Calculations in supersymmetric quantum theories

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CALCULATIONS IN SUPERSYMMETRIC QUANTUM THEORIES

by

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Ros

for companionship and love
along the way toward
seeking only the essential

Contents

ABSTRACT

Supersymmetric models which are based on ordinary quantum-mechanical systems , of the form originally suggested by Witten , are studied . Ground-state energy is chosen as a suitable measure of the extent of spontaneous supersymmetry breaking . Insight into the mechanism by which such breaking can occur is also sought . Several existing methods for estimating ground-state energies are reviewed and compared with the true value , and then some new methods are developed . At first , the canonical Hamiltonian formulation of the models is used , but later the path-integral approach is also considered . In the latter it is shown that the use of a larger family of classical solutions of the equations of motion than customary may be used to improve the normal semiclassical calculation . Analogous solutions exist in CP^{n-1} model field theories . Classical properties of these are discussed , and their potential use in semiclassical calculations similar to the quantum-mechanical ones indicated . Finally we return to supersymmetric quantum mechanics in order to generalise the original structure to cases with many degrees of freedom .

Declaration

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Ph.D. Thesis by Richard Bruce Abbott .

The work for this thesis was carried out at the Department of Mathematical Sciences , University of Durham , Durham , England , between October 1980 and June 1983 . This thesis has not been submitted for any other degree .

The later parts of chapter two are claimed as original . The first part of chapter three was done in collaboration with W.J.Zakrzewski , and all of that chapter is claimed as original , except where otherwise indicated . Chapters four and five are also claimed as original except where otherwise indicated . Where other authors have done work which is similar , or has suggested lines of approach , they have been acknowledged in the text .

The first part of chapter four has been published in Zeitschrift fur Physik C [1] . Much of chapter two , together with the first part of chapter three , is available as Durham University preprints [2,3] . The later sections of chapters three and four are in preparation as a Durham University preprint .

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Chapter 1 - Introduction

Supersymmetric extensions of field theories were proposed a number of years ago [30] , and study of their properties has mushroomed quite recently . There are several reasons for this . Firstly , it provides a way of incorporating fermionic degrees of freedom into bosonic theories in such a manner that both kinds of particle enter on an equal footing . Previously , models had tended to concentrate on just one kind (e.g. ϕ^4 theory and the sine-Gordon model are purely bosonic , whereas the Thirring model or those studied in [14] are purely fermionic) , or else the two sectors were assumed to be only weakly coupled , with the fermions having little effect on the spectrum of the bosonic part . The equality of importance of the two sorts of particle is made particularly manifest in the superspace approach to supersymmetric theories , in which a "normal" (i.e. Lorentzian or Euclidean) space-time is extended by a set of anti-commuting (Grassmannian) coordinates to form superspace . Models are then constructed so as to be manifestly invariant with respect to general coordinate transformations in this superspace . In this picture , supersymmetry may be understood as a global gauge-type invariance including ordinary Lorentz invariance as well as other more general transformations .

The structure of supersymmetry imposes stringent requirements on the bosonic and fermionic parts of the Lagrangian (and on their interaction) , and hence imposes relationships between the various masses and coupling constants of the theory . This

means that in supersymmetric theories we find cancellations of divergences which normally appear in field theories as soon as perturbative calculations are carried out beyond tree level - here infinities arising from bosonic and fermionic determinants cancel exactly , leaving in many cases an essentially finite theory (all divergences may be absorbed into wave-function renormalisation) [39] . Recent work has shown that this sort of cancellation , which was at first thought to be accidental , can in fact be traced to a single root and thus all of the cancellations found so far may be systematically understood . This is fortunate in some ways , but we note the following . First , the construction is carried out order by order in perturbation theory , so potentially harmful nonperturbative aspects are neglected . Secondly the arguments apply only to models with global supersymmetry , i.e. a single global choice may be used to fix the gauge . Models with local supersymmetric gauge freedom (i.e. supergravity models) are not included in this systematic scheme and so are now expected (despite earlier hopes) to exhibit the usual divergences , which can first appear in 3-loop calculations [41] .

The above point is related to the following . A perennial stumbling-block in the construction of grand unified theories (i.e. theories seeking to unify gravitation with the other known interactions) is the so-called hierarchy problem [42] . There are many ways of expressing this problem , one of which is this . The characteristic energy-scale of gravity is the Planck scale $\sim 10^{19}$ GeV and this is presumably a lower bound on energies at which an unbroken supersymmetric model would be of direct relevance . The characteristic energy-scales of

theories which seem to be of immediate relevance are very much lower , say $\sim 10^{3\to 7}$ GeV . If the construction of fundamental supersymmetric theories is to be of any value at all , there must be essentially no new physics between these energy-scales . This in itself seems to be a major and unrealistic assumption , but supposing it to be true we still have to account for the huge ratio between the scales . In traditional models where renormalisation effects are controlled only by explicit cancellation , the (relatively) extremely small energies of "ordinary" processes requires an extraordinarily fine tuning of the parameters in an apparently ad hoc manner . In supersymmetric models , on the other hand , this tuning can arise naturally - any cancellation which is exact at the 1-loop level remains exact to all orders in perturbation theory and can at most change because of nonperturbative effects . These are expected to be exponentially small , i.e. to be suppressed by factors like $\left(\frac{2\pi M_X}{\alpha_{GUT}}\right)^2 e^{-2\pi/\alpha_{GUT}}$ where M_X is the mass-scale of the vector bosons associated with gravity - for M_X of order $10^{15\to 19}$ GeV and $\alpha_{GUT} \sim \frac{1}{25} \to \frac{1}{20}$ this gives a low-energy scale of the order 10^3 GeV [43] . So this leads naturally to very large ratios . Thus several recent proposals for grand unified theories begin with a fundamental supersymmetric theory at extremely high energies , which is broken by nonperturbative effects at low energies (i.e. laboratory scales or at least not significantly higher) .

A further reason for requiring the effective low-energy theory to be broken , is that in unbroken supersymmetric theories the particles come in multiplets in which bosons and fermions have equal masses . A brief glance at some of

the experimentally determined masses of particles shows this not to be the case . Hence the low-energy form of the theory must somehow break the original supersymmetry in order to break up these supermultiplets and give a more varied spectrum . However , the breaking must not be too severe , or as explained above the original theory would be of no value in understanding laboratory physics . Hence it has been suggested [44] that explicit breaking should not be necessary - the breakdown of supersymmetry should occur spontaneously via dynamical aspects of the model . This means that although the Lagrangian for the model is supersymmetrically invariant , the space of physical states is not . In chapter 2 we shall see this mechanism operating in a particular case .

As a final motivation we wish to mention that regarded as purely abstract objects , spaces which are defined in terms of both normal and anticommuting coordinates have an intrinsic mathematical interest . Questions regarding (for instance) the global existence of a supersymmetric structure over the entire space as well as just locally (thus giving a super-manifold structure which generalises that of the more familiar manifold) seem difficult to answer . As with the physical problem with which we are concerned here , the constraints which supersymmetry imposes have major consequences for the geometric structure of such spaces .

The above discussion raises many questions . Supersymmetric models are inherently complicated , particularly if they are intended as realistic models in four (or more) space-time dimensions . Hence there is a shortage of exact results which include nonperturbative as well as perturbative effects . Thus

many of the properties claimed for supersymmetric models are conjectural , or else the result of a necessarily partial analysis of the problem . It is hard to have an intuitive grasp of theories in which masses and coupling constants are so inter-related , and in which bosons and fermions have an equal priority . Therefore exact results concerning any supersymmetric model are of value . For example , how big can we expect nonperturbative effects to be ? For proposed solutions of the hierarchy problem in grand unified theories to be viable , nonperturbative effects must be suppressed to the extent explained above , but it is clearly impossible as yet to perform exact calculations in order to check that this really does happen .

Spontaneous dynamical breakdown of supersymmetry (or indeed any symmetry) is a little-understood phenomenon . It is possible in many cases to assert that it happens , but details of the mechanisms involved remain obscure . It has in the past been suggested that finite action solutions of the classical equations of motion , representing quantum-mechanical tunnelling between classically separate and degenerate vacua , may trigger this , but as yet this is not clear . Since the mechanism forms an essential part of current models proposed as fundamental theories , it would be of value to understand it better by studying solvable models in which it happens .

The particular framework within which we choose to work uses supersymmetric extensions of ordinary quantum-mechanical problems with a single degree of freedom . Such models were first proposed in [43] as a potentially fruitful area of study , and subsequently

there have been some investigations of the models [e.g.18,29,38] . Being simple models they allow exact results to be found in many cases (by direct integration of equations of motion if necessary , as we are dealing with ordinary rather than partial differential equations) . Nevertheless , they can be viewed as special cases of field-theoretic models and so insight may be gained into other more complicated models . Many techniques and approximations used in field theories become particularly easy to use in quantum-mechanical models , and thus we expect to gain experience of the quality of the various methods . By varying suitable parameters in the models we may investigate either unbroken or spontaneously broken cases , and hence hope to see which differences between them are essential . In particular we will be able to see the effect of spontaneous symmetry breakdown on the energy spectrum . Some of the models considered have instanton (vacuum-tunnelling) solutions at the classical level and so we can attempt to learn the role they play in the theory . In short , there are few aspects of field theories which are not reflected in these models , but in a much simpler and computationally tractable form - numerical comparison of various methods of approximate calculation forms a major part of this thesis .

One important feature of field theory which has no counterpart in quantum-mechanical models is renormalisation . This is unfortunate since the special properties of supersymmetric field theories first manifested themselves here , and the effect of nonperturbative effects on renormalisation remains an open question . In this respect , supersymmetric quantum-mechanical models are no more special than ordinary ones , as none are renormalised . Furthermore , this raises important

questions regarding the nature of the interactions which we choose to include in our quantum-mechanical potential . Should we proceed by direct analogy with field theory , and so use interaction potentials to imitate the bare potentials , or should we attempt to investigate renormalisation effects by imitating effective potentials such as those derived from calculations at the 1-loop level ? In the former case the potential is restricted to being a polynomial of fairly low order , whereas in the latter we should include other sorts of expression such as logarithmic terms . This question is still open ; in the main part of this thesis we choose the first approach , reserving a short discussion about the second method until the concluding chapter . Preliminary investigations indicate that the second choice can produce features which are quite different to behaviour anticipated from the first choice .

The models are studied with the intention of gaining intuition about the nature and extent of breakdown of supersymmetry . For this purpose we must choose some readily calculated measure of this breakdown . For problems in quantum mechanics a suitable choice is the value of the ground-state energy - we will explain in chapter 2 why this is suitable for discriminating between broken and unbroken models . In field theories the analogous quantity would be the intrinsic vacuum energy - on a local scale the absolute scale of energy is usually thought to be unobservable and irrelevant , with only relative scales being of importance . However , supersymmetry forces us to look at the theory globally , and on this (cosmological) scale we can discriminate between situations in which the background energy density is positive or zero . Here again we see the

effects of supersymmetry in mixing both scales and physical properties usually thought to be separable . By looking at excited states of the system we can investigate the energy spectrum of the theory and hope to make inferences about particle spectra in field theories .

There are many ways of estimating ground-state energies in quantum mechanics and in the next two chapters we use several of them . First we use methods which are essentially exact (i.e. known to be accurate to a higher precision than that necessary for comparison) , and then a selection of commonly used approximation techniques . As far as possible we restrict ourselves to methods which have a known or possible analogy in field theory . The methods divide into two main categories , the first being based on the Hamiltonian formulation of the theory , and the second using the Lagrangian path-integral approach . The "exact" methods are included in the former , together with other ways . The second chapter contains several variations on semiclassical calculations which use solutions of the classical equations of motion to approximately evaluate a functional integral . The question of how instanton solutions are to be used arises here . Of the two chapters , the second (involving Lagrangian ideas) is the one for which conclusions may more obviously be generalised to field theory - Hamiltonian techniques are seldom used in field theories except in lattice calculations . In both of these chapters , various quantum-mechanical models other than the supersymmetric ones are used to illustrate certain points which arise , or to see whether certain conclusions are generally applicable . Much of this

material appears in [2,3] although some is new .

There are several immediate generalisations of the work in chapters 2 and 3 , and two such generalisations are then considered . In chapter 4 we consider the CP^{n-1} model field theories , which are models in two space-time dimensions which share many properties with four-dimensional Yang-Mills theories . The models considered here are not supersymmetric . The purpose of this chapter is to show that they also share many properties with the quantum-mechanical models previously discussed . Various features of classical solutions of the CP^{n-1} equations of motion are reviewed , both at the purely classical level and also towards their inclusion in semiclassical calculations . In particular we show that the nature of the various solutions is directly analogous to solutions in the quantum-mechanical models , and that they can play the same role in semiclassical calculations in both sets of models . At this level the degree of similarity is remarkable . The classical properties have been discussed in [1] , but use of the solutions in a semiclassical manner has not appeared elsewhere .

In chapter 5 we return to supersymmetric models . This time we generalise the earlier models , which had only one bosonic degree of freedom , to cases with many such degrees of freedom . Thus if the original problem is thought of as representing fields at a single site propagating in time , we now think of fields at a finite collection of sites propagating , where the sites are coupled together in a specific way by the requirement of supersymmetry for the model . By taking a limit in which the total number of sites tends to infinity we may seek to study

the approach towards a field theory . In particular we shall see that the energy spectrum of the model can be significantly altered by the inclusion of the fermions . The effects of altering the geometry of how the sites are linked together is briefly discussed . Previously , this work has not been written up .

After chapter 6 , which contains concluding remarks , there are a number of appendices containing technical work which we do not wish to include in the text , but which aids the understanding of the material .

Chapter 2 - Quantum-mechanical Models - Hamiltonian Methods

2.1 - Description of models used

These fall into two categories - firstly models which exhibit supersymmetry in either its spontaneously broken or unbroken forms , and secondly a series of supplementary models used purely to amplify particular points arising from the study of the supersymmetric cases . The families of supersymmetric models arise in a natural and systematic manner , as will be explained later . We are concerned only with the question of dynamical breaking of supersymmetry and so do not consider models in which breaking is put in by hand .

The idea of studying supersymmetry within the context of quantum mechanics was first proposed in [43] , using techniques for integrating over fermionic degrees of freedom developed some years earlier [9] . There are many reasons for considering such models rather than ordinary "realistic" supersymmetric field theories in 4 or more space-time dimensions , some of which have been discussed in the introduction . The reasons for , and the mechanism of dynamical symmetry breakdown are not at all well understood , although several necessary conditions have been found - for example general arguments have produced a variety of functional methods [15] . These tend , however , not to lend themselves readily to calculation , and in addition give very little information in the case of greatest interest - i.e. the spontaneously broken cases . It is clear that any supersymmetric theory which is to be useful in practise (i.e. at low energies

relative to any postulated grand unification scales) must be broken in some manner , so techniques which produce useful answers in such cases are essential .

We are seeking , then , a readily-computable method of distinguishing between spontaneously broken and unbroken cases , and for broken cases , assessing the extent to which the theory is broken . Several calculational schemes are available for this purpose , and we review these as well as developing new methods which to an extent overcome the problems which emerge from using existing methods . This is another reason for looking at quantum mechanics rather than field theory - all calculations are much easier and quicker , while still giving indications of how similar calculations might proceed in more complicated models . Thus we can test various common beliefs against readily obtained and accurate answers .

There are two principle ways in which the Hamiltonian or Lagrangian appropriate to supersymmetric quantum mechanics may be derived - firstly by dimensional reduction from a supersymmetric field theory (thinking of quantum mechanics as a 0+1 dimensional field theory , or the field theory of a single point or site) , and secondly by constructing the most general supersymmetric action in superspace . Here we briefly sketch this approach - see appendix 2.1 for more details .

The basic objects needed to construct the theory are the so-called supercharges , operators which generate the supersymmetry algebra . In our case there are two of these ;

2.1.1)
$$\begin{cases} \hat{Q}_1 = \frac{1}{\sqrt{2}} \left(\hat{p}\,\hat{\psi}_1 - v'(\hat{q})\,\hat{\psi}_2 \right) \\ \hat{Q}_2 = \frac{1}{\sqrt{2}} \left(\hat{p}\,\hat{\psi}_2 + v'(\hat{q})\,\hat{\psi}_1 \right) \end{cases}$$

where $\hat{q},\hat{p}$ are bosonic position and momentum operators ,

$\hat{\psi}_{1,2}$ are fermion operators satisfying $\{\hat{\psi}_\alpha, \hat{\psi}_\beta\} = \delta_{\alpha\beta}$,

$v(q)$ is a polynomial in q called the superpotential ,

with prime denoting derivative with respect to q .

Note that except in chapter 3 , for convenience we set $\hbar = 1$, and also take our system to be of unit mass . Also there are differences in the literature as to what is called the superpotential - here we follow the conventions of [29,38] .

It is convenient to define ladder operators

2.1.2)
$$\begin{cases} \hat{\psi}_\pm = \frac{1}{\sqrt{2}} (\hat{\psi}_1 \pm i\hat{\psi}_2) \quad ; \; \hat{\psi}_\pm^2 = 0 \,, \{\hat{\psi}_+, \hat{\psi}_-\} = 1 \\ \hat{Q}_\pm = \frac{1}{\sqrt{2}} (\hat{Q}_1 \pm i\hat{Q}_2) = \left(\hat{p} \pm i v'(\hat{q}) \right) \hat{\psi}_\pm \end{cases}$$

in terms of which

2.1.3)
$$\hat{H} = \tfrac{1}{2} \{\hat{Q}_+, \hat{Q}_-\} \quad ; \; [\hat{Q}_\pm, \hat{H}] = 0$$

where $\hat{H}$ is the Hamiltonian , explicitly given by

2.1.4)
$$\hat{H} = \tfrac{1}{2} \hat{p}^2 + \tfrac{1}{2} v'(\hat{q})^2 + [i v'(\hat{q}), \hat{p}][\hat{\psi}_+, \hat{\psi}_-]$$

The corresponding Lagrangian is

2.1.5)
$$L = \tfrac{1}{2} \dot{q}^2 - \tfrac{1}{2} v'(q)^2 + i\psi_+\dot{\psi}_- + \tfrac{1}{2}[\psi_+, \psi_-] v''(q)$$

At this stage we choose a particular representation for the bosonic and fermionic coordinates , rather than keep using the abstract versions above . In fact the choice is easy - we use the canonical irreducible representations in both cases , viz.

2.1.6)
$$\begin{cases} \hat{q} = q \quad , \quad \hat{p} = -i \dfrac{d}{dq} \\ \hat{\psi}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \hat{\psi}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{cases}$$

- the von Neumann uniqueness theorem and its fermionic analogue [32,12] ensure that these choices are essentially unique . The fermionic choice has several consequences - the supercharges and Hamiltonian become 2x2 matrices , acting on 2-component vector wave-functions (each component being a bosonic function acted on by $\hat{q}$ and $\hat{p}$ as above) and this immediately yields an occupation number interpretation for the fermions . If we take the fermion number-operator to be

2.1.7) $$\hat{f} = \hat{\psi}_- \hat{\psi}_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then we may interpret the upper (lower) component as having fermion number zero (one) . With just one site there is a certain amount of arbitrariness in this choice , but it will become more relevant in chapter 4 when we consider more degrees of freedom . At this point it is sufficient to note that $\hat{f}$ commutes with $\hat{H}$ and so is a constant of motion , and that in this representation $\hat{H}$ is diagonal -

2.1.8) $$\hat{H} = \begin{pmatrix} -\frac{1}{2}\frac{d^2}{dq^2} + \frac{1}{2}V_+(q) & 0 \\ 0 & -\frac{1}{2}\frac{d^2}{dq^2} + \frac{1}{2}V_-(q) \end{pmatrix}$$

where $V_\pm(q) = W'(q)^2 \mp W''(q)$

We also define $V_b(q) = W'^2$ - this is known as the bosonic part of the potential and gives bosonic mass and self-interaction terms , with the other terms in $V_\pm$ giving the boson-fermion interaction .

So it would seem that we have two decoupled problems to solve - however the supersymmetric structure permits a further simplification . Note from (2.1.2) and (2.1.3) that $\hat{H} = \frac{1}{2}(\hat{Q}_1^2 + \hat{Q}_2^2)$ and therefore

1) all eigenstates of $\hat{H}$ have non-negative eigenvalues , and

2) a state has energy zero if and only if it is annihilated by both $\hat{Q}_1$ and $\hat{Q}_2$ (equivalently $\hat{Q}_\pm$) .

Also we see from (2.1.3) that if ψ is an eigenstate of $\hat{H}$ with energy $E>0$, then $\hat{Q}_\pm\psi$ are eigenstates with the same energy , unless they vanish . There are two possibilities -

1) all eigenvalues of H are strictly positive and come in degenerate pairs , linked by supersymmetry transformations

2) H has a single non-degenerate zero eigenvalue for its ground-state ,with all the excited states in pairs as above .

Thus , as mentioned in the introduction , we may use the value of the ground-state energy , E_0 , as a measure of the extent of supersymmetry breaking . As will be seen later , this choice is not without problems , but it does have the advantage of being readily computable in a number of different ways . In field theory , in case (1) the degeneracy is attributed to the presence of a zero-mass fermion , the Goldstone fermion . In this case we may choose the eigenstates to be of the form

2.1.9) $$\begin{pmatrix}\psi_u\\0\end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix}0\\\psi_d\end{pmatrix} = \hat{Q}_- \begin{pmatrix}\psi_u\\0\end{pmatrix}$$

In case (2) the excited states are of this form , and there is also a single state of zero energy , annihilated by both $\hat{Q}_+$ and $\hat{Q}_-$. To find the form of this state we solve the simultaneous equations

2.1.10) $$\hat{Q}_\pm \begin{pmatrix}\psi_u\\\psi_d\end{pmatrix} = 0$$

where from (2.1.2) we see that

2.1.10a) $$\hat{Q}_+ = \begin{pmatrix}0 & -i\frac{d}{dq} + iv'(q)\\ 0 & 0\end{pmatrix} \quad , \quad \hat{Q}_- = \begin{pmatrix}0 & 0\\ -i\frac{d}{dq} - iv'(q) & 0\end{pmatrix}$$

This gives the pair of equations

2.1.11) $$\left(\frac{d}{dq} + v'\right)\psi_u = 0 = \left(\frac{d}{dq} - v'\right)\psi_d$$

whose solution is

2.1.11a)
$$\begin{cases} \psi_u(q) = \psi_u(q_0)\, e^{-\int_{q_0}^{q} v'(q')\,dq'} = \psi_u(q_0)\, e^{-(v(q)-v(q_0))} \\ \psi_d(q) = \psi_d(q_0)\, e^{+(v(q)-v(q_0))} \end{cases}$$

For this wavefunction to be admissible we require it to be normalisable , and this depends on the asymptotic behaviour of the superpotential . In order that we are solving a bound-state problem as opposed to a scattering problem , we require that $|v(q)| \to \infty$ as $|q| \to \infty$ and without loss of generality we also require $v(q) \to +\infty$ as $q \to +\infty$ (the opposite choice simply reverses the roles of the upper and lower components of ψ) . There are then two possibilities -

1) $v(q) \to +\infty$ as $q \to -\infty$. In this case ψ_u of (2.1.11a) is normalisable and so (2.1.10) has a solution . Supersymmetry is unbroken .

2) $v(q) \to -\infty$ as $q \to -\infty$. In this case ψ_u of (2.1.11a) is not normalisable and (2.1.10) has no allowed solution . There is no state of zero energy and supersymmetry is spontaneously broken .

We now notice two things - firstly it is the leading asymptotic behaviour of the superpotential which is solely responsible for the breaking or otherwise of supersymmetry , and secondly the potential V_+ contains all the information necessary for a complete determination of eigenstates and energy levels , irrespective of any possible breaking . So , despite initial appearances , we have only a single eigenvalue equation to investigate -

2.1.12)
$$\left(-\frac{1}{2}\frac{d^2}{dq^2} + \frac{1}{2}V_+(q)\right)\psi(q) = E\,\psi(q)$$

corresponding to the upper component of (2.1.8) . See also [38]

in which it is shown that quantisation schemes using 2-component vectors or Grassmann-type variables are equivalent , although the reduction to a single problem as above is not used there .

We may proceed in a similar way from (2.1.5) to obtain a 2x2 diagonal matrix Lagrangian , which again reduces to a single component -

2.1.13) $$L = \tfrac{1}{2}\dot{q}^2 - \tfrac{1}{2}V_+(q)$$

It is clear that there are two generic forms of the superpotential in which the leading power is odd or even , which correspond to spontaneously broken and unbroken models respectively (with our convention of notation) . This simple choice can be expressed in a number of different languages , e.g. in [33] it corresponds to whether or not the Nicolai map is single-valued . The simplest non-trivial case has $v = \frac{1}{2}\omega q^2$ which corresponds to a supersymmetric harmonic oscillator . This problem is exactly soluble and displays the expected energy-level structure for an unbroken model . In fact $V_b = \omega^2 q^2$, $V_+ = \omega^2 q^2 - \omega$ and $V_- = \omega^2 q^2 + \omega$. Thus the eigenstates for the problems are just the harmonic oscillator eigenstates and we have the following structure :

V_+ problem		V_- problem	
ψ_n	E_n	ψ_n	E_n
$\begin{pmatrix} C_n H_n(\sqrt{\omega}q)e^{-\frac{1}{2}\omega q^2} \\ 0 \end{pmatrix}$	$\omega(n+\frac{1}{2}) - \frac{1}{2}\omega$ $= n\omega$	$\begin{pmatrix} 0 \\ C_n H_n(\sqrt{\omega}q)e^{-\frac{1}{2}\omega q^2} \end{pmatrix}$	$\omega(n+\frac{1}{2}) + \frac{1}{2}\omega$ $= \omega(n+1)$

Note that the lowest level of the V_+ problem is at zero , and that the $(n+1)^{th}$ level here is equal to the n^{th} level of the V_- problem . Also

$$\hat{Q}_+ \begin{pmatrix} \psi_n \\ 0 \end{pmatrix} = 0$$

$$\hat{Q}_- \begin{pmatrix} \psi_n \\ 0 \end{pmatrix} = -iC_n \begin{pmatrix} 0 \\ \left(\frac{d}{dq} + \omega q\right)\left(H_n(\sqrt{\omega}q)e^{-\frac{1}{2}\omega q^2}\right) \end{pmatrix}$$

$$\propto \begin{pmatrix} 0 \\ H_{n-1}(\sqrt{\omega}q)e^{-\frac{1}{2}\omega q^2} \end{pmatrix}$$

$$\propto \begin{pmatrix} 0 \\ \psi_{n-1} \end{pmatrix}$$

and so as claimed earlier the supersymmetry transformations transform pairs of eigenstates of the two problems into one another - solving the single V_+ problem allows us to find both the eigenvalues and the eigenstates of both problems .

Another point , first observed in [43] , is illustrated by this model . We have said earlier that the v'^2 part of V_+ may be termed the bosonic part and the v'' part fermionic . It is often stated that in unbroken supersymmetric theories the bosonic and fermionic determinants cancel - here the bosonic "determinant" is given by $\frac{1}{2}\sqrt{\frac{1}{2}V_b''} = \frac{1}{2}\omega$ and the fermionic by $\frac{1}{2}v'' = \frac{1}{2}\omega$ and clearly the two are equal . For this problem there are no higher-order corrections to this and so this cancellation at tree-level remains exact to all orders .

We do not consider this problem further , but turn instead to models which have higher powers in the superpotential and hence are not exactly soluble . The next two cases are the simplest non-trivial examples of the generic cases , and most of the rest of this chapter is devoted to their study . They are

2.1.14a) $v = \frac{1}{3}\Lambda q^3 + Mq$ - broken

2.1.14b) $v = \frac{1}{4}\Lambda q^4 - \frac{1}{2}Mq^2$ - unbroken

in both of which $\Lambda > 0$ with M arbitrary

We also briefly consider $v = \frac{1}{4}\Lambda q^4 + Mq$, for reasons explained later .

