Supersymmetric gauge theories and their supercurrents

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SUPERSYMMETRIC GAUGE THEORIES

AND THEIR SUPERCURRENTS

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ABSTRACT

Using the method of dimensional reduction of N=1 supersymmetric Yang-Mills theories from higher dimensions down to four dimensions, all possible supersymmetric Yang-Mills theories in four dimensions are obtained.

The conserved currents associated with the symmetries of these models are then developed using Noether's theorem in ordinary space-time. By the variation of these conserved currents under supersymmetry transformations the supercurrent multiplets for the different models are obtained.

Supersymmetric gauge theories are then discussed in superspace where differential geometry can be used to obtain Bianchi identities for the supersymmetric field strengths. The constraints on the field strengths that give rise to off-shell representations for each of the different supersymmetric gauge theories are then obtained and off-shell Lagrangians written down. The connection with supersymmetric gauge theories in ordinary space-time is made.

The supercurrents in superspace are then derived using the generalization of Noether's theorem to superspace.

PREFACE

The work presented in this thesis was carried out at the Department of Mathematical Sciences, Durham between October 1981 and July 1984 and has not been submitted previously for any other degree.

No claim of originality is made for the contents of chapter 2. In chapter 3 original work is presented in sections 3.4 and 3.5 , some of which was published in Nuclear Physics B229 (1983) 142 by the author. Except as otherwise noted, the material presented in sections 4.3 and 4.4 of chapter 4 is original work, the contents of the later section forming the material of a Durham University preprint by the author. Original material is also presented in sections 5.2 and 5.3 of chapter 5, this forming the subject of Nuclear Physics B241 (1984) 243.

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CONTENTS

CHAPTER I INTRODUCTION

The idea of a symmetry relating bosonic and fermionic particles is an appealing one in any process of unifying particle interactions. The first discussion of such a symmetry arose in dual resonance models formulated as 2 dimensional field theories [1] . The first model in 4 dimensions that related bosons and fermions by supersymmetry was proposed in [2] and comprised of a Majorana fermion together with scalar and pseudoscalar bosons (although there had already been previous papers [3] on similar ideas) .

Supersymmetric theories with higher spins were obtained later - the supersymmetric gauge theory by [4 , 5] and supergravity by [6] . The amount of supersymmetry can also be increased so that there are N supersymmetry generators instead of just 1 [7] .

Also, supersymmetry gives rise to a non-trivial extension of the Poincare group as the supersymmetry algebra [8] provides an extention of the Poincare algebra, so avoiding the no-go theorems of [9] . This means that for extended supergravity theories the Poincare and internal symmetries are unified into a single symmetry.

The second great attraction of supersymmetric theories is the improved ultraviolet behaviour that results from the supersymmetry. This appears in the form of the non-appearance of infinities that naively should arise. For some N=2 theories [10] and for the N=4 supersymmetric gauge theory [11,12], the supersymmetry restricts the theory so much that they are in fact completely finite to all orders in perturbation theory (although see [13]).

As the supersymmetry algebra is a non-trivial extension of the Poincare algebra, one can represent supersymmetry multiplets over the extension of normal space-time to superspace [14] which includes fermionic coordinates. Writing multiplets in terms of superfields over superspace results in a great simplification of the theory and allows calculations to be performed far more simply. In fact, for the finite theories it is by power counting arguments using superfield perturbation theory that the finiteness is proven [12].

The finiteness properties of supersymmetric theories are intimately connected with the supermultiplet of conserved currents associated with the supersymmetry [11]. The conserved spinor current that arises as a Noether current from the supersymmetry falls into a supermultiplet with the energy-momentum tensor, the other conserved currents for the theory and for extended supersymmetry other auxiliary quantities.

In this work I will discuss in detail supersymmetric gauge theories and their supercurrents in ordinary space-time and also in superspace. This process is started in chapter 2 where the $N = 1$, 2 and 4 supersymmetric gauge theories in ordinary space-time are obtained by the dimensional reduction technique of [15] . The N=2 theory is obtained by the reduction of the 6 dimensional N=1 supersymmetric gauge theory while the N=4 theory is obtained by reduction from 10 dimensions.

Following the normal procedure one now uses Noether's theorem in ordinary space-time to derive the conserved currents for these theories and this is done in chapter 3. The conseved currents are then modified in the standard fashion - the energy-momentum tensor is modified so as to be symmetric and as the theory is conformal, traceless. The spinor supersymmetry current is also modified so as to be pure spin $\frac{3}{2}$ [16] , this being because of the superconformal nature of supersymmetric gauge theories. There are also conserved U(N) (SU(4) for N=4) currents associated with the U(N) (SU(4)) symmetries of the theories.

Using the supersymmetry transfomations obtained in chapter 2, these conserved currents are varied so as to form a complete supersymmetry multiplet. For the N=1 theory, the conserved currents form a closed multiplet under supersymmetry without the need for additional quantities (for a superconformal theory) , but for the N=2 and N=4 theories additional auxiliary quantities are required to complete the multiplet. These auxiliary quantities appear to have no known geometrical significance.

In chapter 4 supersymmetric gauge theories are discussed in superspace. This is done by developing differential geometry in superspace [17] and then imposing constraints on the field strengths developed.

The content of the remaining field strengths are then studied by means of the Bianchi identities. For N=1 and N=2 supersymmetry the constraints imposed upon the field strengths [17 , 18] arise as integrability conditions for coupling lower spin supermultiplets to the supersymmetric gauge theories [19] . For the N=4 theory, using the same constraints as for the N=2 theory [18] leads to the equations of motion [20 , 21] . These constraints can however be relaxed as there are now no lower spin theories than the N=4 supersymmetric gauge theory itself. In 4 dimensions there are two minimal relaxations and both lead to the physical fields being off-shell. The relations resulting from the Bianchi identities are worked out in detail for one of these minimal relaxations and are shown to result in on-shell conditions for the auxiliary fields. So one is led to consider the non-minimal case when only the conventional constraint is applied. Here the self duality condition may be imposed without implying field equations (unlike the case of the minimal constraints where the self duality condition leads immediately to field equations) .

In this case it is possible to use a Lagrange multiplier method to write down a Lagrangian which propagates the physical fields. However such a method involves other fields propagating,so avoiding the counting arguments of [22] which exclude an off-shell representation of this form with just one set of physical fields. As in CP_n models and the action I_1 of [23] , some of the propagating fields have kinetic terms of the wrong sign. These fields must be eliminated by a set of constraints in order that only the physical fields which have the correct sign for the kinetic term actually propagate. The constraint used in this case is a non-linear Lagrange multiplier term as suggested by [24].

Also in this chapter, the connection between the formulation in superspace and the ordinary space-time formulation is made. The independent superfields in superspace have as their $\theta = \bar{\theta} = 0$ components the space-time fields of the theory. For the N=1 and abelian N=2 models, unconstrained formulations in superspace are also discussed as this is necessary in deriving the supercurrents in superspace by the generalization of Noether's theorem to superspace.

In chapter 5 the supercurrent multiplet in superspace is discussed. A superfield that contains all the components of the supercurrent multiplet is easily written down by taking the superfield whose $\theta = \bar{\theta} = 0$ component is the member of the multiplet with lowest dimension. For the N=1 theory this leads to a vector indexed supercurrent [16] with a clear geometrical interpretation, but for the N=2 theories the superfields are scalars [25 , 26] and have no immediate geometrical interpretation.

For a geometrical interpretation one turns to Noether's theorem in superspace. For the N=1 theory this leads [27 , 28] to a derivation of the vector indexed supercurrent of [16] . For the N=2 and N=4 theories, Noether's theorem in superspace again leads to a vector-indexed supercurrent with a clear geometrical interpretation [29] . The vector-indexed supercurrents obtained in this way are simply related to the scalar supercurrents of [25 , 26] .

CHAPTER 2 SUPERSYMMETRIC GAUGE THEORIES

As supersymmetric gauge theories are massless representations of supersymmetry with maximum spin 1, the number of supersymmetry generators must be less than or equal to 4 [30] otherwise higher spin particles arise. Also, the particle content of the N=3 theory is identical to the particle content of the CPT self-conjugate N=4 multiplet. So in 4 dimensions, the only pure supersymmetric gauge theories are those with N = 1 , 2 and 4 supersymmetries.

Of these, the supersymmetric gauge theory first constructed was the supersymmetric extension of the abelian gauge theory with N=1 supersymmetry [4], this later being extended to the non-abelian case by [5]. For the case of extended supersymmetry, the N=2 supersymmetric gauge theory without auxiliary fields was constructed in [31] and with the auxiliary fields by [18]. Finally, the N=4 supersymmetric gauge theory without auxiliary fields was obtained from the 10 dimensional supersymmetric string theory by [32].

2.I Obtaining Supersymmetric Gauge Theories by Dimensional Reduction

Using the dimensional reduction technique of [32] it is possible to obtain all the extended supersymmetric gauge theories by the dimensional reduction of unextended supersymmetric gauge theories from various higher dimensions [15]. The N=2 supersymmetric gauge theory in 4 dimensions is obtained by the dimensional reduction of the 6 dimensional N=1 theory and the N=4 theory in 4 dimensions by the dimensional reduction of the 10 dimensional N=1 theory.

It is this method that will be followed here in order to set up the models to be studied. In D dimensions the N=1 Lagrangian is

$$\mathcal{L} = \mathrm{tr} \left(- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{i}{2} \bar{\lambda} \Gamma \cdot \overleftrightarrow{\mathcal{D}} \lambda \right) \tag{2.1.1}$$

where the metric in D dimensions is

$$g_{\mu\nu} = \mathrm{diag} (-1 , +1 , . . . , +1) \tag{2.1.2}$$

and Γ_μ are the D dimensional gamma matrices.

Both fields are in the adjoint representation of an arbitary gauge group. The field strength and covariant derivative are given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu , A_\nu]$$

$$\mathcal{D}_\mu \lambda = \partial_\mu \lambda + i [A_\mu , \lambda] \tag{2.1.3}$$

The fields are Lie algebra valued

$$F_{\mu\nu} = F^a_{\mu\nu} T^a \quad , \quad \lambda = \lambda^a T^a \tag{2.1.4}$$

where T^a are the generators of the gauge group and satisfy

$$[T^a , T^b] = i f_{abc} T^c \tag{2.1.5}$$

The structure constants f_{abc} are real.

For later use, the equations of motion that result from this Lagrangian are

$$\mathcal{D}^\mu F_{\mu\nu} = \{ \bar{\lambda} , \Gamma_\nu \lambda \}$$

$$\Gamma \cdot \mathcal{D} \lambda = 0 \tag{2.1.6}$$

For the theory to be supersymmetric there must be the same number of degrees of freedom on-shell for the bosonic and fermionic fields. In D dimensions the gauge field A_μ has D-2 degrees of freedom whilst a Dirac spinor has $2^{D/2}$ when D is even and $2^{(D-1)/2}$ when D is odd. As these two numbers are not equal for any D, there must be a restriction on the spinors that reduces the number of fermionic degrees of freedom.

The possible restrictions that can be imposed are a Majorana condition

$$\lambda = C_D \bar{\lambda}^T \tag{2.1.7}$$

which is possible when D is 2 or 4 modulo 8, and a Weyl condition

$$\lambda = \pm \Gamma_{D+1} \lambda \tag{2.1.8}$$

which is possible for any even D. Here C_D is the charge conjugation matrix and

$$\Gamma_{D+1} = \Gamma^0 \Gamma^1 \dots \Gamma^D \tag{2.1.9}$$

It is also possible to impose both the above conditions when D is 2 modulo 8.

So in the cases of interest, in D=4 either the Majorana or Weyl condition can be chosen, for D=6 the Weyl condition must be imposed and in D=10 both the Majorana and Weyl conditions must be imposed.

It can now be checked that under the supersymmetry transformations

$$\delta A_\mu = i(\bar{\alpha}\Gamma_\mu \lambda - \bar{\lambda}\Gamma_\mu \alpha)$$

$$\delta \lambda = F_{\mu\nu}\Sigma^{\mu\nu}\alpha \qquad (2.1.10)$$

where $\Sigma^{\mu\nu} = \frac{1}{4}[\Gamma^\mu, \Gamma^\nu]$, the Lagrangian (2.1.1) transforms by a total derivative (so leaving the action invariant) for the cases of $D = 4$, 6 and 10 only [15].

The D=6 and D=10 theories can now be dimensionally reduced to D=4 whereupon they become N=2 and N=4 theories respectively. The reduction is the trivial one obtained by eliminating all dependence of the fields on any but the first four space-time components.

2.2 *The N=1 Theory*

For later purposes it will be convenient to write the N=1 theory using two component notation for the spinors.

The Lagrangian is

$$\mathcal{L} = \mathrm{tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}\bar{\lambda}_{\dot\alpha}\bar{\sigma}_\mu^{\dot\alpha\alpha}\overleftrightarrow{D}^\mu\lambda_\alpha\right) \qquad (2.2.1)$$

where the SL (2 , C) indices α, $\dot\alpha$ take the values 1 , 2.

The convention for raising and lowering SL (2 , C) indices is

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \quad , \quad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta \qquad (2.2.2)$$

where $\varepsilon^{12} = \varepsilon_{21} = -\varepsilon^{21} = -\varepsilon_{12} = 1$ (and similarly for dotted indices) .

The supersymmetry transformations are

$$\delta A_\mu = i(\bar{\xi}_{\dot\alpha}\bar{\sigma}_\mu^{\dot\alpha\alpha}\lambda_\alpha - \bar{\lambda}_{\dot\alpha}\bar{\sigma}_\mu^{\dot\alpha\alpha}\xi_\alpha)$$

$$\delta\lambda_\alpha = F_{\mu\nu}\sigma^{\mu\nu}{}_\alpha{}^\beta\xi_\beta \qquad (2.2.3)$$

where $\sigma^{\mu\nu}{}_{\alpha}{}^{\beta} = \frac{1}{4}\,(\sigma^{\mu}_{\alpha\dot{\alpha}}\,\bar{\sigma}^{\nu\dot{\alpha}\beta} - \sigma^{\nu}_{\alpha\dot{\alpha}}\,\bar{\sigma}^{\mu\dot{\alpha}\beta})$.

The equations of motion that result from (2.2.1) are

$$\mathcal{D}^{\mu} F_{\mu\nu} = \bar{\sigma}^{\dot{\alpha}\alpha}_{\nu} \{ \bar{\lambda}_{\dot{\alpha}} , \lambda_{\alpha} \}$$

$$\bar{\sigma}^{\dot{\alpha}\alpha}_{\mu}\, \mathcal{D}^{\mu}\, \lambda_{\alpha} = 0 \qquad (2.2.4)$$

2.3 *The N=2 Theory*

In 6 dimensions the Lagrangian is

$$\mathcal{L} = \mathrm{tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{i}{2} \bar{\lambda}\, \Gamma \cdot \overleftrightarrow{\mathcal{D}}\, \lambda \right) \qquad (2.3.1)$$

The spinor satisfies the Weyl condition

$$\lambda = \Gamma_7\, \lambda \qquad (2.3.2)$$

where the positive sign has been chosen for definiteness. Also, the following representation is chosen for the 6 dimensional gamma matrices

$$\Gamma^{\mu} = \gamma^{\mu} \times I_2 = \begin{pmatrix} \gamma^{\mu} & 0 \\ 0 & \gamma^{\mu} \end{pmatrix} , \quad \mu = 0 , 1 , 2 , 3$$

$$\Gamma^{4} = \gamma_5 \times i\sigma^{1} = \begin{pmatrix} 0 & i\gamma_5 \\ i\gamma_5 & 0 \end{pmatrix}$$

$$\Gamma^{5} = -\gamma_5 \times i\sigma^{2} = \begin{pmatrix} 0 & -\gamma_5 \\ \gamma_5 & 0 \end{pmatrix}$$

and so

$$\Gamma_7 = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5$$

$$= \gamma_5 \times \sigma^3$$

$$= \begin{pmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{pmatrix} \qquad (2.3.3)$$

The condition (2.3.2) implies that λ has the structure

$$\lambda = \begin{pmatrix} L\chi \\ R\chi \end{pmatrix} \qquad (2.3.4)$$

where $L = \frac{1}{2}(1 + \gamma_5)$, $R = \frac{1}{2}(1 - \gamma_5)$ and χ is a 4 dimensional Dirac spinor.

Now, defining the fields

$$A_\mu = A_\mu \; , \; \mu = 0, 1, 2, 3$$

$$A_4 = P$$

$$A_5 = S \qquad (2.3.5)$$

gives the Lagrangian in 4 dimensions upon performing a dimensional reduction as

$$\mathcal{L} = \mathrm{tr}\Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \mathcal{D}_\mu P \, \mathcal{D}^\mu P - \frac{1}{2} \mathcal{D}_\mu S \, \mathcal{D}^\mu S - \frac{i}{2} \overline{\chi} \gamma \cdot \overleftrightarrow{\mathcal{D}} \chi$$

$$+ i \overline{\chi} \gamma_5 [P, \chi] + \overline{\chi} [S, \chi] + \frac{1}{2} [P, S][P, S] \Big)$$

$$(2.3.6)$$

The supersymmetry transformations now become

$$\delta A_\mu = i (\overline{\alpha} \gamma_\mu \chi - \overline{\chi} \gamma_\mu \alpha)$$

$$\delta P = - (\overline{\alpha} \gamma_5 \chi - \overline{\chi} \gamma_5 \alpha)$$

$$\delta S = i (\overline{\alpha} \chi - \overline{\chi} \alpha)$$

$$\delta \chi = (F_{\mu\nu} \sigma^{\mu\nu} + i \gamma \cdot \mathcal{D} P \gamma_5 + \gamma \cdot \mathcal{D} S - [P, S] \gamma_5) \alpha$$

$$(2.3.7)$$

where the supersymmetry transformation parameter α is a 4 dimensional Dirac spinor.

The theory can now be written in 2 component notation as this will be more useful later on by using SU(2) doublets of Weyl spinors instead of Dirac spinors.

Taking a Weyl representation of the gamma matrices

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}$$

$$\gamma_5 = i\,\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where

$$\sigma^{\mu} = (1 , \underset{\sim}{\sigma}) \; , \; \bar{\sigma}^{\mu} = (1 , - \underset{\sim}{\sigma}) \qquad (2.3.8)$$

and σ^1, σ^2, σ^3 are the usual Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (2.3.9)$$

Now redefining the fields by

$$\chi = \begin{pmatrix} \lambda_{\alpha 1} \\ i\,\bar{\lambda}^{\dot{\alpha} 2} \end{pmatrix}$$

$$C = \frac{1}{2} (S - i P) \qquad (2.3.10)$$

gives a Lagrangian that has an SU(2) symmetry in the indices 1 and 2 :

$$\mathcal{L} = \mathrm{tr} \Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 2\, \mathcal{D}_{\mu} C^* \mathcal{D}^{\mu} C - \frac{i}{2} \bar{\lambda}^{i}_{\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\alpha}_{\mu} \overleftrightarrow{\mathcal{D}}^{\mu} \lambda_{\alpha i}$$
$$+ i\, g^{12} C^* \{ \lambda^{\alpha}_{i} , \lambda^{i}_{\alpha} \} - i\, g_{12}\, C \{ \bar{\lambda}_{\dot{\alpha} i} , \bar{\lambda}^{\dot{\alpha} i} \}$$
$$- 2 [C^* , C] [C^* , C] \Big) \qquad (2.3.11)$$

Defining

$$\alpha = \begin{pmatrix} \xi_{\alpha 1} \\ i\,\bar{\xi}^{\dot{\alpha}2} \end{pmatrix} \qquad (2.3.12)$$

gives the supersymmetry transformations as

$$\delta A_\mu = i\,(\bar{\xi}^i_{\dot{\alpha}}\,\bar{\sigma}^{\dot{\alpha}\alpha}_\mu\,\lambda_{\alpha i} - \bar{\lambda}^i_{\dot{\alpha}}\,\bar{\sigma}^{\dot{\alpha}\alpha}_\mu\,\xi_{\alpha i})$$

$$\delta C = g_{12}\,\lambda^\alpha_i\,\xi^i_\alpha$$

$$\delta\lambda^i_\alpha = F_{\mu\nu}\,\sigma^{\mu\nu}{}_\alpha{}^\beta\,\xi^i_\beta - 2\,i\,g_{12}\,\sigma^\mu_{\alpha\dot{\alpha}}\,\mathcal{D}_\mu C\,\bar{\xi}^{\dot{\alpha}i} + 2\,i\,[\,C^*, C\,]\,\xi^i_\alpha$$

(2.3.13)

The convention for raising and lowering SU(2) indices is

$$\psi^i = g^{ij}\,\psi_j \quad , \quad \psi_i = g_{ij}\,\psi^j \qquad (2.3.14)$$

where $g \equiv \varepsilon$.

With this convention and (2.2.2) for the spinor indices,

$$(\psi^\alpha_i)^\dagger = \bar{\psi}^{\dot{\alpha}i} \quad , \quad (\bar{\psi}^{\dot{\alpha}i})^\dagger = \psi^\alpha_i \qquad (2.3.15)$$

and so

$$(g^{ij})^* = g_{ij} = -\,g^{ij} \qquad (2.3.16)$$

From this it can be seen that the factors of g^{12} and g_{12} above are necessary in order to ensure that the relevant pieces of $\mathcal{L}$ are real. Under infinitesimal SU(2) transformations these factors are invariant.

It is possible to absorb the factors of g_{12} and g^{12} into the fields C , C^* in the above expressions. This gives rise to the Lagrangian

$$\mathcal{L} = \mathrm{tr}\Big(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + 2\, \mathcal{D}_\mu C^* \mathcal{D}^\mu C - \tfrac{i}{2} \bar{\lambda}^i_{\dot\alpha} \bar{\sigma}^{\dot\alpha\alpha}_\mu \overleftrightarrow{\mathcal{D}}^\mu \lambda_{\alpha i}$$

$$+ i\, C^* \{ \lambda^\alpha_i , \lambda^i_\alpha \} - i\, C \{ \bar{\lambda}_{\dot\alpha i} , \bar{\lambda}^{\dot\alpha i} \}$$

$$- 2 [C^* , C][C^* , C] \Big) \qquad (2.3.17)$$

and the supersymmetry transformations

$$\delta A_\mu = i (\bar{\xi}^i_{\dot\alpha} \bar{\sigma}^{\dot\alpha\alpha}_\mu \lambda_{\alpha i} - \bar{\lambda}^i_{\dot\alpha} \bar{\sigma}^{\dot\alpha\alpha}_\mu \xi_{\alpha i})$$

$$\delta C = \lambda^\alpha_i \xi^i_\alpha$$

$$\delta \lambda^i_\alpha = F_{\mu\nu} \sigma^{\mu\nu}{}_\alpha{}^\beta \xi^i_\beta - 2 i \sigma^\mu_{\alpha\dot\alpha} \mathcal{D}_\mu C \bar{\xi}^{\dot\alpha i} - 2 i [C^* , C] \xi^i_\alpha$$

$$(2.3.18)$$

However, one should note that when the factors of g_{12} and g^{12} are absorbed into C and C* in this way then the sign of the free Lagrangian density for the field C has the opposite sign to normal.

For later reference, the equations of motion that arise from the Lagrangian (2.3.17) are

$$\mathcal{D}^\mu F_{\mu\nu} = - 2 i [C^* , \overleftrightarrow{\mathcal{D}}_\nu C] + \bar{\sigma}^{\dot\alpha\alpha}_\nu \{ \bar{\lambda}^i_{\dot\alpha} , \lambda_{\alpha i} \}$$

$$\mathcal{D}_\mu \mathcal{D}^\mu C = \tfrac{i}{2} \{ \lambda^\alpha_i , \lambda^i_\alpha \} + 2 [C , [C , C^*]]$$

$$\bar{\sigma}^{\dot\alpha\alpha}_\mu \mathcal{D}^\mu \lambda^i_\alpha = - 2 [C , \bar{\lambda}^{\dot\alpha i}] \qquad (2.3.19)$$

2.4 *The N=4 Theory*

In 10 dimensions the Lagrangian is taken to be

$$\mathcal{L} = \mathrm{tr}\Big(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{i}{4} \bar{\lambda} \Gamma \cdot \overleftrightarrow{\mathcal{D}} \lambda \Big) \qquad (2.4.1)$$

with the supersymmetry transformations

$$\delta A_\mu = i\,\bar{\alpha}\,\Gamma_\mu\,\lambda$$

$$\delta\lambda = F_{\mu\nu}\,\Sigma^{\mu\nu}\,\alpha \qquad (2.4.2)$$

The spinors satisfy Majorana-Weyl conditions in 10 dimensions

$$\lambda = C_{10}\,\bar{\lambda}^T$$

$$\lambda = \Gamma_{11}\,\lambda \qquad (2.4.3)$$

where the positive sign in the Weyl condition has again been chosen for definiteness.

The following representation is chosen for the 10 dimensional gamma matrices

$$\Gamma^\mu = \gamma^\mu \times I_8 \qquad \mu = 0, 1, 2, 3$$

$$\Gamma^{i+3} = \Gamma^{i4} + \frac{1}{2}\varepsilon_{i4jk}\,\Gamma^{jk} \qquad i = 1, 2, 3$$

$$i\,\Gamma^{i+6} = \Gamma^{i4} - \frac{1}{2}\varepsilon_{i4jk}\,\Gamma^{jk} \qquad i = 1, 2, 3$$

where

$$\Gamma^{ij} = \gamma_5 \times \begin{pmatrix} 0 & \rho^{ij} \\ \rho_{ij} & 0 \end{pmatrix} \qquad i, j = 1, 2, 3$$

and

$$(\rho^{ij})_{k\ell} = \delta_{ik}\,\delta_{j\ell} - \delta_{i\ell}\,\delta_{jk}$$

$$(\rho_{ij})_{k\ell} = \frac{1}{2}\varepsilon_{ijmn}\,(\rho^{mn})_{k\ell}$$

$$= \varepsilon_{ijk\ell}$$

Also

$$\Gamma_{11} = \Gamma^0\,\Gamma^1 \ldots \Gamma^9 = \gamma_5 \times \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}$$

$$C_{10} = C_4 \times \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} \qquad (2.4.4)$$

where C_4 is the 4 dimensional charge conjugation matrix.

The Majorana-Weyl condition (2.4.3) now means that λ has the structure

$$\lambda = \begin{pmatrix} R\chi^1 \\ R\chi^2 \\ R\chi^3 \\ R\chi^4 \\ L\tilde{\chi}_1 \\ L\tilde{\chi}_2 \\ L\tilde{\chi}_3 \\ L\tilde{\chi}_4 \end{pmatrix} \qquad (2.4.5)$$

where

$$\tilde{\chi}_i = C_4 \bar{\chi}_i^T \qquad (2.4.6)$$

χ^i transform as a 4 of SU(4) and $\tilde{\chi}_i$ as a 4^*.

Now defining the fields

$$A_\mu = A_\mu \qquad \mu = 0, 1, 2, 3$$

$$\phi_{i4} = -\frac{1}{2}(A_{i+3} + i A_{i+6})$$

$$\phi^{jk} = \frac{1}{2}\varepsilon^{jk\ell m}\phi_{\ell m} = (\phi_{jk})^* \qquad (2.4.7)$$

and performing the dimensional reduction to 4 dimensions gives the Lagrangian as

$$\mathcal{L} = \mathrm{tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\phi_{ij}D^\mu\phi^{ij} - \frac{i}{2}\bar{\chi}_i\gamma\cdot\overleftrightarrow{D}R\chi^i$$
$$+ \{ \bar{\tilde{\chi}}^i , R\chi^j \}\phi_{ij} - \{ \bar{\chi}_i , L\tilde{\chi}_j \}\phi^{ij}$$
$$+ \frac{1}{4}[\phi_{ij} , \phi_{k\ell}][\phi^{ij} , \phi^{k\ell}] \Big) \qquad (2.4.8)$$

The supersymmetry transformations now read

$$\delta A_\mu = i\,(\bar{\alpha}_i \gamma_\mu R\chi^i - \bar{\chi}_i \gamma_\mu R\alpha^i)$$

$$\delta\phi_{ij} = i\,(\bar{\alpha}_{[i} L\tilde{\chi}_{j]} - \varepsilon_{ijk\ell}\, \bar{\tilde{\alpha}}^k R\chi^\ell)$$

$$\delta R\chi^i = F_{\mu\nu}\sigma^{\mu\nu} R\alpha^i + 2\gamma\cdot\mathcal{D}\phi^{ij} L\tilde{\alpha}_j + 2i[\phi^{ik}, \phi_{kj}]R\alpha^j \tag{2.4.9}$$

The equations of motion that result from the Lagrangian (2.4.8) are

$$\mathcal{D}^\mu F_{\mu\nu} = i[\phi_{ij}, \mathcal{D}_\nu \phi^{ij}] + \{\bar{\chi}_i, \gamma_\nu R\chi^i\}$$

$$\mathcal{D}_\mu \mathcal{D}^\mu \phi_{ij} = \{\bar{\chi}_i, L\tilde{\chi}_j\} - \tfrac{1}{2}\varepsilon_{ijk\ell}\{\bar{\tilde{\chi}}^k, R\chi^\ell\} + [\phi^{k\ell}, [\phi_{k\ell}, \phi_{ij}]]$$

$$\gamma\cdot\mathcal{D}\chi^i = -2i[\phi^{ij}, \tilde{\chi}_j] \tag{2.4.10}$$

Again, it will be convenient for later purposes to write down this theory in an explicit 2 component formulation. In this case the answer is simply obtained by defining the Weyl spinor $\bar{\lambda}^{\dot{\alpha} i}$ by

$$R\chi^i = \begin{pmatrix} 0 \\ \bar{\lambda}^{\dot{\alpha} i} \end{pmatrix} \tag{2.4.11}$$

Also, the Weyl representation (2.3.8) of the 4 dimensional gamma matrices is taken. In this representation, the charge conjugation matrix C_4 is

$$C_4 = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \tag{2.4.12}$$

So rewriting the Lagrangian (2.4.8) above in 2 component notation gives

$$\mathcal{L} = \mathrm{tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mathcal{D}_\mu\phi_{ij}\,\mathcal{D}^\mu\phi^{ij} - \frac{i}{2}\bar{\lambda}^i_{\dot{\alpha}}\,\bar{\sigma}^{\dot{\alpha}\alpha}_\mu\,\overleftrightarrow{\mathcal{D}}^\mu\,\lambda_{\alpha i}$$

$$+ \{ \bar{\lambda}^i_{\dot{\alpha}} , \bar{\lambda}^{\dot{\alpha} j} \}\,\phi_{ij} - \{ \lambda^\alpha_i , \lambda_{\alpha j} \}\,\phi^{ij}$$

$$+ \frac{1}{4}[\phi_{ij} , \phi_{k\ell}][\phi^{ij} , \phi^{k\ell}] \Big) \qquad (2.4.13)$$

CHAPTER 3 THE SUPERMULTIPLET OF CURRENTS

3.1 *Noether's Theorem and Conserved Currents*

As well as the conserved energy-momentum tensor, the theories derived from the Lagrangian (2.1.1) have various other conserved currents obtained from Noether's theorem that are associated with their symmetries. For the N=1 theory, as well as the energy-momentum tensor $\Theta_{\mu\nu}$ there is a conserved spinor current J_μ associated with the supersymmetry and also an axial current $j_\mu^{(5)}$ associated with the U(1) invariance under chiral transformations of the spinors. As the theory is superconformally invariant, the currents can be improved so that the energy-momentum tensor is traceless, $\Theta_\mu{}^\mu = 0$, and the spinor current J_μ contains only a spin $\frac{3}{2}$ component, $\gamma \cdot J = 0$.

The currents for the superconformal and also for the non-superconformal cases were first studied in [16]. They found that the above currents transformed into each other under supersymmetry transformations to form a supersymmetry multiplet. In the non-superconformal case there are also auxiliary quantities that are necessary to form a multiplet.

For N=2 supersymmetry, the conserved currents again form a multiplet under supersymmetry. This supercurrent was first obtained by [25] for the non-superconformal case. Again, auxiliary quantities are required to form the multiplet in addition to the conserved currents, but in this case they are also required for the superconformal multiplet.

The situation for the case of N=4 supersymmetry is similar, but here the multiplet is automatically superconformal as the theory involved is the N=4 supersymmetric gauge theory which is finite and so superconformally invariant to all orders in perturbation theory[11,12]. The multiplet of currents for this theory was first obtained by [33] .

In this chapter the conserved currents for the N = 1 , 2 and 4 supersymmetric gauge theories of chapter 2 will be derived from Noether's theorem and by studying their transformations under supersymmetry, the full supercurrent multiplets will be obtained for the superconformal case.

Associated with the Lagrangian (2.1.1) is a conserved, gauge-invariant and symmetric energy-momentum tensor $\Theta_{\mu\nu}$ that is obtained by adding improvement terms to the energy-momentum tensor $\tilde{\Theta}_{\mu\nu}$ derived from Noether's theorem.

As the close similarity of Noether's theorem in superspace and Noether's theorem in ordinary space will be exhibited later, it will be convenient to derive Noether's theorem here.

If we consider a symmetry under which the fields transform as

$$\delta\phi = \frac{\delta\phi}{\delta\omega^a}\,\delta\omega^a \qquad (3.1.1)$$

then the transformation of a Lagrangian $\mathcal{L}$ that depends only on the fields ϕ and their derivatives $\partial_\mu\phi$ can be calculated in two ways.

In the first the equations of motion

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu\phi)} = \frac{\partial\mathcal{L}}{\partial\phi} \qquad (3.1.2)$$

are used.

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial^\mu\phi)}\delta\,\partial^\mu\phi$$

$$= \partial^\mu\left(\frac{\partial \mathcal{L}}{\partial\,\partial^\mu\phi}\,\frac{\delta\phi}{\delta\omega^a}\right)\delta\omega^a \qquad (3.1.3)$$

using (3.1.2) .

But the transformation of $\mathcal{L}$ can also be performed directly

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\omega^a}\,\delta\omega^a \qquad (3.1.4)$$

So Noether's theorem states that the current

$$j_a^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\frac{\delta\phi}{\delta\omega^a} \qquad (3.1.5)$$

has the divergence

$$\partial_\mu\, j_a^\mu = -\frac{\delta\mathcal{L}}{\delta\omega^a} \qquad (3.1.6)$$

For example, this can now be applied to the translational invariance of a Lagrangian. Under a translation $x^\mu \to x^\mu + a^\mu$ the field $\phi(x)$ transforms as

$$\delta\phi(x) = a^\mu\,\partial_\mu\,\phi(x) \qquad (3.1.7)$$

and similarly

$$\delta\mathcal{L}(x) = a^{\mu}\,\partial_{\mu}\,\mathcal{L}(x) \tag{3.1.8}$$

So the energy-momentum tensor

$$\tilde{\Theta}_{\mu\nu} = \frac{\partial\,\mathcal{L}}{\partial(\partial^{\mu}\phi)}\,\partial_{\nu}\phi - g_{\mu\nu}\,\mathcal{L} \tag{3.1.9}$$

is conserved by Noether's theorem when the equations of motion are used.

3.2 *The Energy-Momentum Tensor and Spinor Supersymmetry Current in D Dimensions*

When (3.1.9) is applied to the Lagrangian (2.1.1) above, a non-gauge invariant, non-symmetric $\tilde{\Theta}_{\mu\nu}$ is obtained

$$\tilde{\Theta}_{\mu\nu} = \mathrm{tr}\Big(- F_{\mu\rho}\,\partial_{\nu}\,A^{\rho} + \frac{1}{4}\,g_{\mu\nu}\,F_{\rho\lambda}\,F^{\rho\lambda} - \frac{i}{2}\,\bar{\lambda}\,\Gamma_{\mu}\,\overleftrightarrow{\partial}_{\nu}\,\lambda - \frac{i}{2}\,g_{\mu\nu}\,\bar{\lambda}\,\Gamma\cdot\overleftrightarrow{D}\,\lambda\Big) \tag{3.2.1}$$

However, this energy-momentum tensor $\tilde{\Theta}_{\mu\nu}$ can be modified to give a gauge-invariant energy-momentum tensor $\Theta'_{\mu\nu}$. The additional term $\Delta\Theta_{\mu\nu}$ that cancels the gauge dependence of $\tilde{\Theta}_{\mu\nu}$ is

$$\Delta\Theta_{\mu\nu} = \partial^{\rho}\,\mathrm{tr}\,(\,F_{\mu\rho}\,A_{\nu}\,) \tag{3.2.2}$$

This automatically satisfies

$$\partial^{\mu}\,\Delta\Theta_{\mu\nu} = 0 \tag{3.2.3}$$

and also

$$\int d^{3}x\,\Delta\Theta_{0\mu} = 0 \tag{3.2.4}$$

which means that the generators of 4-momentum

$$P^{\mu} = \int d^3x\, \Theta^{0\mu} \qquad (3.2.5)$$

are unaltered by the change $\tilde{\Theta}_{\mu\nu} \to \Theta'_{\mu\nu} = \tilde{\Theta}_{\mu\nu} + \Delta\Theta_{\mu\nu}$.

By using the equations of motion the term $\Delta\Theta_{\mu\nu}$ can be rewritten as

$$\Delta\Theta'_{\mu\nu} = \mathrm{tr}\left(F_{\mu\rho}\,\partial^{\rho} A_{\nu} + i\, F_{\mu\rho}[A^{\rho}, A_{\nu}] + \bar{\lambda}\,\Gamma_{\mu}[A_{\nu}, \lambda] \right) \qquad (3.2.6)$$

Now

$$\begin{aligned}\Theta'_{\mu\nu} &= \tilde{\Theta}_{\mu\nu} + \Delta\Theta'_{\mu\nu} \\ &= \mathrm{tr}\left(F_{\mu\rho} F^{\rho}{}_{\nu} + \frac{1}{4} g_{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} - \frac{i}{2}\bar{\lambda}\,\Gamma_{\mu} \overleftrightarrow{D}_{\nu}\lambda \right. \\ &\qquad \left. + \frac{i}{2} g_{\mu\nu}\bar{\lambda}\,\Gamma\cdot\overleftrightarrow{D}\lambda \right) \qquad (3.2.7)\end{aligned}$$

This is now gauge-invariant and the part of the energy-momentum tensor for the gauge field is also symmetric. However, the part of the energy-momentum tensor for the spinor field is not symmetric. The antisymmetric part of $\bar{\lambda}\Gamma_{\mu}\overleftrightarrow{D}_{\nu}\lambda$ can now be cancelled using the equations of motion by adding on a term that is automatically conserved.

$$\begin{aligned}&\mathrm{tr}\left(\bar{\lambda}\,\Gamma_{[\mu} \overleftrightarrow{D}_{\nu]}\,\lambda \right) \\ &= \partial^{\rho}\,\mathrm{tr}\left(\frac{1}{(D-3)!}\,\varepsilon_{\mu\nu\rho\lambda_1\lambda_2\ldots\lambda_{D-3}}\,\bar{\lambda}\,\Gamma_{D+1}\,\Gamma^{\lambda_1}\Gamma^{\lambda_2}\ldots\Gamma^{\lambda_{D-3}}\lambda \right) \\ &\quad - 2\,\mathrm{tr}\left(D\bar{\lambda}\cdot\Gamma\,\Sigma_{\mu\nu}\lambda + \bar{\lambda}\,\Sigma_{\mu\nu}\,\Gamma\cdot D\lambda \right) \qquad (3.2.8)\end{aligned}$$

which makes use of the following identities for gamma matrices in D dimensions (D even)

$$g^{\mu\rho}\Gamma^{\nu} - g^{\nu\rho}\Gamma^{\mu} = -2\Gamma^{\rho}\Sigma^{\mu\nu} + \frac{1}{(D-3)!}\varepsilon^{\mu\nu\rho\lambda_1\lambda_2\cdots\lambda_{D-3}}\Gamma_{D+1}\Gamma_{\lambda_1}\Gamma_{\lambda_2}\cdots\Gamma_{\lambda_{D-3}}$$

$$g^{\nu\rho}\Gamma^{\mu} - g^{\mu\rho}\Gamma^{\nu} = -2\Sigma^{\mu\nu}\Gamma^{\rho} + \frac{1}{(D-3)!}\varepsilon^{\mu\nu\rho\lambda_1\lambda_2\cdots\lambda_{D-3}}\Gamma_{D+1}\Gamma_{\lambda_1}\Gamma_{\lambda_2}\cdots\Gamma_{\lambda_{D-3}} \tag{3.2.9}$$

So the final form of the energy-momentum tensor becomes

$$\Theta_{\mu\nu} = \mathrm{tr}\left(F_{\mu\rho}F^{\rho}{}_{\nu} + \frac{1}{4}g_{\mu\nu}F_{\rho\lambda}F^{\rho\lambda} - \frac{i}{4}\bar{\lambda}\Gamma_{(\mu}\overleftrightarrow{\mathcal{D}}_{\nu)}\lambda \right) \tag{3.2.10}$$

This is conserved using the equations of motion, gauge-invariant and symmetric.

The Lagrangian (2.1.1) also has an invariance under supersymmetry which gives rise via Noether's theorem to a conserved spinor current.

For supersymmetry transformations

$$\frac{\delta \mathcal{L}}{\delta \omega^{a}} \sim \partial_{\mu}\Lambda^{\mu} \tag{3.2.11}$$

and so the current $J^{\mu} = j^{\mu} + \Lambda^{\mu}$ is conserved,

$$\partial_{\mu}J^{\mu} = 0 \tag{3.2.12}$$

In the case in hand,

$$j^{\mu} = -\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\frac{\delta\phi}{\delta\bar{\alpha}}$$

$$\bar{j}^{\mu} = -\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\frac{\delta\phi}{\delta\alpha} \tag{3.2.13}$$

and

$$\delta \mathcal{L} = \partial_\mu \overline{\Lambda}^\mu \alpha + \overline{\alpha} \partial_\mu \Lambda^\mu \tag{3.2.14}$$

From the Lagrangian (2.1.1), calculating (3.2.13) gives

$$j^\mu = \mathrm{tr}\left(i F^{\mu\nu} \Gamma_\nu \lambda + \frac{i}{2} F_{\nu\rho} \Sigma^{\nu\rho} \Gamma^\mu \lambda \right) \tag{3.2.15}$$

and upon applying a supersymmetry transformation to $\mathcal{L}$

$$\Lambda^\mu = \mathrm{tr}\left(-\frac{i}{2} F_{\nu\rho} \Sigma^{\nu\rho} \Gamma^\mu \lambda + \frac{i}{2} F_{\nu\rho} \frac{1}{(D-3)!} \varepsilon^{\nu\rho\mu\lambda_1\lambda_2\ldots\lambda_{D-3}} \Gamma_{D+1} \Gamma_{\lambda_1} \Gamma_{\lambda_2} \ldots \Gamma_{\lambda_{D-3}} \right) \tag{3.2.16}$$

Combining these gives the conserved spinor supersymmetry current as

$$J^\mu = \mathrm{tr}\left(i F_{\nu\rho} \Sigma^{\nu\rho} \Gamma^\mu \lambda \right) \tag{3.2.17}$$

3.3 *The N=1 Supercurrent Multiplet*

For the N=1 model when D=4 for the Lagrangian (2.1.1), the energy-momentum tensor $\Theta_{\mu\nu}$(3.2.10) reads

$$\Theta_{\mu\nu} = \mathrm{tr}\left(F_{\mu\rho} F^\rho{}_\nu + \frac{1}{4} g_{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} - \frac{i}{4} \overline{\lambda} \gamma_{(\mu} \overleftrightarrow{D}_{\nu)} \lambda \right) \tag{3.3.1}$$

This is conserved, gauge- invariant and symmetric. As the theory is a conformal one it can be adjusted by the addition of terms which vanish by the equations of motion to give a traceless energy-momentum tensor

$$\Theta_{\mu\nu} = \mathrm{tr}\left(F_{\mu\rho} F^\rho{}_\nu + \frac{1}{4} g_{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} - \frac{i}{4} \overline{\lambda} \gamma_{(\mu} \overleftrightarrow{D}_{\nu)} \lambda + \frac{i}{8} g_{\mu\nu} \overline{\lambda} \gamma \cdot \overleftrightarrow{D} \lambda \right) \tag{3.3.2}$$

The conserved spinor supersymmetry current from (3.2.17) is

$$J^{\mu} = \mathrm{tr}\left[i\, F_{\nu\rho}\, \sigma^{\nu\rho}\, \gamma^{\mu}\, \lambda \right] \qquad (3.3.3)$$

This automatically contains only the spin $\frac{3}{2}$ part

$$\gamma \cdot J = 0 \qquad (3.3.4)$$

as is required for a superconformal theory [16] . In this case no improvement term is needed.

The Lagrangian also has a U(1) invariance under a chiral transformation of the spinors in the infinitesimal form

$$\delta\lambda = i\, \beta\, \gamma_5\, \lambda \qquad (3.3.5)$$

This results in the conserved current

$$j^{(5)}_{\mu} = \mathrm{tr}\left[i\, \bar{\lambda}\, \gamma_{\mu}\, \gamma_5\, \lambda \right] \qquad (3.3.6)$$

Under supersymmetry transformations these conserved currents $\Theta_{\mu\nu}$, J_{μ} and $j^{(5)}_{\mu}$ are transformed into each other to form the supercurrent multiplet. In this case no auxiliary quantities are necessary in order to complete the multiplet for the superconformal case.

$$\delta j^{(5)}_{\mu} = 2\, \bar{\alpha}\, \gamma_5\, J_{\mu}$$

$$\delta J_{\mu} = (- 2i\, \gamma^{\nu} \Theta_{\mu\nu} + \frac{i}{2} \gamma \cdot \partial\, \gamma_5\, j^{(5)}_{\mu} - \frac{i}{4} \varepsilon_{\mu\nu\rho\lambda}\, \gamma^{\nu}\, \partial^{\rho}\, j^{(5)\lambda})\, \alpha$$

$$\delta\Theta_{\mu\nu} = - \frac{1}{2} (\bar{\alpha}\, \sigma_{\mu\rho}\, \partial^{\rho} J_{\nu} + \bar{\alpha}\, \sigma_{\nu\rho}\, \partial^{\rho} J_{\mu}) \qquad (3.3.7)$$

When this is extended to the non-superconformal case, in addition to the above currents which now no longer satisfy $\Theta_\mu{}^\mu = \gamma \cdot J = \partial \cdot j^{(5)} = 0$, auxiliary quantities become necessary in order to form a multiplet under supersymmetry.

3.4 *The N=2 Supercurrent Multiplet*

The energy-momentum tensor (3.2.10) in 6 dimensions can be reduced to 4 dimensions to give

$$\tilde{\Theta}_{\mu\nu} = \mathrm{tr}\Big(F_{\mu\rho} F^{\rho}{}_{\nu} + \frac{1}{4} g_{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} - \frac{i}{4} \bar{\lambda}^{i}_{\dot\alpha} \bar{\sigma}_{(\mu}{}^{\dot\alpha\alpha} \overleftrightarrow{\mathcal{D}}_{\nu)} \lambda_{\alpha i} + 2\mathcal{D}_{(\mu} C^* \mathcal{D}_{\nu)} C - 2 g_{\mu\nu} \mathcal{D}_\rho C^* \mathcal{D}^\rho C + 2 g_{\mu\nu} [C^*, C][C^*, C] \Big) \quad (3.4.1)$$

where the field definitions (2.3.4), (2.3.5) and (2.3.10) have been used.

