beta$$

where β is some vector in E^{n+1} and α is a skew-symmetric matrix, hence Z is trivial. The converse can be proved in a similar way.

The previous theorem is quite useful for transferring IID problems from the hyperbolic space to the Euclidean one. The reverse way is given by the following theorem which has a similar proof to that of the previous one

Theorem (I.2.5)

Let Z be the field of IID of the submanifold S = (M,y) in E^{n+1}, then

$$\xi_x = \frac{Z_y + < Z_y, y > e_0}{\sqrt{1 - < y,y >}}$$

is the field of IID of the submanifold $\bar{S}$ = (M,x) defined by

$$x = \frac{y + e_o}{\sqrt{1 - \langle y,y \rangle}}$$

in H. The field ξ is trivial if and only if the field Z is trivial.

(I.2.6) Conclusions

We conclude this section by giving, firstly, some notes on the IID of submanifolds of a Riemannian manifold $(\tilde{M},g)$ which are, in particular, true in hyperbolic spaces.

Definition

A vector field $Z \varepsilon \mathfrak{X}(\tilde{M})$ is called an IID of $\tilde{M}$ if the one-parameter group $\{\phi_t\}$ generated by Z is an infinitesimal isometric transformation group.

Actually, the following proposition shows an important fact concerning this kind of fields just defined.

Proposition (I.2.11) :

Let $(\tilde{M},g)$ be a Riemannian manifold and let $Z \varepsilon \mathfrak{X}(\tilde{M})$ be an IID vector field on $(\tilde{M},g)$, then Z is an isometric deformation vector field (Killing vector field).

Proof :

Let $\{\phi_t\}$ be the one-parameter group of transformations generated by Z, then by definition of IID, we have

$$g_t = g_o + O(t^2)$$

Using the definition of the Lie derivative we get

$$L_Z g = \lim_{t \to o} (g_t - g_o)/t = o \qquad (2.6.1)$$

The following proposition has been proved in ([19] Vol.I p.237)

Proposition (I.2.11)

For a vector field Z on a Riemannian manifold $(\tilde{M},g)$, the following two conditions are mutually equivalent :

(1) Z is a Killing vector field (ID).

(2) $L_Z\, g = o$.

Using equation (2.6.1) together with proposition (I.2.11) we conclude that Z is a Killing vector field (ID field). Hence $\{\phi_t\}$ is a one-parameter group of isometries of $(\tilde{M},g)$.

Corollaries

Let M be a submanifold of $(\tilde{M},g)$ and let Z be an IID of M in $(\tilde{M},g)$

1. If Z can be extended to an IID field $\tilde{Z}$ on $(\tilde{M},g)$ then $\tilde{Z}$ will be an ID field of $(\tilde{M},g)$. (The ID field is defined below).

2. If Z is in $\mathfrak{X}(M)$, then Z is a Killing vector field on M.

Now to explain how to transfer the infinitesimal rigidity problems from the hyperbolic space (represented by the H-model) to the corresponding problem in the Euclidean space E^{n+1} , we mention only two examples and refer the reader to [25].

We start by mentioning some geometric properties of the maps $y : H \to E^{n+1}$ and $x : E^{n+1} \to H$ defined by

$$y = -(x + \langle x,e_o\rangle e_o)/\langle x, e_o\rangle , \quad x = (y + e_o)/\sqrt{1 - \langle y,y\rangle}$$

which have been mentioned in theorems (I.2.4) and (I.2.5), respectively. Clearly, the first map y can be written as

$$y = -(x/\langle x,e_o\rangle) - e_o$$

We notice that the first term in the right hand side is the central projection (Beltrami map) of H into E^{n+1} while the second term represents the parallel translation of H_{e_o} up to the origin 0 of $(\mathbb{R}^{n+2}, \langle , \rangle)$.

The second map x can be written similarly. Since the central projection is a geodesic mapping, then y takes convex bodies in H to convex bodies in E_0^{n+1} while x takes convex bodies in E_0^{n+1} to convex bodies in H (see § 1. chapter III). Under this understanding of the geometries of x and y, theorems (I.2.4) and (I.2.5) are good machines to carry over many a result related to IID of submanifolds in E^{n+1} to those in H.

Example (I.2.3)

Closed convex surface not containing any totally geodesic piece in the 3-dimensional hyperbolic space H is infinitesimally rigid.

Proof :

The proof of this fact depends on a similar one proved in [29] for Euclidean space E^3 which can be stated as followe : Let $M \subset E^3$ be any closed convex surface which does not contain a portion of a plane. Then M is infinitesimally rigid.

Now let $\bar{M} \subset H$ be as in example (I.2.3), then its image $\bar{\bar{M}} : y = -(x + \langle x, e_0 \rangle e_0)/\langle x, e_0 \rangle$, $x \in \bar{M}$ is also a closed convex surface not containing any planar piece in E^3. Let ξ be an IID field of $\bar{M}$ in H and let $Z = -(\xi + \langle \xi . e_0 \rangle e_0)/\langle x, e_0 \rangle$ be IID field of y in E^3. By the previous paragraph Z should be trivial and consequently by theorem (I.2.4) ξ is also trivial and the proof now is complete.

Any surface $F \subset E^3$ whose convex part lies wholly on its convex hull will be called a surface of type T. A similar definition can be stated for surfaces of type T in hyperbolic spaces. Under the above mentioned mappings x and y it is easy to prove that T-surfaces in hyperbolic space go to T-surfaces in Euclidean space and vice versa. It has been proved by Alexandrov [25] that analytic surfaces of type T in E^3 are infinitesimally rigid. Hence similar to example (I.2.3) above we can prove that:

Example (I.2.4)

Analytic surfaces of type T in hyperbolic space are infinitesimally rigid.

We now move to give some notes on the theory of continuous rigidity as a third theory of rigidity. First we recall the definition of isometric deformation (bending). In fact we distinguish between two kinds of isometric deformations as follows :

Consider a C^∞ imbedding $r : M \to \tilde{M}$ of a manifold M into a Riemannian manifold $(\tilde{M}, <,>)$. The isometric deformation of this imbedding is the C^∞ map $\gamma : [0,1] \times M \to \tilde{M}$ such that :

(a) each $\gamma_t : M \to \tilde{M}$ is an imbedding.

(b) $\gamma_0 = r$.

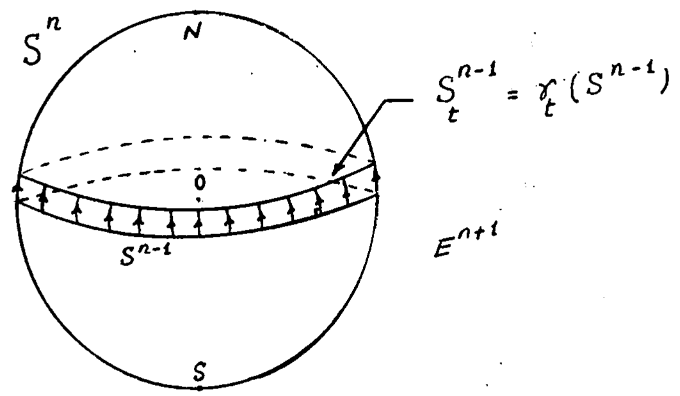
(c) $\gamma_t^* <,> = \gamma_0^* <,>$, for all $t \in [0,1]$.

The isometric deformation (ID) through immersion can be defined similarly. If the isometric deformation $\gamma : [0,1] \times M \to \tilde{M}$ is not purely imbedding or immersion we say that γ is an isometric deformation.

It is clear from the above definition that each isometric deformation is infinitesimally isometric at each $t \in [0,1]$ and the converse is also true (i.e. a deformation which is IID at each $t \in [0,1]$ is an ID). From this argument we see that the crucial difference between dealing with ID and IID is that for ID we should study the behaviour of the deformation for each value of t in its domain of definition while in IID case we study the behaviour of the deformation only at $t = 0$.

To clarify this point, consider the following :

We have proved in proposition (I.2.1) that for each totally geodesic hypersurface M in a Riemannian manifold $\tilde{M}$, any (non-vanishing) normal deformation is an IID one. This fact is no longer true for the isometric deformations according to the following example : Consider the Riemannian manifold $\tilde{M}$ to be the unit n-sphere S^n in E^{n+1} . Let M be the greatest sphere S^{n-1}, say. (see the following figure).



If we push S^{n-1} normally upstairs such that each point $x \in S^{n-1}$ moves along the normal geodesic joining x to the north pole with constant velocity. In this way we get a deformation $\gamma : [o,1] \times S^{n-1} \longrightarrow S^n$ which is IID but not ID.

For some value u of t, $u \in I = [o,1]$, let $Z_u(x)$ denote the tangent to the curve $t \to \gamma(t,x)$ for $x \in M$ at $t = u$. In this way we can define at each $t \in I$ a vector field Z_t which is tangential to $\tilde{M}$ along $\gamma_t(M) = M_t$ and we call it the deformation vector field associated with the deformation $\gamma : I \times M \to \tilde{M}$ at $t \in I$. Following a similar method of proof to that of theorem (I.2.1) it is easy to prove that :

Theorem (I.2.6)

The deformation $\gamma : I \times M \to \tilde{M}$ is an ID if and only if for each $t \in I$

$$< \tilde{\nabla}_{X_t} Z_t , Y_t > + < X_t , \tilde{\nabla}_{Y_t} Z_t > = o$$

for $X_t = \gamma_{t*}(X)$, $Y_t = \gamma_{t*}(Y)$, $X, Y \varepsilon \mathfrak{X}(M)$ and $\tilde{\nabla}$ denotes - as before - the covariant differentiation in $(\tilde{M}, <\ ,\ >)$.

Actually, with this understanding of the ID we can give the following definitions :

1. An isometric deformation $\gamma : I \times M \to \tilde{M}$ is called trivial if each γ_t can be written as $\phi(t) \circ r$ for some continuous curve $\phi(t) \subset I(\tilde{M})$ such that $\phi(o) = 1$. It is called non-trivial if at least one γ_t is not of this form.
2. The submanifold (M,r) is called continuously rigid in $\tilde{M}$ if every ID of (M,r) is trivial.

It can be shown that the isometric deformation $\gamma : I \times M \to H$ of the imbedding (immersion) $r : M \to H$ is trivial if and only if the variation vector field Z_t, at time t of γ is trivial at each $t \varepsilon I$ (The method of proof is similar to that of [29] in E^{n+1})

In what follows we give an example of continuously rigid submanifolds of hyperbolic space. For the next discussion let H denote the three-dimensional hyperbolic space model in the 4-dimensional Minkowski space $(\mathbb{R}^4\ ,\ <\ ,\ >)$.

We proved before that any closed convex surface in H not containing any totally geodesic piece is infinitesimally rigid. Using this result we can prove :

Example (I.2.4) :

Any closed convex surface in H not containing any totally geodesic piece $(K > -1)$ is continuously rigid (unbendable).

Proof :

Let M be a surface in H satisfying all the hypothesis in the example. Let $\gamma : I \times M \to H$ be an isometric deformation through imbedding of M in H. Then for each $t \in I$, $\gamma_t(M) = M_t$ is a closed convex surface of H with $K > -1$ and hence M_t is infinitesimally rigid. This shows that the ID vector field Z_t (which is an IID field) is trivial for each $t \in I$ which means that γ is trivial at each $t \in I$ and hence the result.

This example, in a natural way, gives rise to the question : Is every infinitesimally rigid submanifold continuously rigid? The answer is "yes" on condition that any isometric deformation through imbedding of this submanifold preserves all its geometric properties such as curvature, second fundamental forms,..., etc. To clarify this idea consider the following :

Let $\gamma : I \times S \to H$ be an isometric deformation through imbedding of the geodesic sphere $S = S(p,r)$ of center $p \in H$ and radius $r \in IR$ in the hyperbolic (n+1)-space H. Clearly $\gamma_t(S) = S_t$ is a geodesic sphere for each $t \in I$ and since geodesic spheres in H are infinitesimally rigid (by theorem (I.2.3)) then similar argument to that of example (I.2.4) shows that :

Theorem (I.2.7)

Geodesic spheres in hyperbolic spaces are continuously rigid.

Using theorem (I.2.2) we also can show that :

Theorem (I.2.8) :

The H-model in the Minkowski space (IR^{n+2}, < , >) is continuously rigid.

Depending on the results of R.A. Goldstein and P.J. Ryan [15] we have the following results :

(a) Euclidean sphere S^{n+1} in E^{n+2} is continuously rigid.

(b) Small geodesic spheres in S^{n+1} are continuously rigid.

We finish off this section with mentioning the following two facts :

(1) The set of all IID fields of a submanifold M in $(\tilde{M}, < , >)$ forms a vector space over $\mathbb{R}$. This is clear since equation (2.2.1) is linear in Z.

(2) For future work, the triviality problem of an infinitesimal isometric deformation can be discussed through the dimensional analysis on the vector spaces of IID fields and trivial fields.

CHAPTER II

SUBMANIFOLDS OF HYPERBOLIC SPACES

Section 1 : Isometric immersion with conditional second fundamental form

(II.1.0) - Introduction

J. Simons [28] made an important contribution to the study of minimal submanifolds immersed in a Reimannian manifold by using the derivation of the linear elliptic second order differential equation satisfied by the second fundamental form of each minimal submanifold. An application of this study has been carried out by S.S. Chern, M.P. DoCarmo and S. Kobayashi jointly [9], in the unit (n+p) - sphere S^{n+p} when the length of the second fundamental form of the immersed n-dimensional minimal submanifold M is $\{ n/(2-\frac{1}{p}) \}^{\frac{1}{2}}$.

S. Braidi and C.C. Hsiung [4] jointly extended the results obtained by S.S. Chern, M.P. DoCarmo and S. Kobayashi to compact oriented n-dimensional immersed submanifold M of S^{n+p} whose second fundamental form satisfies certain condition. This assumed condition reduces to the condition above concerning the length of the second fundamental form in case M is minimally immersed. One of the theorems proved in [4] can be stated as follows :

Theorem (II.1.1)

Let M be a compact oriented immersed hypersurface satisfying

$$\int_M [W_1 - (\mathrm{Tr}\, H_{n+1})\Delta(\mathrm{Tr}\, H_{n+1})]\; dv = 0$$

in an (n+1) - dimensional space N of constant sectional curvature 1. Then M is either an n-sphere or locally a Riemannian direct product $M \supset U = V_1 \times V_2$ of spaces V_1 and V_2 of constant sectional curvature, $\dim V_1 = m \geqslant 1$ and $\dim V_2 = n-m \geqslant 1$. In the latter case, with respect to an adapted frame field, the connexion form (ω^A_B)

of N, restricted to M, is given by

$$\left[\begin{array}{c|c|c} \begin{matrix} \omega^1_1 & \cdots & \omega^1_m \\ \vdots & & \vdots \\ \omega^m_1 & \cdots & \omega^m_m \end{matrix} & 0 & \begin{matrix} -\lambda\omega^1 \\ \vdots \\ -\lambda\omega^m \end{matrix} \\ \hline 0 & \begin{matrix} \omega^{m+1}_{m+1} & \cdots & \omega^{m+1}_n \\ \vdots & & \vdots \\ \omega^n_{m+1} & \cdots & \omega^n_n \end{matrix} & \begin{matrix} -\mu\omega^{m+1} \\ \vdots \\ -\mu\omega^n \end{matrix} \\ \hline \lambda\omega^1 \cdots \lambda\omega^m & \mu\omega^{m+1} \cdots \mu\omega^n & 0 \end{array}\right]$$

where Δ is the Laplacion operator, H_{n+1} denotes the symmetric matrix of the second fundamental forms and W_1 is given by

$$W_1 = (S-n)S + (\mathrm{Tr}\, H_{n+1})^2 - (\mathrm{Tr}\, H_{n+1})(\mathrm{Tr}\, H^3_{n+1}), \quad S = \sum_{ij}(h_{ij})^2$$

Although studying some cases of submanifolds in hyperbolic spaces shows consistency with similar cases in elliptic spaces, some others prove great deviations. In what follows we find the modified form of theorem (II.1.1) for hyperbolic spaces. Firstly, we demonstrate the necessary relations. For more details see [4].

(II.1.2) - Basic relations :

In this article we find the expression of the Laplacian for the second fundamental form of a submanifold immersed in a locally symmetric space.

Let M be an n-dimensional Riemannian manifold immersed in an (n+p)-dimensional Riemannian manifold N. Choose a local field of orthonormal frames $e_1,\dots,e_{n+p}$ in N such that, restricted to M, the vectors $e_1,\dots,e_n$ are tangent to M. We shall make the following convention on the ranges of indices :

$$1 \leqslant A,B\dots \leqslant n+p \quad , \quad 1 \leqslant i,j,k\dots \leqslant n \quad \text{and } (n+1) \leqslant \alpha, \beta, \dots \leqslant (n+p)$$

For the following computations we use the Einstein summation convention.

Let $\omega^1,\dots,\omega^{n+p}$ be the coframe field dual to $e_1,\dots,e_{n+p}$ chosen above. Let $h^{e_\alpha}(e_i,e_j) = h^\alpha_{ij} = h^\alpha_{ji}$. Applying in the structural equations with restriction to M, we have

$$\omega^{\alpha} = o \tag{1.1}$$

$$\omega_i^{\alpha} = h_{ij}^{\alpha}\,\omega^j \tag{1.2}$$

Gauss' equation ((2.7) - chapter 0) can also be written in the following forms

$$R_{jk\ell}^{i} = \tilde{R}_{jk\ell}^{i} + \sum_{\alpha} (h_{ik}^{\alpha}\, h_{j\ell}^{\alpha} - h_{i\ell}^{\alpha}\, h_{jk}^{\alpha}) \tag{1.3}$$

$$R_{\beta k\ell}^{\alpha} = \tilde{R}_{\beta k\ell}^{\alpha} + \sum_{i} (h_{ik}^{\alpha}\, h_{i\ell}^{\beta} - h_{i\ell}^{\alpha}\, h_{ik}^{\beta}) \tag{1.4}$$

where $\tilde{R}$ and R represent the curvature tensors of N and M, respectively. Actually $\frac{1}{n}\sum_i h_{ii}^{\alpha} e_{\alpha}$, which is independent of the choice of coordinates [19] , is called the mean curvature vector and an immersion is called minimal if its mean curvature vector vanishes identically, i.e. $\sum_i h_{ii}^{\alpha} = o$ for all α .

Exterior differentation of equation (1.2) and taking

$$h_{ijk}^{\alpha} = dh_{ij}^{\alpha} - h_{i\ell}^{\alpha}\,\omega_j^{\ell} - h_{\ell j}^{\alpha}\,\omega_i^{\ell} + h_{ij}^{\beta}\,\omega_{\beta}^{\alpha} \tag{1.5}$$

give that

$$(h_{ijk}^{\alpha} + \frac{1}{2}\tilde{R}_{ijk}^{\alpha})\,\omega^j \wedge \omega^k = o \tag{1.6}$$

$$h_{ijk}^{\alpha} - h_{ikj}^{\alpha} = \tilde{R}_{ikj}^{\alpha} = -\tilde{R}_{ijk}^{\alpha} \tag{1.7}$$

Similarly, by exterior differentiating (1.5) and defining

$$h_{ijk\ell}^{\alpha}\omega^{\ell} = dh_{ijk}^{\alpha} - h_{\ell jk}^{\alpha}\,\omega_i^{\ell} - h_{i\ell k}^{\alpha}\,\omega_j^{\ell} - h_{ij\ell}^{\alpha}\,\omega_k^{\ell} + h_{ijk}^{\beta}\omega_{\beta}^{\alpha} \tag{1.8}$$

we get

$$(h_{ijk\ell}^{\alpha} - \frac{1}{2} h_{im}^{\alpha} R_{jk\ell}^{m} - \frac{1}{2} h_{mj}^{\alpha} R_{ik\ell} + \frac{1}{2} h_{ij}^{\beta} R_{\beta k\ell}^{\alpha})\,\omega^k \wedge \omega^{\ell} = o \tag{1.9}$$

$$h^\alpha_{ijk\ell} - h^\alpha_{ij\ell k} = h^\alpha_{im} \; R^m_{jk\ell} + h^\alpha_{mj} \; R^m_{ik\ell} - h^\beta_{ij} \; R^\alpha_{\beta k\ell} \tag{1.10}$$

In fact h^α_{ijk} is the covariant derivative of h^α_{ij} while $h^\alpha_{ijk\ell}$ is the covariant derivative of h^α_{ijk}. Looking at $\tilde{R}^\alpha_{ijk}$ as a section of the bundle $T(M)^\perp \otimes T^*(M) \otimes T^*(M) \otimes T^*(M)$, its covariant derivative $\tilde{R}^\alpha_{ijk\ell}$ is defined by

$$\tilde{R}^\alpha_{ijk\ell}\omega^\ell = d\tilde{R}^\alpha_{ijk\ell} - \tilde{R}^\alpha_{mjk}\,\omega^m_{\ i} - \tilde{R}^\alpha_{imk}\,\omega^m_{\ j} - \tilde{R}^\alpha_{ijm}\,\omega^m_{\ k} + \tilde{R}^\beta_{ijk}\omega^\alpha_\beta \tag{1.11}$$

This covariant derivative of $\tilde{R}^\alpha_{ijk}$ must be distinguished from the covariant derivative of $\tilde{R}^A_{BCD}$ as a curvature tensor of N, which will be denoted by $\tilde{R}^A_{BCD;E}$. Restricted to M, $\tilde{R}^\alpha_{ijk;\ell}$ is given by

$$\tilde{R}^\alpha_{ijk;\ell} = \tilde{R}^\alpha_{ijk\ell} - \tilde{R}^\alpha_{\beta jk}h^\beta_{i\ell} \quad \tilde{R}^\alpha_{i\beta k}\, h^\beta_{j\ell} - \tilde{R}^\alpha_{ij\beta}\, h^\beta_{k\ell} + \tilde{R}^m_{ijk} \; h^\alpha_{m\ell} \tag{1.12}$$

Assuming that N is locally symmetric [19], we have

$$\tilde{R}^A_{BCD;E} = 0$$

The Laplacian Δh^α_{ij} of the second fundamental form h^α_{ij} is defined by

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk} \tag{1.14}$$

Covariant differentiating (1.7) and substituting in (1.14) we get that

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ikjk} - \sum_k \tilde{R}^\alpha_{ijkk} = \sum_k h^\alpha_{kijk} - \sum_k \tilde{R}^\alpha_{ijkk} \tag{1.15}$$

Using (1.10) together with (1.3), (1.4), (1.12) and (1.15) we obtain the final expression of Δh^α_{ij} which may be written as

$$\Delta h^\alpha_{ij} = \sum_k (h^\alpha_{kkij} - \tilde{R}^\alpha_{ij\beta}\, h^\beta_{kk} + 2\tilde{R}^\alpha_{\beta ki}\, h^\beta_{jk} - \tilde{R}^\alpha_{k\beta k}\, h^\beta_{ij} + 2\tilde{R}^\alpha_{\beta kj}\, h^\beta_{ki} +$$

$$+ \tilde{R}_{kik}{}^m \; h^\alpha_{mj} + \tilde{R}_{kjk}{}^m \; h^\alpha_{mi} + 2\tilde{R}^m_{ijk}\; h^\alpha_{mk}) + \sum_{\beta,m,k} (h^\alpha_{mi}\, h^\beta_{mj}\, h^\beta_{kk} +$$

$$+ 2h^{\alpha}_{km} h^{\beta}_{ki} h^{\beta}_{mj} - h^{\alpha}_{km} h^{\beta}_{km} h^{\beta}_{ij} - h^{\alpha}_{mi} h^{\beta}_{mk} h^{\beta}_{kj} - h^{\alpha}_{mj} h^{\beta}_{ki} h^{\beta}_{mk}) \qquad (1.16)$$

Moreover, we have

$$\sum_{\alpha,i,j} h^{\alpha}_{ij} \Delta h^{\alpha}_{ij} = \sum_{\alpha,i,j,k} (h^{\alpha}_{ij} h^{\alpha}_{kkij} - \tilde{R}^{\alpha}_{ij\beta} h^{\alpha}_{ij} h^{\beta}_{kk} + 4\tilde{R}^{\alpha}_{\beta ki} h^{\beta}_{jk} h^{\alpha}_{ij} -$$

$$- \tilde{R}^{\alpha}_{k\beta k} h^{\alpha}_{ij} h^{\beta}_{ij} + 2\tilde{R}^{m}_{kik} h^{\alpha}_{mj} h^{\alpha}_{ij} + 2\tilde{R}^{m}_{ijk} h^{\alpha}_{mk} h^{\alpha}_{ij}) -$$

$$- \sum_{\alpha,\beta,\, i,j,k,} \Big[(h^{\alpha}_{ik} h^{\beta}_{jk} - h^{\alpha}_{jk} h^{\beta}_{ik}).(h^{\alpha}_{i\ell} h^{\beta}_{j\ell} - h^{\alpha}_{j\ell} h^{\beta}_{i\ell}) +$$

$$+ h^{\alpha}_{ij} h^{\alpha}_{k\ell} h^{\beta}_{ij} h^{\beta}_{k\ell} - h^{\alpha}_{ij} h^{\alpha}_{ki} h^{\beta}_{kj} h^{\beta}_{\ell\ell} \Big] \qquad (1.17)$$

Assuming that N has constant sectional curvature c, and choosing $e_1, \ldots, e_{n+p}$ such that the symmetric matrix $(S_{\alpha\beta}) = (\sum_{i,j} h^{\alpha}_{ij} h^{\beta}_{ij})$ is diagonalised, we have a simpler form for equation (1.17) as follows :

$$\sum_{\alpha,i,j} h^{\alpha}_{ij} \Delta h^{\alpha}_{ij} = \sum_{\alpha,i,j,k} h^{\alpha}_{ij} h^{\alpha}_{kkij} + ncS - \sum_{\alpha} S^2_{\alpha} + \sum_{\alpha\beta} Tr(H_{\alpha} H_{\beta} - H_{\beta} H_{\alpha})^2 -$$

$$- c \sum_{\alpha} (Tr\, H_{\alpha})^2 + \sum_{\alpha,\beta} (TrH_{\beta})(Tr(H_{\alpha} H_{\beta} H_{\alpha})) \qquad (1.18)$$

where $S = \sum_{\alpha} S_{\alpha\alpha}$ and $S_{\alpha} = S_{\alpha\alpha}$.