The corresponding potentials are

2.1.15a) $V_b = (\Lambda q^2 + M)^2$, and $V_+ = (\Lambda q^2 + M)^2 - 2\Lambda q$

2.1.15b) $V_b = (Lq^3 - Mq)^2$, and $V_+ = (Lq^3 - Mq)^2 - (3Lq^2 + M)$

Both families of superpotentials have in common the fact that they have two arbitrary parameters (one of which may be scaled out of the classical theory) , and the interplay between these gives rise to a number of interesting effects . For various ranges of the parameters one or both of V_b or V_+ have degenerate minima , giving rise to finite action (instanton) solutions of the Euler-Lagrange classical equations of motion , as shown later .

As mentioned earlier , we also use other quantum-mechanical models from time to time in order to elucidate certain features of the methods used in this thesis . By analogy with (2.1.12) , these all have Hamiltonians of the general form

$$\hat{H} = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{1}{2}V(q)$$

2.1.16a) where 1) $V(q) = (q^2 - f^2)^4$ - the octic oscillator ($f^2 > 0$) , or

2.1.16b) 2) $V(q) = (q^2 - \frac{1}{2}\mu^2)^2$ - the quartic oscillator (μ^2 arbitrary) .

Reasons for the use of these models are given at the appropriate point in the text .

Note that there is a (not essential) difference between the supersymmetric models and the supplementary ones . The latter are purely bosonic models from the outset , whereas the former are properly models containing both bosonic and fermionic degrees of freedom . However , by use of the supersymmetric structure and a particular (but canonical) choice of representation we may reduce these to effective bosonic theories in order to compare results . No information is lost

in so doing , as the full spectrum of the Hamiltonian is implicit in the single equation (2.1.12) and we can readily pass from one formalism to the other . So , although we seem to have discarded the fermionic degrees of freedom , it is our belief that in doing this the structure and consequences of supersymmetry are more clearly revealed .

The remainder of this chapter is in three parts , in which we consider a number of various approaches , all of which work directly with the Hamiltonian or Schrodinger's equation in one form or another , and so may be considered as full quantum-mechanical approaches . In the next chapter we work from the alternative , Lagrangian point of view , using solutions to classical equations of motion to build up a semiclassical picture via instanton calculations and generalisations of these . It will be seen that although the Hamiltonian approaches can be expected to reflect the physical structure more closely (or at least more intuitively) , it is less easy to see how they may be applied to field-theoretic calculations in which semiclassical calculations using functional integrals presently hold sway .

As explained above , we will concentrate always on estimating the value of E_0 , the ground-state energy , as our measure of supersymmetry breakdown .

2.2 - Some existing methods of estimating E_0

We consider three families of models in this section - the

two supersymmetric cases given in (2.1.14) and also the quartic oscillator of (2.1.16) . First we display the various generic types of the potentials involved .

a) Broken supersymmetric case .

Recall that the superpotential is

2.2.1) $$W = \frac{1}{3} L q^3 + M q = \frac{1}{3} (L^{1/3} q)^3 + \mu (L^{1/3} q)$$
$$= \frac{1}{3} Q^3 + \mu Q$$

where we have defined $\mu = M/L^{1/3}$, $Q = L^{1/3} q$

So classically we may scale L out of the problem and are left with a single free parameter μ .

There are two generic forms of the bosonic potential

2.2.2a) $$V_b = W'^2 = (Q^2 + \mu)^2 L^{2/3}$$

according as μ is positive or negative - see fig. 2.1a . Customarily it is said that for $\mu > 0$ the theory is broken perturbatively (since the minimum of the of the potential is strictly positive) , while for $\mu < 0$ it is broken by non-perturbative effects such as instantons - in another language it might be said that the tunneling between the degenerate minima breaks the supersymmetry . In [38] the $\mu < 0$ case is extensively treated , and an estimate for E_0 is derived using instanton techniques which we review in section 3.1 .

If we look at the full potential

2.2.2b) $$V_+ = W'^2 - W'' = ((Q^2 + \mu)^2 - 2Q) L^{2/3}$$

leading to the Schrodinger equation

2.2.2c) $$\left(-\frac{1}{2} \frac{d^2}{dQ^2} + \frac{1}{2} V_+ (Q) \right) \psi = \frac{E}{L^{2/3}} \psi$$

then we see that it too has two generic forms - see fig. 2.1b - however they are separated at $\mu_c = -\frac{3}{4} \cdot 4^{1/3}$, i.e. the parameter-space is partitioned differently . No special

BROKEN

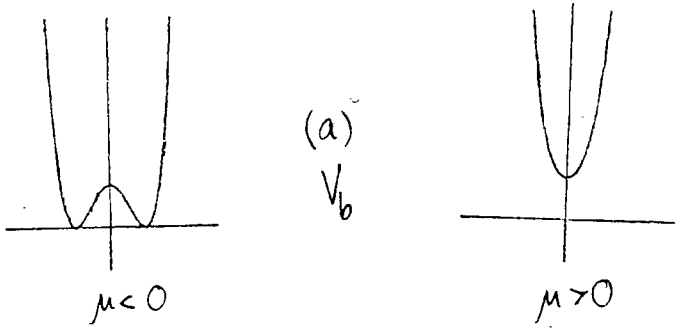


FIG. 2.1

UNBROKEN

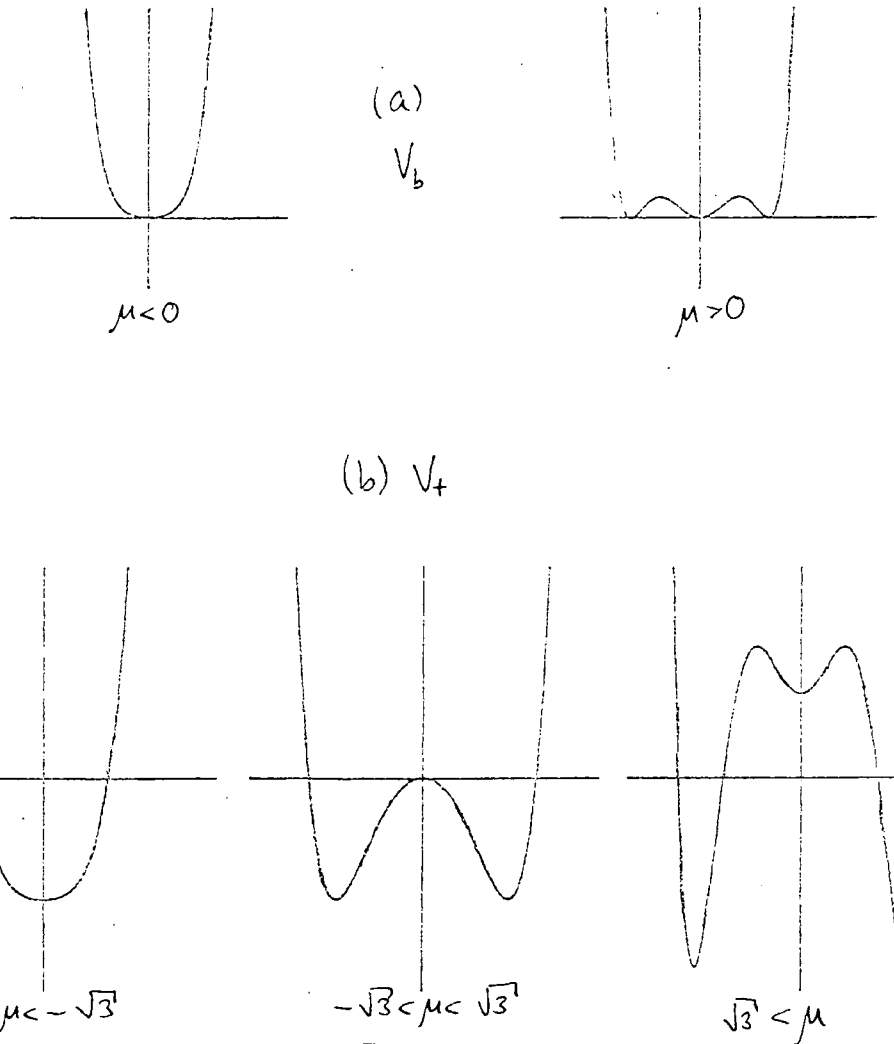


FIG. 2.2

behaviour is observed near $\mu = 0$. The double-well structure of V_b is remembered for $\mu < \mu_c$ but not very exactly - in fact as $\mu \to -\infty$ the wells do not become degenerate but diverge to $\pm\infty$. Note that L merely provides an overall scale - in fact it sets the quantum-mechanical scale - and in the following we fix L = 0.3 for convenience (the small value of L makes the perturbative calculations which follow easier and quicker to converge) .

b) Unbroken supersymmetric case .

Here $$U = \frac{1}{4} L q^4 - \frac{1}{2} M q^2 = \frac{1}{4} (L^{1/4} q)^4 - \frac{1}{2} \mu (L^{1/4} q)^2$$

2.2.3) $$= \frac{1}{4} Q^4 - \frac{1}{2} \mu Q^2$$

where now $\mu = M / L^{1/2}$, $Q = L^{1/4} q$

So

2.2.4a) $$V_b = (Q^3 - \mu Q)^2 \, L^{1/2}$$

which again has two generic forms according as μ is positive or negative (fig. 2.2a) . Since the bottom of the potential is always at zero , it would normally be said that the model is not broken perturbatively - however for $\mu > 0$ there are degenerate minima , hence instantons , and one is led to wonder why they do not break supersymmetry here . This will be considered more fully later .

Note also

2.2.4b) $$V_+ = ((Q^3 - \mu Q)^2 - (3Q^2 - \mu)) \, L^{1/2}$$

giving

2.2.4c) $$\left(-\frac{1}{2} \frac{d^2}{dQ^2} + \frac{1}{2} V_+(Q) \right) \psi = \frac{E}{L^{1/2}} \psi$$

V_+ has three generic forms (fig 2.2b) , with the divisions between them at $\mu = \pm \sqrt{3}$ - again a different partitioning of the parameter-space . Again L just sets the scale and this time we fix L = 1 for convenience .

c) Quartic anharmonic oscillator .

Here

2.2.5a) $$V = \left(q^2 - \tfrac{1}{2}\mu^2\right)^2$$

and so

2.2.5b) $$\left(-\tfrac{1}{2}\frac{d^2}{dq^2} + \tfrac{1}{2}V(q)\right)\psi = E\psi$$

The two generic shapes of V are as in fig. 2.1a .

Before continuing we wish to point out that very little of the rest of this thesis would exist in its present form without the work and help of the operating staff of the NUMAC computing network . Specific references are not made in the text , as they would be too numerous [34] .

We now establish to a reasonable degree of accuracy the true value of E_0 in order to assess the quality of the various estimation schemes - for quantum-mechanical models this is relatively easy , though not , of course , in field theories . It is particularly easy for case (b) - since the supersymmetry remains unbroken we know $E_0 = 0$ for all values of the parameters . For case (c) we refer to [11] in which the value of E_0 as a function of μ^2 is given to rather more accuracy than we need . A plot of these results may be seen in fig. 2.3b .

Case (a) remains . Upper and lower bounds on E_0 were found by a combination of two methods - standard Rayleigh-Ritz methods for upper and an adaptation of this described in [7] for lower bounds . The potential V_+ is split into two parts , a quadratic term and a positive semidefinite remainder . The truncated Hamiltonian (of standard harmonic oscillator form and hence soluble) clearly has eigenvalues less than those of the full

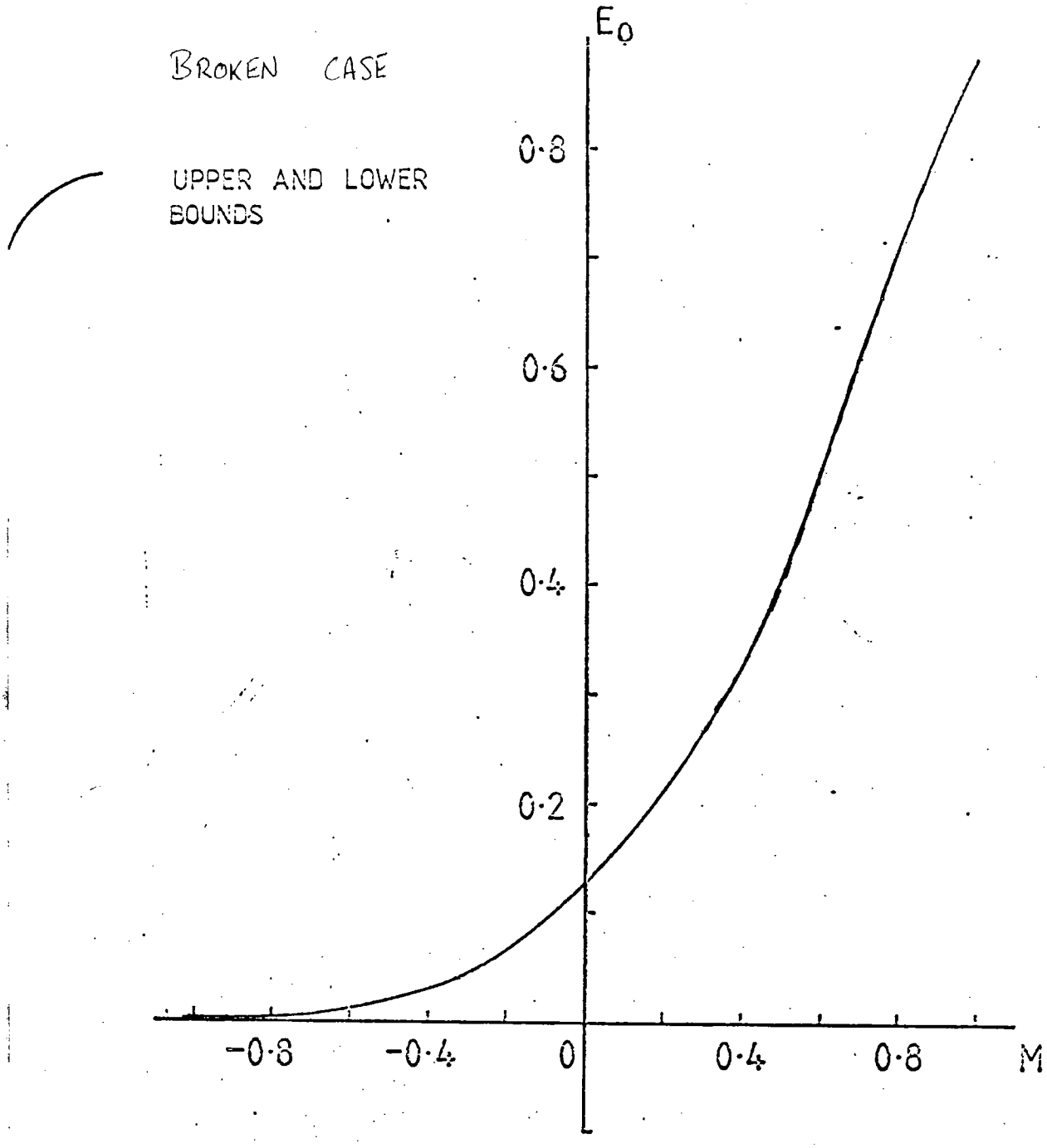


FIG. 2.3a

QUARTIC OSCILLATOR

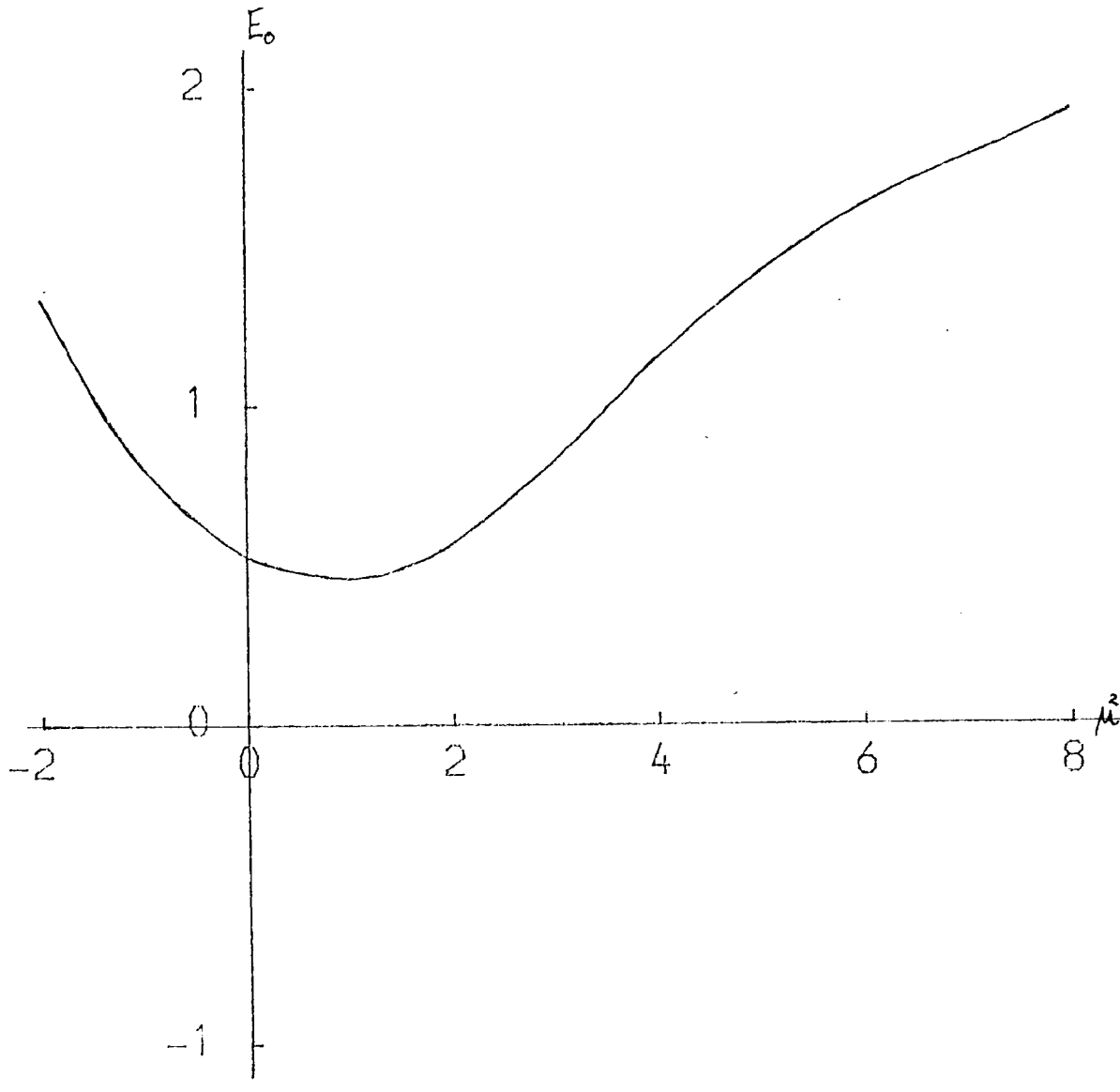


FIG. 2.36

one , and the method of [7] refines values upwards from these lower values towards the true ones , much as the Rayleigh-Ritz method refines values downwards . The trial basis states were chosen to be the eigenstates of the oscillator problem , and by taking sufficiently many of these the bounds may be made arbitrarily tight - we found that taking the first 14 states gave sufficient accuracy (the lower bound accuracy improves more slowly) . Results are shown in fig. 2.3a . It will be seen that E_0 increases smoothly and steadily with μ and at no point is there a sudden transition between different regimes .

For comparison , we have estimated E_0 for case (a) via perturbation theory , by taking the unperturbed Hamiltonian to consist of the kinetic term plus the quadratic part of V_+ , and the perturbation to be the rest of V_+ . This gives the problem

2.2.6) $$\tfrac{1}{2}\left(-\frac{d^2}{dq'^2}+q'^2\right)\psi+\tfrac{1}{2}\left(\lambda q'^3+\mu q'^4\right)\psi = E_0'\psi$$

(q', E_0' are rescaled versions of q, E_0)

We set

2.2.7) $$\begin{cases} \psi(q') = \sum_{ij} B_{ij}(q')\lambda^i\mu^j e^{-q'^2/2} = \sum_{ijk} B_{ijk} q'^k \lambda^i \mu^j e^{-q'^2/2} \\ E_0' = \sum_{ij} A_{ij}\lambda^i\mu^j \end{cases}$$

Following [8] we obtain recurrence relations between the B_{ijk} and the A_{ij} , and use these to calculate any desired number of terms in the series for E_0 - we used up to $i = j = 15$, $k = 105$. As noted in [8] the series for E_0 is asymptotic rather than convergent , so the Padé method of resumming was used to extract physically useful answers . The results are not shown graphically as they are not distinguishable from the exact bounds

shown in fig. 2.3a . This came as a surprise as this method was only expected to give reasonable answers in the $\mu > 0$ region . However it proved to give highly accurate results for all values of μ including regions where perturbative methods are generally thought to be of little value . This is no doubt due to the large number of terms of the original series available for resumming - these higher order terms can probe the long-distance structure of the potential sufficiently well to allow for the secondary minimum .

Another major way of estimating ground-state energies is by some form of quadratic approximation to the potential around its minimum q_0 - i.e. the traditional zero-point energy induced by quantum fluctuations . The usual way of doing this is to evaluate the curvature of the potential at its minimum and then estimate E_0 by $\frac{1}{2}\sqrt{V_+''|_{q_0}}$. Thus higher-order terms in the potential are simply discarded . Results for the three cases are shown in fig. 2.4 compared with the true values of E_0 . Note the cusps in figs. 2.4b and c where the nature of the global minimum changes as shown in figs. 2.2b and 1a .

a) The estimate is good for large positive μ , but its accuracy decreases as μ decreases and becomes negative . Thus the expectation that cases $\mu > 0$ are broken perturbatively does indeed hold true for large μ - however there is a range of small positive values of μ for which some other mechanism must be operating . In chapter 3 we find a similar result , where the nonperturbative effects which are supposed to break supersymmetry for $\mu < 0$ only really work for large negative μ , with a region near zero where