This can be rewritten using the equations of motion as

$$\Theta'_{\mu\nu} = \mathrm{tr}\Big(F_{\mu\rho} F^{\rho}{}_{\nu} + \frac{1}{4} g_{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} - \frac{i}{4} \bar{\lambda}^{i}_{\dot\alpha} \bar{\sigma}_{(\mu}{}^{\dot\alpha\alpha} \overleftrightarrow{\mathcal{D}}_{\nu)} \lambda_{\alpha i} + \frac{i}{8} g_{\mu\nu} \bar{\lambda}^{i}_{\dot\alpha} \bar{\sigma}_{\rho}^{\dot\alpha\alpha} \overleftrightarrow{\mathcal{D}}^{\rho} \lambda_{\alpha i} + 2\mathcal{D}_{(\mu} C^* \mathcal{D}_{\nu)} C - g_{\mu\nu} \big(2\mathcal{D}_\rho C^* \mathcal{D}^\rho C + \frac{1}{2} (C^* \mathcal{D}_\rho \mathcal{D}^\rho C + C \mathcal{D}_\rho \mathcal{D}^\rho C^*) \big) \Big) \quad (3.4.2)$$

This energy-momentum tensor is not yet traceless. However, a term

$$\Delta\Theta_{\mu\nu} = (\partial^\mu \partial^\nu - g^{\mu\nu} \partial_\rho \partial^\rho)\, \mathrm{tr}(C^* C) \quad (3.4.3)$$

which is automatically conserved can be added on to this without altering P^μ (3.2.5).

Computing the trace of $\Theta'_{\mu\nu} + \alpha\,\Delta\Theta_{\mu\nu}$ gives a zero answer for $\alpha = -2/3$, so the conserved, gauge-invariant, symmetric and traceless energy-momentum tensor is

$$\Theta_{\mu\nu} = \mathrm{tr}\Big(F_{\mu\rho} F^{\rho}{}_{\nu} + \tfrac{1}{4} g_{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} - \tfrac{i}{4} \bar{\lambda}^{i}_{\dot\alpha} \bar{\sigma}_{(\mu}^{\dot\alpha\alpha} \overleftrightarrow{D}_{\nu)} \lambda_{\alpha i}$$
$$+ \tfrac{i}{8} g_{\mu\nu} \bar{\lambda}^{i}_{\dot\alpha} \bar{\sigma}^{\dot\alpha\alpha}_{\rho} \overleftrightarrow{D}^{\rho} \lambda_{\alpha i} + \tfrac{4}{3} D_{(\mu} C^{*} D_{\nu)} C$$
$$- \tfrac{2}{3} (C^{*} D_{\mu} D_{\nu} C + C D_{\mu} D_{\nu} C^{*}) - g_{\mu\nu} (\tfrac{2}{3} D_{\rho} C^{*} D^{\rho} C$$
$$- \tfrac{1}{6} (C^{*} D_{\rho} D^{\rho} C + C D_{\rho} D^{\rho} C^{*})) \Big) \qquad (3.4.4)$$

Similarly, reducing (3.2.17) from 6 dimensions to 4 dimensions gives the spinor supersymmetry current

$$\tilde{J}^{\mu i}_{\alpha} = \mathrm{tr}\Big(i F_{\nu\rho} \sigma^{\nu\rho}{}_{\alpha}{}^{\beta} \sigma^{\mu}_{\beta\dot\alpha} \bar{\lambda}^{\dot\alpha i} - 2 D_{\nu} C^{*} \sigma^{\nu}_{\alpha\dot\alpha} \bar{\sigma}^{\mu\dot\alpha\beta} \lambda^{i}_{\beta}$$
$$- 2 \sigma^{\mu}_{\alpha\dot\alpha} C^{*} [C , \bar{\lambda}^{\dot\alpha i}] \Big) \qquad (3.4.5)$$

The last term can be rewritten using the equations of motion (2.3.19) to give

$$J^{\mu i \prime}_{\alpha} = \mathrm{tr}\Big(i F_{\nu\rho} \sigma^{\nu\rho}{}_{\alpha}{}^{\beta} \sigma^{\mu}_{\beta\dot\alpha} \bar{\lambda}^{\dot\alpha i} - 2 D_{\nu} C^{*} \sigma^{\nu}_{\alpha\dot\alpha} \bar{\sigma}^{\mu\dot\alpha\beta} \lambda^{i}_{\beta}$$
$$+ C^{*} \sigma^{\mu}_{\alpha\dot\alpha} \bar{\sigma}^{\nu\dot\alpha\beta} D_{\nu} \lambda^{i}_{\beta} \Big) \qquad (3.4.6)$$

This does not satisfy $\bar{\sigma}^{\dot\alpha\alpha}_{\mu} J^{\mu i \prime}_{\alpha} = 0$ as requires by superconformal invariance. However, to (3.4.6) can be added

$$\Delta J^{\mu i}_{\alpha} = \sigma^{\mu\nu}{}_{\alpha}{}^{\beta} \partial_{\nu} \mathrm{tr} (C^{*} \lambda^{i}_{\beta}) \qquad (3.4.7)$$

which is automatically conserved and leaves the generator of supersymmetry transformations,

$$Q^{i}_{\alpha} = \int d^{3}x \, J_{0\alpha}^{\;i} \qquad (3.4.8)$$

unchanged.

Computing $\bar{\sigma}^{\dot\alpha\alpha}_{\mu} J^{\mu i}_{\alpha}$ for $J^{\mu i}_{\alpha} = J^{\mu i \prime}_{\alpha} + \alpha \Delta J^{\mu i}_{\alpha}$ gives zero for $\alpha = -8/3$, so giving the spinor supersymmetry current as

$$J^{\mu i}_{\alpha} = \mathrm{tr}\Big(i\, F_{\nu\rho}\, \sigma^{\nu\rho}{}_{\alpha}{}^{\beta}\, \sigma^{\mu}_{\beta\dot\alpha}\, \bar\lambda^{\dot\alpha i} - 2\, \mathcal{D}_{\nu} C^{*}\, \sigma^{\nu}_{\alpha\dot\alpha}\, \bar\sigma^{\mu\dot\alpha\beta}\, \lambda^{i}_{\beta}$$

$$+ C^{*}\, \sigma^{\mu}_{\alpha\dot\alpha}\, \bar\sigma^{\nu\dot\alpha\beta}\, \mathcal{D}_{\nu}\, \lambda^{i}_{\beta} - \frac{8}{3}\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\, \mathcal{D}_{\nu}\,(C^{*} \lambda^{i}_{\beta}) \Big) \qquad (3.4.9)$$

The Lagrangian (2.3.17) also has U(1) and SU(2) invariances which lead to conserved currents via Noether's theorem. The U(1) invariance is under a simultaneous chiral transformation of λ^{i}_{α} and phase transformation of C which reads in infinitesimal form

$$\delta C = i\,\alpha\, C$$

$$\delta\lambda^{i}_{\alpha} = \frac{i}{2}\,\alpha\,\lambda^{i}_{\alpha} \qquad (3.4.10)$$

This gives rise to the conserved current

$$j^{(5)}_{\mu} = -\,\mathrm{tr}\,(2\, i\, C^{*} \overleftrightarrow{\mathcal{D}}_{\mu}\, C + \frac{1}{2}\bar\lambda_{\dot\alpha i}\, \bar\sigma^{\dot\alpha\alpha}_{\mu}\, \lambda^{i}_{\alpha}) \qquad (3.4.11)$$

The SU(2) transformations in infinitesimal form are

$$\delta\lambda^{i}_{\alpha} = i\, \varepsilon_{A}\, \tau_{A}{}^{i}{}_{j}\, \lambda^{j}_{\alpha} \qquad (3.4.12)$$

where τ_{A} are the Pauli matrices (2.3.9).

This invariance gives rise to the conserved current

$$T_{\mu i}{}^{j} = \mathrm{tr}\,(\bar\lambda_{\dot\alpha i}\, \bar\sigma^{\dot\alpha\alpha}_{\mu}\, \lambda^{j}_{\alpha} - \frac{1}{2}\,\delta^{j}_{i}\, \bar\lambda_{\dot\alpha k}\, \bar\sigma^{\dot\alpha\alpha}_{\mu}\, \lambda^{k}_{\alpha}) \qquad (3.4.13)$$

Under supersymmetry transformations these conserved currents do not form a closed multiplet. The additional auxiliary quantities

$$D_{1} = \mathrm{tr}\,(C^{*} C)$$

$$\chi_{1\alpha}{}^{i} = \mathrm{tr}\,(C^{*} \lambda^{i}_{\alpha})$$

$$t_{1\alpha}{}^{\beta} = \mathrm{tr}\,(\,C^{*} F_{\mu\nu}\,\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\,) \qquad (3.4.14)$$

are required to form a multiplet under supersymmetry with the following transformations under supersymmetry

$$\delta D_1 = -\bar{\chi}_{1\dot{\alpha} i}\,\bar{\xi}^{\dot{\alpha} i} - \xi_i^{\alpha}\,\chi_{1\alpha}^{\ i}$$

$$\delta\chi_{1\alpha}^{\ i} = t_{1\alpha}{}^{\beta}\,\xi_{\beta}^{i} - \frac{1}{2}\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,j^{(5)}_{\mu}\,\bar{\xi}^{\dot{\alpha} i} - \frac{1}{2}\,T_{\mu j}{}^{i}\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,\bar{\xi}^{\dot{\alpha} j}$$
$$+\, i\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,\partial_{\mu} D_1\,\bar{\xi}^{\dot{\alpha} i}$$

$$\delta t_{1\alpha}{}^{\beta} = \frac{i}{8}\,(\,J_{\mu i}^{\ \beta}\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,\bar{\xi}^{\dot{\alpha} i} - \bar{\xi}_{\dot{\alpha} i}\,\bar{\sigma}^{\dot{\alpha}\beta}_{\mu}\,J^{\mu i}_{\ \ \alpha}\,)$$
$$-\frac{2}{3}\,i\,(\,\bar{\xi}_{\dot{\alpha} i}\,\bar{\sigma}^{\dot{\alpha}\beta}_{\mu}\,\partial^{\mu}\,\chi_{1\alpha}^{\ i} - \partial_{\mu}\,\chi_{1i}^{\ \beta}\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,\bar{\xi}^{\dot{\alpha} i}\,)$$

$$\delta j^{(5)}_{\mu} = \frac{i}{2}\,\xi_i^{\alpha}\,J_{\mu\alpha}^{\ \ i} + \frac{i}{2}\,\bar{\xi}_{\dot{\alpha} i}\,\bar{J}^{\dot{\alpha} i}_{\mu} + \frac{8}{3}\,i\,\xi_i^{\alpha}\,\sigma_{\mu\nu\alpha}{}^{\beta}\,\partial^{\nu}\,\chi_{1\beta}^{\ i}$$
$$+\frac{8}{3}\,i\,\bar{\xi}_{\dot{\alpha} i}\,\bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}}\,\partial^{\nu}\,\bar{\chi}_1^{\dot{\beta} i}$$

$$\delta T_{\mu i}{}^{j} = -\,i\,J_{\mu i}^{\ \alpha}\,\xi_{\alpha}^{j} + i\,\bar{\xi}_{\dot{\alpha} i}\,\bar{J}^{\dot{\alpha} j}_{\mu} + \frac{1}{2}\,\delta_i^j\,(\,i\,J_{\mu k}^{\ \alpha}\,\xi_{\alpha}^{k} - i\,\bar{\xi}_{\dot{\alpha} k}\,\bar{J}^{\dot{\alpha} k}_{\mu}\,)$$
$$+\frac{4}{3}\,i\,\xi_i^{\alpha}\,\sigma_{\mu\nu\alpha}{}^{\beta}\,\partial^{\nu}\,\chi_{1\beta}^{\ j} - \frac{4}{3}\,i\,\partial^{\nu}\,\chi_{1i}^{\ \alpha}\,\sigma_{\mu\nu\alpha}{}^{\beta}\,\xi_{\beta}^{j}$$
$$+\frac{4}{3}\,i\,\partial^{\nu}\,\bar{\chi}_{1\dot{\alpha} i}\,\bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}}\,\bar{\xi}^{\dot{\beta} j} - \frac{4}{3}\,i\,\bar{\xi}_{\dot{\alpha} i}\,\bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}}\,\partial^{\nu}\,\bar{\chi}_1^{\dot{\beta} j}$$

$$\delta J^{\mu i}_{\ \ \alpha} = -\,2\,i\,\sigma_{\nu\alpha\dot{\alpha}}\,\bar{\xi}^{\dot{\alpha} i}\,\Theta^{\mu\nu} - 4\,(\,\partial_{\nu}\,t_{1\alpha}{}^{\beta}\,\sigma^{\mu\nu}{}_{\beta}{}^{\gamma} + \frac{1}{3}\,\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\,\partial_{\nu}\,t_{1\beta}{}^{\gamma}\,)\,\xi_{\gamma}^{i}$$
$$+\frac{1}{3}\,(\,\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\,\sigma^{\rho}_{\beta\dot{\alpha}} - 3\,\sigma^{\rho}_{\alpha\dot{\beta}}\,\bar{\sigma}^{\mu\nu\dot{\beta}}{}_{\dot{\alpha}}\,)\,\bar{\xi}^{\dot{\alpha} j}\,(\,-\,\delta_j^i\,\partial_{\nu}\,j^{(5)}_{\rho} + 2\,\partial_{\nu}\,T_{\rho j}{}^{i}\,)$$

$$\delta\Theta^{\mu\nu} = -\frac{1}{2}\,\xi_i^{\alpha}\,(\,\sigma^{\mu\rho}{}_{\alpha}{}^{\beta}\,\partial_{\rho}\,J^{\nu i}_{\ \ \beta} + \sigma^{\nu\rho}{}_{\alpha}{}^{\beta}\,\partial_{\rho}\,J^{\mu i}_{\ \ \beta}\,)$$
$$+\frac{1}{2}\,\bar{\xi}_{\dot{\alpha} i}\,(\,\bar{\sigma}^{\mu\rho\dot{\alpha}}{}_{\dot{\beta}}\,\partial_{\rho}\,\bar{J}^{\nu\dot{\beta} i} + \bar{\sigma}^{\nu\rho\dot{\alpha}}{}_{\dot{\beta}}\,\partial_{\rho}\,\bar{J}^{\mu\dot{\beta} i}\,) \qquad (3.4.15)$$

3.5 *The N=4 Supercurrent Multiplet*

Similarly to the above cases, one can obtain a conserved, gauge-invariant, symmetric and traceless energy-momentum tensor by reducing (3.2.10) from 10 to 4 dimensions, adding on terms that are automatically conserved and then using the equations of motion to rewrite it. The answer is

$$\Theta_{\mu\nu} = \mathrm{tr}\Big(F_{\mu\rho} F^{\rho}{}_{\nu} + \tfrac{1}{4} g_{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} - \tfrac{i}{4} \bar{\chi}_i \gamma_{(\mu} \overleftrightarrow{\mathcal{D}}_{\nu)} R\chi^i + \tfrac{i}{8} g_{\mu\nu} \bar{\chi}_i \gamma \cdot \overleftrightarrow{\mathcal{D}} R\chi^i - \tfrac{2}{3} \mathcal{D}_\mu \phi_{ij} \mathcal{D}_\nu \phi^{ij} + \tfrac{1}{3} \phi_{ij} \mathcal{D}_\mu \mathcal{D}_\nu \phi^{ij} + g_{\mu\nu} \big(\tfrac{1}{6} \mathcal{D}_\rho \phi_{ij} \mathcal{D}^\rho \phi^{ij} - \tfrac{1}{12} \phi_{ij} \mathcal{D}_\rho \mathcal{D}^\rho \phi^{ij} \big) \Big) \tag{3.5.1}$$

Similarly, the spinor supersymmetry current is obtained by reducing (3.2.17) from 10 to 4 dimensions, adding on improvement terms that are automatically conserved and then using the equations of motion to rewrite it.

$$J^{\mu i} = \mathrm{tr}\Big(i F_{\nu\rho} \sigma^{\nu\rho} \gamma^\mu R\chi^i + 2 i \mathcal{D}_\nu \phi^{ij} \gamma^\nu \gamma^\mu L\tilde{\chi}_j - i \phi^{ij} \gamma^\mu \gamma^\nu \mathcal{D}_\nu L\tilde{\chi}_j + \tfrac{8}{3} i \sigma^{\mu\nu} \mathcal{D}_\nu (\phi^{ij} L\tilde{\chi}_j) \Big) \tag{3.5.2}$$

This is conserved using the equations of motion and satisfied $\gamma^\mu J^i_\mu = 0$.

The Lagrangian (2.4.8) also has an SU(4) invariance which gives rise to the conserved currents

$$T^\mu{}_i{}^j = \mathrm{tr}\Big(\bar{\chi}_i \gamma^\mu R\chi^j - \tfrac{1}{4} \delta_i^j \bar{\chi}_k \gamma^\mu R\chi^k - i \phi_{ik} \overleftrightarrow{\mathcal{D}}^\mu \phi^{kj} \Big) \tag{3.5.3}$$

For this theory there is however no U(1) invariance of the Lagrangian.

Again auxiliary quantities are required to form a multiplet under supersymmetry, these being

$$d^{ij}{}_{k\ell} = \mathrm{tr}\,(\,\phi^{ij}\,\phi_{k\ell} - \tfrac{1}{12}\,\delta^{[i}_{k}\,\delta^{j]}_{\ell}\,\phi_{mn}\,\phi^{mn}\,)$$

$$\chi^{ij}{}_{k} = \tfrac{1}{2}\,\varepsilon^{ij\ell m}\,\mathrm{tr}\,(\,\phi_{\ell m}\,L\tilde{\chi}_k + \phi_{km}\,L\tilde{\chi}_\ell\,)$$

$$t^{\mu\nu}_{ij} = \mathrm{tr}\,(\,i\,\phi_{ij}\,(\,F^{\mu\nu} + \tfrac{i}{2}\,\varepsilon^{\mu\nu\rho\lambda}\,F_{\rho\lambda}\,) + \bar{\chi}_i\,\sigma^{\mu\nu}\,L\tilde{\chi}_j\,)$$

$$e_{ij} = \mathrm{tr}\,(\,\bar{\chi}_i\,L\tilde{\chi}_j - \tfrac{4}{3}\,\phi_{ik}\,[\,\phi^{k\ell}\,,\,\phi_{\ell j}\,]\,)$$

$$\lambda_i = \mathrm{tr}\,(\,F_{\mu\nu}\,\sigma^{\mu\nu}\,L\tilde{\chi}_i + 2\,i\,\phi_{ij}\,[\,\phi^{jk}\,,\,L\tilde{\chi}_k\,]\,)$$

$$C = \mathrm{tr}\,(\,\tfrac{1}{2}\,F_{\mu\nu}\,(\,F^{\mu\nu} - \tfrac{i}{2}\,\varepsilon^{\mu\nu\rho\lambda}\,F_{\rho\lambda}\,) + 2\,\bar{\tilde{\chi}}^{i}\,[\,R\chi^{j}\,,\,\phi_{ij}\,]$$
$$+ \tfrac{1}{2}\,[\,\phi_{ij}\,,\,\phi_{k\ell}\,]\,[\,\phi^{ij}\,,\,\phi^{k\ell}\,]\,) \qquad (3.5.4)$$

Under supersymmetry transformations these quantities together with the conserved currents transform into each other as

$$\delta d^{ij}{}_{k\ell} = \tfrac{4}{3}\,i\,\bar{\alpha}_{[k}\,\chi^{ij}{}_{\ell]} - \tfrac{2}{3}\,i\,\delta^{[i}_{[k}\,\bar{\alpha}_m\,\chi^{mj]}{}_{\ell]} + \mathrm{h.c.}$$

$$\delta\chi^{ij}{}_{k} = -\tfrac{3}{4}\,i\,\varepsilon^{ij\ell m}\,\sigma\cdot t_{k\ell}\,L\tilde{\alpha}_m - \tfrac{1}{2}\,i\,\delta^{[i}_{k}\,\sigma\cdot t^{j]\ell}\,L\tilde{\alpha}_\ell$$
$$+ \tfrac{3}{4}\,i\,\varepsilon^{ij\ell m}\,e_{k\ell}\,L\tilde{\alpha}_m - \tfrac{3}{4}\,i\,\gamma\cdot T_k{}^{[i}\,R\alpha^{j]}$$
$$- \tfrac{1}{4}\,i\,\delta^{[i}_{k}\,\gamma\cdot T_\ell{}^{j]}\,R\alpha^\ell - \tfrac{3}{2}\,\not{\partial}\,d^{ij}{}_{k\ell}\,R\alpha^\ell$$

$$\delta e_{ij} = -\bar{\alpha}_{(i}\,\lambda_{j)} - \tfrac{2}{3}\,\varepsilon_{k\ell m(i}\,\bar{\tilde{\alpha}}^{k}\,\not{\partial}\,\chi^{\ell m}{}_{j)}$$

$$\delta t^{\mu\nu}_{ij} = -\tfrac{1}{2}\,i\,\varepsilon_{ijk\ell}\,\bar{\tilde{\alpha}}^{k}\,\gamma^\rho\,\sigma^{\mu\nu}\,J^{\ell}_{\rho} + \bar{\alpha}_{[i}\,\sigma^{\mu\nu}\,\lambda_{j]}$$
$$- \tfrac{2}{3}\,\varepsilon_{ijk\ell}\,\bar{\tilde{\alpha}}^{m}\,\gamma^\rho\,\sigma^{\mu\nu}\,\partial_\rho\,\chi^{k\ell}{}_{m}$$

$$\delta\lambda_i = - C^* L\tilde{\alpha}_i + i\, \sigma_{\mu\nu} \not\partial\, t^{\mu\nu}_{ij} R\alpha^j - i \not\partial\, e_{ij} R\alpha^j$$

$$\delta\Theta^{\mu\nu} = - \frac{1}{2} \overline{\alpha}_i\, \sigma^{(\mu\rho}\, \partial_\rho\, J^{\nu)i} + \text{h.c.}$$

$$\delta C = 2\, i\, \overline{\alpha}_i \not\partial\, \tilde{\lambda}^i \qquad (3.5.5)$$

CHAPTER 4 SUPERSYMMETRIC GAUGE THEORIES IN SUPERSPACE

The idea of defining supersymmetric theories over superspace instead of ordinary space-time, and the great simplification of the theory that results, was originated by [14]. This was applied to the unextended supersymmetric gauge theory by [5] . [17] used differential geometry in superspace to reach a better understanding of supersymmetric gauge theories in superspace, and using these techniques the off-shell N=2 supersymmetric gauge theory was first derived in superspace by [18].

The extension of these methods to find the off-shell N=4 supersymmetric gauge theory was attempted by [20 , 21] but the constraints considered there were found to put the theory on-shell automatically.

In this chapter differential geometry in superspace will be reviewed and the resulting independent Bianchi identities will be written down [20]. Constraints will then be presented for the N=1 and N=2 cases that give rise to the off-shell N=1 and N=2 supersymmetric gauge theories. For the N=4 case the relaxation of the constraints of [20 , 21] will be considered and the resulting theories studied.

4.1 *Differential Geometry in Superspace*

Superspace consists of ordinary space-time with coordinates x^{μ} and the 4N anticommuting coordinates θ^{α}_{i} , $\bar{\theta}^{\dot{\alpha} j}$

where α , $\dot{\alpha}$ are 2 component spinor indices and $i , j = 1 , \ldots , N$ are SU(N) indices.

The algebra of N-extended supersymmetry (with no central charges) is

$$\{ Q^i_\alpha , Q^j_\beta \} = \{ \overline{Q}_{\dot{\alpha} i} , \overline{Q}_{\dot{\beta} j} \} = [P_\mu , P_\nu] = [Q^i_\alpha , P_\mu] = [\overline{Q}_{\dot{\alpha} i} , P_\mu] = 0$$

$$\{ Q^i_\alpha , \overline{Q}_{\dot{\alpha} j} \} = 2 \delta^i_j \sigma^\mu_{\alpha\dot{\alpha}} P_\mu \qquad (4.1.1)$$

This is represented in superspace by the differential operators

$$P_\mu = i \partial_\mu$$

$$Q^i_\alpha = \frac{\partial}{\partial \theta^\alpha_i} - i \sigma^\mu_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha} i} \partial_\mu$$

$$\overline{Q}_{\dot{\alpha} i} = - \frac{\partial}{\partial \overline{\theta}^{\dot{\alpha} i}} + i \theta^\alpha_i \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \qquad (4.1.2)$$

A superfield transforms under supersymmetry transformations as

$$\delta\phi = (\xi^\alpha_i Q^i_\alpha + \overline{Q}_{\dot{\alpha} i} \overline{\xi}^{\dot{\alpha} i}) \phi \qquad (4.1.3)$$

The 'covariant' spinor derivatives in superspace must anticommute with the supersymmetry generators

$$\{ D^i_\alpha , Q^j_\beta \} = \{ D^i_\alpha , \overline{Q}_{\dot{\alpha} j} \} = \{ \overline{D}_{\dot{\alpha} i} , Q^j_\beta \} = \{ \overline{D}_{\dot{\alpha} i} , \overline{Q}_{\dot{\beta} j} \} = 0 \qquad (4.1.4)$$

They are represented by

$$D^i_\alpha = \frac{\partial}{\partial \theta^\alpha_i} + i \sigma^\mu_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha} i} \partial_\mu$$

$$\overline{D}_{\dot{\alpha} i} = - \frac{\partial}{\partial \overline{\theta}^{\dot{\alpha} i}} - i \theta^\alpha_i \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \qquad (4.1.5)$$

and so satisfy

$$\{D^i_\alpha , \overline{D}_{\dot{\alpha} j}\} = -2i\,\delta^i_j\,\sigma^\mu_{\alpha\dot{\alpha}}\,\partial_\mu \qquad (4.1.6)$$

Under a gauge transformation a covariant superfield transforms as

$$\phi \rightarrow e^{i\Lambda}\,\phi\,e^{-i\Lambda} \qquad (4.1.7)$$

where the gauge parameter Λ is Lie algebra valued

$$\Lambda = T^a \lambda^a (x , \theta , \overline{\theta}) \qquad (4.1.8)$$

T^a being the generators of the gauge group.

As usual, gauge potentials A_μ, A^i_α, $\overline{A}_{\dot{\alpha} i}$ are introduced which transform under gauge transformations as

$$A_\mu \rightarrow e^{i\Lambda} A_\mu e^{-i\Lambda} - i\,e^{i\Lambda}\,\partial_\mu\,e^{-i\Lambda}$$

$$A^i_\alpha \rightarrow e^{i\Lambda} A^i_\alpha e^{-i\Lambda} - i\,e^{i\Lambda}\,D^i_\alpha\,e^{-i\Lambda}$$

$$\overline{A}_{\dot{\alpha} i} \rightarrow e^{i\Lambda} \overline{A}_{\dot{\alpha} i} e^{-i\Lambda} - i\,e^{i\Lambda}\,\overline{D}_{\dot{\alpha} i}\,e^{-i\Lambda} \qquad (4.1.9)$$

These transformations are chosen so that, introducing the gauge covariant derivatives $\mathcal{D}_\mu$, $\mathcal{D}^i_\alpha$, $\overline{\mathcal{D}}_{\dot{\alpha} i}$

$$\mathcal{D}_\mu \phi = \partial_\mu \phi + i [A_\mu , \phi]$$

$$\mathcal{D}^i_\alpha \phi = D^i_\alpha \phi + i [A^i_\alpha , \phi]$$

$$\overline{\mathcal{D}}_{\dot{\alpha} i} \phi = \overline{D}_{\dot{\alpha} i} \phi + i [\overline{A}_{\dot{\alpha} i} , \phi] \qquad (4.1.10)$$

Then under a gauge transformation

$$\mathcal{D}_\mu \phi \rightarrow e^{i\Lambda}\,\mathcal{D}_\mu \phi\,e^{-i\Lambda}$$

$$\mathcal{D}^i_\alpha \phi \rightarrow e^{i\Lambda}\,\mathcal{D}^i_\alpha \phi\,e^{-i\Lambda}$$

$$\overline{\mathcal{D}}_{\dot{\alpha} i} \phi \rightarrow e^{i\Lambda}\,\overline{\mathcal{D}}_{\dot{\alpha} i} \phi\,e^{-i\Lambda} \qquad (4.1.11)$$

The graded commutator of two gauge covariant derivatives give field strengths in the usual fashion

$$\{ \mathcal{D}^i_\alpha , \mathcal{D}^j_\beta \} = i \mathcal{F}^{ij}_{\alpha\beta}$$

$$\{ \bar{\mathcal{D}}_{\dot\alpha i} , \bar{\mathcal{D}}_{\dot\beta j} \} = i \mathcal{F}_{\dot\alpha i \dot\beta j}$$

$$\{ \mathcal{D}^i_\alpha , \bar{\mathcal{D}}_{\dot\alpha j} \} = i \mathcal{F}^i_{\alpha\dot\alpha j} - 2 i \sigma^\mu_{\alpha\dot\alpha} \delta^i_j \mathcal{D}_\mu$$

$$[\mathcal{D}_\mu , \mathcal{D}^i_\alpha] = i \mathcal{F}_{\mu\alpha}{}^i$$

$$[\mathcal{D}_\mu , \bar{\mathcal{D}}_{\dot\alpha i}] = i \mathcal{F}_{\mu\dot\alpha i}$$

$$[\mathcal{D}_\mu , \mathcal{D}_\nu] = i \mathcal{F}_{\mu\nu} \qquad (4.1.12)$$

These are again Lie algebra valued superfields

$$\mathcal{F}^{ij}_{\alpha\beta} = D^i_\alpha \mathcal{A}^j_\beta + D^j_\beta \mathcal{A}^i_\alpha + i \{ \mathcal{A}^i_\alpha , \mathcal{A}^j_\beta \}$$

$$\mathcal{F}_{\dot\alpha i \dot\beta j} = \bar{D}_{\dot\alpha i} \bar{\mathcal{A}}_{\dot\beta j} + \bar{D}_{\dot\beta j} \bar{\mathcal{A}}_{\dot\alpha i} + i \{ \bar{\mathcal{A}}_{\dot\alpha i} , \bar{\mathcal{A}}_{\dot\beta j} \}$$

$$\mathcal{F}^i_{\alpha\dot\alpha j} = D^i_\alpha \bar{\mathcal{A}}_{\dot\alpha j} + \bar{D}_{\dot\alpha j} \mathcal{A}^i_\alpha + i \{ \mathcal{A}^i_\alpha , \bar{\mathcal{A}}_{\dot\alpha j} \} + 2 i \sigma^\mu_{\alpha\dot\alpha} \delta^i_j \mathcal{A}_\mu$$

$$\mathcal{F}_{\mu\alpha}{}^i = \partial_\mu \mathcal{A}^i_\alpha - D^i_\alpha \mathcal{A}_\mu + i [\mathcal{A}_\mu , \mathcal{A}^i_\alpha]$$

$$\mathcal{F}_{\mu\dot\alpha i} = \partial_\mu \bar{\mathcal{A}}_{\dot\alpha i} - \bar{D}_{\dot\alpha i} \mathcal{A}_\mu + i [\mathcal{A}_\mu , \bar{\mathcal{A}}_{\dot\alpha i}]$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i [\mathcal{A}_\mu , \mathcal{A}_\nu] \qquad (4.1.13)$$

They are gauge covariant superfields and transform under gauge transformations according to (4.1.7).

By means of generalized Jacobi identities, the following Bianchi identities may be obtained

$$\mathcal{D}^i_\alpha \mathcal{F}^{jk}_{\beta\gamma} + \mathcal{D}^j_\beta \mathcal{F}^{ki}_{\gamma\alpha} + \mathcal{D}^k_\gamma \mathcal{F}^{ij}_{\alpha\beta} = 0$$

$$\bar{\mathcal{D}}_{\dot\alpha i} \mathcal{F}_{\dot\beta j \dot\gamma k} + \bar{\mathcal{D}}_{\dot\beta j} \mathcal{F}_{\dot\gamma k \dot\alpha i} + \bar{\mathcal{D}}_{\dot\gamma k} \mathcal{F}_{\dot\alpha i \dot\beta j} = 0$$

$$\bar{\mathcal{D}}_{\dot\alpha i}\,\mathcal{F}^{jk}_{\beta\gamma} + \mathcal{D}^j_\beta\,\mathcal{F}^k_{\gamma\dot\alpha i} + \mathcal{D}^k_\gamma\,\mathcal{F}^j_{\beta\dot\alpha i} + 2\,i\,\sigma^\mu_{\beta\dot\alpha}\,\delta^j_i\,\mathcal{F}_{\mu\gamma}{}^k + 2\,i\,\sigma^\mu_{\gamma\dot\alpha}\,\delta^k_i\,\mathcal{F}_{\mu\beta}{}^j = 0$$

$$\mathcal{D}^i_\alpha\,\mathcal{F}_{\dot\beta j\dot\gamma k} + \bar{\mathcal{D}}_{\dot\beta j}\,\mathcal{F}^i_{\alpha\dot\gamma k} + \bar{\mathcal{D}}_{\dot\gamma k}\,\mathcal{F}^i_{\alpha\dot\beta j} + 2\,i\,\sigma^\mu_{\alpha\dot\beta}\,\delta^i_j\,\mathcal{F}_{\mu\dot\gamma k} + 2\,i\,\sigma^\mu_{\alpha\dot\gamma}\,\delta^i_k\,\mathcal{F}_{\mu\dot\beta j} = 0$$

$$\mathcal{D}_\mu\,\mathcal{F}^{ij}_{\alpha\beta} - \mathcal{D}^i_\alpha\,\mathcal{F}_{\mu\beta}{}^j - \mathcal{D}^j_\beta\,\mathcal{F}_{\mu\alpha}{}^i = 0$$

$$\mathcal{D}_\mu\,\mathcal{F}_{\dot\alpha i\dot\beta j} - \bar{\mathcal{D}}_{\dot\alpha i}\,\mathcal{F}_{\mu\dot\beta j} - \bar{\mathcal{D}}_{\dot\beta j}\,\mathcal{F}_{\mu\dot\alpha i} = 0$$

$$\mathcal{D}_\mu\,\mathcal{F}^i_{\alpha\dot\beta j} - \mathcal{D}^i_\alpha\,\mathcal{F}_{\mu\dot\beta j} - \bar{\mathcal{D}}_{\dot\beta j}\,\mathcal{F}_{\mu\alpha}{}^i - 2\,i\,\sigma^\nu_{\alpha\dot\beta}\,\delta^i_j\,\mathcal{F}_{\mu\nu} = 0$$

$$\mathcal{D}_\mu\,\mathcal{F}_{\nu\alpha}{}^i - \mathcal{D}_\nu\,\mathcal{F}_{\mu\alpha}{}^i + \mathcal{D}^i_\alpha\,\mathcal{F}_{\mu\nu} = 0$$

$$\mathcal{D}_\mu\,\mathcal{F}_{\nu\dot\alpha i} - \mathcal{D}_\nu\,\mathcal{F}_{\mu\dot\alpha i} + \bar{\mathcal{D}}_{\dot\alpha i}\,\mathcal{F}_{\mu\nu} = 0$$

$$\mathcal{D}_\mu\,\mathcal{F}_{\nu\rho} + \mathcal{D}_\nu\,\mathcal{F}_{\rho\mu} + \mathcal{D}_\rho\,\mathcal{F}_{\mu\nu} = 0 \qquad (4.1.14)$$

However, not all of these Bianchi identities are independent and in [20] the independent identities are obtained for the N = 1 , 2 and 4 theories.

4.2 *The N=1 Supersymmetric Gauge Theory in Superspace*

For the N=1 theory, the independent Bianchi identities are

$$\mathcal{D}_\alpha\,\mathcal{F}_{\beta\gamma} + \mathcal{D}_\beta\,\mathcal{F}_{\gamma\alpha} + \mathcal{D}_\gamma\,\mathcal{F}_{\alpha\beta} = 0$$

$$\bar{\mathcal{D}}_{\dot\alpha}\,\mathcal{F}_{\dot\beta\dot\gamma} + \bar{\mathcal{D}}_{\dot\beta}\,\mathcal{F}_{\dot\gamma\dot\alpha} + \bar{\mathcal{D}}_{\dot\gamma}\,\mathcal{F}_{\dot\alpha\dot\beta} = 0$$

$$8\,i\,\mathcal{F}_{\mu\alpha} + 2\,i\,(\sigma_\mu\bar\sigma^\nu)_\alpha{}^\beta\,\mathcal{F}_{\nu\beta} = (\bar{\mathcal{D}}\bar\sigma_\mu)^\beta\,\mathcal{F}_{\alpha\beta} + (\bar\sigma_\mu\mathcal{D})^{\dot\alpha}\,\mathcal{F}_{\alpha\dot\alpha} + \mathcal{D}_\alpha\,\bar\sigma_\mu^{\dot\beta\beta}\,\mathcal{F}_{\beta\dot\beta}$$

$$8\,i\,\mathcal{F}_{\mu\dot\alpha} + 2\,i\,\mathcal{F}_{\nu\dot\beta}\,(\bar\sigma^\nu\sigma_\mu)^{\dot\beta}{}_{\dot\alpha} = (\bar{\mathcal{D}}\bar\sigma_\mu)^\beta\,\mathcal{F}_{\beta\dot\alpha} + (\bar\sigma_\mu\mathcal{D})^{\dot\beta}\,\mathcal{F}_{\dot\alpha\dot\beta} + \bar{\mathcal{D}}_{\dot\alpha}\,\bar\sigma_\mu^{\dot\beta\beta}\,\mathcal{F}_{\beta\dot\beta}$$

$$\mathcal{D}_\mu\,\mathcal{F}_{\alpha\beta} - \mathcal{D}_\alpha\,\mathcal{F}_{\mu\beta} - \mathcal{D}_\beta\,\mathcal{F}_{\mu\alpha} = 0$$

$$\mathcal{D}_\mu\,\mathcal{F}_{\dot\alpha\dot\beta} - \bar{\mathcal{D}}_{\dot\alpha}\,\mathcal{F}_{\mu\dot\beta} - \bar{\mathcal{D}}_{\dot\beta}\,\mathcal{F}_{\mu\dot\alpha} = 0$$

$$8\,i\,\mathcal{F}_{\mu\nu} = \bar{\sigma}_{[\mu}^{\dot{\beta}\alpha}\,(\,\mathcal{D}_{\nu]}\,\mathcal{F}_{\alpha\dot{\beta}} - \mathcal{D}_{\alpha}\,\mathcal{F}_{\nu]\dot{\beta}} - \bar{\mathcal{D}}_{\dot{\beta}}\,\mathcal{F}_{\nu]\alpha}\,)$$

$$0 = \bar{\sigma}^{\mu\dot{\beta}\alpha}\,(\,\mathcal{D}_{\mu}\,\mathcal{F}_{\alpha\dot{\beta}} - \mathcal{D}_{\alpha}\,\mathcal{F}_{\mu\dot{\beta}} - \bar{\mathcal{D}}_{\dot{\beta}}\,\mathcal{F}_{\mu\alpha}\,) \tag{4.2.1}$$

To simplify the theory the constraints

$$\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\dot{\alpha}\dot{\beta}} = 0 \tag{4.2.2a}$$

$$\mathcal{F}_{\alpha\dot{\beta}} = 0 \tag{4.2.2b}$$

are imposed. (4.2.2a) arise as integrability conditions for coupling chiral matter superfields to the theory [19] and (4.2.2b) is a conventional constraint that allows one to solve for A_μ in terms of A^i_α and $\bar{A}_{\dot{\alpha}i}$

$$A_\mu = \frac{i}{4}\,\bar{\sigma}_{\mu}^{\dot{\beta}\alpha}\,(\,D_{\alpha}\,\bar{A}_{\dot{\beta}} + \bar{D}_{\dot{\beta}}\,A_{\alpha} + i\,\{\,A_{\alpha}\,,\bar{A}_{\dot{\beta}}\}\,) \tag{4.2.3}$$

The Bianchi identities now read

$$8\,i\,\mathcal{F}_{\mu\alpha} + 2\,i\,(\,\sigma_{\mu}\,\bar{\sigma}^{\nu}\,)_{\alpha}{}^{\beta}\,\mathcal{F}_{\nu\beta} = 0$$

$$8\,i\,\mathcal{F}_{\mu\dot{\alpha}} + 2\,i\,\mathcal{F}_{\nu\dot{\beta}}\,(\,\bar{\sigma}^{\nu}\,\sigma_{\mu}\,)^{\dot{\beta}}{}_{\dot{\alpha}} = 0$$

$$\mathcal{D}_{\alpha}\,\mathcal{F}_{\mu\beta} + \mathcal{D}_{\beta}\,\mathcal{F}_{\mu\alpha} = 0$$

$$\bar{\mathcal{D}}_{\dot{\alpha}}\,\mathcal{F}_{\mu\dot{\beta}} + \bar{\mathcal{D}}_{\dot{\beta}}\,\mathcal{F}_{\mu\dot{\alpha}} = 0$$

$$8\,i\,\mathcal{F}_{\mu\nu} = \bar{\sigma}_{[\nu}^{\dot{\beta}\alpha}\,(\,\mathcal{D}_{\alpha}\,\mathcal{F}_{\mu]\dot{\beta}} + \bar{\mathcal{D}}_{\dot{\beta}}\,\mathcal{F}_{\mu]\alpha}\,)$$

$$\bar{\sigma}^{\mu\dot{\beta}\alpha}\,(\,\mathcal{D}_{\alpha}\,\mathcal{F}_{\mu\dot{\beta}} + \bar{\mathcal{D}}_{\dot{\beta}}\,\mathcal{F}_{\mu\alpha}\,) = 0 \tag{4.2.4}$$

The first two of these identities have the solution

$$\mathcal{F}_{\mu\alpha} = -\frac{1}{8}\,\sigma_{\mu\alpha\dot{\beta}}\,\bar{\mathcal{W}}^{\dot{\beta}}$$

$$\mathcal{F}_{\mu\dot{\alpha}} = -\frac{1}{8}\,\sigma_{\mu\beta\dot{\alpha}}\,\mathcal{W}^{\beta} \tag{4.2.5}$$

The remaining identities now take the form

$$\mathcal{D}_\alpha \overline{\mathcal{W}}^{\dot\beta} = 0 = \overline{\mathcal{D}}_{\dot\alpha} \mathcal{W}^\beta$$

$$\mathcal{D}_\alpha \mathcal{W}^\alpha + \overline{\mathcal{D}}_{\dot\alpha} \overline{\mathcal{W}}^{\dot\alpha} = 0$$

$$\mathcal{F}_{\mu\nu} = \frac{i}{6} \left(\mathcal{D}^\alpha \sigma_{\mu\nu\alpha}{}^\beta \mathcal{W}_\beta - \overline{\mathcal{D}}_{\dot\alpha} \overline{\sigma}_{\mu\nu}{}^{\dot\alpha}{}_{\dot\beta} \overline{\mathcal{W}}^{\dot\beta} \right) \qquad (4.2.6)$$

Using the Bianchi identities, the theory can now be written completely in terms of the superfields $\mathcal{W}^\alpha$, $\mathcal{D}_\alpha \mathcal{W}^\alpha$, $\mathcal{F}^{\mu\nu}$ and covariant space-time derivatives of these superfields.

$$\mathcal{D}_\alpha \mathcal{W}^\beta = \frac{1}{2} \delta_\alpha^\beta \mathcal{D}_\gamma \mathcal{W}^\gamma - 8 i \sigma^{\mu\nu}{}_\alpha{}^\beta \mathcal{F}_{\mu\nu}$$

$$\mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{W}^\beta = 2 i \sigma^\mu_{\alpha\dot\alpha} \mathcal{D}_\mu \overline{\mathcal{W}}^{\dot\alpha}$$

$$\overline{\mathcal{D}}_{\dot\alpha} \mathcal{D}_\beta \mathcal{W}^\beta = 2 i \sigma^\mu_{\alpha\dot\alpha} \mathcal{D}_\mu \mathcal{W}^\alpha$$

$$\mathcal{D}_\alpha \mathcal{F}_{\mu\nu} = \frac{1}{8} \sigma_{[\mu\alpha\dot\alpha} \mathcal{D}_{\nu]} \overline{\mathcal{W}}^{\dot\alpha} \qquad (4.2.7)$$

Taking

$$\mathcal{L} = \mathrm{tr} \left(-\frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} - \frac{i}{128} \mathcal{W}^\alpha \sigma^\mu_{\alpha\dot\alpha} \overleftrightarrow{\mathcal{D}}_\mu \overline{\mathcal{W}}^{\dot\alpha} + \frac{1}{512} \mathcal{D}_\alpha \mathcal{W}^\alpha \mathcal{D}_\beta \mathcal{W}^\beta \right) \qquad (4.2.8)$$

one finds that $D_\alpha \mathcal{L}$ is a total derivative

$$D_\alpha \mathcal{L} = \partial_\mu \mathrm{tr} \left(\frac{1}{16} \mathcal{F}^{\mu\nu} \sigma_{\nu\alpha\dot\alpha} \overline{\mathcal{W}}^{\dot\alpha} - \frac{i}{32} \varepsilon^{\mu\nu\rho\lambda} \mathcal{F}_{\nu\rho} \sigma_{\lambda\alpha\dot\alpha} \overline{\mathcal{W}}^{\dot\alpha} + \frac{i}{256} \mathcal{D}_\beta \mathcal{W}^\beta \sigma^\mu_{\alpha\dot\alpha} \overline{\mathcal{W}}^{\dot\alpha} \right) \qquad (4.2.9)$$

So taking the $\theta = \overline{\theta} = 0$ component of (4.2.8) gives a supersymmetric Lagrangian as

$$Q_\alpha \mathcal{L} \big|_{\theta=\overline{\theta}=0} = D_\alpha \mathcal{L} \big|_{\theta=\overline{\theta}=0} \qquad (4.2.10)$$

With the field definitions

$$\frac{1}{8} \mathcal{W}^\alpha \big|_{\theta=\overline{\theta}=0} = i \lambda^\alpha$$

$$\mathcal{F}^{\mu\nu}\Big|_{\theta=\bar{\theta}=0} = F^{\mu\nu}$$

$$-\frac{1}{16}\mathcal{D}_\alpha \mathcal{W}^\alpha\Big|_{\theta=\bar{\theta}=0} = D \qquad (4.2.11)$$

this gives the Lagrangian

$$\mathcal{L} = \mathrm{tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}\lambda^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \overleftrightarrow{\mathcal{D}}_\mu \bar{\lambda}^{\dot{\alpha}} + \frac{1}{2}D^2 \right) \qquad (4.2.12)$$

However, in superspace one wishes a local Lagrangian density, at least over some subspace of superspace [34]. For this theory, the Lagrangian can be written as an integral over the chiral subspace of superspace of the chiral superfield $\mathrm{tr}\,(\mathcal{W}^\alpha \mathcal{W}_\alpha)$

$$\begin{aligned} I &= \int d^4x\, d^2\theta\, \mathrm{tr}\,(\mathcal{W}^\alpha \mathcal{W}_\alpha) \\ &= \int d^4x\, D^\alpha D_\alpha\, \mathrm{tr}\,(\mathcal{W}^\beta \mathcal{W}_\beta) \\ &= 2\int d^4x\, \mathrm{tr}\,(\mathcal{D}^\alpha \mathcal{D}_\alpha \mathcal{W}^\beta \mathcal{W}_\beta + \mathcal{D}_\alpha \mathcal{W}^\beta \mathcal{D}^\alpha \mathcal{W}_\beta) \\ &= -512\int d^4x\, \mathrm{tr}\left(-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{i}{128}\mathcal{W}^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \overleftrightarrow{\mathcal{D}}_\mu \bar{\mathcal{W}}^{\dot{\alpha}} \right. \\ &\qquad \left. + \frac{1}{512}\mathcal{D}_\alpha \mathcal{W}^\alpha \mathcal{D}_\beta \mathcal{W}^\beta \right) \qquad (4.2.13) \end{aligned}$$

using (4.2.7) above.