S. Braidi and C.C. Hsiung [4] proved that

$$- \sum_{\alpha,i,j} h^{\alpha}_{ij} \Delta h^{\alpha}_{ij} \leqslant W_p - \sum_{\alpha,i,j,k} h^{\alpha}_{ij} h^{\alpha}_{kkij}$$

where

$$W_p = [(2 - \frac{1}{p})S - nc]\; S + c \sum_{\alpha} (TrH_{\alpha})^2 - \sum_{\alpha,\beta} (TrH_{\beta})(Tr(H_{\alpha}H_{\beta}H_{\alpha})) \qquad (1.19)$$

The following inequalities have been also proved in [4] :

1. If M is a compact oriented n-manifold immersed in an (n+p) - dimensional Riemannian manifold N, then

$$\int_M \sum_{\alpha,i,j} h^{\alpha}_{ij} \, \Delta h^{\alpha}_{ij} \, dv = -\int_M \sum_{\alpha,i,j,k} (h^{\alpha}_{ijk})^2 \, dv \leqslant o \qquad (1.20)$$

2. If in addition N has constant sectional curvature c, then

$$\int_M [\, W_p - \sum_{\alpha}(TrH_{\alpha}) \, \Delta (TrH_{\alpha}) \,] \, dv \geqslant \int_M \sum_{\alpha,i,j} (h^{\alpha}_{ijk})^2 \, dv \geqslant o \qquad (1.21)$$

and consequently if

$$W_p - \sum_{\alpha} (Tr\, H_{\alpha}) \, \Delta \, (Tr\, H_{\alpha}) \leqslant o$$

everywhere on M, then

$$W_p - \sum_{\alpha} (Tr\, H_{\alpha}) \, \Delta \, (Tr\, H_{\alpha}) = o$$

everywhere on M.

(II.1.3) - Main theorem :

Throughout the present work we assume that M is a compact orientable n-manifold immersed in an (n+1)-dimensional hyperbolic space of sectional curvature -1. We also assume that M has the property that

$$\int_M [W_1 - (Tr\, H_{n+1}) \, \Delta \, (Tr\, H_{n+1})] \, dv = o \qquad (1.22)$$

The case of M being a totally geodesic hypersurface is possible in spherical spaces but is impossible in hyperbolic spaces. The reason is that in hyperbolic spaces totally geodesic submanifolds are non-compact. This may show the first deviation from theorem (II.1.1).

Equations (1.21) and (1.22) give that

$$h^{n+1}_{ijk} = o \quad \text{for all } i,j,k \qquad (1.23)$$

Using (1.14) together with (1.23) we have

$$\Delta h^{n+1}_{ij} = o \tag{1.24}$$

For simplicity let

$$h^{n+1}_{ij} = h_{ij} \quad , \quad h_{ii} = h_i \tag{1.25}$$

and choose the frame field $e_1, \dots, e_{n+1}$ such that

$$h_{ij} = o \quad \text{for} \quad i \neq j$$

Lemma (II.1.1) :

After a suitable renumbering of the basis $e_1, \dots, e_n$ we have either

(i) $h_1 = h_2 = \dots = h_n$ = constant, $|h_i| > 1$ for all i.

or

(ii) $h_1 = h_2 = \dots = h_m = \lambda$ = constant $1 < m < n$

$h_{m+1} = \dots = h_n = \mu =$ constant , $\lambda\mu = 1$, $\omega^i_j = o$ for $1 \leqslant i \leqslant m$,

$m + 1 \leqslant j \leqslant n$.

Proof :

Putting $i = j$ and $\alpha = n+1$ in equation (1.5) and using (1.25) and (1.26), we get

$$dh_{ii} = o \tag{1.27}$$

which shows that $h_{ii} = h_i$ = constant.

For $i \neq j$, equation (1.5) becomes

$$(h_i - h_j)\, \omega^i_j = o \tag{1.28}$$

from which it follows that $\omega^i_j = o$ whenever $h_i \neq h_j$. Thus if $h_i \neq h_j$, the equations of structure give that

$$o = d\omega^i_j = - \omega^i_k \wedge \omega^k_j - \omega^i_{n+1} \wedge \omega^{n+1}_j - \omega^i \wedge \omega^j \tag{1.29}$$

From (1.2) and (1.29) we obtain

$$(1 - h_{ii}h_{jj})\,\omega^i \wedge \omega^j = o \qquad (1.30)$$

which shows that if $h_i \neq h_j$, then $h_i h_j = 1$. Set $h_1 = \lambda$ then we have as a first possibility that $h_1 = \dots = h_n = \lambda$ which proves part (i) of lemma (II.1.1) partially.

If $h_2 \neq h_1$, then $h_2 = 1/\lambda = \mu$. If $h_2 \neq h_3$, then $h_3 = \lambda$. Repeating similar discussion with all h_i's we have under suitable renumbering of $e_1,\dots,e_n$ that $h_1 = h_2 = \dots = h_m = \lambda$ and $h_{m+1} = \dots = h_n = \mu$ where $m \geqslant 2$. In this case and from (1.28) we have $\omega^i_j = o$ for $1 \leqslant i \leqslant m$ and $m+1 \leqslant j \leqslant n$. This completes the proof of the lemma except for part (i) which can be completed as follows:

In case (i) the sectional curvature of M may be written as

$$K(e_i \wedge e_j) = -1 + \lambda^2 = \text{constant}$$

Applying Amaral's theorem (I.1.2) we conclude that M should have a point with $K(e_i \wedge e_j) > o$ for all i,j, hence $|\lambda| > 1$.

In case (i) also, M is totally umbilical and hence M is a geodesic sphere (proposition (6.1) chapter 0).

In case (ii) we have for $1 \leqslant i \leqslant m$, $m + 1 \leqslant j \leqslant n$ that

$$K(e_i \wedge e_j) = -1 + h_i h_j = o$$

which contradicts Amalaral's theorem (I.1.2).

From the above argument we can state the main theorem of this section as follows:

<u>Theorem (II.1.2)</u>

Let M be a compact, oriented immersed hypersurface in an (n+1)-dimensional hyperbolic space with curvature K = -1. Let M have the property that

$$\int_M [W_1 - (\mathrm{Tr}\, H_{n+1}) \Delta (\mathrm{Tr}\, H_{n+1})]\, dv = 0$$

then M is an n-geodesic sphere.

Section 2 : On the Gauss mapping for hypersurface of constant mean curvature

(II.2.0) - Introduction

The aim of this section is to prove the following two theorems :

Theorem (II.2.1)

Let M be a complete, orientable, Riemannian manifold of dimension $n \geqslant 2$ isometrically immersed in the H-model of the (n+1)-hyperbolic spaces and let $\Phi : M \to H^*$ be the associated Gauss mapping into the conjugate hypersurface H^* of H (see definition below)

i) If $\Phi(M)$ is contained in a compact hypersphere of H^*, i.e. $\Phi(M) \subset L^{n+1} \cap H^*$ where L^{n+1} is a hyperplane in the Minkowski space $(\mathbb{R}^{n+2}, < , >)$, then M is imbedded as a geodesic sphere.

ii) If $\Phi(M)$ is contained in a hypersphere of H^* whose plane is asymptotic to H, then M is imbedded as a horosphere of H.

iii) The image $\Phi(M)$ is a single point of H^* if and only if M is imbedded as a totally geodesic hypersurface of H.

Theorem (II.2.2)

Let M be a compact, connected, orientable n-manifold immersed in the (n+1)-model H. Let M have constant mean curvature. If the Gauss image $\Phi(M)$ lies in a closed hemisphere of H^*, then M imbeds as an n-geodesic sphere in H.

In fact, K. Nomizu and B. Smith [24] proved the corresponding two theorems in the Euclidean sphere S^{n+1} in E^{n+2}. In chapter III we define other types of Gauss mappings different from that given in this section.

(II.2.1) - The Gauss mapping

The conjugate hypersurface H^* of H in the Minkowski space $(\mathbb{R}^{n+2}, <,>)$ is defined by

$$H^* = \{ x \in \mathbb{R}^{n+2} \ ; \ <x,x> = 1 \}$$

It is to be noted that H^* is a Lorentz manifold whose induced metric has index 1. It can be shown, similar to H, that H^* has constant sectional curvature $K = 1$. As we mentioned above that a hypersphere in H^* means $L^{n+1} \cap H^*$ where L^{n+1} is a hyperplane in $(\mathbb{R}^{n+2}, <,>)$. If L^{n+1} passes through the origin 0 of $(R^{n+2}, <,>)$ we call $L^{n+1} \cap H^*$ great hypersphere.

As in chapter 0, let D denote the covariant differentiation operator of the Riemannian connexion on $(\mathbb{R}^{n+2}, <,>)$. Let $\tilde{M}$ be a hypersurface of $(\mathbb{R}^{n+2}, <,>)$ with induced covariant differentiation operator $\tilde{\nabla}$ and let M be an immersed hypersurface of $\tilde{M}$. If ξ is the field of unit normal vectors of M as a hypersurface of $\tilde{M}$, we have

$$\tilde{\nabla}_X \xi = -A_\xi X \qquad (2.1)$$

where A_ξ is the second fundamental tensor of the immersion of M into $\tilde{M}$ and $X \in \mathfrak{X}(M)$. We have also that

$$D_X \xi = \tilde{\nabla}_X \xi + \tilde{h}(X, \xi)\eta \qquad (2.2)$$

where η is the unit normal vector field of $\tilde{M}$ and $\tilde{h}$ is the second fundamental form of $\tilde{M}$ as a submanifold of $(\mathbb{R}^{n+2}, <,>)$. If we project $D_X\xi$ orthogonally on $\tilde{M}_x$ for some $x \in M$ we obtain by using (2.1) an orthogonal projection on M_x in the same time. Hence if P denotes the orthogonal projection mapping, then we have

$$P(D_X \xi) = \tilde{\nabla}_X \xi = -A_\xi X \qquad (2.3)$$

Consider 0 to be the origin of $(\mathbb{R}^{n+2}, <, >)$ and γ to be the straight line segment joining 0 and $x \in M$. Now parallel translate ξ along γ to 0, we obtain a unit vector at 0 and we call it $\Phi(\xi)$. If we identify each $x \in M$ with ξ_x, we obtain the desired Gauss mapping $\Phi : M \to H^*$ or $\xi : M \to H^*$. Direct computations show that the differential ξ_*(or Φ_*) of this Gauss mapping is given by

$$D_X \xi = \xi_*(X) \tag{2.4}$$

for $X \in \mathfrak{X}(M)$. From (2.3) and (2.4) we have

$$P \circ \Phi_* = -A \tag{2.5}$$

where we put $A_\xi = A$ for simplicity.

Notice that if M is a hypersurface of $(\mathbb{R}^{n+2}, <,>)$ itself, then equation (2.5) becomes simply

$$\Phi_* = -A$$

<u>Lemma (II.2.1)</u> :

Under the above notations if ξ is a constant vector field, i.e. $D_X \xi = o$ for each $X \in \mathfrak{X}(M)$, the M is a totally geodesic hypersurface of $\tilde{M}$.

<u>Proof</u> :

Since $D_X \xi = o$, then $P(D_X \xi) = o$ for each $X \in \mathfrak{X}(M)$. From (2.5) we have that $A = o$ and hence M is a totally geodesic hypersurface of $\tilde{M}$.

This lemma says that if $\Phi(M)$ is a single point of H^* then M is a totally geodesic hypersurface of $\tilde{M}$.

Now we specialize to the case when $\tilde{M}$ is the H-model. Equation (2.2) together with equation ((2.4) - chapter 0) give

$$D_X \xi \quad = \quad \tilde{\nabla}_X \xi \quad = \quad -AX \tag{2.6}$$

as $\tilde{h}(X, \xi) = \langle X, \xi \rangle = o$ and $X \varepsilon \mathfrak{X}(M)$, hence

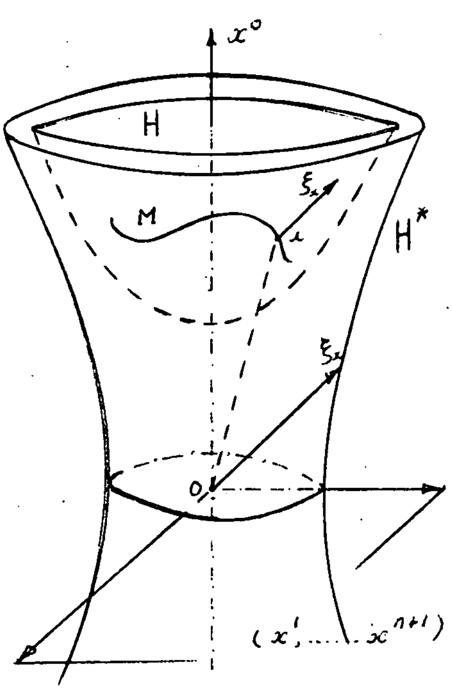
$$\Phi_* \quad = \quad -A \tag{2.7}$$

Lemma (II.2.2)

Let M be a hypersurface of H. Then M is a totally geodesic hypersurface if and only if $\Phi(M)$ is a single point of H^*

Proof :

The necessity part is clear by lemma (II.2.1) when taking $\tilde{M} = H$. Conversely, if $\Phi(M)$ is a single point of H^*, then $\Phi_* = o$ and by (2.7) we have $A = o$. Hence M is totally geodesic and the proof is complete.



(II.2.2) Proof of theorem (II.2.1)

It is clear that if $\Phi(M)$ lies in a hypersphere $L^{n+1} \cap H^*$ then there exists a vector $a \in (\mathbb{R}^{n+1}, <\ ,\ >)$ such that $< \xi, a > =$ constant. If the hypersphere is a great one then $< \xi, a > = 0$ and if it is a small one which passes through some point c then $< \xi,a > = < c,a >$.

Differentiating the relation $<\xi,a> =$ constant, we have that

$$< D_X \xi , a > = 0 \qquad \text{for } X \in \mathfrak{X}(M)$$

Using (2.7) we have

$$< AX, a > = 0$$

Since X is an arbitrary vector field on M and A maps $\mathfrak{X}(M)$ to $\mathfrak{X}(M)$ then M itself should lie in $\tilde{L}^{n+1} \cap H$ where $\tilde{L}^{n+1}$ is a hyperplane in $(\mathbb{R}^{n+2}, <\ ,\ >)$ with "a" as its normal, i.e. L^{n+1} and $\tilde{L}^{n+1}$ are parallel hyperplanes.

Now suppose that the hypersurface $L^{n+1} \cap H^*$ is a compact hypersphere, then $\tilde{L}^{n+1} \cap H$ should be also compact and by § 6 - chapter 0, M is imbedded as a geodesic sphere of H which proves part (i).

As $\tilde{L}^{n+1}$ is always parallel to L^{n+1} and in part (ii) L^{n+1} is assumed to be asymptotic to H, hence from the geometry of horospheres of the H-model (§ 6-chapter 0) we have that M is imbedded as a horosphere of H. Notice that M in this case should be non-compact.

Part (iii) of the theorem is proved by lemma (II.2.2).

(II.2.3) - Proof of theorem (II.2.2) :

For the proof of theorem (II.2.2) we need to find the Laplacian Δf of some differentiable function f.

Let M be a Riemannian k-manifold. For any differentiable function $f \in \mathcal{F}(M)$ it is known that Δf can be written as [19] :

$$(\Delta f)(p) = \sum_{i=1}^{k} (\nabla^2 f)(e_i, e_i) \tag{2.8}$$

for $e_1, \ldots, e_k$ orthonormal basis of M_p , $p \in M$, where

$$\nabla^2 f = X(Yf) - (\nabla_X Y)f \tag{2.9}$$

and ∇ denotes the covariant differentiation in M.

Let M be a k-dimensional submanifold of the (n+1) - dimensional model H in the Minkowski space $(\mathbb{R}^{n+2}, < , >)$. In addition to the normal x to H, choose n - k + 1 vector fields $\{\xi_i\}$ which are normal to M and tangent to H such that $\xi_1, \ldots, \xi_{n-k+1}$ are othonormal at every point of M. As before, we denote by D, $\tilde{\nabla}$ and ∇ the Riemannian connexions of $(\mathbb{R}^{n+2}, < , >)$, H and M, respectively. For vector fields X and Y tangent to M we write

$$D_X Y = \tilde{\nabla}_X Y + < X,Y > x \tag{2.10}$$

and

$$\tilde{\nabla}_X Y = \nabla_X Y + \sum_{j=1}^{n-k+1} h^j(X,Y)\, \xi_j \tag{2.11}$$

so that

$$D_X Y = \nabla_X Y + <X,Y>\, x + \sum_{j=1}^{n-k+1} h^j(X,Y)\xi_j \tag{2.12}$$

where $h^1, \ldots, h^{n-k+1}$ are the second fundamental forms of M as a submanifold of H.

For a constant vector "a" in $(\mathbb{R}^{n+2}, <,>)$, consider $f(x) = <x,a>$, $x \in M$, as a function on M. For X, $Y \in \mathfrak{X}(M)$ we have

$$X(Y <x,a>) = <\nabla_X Y + \sum_{j=1}^{n-k+1} h^j(X,Y)\, \xi_j , a> + <X,Y><x,a> \tag{2.13}$$

We have also that

$$(\nabla_X Y) <x,a> = <\nabla_X Y, a> \tag{2.14}$$

From equations (2.8), (2.13) and (2.14) we have

$$(\Delta f)(x) = \left< \sum_{i=1}^{k} \sum_{j=1}^{n-k+1} h^j (X_i, X_i)\xi_j + kx, a \right> \tag{2.15}$$

where $X_1, \ldots, X_k$ are orthonormal basis of the tangent space M_x.

Following the same notations of § 1 we can write

$$k\zeta = \sum_{i=1}^{k} \sum_{j=1}^{n-k+1} h^j_{ii} \xi_j \tag{2.16}$$

where ζ denotes the mean curvature vector of M as a submanifold of H. From (2.15 and (2.16) we have

$$\Delta < x,a > = k < x,a > + k < \zeta ,a > \tag{2.17}$$

If M is a hypersurface of H, then equation (2.17) takes the form

$$\Delta < x,a > = (TrA) < \xi,a > + n < x,a > \tag{2.18}$$

Where A is the second fundamental tensor of M and ξ is the field of unit normal vectors of M as a hypersurface of H. In this case and for $f = <\xi,a>$, similar computations show that

$$\Delta < \xi,a > = - \sum_{i=}^{n} <\text{grad } (TrA) , a > - (TrA^2) < \xi ,a > - (TrA) < x,a > \tag{2.19}$$

We are now in a position to prove theorem (II.2.2). Since we are concerned with hypersurfaces of constant mean curvature (i.e. TrA = constant on M), we write equation (2.19) as

$$\Delta < \xi,a > = -TrA^2 < \xi,a > - (TrA) < x,a > \tag{2.20}$$

Combining (2.18) and (2.20) together we obtain

$$\begin{aligned} \Delta < n\xi + (TrA)\, x,a > &= -\{ n(TrA^2) - (TrA)^2 \} < \xi ,a > \\ &= - \sum_{i<j} (\lambda_i - \lambda_j)^2 < \xi,a > \end{aligned} \tag{2.21}$$

where $\lambda_1, \ldots, \lambda_n$ denote the characteristic roots of A.

The assumption on the Gauss image $\Phi(M)$ of M is equivalent to the existence of a constant unit vector "a" in $(\mathbb{R}^{n+2}, <\ ,\ >)$ for which $<\xi, a> \geqslant o$ on M. By virtue of (2.21), we have

$$\Delta <n\xi + (\mathrm{Tr}\,A)x, a> \leqslant o \qquad (2.22)$$

Using (2.22) in Hopf's lemma [19] we have that $<n\xi + (\mathrm{Tr}\,A)x, a>$ is constant on M. If M is minimal $<\xi, a>$ is constant on M and in this case M is a geodesic sphere in H by virtue of theorem (II.2.1).

We now assume that $\mathrm{Tr}\,A \neq o$. By equation (2.21) and letting $f : M \to H$ denotes the isometric immersion under consideration, every point of

$$W = \{ p \in M \ ; \ <\xi_{f(p)}, a> > o \}$$

is an umbilic as in this case $\lambda_i = \lambda_j$ for all i,j. However, $<n\xi + (\mathrm{Tr}\,A)x, a>$ being constant on M, it is clear that $<x, a>$ is constant on $M \setminus \bar{W}$. Therefore $M \setminus \bar{W}$ immerses into a hypersphere of H so that $M \setminus \bar{W}$ is totally umbilic. Thus M immerses totally umbilically in H and by virtue of proposition (6.1) - chapter 0 M is an imbedded geodesic sphere in H. Thus the proof of theorem (II.2.2) is now complete.

Section 3 : Stability of minimal surfaces

(II.3.0) - Introduction :

Since the paper of B.Y. Chen [8], the stability problem of submanifolds has become an interesting area of research. T.J. Willmore and C.S. Jhaveri [32] generalized the concept of stability adopted by B.Y. Chen in Euclidean spaces to a general Riemannian manifold. The stability problem has been developed by H.Mori [23] when he studied the stability of minimal surfaces in the 3-dimensional Euclidean sphere S^3 of unit radius in E^4. In fact, H. Mori proved the following :

Theorem (II.3.1)

Let S^3 be the 3-dimensional unit sphere in E^4 with the canonical Riemannian metric and let $f : M \to S^3$ be a minimal immersion of a compact orientable surface M with piecewise smooth boundary ∂M. Suppose that there is a constant $a (\geq 4)$ such that the Gaussian curvature K of M satisfies $K \leq (a-4)/(a-2)$. Then if $\int_M (1-K)dV \leq \frac{1}{27\pi a}$ (M,f) is stable.