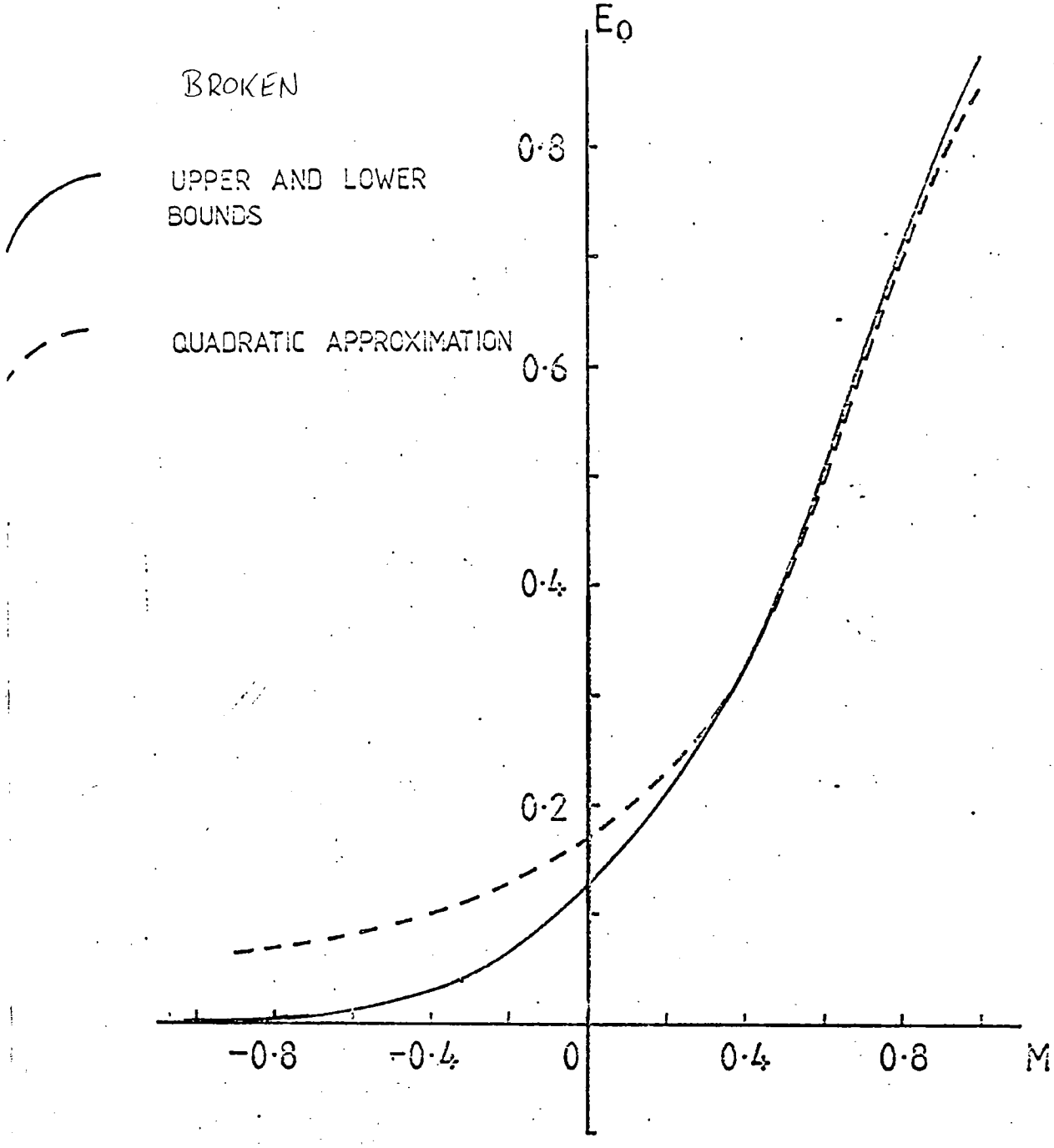

BROKEN
UPPER AND LOWER BOUNDS
QUADRATIC APPROXIMATION
E_0
0·8
0·6
0·4
0·2
−0·8
−0·4
0
0·4
0·8
M

FIG. 2.4a

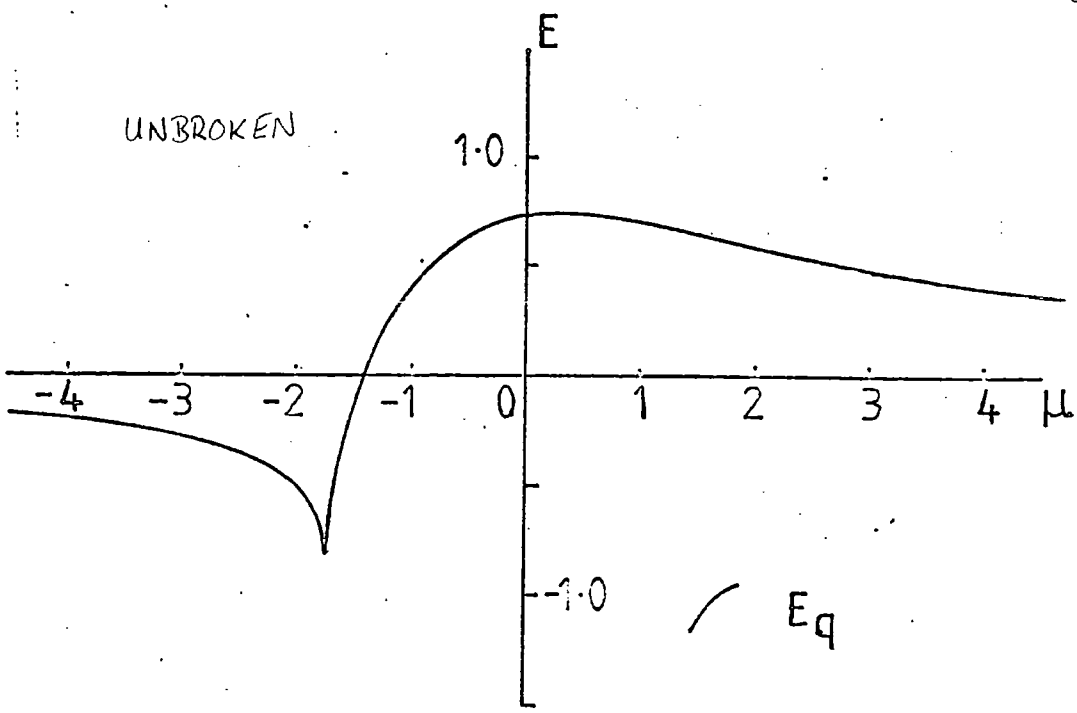


FIG. 2.4b

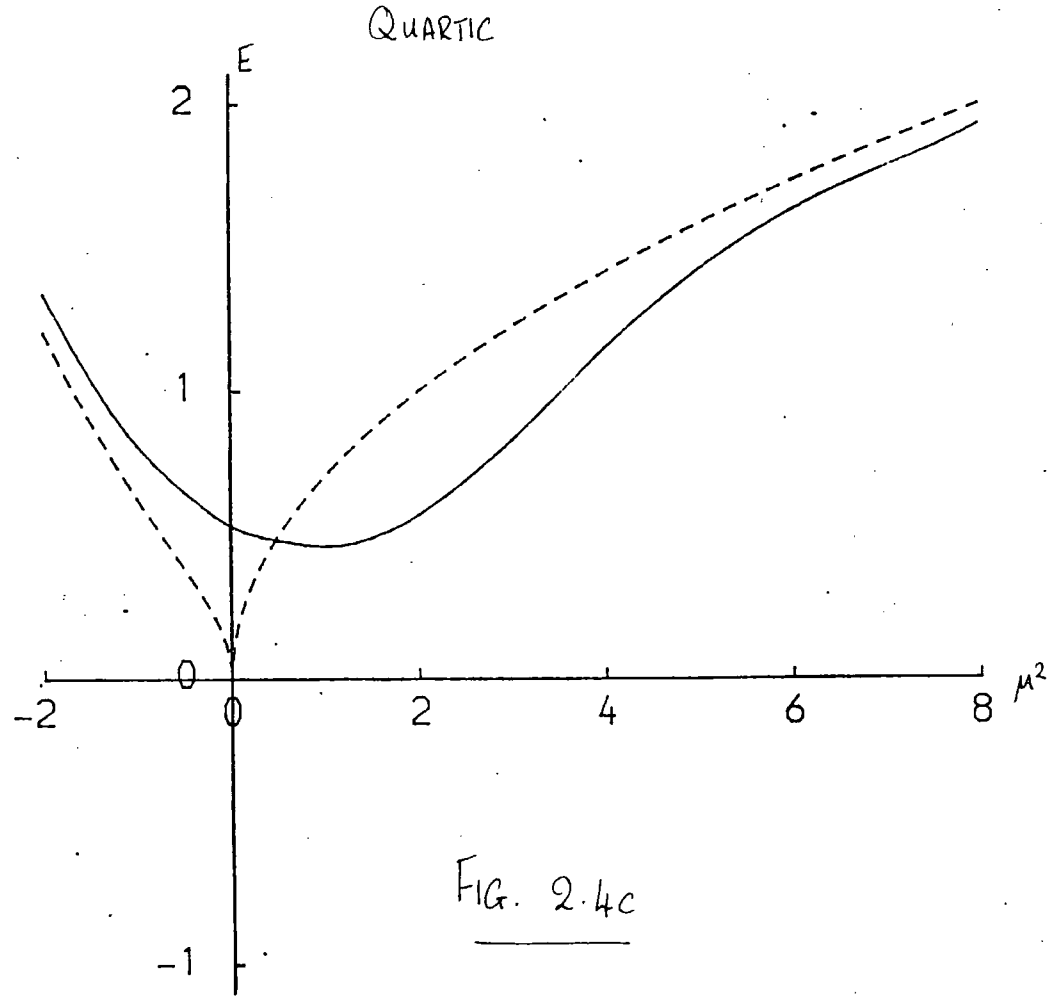


FIG. 2.4c

they fail . Taken together , these results suggest that the usually sharply-drawn distinction between perturbative and nonperturbative effects is too simple - the true breaking mechanism must be a blend of both , with one or the other perhaps dominating in certain asymptotic regions .

b) and c) In both of these cases the estimate is good for both large positive and negative values of the relevant parameter , but poor in the central region around zero . In both cases it is particularly bad at the transitions between the various generic shapes of V_+ at which the curvature V_+'' vanishes and we are left with just the classical contribution V_+^{min} . At such points we are making no allowance for quantum-mechanical effects , which are contained in the discarded part of the potential .

We have tried to get round this problem in the following way - the procedure is analogous but not identical to that described in [36] . For simplicity we only discuss model (c) . Using the above method is equivalent to approximating the true wave-function ψ by a Gaussian $\tilde{\psi}$, whose spread is fixed solely by the curvature at q_0 . Now we retain the Gaussian shape ,

2.2.8) $$\tilde{\psi} \sim e^{-\lambda(q-q_0)^2/2}$$

but allow higher-order terms in the potential to affect our choice of λ . The energy functional is given by

2.2.9) $$E[\tilde{\psi}] = \langle \tilde{\psi}, \hat{H}\tilde{\psi} \rangle \; / \langle \tilde{\psi}, \tilde{\psi} \rangle$$

and is evidently a function of λ (as well as μ^2) . Evaluation of the integrals in (2.2.9) is easy as they are of the form (power.Gaussian) , and gives the answer

2.2.10) $$E(\lambda) = \frac{\lambda}{4} + \frac{\mu^2}{2\lambda} + \frac{3}{8\lambda^2}$$

- it will be seen that $E(\lambda) \to \infty$ as $\lambda \to 0$ and ∞. In fact

FROM (2.2.8)

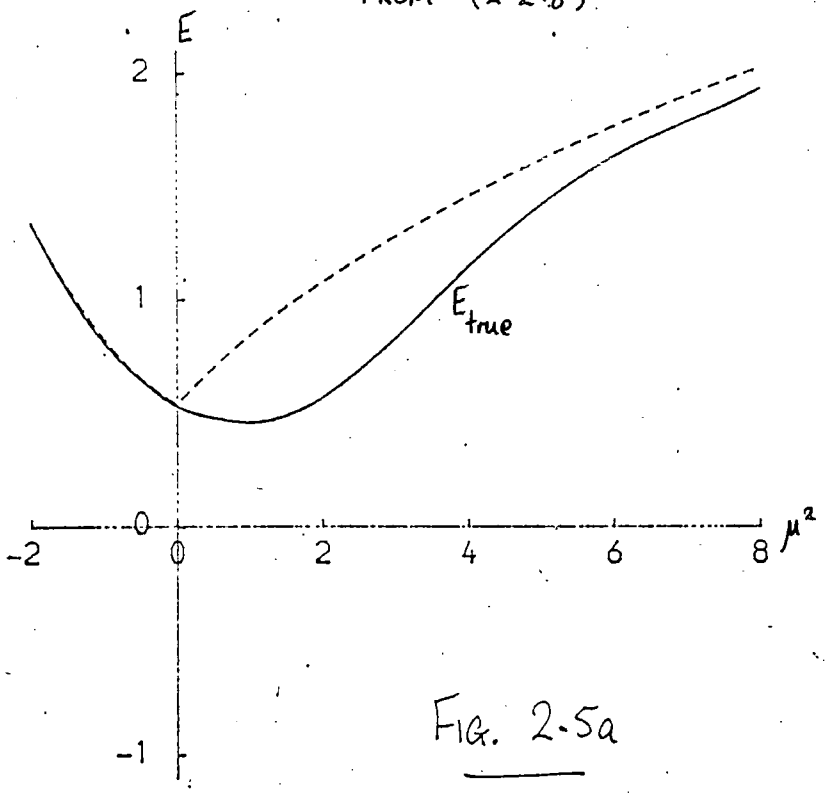


FIG. 2.5a

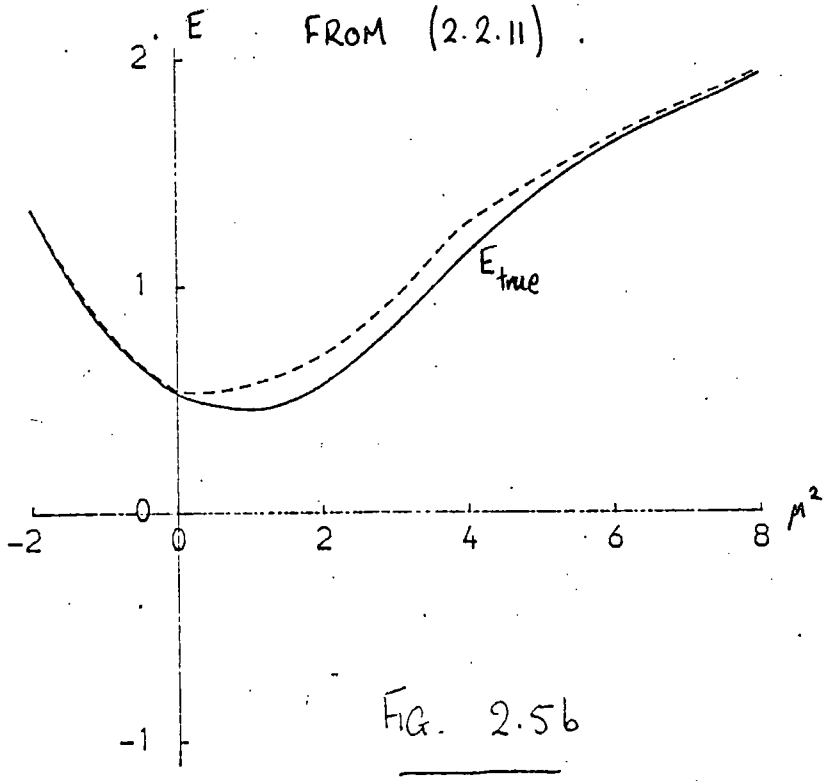


FIG. 2.5b

$E(\lambda)$ has a single minimum at $\tilde{\lambda}$ say , which we choose in our trial $\tilde{\psi}$. Results are shown in fig. 2.5a .

This procedure replaces the original λ (given by just the quadratic or mass term) by an effective value $\tilde{\lambda}$ incorporating some of the higher-order terms in the potential . The usual problem with zero-point energy estimates is that they probe only the short-range structure round the minimum and so neglect all long-range effects , and this procedure is an attempt to redress this . However , it takes into account only even-order terms in the potential as the odd powers vanish when integrated . We may take these odd powers into account in the following way - instead of (2.2.8) we fit the following trial wave-function -

2.2.11) $$\tilde{\psi} \sim \begin{cases} \exp\left[-\lambda_1 (q-q_0)^2/2\right] \ldots\ldots q > q_0 \\ \exp\left[-\lambda_2 (q-q_0)^2/2\right] \ldots\ldots q < q_0 \end{cases}$$

- and now minimise the two energy functionals in the two half-ranges over the parameters $\lambda_{1,2}$. Results are shown in fig. 2.5b . We have now taken notice of the fact (clearly visible in fig. 2.1a) that the potential is less steep to one side of the minimum than the other . Results are surprisingly good - recall that we have not explicitly taken into account the fact that there are two degenerate wells for $\mu^2 > 0$. Figs. 2.5a and b are identical for $\mu^2 < 0$ as the potential is symmetric here . It is our belief that allowing for this asymmetry is one of the most important modifications to the ordinary Gaussian approach - perhaps more important than allowing for higher-order terms . In section 2.3 we shall encounter similar considerations .

2.3 - The mixed wave-function method

The motivation for this method arose in the following way - we wish to develop a method of estimating E_o which in some way blends both perturbative and nonperturbative effects , i.e. both the short and long distance structure of the potential . We also want the method to be usable for all values of the various parameters in the superpotential . A property of the Schrodinger potential V_+ which is independent of specific parameter values is the number of stationary points it has in the complex plane - for the broken models of (2.1.14a) there are three , while for the broken models of (2.1.14b) there are five . The locations and character of these saddles change as the parameters change , but the total number is fixed solely by the asymptotic growth of the superpotential , i.e. its leading power . So the method will use information from all these stationary points , both those on the real axis and those in the complex plane .

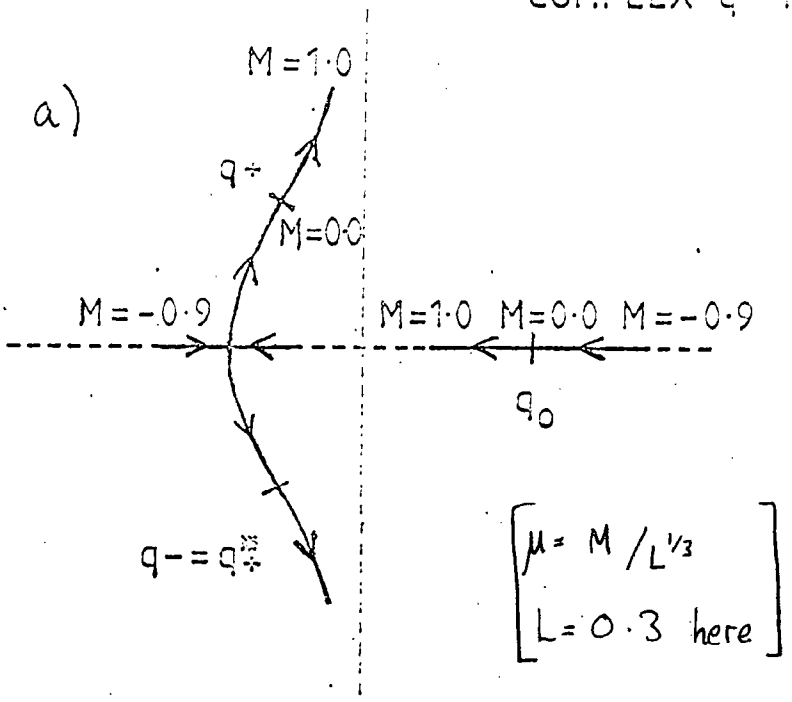
Since we are presently using a Hamiltonian framework we must now use this information to construct wave-functions in a systematic manner . We do this by first noting that around each saddle there exists a characteristic direction along which the Hessian matrix (matrix of second derivatives) is most positive , i.e. the potential is most a minimum . The only exception to this comes when the saddle is degenerate , so the second derivatives vanish - such cases lead to special problems which will be dealt with later (they are analogous to the cusps seen in figs. 2.4b and c) . Wave-functions around each saddle are then constructed by solving in some approximate way the associated Schrodinger equation along this characteristic

BEHAVIOUR OF SADDLES

FIG. 2·6

BROKEN CASE

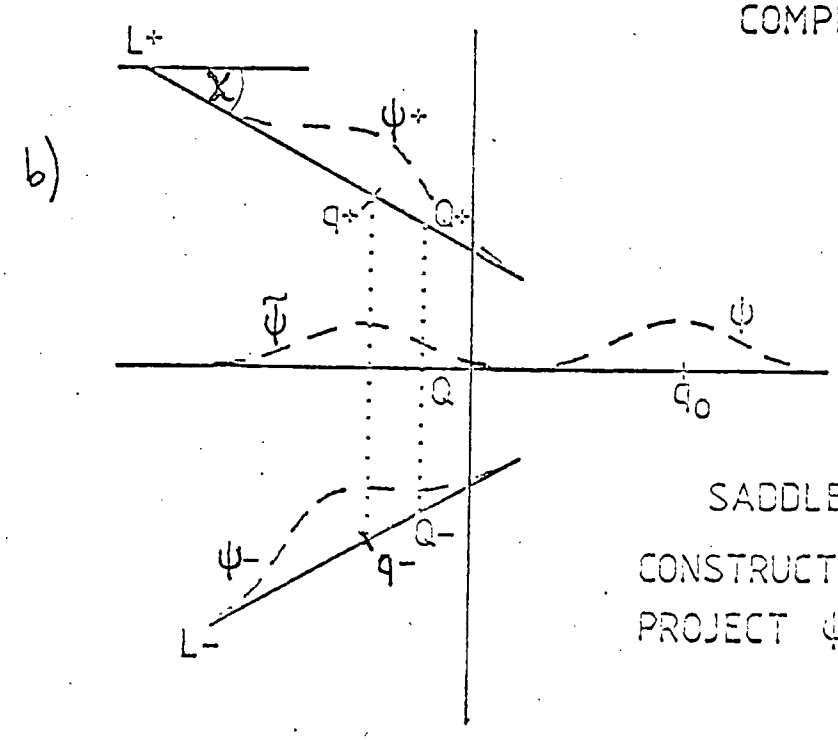
COMPLEX q- PLANE



$\left[\mu = M/L^{1/3}\right.$
$\left. L = 0·3 \text{ here}\right]$

NOTATION

COMPLEX q-PLANE



SADDLES AT q_0, q_+, q_-.

CONSTRUCT WAVEFUNCTIONS ψ_-, ψ_+

PROJECT ψ_+, ψ_- TO GIVE $\tilde{\psi}$

direction . To then get a real wave-function we then if necessary project onto the real axis .

For definiteness we summarise the procedure for case (a) - see fig. 2.6a for the behaviour of the stationary points as μ varies , and fig. 2.6b and also appendix 2.2 to establish notation .

1) Locate all stationary points in the complex plane (here at q_0 , $q_\pm$) .

2) Establish the characteristic direction at each stationary point .

3) Construct wave-functions ψ_0 , $\psi_\pm$ at their respective locations by some means (see later) .

4) ψ_0 is already real - project $\psi_\pm$ onto the real axis to get a single representative $\tilde{\psi}$ (note that $q_\pm$ are complex conjugates so their effects may be combined in this way) .

The wave-functions ψ_0 , $\psi_\pm$ may be obtained by a variety of methods . Perhaps the simplest is to choose them to be Gaussian humps , whose spread is fixed by the curvature at the stationary point (which matches quadratic terms in the approximate Schrodinger equation , as in ordinary zero-point calculations) . Other methods tried include refining this by multiplying the Gaussian by a polynomial , where the coefficients are found by matching higher-order terms in the equation , or else allowing the various parameters to vary in an optimisation scheme . Further details may be found in appendix 2.2 .

We now construct a family of trial wave-functions by taking linear combinations

2.3.1) $$\hat{\psi} = \psi_0 \cdot \sin\alpha + \tilde{\psi} \cdot \cos\alpha$$

where $0 \leq \alpha < \frac{\pi}{2}$. As usual the energy functional is

2.3.2) $$E[\hat{\psi}] = \langle \hat{\psi}, \hat{H}\hat{\psi} \rangle / \langle \hat{\psi}, \hat{\psi} \rangle$$

and we minimise this over the family (2.3.1) - minimisation may be just over the mixing angle α , or we can allow some or all of the wave-function parameters to vary as well .

The procedure blends perturbative and nonperturbative effects in this way . By expanding trial wave-functions about just the global minimum q_0 (which corresponds to fixing $\alpha = \pi/2$ in (2.3.1)) we are performing a perturbative expansion . On the other hand , since we include contributions from the other stationary points as well , the procedure uses information about the long distance behaviour of the potential from an early stage - as we saw in the last section , resummed perturbation theory can do this but only when high-order terms can be calculated . The method described here needs only a few terms in order to get accurate answers .

Results are shown in figs. 2.7a,b . (a) shows the difference between fixing $\alpha = \frac{\pi}{2}$ and allowing α to vary , when wave-functions around stationary point q_s are chosen of the form $\psi_s \sim (1 + a_s(q-q_s))e^{-\lambda_s(q-q_s)^2/2}$. (b) shows the difference between fixing $a_0 = \tilde{a} = 0$ and using values given in appendix 2.2 . We have also investigated the improvement if higher powers are included in the expansions of $\psi_0, \psi_\pm$, but little difference can be seen . In both cases λ_0 and $\tilde{\lambda}$ are calculated as in appendix 2.2 .

Several points may be noted . Firstly , as hoped , there is a general improvement in the estimate by including contributions from the complex saddles and this improvement is best in the middle range of values of μ when neither the zero-point estimate , nor (as we shall see later) the instanton methods of [38] give good answers . Secondly , as we might have expected

BROKEN CASE

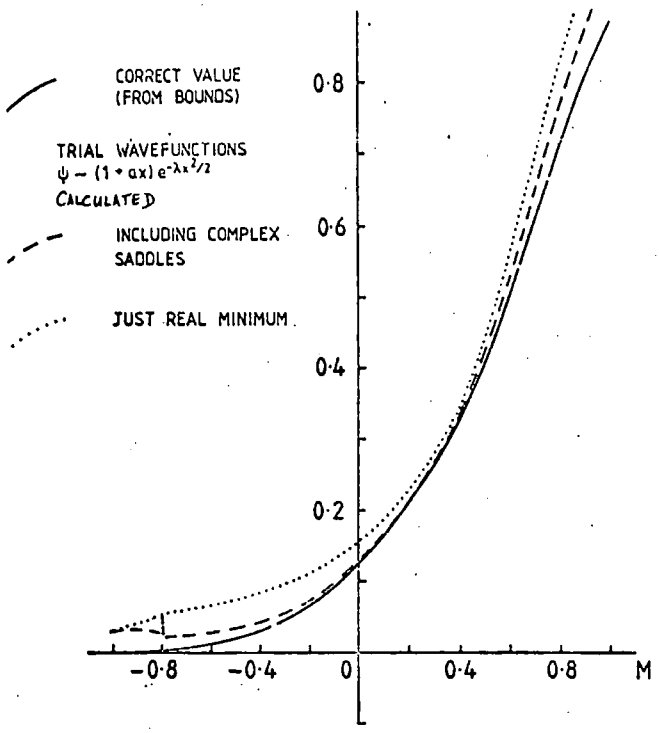


FIG. 2.7a

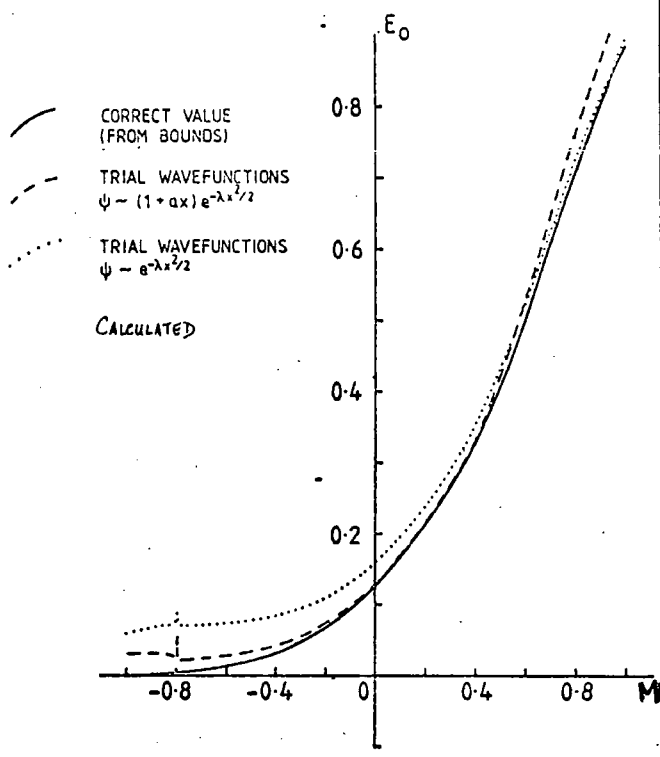


FIG. 2.7b

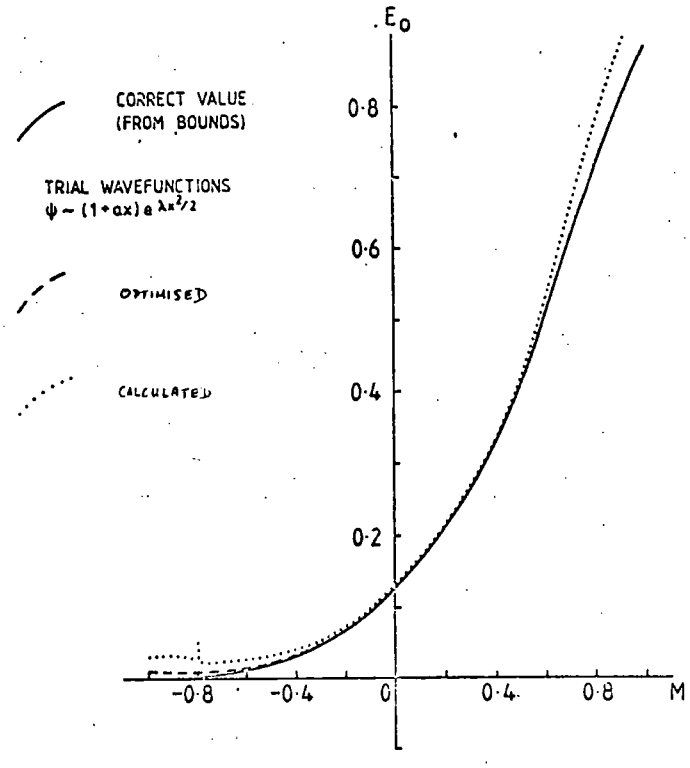


FIG. 2.8a

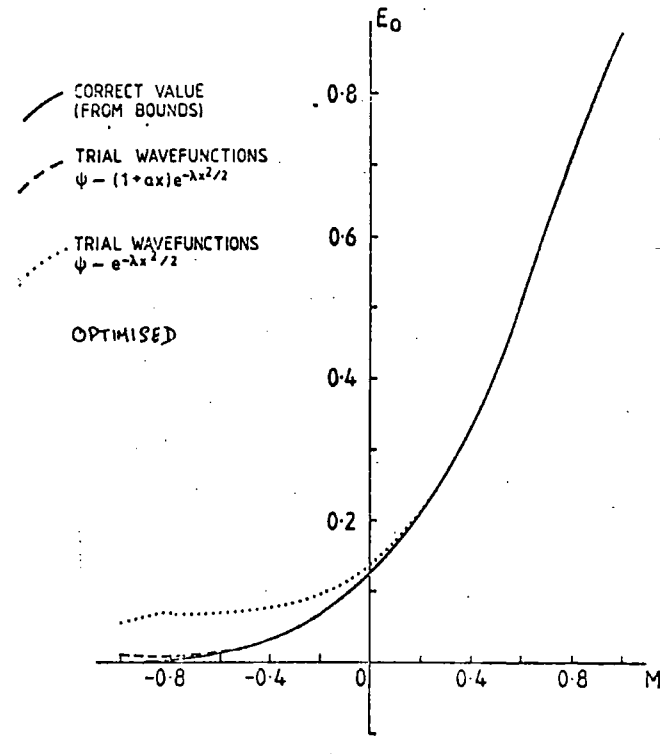


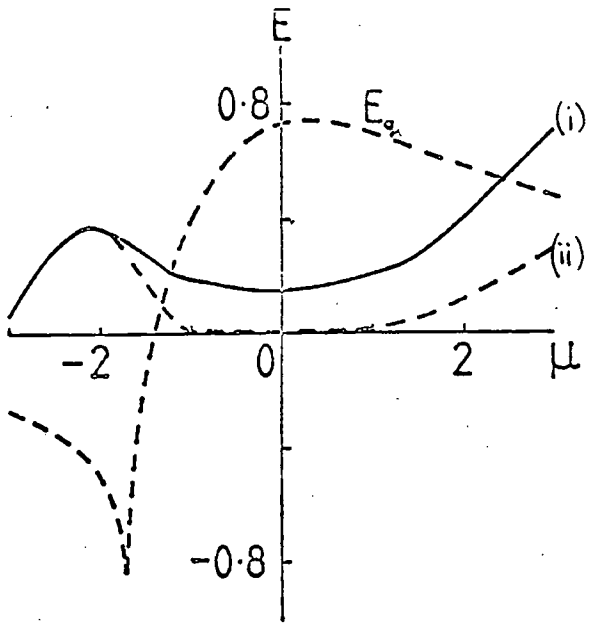
FIG. 2.8b

from figs. 2.4b and c , there is a problem at the critical value of μ at which the two saddles in the complex plane merge and then separate along the real axis - at such values the saddles are degenerate in the sense explained above . Inspection of the mixing angle α shows that it is equal to $\pi/2$ here , so there is no contribution from the complex saddle and the method reduces to an ordinary expansion around q_0 . Clearly what is needed is a method analogous to that used last section for replacing calculated values of a's and λ's by effective values which take account of higher powers in the potential . One way of doing this would be to repeat the calculations explained at the end of section (2.2) , but in fact we simply allowed the parameters to vary in the minimisation of (2.3.2) just as varies - so we are now optimising in a multi-dimensional parameter space .