The constraints (4.2.2a) have the solution

$$\mathcal{A}_\alpha = -i\, e^{-V} D_\alpha e^V$$

$$\bar{\mathcal{A}}_{\dot{\alpha}} = -i\, e^U \bar{D}_{\dot{\alpha}} e^{-U} \qquad (4.2.14)$$

where U , V are arbitary superfields. Under gauge transformations these transform as

$$e^V \rightarrow e^{S^\dagger} e^V e^{-i\Lambda} \qquad (4.2.15a)$$

$$e^{-U} \rightarrow e^{-T} e^{-U} e^{-i\Lambda} \qquad (4.2.15b)$$

where S , T are chiral superfields

$$\bar{D}_{\dot\alpha} S = 0 = \bar{D}_{\dot\alpha} T \tag{4.2.16}$$

Under the gauge transformations (4.2.15) the gauge potentials transform in the usual fashion (4.1.9).

To further restrict the theory the gauge potentials $\mathcal{A}_\mu$, $\mathcal{A}_\alpha$, $\bar{\mathcal{A}}_{\dot\alpha}$ are restricted to be real up to gauge transformations

$$\begin{aligned}
(\mathcal{A}^\dagger)_{\dot\alpha} &= -i\, e^{V^\dagger} \bar{D}_{\dot\alpha} e^{-V^\dagger} \\
&= e^X \bar{\mathcal{A}}_{\dot\alpha} e^{-X} - i\, e^X \bar{D}_{\dot\alpha} e^{-X} \\
&= -i\, e^X e^U \bar{D}_{\dot\alpha} e^{-U} e^{-X} - i\, e^X \bar{D}_{\dot\alpha} e^{-X} \\
&= -i\, e^X e^U \bar{D}_{\dot\alpha} (e^{-U} e^{-X}) \\
(\bar{\mathcal{A}}^\dagger)_\alpha &= -i\, e^{-U^\dagger} D_\alpha e^{U^\dagger} \\
&= e^{X^\dagger} \mathcal{A}_\alpha e^{-X^\dagger} - i\, e^{X^\dagger} D_\alpha e^{-X^\dagger} \\
&= -i\, e^{X^\dagger} e^{-V} D_\alpha (e^V e^{-X^\dagger})
\end{aligned} \tag{4.2.17}$$

Now, using (4.2.3) and (4.2.17), the condition that $\mathcal{A}_\mu$ is real up to a gauge transformation is that $X^\dagger = X$ and so

$$\mathcal{A}^\dagger_\mu = e^X \mathcal{A}_\mu e^{-X} - i\, e^X \partial_\mu e^{-X} \tag{4.2.18}$$

where

$$e^{V^\dagger} e^{-U} = e^X = e^{-U^\dagger} e^V \tag{4.2.19}$$

Now consider the transformation of the two expressions (4.2.19) using (4.2.15)

$$e^{V^\dagger} e^{-U} \to e^{i\Lambda^\dagger} e^{V^\dagger} e^{S} e^{-T} e^{-U} e^{-i\Lambda}$$

$$e^{-U^\dagger} e^{V} \to e^{i\Lambda^\dagger} e^{-U^\dagger} e^{-T^\dagger} e^{S^\dagger} e^{V} e^{-i\Lambda} \qquad (4.2.20)$$

leads one to the conclusion $e^S e^{-T} = 1$ and so $S = T$.

One can further restrict the theory by making the gauge choice $U = 0$. This has the restricted gauge invariance from (4.2.15b) of $T = -i\Lambda$. This means that V transforms as

$$e^{V} \to e^{i\bar{\Lambda}} e^{V} e^{-i\Lambda} \qquad (4.2.21)$$

and Λ satisfies

$$\bar{D}_{\dot\alpha} \Lambda = 0 \qquad (4.2.22)$$

In this gauge (4.2.19) gives $V^\dagger = V$.

This gauge choice is called the chiral representation of the theory. In this gauge the gauge potentials are

$$\mathcal{A}_\alpha = -i\, e^{-V} D_\alpha e^{V}$$

$$\bar{\mathcal{A}}_{\dot\alpha} = 0$$

$$\mathcal{A}_\mu = \frac{1}{4} \bar{\sigma}_\mu^{\dot\beta\alpha} \bar{D}_{\dot\beta} (e^{-V} D_\alpha e^{V}) \qquad (4.2.23)$$

and from (4.2.5)

$$\mathcal{W}^\alpha = 2 \mathcal{F}_{\mu\dot\alpha} \bar{\sigma}^{\mu\dot\alpha\alpha}$$

$$= -2 \bar{\sigma}^{\mu\dot\alpha\alpha} \bar{D}_{\dot\alpha} \mathcal{A}_\mu$$

$$= \bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha} (e^{-V} D^\alpha e^{V})$$

$$\equiv W^\alpha$$

$$\bar{\mathcal{W}}^{\dot\alpha} = 2 \bar{\sigma}^{\mu\dot\alpha\alpha} \mathcal{F}_{\mu\alpha}$$

$$= -2\,\overline{\sigma}^{\mu\dot{\alpha}\alpha}\, e^{-V} D_\alpha A^\dagger_\mu\, e^V$$

$$= e^{-V} D^\alpha D_\alpha\, (\overline{D}^{\dot{\alpha}} e^V e^{-V})\, e^V$$

$$= e^{-V}\, \overline{W}^{\dot{\alpha}}\, e^V \qquad (4.2.24)$$

where

$$\overline{W}^{\dot{\alpha}} = (W^\dagger)^{\dot{\alpha}} \qquad (4.2.25)$$

The transformation (4.2.21) can be written as

$$\delta V = -\frac{1}{2} i\, L_V\, (\overline{\Lambda} + \Lambda + \coth \frac{1}{2} L_V\, (\Lambda - \overline{\Lambda}))$$

$$= i\, (\overline{\Lambda} - \Lambda) - \frac{1}{2} i\, [V , \overline{\Lambda} + \Lambda] + 0\, (V^2) \qquad (4.2.26)$$

where

$$L_V X = [V , X] \qquad (4.2.27)$$

The superfield V can be expanded as

$$V = C + i\,\theta^\alpha \psi_\alpha - i\,\overline{\theta}_{\dot{\alpha}} \overline{\psi}^{\dot{\alpha}} + \frac{i}{2}\theta^2 H - \frac{i}{2}\overline{\theta}^2 \overline{H} - 2\,\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha}} A_\mu$$

$$- i\,\overline{\theta}^2 \theta^\alpha\, (2\lambda_\alpha + \frac{i}{2} \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \overline{\psi}^{\dot{\alpha}}) + i\,\theta^2 \overline{\theta}_{\dot{\alpha}}\, (2\overline{\lambda}^{\dot{\alpha}} + \frac{i}{2} \overline{\sigma}^{\dot{\alpha}\alpha}_\mu \partial^\mu \psi_\alpha)$$

$$+ \frac{1}{2}\theta^2 \overline{\theta}^2\, (2D - \frac{1}{2}\Box C) \qquad (4.2.28)$$

and the gauge parameter superfield Λ as

$$\Lambda = \Lambda_0 (y_+) + \theta^\alpha \Lambda_\alpha (y_+) + \frac{1}{2}\theta^2 \Lambda_2 (y_+) \qquad (4.2.29)$$

where

$$y^\mu_\pm = x^\mu \pm i\,\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha}} \qquad (4.2.30)$$

Under the gauge transformation (4.2.26) the component fields as defined by (4.2.28) transform as

$$\delta C = i\, (\overline{\Lambda}_0 - \Lambda_0) + 0\, (V)$$

$$\delta\psi_\alpha = -\Lambda_\alpha + 0\, (V)$$

$$\delta\bar{\psi}_{\dot{\alpha}} = -\bar{\Lambda}_{\dot{\alpha}} + 0(V)$$

$$\delta H = -\Lambda_2 + 0(V)$$

$$\delta\bar{H} = -\bar{\Lambda}_2 + 0(V)$$

$$\delta A_\mu = -\frac{1}{2}\partial_\mu(\Lambda_0 + \bar{\Lambda}_0) - \frac{i}{2}[A_\mu, \Lambda_0 + \bar{\Lambda}_0] + 0(V^2)$$

$$\delta\lambda_\alpha = -\frac{1}{2} i [\lambda_\alpha, \Lambda_0 + \bar{\Lambda}_0] + 0(V^2)$$

$$\delta\bar{\lambda}_{\dot{\alpha}} = -\frac{1}{2} i [\bar{\lambda}_{\dot{\alpha}}, \Lambda_0 + \bar{\Lambda}_0] + 0(V^2)$$

$$\delta D = -\frac{1}{2} i [D, \Lambda_0 + \bar{\Lambda}_0] + 0(V^2) \qquad (4.2.31)$$

So one can make a gauge transformation to the Wess-Zumino gauge where $C = \psi_\alpha = \bar{\psi}_{\dot{\alpha}} = H = \bar{H} = 0$. This is preserved by gauge transformations with $\Lambda = \bar{\Lambda}$, ie. $\Lambda_0 = \bar{\Lambda}_0$, $\Lambda_\alpha = \bar{\Lambda}_{\dot{\alpha}} = \Lambda_2 = \bar{\Lambda}_2 = 0$.

So in this gauge V takes the form

$$V = -2\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} A_\mu - 2i\bar{\theta}^2 \theta^\alpha \lambda_\alpha + 2i\theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} + \theta^2 \bar{\theta}^2 D \qquad (4.2.32)$$

This gauge choice is not preserved by a supersymmetry transformation, but it can be restored if a compensating gauge transformation is applied after the supersymmetry transformation. The resulting transformation is the covariant supersymmetry transformation for the fields in (4.2.32).

Under a supersymmetry transformation V transforms as

$$\delta V = (\xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}) V \qquad (4.2.33)$$

This gives rise to components that depend only upon θ or $\bar{\theta}$. The terms that depend only upon θ are

$$(\xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}) V \big|_{\bar{\theta}=0} \qquad (4.2.34)$$

These can be removed by a gauge transformation with parameter

$$- i \Xi (x) = - (\xi^{\alpha} Q_{\alpha} + \bar{\xi}_{\dot\alpha} \bar{Q}^{\dot\alpha}) V \big|_{\bar\theta=0}$$

$$= - \bar{\xi}^{\dot\alpha} \bar{\partial}_{\dot\alpha} V \big|_{\bar\theta=0}$$

$$= (2 \theta^{\alpha} \sigma^{\mu}_{\alpha\dot\alpha} A_{\mu} - 2 i \theta^{2} \bar{\lambda}_{\dot\alpha}) \bar{\xi}^{\dot\alpha} \quad (4.2.35)$$

So the covariant supersymmetry transformation is

$$\delta V = (\xi Q + \bar{\xi} \bar{Q}) V + i (\bar{\Xi} (y_{-}) - \Xi (y_{+}))$$

$$- \frac{1}{2} i \left[V , \bar{\Xi} (y_{-}) + \Xi (y_{+}) \right] + \ldots \quad (4.2.36)$$

For the components defined in (4.2.32) this gives

$$\delta A^{\mu} = i (\xi^{\alpha} \sigma^{\mu}_{\alpha\dot\alpha} \bar{\lambda}^{\dot\alpha} - \lambda^{\alpha} \sigma^{\mu}_{\alpha\dot\alpha} \bar{\xi}^{\dot\alpha})$$

$$\delta \lambda_{\alpha} = F_{\mu\nu} \sigma^{\mu\nu}{}_{\alpha}{}^{\beta} \xi_{\beta} + i D \xi_{\alpha}$$

$$\delta D = - (\xi^{\alpha} \sigma^{\mu}_{\alpha\dot\alpha} \mathcal{D}_{\mu} \bar{\lambda}^{\dot\alpha} + \mathcal{D}_{\mu} \lambda^{\alpha} \sigma^{\mu}_{\alpha\dot\alpha} \bar{\xi}^{\dot\alpha}) \quad (4.2.37)$$

The expansion of W^{α} can be obtained directly from (4.2.24). However, it is more useful to obtain it using the expressions (4.2.7) and also the equations

$$\mathcal{D}_{\alpha} \mathcal{D}_{\beta} \mathcal{W}^{\gamma} = - 2 i \varepsilon_{\alpha\beta} \mathcal{D}^{\mu} \bar{\mathcal{W}}_{\dot\alpha} \bar{\sigma}^{\dot\alpha\gamma}_{\mu}$$

$$\mathcal{D}_{\alpha} \mathcal{D}_{\beta} \mathcal{D}_{\gamma} \mathcal{W}^{\delta} = 0 \quad (4.2.38)$$

which are simply derived from them.

In the gauge U = 0 one has

$$\mathcal{D}_{\alpha} \mathcal{W}^{\beta} = e^{-V} D_{\alpha} (e^{V} W^{\beta} e^{-V}) e^{V}$$

$$= \frac{1}{2} \delta^{\beta}_{\alpha} \mathcal{D}_{\gamma} \mathcal{W}^{\gamma} - 8 i \sigma^{\mu\nu}{}_{\alpha}{}^{\beta} \mathcal{F}_{\mu\nu}$$

$$\mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{W}^\gamma = e^{-V} D_\alpha D_\beta (e^V W^\gamma e^{-V}) e^V$$

$$= -2 i \varepsilon_{\alpha\beta} \mathcal{D}_\nu \overline{\mathcal{W}}_{\dot\alpha} \overline{\sigma}^{\nu\dot\alpha\gamma} \qquad (4.2.39)$$

When V is in the Wess-Zumino gauge (4.2.32) one has, taking the $\theta = \overline{\theta} = 0$ components and using the field definitions (4.2.11)

$$W^\alpha \big|_{\theta=\overline{\theta}=0} = 8 i \lambda^\alpha$$

$$\partial_\alpha W^\beta \big|_{\theta=\overline{\theta}=0} = -8 \delta_\alpha^\beta D - 8 i \sigma^{\mu\nu}{}_\alpha{}^\beta F_{\mu\nu}$$

$$\partial_\alpha \partial_\beta W^\gamma \big|_{\theta=\overline{\theta}=0} = -16 \varepsilon_{\alpha\beta} \mathcal{D}^\mu \overline{\lambda}_{\dot\alpha} \overline{\sigma}_\mu^{\dot\alpha\gamma} \qquad (4.2.40)$$

which gives the expansion of W^α as

$$\frac{1}{8} W^\alpha = i \lambda^\alpha(y_+) - D(y_+)\theta^\alpha - i \theta^\beta \sigma^{\mu\nu}{}_\beta{}^\alpha F_{\mu\nu}(y_+) + \theta^2 \mathcal{D}^\mu \overline{\lambda}_{\dot\alpha}(y_+) \overline{\sigma}_\mu^{\dot\alpha\alpha}$$

$$(4.2.41)$$

Finally, from (4.2.13) it is easy to derive the equations of motion in superspace [35]

$$I = \int d^4x\, d^2\theta\, \mathrm{tr}\, (W^\alpha W_\alpha)$$

$$= \int d^4x\, d^2\theta\, d^2\overline{\theta}\, \mathrm{tr}\, (e^{-V} D^\alpha e^V W_\alpha) \qquad (4.2.42)$$

Taking the variation of this gives

$$0 = \delta I = 2 \int d^4x\, d^2\theta\, d^2\overline{\theta}\, \mathrm{tr}\, (\delta e^{-V} D^\alpha e^V W_\alpha + e^{-V} D^\alpha \delta e^V W_\alpha)$$

$$= -2 \int d^4x\, d^2\theta\, d^2\overline{\theta}\, \mathrm{tr}\, (\delta e^V e^{-V} D^\alpha (e^V W_\alpha e^{-V}))$$

$$(4.2.43)$$

This implies the equations of motion are

$$D^\alpha (e^V W_\alpha e^{-V}) = 0 \qquad (4.2.44)$$

Note that (4.2.42) is real, as can be shown using the Bianchi identity (4.2.6b) which reads in the gauge U = 0

$$D^{\alpha}(e^{V}W_{\alpha}e^{-V}) - e^{V}\overline{D}_{\dot\alpha}(e^{-V}\overline{W}^{\dot\alpha}e^{V})e^{-V} = 0 \qquad (4.2.45)$$

4.3 The N=2 Supersymmetric Gauge Theory in Superspace

For the N=2 theory, the independent Bianchi identities are [20]

$$\mathcal{D}^i_{\alpha}\mathcal{F}^{jk}_{\beta\gamma} + \mathcal{D}^j_{\beta}\mathcal{F}^{ki}_{\gamma\alpha} + \mathcal{D}^k_{\gamma}\mathcal{F}^{ij}_{\alpha\beta} = 0$$

$$\overline{\mathcal{D}}_{\dot\alpha i}\mathcal{F}_{\dot\beta j\dot\gamma k} + \overline{\mathcal{D}}_{\dot\beta j}\mathcal{F}_{\dot\gamma k\dot\alpha i} + \overline{\mathcal{D}}_{\dot\gamma k}\mathcal{F}_{\dot\alpha i\dot\beta j} = 0$$

$$12\,i\,\mathcal{F}_{\mu\alpha}^{\ \ i} + 2\,i\,(\sigma_{\mu}\overline{\sigma}^{\nu})_{\alpha}^{\ \beta}\mathcal{F}_{\nu\beta}^{\ \ i}$$

$$= (\overline{\mathcal{D}}_j\overline{\sigma}_{\mu})^{\beta}\mathcal{F}^{ij}_{\alpha\beta} + (\overline{\sigma}_{\mu}\mathcal{D}^j)^{\dot\alpha}\mathcal{F}^{i}_{\alpha\dot\alpha j} + \mathcal{D}^i_{\alpha}\overline{\sigma}^{\dot\beta\beta}_{\mu}\mathcal{F}^{j}_{\beta\dot\beta j}$$

$$12\,i\,\mathcal{F}_{\mu\dot\alpha i} + 2\,i\,\mathcal{F}_{\nu\dot\beta i}(\overline{\sigma}^{\nu}\sigma_{\mu})^{\dot\beta}_{\ \dot\alpha}$$

$$= (\overline{\mathcal{D}}_j\overline{\sigma}_{\mu})^{\beta}\mathcal{F}^{j}_{\beta\dot\alpha i} + (\overline{\sigma}_{\mu}\mathcal{D}^j)^{\dot\beta}\mathcal{F}_{\dot\alpha i\dot\beta j} + \overline{\mathcal{D}}_{\dot\alpha i}\overline{\sigma}^{\dot\beta\beta}_{\mu}\mathcal{F}^{j}_{\beta\dot\beta j}$$

$$(\varepsilon\sigma_{\mu\nu})^{\beta\gamma}\left[\overline{\mathcal{D}}_{\dot\alpha i}\mathcal{F}^{jk}_{\beta\gamma} + \mathcal{D}^k_{\gamma}\mathcal{F}^{j}_{\beta\dot\alpha i} + \mathcal{D}^j_{\gamma}\mathcal{F}^{k}_{\beta\dot\alpha i}\right]$$

$$= \frac{1}{3}\delta^{(j}_{i}(\varepsilon\sigma_{\mu\nu})^{\beta\gamma}\left[\overline{\mathcal{D}}_{\dot\alpha\ell}\mathcal{F}^{\ell k)}_{\beta\gamma} + \mathcal{D}^{k)}_{\gamma}\mathcal{F}^{\ell}_{\beta\dot\alpha\ell} + \mathcal{D}^{\ell}_{\gamma}\mathcal{F}^{k)}_{\beta\dot\alpha\ell}\right]$$

$$(\overline{\sigma}_{\mu\nu}\varepsilon)^{\dot\beta\dot\gamma}\left[\mathcal{D}^i_{\alpha}\mathcal{F}_{\dot\beta j\dot\gamma k} + \overline{\mathcal{D}}_{\dot\beta j}\mathcal{F}^{i}_{\alpha\dot\gamma k} + \overline{\mathcal{D}}_{\dot\beta k}\mathcal{F}^{i}_{\alpha\dot\gamma j}\right]$$

$$= \frac{1}{3}\delta^{\ i}_{(j}(\overline{\sigma}_{\mu\nu}\varepsilon)^{\dot\beta\dot\gamma}\left[\mathcal{D}^{\ell}_{\alpha}\mathcal{F}_{\dot\beta\ell\dot\gamma k)} + \overline{\mathcal{D}}_{\dot\beta\ell}\mathcal{F}^{\ell}_{\alpha\dot\gamma k)} + \overline{\mathcal{D}}_{\dot\beta k)}\mathcal{F}^{\ell}_{\alpha\dot\gamma\ell}\right]$$

$$16\,i\,\mathcal{F}_{\mu\nu} = -\overline{\sigma}^{\dot\beta\alpha}_{[\nu}(\mathcal{D}_{\mu]}\mathcal{F}^{i}_{\alpha\dot\beta i} - \mathcal{D}^i_{\alpha}\mathcal{F}_{\mu]\dot\beta i} - \overline{\mathcal{D}}_{\dot\beta i}\mathcal{F}_{\mu]\alpha}^{\ \ \ i})$$

$$0 = \overline{\sigma}^{\mu\dot\beta\alpha}(\mathcal{D}_{\mu}\mathcal{F}^{i}_{\alpha\dot\beta j} - \mathcal{D}^i_{\alpha}\mathcal{F}_{\mu\dot\beta j} - \overline{\mathcal{D}}_{\dot\beta j}\mathcal{F}_{\mu\alpha}^{\ \ i} - \text{trace in } i, j)$$

$$(4.3.1)$$

The constraints used to restrict the theory are [18]

$$\mathcal{F}^{(ij)}_{\alpha\beta} = 0 = \mathcal{F}_{\dot\alpha(i\dot\beta j)} \qquad (4.3.2a)$$

$$\mathcal{F}^{i}_{\alpha\dot\beta j} = 0 \qquad (4.3.2b)$$

The first two constraints (4.3.2a) and the traceless part of (4.3.2b) arise as integrability conditions for coupling the theory to N=2 matter superfields whilst the remaining part of (4.3.2b) is the conventional constraint that allows one to solve for $\mathcal{A}_\mu$ in terms of $\mathcal{A}^i_\alpha$ and $\bar{\mathcal{A}}_{\dot{\alpha} i}$

$$\mathcal{A}_\mu = \frac{i}{8}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\left(D^i_\alpha \bar{\mathcal{A}}_{\dot{\alpha} i} + \bar{D}_{\dot{\alpha} i}\mathcal{A}^i_\alpha + i\{ \mathcal{A}^i_\alpha , \bar{\mathcal{A}}_{\dot{\alpha} i}\}\right) \qquad (4.3.3)$$

The remaining Bianchi identities can now be written down in terms of the superfields $\mathcal{W}$, $\bar{\mathcal{W}}$ defined by

$$\mathcal{F}^{ij}_{\alpha\beta} = i\,\varepsilon_{\alpha\beta}\, g^{ij}\,\bar{\mathcal{W}}$$

$$\mathcal{F}_{\dot{\alpha} i\dot{\beta} j} = i\,\varepsilon_{\dot{\alpha}\dot{\beta}}\, g_{ij}\,\mathcal{W} \qquad (4.3.4)$$

They are

$$\mathcal{D}^i_\alpha \bar{\mathcal{W}} = 0 = \bar{\mathcal{D}}_{\dot{\alpha} i}\mathcal{W}$$

$$\mathcal{F}_{\mu\alpha}{}^i = -\frac{1}{4}\sigma_{\mu\alpha\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha} i}\bar{\mathcal{W}}$$

$$\mathcal{F}_{\mu\dot{\alpha} i} = -\frac{1}{4}\sigma_{\mu\alpha\dot{\alpha}}\mathcal{D}^\alpha_i \mathcal{W}$$

$$\mathcal{F}_{\mu\nu} = -\frac{i}{16}\left(\mathcal{D}^\alpha_i \sigma_{\mu\nu\alpha}{}^\beta \mathcal{D}^i_\beta \mathcal{W} + \bar{\mathcal{D}}_{\dot{\alpha} i}\bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\mathcal{D}}^{\dot{\beta} i}\bar{\mathcal{W}}\right)$$

$$\mathcal{D}^{\alpha(i}\mathcal{D}^{j)}_\alpha \mathcal{W} = \bar{\mathcal{D}}^{(i}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha} j)}\bar{\mathcal{W}} \qquad (4.3.5)$$

Using these Bianchi identities, the theory can be written completely in terms of the superfields $\mathcal{W}$, $\mathcal{D}^i_\alpha \mathcal{W}$, $\mathcal{D}^{\alpha(i}\mathcal{D}^{j)}_\alpha \mathcal{W}$, $\mathcal{F}_{\mu\nu}$ and covariant space-time derivatives of these superfields

$$\mathcal{D}^i_\alpha \mathcal{D}^j_\beta \mathcal{W} = \frac{1}{4}\varepsilon_{\alpha\beta}\mathcal{D}^{\gamma(i}\mathcal{D}^{j)}_\gamma \mathcal{W} + 4i\, g^{ij}\left(\sigma^{\mu\nu}\varepsilon\right)_{\alpha\beta}\mathcal{F}_{\mu\nu}$$

$$-\frac{1}{2}\varepsilon_{\alpha\beta}\, g^{ij}\left[\bar{\mathcal{W}}, \mathcal{W}\right]$$

$$\mathcal{D}^i_\alpha \mathcal{D}^{\beta(j} \mathcal{D}^{k)}_\beta \mathcal{W} = 4\, i\, g^{i(j} \sigma^\mu_{\alpha\dot\alpha} \mathcal{D}_\mu \bar{\mathcal{D}}^{\dot\alpha k)} \bar{\mathcal{W}} + 2\, g^{i(j} [\, \bar{\mathcal{W}} \,,\, \mathcal{D}^{k)}_\alpha \mathcal{W} \,]$$

$$\mathcal{D}^i_\alpha \mathcal{F}_{\mu\nu} = -\frac{1}{4} \sigma_{[\mu\alpha\dot\alpha} \mathcal{D}_{\nu]} \bar{\mathcal{D}}^{\dot\alpha i} \bar{\mathcal{W}} \qquad (4.3.6)$$

Using these, one finds that $D^i_\alpha \mathcal{L}$ is a total derivative where

$$\mathcal{L} = \mathrm{tr} \Big(-\frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + \frac{i}{32} \bar{\mathcal{D}}_{\dot\alpha i} \bar{\mathcal{W}} \bar\sigma^{\dot\alpha\alpha}_\mu \overleftrightarrow{\mathcal{D}}^\mu \mathcal{D}^i_\alpha \mathcal{W} + \frac{1}{8} \mathcal{D}_\mu \bar{\mathcal{W}} \mathcal{D}^\mu \mathcal{W}$$
$$- \frac{1}{1024} \mathcal{D}^\alpha_{(i} \mathcal{D}_{\alpha j)} \mathcal{W}\; \mathcal{D}^{\beta(i} \mathcal{D}^{j)}_\beta \mathcal{W} + \frac{1}{64} \mathcal{W} \{ \bar{\mathcal{D}}_{\dot\alpha i} \bar{\mathcal{W}} \,,\, \bar{\mathcal{D}}^{\dot\alpha i} \bar{\mathcal{W}} \}$$
$$+ \frac{1}{64} \bar{\mathcal{W}} \{ \mathcal{D}^\alpha_i \mathcal{W} \,,\, \mathcal{D}^i_\alpha \mathcal{W} \} - \frac{1}{128} [\bar{\mathcal{W}}, \mathcal{W}][\bar{\mathcal{W}}, \mathcal{W}] \Big) \qquad (4.3.7)$$

So again, the $\theta = \bar\theta = 0$ component of this equation gives a supersymmetric Lagrangian as

$$Q^i_\alpha \mathcal{L} \big|_{\theta=\bar\theta=0} = D^i_\alpha \mathcal{L} \big|_{\theta=\bar\theta=0} \qquad (4.3.8)$$

With the field definitions

$$\frac{1}{4} \bar{\mathcal{W}} \big|_{\theta=\bar\theta=0} = -\, i\, C^*$$

$$\frac{1}{4} \mathcal{W} \big|_{\theta=\bar\theta=0} = i\, C$$

$$\frac{1}{4} \mathcal{D}^i_\alpha \mathcal{W} \big|_{\theta=\bar\theta=0} = -\, i\, \lambda^i_\alpha$$

$$\frac{1}{4} \bar{\mathcal{D}}_{\dot\alpha i} \bar{\mathcal{W}} \big|_{\theta=\bar\theta=0} = i\, \bar\lambda_{\dot\alpha i}$$

$$\frac{1}{32} \mathcal{D}^{\alpha(i} \mathcal{D}^{j)}_\alpha \mathcal{W} \big|_{\theta=\bar\theta=0} = i\, D^{ij}$$

$$\mathcal{F}_{\mu\nu} \big|_{\theta=\bar\theta=0} = F_{\mu\nu} \qquad (4.3.9)$$

this gives the Lagrangian

$$\mathcal{L} = \mathrm{tr} \Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar\lambda_{\dot\alpha i} \bar\sigma^{\dot\alpha\alpha}_\mu \overleftrightarrow{\mathcal{D}}^\mu \lambda^i_\alpha + 2\, \mathcal{D}_\mu C^* \mathcal{D}^\mu C + D_{ij} D^{ij}$$
$$-\, i\, C \{ \bar\lambda_{\dot\alpha i} \,,\, \bar\lambda^{\dot\alpha i} \} + i\, C^* \{ \lambda^\alpha_i \,,\, \lambda^i_\alpha \} - 2 [C^*, C][C^*, C] \Big) \qquad (4.3.10)$$

This is the Lagrangian given in (2.3.17) above.

The Lagrangian can be written as an integral over the chiral subspace of superspace of the chiral superfield tr $(\mathcal{W} \mathcal{W})$

$$
\begin{aligned}
I &= \int d^4x\, d^4\theta\, \mathrm{tr}\, (\mathcal{W} \mathcal{W}) \\
&= \int d^4x\, D_i^\alpha D_\alpha^j D_j^\beta D_\beta^i\, \mathrm{tr}\, (\mathcal{W} \mathcal{W}) \\
&= 2 \int d^4x\, \mathrm{tr}\, (\mathcal{D}_i^\alpha \mathcal{D}_\alpha^j \mathcal{D}_j^\beta \mathcal{D}_\beta^i \mathcal{W} \mathcal{W} - \mathcal{D}_i^\alpha \mathcal{D}_\alpha^j \mathcal{D}_\beta^i \mathcal{W} \mathcal{D}_j^\beta \mathcal{W} + \mathcal{D}_i^\alpha \mathcal{D}_\alpha^j \mathcal{D}_j^\beta \mathcal{W} \mathcal{D}_\beta^i \mathcal{W} \\
&\qquad - \mathcal{D}_\alpha^j \mathcal{D}_j^\beta \mathcal{D}_\beta^i \mathcal{W} \mathcal{D}_i^\alpha \mathcal{W} + \mathcal{D}_i^\alpha \mathcal{D}_j^\beta \mathcal{D}_\beta^i \mathcal{W} \mathcal{D}_\alpha^j \mathcal{W} + \mathcal{D}_i^\alpha \mathcal{D}_j^\beta \mathcal{W} \mathcal{D}_\alpha^i \mathcal{D}_\beta^j \mathcal{W} \\
&\qquad + \mathcal{D}_i^\alpha \mathcal{D}_\beta^i \mathcal{W} \mathcal{D}_\alpha^j \mathcal{D}_j^\beta \mathcal{W} + \mathcal{D}_i^\alpha \mathcal{D}_\alpha^j \mathcal{W} \mathcal{D}_j^\beta \mathcal{D}_\beta^i \mathcal{W}) \\
&= -768 \int d^4x\, \mathrm{tr} \Big(-\tfrac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + \tfrac{i}{32} \bar{\mathcal{D}}_{\dot\alpha i} \bar{\mathcal{W}} \bar{\sigma}_\mu^{\dot\alpha\alpha} \overleftrightarrow{\mathcal{D}}^\mu \mathcal{D}_\alpha^i \mathcal{W} \\
&\qquad + \tfrac{1}{8} \mathcal{D}_\mu \bar{\mathcal{W}} \mathcal{D}^\mu \mathcal{W} - \tfrac{1}{1024} \mathcal{D}_{(i}^{\alpha} \mathcal{D}_{\alpha j)} \mathcal{W} \mathcal{D}^{\beta(i} \mathcal{D}_\beta^{j)} \mathcal{W} \\
&\qquad + \tfrac{1}{64} \mathcal{W} \{ \bar{\mathcal{D}}_{\dot\alpha i} \bar{\mathcal{W}} , \bar{\mathcal{D}}^{\dot\alpha i} \bar{\mathcal{W}} \} + \tfrac{1}{64} \bar{\mathcal{W}} \{ \mathcal{D}_i^\alpha \mathcal{W} , \mathcal{D}_\alpha^i \mathcal{W} \} \\
&\qquad - \tfrac{1}{128} [\bar{\mathcal{W}} , \mathcal{W}] [\bar{\mathcal{W}} , \mathcal{W}] \Big)
\end{aligned}
\qquad (4.3.11)
$$

The constraints (4.3.2a) have the solution

$$
\begin{aligned}
\mathcal{A}_\alpha^i &= \tfrac{i}{2} e^{-U} \theta_\alpha^i \bar{W} e^U - i\, e^{-U} D_\alpha^i e^U \\
\bar{\mathcal{A}}_{\dot\alpha i} &= -\tfrac{i}{2} e^{-V} \bar\theta_{\dot\alpha i} W e^V - i\, e^{-V} \bar{D}_{\dot\alpha i} e^V
\end{aligned}
\qquad (4.3.12)
$$

where

$$ D_\alpha^i \bar{W} = 0 = \bar{D}_{\dot\alpha i} W \qquad (4.3.13) $$

The solutions (4.3.12) then give $\mathcal{W}$, $\bar{\mathcal{W}}$ as

$$
\begin{aligned}
\bar{\mathcal{W}} &= e^{-U} \bar{W} e^U \\
\mathcal{W} &= e^{-V} W e^V
\end{aligned}
\qquad (4.3.14)
$$

If one chooses the gauge parameter to be $\bar{\Lambda}$ instead of Λ, so that

$$\phi \to e^{i\bar{\Lambda}} \phi e^{-i\bar{\Lambda}} \qquad (4.3.15)$$

for covariant superfields, then the superfields U , V transform under gauge transformations according to

$$e^{U} \to e^{-T^{\dagger}} e^{U} e^{-i\bar{\Lambda}}$$

$$e^{V} \to e^{S} e^{V} e^{-i\bar{\Lambda}} \qquad (4.3.16)$$

Under a gauge transformation W and $\bar{W}$ transform as

$$W \to e^{S} W e^{-S}$$

$$\bar{W} \to e^{-T^{\dagger}} \bar{W} e^{T^{\dagger}} \qquad (4.3.17)$$

where

$$\bar{D}_{\dot{\alpha} i} S = 0 = D^{i}_{\alpha} T^{\dagger} \qquad (4.3.18)$$

Again, as in the case of unextended supersymmetry, the theory is restricted by the condition that the gauge potentials A_{μ} , A^{i}_{α} , $\bar{A}_{\dot{\alpha} i}$ are restricted to be real up to gauge transformations

$$(A^{\dagger})_{\dot{\alpha} i} = -\frac{i}{2} e^{U^{\dagger}} \bar{\theta}_{\dot{\alpha} i} W e^{-U^{\dagger}} - i e^{U^{\dagger}} \bar{D}_{\dot{\alpha} i} e^{-U^{\dagger}}$$

$$= e^{X} \bar{A}_{\dot{\alpha} i} e^{-X} - i e^{X} \bar{D}_{\dot{\alpha} i} e^{-X}$$

$$= -\frac{i}{2} e^{X} e^{-V} \bar{\theta}_{\dot{\alpha} i} W e^{V} e^{-X} - i e^{X} e^{-V} \bar{D}_{\dot{\alpha} i} (e^{V} e^{-X})$$

$$(\bar{A}^{\dagger})^{i}_{\alpha} = \frac{i}{2} e^{V^{\dagger}} \theta^{i}_{\alpha} \bar{W} e^{-V^{\dagger}} - i e^{V^{\dagger}} D^{i}_{\alpha} e^{-V^{\dagger}}$$

$$= e^{X^{\dagger}} A^{i}_{\alpha} e^{-X^{\dagger}} - i e^{X^{\dagger}} D^{i}_{\alpha} e^{-X^{\dagger}}$$

$$= \frac{i}{2} e^{X^{\dagger}} e^{-U} \theta^{i}_{\alpha} \bar{W} e^{U} e^{-X^{\dagger}} - i e^{X^{\dagger}} e^{-U} D^{i}_{\alpha} (e^{U} e^{-X^{\dagger}})$$

$$(4.3.19)$$

The condition that $\mathcal{A}_\mu$ is real up to a gauge transformation is again that $X^\dagger = X$. This then gives

$$\mathcal{A}_\mu^\dagger = e^X \mathcal{A}_\mu e^{-X} - i\, e^X \partial_\mu e^{-X} \qquad (4.3.20)$$

Also, using $X^\dagger = X$ and (4.3.19), one finds that

$$e^{U^\dagger} e^V = e^X = e^{V^\dagger} e^U \qquad (4.3.21)$$

These again lead one to the expression

$$S = T \qquad (4.3.22)$$

As in the unextended case, the gauge choice $U = 0$ can be made. This restricts the gauge group to $T = i\Lambda$. This gives the gauge transformations of the fields as

$$\begin{aligned} e^V &\to e^{i\Lambda} e^V e^{-i\bar\Lambda} \\ W &\to e^{i\Lambda} W e^{-i\Lambda} \\ \bar W &\to e^{i\bar\Lambda} \bar W e^{-i\bar\Lambda} \end{aligned} \qquad (4.3.23)$$

where

$$D^i_\alpha \bar\Lambda = 0 \qquad (4.3.24)$$

In this gauge $V = V^\dagger$ and the gauge potentials are

$$\begin{aligned} \mathcal{A}^i_\alpha &= \tfrac{i}{2}\theta^i_\alpha \bar W \\ \bar{\mathcal{A}}_{\dot\alpha i} &= -\tfrac{i}{2} e^{-V} \bar\theta_{\dot\alpha i} W e^V - i\, e^{-V} \bar D_{\dot\alpha i}\, e^V \end{aligned} \qquad (4.3.25)$$

From these one has

$$\begin{aligned} \bar{\mathcal{W}} &= \bar W \\ \mathcal{W} &= e^{-V} W e^V \end{aligned} \qquad (4.3.26)$$

The gauge transformation (4.3.23a) for V means that a gauge transformation to the Wess-Zumino gauge can be performed so that its expansion in θ , $\bar{\theta}$ has no terms where only θ's or $\bar{\theta}$'s occur.

However, unlike the unextended case, the superfield V is not unconstrained [36]. From $\mathcal{F}^{i}_{\alpha\dot{\beta}j} = 0$ the following constraint arises

$$D^{(i}_{\alpha} \bar{\mathcal{A}}^{j)}_{\dot{\beta}} + \bar{D}^{(i}_{\dot{\beta}} \mathcal{A}^{j)}_{\alpha} + i\{ \mathcal{A}^{(i}_{\alpha} , \bar{\mathcal{A}}^{j)}_{\dot{\beta}} \} = 0 \qquad (4.3.27)$$

In the gauge U = 0 this reads

$$D^{(i}_{\alpha} (-\tfrac{i}{2} e^{-V} \bar{\theta}^{j)}_{\dot{\alpha}} W e^{V} - i e^{-V} \bar{D}^{j)}_{\dot{\alpha}} e^{V}) + \bar{D}^{(i}_{\dot{\alpha}} (\tfrac{i}{2} \theta^{j)}_{\alpha} \bar{W})$$

$$+ i \{ \tfrac{i}{2} \theta^{(i}_{\alpha} \bar{W} , -\tfrac{i}{2} e^{-V} \bar{\theta}^{j)}_{\dot{\alpha}} W e^{V} - i e^{-V} \bar{D}^{j)}_{\dot{\alpha}} e^{V} \} = 0 \qquad (4.3.28)$$

For V in the Wess-Zumino gauge, this reads at the $\theta = \bar{\theta} = 0$ level as

$$\partial^{(i}_{\alpha} \bar{\partial}^{j)}_{\dot{\alpha}} V \big|_{\theta=\bar{\theta}=0} = 0 \qquad (4.3.29)$$

It is now easy to obtain the component expansion of W when V is in a Wess-Zumino gauge in the same way as the components of W^{α} were obtained in the unextended case.

Taking the $\theta = \bar{\theta} = 0$ components of (4.3.26b) and using (4.3.9) gives

$$W \big|_{\theta=\bar{\theta}=0} = 4 i C \qquad (4.3.30)$$

Using (4.3.25) one has

$$\mathcal{D}^{i}_{\alpha} \mathcal{W} = D^{i}_{\alpha} (e^{-V} W e^{V}) - \tfrac{1}{2} \theta^{i}_{\alpha} [\bar{W} , e^{-V} W e^{V}] \qquad (4.3.31)$$

Taking $\theta = \bar{\theta} = 0$ components and using (4.3.9) gives

$$\partial^{i}_{\alpha} W \big|_{\theta=\bar{\theta}=0} = - 4 i \lambda^{i}_{\alpha} \qquad (4.3.32)$$

Again, using (4.3.25) and (4.3.6a)

$$\mathcal{D}^{i}_{\alpha} \mathcal{D}^{j}_{\beta} \mathcal{W} = \tfrac{1}{4} \varepsilon_{\alpha\beta} \mathcal{D}^{\gamma(i} \mathcal{D}^{j)}_{\gamma} \mathcal{W} + 4 i g^{ij} (\sigma^{\mu\nu} \varepsilon)_{\alpha\beta} \mathcal{F}_{\mu\nu}$$

$$- \frac{1}{2} \varepsilon_{\alpha\beta} g^{ij} [\mathcal{\overline{W}} , \mathcal{W}]$$

$$= D^i_\alpha D^j_\beta (e^{-V} W e^V) - \frac{1}{2} g^{ij} \varepsilon_{\alpha\beta} [\overline{W}, e^{-V} W e^V]$$

$$+ \frac{1}{2} \theta^j_\beta [\overline{W} , D^i_\alpha (e^{-V} W e^V)] - \frac{1}{2} \theta^i_\alpha [\overline{W} , D^j_\beta (e^{-V} W e^V)]$$

$$+ \frac{1}{4} \theta^i_\alpha \theta^j_\beta [\overline{W} , [\overline{W} , e^{-V} W e^V] \qquad (4.3.33)$$

Taking $\theta = \overline{\theta} = 0$ components and using (4.3.9) gives

$$\partial^i_\alpha \partial^j_\beta W \big|_{\theta=\overline{\theta}=0} = 8 i \varepsilon_{\alpha\beta} D^{ij} + 4 i g^{ij} (\sigma^{\mu\nu} \varepsilon)_{\alpha\beta} F_{\mu\nu} \qquad (4.3.34)$$

Using (4.3.6) and (4.3.25)

$$\mathcal{D}^i_\alpha \mathcal{D}^j_\beta \mathcal{D}^k_\gamma \mathcal{W} = i \varepsilon_{\beta\gamma} g^{i(j} \sigma^\mu_{\alpha\dot{\alpha}} \mathcal{D}_\mu \overline{\mathcal{D}}^{\dot{\alpha}k)} \overline{\mathcal{W}} - i g^{jk} \varepsilon_{\alpha(\beta} \sigma^\mu_{\gamma)\dot{\alpha}} \mathcal{D}_\mu \overline{\mathcal{D}}^{\dot{\alpha}i} \overline{\mathcal{W}}$$

$$+ g^{ij} \varepsilon_{\beta\gamma} [\overline{\mathcal{W}}, \mathcal{D}^k_\alpha \mathcal{W}]$$

$$= D^i_\alpha D^j_\beta D^k_\gamma (e^{-V} W e^V) - \frac{1}{2} g^{jk} \varepsilon_{\beta\gamma} [\overline{W} , D^i_\alpha (e^{-V} W e^V)]$$

$$+ \frac{1}{2} g^{ik} \varepsilon_{\alpha\gamma} [\overline{W} , D^j_\beta (e^{-V} W e^V)]$$

$$- \frac{1}{2} g^{ij} \varepsilon_{\alpha\beta} [\overline{W} , D^k_\gamma (e^{-V} W e^V)]$$

$$+ \text{higher order terms in } \theta \qquad (4.3.35)$$

Taking $\theta = \overline{\theta} = 0$ components and using (4.3.9) gives

$$\partial^i_\alpha \partial^j_\beta \partial^k_\gamma W \big|_{\theta=\overline{\theta}=0} = - 4 \varepsilon_{\beta\gamma} g^{i(j} \sigma^\mu_{\alpha\dot{\alpha}} \mathcal{D}_\mu \overline{\lambda}^{\dot{\alpha}k)} + 4 g^{jk} \varepsilon_{\alpha(\beta} \sigma^\mu_{\gamma)\dot{\alpha}} \mathcal{D}_\mu \overline{\lambda}^{\dot{\alpha}i}$$

$$- 4 g^{i(j} \varepsilon_{\beta\gamma} [C^* , \lambda^{k)}_\alpha] + 4 g^{jk} \varepsilon_{\alpha(\beta} [C^* , \lambda^i_{\gamma)}] \qquad (4.3.36)$$

Lastly, using (4.3.9) and (4.3.25)

$$\mathcal{D}^i_\alpha \mathcal{D}^j_\beta \mathcal{D}^k_\gamma \mathcal{D}^\ell_\delta \mathcal{W} = (g^{ij} g^{k\ell} \varepsilon_{\alpha(\gamma} \varepsilon_{\delta)\beta} - \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} g^{i(k} g^{\ell)j})$$

$$\times (2 \mathcal{D}_\mu \mathcal{D}^\mu \overline{\mathcal{W}} - \frac{1}{4} \{ \overline{\mathcal{D}}_{\dot{\alpha}m} \overline{\mathcal{W}} , \overline{\mathcal{D}}^{\dot{\alpha}m} \overline{\mathcal{W}} \})$$

$$-\frac{1}{2}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\,g^{i\ell}\,g^{jk}[\,\bar{\mathcal{W}}\,,[\,\bar{\mathcal{W}}\,,\mathcal{W}\,]\,]$$

$$+\frac{1}{4}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\,g^{jk}[\,\bar{\mathcal{W}}\,,\mathcal{D}^{\varepsilon(i}\,\mathcal{D}^{\ell)}_{\varepsilon}\,\mathcal{W}\,]$$

$$+2\,i\,g^{ij}\,g^{k\ell}\,(\,\sigma^{\mu\nu}\,\varepsilon\,)_{\alpha\beta}\,\varepsilon_{\gamma\delta}[\,\bar{\mathcal{W}}\,,\mathcal{F}_{\mu\nu}\,]$$

$$-2\,i\,g^{ij}\,g^{k\ell}\,(\,\sigma^{\mu\nu}\,\varepsilon\,)_{\alpha(\gamma}\,\varepsilon_{\delta)\beta}[\,\bar{\mathcal{W}}\,,\mathcal{F}_{\mu\nu}\,]$$

$$=D^{i}_{\alpha}\,D^{j}_{\beta}\,D^{k}_{\gamma}\,D^{\ell}_{\delta}\,(\,e^{-V}\,W\,e^{V}\,)-\frac{1}{2}g^{k\ell}\,\varepsilon_{\gamma\delta}[\,\overline{W}\,,D^{i}_{\alpha}\,D^{j}_{\beta}\,(\,e^{-V}\,W\,e^{V}\,)\,]$$

$$+\frac{1}{2}g^{j\ell}\,\varepsilon_{\beta\delta}[\,\overline{W}\,,D^{i}_{\alpha}\,D^{k}_{\gamma}\,(\,e^{-V}\,W\,e^{V}\,)\,]$$

$$-\frac{1}{2}g^{i\ell}\,\varepsilon_{\alpha\delta}[\,\overline{W}\,,D^{j}_{\beta}\,D^{k}_{\gamma}\,(\,e^{-V}\,W\,e^{V}\,)\,]$$

$$-\frac{1}{2}g^{jk}\,\varepsilon_{\beta\gamma}[\,\overline{W}\,,D^{i}_{\alpha}\,D^{\ell}_{\delta}\,(\,e^{-V}\,W\,e^{V}\,)\,]$$

$$+\frac{1}{2}g^{ik}\,\varepsilon_{\alpha\gamma}[\,\overline{W}\,,D^{j}_{\beta}\,D^{\ell}_{\delta}\,(\,e^{-V}\,W\,e^{V}\,)\,]$$

$$-\frac{1}{2}g^{ij}\,\varepsilon_{\alpha\beta}[\,\overline{W}\,,D^{k}_{\gamma}\,D^{\ell}_{\delta}\,(\,e^{-V}\,W\,e^{V}\,)\,]$$

$$+\frac{1}{4}g^{i\ell}\,g^{jk}\,\varepsilon_{\beta\gamma}\,\varepsilon_{\alpha\delta}[\,\overline{W}\,,[\,\overline{W}\,,e^{-V}\,W\,e^{V}\,]\,]$$

$$-\frac{1}{4}g^{ik}\,g^{j\ell}\,\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta\delta}[\,\overline{W}\,,[\,\overline{W}\,,e^{-V}\,W\,e^{V}\,]\,]$$

$$+\frac{1}{4}g^{ij}\,g^{k\ell}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}[\,\overline{W}\,,[\,\overline{W}\,,e^{-V}\,W\,e^{V}\,]\,]$$

$$+\text{ higher order terms in }\theta \qquad (4.3.37)$$

Taking $\theta=\overline{\theta}=0$ components gives

$$\partial^{i}_{\alpha}\,\partial^{j}_{\beta}\,\partial^{k}_{\gamma}\,\partial^{\ell}_{\delta}\,W\big|_{\theta=\overline{\theta}=0}=(\,\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\,g^{i(k}\,g^{\ell)j}-g^{ij}\,g^{k\ell}\,\varepsilon_{\alpha(\gamma}\,\varepsilon_{\delta)\beta}\,)$$

$$\times\,(\,8\,i\,\mathcal{D}_{\mu}\,\mathcal{D}^{\mu}\,C^{*}-4\{\overline{\lambda}_{\dot{\alpha}m}\,,\overline{\lambda}^{\dot{\alpha}m}\}-8\,i[\,C^{*}\,,[\,C^{*}\,,C\,]\,]\,)$$

$$(4.3.38)$$

Taking (4.3.30) , (4.3.32) , (4.3.34) , (4.3.36) and (4.3.38) together gives

$$\frac{1}{4}W(y_+) = i\,C(y_+) - i\,\theta^\alpha_i\,\lambda^i_\alpha(y_+) - i\,\theta^\alpha_i\,\theta_{\alpha j}\,D^{ij}(y_+)$$

$$-\frac{i}{2}\,\theta^\alpha_i\,\sigma^{\mu\nu}{}_\alpha{}^\beta\,\theta^i_\beta\,F_{\mu\nu}(y_+) - 2\,\chi^\alpha_i\,(\,\sigma^\mu_{\alpha\dot\alpha}\,\mathcal{D}_\mu\,\overline{\lambda}^{\dot\alpha i}(y_+)$$

$$+[\,C^*(y_+)\,,\,\lambda^i_\alpha(y_+)]\,) + u\,(\,4\,i\,\mathcal{D}_\mu\,\mathcal{D}^\mu\,C^*(y_+)$$

$$-2\,\{\,\overline{\lambda}_{\dot\alpha i}(y_+)\,,\,\overline{\lambda}^{\dot\alpha i}(y_+)\} - 4\,i\,[C^*(y_+)\,,[C^*(y_+)\,,\,C(y_+)]]\,)$$

(4.3.39)

Here, the products of θ's are [37]

$$\chi^i_\alpha = \partial^i_\alpha\,u$$

$$u = \theta^{\alpha=1}_{i=1}\,\theta^1_2\,\theta^2_1\,\theta^2_2$$

and so

$$\overline{\chi}_{\dot\alpha i} = -\,\frac{\partial}{\partial\overline{\theta}^{\dot\alpha i}}\,\underline{u}$$

$$\underline{u} = \overline{\theta}^{\dot 1 1}\,\overline{\theta}^{\dot 1 2}\,\overline{\theta}^{\dot 2 1}\,\overline{\theta}^{\dot 2 2} \qquad (4.3.40)$$

This gives

$$\frac{1}{64}\,\mathrm{tr}\,(\,W^2\,)\big|_u = \mathrm{tr}\,(\,-\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{i}{2}\,\overline{\lambda}_{\dot\alpha i}\,\overline{\sigma}^{\dot\alpha\alpha}_\mu\,\overleftrightarrow{\mathcal{D}}^\mu\,\lambda^i_\alpha + 2\,\mathcal{D}_\mu C^*\,\mathcal{D}^\mu C$$

$$+ D_{ij}\,D^{ij} + i\,C^*\{\,\lambda^\alpha_i\,,\,\lambda^i_\alpha\} - i\,C\{\,\overline{\lambda}_{\dot\alpha i}\,,\,\overline{\lambda}^{\dot\alpha i}\}$$

$$-2\,[\,C^*\,,\,C]\,[C^*\,,\,C]\,) \qquad (4.3.41)$$

The superfield W transforms under supersymmetry transformations as

$$\delta W(y_+) = (\,\xi^\alpha_i\,\frac{\partial}{\partial\theta^\alpha_i} + 2\,i\,\theta^\alpha_i\,\sigma^\mu_{\alpha\dot\alpha}\,\overline{\xi}^{\dot\alpha i}\,\frac{\partial}{\partial y^\mu_+}\,)\,W(y_+) \qquad (4.3.42)$$

However, the supersymmetry transformations destroy the Wess-Zumino gauge for V, as in the unextended case. Similarly to before, the transformation law (4.3.42) has to be modified by a gauge transformation in order to give the covariant supersymmetry transformations.