In this section we prove a similar theorem concerning stability of minimally immersed surfaces in 3-dimensional hyperbolic space. The model which we use is the H-model in the 4-dimensional Minkowski space $(\mathbb{R}^4, \langle\,,\,\rangle)$.

(II.3.1) - Basic Relations :

Let (M,f) be an immersed n-submanifold in the Riemannian (n+p)-manifold $(\tilde{M},g)$. Let - as before - $T(M)^{\perp}$ represent the normal bundle of M as a submanifold of $\tilde{M}$. Let $\tilde{\nabla}$, ∇ and $\tilde{\nabla}^{\perp}$ denote the Riemannian connexions on $\tilde{M}$, M and in $T(M)^{\perp}$, respectively. Let $\Gamma(M)$ denote the space of cross-sections of $T(M)^{\perp}$. Suppose that ν is a section of $\Gamma(M)$, then the Laplacian $\Delta\nu$ of ν can be written as follows [28] :

$$(\Delta\nu)(p) = \sum_{i=1}^{n} \left(\tilde{\nabla}^{\perp}_{e_i} \tilde{\nabla}^{\perp}_{e_i} - \tilde{\nabla}^{\perp}_{\nabla_{e_i} e_i} \nu \right)(p) \qquad (3.1)$$

for $p \in M$ where $e_1, \ldots, e_n$ are an orthonormal basis of M_p. If M is compact and closed or if M is compact with boundary ∂M, and Ψ, ϕ are two cross-sections in $\Gamma(M)$ which vanish on ∂M, we have [19] :

$$\int_M \bar{g}(\Delta\Psi, \phi)dv = \int_M \bar{g}(\Psi, \Delta\phi)dv = -\int_M \bar{g}(\Delta\Psi, \Delta\phi)dv \qquad (3.2)$$

where $\bar{g}$ is the induced Riemannian metric on $T(M)^{\perp}$.

D. Hoffman and J. Spruck [18] considered the isometric immersion $f : M \to \tilde{M}$ of Riemannian manifolds M and $\tilde{M}$. Using the following notations :

$\tilde{K}$ = sectional curvature of $\tilde{M}$.

ξ = mean curvature vector field of immersion

$\tilde{r}(M)$ = the injectivity radius (*) of $\tilde{M}$ restricted to M

ω_m = the volume of the unit ball in E^m

b = a positive real number or pure imaginary one

they proved the following two theorems:

<u>Theorem (II.3.2)</u>

Assume $\tilde{K} \leqslant b^2$ and let h be a non-negative C^1 function on M vanishing on ∂M. Then

$$\left\{ \int_M h^{m/(m-1)} dv \right\}^{(m-1)/m} \leqslant c(m) \int_M \left(\|\text{grad } h\| + h\|\xi\| \right) dv$$

(*) The injectivity radius $\tilde{r}(M)$ of a Riemannian manifold M is the largest r such that for all $p \in M$, $\exp_p$ is an imbedding of the open ball B(0,m) of center 0 and radius r in M_p.

provided

$$b^2(1-\alpha)^{-2/m}\ (\omega_m^{-1}\operatorname{Vol}(\operatorname{supp} h))^{2/m} \leq 1$$

and

$$2\rho_0 \leq \tilde{r}(M)$$

where

$$\rho_0 = \begin{cases} b^{-1}\sin^{-1} b(1-\alpha)^{-1/m}\ (\omega_m^{-}\operatorname{Vol}(\operatorname{supp} h))^{1/m}. & \text{for b real.} \\ (1-\alpha)^{-1/m}\ (\omega_m^{-1}\ \operatorname{Vol}(\operatorname{supp} h))^{1/m} & \text{, for b imaginery} \end{cases}$$

Here $o < \alpha < 1$ is a free parameter, $\dim M = m$ and

$$c(m) = c(m,\alpha) = \frac{1}{2}\pi\ 2^{m-2}\ \alpha^{-1}(1-\alpha)^{-1/m}\ \frac{m}{m-1}\ \omega_m^{1/m}$$

for b imagery, we may omit the factor $\frac{1}{2}\pi$ in the definition of c.

Theorem (II.3.3)

Let M be compact with boundary ∂M and assume $\tilde{K} \leq b^2$ then

$$(\operatorname{Vol} M)^{(m-1)/m} \leq c(m)\quad (\operatorname{Vol}\ \partial M + \int_M \|\xi\|\ dv)$$

provided

$$b^2\ (1-\alpha)^{-2/m}\ (\omega_m^{-1}\ \operatorname{Vol} M)^{2/m} \leq 1$$

and

$$2\rho_0 \leq \tilde{r}(M)$$

where

$$\rho_0 = \begin{cases} b^{-1}\ \sin^{-1} b(1-\alpha)^{-1/m}\ (\omega_m^{-1}\operatorname{Vol} M)^{1/m} & \text{, for b real} \\ (1-\alpha)^{-1/m}\ (\omega_m^{-1}\ \operatorname{Vol} M)^{1/m} & \text{, for b imaginery} \end{cases}$$

(II.3.2) - Variations and Stability :

Let $\{ f_t \}$ be a 1-parameter family of immersions of the p-manifold M into the Riemannian n-manifold $(\tilde{M},g)$ with the property that $f_0 = f$, where $f : M \to \tilde{M}$ is a C^∞ immersion, and the map $F : [0,1] \times M \to \tilde{M}$, defined by $F(t,m) = f_t(m)$ is C^∞. Then $\{ f_t \}$ is called a variation of f.

The variation $\{ f_t \}$ of $f : M \to \tilde{M}$ induces a vector field in $\tilde{M}$ defined along the image f(M) of M. We denote this field by $\tilde{E}$ and it is constructed as follows : let $\frac{\partial}{\partial t}$ be the standard vector field in $[0,1] \times M$. We set $\tilde{E}(m) = F_*\left(\frac{\partial}{\partial t}(0,m)\right)$. The field E gives rise to C^∞ cross-sections $\tilde{E}^N$ and $\tilde{E}^T$ in $T(M)^\perp$ and T(M), respectively, by orthogonally projecting $\tilde{E}$ into the appropriate space.

Since $\tilde{E}^T$ is a vector field on M, $\tilde{E}^T$ corresponds to a differential (p-1)-form $\theta_{\tilde{E}^T}$ on M defined by

$$\theta_{\tilde{E}^T}(x_1,\dots, x_{p-1}) = g(\tilde{E}^T \wedge x_1 \wedge \dots \wedge x_{p-1}, e_1 \wedge e_2 \wedge \dots \wedge e_p)$$

where $e_1,\dots, e_p$ is any positively oriented frame on M_m.

The following important theorems have been proved by J. Simons [28] .

Theorem (II.3.4)

Suppose that M is compact. Let Q(t) = p-dimensional area of $f_t(M)$. Then Q(t) is a C^∞ function on $f_t(M)$ and

$$Q'(0) = -\int_M g(\tilde{E}^N, \xi) + \int_{\partial M} \theta_{\tilde{E}^T}$$

where ξ is the mean curvature vector of M as a submanifold of $(\tilde{M},g)$.

Theorem (II.3.5)

Let $f : M \to \tilde{M}$ be a minimal immersion in $(\tilde{M},g)$. Let $\{ f_t \}$ be a variation of f. Suppose that for all $t \in [o,1]$, $f_t (\partial M) = f(\partial M)$. Then if M is compact $Q'(o) = o$.

Theorem (II.3.5)

Let $\{f_t\}$ be a variation of f such that $f_t (\partial M) = f(\partial M)$. Suppose that M is compact. Set $V = E^N$. If f_0 is a minimal immersion then

$$Q''(o) = \int_M g(-\Delta V + \bar{R}(V) - \tilde{A}(V), V) \quad (3.3)$$

where

$$\bar{R}(V) = \sum_{i=1}^{p} (\tilde{R}(e_i, V) e_i)^N ,$$

$\tilde{R}$ is the curvature tensor of $\tilde{M}$ and $\tilde{A} = A^* \circ A$, A is the second fundamental tensor of the immersion f.

Let M be a p-dimensional, compact, orientable, C^∞ manifold with boundary ∂M and let $f : M \to \tilde{M}$ be a minimal immersion of M into the Riemannian manifold $(\tilde{M},g)$. It is known (by theorem (II.3.5)) that M is stationary with respect to the n-dimensional volume Q(t). We say that M is stable if $Q''(o) > o$, i.e. the volume of f(M) is a strict minimum of all variations $\{ f_t \}$ corresponding to Q(t).

Now, we specialize to the following case : let $f : M \to H$ be a minimal immersion of a compact orientable surface M with piecewise smooth boundary ∂M into the 3-dimensional hyperbolic space model H. By assumption on M we have a unique (global) unit normal vector field ν (up to a sign) on M, and this field ν is parallel with respect to the induced connexion on $T(M)^\perp$.

(II.3.3) - Main Theorem

The aim of this section is to prove the following:

Theorem (II.3.7)

Let H be the 3-dimensional hyperbolic space of sectional curvature $\tilde{K} = -1$ in the Minkowski space $(\mathbb{R}^4, \langle\, ,\, \rangle)$. Let $f : M \to H$ be a minimal immersion of a compact, orientable, surface M with piecewise smooth boundary ∂M. Suppose that there is a constant $a \in \mathbb{R}$ $(a \in \mathbb{R} \setminus (2,4))$ such that the Gaussian curvature K of M satisfies $K \leqslant (4-a)/(a-2)$. Then if

$$\int_M (1+K)\, dV > -1/(27\pi a),$$

(M,f) is stable.

For the proof of this theorem we need the following lemmas.

Lemma (II.3.1)

Let $f : M \to H$ be as in the above mentioned theorem. Then for a normal variation with compact support and with variation vector field $u\nu$, the second variation of the 2-dimensional area Q(t) is given by

$$Q''(o) = -\int_M \{ u\Delta u + (\|B\|^2 - 2)\, u^2 \}\, dV$$
$$= \int_M \{ \|\text{grad } u\|^2 - (\|B\|^2 - 2)u^2 \}\, dV$$

where $\|B\|$ denotes the length of the second fundamental form of (M,f), $u \in \mathcal{F}(M)$ and ν is the field of unit normals of (M,f).

Proof :

If we consider $V = u\nu$ where $\|V\| = |u|$, $u|_{\partial M} = o$, relation (3.3) takes the form

$$\left.\begin{aligned} Q''(o) &= -\int_M \{<\Delta u\nu\,,\, u\nu> + u^2(\sum_{i,\alpha}\tilde{R}_{i\alpha i\alpha} + \|B\|^2)\}\, dV \\ &= -\int_M \{u\Delta u + u^2<\nu\,,\Delta\nu> + u^2(\sum_{i,\alpha}\tilde{R}_{i\alpha i\alpha} + \|B\|^2)\}\, dV \end{aligned}\right\} \quad (3.4)$$

where $\tilde{R}$ is the curvature tensor of H.

From relation (3.1), we have by direct computations that

$$<\nu,\Delta\nu> \;=\; -\sum\|\tilde{\nabla}^{\perp}_{e_j}\;\nu\|^2 \qquad (3.5)$$

where e_1 , e_2 are orthonormal basis for the tangent space of M at a point. From equations (3.4) and (3.5) we obtain

$$Q''(o) = -\int_M \{u\Delta u - u^2\sum_{j=1}^{2}\|\tilde{\nabla}^{\perp}_{e_j}\nu\|^2 + u^2(\sum_{i,\alpha}\tilde{R}_{i\alpha i\alpha} + \|B\|^2)\}\, dV \qquad (3.6)$$

Using Stokes' theorem, we have

$$\int_M u\Delta u = \int_M \|\text{grad}\;\; u\|^2 \qquad (3.7)$$

which together with the fact that $\tilde{\nabla}^{\perp}\nu = o$ give

$$Q''(o) = \int_M \{\|\text{grad}\;\; u\|^2 - u^2(\sum_{i,\alpha}\tilde{R}_{i\alpha i\alpha} + \|B\|^2)\}\, dV \qquad (3.8)$$

Taking into account that α = 3 and i takes the values 1 and 2 we have $\sum_{i,\alpha}\tilde{R}_{i\alpha i\alpha} = -2$ and now (3.8) becomes

$$Q''(o) = \int_M \{\|\text{grad}\;\; u\|^2 - u^2(\|B\|^2 - 2)\}\; dV$$

which completes the proof of the lemma.

Lemma (II.3.2)

The Gauss mapping Φ : $M \to H^*$ (defined in section 2) for the two-dimensional minimal submanifold M of H is conformal.

Proof :

We have seen before (in § 2) that for a hypersurface M in H, the

Gauss mapping $\Phi : M \to H^*$ has the property

$$\Phi_* = -A$$

where A is the second fundamental tensor of M as a submanifold of H. Suppose that λ_1 and λ_2 are the two eigenvalues of A. Since M is a minimally immersed surface in H then

$$\lambda_1 + \lambda_2 = 0 \quad \text{or} \quad \lambda_1 = -\lambda_2 \tag{3.9}$$

Suppose that e_1 and e_2 are the corresponding orthonormal eigenvectors of A at the point $p \in M$. For any pair of vectors $\omega, v \in M_p$, we have

$$< \Phi_* \omega , \Phi_* v > = < A\omega , Av >$$

Expressing ω, v as linear combinations of e_1, e_2, and using (3.9), we get

$$< \Phi_* \omega, \Phi_* v > = \lambda_1^2 (a_1 b_1 + a_2 b_2) = \lambda_1^2 < \omega, v > \tag{3.10}$$

where (a_1, b_1), (a_2, b_2) are the components of ω and v, respectively. From (3.10) it is clear that Φ is conformal with scale function λ_1^2 or λ_2^2.

Corollary (II.3.1)

There is no compact minimal surface (without boundary) in the 3-dimensional hyperbolic spaces.

The reason is that if we apply Gauss equation ((2.8) - chapter 0) we get that for a hypersurface M in the 3-dimensional hyperbolic space the Gaussian curvature is

$$K = -1 + \lambda_1 \lambda_2 = -1 - \lambda_1^2 < 0$$

If we assume that M is compact without boundary we get a contradiction to Amaral's theorem (I.1.2). Similar argument shows that the above corollary is valid also for compact hypersurfaces (without boundary) in the n-dimensional hyperbolic space.

Lemma (II.3.3)

Using the same notations and under the induced metric, the volume $V(\Phi(M))$ of the Gauss image $\Phi(M)$ of M is

$$\int_M dS = V(\Phi(M)) = \int_M (\omega^2 - 1)\, dV$$

where $K = -\omega^2$ and dS denotes the volume element of the image $\Phi(M)$.

Proof :

The volume $V(\Phi(M))$ can be written as

$$V(\Phi(M) = \int_M \sqrt{\det(\langle \Phi_*(e_i), \Phi_*(e_j) \rangle)}\, dV \tag{3.12}$$

From (3.10) and (3.12) we have

$$V(\Phi(M)) = \int_M \lambda_1^2\, dV$$

From (3.11) we obtain

$$V(\Phi(M)) = - \int_M (1+K)\, dV = \int_M (\omega^2 - 1)\, dV$$

and the proof is complete.

Proof of theorem (II.3.7)

To show that M is stable under the hypothesis in the theorem, it is to show that $Q''(o) > o$ which is equivalent, by (lemma (II.3.1)), to show that

$$\int_M (\|B\|^2 - 2) u^2\, dV < \int_M \|\text{grad } u\|^2\, dV \tag{3.13}$$

Using equations (3.9) and (3.11), we get

$$\|B\|^2 = \lambda_1^2 + \lambda_2^2 = 2\lambda_1^2$$

Hence

$$\|B\|^2 - 2 = 2(\omega^2 - 2) \tag{3.14}$$

From equation (3.14) inequality (3.13) takes the form

$$\int_M 2u^2 \; (\omega^2 - 2) \; dV < \int_M \|\text{grad} \;\; u\|^2 \;\; dV \tag{3.15}$$

Applying theorems (II.3.2 - 3) for $\tilde{K} = -1$ and $m = 2$ we have

$$\left\{ \int_M u^2 \; dS \right\}^{1/2} \leqslant c_1 \int_M \|\text{grad} \;\; u\| \;\; dS \tag{3.16}$$

where

$$c_1 = 2 \, \pi^{1/2} \alpha^{-1} \, (1 - \alpha)^{-1/2}, \quad 0 < \alpha < 1 \tag{3.17}$$

As c_1 is positive, (3.16) can be written as

$$\int_M u^2 \; dS \leqslant c_1^2 \; \left(\int_M \|\text{grad} \;\; u\| \;\; dV \right)^2 \tag{3.18}$$

Using Schwartz inequality, (3.18) gives

$$\int_M u^2 \; dS \leqslant c_1^2 \int_M \|\text{grad} \;\; u\|^2 \; dV \; . \; \text{Vol}(\, \Phi \, (M)) \tag{3.19}$$

From lemma (II.3.3), inequality (3.19) becomes

$$\int_M u^2 \; (\omega^2 - 1) \; dV \leqslant c_1^2 \int_M (\omega^2 - 1) \; dV. \int_M \|\text{grad} \;\; u\|^2 \; dV \tag{3.20}$$

Since $2(\omega^2 - 2) < (\omega^2 - 1)a$, then inequality (3.20) takes the form

$$\int_M 2u^2 \; (\omega^2 - 1) \; dV \leqslant ac_1^2 \int_M (\omega^2 - 1) \; dV. \int_M \|\text{grad} \;\; u\|^2 \; dV \tag{3.21}$$

For inequality (3.15) to be satisfied, $ac_1^2 \int_M (\omega^2 - 1) \; dV$ should be less than 1. The idea now is to find the value of α which makes $1/ac_1^2$ maxima. Computations on (3.17) show that the required value is $\alpha = 2/3$ hence $c_1 = 27 \, \pi$. This argument shows that the stability condition is

$$- \int_M (1+K) \; dV < (1/27 \, \pi \, a)$$

which completes the proof of the main theorem.

Problem

What is the corresponding theorem for hypersurfaces (or, even worse, for submanifolds) in the (n+1)-dimensional hyperbolic space, $n \geqslant 3$?

CHAPTER III

CONDITIONAL IMMERSIONS INTO MANIFOLDS WITHOUT CONJUGATE POINTS

Section 1 : Convexity

Relation between the second fundamental form and the Gaussian curvature of hypersurfaces in Euclidean spaces on one hand and the convexity on the other hand has been proved to be a fruitful area of research. Throughout this section we generalize the concept of convexity to general Riemannian manifolds. When restricting to manifolds without conjugate points more results are obtained. For this section all manifolds are complete, connected, C^∞ Riemannian.

We start by recalling some definitions :

1. A set B in a manifold $\tilde{M}$ is called convex if for each pair of points $p, q \in B$, there is a unique minimal geodesic segment from p to q and this segment is in B. An open (closed) convex set which is a submanifold of $\tilde{M}$ of maximal dimension is called open (closed) convex body. For the rest of this section convex bodies are assumed to have smooth boundaries.

2. A hypersurface M of $\tilde{M}$ is said to be convex at a point $x \in M$ if the geodesic hypersurface of $\tilde{M}$ tangent to M at x does not separate a neighbourhood of x in M into two (or more) parts. Moreover, if x is the only point of a neighbourhood of M which lies on the geodesic hypersurface tangent to M at x then M is said to be strictly convex at x. If these properties are satisfied for each $x \in M$, then M is called locally convex, strictly locally convex, respectively.

3. If for every $x \in M$ the tangent geodesic hypersurface of $\tilde{M}$ to M at x does not separate M into two (or more) parts, then M is said to be convex. Moreover, if for every $x \in M$, x is the only point of M which lies on the tangent geodesic hypersurface at x, then M is said to be strictly convex.

A strictly convex hypersurface M in Euclidean space is always orientable and its orientation can be given in the following way : Choose at each point $x \in M$ the unit normal vector pointing to the opposite side of M with respect to its tangent geodesic hypersurface (hyperplane) at x. We obtain a continuous field of unit normal vectors (orientation).

It is worth mentioning that the above mentioned way of giving an orientation to strictly convex hypersurfaces in Euclidean spaces fails in the case of convex hypersurfaces. Our indicative example of this situation is the 2-disc in E^3.

It can be shown, similar to Euclidean spaces, that a hypersurface M in $\tilde{M}$ is strictly convex at a point $x \in M$ if its second fundamental form is definite at x. It is also easy to see that each convex hypersurface is locally convex but the converse,in general, is not true. In the following, Int (A) for any subset $A \subseteq \tilde{M}$ will denote the interior of A and $\bar{A}$ will denote the closure of A.

It should be noted also that if A is a closed convex body in a manifold $\tilde{M}$, then Int (A) is also convex. Unfortunately, the converse of this fact is not true, i.e. if B is an open convex body in $\tilde{M}$, then $\bar{B} = B \cup \partial B$ is not necessarily convex. An obvious example for this case is the open hemisphere in the Euclidean sphere $S^n \subset E^{n+1}$.

It can be shown easily that

Proposition (III.1.1)

In a Riemannian manifold $\tilde{M}$, the intersection of two convex bodies is a convex body.

Before proving the next proposition, we define the geodesic cone as follows :

Definition :

A (truncated) geodesic cone in a manifold M with vertex $p \in M$ is a solid body in M which is the image under $\exp_p : M_p \to M$ of a (truncated) cone in M_p with vertex at 0.

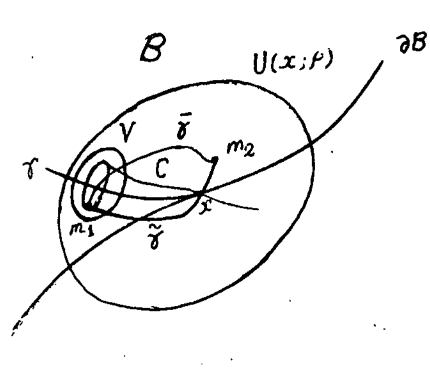
Proposition (III.1.2-a)

The boundary ∂B of an open convex body B in a complete Reimannian manifold M is a locally convex hypersurface of M.

Proof :

For the proof of this proposition we need the following theorem ([19] , Vol. I, p.166) : Let $(x^1,....,x^n)$ be a normal coordinate system at x of a Riemannian manifold M. There exists a positive number "a" such that, if $o < \rho < a$ then any two points of $U(x;\rho) = \{p \in M;\ d(x,p) < \rho\}$ can be joined by a unique minimizing geodesic, and it is the unique geodesic joining the two points and lying in $U(x;\rho)$.

Now suppose that B is an open convex body in M and assume that the boundary ∂B of B is not locally convex. Then there exists a point $x \in \partial B$ such that the tangent geodesic hypersurface at x to ∂B divides every neighbourhood of x in ∂B into (at least) two parts. Consequently, there exists at least one tangential geodesic, $\gamma : (o,1) \to M$, to ∂B from x which goes inside B, that is $\gamma(o) = x$ and $\gamma(t)$ is an anterior point of B for all $t \in (o,1)$. Now $U(x;\rho) \cap B$ is open in M. Consider the point $\gamma(t_0)$, $t_0 \in (o,1)$, such that $\gamma(t_0) \in U(x;\rho) \cap B$. Clearly there exists an open neighbourhood $V \subset U(x;\rho) \cap B$ around $\gamma(t_0)$. (See the following figure).