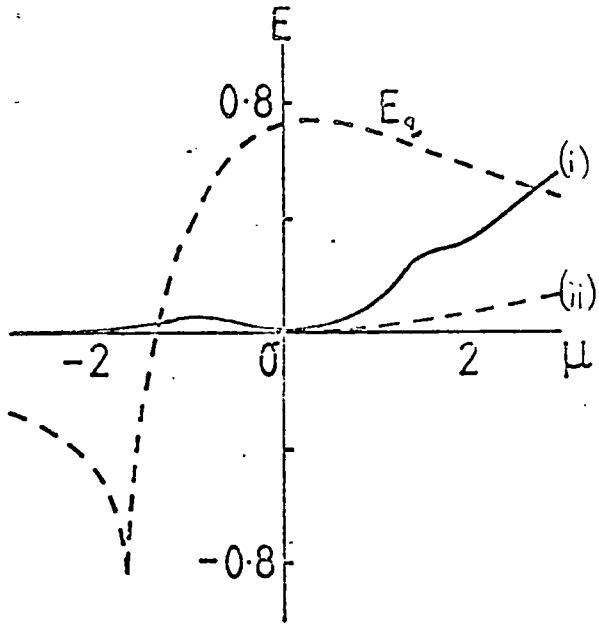
Results are shown in figs. 2.8a,b . (a) shows the difference between calculated and optimised values with trial wave-functions chosen as for fig. 2.7a . (b) is the equivalent of fig. 2.7b but again with parameters optimised . It will be seen that there is an improvement over the entire length of the curve compared to figs. 2.7 , particularly in the positive μ region . The problem encountered at the degenerate stationary points has been overcome , as expected from section (2.2) . We have investigated the effect of including higher order terms in the perturbative expansions , but again this makes little difference . As may be seen from fig. 2.8b , in the negative μ region it is the inclusion of linear corrections which is important - these allow for asymmetry of the potential around its stationary points , which also turned out to be a crucial consideration

UNBROKEN CASE

FIG. 2·9

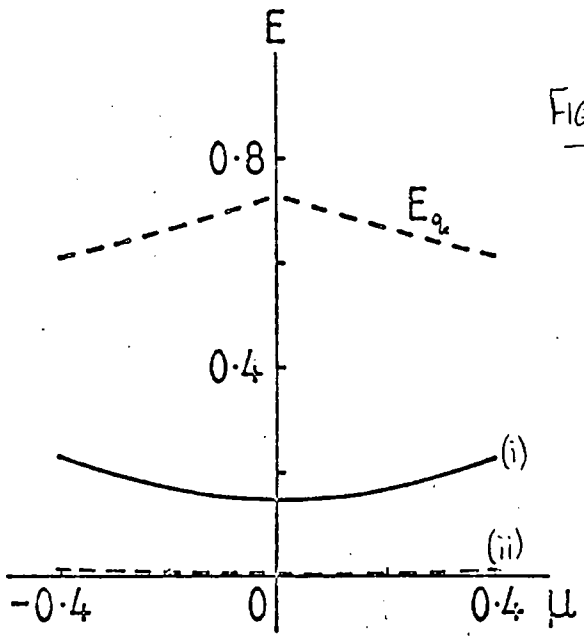


a) CALCULATED

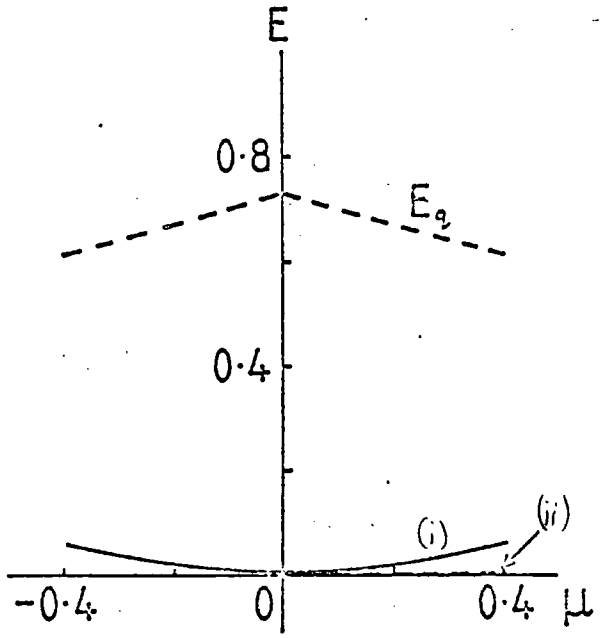


b) OPTIMISED

FIG. 2·10



a) CALCULATED



b) OPTIMISED

(i) – WAVEFUNCTIONS $\psi \sim e^{-\lambda x^2/2}$

(ii) – WAVEFUNCTIONS $\psi \sim (1+ax)\, e^{-\lambda x^2/2}$

in the Gaussian calculations of last section . For a comparison of calculated and optimised parameter values , see appendix 2.2 .

We have repeated the equivalent calculations in various unbroken cases as well , using superpotentials with leading power quartic . In such cases there are five stationary points , and considerations of symmetry mean that we now have either two or three wavefunctions making up the linear combination $\hat{\psi}$. The procedure differs only in detail from the above , so we simply present the results here in figs. 2.9a,b and 2.10a,b .

Fig. 2.9 shows results for the superpotential of (2.2.3) . (a) contains cases where the parameters are calculated , comparing the inclusion or otherwise of linear corrections to the wave-functions . (b) is the same , but the parameters are optimised . For comparison , the curvature quadratic estimate E_q is also shown .

Fig. 2.10 shows the same plots for superpotentials $W = \frac{1}{4}q^4 + \mu q$, again with E_q for comparison . This case was considered because the linear term in v means that neither V_b nor V_+ have any symmetry properties permitting the use of instanton techniques - thus the only existing method of calculation other than resummed perturbation theory would be the quadratic estimates .

As in the broken case , considerable improvement in the estimation of E_0 is achieved by using the mixed wave-function method - in some cases up to three orders of magnitude better than E_q . In all cases it seems that the majority of the improvement comes at the step from using just Gaussian humps to the modification of these to allow for asymmetry in the potential - this not only confirms what was found last section

with quadratic estimates using only the global minimum , but is also particularly convenient in terms of computer-time . We do not have to perform large perturbative expansions around each stationary point , but need only include a few terms in order to extract the basic shape . The pictures show clearly that the gain in accuracy obtained by using the information about the long-range behaviour of the potential given by stationary points other than the global minimum is substantial .

Finally we have applied the method to the octic problem of (2.1.16a) . This was chosen because neither the ordinary quadratic nor instanton estimates are of any use here . The potential has been chosen to have vanishing curvature round each minimum , so the ordinary quadratic correction vanishes . Further , as explained in [35] , the ordinary semiclassical expansion around instanton solutions fails . Hence this problem has been used as a testing-ground for various alternative approaches [31] . Our results are as follows , compared with the exact results . In all cases parameters have been optimised rather than calculated . In this problem their are only two (real) stationary points and the symmetry of the problem also constrains some of the parameters .

	Exact	Terms in perturbative expansion		
	result	Constant	Const+linear	Const+lin+quadratic
$f^2 = 2$	1.37	2.36	1.93	1.93
$f^2 = 4$	3.26	3.54	3.35	3.31

Again results are good , and they also show another point of interest . The rate of convergence would be better if we input one more physical fact - the asymptotic rate of growth of the potential , or equivalently the decay-rate of the wave-function . With our ansatz , the wave-function can fall off no more quickly than as $\exp(-k.x^2)$, whereas a quick estimate shows here that it should fall off as $\exp(-k.|x|^5)$ for large $|x|$. Perhaps using this knowledge would be advantageous . We have also repeated the methods outlined at the end of last section for effective quadratic methods , and results are as follows :

$f^2 = 2$	Even powers only	2.36	Even and odd powers	1.95
$f^2 = 4$		3.54		3.34

The correspondence between these results and the first two columns of the mixed wave-function method results is striking .

2.4 - Comparison and criticism

We wish first to repeat that it is not yet obvious how any scheme based on Hamiltonian techniques will be applicable in field theories , where Lagrangian functional methods are customarily used . Some work has been done on the use of such an approach , particularly when considering lattice approximations to continuum models (e.g. [37]) , but so far as we know results are not yet conclusive . Since , however , their use in quantum mechanics is intuitively clearer , we have investigated various approaches .

For the spontaneously broken cases considered , we have seen that the usual assumption that there is a neat division between perturbatively and nonperturbatively broken models is not supported by plots of the actual value of E_o - such assertions hold only in asymptotic regions , and there is a sizeable central region in which a mixture of both effects must operate .

It has also become apparent that ground-state energy is not a particularly good quantity to look at if one is unsure whether a given model is unbroken or slightly broken . Of necessity numerical methods give approximate answers , and so can never yield the exact answer zero . However , most methods which have been developed until now give almost no information for cases which are broken , and numerical estimates of E_o can reveal the extent to which a given model is dynamically broken .

A variety of Gaussian-type methods were investigated . There are strong indications that great improvements over the usual method of matching curvature may be gained simply by allowing for asymmetry of the potential about its minimum . Such asymmetry is caused by cubic or higher odd powers , and are therefore usually neglected . Taking higher even powers into account (a simpler procedure) can also give good improvements .The underlying theme in all these variations is the replacement of the simple curvature estimate by an effective value incorporating longer-distance effects , just as in mean-field techniques .

The mixed wave-function method described last section is an alternative way of including higher-order effects , involving use of all stationary points of the potential in the complex plane , not just the global minimum along the real axis . We suspect that there is a connection between this method and the

use of complex solutions of the equations of motion [6] (i.e. stationary points of the action density in complex space) , but we have not been able to find any explicit correspondence . Direct comparison is difficult because of the inherent differences between Hamiltonian real-time and Lagrangian imaginary-time methods . The results for the octic potential indicate that there is also a strong connection between this method and the improved Gaussian methods described above , at least when the mixed-wavefunction method has only two basis functions and a high degree of symmetry .

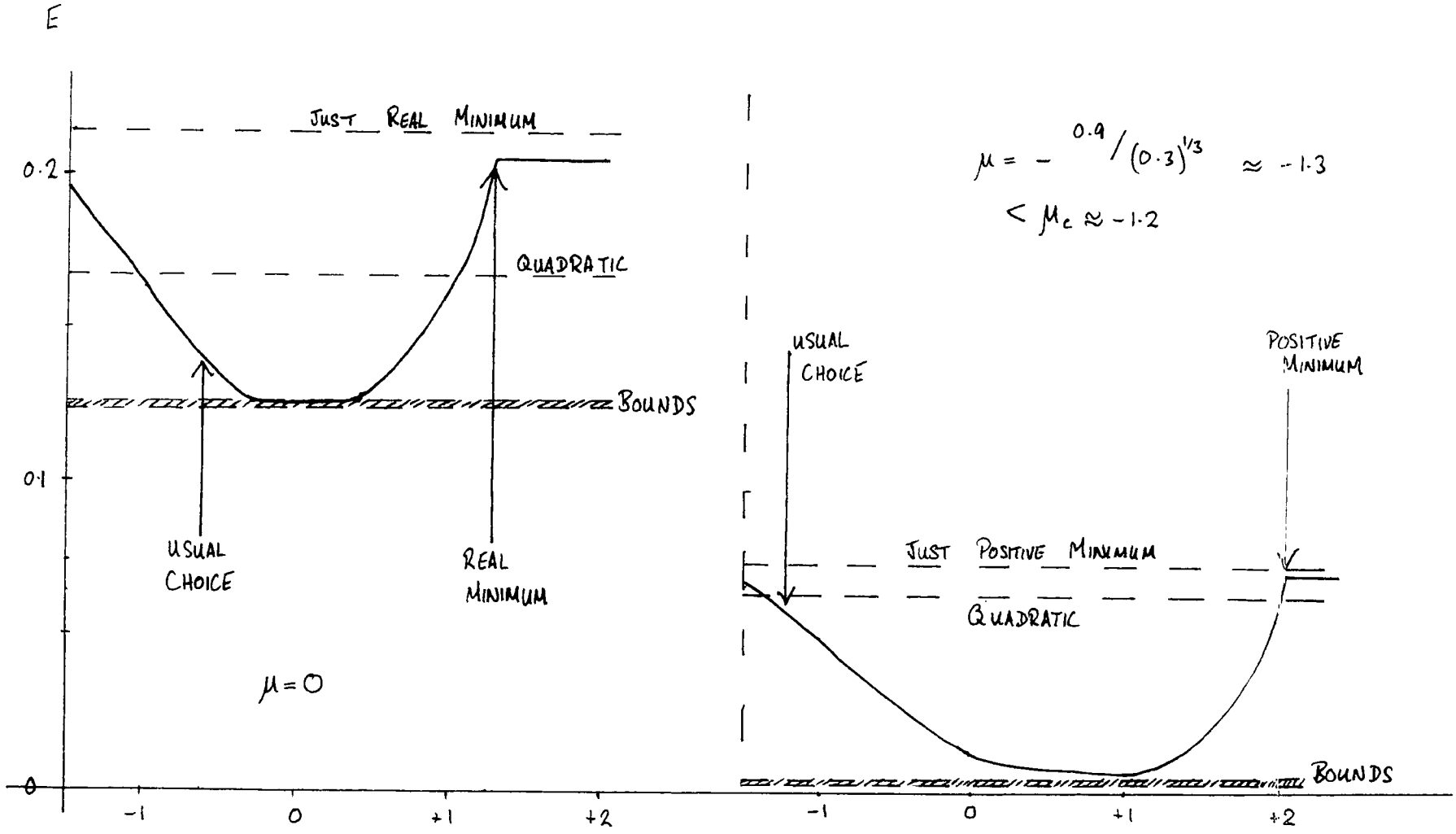
We have also investigated the effect of shifting the location of the secondary wave-function $\tilde{\psi}$ to see whether the position of the complex saddles does indeed give a sensible guide to this . Direct comparison of results is of course difficult , because the size of shift of location becomes another parameter to be varied in the optimisation process , so we cannot end up with a worse answer . However , it is of interest to see how far the original location is from the one which gives the smallest answer , and the extent of variation in the answers .

For basis wave-functions chosen as simple Gaussians figs. 2.11a and b show the results for two values of μ . In neither case does the original choice give the smallest answer , although for $\mu = 0$ the two values are close . In both cases the secondary minimum should be closer to the primary one . If we use basis wave-functions which include a linear correction then a slightly different picture emerges . These results are not shown because they are not instructive - all the values are very close to the exact bounds , increasing only very slowly towards the asymptotic values . Once again the location

(a)

FIG. 2·11

(b)



LOCATION OF SECONDARY WAVE-FUNCTION
(BROKEN CASE)

giving the minimum value is shifted towards the primary location , but this time the original choice is in both cases close to the best choice . We also notice that the lowest value obtained by using just Gaussians and shifting the secondary location is virtually the same as the lowest value when linear corrections are included . This is an echo of fig. 2.8b - a big improvement is gained by including linear corrections , with very little benefit from adding higher-power corrections . It seems that allowing the location of the secondary minimum to vary in this way is equivalent to including the next-higher power corrections to the wave-functions (and is much cheaper in terms of computer-time) . In both cases the principal wave-function is given the opportunity of being asymmetric in order to match asymmetry in the potential .

In the next chapter we turn to the other main approach to estimating quantities in quantum theories - Lagrangian methods using functional integrals to quantise the theory . These methods are more easily translated into field-theoretic terminology , and we shall partially do this in chapter 4 .

Chapter 3 - Quantum-mechanical Models - Lagrangian Methods

3.1 - Semiclassical method using instantons

Quantising a theory by the path-integral approach was originally suggested by Feynman [25] and has since become the standard field-theoretic approach , despite ambiguities of proper definition . It has strong connections with the use of the partition function in statistical mechanics , and rather weaker links with the (respectably defined) theory of Wiener and other stochastic processes .

The basic object is the generating function

3.1.1) $$Z_M = \int [dx]\, e^{iS_M[x]/\hbar} = \int [dx]\, e^{i\int L_M[x]dt/\hbar}$$

where the integral is over all paths $x(t)$ satisfying the prescribed boundary conditions . Subscript M indicates the use of Minkowski space . Note that in order to define the normalisation of the measure correctly , either some form of limiting procedure must be used [4] or else an entirely different conceptual framework [22] .

Z is the generating functional for all the n-point functions of the theory . For purposes of calculation we analytically continue to Euclidean space -

3.1.2) $$Z_E = \int [dx]\, e^{-\int L_E[x]dt/\hbar}$$

For convenience we will not henceforth distinguish between Euclidean and Minkowski space quantities , but will work exclusively in the former .

It is now assumed that we may apply the infinite-dimensional equivalent of the stationary-phase approximation - then the

stationary points of the integrand are assumed to dominate the integral so we need only calculate fluctuations in the immediate vicinity of these . It is readily seen that these stationary functions are precisely those functions $x(t)$ which are solutions of the classical (Euler-Lagrange) equations of motion , where as before we also impose suitable boundary conditions . Thus an essential step is to ascertain all such solutions .

If our Hamiltonian is of the standard form

3.1.3) $$\hat{H} = \frac{1}{2m}\hat{P}^2 + V(\hat{q})$$

- note that now we retain all dimensional parameters for purposes explained later - then the Euclidean-space equations of motion are

3.1.4) $$\frac{m}{2}\left(\frac{dq}{dt}\right)^2 = V(q) - \text{constant}$$

where the constant is chosen to fix the boundary conditions and is usually given by the minimum of the potential . We must now choose what our potential is to be - normally this is straightforward , but for supersymmetric quantum mechanics we have two choices - V_b or V_+ . The authors of [38] use V_b as they believe this to be the more fundamental , and in consequence their instanton solutions represent a tunneling between two degenerate wells together with a change of fermion number - their procedure is more complicated than ours . We believe that the full potential V_+ retains all of the super-symmetric character of the problem , despite the reduction to an effective bosonic theory , and also makes the calculations more transparent .

For the broken case with superpotential (2.1.14a) there is in

fact only one choice . For $\mu<0$ (fig. 2.1a) an instanton calculation can be performed using V_b (this calculation is given in detail in [38]) , whereas no value of permits any such calculation using V_+ , as there is never any degeneracy of minima . We have only one criticism of the calculations of [38] - it is claimed there that the instanton contribution to E_0 is damped by a factor of order $e^{-1/\hbar}$ and hence is clearly exponentially small and nonperturbative in origin . However , a careful dimensional analysis shows this not to be the case - the damping factor is in fact of order $e^{-|\mu|^{3/2}}$ and all dimensional parameters such as $\hbar$ cancel . This will be demonstrated more clearly for the unbroken case shortly . The results of [38] are shown in fig. 3.1 , compared with the exact bounds on E_0 . As explained in chapter 2 , they are extremely good for large negative μ (the asymptotic non-perturbative region) but become poor near $\mu=0$. Clearly this calculation is impossible for $\mu>0$ but presumably the authors would continue their estimate in this region with some form of perturbative calculation using V_b .

We now turn to the unbroken cases . We have not tried to repeat the above method in this case - for $\mu>0$ (fig. 2.2a) there are two sorts of instantons tunneling between the central minimum and one or other of the outer ones . The action of any of these is

3.1.5) $$S[\bar{q}] = \mu^2\hbar/4$$

- we will comment on this later .

If we use V_+ instead of V_b there are instanton solutions for $\mu>-\sqrt{3}$, found by solving

BROKEN CASE

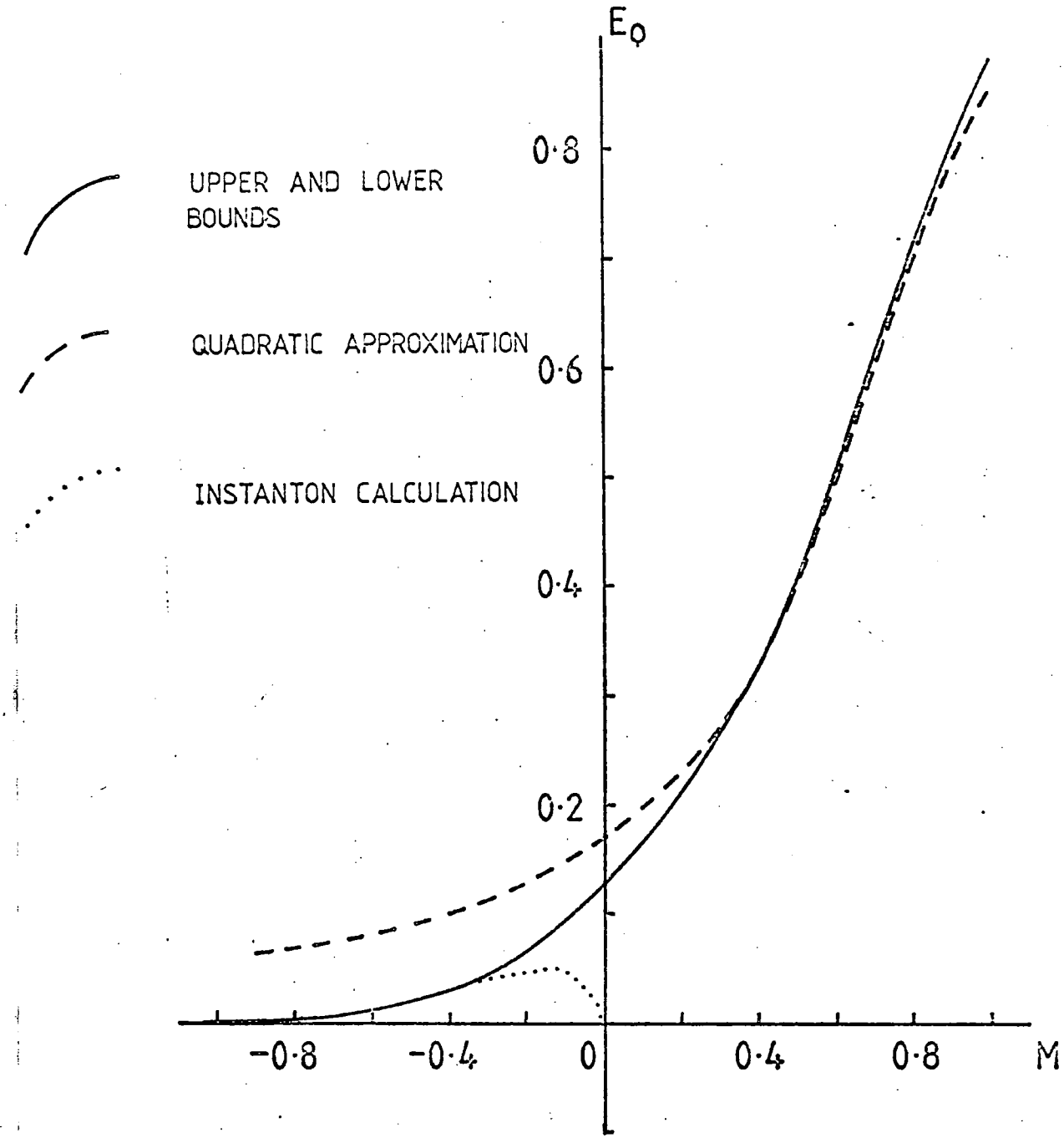


FIG. 3.1

$$\tfrac{1}{2}m\dot{q}^2 = \frac{\hbar^2}{2m}\left(V_+(q) - V_{min}\right) \quad \text{subject to} \quad q(T) = -q(-T) = q_0$$

where $V_+ = \left(Q^6 - 2\mu Q^4 + (\mu^2 - 3)Q^2 + \mu\right)L^{1/2}$; $V_{min} = V_+(q_0)$

The solution of this is

3.1.6) $$\bar{q}(t) = \alpha \cdot \tanh \omega(t - t_0) \Big/ \sqrt{1 + \beta \cdot \operatorname{sech}^2 \omega(t - t_0)}$$

where for $\mu = 0$,
$$\begin{cases} \alpha = L^{-1/4} \\ \omega = \sqrt{3}\ (\hbar\sqrt{L}/m) \\ \beta = 1/2 \end{cases}$$

- corresponding expressions for general μ may be found in appendix 3.1 . Fig. 3.2 shows plots of the solutions and their time derivatives for two values of μ - note the change of shape for $\mu > \sqrt{3}$. To easily visualise the behaviour of Euclidean-time solutions of a potential V , one may imagine the particle moving in normal time in a potential -V , i.e. moving in the inverted potential (see [17] for a discussion of this) . The arbitrariness of choice of origin of time gives rise to a zero-mode in a determinant about to be calculated , so henceforth we fix $t_0 = 0$ and use the Faddeev-Popov method to cope with this mode .