When this is done one finds

$$\delta C = \lambda^{\alpha}_{i}\, \xi^{i}_{\alpha}$$

$$\delta\lambda^{i}_{\alpha} = 2\, D^{ij}\, \xi_{\alpha j} + \sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\, \xi^{i}_{\beta}\, F_{\mu\nu} - 2\, i\, \sigma^{\mu}_{\alpha\dot{\alpha}}\, \bar{\xi}^{\dot{\alpha} i}\, \mathcal{D}_{\mu} C - 2\, i\, [C^{*}, C]\, \xi^{i}_{\alpha}$$

$$\delta F^{\mu\nu} = i\, (\, \xi^{\alpha}_{i}\, \sigma^{[\nu}_{\alpha\dot{\alpha}}\, \mathcal{D}^{\mu]}\, \bar{\lambda}^{\dot{\alpha} i} - \mathcal{D}^{[\mu}\, \lambda^{\alpha}_{i}\, \sigma^{\nu]}_{\alpha\dot{\alpha}}\, \bar{\xi}^{\dot{\alpha} i}\,)$$

$$\delta D^{ij} = -\frac{i}{2}\, (\, \xi^{\alpha(i}\, \sigma^{\mu}_{\alpha\dot{\alpha}}\, \mathcal{D}_{\mu}\, \bar{\lambda}^{\dot{\alpha} j)} + \mathcal{D}_{\mu}\, \lambda^{\alpha(i}\, \sigma^{\mu}_{\alpha\dot{\alpha}}\, \bar{\xi}^{\dot{\alpha} j)}\,)$$

$$- i\, (\, \xi^{\alpha(i}\, [C^{*}, \lambda^{j)}_{\alpha}] - \bar{\xi}^{(i}_{\dot{\alpha}}\, [C, \bar{\lambda}^{\dot{\alpha} j)}]\,) \qquad (4.3.43)$$

These transformation laws can also be obtained in their exact form from

$$\delta\phi = (\, \xi^{\alpha}_{i}\, Q^{i}_{\alpha} - \bar{\xi}_{\dot{\alpha} i}\, \bar{Q}^{\dot{\alpha} i}\,)\, \phi \big|_{\theta=\bar{\theta}=0}$$

$$= (\, \xi^{\alpha}_{i}\, \mathcal{D}^{i}_{\alpha} - \bar{\xi}_{\dot{\alpha} i}\, \bar{\mathcal{D}}^{\dot{\alpha} i}\,)\, \phi \big|_{\theta=\bar{\theta}=0} \qquad (4.3.44)$$

when ϕ is the $\theta = \bar{\theta} = 0$ component of the superfields $\mathcal{W}$, $\mathcal{D}^{i}_{\alpha}\, \mathcal{W}$, $\mathcal{F}^{\mu\nu}$, $\mathcal{D}^{\alpha(i}\, \mathcal{D}^{j)}_{\alpha}\, \mathcal{W}$.

For the abelian theory the constraints on W read

$$D^{i}_{\alpha}\, \bar{W} = 0 = \bar{D}_{\dot{\alpha} i}\, W$$

$$D^{\alpha i}\, D^{j}_{\alpha}\, W = \bar{D}^{i}_{\dot{\alpha}}\, \bar{D}^{\dot{\alpha} j}\, \bar{W} \qquad (4.3.45)$$

These can be solved in terms of a real superfield V^{ij}, symmetric in its SU(2) indices [38]

$$W = \bar{D}^{4}\, D^{\alpha i}\, D^{j}_{\alpha}\, V_{ij} \qquad (4.3.46)$$

where

$$\bar{D}^{4}\, \underline{u} = 1 \qquad (4.3.47)$$

The superfield V_{ij} has the gauge invariance

$$V_{ij} = D^{\alpha k}\chi_{\alpha ijk} + \overline{D}^{\dot\alpha k}\overline{\chi}_{\dot\alpha ijk} \qquad (4.3.48)$$

where $\chi_{\alpha ijk}$ is totally symmetric in ijk . This gauge invariance can be used to transform V^{ij} to the form

$$\begin{aligned}
V^{ij} &= -4iu\underline{u}D^{ij}(x) - \tfrac{2}{3}i\underline{u}\chi^{\alpha(i}\lambda^{j)}_{\alpha}(x) + \tfrac{2}{3}iu\overline{\chi}^{(i}_{\dot\alpha}\overline{\lambda}^{\dot\alpha j)}(x) \\
&\quad - \tfrac{1}{6}\chi^{\alpha(i}\sigma^{\mu}_{\alpha\dot\alpha}\overline{\chi}^{\dot\alpha j)}A_{\mu}(x) - \tfrac{i}{3}\underline{u}\theta^{\alpha i}\theta^{j}_{\alpha}C(x) + \tfrac{i}{3}u\overline{\theta}^{i}_{\dot\alpha}\overline{\theta}^{\dot\alpha j}C^*(x) \\
&= -4iu\underline{u}D^{ij}(y_+) - \tfrac{2}{3}i\underline{u}\chi^{\alpha(i}\lambda^{j)}_{\alpha}(y_+) + \tfrac{2}{3}iu\overline{\chi}^{(i}_{\dot\alpha}\overline{\lambda}^{\dot\alpha j)}(y_+) \\
&\quad - \tfrac{1}{6}\chi^{\alpha(i}\sigma^{\mu}_{\alpha\dot\alpha}\overline{\chi}^{\dot\alpha j)}A_{\mu}(y_+) - \tfrac{i}{3}\underline{u}\theta^{\alpha i}\theta^{j}_{\alpha}C(y_+) \\
&\quad + \tfrac{i}{3}u\overline{\theta}^{i}_{\dot\alpha}\overline{\theta}^{\dot\alpha j}C^*(y_+) \qquad (4.3.49)
\end{aligned}$$

It is now easy to see that using

$$\overline{D}_{\dot\alpha i} = -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha i}}$$

$$D^{i}_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}_{i}} + 2i\sigma^{\mu}_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha i}\frac{\partial}{\partial y^{\mu}_{+}} \qquad (4.3.50)$$

and so

$$D^{\alpha}_{i}D_{\alpha j} = \partial^{\alpha}_{i}\partial_{\alpha j} - 2i\overline{\theta}_{\dot\alpha(i}\overline{\sigma}^{\dot\alpha\alpha}_{\mu}\partial_{\alpha j)}(\partial_{y_+})^{\mu} - 4\overline{\theta}_{\dot\alpha i}\overline{\theta}^{\dot\alpha}_{j}(\partial_{y_+})^{2} \qquad (4.3.51)$$

that W as calculated from (4.3.46) is the same as given in (4.3.39).

It is possible to extend the unconstrained formulation to the non-abelian case as is done by [39], but the solution in ordinary superspace is not an explicit one and has no geometrical interpretation. However, it is possible to formulate the theory in N=2 harmonic superspace, and there the unconstrained formulation and its geometric interpretation are very simple [40].

4.4 The N=4 Supersymmetric Gauge Theory in Superspace

For the N=4 theory, the independent Bianchi identities are [20]

$$D^i_\alpha \mathcal{F}^{jk}_{\beta\gamma} + D^j_\beta \mathcal{F}^{ki}_{\gamma\alpha} + D^k_\gamma \mathcal{F}^{ij}_{\alpha\beta} = 0$$

$$\bar{D}_{\dot\alpha i} \mathcal{F}_{\dot\beta j \dot\gamma k} + \bar{D}_{\dot\beta j} \mathcal{F}_{\dot\gamma k \dot\alpha i} + \bar{D}_{\dot\gamma k} \mathcal{F}_{\dot\alpha i \dot\beta j} = 0$$

$$20\, i\, \mathcal{F}_{\mu\alpha}{}^{i} + 2\, i\, (\sigma_\mu \bar\sigma^\nu)_\alpha{}^\beta \mathcal{F}_{\nu\beta}{}^{i}$$

$$= (\bar{D}_j \bar\sigma_\mu)^\beta \mathcal{F}^{ij}_{\alpha\beta} + (\bar\sigma_\mu D^j)^{\dot\alpha} \mathcal{F}^{i}_{\alpha\dot\alpha j} + D^i_\alpha \bar\sigma_\mu^{\dot\beta\beta} \mathcal{F}^{j}_{\beta\dot\beta j}$$

$$20\, i\, \mathcal{F}_{\mu\dot\alpha i} + 2\, i\, \mathcal{F}_{\nu\dot\beta i} (\bar\sigma^\nu \sigma_\mu)^{\dot\beta}{}_{\dot\alpha}$$

$$= (\bar{D}_j \bar\sigma_\mu)^\beta \mathcal{F}^{j}_{\beta\dot\alpha i} + (\bar\sigma_\mu D^j)^{\dot\beta} \mathcal{F}_{\dot\alpha i \dot\beta j} + \bar{D}_{\dot\alpha i} \bar\sigma_\mu^{\dot\beta\beta} \mathcal{F}^{j}_{\beta\dot\beta j}$$

$$(\varepsilon \sigma_{\mu\nu})^{\beta\gamma} \left(\bar{D}_{\dot\alpha i} \mathcal{F}^{jk}_{\beta\gamma} + D^k_\gamma \mathcal{F}^{j}_{\beta\dot\alpha i} + D^j_\gamma \mathcal{F}^{k}_{\beta\dot\alpha i} \right)$$

$$= \frac{1}{5} \delta^{(j}_{i} (\varepsilon \sigma_{\mu\nu})^{\beta\gamma} \left(\bar{D}_{\dot\alpha \ell} \mathcal{F}^{\ell k)}_{\beta\gamma} + D^{k)}_\gamma \mathcal{F}^{\ell}_{\beta\dot\alpha\ell} + D^{\ell}_\gamma \mathcal{F}^{k)}_{\beta\dot\alpha\ell} \right)$$

$$(\bar\sigma_{\mu\nu} \varepsilon)^{\dot\beta\dot\gamma} \left(D^i_\alpha \mathcal{F}_{\dot\beta j \dot\gamma k} + \bar{D}_{\dot\beta j} \mathcal{F}^{i}_{\alpha\dot\gamma k} + \bar{D}_{\dot\beta k} \mathcal{F}^{i}_{\alpha\dot\gamma j} \right)$$

$$= \frac{1}{5} \delta^{i}_{(j} (\bar\sigma_{\mu\nu} \varepsilon)^{\dot\beta\dot\gamma} \left(D^{\ell}_\alpha \mathcal{F}_{\dot\beta\ell\dot\gamma k)} + \bar{D}_{\dot\beta\ell} \mathcal{F}^{\ell}_{\alpha\dot\gamma k)} + \bar{D}_{\dot\beta k)} \mathcal{F}^{\ell}_{\alpha\dot\gamma\ell} \right)$$

$$\varepsilon^{\beta\gamma} \left(\bar{D}_{\dot\alpha i} \mathcal{F}^{jk}_{\beta\gamma} + D^k_\gamma \mathcal{F}^{j}_{\beta\dot\alpha i} - D^j_\gamma \mathcal{F}^{k}_{\beta\dot\alpha i} \right)$$

$$= \frac{1}{3} \delta^{[j}_{i} \varepsilon^{\beta\gamma} \left(\bar{D}_{\dot\alpha\ell} \mathcal{F}^{\ell k]}_{\beta\gamma} + D^{k]}_\gamma \mathcal{F}^{\ell}_{\beta\dot\alpha\ell} - D^{\ell}_\gamma \mathcal{F}^{k]}_{\beta\dot\alpha\ell} \right)$$

$$\varepsilon^{\dot\beta\dot\gamma} \left(D^i_\alpha \mathcal{F}_{\dot\beta j \dot\gamma k} + \bar{D}_{\dot\beta j} \mathcal{F}^{i}_{\alpha\dot\gamma k} - \bar{D}_{\dot\beta k} \mathcal{F}^{i}_{\alpha\dot\gamma j} \right)$$

$$= \frac{1}{3} \delta^{i}_{[j} \varepsilon^{\dot\beta\dot\gamma} \left(D^{\ell}_\alpha \mathcal{F}_{\dot\beta\ell\dot\gamma k]} + \bar{D}_{\dot\beta\ell} \mathcal{F}^{\ell}_{\alpha\dot\gamma k]} - \bar{D}_{\dot\beta k]} \mathcal{F}^{\ell}_{\alpha\dot\gamma\ell} \right)$$

$$32\, i\, \mathcal{F}_{\mu\nu} = \bar\sigma^{\dot\beta\alpha}_{[\mu} (D_{\nu]} \mathcal{F}^{i}_{\alpha\dot\beta i} - D^i_\alpha \mathcal{F}_{\nu]\dot\beta i} - \bar{D}_{\dot\beta i} \mathcal{F}_{\nu]\alpha}{}^{i}) \quad (4.4.1)$$

The same constraints as in the N=2 theory (4.3.2) can now be imposed. However it should be noted that as there are no matter multiplets to couple the gauge theory to, they are not necessary as integrability conditions.

(4.3.2) have the solution

$$\mathcal{F}^{ij}_{\alpha\beta} = i\,\varepsilon_{\alpha\beta}\,\overline{W}^{ij} \quad , \quad \overline{W}^{ij} = -\overline{W}^{ji}$$

$$\mathcal{F}_{\dot{\alpha}i\dot{\beta}j} = i\,\varepsilon_{\dot{\alpha}\dot{\beta}}\,W_{ij} \quad , \quad W_{ij} = -W_{ji} \qquad (4.4.2)$$

The Bianchi identities can now be written down in terms of the superfields $\overline{W}^{ij}$, W_{ij} as

$$D^{(i}_{\alpha}\overline{W}^{j)k} = 0 = \overline{D}_{\dot{\alpha}(i}W_{j)k}$$

$$\mathcal{F}_{\mu\alpha}^{\ \ i} = \frac{1}{12}\sigma_{\mu\alpha\dot{\alpha}}\overline{D}^{\dot{\alpha}}_{j}\overline{W}^{ji}$$

$$\mathcal{F}_{\mu\dot{\alpha}i} = \frac{1}{12}\sigma_{\mu\alpha\dot{\alpha}}D^{\alpha j}W_{ji}$$

$$\overline{D}_{\dot{\alpha}i}\overline{W}^{jk} = \frac{1}{3}\delta^{[j}_{i}\overline{D}_{\dot{\alpha}\ell}\overline{W}^{\ell k]}$$

$$\mathcal{F}_{\mu\nu} = \frac{i}{96}\left(D^{\alpha i}\sigma_{\mu\nu\alpha}{}^{\beta}D^{j}_{\beta}W_{ij} - \overline{D}_{\dot{\alpha}i}\overline{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}}\overline{D}^{\dot{\beta}}_{j}\overline{W}^{ij}\right) \qquad (4.4.3)$$

The additional constraint

$$\overline{W}^{ij} = \frac{1}{2}\varepsilon^{ijk\ell}W_{k\ell} \qquad (4.4.4)$$

can be imposed, this restricting the multiplet to be CPT self conjugate.

Using the Bianchi identities, the theory can now be written completely in terms of the superfields W_{ij} , $D^{j}_{\alpha}W_{ji}$, $\mathcal{F}_{\mu\nu}$ and covariant space-time derivatives of these superfields

$$D^{i}_{\alpha}D^{\ell}_{\beta}W_{\ell j} = 12\,i\,\delta^{i}_{j}\,(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\mathcal{F}_{\mu\nu} - \frac{3}{2}\varepsilon_{\alpha\beta}\left[\overline{W}^{i\ell}, W_{\ell j}\right]$$

$$D^{i}_{\alpha}\overline{D}_{\dot{\beta}\ell}\overline{W}^{\ell j} = -6\,i\,\sigma^{\mu}_{\alpha\dot{\beta}}D_{\mu}\overline{W}^{ij}$$

$$D^{i}_{\alpha}\mathcal{F}^{\mu\nu} = \frac{1}{12}\sigma^{[\mu}_{\alpha\dot{\alpha}}D^{\nu]}\overline{D}^{\dot{\alpha}}_{j}\overline{W}^{ji}$$

$$\mathcal{D}^i_\alpha \mathcal{W}_{jk} = \frac{1}{3}\delta^{\ i}_{[j}\, \mathcal{D}^\ell_\alpha \mathcal{W}_{\ell k]}$$

$$\mathcal{D}^i_\alpha \overline{\mathcal{W}}^{jk} = \frac{1}{3}\varepsilon^{ijk\ell}\, \mathcal{D}^m_\alpha \mathcal{W}_{m\ell} \qquad (4.4.5)$$

As in the previous cases, it is again possible to write down an $\mathcal{L}$ for which $D^i_\alpha \mathcal{L}$ is a total derivative

$$\mathcal{L} = \mathrm{tr}\Big(-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{i}{288}\overline{\mathcal{D}}_{\dot\alpha j}\overline{\mathcal{W}}^{jk}\bar\sigma^{\dot\alpha\beta}_\mu \overleftrightarrow{\mathcal{D}}^\mu \mathcal{D}^\ell_\beta \mathcal{W}_{\ell k} - \frac{1}{32}\mathcal{D}_\mu \mathcal{W}_{jk}\mathcal{D}^\mu \overline{\mathcal{W}}^{jk} - \frac{1}{576}\mathcal{W}_{ij}\{\overline{\mathcal{D}}_{\dot\alpha k}\overline{\mathcal{W}}^{ki}, \overline{\mathcal{D}}^{\dot\alpha}_\ell \overline{\mathcal{W}}^{\ell j}\} + \frac{1}{576}\overline{\mathcal{W}}^{ij}\{\mathcal{D}^{\alpha k}\mathcal{W}_{ki}, \mathcal{D}^\ell_\alpha \mathcal{W}_{\ell j}\} - \frac{1}{1024}[\mathcal{W}_{ij}, \mathcal{W}_{k\ell}][\overline{\mathcal{W}}^{ij}, \overline{\mathcal{W}}^{k\ell}]\Big) \qquad (4.4.6)$$

So, the $\theta = \bar\theta = 0$ component of this equation gives a supersymmetric Lagrangian.

With the field definitions

$$F_{\mu\nu} = \mathcal{F}_{\mu\nu}\big|_{\theta=\bar\theta=0}$$

$$\bar\lambda^i_{\dot\alpha} = \frac{i}{12}\overline{\mathcal{D}}_{\dot\alpha j}\overline{\mathcal{W}}^{ji}\big|_{\theta=\bar\theta=0}$$

$$\lambda_{\alpha i} = -\frac{i}{12}\mathcal{D}^j_\alpha \mathcal{W}_{ji}\big|_{\theta=\bar\theta=0}$$

$$\phi_{ij} = \frac{1}{4}\mathcal{W}_{ij}\big|_{\theta=\bar\theta=0}$$

$$\phi^{ij} = \frac{1}{4}\overline{\mathcal{W}}^{ij}\big|_{\theta=\bar\theta=0} \qquad (4.4.7)$$

(4.4.6) gives the Lagrangian derived before

$$\mathcal{L} = \mathrm{tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}\bar\lambda^i_{\dot\alpha}\bar\sigma^{\dot\alpha\alpha}_\mu \overleftrightarrow{\mathcal{D}}^\mu \lambda_{\alpha i} - \frac{1}{2}\mathcal{D}_\mu \phi_{ij}\mathcal{D}^\mu \phi^{ij} + \phi_{ij}\{\bar\lambda^i_{\dot\alpha}, \bar\lambda^{\dot\alpha j}\} - \phi^{ij}\{\lambda^\alpha_i, \lambda_{\alpha j}\} - \frac{1}{4}[\phi_{ij},\phi_{k\ell}][\phi^{ij},\phi^{k\ell}]\Big) \qquad (4.4.8)$$

Again, the above Lagrangian can be written as an integral over a subspace of superspace [34]

$$I = \int d^4x \, D^{[ij],[k\ell]} \, K_{[ij],[k\ell]} \qquad (4.4.9)$$

where

$$K_{[ij],[k\ell]} = \mathrm{tr}\,(\mathcal{W}_{ij} \mathcal{W}_{k\ell} - \frac{1}{12} \varepsilon_{ijk\ell} \mathcal{W}_{mn} \overline{\mathcal{W}}^{mn})$$

$$D^{[ij],[k\ell]} = D^{[i}_{(\alpha} D^{j]}_{\beta)} D^{[k}_{(\gamma} D^{\ell]}_{\delta)} \varepsilon^{\alpha\gamma} \varepsilon^{\beta\delta} \qquad (4.4.10)$$

Once again, one can derive the supersymmetry transformations from the Bianchi identities (4.4.3). These give

$$\delta\phi_{ij} = i\,(\xi^{\alpha}_{[i} \lambda_{\alpha j]} - \varepsilon_{ijk\ell} \overline{\xi}^{\dot\alpha k} \overline{\lambda}^{\ell}_{\dot\alpha})$$

$$\delta\lambda_{\alpha i} = F_{\mu\nu} \sigma^{\mu\nu}{}_{\alpha}{}^{\beta} \xi_{\beta i} - 2\, \sigma^{\mu}_{\alpha\dot\alpha} \overline{\xi}^{\dot\alpha j} \mathcal{D}_{\mu} \phi_{ij} + 2i [\phi_{ij}, \phi^{jk}] \xi_{\alpha k}$$

$$\delta F^{\mu\nu} = i\,(\xi^{\alpha}_{i} \sigma^{[\nu}_{\alpha\dot\alpha} \mathcal{D}^{\mu]} \overline{\lambda}^{\dot\alpha i} - \mathcal{D}^{[\mu} \lambda^{\alpha}_{i} \sigma^{\nu]}_{\alpha\dot\alpha} \overline{\xi}^{\dot\alpha i}) \qquad (4.4.11)$$

The constraints (4.3.2a) can be solved similarly to the N=1 and N=2 cases. For a particular solution one can try

$$\mathcal{A}^{i}_{\alpha} = \frac{i}{2} \theta_{\alpha j} \overline{W}^{ij} \qquad (4.4.12)$$

where the superfield $\overline{W}^{ij}$ satisfies

$$D^{(i}_{\alpha} \overline{W}^{j)k} = 0 \qquad (4.4.13)$$

Using this one finds

$$\mathcal{F}^{ij}_{\alpha\beta} = i\, \varepsilon_{\alpha\beta} (\overline{W}^{ij} + \frac{1}{2} \theta^{\gamma}_{k} D^{i}_{\gamma} \overline{W}^{jk} - \frac{1}{8} \theta^{\gamma}_{k} \theta_{\gamma\ell} [\overline{W}^{ik}, \overline{W}^{j\ell}])$$

$$- \frac{i}{8} \theta_{(\alpha k} \theta_{\beta)\ell} [\overline{W}^{ik}, \overline{W}^{j\ell}] \qquad (4.4.14)$$

So, unlike the N=2 case, (4.4.12) goes wrong at the θ^2 level. However, one can improve $\mathcal{A}^{i}_{\alpha}$ by a θ^3 term, so as to give $\mathcal{F}^{ij}_{\alpha\beta}$ of the right form at the θ^2 level. When one tries

$$\mathcal{A}^{i}_{\alpha} = \frac{i}{2}\theta_{\alpha j}\overline{W}^{ij} + \frac{i}{8}\theta^{\beta}_{j}\theta_{\beta k}\theta_{\alpha\ell}\left[\overline{W}^{j[k}, \overline{W}^{i]\ell}\right] \qquad (4.4.15)$$

then now one finds

$$\mathcal{F}^{ij}_{\alpha\beta} = i\,\varepsilon_{\alpha\beta}\left(\overline{W}^{ij} + \frac{1}{2}\theta^{\gamma}_{k}D^{i}_{\gamma}\overline{W}^{jk} - \frac{1}{2}\theta^{\gamma}_{k}\theta_{\gamma\ell}\left[\overline{W}^{ik}, \overline{W}^{j\ell}\right]\right)$$

$$+ \text{ higher order terms in } \theta \qquad (4.4.16)$$

This process can be continued to higher orders to find the complete solution. One can then obtain the general solution to the constraints from this particular solution by a gauge transformation. When a Wess-Zumino gauge is chosen for the gauge transformation superfield V (again one can chose the gauge U = 0) then it would be possible to find the expansions of $\overline{W}^{ij}$ and W_{ij} order by order in θ , $\overline{\theta}$. However, for later purposes, only the expansion for the abelian case is considered, where the calculation is far simpler.

$$W_{ij}\big|_{\theta=\overline{\theta}=0} = 4\,\phi_{ij}$$

$$\partial^{i}_{\alpha}W_{jk}\big|_{\theta=\overline{\theta}=0} = 4\,i\,\delta^{i}_{[j}\lambda_{\alpha k]}$$

$$\overline{\partial}_{\dot\alpha i}W_{jk}\big|_{\theta=\overline{\theta}=0} = 4\,i\,\varepsilon_{ijk\ell}\overline{\lambda}^{\ell}_{\dot\alpha}$$

$$\partial^{i}_{\alpha}\partial^{j}_{\beta}W_{k\ell}\big|_{\theta=\overline{\theta}=0} = -4\,i\,\delta^{i}_{[k}\delta^{j}_{\ell]}(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}F_{\mu\nu}$$

$$\overline{\partial}_{\dot\alpha i}\overline{\partial}_{\dot\beta j}W_{k\ell}\big|_{\theta=\overline{\theta}=0} = -4\,i\,\varepsilon_{ijk\ell}(\varepsilon\overline{\sigma}^{\mu\nu})_{\dot\alpha\dot\beta}F_{\mu\nu}$$

$$\partial^{i}_{\alpha}\overline{\partial}_{\dot\alpha j}W_{k\ell}\big|_{\theta=\overline{\theta}=0} = 4\,i\,\delta^{i}_{j}\sigma^{\mu}_{\alpha\dot\alpha}\partial_{\mu}\phi_{k\ell} + 8\,i\,\delta^{i}_{[k}\sigma^{\mu}_{\alpha\dot\alpha}\partial_{\mu}\phi_{\ell]j}$$

$$\partial^{i}_{\alpha}\partial^{j}_{\beta}\partial^{k}_{\gamma}W_{\ell m}\big|_{\theta=\overline{\theta}=0} = 0$$

$$\partial^i_\alpha \partial^j_\beta \bar\partial_{\dot\alpha k} W_{\ell m}\Big|_{\theta=\bar\theta=0} = 2\,\delta^{[i}_k \delta^{j]}_{[\ell}\, \sigma^\mu_{(\alpha\dot\alpha}\partial_\mu \lambda_{\beta)m]} + 4\,\delta^{[i}_\ell \delta^{j]}_m\, \sigma^\mu_{(\alpha\dot\alpha}\partial_\mu \lambda_{\beta)k}$$

$$\partial^i_\alpha \bar\partial_{\dot\alpha j} \bar\partial_{\dot\beta k} W_{\ell m}\Big|_{\theta=\bar\theta=0} = 4\,\varepsilon_{jk\ell m}\, \sigma^\mu_{\alpha(\dot\alpha}\partial_\mu \bar\lambda^i_{\dot\beta)} + 2\,\delta^i_{[j}\varepsilon_{k]\ell mn}\, \sigma^\mu_{\alpha(\dot\alpha}\partial_\mu \bar\lambda^n_{\dot\beta)}$$

$$\bar\partial_{\dot\alpha i} \bar\partial_{\dot\beta j} \bar\partial_{\dot\gamma k} W_{\ell m}\Big|_{\theta=\bar\theta=0} = 0$$

$$\partial^i_\alpha \partial^j_\beta \partial^k_\gamma \partial^\ell_\delta W_{mn}\Big|_{\theta=\bar\theta=0} = 0$$

$$\partial^i_\alpha \partial^j_\beta \partial^k_\gamma \bar\partial_{\dot\alpha\ell} W_{mn}\Big|_{\theta=\bar\theta=0} = 4\,\varepsilon^{ijkp}\varepsilon_{\ell mnp}\,\partial_\mu F_{\nu\lambda}\big(\sigma^\mu_{\alpha\dot\alpha}(\sigma^{\nu\lambda}\varepsilon)_{\beta\gamma} + \sigma^\mu_{\beta\dot\alpha}(\sigma^{\nu\lambda}\varepsilon)_{\gamma\alpha} + \sigma^\mu_{\gamma\dot\alpha}(\sigma^{\nu\lambda}\varepsilon)_{\alpha\beta}\big)$$

$$\begin{aligned}\partial^i_\alpha \partial^j_\beta \bar\partial_{\dot\alpha k} \bar\partial_{\dot\beta\ell} W_{mn}\Big|_{\theta=\bar\theta=0} &= 4\,\varepsilon_{k\ell mn}\,\sigma^\mu_{\alpha(\dot\alpha}\sigma^\nu_{\beta\dot\beta)}\partial_\mu\partial_\nu \phi^{ij} \\ &\quad + 4\,\delta^{[i}_m \delta^{j]}_n\, \sigma^\mu_{\alpha(\dot\alpha}\sigma^\nu_{\beta\dot\beta)}\partial_\mu\partial_\nu\phi_{k\ell} \\ &\quad - 2\,\delta^{[i}_{[k}\,\sigma^\mu_{\alpha(\dot\alpha}\varepsilon_{\ell]mnp}\,\sigma^\nu_{\beta\dot\beta)}\partial_\mu\partial_\nu\phi^{j]p} \\ &\quad + 2\,\delta^{[i}_k \delta^{j]}_\ell\, \sigma^\mu_{\alpha(\dot\alpha}\sigma^\nu_{\beta\dot\beta)}\partial_\mu\partial_\nu\phi_{mn} \\ &\quad - 2\,\delta^{[i}_{[k}\delta^{j]}_{[m}\, \sigma^\mu_{\alpha(\dot\alpha}\sigma^\nu_{\beta\dot\beta)}\partial_\mu\partial_\nu\phi_{\ell]n]}\end{aligned}$$

$$\begin{aligned}\partial^i_\alpha \bar\partial_{\dot\alpha j} \bar\partial_{\dot\beta k} \bar\partial_{\dot\gamma\ell} W_{mn}\Big|_{\theta=\bar\theta=0} &= -4\,\delta^i_j\,\varepsilon_{k\ell mn}\big(\sigma^\mu_{\alpha\dot\alpha}(\varepsilon\bar\sigma^{\nu\lambda})_{\dot\beta\dot\gamma} + \dot\beta\dot\gamma\dot\alpha + \dot\gamma\dot\alpha\dot\beta\big)\partial_\mu F_{\nu\lambda} \\ &\quad - 4\,\delta^i_k\,\varepsilon_{\ell jmn}\big(\sigma^\mu_{\alpha\dot\alpha}(\varepsilon\bar\sigma^{\nu\lambda})_{\dot\beta\dot\gamma} + \dot\beta\dot\gamma\dot\alpha + \dot\gamma\dot\alpha\dot\beta\big)\partial_\mu F_{\nu\lambda} \\ &\quad - 4\,\delta^i_\ell\,\varepsilon_{jkmn}\big(\sigma^\mu_{\alpha\dot\alpha}(\varepsilon\bar\sigma^{\nu\lambda})_{\dot\beta\dot\gamma} + \dot\beta\dot\gamma\dot\alpha + \dot\gamma\dot\alpha\dot\beta\big)\partial_\mu F_{\nu\lambda}\end{aligned}$$

$$\bar\partial_{\dot\alpha i} \bar\partial_{\dot\beta j} \bar\partial_{\dot\gamma k} \bar\partial_{\dot\delta\ell} W_{mn}\Big|_{\theta=\bar\theta=0} = 0 \qquad (4.4.17)$$

From these, one can calculate that to 4^{th} order in θ, $\bar\theta$ that W_{ij} has the expansion

$$\tfrac{1}{4}W_{ij} = \phi_{ij} + i\,\theta_{[i}\lambda_{\alpha\, j]} + i\,\varepsilon_{ijk\ell}\,\bar\lambda^k_{\dot\alpha}\bar\theta^{\dot\alpha\ell} + i\,(\theta^\alpha_i\,\sigma^{\mu\nu}{}_\alpha{}^\beta\,\theta_{\beta j}$$

$$- \frac{1}{2} \varepsilon_{ijk\ell} \, \bar\theta^k_{\dot\alpha} \, \bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta} \, \bar\theta^{\dot\beta\ell} \,) \, F_{\mu\nu} - i \, \theta^\alpha_k \, \sigma^\mu_{\alpha\dot\alpha} \, \bar\theta^{\dot\alpha k} \, \partial_\mu \, \phi_{ij}$$

$$- 2 i \, \theta^\alpha_{[i} \, \sigma^\mu_{\alpha\dot\alpha} \, \bar\theta^{\dot\alpha k} \, \partial_\mu \, \phi_{j]k} - \varepsilon_{ijk\ell} \, \theta^\beta_m \, \sigma^\mu_{\beta\dot\beta} \, \bar\theta^{\dot\beta m} \, \partial_\mu \, \bar\lambda^k_{\dot\alpha} \, \bar\theta^{\dot\alpha\ell}$$

$$+ \varepsilon_{ijk\ell} \, \theta^\alpha_m \, \sigma^\mu_{\alpha\dot\alpha} \, \bar\theta^{\dot\alpha k} \, \partial_\mu \, \bar\lambda^m_{\dot\beta} \bar\theta^{\dot\beta\ell} + \theta^\beta_k \, \sigma^\mu_{\beta\dot\beta} \, \bar\theta^{\dot\beta k} \, \theta^\alpha_{[i} \, \partial_\mu \, \lambda_{\alpha j]}$$

$$+ \theta^\alpha_{[i} \, \sigma^\mu_{\alpha\dot\alpha} \, \bar\theta^{\dot\alpha k} \, \theta^\beta_{j]} \, \partial_\mu \, \lambda_{\beta \, k} + \theta^\alpha_i \, \sigma^{\mu\nu}{}_\alpha{}^\beta \, \theta_{\beta j} \, \theta^\gamma_k \, \sigma^\lambda_{\gamma\dot\alpha} \, \bar\theta^{\dot\alpha k} \, \partial_\lambda \, F_{\mu\nu}$$

$$+ \frac{1}{2} \varepsilon_{ijk\ell} \, \bar\theta^k_{\dot\alpha} \, \bar\sigma^{\mu\nu\dot\alpha}{}_{\dot\beta} \, \bar\theta^{\dot\beta\ell} \, \theta^\alpha_m \, \sigma^\lambda_{\alpha\dot\gamma} \, \bar\theta^{\dot\gamma m} \, \partial_\lambda \, F_{\mu\nu}$$

$$- 2 \, \theta^\alpha_i \, \sigma^\mu_{\alpha\dot\alpha} \, \bar\theta^{\dot\alpha k} \, \theta^\beta_j \, \sigma^\nu_{\beta\dot\beta} \, \bar\theta^{\dot\beta\ell} \, \partial_\mu \, \partial_\nu \, \phi_{k\ell}$$

$$- \frac{1}{2} \varepsilon_{ijk\ell} \, \theta^\alpha_m \, \sigma^\mu_{\alpha\dot\alpha} \, \bar\theta^{\dot\alpha k} \, \theta^\beta_n \, \sigma^\nu_{\beta\dot\beta} \, \bar\theta^{\dot\beta\ell} \, \partial_\mu \, \partial_\nu \, \phi^{mn}$$

$$+ \frac{1}{2} \theta^\alpha_k \, \sigma^\mu_{\alpha\dot\alpha} \, \bar\theta^{\dot\alpha k} \, \theta^\beta_\ell \, \sigma^\nu_{\beta\dot\beta} \, \bar\theta^{\dot\beta\ell} \, \partial_\mu \, \partial_\nu \, \phi_{ij}$$

$$+ \frac{1}{2} \varepsilon_{[ik\ell m} \, \theta^\alpha_{j]} \, \sigma^\mu_{\alpha\dot\alpha} \, \bar\theta^{\dot\alpha\ell} \, \theta^\beta_n \, \sigma^\nu_{\beta\dot\beta} \, \bar\theta^{\dot\beta m} \, \partial_\mu \, \partial_\nu \, \phi^{nk}$$

$$+ \theta^\alpha_k \, \sigma^\mu_{\alpha\dot\alpha} \, \bar\theta^{\dot\alpha k} \, \theta^\beta_{[i} \, \sigma^\nu_{\beta\dot\beta} \, \bar\theta^{\dot\beta\ell} \, \partial_\mu \, \partial_\nu \, \phi_{\ell j]}$$

$$+ \text{higher order terms in } \theta \, , \, \bar\theta \qquad (4.4.18)$$

However, this formulation of the N=4 theory in superspace suffers from the drawback that the constraints (4.3.2) imply the field equations for the theory [20 , 21]. This is easiest to see in the abelian case where the Bianchi identities (4.4.3) can be used to obtain the field equations

$$\sigma^\mu_{\alpha\dot\alpha} \, \partial_\mu \, \bar D^{\dot\alpha}_j \, \bar W^{ji} = 0 \, , \quad \Box \bar W^{ij} = 0 \, , \quad \partial^\mu \, F_{\mu\nu} = 0 \qquad (4.4.19)$$

However, as the constraints (4.3.2) are not forced upon the theory in order to be able to couple it to lower spin matter theories (the N=4 supersymmetric gauge theory being the theory with the minimum spin for N=4) , it is possible to consider weakening them in order to produce an off-shell theory.

There are two minimal relaxations of these constraints. The first is to relax the constraint (4.3.2b). The constraints then read

$$(\text{I}) \qquad \mathcal{F}^{(ij)}_{\alpha\beta} = 0 = \mathcal{F}_{\dot\alpha(i\dot\beta j)}$$

$$\mathcal{F}^{i}_{\alpha\dot\alpha i} = 0 \qquad (4.4.20)$$

The second constraint still allows one to solve for A_μ in terms of A^i_α and $\bar{A}_{\dot\alpha i}$, but the traceless part of $\mathcal{F}^i_{\alpha\dot\alpha j}$ is no longer constrained.

The second minimal possibility of taking the N=4 theory off-shell is to relax the constraint (4.3.2a) so that

$$(\text{II}) \qquad \mathcal{F}^{i}_{\alpha\dot\alpha j} = 0 \qquad (4.4.21)$$

Finally, one can consider relaxing both constraints (4.3.2) so that now only

$$(\text{III}) \qquad \mathcal{F}^{i}_{\alpha\dot\alpha i} = 0 \qquad (4.4.22)$$

One could also have considered relaxing the conventional constraint (4.4.22) but the resulting superfield can be absorbed into A_μ.