Draw the truncated geodesic cone C in M with vertex x, and axis γ such that its base A is completely inside V. From the above figure, there exists a geodesic $\tilde{\gamma} \in C$ going outside B and intersecting ∂B at x transversally. Consequently, there exist two points $m_1, m_2 \in \tilde{\gamma}$ such that m_1, $m_2 \in U(x;\rho) \cap B$. By the above mentioned theorem $\tilde{\gamma}$ is minimal from m_1 to m_2. Since B is convex then there exists a unique minimal geodesic, $\bar{\gamma}$ say, joining m_1 and m_2 and $\bar{\gamma} \subset B$. Clearly $\tilde{\gamma} \neq \bar{\gamma}$ and hence we obtain a contradiction showing that ∂B should be locally convex.

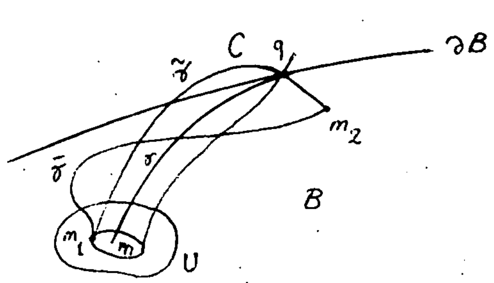
The following proposition shows that the boundary ∂B of a convex body B is convex provided that some extra conditions are assumed for the ambient manifold.

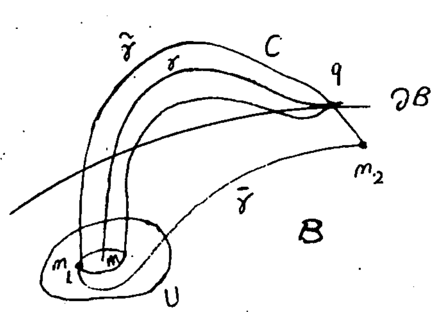
Proposition (III.1.2-b)

The boundary ∂B of the open convex body B in a complete, simply connected, Riemannian manifold W without conjugate points is convex.

Proof :

Let B and ∂B be as in the proposition. Let the boundary ∂B of B is not a convex hypersurface in W, hence there exists a point $q \in \partial B$ where the tangent geodesic hypersurface of ∂B at q separates ∂B into (at least) two pieces. Consequently, there exists a geodesic γ tangential to ∂B at q which goes inside B. Let m be a point on the geodesic γ such that $m \in B$. Since B is open, then there exists an open neighbourhood U of m such that $U \subset B$. (The figures below give some possible cases).





Draw the geodesic cone C with vertex q and axis γ such that its base around m is included in U. Similar to proposition (II.2.1-a), there exist two points $m_1, m_2 \varepsilon B$ with joining geodesic $\tilde{\gamma}$ which does not lie completely inside B.

Since W is complete, simply connected without conjugate points then there is only one geodesic segment $\bar{\gamma}$ joining m_1 and m_2 and by convexity of B, $\bar{\gamma}$ should be included in B. Now, we have two different geodesic segments $\bar{\gamma}$ and $\tilde{\gamma}$ joining m_1 and m_2 which is a contradiction. Hence ∂B is convex.

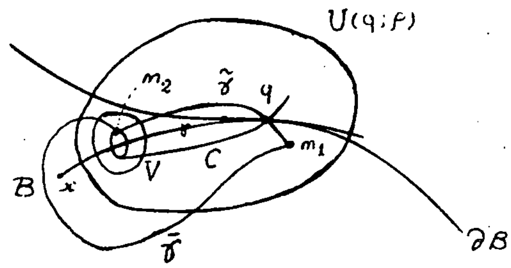
Proposition (III.1.2-c)

Let B be a convex body in a complete Riemannian manifold M. Let x be an interior point of B, $x \varepsilon$ Int (B). Then any geodesic ray γ from x which intersects ∂B will do so transversally (not tangentially).

Proof :

Under the same notations of the proposition, let γ be a geodesic ray from x which intersects ∂B. Suppose that γ intersects ∂B tangentially. If $\gamma \cap \partial B = \{ p \}$ is a single point, we can show that ∂B is not locally convex at p which contradicts proposition (III.1.2-a).

If $A = \gamma \cap \partial B$ is a continuous subset of γ, let $q \varepsilon \partial A$ be the nearest point to x. Consequently, all points of γ between x and q are interior points of B. Let $U(q\,;\rho)$ be the convex neighbourhood of q in M as mentioned before in the proof of proposition (III.1.2-a). Let $m \varepsilon U(q\,;\rho) \cap \gamma$ such that $m \varepsilon B$. Let V be a neighbourhood of m such that $V \subset U(q\,;\rho) \cap B$. Adapting the idea of geodesic cones as before we draw the geodesic cone C with vertex at q and axis γ such that its base is included in V.



It is clear that there exist two points $m_1, m_2 \varepsilon U(q\ ;\rho) \cap B$ which are connected by a geodesic segment $\tilde{\gamma}$ not included in B. By the theorem mentioned before in the proof of proposition (III.1.2-a), the geodesic $\tilde{\gamma}$ is minimal. Since B is convex, then there exists a unique minimal geodesic segment $\bar{\gamma}$ joining m_1 and m_2 and $\bar{\gamma} \subset B$. This contradiction shows that γ should intersect ∂B transversally and the proof is complete.

The following fact will be helpful in concluding some nice results : Let B be a bounded convex body with smooth boundary ∂B in a Riemannian manifold M and let γ be a geodesic ray in M which intersects ∂B transversally, then γ intersects ∂B at least twice. The exponential map can be used successfully in proving this fact.

The study of convexity in complete, simply connected, Riemannian manifolds without conjugate points is more fruitful than general Riemannian manifolds. The reason is that any pair of points in a complete, simply connected, Riemannian manifold without conjugate points has a unique connecting geodesic segment.

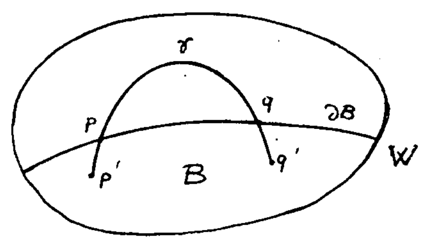
Proposition (III.1.3)

Let W be a complete, simply connected, Riemannian manifold without conjugate points and let $B \subset W$ be an open convex body. Then the closure $\bar{B} = B \cup \partial B$ of B is also a convex body.

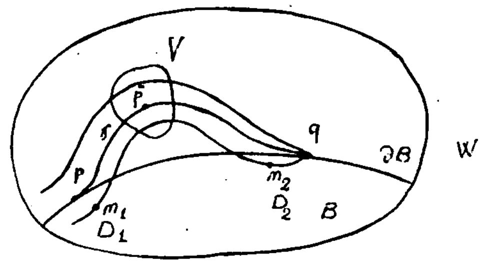
Proof :

Suppose that the closure $\bar{B}$ is not a convex body in W, then there exist two points p, $q \varepsilon \partial B$ whose joining geodesic, γ say, does not lie completely inside $\bar{B}$. Suppose that γ meets ∂B (i) transversally at p and q (ii) tangentially at p and q (iii) tangentially at p and transversally at q. We study each case separately.

(i) Slightly extend γ beyond p and q. Now there exist two points p', $q' \varepsilon B$ with connecting (unique) geodesic γ not included inside B which contradicts the hypothesis that B is a convex body.



(ii) Since γ is not included completely inside $\bar{B}$, then there exists a point $\bar{p} \varepsilon \gamma$ such that $\bar{p} \varepsilon W \setminus \bar{B}$. As $W \setminus \bar{B}$ is open, there exists a neighbourhood $V \subset W \setminus \bar{B}$ of $\bar{p}$. Draw the geodesic cone C with vertex q, axis γ and such that C goes through V. Now $C \cap B$ will be more than one component. Without loss of generality let $C \cap B$ have two components D_1 and D_2 as indicated in the following figure. Let $m_1 \varepsilon D_1$



and $m_2 \varepsilon D_2$. Now we have three possibilities :

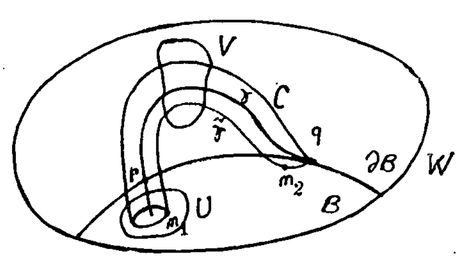
(a) If there is a geodesic $\tilde{\gamma} \varepsilon C$ from m_1 to m_2 we get a contradiction with the hypothesis that B is convex.

(b) If there is no geodesic in C from m_1 to any point of D_2 , we have either :

(b.1) If the geodesic in C from m_1 to q meets ∂B at q transversally, we get case (i).

(b.2) If the geodesic in C from m_1 to q meets ∂B at q tangentially, we get case (iii) below.

(iii) Slightly extend γ beyond p and let $p' \varepsilon \gamma$ be an interior point of B. The point p' has, therefore, a neighbourhood $U \subset B$. Similar to (ii) we can draw a geodesic cone C with base in U as shown in the following figure. Now it is clear that there exist two points $m_1, m_2 \varepsilon B$ with connecting geodesic $\tilde{\gamma}$ not included in B. This is again a contradiction to the hypothesis that B is convex.

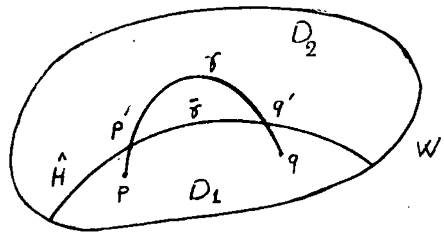


Proposition (III.1.4)

In complete, simply connected, Riemannian manifold W without conjugate points, totally geodesic hypersurface (when exists) divides W into two convex bodies.

Proof

Let $\hat{H}$ be a totally geodesic hypersurface in W and let $\hat{H}$ divide W into two parts D_1 and D_2 . Suppose that D_1 is not a convex body in W, hence there exist two points p, q∈D_1 , p ≠ q, whose joining geodesic segment γ does not lie completely inside D_1 .



Suppose that γ intersects $\hat{H}$ in p′,q′. As we know $\hat{H}$ is a geodesic submanifold at any of its points. Draw all geodesics from p′ spanning $\hat{H}$. Clearly one of these geodesics, $\bar{\gamma}$ say, will pass through q′ . This argument shows that there exist two different points p′, q′ ε W which are joined together with two different geodesics γ and $\bar{\gamma}$. This will contradict the hypothesis on W and so D_1 is a convex body in W. Similar argument shows that D_2 is also a convex body in W and the proof is complete.

Proposition (III.1.5)

Let $\bar{W}$ be a complete, simply connected, Riemannian manifold without focal points. Then geodesic balls are convex bodies in $\bar{W}$.

Proof :

Consider the geodesic ball

$$B_m(r) = B(m,r) = \{ x \in \bar{W} \;;\; d(m,x) < r \}$$

where m ε $\bar{W}$ and r > o is a real number. Suppose that B(m,r) is not a convex body in $\bar{W}$ then there exist p, qε B(m,r) , p ≠ q, such that the geodesic segment γ joining p and q is not included in B(m,r). Suppose that γ intersects ∂B(m,r) = S(m,r) at p′ and q′. Expand S(m,r) radially to the geodesic sphere S(m,$\bar{r}$) which contains γ and such that $\gamma \cap S(m,\bar{r}) \neq \Phi$. It is clear that $\gamma \cap S(m,\bar{r})$ will be a set of isolated points otherwise $\bar{W}$ will have focal points. It is also easy to see that all points in $\gamma \cap S(m,\bar{r})$ are critical points of the distance function $L_m(\gamma(t)) = d(m,\gamma(t))$ with index 1. This argument shows that γ has focal points on each geodesic from m to the points of $\gamma \cap S(m,\bar{r})$. This will give that $\bar{W}$ has focal points which is a contradiction. Hence $B(m,r)$ is a convex body in $\bar{W}$.

As proposition (III.1.5) is true for all geodesic balls in $\bar{W}$ which has no focal points and as horodisc is a limit of a sequence of geodesic balls, we have

Corollary (III.1.1)

In complete, simply connected, Riemannian manifolds without focal points, horodiscs are convex bodies.

Using proposition (III.1.2) together with the last corollary, we have

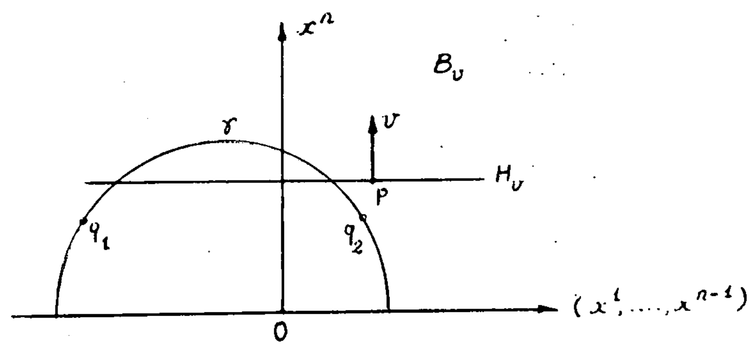
Corollary (III.1.2)

In complete, simply connected, Riemannian manifolds without focal points, horospheres are convex hypersurfaces.

Remark (1) :

In complete, simply connected, Riemannian manifold without focal points, the complement of a horodisc is not necessarily convex. The following counter example demonstrates this fact. Consider the half-space model $\mathbb{R}^{n+}$ of hyperbolic spaces. Let v be a vector in $\mathbb{R}^{n+}_p$ parallel to the x^n - axis. The horosphere H_v at p is the Euclidean hyperplane $x^n = x^n(p)$ and the upper region bounded by H_v is the corresponding horodisc B_v. (Look at the following figure). Clearly, the (unique) geodesic segment γ joining q_1, $q_2 \in \mathbb{R}^{n+} \setminus B_v$

is not included inside $\mathbb{R}^{n+} \setminus B_v$.



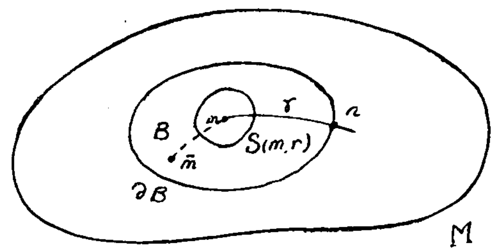
Proposition (III.1.6-a) :

Let M be a complete Riemannian (n+1)-manifold. Let B be a bounded convex body in M with smooth boundary ∂B. If $\bar{B} = B \cup \partial B$ is a convex body in M, then ∂B is diffeomorphic to the unit n-sphere S^n.

Proof :

For the proof of this proposition we use the fact that $\exp_m : M_m \to M$ is a diffeomorphism onto a neighbourhood of m for each $m \in M$. Consequently, a geodesic sphere S(m,r) in M with sufficiently small radius r such that S(m,r) is contained inside the normal coordinate neighbourhood of m is diffeomorphic to an Euclidean n-sphere of radius r.

Now let B and ∂B be as in the proposition. Using the last paragraph and considering m to be an arbitrary interior point of B, $m \in \mathrm{Int}(B)$, then there exists a small geodesic sphere $S(m,r) \subset B$ around m in M which is diffeomorphic to an Euclidean n-sphere of radius r. By Gauss lemma and since $B \cup \partial B$ is convex, then each minimal geodesic segment in $B \cup \partial B$ from m to any point of ∂B will intersect S(m,r) orthogonally and only once. This geodesic ray will intersect ∂B transversally (by proposition (III.1.2)).



Consider the mapping p_m : $\partial B \to S(m,r)$ which is (geometrically) the central projection through minimal geodesic rays from m. The convexity and boundedness of B guarantee that p_m is a homeomorphism. In fact p_m is so for any arbitrary interior point m.

It can be shown that the critical points of the map p_m^{-1} are the points of ∂B which are conjugate to m. To show that p_m^{-1} is a diffeomorphism, it is sufficient to show that ∂B has no conjugate points of m. Now let $n \varepsilon \partial B$ be a conjugate point of m. Slightly extend the minimal geodesic segment γ, joining m and n, beyond m to $\bar{m} \varepsilon \text{Int}(B)$. Draw the small geodesic sphere $S(\bar{m},\bar{r})$ which has the same properties as S(m,r) above. Taking into account the two facts that γ does not minimize distance between n and $\bar{m}$ and uniqueness of geodesics, we conclude that $p_{\bar{m}}^{-1}$ is not injective. This contradiction shows that ∂ B has no conjugate points to any interior point of B and hence by using the inverse function theorem p_m is a diffeomorphism.

Now lift S(m,r) into M_m by using $\exp_m^{-1} : M \to M_m$. Under the properties of the exponential map mentioned above, S(m,r) will be diffeomorphic to the Euclidean n-sphere $\tilde{S}(o,r) = \exp_m^{-1} S(m,r)$ of radius r and center $0 = \exp_m^{-1}$ m. Central project $\tilde{S}(o,r)$ onto $\tilde{S}(o,1) = S^n$ in M_m and call this map $\tilde{p}_o : \tilde{S}(o,r) \to \tilde{S}(o,1)$. It is clear that $\tilde{p}_o$ is a diffeomorphism as well. Composing all the above mentioned maps together, we have

$$\tilde{p}_o \circ \exp_m^{-1} \circ p_m \quad : \partial B \to S^n$$

As each map in the composition is a diffeormorphism, this completes the proof.

Remark (2)

1. The converse of the proposition (III.1.6-a) is not true in general since the geodesic circle of radius $> \ell\pi/2$, on a 2-dimensional cylinder of radius ℓ in E^3, which is diffeo-morphic to S^1 does not bound a convex body.

2. The previous proposition will be easier in proof in case of complete, simply connected, Riemannian manifold without conjugate points W and in this case we have :

Proposition (III.1.6-b)

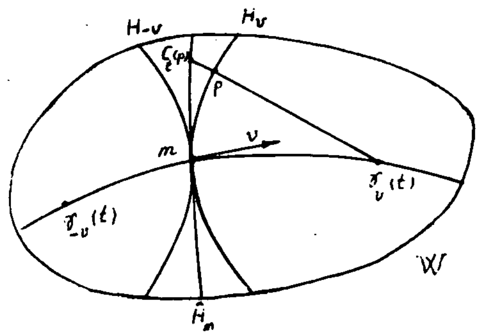
Let B be a bounded convex body in W. Then ∂B is diffeomorphic to S^n.

Proposition (III.1.7)

Let W be a complete, simply connected, Riemannian manifold without conjugate points and let $\hat{H}_m$ be a geodesic hypersurface at $m \in W$. Then $\hat{H}_m$ for each $m \in W$ lies between its tangential horospheres at m if and only if W has no focal points.

Proof

First let W have no conjugate points and such that for each $m \in W$, $\hat{H}_m$ lies between its tangential horospheres H_v and H_{-v} where $v \in W_m$ is a unit vector perpendicular to $\hat{H}_m$.



Let $\gamma_v(t)$ be a unit speed geodesic with $\gamma_v(0) = m$ and $\gamma_v'(0) = v$. Consider the following map $C_t : H_v \to \hat{H}_m$ defined as follows : draw the unique geodesic from $\gamma_v(t)$ to the point $p \in H_v$ and extend it to intersect $\hat{H}_m$ at $C_t(p)$. This map C_t is defined at least on small neighbourhoods of m on H_v. If we consider the distance function $L_{\gamma_v(t)}(x) = d(\gamma_v(t),x)$ for $x \in W$, then by restriction to H_v m is a critical point of $L_{\gamma_v(t)} \mid H_v$. In fact m is a minimum point. From the nature of the mapping C_t, we have

$$L_{\gamma_v(t)}(p) \leqslant L_{\gamma_v(t)}(C_t(p))$$

which means that the distance function $L_{\gamma_v(t)} \mid \hat{H}_m$ has a critical point at m which is also a minimum point. This argument shows that $\hat{H}_m$ has no focal points on the geodesic segment $\gamma_v(s)$, $s \in [o,t]$ (see lemma (III.2.7). Letting $t \to \infty$ we obtain that $\hat{H}_m$ has no focal points on $\gamma_{\pm v}(t)$, $t \geqslant o$. As the above discussion is true at each point in W, hence W has no focal points.

Conversely, let W have no focal points and let B_v and B_{-v} denote the horodiscs corresponding to $v \in SW$ which are convex bodies in W by corollary (III.1.1). As $H_v = \partial B_v$ and $H_{-v} = \partial B_{-v}$ are both convex hypersurfaces in W then if $p \in H_v \cap H_{-v}$, any geodesic ray from p and tangent to both H_v and H_{-v} lies globally outside B_v and B_{-v}. As this is true for every geodesic of this kind we have that the tangent geodesic hypersurface at any point in $H_v \cap H_{-v}$ lies between

H_v and H_{-v}. Because this is true for each $v \in SW$ the proof is complete.

Corollary (III.1.3)

Let W be a complete, simply connected, Riemannian manifold without conjugate points. Then W has no focal points if and only if each horodisc is a convex body.

Proposition (III.1.8)

Let $\bar{W}$ be as in proposition (III.1.5). Then the interior and exterior of each horodisc are convex bodies if and only if $\bar{W}$ is the Euclidean space.

Proof :

Let H_v and H_{-v} denote, as before, the horospheres corresponding to v, $-v \in S\bar{W}$, respectively. As the horodisc B_v is a convex body then H_v is a convex hypersurface. Let p, $q \in H_v$ be two arbitrary points. By the convexity of $\bar{B}_v$ and $\bar{W} \setminus B_v$ (proposition (III.1.3), p and q should be joined by two minimal geodesic segments γ_1 and γ_2 such that $\gamma_1 \subset \bar{B}_v$ and $\gamma_2 \subset \bar{W} \setminus B_v$. By conditions on $\bar{W}$ these two geodesics coincide and both of them lie in H_v. Repeating the same argument with all points of H_v we obtain that H_v is a totally geodesic hypersurface in $\bar{W}$. Similarly, H_{-v} is a totally geodesic hypersurface in $\bar{W}$. Taking into account that a geodesic is defined uniquely by its initial point and velocity we get that $H_v \equiv H_{-v}$.

Since the above argument is true at each point of $\bar{W}$ and by (section 2 - chapter 0) $\bar{W}$ is a space form. Now $\bar{W}$ could not be a hyperbolic space by remark 1. Also, $\bar{W}$ is not isometric to a sphere since the latter has conjugate points. The only possibility is that $\bar{W}$ is the Euclidean space.

The converse is easy from the properties of the Euclidean space.

Definition

In a complete Riemannian manifold M two unit speed geodesics α , β are called bi-asymptotic if there exists a real number c, $o < c < \infty$, such that $d(\alpha(t), \beta(t)) \leqslant c$ for all $t \in \mathbb{R}$.

In Euclidean spaces any two asymptotic geodesics are bi-asymptotic while this is no longer true for hyperbolic spaces. In the next few propositions we prove some results concerned with the concept just defined. The next proposition which has been proved by J.H. Eschenburg [14] is helpful.

Proposition (III.1.9)

Let W be a complete, simply connected Riemannian manifold without conjugate points. Assume that the stable Jacobi tensor of W is continuous. Then for each $v \in SW$

(i) $H_v \cap H_{-v} = \bar{B}_v \cap \bar{B}_{-v}$ (ii) $H_v \cap H_{-v}$ is a connected set.

(iii) Exactly the geodesics intersecting H_v perpendicularly at points of $H_v \cap H_{-v}$ are bi-asymptotic to γ_v.

Corollary (III.1.4)

Let W be as in (proposition (III.1.9). Let W in addition have the property that no two asymptotic geodesics are bi-asymptotic. Then $H_v \cap H_{-v}$ for any $v \in SW$ is a single point.

Proposition (III.1.10)

Let W be as in corollary (III.1.4). Then $b_v| H_{-v}$ has no maximum or minimum points except the point $H_v \cap H_{-v}$, $v \in SW$.