The action of these solutions is

$$S[\bar{q}] = \int_{-T}^{T} L[\bar{q}]\, dt$$

3.1.7) $$\simeq \frac{m\alpha^2\omega}{4\beta}\left\{2\beta - 1 + \frac{4\beta+1}{\sqrt{\beta(1+\beta)}}\ln\left[\sqrt{\beta} + \sqrt{1+\beta}\right]\right\} + 2 \cdot E_{min} \cdot T \quad \text{for large } T$$

$$= \frac{3\hbar}{2}\ln(2+\sqrt{3}) + \frac{2\hbar^2\sqrt{L}}{m}T \quad \text{at } \mu = 0$$

Note that $E_{min} = \frac{\hbar^2}{2m}V_{min}$. The fact that S contains a piece rising linearly with T reflects the fact that the bottom of the potential is not at zero - it is the classical value of the ground energy . The semiclassical estimate of ground-state energy contains three terms

3.1.8) $$E_{tot} = E_{min} + E_z - \Delta E \equiv E_q - \Delta E$$

which are respectively the classical value , zero-point energy from quantum fluctuations , and the energy splitting due to

tunneling . Full details of the derivation of these terms can be found in appendix 3.1 and here we simply remark

3.1.9) $$\Delta E = \hbar K e^{-S'[\bar{q}]/\hbar} \quad (S' \text{ excludes piece linear in } T)$$

where K is a factor containing firstly a numerical factor representing the amount of overlap between two wave-functions centred in each well (i.e. the extent to which the wells are separated) and secondly the quantum-mechanical fluctuations about one-instanton configurations . Only single instantons are used because , as explained in appendix 3.1 , we have made use of the so-called dilute gas approximation . This essentially assumes that multi-instanton quasi-solutions (i.e. multiple traversals from one well to the other) can be taken into account by superposing appropriate single instantons and anti-instantons - in other words the locations of the crossings are well-separated in time as compared to the time taken to cross . There are other calculational schemes which do not make this assumption , but they tend to be considerably harder to use . The fluctuation part of ΔE may be expressed essentially as a ratio of determinants , which can be calculated in a number of ways - so for example [17] and [27] give two superficially different ways . As explained in [3] both methods in fact give the same answer . Further explanation may be found in appendix 3.1 , together with some of the steps of calculation involved . The result is

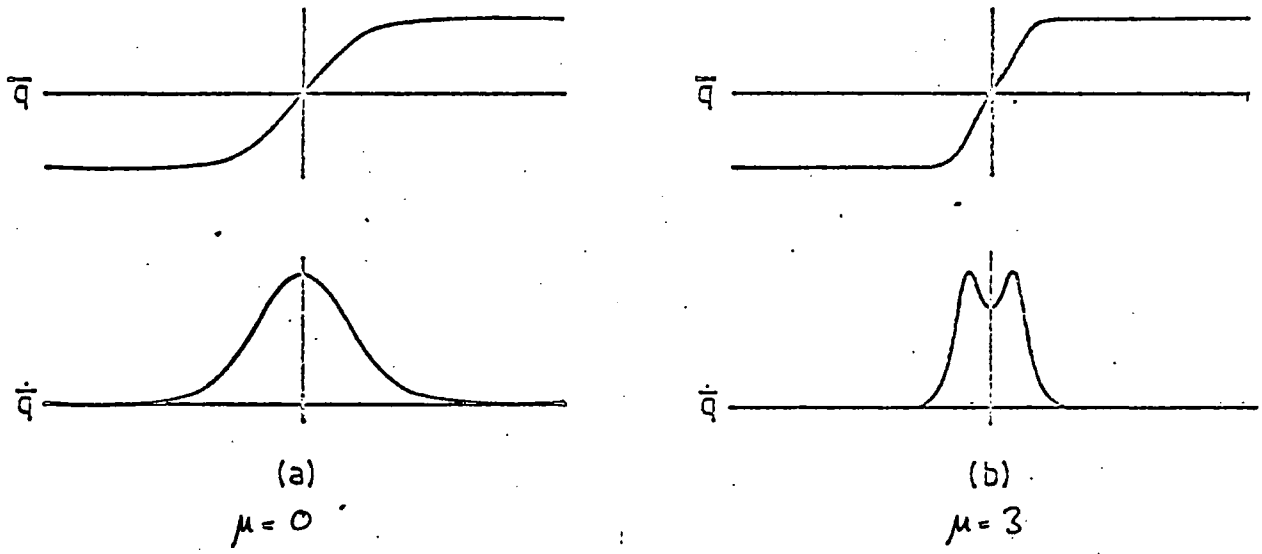
3.1.10) $$\Delta E = 8\alpha\omega(1+\beta)\sqrt{\frac{m\hbar\omega}{2\pi}}\, e^{-S'[\bar{q}]/\hbar}$$

We may now substitute explicit values into (3.1.8) for $\mu = 0$, giving

3.1.11)
$$\begin{aligned} E_{tot} &= \frac{\hbar^2\sqrt{L}}{m}(-1) + \frac{\hbar^2\sqrt{L}}{m}\sqrt{3} - \frac{\hbar^2\sqrt{L}}{m}\cdot\frac{36}{\sqrt{2\pi\sqrt{3}}}\exp\left[-\tfrac{3}{2}\ln(2+\sqrt{3})\right] \\ &= \frac{\hbar^2\sqrt{L}}{m}\left\{-1+\sqrt{3}-\frac{36}{\sqrt{2\pi\sqrt{3}}}\exp\left[-\tfrac{3}{2}\ln(2+\sqrt{3})\right]\right\} \\ &\approx \frac{\hbar^2\sqrt{L}}{m}\left\{-1+1.73-1.51\right\} \\ &= -0.78\,\frac{\hbar^2\sqrt{L}}{m} \end{aligned}$$

FIG. 3·2

INSTANTONS FOR UNBROKEN CASE



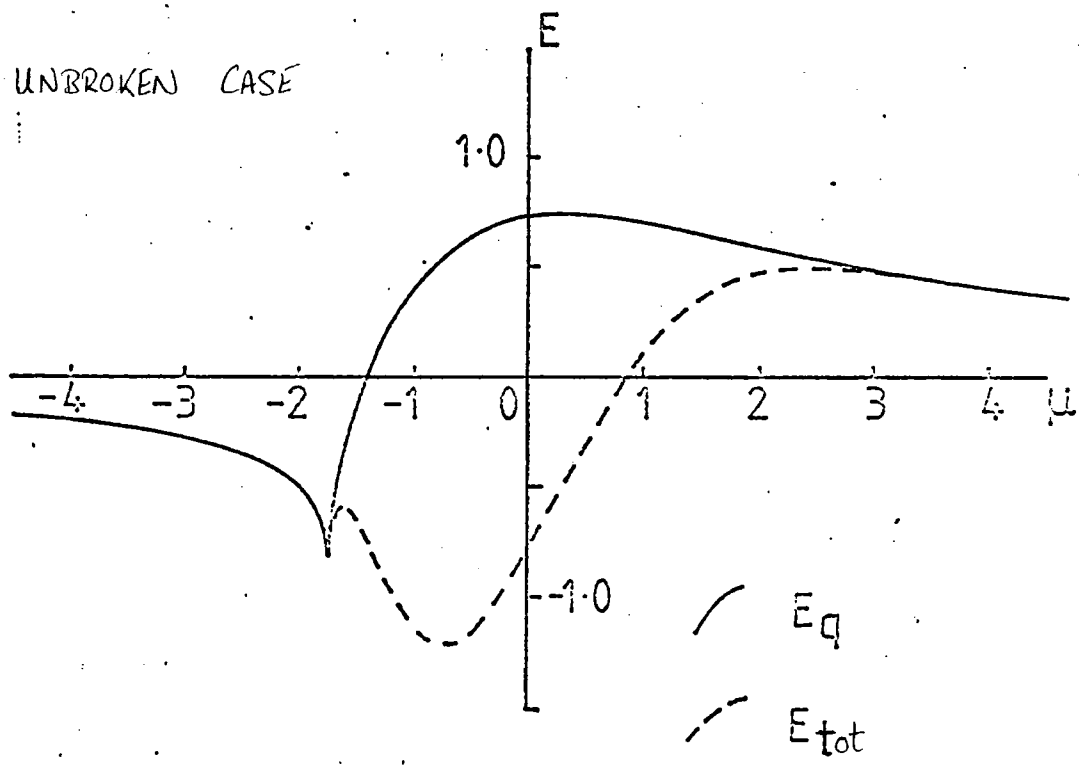


FIG. 3·3

This should be compared with the exact value (zero) and with other estimates - in particular notice that the instanton correction has not improved the curvature estimate

3.1.12) $$E_q = E_{min} + E_2 \approx 0.73 \frac{\hbar^2\sqrt{L}}{m}$$

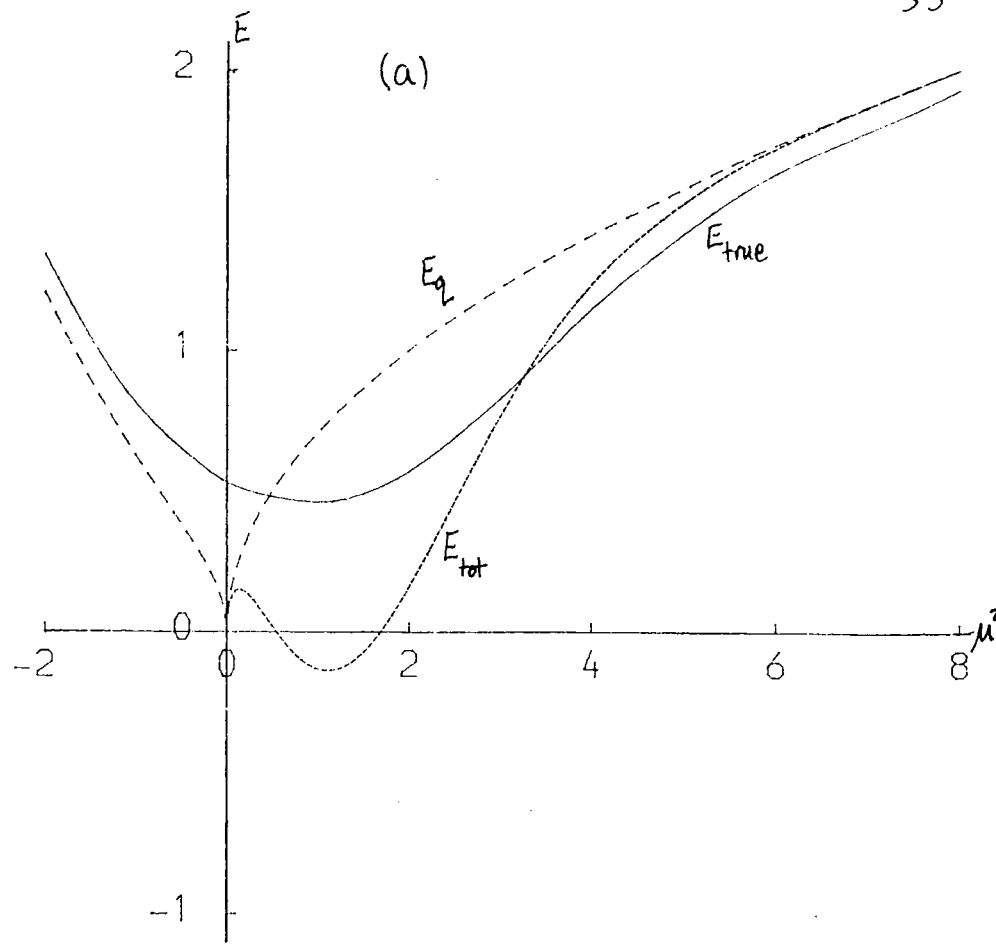
in magnitude . In fact all three terms are comparable in magnitude , with the instanton contribution not suppressed either by powers of $\hbar$ or exponentials of $-1/\hbar$, just as in the broken case .

Fig. 3.3 shows the full dependence of E_q and E_{tot} on μ - we have removed the overall factor $\hbar^2\sqrt{L}/m$ which we now see as being present only for dimensional reasons . Note that nothing special happens at $\mu=0$ and that the results depend on μ in a relatively arbitrary way . There is a value of μ for which $E_{tot} = 0$, but E_{tot} varies rapidly around this point and there seems nothing special about the value . Note that asymptotically for large positive μ ,

3.1.13)
$$E_q \sim 1/\mu$$
$$S[\bar{q}] \sim \tfrac{1}{2}\mu^2$$
$$\Delta E \sim \mu^4 \exp\left[-\tfrac{1}{2}\mu^2\right]$$
$$\longrightarrow E_{tot} \sim E_q \sim 1/\mu$$

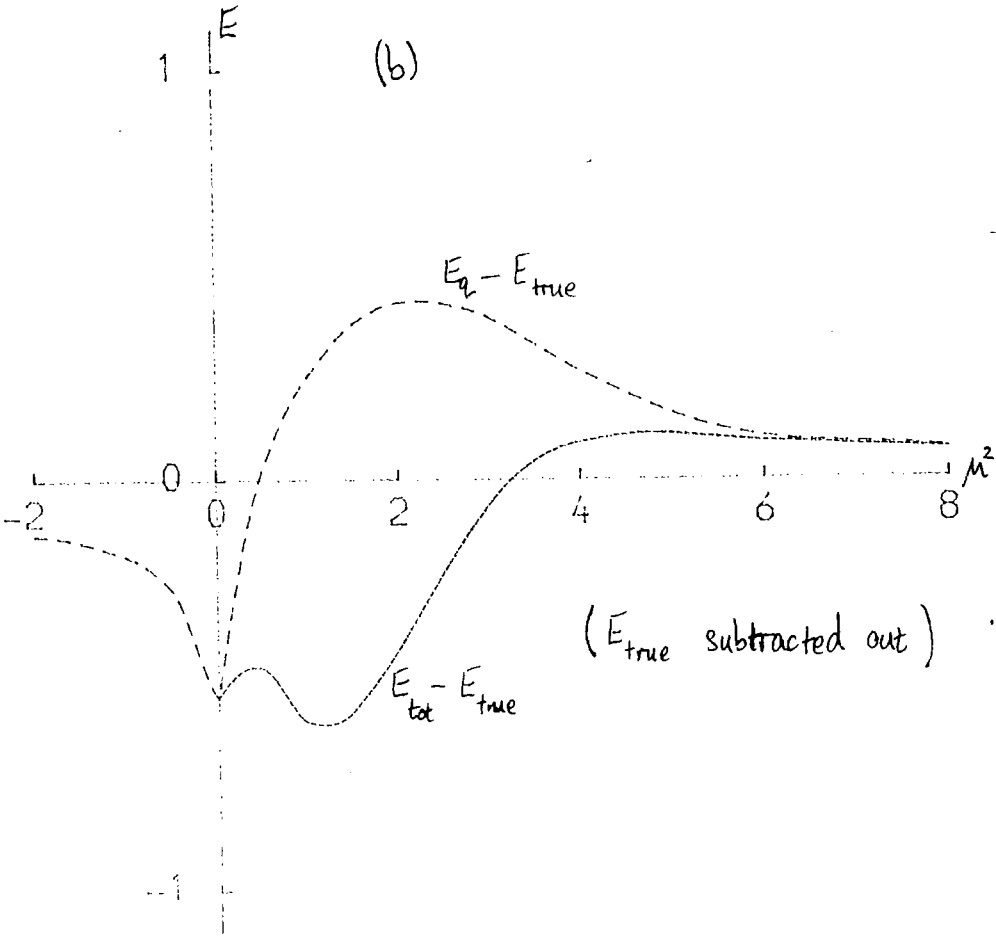
Compare (3.1.5) with the corresponding behaviour in (3.1.13) . This is a further reason why we have not attempted to duplicate the calculations of [38] - there are no indications that the results will be any better overall than the results found here .

The poor quality of the results derived from the semiclassical method came as something of a surprise , so we have obtained the equivalent results for the quartic anharmonic oscillator with potential as in (2.5a) . These results are shown in fig. 3.4a (as given in [27]) compared with the true value (from [11])



QUARTIC OSCILLATOR

FIG 3.4



and the corresponding E_q . To allow better comparison with fig. 3.3 , fig. 3.4b shows the same plots but with the true value subtracted out (this is equivalent to arranging that the true value is always zero) . It should be noted that in this case , just as in the unbroken supersymmetric case , all dimensional parameters scale out of the problem and the instanton correction is of the same order as the other estimates - appendix 3.2 contains a brief discussion of the dimensional behaviour . We can clearly see from figs. 3.3,4 that the qualitative behaviour of the semiclassical answer E_{tot} is exactly the same in both the unbroken supersymmetric and quartic oscillator cases . This again came as a surprise , particularly as the estimates are not very good .

Some comments concerning these results are in order . The whole instanton calculation and in particular the dilute gas approximation are only expected to give reliable answers in the so-called weak-coupling regime , which for the quartic case corresponds to large positive μ^2 . For the supersymmetric case it is less clear , because of the way the coefficients in the potential are related . For the time being we assume that once again , weak coupling means large values of μ . Thus , continuation of the results down to $\mu = 0$ is equivalent to going from a weak to a strong coupling regime , and it could be argued that applying these techniques in such a situation is wrong .

However , we feel that the results are significant for the following reasons . For a general theory it is not normally possible to predict in advance where strong and weak coupling

regions may be expected to lie as this depends on the detailed interplay between parameters in the theory - the precise nature of boundary conditions chosen can also confuse this . In such a situation one would in practise use a calculational scheme such as this in the hope that the answers obtained would be indicative of the correct value . It is therefore of interest to assess how accurate instanton calculations are over the entire range of variation of parameters . Secondly , there is a great interest at present in theories attempting to describe particles which classically have zero mass and acquire a small mass through quantum fluctuations - the analogy in both cases here is for the dimensionless parameter to have a value in a region where the instanton answer is at its least accurate . The third point is closely related . In ordinary models we are accustomed to being able to tune masses and coupling constants independently and so , for example , consider theories which are arranged to display weakly-coupled massive particles . In supersymmetric models this is no longer possible , as we saw above - because of the highly constrained structure of the Lagrangian , masses and coupling constants become inextricably bound together , and independent tuning becomes difficult or impossible . Hence we may be forced to work in a region of parameter-space where we cannot be sure that our approximations are valid . We therefore seek some refinement of the normal procedure which produces more reliable answers over the whole range of μ - in short , one that is more robust .

Before commenting further on these results , we wish to mention one point . The part of the factor K in (3.1.9) (and its equivalent in the quartic case) representing fluctuations

about instantons turns out to have quite a simple physical interpretation - it is precisely the curvature of the potential about its minimum , or equivalently the square of the asymptotic (exponential) approach rate of the instanton to its limiting value . We will return to this point in the next section .

We believe the following to be the reason why the above semi-classical methods are not good . Semiclassical techniques attempt to give a halfway stage between the classical answer E_{min} and the full quantum-mechanical answer given in terms of a path-integral which cannot in general be computed . To this end , solutions of the equations of motion are taken and Gaussian fluctuations around these calculated . Thus quantum-mechanical effects enter only through these small fluctuations (and of course in having a path-integral representation in the first place) , and the choice of path and boundary conditions are entirely classical in origin . Take for example the boundary conditions - these require the particle to be asymptotically at one or other of the degenerate minima of the potential , and simultaneously (if we are to use the instanton solution) to have zero momentum . We believe that this requirement of exact localisability should be relaxed to a condition more appropriate to the quantum-mechanical aspects of the problem . The precise nature of this relaxation may be seen in a number of ways . Computationally the easiest is to allow for a spread of values in position (momentum) around the bottom of the well (zero) - this could be done to a first approximation by using the curvature-Gaussian wave-function and its Fourier transform . A refined approximation would use the effective value of this

"mass" parameter obtained from (2.2.10) or similar methods . Conceptually we can view the procedure in the following way - we are fixing the value of the (Euclidean) energy of the trajectory at times $\pm T$ by requiring the particle to have zero momentum at the top of the hump . Quantum-mechanically there will be an uncertainty in the time at which we measure , and correspondingly , a spread in the value of the conjugate parameter , energy . Of these , the spread in the value of T becomes unimportant for large T since the particle spends most of its time near one or other well as opposed to tunnelling , but the spread in energy has a greater effect , as we shall see later . An alternative conceptual aid is that as the wells get closer and thus more strongly coupled , the (usual) expansion of the path-integral in terms of a basis of position eigenstates becomes unreliable computationally (while still being exact in an abstract sense) since in this region there is almost a degeneracy of many neighbouring position eigenstates . Thus we do better in this region to choose a different basis , more suited to numerical calculations - such a basis may be provided by the harmonic-oscillator-type basis states mentioned above .

However we choose to regard this procedure , the practical effect is the same . This relaxation of the boundary conditions means that solutions of (3.1.4) other than the instanton become important . There are two families , depending on whether the constant in (3.1.4) is greater or less than the minimum of the potential . This constant , $\mathcal{E}$ say , is the Euclidean-space equivalent of the energy - as usual we consider the motion of the particle in the inverted potential and fig. 3.5a shows the various possibilities allowed .

Explicit formulae for these non-instanton solutions are :

Quartic oscillator , $V = \left(q^2 - \frac{1}{2}\mu^2\right)^2$

i) oscillatory solutions , $-\frac{1}{4}\mu^4 \leq \varepsilon < 0$

3.1.14a) $$q_\varepsilon(t) = \alpha \cdot sn\ \omega t \quad ; \quad \alpha^2 = \tfrac{1}{2}\mu^2 - \sqrt{-\varepsilon}\ , \ \omega^2 = \tfrac{1}{2}\mu^2 + \sqrt{-\varepsilon}\ , \ k^2 = \alpha^2/\omega^2$$

ii) direct solutions , $0 < \varepsilon$

$$q_\varepsilon(t) = \alpha \cdot \frac{sn\ \omega t \cdot dn\ \omega t}{cn\ \omega t} \quad ; \quad \alpha^4 = \omega^4 = \tfrac{1}{4}\mu^4 + \varepsilon \ , \ k^2 = \frac{\frac{1}{2}\mu^2 + \alpha^2}{2\alpha^2}$$

- the instanton solution is the borderline case with $\varepsilon = 0$, given by $\bar{q}(t) = \sqrt{\frac{\mu^2}{2}} \cdot \tanh \sqrt{\frac{\mu^2}{2}}\, t$

sn , cn , dn are elliptic functions with parameter k - see [5] for properties .

Unbroken supersymmetric case , $V = V_+ = q^6 - 3q^2$

- we consider only the $\mu = 0$ case as the general case is similar but more complicated .

i) oscillatory solutions , $0 \leq \varepsilon < 2$

$$q_\varepsilon(t) = \frac{\alpha \cdot sn\ \omega t \cdot cn\ \omega t}{\sqrt{\gamma\, dn^2 \omega t - sn^2 \omega t \cdot cn^2 \omega t}} \quad ; \quad \alpha = \sqrt{-2\cos\chi}\ , \ \gamma = 3R \Big/ k^2(1 - 3\tan^2\chi)$$

3.1.14b)
$$\chi = \tfrac{1}{3}(\theta + 2\pi) \ , \ e^{i\theta} = \frac{-\varepsilon - i\sqrt{4-\varepsilon^2}}{2} \ , \ \omega = -\cos\chi \sqrt{\tfrac{3}{2}(1 + R - \tan^2\chi)}$$
$$R^2 = 1 - \tfrac{10}{3}\tan^2\chi + \tan^4\chi \ , \ k^2 = 2R \,/\, (1 + R - \tan^2\chi)$$

ii) direct solutions ,

$$q_\varepsilon(t) = \frac{\alpha \cdot sn\ \omega t \cdot dn\ \omega t}{\sqrt{\gamma\, cn^2 \omega t - sn^2 \omega t \cdot dn^2 \omega t}} \quad ; \quad \alpha = \sqrt{-S}\ , \ \gamma = 3R \,/\, (1 + 3T^2/S^2)$$

$$S = s_1 + s_2 \ , \ T = s_1 - s_2 \ , \ s_1 = \left(\frac{-\varepsilon + \sqrt{\varepsilon^2 - 4}}{2}\right)^{1/3} \ , \ s_2 = \left(\frac{-\varepsilon - \sqrt{\varepsilon^2 - 4}}{2}\right)^{1/3}$$

$$R^2 = 1 + \tfrac{10}{3}\frac{T^2}{S^2} + \frac{T^4}{S^4} \ , \ k^2 = (1 + R + T^2/S^2) \,/\, 2R$$

- again the instanton is the borderline case , here with $\varepsilon = 2$.

Figure 3.5b shows the qualitative nature of the two families of solutions . As mentioned earlier , specification of the energy parameter ε may be used to uniquely determine a solution in these families , provided that we require that the

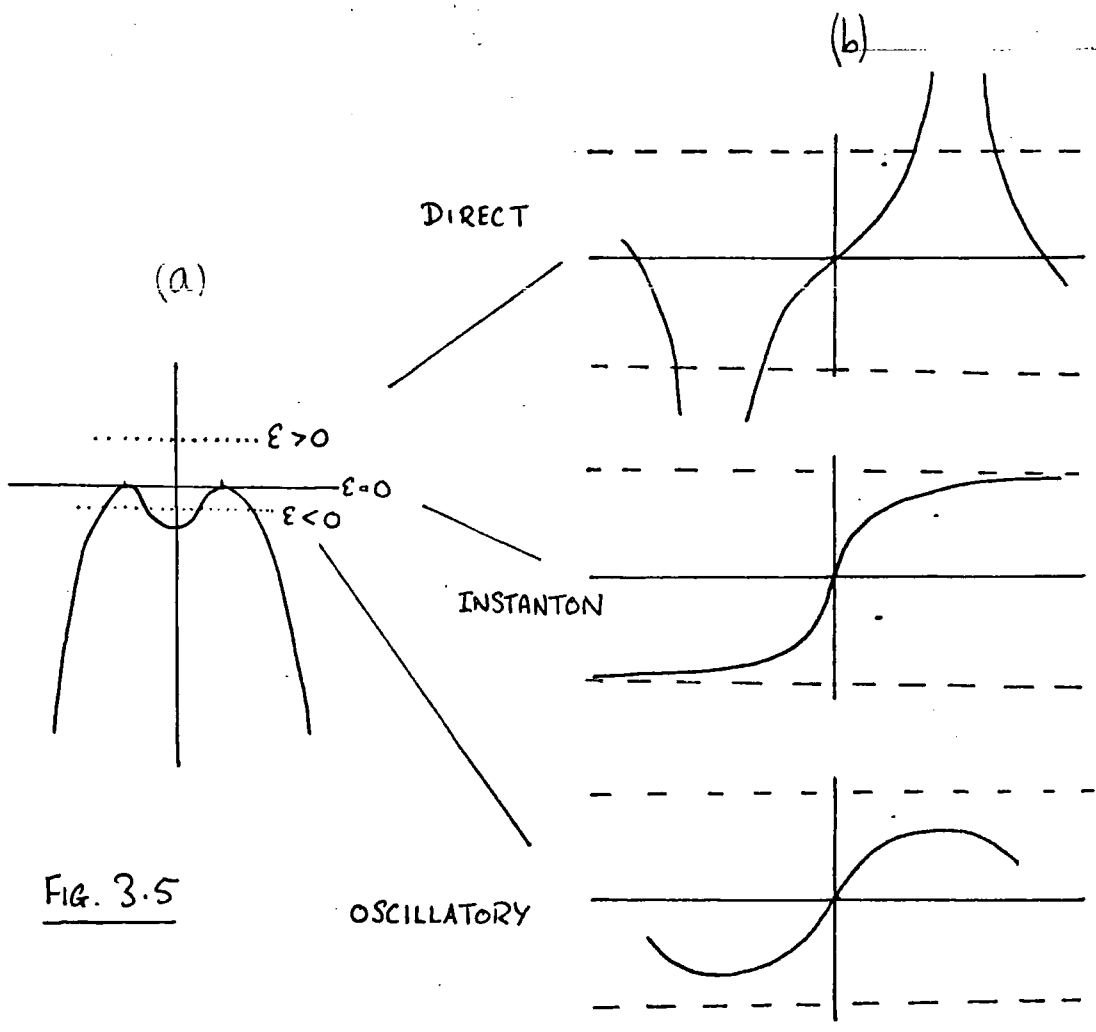


FIG. 3.5

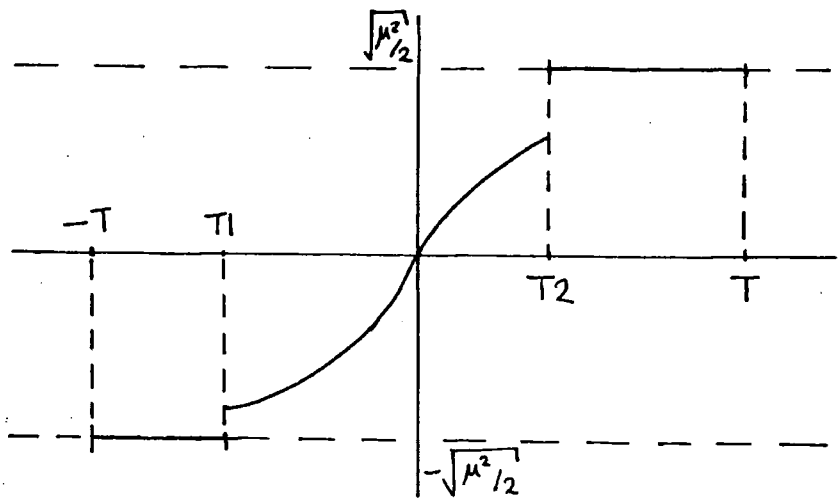


FIG. 3.6

particle generally moves from left to right . There are other solutions corresponding to motion , say , from the left-hand hump off towards negative infinity , but we do not consider these . The instanton is given by a specific value of ε , and allowing for a spread in ε means that we must include other members of the family . The use of these solutions in an extended semiclassical calculation will now be developed .