Cases I , II and III will now be studied in detail

Case I

The Bianchi identities can now be written down in terms of the superfields W_{ij}, $\bar{W}^{ij}$ and $W_\mu{}^i{}_j$ where

$$\mathcal{F}^{ij}_{\alpha\beta} = i\,\varepsilon_{\alpha\beta}\bar{W}^{ij} \quad , \quad \bar{W}^{ij} = -\bar{W}^{ji}$$

$$\mathcal{F}_{\dot\alpha i\dot\beta j} = i\,\varepsilon_{\dot\alpha\dot\beta}W_{ij} \quad , \quad W_{ij} = -W_{ji}$$

$$\mathcal{F}^{i}_{\alpha\dot\alpha j} = i\,\sigma^{\mu}_{\alpha\dot\alpha}\,\omega_{\mu}{}^{i}{}_{j} \quad , \quad \omega_{\mu}{}^{i}{}_{i} = 0 \tag{4.4.23}$$

The Bianchi identities (4.4.1) now read as

$$\mathcal{D}^{(i}_{\alpha}\,\overline{\mathcal{W}}^{j)k} = 0 = \overline{\mathcal{D}}_{\dot\alpha(i}\,\mathcal{W}_{j)k}$$

$$\mathcal{F}_{\mu\alpha}{}^{i} = \frac{1}{12}\,\sigma_{\mu\alpha\dot\alpha}\,\overline{\mathcal{D}}^{\dot\alpha}_{j}\,\overline{\mathcal{W}}^{ji} - \frac{1}{60}\Big[\,(\,\sigma_{\mu}\,\overline{\sigma}^{\nu} - 3\,\sigma^{\nu}\,\overline{\sigma}_{\mu}\,)\,\mathcal{D}\Big]^{j}_{\alpha}\,\omega_{\nu}{}^{i}{}_{j}$$

$$\mathcal{F}_{\mu\dot\alpha i} = \frac{1}{12}\,\sigma_{\mu\alpha\dot\alpha}\,\mathcal{D}^{\alpha j}\,\mathcal{W}_{ji} - \frac{1}{60}\Big(\,\overline{\mathcal{D}}\,(\,\overline{\sigma}^{\nu}\,\sigma_{\mu} - 3\,\overline{\sigma}_{\mu}\,\sigma^{\nu}\,)\Big)_{\dot\alpha j}\,\omega_{\nu}{}^{j}{}_{i}$$

$$(\,\overline{\sigma}^{\rho}\,\sigma^{\mu\nu}\,)^{\dot\alpha\alpha}\,\mathcal{D}^{(j}_{\alpha}\,\omega_{\rho}{}^{k)}{}_{i} = \frac{1}{5}\,\delta^{(j}_{i}\,(\,\overline{\sigma}^{\rho}\,\sigma^{\mu\nu}\,)^{\dot\alpha\alpha}\,\mathcal{D}^{\ell}_{\alpha}\,\omega_{\rho}{}^{k)}{}_{\ell}$$

$$\overline{\mathcal{D}}_{\dot\alpha(j}\,(\,\overline{\sigma}^{\mu\nu}\,\overline{\sigma}^{\rho}\,)^{\dot\alpha\alpha}\,\omega_{\rho}{}^{i}{}_{k)} = \frac{1}{5}\,\delta_{(j}^{i}\,\overline{\mathcal{D}}_{\dot\alpha\ell}\,(\,\overline{\sigma}^{\mu\nu}\,\overline{\sigma}^{\rho}\,)^{\dot\alpha\alpha}\,\omega_{\rho}{}^{\ell}{}_{k)}$$

$$\overline{\mathcal{D}}_{\dot\alpha i}\,\overline{\mathcal{W}}^{jk} = \frac{1}{3}\,\delta^{[j}_{i}\,\overline{\mathcal{D}}_{\dot\alpha\ell}\,\overline{\mathcal{W}}^{\ell k]} - \frac{1}{2}\,\sigma^{\mu}_{\alpha\dot\alpha}\,(\,\mathcal{D}^{\alpha[j}\,\omega_{\mu}{}^{k]}{}_{i} - \frac{1}{3}\,\delta^{[j}_{i}\,\mathcal{D}^{\alpha\ell}\,\omega_{\mu}{}^{k]}{}_{\ell})$$

$$\mathcal{D}^{i}_{\alpha}\,\mathcal{W}_{jk} = \frac{1}{3}\,\delta^{i}_{[j}\,\mathcal{D}^{\ell}_{\alpha}\,\mathcal{W}_{\ell k]} - \frac{1}{2}\,\sigma^{\mu}_{\alpha\dot\alpha}\,(\,\overline{\mathcal{D}}^{\dot\alpha}_{[j}\,\omega_{\mu}{}^{i}{}_{k]} - \frac{1}{3}\,\delta^{i}_{[j}\,\overline{\mathcal{D}}^{\dot\alpha}_{\ell}\,\omega_{\mu}{}^{\ell}{}_{k]}\,)$$

$$\begin{aligned}\mathcal{F}_{\mu\nu} = {} & \frac{i}{96}\,(\,\mathcal{D}^{\alpha i}\,\sigma_{\mu\nu\alpha}{}^{\beta}\,\mathcal{D}^{j}_{\beta}\,\mathcal{W}_{ij} - \overline{\mathcal{D}}_{\dot\alpha i}\,\overline{\sigma}_{\mu\nu}{}^{\dot\alpha}{}_{\dot\beta}\,\overline{\mathcal{D}}^{\dot\beta}_{j}\,\overline{\mathcal{W}}^{ij}\,) \\ & - \frac{1}{240}\,\varepsilon_{\mu\nu\rho\lambda}\,\overline{\sigma}^{\rho\dot\alpha\alpha}\big[\,\mathcal{D}^{i}_{\alpha}\,,\,\overline{\mathcal{D}}_{\dot\alpha j}\big]\,\omega^{\lambda j}{}_{i} \\ & + \frac{i}{240}\,\big[\omega_{\mu}{}^{i}{}_{j}\,,\,\omega_{\nu}{}^{j}{}_{i}\big]\end{aligned} \tag{4.4.24}$$

Here the additional constraint (4.4.4) cannot be imposed as the Bianchi identities then give rise to equations of motion.

The Bianchi identities (4.4.24) can now be used to derive the relations

$$\mathcal{D}^{i}_{\alpha}\,\overline{\mathcal{W}}^{jk} = \frac{1}{3}\,\varepsilon^{ijk\ell}\,\lambda_{2\alpha\ell}$$

where $\lambda_{2\alpha i} = -\frac{1}{2}\,\varepsilon_{ijk\ell}\,\mathcal{D}^{j}_{\alpha}\,\overline{\mathcal{W}}^{k\ell}$

$$\mathcal{D}^{i}_{\alpha}\,\mathcal{W}_{jk} = \frac{1}{3}\,\delta^{i}_{[j}\,\lambda_{1\alpha k]} - \frac{1}{2}\,\sigma^{\mu}_{\alpha\dot\alpha}\,\overline{\chi}_{1\mu jk}{}^{\dot\alpha i}$$

where $\lambda_{1\alpha i} = \mathcal{D}^j_\alpha \, \omega_{ji}$

$$\bar{\chi}_{1\mu\dot{\alpha}jk}{}^i = \bar{\mathcal{D}}_{\dot{\alpha}[j} \, \omega_\mu{}^i{}_{k]} - \frac{1}{3}\delta^i_{[j} \bar{\mathcal{D}}_{\dot{\alpha}\ell} \, \omega_\mu{}^\ell{}_{k]}$$

$$\mathcal{D}^i_\alpha \lambda_{1\beta j} = 12\, i\, \delta^i_j \, (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta} \mathcal{F}_{\mu\nu} - \frac{3}{2}\varepsilon_{\alpha\beta}[\bar{\omega}^{ik}, \omega_{kj}] - \frac{1}{4}\delta^i_j \varepsilon_{\alpha\beta}[\bar{\omega}^{k\ell}, \omega_{k\ell}]$$

$$+ \frac{1}{20}\delta^i_j \, \sigma^\mu_{(\alpha\dot{\alpha}}[\mathcal{D}^k_{\beta)}, \bar{\mathcal{D}}^{\dot{\alpha}}_\ell]\omega_\mu{}^\ell{}_k + \frac{1}{20}\delta^i_j \, (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}[\omega_\mu{}^k{}_\ell, \omega_\nu{}^\ell{}_k]$$

$$+ \frac{9}{16}\sigma^\mu_{\alpha\dot{\alpha}} \mathcal{D}^k_\beta \bar{\chi}_{1\mu}{}^{\dot{\alpha}}{}_{kj}{}^i - \frac{3}{16}\sigma^\mu_{\beta\dot{\alpha}} \mathcal{D}^k_\alpha \bar{\chi}_{1\mu}{}^{\dot{\alpha}}{}_{kj}{}^i$$

$$\mathcal{D}^i_\alpha \bar{\lambda}^j_{1\dot{\alpha}} = -6\, i\, \sigma^\mu_{\alpha\dot{\alpha}} \mathcal{D}_\mu \bar{\omega}^{ij} - \frac{4}{5}\sigma^\mu_{\alpha\dot{\alpha}}[\omega_\mu{}^i{}_k, \bar{\omega}^{kj}] - \frac{1}{5}\sigma^\mu_{\alpha\dot{\alpha}}[\omega_\mu{}^j{}_k, \bar{\omega}^{ki}]$$

$$+ \frac{3}{10}\sigma^\mu_{\beta\dot{\alpha}} \mathcal{D}^k_\alpha \chi_{1\mu}{}^{\beta ij}{}_k$$

$$\mathcal{D}^i_\alpha \lambda_{2\beta j} = 12\, i\, \delta^i_j \, (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta} \mathcal{G}_{\mu\nu} + \frac{3}{4}\varepsilon_{jk\ell m}\varepsilon_{\alpha\beta}[\bar{\omega}^{ik}, \bar{\omega}^{\ell m}]$$

where $\mathcal{G}_{\mu\nu} = \frac{i}{192}\left(\varepsilon_{ijk\ell} \mathcal{D}^{\alpha i} \sigma_{\mu\nu\alpha}{}^\beta \mathcal{D}^j_\beta \bar{\omega}^{k\ell} - \varepsilon^{ijk\ell} \bar{\mathcal{D}}_{\dot{\alpha}i} \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\mathcal{D}}^{\dot{\beta}}_j \omega_{k\ell}\right)$

$$\mathcal{D}^i_\alpha \bar{\lambda}^j_{2\dot{\alpha}} = -3\, i\, \varepsilon^{ijk\ell} \sigma^\mu_{\alpha\dot{\alpha}} \mathcal{D}_\mu \omega_{k\ell} + \frac{7}{10}\varepsilon^{jk\ell m}\sigma^\mu_{\alpha\dot{\alpha}}[\omega_\mu{}^i{}_k, \omega_{\ell m}]$$

$$+ \frac{3}{5}\varepsilon^{ijk\ell}\sigma^\mu_{\alpha\dot{\alpha}}[\omega_\mu{}^m{}_k, \omega_{\ell m}] - \frac{1}{5}\varepsilon^{ik\ell m}\sigma^\mu_{\alpha\dot{\alpha}}[\omega_\mu{}^j{}_k, \omega_{\ell m}]$$

$$- \frac{7}{20}\varepsilon^{jk\ell m}\sigma^\mu_{\alpha\dot{\beta}} \bar{\mathcal{D}}_{\dot{\alpha}k} \bar{\chi}_{1\mu}{}^{\dot{\beta}}{}_{\ell m}{}^i + \frac{1}{10}\varepsilon^{ik\ell m}\sigma^\mu_{\alpha\dot{\beta}} \bar{\mathcal{D}}_{\dot{\alpha}k} \bar{\chi}_{1\mu}{}^{\dot{\beta}}{}_{\ell m}{}^j$$

$$\mathcal{D}^i_\alpha \mathcal{F}_{\mu\nu} = \frac{1}{12}\sigma_{[\mu\alpha\dot{\alpha}} \mathcal{D}_{\nu]} \bar{\lambda}^{\dot{\alpha}i}_1 - \frac{1}{60}\left(\sigma_{[\mu\alpha\dot{\alpha}} \bar{\sigma}^{\lambda\dot{\alpha}\beta} - 3\sigma^\lambda_{\alpha\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\beta}_{[\mu}\right) \mathcal{D}_{\nu]} \mathcal{D}^j_\beta \omega_\lambda{}^i{}_j$$

$$\mathcal{D}^i_\alpha \mathcal{G}_{\mu\nu} = \frac{1}{12}\sigma^\lambda_{\alpha\dot{\alpha}} \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}} \mathcal{D}_\lambda \bar{\lambda}^{\dot{\beta}i}_2 - \frac{i}{24}\sigma_{\mu\nu\alpha}{}^\beta[\bar{\omega}^{ij}, \lambda_{2\beta j}]$$

$$- \frac{7}{360}\, i\, \sigma^\lambda_{\alpha\dot{\alpha}} \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}}[\omega_\lambda{}^i{}_j, \bar{\lambda}^{\dot{\beta}j}_2]$$

$$- \frac{i}{960}\sigma^\lambda_{\alpha\dot{\alpha}} \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}} \varepsilon^{ik\ell m}[\bar{\mathcal{D}}^{\dot{\beta}}_j \omega_\lambda{}^j{}_k, \omega_{\ell m}]$$

$$- \frac{7}{960}\, i\, \sigma^\lambda_{\alpha\dot{\alpha}} \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}} \varepsilon^{jk\ell m}[\bar{\mathcal{D}}^{\dot{\beta}}_j \omega_\lambda{}^i{}_k, \omega_{\ell m}]$$

$$- \frac{i}{960}\sigma^\lambda_{\alpha\dot{\alpha}} \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}} \varepsilon^{ijk\ell}[\bar{\mathcal{D}}^{\dot{\beta}}_j \omega_\lambda{}^m{}_k, \omega_{\ell m}]$$

$$+ \frac{i}{1920}\, \sigma^{\lambda}_{\alpha\dot{\gamma}}\, (\bar{\sigma}_{\mu\nu}\, \varepsilon)^{\dot{\alpha}\dot{\beta}}\, \bar{\mathcal{D}}_{\dot{\alpha} j}\, \bar{\mathcal{D}}_{\dot{\beta} k}\, (7\, \varepsilon^{jk\ell m}\, \bar{\chi}_{1\lambda\ell m}{}^{\dot{\gamma}\, i} - 2\, \varepsilon^{ik\ell m}\, \bar{\chi}_{1\lambda\ell m}{}^{\dot{\gamma}\, j}) \tag{4.4.25}$$

From these identities it follows that the component field content of the theory can be expressed as the $\theta = \bar{\theta} = 0$ components of the superfields $\bar{\mathcal{W}}^{ij}$, $\mathcal{W}_{ij}$, $\lambda_{1\alpha i}$, $\lambda_{2\alpha i}$, $\mathcal{F}_{\mu\nu}$, $\mathcal{G}_{\mu\nu}$, $\mathcal{W}_{\mu}{}^{i}{}_{j}$, covariant spinor derivatives of $\mathcal{W}_{\mu}{}^{i}{}_{j}$ and covariant space-time derivatives of these superfields.

To find the component field content of the superfield $\mathcal{W}_{\mu}{}^{i}{}_{j}$ it is easiest to study the abelian case with the aid of the tableaux calculus developed by [34]. Acting on the superfield $W_{\mu}{}^{i}{}_{j}$ with the covariant spinor derivative, one can split the result up into the irreducible representations of SU(4) as follows

$$D^{i}_{\alpha} W_{\mu}{}^{j}{}_{k} = \frac{1}{2} \chi_{1\mu\alpha}{}^{ij}{}_{k} + \frac{4}{15} \delta^{i}_{k}\, \chi_{2\mu\alpha}{}^{j} - \frac{1}{15} \delta^{j}_{k}\, \chi_{2\mu\alpha}{}^{i} + \frac{1}{2} \chi_{3\mu\alpha}{}^{ij}{}_{k} \tag{4.4.26}$$

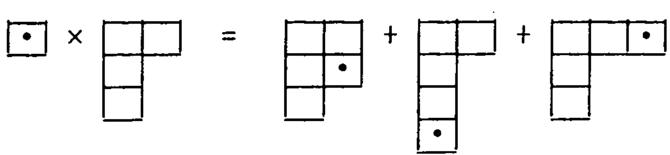
where

$$\chi_{1\mu\alpha}{}^{ij}{}_{k} = D^{[i}_{\alpha} W_{\mu}{}^{j]}{}_{k} - \frac{1}{3}\, \delta^{[i}_{k}\, D^{\ell}_{\alpha} W_{\mu}{}^{j]}{}_{\ell}$$

$$\chi_{2\mu\alpha}{}^{i} = D^{j}_{\alpha} W_{\mu}{}^{i}{}_{j}$$

$$\chi_{3\mu\alpha}{}^{ij}{}_{k} = D^{(i}_{\alpha} W_{\mu}{}^{j)}{}_{k} - \frac{1}{5}\, \delta^{(i}_{k}\, D^{\ell}_{\alpha} W_{\mu}{}^{j)}{}_{\ell} \tag{4.4.27}$$

(4.4.26) is represented by the tableaux equation



(4.4.28)

From the Bianchi identity (4.4.24d), the last of these terms must be put equal to zero. One might also consider restricting $W_{\mu}{}^{i}{}_{j}$ further by imposing the constraint

$$D^{j}_{\alpha} W_{\mu}{}^{i}{}_{j} = 0 \qquad (4.4.29)$$

However, this constraint again leads to equations of motion using the resulting Bianchi identities. One has from (4.4.26) and (4.4.25) that

$$D^{i}_{\alpha} \bar{D}_{\dot{\alpha} k} \bar{W}^{kj} = -6\,i\,\sigma^{\mu}_{\alpha\dot{\alpha}}\, \partial_{\mu} \bar{W}^{ij} + \frac{3}{10}\, \sigma^{\mu}_{\beta\dot{\alpha}}\, D^{k}_{\alpha} D^{\beta\,[i} W_{\mu}{}^{j]}{}_{k}$$

$$= -6\,i\,\sigma^{\mu}_{\alpha\dot{\alpha}}\, \partial_{\mu} \bar{W}^{ij} \qquad (4.4.30)$$

One can now derive the following chain of equations

$$-2\,i\,\sigma^{\mu}_{\alpha\dot{\alpha}}\, \partial_{\mu} \bar{D}^{\dot{\alpha}}_{k} \bar{W}^{ki} = \{ D^{i}_{\alpha} , \bar{D}_{\dot{\beta} j} \} \bar{D}^{\dot{\beta}}_{k} \bar{W}^{kj}$$

$$= \bar{D}_{\dot{\beta} j}\, D^{i}_{\alpha}\, \bar{D}^{\dot{\beta}}_{k} \bar{W}^{kj}$$

$$= -6\,i\,\sigma^{\mu}_{\alpha\dot{\alpha}}\, \partial_{\mu} \bar{D}^{\dot{\alpha}}_{k} \bar{W}^{ki}$$

$$= 0 \qquad (4.4.31)$$

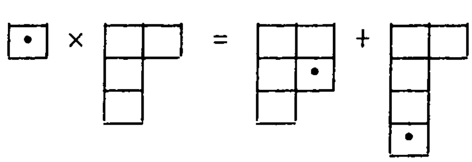
as $\bar{D}_{\dot{\beta} j} \bar{D}^{\dot{\beta}}_{k} \bar{W}^{kj} = 0$.

This is however the field equation for the spinor field $D^{j}_{\alpha} W_{ji}$.

As the first part of (4.4.26) cannot be put to zero without decoupling the superfields $\bar{W}^{ij}$ and W_{ij} from $W_{\mu}{}^{i}{}_{j}$, one is led to the unique constraint

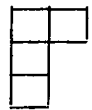
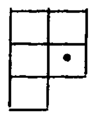
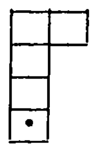
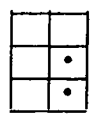
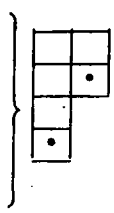
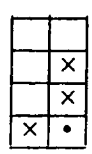
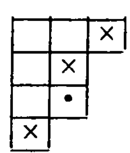
$$D^{i}_{\alpha} W_{\mu}{}^{j}{}_{k} = \frac{1}{2} \chi_{1\mu\alpha}{}^{ij}{}_{k} + \frac{4}{15}\, \delta^{i}_{k}\, \chi_{2\mu\alpha}{}^{j} - \frac{1}{15}\, \delta^{j}_{k}\, \chi_{2\mu\alpha}{}^{i} \qquad (4.4.32)$$

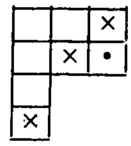
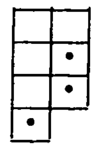
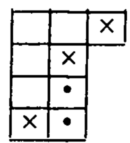
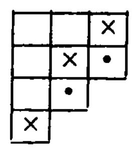
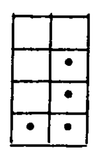
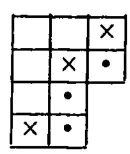
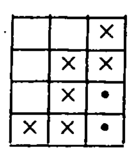
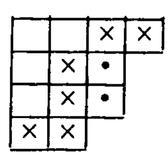
which is represented in tableaux form by

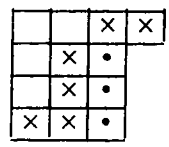
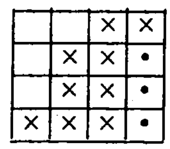


(4.4.33)

One can now calculate the components of the superfield $W_\mu{}^i{}_j$ subject to this constraint using the tableaux calculus. The result is obtained in table 1 of [34] except for the additional space-time index and is

Field	SU(4) Representation	SL(2,C) Representation	Tableaux	Components
$W_\mu{}^i{}_j$	15	$(\frac{1}{2},\frac{1}{2})\times(0,0)$		15×4
$\chi_{1\mu\alpha}{}^{ij}{}_k$	20 + 20*	$(\frac{1}{2},\frac{1}{2})\times\left[(\frac{1}{2},0)+(0,\frac{1}{2})\right]$		80×4
$\chi_{2\mu\alpha}{}^i$	4 + 4*	$(\frac{1}{2},\frac{1}{2})\times\left[(\frac{1}{2},0)+(0,\frac{1}{2})\right]$		16×4
$A_{1\mu\alpha\beta ij}$	10 + 10*	$(\frac{1}{2},\frac{1}{2})\times\left[(1,0)+(0,1)\right]$		60×4
$A_{2\mu\alpha\beta}{}^{ij}$	6 + 6	$(\frac{1}{2},\frac{1}{2})\times\left[(1,0)+(0,1)\right]$		36×4
$A_{3\mu}{}^{ij}$	6 + 6	$(\frac{1}{2},\frac{1}{2})\times(0,0)$		12×4
$A_{4\mu\alpha\dot{\alpha}}$	1	$(\frac{1}{2},\frac{1}{2})\times(\frac{1}{2},\frac{1}{2})$		$(4-1)\times 4$
$A_{5\mu\alpha\dot{\alpha}}{}^i{}_j$	15	$(\frac{1}{2},\frac{1}{2})\times(\frac{1}{2},\frac{1}{2})$		$(60-15)\times 4$

Field	*SU(4) Representation*	*SL(2,C) Representation*	*Tableaux*	*Components*
$A_{6\mu\alpha\dot\alpha}{}^{ij}{}_{k\ell}$	20'	$(\frac{1}{2},\frac{1}{2})\times(\frac{1}{2},\frac{1}{2})$		(80-20) × 4
$\Psi_{1\mu\alpha\beta\gamma i}$	4 + 4*	$(\frac{1}{2},\frac{1}{2})\times\left[(\frac{3}{2},0)+(0,\frac{3}{2})\right]$		32 × 4
$\Psi_{2\mu\alpha i}$	4 + 4*	$(\frac{1}{2},\frac{1}{2})\times\left[(\frac{1}{2},0)+(0,\frac{1}{2})\right]$		16 × 4
$\Psi_{3\mu\alpha\beta\dot\alpha}{}^{i}$	4 + 4*	$(\frac{1}{2},\frac{1}{2})\times\left[(1,\frac{1}{2})+(\frac{1}{2},1)\right]$		(48-16) × 4
$\Psi_{4\mu\alpha\beta\dot\alpha}{}^{ij}{}_{k}$	20 + 20*	$(\frac{1}{2},\frac{1}{2})\times\left[(1,\frac{1}{2})+(\frac{1}{2},1)\right]$		(240-80) × 4
$B_{1\mu\alpha\beta}$	1 + 1	$(\frac{1}{2},\frac{1}{2})\times\left[(1,0)+(0,1)\right]$		6 × 4
$B_{2\mu\alpha\beta\gamma\dot\alpha}{}^{ij}$	6 + 6	$(\frac{1}{2},\frac{1}{2})\times\left[(\frac{3}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})\right]$		(96-36) × 4
$B_{3\mu\alpha\beta\dot\alpha\dot\beta}$	1	$(\frac{1}{2},\frac{1}{2})\times(1,1)$		(9-4) × 4
$B_{4\mu\alpha\beta\dot\alpha\dot\beta}{}^{i}{}_{j}$	15	$(\frac{1}{2},\frac{1}{2})\times(1,1)$		(135-60) × 4

Field	*SU(4) Representation*	*SL(2,C) Representation*	*Tableaux*	*Components*
$\xi_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}}{}^{i}$	$4 + 4^{*}$	$(\frac{1}{2},\frac{1}{2})\times\left[(\frac{3}{2},1)+(1,\frac{3}{2})\right]$		$(96-48)\times 4$
$C_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}}$	1	$(\frac{1}{2},\frac{1}{2})\times(\frac{3}{2},\frac{3}{2})$		$(16-9)\times 4$

(4.4.34)

From the constraint (4.4.32) it follows that the fields with mixed indices are conserved

$$\bar{\sigma}^{\nu\dot{\alpha}\alpha}\,\partial_\nu A_{4\mu\alpha\dot{\alpha}} = 0$$

$$\bar{\sigma}^{\nu\dot{\alpha}\alpha}\,\partial_\nu A_{5\mu\alpha\dot{\alpha}}{}^{i}{}_{j} = 0$$

$$\bar{\sigma}^{\nu\dot{\alpha}\alpha}\,\partial_\nu A_{6\mu\alpha\dot{\alpha}}{}^{ij}{}_{k\ell} = 0$$

$$\bar{\sigma}^{\nu\dot{\alpha}\beta}\,\partial_\nu \psi_{3\mu\alpha\beta\dot{\alpha}}{}^{i} = 0$$

$$\bar{\sigma}^{\nu\dot{\alpha}\beta}\,\partial_\nu \psi_{4\mu\alpha\beta\dot{\alpha}}{}^{ij}{}_{k} = 0$$

$$\bar{\sigma}^{\nu\dot{\alpha}\gamma}\,\partial_\nu B_{2\mu\alpha\beta\gamma\dot{\alpha}}{}^{ij} = 0$$

$$\bar{\sigma}^{\nu\dot{\beta}\beta}\,\partial_\nu B_{3\mu\alpha\beta\dot{\alpha}\dot{\beta}} = 0$$

$$\bar{\sigma}^{\nu\dot{\beta}\beta}\,\partial_\nu B_{4\mu\alpha\beta\dot{\alpha}\dot{\beta}}{}^{i}{}_{j} = 0$$

$$\bar{\sigma}^{\nu\dot{\beta}\gamma}\,\partial_\nu \xi_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}}{}^{i} = 0$$

$$\bar{\sigma}^{\nu\dot{\gamma}\gamma}\,\partial_\nu C_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}} = 0 \qquad (4.4.35)$$

The fields in (4.4.34) can be expressed in terms of $W_\mu{}^{i}{}_{j}$ as follows

$$\chi_{1\mu\alpha}{}^{ij}{}_{k} = D^{[i}_{\alpha} W_{\mu}{}^{j]}{}_{k} - \frac{1}{3}\delta^{[i}_{k} D^{\ell}_{\alpha} W_{\mu}{}^{j]}{}_{\ell}$$

$$\chi_{2\mu\alpha}{}^{i} = D^{j}_{\alpha} W_{\mu}{}^{i}{}_{j}$$

$$A_{1\mu\alpha\beta ij} = \varepsilon_{(ik\ell m} D^{k}_{\alpha} D^{\ell}_{\beta} W_{\mu}{}^{m}{}_{j)}$$

$$A_{2\mu\alpha\beta}{}^{ij} = \frac{1}{2} D^{i}_{(\alpha} D^{k}_{\beta)} W_{\mu}{}^{j}{}_{k}$$

$$A_{3\mu}{}^{ij} = \frac{1}{2} D^{\alpha i} D^{k}_{\alpha} W_{\mu}{}^{j}{}_{k}$$

$$A_{4\mu\alpha\dot{\alpha}} = [D^{i}_{\alpha}, \bar{D}_{\dot{\alpha}j}] W_{\mu}{}^{j}{}_{i}$$

$$A_{5\mu\alpha\dot{\alpha}}{}^{i}{}_{j} = [D^{k}_{\alpha}, \bar{D}_{\dot{\alpha}k}] W_{\mu}{}^{i}{}_{j}$$

$$A_{6\mu\alpha\dot{\alpha}}{}^{ij}{}_{k\ell} = [D^{[i}_{\alpha}, \bar{D}_{\dot{\alpha}[k}] W_{\mu}{}^{j]}{}_{\ell]} + \frac{1}{6}\delta^{i}_{[k}\delta^{j}_{\ell]} [D^{m}_{\alpha}, \bar{D}_{\dot{\alpha}n}] W_{\mu}{}^{n}{}_{m}$$
$$- \frac{7}{4}\delta^{[i}_{[k} [D^{m}_{\alpha}, \bar{D}_{\dot{\alpha}m}] W_{\mu}{}^{j]}{}_{\ell]}$$

$$\psi_{1\mu\alpha\beta\gamma i} = \varepsilon_{jk\ell m} D^{j}_{\alpha} D^{k}_{\beta} D^{\ell}_{\gamma} W_{\mu}{}^{m}{}_{i}$$

$$\psi_{2\mu\alpha i} = \varepsilon_{ijk\ell} D^{j}_{\alpha} D^{\beta k} D^{m}_{\beta} W_{\mu}{}^{\ell}{}_{m}$$

$$\psi_{3\mu\alpha\beta\dot{\alpha}}{}^{i} = D^{i}_{(\alpha}[D^{j}_{\beta)}, \bar{D}_{\dot{\alpha}k}] W_{\mu}{}^{k}{}_{j}$$

$$\psi_{4\mu\alpha\beta\dot{\alpha}}{}^{ij}{}_{k} = D^{[i}_{(\alpha}[D^{\ell}_{\beta)}, \bar{D}_{\dot{\alpha}\ell}] W_{\mu}{}^{j]}{}_{k} - \frac{1}{3}\delta^{[i}_{k} D^{\ell}_{(\alpha}[D^{m}_{\beta)}, \bar{D}_{\dot{\alpha}m}] W_{\mu}{}^{j]}{}_{\ell}$$

$$B_{1\mu\alpha\beta} = \varepsilon_{ijk\ell} D^{i}_{\alpha} D^{j}_{\beta} D^{\gamma k} D^{m}_{\gamma} W_{\mu}{}^{\ell}{}_{m}$$

$$B_{2\mu\alpha\beta\gamma\dot{\alpha}}{}^{ij} = D^{i}_{(\alpha} D^{j}_{\beta}[D^{k}_{\gamma)}, \bar{D}_{\dot{\alpha}\ell}] W_{\mu}{}^{\ell}{}_{k}$$

$$B_{3\mu\alpha\beta\dot{\alpha}\dot{\beta}} = [D^{i}_{(\alpha}, \bar{D}_{(\dot{\alpha}i}][D^{j}_{\beta)}, \bar{D}_{\dot{\beta})k}] W_{\mu}{}^{k}{}_{j}$$

$$B_{4\mu\alpha\beta\dot{\alpha}\dot{\beta}}{}^{i}{}_{j} = [D^{k}_{(\alpha}, \bar{D}_{(\dot{\alpha}k}][D^{\ell}_{\beta)}, \bar{D}_{\dot{\beta})\ell}] W_{\mu}{}^{i}{}_{j}$$

$$\xi_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}}{}^{i} = D^{i}_{(\alpha}[D^{j}_{\beta}, \bar{D}_{(\dot{\alpha}j}][D^{k}_{\gamma)}, \bar{D}_{\dot{\beta})\ell}] W_{\mu}{}^{\ell}{}_{k}$$

$$C_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}} = [D^{i}_{(\alpha}, \bar{D}_{(\dot{\alpha}i}][D^{j}_{\beta}, \bar{D}_{\dot{\beta}j}][D^{k}_{\gamma)}, \bar{D}_{\dot{\gamma})\ell}] W_{\mu}{}^{\ell}{}_{k} \quad (4.4.36)$$

From these superfields the supersymmetry transformations of the component fields can be calculated by the usual method of acting on the above by D^i_α and $\bar{D}_{\dot\alpha i}$. One finds in the abelian case

$$D^i_\alpha \bar{W}^{jk} = \frac{1}{3}\varepsilon^{ijk\ell}\lambda_{2\alpha\ell}$$

$$D^i_\alpha W_{jk} = \frac{1}{3}\delta^{\,i}_{[j}\lambda_{1\alpha k]} - \frac{1}{2}\sigma^\mu_{\alpha\dot\alpha}\bar{\chi}_{1\mu jk}{}^{\dot\alpha\, i}$$

$$D^i_\alpha W_\mu{}^j{}_k = \frac{1}{2}\chi_{1\mu\alpha}{}^{ij}{}_k + \frac{4}{15}\delta^i_k\chi_{2\mu\alpha}{}^j - \frac{1}{15}\delta^j_k\chi_{2\mu\alpha}{}^i$$

$$D^i_\alpha \lambda_{1\beta j} = 12\,i\,\delta^i_j(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}F_{\mu\nu} - \frac{1}{20}\delta^i_j A_{4\mu(\alpha\dot\alpha}(\bar\sigma^\mu\varepsilon)^{\dot\alpha}{}_{\beta)}$$
$$- \frac{1}{4}A_{5\mu\alpha\dot\alpha}{}^i{}_j(\bar\sigma^\mu\varepsilon)^{\dot\alpha}{}_\beta + \frac{3}{4}A_{5\mu\beta\dot\alpha}{}^i{}_j(\bar\sigma^\mu\varepsilon)^{\dot\alpha}{}_\alpha$$
$$+ i(\sigma^\mu\bar\sigma^\nu\varepsilon)_{\alpha\beta}\partial_\mu W_\nu{}^i{}_j - 3\,i(\sigma^\mu\bar\sigma^\nu\varepsilon)_{\beta\alpha}\partial_\mu W_\nu{}^i{}_j$$

$$D^i_\alpha \bar\lambda_{1\dot\alpha}^j = -6\,i\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar{W}^{ij} + \frac{4}{5}A_{2\mu\alpha\beta}{}^{ij}(\varepsilon\sigma^\mu)^\beta{}_{\dot\alpha} - \frac{2}{5}A_{3\mu}{}^{ij}\sigma^\mu_{\alpha\dot\alpha}$$

$$D^i_\alpha \lambda_{2\beta j} = 12\,i\,\delta^i_j(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}G_{\mu\nu}$$

$$D^i_\alpha \bar\lambda_{2\dot\alpha}^j = -3\,i\,\varepsilon^{ijk\ell}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu W_{k\ell} + \frac{1}{4}(\sigma^\mu\varepsilon)_\alpha{}^{\dot\beta}\bar{A}_{1\mu\dot\alpha\dot\beta}{}^{ij}$$
$$+ \frac{3}{5}(\sigma^\mu\varepsilon)_\alpha{}^{\dot\beta}\varepsilon^{ijk\ell}\bar{A}_{2\mu\dot\beta\dot\alpha k\ell} + \frac{3}{10}\sigma^\mu_{\alpha\dot\alpha}\varepsilon^{ijk\ell}\bar{A}_{3\mu k\ell}$$

$$D^i_\alpha \chi_{1\mu\beta}{}^{jk}{}_\ell = -\frac{1}{6}\varepsilon^{ijkm}A_{1\mu\alpha\beta m\ell} - \frac{4}{5}\left(\delta^i_\ell A_{2\mu\alpha\beta}{}^{jk} + \frac{1}{3}\delta^{[j}_\ell A_{2\mu\alpha\beta}{}^{k]i}\right)$$
$$+ \frac{2}{5}\varepsilon_{\alpha\beta}\left(\delta^i_\ell A_{3\mu}{}^{jk} + \frac{1}{3}\delta^{[j}_\ell A_{3\mu}{}^{k]i}\right)$$

$$D^i_\alpha \bar\chi_{1\mu\dot\alpha jk}{}^\ell = \frac{1}{4}A_{6\mu\alpha\dot\alpha}{}^{i\ell}{}_{jk} + \frac{1}{2}\delta^{\,i}_{[j}A_{5\mu\alpha\dot\alpha}{}^\ell{}_{k]} - \frac{1}{6}\delta^{\,\ell}_{[j}A_{5\mu\alpha\dot\alpha}{}^i{}_{k]}$$
$$- 2\,i\,\delta^{\,i}_{[j}\sigma^\nu_{\alpha\dot\alpha}\partial_\nu W_\mu{}^\ell{}_{k]} + \frac{2}{3}i\,\delta^{\,\ell}_{[j}\sigma^\nu_{\alpha\dot\alpha}\partial_\nu W_\mu{}^i{}_{k]}$$

$$D^i_\alpha \chi_{2\mu\beta}^j = A_{2\mu\alpha\beta}{}^{ij} + \varepsilon_{\alpha\beta}A_{3\mu}{}^{ij}$$

$$D^i_\alpha \bar\chi_{2\mu\dot\alpha j} = -\frac{5}{8}A_{5\mu\alpha\dot\alpha}{}^i{}_j + \frac{1}{8}\delta^i_j A_{4\mu\alpha\dot\alpha} - 5\,i\,\sigma^\nu_{\alpha\dot\alpha}\partial_\nu W_\mu{}^i{}_j$$

$$D^i_\alpha F^{\mu\nu} = \frac{1}{12}\,\sigma^{[\mu}_{\alpha\dot\alpha}\,\partial^{\nu]}\,\bar\lambda^{\dot\alpha i}_1 - \frac{1}{60}\,(\,\sigma^{[\mu}\bar\sigma^\lambda - 3\,\sigma^\lambda\bar\sigma^{[\mu}\,)_\alpha{}^\beta\,\partial^{\nu]}\,\chi_{2\lambda\beta}{}^i$$

$$D^i_\alpha G^{\mu\nu} = \frac{1}{12}\,(\,\sigma^\lambda\bar\sigma^{\mu\nu}\,)_{\alpha\dot\alpha}\,\partial_\lambda\,\bar\lambda^{\dot\alpha i}_2 - \frac{i}{180}\,(\,\sigma^\lambda{}_\varepsilon\,)_\alpha{}^{\dot\alpha}\,(\,\bar\sigma^{\mu\nu}{}_\varepsilon\,)^{\dot\beta\dot\gamma}\,\bar\Psi_{1\lambda\dot\alpha\dot\beta\dot\gamma}{}^i$$

$$- \frac{i}{720}\,(\,\sigma^\lambda\bar\sigma^{\mu\nu}\,)_{\alpha\dot\alpha}\,\bar\Psi^{\dot\alpha i}_2$$

$$D^i_\alpha A_{1\mu\beta\gamma jk} = \frac{4}{15}\,\delta^{\,i}_{(j}\,\psi_{1\mu\alpha\beta\gamma k)} - \frac{1}{15}\,\delta^{\,i}_{(j}\,\varepsilon_{\alpha(\beta}\,\psi_{2\mu\gamma)k)}$$

$$D^i_\alpha A_{2\mu\beta\gamma}{}^{jk} = -\frac{1}{18}\,\varepsilon^{ijk\ell}\,\psi_{1\mu\alpha\beta\gamma\ell} - \frac{1}{36}\,\varepsilon^{ijk\ell}\,\varepsilon_{\alpha(\beta}\,\psi_{2\mu\gamma)\ell}$$

$$D^i_\alpha A_{3\mu}{}^{jk} = -\frac{1}{12}\,\varepsilon^{ijk\ell}\,\psi_{2\mu\alpha\ell}$$

$$D^i_\alpha A_{4\mu\beta\dot\alpha} = \frac{1}{2}\,\psi_{3\mu\alpha\beta\dot\alpha}{}^i - 10\,i\,\varepsilon_{\alpha\beta}\,\partial\!\!\!/_{\gamma\dot\alpha}\,\chi^{\gamma i}_{2\mu}$$

$$D^i_\alpha A_{5\mu\beta\dot\alpha}{}^j{}_k = \frac{1}{4}\,\psi_{4\mu\alpha\beta\dot\alpha}{}^{ij}{}_k + \frac{2}{15}\,(\,\delta^i_k\,\psi_{3\mu\alpha\beta\dot\alpha}{}^j - \frac{1}{4}\,\delta^j_k\,\psi_{3\mu\alpha\beta\dot\alpha}{}^i\,)$$

$$- 2\,i\,\varepsilon_{\alpha\beta}\,\partial\!\!\!/_{\gamma\dot\alpha}\,\chi_{1\mu}{}^{\gamma ij}{}_k - \frac{16}{15}\,i\,(\,\delta^i_k\,\partial\!\!\!/_{(\alpha\dot\alpha}\,\chi_{2\mu\beta)}{}^j$$

$$- \frac{1}{4}\,\delta^j_k\,\partial\!\!\!/_{(\alpha\dot\alpha}\,\chi_{2\mu\beta)}{}^i\,) + \frac{8}{15}\,i\,\varepsilon_{\alpha\beta}\,(\,\delta^i_k\,\partial\!\!\!/_{\gamma\dot\alpha}\,\chi^{\gamma j}_{2\mu}$$

$$- \frac{1}{4}\,\delta^j_k\,\partial\!\!\!/_{\gamma\dot\alpha}\,\chi^{\gamma i}_{2\mu}\,)$$

$$D^i_\alpha A_{6\mu\beta\dot\alpha}{}^{jk}{}_{\ell m} = -\,(\,\delta^{\,i}_{[\ell}\,\psi_{4\mu\alpha\beta\dot\alpha}{}^{jk}{}_{m]} + \frac{1}{2}\,\delta^{[j}_{[\ell}\,\psi_{4\mu\alpha\beta\dot\alpha}{}^{k]i}{}_{m]}\,)$$

$$+ 8\,i\,(\delta^{\,i}_{[\ell}\,\partial\!\!\!/_{\alpha\dot\alpha}\,\chi_{1\mu\beta}{}^{jk}{}_{m]} + \frac{1}{2}\,\delta^{[j}_{[\ell}\,\partial\!\!\!/_{\alpha\dot\alpha}\,\chi_{1\mu\beta}{}^{k]i}{}_{m]}\,)$$

$$\bar D_{\dot\alpha i}\,A_{1\mu\alpha\beta jk} = \frac{3}{8}\,\varepsilon_{i(j\ell m}\,\psi_{4\mu\alpha\beta\dot\alpha}{}^{\ell m}{}_{k)}$$

$$\bar D_{\dot\alpha i}\,A_{2\mu\alpha\beta}{}^{jk} = -\frac{5}{32}\,\psi_{4\mu\alpha\beta\dot\alpha}{}^{jk}{}_i - \frac{1}{12}\,\delta^{[j}_i\,\psi_{3\mu\alpha\beta\dot\alpha}{}^{k]} + \frac{5}{4}\,i\,\partial\!\!\!/_{(\alpha\dot\alpha}\,\chi_{1\mu\beta)}{}^{jk}{}_i$$

$$+ \frac{1}{3}\,i\,\delta^{[j}_i\,\partial\!\!\!/_{(\alpha\dot\alpha}\,\chi_{2\mu\beta)}{}^{k]}$$

$$\bar D_{\dot\alpha i}\,A_{3\mu}{}^{jk} = \frac{5}{2}\,i\,\partial\!\!\!/_{\alpha\dot\alpha}\,\chi_{1\mu}{}^{\alpha jk}{}_i + \frac{4}{3}\,i\,\delta^{[j}_i\,\partial\!\!\!/_{\alpha\dot\alpha}\,\chi^{\alpha k]}_{2\mu}$$

$$D^i_\alpha\,\psi_{1\mu\beta\gamma\delta j} = \frac{1}{16}\,(\,\varepsilon_{\alpha\beta}\,B_{1\gamma\delta} + \gamma\delta\beta + \delta\beta\gamma\,)\,\delta^i_j$$

$$D^i_\alpha \psi_{2\mu\beta j} = \frac{1}{4}\delta^i_j B_{1\mu\alpha\beta}$$

$$\bar{D}_{\dot\alpha i}\psi_{1\mu\alpha\beta\gamma j} = \frac{3}{8}\varepsilon_{ijk\ell} B_{2\mu\alpha\beta\gamma\dot\alpha}{}^{k\ell} - 6i\varepsilon_{ijk\ell}(\not\partial_{\alpha\dot\alpha} A_{2\mu\beta\gamma}{}^{k\ell} + \beta\gamma\alpha + \gamma\alpha\beta)$$

$$- \frac{5}{4} i(\not\partial_{\alpha\dot\alpha}A_{1\mu\beta\gamma ij} + \beta\gamma\alpha + \gamma\alpha\beta)$$

$$\bar{D}_{\dot\alpha i}\psi_{2\mu\alpha j} = 12 i \varepsilon_{ijk\ell}\not\partial^\beta{}_{\dot\alpha} A_{2\mu\alpha\beta}{}^{k\ell} + 6i\varepsilon_{ijk\ell}\not\partial_{\alpha\dot\alpha}A_{3\mu}{}^{k\ell}$$

$$+ 5i\not\partial^\beta{}_{\dot\alpha}A_{1\mu\alpha\beta ij}$$

$$D^i_\alpha \psi_{3\mu\beta\gamma\dot\alpha}{}^j = \frac{1}{3} B_{2\mu\alpha\beta\gamma\dot\alpha}{}^{ij} + \frac{20}{3} i\varepsilon_{\alpha(\beta}\not\partial^\delta{}_{\dot\alpha}A_{2\mu\gamma)\delta}{}^{ij} + \frac{20}{3}i\varepsilon_{\alpha(\beta}\not\partial_{\gamma)\dot\alpha}A_{3\mu}{}^{ij}$$

$$D^i_\alpha\psi_{4\mu\beta\gamma\dot\alpha}{}^{jk}{}_\ell = -\frac{4}{15}(\delta^i_\ell B_{2\mu\alpha\beta\gamma\dot\alpha}{}^{jk} + \frac{1}{3}\delta^{[j}_\ell B_{2\mu\alpha\beta\gamma\dot\alpha}{}^{k]i})$$

$$+ \frac{128}{15} i\not\partial_{\alpha\dot\alpha}(\delta^i_\ell A_{2\mu\beta\gamma}{}^{jk} + \frac{1}{3}\delta^{[j}_\ell A_{2\mu\beta\gamma}{}^{k]i})$$

$$+ \frac{32}{15}i\not\partial_{(\beta\dot\alpha}(\delta^i_\ell A_{2\mu\gamma)\alpha}{}^{jk} + \frac{1}{3} \delta^{[j}_\ell A_{2\mu\gamma)\alpha}{}^{k]i})$$

$$+ \frac{16}{15} i\varepsilon_{\alpha(\beta}\not\partial_{\gamma)\dot\alpha}(\delta^i_\ell A_{3\mu}{}^{jk} + \frac{1}{3}\delta^{[j}_\ell A_{3\mu}{}^{k]i})$$

$$- \frac{4}{3} i\varepsilon^{ijkm}\not\partial_{\alpha\dot\alpha}A_{1\mu\beta\gamma m\ell}$$

$$\bar{D}_{\dot\alpha i}\psi_{3\mu\alpha\beta\dot\beta}{}^j = \frac{5}{16}B_{4\mu\alpha\beta\dot\alpha\dot\beta}{}^j{}_i - \frac{1}{16}\delta^j_i B_{3\mu\alpha\beta\dot\alpha\dot\beta} - \frac{1}{2} i\delta^j_i\not\partial_{(\alpha(\dot\alpha}A_{4\mu\beta)\dot\beta)}$$

$$+ 20\not\partial_{(\alpha(\dot\alpha}\not\partial_{\beta)\dot\beta)}W_\mu{}^j{}_i - \frac{25}{4} i\varepsilon_{\dot\alpha\dot\beta}\not\partial_{(\alpha}{}^{\dot\gamma}A_{5\mu\beta)\dot\gamma}{}^j{}_i$$

$$+ \frac{1}{4} i\varepsilon_{\dot\alpha\dot\beta}\delta^j_i\not\partial_{(\alpha}{}^{\dot\gamma}A_{4\mu\beta)\dot\gamma}$$

$$\bar{D}_{\dot\alpha i}\psi_{4\mu\alpha\beta\dot\beta}{}^{jk}{}_\ell = -\frac{1}{4}(\delta^{[j}_i B_{4\mu\alpha\beta\dot\alpha\dot\beta}{}^{k]}{}_\ell - \frac{1}{3}\delta^{[j}_\ell B_{4\mu\alpha\beta\dot\alpha\dot\beta}{}^{k]}{}_i)$$

$$- i\not\partial_{(\alpha(\dot\alpha}(\delta^{[j}_i A_{5\mu\beta)\dot\beta)}{}^{k]}{}_\ell - \frac{1}{3}\delta^{[j}_\ell A_{5\mu\beta)\dot\beta)}{}^{k]}{}_i)$$

$$+ i\varepsilon_{\dot\alpha\dot\beta}\not\partial_{(\alpha}{}^{\dot\gamma}A_{6\mu\beta)\dot\gamma}{}^{jk}{}_{i\ell}$$

$$D^i_\alpha B_{1\mu\beta\gamma} = 0$$

$$\bar{D}_{\dot\alpha i}B_{1\mu\alpha\beta} = -\frac{32}{3}i\not\partial^\gamma{}_{\dot\alpha}\psi_{1\mu\alpha\beta\gamma i} - \frac{8}{3} i\not\partial_{(\alpha\dot\alpha}\psi_{2\mu\beta)i}$$