Proof :

It is clear that under the hypothesis of corollary (III.1.4), $b_v \mid H_{-v} \leqslant o$ and the point $H_v \cap H_{-v}$ is a maximum point. Suppose that

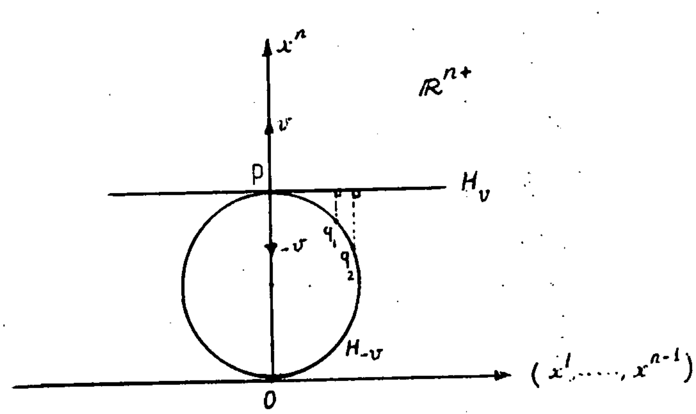
$b_v | H_{-v}$ has $q \varepsilon H_{-v}$ as another maximum point. Then there exists a geodesic segment γ from q to H_v which intersects both H_v and H_{-v} orthogonally. This means that the geodesic ray γ is bi-asymptotic to γ_v which is a contradiction to the hypothesis on W. The same argument can be carried out if q is a minimum point which completes the proof.

Lemma (III.1.1)

In hyperbolic spaces $b_v | H_{-v}$ is strictly decreasing along any geodesic of H_{-v} from the point $H_v \cap H_{-v}$. Moreover, $b_v | H_{-v}$ runs from 0 to $-\infty$.

Proof :

The half-space model IR^{n+} is the most convenient one to carry out the proof of this lemma. Without loss of generality consider the unit vector $v \varepsilon IR_p^{n+}$, $p \varepsilon IR^{n+}$, such that v is parallel to x^n-axis in the positive direction. Since horosphere in hyperbolic space is a complete, flat manifold, it is isometric to E^{n-1}. This shows that geodesics of H_v are straight lines in the plane $x^n = x^n(p)$. It can be shown that the geodesics of H_{-v} from p are great circles. Consider $\gamma(s)$ to be one of these geodesics which is parametrized by arc-length s such that $\gamma(o) = p$. Let $q_1 = \gamma(s_1)$ and $q_2 = \gamma(s_2)$ where $s_2 > s_1$. It is clear from the figure below that $o > b_v(q_1) > b_v(q_2)$ and $b_v(p) = o$ which completes the proof.



Proposition (III.1.11)

In hyperbolic spaces, the horosphere $H_v = b_v^{-1}(o)$ intersects all the horospheres $b_v^{-1}(k)$ for $k \leqslant o$.

Proof :

Under the same notations given in the proposition let $r \leqslant o$ be a finite real number. Following the same argument as in lemma (III.1.1) we can show that $b_v(b_{-v}^{-1}(r))$ is strictly decreasing along geodesics of $b_{-v}^{-1}(r)$ from the point $\gamma_v(r)$ where $\gamma_v(s)$ is an arc-length parametrized geodesic such that $\gamma_v(o) \in H_v$ and $\gamma_v'(o) = v$. Moreover, $b_v(b_{-v}^{-1}(r))$ runs from $-r$ to $-\infty$. By the continuity of $b_v \mid H_{-v}$ we conclude that $b_{-v}^{-1}(r)$ has points with $b_v(b_{-v}^{-1}(r)) = o$ which are themselves points of $b_v^{-1}(o) = H_v$. Since r is an arbitrary non-positive real number the proof is now complete.

We finish off this section by mentioning the effect of geodesic mappings on convex bodies. In fact geodesic mapping may be defined as follows:

Definition

A homeomorphism $\Phi : M \to N$ from the manifold M into the manifold N is called a geodesic mapping if for every geodesic γ of M the composition $\Phi \circ \gamma$ is a reparametrization of a geodesic of N.

Notice that a geodesic mapping $\Phi : M \to N$ takes totally geodesic submanifolds of M to totally geodesic submanifolds of N. When M and N are of the same dimensions Φ turns out to be a diffeomorphism. It can be shown that the inverse of a geodesic mapping between manifolds of the same dimensions is also a geodesic mapping. For more details about this sort of mappings see [29] .

Proposition (III.1.12)

Let W_1 and W_2 be two complete, simply connected, Riemannian n-manifolds without conjugate points and let $\Phi: W_1 \to W_2$ be a geodesic mapping. Then Φ takes convex bodies in W_1 to convex bodies in W_2.

Proof:

Let B be a convex body in W_1, then every two different points $p, q \in B$ have their unique connecting geodesic segment γ to be contained completely inside B. Let $\tilde{B} = \Phi(B)$. Using the map Φ it is clear that $\tilde{\gamma} = \Phi \circ \gamma$ is a geodesic in W_2 which is contained inside $\tilde{B}$ and joining $\Phi(p)$ and $\Phi(q)$. Since there is no other geodesic segment of W_2 joining $\Phi(p)$ and $\Phi(q)$ we conclude that $\tilde{B}$ is a convex body in W_2.

Using proposition (III.1.3) together with the last proposition, we have

Corollary (III.1.4)

Let W_1, W_2 and Φ be as in proposition (III.1.12), then Φ takes convex hypersurfaces of W_1 to convex hypersurfaces of W_2.

Section 2 : Tight and taut immersions

(III.2.0) Introduction

In this section we try to generalize and study the concepts of taut and tight immersions in complete, simply connected, Riemannian manifolds without conjugate points. Actually, tight and taut immersions have been introduced for the first time by S.S. Chern and R.K. Lashof [10,11] and studied by N. Kuiper [20,21] and many others [3,5,6,32]. For instance, T.E. Cecil and P.J.Ryan [6] generalized the above mentioned concepts for hyperbolic spaces and they proved many related results. A successful trial has been carried out by J. Bolton [3] when he generalized the concept of tight immersion for complete, simply connected, Riemannian manifolds without conjugate points and he proved two theorems corresponding to those which have been already proved in [10] in Euclidean spaces.

The concepts of (spherical)-two-piece property have been generalized as well. The relations between these concepts and tight and taut immersions have been found.

For the present section we need oftenly the Morse inequalities which can be stated as follows [32] :

Let f be a C^2 function on a compact C^∞ n-manifold M with no degenerate critical points. Let μ_λ be the number of critical points of f on M with index λ and let β_λ be the λ-dimensional Betti number of M. Then we have the following inequalities :

$$\mu_0 \geqslant \beta_0$$

$$\mu_1 - \mu_0 \geqslant \beta_1 - \beta_0$$

$$\mu_i - \mu_{i-1} + \ldots + (-1)^i \mu_0 \geqslant \beta_i - \beta_{i-1} + \ldots + (-1)^i \beta_0 , i \leqslant i \leqslant n-1$$

$$\mu_n - \mu_{n-1} + \ldots + (-1)^n \mu_0 \geqslant \beta_n \quad \beta_{n-1} + \ldots + (-1)^n \beta_0$$

In particular we have $\mu_k \geqslant \beta_k$ for all k.

(III.2.1) Notations and definitions :

Let M be a connected m-manifold without boundary.

Definition (1.1)

We say that the C^2 function $\phi : M \to \mathbb{R}$ is a T-function if

(i) ϕ is non-degenerate.

(ii) $M_r(\phi) = \{ p \in M;\ \phi(p) \leqslant r \}$ is compact for all $r \in \mathbb{R}$.

(iii) There exists a field F such that for all $r \in \mathbb{R}$ and integers k the number of critical points of ϕ with index k which lie in $M_r(\phi)$ is equal to $\dim H_k(M_r(\phi)\,;F)$ where $H_k(M_r(\phi)\,;F)$ denotes the $k^{\underline{th}}$ homology group of $M_r(\phi)$ with respect to the field F.

Let us now consider an immersion $f : M \to W$ where W is a complete, simply connected, Riemannian n-manifold without conjugate points and from now on let W always have these properties. In addition, let all maps be C^k, $k \geqslant 2$, unless otherwise stated.

Definition (1.2)

An immersion $f : M \to W$ is taut if, for every $x \in W$ the distance function $L_x : M \to \mathbb{R}$ defined by $L_x(p) = d(x,f(p))$ is either degenerate or a T-function.

Definition (1.3)

An immersion $f : M \to W$ is called proper if the inverse image under f of every compact subset is compact.

The definition (1.2) of tautness has been reformulated in terms of homology homomorphisms by N. Kuiper [20] as follows:

Lemma (III.2.1)

Let $\phi : M \to \mathbb{R}$ be a real-valued function such that ϕ is non-degenerate and $M_r(\phi)$ is compact for all $r \in \mathbb{R}$. Then ϕ is a T-function if and only if there exists a field F such that the map $H_*(M_r(\phi)\ ;\ F) \to H_*(M\ ;\ F)$ induced by the inclusion $M_r(\phi) \subset M$ is injective for all $r \in \mathbb{R}$.

Let $B = B(m,r)$ and $S = S(m,r) = \partial B(m,r)$ denote the open geodesic ball and geodesic sphere centered at m and have radius $r \in \mathbb{R}$, respectively. Now lemma (2.8)[5] can be modified in the following way :

Lemma (III.2.2) :

Let $f : M \to W$ be a proper immersion. Then f is taut if and only if there exists a field F such that for every closed geodesic ball $\bar{B} \subset W$ for which f is transversal to the round geodesic sphere ∂B the map $H_*(f^{-1}(\bar{B})\ ;\ F) \to H_*(M\ ;\ F)$ induced by inclusion is injective.

Proof :

Before being involved in the proof we notice that for the closed geodesic ball $\bar{B} = \bar{B}(x,r)$ we have

$$f^{-1}(\bar{B}) = M_r(L_x) = \{\, p \in M\ ;\ L_x \circ f(p) \leqslant r \,\}$$

Also f is transversal to ∂B if and only if r is not a critical value of L_x. We shall write for simplicity $M(x,r) = M_r(L_x) = f^{-1}(\bar{B})$ and $J(x,r) : H_*(M(x,r)\ ;\ F) \to H_*(M;F)$ the homology map induced by the inclusion $M(x,r) \subset M$.

(a) Firstly, let f be a taut immersion, then every L_x, $x \in W$, is either degenerate or T-function. Suppose that L_x, for some $x \in W$, is a T-function (i.e. $x \notin C_f$) then by lemma (III.1.1) $J(x,r)$ is an injective map.

For the second possibility where L_{x_0}, for some $x_0 \in W$, is a degenerate function (i.e. $x \in C_f$) we make use of the exponential map of W as it is a global diffeomorphism. If we consider $\exp_p^{-1} : W \to W_p$, for $p \in W$, it is easy to see that the point q of W_p is a critical point of $\tilde{L}_p = L_o \circ \exp_p^{-1}$ if and only if $\exp_p q$ is a critical point of L_p. The contact order is also preserved by the exponential map. Moreover, the signature of L_p and $\tilde{L}_p$ are the same.

Taking all these properties into account we can transfer the situation from W to E^{m+k} (m+k = dimW) and so we can benefit from the proof of the corresponding lemma in [5].

(b) For the converse, suppose that J(x,r) is injective whenever r is not a critical value of L_x for any $x \in W$. By lemma (III.1.1) we have to show that J(x,r) is still injective even if r is a critical value of L_x but L_x is non-degenerate. It is known [22] that if L_x is non-degenerate, its critical points are isolated and hence there is only a finite number in M(x,r). Consequently, given a critical value r_o of L_x, there is $r > r_o$ so that r is not a critical value of L_x and $M(x,r_o) \subset M(x,r)$ is a strong deformation retract. Since J(x,r) is injective then $J(x,r_o)$ is also injective which completes the proof.

The fact that horospheres can be regarded as "geodesic spheres of infinite radius" makes the following generalization of tightness usual.

Definition (1.4) :

Let $f : M \to \tilde{W}$ be an immersion where $\tilde{W}$ is a complete, simply connected, Riemannian manifold without conjugate points, with bounded asymptote and with sectional curvature bounded from below. Then f is called h-tight if, for every $v \in T(M)^{\perp}$ the function $b_v \circ f$ is either degenerate or a T-function.

From now on $\tilde{W}$ will denote a complete, simply connected, Riemannian manifold without conjugate points, with bounded asymptote and with sectional curvature bounded from below. In a natural way, lemma (2.9) in [5] can be restated for $\tilde{W}$ as follows :

Lemma (III.2.3) :

Let $f : M \to \tilde{W}$ be an immersion of a compact manifold M. Then f is h-tight if and only if there exists a field F such that for every horosphere H_v, $v \in S\tilde{W}$, to which f is transversal, the map $H_*(f^{-1}(\tilde{W} \backslash B_v); F) \to H_*(M; F)$ induced by the inclusion is injective, where B_v is - as before - the open horodisc associated with v.

The proof of the above lemma depends originally on the following result which has been proved in [3] through applying Sard's theorem to the generalized Gauss map (see § 3) : the set $\{ v \in S_p\tilde{W} : b_v$ is non-degenerate $\}$ is dense in $\tilde{W}$.

(III.2.2) - General theorems on taut immersions

We start by proving the following theorem

Theorem (III.2.1)

Any taut immersion $f : M \to W$ is an imbedding.

Proof:

Since the immersion $f : M \to W$ is proper we show that f is injective. Suppose that f is not injective hence there exist two points $p, q \in M$, $p \neq q$, such that $f(p) = f(q)$. Then there is a closed geodesic ball $\bar{B}(f\,(p), r)$ with sufficiently small radius r such that f intersects ∂B transversally and p, q lie in different components of $f^{-1}(\bar{B})$. Since M is connected $H_0(M;F)$ has dimension 1 while $H_0(f^{-1}(\bar{B});F)$ has dimension, at least, 2. So the map $H_0(f^{-1}(\bar{B});F) \to H_0(M;F)$ induced by the inclusion cannot be injective and hence by lemma (III.2.2) f is not taut which is a contradiction. This completes the proof of the theorem.

Definitions (2.1)

(a) An immersion $f : M \to W$ is said to have the spherical two-piece property (STPP) if for any geodesic sphere S or horosphere H_v, $v \in SW$, the set $f^{-1}(W \setminus S)$ or $f^{-1}(W \setminus H_v)$ has at most two connected components.

(b) An immersion $f : M \to \tilde{W}$ has the h-two-piece property (hTPP) if for any horosphere H_v, $v \in S\tilde{W}$, the set $f^{-1}(\tilde{W} \setminus H_v)$ has at most two connected components.

Notice that if $W = \tilde{W} = E^{n+k}$ the above two definitions reduce to the STPP and TPP which have been introduced by T.E. Banchoff [31]. Adopting definitions (2.1- a,b) we prove the following :

Proposition (III.2.1)

Any taut imbedding $f : M \to W$ of a compact manifold M has the STPP.

Proof:

Since M is compact, then L_x, for any $x \in W$ when restricted to M, attains its maximum and minimum. By lemma (III.2.2), which is true for the complement $W \setminus B$ of a round open geodesic ball B, together with the fact that $H_0(f^{-1}(\bar{B}) ; F) \to H_0(M ; F)$ is injective then $f^{-1}(\bar{B})$ is connected and so is $f^{-1}(W \setminus B)$, hence the result.

In a similar way of proof of the last proposition and using lemma (III.2.3) we can prove the following:

Proposition (III.2.2)

Any h-tight immersion $f : M \to \tilde{W}$ of a compact manifold M has hTPP.

For the following propositions we need the next lemma

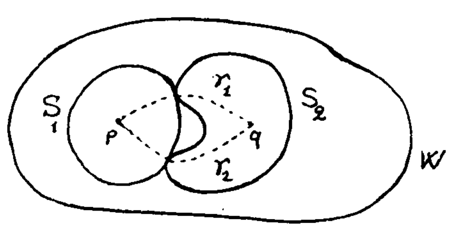
Lemma (III.2.4)

Let W (as before) be a complete, simply connected, Riemannian manifold without conjugate points. Let $S_1 = S(p,r_1)$, $S_2 = S(q,r_2)$ be two different geodesic spheres. Then if S_1 is tangent to S_2 at all points of intersection, $S_1 \cap S_2$ is a single point.

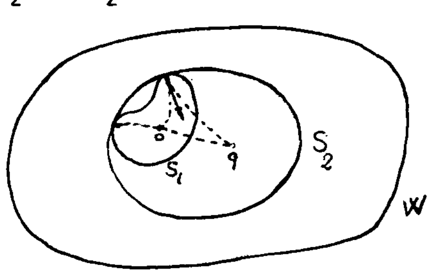
Proof:

First of all we notice that, under the hypotheses mentioned in the lemma, the open geodesic balls $B_1 = B(p,r_1)$ and $B_2 = B(q,r_2)$ have the property that $B_1 \cap B_2 = \Phi$ if $p \notin B_2$, $B_1 \cap B_2 = B_1$ if $p \in B_2$ and finally $B_1 \cap B_2 = B_2$ if $q \in B_1$.

Now suppose that $B_1 \cap B_2 = \Phi$ and $S_1 \cap S_2$ is a set which contains more than one point. In this case W should have, at least, two different geodesics γ_1, γ_2 from p to q intersecting twice which contradicts the hypotheses on W.



Secondly, suppose that $B_1 \cap B_2 = B_1$ and let $S_1 \cap S_2$ - as before - be a set containing more than one point. In this case the uniqueness of geodesics will be no longer true as indicated in the following figure. If $B_1 \cap B_2 = B_2$ we have a similar situation.



The last lemma becomes false if W is a general Riemannian manifold. The reason is that in S^2 geodesic spheres $S_1 = S(N, \pi/2)$, $S_2 = S(S, \pi/2)$ where N, S denote the north and south poles, respectively, intersect in the equator S_1 or S_2.

In the following $\bar{W}$ denotes a complete, simply connected, Riemannian manifold without focal points.

Proposition (III.2.3)

In $\bar{W}$ geodesic spheres have the STPP.

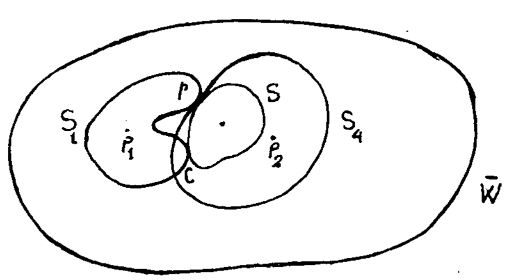
Proof :

Assume that there exists a geodesic sphere $S_1 = S(p_1,r_1)$, $p \in W$, which does not have the STPP. Consequently, there exists another geodesic sphere $S_2 = S(p_2,r_2)$, $p_2 \in \bar{W}$, or a horosphere H_V, $v \in S\bar{W}$, which divides S_2 into more than two connected components. Without loss of generality the number of components of S_1 contained inside B_2 (or B_V) may be assumed two.

Firstly, we consider the case of geodesic sphere S_2. Now contract S_2 radially keeping p_2 fixed. Two possibilities might happen :

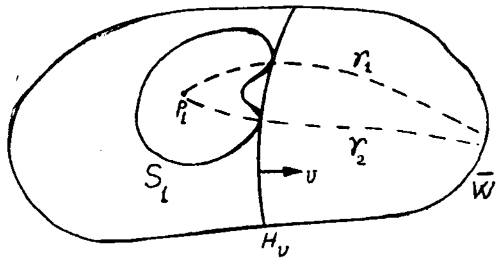
(i) S_2 contracts to a geodesic sphere $S_3 = S(p_2,r)$ tangent to S_1 in more than one point which is impossible by the last lemma.

(ii) S_2 contracts to a geodesic sphere $S_4 = S(p_2, \bar{r})$ as follows:



Since the component C of S_1 inside S_4 is compact, there exists a sequence of geodesic spheres tangent to S_1 at p whose limit is a geodesic sphere S tangent to S_1 in more than one point. This is again a contradiction to lemma (III.2.4)

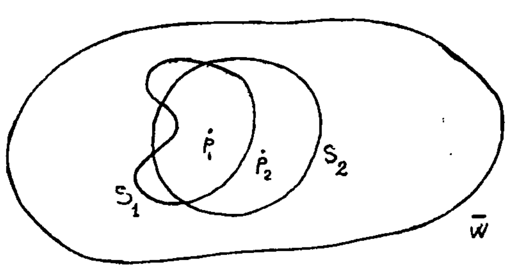
Following similar argument with H_V instead of S_2 we reach to the following situation.

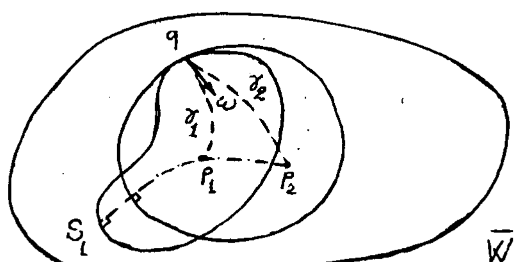


In this case there exist two different geodesic rays γ_1 and γ_2 from p_1 which are asymptotic to each other. This is a contradiction to theorem (3.3) - chapter 0.

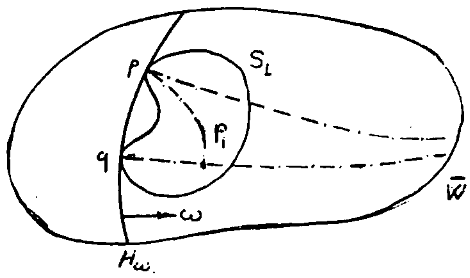
To complete the proof we should consider the case when the connected components of S_1 outside B_2 (or B_v) are two.

Now, suppose that $\bar{W}\setminus B_2$ contains two connected components of S_1 as in the following figure. Clearly, the centers p_1 and p_2 of S_1 and S_2, respectively, are different points. Keep p_2 fixed and let S_2 expand radially till it has a point of tangency with S_1, q say. In this case we have two different geodesic segments γ_1 and γ_2 from q to p_1 and p_2, respectively, with the same initial velocity ω. This is now a contradiction to the uniqueness fact of geodesics.





When considering the case of H_v instead of S_2 , for some $v \varepsilon S\bar{W}$, and using a similar technique we arrive to such a situation given in the following figure.



Clearly, the uniqueness of geodesics is broken at p or $q \varepsilon S_1 \cap H_\omega$ for some $\omega \varepsilon S\bar{W}$. This is again a contradiction which completes the proof of the proposition.

In the manifold $\bar{W}$, which has no focal points, horospheres do not have the STPP in general. Putting some restriction on $\bar{W}$ such as, in $\bar{W}$ there are no two bi-asymptotic geodesics, we can show that H_v, for each $v \varepsilon S\bar{W}$, has the STPP.

Theorem (III.2.2) :

Any taut immersion $f : M \to \tilde{W}$ is h-tight. (M is compact)

Proof:

In view of lemma (III.2.3) it is sufficient to consider a horodisc B_v, $v \varepsilon S\tilde{W}$, bounded by the horosphere $H_v = \partial B_v$ such that f is transversal to H_v and show that if f is taut then there is some closed geodesic ball $\bar{B} = \bar{B}(x,r)$ such that f is transversal to ∂B and $f^{-1}(\bar{B}_v) \subset f^{-1}(\bar{B})$ is a strong deformation retract. This can be carried out similar to theorem (2.3) [5] taking into account that $\tilde{W} \setminus C_f$ is dense in $\tilde{W}$.

Lemma (III.2.5)

Let $f : M \to W$ be an immersion and let $U \subset M$ be open. For $x \varepsilon W$ let L_x have precisely one critical point $p \varepsilon U$ and let p be non-degenerate with index ℓ. Then there exists a neighbourhood E of x in W such that if $y \in E$ then L_y has a critical point q in U and q is non-degenerate with index ℓ.

Proof :

Since L_x has precisely one critical point $p \in U$ which is non-degenerate, then $x = \exp_{f(p)} n$ for some $n \in T(M)^{\perp}$ and x is not a focal point. As the focal set C_f is closed then there exists an open neighbourhood E of x which contains no focal points. We denote the pre-image of E in $T(M)^{\perp}$ by V, hence $\exp_{f(p)}|V$ is a diffeomorphism.