3.2 - Phase-space semiclassical method

In this section we work entirely with the quartic oscillator so as to get a feel for the method . In addition the calculations take substantially longer with the supersymmetric problem (since the functional form of the solutions is much more complicated) and we have not had time to thoroughly investigate this case .

The constituent parts of the approach are firstly that the boundary conditions should be relaxed from the strict classical ones to allow for uncertainty , and secondly that classical solutions other than the instanton may then be used . Although we have seen that there are several ways of visualising the method , the easiest from the point of view of calculation seems to be to work with Gaussian spreads in position and momentum . In accordance with the usual calculation , we work inside a finite box (i.e. a finite time-interval $[-T,T]$) in order to cope with the zero-mode . We propose to use the following family of solutions -

3.2.1)
$$\bar{q}(t) = \begin{cases} -\sqrt{\mu^2/2} & \ldots\ldots \quad -T < t < T1 < 0 \\ q_\varepsilon(t) & \ldots\ldots \quad T1 < t < T2 \\ +\sqrt{\mu^2/2} & \ldots\ldots \quad 0 < T2 < t < T \end{cases}$$

where $q_\varepsilon(t)$ is any of the non-instanton solutions of (3.1.14a) . Thus the normal calculation uses the single member of this family with $T1=T2=T$ and $\varepsilon=0$. Fig. 3.6 shows the form of this choice of $\bar{q}$. Note that $\bar{q}$ is always a solution of the equations of motion and so an extreme point of the equations of motion . There are two "jumps" in the solution at T1,T2 in accordance with the relaxed boundary conditions - these are present purely to give a definite functional form for $\bar{q}$ to insert into the functional integral , and reflect the mixture of classical and quantum ideas here . The essential content of the solution is that the middle section does not start and finish <u>at</u> the minima of V , but only <u>near</u> them , where nearness is measured by the uncertainty in position and momentum . The particle should be imagined during the first and last time-intervals not as residing at these minima , but around them , and with this picture the solution has no discontinuities . In a full quantum approach only the complete amplitude is important , and our selection of a particular path or family of paths is only a tool to aid calculation of this amplitude .

We may isolate the specific classical and quantum elements in the scheme as follows . Classical ideas enter in that

1) we choose representative paths which are piecewise solutions of the classical equations of motion , and
2) we assume that for a particular path we may work with definite values of position and momentum in order to calculate the associated amplitude .

The significance of the second point is unclear , as we will integrate the various contributions over the whole of phase-space anyway , and so conceptually need only think of taking

representative values of q and p from a small but finite region of phase-space in accordance with the uncertainty principle . Also the assumption is implicitly made in the standard semi-classical approach anyway .

Quantum-mechanical ideas enter in that

1) we calculate Gaussian fluctuations around the classical paths as before , and
2) the family of allowed paths is extended because of the spread of values in the energy parameter , given by uncertainty .

Thus quantum-mechanical aspects of the problem are incorporated here to a greater extent than with the usual method .

To integrate over this family of paths we must first define a measure , which we take to be given by the spread of the wave-function in phase-space .If we temporarily regard the wells as separate and construct trial Gaussian wave-functions in each well according to the usual curvature method , then we may reinterpret the jumps in (3.2.1) as being probabilities of the particle being at that particular point of phase-space , given these trial wave-functions in position and momentum space . Thus we have two integrals over phase-space , corresponding to the jumps at T1 and T2 . T1 and T2 are given implicitly by the values of the jumps together with the requirement that at $t = 0$, $\bar{q} = 0$. This would seem to give a 4-dimensional integral to perform , but there is a constraint that the value of $\mathcal{E}$ should be constant in [T1,T2] - from the equation of motion we see that

3.2.2) $$p_1^2 - \left(\frac{M^2}{2} - q_1^2\right)^2 = \mathcal{E} = p_2^2 - \left(\frac{M^2}{2} - q_2^2\right)^2$$

where $q_{1,2}$ ($p_{1,2}$) are the jumps in position (momentum) at T1,T2 . We use this to fix the value of p_2 .

REGION OF PHASE-SPACE IN WHICH INTEGRAND IS NON-ZERO

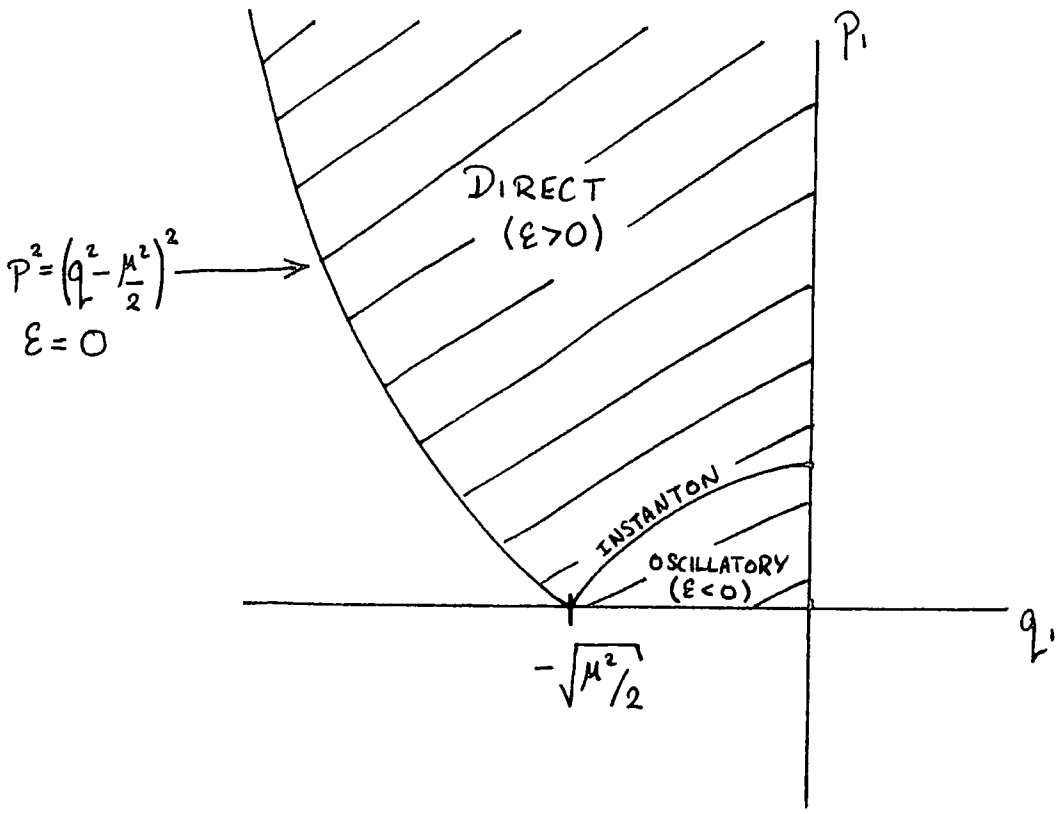


FIG. 3.7

If we call the value of the correction due to the solution (3.2.1) $K(T1,T2;\varepsilon)$, then the full calculation we wish to do is

3.2.3) $$\overline{\Delta E} = \frac{\int dq_1 \int dp_1 \int dq_2 \int dp_2 \; K(T1,T2;\varepsilon)\, P_1(q_1,p_1)\, P_2(q_2,p_2)\, \delta\left(p_2^2 - p_1^2 - \left(\frac{\mu^2}{2} - q_2^2\right)^2 + \left(\frac{\mu^2}{2} - q_1^2\right)^2\right)}{\int dq_1 \cdots \int dp_2 \; P_1 \cdot P_2 \cdot \delta(\cdots)}$$

The region of $q_1 - p_1$ phase-space over which the integrals are to be performed is shown in fig. 3.7 - for q_2 and p_2 the shape is analogous . Calculation of K for a given solution proceeds in fact exactly the same as for the ordinary calculation - details are given in appendix 3.3 - except that usually now the integrals cannot be done analytically and we must resort to numerical methods . Note that for this calculation we retain use of the dilute gas approximation , rather than discarding it and attempting to use a different scheme , as we feel that allowing solutions other than the instanton is a more important physical effect to include . Furthermore it is shown in [6] that the inclusion of quasi-solutions of the equations of motion is equivalent to the complex saddle-point method , which gives good results in situations in which the dilute gas approximation is known to fail . Although the solutions we use are not what is meant by quasi-solutions in these references , we believe it possible that the same effect is being probed - the answers obtained will provide some justification for this viewpoint .

The weighting factors are given by

3.2.4) $$P_1(q_1,p_1) = \exp\left[-2\sqrt{\frac{\mu^2}{2}}\left(q_1 + \sqrt{\frac{\mu^2}{2}}\right)^2 - p_1^2/2\sqrt{\mu^2/2}\right]$$

$$P_2(q_2,p_2) = \exp\left[-2\sqrt{\frac{\mu^2}{2}}\left(q_2 - \sqrt{\frac{\mu^2}{2}}\right)^2 - p_2^2/2\sqrt{\mu^2/2}\right]$$

The complete answer is given as before by three terms

3.2.5) $$E_{full} = E_{min} + E_2 - \overline{\Delta E}$$

where here $E_{min} = 0$ and E_2 is the usual quadratic term (as in (3.1.8)) .

QUARTIC OSCILLATOR

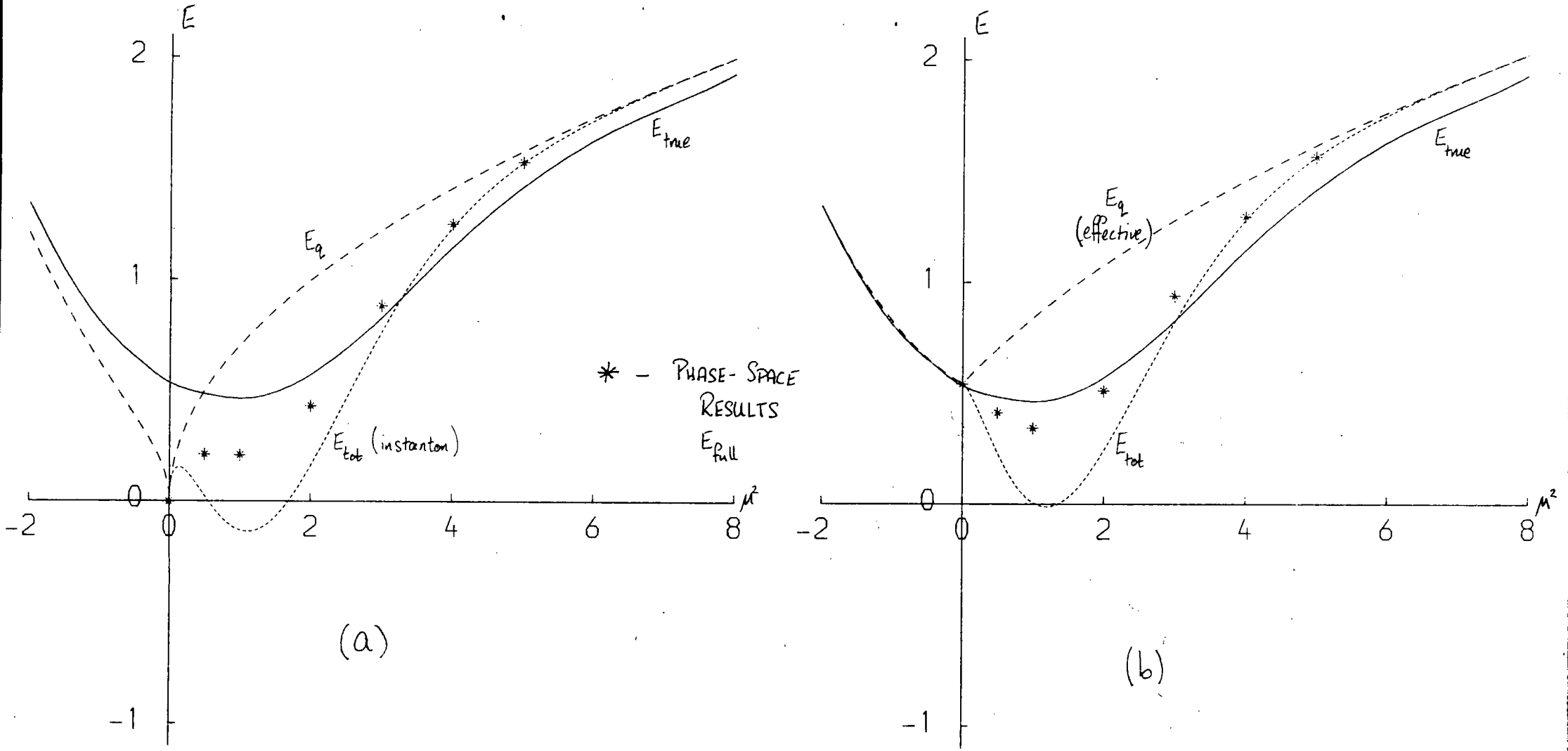


FIG 3.8

Results are shown in fig. 3.8a and are seen to be more reliably accurate than the ordinary calculation , though still by no means perfect . We surmise that using trial wave-functions for (3.2.4) which better reflect the true shapes would improve results further , but have not had time to check this . Notice that the reason for the failure of both the ordinary instanton calculation and this extended method to give good answers near $\mu^2=0$ is the poor quality of the quadratic term E_2 here . Fig. 3.8b shows results if instead of E_2 we use the refined version calculated in section 2.2 and displayed in fig. 2.5a . This has most effect near $\mu^2=0$ and the results are very encouraging - both the usual answer and the new one are improved , but the full phase-space calculations to a much greater extent . It is plausible that the remaining discrepancies are due to the neglected higher-order terms in the evaluation of K , P_1 , and P_2 , rather than showing the need for another new calculational method .

We have performed the same calculation for the unbroken supersymmetric case with $\mu=0$ - as explained above there has not been time to repeat it in the general case . Clearly the working is no different in essence from the above situation , although there are many differences in detail . Recall that the usual calculation gave (neglecting overall scales)

3.2.6) $$\Delta E \approx 1.51 \longrightarrow E_{tot} = -1+\sqrt{3}-1.51 \approx -0.78$$

The full phase-space integral answer is

3.2.7) $$\overline{\Delta E} \approx 0.68 \longrightarrow E_{Full} \approx 0.05$$

- evidently a substantial improvement .

We wish to make the following point clear . It has been thought that only the instanton solution need be used because 1) it is easy to calculate , 2) the action of the instanton is the least value of action of all paths satisfying the boundary conditions , and 3) it is continuous everywhere . Another reason will be discussed next chapter . The first point is indubitable , though one relic of the simplicity of the instanton calculation does persist in the full phase-space calculation - we will turn to this next section . The second point is partially true in that ΔE consists of two parts - the action (for which the claim is true) and the ratio of determinants (for which it is not) . Thus the instanton need not give the dominant piece in ΔE , and by the time we have averaged over phase-space it is far from clear where we expect the major contribution to originate . The final point is not , we believe , an important one . This is partly because the idea of a single continuous path followed by the particle is a classical one , and may be inappropriate here , and partly also because of the following . When defining the path-integral by discretising the time-interval , paths which are only piecewise smooth are considered (e.g. piecewise linear in position and constant in momentum , or some similar prescription) and only in some limit does one arrive at completely smooth paths . Here we have split the interval into three sub-intervals and taken paths which are piecewise solutions of the equations of motion . Presumably the equivalent of the above process would be to take more and more subdivisions of the main interval , with a "jump" (and thus a constrained phase-space integral) at each division . This would entail progressively higher-dimensional integrals and would rapidly

become impractical computationally . Fortunately the results seem to indicate that we do not need to do this .

As the closing part of this chapter we wish to report a rather curious fact which emerged from the calculations . In terms both of computing and analytic work , it is much better not to have to calculate the part of ΔE which comes from the ratio of determinants (see comments after (3.1.9)) . This would be even more the case in field theory where there is a long-standing controversy concerning this calculation [10] . For the usual instanton calculation it was found that there was a physical interpretation for this ratio , and numerically it supplies a factor of $2\mu^2$. For a given member of the family of solutions in (3.2.1) this is no longer true . However having integrated over all such solutions this pattern is recovered :

	Ratio of determinants		
μ^2	included	omitted	col.2 / $2\mu^2$
0.5	0.29	0.32	0.29
1.0	0.50	0.33	0.25
2.0	0.57	0.17	0.14
3.0	0.34	0.077	0.057
4.0	0.16	0.022	0.020
5.0	0.050	0.0051	0.0050

Similarly the unbroken supersymmetric case with $\mu = 0$ gives

(included)	(omitted)	(inc) / 6
0.68	0.010	0.011

We feel that there must be some underlying physical reason for this result and if this were properly understood , similar calculations in field theory would be made much easier . In terms of the construction of appendix 3.3 , the omission of this ratio seems to correspond to using a solution which at t=0 jumps directly from one well to the other rather than smoothly making this transition .

We have seen that this extended semiclassical calculation , in which more of the quantum-mechanical nature of the system is taken into account , provides answers which are considerably more reliable than the original version . The results are even better if the original quadratic estimate of the zero-point energy is replaced by an effective value as derived in section 2.2 . If we regard the original semiclassical method as a weak-coupling approximation (large) , then this extended method , when used with an effective quadratic estimate for negative μ^2 , may be seen as an attempt to find a strong-coupling approximation while still retaining the dilute gas approximation , which is easy to use . From this point of view , the results shown in fig. 3.8b are very promising . There are drawbacks to the method . Firstly one needs a much greater knowledge of the solutions of the equations of motion , although in the next section in which we turn to CP^{n-1} model field theories , we shall see that this knowledge is already available , at least in certain cases . Secondly it requires the use of a computer , as very few of the intermediate integrals can be performed analytically . In view of the greater physical and numerical accuracy of the calculations , we feel that these are not serious problems .

Chapter 4 - CP^{n-1} Model Field Theories

4.1 - Classical properties of general CP^{n-1} solutions

The work in this chapter is based on [1] , although at the time of writing that the possible significance of the solutions was not known . The models considered here are not supersymmetric (although supersymmetric versions of the CP^{n-1} models do exist [20]) , since we wish only to illustrate that a semi-classical calculation incorporating general (non-instanton) solutions is possible here in direct analogy with that of the previous chapter .

CP^{n-1} field theories [19] are models in two space-time dimensions , defined by a Lagrangian density

4.1.1) $$\mathcal{L} = \tfrac{1}{2}(D_\mu z)^\dagger . D_\mu z \quad , \quad D_\mu z = \partial_\mu z - A_\mu z \quad , \quad A_\mu = z^\dagger . \partial_\mu z$$

where z is an n-component complex vector satisfying $z^\dagger . z = 1$. The model has local U_1 and global SU_n gauge invariances , represented by multiplication of z by a phase and pre-multiplication by an SU_n nxn matrix . Subscript μ runs over Lorentz indices 1,2 . As in chapter 3 we shall work in Euclidean space , and to this end define auxiliary variables

4.1.2) $$x_\pm = x \pm it$$
$$\partial_\pm = \tfrac{1}{2}(\partial_x \mp i \partial_t)$$

One can also define a topological charge density

4.1.3) $$q = \frac{1}{2\pi i} \varepsilon_{\mu\nu} (D_\mu z)^\dagger . D_\nu z$$

where as usual the total charge $Q = \iint q \, dx \, dt$ takes integer values and thus partitions field configurations into disjoint classes . These models have been extensively studied because they display very similar properties to Yang-Mills models in

four dimensions , while being much simpler .

As explained in [1] , a variety of solutions of the classical equations of motion have been found , among which are instanton solutions . One of the conclusions of [1] was that for CP^1 one could find a family of neighbouring solutions joining smoothly to an arbitrary instanton in any charge sector .
For general charge in general CP^{n-1} such families have not yet been found , so that some instanton solutions appear isolated at present - we conjecture that they are not in fact isolated , and that the required families have simply not yet been found . For this reason we concentrate here on CP^1 , although in the charge one sector this is not necessary .

For CP^1 one may parametrise the charge q (> 0) instanton solutions by [24]

4.1.4) $$z = N\left(\prod_{j=1}^{q}\left(\frac{x_+ - a_j^-}{x_+ - a_j^+}\right)\right) ; \quad R$$

$\begin{cases} a_j^{\pm} \text{ are complex constants,} \\ R \text{ is a real scale,} \\ N \text{ normalises} \end{cases}$

- for negative charge (anti-instantons) replace x_+ by x_- .
For an instanton of charge q the action is $\pi.|q|$, and it can be shown that all continuous field configurations of charge q have action strictly greater than this . Both action and charge density are strongly localised around the points $\frac{1}{2}(a_j^+ + a_j^-)$ with spread of order $\frac{1}{2}|a_j^+ - a_j^-|$ - these are referred to as the location and scale-size of the instanton .

To determine instanton solutions it is only necessary to consider the self- duality (or anti-self-duality) equations $D_{\pm} z = 0$, which are first order in derivatives , and it is readily seen that these equations have no other solutions (up to reparametrisation) . In order to find other solutions such as those discussed in chapter 3 we must turn to the full

equations of motion derived from the Lagrangian (4.1.1) ,

4.1.5) $$D_\mu D_\mu z + \left[(D_\mu z)^\dagger . D_\mu z \right] z = 0$$

which are second order and hence much harder to solve . The issue is further complicated by the gauge invariances of the theory as some solutions which at first sight seem to be new , prove on closer inspection to be gauge-equivalent to earlier ones . Historically the first non-instanton solution to be found was the meron [28] ;

4.1.6a) $$z = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta} \\ 1 \end{pmatrix} \quad ; \quad \theta = \arg \prod_{j=1}^{q} \left(\frac{x_+ - a_j^-}{x_+ - a_j^+} \right) .$$

This depends on the single function $\theta(x_+)$. In order to find solutions which join smoothly between the instanton and meron solutions we must include some dependence on a second function $\varphi(x_+)$ such that

$$\partial^2 \varphi = \partial_\mu \varphi \partial_\mu \theta = \partial^2 \theta = 0 \quad ; \quad \partial_\mu \varphi \partial_\mu \varphi = \partial_\mu \theta \partial_\mu \theta$$

Such a function is given by

$$\varphi = \ln \left| \prod_{j=1}^{q} \left(\frac{x_+ - a_j^-}{x_+ - a_j^+} \right) \right|$$

and the required families of solutions are

4.1.6b) $$z = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + k.sn\varphi} \; e^{i\theta} \\ \sqrt{1 - k.sn\varphi} \end{pmatrix}$$

and

4.1.6c) $$z = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + sn\,\varphi/k} \; e^{i\theta} \\ \sqrt{1 - sn\,\varphi/k} \end{pmatrix}$$

where sn is an elliptic function with parameter k , $0 \leq k \leq 1$, [5] .

It may be shown that as $k \to 1$, both families of solutions merge smoothly onto the instanton solution of (4.1.4) . Also , as $k \to 0$ both families tend to the meron solution above . All of these

solutions have charge q , as explained later . By changing x_+ to x_- we may obtain solutions of negative charge .

In Euclidean space , all of these solutions except the instanton have infinite action - this is due to the vortex-like behaviour around the points $a_j^{\pm}$ and not because the solutions are any less localised than the instanton . However , for this reason they have often been thought not to play a role in the theory .

The later part of this chapter will use these solutions to develop the first stages of an extended semiclassical calculation for the CP^1 model , as was done for quantum-mechanical problems last chapter . Before doing that , however , we wish to describe various features of the above solutions , viewed at the purely classical level . Further details of this work may be found in [1] .

Firstly we can ask whether these solutions exhaust the possible solutions of the equations of motion . At first sight this seems not to be the case - if we parametrise field configurations by

4.1.7a) $$z = N \begin{pmatrix} re^{i\theta} \\ 1 \end{pmatrix}$$

then the equations of motion are

4.1.7b) $$\partial_\mu \left(\frac{r^2 \partial_\mu \theta}{(1+r^2)^2} \right) = 0 \; ; \; \partial_\mu \left(\frac{\partial_\mu r}{1+r^2} \right) = \frac{r(1-r^2)}{(1+r^2)^2} \partial_\mu \theta \partial_\mu \theta$$

The first is satisfied by choosing $r = r(\varphi)$ where $\partial^2 \varphi = \partial^2 \theta = \partial_\mu \varphi \partial_\mu \theta = 0$. Then if also $|\partial_\mu \theta|^2 = |\partial_\mu \varphi|^2$ and we set $r = \tan s$, the second becomes $s'' = \frac{1}{4} \sin 4s$, which clearly has elliptic solutions . At first sight our solutions appear not to depend on enough arbitrary constants of integration (the only manifest one being k) .

However , some of the arbitrariness corresponds only to choice of origin - we could have written sn $(\varphi-\varphi_0)$ with φ_0 arbitrary - and some is absorbed by the U_1 gauge - we could have written $e^{i(\theta-\theta_0)}$ with θ_0 arbitrary . However , there is one parameter which we have fixed to be zero - it seems to be linked with the choice of (global) SU_n gauge and is therefore unimportant . Certainly for the meron case it may be explicitly shown that the parameter is not a genuine degree of freedom . In addition the general arguments of [26] , based on the properties of harmonic maps , indicate that these solutions are exhaustive , at least in the charge one sector . We therefore feel justified in assuming that the family of solutions in (4.1.6) is complete .

Next we turn to the question of charge distribution . As explained earlier the single instanton solution exhibits a cloud of charge density with a well- defined orientation in parameter-space , symmetric in position-space , and vanishing rapidly away from its centre . Multi-instantons are similar , but with several centres and hence a more complicated structure . As we reduce the value of k from one , two point sources of charge each of intensity $\frac{1}{2}$ appear at locations $a^{\pm}$ - the diffuse cloud of charge is also present , but because of the periodicity of the elliptic functions it carries zero <u>total</u> charge - thus the total charge of the configuration is conserved . Once again the picture for higher charges is qualitatively the same . As $k \rightarrow 0$ the cloud vanishes and for the meron we are left with just the two point sources . The relationship between meron (a function of x_+ with charge +1) and anti-meron (a function of x_- with charge -1) is unclear . The two solutions are formally U_1 gauge-equivalent , but the necessary transformation

is singular at the points $a^{\pm}$, and this causes the apparent ambiguity in the sign of the charge . We regard the two solutions as being essentially different since elliptic ($k \neq 0$) solutions arbitrarily close on either side cannot be mapped into one another by this method . Another artifact of an invariance , this time the conformal invariance , has given rise to the idea that merons (and by extension , the elliptic solutions) can exist as individual point sources as opposed to always being linked in pairs . This uses a conformal transformation to send one of the location parameters $a^{\pm}$ to the origin and the other to infinity - this gives

$$\theta = \arg x_+ \quad , \quad \varphi = \ln |x_+|$$

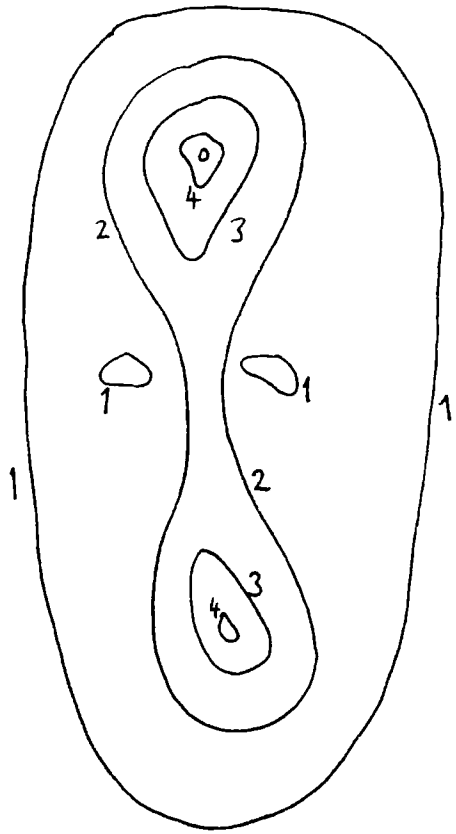
which gives the solutions an axial symmetry . Because of the concurrent loss of information at infinity , determination of the properties of these solutions requires care and this has led to apparent ambiguities (such as the claim that the charge of the elliptic solutions was not localised) . However , this choice of solution is convenient for the investigation of conformally invariant quantities , because of its simplicity .