$$D^i_\alpha B_{2\mu\beta\gamma\delta\dot{\alpha}}{}^{jk} = -\frac{8}{3} i\, \varepsilon^{ijk\ell} \not\partial_{\alpha\dot{\alpha}} \psi_{1\mu\beta\gamma\delta\ell} - \frac{4}{9} i\, \varepsilon^{ijk\ell} (\not\partial_{\beta\dot{\alpha}} \psi_{1\mu\gamma\delta\alpha\ell} + \gamma\delta\beta + \delta\beta\gamma) - \frac{5}{9} i\, \varepsilon^{ijk\ell} (\varepsilon_{\alpha\beta} \not\partial_{(\gamma\dot{\alpha}} \psi_{2\mu\delta)\ell} + \gamma\delta\beta + \delta\beta\gamma)$$

$$D^i_\alpha B_{4\mu\beta\gamma\dot{\alpha}\dot{\beta}}{}^j{}_k = \frac{4}{45} (\delta^i_k \xi_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}}{}^j - \frac{1}{4} \delta^j_k \xi_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}}{}^i)$$

$$+ \frac{2}{3} i (\not\partial_{\alpha(\dot{\alpha}} \psi_{4\mu\beta\gamma\dot{\beta})}{}^{ij}{}_k + \beta\gamma\alpha + \gamma\alpha\beta)$$

$$+ \frac{5}{3} i\, \varepsilon_{\alpha(\beta} \not\partial^\delta{}_{(\dot{\alpha}} \psi_{4\mu\gamma)\delta\dot{\beta})}{}^{ij}{}_k$$

$$- \frac{32}{45} i \not\partial_{\alpha(\dot{\alpha}} (\delta^i_k \psi_{3\mu\beta\gamma\dot{\beta})}{}^j - \frac{1}{4} \delta^j_k \psi_{3\mu\beta\gamma\dot{\beta})}{}^i) + \beta\gamma\alpha + \gamma\alpha\beta$$

$$- \frac{8}{45} i\, \varepsilon_{\alpha(\beta} \not\partial^\delta{}_{(\dot{\alpha}} (\delta^i_k \psi_{3\mu\gamma)\delta\dot{\beta})}{}^j - \frac{1}{4} \delta^j_k \psi_{3\mu\gamma)\delta\dot{\beta})}{}^i)$$

$$+ 8\, \varepsilon_{\alpha(\beta} \not\partial_{\gamma)(\dot{\alpha}} \not\partial^\delta{}_{\dot{\beta})} \chi_{1\mu\delta}{}^{ij}{}_k$$

$$- \frac{256}{45} \not\partial_{\alpha(\dot{\alpha}} \not\partial_{(\beta\dot{\beta})} (\delta^i_k \chi_{2\mu\gamma)}{}^j - \frac{1}{4} \delta^j_k \chi_{2\mu\gamma)}{}^i) + \beta\gamma\alpha + \gamma\alpha\beta$$

$$+ \frac{32}{45} \varepsilon_{\alpha(\beta} \not\partial_{\gamma)(\dot{\alpha}} \not\partial^\delta{}_{\dot{\beta})} (\delta^i_k \chi_{2\mu\delta}{}^j - \frac{1}{4} \delta^j_k \chi_{2\mu\delta}{}^i)$$

$$D^i_\alpha B_{3\mu\beta\gamma\dot{\alpha}\dot{\beta}} = \frac{1}{3} \xi_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}}{}^i + 6 i\, \varepsilon_{\alpha(\beta} \not\partial^\delta{}_{(\dot{\alpha}} \psi_{3\mu\delta\gamma)\dot{\beta})}{}^i$$

$$+ 40\, \varepsilon_{\alpha(\beta} \not\partial_{\gamma)(\dot{\alpha}} \not\partial^\delta{}_{\dot{\beta})} \chi_{2\mu\delta}{}^i$$

$$\bar{D}_{\dot{\alpha}i} B_{2\mu\alpha\beta\gamma\dot{\beta}}{}^{jk} = - \frac{1}{12} \delta^{[j}_i \xi_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}}{}^{k]} - \frac{5}{8} i \not\partial_{\alpha(\dot{\alpha}} \psi_{4\mu\beta\gamma\dot{\beta})}{}^{jk}{}_i + \beta\gamma\alpha + \gamma\alpha\beta$$

$$- \frac{1}{3} i\, \delta^{[j}_i \not\partial_{\alpha(\dot{\alpha}} \psi_{3\mu\beta\gamma\dot{\beta})}{}^{k]} + \beta\gamma\alpha + \gamma\alpha\beta$$

$$+ \frac{25}{8} i\, \varepsilon_{\dot{\alpha}\dot{\beta}} \not\partial_\alpha{}^{\dot{\gamma}} \psi_{4\mu\beta\gamma\dot{\gamma}}{}^{jk}{}_i + \beta\gamma\alpha + \gamma\alpha\beta$$

$$+ \frac{2}{3} i\, \varepsilon_{\dot{\alpha}\dot{\beta}}\, \delta^{[j}_i \not\partial_\alpha{}^{\dot{\gamma}} \psi_{3\mu\beta\gamma\dot{\gamma}}{}^{k]} + \beta\gamma\alpha + \gamma\alpha\beta$$

$$- 10 \not\partial_{\alpha(\dot{\alpha}} \not\partial_{(\beta\dot{\beta})} \chi_{1\mu\gamma)}{}^{jk}{}_i + \beta\gamma\alpha + \gamma\alpha\beta$$

$$D^i_\alpha \xi_{\mu\beta\gamma\delta\dot{\alpha}\dot{\beta}}{}^j = 12 i \not\partial_{\alpha(\dot{\alpha}} B_{2\mu\beta\gamma\delta\dot{\beta})}{}^{ij} + 48 \not\partial_{\beta(\dot{\alpha}} \not\partial_{(\gamma\dot{\beta})} A_{2\mu\delta)\alpha}{}^{ij} + \gamma\delta\beta + \delta\beta\gamma$$

$$+ 96 \not\partial_{\alpha(\dot{\alpha}} \not\partial_{\beta\dot{\beta})} A_{2\mu\gamma\delta}{}^{ij} + \gamma\delta\beta + \delta\beta\gamma$$

$$\overline{D}_{\dot{\alpha} i}\, \xi_{\mu\alpha\beta\gamma\dot{\beta}\dot{\gamma}}{}^{j} = -\frac{1}{24}\, \delta^{j}_{i}\, C_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}} - \frac{1}{3}\, i\, \delta^{j}_{i}\, (\, \partial\!\!\!/_{\alpha\dot{\alpha}}\, B_{3\mu\beta\gamma\dot{\beta}\dot{\gamma}} + \beta\gamma\alpha + \gamma\alpha\beta\,) + \dot{\beta}\dot{\gamma}\dot{\alpha} + \dot{\gamma}\dot{\alpha}\dot{\beta}$$

$$+ 10\, (\, \partial\!\!\!/_{\alpha\dot{\alpha}}\, \partial\!\!\!/_{(\beta(\dot{\beta}}\, A_{5\mu\gamma)\dot{\gamma})}{}^{j}{}_{i} + \beta\gamma\alpha + \gamma\alpha\beta\,) + \dot{\beta}\dot{\gamma}\dot{\alpha} + \dot{\gamma}\dot{\alpha}\dot{\beta}$$

$$- \frac{15}{4}\, i\, \varepsilon_{\dot{\alpha}(\dot{\beta}}\, \partial\!\!\!/_{\alpha}{}^{\dot{\delta}}\, B_{4\mu\beta\gamma\dot{\gamma})\dot{\delta}}{}^{j}{}_{i} + \beta\gamma\alpha + \gamma\alpha\beta$$

$$+ \frac{1}{12}\, i\, \delta^{j}_{i}\, \varepsilon_{\dot{\alpha}(\dot{\beta}}\, \partial\!\!\!/_{\alpha}{}^{\dot{\delta}}\, B_{3\mu\beta\gamma\dot{\gamma})\dot{\delta}} + \beta\gamma\alpha + \gamma\alpha\beta$$

$$+ 25\, \varepsilon_{\dot{\alpha}(\dot{\beta}}\, \partial\!\!\!/_{(\alpha}{}^{\dot{\delta}}\, \partial\!\!\!/_{\beta)\dot{\gamma})}\, A_{5\mu\gamma\dot{\delta}}{}^{j}{}_{i} + \beta\gamma\alpha + \gamma\alpha\beta$$

$$+ \varepsilon_{\dot{\alpha}(\dot{\beta}}\, \delta^{j}_{i}\, \partial\!\!\!/_{(\alpha}{}^{\dot{\delta}}\, \partial\!\!\!/_{\beta)\dot{\gamma})}\, A_{4\mu\gamma\dot{\delta}} + \beta\gamma\alpha + \gamma\alpha\beta$$

$$D^{i}_{\alpha}\, C_{\mu\beta\gamma\delta\dot{\alpha}\dot{\beta}\dot{\gamma}} = 16\, i\, \partial\!\!\!/_{\alpha\dot{\alpha}}\, \xi_{\mu\beta\gamma\delta\dot{\beta}\dot{\gamma}}{}^{i} + \dot{\beta}\dot{\gamma}\dot{\alpha} + \dot{\gamma}\dot{\alpha}\dot{\beta}$$

$$- \frac{8}{3}\, i\, (\, \partial\!\!\!/_{\beta\dot{\alpha}}\, \xi_{\mu\gamma\delta\alpha\dot{\beta}\dot{\gamma}}{}^{i} + \gamma\delta\beta + \delta\beta\gamma\,) + \dot{\beta}\dot{\gamma}\dot{\alpha} + \dot{\gamma}\dot{\alpha}\dot{\beta}$$

$$+ 96\, (\, \partial\!\!\!/_{\alpha\dot{\alpha}}\, \partial\!\!\!/_{\beta(\dot{\beta}}\, \psi_{3\mu\gamma\delta\dot{\gamma})}{}^{i} + \gamma\delta\beta + \delta\beta\gamma\,) + \dot{\beta}\dot{\gamma}\dot{\alpha} + \dot{\gamma}\dot{\alpha}\dot{\beta}$$

$$- 24\, (\, \partial\!\!\!/_{(\beta(\dot{\alpha}}\, \partial\!\!\!/_{\gamma)\dot{\beta})}\, \psi_{3\mu\delta\alpha\dot{\gamma}}{}^{i} + \gamma\delta\beta + \delta\beta\gamma\,) + \dot{\beta}\dot{\gamma}\dot{\alpha} + \dot{\gamma}\dot{\alpha}\dot{\beta}$$

$$+ 224\, i\, (\, \partial\!\!\!/_{(\beta(\dot{\alpha}}\, \partial\!\!\!/_{\gamma)\dot{\beta})}\, \partial\!\!\!/_{\delta\dot{\gamma}}\, \chi_{2\mu\alpha}{}^{i} + \gamma\delta\beta + \delta\beta\gamma\,) + \dot{\beta}\dot{\gamma}\dot{\alpha} + \dot{\gamma}\dot{\alpha}\dot{\beta}$$

$$- 608\, i\, (\, \partial\!\!\!/_{\alpha\dot{\alpha}}\, \partial\!\!\!/_{(\beta(\dot{\beta}}\, \partial\!\!\!/_{\gamma)\dot{\gamma})}\, \chi_{2\mu\delta}{}^{i} + \gamma\delta\beta + \delta\beta\gamma\,) + \dot{\beta}\dot{\gamma}\dot{\alpha} + \dot{\gamma}\dot{\alpha}\dot{\beta} \qquad (4.4.37)$$

These fields are however not all independent but must satisfy certain identities. At lowest order the first constraint could come from

$$\{\, D^{i}_{\alpha}\, ,\, \overline{D}_{\dot{\alpha} j}\} \,\overline{W}^{k\ell} = -\, 2\, i\, \delta^{i}_{j}\, \sigma^{\mu}_{\alpha\dot{\alpha}}\, \partial_{\mu}\, \overline{W}^{k\ell} \qquad (4.4.38)$$

but using (4.4.37) above, this is found to be an automatic identity.

The first constraints arise at the next order where

$$\{ D^i_\alpha , \bar{D}_{\dot{\alpha} j} \} \bar{\lambda}_{1\dot{\beta}}{}^k = -2 i \delta^i_j \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \bar{\lambda}_{1\dot{\beta}}{}^k \qquad (4.4.39)$$

gives the identities

$$\sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \bar{\lambda}_1^{\dot{\alpha} i} = -\frac{i}{20} \bar{\sigma}^{\mu\dot{\alpha}\beta} \psi_{3\mu\alpha\beta\dot{\alpha}}{}^i + \frac{1}{5} \partial^\mu \chi_{2\mu\alpha}{}^i - \frac{6}{5} \sigma^{\mu\nu}{}_\alpha{}^\beta \partial_\mu \chi_{2\nu\beta}{}^i \qquad (4.4.40)$$

$$\bar{\sigma}^{\mu\dot{\alpha}\beta} \psi_{4\mu\alpha\beta\dot{\alpha}}{}^{ij}{}_k = -8 i \partial^\mu \chi_{1\mu\alpha}{}^{ij}{}_k + 16 i \sigma^{\mu\nu}{}_\alpha{}^\beta \partial_\mu \chi_{1\nu\beta}{}^{ij}{}_k \qquad (4.4.41)$$

while

$$\{ D^i_\alpha , \bar{D}_{\dot{\alpha} j} \} \bar{\lambda}_{2\dot{\beta}}^k = -2 i \delta^i_j \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \bar{\lambda}_{2\dot{\beta}}^k \qquad (4.4.42)$$

gives

$$\sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \bar{\lambda}_2^{\dot{\alpha} i} = \frac{i}{20} \sigma^\mu_{\alpha\dot{\alpha}} \bar{\psi}_{2\mu}^{\dot{\alpha} i} \qquad (4.4.43)$$

At the next order one can derive constraints by acting on (4.4.40), (4.4.41) and (4.4.43) with spinor derivatives and using (4.4.37). This gives the constraints

$$\Box W^{ij} = -\frac{2}{5} i (\varepsilon \sigma^{\mu\nu})^{\alpha\beta} \partial_\mu A_{2\nu\alpha\beta}{}^{ij} + \frac{1}{5} i \partial^\mu A_{3\mu}{}^{ij} \qquad (4.4.44)$$

$$\bar{\sigma}^{\mu\dot{\alpha}\gamma} B_{2\mu\alpha\beta\gamma\dot{\alpha}}{}^{ij} = 64 i \sigma^{\mu\nu}{}_{(\alpha}{}^\gamma \partial_\mu A_{2\nu\beta)\gamma}{}^{ij} - 80 i \partial^\mu A_{2\mu\alpha\beta}{}^{ij} + 16 i (\sigma^{\mu\nu} \varepsilon)_{\alpha\beta} \partial_\mu A_{3\nu}{}^{ij} \qquad (4.4.45)$$

$$\sigma^\nu_{\alpha\dot{\alpha}} \partial^\mu F_{\mu\nu} = -\frac{1}{3840} \bar{\sigma}^{\mu\dot{\beta}\beta} B_{3\mu\alpha\beta\dot{\alpha}\dot{\beta}} - \frac{1}{160} \varepsilon^{\mu\nu\rho\lambda} \sigma_{\mu\alpha\dot{\alpha}} \bar{\sigma}_\nu^{\dot{\beta}\beta} \partial_\rho A_{4\lambda\beta\dot{\beta}} \qquad (4.4.46)$$

$$\bar{\sigma}^{\mu\dot{\beta}\beta} B_{4\mu\alpha\beta\dot{\alpha}\dot{\beta}}{}^i{}_j = -8 \varepsilon^{\mu\nu\rho\lambda} \sigma_{\mu\alpha\dot{\alpha}} \bar{\sigma}_\nu^{\dot{\beta}\beta} \partial_\rho A_{5\lambda\beta\dot{\beta}}{}^i{}_j - 64 \sigma^\mu_{\alpha\dot{\alpha}} \Box W_\mu{}^i{}_j + 128 \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \partial^\nu W_\nu{}^i{}_j \qquad (4.4.47)$$

$$(\varepsilon \sigma^{\mu\nu})^{\alpha\beta} \partial_\mu A_{1\nu\alpha\beta ij} = 0 \qquad (4.4.48)$$

$$\partial_\mu \bar{\sigma}^{\nu\dot{\alpha}\alpha} A_{6\nu\alpha\dot{\alpha}}{}^{ij}{}_{k\ell} = 0 \tag{4.4.49}$$

$$(\sigma^{\nu\lambda} \sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu G_{\nu\lambda} = - \frac{1}{960} (\varepsilon\sigma^\mu)^\beta{}_{\dot{\alpha}} B_{1\mu\alpha\beta} \tag{4.4.50}$$

At this order the constraints

$$\{ D^i_\alpha , \bar{D}_{\dot{\alpha}j} \} G_{\mu\nu} = - 2 i \delta^i_j \sigma^\lambda_{\alpha\dot{\alpha}} \partial_\lambda G_{\mu\nu}$$

$$\{ D^i_\alpha , \bar{D}_{\dot{\alpha}j} \} F_{\mu\nu} = - 2 i \delta^i_j \sigma^\lambda_{\alpha\dot{\alpha}} \partial_\lambda F_{\mu\nu} \tag{4.4.51}$$

give nothing new.

The remaining constraints can now be obtained by acting on (4.4.44) to (4.4.50) with spinor derivatives.

D^i_α (4.4.44) gives

$$2 (\varepsilon \sigma^{\mu\nu})^{\beta\gamma} \partial_\mu \psi_{1\nu\alpha\beta\gamma i} = \sigma^{\mu\nu}{}_\alpha{}^\beta \partial_\mu \psi_{2\nu\beta i} \tag{4.4.52}$$

while $\bar{D}_{\dot{\alpha}i}$ (4.4.44) gives

$$\Box \bar{\lambda}_{1\dot{\alpha}}^i = \frac{i}{10} (\varepsilon \sigma^{\mu\nu})^{\alpha\beta} \partial_\mu \psi_{3\nu\alpha\beta\dot{\alpha}}{}^i - \frac{2}{5} \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \partial^\nu \chi^{\alpha i}_{2\nu}$$
$$- \frac{2}{5} \sigma^\mu_{\alpha\dot{\alpha}} \Box \chi^{\alpha i}_{2\mu} \tag{4.4.53}$$

However, from (4.4.40) one finds

$$\Box \bar{\lambda}_{1\dot{\alpha}}^i = \frac{i}{10} (\varepsilon \sigma^{\mu\nu})^{\alpha\beta} \partial_\mu \psi_{3\nu\alpha\beta\dot{\alpha}}{}^i - \frac{2}{5} \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \partial^\nu \chi^{\alpha i}_{2\nu}$$
$$+ \frac{3}{5} \sigma^\mu_{\alpha\dot{\alpha}} \Box \chi^{\alpha i}_{2\mu} \tag{4.4.54}$$

Comparing thses one finds an 'on-shell' type condition

$$\sigma^\mu_{\alpha\dot{\alpha}} \Box \chi^{\alpha i}_{2\mu} = 0 \tag{4.4.55}$$

$\bar{D}_{\dot{\alpha}i}$ (4.4.44) also gives

$$(\varepsilon \sigma^{\mu\nu})^{\alpha\beta} \partial_\mu \psi_{4\nu\alpha\beta\dot{\alpha}}{}^{ij}{}_k = 0 \tag{4.4.56}$$

However, from (4.4.41) one has

$$(\varepsilon \sigma^{\mu\nu})^{\beta\gamma} \partial_\mu \psi_{4\nu\beta\gamma\dot{\alpha}}{}^{ij}{}_k = 4 i \sigma^\mu_{\alpha\dot{\alpha}} \Box \chi_{1\mu}{}^{\alpha ij}{}_k \tag{4.4.57}$$

and so

$$\sigma^{\mu}_{\alpha\dot{\alpha}} \Box \chi_{1\mu}{}^{\alpha ij}{}_{k} = 0 \qquad (4.4.58)$$

D^{i}_{α} (4.4.45) gives (4.4.52) while $\overline{D}_{\dot{\alpha}i}$ (4.4.45) gives

$$\sigma^{\mu}_{(\alpha\dot{\alpha}} \Box \chi_{1\mu\beta)}{}^{ij}{}_{k} = 0 \qquad (4.4.59)$$

after using (4.4.41), and also

$$\begin{aligned}\bar{\sigma}^{\mu\dot{\beta}\gamma} \xi_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}}{}^{i} &= -36 i \partial^{\mu} \psi_{3\mu\alpha\beta\dot{\alpha}}{}^{i} + 48 i \sigma^{\mu\nu}{}_{(\alpha}{}^{\gamma} \partial_{\mu} \psi_{3\nu\beta)\gamma\dot{\alpha}}{}^{i} \\ &- 24 i \bar{\sigma}^{\mu\nu\dot{\beta}}{}_{\dot{\alpha}} \partial_{\mu} \psi_{3\nu\alpha\beta\dot{\beta}}{}^{i} + 64 \sigma^{\mu}_{(\alpha\dot{\alpha}} \Box \chi_{2\mu\beta)}{}^{i} \\ &- 384 \sigma^{\nu}_{(\alpha\dot{\alpha}} \partial_{\nu} \partial^{\mu} \chi_{2\mu\beta)}{}^{i} + 384 \sigma^{\mu\nu}{}_{(\alpha}{}^{\gamma} \sigma^{\lambda}_{\beta)\dot{\alpha}} \partial_{\lambda} \partial_{\mu} \chi_{2\nu\gamma}{}^{i}\end{aligned} \qquad (4.4.60)$$

From D^{i}_{α} (4.4.46) one finds (4.4.53) and

$$\sigma^{\mu}_{(\alpha\dot{\alpha}} \Box \chi_{2\mu\beta)}{}^{i} = 0 \qquad (4.4.61)$$

using (4.4.60).

The identities obtained from (4.4.47) to (4.4.50) are all implied by those already obtained and so summarizing, we have at this order

$$2 (\epsilon \sigma^{\mu\nu})^{\beta\gamma} \partial_{\mu} \psi_{1\nu\alpha\beta\gamma i} = \sigma^{\mu\nu}{}_{\alpha}{}^{\beta} \partial_{\mu} \psi_{2\nu\beta i}$$

$$(\epsilon \sigma^{\mu\nu})^{\alpha\beta} \partial_{\mu} \psi_{4\nu\alpha\beta\dot{\alpha}}{}^{ij}{}_{k} = 0$$

$$\Box \chi_{1\mu\alpha}{}^{ij}{}_{k} = 0$$

$$\Box \chi_{2\mu\alpha}{}^{i} = 0$$

$$\begin{aligned}\bar{\sigma}^{\mu\dot{\beta}\gamma} \xi_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}}{}^{i} &= -36 i \partial^{\mu} \psi_{3\mu\alpha\beta\dot{\alpha}}{}^{i} + 48 i \sigma^{\mu\nu}{}_{(\alpha}{}^{\gamma} \partial_{\mu} \psi_{3\nu\beta)\gamma\dot{\alpha}}{}^{i} \\ &- 24 i \bar{\sigma}^{\mu\nu\dot{\beta}}{}_{\dot{\alpha}} \partial_{\mu} \psi_{3\nu\alpha\beta\dot{\beta}}{}^{i} - 384 \sigma^{\nu}_{(\alpha\dot{\alpha}} \partial_{\nu} \partial^{\mu} \chi_{2\mu\beta)}{}^{i} \\ &+ 384 \sigma^{\mu\nu}{}_{(\alpha}{}^{\gamma} \sigma^{\lambda}_{\beta)\dot{\alpha}} \partial_{\lambda} \partial_{\mu} \chi_{2\nu\gamma}{}^{i}\end{aligned} \qquad (4.4.62)$$

At the next order the identities obtained from these are

$$\Box A_{1\mu\alpha\beta ij} = 0$$

$$\Box A_{2\mu\alpha\dot{\beta}}{}^{ij} = 0$$

$$\Box A_{3\mu}{}^{ij} = 0$$

$$\Box A_{4\mu\alpha\dot{\alpha}} = 0$$

$$\Box A_{5\mu\alpha\dot{\alpha}}{}^{i}{}_{j} = 0$$

$$\Box A_{6\mu\alpha\dot{\alpha}}{}^{ij}{}_{k\ell} = 0$$

$$\partial^{\mu} \Box W_{\nu}{}^{i}{}_{j} = 0$$

$$\bar{\sigma}^{\mu\dot{\gamma}\gamma} C_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}} = 56\, i\, \sigma^{\mu\nu}{}_{(\alpha}{}^{\gamma} \partial_{\mu} B_{3\nu\beta)\gamma\dot{\alpha}\dot{\beta}} - 40\, i\, \partial^{\mu} B_{3\mu\alpha\beta\dot{\alpha}\dot{\beta}}$$
$$+ 48\, \bar{\sigma}^{\mu\nu\dot{\gamma}}{}_{(\dot{\alpha}}\, \sigma^{\lambda}{}_{(\alpha\dot{\beta})}\, \partial_{\lambda} \partial_{\mu} A_{4\nu\beta)\dot{\gamma}} + 264\, \sigma^{\mu}{}_{(\alpha(\dot{\alpha}}\, \partial_{\mu} \partial^{\nu} A_{4\nu\beta)\dot{\beta})} \qquad (4.4.63)$$

The higher order constraints are that box acting on the components ψ , B , ξ and C of $W_{\mu}{}^{i}{}_{j}$ gives zero.

So from these relations we can see that the physical fields contained in W_{ij} and $\bar{W}^{ij}$ are now off-shell, but the auxiliary fields in $W_{\mu}{}^{i}{}_{j}$ appear to satisfy on-shell type constraints.

<u>Case II</u>

The Bianchi identities may now be written down in terms of the superfields $\mathcal{W}_{ij}$, $\bar{\mathcal{W}}^{ij}$, $\chi^{ij}_{\alpha\beta}$ and $\bar{\chi}_{\dot{\alpha}\dot{\beta}ij}$ defined by

$$\mathcal{F}^{ij}_{\alpha\beta} = i\, \varepsilon_{\alpha\beta}\, \bar{\mathcal{W}}^{ij} + i\, \chi^{ij}_{\alpha\beta}$$

$$\mathcal{F}_{\dot{\alpha}i\dot{\beta}j} = i\, \varepsilon_{\dot{\alpha}\dot{\beta}}\, \mathcal{W}_{ij} + i\, \bar{\chi}_{\dot{\alpha}\dot{\beta}ij} \qquad (4.4.64)$$

where

$$W_{ij} = - W_{ji} \quad , \quad \bar{W}^{ij} = - \bar{W}^{ji}$$

$$\chi^{ij}_{\alpha\beta} = \chi^{ji}_{\alpha\beta} = \chi^{ij}_{\beta\alpha}$$

$$\bar{\chi}_{\dot{\alpha}i\dot{\beta}j} = \bar{\chi}_{\dot{\alpha}j\dot{\beta}i} = \bar{\chi}_{\dot{\beta}i\dot{\alpha}j} \qquad (4.4.65)$$

The Bianchi identities read as

$$D^{(i}_{\alpha} \bar{W}^{j)k} = D^{\beta k} \chi^{ij}_{\beta\alpha}$$

$$D^{(i}_{\alpha} \chi^{j)k}_{\beta\gamma} + D^{(j}_{\alpha} \chi^{k)i}_{\beta\gamma} + D^{(k}_{\alpha} \chi^{i)j}_{\beta\gamma} = 0$$

$$\bar{D}_{\dot{\alpha}(i} W_{j)k} = \bar{D}^{\dot{\beta}}_{k} \bar{\chi}_{\dot{\beta}\dot{\alpha}ij}$$

$$\bar{D}_{\dot{\alpha}(i} \bar{\chi}_{\dot{\beta}\dot{\gamma}j)k} + \bar{D}_{\dot{\alpha}(j} \bar{\chi}_{\dot{\beta}\dot{\gamma}k)i} + \bar{D}_{\dot{\alpha}(k} \bar{\chi}_{\dot{\beta}\dot{\gamma}i)j} = 0$$

$$F_{\mu\alpha}{}^{i} = \frac{1}{12} \sigma_{\mu\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}}_{j} \bar{W}^{ji} + \frac{1}{20} \bar{D}_{\dot{\alpha}j} \bar{\sigma}^{\dot{\alpha}\beta}_{\mu} \chi^{ij}_{\alpha\beta}$$

$$F_{\mu\dot{\alpha}i} = \frac{1}{12} \sigma_{\mu\alpha\dot{\alpha}} D^{\alpha j} W_{ji} + \frac{1}{20} D^{j}_{\alpha} \bar{\sigma}^{\dot{\beta}\alpha}_{\mu} \bar{\chi}_{\dot{\alpha}\dot{\beta}ij}$$

$$\bar{D}_{\dot{\alpha}i} \chi^{jk}_{\alpha\beta} = \frac{1}{5} \delta^{(j}_{i} \bar{D}_{\dot{\alpha}\ell} \chi^{\ell k)}_{\alpha\beta}$$

$$D^{i}_{\alpha} \bar{\chi}_{\dot{\alpha}\dot{\beta}jk} = \frac{1}{5} \delta^{\ i}_{(j} D^{\ell}_{\alpha} \bar{\chi}_{\dot{\alpha}\dot{\beta}\ell k)}$$

$$\bar{D}_{\dot{\alpha}i} \bar{W}^{jk} = \frac{1}{3} \delta^{[j}_{i} \bar{D}_{\dot{\alpha}\ell} \bar{W}^{\ell k]}$$

$$D^{i}_{\alpha} W_{jk} = \frac{1}{3} \delta^{\ i}_{[j} D^{\ell}_{\alpha} W_{\ell k]}$$

$$F_{\mu\nu} = \frac{i}{96} \left(D^{\alpha i} \sigma_{\mu\nu\alpha}{}^{\beta} D^{j}_{\beta} W_{ij} - \bar{D}_{\dot{\alpha}i} \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{D}^{\dot{\beta}}_{j} \bar{W}^{ij} \right)$$
$$- \frac{i}{320} \left((\bar{\sigma}_{\mu\nu} \varepsilon)^{\dot{\alpha}\dot{\beta}} D^{\alpha i} D^{j}_{\alpha} \bar{\chi}_{\dot{\alpha}\dot{\beta}ij} - (\varepsilon \sigma_{\mu\nu})^{\alpha\beta} \bar{D}_{\dot{\alpha}i} \bar{D}^{\dot{\alpha}}_{j} \chi^{ij}_{\alpha\beta} \right)$$

(4.4.66)

Again, the self-duality constraint (4.4.4) on the field strengths W_{ij} and $\bar{W}^{ij}$ cannot be imposed as this would decouple the W's from the χ's and so lead to equations of motion.

To study the component content of the theory the abelain gauge theory is again considered for simplicity. From the Bianchi identities (4.4.66) one can derive the relations

$$D^i_\alpha \overline{W}^{jk} = \frac{1}{3}\varepsilon^{ijk\ell}\lambda_{2\alpha\ell} - \frac{1}{3}D^{\beta[j} X^{ik]}_{\beta\alpha}$$

$$D^i_\alpha W_{jk} = \frac{1}{3}\delta^{\ i}_{[j}\lambda_{1\alpha k]}$$

$$D^i_\alpha \lambda_{1\beta j} = 12\, i\, \delta^i_j (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta} F_{\mu\nu} - \frac{3}{40}\delta^i_j \overline{D}_{\dot\alpha k}\overline{D}^{\dot\alpha}_\ell X^{k\ell}_{\alpha\beta}$$

$$\overline{D}_{\dot\alpha i}\lambda_{1\alpha j} = -6\, i\, \sigma^\mu_{\alpha\dot\alpha}\partial_\mu W_{ij} - \frac{4}{5}D^k_\alpha \overline{D}^{\dot\beta}_j \overline{X}_{\dot\beta\dot\alpha ik} - \frac{1}{5}D^k_\alpha \overline{D}^{\dot\beta}_i \overline{X}_{\dot\beta\dot\alpha jk}$$

$$D^i_\alpha \lambda_{2\beta j} = 12\, i\, \delta^i_j (\sigma^{\mu\nu}\varepsilon)_{\alpha\beta} G_{\mu\nu} - \frac{3}{8}\varepsilon_{jk\ell m} D^k_\beta D^{\gamma\ell} X^{im}_{\alpha\gamma} + \frac{1}{8}\varepsilon_{jk\ell m} D^k_\alpha D^{\gamma\ell} X^{im}_{\beta\gamma}$$

$$\overline{D}_{\dot\alpha i}\lambda_{2\alpha j} = -3\, i\, \varepsilon_{ijk\ell}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \overline{W}^{k\ell} + \frac{1}{5}\varepsilon_{ijk\ell}\overline{D}_{\dot\alpha m} D^{\beta k} X^{\ell m}_{\beta\alpha}$$

$$D^i_\alpha F_{\mu\nu} = \frac{1}{12}\sigma_{[\mu\alpha\dot\alpha}\partial_{\nu]}\overline{\lambda}^{\dot\alpha i}_1 + \frac{1}{20}\overline{\sigma}^{\dot\alpha\beta}_{[\mu}\partial_{\nu]}\overline{D}_{\dot\alpha j} X^{ij}_{\alpha\beta}$$

$$D^i_\alpha G_{\mu\nu} = \frac{1}{12}\sigma^\lambda_{\alpha\dot\alpha}\overline{\sigma}_{\mu\nu}{}^{\dot\alpha}{}_{\dot\beta}\partial_\lambda \overline{\lambda}^{\dot\beta i}_2 - \frac{i}{480}(\varepsilon\sigma_{\mu\nu})^{\beta\gamma}(\varepsilon_{jk\ell m} D^j_\alpha D^k_\beta D^{\delta\ell} X^{im}_{\gamma\delta} - 2\varepsilon_{jk\ell m} D^j_\beta D^k_\gamma D^{\delta\ell} X^{im}_{\alpha\delta}) - \frac{i}{480}(\overline{\sigma}_{\mu\nu}\varepsilon)^{\dot\alpha\dot\beta}\varepsilon^{ijk\ell}\overline{D}_{\dot\alpha j} D^m_\alpha \overline{D}^{\dot\gamma}_k \overline{X}_{\dot\gamma\dot\beta\ell m}$$

(4.4.67)

So again, the component field content of the theory can be expressed as the $\theta = \overline{\theta} = 0$ components of the superfields W_{ij}, $\overline{W}^{ij}$, $\lambda_{1\alpha i}$, $\lambda_{2\alpha i}$, $F_{\mu\nu}$, $G_{\mu\nu}$, $X^{ij}_{\alpha\beta}$, spinor derivatives of $X^{ij}_{\alpha\beta}$ and space-time derivatives of these superfields.

To find the component content of the superfield $X^{ij}_{\alpha\beta}$ the tableaux calculus of [34] is again used. Acting on superfield $X^{ij}_{\alpha\beta}$ by the spinor derivative, one can split the result up into the irreducible representations of SU(4) as follows

$$D^{i}_{\alpha} X^{jk}_{\beta\gamma} = \frac{1}{3} X_{1\alpha\beta\gamma}^{i(jk)} + \frac{1}{3} X_{3\alpha\beta\gamma}^{ijk} \tag{4.4.68}$$

where

$$X_{1\alpha\beta\gamma}^{ijk} = D^{[i}_{\alpha} X^{j]k}_{\beta\gamma}$$

$$X_{3\alpha\beta\gamma}^{ijk} = D^{i}_{\alpha} X^{jk}_{\beta\gamma} + D^{j}_{\alpha} X^{ki}_{\beta\gamma} + D^{k}_{\alpha} X^{ij}_{\beta\gamma} \tag{4.4.69}$$

and

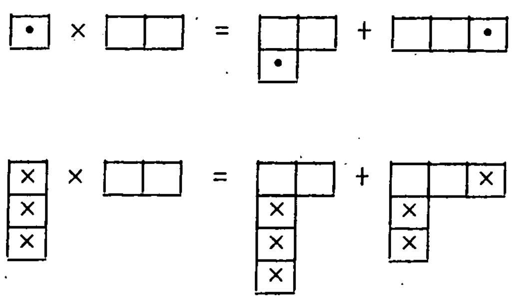
$$\bar{D}_{\dot\alpha i} X^{jk}_{\alpha\beta} = \frac{1}{5} \delta^{(j}_{i} \bar{X}_{2\dot\alpha\alpha\beta}{}^{k)} + \bar{X}_{4\dot\alpha\alpha\beta}{}^{jk}{}_{i} \tag{4.4.70}$$

where

$$\bar{X}_{2\dot\alpha\alpha\beta}{}^{i} = \bar{D}_{\dot\alpha j} X^{ji}_{\alpha\beta}$$

$$\bar{X}_{4\dot\alpha\alpha\beta}{}^{ij}{}_{k} = \bar{D}_{\dot\alpha k} X^{ij}_{\alpha\beta} - \frac{1}{5} \delta^{(i}_{k} \bar{D}_{\dot\alpha \ell} X^{\ell j)}_{\alpha\beta} \tag{4.4.71}$$

These are represented by the tableaux equations



(4.4.72)

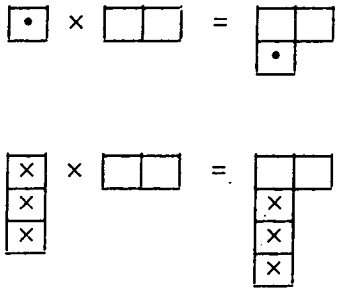
respectively.

From the Bianchi identities, the last tableaux of each of these equations must be put equal to zero. So the constraint equations read

$$D^{i}_{\alpha} X^{jk}_{\beta\gamma} = \frac{1}{3} X_{1\alpha\beta\gamma}^{i(jk)}$$

$$\bar{D}_{\dot\alpha i} X^{jk}_{\beta\gamma} = \frac{1}{5} \delta^{(j}_{i} \bar{X}_{2\dot\alpha\alpha\beta}{}^{k)} \tag{4.4.73}$$

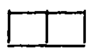
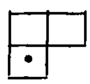
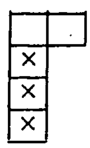
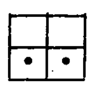
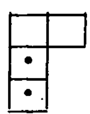
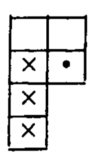
which are represented by the tableaux equations

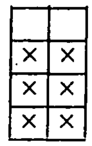
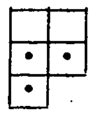
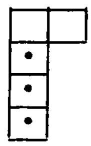
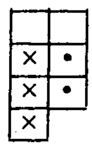
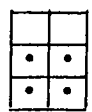
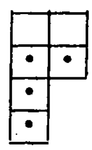
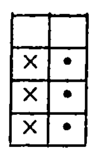
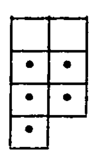


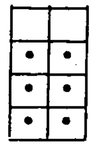
(4.4.74)

respectively.

One can now calculate the components of this superfield subject to these constraints using the tableaux calculus. The result is

Field	*SU(4)* *Representation*	*SL(2,C)* *Representation*	*Tableaux*	*Components*
$X^{ij}_{\alpha\beta}$	10	$(1\ ,\ 0) \times (0\ ,\ 0)$		10×3
$X_{1\alpha\beta\gamma}^{ijk}$	20	$(1\ ,\ 0) \times (\frac{1}{2}\ ,\ 0)$		40×3
$\bar{X}_{2\dot{\alpha}\beta\gamma}{}^{i}$	4	$(1\ ,\ 0) \times (0\ ,\ \frac{1}{2})$		8×3
$A_{1\alpha\beta}^{ijk\ell}$	20'	$(1\ ,\ 0) \times (0\ ,\ 0)$		20×3
$A_{2\alpha\beta\gamma\delta}{}^{i}{}_{j}$	15	$(1\ ,\ 0) \times (1\ ,\ 0)$		45×3
$A_{3\alpha\dot{\alpha}\beta\gamma}^{ij}$	6	$(1\ ,\ 0) \times (\frac{1}{2}\ ,\ \frac{1}{2})$		$(24 - 6) \times 3$

Field	*SU(4) Representation*	*SL(2,C) Representation*	*Tableaux*	*Components*
$A_{4\alpha\beta}$	1	$(1,0)\times(0,0)$		1×3
$\psi_{1\alpha\beta\gamma}{}^{ij}{}_{k}$	20	$(1,0)\times(\frac{1}{2},0)$		40×3
$\psi_{2\alpha\beta\gamma\delta\epsilon}{}^{i}$	4	$(1,0)\times(\frac{3}{2},0)$		16×3
$\psi_{3\alpha\beta\dot{\alpha}\gamma\delta i}$	4*	$(1,0)\times(1,\frac{1}{2})$		$(24-8)\times 3$
$B_{1\alpha\beta ij}$	10*	$(1,0)\times(0,0)$		10×3
$B_{2\alpha\beta\gamma\delta}{}^{ij}$	6	$(1,0)\times(1,0)$		18×3
$B_{3\alpha\beta\gamma\dot{\alpha}\delta\epsilon}$	1	$(1,0)\times(\frac{3}{2},\frac{1}{2})$		$(8-3)\times 3$
$\xi_{\alpha\beta\gamma i}$	4*	$(1,0)\times(\frac{1}{2},0)$		8×3

Field	*SU(4) Representation*	*SL(2,C) Representation*	*Tableaux*	*Components*
$C_{\alpha\beta}$	1	$(1\,,0)\times(0\,,0)$		1×3

(4.4.75)

In addition to these fields there are of course the conjugate fields to these.

The fields with mixed indices are again conserved

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha}\,\partial_\mu\, A_{3\alpha\dot{\alpha}\beta\gamma}^{\;ij} = 0$$

$$\bar{\sigma}^{\mu\dot{\alpha}\beta}\,\partial_\mu\, \psi_{3\alpha\beta\dot{\alpha}\gamma\delta i} = 0$$

$$\bar{\sigma}^{\mu\dot{\alpha}\gamma}\,\partial_\mu\, B_{3\alpha\beta\gamma\dot{\alpha}\delta\varepsilon} = 0 \qquad (4.4.76)$$

The fields in (4.4.75) can be expressed in terms of $X^{ij}_{\alpha\beta}$ as follows

$$\chi_{1\alpha\beta\gamma}^{\;ijk} = D^{[i}_{\alpha}\, X^{j]k}_{\beta\gamma}$$

$$\bar{\chi}_{2\dot{\alpha}\alpha\beta}^{\;i} = \bar{D}_{\dot{\alpha}j}\, X^{ji}_{\alpha\beta}$$

$$A_{1\alpha\beta}^{\;ijk\ell} = D^{\gamma i}\, D^{[j}_{\gamma}\, X^{k]\ell}_{\alpha\beta}$$

$$A_{2\alpha\beta\gamma\delta}{}^{i}{}_{j} = \varepsilon_{jk\ell m}\, D^{k}_{\alpha}\, D^{\ell}_{\beta}\, X^{mi}_{\gamma\delta}$$

$$A_{3\alpha\dot{\alpha}\beta\gamma}^{\;ij} = D^{[i}_{\alpha}\, \bar{D}_{\dot{\alpha}k}\, X^{kj]}_{\beta\gamma}$$

$$A_{4\alpha\beta} = \bar{D}_{\dot{\alpha}i}\, \bar{D}^{\dot{\alpha}}_{j}\, X^{ij}_{\alpha\beta}$$

$$\psi_{1\alpha\beta\gamma}{}^{ij}{}_{k} = \varepsilon_{k\ell mn}\, D^{\delta[i}\, D^{\ell}_{\delta}\, D^{m}_{\alpha}\, X^{nj]}_{\beta\gamma} - \text{trace}$$

$$\psi_{2\alpha\beta\gamma\delta\varepsilon}^{\;i} = \varepsilon_{jk\ell m}\, D^{j}_{\alpha}\, D^{k}_{\beta}\, D^{\ell}_{\gamma}\, X^{mi}_{\delta\varepsilon}$$

$$\psi_{3\alpha\beta\dot{\alpha}\gamma\delta i} = \varepsilon_{ijk\ell}\, D^j_\alpha\, D^k_\beta\, \overline{D}_{\dot{\alpha}m}\, X^{m\ell}_{\gamma\delta}$$

$$B_{1\alpha\beta ij} = \varepsilon_{ik\ell m}\, \varepsilon_{jnpq}\, D^{\gamma k}\, D^n_\gamma\, D^{\delta\ell}\, D^p_\delta\, X^{mq}_{\alpha\beta}$$

$$B_{2\alpha\beta\gamma\delta}{}^{ij} = \varepsilon_{k\ell mn}\, D^{\varepsilon i}\, D^k_\varepsilon\, D^\ell_\alpha\, D^m_\beta\, X^{nj}_{\gamma\delta}$$

$$B_{3\alpha\beta\gamma\dot{\alpha}\delta\varepsilon} = \varepsilon_{ijk\ell}\, D^i_\alpha\, D^j_\beta\, D^k_\gamma\, \overline{D}_{\dot{\alpha}m}\, X^{m\ell}_{\delta\varepsilon}$$

$$\xi_{\alpha\beta\gamma i} = \varepsilon_{ijk\ell}\, \varepsilon_{mnpq}\, D^m_\alpha\, D^{\delta j}\, D^n_\delta\, D^{\varepsilon k}\, D^p_\varepsilon\, X^{\ell q}_{\beta\gamma}$$

$$C_{\alpha\beta} = \varepsilon_{ijk\ell}\, \varepsilon_{mnpq}\, D^{\gamma i}\, D^m_\gamma\, D^{\delta j}\, D^n_\delta\, D^{\varepsilon k}\, D^p_\varepsilon\, X^{\ell q}_{\alpha\beta} \qquad (4.4.77)$$

From these superfields the supersymmetry transformations of the component fields can be calculated by acting on the above with D^i_α and $\overline{D}_{\dot{\alpha}i}$. The fields will also satisfy identities similar to the previous case.