If $y \in E$, then $y = \exp_{f(q)} u$ for some $u \in V$ whose base point is q. If we join u to n in V, we see that the connecting arc will pass through no critical normals which ensures that the index of q as a critical point of L_y is also ℓ (see [22] p.85 and lemma (III.2.7) below).

Lemma (III.2.6)

Let $f : M \to W$ be a taut immersion and suppose that for $(p,n) \in T(M)^{\perp}$ there is no focal point of the form $\exp_{f(p)} tn$, $o < t < 1$. Let $x = \exp_{f(p)} n$. Then $L_x(q) \geqslant L_x(p)$ for all $q \in M$. Further if (p,n) is not a critical normal, equality holds only when p = q.

Proof :

Suppose that $(p,n) \in T(M)^{\perp}$ is not a critical normal (i.e. $x = \exp_{f(p)} n \notin C_f$). This means that f(p) is a non-degenerate critical point of the function L_x with 0 index. This argument shows that f(p) is an isolated point in $M_{\|n\|}(L_x)$. Since the immersion is taut, $M_{\|n\|}(L_x)$ must be connected and hence $M_{\|n\|}(L_x)$ consists of f(p) alone and so $L_x(q) > L_x(p)$ for all $q \in M$ and $q \neq p$.

Suppose that (p,n) is a critical normal (i.e. $x = \exp_{f(p)} n \in C_f$). We repeat the above argument but with $M_{t\|n\|}(L_{x_t})$ where $x_t = \exp_{f(p)} tn$,

$o < t < 1$ and so we get that f(M) lies outside the open geodesic ball $B(x_t , t\|n\|)$. Hence f(M) lies outside the union of such open balls which implies that f(M) lies outside the open geodesic ball $B(x, \|n\|)$ i.e. $L_x(q) \geq L_x(p)$ for all $q \in M$.

If we consider the complement case when the index is maximal, we have the following :

Lemma (III.2.7)

Let $f : M \to W$ be a taut immersion of a compact m-manifold M and suppose that $(p,n) \in T(M)^{\perp}$ is such that the multiplicities of the focal points of the form $\exp_{f(p)} tn$, $o < t \leq 1$, is m, then $L_x(q) \leq L_x(p)$ for all $q \in M$, where $x = \exp_{f(p)} n$. Further if (p,n) is not a critical normal, equality holds only when p = q.

Proof :

The proof of this lemma depends on the fact that any normal geodesic to f(M) from f(P) could not have more than m focal points (taken with their multiplicities). Actually the proof of this result is not so easy and therefore we give it in the following. The proof depends deeply on the assumption that W has no conjugate points.

Let τ be a normal geodesic to f(M) at $f(p) = \tau(o)$. Let $\mathcal{L}$ be the vector space of all C^∞ broken vector fields along τ vanishing at $\tau(b)$, $b \neq o$, and perpendicular to τ. Let $\mathcal{L}^-$ be a subspace of $\mathcal{L}$ with maximal dimension on which the index form I is negative semi-definite.

The vector space $\mathcal{L}^-$ has an important property which has been given in the following lemma.

Lemma (α)

Let V_1 and V_2 be two different vector fields in $\mathcal{L}^-$. Then $V_1(o) \neq V_2(o)$.

Proof:

Assume that $V_1(o) = V_2(o)$. Then the vector field $V_3 = V_1 - V_2$ which is in $\mathcal{L}^-$ has the property that $V_3(b) = o$ and $V_3(o) = o$. Since τ has no conjugate points, then by the Morse index theorem we have $I(V_3,V_3) > o$ which is a contradiction showing that $V_3(t) = o$ for all $t \in [o,b]$, i.e. $V_1 = V_2$. This contradiction again gives that $V_1(o)$ should be different from $V_2(o)$.

Consider the map $g : \mathcal{L}^- \to M_p$ which assigns to each element $V \in \mathcal{L}^-$ its initial value $V(o)$. In the light of the above lemma we can easily see that :

Corollary :

The map $g : \mathcal{L}^- \to M_p$, defined above, is injective.

This last corollary shows that $\dim \mathcal{L}^- \leqslant m$. The following lemma which has been proved in [2] is needed for the proof.

Lemma (β)

Suppose that there is no conjugate points of $\tau(o)$ on $\tau((o,b])$. For $V \in \mathcal{L}$ there is a unique Jacobi field Y such that $Y(o) = V(o)$ and $Y(b) = o$. Moreover, $I(V,V) \geqslant I(Y,Y)$ and equality occurs if and only if $V = Y$.

Consider $\mathcal{J}$ to be the vector space of all Jacobi vector fields along τ vanishing at $\tau(b)$ and have their initial values in M_p. Since $\tau((o,b]))$ contains no conjugate points, we can show easily that

$$g|\mathcal{J} : \mathcal{J} \to M_p$$

is a linear isomorphism onto M_p.

Using lemma (β) we can define the map $h : \mathcal{L}^- \to \mathcal{J}$ by $h(V) = Y$ where $Y(o) = V(o)$. It is clear that the map h is injective.

If we let $\mathcal{J}^-$ denote the subspace of $\mathcal{J}$ on which I is negative semi-definite, we conclude from lemma (β) and taking into account that $\mathcal{J}^- \subset \mathcal{L}^-$ that $h(\mathcal{L}^-) = \mathcal{J}^-$ (i.e. $\dim \mathcal{L}^- = \dim \mathcal{J}^-$). Consequently,

$$\text{index of } I|\ \mathcal{L} \times \mathcal{L} = \text{index of } I\ |\ \mathcal{J} \times \mathcal{J} \leq m.$$

Now the index of $I\ |\mathcal{J} \times \mathcal{J}$ is, by definition, the index of the hessian matrix of the length function when restricted to the space of all geodesic paths from $\tau(b)$ to $f(M)$. Using the fact that any point of $f(M)$ has a unique connecting geodesic segment with $\tau(b)$, we can identify each geodesic segment from $\tau(b)$ with its point of intersection with $f(M)$. Identifying each $Y \in \mathcal{J}$ with its initial value in M_p it becomes clear that index of $I|\ \mathcal{J} \times \mathcal{J}$ is exactly the index of the hessian of the distance function $L_{\tau(b)} : M \to \mathbb{R}$ at $\tau(o)$ as a critical point. (i.e. the index of the point $\tau(o)$ as a critical point of $L_{\tau(b)}$ equals the number of focal points of $f(M)$ on $\tau((o,b\,])$ which is at most $m = \dim M$). Since $\tau(b)$ is an arbitrary point on τ the proof is complete.

Now, we return back to the proof of lemma (III.2.7)

Suppose that $(p,n) \in T(M)^\perp$ is not a critical normal (i.e. $x = \exp_{f(p)} n \notin C_f$), hence $f(p)$ is a non-degenerate critical point of the function L_x with maximal index m (i.e. p is a maximum point of L_x). Since M is compact then the distance function L_y, for any $y \in W$, attains its maximum $L(y)$ and minimum $\ell(y)$, i.e. $\ell(y) \leq L_y(q) \leq L(y)$ for each $q \in M$. Clearly, the function $\mathcal{L}_x = L(x) - L_x$ has p as a non-degenerate critical point with index 0 (i.e. p is a minimum point of $\mathcal{L}_x$).

Since f is taut then p is an isolated point in $M_{L(x)-\|n\|}(\mathcal{L}_x)$ and $M_{M(x)-\|n\|}(\mathcal{L}_x)$ is connected. Consequently, $M_{M(x)-\|n\|}(\mathcal{L}_x)$ consists of $f(p)$ alone and so $\mathcal{L}_x(q) > \mathcal{L}_x(p)$ for all $q \in M$.

Equivalently, $L_x(q) < L_x(p)$ for each $q \in M$.

If x is a focal point then there is no other focal point of M along the geodesic $\gamma(t) = \exp_{f(p)} tn$, $t \geq o$ for $t > 1$. Consider $\bar{x} = \exp_{f(p)}(1+\varepsilon)n$ for sufficiently small positive real number ε. It is clear that f(p) is a critical point of $L_{\bar{x}}$ with index m. Similar argument shows that $L_{\bar{x}}(q) < L_{\bar{x}}(p)$ for all $q \in M$. Taking the limit as $\varepsilon \to o$, we obtain that $L_x(q) \leq L_x(p)$ for all $q \in M$ which completes the proof.

Corollary (III.2.1)

Let $f : M \to W$ be a taut imbedding and let $p \in M$ have a normal ray $\gamma(t) = \exp_{f(p)} tn$, $t \geq o$, $n \in T(M)^{\perp}$, on which there are no focal points of f(M). Then f(M) lies in the closed region $W \setminus B_n$ bounded by the horosphere H_n through f(p) with inward pointing normal n.

Proof :

The proof is a direct conclusion of lemma (III.2.6). Consider the point $\gamma(\bar{t}) = \exp_{f(p)} \bar{t}n$, $\bar{t} \geq o$. It is clear that f(M) does not have any focal point on the geodesic segment from $\gamma(o)$ to $\gamma(\bar{t})$ and consequently f(M) does not intersect the open geodesic ball $B(\gamma(\bar{t}), \bar{t} \|n\|)$. Since this is true for each $\bar{t} \geq o$ and as the union of all the open geodesic balls $B(\gamma(\bar{t}), \bar{t} \|n\|)$ is the open horodisc B_n we obtain the required result.

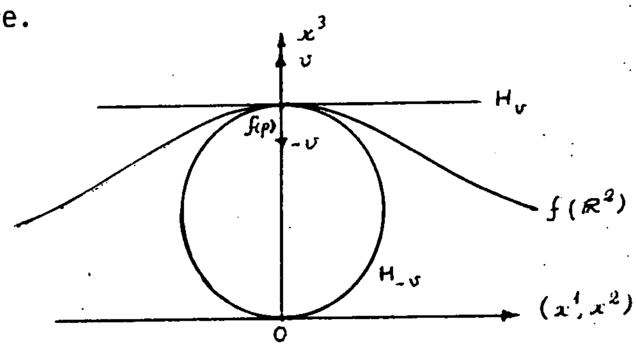
In a natural way we can generalize the concept of substantial immersion (imbedding) in Euclidean spaces to manifolds without conjugate points as follows :

Definition (2.2)

An immersion (imbedding) $f : M \to W$ is called h-substantial if f(M) is not included in any horosphere of W.

It has been proved in [5] p.709 that if $f : M \to E^n$ is a substantial taut imbedding then there is a critical normal on every normal line. This result is no longer true, generally, in manifolds without conjugate points according to the following example :

Consider the h-substantial taut imbedding $f : \mathbb{R}^2 \to \mathbb{R}^{3+}$ in the half-space model of hyperbolic 3-spaces as indicated in the following figure.



It is clear that the normal geodesic at f(p) which coincides with x^3-axis has no focal points of $f(\mathbb{R}^2)$.

Definition (2.3)

(a) An immersion $f : M \to W$ is called spherical if f(M) lies on a geodesic sphere of W.

(b) An umbilic of f(M) is defined to be a focal point of f(M) with multiplicity = dim M.

Corollary (III.2.2) :

Let $f : M \to W$ be a taut imbedding with an umbilic. Then f is spherical.

Proof :

Consider the point $x \in W$ such that $x = \exp_{f(p)} n$, for $p \in M$ and $n \in T(M)^{\perp}$, to be the umbilic point of f(M). This means that x is a focal point of f(M) with multiplicity equal to dim M. Applying lemmas (III.2.6-7) we have that

$$L_x(q) \geqslant L_x(p)$$

and
$$L_x(q) \leqslant L_x(p)$$

for all $q \in M$. This implies that $L_x(p) = L_x(q)$ for all $q \in M$ and hence $f(M)$ lies on the geodesic sphere $S = S(x, \|n\|)$, i.e. f is spherical.

Corollary (III.2.3)

Let $f : M \to \bar{W}$ be a taut imbedding of a compact manifold M as a hypersurface of $\bar{W}$. If f (M) has an umbilic , then f(M) is diffeomorphic to S^m, where dim M = m.

Proof :

From corollary (III.2.2), f(M) will be imbedded as a geodesic sphere in $\bar{W}$. Using propositions (III.1.5-6) we obtain the required result.

Proposition (III.2.4)

Let $f : M \to W$ be a taut imbedding such that there is at most one focal point on each normal geodesic ray. Then $f(M) \cap C_f = \Phi$ and $W \setminus C_f$ is arcwise connected.

Proof :

Suppose that γ is a normal geodesic ray to f(M) starting at f(p) for some $p \in M$. Then for points of γ very close to f(p) we can pick $x \in W$ such that $L_x | f(M)$ has f(p) as a non-degenerate critical point with index o (i.e. there are no focal points of f(M) on γ between x and f(p)). Applying lemma (III.2.6) we have that $L_x(p) < L_x(q)$ for all $q \in M$.

Suppose that x is a focal point of f(M), then x can be written as $x = \exp_{f(q)} tn$ for some $q \in M$, $t \in \mathbb{R}$ and $n \in T(M)^{\perp}$. By hypotheses in the proposition x is the only focal point on the geodesic ray $\bar{\gamma}(t) = \exp_{f(q_1)} tn$. Using lemma (III.2.7) we have that $L_x(q_1) \leqslant L_x(q)$ for all $q \in M$. In particular $L_x(q_1) \leqslant L_x(p)$ which is a contradiction showing that $x \notin C_f$.

Push x along γ nearer to f(M) and eventually we obtain that f(M) contains no focal points of itself. Hence $f(M) \cap C_f = \Phi$ which proves the first claim in the proposition.

For proving the second part suppose that $x, y \in W \setminus C_f$. As f is a taut imbedding and M is connected then L_x, L_y have unique critical points p,q, respectively, of index o. Consider the path consisting of the geodesic segment γ_p joining x to f(p), any path τ in f(M) joining f(p) to f(q) and finally the geodesic segment γ_q joining f(q) to y. For any point $x' \in \gamma_p$, p is a non-degenerate critical point of $L_{x'}$ and hence $x' \notin C_f$; thus $\gamma_p \cap C_f = \Phi$. Similarly, $\gamma_q \cap C_f = \Phi$. We have already shown that $f(M) \cap C_f = \Phi$ so $\tau \cap C_f = \Phi$. In this way the path described above which joins x and y is contained in $W \setminus C_f$. As this argument is true for any arbitrary pair of points $x, y \in W \setminus C_f$ we obtain the result.

<u>Proposition (III.2.5)</u>

Let $f : M \to W$ be a taut immersion of a compact manifold M as a hypersurface of W. Then every normal geodesic has focal points of f(M).

<u>Proof</u> :

Since f(M) is compact and f is an imbedding then any inward normal geodesic ray $\gamma(t)$, $t \geqslant o$, from $f(p) = \gamma(o)$, $p \in M$, intersects f(M) at least once (let $\gamma(a)$, $a > o$ be the first intersection). Suppose that the geodesic $\gamma_v(t)$, $\gamma_v(o) \in f(M)$ and $v \in T(M)^{\perp}$ is free from focal points of f(M), then by corollary (III.2.1) f(M) lies completely in $W\setminus B_v$ since $\gamma_v(t)$ for $t \geqslant o$ has no focal points of $f(M)$. Similarly, since the geodesic ray $\gamma_v(t)$, $t \leqslant o$ has no focal points of $f(M)$, $f(M) \subset W\setminus B_{-v}$. The result now is that $\gamma_{v}(a) \in f(M) \cap \gamma_{v}(t)$, $a \neq o$, coincides with $\gamma_v(o)$ which contradicts the hypothesis that f is an imbedding, hence the result.

Theorem (III.2.3)

Let $f : M \to W$ be a taut imbedding of an $(m-1)$-connected compact $2m$-manifold M with

$$H_m (M;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots\ldots \oplus \mathbb{Z} \text{ (m terms)}.$$

Let $\dim W = 2m+1$. Then $k = o$ or 2.

Proof :

Let v be an outward normal of M at p. Since the geodesic ray $\gamma_{-v}(t) = \exp_p(-tv)$, $t \geqslant o$, goes to the inside of $f(M)$ it must meet $f(M)$ again and by the last proposition there is a focal point of $f(M)$ on $\gamma_{-v}(t)$, $t \geqslant o$. Let $q = \exp_p(-v)$ be the first focal point on γ_{-v} with multiplicity $\mu > o$. From the index properties ([22] p.85) the point $q' = \exp_p(-tv)$ for $t > 1$ and sufficiently near 1, is not a focal point of $f(M)$ and hence p is a non-degenerate critical point of $L_{q'}|M$ with the same index μ.

Considering any T-function on M and taking into account that M is compact, the negative of a T-function is also T-function and from the hypothesis on the homology groups of M we have : A T-function on M has one minimum, one maximum and k critical points each of index m. Consequently, if a distance function L_x , $x \in W$, has a non-degenerate critical point then its index must be o, m or 2m. This argument shows that $\mu = m$ or $\mu = 2m$.

Now if $\mu = 2m$ for any point $p \in f(M)$, then M is tautly imbedded with an umbilic and by corollary (III.2.2) f(M) is spherical. Using the codimension 1 property, f(M) will be an imbedded geodesic sphere in W. Since the exponential mapping is a global diffeomorphism on W, then $\exp_{\bar{p}}^{-1} \circ f : M \to S^{2m}(r)$, where $\bar{p}$ and r are the center and the radius of f(M), respectively, is a diffeomorphism of M onto the 2m-sphere $S^{2m}(r) \subset E^{2m+1}$ of radius r. If $\mathfrak{p} : S^{2m}(r) \to S^{2m}(1)$ denotes the central projection in $W_{\bar{p}}$ then $\mathfrak{p} \circ \exp_{\bar{p}}^{-1} \circ f :$ $M \to S^{2m}(1)$ is a diffeomorphism and so k = o.

In the case k ≠ o, i.e. $\mu = m$ we find that if there is another focal point on $\gamma_{-v}(t)$. This must have multiplicity m. Let

$$d = \sup \{ \|n\| \ ; n \in T(M)^{\perp}, \exp_p n \text{ is the first focal point on the inward-pointing geodesic ray from } p, p \in M \}.$$

$$e = \operatorname{Inf} \{ \|n\| \ ; n \in T(M)^{\perp}, \exp_p n \text{ is the second focal point on the inward-pointing geodesic ray from } p, p \in M \}.$$

Following similar argument as in [5] we can prove that e is well defined and that d < e.

If we choose d < c < e , for every $p \in M$ define $\theta(p)$ to be the inward-pointing normal at p with length c. Then $\theta : M \to T(M)^{\perp}$ is a smooth imbedding. It is clear now that $\theta(M)$ contains no critical normals which means that $\exp_f | \theta(M)$ has no critical points. Hence

$$g = \exp_f \circ \theta : M \to W$$

is an immersion.

Now consider the map $\bar{\theta} : T(g(M))^{\perp} \to T(f(M))^{\perp}$ which is defined by

$$\bar{\theta}(n) = (1 \pm \frac{\|n\|}{C})\, \theta(p) \quad , \quad C = \|\theta(p)\|$$

where p is the base point of $\bar{\theta}(n)$ and the + and - signs denote the case when n is the initial velocity of an inward or outward-pointing geodesic ray. The map $\bar{\theta}$ is a fibre-preserving diffeomorphism.

From the nature of the mapping $\bar{\theta}$, it is clear that $\exp_{f(p)} \circ \bar{\theta}(n) = \exp_{g(p)} n$, hence $\exp_g = \exp_f \circ \bar{\theta}$. Accordingly, critical points of $\exp_g$ are mapped onto the critical points of $\exp_f$ with multiplicity preserving. Also the focal sets C_f and C_g coincide.

If we consider the point $x \in W \setminus C_f$ $(= W \setminus C_g)$ and choose $L_x(p) = d(x, f(p))$ and $L'_x(p) = d(x, g(p))$, then as $C_f \equiv C_g$ the non-degeneracy of L_x implies the non-degeneracy of L'_x. Moreover, the nature of g gives rise to the following fact : $p \in M$ is a critical point of L'_x if and only if p is a critical point of L_x. Since f is taut, then L_x is a T-function and hence L_x has precisely k + 2 critical points, one with index $0, p_0$ say, one with index 2m, p_1 say, and k with index m, $q_1, \ldots, q_k$. It can be shown graphically that $g(p_0)$ is a critical point of L'_x with index m, $g(p_1)$ is a critical point of L'_x with index m and $g(q_i)$, $i = 1, \ldots, k$ is a critical point of L'_x with index 0 or 2m. Applying Morse inequalities as it has been done in [5] we get that k = 2 which completes the proof of the theorem.

(III.2.3) - More about taut and tight immersions:

In the case of the immersion $f : M \to W$ of a compact manifold M into a complete, simply connected, Riemannian manifold W without conjugate points, we can give the following definition of taut immersion similar to that one given in the Euclidean space [31].

Definition (3.1)

The immersion $f : M \to W$ is called taut if every non-degenerate distance function L_x, $x \in W$, has the minimal number of critical points required by the Morse inequalities.

In [3], J. Bolton defined the h-tight immersion $f : M \to \tilde{W}$ of a compact manifold M into the complete, simply connected, Riemannian manifold $\tilde{W}$ without conjugate points, with bounded asymptote and with sectional curvature bounded from below as follows:

Definition (3.2)

An immersion $f : M \to \tilde{W}$ is called h-tight if every non-degenerate $b_v \circ f$, $v \in S\tilde{W}$, has the minimal number of critical points required by the Morse inequalities.

It seems quite interesting to find the relations between the definitions of tautness and h-tightness mentioned in (III.2.1) and the above ones. To make the following discussion clear, we mention the basic ideas about the so called convex mappings. These ideas are originally due to N. Kuiper [20].

Let M be a compact manifold without boundary (closed) and let $\phi : M \to \mathbb{R}$ be a non-degenerate C^2 real-valued function on M. Let $\mu_k(M, \phi)$ be the number of critical points of ϕ of index k and let also $\mu(M, \phi) = \sum_k \mu_k(M, \phi)$ represents the total number of critical points of ϕ. Let further $\mu(M) = \mathrm{Inf}_\phi\, \mu(M, \phi)$ be the minimal number of critical points a non-degenerate function on M can have. (Infimum is taken over all non-degenerate functions ϕ).

Definition (3.3)

A smooth map $\phi : M \to \mathbb{R}$ will be called convex if it is non-degenerate and if moreover the minimal number of critical points is attained, i.e. $\mu(M, \phi) = \mu(M)$.

It is clear now from definitions (3.1 - 3.3) that in case of taut immersion of a compact manifold M into W, any non-degenerate distance function L_x, $x \in W$, is a convex map on M. Similarly, $b_v \circ f$, $v \in S\tilde{W}$, is convex in the case of h-tightness of M into $\tilde{W}$.

In terms of T-functions, lemma on page 153 [20] can be rewritten as follows:

Lemma (III.2.8)

Let $\phi : M \to IR$ be a non-degenerate function on a closed manifold M which fulfills the following condition:
There exists a field F such that $\mu(M) = \sum_k \beta_k(M ; F)$ (and consequently, for any function ϕ for which $\mu(M) = \mu(M, \phi)$, $\mu_k(M) = \mu_k(M,\phi) = \beta_k(M;F)$).
Then ϕ is convex if and only if ϕ is a T-function.

In terms of this lemma we conclude that :

Corollary (III.2.4)

(a) Let $f : M \to W$ be an immersion of a closed manifold M and let M have the property that $\mu(M) = \sum_k \beta_k(M;F)$ for some field F. Then the two concepts of taut immersions (Definitions (1.2) and (3.1)) are coincident.

(b) Let $f : M \to \tilde{W}$ be an immersion of a closed manifold M where M has the same property as in (a). Then the two definitions (1.4) and (3.2) coincide.