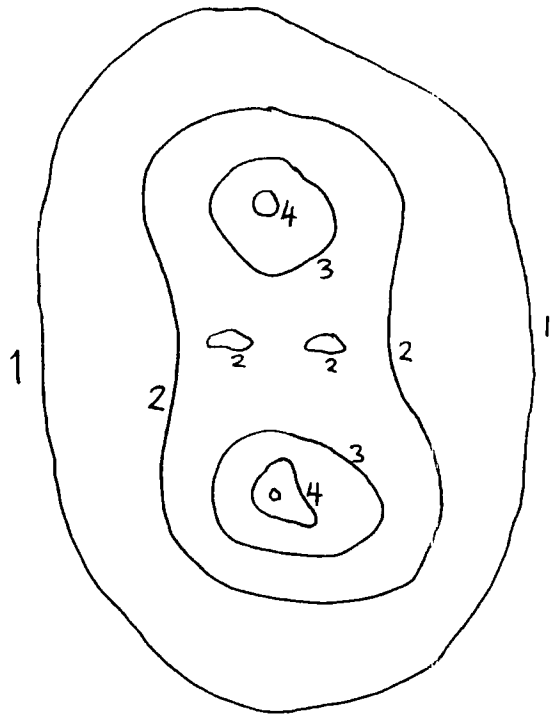
The existence of a smooth one-parameter family of solutions linking the instanton (a single extended object) with a meron (two point sources of charge) has been used as a conjectural model of the process of confinement [13] ; the analogous situation would be that a nucleon (extended) behaves under suitable conditions like a collection of quarks (point charges) . There has been speculation as to whether the number of "constituents" (used in the above rather restricted sense) of instantons in CP^{n-1} models is constant or increases with n - in the former case merons would be sufficient , while in the

FIG. 4.1

CONTOURS OF ACTION DENSITY

(NUMBERS ARE INDICATORS, NOT ACTUAL VALUES)





latter , objects called instanton-quarks have been suggested [24] . We have studied the behaviour of the action density of 2-instanton solutions of CP^1 and CP^2 models in order to see if there is any qualitative difference visible at the classical level between them . If merons are sufficient the two pictures should not differ , whereas if instanton-quarks are required the CP^2 case should show more intricate structure . A typical plot of action density (or equivalently charge density , as the two are proportional for instanton solutions) is shown in fig. 4.1 . It will be seen that there is no qualitative difference - at the classical level , merons seem sufficient to account for the structure of instantons .

For the model of confinement proposed in [13] , the interaction between merons was investigated in [45] . Because of oscillatory singularities at $a^{\pm}$, we must regularise the solutions in some way , and in that paper this was done by the insertion of instanton cores around $a^{\pm}$, each of charge $\frac{1}{2}$ in order to mimic the point source of charge $\frac{1}{2}$. Thus such a "smeared" meron would be

4.1.8)
$$z = \begin{cases} N_1 \left(\frac{\Omega}{R_1}\right) & \cdots\cdots \text{ instanton scale-size } R_1 \cdots |\Omega| < r_1 \\ \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i \arg \Omega} \\ 1 \end{pmatrix} & \cdots\cdots \text{ meron } \cdots r_1 < |\Omega| < r_2 \\ N_2 \left(\frac{\Omega}{R_2}\right) & \cdots\cdots \text{ instanton scale-size } R_2 \cdots r_2 < |\Omega| \end{cases}$$

where $\Omega = \frac{x_+ - a^-}{x_+ - a^+}$ and $r_1 < |\Omega| < r_2$

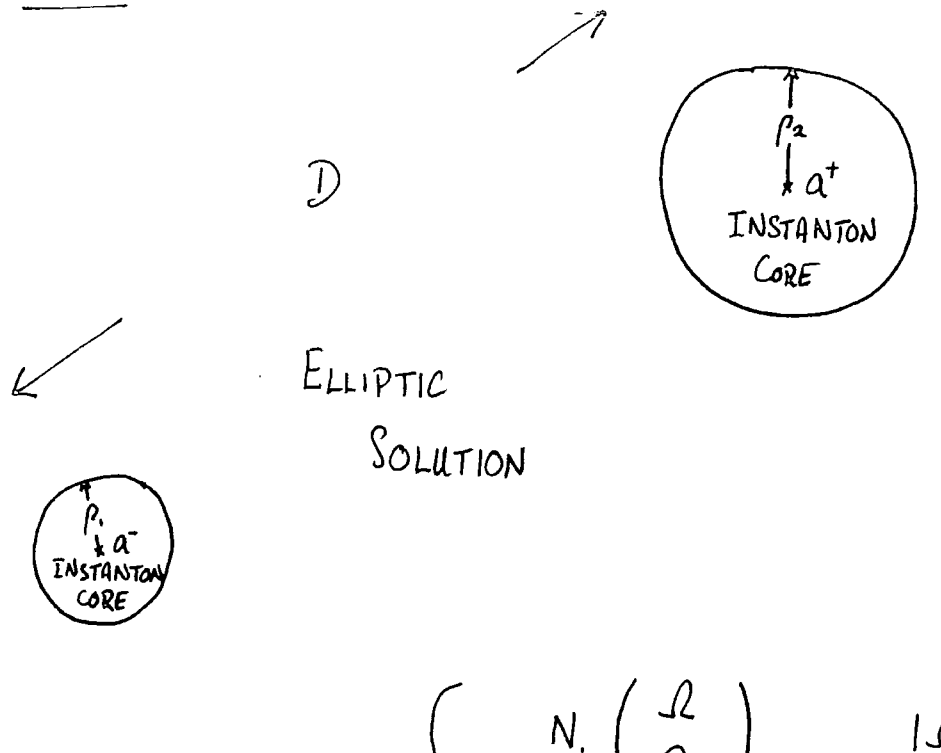
For continuity of z we require $R_i = r_i$, $i = 1,2$. To similarly investigate the interaction of elliptic solutions we do the same , except that the central portion is now

$$z = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + k\, sn(\ln|\Omega|)} \\ \sqrt{1 - k\, sn(\ln|\Omega|)} \end{pmatrix} e^{i \arg \Omega}$$

and the continuity conditions become

FIG. 4.2a

SMEARING :



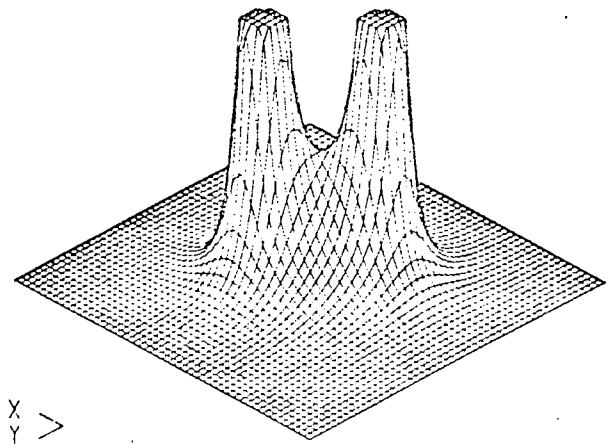
$$Z = \begin{cases} N_1 \begin{pmatrix} \Omega \\ R_1 \end{pmatrix} & |\Omega| < r_1 \\ \frac{1}{\sqrt{2}} \left(\frac{\sqrt{1 + sn \cdot \ln|\Omega|}}{\sqrt{1 - sn \cdot \ln|\Omega|}} e^{i \cdot \arg \Omega} \right) & r_1 < |\Omega| < r_2 \\ N_2 \begin{pmatrix} \Omega \\ R_2 \end{pmatrix} & r_2 < |\Omega| \end{cases}$$

$$\Omega = \frac{x_+ - a^-}{x_+ - a^+}$$

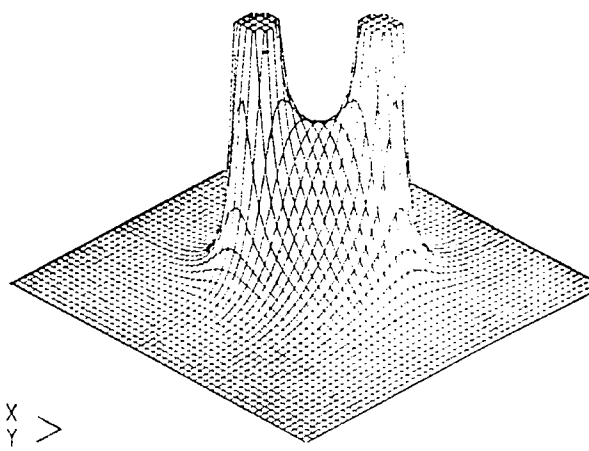
$$\rho_1 = \frac{|a^+ - a^-| r_1}{1 - r_1^2} \quad , \quad \rho_2 = \frac{|a^+ - a^-| r_2}{r_2^2 - 1}$$

$$D = \frac{|a^+ - a^-| (r_2^2 - r_1^2)}{(1 - r_1^2)(r_2^2 - 1)}$$

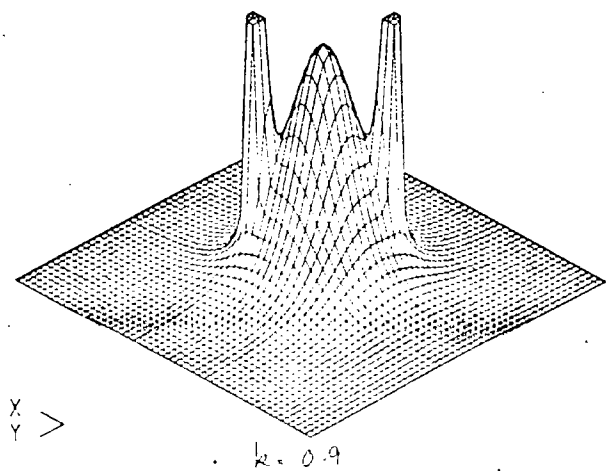
ACTION DENSITY – SMEARED ELLIPTIC SOLUTIONS
(ON SAME SCALE AS INSTANTON)
SMEARING RADII $\frac{1}{10}$, 10.
LOCATION PARAMETERS $a^{\pm}$ SAME THROUGHOUT.



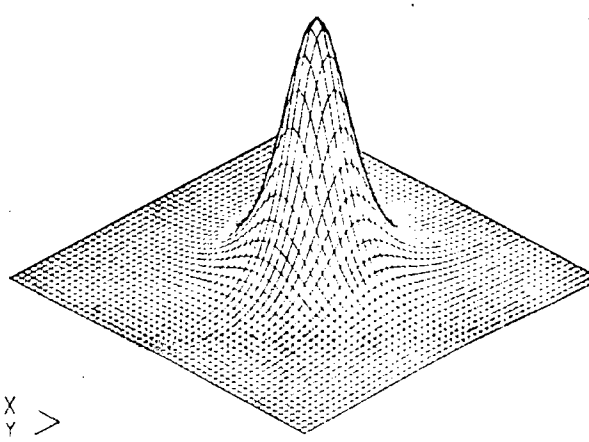
$k=0$ (ie meron)



$k=0.5$



$k=0.9$

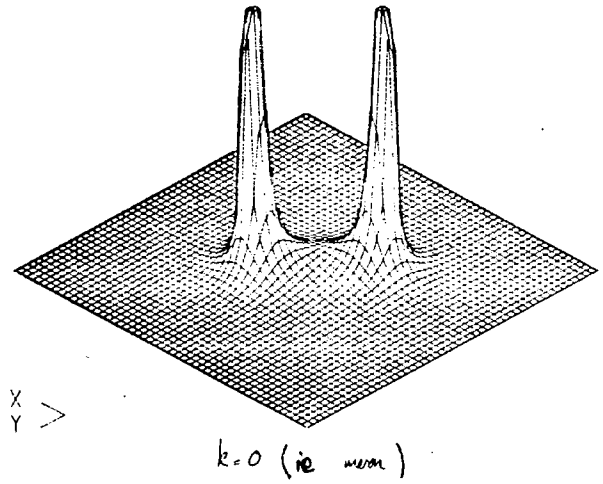


$k=1$ (ie instanton)

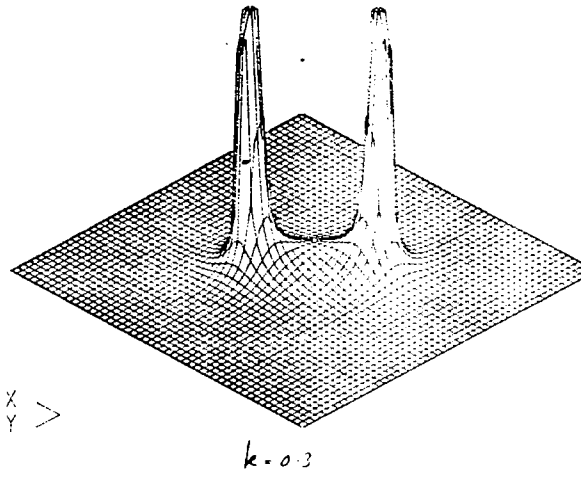
FIG. 4.2 b(i)

Action Density – Smeared Elliptic Solutions

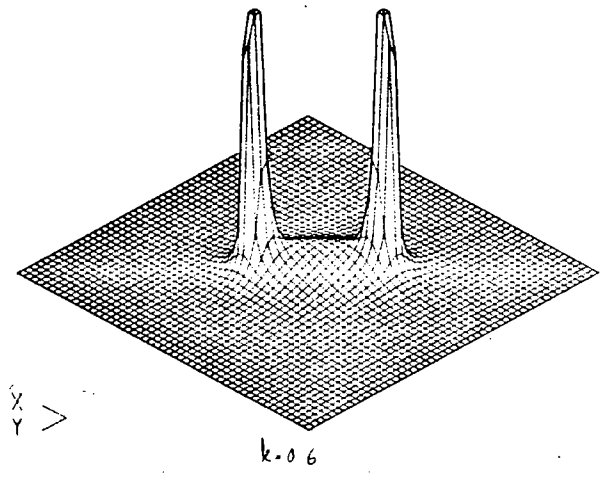
Smearing Radii $\frac{1}{10}$, 10.



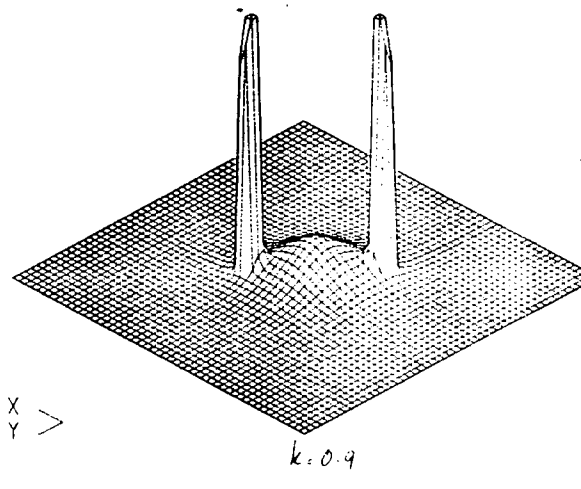
k = 0 (ie mean)



k = 0·3



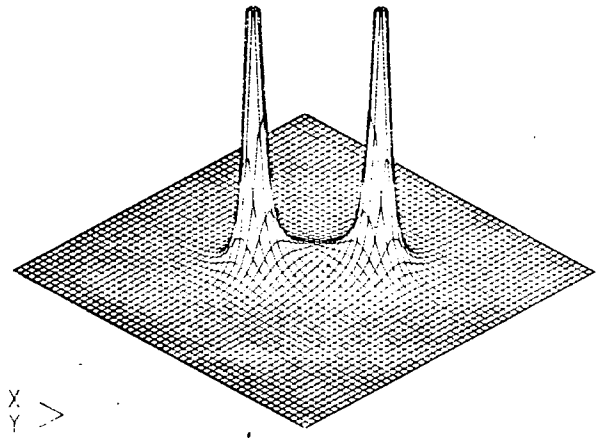
k = 0·6



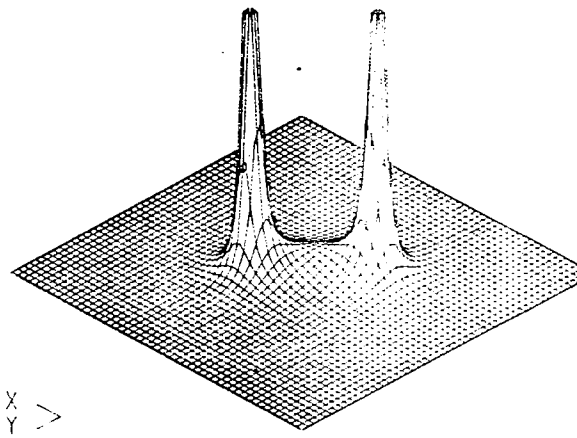
k = 0·9

Fig. 4.2b.(ii)

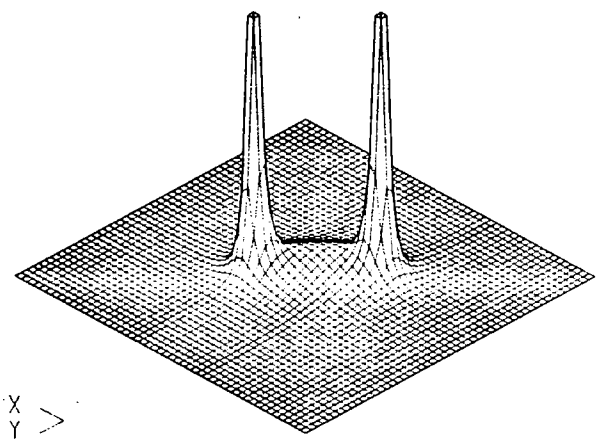
ACTION DENSITY – UNSMEARED ELLIPTIC SOLUTIONS



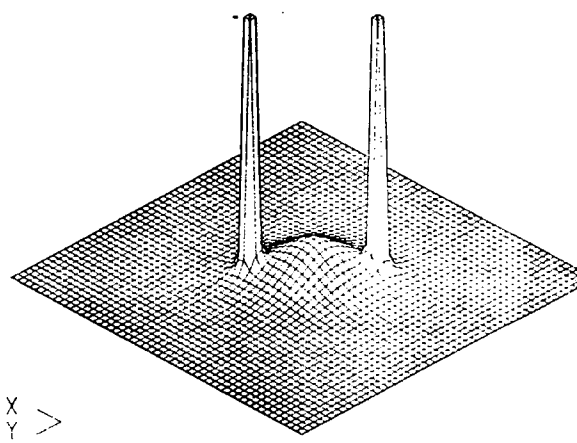
$k = 0$ (a maxima)



$k = 0.3$



$k = 0.6$



$k = 0.9$

FIG 4.2.6(iii)

4.1.9) $$k\, sn(\ln r_i) = \frac{r_i^2 - R_i^2}{r_i^2 + R_i^2} \quad , \quad i = 1,2$$

Since $-1 \leq sn\, x \leq +1$, this implies

$$\frac{1-k}{1+k} \leq \frac{r_i^2}{R_i^2} \leq \frac{1+k}{1-k}$$

which for k = 0 recovers the original condition . So in the general case we may use a wider collection of part-instantons to smooth the solution . A schematic picture of such a solution is shown in fig. 4.2a , and the action densities for smeared and unsmeared solutions for a variety of values of k are shown in fig. 4.2b .

The total action of the configuration is $S_{tot} = S_{inst} + S_{ell}$

$$S_{inst} = \pi \left\{ \frac{r_1^2/R_1^2}{1 + r_1^2/R_1^2} + \frac{1}{1 + r_2^2/R_2^2} \right\}$$

4.1.10a) where $$S_{ell} = \frac{\pi}{2} \int_{\ln r_1}^{\ln r_2} d\varphi\, dn^2\varphi - \frac{\pi}{4}(1-k^2) \ln r_2/r_1$$

Since the action of a single instanton is π , look at

4.1.10b) $$\frac{S_{tot} - \pi}{\pi} = \frac{1}{2} \int_{\ln r_1}^{\ln r_2} d\varphi\, dn^2\varphi - \frac{1-k^2}{4} \ln \frac{r_2}{r_1} + \frac{1}{1 + r_2^2/R_2^2} - \frac{1}{1 + r_1^2/R_1^2}$$

$$= \frac{1}{2} \left\{ F(-\ln r_1, k) + F(\ln r_2, k) \right\}$$

where $$F(t,k) = \int_0^t d\varphi\, dn^2\varphi - \frac{1}{2}(1-k^2)t - k\, sn\, t = \mathcal{D}_n\, t - \frac{1-k^2}{2} t - k\, sn\, t$$

for which we have used the continuity condition (4.1.9) and the requirement $r_1 < 1 < r_2$. It may be shown that $F(t,k) \geq 0 \quad \forall t, k \geq 0$. Note also that $F(t,0) = \frac{1}{2} t$ which gives

4.1.10c) $$S_{tot}^{meron} = \pi \left(1 + \frac{1}{4} \ln \frac{r_2}{r_1} \right)$$

Thus we recover the logarithmic interaction between (smeared) merons derived in [45] . For general k the behaviour of F is shown in fig. 4.3 . Thus the interaction between smeared elliptic solutions is weaker than that between merons . The interaction vanishes for k = 1 , so that instantons do not interact (this is because they have least action in their charge sector and so have no tendency to evolve into other configurations) .Since the potential for k = 1 does increase with distance , these solutions can be expected to confine each

other in the way that merons do , as explained in [45] .

The total charge of the smeared configuration is

$$Q_{tot} = Q_{inst} + Q_{ell}$$

4.1.11) $$= \frac{r_1^2/R_1^2}{1+r_1^2/R_1^2} + \frac{1}{1+r_2^2/R_2^2} + \frac{k}{2}\int_{\ln r_1}^{\ln r_2} d\varphi \, \frac{d}{d\varphi}(\sin\varphi)$$

$$= 1$$

where again we have used the continuity condition . Thus as stated earlier , the total charge is conserved by the smearing process by dividing it between the instanton cores and the diffuse cloud . For smeared merons the cloud vanishes and we are left with just the instanton cores which mimic the point sources.

4.2 - Semiclassical use of the solutions

We now turn to the development of the extended semiclassical calculation . For this purpose we will only use solutions in the charge one sector of the theory , on the assumption that the dilute gas approximation is applicable as in the quantum-mechanical cases last chapter . There are indications that for the usual calculation this is not the case [46] , but we saw last chapter that inclusion of the additional solutions seemed to overcome such problems . Once again we refer to [6] for supporting evidence for retaining the dilute gas approximation here . For convenience we repeat equations (4.1.6) restricted to the case q = 1 .

Instanton : $$Z = N \left((x_+ - a^-) / (x_+ - a^+) \right)^{1}$$

4.2.1) Elliptic families :

$$Z = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{1+k\,sn\varphi}}{\sqrt{1-k\,sn\varphi}} e^{i\theta} \right) \quad \text{and} \quad \frac{1}{\sqrt{2}} \left(\frac{\sqrt{1+sn\varphi/k}}{\sqrt{1-sn\varphi/k}} e^{i\theta} \right)$$

with

$$\varphi = \ln \left| \frac{x_+ - a^-}{x_+ - a^+} \right| \quad , \quad \theta = \arg\left(\frac{x_+ - a^-}{x_+ - a^+} \right)$$

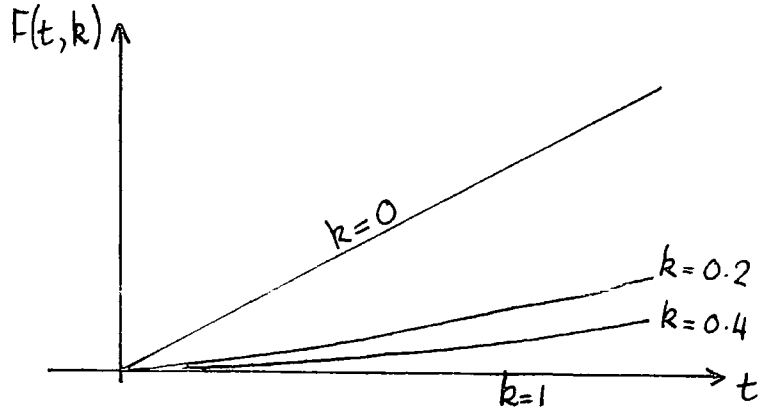


FIG. 4-3

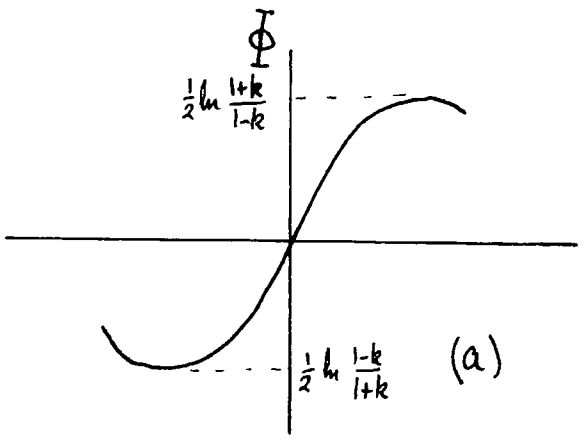
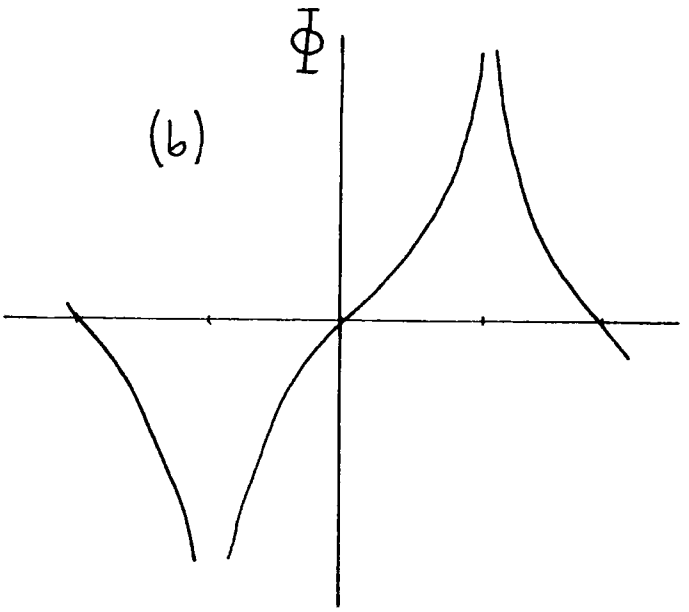


FIG. 4.4

$$\Phi = \ln \left| \frac{z_1}{z_2} \right|$$



cf. FIG. 3.5a,b

Asymptotically (large $|x_4|$) all these solutions tend to the same vacuum configuration $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$.

Note that all instantons of charge one in any CP^{n-1} model have corresponding elliptic families (and thus the work which follows may be repeated) as it is only in the higher charge sectors that problems arise .

In [1] it was thought that the two families (4.1.6b,c) were not essentially different - with the benefit of the experience of chapter 3 we realise that this was wrong . The elliptic function sn is periodic , and we will examine its behaviour over a single period . Consider first (4.1.6b) . Since $k<1$ and $|sn\varphi|\leq 1$ neither component of z ever vanishes ; instead they oscillate between $\sqrt{1\pm k}$. For (4.1.6c) , however , at alternate half-periods the solution takes the form $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$. If we plot the logarithm of the ratio of the two components ,

4.2.2) $$\Phi = \ln\sqrt{\frac{1+k\,sn\varphi}{1-k\,sn\varphi}} \quad \text{and} \quad \ln\sqrt{\frac{1+sn\varphi/k}{1-sn\varphi/k}}$$

we see the two figures 4.4a,b . Using this we identify the family (4.1.6b) with the oscillatory solutions of chapter 3 , and (4.1.6c) with the direct solutions . The vacuum configurations which they interpolate are $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$ - in CP^{n-1} models any constant vector (up to U_1 gauge freedom) is an acceptable vacuum as these are all equivalent under choice of global SU_n gauge . Since the elliptic solutions of (4.1.6) are essentially the only solutions of CP^1 , we have all the necessary ingredients for an extended semiclassical calculation here , as was performed for quantum mechanics in chapter 3 . As explained above , there probably are other solutions in higher CP^{n-1} models , though perhaps not in the

charge one sector . We have not explicitly performed this calculation , but present here some of the intermediary steps .