Case III

The Bianchi identities may now be written down in terms of the superfields $\mathcal{W}_{ij}$, $\overline{\mathcal{W}}^{ij}$, $\chi^{ij}_{\alpha\beta}$, $\overline{\chi}_{\dot{\alpha}\dot{\beta}ij}$ and $\mathcal{W}_\mu{}^i{}_j$ defined by

$$\mathcal{F}^{ij}_{\alpha\beta} = i\, \varepsilon_{\alpha\beta}\, \overline{\mathcal{W}}^{ij} + i\, \chi^{ij}_{\alpha\beta}$$

$$\mathcal{F}_{\dot{\alpha}i\dot{\beta}j} = i\, \varepsilon_{\dot{\alpha}\dot{\beta}}\, \mathcal{W}_{ij} + i\, \overline{\chi}_{\dot{\alpha}\dot{\beta}ij}$$

$$\mathcal{F}^i_{\alpha\dot{\alpha}j} = i\, \sigma^\mu_{\alpha\dot{\alpha}}\, \mathcal{W}_\mu{}^i{}_j \qquad (4.4.78)$$

where

$$\mathcal{W}_{ij} = -\mathcal{W}_{ji}\ , \quad \overline{\mathcal{W}}^{ij} = -\overline{\mathcal{W}}^{ji}$$

$$\chi^{ij}_{\alpha\beta} = \chi^{ji}_{\alpha\beta} = \chi^{ij}_{\beta\alpha}$$

$$\overline{\chi}_{\dot{\alpha}i\dot{\beta}j} = \overline{\chi}_{\dot{\alpha}j\dot{\beta}i} = \overline{\chi}_{\dot{\beta}i\dot{\alpha}j}$$

$$\mathcal{W}_\mu{}^i{}_i = 0 \qquad (4.4.79)$$

The Bianchi identities now read as

$$
\begin{aligned}
&\mathcal{D}^{(i}_{\alpha} \bar{W}^{j)k} = \mathcal{D}^{\beta k} \chi^{ij}_{\beta\alpha} \\
&\mathcal{D}^{(i}_{\alpha} \chi^{jk)}_{\beta\gamma} = 0 \\
&\bar{\mathcal{D}}_{\dot\alpha(i} W_{j)k} = \bar{\mathcal{D}}^{\dot\beta}_{k} \bar{\chi}_{\dot\beta\dot\alpha ij} \\
&\bar{\mathcal{D}}_{\dot\alpha(i} \bar{\chi}_{\dot\beta\dot\gamma jk)} = 0 \\
&\mathcal{F}_{\mu\alpha}{}^{i} = \frac{1}{12}\sigma_{\mu\alpha\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}_{j}\bar{W}^{ji} + \frac{1}{20}\bar{\mathcal{D}}_{\dot\alpha j}\bar{\sigma}^{\dot\alpha\beta}_{\mu}\chi^{ij}_{\alpha\beta} - \frac{1}{60}\Big[(\sigma_{\mu}\bar{\sigma}^{\nu} - 3\sigma^{\nu}\bar{\sigma}_{\mu})\mathcal{D}\Big]^{j}_{\alpha} W_{\nu}{}^{i}{}_{j} \\
&\mathcal{F}_{\mu\dot\alpha i} = \frac{1}{12}\sigma_{\mu\alpha\dot\alpha}\mathcal{D}^{\alpha j} W_{ji} + \frac{1}{20}\mathcal{D}^{j}_{\alpha}\bar{\sigma}^{\dot\beta\alpha}_{\mu}\bar{\chi}_{\dot\alpha\dot\beta ij} - \frac{1}{60}\Big[\bar{\mathcal{D}}(\bar{\sigma}^{\nu}\sigma_{\mu} - 3\bar{\sigma}_{\mu}\sigma^{\nu})\Big]_{\dot\alpha j} W_{\nu}{}^{j}{}_{i} \\
&(\epsilon\sigma_{\mu\nu})^{\beta\gamma}\big(\bar{\mathcal{D}}_{\dot\alpha i}\chi^{jk}_{\beta\gamma} + \mathcal{D}^{(j}_{\beta}\sigma^{\lambda}_{\gamma\dot\alpha} W_{\lambda}{}^{k)}{}_{i}\big) \\
&\qquad = \frac{1}{5}\delta^{(j}_{i}(\epsilon\sigma_{\mu\nu})^{\beta\gamma}\big(\bar{\mathcal{D}}_{\dot\alpha\ell}\chi^{\ell k)}_{\beta\gamma} + \mathcal{D}^{\ell}_{\beta}\sigma^{\lambda}_{\gamma\dot\alpha} W_{\lambda}{}^{k)}{}_{\ell}\big) \\
&(\bar{\sigma}_{\mu\nu}\epsilon)^{\dot\beta\dot\gamma}\big(\mathcal{D}^{i}_{\alpha}\bar{\chi}_{\dot\beta\dot\gamma jk} + \bar{\mathcal{D}}_{\dot\beta(j}\sigma^{\lambda}_{\alpha\dot\gamma} W_{\lambda}{}^{i}{}_{k)}\big) \\
&\qquad = \frac{1}{5}\delta^{i}_{(j}(\bar{\sigma}_{\mu\nu}\epsilon)^{\dot\beta\dot\gamma}\big(\mathcal{D}^{\ell}_{\alpha}\bar{\chi}_{\dot\beta\dot\gamma\ell k)} + \bar{\mathcal{D}}_{\dot\beta\ell}\sigma^{\lambda}_{\alpha\dot\gamma} W_{\lambda}{}^{\ell}{}_{k)}\big) \\
&\bar{\mathcal{D}}_{\dot\alpha i}\bar{W}^{jk} = \frac{1}{3}\delta^{[j}_{i}\bar{\mathcal{D}}_{\dot\alpha\ell}\bar{W}^{\ell k]} - \frac{1}{2}\sigma^{\mu}_{\alpha\dot\alpha}\big(\mathcal{D}^{\alpha[j} W_{\mu}{}^{k]}{}_{i} - \frac{1}{3}\delta^{[j}_{i}\mathcal{D}^{\alpha\ell} W_{\mu}{}^{k]}{}_{\ell}\big) \\
&\mathcal{D}^{i}_{\alpha} W_{jk} = \frac{1}{3}\delta^{i}_{[j}\mathcal{D}^{\ell}_{\alpha} W_{\ell k]} - \frac{1}{2}\sigma^{\mu}_{\alpha\dot\alpha}\big(\bar{\mathcal{D}}^{\dot\alpha}_{[j} W_{\mu}{}^{i}{}_{k]} - \frac{1}{3}\delta^{i}_{[j}\bar{\mathcal{D}}^{\dot\alpha}_{\ell} W_{\mu}{}^{\ell}{}_{k]}\big) \\
&\mathcal{F}_{\mu\nu} = \frac{i}{96}\big(\mathcal{D}^{\alpha i}\sigma_{\mu\nu\alpha}{}^{\beta}\mathcal{D}^{j}_{\beta} W_{ij} - \bar{\mathcal{D}}_{\dot\alpha i}\bar{\sigma}_{\mu\nu}{}^{\dot\alpha}{}_{\dot\beta}\bar{\mathcal{D}}^{\dot\beta}_{j}\bar{W}^{ij}\big) \\
&\qquad - \frac{1}{240}\epsilon_{\mu\nu\rho\lambda}\bar{\sigma}^{\rho\dot\alpha\alpha}\big[\mathcal{D}^{i}_{\alpha}, \bar{\mathcal{D}}_{\dot\alpha j}\big] W^{\lambda j}{}_{i} + \frac{i}{240}\big[W_{\mu}{}^{i}{}_{j}, W_{\nu}{}^{j}{}_{i}\big] \\
&\qquad - \frac{i}{320}\big((\bar{\sigma}_{\mu\nu}\epsilon)^{\dot\alpha\dot\beta}\mathcal{D}^{\alpha i}\mathcal{D}^{j}_{\alpha}\bar{\chi}_{\dot\alpha\dot\beta ij} - (\epsilon\sigma_{\mu\nu})^{\alpha\beta}\bar{\mathcal{D}}_{\dot\alpha i}\bar{\mathcal{D}}^{\dot\alpha}_{j}\chi^{ij}_{\alpha\beta}\big)
\end{aligned}
\tag{4.4.80}
$$

The self-duality constraint (4.4.4) can now be imposed without implying equations of motion. It now only relates $\chi^{ij}_{\alpha\beta}$ and $W_{\mu}{}^{i}{}_{j}$ by

$$\frac{1}{3}\mathcal{D}^{\beta[i}\chi^{j]k}_{\beta\alpha} = \frac{1}{4}\varepsilon^{ij\ell m}\sigma^{\mu}_{\alpha\dot\alpha}\left(\bar{\mathcal{D}}^{\dot\alpha}_{[\ell}\,\mathcal{W}_{\mu}{}^{k}{}_{m]} - \frac{1}{3}\delta^{k}_{[\ell}\bar{\mathcal{D}}^{\dot\alpha}_{n}\,\mathcal{W}_{\mu}{}^{n}{}_{m]}\right) \tag{4.4.81}$$

The Bianchi identities can be used to express the theory in terms of the superfields $\mathcal{W}_{ij}$, $\lambda_{\alpha i}$, $\mathcal{F}_{\mu\nu}$, $\mathcal{W}_{\mu}{}^{i}{}_{j}$, $\chi^{ij}_{\alpha\beta}$ and covariant spinor derivatives of the superfields $\mathcal{W}_{\mu}{}^{i}{}_{j}$ and $\chi^{ij}_{\alpha\beta}$. The component content of $\chi^{ij}_{\alpha\beta}$ is as in case II as it satisfies the constraint

$$\mathcal{D}^{(i}_{\alpha}\chi^{j)k}_{\beta\gamma} + \mathcal{D}^{(j}_{\alpha}\chi^{k)i}_{\beta\gamma} + \mathcal{D}^{(k}_{\alpha}\chi^{i)j}_{\beta\gamma} = 0 \tag{4.4.82}$$

while the following combination can be expressed in terms of spinor derivatives acting on $\mathcal{W}_{\mu}{}^{i}{}_{j}$

$$\bar{\mathcal{D}}_{\dot\alpha i}\,\chi^{jk}_{\alpha\beta} - \frac{1}{5}\delta^{(j}_{i}\,\bar{\mathcal{D}}_{\dot\alpha\ell}\,\chi^{\ell k)}_{\alpha\beta} = -\frac{1}{2}\sigma^{\mu}_{(\alpha\dot\alpha}\left(\mathcal{D}^{(j}_{\beta)}\,\mathcal{W}_{\mu}{}^{k)}{}_{i} - \frac{1}{5}\delta^{(j}_{i}\,\mathcal{D}^{\ell}_{\beta)}\,\mathcal{W}_{\mu}{}^{k)}{}_{\ell}\right) \tag{4.4.83}$$

The superfield $\mathcal{W}_{\mu}{}^{i}{}_{j}$ in this case is unconstrained.

From the expansion of the superfields $W_{\mu}{}^{i}{}_{j}$ and $X^{ij}_{\alpha\beta}$ given by (4.4.34) and (4.4.75), we can see that no quadratic Lagrangian is possible even for the minimal relaxations I and II, as fields with dimension up to 4 appear. However one could consider the Lagrangian where a Lagrange multiplier superfield is used to put $W_{\mu}{}^{i}{}_{j}$ or $X^{ij}_{\alpha\beta}$ equal to zero in one of the minimal cases, so implying the equations of motion for the physical fields contained in W_{ij} and $\overline{W}^{ij}$.

The highest dimension components of $W_{\mu}{}^{i}{}_{j}$ and $X^{ij}_{\alpha\beta}$ in the minimal formulations are $C_{\mu\alpha\beta\gamma\dot\alpha\dot\beta\dot\gamma}$ and $C_{\alpha\beta}$ respectively. A Lagrange multiplier method would be to take a dimension zero superfield $D_{\mu\alpha\beta\gamma\dot\alpha\dot\beta\dot\gamma}$ or $D_{\alpha\beta}$ and then by making it satisfy certain constraints, obtain a 'contragradient' superfield [23] to $W_{\mu\ j}^{\ i}$ or $X_{\alpha\beta}^{ij}$. This would contain the same components as $W_{\mu\ j}^{\ i}$ or $X_{\alpha\beta}^{ij}$ but in the reverse order and with dimension 0 to 3. For example, the superfield $D_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}}$ would have to satisfy the constraint

$$\bar{D}_{\dot{\alpha}i}\, D_{\mu\alpha\beta\gamma\dot{\beta}\dot{\gamma}\dot{\delta}} \sim \varepsilon_{\dot{\alpha}\dot{\beta}}\, \zeta_{\mu\alpha\beta\gamma\dot{\gamma}\dot{\delta}i} + \dot{\gamma}\dot{\delta}\dot{\beta} + \dot{\delta}\dot{\beta}\dot{\gamma} \qquad (4.4.84)$$

so that part of the variation under $\bar{D}_{\dot{\alpha}i}$ of

$$\zeta^{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}}{}_{i}\, \xi_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}}{}^{i} \qquad (4.4.85)$$

cancels the relevant part of the variation of

$$D^{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}}\, C_{\mu\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}} \qquad (4.4.86)$$

In the non-minimal case III, the situation is far simpler. Here the superfield $X_{\alpha\beta}^{ij}$ still satisfies the same constraints, but now $W_{\mu\ j}^{\ i}$ is totally unconstrained. For an unconstrained superfield the Lagrange multiplier method is now straightforward as an unconstrained Lagrange multiplier can be used to put it equal to zero. This can be seen as follows - the unconstrained superfield $W_{\mu\ j}^{\ i}$ has the components

$$W_{\mu\ j}^{\ i},\ \chi_{\mu\alpha\ \ k}^{\ \ ij} = D_{\alpha}^{i} W_{\mu\ k}^{\ j}, \ldots, \xi_{\mu\alpha i\ k}^{\ \ \ j} = D_{\alpha i}^{7} \bar{D}^{8} W_{\mu\ k}^{\ j},\ C_{\mu\ j}^{\ i} = D^{8}\bar{D}^{8} W_{\mu\ j}^{\ i}$$

while the unconstrained dimension -5 Lagrange multiplier superfield $D_{\mu\ j}^{\ i}$ has the components

$$D_{\mu\ j}^{\ i},\ \Psi_{\mu\alpha\ \ k}^{\ \ ij} = D_{\alpha}^{i} D_{\mu\ k}^{\ j}, \ldots, \zeta_{\mu\alpha i\ k}^{\ \ \ j} = D_{\alpha i}^{7} \bar{D}^{8} D_{\mu\ k}^{\ j},\ E_{\mu\ j}^{\ i} = D^{8}\bar{D}^{8} D_{\mu\ j}^{\ i}$$

So the supersymmetric Lagrangian is

$$\mathcal{L} = E_{\mu\ j}^{\ i}\, W^{\mu j}{}_{i} + \zeta_{\mu\alpha i\ k}^{\ \ \ j}\, \chi^{\mu\alpha ik}{}_{j} + \ldots + \Psi_{\mu\alpha\ \ k}^{\ \ ij}\, \xi^{\mu\alpha\ k}{}_{i\ j} + D_{\mu\ j}^{\ i}\, C^{\mu j}{}_{i} \qquad (4.4.87)$$

where the numerical factors multiplying each of the terms have been suppressed.

For this to imply the field equations the self-duality constraint (4.4.4) must be used so that (4.4.81) may be used to express the components of $X^{ij}_{\alpha\beta}$ that occur in terms of components of $W_{\mu}{}^{i}{}_{j}$. Explicitly, in the abelian case one finds

$$D^{i}_{\alpha}\,\bar{D}_{\dot{\alpha}k}\,\bar{W}^{kj} = -6i\,\partial\!\!\!/_{\alpha\dot{\alpha}}\,\bar{W}^{ij} + \frac{3}{10}\,\sigma^{\mu}_{\beta\dot{\alpha}}\,D^{k}_{\alpha}\,X_{1\mu}{}^{\beta ij}{}_{k} + \frac{9}{40}\,\varepsilon^{ij\ell m}\,\sigma^{\mu}_{\alpha\dot{\beta}}\,\bar{D}_{\dot{\alpha}k}\,\bar{X}_{1\mu\ell m}{}^{\dot{\beta}}{}^{k} \qquad (4.4.88)$$

where as before

$$X_{1\mu\alpha}{}^{ij}{}_{k} = D^{[i}_{\alpha}\,W_{\mu}{}^{j]}{}_{k} - \frac{1}{3}\,\delta^{[i}_{k}\,D^{\ell}_{\alpha}\,W_{\mu}{}^{j]}{}_{\ell} \qquad (4.4.89)$$

So now

$$\begin{aligned} -2i\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,\partial_{\mu}\,\bar{D}^{\dot{\alpha}}_{k}\,\bar{W}^{ki} &= \{D^{i}_{\alpha}\,,\,\bar{D}_{\dot{\alpha}j}\}\,\bar{D}^{\dot{\alpha}}_{k}\,\bar{W}^{kj} \\ &= -\bar{D}^{\dot{\alpha}}_{j}\,D^{i}_{\alpha}\,\bar{D}_{\dot{\alpha}k}\,\bar{W}^{kj} \\ &= -6i\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,\partial_{\mu}\,\bar{D}^{\dot{\alpha}}_{k}\,\bar{W}^{ki} - \bar{D}^{\dot{\alpha}}_{j}\,\Big(\frac{3}{10}\,\sigma^{\mu}_{\beta\dot{\alpha}}\,D^{k}_{\alpha}\,X_{1\mu}{}^{\beta ij}{}_{k} \\ &\quad + \frac{9}{40}\,\varepsilon^{ij\ell m}\,\sigma^{\mu}_{\alpha\dot{\beta}}\,\bar{D}_{\dot{\alpha}k}\,\bar{X}_{1\mu\ell m}{}^{\dot{\beta}}{}^{k}\Big) \qquad (4.4.90) \end{aligned}$$

So we can see that the equations of motion for the physical fields contained in W_{ij} hold when $W_{\mu}{}^{i}{}_{j}$ is zero.

Therefore the Lagrangian (4.4.87) propagates the physical fields in W_{ij}. However, it also propagates other fields, for example, in the abelian case it also propagates another set of the physical fields. This is because the terms B^{ν}, Ψ^{α}_{i} and M^{ij} in the Lagrange multiplier superfield that multiply the components $\partial^{\mu}F_{\mu\nu}$, $\sigma^{\mu}_{\alpha\dot{\alpha}}\,\partial_{\mu}\,\bar{\lambda}^{\dot{\alpha}i}$

and $\Box W_{ij}$ of $W_{\mu}{}^{i}{}_{j}$ (see (4.4.40) and (4.4.46) above†) also propagate with the field equations $\partial^{\mu} (\partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu}) = 0$, $\partial_{\mu} \Psi^{\alpha}_{i} \sigma^{\mu}_{\alpha\dot{\alpha}} = 0$ and $\Box M^{ij} = 0$.

It is this increase in the number of propagating fields that avoids the arguments of [22] which ruled out an off-shell representation with just 1 spin 1, 4 spin ½ and 6 spin 0 fields propagating for a quadratic Lagrangian. However, as in the case of the relaxed hypermultiplet when only the action I_1 of [23] is considered, some of the propagating fields have kinetic terms with the wrong sign. For example, for the term in the Lagrangian (4.4.87)

$$G^{\mu\nu} F_{\mu\nu} \qquad (4.4.91)$$

where

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$$

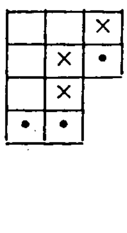
$$G_{\mu\nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu} \qquad (4.4.92)$$

the following redefinition of fields may be made

$$F_{\mu\nu} = F_{1\mu\nu} + F_{2\mu\nu}$$

$$G_{\mu\nu} = F_{1\mu\nu} - F_{2\mu\nu} \qquad (4.4.93)$$

†When the superfield $W_{\mu}{}^{i}{}_{j}$ is unconstrained it contains a component $N_{\mu\alpha\dot{\alpha}ij}$ with the tableaux



Now $\bar{\sigma}^{\mu\dot{\alpha}\alpha} N_{\mu\alpha\dot{\alpha}ij}$ appears on the right hand side of (4.4.44)

So in the Lagrangian we have

$$F_{1\mu\nu} F_1^{\mu\nu} - F_{2\mu\nu} F_2^{\mu\nu} \tag{4.4.94}$$

The problem of kinetic terms in the Lagrangian with the wrong sign can be solved by the introduction of constraints using a non-linear Lagrange multiplier, as in the case of CP_n models. For the case of (4.4.91) one takes

$$G^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \lambda_{\mu\nu\rho\lambda} (G^{\mu\nu} - F^{\mu\nu})(G^{\rho\lambda} - F^{\rho\lambda}) \tag{4.4.95}$$

where $\lambda_{\mu\nu\rho\lambda} = \lambda_{\rho\lambda\mu\nu}$

This gives the field equations

$$G^{\mu\nu} = F^{\mu\nu}$$

$$\partial^\mu (F_{\mu\nu} + \lambda_{\mu\nu\rho\lambda} (G^{\rho\lambda} - F^{\rho\lambda})) = 0$$

$$\partial^\mu (G_{\mu\nu} - \lambda_{\mu\nu\rho\lambda} (G^{\rho\lambda} - F^{\rho\lambda})) = 0 \tag{4.4.96}$$

The supersymmetrization of the second part of (4.4.95) now has to be obtained. The components and supersymmetry transformations of the supersymmetry multiplet which contains $\lambda_{\mu\nu\rho\lambda}$ will be determined by the supersymmetry transformations of $G_{\mu\nu}$ and $F_{\mu\nu}$ in order that the non-linear Lagrange multiplier term of the Lagrangian is invariant under supersymmetry.

CHAPTER 5 THE SUPERCURRENT IN SUPERSPACE

The supercurrent multiplet of conserved currents and auxiliary quantities are the components of a supercurrent superfield. For unextended supersymmetry, the supercurrent superfield is a vector indexed object J_μ with the axial current $j_\mu^{(5)}$ as its $\theta = \overline{\theta} = 0$ component, this being the member of the supercurrent multiplet of lowest dimension [16] .

For extended supersymmetry, the N=2 non- superconformal supercurrent multiplet was first obtained by [25], and again these fit into a superfield J (this time a scalar superfield) which has the lowest dimension component of the supercurrent multiplet as its $\theta = \overline{\theta} = 0$ component. For the supersymmetric N=2 gauge theory the supercurrent superfield J was written down by [26] . In the case of N=4 supersymmetry one has a superconformal supercurrent multiplet [33] and one can again fit the components into a scalar superfield $J_{ij,k\ell}$ which has as its $\theta = \overline{\theta} = 0$ component the member of the supercurrent multiplet of lowest dimension [26] .

However, the supercurrents J_μ , J and $J_{ij,k\ell}$ appear somewhat arbitary and one can envisage other superfields, satisfying different constraints, but containing the same supercurrent components. One would like a clear geometrical interpretation of the supercurrent superfield, and to obtain this one turns to Noether's theorem in superspace.

For unextended supersymmetry, Noether's theorem in superspace can be used to obtain a geometrical derivation of the supercurrent J_μ [27, 28]. However, for extended supersymmetry, one finds that Noether's theorem in superspace does not give rise to the scalar supercurrent superfields J and $J_{ij,k\ell}$. As for the unextended supersymmetry, Noether's theorem in superspace gives rise to vector indexed supercurrent superfields $J_\mu{}^i{}_j$ with a clear geometrical interpretation [29]. These supercurrent superfields $J_\mu{}^i{}_j$ are simply related to the scalar supercurrent superfields J and $J_{ij,k\ell}$, and of course have the same component content.

5.1 *The N=1 Supercurrent in Superspace*

For the N=1 superconformal supercurrent multiplet, the lowest dimension component is the axial vector current $j^{(5)}_\mu$. One then looks for the gauge invariant superfield constructed from $\mathcal{W}_\alpha$ and $\overline{\mathcal{W}}_{\dot\alpha}$ that has this as the $\theta = \overline{\theta} = 0$ component. This is

$$J_{\alpha\dot\alpha} = \mathrm{tr}\,(\mathcal{W}_\alpha \overline{\mathcal{W}}_{\dot\alpha})$$

$$= \mathrm{tr}\,(W_\alpha\, e^{-V}\, \overline{W}_{\dot\alpha}\, e^{V}) \qquad (5.1.1)$$

where the second form is in the gauge where $U = 0$.

From (4.2.41) and (4.2.32) one can write down the expansion of $J_{\alpha\dot\alpha}$ with V in the Wess-Zumino gauge as

$$J_\mu = \overline{\sigma}^{\dot\alpha\alpha}_\mu J_{\alpha\dot\alpha}$$

$$= 64\,(- j^{(5)}_\mu + i\,\theta^\alpha J_{\mu\alpha} - i\,\overline{\theta}_{\dot\alpha}\,\overline{J}^{\dot\alpha}_\mu + 2\,\theta^\alpha \sigma^\nu_{\alpha\dot\alpha}\,\overline{\theta}^{\dot\alpha}\,\Theta_{\mu\nu}$$

$$- \tfrac{1}{2}\,\varepsilon_{\mu\nu\rho\lambda}\,\theta^\alpha \sigma^\nu_{\alpha\dot\alpha}\,\overline{\theta}^{\dot\alpha}\,\partial^\rho j^{(5)\lambda} + \tfrac{i}{2}\,\theta^2\,\overline{\theta}_{\dot\alpha}\,\overline{\sigma}^{\dot\alpha\alpha}_\nu\,\partial^\nu J_{\mu\alpha}$$

$$+ \frac{i}{2}\bar{\theta}^2 \theta^\alpha \sigma^\nu_{\alpha\dot{\alpha}} \partial_\nu \bar{J}^{\dot{\alpha}}_\mu - \frac{1}{4} \Box j^{(5)}_\mu \theta^2 \bar{\theta}^2) \qquad (5.1.2)$$

where the currents are

$$j^{(5)}_\mu = \mathrm{tr}\,(\bar{\lambda}_{\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\alpha}_\mu \lambda_\alpha)$$

$$J^\mu_\alpha = \mathrm{tr}\,(i\, \sigma^{\nu\rho}{}_\alpha{}^\beta \sigma^\mu_{\beta\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} F_{\nu\rho})$$

$$\Theta^{\mu\nu} = \mathrm{tr}\,(F^{\mu\rho} F_\rho{}^\nu + \frac{1}{4} g^{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} - \frac{i}{4} \bar{\lambda}_{\dot{\alpha}} \bar{\sigma}^{(\mu\dot{\alpha}\alpha} \overleftrightarrow{\mathcal{D}}^{\nu)} \lambda_\alpha$$

$$+ \frac{i}{8} g^{\mu\nu} \bar{\lambda}_{\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\alpha}_\rho \overleftrightarrow{\mathcal{D}}^\rho \lambda_\alpha) \qquad (5.1.3)$$

From the equations of motion

$$\mathcal{D}^\alpha W_\alpha = 0 = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \qquad (5.1.4)$$

and the Bianchi identities (4.2.6), the superconformal supercurrent satisfies

$$D^\alpha J_{\alpha\dot{\alpha}} = 0 = \bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} \qquad (5.1.5)$$

The geometrical interpretation of this superfield is obvious. It satisfies

$$\partial^\mu J_\mu(z) = 0 \qquad (5.1.6)$$

and so the supercharge

$$Q = \int d^3x\, J_0 \qquad (5.1.7)$$

contains the charges for the theory

$$Q = 64\,(- Q^{(5)} + i\, \theta^\alpha Q_\alpha - i\, \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}} + 2\, \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} P_\mu) \qquad (5.1.8)$$

However, the supercurrent J_μ has not been obtained directly from the superconformal symmetry of the theory. To do this one uses the generalization of Noether's theorem to superspace [27, 28]

Noether's theorem in superspace is obtained by equating the variation of the local Lagrangian density in superspace obtained in two different ways. The first variation is obtained by using the symmetry transformations of the superfields that make up L, and then using the equations of motion to write this in the form

$$\delta_1 L = \partial_\mu u^\mu + D^\alpha u_\alpha + \overline{D}_{\dot{\alpha}} \overline{u}^{\dot{\alpha}} \qquad (5.1.9)$$

The second variation is the transformation of L itself. For most symmetries this will be zero, but it may also be the equivalent of a total derivative in superspace

$$\delta_2 L = -\partial^\mu v_\mu - D^\alpha v_\alpha - \overline{D}_{\dot{\alpha}} \overline{v}^{\dot{\alpha}} \qquad (5.1.10)$$

So equating these two forms of the variation one obtains Noether's theorem in superspace which says that the current (j_μ , j_α , $\overline{j}^{\dot{\alpha}}$) satisfies the generalized conservation equation

$$\partial_\mu j^\mu + D^\alpha j_\alpha + \overline{D}_{\dot{\alpha}} \overline{j}^{\dot{\alpha}} = 0 \qquad (5.1.11)$$

using the equations of motion, where

$$j_\mu = u_\mu + v_\mu$$

$$j_\alpha = u_\alpha + v_\alpha$$

$$\overline{j}^{\dot{\alpha}} = \overline{u}^{\dot{\alpha}} + \overline{v}^{\dot{\alpha}} \qquad (5.1.12)$$

To see how this works for a specific example, consider the N=1 supersymmetric abelian gauge theory under a superconformal transformation. The local Lagrangian density in superspace is from (4.2.42)

$$L = L_1 + L_2$$

where

$$L_1 = \frac{1}{256} W^\alpha W_\alpha \delta(\bar{\theta})$$

$$L_2 = L_1^\dagger \qquad (5.1.13)$$

The delta function over the $\bar{\theta}$ variables is normalized so that

$$\delta(\bar{\theta}) = \bar{\theta}^2 \qquad (5.1.14)$$

The field strength W_α is now expressed in terms of the unconstrained superfield V

$$W_\alpha = \bar{D}^2 D_\alpha V \qquad (5.1.15)$$

One can now write down the variation of L using the variation of the superfield V under a superconformal transformation

$$\delta L = \frac{1}{128} W^\alpha \bar{D}^2 D_\alpha \delta V \delta(\bar{\theta}) + \text{h.c.} \qquad (5.1.16)$$

The equations of motion (5.1.4) and the Bianchi identities (4.2.6) can now be used to express the variation in the desired form

$$\delta L = D^\alpha u_\alpha + \bar{D}_{\dot{\alpha}} \bar{u}^{\dot{\alpha}} \qquad (5.1.17)$$

where

$$\bar{u}^{\dot{\alpha}} = -\frac{1}{128} (W^\alpha \bar{D}^{\dot{\alpha}} D_\alpha \delta V \delta(\bar{\theta}) + 2 W^\alpha D_\alpha \delta V \bar{\theta}^{\dot{\alpha}} + 4 \bar{W}^{\dot{\alpha}} \delta V) \qquad (5.1.18)$$

From appendix B, the variation δV under a superconformal transformation for the dimension zero superfield V is given by

$$\delta V(z) = f^\mu(z) \partial_\mu V(z) + \xi^\alpha(z) D_\alpha V(z) + \bar{\xi}_{\dot{\alpha}}(z) \bar{D}^{\dot{\alpha}} V(z) \qquad (5.1.19)$$

where the superconformal parameters are discussed in appendix A.

For the second form of the variation, the explicit transformation of the superfield L under the superconformal group from appendix B is used

$$\delta L(z) = \partial^{\mu} (f_{\mu}(z) L(z)) + D^{\alpha} (\xi_{\alpha}(z) L_1(z)) + \overline{D}_{\dot\alpha} (\overline{\xi}^{\dot\alpha}(z) L_2(z))$$

$$= - D^{\alpha} v_{\alpha} - \overline{D}_{\dot\alpha} \overline{v}^{\dot\alpha} \qquad (5.1.20)$$

where

$$\overline{v}^{\dot\alpha} = \frac{1}{4} i f^{\mu}(z) \overline{\sigma}_{\mu}^{\dot\alpha\alpha} D_{\alpha} L(z) + \overline{\xi}^{\dot\alpha}(z) (L(z) + L_1(z)) \qquad (5.1.21)$$

In obtaining this, the properties of the superconformal parameters (A.7) and (A.9) have been used.

Equating the two forms of the variation, Noether's theorem states that the current (j'_{μ} , j'_{α} , $\overline{j}^{\dot\alpha\prime}$) where

$$j'_{\mu} = 0$$

$$j'_{\alpha} = u_{\alpha} + v_{\alpha}$$

$$\overline{j}^{\dot\alpha\prime} = \overline{u}^{\dot\alpha} + \overline{v}^{\dot\alpha} \qquad (5.1.22)$$

satisfies the generalized conservation equation

$$\partial^{\mu} j'_{\mu} + D^{\alpha} j'_{\alpha} + \overline{D}_{\dot\alpha} \overline{j}^{\dot\alpha\prime} = 0 \qquad (5.1.23)$$

As in the case of ordinary gauge theories, the current j'_{α} obtained by Noether's theorem is gauge dependent and needs to be modified to give a gauge invariant current j_{α} which contains as components the modified currents of the theory, ie. a gauge invariant, symmetric and traceless energy-momentum tensor, a pure spin $\frac{3}{2}$ spinor supersymmetry current and a conserved axial current.

Unfortunately finding the correct improvement term is not at all obvious, unlike ordinary gauge theories. Here it is far easier to find a gauge invariant current j_α of the right form which satisfies

$$D^\alpha j_\alpha + \overline{D}_{\dot\alpha} \overline{j}^{\dot\alpha} = 0 \tag{5.1.24}$$

when the equations of motion are used. Once this has been obtained, showing that j'_α becomes j_α when the currents are improved becomes straightforward.

One takes as j_α a combination of terms of the form

$$\begin{pmatrix} \text{superconformal} \\ \text{parameter} \end{pmatrix} \times \begin{pmatrix} \text{powers of} \\ \theta\,,\,\overline{\theta} \end{pmatrix} \times \begin{pmatrix} \text{supercurrent} \\ J_{\alpha\dot\alpha} \end{pmatrix} \tag{5.1.25}$$

that are of the correct dimension.

$$j_\alpha = A\, J_{\alpha\dot\alpha}\, f^{\dot\alpha\beta}\, \theta_\beta + B\, J_{\alpha\dot\alpha}\, \overline{\xi}^{\dot\alpha}\, \theta^2 + C\, J_{\alpha\dot\alpha}\, \overline{\xi}^{\dot\alpha}\, \overline{\theta}^2 + D\, J_{\alpha\dot\alpha}\, \xi^\beta\, \theta_\beta\, \overline{\theta}^{\dot\alpha} \tag{5.1.26}$$

where the index α must be on the supercurrent $J_{\alpha\dot\alpha}$ in order to use the equations of motion in the form (5.1.5).

Imposing the conservation equation (5.1.24) leads to

$$0 = (2iA - 2B)\, J_{\alpha\dot\alpha}\, \overline{\xi}^{\dot\alpha} \theta^\alpha - A\, J_{\alpha\dot\alpha}\, f^{\dot\alpha\alpha} + D\, J_{\alpha\dot\alpha}\, D^\alpha\, \xi^\beta\, \theta_\beta\, \overline{\theta}^{\dot\alpha}$$

$$+ D\, J_{\alpha\dot\alpha}\, \xi^\alpha\, \overline{\theta}^{\dot\alpha} + \text{h.c.} \tag{5.1.27}$$

using the identities (A.7) for the superconformal parameters and (5.1.5).

So

$$D = 0 \quad , \quad A + A^* = 0 \quad , \quad iA - B = 0 \tag{5.1.28}$$

One can now write (5.1.26) as

$$j_\alpha = A\, J_{\alpha\dot\alpha} \left(f^{\dot\alpha\beta}\, \theta_\beta + i\, \bar\xi^{\dot\alpha}\, \theta^2 \right) \qquad (5.1.29)$$

The connection with the currents in ordinary space-time is as follows. Defining

$$j_\mu(x) = \int d\theta\, d\bar\theta\, \tilde{j}_\mu(z) \qquad (5.1.30)$$

where

$$\tilde{j}_\mu(z) = j_\mu(z) + i\, \theta^\alpha\, \sigma_{\mu\alpha\dot\alpha}\, \bar{j}^{\dot\alpha}(z) + i\, \bar\theta_{\dot\alpha}\, \bar\sigma_\mu^{\dot\alpha\alpha}\, j_\alpha(z) \qquad (5.1.31)$$

Then

$$\partial^\mu\, j_\mu(x) = 0 \qquad (5.1.32)$$

follows from

$$\partial^\mu\, j_\mu(z) + D^\alpha\, j_\alpha(z) + \bar{D}_{\dot\alpha}\, \bar{j}^{\dot\alpha}(z) = 0 \qquad (5.1.33)$$

Using the expansions of the superconformal parameters (A.3) and (5.1.2) gives

$$j_\mu(x) = 64\, i\, A \left(\alpha\, j_\mu^{(5)} + \xi^\alpha(x)\, J_{\mu\alpha} + \bar\xi_{\dot\alpha}(x)\, \bar{J}_\mu^{\dot\alpha} - \xi^\nu(x)\, \Theta_{\mu\nu} \right) \qquad (5.1.34)$$

To see that the change from j'_α to j_α corresponds to improving the currents then the component content of the conservation equations must be compared. For j'_α the conservation equation is

$$D^\alpha\, j'_\alpha + \bar{D}_{\dot\alpha}\, \bar{j}^{\dot\alpha\prime} = -\frac{1}{16}\, D^\alpha\, W_\alpha\, \delta V \qquad (5.1.35)$$

where the Bianchi identity (4.2.6b) has been used.

From (5.1.19) this has at the $\theta\bar\theta$ level the gauge dependent term

$$\frac{1}{8}\, \theta^\alpha\, \sigma^\nu_{\alpha\dot\alpha}\, \bar\theta^{\dot\alpha}\, D^\beta\, W_\beta\, \xi^\mu\, \partial_\mu\, A_\nu \qquad (5.1.36)$$

Now looking at the conservation equation for j_α

$$D^{\alpha} j_{\alpha} + \overline{D}_{\dot\alpha} \overline{j}^{\dot\alpha} = A\, D^{\alpha} W_{\alpha} \Big[\overline{W}_{\dot\alpha} (f^{\dot\alpha\beta} \theta_{\beta} + i\, \overline{\xi}^{\dot\alpha} \theta^2) - (\overline{\theta}_{\dot\alpha} f^{\dot\alpha\beta} - i\, \xi^{\beta} \theta^2) W_{\beta} \Big] \qquad (5.1.37)$$

this has at the $\theta\overline{\theta}$ level the component

$$8\, i\, A\, \theta^{\alpha} \sigma^{\nu}_{\alpha\dot\alpha} \overline{\theta}^{\dot\alpha} D^{\beta} W_{\beta} \xi^{\mu} F_{\mu\nu} \qquad (5.1.38)$$

So for $A = -\frac{i}{64}$ (cf. (5.1.34)) then we can see that modifying j'_{α} gives j_{α} in the same way that for an ordinary gauge theory where

$$\Theta'_{\mu\nu} = - F_{\mu\rho} \partial_{\nu} A^{\rho} + \frac{1}{4} g_{\mu\nu} F_{\rho\lambda} F^{\rho\lambda}$$

$$\Theta_{\mu\nu} = F_{\mu\rho} F^{\rho}{}_{\nu} + \frac{1}{4} g_{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} \qquad (5.1.39)$$

then the conservation equations for the two energy-momentum tensors are

$$\partial^{\mu} \Theta'_{\mu\nu} = - \partial^{\mu} F_{\mu\rho} \partial_{\nu} A^{\rho}$$

$$\partial^{\mu} \Theta_{\mu\nu} = \partial^{\mu} F_{\mu\rho} F^{\rho}{}_{\nu} \qquad (5.1.40)$$

5.2 *The N=2 Supercurrent in Superspace*

For the N=2 superconformal supercurrent multiplet, the lowest dimension component is the auxiliary quantity D_1 (3.4.14). Again, one then looks for the gauge invariant superfield constructed from $\mathcal{W}$ and $\overline{\mathcal{W}}$ that has this as $\theta = \overline{\theta} = 0$ component. This is

$$J = \mathrm{tr}\, (\mathcal{W} \overline{\mathcal{W}})$$

$$= \mathrm{tr}\, (e^{-V(x)} W(y_{+})\, e^{V(x)} \overline{W}(y_{-})) \qquad (5.2.1)$$

where the second form is in the gauge $U = 0$. Again, choosing a Wess-Zumino gauge for V, the expansion of J is

$$J = \frac{1}{16} (D_1 - \theta_i^\alpha \chi_{1\alpha}{}^i - \bar{\chi}_{1\dot{\alpha}i} \bar{\theta}^{\dot{\alpha}i} - \frac{1}{2} \theta_i^\alpha t_{1\alpha}{}^\beta \theta_\beta^i - \frac{1}{2} \bar{\theta}_{\dot{\alpha}i} \bar{t}_1{}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}i}$$

$$- \frac{1}{2} \theta_i^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} j^{(5)}_\mu + \frac{1}{2} \theta_i^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}j} T_\mu{}^i{}_j$$

$$+ \frac{i}{2} \theta_i^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} \theta_j^\beta (J_{\mu\beta}^j - \frac{4}{3} \sigma_{\mu\nu\beta}{}^\gamma \partial^\nu \chi_{1\gamma}^j)$$

$$+ \frac{i}{2} \theta_i^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} (\bar{J}_{\mu\dot{\beta}j} + \frac{4}{3} \partial^\nu \bar{\chi}_{1\dot{\gamma}j} \bar{\sigma}_{\mu\nu}{}^{\dot{\gamma}}{}_{\dot{\beta}}) \bar{\theta}^{\dot{\beta}j}$$

$$+ \frac{1}{2} \theta_i^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} \theta_j^\beta \sigma^\nu_{\beta\dot{\beta}} \bar{\theta}^{\dot{\beta}j} (\Theta_{\mu\nu} - \frac{1}{3} (\partial_\mu \partial_\nu - g_{\mu\nu} \Box) D_1)$$

$$- \frac{1}{4} \varepsilon^{\mu\nu\rho\lambda} \theta_i^\alpha \sigma_{\mu\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} \theta_j^\beta \sigma_{\nu\beta\dot{\beta}} \bar{\theta}^{\dot{\beta}k} \partial_\rho T_\lambda{}^j{}_k$$

$$+ i \chi_i^\alpha \partial_\mu t_{1\alpha}{}^\beta \sigma^\mu_{\beta\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} - i \theta_i^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \bar{t}_1{}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\chi}^{\dot{\beta}i}$$

$$+ \text{higher order terms in } \theta , \bar{\theta}) \qquad (5.2.2)$$

from (4.3.39) and

$$V = 2 \theta_i^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} A_\mu + i \theta_i^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} \theta_j^\beta \sigma_{\mu\beta\dot{\beta}} \bar{\lambda}^{\dot{\beta}j} - i \theta_i^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} \lambda_j^\beta \sigma_{\mu\beta\dot{\beta}} \bar{\theta}^{\dot{\beta}j}$$

$$- \theta_i^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} \theta_j^\beta \sigma_{\mu\beta\dot{\beta}} \bar{\theta}^{\dot{\beta}j} [C^* , C] + 4 \chi_i^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} \mathcal{D}_\mu C^*$$

$$+ 4 \bar{\chi}_{\dot{\alpha}i} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \theta_\alpha^i \mathcal{D}^\mu C + \ldots \qquad (5.2.3)$$

using the expansion

$$\text{tr} (e^{-V} W e^V \bar{W}) = \text{tr} (W \bar{W}) - \text{tr} ([V , W] \bar{W})$$

$$- \frac{1}{2} \text{tr} ([V , W] [V , \bar{W}]) - \frac{1}{6} \text{tr} ([V , W] [V , [V , \bar{W}]]) + \ldots$$

$$(5.2.4)$$

For the N=2 theory the equations of motion are

$$\mathcal{D}^{\alpha i} \mathcal{D}^j_\alpha \mathcal{W} = 0 = \bar{\mathcal{D}}^i_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}j} \bar{\mathcal{W}} \qquad (5.2.5)$$

and so the supercurrent J satisfies

$$D^{\alpha i} D^j_\alpha J = 0 = \bar{D}^i_{\dot{\alpha}} \bar{D}^{\dot{\alpha}j} J \qquad (5.2.6)$$

using the Bianchi identities (4.3.5).

However, in this case, the geometrical interpretation of the superfield J is not obvious. This is because the supercurrent J has not been obtained by a geometrical method. To do this, one again turns to Noether's theorem in superspace.

For simplicity, the abelian theory is considered. The local Lagrangian in superspace is

$$L = L_1 + L_2$$

where

$$L_1 = \frac{1}{128} W^2 \delta(\bar{\theta})$$

$$L_2 = L_1^{\dagger} \qquad (5.2.7)$$

The delta function over the $\bar{\theta}$ variables has been normalized so that

$$\delta(\bar{\theta}) = \underline{u} \qquad (5.2.8)$$

The field strength W can now be expressed in terms of an unconstrained prepotential for the abelian theory as in (4.3.46).

The variation of L can now be expressed in terms of the variation of the superfield V_{ij} under a superconformal transformation

$$L = \frac{1}{64} W \bar{D}^4 D^{\alpha i} D^j_{\alpha} \delta V_{ij} \delta(\bar{\theta}) + \text{h.c.} \qquad (5.2.9)$$

Again, the Bianchi identities (4.3.5) and the equations of motion (5.2.5) can be used to express this variation in the form

$$\delta L = D^{\alpha}_i u^i_{\alpha} - \bar{D}_{\dot{\alpha} i} \bar{u}^{\dot{\alpha} i} \qquad (5.2.10)$$

where

$$u^i_\alpha = \frac{1}{64}\Big(\frac{1}{12}\overline{W}\, D_{\alpha j}\, D^{\beta i}\, D^j_\beta\, \overline{D}^k_{\dot\alpha}\, \overline{D}^{\dot\alpha \ell}\, \delta V_{k\ell}\, u + \frac{1}{12}\overline{W}\, D^{\beta i}\, D_{\beta j}\, \overline{D}^k_{\dot\alpha}\, \overline{D}^{\dot\alpha \ell}\, \delta V_{k\ell}\, \chi^j_\alpha$$

$$- \frac{1}{12}\overline{W}\, D^j_\alpha\, \overline{D}^k_{\dot\alpha}\, \overline{D}^{\dot\alpha\ell}\, \delta V_{k\ell}\, \theta^{\beta i}\, \theta_{\beta j} + \frac{1}{4}\overline{W}\, \overline{D}^k_{\dot\alpha}\, \overline{D}^{\dot\alpha\ell}\, \delta V_{k\ell}\, \theta^i_\alpha$$

$$+ W\, D^j_\alpha\, \delta V^i{}_j - D^j_\alpha\, W\, \delta V^i{}_j \Big) \qquad (5.2.11)$$

Here the variation δV_{ij} is more complicated than the N=1 case as V_{ij} is not dimensionless and also has SU(2) indices. It can be worked out from the variation of a general extended superfield under a superconformal transformation as given in appendix B. However, all that is necessary here is that it contains the term

$$\delta V_{ij}(z) = f^\mu(z)\, \partial_\mu\, V_{ij}(z) + \ldots \qquad (5.2.12)$$

as before.

For the second form of the variation, the explicit transformation of the superfield L under the superconformal group is

$$\delta L(z) = \partial^\mu\, (\, f_\mu(z)\, L(z)\,) - D^\alpha_i\, (\, \xi^i_\alpha(z)\, L_1(z)\,) + \overline{D}_{\dot\alpha i}\, (\, \overline{\xi}^{\dot\alpha i}(z)\, L_2(z)\,)$$

$$= - D^\alpha_i\, v^i_\alpha + \overline{D}_{\dot\alpha i}\, \overline{v}^{\dot\alpha i} \qquad (5.2.13)$$

where

$$v^i_\alpha = \frac{i}{4}\, f_{\alpha\dot\alpha}\, \overline{D}^{\dot\alpha i}\, L - \xi^i_\alpha\, (\, L + L_2\,) \qquad (5.2.14)$$

using the properties (A.11) and (A.12) of the superconformal parameters.

So equating these two forms of the variation leads to the current (j'_μ , $j^{i\,\prime}_\alpha$, $\overline{j}^{\dot\alpha i\,\prime}$) where

$$j'_\mu = 0$$

$$j^{i\,\prime}_\alpha = u^i_\alpha + v^i_\alpha$$

$$\bar{j}^{\dot\alpha i\,\prime} = \bar{u}^{\dot\alpha i} + \bar{v}^{\dot\alpha i} \tag{5.2.15}$$

that satisfies the generalized conservation equation

$$\partial^\mu j_\mu{}' + D^\alpha_i j^{i\,\prime}_\alpha - \bar{D}_{\dot\alpha i} \bar{j}^{\dot\alpha i\,\prime} = 0 \tag{5.2.16}$$

The current (j'_μ , $j^{i\,\prime}_\alpha$, $\bar{j}^{\dot\alpha i\,\prime}$) obtained from Noether's theorem in this way is gauge-dependent as before and needs to be modified to give a gauge-invariant current (j_μ , j^i_α , $\bar{j}^{\dot\alpha i}$) which contains as components the modified currents of the theory and also the auxiliary quantities which are also contained in the supercurrent multiplet.

As before, it is easiest to find a gauge-invariant current j^i_α of the right form which satisfies

$$D^\alpha_i j^i_\alpha - \bar{D}_{\dot\alpha i} \bar{j}^{\dot\alpha i} = 0 \tag{5.2.17}$$

when the equations of motion are used.