Other kinds of taut (h-tight) immersions may be given as follows:
A non-degenerate function $\phi : M \to IR$ on a manifold M is said to be k-convex if $\mu_i(M, \phi) = \mu_i(M)$ for $i = o, 1, \ldots, k$. The function ϕ is called convex if $\mu(M,\phi) = \mu(M)$ as mentioned before. Evidently, if ϕ is k-convex for all k, then ϕ is convex. It has been proved by M.Morse that if ϕ is convex, then it is 0-convex. However, whether ϕ convex implies k-convex for all k seems unlikely. For more details see [31] .

Definition (3.4)

(a) Let $f : M \to W$ be an immersion of a closed manifold M in a complete, simply connected, Riemannian manifold W without conjugate points. The mapping f is called k-taut (taut) if every non-degenerate distance function L_x, $x \in W$, is k-convex (convex).

(b) Let $f : M \to \tilde{W}$ be an immersion of a closed manifold M into a complete, simply connected, Riemannian manifold $\tilde{W}$ without conjugate points, with bounded asymptote and with sectional curvature bounded from below. The mapping f is called k-h-tight (h-tight) if every non-degenerate function $b_v \circ f$, $v \in S\tilde{W}$, is k-convex (convex).

From the above argument we see that if M, beside being closed, has the property that $\mu(M) = \sum_k \beta_k(M;F)$ for some field F, then any taut immersion $f : M \to W$ (or h-tight immersion $f : M \to \tilde{W}$) is k-taut (or k-h-tight) for every k, where W and $\tilde{W}$ are as mentioned in definitions (3.3-a) and (3.3-b), respectively.

For the rest of this part let W, $\tilde{W}$ and M denote manifolds as mentioned above. From the above mentioned result of M. Morse we conclude that, any taut immersion $f : M \to W$ (h-tight immersion $f : M \to \tilde{W}$) is o-taut (o-h-tight). This means that any non-degenerate function $L_x(b_v \circ f)$, $x \in W$, $v \in S\tilde{W}$, is o-convex, i.e. $\mu_0(M,L_x) = \mu_0(M)$ $[\mu_0(M,b_v \circ f) = \mu_0(M)]$. But from Morse inequalities we have

$$\mu_0(M,L_x) \geq \beta_0(M;F) \quad \text{and} \quad \mu_0(M,b_v \circ f) \geq \beta_0(M;F) \tag{1}$$

If M is connected then $\beta_0(M;F) = 1$ and hence

$$\mu_0(M,L_x) \geq 1 \quad \text{and} \quad \mu_0(M,b_v \circ f) \geq 1 \tag{2}$$

Again if M has the property $\mu(M) = \sum_k \beta_k(M;F)$ then

$$\mu_0(M,L_x) = 1 \quad \text{and} \quad \mu_0(M,b_v \circ f) = 1 \tag{3}$$

<u>Proposition (III.2.6)</u>

Let $f : M \to W$ be a o-taut immersion of a compact (closed without boundary) manifold M. Suppose that every non-degenerate distance function

$L_x (x \notin C_f)$ has only one local minimum, then M has the STPP.

Proof :

As the non-degenerate distance function $L_x (x \notin C_f)$ has only one local minimum and since f is o-taut then the number of critical points of L_x over the whole of M is exactly one, i.e. $\mu_o(M,L_x) = 1$. Also this local minimum of L_x coincides with the global one.

Consider now the submanifold

$$M_c(L_x) = \{ p \in M ; L_x(p) \leqslant c \ , a \leqslant c \leqslant b \}$$

where "a" and "b" are the (global) minimum and maximum values of L_x, respectively. The number of critical points of L_x with index o on $M_c(L_x)$ is exactly one which is the global minimum. Applying Morse inequalities, we obtain

$$\mu_o(M_c(L_x) \ , L_x) = 1 \geqslant \beta_o(M_c(L_x) \ ; F)$$

which shows that $\beta_o(M_c(L_x):F) = 1$ and so $M_c(L_x)$ is connected for each $a \leqslant c \leqslant b$.

Considering $(-L_x)$ instead of L_x and repeating similar argument, we have that $\beta_o(M_d(-L_x),F) = 1$ and hence $M_d(-L_x)$ is connected for each $-b \leqslant d \leqslant -a$. Hence the STPP is now proved.

A similar proposition can be stated for h-tightness in the following way.

Proposition (III.2.7)

Let $f : M \to \tilde{W}$ be a o-h-tight immersion of a compact (closed without boundary) manifold M. Suppose that every non-degenerate function $b_v \circ f$, $v \in S\tilde{W}$, has only one local minimum, then M has the hTPP.

Remark (3)

1 - Using the compactness of M, the word "local minimum" in the last two propositions can be replaced by "local maximum".

2 - The last two propositions are naturally true for tautness (h-tightness) as every taut (h-tight) immersion is o-taut (o-h-tight)

(III.2.4) - On supporting of submanifolds :

In what follows we generalize and study some concepts of supporting imbedded submanifolds of a complete, simply connected, Riemannian manifold $\tilde{W}$ without conjugate points, with bounded asymptote and with sectional curvature bounded from below [19] . We have shown in (chapter 0) that in such manifold horospheres are complete, non-compact equidistant hypersurfaces.

Definition (4.1)

(a) For a given immersion $f : M \to \tilde{W}$, the closed horodisc $\bar{B}^c_v = \{ p \in \tilde{W} ; b_v(p) \geq c \}$ for some $v \in S\tilde{W}$ and $c \in IR$ is called supporting of f(M) if the boundary $\partial \bar{B}^c_v = H^c_v = \{ p \in \tilde{W}; b_v(p) = c \}$ but not $Int(\bar{B}^c_v)$ has points in common with f(M).

(b) The subset $M_v = f^{-1}(\bar{B}^c_v) = f^{-1}(H^c_v) \neq \Phi$ of M is called the top-set in case of $M_v \neq M$.

Clearly if $M_v = M$ for some $v \in S\tilde{W}$, then f is not h-substantial. It can be shown easily that in a complete, simply connected, Riemannian manifold $\tilde{W}$ without focal points, geodesic sphere has single point top-set.

One of the important characterizations of the Euclidean space E^n is that the two horospheres H_v and H_{-v}, for some $v \in SE^n$, are coincident and consequently any substantial immersed submanifold M of E^n which is supported at one of its points, p say, in v-direction, for $v \in SE^n$, could not be supported at the same point p in (-v)-direction. In $\tilde{W}$ this fact is no longer true (see example part (III.2.2)). If M is imbedded as a compact (closed without boundary) hypersurface of $\tilde{W}$ then M can be supported only once at any of its points where supporting is available.

Proposition (III.2.8)

Let M be an imbedded hypersurface of $\tilde{W}$. If M is supported twice at some point $p \in M$ in v and (-v)-directions, for some $v \in S\tilde{W}$, then M has no focal points along the normal geodesic from p with initial velocity v.

Proof:

Let M be supported twice at p in v and (-v)-directions. Let $\gamma_v(t)$ be a geodesic in $\tilde{W}$ such that $\gamma_v(o) = p$ and $\gamma_v'(o) = v$.

Firstly consider the supporting closed horodisc $\bar{B}_v$. Define the map $C_{t_o} : M \to H_v$ as follows : Draw the unit speed geodesic $\gamma_\omega(s)$ such that $\gamma_\omega(o) = \gamma_v(t_o)$ and $\gamma_\omega'(o) = \omega \in \tilde{W}_{\gamma_v(t_o)}$. This geodesic $\gamma_\omega(s)$ intersects M at some point, $\gamma_\omega(\bar{s})$ say, and intersects H_v at $\gamma_\omega(r)$, $\bar{s} \geqslant r$. Put $\gamma_\omega(r) = C_{t_o}(\gamma_\omega(\bar{s}))$. The map C_{t_o} is defined (at least) locally at p. Considering the distance function $L_{\gamma_v(t_o)}$, we have $L_{\gamma_v(t_o)} \geqslant L_{\gamma_v(t_o)} \circ C_{t_o}$. Using Morse index theorem and taking into account that H_v has no focal points on the geodesic segment from p to $\gamma_v(t_o)$ we obtain that p is a non-degenerate critical point of $L_{\gamma_v(t_o)}$ with index o. Hence M has no focal points along $\gamma_v(t)$, $o \leqslant t \leqslant t_o$. For t_o large enough we have that M has no focal points along $\gamma_v(t)$, $t \geqslant o$.

Similar agument can be carried out when considering H_{-v} and $\gamma_v(t)$, $t \leqslant o$, which completes the proof.

The main theorem of this part which is a generalization of a corresponding one in Euclidean space [21] can be stated as follows:

Theorem (III.2.4)

Let $\tilde{W}$ be a complete, simply connected, Riemannian manifold without conjugate points with bounded asymptote and with sectional curvature bounded from below. Let $f : M \to \tilde{W}$ be an h-tight imbedding of a compact manifold M as a hypersurface of $\tilde{W}$. Let M_v be the top-set of the imbedding f, $v \in S\tilde{W}$. Then the induced map $H_*(M_v;F) \to H_*(M;F)$ is injective, and the top-map $f' = f\,|\,M_v : M_v \to \tilde{W}$ is h-tight as well.

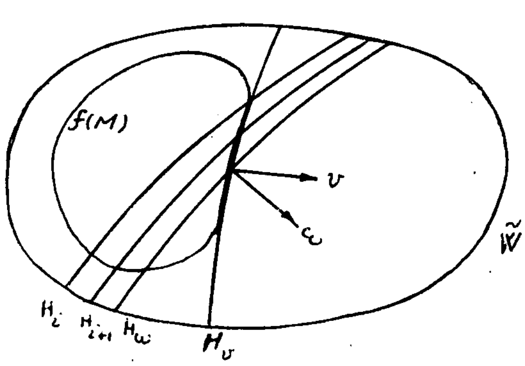
Proof

Since f is an imbedding and f(M) is a compact hypersurface of $\tilde{W}$, then f(M) can be supported only once from any of its points. This advantage enables us to deal only with one supporting horodisc $\bar{B}_v$ or $\bar{B}_{-v}$ for some $v \in S\tilde{W}$.

Suppose that f(M) is supported by $\bar{B}_v$. The first statement of the theorem is naturally true from the definition of h-tightness.

Let M_v be the top-set in the horosphere H_v. Consider any other horosphere H_ω passing through $p \in M_v$, $\omega \in \tilde{W}_p$. The idea of the proof is to show that the induced homomorphism $H_*(M_v \cap \bar{B}_\omega;F) \to H_*(M_v;F)$ is injective for each $\bar{B}_\omega$. A sequence of horodiscs $\bar{B}_i = \bar{B}_{u_i}$, $u_i \in \tilde{W}_p$ can be found (as in the following figure) making $M_v \cap \bar{B}_i$ converge to $M_v \cap \bar{B}_\omega$. Notice that $M \cap \bar{B}_i$ is a connected component. Choose u_i's such that

$$M \cap \bar{B}_i \supset M \cap \bar{B}_{i+1} \supset \bigcap_{j=1}^{\infty} \bar{B}_j \cap M = \bar{B}_\omega \cap M$$



Depending on the continuity of H_* in $\tilde{W}$ and following similar way to that of N. Kuiper [19] , we have

$$\lim_{j\to\infty} H_*(M \cap \bar{B}_j \,;\, F) = H_*(M_V \cap \bar{B}_\omega \,; F)$$

We now obtain a commutative diagram of morphisms in homology, with injective morphisms denoted by $\xrightarrow{c}$, from a corresponding diagram of inclusions as follows:

$$\begin{array}{ccccc}
H_*(M_V \cap \bar{B}_V) & \longrightarrow & H_*(M \cap \bar{B}_{i+1}) & \longrightarrow & H_*(M \cap \bar{B}_i) \\
 & \searrow^{u} & & \searrow^{c} & \Big\downarrow c \\
 & & H_*(M_V) & \xrightarrow[\text{h-tight}]{c} & H_*(M)
\end{array}$$

Then the morphism u is also injective and the proof is complete.

Section 3 : Total curvature of immersed submanifolds of hyperbolic spaces

(III.3.0) Introduction

The concept of total absolute curvature of immersed manifolds in Euclidean spaces has been introduced, for the first time, by S.S. Chern and R.K. Lashof [10]. Many trials have been done since this paper has been published in (1957) to extend the concept of total curvature to immersions in non-Euclidean spaces. For instance, J.L. Weiner [31] considered the case in which the ambient space is spherical S^n. Defining a convenient Gauss mapping and adapting the conformal mapping (stereographic projection) between spherical and Euclidean spaces, J.L.Weiner was able to use the results of S.S. Chern - R.K. Lashof to obtain similar ones in spherical space.

In the following part (III.3.1) we share this sort of studies and prove similar results in hyperbolic spaces. The scheme used here is analogous to that of J.L. Weiner. Since hyperbolic space is a complete, simply connected, Riemannian manifold without conjugate points, our study is much easier than that of spherical space.

For the rest of this section let M be a compact, oriented, C^∞ n-manifold immersed in (n+k)-hyperbolic space. The model used in (III.3.1) is the H-model described before. Let p be the north pole of H and $S(M)^\perp$ be the bundle of unit vectors normal to M as a submanifold of H. We define the Gauss mapping $e_p : S(M)^\perp \to S_pH$ where S_pH is the unit sphere in the tangent space H_p of H at p. We shall use $\langle \, , \rangle$ to denote the induced Riemannian metric on H or any submanifold.

(III.3.1) Total curvature and conventional Gauss map:

1. Definitions and basic material :

Let M be an immersed submanifold of H. Define $e_p : S(M)^{\perp} \to S_pH$ as follows : Let $v \in S_q(M)^{\perp}$ ($S_q(M)$ is the fibre of $S(M)^{\perp}$ over $q \in M$), then $e_p(v)$ is defined to be the parallel translation of v to p along the (unique) geodesic segment γ from p to q. This map e_p is now well-defined and it is easy to prove that it is continuous differentiable (C^{∞}) over the whole $S(M)^{\perp}$. As M is compact oriented we may globally define the Gauss mapping e_p on M with respect to any base point $p \in H$. If H is replaced by E^{n+k}, e_p turns out to be the usual Gauss mapping and in this case e_p is independent of the choice of the base point p. Let $d\alpha^{n+k-1}$ denote the volume element of S_pH normalized so that

$$\int_{S_pH} d\alpha^{n+k-1} = 1$$

Definition (1) :

Set

$$\kappa_p(M) = \int_{S(M)^{\perp}} e_p^*(d\alpha^{n+k-1}) \quad , \quad \tau_p(M) = \int_{S(M)^{\perp}} \| e_p^*(d\alpha^{n+k-1}) \|$$

We call $\kappa_p(M)$ the total (algebraic) curvature of M with respect to p and $\tau_p(M)$ the total absolute curvature of M with respect to p.

Actually, $\kappa_p(M)$ equals the algebraic normalized volume covered by e_p while $\tau_p(M)$ is the normalized volume covered by e_p. Because the volume is normalized, $\tau_p(M)$ equals the average number of times any vector in S_pH is taken on by e_p. Since M is compact then $\kappa_p(M)$ equals the degree of the mapping e_p.

Before being involved in any other details we state the main results of S.S. Chern and R.K. Lashof [10, 11] .

Theorem (III.3.1)

Let $x : M \to E^{n+k}$ be an immersion of an n-dimensional closed manifold M. Then the total absolute curvature of x satisfies the following inequality

$$\tau(x) \geqslant \beta(M) = \text{sum of the Bitti numbers of } M.$$

Theorem (III.3.2)

Let $x : M \longrightarrow E^{n+k}$ be as before:

(i) If $\tau(x) < 3$, then M is homeomorphic to an n-sphere.

(ii) $\tau(x) = 2$ if and only if x is an imbedding and x(M) is a convex hypersurface in an (n+1)-dimensional linear subspace of E^{n+k}.

(2) Main theorem :

Let $\sigma_p : H \to E^{n+k}$ be the steroegraphic projection as described in chapter 0. Now the tangent space H_p of H at p can be identified with the Euclidean space E^{n+k} through parallel translation in the Minkowski space $(\mathbb{R}^{n+k+1}, < , >)$. Let $M(p) = \sigma_p(M)$ and let M(p) carry the metric induced from E^{n+k}. Similar to [31] the following is easy to prove:

Lemma (III.3.1)

Let M be an immersed submanifold of H. Then the following diagram is commutative

$$\begin{array}{ccc} S(M)^{\perp} & \xrightarrow{\ e_p\ } & S_pH \\ {\scriptstyle (1 - \langle x,p \rangle)\,\sigma_{p*}} \downarrow & & \downarrow {\scriptstyle id} \\ S(M(p))^{\perp} & \xrightarrow{\ e\ } & S^{n+k-1} \end{array}$$

where $x \in M$ and e denotes the usual Gauss mapping in E^{n+k}.

If $M(p)$ is given the orientation induced from M by σ_p, the algebraic volume covered by e and e_p are equal and consequently

$$\kappa(M(p)) = \kappa_p(M) \tag{1}$$

Similarly

$$\tau(M(p)) = \tau_p(M) \tag{2}$$

Theorem (III.3.3)

Let M be a compact oriented immersed n-submanifold of H. Then $\kappa_p(M) = \chi(M)$ where $\chi(M)$ is the Euler characteristic number of M.

Proof :

Since $\sigma_p : H \to E^{n+k}$ is a diffeomorphism then in particular M and $M(p)$ are topologically equivalent. Hence $\chi(M) = \chi(M(p))$. From Allendoerfer's theorem [19] we have $\kappa(M(p)) = \chi(M(p))$. So $\kappa_p(M) = \kappa(M(p)) = \chi(M(p)) = \chi(M)$ which completes the proof.

In what follows we say that a submanifold N of H is an m-sphere if $N = H \cap L^{m+1}$ where L^{m+1} is an $(m+1)$-dimensional plane in the Minkowski space $(\mathbb{R}^{m+k+1}, <,>)$. In case of co-dimension 1, m-sphere turns out to be a geodesic sphere of H when being compact.

The main theorem of this part may be stated as follows :

Theorem (III.3.4)

Let M be a compact, oriented, immersed n-submanifold of the $(n+k)$-hyperbolic space H. Let p be the north pole of H, then

(i) $\tau_p(M) \geq \beta(M)$

(ii) $\tau_p(M) < 3$ implies that M is homeomorphic to S^n.

(iii) $\tau_p(M) = 2$ implies that M is imbedded as a hypersurface of an $(n+1)$-sphere of H.

Proof :

(i) We know that M and M(p) are topologically equivalent under σ_p.

So

$$\beta(M) = \beta(M(p)) \tag{3}$$

From equations (2) and (3) together with theorem (III.3.1) we have

$$\tau_p(M) = \tau(M(p)) \geqslant \beta(M(p)) = \beta(M)$$

Hence

$$\tau_p(M) \geqslant \beta(M)$$

which proves part (i) of the theorem.

(ii) For part (ii) we have by equation (2) that

$$\tau(M(p)) = \tau_p(M) < 3 \tag{4}$$

Using this inequality together with theorem (III.3.2) we see that M(p) is homeomorphic to S^n. Since M and M(p) are homeomorphic under σ_p then M is also homeomorphic to S^n.

(iii) For $\tau_p(M) = 2$ we have $\tau(M(p)) = 2$. Hence by theorem (III.3.2) part (ii) we see that M(p) is an imbedded hypersurface in an (n+1)-dimensional linear subspace E^{n+1} in E^{n+k}. Under σ_p^{-1}, E^{n+1} will be mapped onto an (n+1)-sphere in H. If M passes through the north pole p, then M is imbedded as a hypersurface of a totally geodesic submanifold of dimension (n+1) in H.

Related to the concept of total curvature, the following is true:

Lemma (III.3.2)

Let M be a compact oriented immersed n-submanifold of a Euclidean space E^{n+k} $(k \geqslant 1)$. Suppose that there exists an $(n+\ell)$-plane $(1 \leqslant \ell \leqslant k)$ in E^{n+k} which contains M. Then the total absolute curvatures of M as a submanifold of $N^{n+\ell}$ and E^{n+k} are equal.

A corresponding statement can be proved in hyperbolic spaces with some restrictions as follows :

Theorem (III.3.5)

Let M be a compact, oriented, immersed n-submanifold of the (n+k)-hyperbolic space H. Let M be contained in a totally geodesic submanifold Σ of dimension $(n+\ell)$ which passes through $p \in H$. Then the total absolute curvatures of M as a submanifold of H and Σ are the same.

Proof :

The idea is to stereoegraphic project H on E^{n+k} using σ_p. Clearly $\sigma_p(M) = M(p)$ will be contained in $E^{n+\ell}$ in E^{n+k}. Applying the above lemma and using relation (2) we get the result.

(III.3.2) Total curvature and generalized Gauss map :

Assume that $\tilde{W}$ - as before - is a complete, simply connected, Riemannian manifold without conjugate points, with bounded asymptote and with sectional curvature bounded from below. Each $v \in S\tilde{W}$ determines a family of horospheres orthogonal to the unit vector field grad b_v. If u = grad $b_v(q)$, $q \in \tilde{W}$, then we say that u is asymptotic to v and consequently grad b_u = grad b_v. It can be shown that "asymptotic" is an equivalence relation on $S\tilde{W}$. The equivalence classes form a regular continuous foliation $\mathcal{F}$ of $S\tilde{W}$ each leaf of which is a C^1 vector field on $\tilde{W}$ of the form grad b_v [3] .

The definition of the generalized Gauss map $G : S(M)^{\perp} \to S_p\tilde{W}$ of an immersion $x : M \to \tilde{W}$, where $S_p\tilde{W}$ is the sphere of unit vectors at some point $p \in \tilde{W}$ and M is a C^{∞} manifold, is given as follows [3] :

Let $u \in S(M)^{\perp}$, then G(u) will be the element of $S_p\tilde{W}$ asymptotic to u, i.e. $G(u) = \text{grad } b_u(p)$.

It has been proved in [3] that:

(i) If $\mathcal{F}$ is a C^1 foliation of $S\tilde{W}$ then $G : S(M)^{\perp} \to S_p\tilde{W}$ is also C^1.

(ii) Let $v \in S_p\tilde{W}$. Then $b_v \circ x$ is non-degenerate if and only if v is a regular value of G.

(iii) If $\mathcal{F}$ is C^1, then the set $\{ v \in S_p\tilde{W} \; ; \; b_v \circ x \text{ is non-degenerate} \}$ is dense in $S_p\tilde{W}$.

When specializing to the case when $\tilde{W}$ is a hyperbolic space and adopting the above mentioned mapping G we prove the following theorem which is analogous to theorem (III.3.2) part (ii) :

Theorem (III.3.6) :

Let $x : M \to H$ be an imbedding of a compact C^∞ n-manifold M as a hypersurface of the (n+1)-hyperbolic space H. Let $c_n = \int_{S^n} d\alpha$ be the area of the unit sphere S^n in E^{n+1}. Then

$$\tau(x) = \int_{S(M)^{\perp}} \| G^*(d\alpha) \| = 2c_n$$

if and only if x(M) lies on one side of each tangent horosphere.

For the proof of this theorem we need the following lemma:

Lemma (III.3.3)

Let $x : M \to H$ be as in the theorem and let $G : M \to S_qH$ be the generalized Gauss mapping. Let J(p) be the Jacobian of G at p, and let $U_m = \{ p \in M \; ; \; \text{rank } J(p) = n-m \}$. Then if U_m contains an open set V, its image under x is generated by m-dimensional totally geodesic submanifold of the tangential horosphere. Every boundary point of U_m, which is at the same time a limit point of an m-dimensional generating totally geodesic submanifold of the tangential horosphere, belongs to U_m.

Proof :

Since M is imbedded as a hypersurface of H, then G turns out to be a map $G : M \to S_qH$ of M itself onto the unit sphere S_qH in H_q. In [3], J. Bolton constructed a bilinear form Q_u as follows : For $u \in SH$, let $\tilde{H}_u$ be the tensor $\tilde{H}_u = \tilde{\nabla}_X \text{grad } b_u$, where $\tilde{\nabla}$ is the covariant differentiation on H. If u is orthogonal to M at q, let S_u be the second fundamental tensor of M at q and define the bilinear form Q_u on the tangent space T_qM to M at q by

$$Q_u(X,Y) = \langle \tilde{H}_u X - S_u X, Y \rangle \qquad (1)$$

We notice that the restriction of $\tilde{H}_u$ to vectors orthogonal to grad b_u gives the second fundamental tensors of the horospheres which are the level hypersurfaces of b_u.

Now the kernel of G_* can be defined as follows:

$$\ker G_*(u) = \{ X \in M_q \,;\, \tilde{H}_u X - S_u X \in T_q(M)^{\perp} \}$$

$$= \{ X \in M_q \,;\, Q_u(X,Y) = 0 \text{ for all } Y \in M_q \}$$

and hence if $X \in \ker G_*(u)$ then

$$S_u X = \tilde{H}_u X$$

As we know (§ 6-chapter 0), in hyperbolic space $\tilde{H}_u$ = identity map and consequently if $X \in \ker G_*(u)$ then by equation (2) we have

$$S_u X = X \qquad (3)$$

and so S_u = identity map as well.

The following lemma is needed to complete the proof :

Lemma (III.3.4) :

$\ker G_*$ is an involutive distribution.

Proof :

Let $X,Y \in \ker G_*$, the idea is to show that $[X,Y]$ is also in $\ker G_*$. The Mainardi-Codazzi equation (see chapter 0), may be written as

$$< \tilde{R}(X,Y)Z , \nu > = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) \qquad (4)$$

for $X,Y,Z \in \mathfrak{X}(M)$, $\nu \in S(M)^{\perp}$, $\tilde{R}$ is the curvature tensor of H and $h(X,Y) = < S_\nu X,Y> = < X,Y >$ is the second fundamental form of M. Since H has constant sectional curvature -1, then

$$\tilde{R}(X,Y)Z = -\{ < Y,Z > X - < X,Z > Y \} \qquad (5)$$

Since $< X, \nu> = < Y,\nu > = o$, then $< \tilde{R}(X,Y)Z,\nu > = o$ and so (4) has the form

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z) \qquad (6)$$

It is also known that

$$X(h(Y,Z)) = (\bar{\nabla}_X h)(Y,Z) + h(\nabla_X Y,Z) + h(Y,\nabla_X Z) \qquad (7)$$

$$Y(h(X,Z)) = (\bar{\nabla}_Y h)(X,Z) + h(\nabla_Y X,Z) + h(X,\nabla_Y Z) \qquad (8)$$

where ∇ is the induced covariant differentiation on M. From (6) - (8) we have

$$h(\nabla_X Y,Z) - h(\nabla_Y X,Z) = X(h(Y,Z)) - Y(h(X,Z)) - h(Y,\nabla_X Z) + h(X,\nabla_Y Z) \qquad (9)$$

But as $X,Y \in \ker G_*$, then by using Gauss formula((2.3) - chapter 0), we have

$$h(\nabla_X Y,Z) - h(\nabla_Y X,Z) = h(\nabla_X Y - \nabla_Y X,Z) = < S(\nabla_X Y - \nabla_Y X),Z > =$$

$$= < S([X,Y]),Z > = <[X,Y] ,Z >$$

and so

$$S([X,Y]) = [X,Y]$$

i.e. $[X,Y]$ is in $\ker G_*$. This argument shows that $\ker G_*$ is an involutive distribution and the proof of lemma (III.3.4) is now complete.

At any interior point $p \varepsilon U_m$ the assumption on the rank of J(p) implies that we may choose coordinates on M in a neighbourhood of p say $(t^1,\ldots,t^n)$ such that, if ν is the unit normal vector of M at p then

$$\tilde{\nabla}_{\partial/\partial t\alpha}\,\nu = -S_\nu(\partial/\partial t^\alpha) = \partial/\partial t^\alpha, \alpha = 1,\ldots,m \qquad (10)$$

Let $a = m+1,\ldots,n$, we have, by using (10), that

$$(\partial/\partial t^{a})<\nu, \partial/\partial t^\alpha> = <\tilde{\nabla}_{\partial/\partial t^a}\nu, \partial/\partial t^\alpha> + <\nu, \tilde{\nabla}_{\partial/\partial t^a}\partial/\partial t^\alpha>$$

$$= <\tilde{\nabla}_{\partial/\partial t^a}\nu - \partial/\partial t^{a}, \partial/\partial t^\alpha> = o$$

and so $\tilde{\nabla}_{\partial/\partial t^a}\nu - \partial/\partial t^\alpha$ is perpendicular to the integral manifold $\mathcal{R}$ of ker G_* at p. We can write

$$\tilde{\nabla}_{\partial/\partial t^a}\nu - \partial/\partial t^a = (S_\nu - I)\ \partial/\partial t^a$$

and as $S_\nu - I$ is a symmetric linear transformation of M_p, then M_p may be expressed as

$$M_p = \ker\,(S_\nu - I) \oplus V$$

where V is the orthogonal complement of ker $(S_\nu - I)$. It can be shown easily that $(S_\nu - I)$ takes the subspace spanned by $\{\partial/\partial t^a\}$bijectively to V.

Now any normal vector u to the integral manifold $\mathcal{R}$ can be written as

$$u = \sum_{a=m+1}^{n} u^{a}(S_\nu - I)(\partial/\partial t^a)$$

where $\{u^{a}\}$represent the components of u. The next step is to show that the integral manifold is a geodesic submanifold of M at p. This can be done through showing that $<\nabla_X u, Y> = o$ for all $X,Y \varepsilon T_p\mathcal{R}$ and u is any normal vector to the integral manifold as a submanifold of M. Without loss of generality we can prove this fact for $X = \partial/\partial t^\alpha$,

$Y = \partial/\partial t^\beta$ and $u = \tilde{\nabla}_{\partial/\partial t^a}\nu - \partial/\partial t^a$. For simplicity let

$$\Lambda = < \nabla_{\partial/\partial t^\alpha}(\tilde{\nabla}_{\partial/\partial t^a}\nu - \partial/\partial t^a), \partial/\partial t^\beta >$$

Applying Gauss' formula ((2.3) - chapter o), we have

$$\Lambda = < \tilde{\nabla}_{\partial/\partial t^\alpha} \tilde{\nabla}_{\partial/\partial t^a}\nu - \tilde{\nabla}_{\partial/\partial t^\alpha} \partial/\partial t^a , \partial/\partial t^\beta >$$

and since $[\partial/\partial t^i , \partial/\partial t^j] = o$, then

$$\tilde{R}(\partial/\partial t^\alpha , \partial/\partial t^a)\nu = \tilde{\nabla}_{\partial/\partial t^\alpha} \tilde{\nabla}_{\partial/\partial t^a}\nu - \tilde{\nabla}_{\partial/\partial t^a} \tilde{\nabla}_{\partial/\partial t^\alpha}\nu$$

hence

$$\Lambda = < \tilde{R}(\partial/\partial t^\alpha , \partial/\partial t^a)\nu + \tilde{\nabla}_{\partial/\partial t^a} \tilde{\nabla}_{\partial/\partial t^\alpha}\nu - \tilde{\nabla}_{\partial/\partial t^\alpha}\partial/\partial t^a, \partial/\partial t^\beta >$$

Using equation (5), we get

$$\Lambda = < \tilde{\nabla}_{\partial/\partial t^a} \partial/\partial t^\alpha , \partial/\partial t^\beta > - < \tilde{\nabla}_{\partial/\partial t^\alpha}\partial/\partial t^a , \partial/\partial t^\beta >$$

$$= <[\partial/\partial t^a , \partial/\partial t^\alpha], \partial/\partial t^\beta > = o$$

Hence $\mathcal{R}$ is a geodesic submanifold of M at p. In the same time $\mathcal{R}$ is a geodesic submanifold of the tangent horosphere H_ν at p. Repeating the same argument with neighbourhoods of $t^\alpha = o$ we get that t^a = constant are totally geodesic submanifolds of both M and tangent horospheres which means that along any curve in $\mathcal{R}$, the tangent horosphere of M remains constant. This finishes off the first part of lemma (III.3.3).

For the last part we consider the orthonormal frame $e_1,\ldots,e_{n+1}$ such that $e_1,\ldots,e_m$ are tangential to $\mathcal{R}$, $e_{m+1},\ldots,e_n$ are tangential to M and e_{n+1} is perpendicular to M. Let $\omega^1,\ldots,\omega^{n+1}$ be the co-frame. From the structural equations (§ 1-chapter 0) we have that

$$\omega_i^{n+1} = \sum_{j=1}^{n} h_{ij}\,\omega^j \quad \text{or } h_{ij} = \omega_i^{n+1}(e_j) \quad 1 \leqslant i,j \leqslant n$$

But since $h_{\alpha j} = \delta_{\alpha j}$, $1 \leq \alpha \leq m$, then

$$\omega_{\alpha}^{n+1} = \sum_{j=1}^{n} \delta_{\alpha j} \omega^{j} = \omega^{\alpha}, \quad \omega_{a}^{n+1} = \sum_{b=m+1}^{n} h_{ab} \omega^{b}, m+1 \leq a,b \leq n \qquad (11)$$

and consequently the second fundamental form matrix of M restricted to U_m may be written as

$$(h_{ij}) = \begin{bmatrix} I_m & 0 \\ 0 & h_{ab} \end{bmatrix}$$

In a similar way, the second fundamental form matrix of a horosphere in hyperbolic space is of the form

$$(\ell_{ij}) = \begin{bmatrix} I_m & 0 \\ 0 & I_{n-m} \end{bmatrix}$$

Now the proof of the lemma will be complete through studying the behaviour of the matrix

$$(P_{ij}) = (h_{ij} - \ell_{ij}) = \begin{bmatrix} 0 & 0 \\ 0 & h_{ab} - I_{n-m} \end{bmatrix}$$

along the m-dimensional generating totally geodesic submanifolds.

We know that $D = \det(h_{ab} - I_{n-m}) \neq 0$ because if $D = 0$ then $\ker G_*$ will be of dimension $>m$. From the structural equations, we have

$$d\omega_{\alpha}^{n+1} = -\sum_{C=1}^{n+1} \omega_{C}^{n+1} \wedge \omega_{\alpha}^{C} + \Phi_{\alpha}^{n+1}$$

But

$$d\omega_{\alpha}^{n+1} = d\omega^{\alpha} = -\sum_{B=1}^{n+1} \omega_{B}^{\alpha} \wedge \omega^{B}$$

hence

$$-\sum_{C=1}^{n+1} \omega_{C}^{n+1} \wedge \omega_{\alpha}^{C} + \Phi_{\alpha}^{n+1} = -\sum_{C=1}^{n+1} \omega_{C}^{\alpha} \wedge \omega^{C} - \sum_{i=1}^{n} \omega_{i}^{n+1} \wedge \omega_{\alpha}^{i} +$$

$$+ \frac{1}{2} \sum_{C,D=1}^{n+1} K_{\alpha CD}^{n+1} \omega^{C} \wedge \omega^{D} = \sum_{i=1}^{n} \omega_{\alpha}^{i} \wedge \omega^{i}$$

Direct computations show that

$$- \sum_{a,b=m+1}^{n} h_{ab} \, \omega_\alpha^a \wedge \omega^b + \sum_{b=m+1}^{n} \omega^b \wedge \omega_\alpha^b = \tfrac{1}{2} \sum_{i,j=1}^{n} K^{n+1}_{\alpha ij} \, \omega^i \wedge \omega^j \qquad (12)$$

We have for hyperbolic spaces $K^{n+1}_{\alpha ij} = 0$ and so from equation

(12) we get

$$\sum_{a,b=m+1}^{n} (h_{ab} - \delta_{ab}) \, \omega_\alpha^a \wedge \omega^b = 0 \qquad (13)$$

Putting $h_{ab} - \delta_{ab} = A_{ab}$ in (13) we obtain

$$\sum_{a,b=m+1}^{n} A_{ab} \, \omega_\alpha^a \wedge \omega^b = 0$$

which is very similar to the corresponding relation in E^{n+1}, [10].

Proceeding exactly as in [10] we have

$$\sum_{a=m+1}^{n} A_{ab} \, \omega_\alpha^a \wedge \prod_c \omega^c = 0$$

Since $\det (A_{ab}) \neq 0$ we get

$$\omega_\alpha^a \wedge \prod_c \omega^c = 0$$

hence we may write

$$\omega_\alpha^a = \sum_{b=m+1}^{n} \lambda_{\alpha ab} \, \omega^b , \quad \tilde{\omega}_a^{n+1} = \omega_a^{n+1} - \omega_a^a \qquad (14)$$

This choice gives that

$$\prod_a \tilde{\omega}_a^{n+1} = D \prod_c \omega^c \qquad (15)$$

Exterior differentiating equation (15), we have

$$d\tilde{\omega}_a^{n+1} = \sum_{b=m+1}^{n} \tilde{\omega}_b^{n+1} \wedge \omega_b^a \qquad (16)$$

Also

$$d\tilde{\omega}_a^{n+1} = \sum_{\alpha=1}^{m} \omega^\alpha \wedge \omega_\alpha^a + \sum_{b=m+1}^{n} \omega^b \wedge \omega_b^a \qquad (17)$$

Substituting from (14) into (17) we have

$$d\omega^{a} = \sum_{\alpha,\bar{a}} \lambda_{\alpha ab}\, \omega^{\alpha} \wedge \omega^{b} + \sum_{b=m+1}^{n} \omega^{b} \wedge \omega_{b}^{a} \tag{18}$$

Using (18) we get

$$dD \wedge \prod_{c} \omega^{c} + D\Big(\sum_{a,\alpha} \lambda_{\alpha a a}\, \omega^{\alpha} \wedge \prod_{c} \omega^{c} \Big) = o$$

or

$$dD + D\Big(\sum_{a,\alpha} \lambda_{\alpha aa}\, \omega^{\alpha} \Big) = o \quad \mod \omega^{c}$$

Let $p \in M$ be a boundary point of U_m such that $x(p)$ is a limit point of a generating m-dimensional totally geodesic submanifold L of the tangent horosphere H_ν . Choose a neighbourhood U of p such that $x^{-1}(L) \subset U$. Let $\tilde{e}_1(q),\ldots,\tilde{e}_{n+1}(q)$, $q \in U$, be a local cross-section of U in $T(M)^{\perp}$, such that, for $q \in x^{-1}(L)$, $\tilde{e}_1(q),\ldots,\tilde{e}_m(q)$ span L. If $\bar{\omega}^{i}$, $\bar{\omega}_{j}^{i}$ are the restrictions of ω^{i}, $\tilde{\omega}_{j}^{i}$, respectively, to this cross-section, then $\bar{\omega}^{i}$ are linearly independent and we will have

$$\bar{\omega}_{a}^{\alpha} = \sum_{b=m+1}^{n} \lambda_{\alpha ab}\, \omega^{b}$$

Let γ be a curve in x^{-1} (L) abutting at p. Along γ we have

$$dD + D\Big(\sum_{a,\alpha} \lambda_{\alpha aa}\, \omega^{\alpha} \Big) = o$$

and by integration we obtain

$$D(q) = D_o \exp\Big(- \int \sum_{a,\alpha} \lambda_{\alpha aa}\, \omega^{\alpha} \Big)$$

for $q \in \gamma - p_o$, where $D_o \neq o$ is the value of D at a fixed point of γ . As D(q) is a continuous function and since $\lambda_{\alpha aa}$ is bounded then we have $D(p) \neq o$ which means that $p \in U_m$ and the proof of the lemma is now complete.

We now complete the proof of the theorem. Let E represent the space of all horospheres in H. A tangent horosphere is said to be of rank m if it is tangent to x(M) at a point of $x(U_m)$ and at no point of $x(U_\ell)$, $\ell < m$. A tangent horosphere of rank zero does not separate x(M) otherwise the total absolute curvature $\tau(x)$ will be greater than $2c_n$ contradicting the hypothesis of the theorem.

We will show that in every neighbourhood $\tilde{U}$ in E of a tangent horosphere Π of x(M) there is a tangent horosphere of rank zero. Let x(p), $p \in U_m$, be a point of contact of Π. Either there is a neighbourhood of p in M which belongs completely to U_m or there are points of U_ℓ, $\ell < m$, in every neighbourhood of p. In both cases there exists a point p_1 such that the tangent horosphere Π_1 at $x(p_1)$ belongs to $\tilde{U}$ and such that p_1 has a neighbourhood in M which belongs completely to U_ℓ, $\ell \leq m$. The image under x of this neighbourhood of p_1 is generated by ℓ-dimensional totally geodesic submanifolds and the tangent horosphere to x(M) along the ℓ-dimensional totally geodesic generating submanifold through $x(p_1)$ is Π_1. If $x(p_2)$, $p_2 \in M$, is a boundary point of this ℓ-submanifold, p_2 belongs to U_ℓ by the last lemma and is not an interior point of U_ℓ. Hence there exists in every neighbourhood about p_2 an open set whose points are in U_k, $k < \ell$, and which contains a point $p_3 \in U_k$ such that the tangent horosphere at $x(p_3)$ is in $\tilde{U}$. Continuation of this process gives that $\tilde{U}$ contains a tangent horosphere of rank zero of x(M).

This argument shows that every neighbourhood of Π in E contains a tangent horosphere such that x(M) lies on one side. It follows that the same is true for Π itself which proves the necessity part of the theorem.

Conversely, let x(M) lies on one side of each tangent horosphere. It can be shown that the degree of the generalized Gauss map is exactly 1 which gives that $\tau(x) = 2c_n$.

APPENDICES

Appendix (i)

Let $(P, GL(n,IR), M)$ be the principal bundle of basis with an affine connection Γ and let $(T(M), GL(n,IR), IR^n, M)$ be the associated bundle with typical fibre IR^n.

The bundle of basis - as a principal bundle - can be looked at as the set of maps $p : IR^n \to M_m$, $m \in M$, defined as follows [2]: If $p = (m, e_1, \ldots, e_n) \in P$ and $f = (f_1, \ldots, f_n) \in IR^n$, then

$$p(f) = (m, \sum_{i=1}^{n} f_i e_i)$$

Let π and π' be the natural projections of P and T(M), respectively. Let γ be a broken C^∞ curve in M, $b \in \pi'^{-1}(\gamma(o))$. We define a lifting $\bar{\gamma}$ of γ into T(M) which will turn out to be horizontal in the sense below.

Let $f \in IR^n$ and $p \in P$ be such that $\pi(p) = \gamma(o)$ and $pf = b$. We know that there is a horizontal lifting $\tilde{\gamma}$ of γ into P with $\tilde{\gamma}(o) = p$. Now define $\bar{\gamma}(t) = \tilde{\gamma}(t) \circ f$. It is clear that $\gamma(o) = \tilde{\gamma}(o) \circ f = pf = b$. We define the parallel translation $\bar{\phi}_t$, in the tangent bundle T(M), along γ from $\pi'^{-1}(\gamma(o))$ to $\pi'^{-1}(\gamma(t))$ to be

$$\bar{\phi}_t = \phi_t(p) \circ p^{-1}$$

where ϕ_t denotes the parallel translation in P, so $\bar{\phi}_t$ is a diffeomorphism.

Take $b \in T(M)$, $p \in P$ such that $\pi'(b) = \pi(p)$. We may view P_p as a subspace of $(P \times IR^n)_{(p,f)}$ where $f \in IR^n$ is such that $pf = b$. Let the map $\lambda : P \times IR^n \to T(M)$ be defined by $\lambda(p,f) = pf$ and define $H'_b = \lambda_*(H_p)$. This definition is independent of p in view of the right invariance of H, while it is clear that the lift defined above is horizontal with respect to H , if the definition of the map

$\phi : \mathbb{R}^n \to \pi'^{-1}(\pi(p))$ is recalled. This argument shows that there is a distribution H' on T(M) which at each point complements the vertical tangent space.

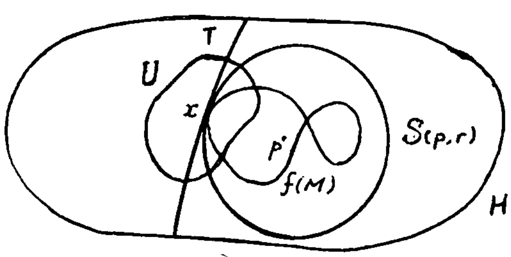
The bundle viewpoint provides a natural "jumping off" for generalizations to connexions in all kinds of bundles and much of the research in differential geometry at this time uses these concepts.

Appendix (ii)

L. Amaral [1] proved his theorem (I.1.2) through considering the H-model of hyperbolic spaces but his proof seems to be difficult in computations. In the following we give an easy proof of this theorem which is valid in hyperbolic spaces in general not for a special model.

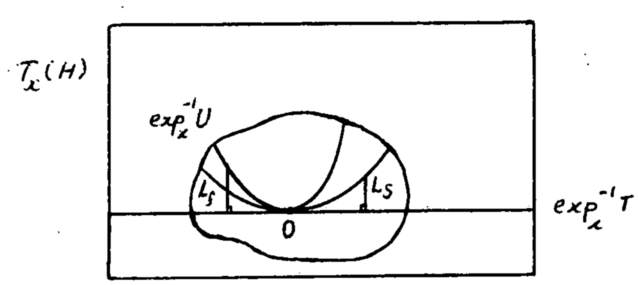
Let $f : M \to H$ be an isometric immersion of the compact n-manifold M into the (n+1)-hyperbolic space H of sectional curvature -1. Since f(M) is a compact immersed hypersurface of H, then there exists a geodesic sphere S(p,r) of finite radius r and center $p \in H$ which contains f(M) and such that $f(M) \cap S(p,r) \neq \Phi$.

Since H is a complete, simply connected, Riemannian manifold without focal points, then the study in (§1 - chapter III) shows that the geodesic sphere S(p,r) is a convex hypersurface of H and hence lies on one side of each tangent totally geodesic hypersurface T at $x \in f(M) \cap S(p\ ,r)$. (see figure 1 below).



(Fig. 1)

Consider U to be a small neighbourhood of x in H. By using the map $\exp_x^{-1} : H \to T_x H$ we get the following picture in $T_x H$.



(Fig.2)

It is clear from (Fig.2) that the height functions L_S, L_f of both S(p,r) and f(M) above $\exp_x^{-1} T$ have 0 as a critical point (maximum or minimum according to the chosen orientation). Now choose a convenient orientation of f(M) such that the eigenvalues of the hessian matrices of L_S and L_f are all positive. Let λ be the eigenvalue for S(p,r) (The reason is that S(p,r) is an unbilical hypersurface of H - see § 6 - chapter 0) while $\lambda_1, \ldots, \lambda_n$ are the eigenvalues for f(M) at x.

From (Fig.2) we can see easily that $\lambda_i \geqslant \lambda$ for all i, $1 \leqslant i \leqslant n$. Applying in Gauss equation ((2.7) § 2-chapter 0) we have

$$K_M(e_i, e_j) = -1 + \lambda_i \lambda_j \geqslant -1 + \lambda^2 \qquad (1)$$

where K_M denotes the sectional curvature of the immersion f at x and $\{e_1, \ldots, e_n\}$ denote the orthonormal basis of $T_x(M)$ which are themselves the eigenvectors corresponding to $\{\lambda_1, \ldots, \lambda_n\}$, respectively. As we know (§ 6- chapter 0) S(p,r) has constant positive sectional curvature K_S and

$$K_S(e_i, e_j) = -1 + \lambda^2 = 1/\sinh^2 r > o \qquad (2)$$

It is clear from equations (1) and (2) that

$$K_M(e_i, e_j) > o$$

for all $1 \leqslant i, j \leqslant n$ and this completes the proof of the theorem.

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