As the equations of motion (5.2.5) are second order in spinor derivatives, then taking a j^i_α of the form

$$\begin{pmatrix} \text{superconformal} \\ \text{parameter} \end{pmatrix} \times \begin{pmatrix} \text{powers of} \\ \theta \, , \, \bar{\theta} \end{pmatrix} \times \begin{pmatrix} \text{supercurrent} \\ J \end{pmatrix} \tag{5.2.18}$$

will obviously not be sufficient as it will not be possible to use (5.2.6).

So consider the supercurrent

$$J_{\alpha\dot\alpha}{}^i{}_j = [D^i_\alpha , \bar{D}_{\dot\alpha j}] \, J \tag{5.2.19}$$

which satisfies the simple equation

$$D^{\alpha}_{i} J_{\alpha\dot{\alpha}}{}^{j}{}_{k} = 2\, g_{ik}\, \bar{X}^{j}_{\dot{\alpha}} + \delta^{j}_{k}\, \bar{X}_{\dot{\alpha}i} + 2\, \bar{D}_{\dot{\alpha}k}\, D^{\alpha}_{i}\, D^{j}_{\alpha}\, J \qquad (5.2.20)$$

where $\bar{X}_{\dot{\alpha}i} = -\, 2\, i\, \sigma^{\mu}_{\alpha\dot{\alpha}}\, \partial_{\mu}\, D^{\alpha}_{i}\, J$

so allowing use of the equations of motion in a similar fashion to the unextended case.

Now take a linear combination of terms of the form

$$\begin{pmatrix} \text{superconformal} \\ \text{parameter} \end{pmatrix} \times \begin{pmatrix} \text{powers of} \\ \theta\,,\, \bar{\theta} \end{pmatrix} \times \begin{pmatrix} \text{supercurrent} \\ J_{\alpha\dot{\alpha}}{}^{i}{}_{j} \end{pmatrix} \qquad (5.2.21)$$

of the correct dimension.

$$\begin{aligned} j^{i}_{\alpha} = {} & A_1\, (J_{\alpha\dot{\alpha}}{}^{i}{}_{j} + J_{\alpha\dot{\alpha}j}{}^{i})\, f^{\dot{\alpha}\beta}\, \chi^{k}_{\beta}\, \bar{\theta}_{\dot{\beta}k}\, \bar{\theta}^{\dot{\beta}j} \\ & + A_2\, (J_{\alpha\dot{\alpha}}{}^{j}{}_{k} + J_{\alpha\dot{\alpha}k}{}^{j})\, f^{\dot{\alpha}\beta}\, \chi^{k}_{\beta}\, \bar{\theta}_{\dot{\beta}j}\, \bar{\theta}^{\dot{\beta}i} \\ & + A_3\, J_{\alpha\dot{\alpha}}{}^{j}{}_{k}\, f^{\dot{\alpha}\beta}\, \chi^{i}_{\beta}\, \bar{\theta}_{\dot{\beta}j}\, \bar{\theta}^{\dot{\beta}k} \\ & + A_4\, J_{\alpha\dot{\alpha}}{}^{k}{}_{k}\, f^{\dot{\alpha}\beta}\, \chi^{j}_{\beta}\, \bar{\theta}_{\dot{\beta}j}\, \bar{\theta}^{\dot{\beta}i} \\ & + B_1\, (J_{\alpha\dot{\alpha}}{}^{i}{}_{j} + J_{\alpha\dot{\alpha}j}{}^{i})\, \bar{\theta}^{\dot{\alpha}k}\, \bar{\theta}_{\dot{\beta}k}\, f^{\dot{\beta}\beta}\, \chi^{j}_{\beta} \\ & + B_2\, J_{\alpha\dot{\alpha}}{}^{j}{}_{j}\, \bar{\theta}^{\dot{\alpha}k}\, \bar{\theta}_{\dot{\beta}k}\, f^{\dot{\beta}\beta}\, \chi^{i}_{\beta} \\ & + C_1\, (J_{\alpha\dot{\alpha}}{}^{i}{}_{j} + J_{\alpha\dot{\alpha}j}{}^{i})\, \bar{\xi}^{\dot{\alpha}k}\, u\, \bar{\theta}_{\dot{\beta}k}\, \bar{\theta}^{\dot{\beta}j} \\ & + C_2\, (J_{\alpha\dot{\alpha}}{}^{j}{}_{k} + J_{\alpha\dot{\alpha}k}{}^{j})\, \bar{\xi}^{\dot{\alpha}k}\, u\, \bar{\theta}_{\dot{\beta}j}\, \bar{\theta}^{\dot{\beta}i} \\ & + C_3\, J_{\alpha\dot{\alpha}}{}^{j}{}_{k}\, \bar{\xi}^{\dot{\alpha}i}\, u\, \bar{\theta}_{\dot{\beta}j}\, \bar{\theta}^{\dot{\beta}k} \\ & + C_4\, J_{\alpha\dot{\alpha}}{}^{k}{}_{k}\, \bar{\xi}^{\dot{\alpha}j}\, u\, \bar{\theta}_{\dot{\beta}j}\, \bar{\theta}^{\dot{\beta}i} \\ & + D_1\, (J_{\alpha\dot{\alpha}}{}^{i}{}_{j} + J_{\alpha\dot{\alpha}j}{}^{i})\, \bar{\theta}^{\dot{\alpha}k}\, \bar{\theta}_{\dot{\beta}k}\, u\, \bar{\xi}^{\dot{\beta}j} \\ & + D_2\, J_{\alpha\dot{\alpha}}{}^{k}{}_{k}\, \bar{\theta}^{\dot{\alpha}j}\, \bar{\theta}_{\dot{\beta}j}\, u\, \bar{\xi}^{\dot{\beta}i} \end{aligned} \qquad (5.2.22)$$

Now imposing (5.2.17) leads to the conditions

$$3 A_1 - A_2 + A_3 - 2 A_4 = 0$$

$$3 B_1 - 2 B_2 = 0$$

$$3 C_1 - C_2 + C_3 - 2 C_4 = 0$$

$$3 D_1 - 2 D_2 = 0$$

$$-(A_1 - A_2) = (A_1 - A_2)^*$$

$$A_4 = - A_4^*$$

$$B_2 = B_2^*$$

$$- B_1 = A_1 + A_2 + A_3$$

$$C_1 + 4 i A_2 = 0$$

$$C_2 + 4 i A_1 = 0$$

$$C_3 + 4 i A_3 = 0$$

$$C_4 - 4 i A_4 = 0$$

$$D_1 - 4 i B_1 = 0$$

$$D_2 + 4 i B_2 = 0 \qquad (5.2.23)$$

using the identities (A.11) for the superconformal parameters and (5.2.20).

These simplify to

$$B_i = D_i = 0$$

$$A_3 = - (A_1 + A_2)$$

$$A_4 = A_1 - A_2 \qquad (5.2.24)$$

So

$$j^i_\alpha = A\,(J_{\alpha\dot\alpha}{}^i{}_j + J_{\alpha\dot\alpha j}{}^i)\,f^{\dot\alpha\beta}\,\chi^k_\beta\,\overline\theta_{\dot\beta k}\,\overline\theta^{\dot\beta j}$$
$$+ B\,(J_{\alpha\dot\alpha}{}^j{}_k + J_{\alpha\dot\alpha k}{}^j)\,f^{\dot\alpha\beta}\,\chi^k_\beta\,\overline\theta_{\dot\beta j}\,\overline\theta^{\dot\beta i}$$
$$-(A+B)\,J_{\alpha\dot\alpha}{}^j{}_k\,f^{\dot\alpha\beta}\,\chi^i_\beta\,\overline\theta_{\dot\beta j}\,\overline\theta^{\dot\beta k}$$
$$+(A-B)\,J_{\alpha\dot\alpha}{}^j{}_j\,f^{\dot\alpha\beta}\,\chi^k_\beta\,\overline\theta_{\dot\beta k}\,\overline\theta^{\dot\beta i}$$
$$-4iB\,(J_{\alpha\dot\alpha}{}^i{}_j + J_{\alpha\dot\alpha j}{}^i)\,\overline\xi^{\dot\alpha k}\,u\,\overline\theta_{\dot\beta k}\,\overline\theta^{\dot\beta j}$$
$$-4iA\,(J_{\alpha\dot\alpha}{}^j{}_k + J_{\alpha\dot\alpha k}{}^j)\,\overline\xi^{\dot\alpha k}\,u\,\overline\theta_{\dot\beta j}\,\overline\theta^{\dot\beta i}$$
$$+4i\,(A+B)\,J_{\alpha\dot\alpha}{}^j{}_k\,\overline\xi^{\dot\alpha i}\,u\,\overline\theta_{\dot\beta j}\,\overline\theta^{\dot\beta k}$$
$$+4i\,(A-B)\,J_{\alpha\dot\alpha}{}^k{}_k\,\overline\xi^{\dot\alpha j}\,u\,\overline\theta_{\dot\beta j}\,\overline\theta^{\dot\beta i} \qquad (5.2.25)$$

where

$$-(A-B)^* = (A-B) \qquad (5.2.26)$$

The connection with the ordinary space-time currents is as before. Defining

$$j_\mu(x) = \int d\theta\, d\overline\theta\, \tilde{j}_\mu(z) \qquad (5.2.27)$$

where

$$\tilde{j}_\mu(z) = j_\mu(z) + i\,\theta^\alpha_i\,\sigma_{\mu\alpha\dot\alpha}\,\overline{j}^{\dot\alpha i}(z) - i\,\overline\theta_{\dot\alpha i}\,\overline\sigma^{\dot\alpha\alpha}_\mu\,j^i_\alpha(z) \qquad (5.2.28)$$

Then

$$\partial^\mu j_\mu(x) = 0 \qquad (5.2.29)$$

follows from

$$\partial^\mu j_\mu(z) + D^\alpha_i\, j^i_\alpha(z) - \overline D_{\dot\alpha i}\,\overline{j}^{\dot\alpha i}(z) = 0 \qquad (5.2.30)$$

Using the expansions (A.10) for the superconformal parameters and the expansion of $J_{\alpha\dot\alpha}{}^{i}{}_{j}$ as obtained from (5.2.2)

$$
\begin{aligned}
J_{\mu}{}^{i}{}_{j} &= \bar\sigma^{\dot\alpha\alpha}_{\mu}\left[D^{i}_{\alpha}\,,\,\bar D_{\dot\alpha j}\right]J \\
&= 16\,(\,-2\,T_{\mu}{}^{i}{}_{j} + 2\,\delta^{i}_{j}\,j^{(5)}_{\mu} - 2\,i\,\delta^{i}_{j}\,\theta^{\alpha}_{k}\,J_{\mu\alpha}{}^{k} + 2\,i\,\theta^{\alpha}_{j}\,J_{\mu\alpha}{}^{i} \\
&\quad + \tfrac{8}{3}\,i\,\delta^{i}_{j}\,\theta^{\alpha}_{k}\,\sigma_{\mu\nu\alpha}{}^{\beta}\,\partial^{\nu}\,\chi_{1\beta}{}^{k} + \tfrac{16}{3}\,i\,\theta^{\alpha}_{j}\,\sigma_{\mu\nu\alpha}{}^{\beta}\,\partial^{\nu}\,\chi_{1\beta}{}^{i} \\
&\quad + 2\,i\,\delta^{i}_{j}\,\bar J_{\mu\dot\alpha k}\,\bar\theta^{\dot\alpha k} - 2\,i\,\bar J_{\mu\dot\alpha j}\,\bar\theta^{\dot\alpha i} + \tfrac{8}{3}\,i\,\delta^{i}_{j}\,\partial^{\nu}\,\bar\chi_{1\dot\alpha k}\,\bar\sigma_{\mu\nu}{}^{\dot\alpha}{}_{\dot\beta}\,\bar\theta^{\dot\beta k} \\
&\quad + \tfrac{16}{3}\,i\,\partial^{\nu}\,\bar\chi_{1\dot\alpha j}\,\bar\sigma_{\mu\nu}{}^{\dot\alpha}{}_{\dot\beta}\,\bar\theta^{\dot\beta i} - 4\,\delta^{i}_{j}\,\theta^{\alpha}_{k}\,\sigma^{\nu}_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha k}\,\Theta_{\mu\nu} + 4\,\theta^{\alpha}_{j}\,\sigma^{\nu}_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha i}\,\Theta_{\mu\nu} \\
&\quad + \tfrac{4}{3}\,\delta^{i}_{j}\,\theta^{\alpha}_{k}\,\sigma^{\nu}_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha k}\,(\,\partial_{\mu}\,\partial_{\nu} - g_{\mu\nu}\Box\,)\,D_{1} \\
&\quad + \tfrac{8}{3}\,\theta^{\alpha}_{j}\,\sigma^{\nu}_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha i}\,(\,\partial_{\mu}\,\partial_{\nu} - g_{\mu\nu}\Box\,)\,D_{1} - 2\,\varepsilon_{\mu\nu\rho\lambda}\,\theta^{\alpha}_{i}\,\sigma^{\nu}_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha i}\,\partial^{\rho}\,j^{(5)\lambda} \\
&\quad + 2\,\delta^{i}_{j}\,\varepsilon_{\mu\nu\rho\lambda}\,\theta^{\alpha}_{k}\,\sigma^{\nu}_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha\ell}\,\partial^{\rho}\,T^{\lambda k}{}_{\ell} + 2\,i\,\theta^{\alpha}_{k}\,\sigma^{\nu}_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha i}\,\partial_{\nu}\,T_{\mu}{}^{k}{}_{j} \\
&\quad - 2\,i\,\theta^{\alpha}_{j}\,\sigma^{\nu}_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha k}\,\partial_{\nu}\,T_{\mu}{}^{i}{}_{k} - 4\,i\,\theta^{\alpha}_{j}\,\sigma^{\nu}_{\alpha\dot\alpha}\,\bar\sigma^{\dot\alpha\beta}_{\mu}\,\partial_{\nu}\,t_{\beta}{}^{\gamma}\,\theta^{i}_{\gamma} \\
&\quad - 2\,i\,\delta^{i}_{j}\,\theta^{\alpha}_{k}\,\partial_{\mu}\,t_{\alpha}{}^{\beta}\,\theta^{k}_{\beta} + 4\,i\,\bar\theta_{\dot\alpha j}\,\partial_{\nu}\,\bar t^{\dot\alpha}{}_{\dot\beta}\,\bar\sigma^{\dot\beta\alpha}_{\mu}\,\sigma^{\nu}_{\alpha\dot\gamma}\,\bar\theta^{\dot\gamma i} \\
&\quad + 2\,i\,\delta^{i}_{j}\,\bar\theta_{\dot\alpha k}\,\partial_{\mu}\,\bar t^{\dot\alpha}{}_{\dot\beta}\,\bar\theta^{\dot\beta k} + \text{higher order terms in } \theta\,,\bar\theta\,)
\end{aligned}
\tag{5.2.31}
$$

Then calculating (5.2.27)

$$
\begin{aligned}
j_{\mu}(x) = 9\,.\,16^{2}\,(\,A - B\,)\,i\,(\,&\xi\,j^{(5)}_{\mu} + \xi_{i}{}^{j}\,T_{\mu}{}^{i}{}_{j} + \xi^{\alpha}_{i}(x)\,J_{\mu\alpha}{}^{i} \\
&- \bar\xi_{\dot\alpha i}(x)\,\bar J^{\dot\alpha i}_{\mu} - \xi^{\nu}(x)\,\Theta_{\mu\nu}\,)
\end{aligned}
\tag{5.2.32}
$$

Note that there is no dependence on the arbitary complex constant (A + B) from (5.2.25) and so it can be put equal to zero.

The superfield $J_{\mu}{}^{i}{}_{j}$ has a clear geometrical interpretation in that

$$\partial^{\mu} J_{\mu}{}^{i}{}_{j} = 0 \qquad (5.2.33)$$

and so the supercharge

$$Q^{i}{}_{j} \doteq \int d^3x \, J_0{}^{i}{}_{j} \qquad (5.2.34)$$

contains the charges for the theory

$$Q^{i}{}_{j} = 16 \, (2 \, \delta^{i}_{j} \, Q^{(5)} - 2 \, T^{i}{}_{j} - 2 \, i \, \delta^{i}_{j} \, \theta^{\alpha}_{k} \, Q^{k}_{\alpha} + 2 \, i \, \theta^{\alpha}_{j} \, Q^{i}_{\alpha} + 2 \, i \, \delta^{i}_{j} \, \overline{Q}_{\dot\alpha k} \, \overline{\theta}^{\dot\alpha k}$$
$$- 2 \, i \, \overline{Q}_{\dot\alpha j} \, \overline{\theta}^{\dot\alpha i} - 4 \, \delta^{i}_{j} \, \theta^{\alpha}_{k} \, \sigma^{\mu}_{\alpha\dot\alpha} \, \overline{\theta}^{\dot\alpha k} \, P_{\mu} + 4 \, \theta^{\alpha}_{j} \, \sigma^{\mu}_{\alpha\dot\alpha} \, \overline{\theta}^{\dot\alpha i} \, P_{\mu} \,) \qquad (5.2.35)$$

To see that the supercurrent j^{i}_{α} above does result from improving $j^{i}_{\alpha}{}'$ obtained from Noether's theorem, the components of the conservation equations must be compared.

For $j^{i}_{\alpha}{}'$ the conservation equation is

$$D^{\alpha}_{i} \, j^{i}_{\alpha}{}' - \overline{D}_{\dot\alpha i} \, \overline{j}^{\dot\alpha i}{}' = \frac{1}{32} D^{\alpha i} \, D^{j}_{\alpha} \, W \, \delta V_{ij} \qquad (5.2.36)$$

This has at the $\theta^3 \overline{\theta}^3$ level the term

$$- \frac{1}{96} \chi^{\alpha}_{i} \, \sigma^{\nu}_{\alpha\dot\alpha} \, \overline{\chi}^{\dot\alpha}_{j} \, D^{\beta i} \, D^{j}_{\beta} \, W \, \xi^{\mu} \, \partial_{\mu} A_{\nu} \qquad (5.2.37)$$

using (5.2.12) and (4.3.49).

For j^{i}_{α} the conservation equation is

$$D^{\alpha}_{i} \, (j^{i}_{\alpha} + w^{i}_{\alpha}) - \overline{D}_{\dot\alpha i} \, (\overline{j}^{\dot\alpha i} + \overline{w}^{\dot\alpha i})$$
$$= - 2 \, (A - B) \, \overline{D}_{\dot\alpha k} \, \overline{W} \, D^{\alpha}_{i} \, D^{j}_{\alpha} \, W \, (f^{\dot\alpha\beta} \, \chi^{i}_{\beta} + 4 \, i \, \overline{\xi}^{\dot\alpha i} \, u) \, \overline{\theta}_{\dot\beta j} \, \overline{\theta}^{\dot\beta k}$$
$$- \frac{2}{3} (A - B) \, \overline{D}_{\dot\alpha j} \, \overline{W} \, D^{\alpha}_{i} \, D^{j}_{\alpha} \, W \, (f^{\dot\alpha\beta} \, \chi^{k}_{\beta} + 4 \, i \, \overline{\xi}^{\dot\alpha k} \, u) \, \overline{\theta}_{\dot\beta k} \, \overline{\theta}^{\dot\beta i}$$
$$- \frac{2}{3} (A - B) \, \overline{W} \, D^{\alpha}_{i} \, D^{j}_{\alpha} \, W \Big[(- 8 \, i \, \xi^{\beta}_{j} \, \chi^{k}_{\beta} + 4 \, i \, \overline{D}_{\dot\alpha j} \, \overline{\xi}^{\dot\alpha k} \, u) \, \overline{\theta}_{\dot\beta k} \, \overline{\theta}^{\dot\beta i}$$
$$+ (f^{\dot\alpha\beta} \, \chi^{i}_{\beta} + 4 \, i \, \overline{\xi}^{\dot\alpha i} \, u) \overline{\theta}_{\dot\alpha j} \Big] + \text{h.c.} \qquad (5.2.38)$$

where

$$\bar{w}^{\dot{\alpha}i} = \frac{2}{3}(A - B)\,\bar{W} D^{\alpha}_{j} D^{i}_{\alpha} W\,(f^{\dot{\alpha}\beta}\chi^{k}_{\beta} + 4\,i\,\bar{\xi}^{\dot{\alpha}k}u)\,\bar{\theta}_{\dot{\beta}k}\,\bar{\theta}^{\dot{\beta}j} \tag{5.2.39}$$

which vanishes on-shell.

At the $\theta^3\bar{\theta}^3$ level this has the component

$$-24\,(A - B)\,i\,\chi^{\alpha}_{i}\,\sigma^{\nu}_{\alpha\dot{\alpha}}\,\bar{\chi}^{\dot{\alpha}}_{j}\,D^{\beta i} D^{j}_{\beta} W\,\xi^{\mu} F_{\mu\nu} \tag{5.2.40}$$

Comparing (5.2.37) with (5.2.40), one sees that with $24\,(A - B)\,i = \frac{1}{96}$ (cf. (5.2.32)) that improving the current $j^{i\prime}_{\alpha}$ leads to j^{i}_{α}.

5.3 *The N=4 Supercurrent in Superspace*

For the N=4 supercurrent multiplet, the lowest dimension component is the auxiliary quantity $d_{ij}{}^{k\ell}$ (3.5.4). The gauge invariant superfield constructed from $\mathcal{W}_{ij}$ and $\bar{\mathcal{W}}^{ij}$ that has this as $\theta = \bar{\theta} = 0$ component is

$$J_{ij}{}^{k\ell} = \mathrm{tr}\,(\mathcal{W}_{ij}\,\bar{\mathcal{W}}^{k\ell} - \frac{1}{12}\,\delta^{k}_{[i}\,\delta^{\ell}_{j]}\,\mathcal{W}_{mn}\,\bar{\mathcal{W}}^{mn}) \tag{5.3.1}$$

Again this superfield has no clear geometrical interpretation. Although Noether's theorem for N=4 supersymmetry would need an off-shell treatment of the N=4 supersymmetric gauge theory, it is possible by generalizing the N=2 result to write down a vector indexed superfield that can be used in forming a conserved current $j_{\alpha i}$.

So consider the vector indexed superfield

$$J_{\alpha\dot{\alpha}}{}^{i}{}_{j} = [D^{k}_{\alpha}\, , \overline{D}_{\dot{\alpha}\ell}] J_{jk}{}^{i\ell} \qquad (5.3.2)$$

This has the following expansion

$$J_{\mu}{}^{i}{}_{j} = \overline{\sigma}^{\dot{\alpha}\alpha}_{\mu} J_{\alpha\dot{\alpha}}{}^{i}{}_{j}$$

$$= 16 \Big(\frac{64}{3} T_{\mu j}{}^{i} + \frac{64}{3} i \,(\theta^{\alpha}_{j} J_{\mu\alpha}{}^{i} - \frac{1}{4} \delta^{i}_{j} \theta^{\alpha}_{k} J_{\mu\alpha}{}^{k}) - \frac{64}{3} i \,(\overline{J}_{\mu\dot{\alpha}j} \overline{\theta}^{\dot{\alpha}i}$$

$$- \frac{1}{4} \delta^{i}_{j} \overline{J}_{\mu\dot{\alpha}k} \overline{\theta}^{\dot{\alpha}k}) + \frac{128}{3} (\theta^{\alpha}_{j} \sigma^{\nu}_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha}i} - \frac{1}{4} \delta^{i}_{j} \theta^{\alpha}_{k} \sigma^{\nu}_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha}k}) \Theta_{\mu\nu}$$

$$+ \text{ derivative terms} \Big) \qquad (5.3.3)$$

It satisfies

$$J_{\mu}{}^{i}{}_{i} = 0 \qquad (5.3.4)$$

$$\partial^{\mu} J_{\mu}{}^{i}{}_{j} = 0 \qquad (5.3.5)$$

and so the supercharge

$$Q^{i}{}_{j} = \int d^{3}x\, J_{0}{}^{i}{}_{j} \qquad (5.3.6)$$

contains the charges for the theory

$$Q^{i}{}_{j} = 16 \Big(\frac{64}{3} T_{j}{}^{i} + \frac{64}{3} i \,(\theta^{\alpha}_{j} Q^{i}_{\alpha} - \frac{1}{4} \delta^{i}_{j} \theta^{\alpha}_{k} Q^{k}_{\alpha}) - \frac{64}{3} i \,(\overline{Q}_{\dot{\alpha}j} \overline{\theta}^{\dot{\alpha}i}$$

$$- \frac{1}{4} \delta^{i}_{j} \overline{Q}_{\dot{\alpha}k} \overline{\theta}^{\dot{\alpha}k}) + \frac{128}{3} (\theta^{\alpha}_{j} \sigma^{\mu}_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha}i} - \frac{1}{4} \delta^{i}_{j} \theta^{\alpha}_{k} \sigma^{\mu}_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha}k}) P_{\mu} \Big)$$

$$(5.3.7)$$

$J_{\alpha\dot{\alpha}}{}^{i}{}_{j}$ also satisfies the simple relations

$$D^{\alpha i} J_{\alpha\dot{\alpha}}{}^{j}{}_{i} = 0$$

$$D^{\alpha(i} J_{\alpha\dot{\alpha}}{}^{j)}{}_{k} = 0 \qquad (5.3.8)$$

as can be shown as follows

$$D^{\alpha i} J_{\alpha\dot{\alpha}}{}^{j}{}_{k} = - 4 i \sigma^{\mu}_{\alpha\dot{\alpha}} \partial_{\mu} D^{\alpha\ell} J_{\ell k}{}^{ij} + 2 \overline{D}_{\dot{\alpha}m} D^{\alpha i} D^{\ell}_{\alpha} J_{k\ell}{}^{jm}$$

$$(5.3.9)$$

where

$$D^{\alpha i} D^{\ell}_{\alpha} J_{k\ell}{}^{jm} = \frac{4}{3} D^{\alpha\ell} W_{k\ell} D^{i}_{\alpha} \overline{W}^{jm} \qquad (5.3.10)$$

using the Bianchi identities (4.4.3). As (5.3.10) is antisymmetric in i , j and traceless in i , k from

$$D^{i}_{\alpha} \overline{W}^{jm} = \frac{1}{3} \varepsilon^{ijmn} D^{p}_{\alpha} W_{pn} \qquad (5.3.11)$$

then both terms on the right hand side of (5.3.9) satisfy (5.3.8).

So again taking a linear combination of terms of the form

$$\begin{pmatrix} \text{superconformal} \\ \text{parameter} \end{pmatrix} \times \begin{pmatrix} \text{powers of} \\ \theta\,,\,\overline{\theta} \end{pmatrix} \times \begin{pmatrix} \text{supercurrent} \\ J_{\alpha\dot{\alpha}}{}^{i}{}_{j} \end{pmatrix} \qquad (5.3.12)$$

of the correct dimension

$$\begin{aligned} j_{\alpha i} = {} & A_1 J_{\alpha\dot{\alpha}}{}^{j}{}_{i} f^{\dot{\alpha}\beta} \chi^{k}_{\beta} \underline{B}_{jk} + A_2 J_{\alpha\dot{\alpha}}{}^{j}{}_{k} f^{\dot{\alpha}\beta} \chi^{k}_{\beta} \underline{B}_{ij} \\ & + B_1 J_{\alpha\dot{\alpha}}{}^{j}{}_{i} \underline{A}^{\dot{\alpha}}{}_{\dot{\beta}jk} f^{\dot{\beta}\beta} \chi^{k}_{\beta} + C_1 J_{\alpha\dot{\alpha}}{}^{j}{}_{i} \overline{\xi}^{\dot{\alpha}k} \underline{B}_{jk} u \\ & + C_2 J_{\alpha\dot{\alpha}}{}^{j}{}_{k} \overline{\xi}^{\dot{\alpha}k} \underline{B}_{ij} u + D_1 J_{\alpha\dot{\alpha}}{}^{j}{}_{i} \underline{A}^{\dot{\alpha}}{}_{\dot{\beta}jk} \overline{\xi}^{\dot{\beta}k} u \end{aligned} \qquad (5.3.13)$$

where the products of θ's used here are

$$\begin{aligned} u &= \theta^{1}_{1} \theta^{1}_{2} \theta^{1}_{3} \theta^{1}_{4} \theta^{2}_{1} \theta^{2}_{2} \theta^{2}_{3} \theta^{2}_{4} \\ \chi^{i}_{\alpha} &= \partial^{i}_{\alpha} u \\ \underline{A}^{\dot{\alpha}}{}_{\dot{\beta}ij} &= -\overline{\partial}^{\dot{\alpha}}_{[i} \overline{\partial}_{\dot{\beta}j]} u \\ \underline{B}_{ij} &= \overline{\partial}_{\dot{\alpha}i} \overline{\partial}^{\dot{\alpha}}_{j} \underline{u} \end{aligned} \qquad (5.3.14)$$

Now imposing the conservation equation

$$D^{\alpha i} j_{\alpha i} + \overline{D}_{\dot{\alpha} i} \overline{j}^{\dot{\alpha} i} = 0 \qquad (5.3.15)$$

requires

$$B_1 = 0$$

$$D_1 = 0$$

$$8\,i\,A_1 - C_2 = 0$$

$$8\,i\,A_2 - C_1 = 0$$

$$A_1 - A_2 + B_1^* = 0$$

$$(A_1 + A_2) + (A_1 + A_2)^* = 0$$

$$B_1 + B_1^* = 0 \qquad (5.3.16)$$

using the identities (A.14) satisfied by the superconformal parameters and (5.3.8). Simplifying gives

$$A_1 = A_2 = A$$

$$C_1 = C_2 = 8\,i\,A$$

$$A + A^* = 0 \qquad (5.3.17)$$

and so (5.3.13) can be written as

$$j_{\alpha i} = A\,(\,J_{\alpha\dot\alpha}{}^{j}{}_{i}\,f^{\dot\alpha\beta}\,\chi_\beta^k\,\underline{B}_{jk} + J_{\alpha\dot\alpha}{}^{j}{}_{k}\,f^{\dot\alpha\beta}\,\chi_\beta^k\,\underline{B}_{ij} + 8\,i\,J_{\alpha\dot\alpha}{}^{j}{}_{i}\,\bar\xi^{\dot\alpha k}\,u\,\underline{B}_{jk} + 8\,i\,J_{\alpha\dot\alpha}{}^{j}{}_{k}\,\bar\xi^{\dot\alpha k}\,u\,\underline{B}_{ij}\,) \qquad (5.3.18)$$

Again, defining

$$j_\mu(x) = \int d\theta\, d\bar\theta\, \tilde{j}_\mu(z) \qquad (5.3.19)$$

where

$$\tilde{j}_\mu(z) = j_\mu(z) - i\,\bar\theta^i_{\dot\alpha}\,\bar\sigma_\mu^{\dot\alpha\alpha}\,j_{\alpha i} + i\,\bar{j}^i_{\dot\alpha}\,\bar\sigma_\mu^{\dot\alpha\alpha}\,\theta_{\alpha i} \qquad (5.3.20)$$

gives the ordinary space-time currents as before. It satisfies

$$\partial^{\mu} j_{\mu}(x) = 0 \qquad (5.3.21)$$

due to the conservation equation

$$\partial^{\mu} j_{\mu}(x) + D^{\alpha i} j_{\alpha i}(z) + \overline{D}_{\dot{\alpha} i} \overline{j}^{\dot{\alpha} i}(z) = 0 \qquad (5.3.22)$$

From the expansion (5.3.3) and the expressions (A.13) for the superconformal parameters one finds for (5.3.19) that

$$j_{\mu}(x) = 3 \, . \, 2^{15} \, i \, A \, (\xi_{i}{}^{j} \, T_{\mu j}{}^{i} + \xi_{i}^{\alpha}(x) \, J_{\mu\alpha}{}^{i} + \overline{J}_{\mu\dot{\alpha} i} \, \overline{\xi}^{\dot{\alpha} i}(x) - \xi^{\nu}(x) \, \Theta_{\mu\nu}) \qquad (5.3.23)$$

CHAPTER 6 DISCUSSION

The first main result of this work is the extension of the methods of [27] used to derive the N=1 supercurrent to the case of extended supersymmetry. Again one finds vector indexed supercurrents arise with obvious geometrical interpretations. These superfields make a clear distinction between the conserved currents which give rise to the invariant charges upon integration over 3-space and the auxiliary quantities which occur only as derivatives and not directly. When the 3-space integration is performed to obtain the supercharge, the auxiliary quantities drop out altogether. This has to be compared to the scalar superfield where all the components, auxiliary as well as conserved currents, appear directly in the θ , $\bar{\theta}$ expansion on an equal footing.

The second main result of this work is the study of the off-shell constraints for N=4 supersymmetric Yang-Mills theory. Here it is found that the two minimal relaxations of the constraints of [20 , 21] give rise to off-shell physical fields but it is shown for one of these minimal relaxations that the auxiliary fields now satisfy on-shell type constraints. This leads one to consider relaxing all the constraints except the conventional constraint. Here the self-duality condition can be applied and when it is we can write down a Lagrangian using a linear Lagrange multiplier method.

This Lagrangian propagates fields other than those of the physical multiplet (1 spin 1 , 4 spin $\frac{1}{2}$ and 6 spin 0) and so avoids the counting arguments of [22] which ruled out a quadratic Lagrangian for an off-shell representation of supersymmetry which propagates only the physical fields.

As in the case of CP_n models and the action I_1 of the relaxed hypermultiplet [23] , some of the propagating fields have kinetic terms of the wrong sign. This is a signal that they have to be eliminated by constraints. In the case of the relaxed hypermultiplet the addition of a linear Lagrange multiplier term I_2 eliminates the unwanted fields. However, for the CP_n model and also for the case of the N=4 supersymmetric gauge theory, the additional fields have to be eliminated by a non-linear Lagrange multiplier, as suggested in [24].

How such a non-linear Lagrange multiplier can eliminate the unwanted fields from the Lagrangian is shown in section 4.4 . From the components and supersymmetry transformations of the off-shell superfields W_{ij} , $W_{\mu}{}^{i}{}_{j}$, $X^{ij}_{\alpha\beta}$ and $D_{\mu}{}^{i}{}_{j}$ the components and supersymmetry transformations of the non-linear Lagrange multiplier multiplet can be obtained as the non-linear Lagrange multiplier part of the action has to be invariant under supersymmetry.

A detailed analysis of the Lagrangian now needs to be carried out to find out whether the fields propagated by this Lagrangian are the physical fields alone. The supersymmetrization of the non-linear Lagrange multiplier term of the Lagrangian also has to be studied further to ascertain whether the Lagrange multiplier multiplet is a representation of supersymmetry.

As the impossibility of finding an off-shell representation whose Lagrangian is quadratic also applies to other $N > 3$ supersymmetric theories [41], non-linear Lagrange multiplier terms may also be involved in the resolution of these problems.

APPENDIX A TRANSFORMATIONS IN SUPERSPACE

The infinitesimal superconformal transformations

$$x'^{\mu} = x^{\mu} - \xi^{\mu}(z)$$

$$\theta'^{\alpha}_{i} = \theta^{\alpha}_{i} - \xi^{\alpha}_{i}(z)$$

$$\bar{\theta}'^{\dot{\alpha}i} = \bar{\theta}^{\dot{\alpha}i} - \bar{\xi}^{\dot{\alpha}i}(z) \qquad (A.1)$$

can be obtained from the representation of the superconformal algebra [42]. Defining

$$f^{\mu}(z) = \xi^{\mu}(z) + i\,\xi^{\alpha}_{i}(z)\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}i} - i\,\theta^{\alpha}_{i}\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,\bar{\xi}^{\dot{\alpha}i}(z) \qquad (A.2)$$

then for N=1 supersymmetry the superconformal parameters can be expanded as

$$f^{\mu}(z) = \xi^{\mu}(x) - 2i\,\xi^{\alpha}(x)\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}} + 2i\,\theta^{\alpha}\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,\bar{\xi}^{\dot{\alpha}}(x) + 2\alpha\,\theta^{\alpha}\,\sigma^{\mu}_{\alpha\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}}$$
$$- \frac{1}{4}\,\varepsilon^{\mu\nu\rho\lambda}\,\theta^{\alpha}\,\sigma_{\nu\alpha\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}}\,\xi_{\rho\lambda}(x) + \ldots$$

$$\xi^{\alpha}(z) = \xi^{\alpha}(x) + \theta^{\alpha}\left(\frac{1}{8}\partial^{\mu}\xi_{\mu}(x) + i\alpha\right) - \frac{1}{4}\theta^{\beta}\,\sigma^{\mu\nu}{}_{\beta}{}^{\alpha}\,\xi_{\mu\nu}(x) + \ldots$$

$$(A.3)$$

where

$$\xi_{\mu\nu}(x) = \partial_{\mu}\xi_{\nu}(x) - \partial_{\nu}\xi_{\mu}(x) \qquad (A.4)$$

The infinitesimal transformations of the conformal group are

$$\xi_{\mu}(x) = c_{\mu} + \omega_{\mu\nu}x^{\nu} + \varepsilon x_{\mu} + a_{\mu}x^{2} - 2(a\cdot x)x_{\mu} \qquad (A.5)$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$, whilst the superconformal group has the additional infinitesimal transformations

$$\xi^{\alpha}(x) = \xi^{\alpha} - x^{\nu}(\bar{\eta}\bar{\sigma}_{\nu})^{\alpha} \qquad (A.6)$$

where the parameters c, $\frac{1}{2}\omega$, ε, a, ξ, η and α correspond to the generators P , M , D , K , Q , S and $Q^{(5)}$ of the superconformal group.

The superconformal parameters satisfy the identities

$$D^{\alpha} f^{\dot{\alpha}\beta} = \frac{1}{2} \varepsilon^{\alpha\beta} D_{\gamma} f^{\dot{\alpha}\gamma}$$

$$D_{\alpha} f^{\dot{\alpha}\alpha} = 4 i \bar{\xi}^{\dot{\alpha}}$$

$$D^{\alpha} \bar{\xi}^{\dot{\alpha}} = 0 \qquad (A.7)$$

where

$$f^{\dot{\alpha}\alpha} = -\frac{1}{2} \bar{\sigma}_{\mu}^{\dot{\alpha}\alpha} f^{\mu} \qquad (A.8)$$

From these one can derive the additional identities

$$D^{\alpha} \xi_{\alpha} + \bar{D}_{\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} = -\frac{1}{2} \partial^{\mu} f_{\mu}$$

$$\partial^{\mu} f^{\nu} + \partial^{\nu} f^{\mu} - \frac{1}{2} g^{\mu\nu} \partial^{\rho} f_{\rho} = 0$$

$$\partial_{\mu} \xi^{\alpha} + \frac{1}{4} \partial_{\nu} \xi^{\beta} \sigma^{\nu}_{\beta\dot{\alpha}} \bar{\sigma}_{\mu}^{\dot{\alpha}\alpha} = 0 \qquad (A.9)$$

For N=2 supersymmetry the superconformal parameters can be expanded as

$$f^{\mu}(z) = \xi^{\mu}(x) - 4 i \xi_{i}^{\alpha}(x) \sigma^{\mu}_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} + 4 i \theta_{i}^{\alpha} \sigma^{\mu}_{\alpha\dot{\alpha}} \bar{\xi}^{\dot{\alpha}i}(x)$$

$$+ 3 \theta_{i}^{\alpha} \sigma^{\mu}_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} \xi + 9 \theta_{i}^{\alpha} \sigma^{\mu}_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}j} \xi_{j}{}^{i} - \frac{1}{4} \varepsilon^{\mu\nu\rho\lambda} \theta_{i}^{\alpha} \sigma_{\nu\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} \xi_{\rho\lambda}(x)$$

$$+ \ldots$$

$$\xi_{i}^{\alpha}(z) = \xi_{i}^{\alpha}(x) + \theta_{j}^{\alpha} \left(\frac{1}{16} \partial^{\mu} \xi_{\mu}(x) \delta_{i}^{j} + \frac{3}{4} i \xi \delta_{i}^{j} + \frac{9}{4} i \xi_{i}{}^{j} \right)$$

$$- \frac{1}{8} \theta_{i}^{\beta} \sigma^{\mu\nu}{}_{\beta}{}^{\alpha} \xi_{\mu\nu}(x) + \ldots \qquad (A.10)$$

Now the superconformal parameters satisfy

$$D^{\alpha i} f^{\dot{\alpha}\beta} = \frac{1}{2} \varepsilon^{\alpha\beta} D^{i}_{\gamma} f^{\dot{\alpha}\gamma}$$

$$D^i_\alpha f^{\dot\alpha\alpha} = 8i\,\bar\xi^{\dot\alpha i}$$

$$D^i_\alpha \bar\xi^{\dot\alpha j} = 0 \qquad (A.11)$$

and so

$$D^\alpha_i \xi^i_\alpha - \bar D_{\dot\alpha i}\bar\xi^{\dot\alpha i} = \frac{1}{2}\partial^\mu f_\mu$$

$$\partial^\mu f^\nu + \partial^\nu f^\mu - \frac{1}{2}g^{\mu\nu}\partial^\rho f_\rho = 0$$

$$\partial_\mu \xi^\alpha_i + \frac{1}{4}\partial_\nu \xi^\beta_i \sigma^\nu_{\beta\dot\alpha}\bar\sigma^{\dot\alpha\alpha}_\mu = 0 \qquad (A.12)$$

For N=4 supersymmetry the superconformal parameters are

$$f^\mu(z) = \xi^\mu(x) - 8i\,\xi^\alpha_i(x)\,\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha i} + 8i\,\theta^\alpha_i\sigma^\mu_{\alpha\dot\alpha}\bar\xi^{\dot\alpha i}(x)$$

$$- 30\,\theta^\alpha_j\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha i}\xi_i{}^j - \frac{1}{4}\varepsilon^{\mu\nu\rho\lambda}\theta^\alpha_i\sigma_{\nu\alpha\dot\alpha}\bar\theta^{\dot\alpha i}\xi_{\rho\lambda}(x) + \ldots$$

$$\xi^\alpha_i(z) = \xi^\alpha_i(x) - \frac{15}{4}i\,\theta^\alpha_j\xi_i{}^j + \frac{1}{32}\theta^\alpha_i\partial^\mu\xi_\mu(x) - \frac{1}{16}\theta^\beta_i\sigma^{\mu\nu}{}_\beta{}^\alpha\xi_{\mu\nu}(x)$$

$$+ \ldots \qquad (A.13)$$

and satisfy the identities

$$D^{\alpha i} f^{\dot\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta}D^i_\gamma f^{\dot\alpha\gamma}$$

$$D^i_\alpha f^{\dot\alpha\alpha} = 16i\,\bar\xi^{\dot\alpha i}$$

$$D^i_\alpha\bar\xi^{\dot\alpha j} = 0 \qquad (A.14)$$

From these, the following identities may be derived

$$D^i_\alpha\xi^\alpha_i - \bar D_{\dot\alpha i}\bar\xi^{\dot\alpha i} = \frac{1}{2}\partial^\mu f_\mu$$

$$\partial^\mu f^\nu + \partial^\nu f^\mu - \frac{1}{2}g^{\mu\nu}\partial^\rho f_\rho = 0$$

$$\partial_\mu\xi^\alpha_i + \frac{1}{4}\partial_\nu\xi^\beta_i\sigma^\nu_{\beta\dot\alpha}\bar\sigma^{\dot\alpha\alpha}_\mu = 0 \qquad (A.15)$$

APPENDIX B TRANSFORMATION PROPERTIES OF SUPERFIELDS

The transformation behaviour of superfields under the superconformal group can be expressed in terms of the parameters of the superconformal group.

For an N=1 superfield that is a finite dimensional representation and is also an irreducible representation in its Lorentz indices

$$\delta V(z) = \Big[f^A(z)\, D_A - \tfrac{i}{2}\Delta\, (\, D^A f_A(z)\,) - \tfrac{1}{3}p\,(\, D\,\xi(x) - \overline{D}\,\overline{\xi}(z)\,) + \tfrac{i}{4}\Sigma^{\mu\nu}\,(\, \partial_\mu f_\nu(z) - \partial_\nu f_\mu(z)\,)\Big]\, V(z) \qquad \text{(B.1)}$$

where

$$f^A D_A = f^\mu \partial_\mu + \xi^\alpha D_\alpha + \overline{\xi}_{\dot\alpha}\, \overline{D}^{\dot\alpha}$$

$$D^A f_A = \partial^\mu f_\mu + D\,\xi + \overline{D}\,\overline{\xi} = \tfrac{1}{2}\partial^\mu f_\mu \qquad \text{(B.2)}$$

If d is the scale dimension in momentum space then $\Delta = i\,d$. The integer p is the γ^5 weight and $\Sigma^{\mu\nu}$ is the representation of the Lorentz generators that acts on the Lorentz indices of V.

For a vector superfield (without external indices) then

$$\Sigma^{\mu\nu} = p = 0 \qquad \text{(B.3)}$$

For a superfield with external spinor index α or $\dot\alpha$ then

$$(\,\Sigma^{\mu\nu}\,)_\alpha{}^\beta = -\tfrac{1}{2}(\,\sigma^{\mu\nu}\,)_\alpha{}^\beta\,,\quad (\,\Sigma^{\mu\nu}\,)^{\dot\alpha}{}_{\dot\beta} = \tfrac{1}{2}(\,\overline{\sigma}^{\mu\nu}\,)^{\dot\alpha}{}_{\dot\beta} \qquad \text{(B.4)}$$

For a vector index

$$(\,\Sigma^{\mu\nu}\,)_{\rho\lambda} = -\,i\,(\,\delta^\mu_\rho\,\delta^\nu_\lambda - \delta^\mu_\lambda\,\delta^\nu_\rho\,) \qquad \text{(B.5)}$$

For chiral superfields that satisfy $\overline{D}\,V = 0$ or $D\,V = 0$ then

$$\Sigma^{\mu\nu} \text{ self-dual}, \; \Delta + 2ip = 0$$

$$\Sigma^{\mu\nu} \text{ antiself-dual}, \; \Delta - 2ip = 0 \qquad (B.6)$$

For an N=2 superfield that is a finite dimensional representation and is an irreducible representation in its Lorentz indices

$$\delta V(z) = \Big[f^A(z)\, D_A - \tfrac{i}{2}\Delta\,(D^A f_A(z)) - \tfrac{1}{3} p\,(D\xi + \overline{D}\,\overline{\xi})$$

$$- \tfrac{1}{3} b_i{}^j\,(D^\alpha_j \xi^i_\alpha - \overline{D}_{\dot\alpha j}\,\overline{\xi}^{\dot\alpha i}) \Big] V(z) \qquad (B.7)$$

where now

$$f^A D_A = f^\mu \partial_\mu + \xi^\alpha_i D^i_\alpha - \overline{\xi}_{\dot\alpha i}\,\overline{D}^{\dot\alpha i}$$

$$D^A f_A = \partial^\mu f_\mu - D^\alpha_i \xi^i_\alpha + \overline{D}_{\dot\alpha i}\,\overline{\xi}^{\dot\alpha i} \qquad (B.8)$$

and $b_i{}^j$ is the representation of the SU(2) generators that acts on the SU(2) indices of V. In the case of V being a 3 of SU(2) (as in the prepotential (4.3.49)) then $(b_A)_B{}^C$ is proportional to ε_{ABC} where the indices $A, B, C = 1, 2, 3$ and so that part of δV becomes

$$\delta V_A = -\tfrac{1}{3} q\, \varepsilon_{ABC}\,(D\,\tau_B\,\xi - \overline{D}\,\tau_B\,\overline{\xi})\, V_C \qquad (B.9)$$

where τ_A are the Pauli matrices.

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