Notice that by a conformal transformation we may map any solution in (4.2.1) into any other with the same value of k - the Lagrangian (4.1.1) is invariant under the action of the conformal group in two dimensions . This gives rise to zero-modes of the fluctuation determinant , just as before the arbitrariness of the origin of time did . In order to make use of the Faddeev-Popov technique we must choose a particular representative of the family to work with by making an arbitrary choice of $a^{\pm}$. This explicitly breaks the conformal invariance ; the arbitrariness and hence the zero-modes are then absorbed into harmless volume factors . For simplicity we make the choice $a^{\pm} = \pm \frac{\lambda}{2}$ with λ an arbitrary (but definitely chosen) scale-length .

In the quantum-mechanical calculation we divided the complete time-interval into three sections , in each of which a solution of the equations of motion was used . The three solutions were separated by jumps in phase-space , and the whole patched together by integrations over phase-space . The analogue of this procedure in CP^1 is as follows . The purpose of truncating the elliptic parts was firstly to establish the correct boundary conditions and secondly to avoid the infinite action which arises from the periodicity of elliptic functions .
So here too we adopt a truncation procedure , but instead of jumping at two points we jump on two closed curves (these are still of measure zero compared with the whole space) . Once again this is done to avoid infinite action from oscillatory

behaviour . In the previous section we described a regularisation method in which the divergent centres of the elliptic or meron solutions were replaced by part-instanton solutions , a process known as smearing ; this was used to define the interaction between such solutions . This time we smear not with instanton cores but with constant (vacuum) solutions . For example , a solution continuous in z (but not its derivative) might be

4.2.3a)
$$z = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \cdots\cdots \quad \varphi < -k.K(k) \\ \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+sn\varphi/k} \\ \sqrt{1-sn\varphi/k}\, e^{i\theta} \end{pmatrix} & \cdots\cdots \quad -k.K(k) < \varphi < k.K(k) \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \cdots\cdots \quad k.K(k) < \varphi \end{cases} \qquad \begin{cases} \varphi = \ln \left| \frac{x_+ + \frac{\lambda}{2}}{x_+ - \frac{\lambda}{2}} \right| \\ \theta = \arg \left(\frac{x_+ + \frac{\lambda}{2}}{x_+ - \frac{\lambda}{2}} \right) \end{cases}$$

where the period of sn is 4.K(k) (K is the complete elliptic integral of the first kind) [5] . This solution is shown schematically in fig. 4.5a , while fig. 4.5b shows the variation of $|z_1|$ along $t = 0$. Around points $\pm \frac{\lambda}{2}$ are circular vacuum regions , with the elliptic solution filling the rest of space - as mentioned before , asymptotically in all directions we reach the vacuum $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This corresponds to one of the direct solutions of chapter 3 . The action of this configuration is

4.2.3b)
$$S = \frac{\pi}{2k} \left(2.E(k) - (1-k^2) K(k) \right)$$

where E(k) is the complete elliptic integral of the second kind [5] . S is evidently finite , and it is also positive . Note that in the limit $k \to 1$, the smearing circles shrink to zero radius and we are left with a single instanton solution filling all of space . As expected the action tends to $S = \pi$.

The generalised direct solutions including jumps are

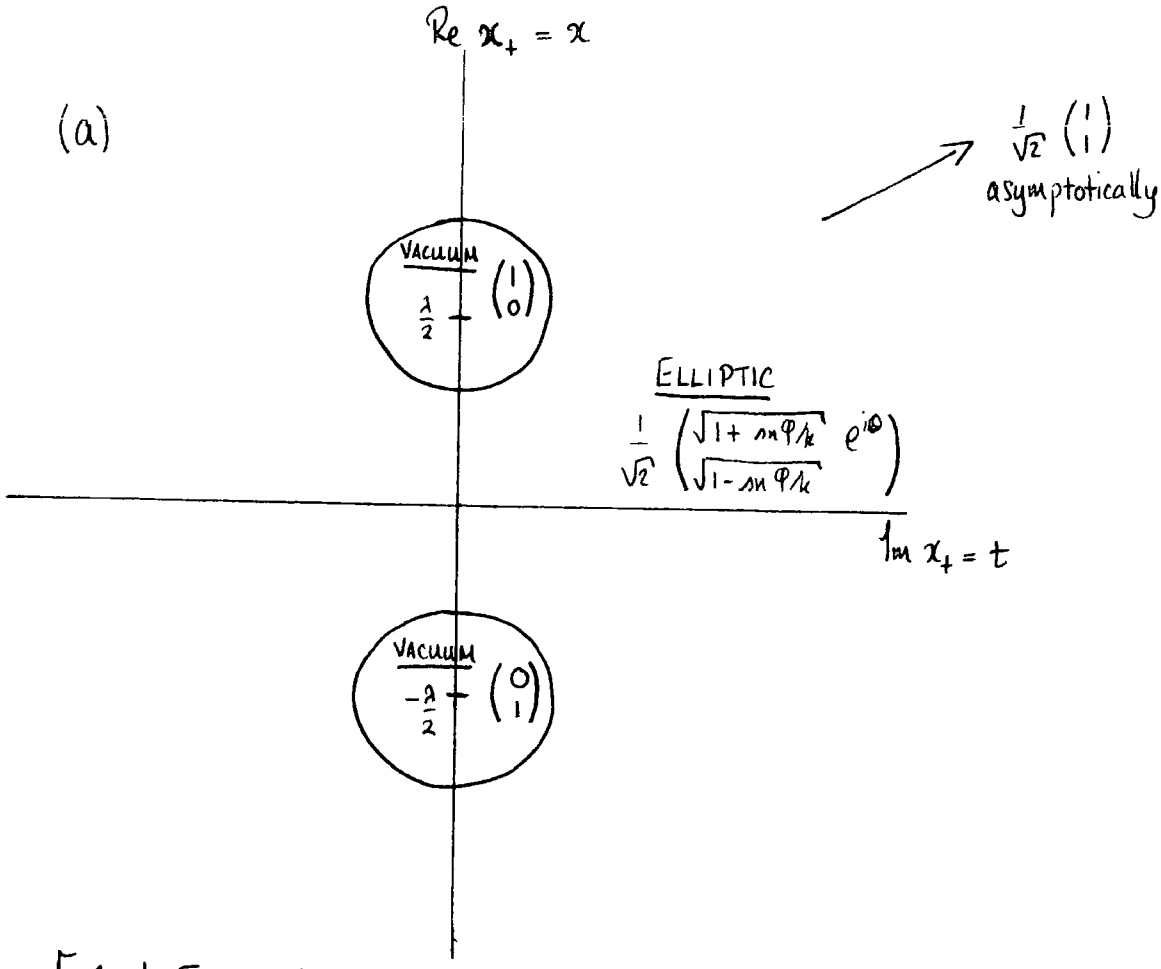
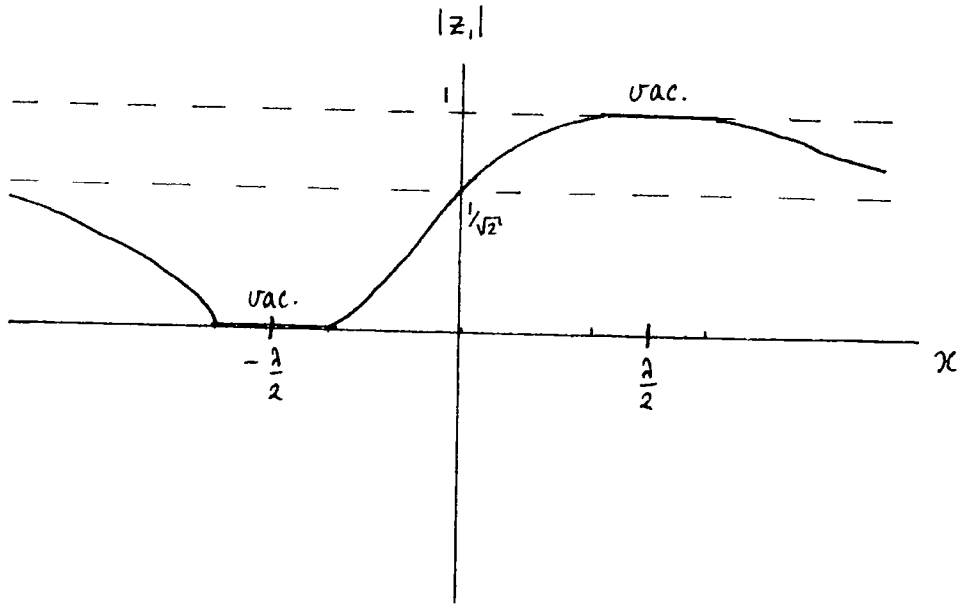


FIG. 4.5 VACUUM-SMEARED ELLIPTIC SOLUTION (4.2.3)



4.2.4a) $$Z = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \cdots\cdots \quad \varphi < \ln r_1 < 0 \\ \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+sn\,\varphi/k}\; e^{i\theta} \\ \sqrt{1-sn\,\varphi/k} \end{pmatrix} & \cdots \quad \ln r_1 < \varphi < \ln r_2 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \cdots\cdots \quad 0 < \ln r_2 < \varphi \end{cases}$$

with action

4.2.4b) $$S = \frac{\pi}{2k}\left(E\left(\frac{\ln r_2}{k}\right) + E\left(-\frac{\ln r_1}{k}\right)\right) - \frac{\pi(1-k^2)}{4k^2} \ln \frac{r_2}{r_1}$$

(E(.) is now an incomplete elliptic integral of the second kind) .

Similarly the generalised oscillatory solutions including jumps are

4.2.5a) $$Z = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \cdots\cdots \quad \varphi < \ln r_1 < 0 \\ \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+k\,sn\,\varphi}\; e^{i\theta} \\ \sqrt{1-k\,sn\,\varphi} \end{pmatrix} & \cdots \quad \ln r_1 < \varphi < \ln r_2 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \cdots\cdots \quad 0 < \ln r_2 < \varphi \end{cases}$$

with action

4.2.5b) $$S = \frac{\pi}{2}\left(E(\ln r_2) + E(-\ln r_1)\right) - \frac{\pi}{4}(1-k^2) \ln \frac{r_2}{r_1}$$

Two things remain to be done ; calculation of the fluctuation determinant around such solutions and choosing a measure on the appropriate phase-space . It is at this point that the calculation in this thesis stops . As mentioned above there is a long-standing and unresolved debate about evaluation of the determinant , concerning the choice of boundary conditions at infinity . The classical conformal invariance implies that choice of conditions at infinity is important (there is no intrinsic length-scale by which to define large distances) , and it is unclear which choice is more appropriate physically .

The scale invariance is expected to be broken at the full quantum level , when quantum fluctuations establish a fundamental scale , but unfortunately at the semiclassical level it still causes problems . The direct analogy from chapter 3 would be to calculate initially in a finite box (large compared to λ) , and then pass to the infinite volume limit . This seems attractive as the procedure is well-defined and respects the order in which limits are taken , however this would explicitly break the Lorentz invariance of the theory and one would have to demonstrate that this was unimportant in the limit . For this reason , calculations in conformally invariant models are beset with difficulties .

Two more hopeful points are the following . Firstly there is the possibility raised in chapter 3 , that explicit calculation of the fluctuation determinant may not be necessary . If one could understand physically why this ratio seems always to be a simple factor related to properties of the potential , then this knowledge could be used in field theories where the determinants are both elusive and ambiguous . This is another reason for postponing the calculation started here - there may be very much easier ways of getting to the same answer .

Secondly there is the point that the elliptic solutions seem to exhaust the range of solutions necessary to perform the extended semiclassical integrals , as only the charge one sector is used . Thus the above approach is applicable to all CP^{n-1} models , despite the lack of full knowledge of solutions in the higher charge sectors . In addition the corresponding family of elliptic solutions has been found in the charge one sector of four-dimensional Yang-Mills theories [16] . Once

again the situation in higher charge sectors is less clear , and it is probable that more solutions here do exist - but for the purposes of this calculation this is not important . Thus we believe that all the information necessary to perform the analogous calculation in Yang-Mills is already available and it is simply a matter of piecing it all together - and learning how to take account of quantum fluctuations easily .

Chapter 5 - Multi-site Supersymmetric Quantum-mechanics

The supersymmetric quantum-mechanical models used in chapters 2 and 3 used a single pair of each of bosonic and fermionic variables , or in other words considered the field theory of fields at a single site propagating in time . In chapter 4 we considered a family of field theories of vectors propagating in one space and one time dimension , in order to demonstrate the general utility of the extended semiclassical calculations developed in chapter 3 . In this chapter we seek to fill in some of the gap between these cases , by considering supersymmetric quantum mechanics with several pairs of bosonic and fermionic degrees of freedom - the theory of fields at a finite number of sites propagating in time . This approach could be pursued to study the approach towards field theory by progressively increasing the number of sites , which as we shall see need not be straightforward . We retain the supersymmetric nature of the problem in order to study the behaviour of the lowest energy state of the extended system . Work with similar intent appears in [21] , but that investigation is from a rather different viewpoint .

The starting point here is the observation that the content of (2.1.2) and (2.1.10a) may be expressed as

5.1) $$\hat{Q}_\pm = \hat{B}_\pm \otimes \hat{F}_\pm = (\text{bosonic operator}) \otimes (\text{fermionic operator})$$
$$\text{with } \hat{B}_+ = \hat{B}_-^\dagger = \hat{p} + iw'(\hat{q}) \quad , \quad \hat{F}_\pm = \hat{\psi}_\pm$$

where the fermionic operators have the form of ladder operators in that they change fermion number by one . Since previously our model was defined at a single site $\hat{B}_\pm$ and $\hat{F}_\pm$ were ordinary

operators then .

To generalise to variables defined at n sites , we retain the form $\hat{Q}_\pm = \hat{B}_\pm \otimes \hat{F}_\pm$ in which $\hat{F}_+$ ($\hat{F}_-$) is still required to decrease (increase) fermion number by one , and $\hat{B}_\pm$ ($\hat{F}_\pm$) still act solely on bosonic (fermionic) degrees of freedom . However , $\hat{B}_\pm$ are now nxn matrices of operators , due to the fact that for each original boson variable we now have n . We shall make as few assumptions as possible about the detailed form of $\hat{B}_\pm$, simply assuming that they have eigenstates and energy levels which could be found if desired . We shall concentrate on the effect that the existence of the fermions has on such levels .

We have now to establish specific forms for $\hat{F}_\pm$ - these are constrained by the requirements that $\{\hat{Q}_\pm , \hat{Q}_\pm\} = 0$, and that $\hat{F}_\pm$ should alter total fermion number by one . We also retain

5.2) $$\hat{H} = \tfrac{1}{2} \{\hat{Q}_+ , \hat{Q}_-\}$$

There are two distinct ways of generalising fermions to n sites , which will be shown to coincide for n = 1 . The ways correspond to whether we want the model to describe just fermions , or a mixture of fermions and anti-fermions - do we want the fermion number-operator $\hat{f}$ to have only positive eigenvalues , or will we allow negative eigenvalues as well ? One way to see that the distinction is unimportant for n = 1 is that here there is nothing to compare the original site with , and so addition of a constant to $\hat{f}$ passes from one formulation to the other without effect . With many sites , and the opportunity to compare one site with another , this freedom in $\hat{f}$ is largely removed . Other consequences of the difference will be noted as they arise .

The first alternative (using fermions only) is the more obvious generalisation of the original problem , and is also easier to study (and by the same token less interesting !) . We therefore consider this first , discussing the n = 2 case in detail and larger n cases more briefly .

With two sites there are two and only two ways in which we may reduce fermion number , by taking away a fermion from one site or the other . If we label the sites a,b , this gives

5.3a) $$\hat{F}_{\pm} = \frac{1}{\sqrt{2}} \left(\psi_{\pm}^{a} \cdot 1^{b} + 1^{a} \cdot \psi_{\pm}^{b} \right)$$

where in order to get the correct anticommutation properties ,

5.3b) $$\psi_{\pm}^{a} = \underbrace{\psi_{\pm}}_{\text{acting at a}} \otimes \underbrace{1_2}_{\text{acting at b}} \; ; \; \psi_{\pm}^{b} = \sigma_3 \otimes \psi_{\pm} \; ; \; 1^{a} = 1^{b} = 1_2 \otimes 1_2$$

$$\left[\psi_{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \psi_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right]$$

Explicitly this gives

5.3c) $$\hat{F}_{+} = \hat{F}_{-}^{T} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It will be recalled that for the n = 1 case , the wave-functions were 2-component objects , with the upper (lower) component corresponding to f = 0 (1) . Here , wave-functions are 4-component objects , with each component being itself a 2-component vector . The four components refer to the four possible distributions of fermion number ((0,0),(0,1),(1,0), (1,1) or equivalently , $1\otimes1, 1\otimes\psi, \psi\otimes1, \psi\otimes\psi$) , with the two components being the associated bosonic wave-functions for each site . As explained earlier , we are primarily concerned with the effect of the fermions and so will not study the form of these bosonic wave-functions . For a basis of the fermion states we choose those combinations which are eigenstates of the Hamiltonian - these are

$$1\otimes1 \; , \; \frac{1}{\sqrt{2}}\left(1\otimes\psi \pm \psi\otimes1\right) \; , \; \psi\otimes\psi$$

It may readily be seen that

$$\begin{cases} \hat{F}_{+} \cdot \psi\otimes\psi = \frac{1}{\sqrt{2}}\left(1\otimes\psi - \psi\otimes1\right) \\ \hat{F}_{+} \cdot \frac{1}{\sqrt{2}}\left(1\otimes\psi + \psi\otimes1\right) = 1\otimes1 \end{cases}$$

and so these pairs of states are degenerate in energy , unless the bosonic part of $\hat{Q}_+$ annihilates the states on the right-hand sides (because of the supersymmetric structure , if one of the pair is annihilated , they both will be) . Furthermore , by examining the Hamiltonian derived from $\hat{Q}_\pm$, $\hat{H} = \frac{1}{2}\{\hat{Q}_+ , \hat{Q}_-\}$, it becomes evident that the following pairs of states are also degenerate : $(\psi \otimes \psi , \frac{1}{\sqrt{2}}(1\otimes\psi + \psi\otimes 1))$, $(\frac{1}{\sqrt{2}}(1\otimes\psi - \psi\otimes 1), 1\otimes 1)$. This degeneracy is only loosely related to the supersymmetry and is not affected by the specific forms of $\hat{Q}_\pm$.

Suppose then that solving of the bosonic part of the problem gives a bosonic wave-function b with energy E_b - as with the $n = 1$ case we have $E_b \geq 0$. Then we have the following two possibilities :

1) $E_b = 0$. The presence of the fermions induces a 2-fold degeneracy , with full wave-functions $b\otimes(1\otimes 1)$, $b\otimes\frac{1}{\sqrt{2}}(1\otimes\psi + \psi\otimes 1)$. The other fermionic states are annihilated .

2) $E_b > 0$. The presence of the fermions induces a 4-fold degeneracy , with all four possible fermion states allowed .

In neither case is the bosonic spectrum altered - the only contribution made by the fermions is to increase degeneracy of the levels as given above .

For n sites the situation is directly analogous . We have

5.4) $$\hat{F}_\pm = \frac{1}{\sqrt{n}}\left(\psi_\pm^a . 1^b \ldots 1^n + \text{cyclic perms.} \right)$$

$$\psi_\pm^j = \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \psi_\pm \otimes 1 \otimes \cdots \otimes 1 \qquad (\psi_\pm \text{ in } j^{th} \text{ place})$$

There are 2^n fermion states , and choosing as basis those which

are eigenstates of the Hamiltonian we find that supersymmetry transformations relate these in pairs . As before , there are only two superficially different Schrodinger equations , and once again we are led to the same conclusions - a bosonic eigenstate of energy E_b gives rise to one or other of

1) a 2^{n-1}-fold degeneracy induced by the fermions if $E_b = 0$,

2) a 2^n-fold degeneracy induced by the fermions if $E_b > 0$.

In both cases the bosonic spectrum is unchanged except for degeneracy .

Clearly this is not an interesting generalisation of the $n = 1$ case as the fermions play essentially no part in the model . The apparent extra degrees of freedom are not genuine ones as no new physical effects are thereby introduced . The picture is , however , different if we allow anti-fermions in the theory . This is done in the following way . Instead of just allowing the fermionic variables $(1,\psi)$ at a given site , we allow the four alternatives $(\bar{\psi},1,\bar{\psi}\psi,\psi)$ with fermion number $(-1,0,0,1)$ respectively . So we now associate a 4-component vector with the fermionic content of each site ;

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 \\ \bar{\psi} \\ \psi \\ \bar{\psi}\psi \end{pmatrix}$$

so that for example $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ represents $1+\bar{\psi}\psi$. This extends the range of fermion interactions by allowing up to four fermion operators in a given term in the Lagrangian - it is such terms which render the supersymmetric CP^{n-1} models difficult [23] .

As before we consider the case $n = 2$ in detail . First we must construct the operator $\hat{F}_+$. This has two parts - an "active" part which reduces the fermion number at a site by one , and a "passive" part which leaves it unaltered . The

active part operates either by destroying a ψ or creating a $\bar{\psi}$. The various annihilation and creation operators are chosen to be

5.5)
$$\psi^a_\pm = (1\otimes\psi_\pm)\otimes(1\otimes 1) \quad ; \quad \psi^b_\pm = (\sigma_3\otimes\sigma_3)\otimes(1\otimes\psi_\pm)$$
$$\bar{\psi}^a_\pm = \underbrace{(\psi_\pm\otimes\sigma_3)}_{\text{acts at } a}\otimes\underbrace{(1\otimes 1)}_{\text{acts at } b} \quad ; \quad \bar{\psi}^b_\pm = (\sigma_3\otimes\sigma_3)\otimes(\psi_\pm\otimes\sigma_3)$$

where for example ψ^a_+ annihilates a fermion at a , $\bar{\psi}^b_-$ creates an antifermion at b , etc. . (Recall that we now have a 4-component fermion representation at each site and hence 4x4 matrix operators , where before they were 2x2) . It is easily checked that these operators have the correct anticommutation rules . So the active (normalised) part of $\hat{F}_+$ is just $\frac{1}{\sqrt{2}}(\psi_+ + \bar{\psi}_-)$ at a particular site . The passive part is slightly more complicated as there are various bilinear terms in the fermions which could be used - in general we could have

$$I = 1 + \alpha\psi_-\psi_+ + \beta\bar{\psi}_-\bar{\psi}_+ + \gamma\psi_+\bar{\psi}_+ + \delta\psi_-\bar{\psi}_-$$

with $\alpha,\beta,\gamma,\delta$ arbitrary . This arbitrariness is reduced first by invoking the requirement that $\{\hat{F}_+,\hat{F}_+\}=0$. If we also impose the physical requirement that "nonlocal" interactions are to be excluded (in other words we exclude the possibility that fermion number may be conserved by the simultaneous creation of a ψ and a $\bar{\psi}$ at different sites) then we find that only one non-trivial possibility remains ,

$$I = 1 + \psi_-\psi_+ + \bar{\psi}_-\bar{\psi}_+ + \psi_+\bar{\psi}_+ + \psi_-\bar{\psi}_-$$

Thus
$$\hat{F}_+ = \frac{1}{4}\left[(\psi_+ + \bar{\psi}_-)^a I^b + I^a(\psi_+ + \bar{\psi}_-)^b\right] = \frac{1}{4}\left[\begin{pmatrix}0&0&1&0\\1&0&0&1\\0&0&0&0\\0&0&-1&0\end{pmatrix}\otimes I_1 + I_\sigma\otimes\begin{pmatrix}0&0&1&0\\1&0&0&1\\0&0&0&0\\0&0&-1&0\end{pmatrix}\right]$$

5.6)
$$I_1 = \begin{pmatrix}1&0&0&-1\\0&2&0&0\\0&0&2&0\\1&0&0&3\end{pmatrix} \quad , \quad I_\sigma = I_1\cdot(\sigma_3\otimes\sigma_3) = \begin{pmatrix}1&0&0&-1\\0&-2&0&0\\0&0&-2&0\\1&0&0&3\end{pmatrix}$$

As usual , $\hat{F}_- = \hat{F}_+^T$, so we may now construct $\hat{Q}_\pm = \hat{B}_\pm\otimes\hat{F}_\pm$ and hence $\hat{H}$. The fermion number operator is $\hat{f} = \sum_{\text{sites}}(\psi_-\psi_+ - \bar{\psi}_-\bar{\psi}_+)$ which should be compared with the above form of I . As before

it commutes with $\hat{H}$ and is therefore a constant of motion .

There are 16 fermion states , with fermion number ranging from -2 to 2 , and with some determination we may find the basis of these which are eigenstates of $\hat{H}$. Appendix 5.1 gives explicit details of this basis and the properties of the various states . As before we find that the supersymmetry transformations relate these in pairs . However , by examining the detailed structure of the action of the Hamiltonian on the basis states we find the following behaviour . As usual we take a bosonic eigenstate of energy E_b .

1) $E_b = 0$. The presence of the fermions induces an 8-fold degeneracy , and the other 8 fermion states are annihilated by $\hat{Q}_+$.

2) $E_b > 0$. This level becomes split in the following way ;
   - a 4-fold degenerate level at $\left(\frac{\sqrt{5}-1}{2}\right)^2 E_b \equiv E_b/\lambda$;
     2 of these states have even fermion number and two odd ,
   - an 8-fold degenerate level at E_b ;
     4 each even and odd fermion number states ,
   - a 4-fold degenerate level at $\left(\frac{\sqrt{5}+1}{2}\right)^2 E_b = E_b \cdot \lambda$;
     2 each of even and odd fermion number .

Thus the existence of the fermions has a non-trivial effect on the spectrum of the theory - the original single level becomes split into three , of different degeneracies . Further explanation of the nature of the fermionic content of the remaining degeneracy is given in appendix 5.1 and also in the following chapter . In spite of the splitting , the entire spectrum may be recovered from knowing just the bosonic value E_b , as splitting is given in terms of ratios rather than

differences . It is of interest to note that the state with lowest fermion number , $\bar{\psi}\otimes\bar{\psi}$, is one of those which remain at E_b ; in full for this state we find

5.7) $\hat{H}\,.\,b\otimes(\bar{\psi}\otimes\bar{\psi}) = \frac{1}{4}\left[\left(\{\hat{B}_+,\hat{B}_-\}+[\hat{B}_+,\hat{B}_-]\right)b\right]\otimes(\bar{\psi}\otimes\bar{\psi}) = E_b\,.\,b\otimes(\bar{\psi}\otimes\bar{\psi})$

For the n = 1 case we had $\frac{1}{4}\{\hat{B}_+,\hat{B}_-\} = -\frac{1}{2}\partial^2+\frac{1}{2}v'^2$; $\frac{1}{4}[\hat{B}_+,\hat{B}_-] = -\frac{1}{2}v''$ so here we find precisely the same as in the original case , that even though when solving for b we are treating only an effective bosonic theory , we must still incorporate the "fermionic part" of the potential , and not just use the "bosonic part" . We feel that this is supporting evidence for the use in chapters 2 and 3 of the full potential V_+ and not just V_b .

Notice the following - for just one state the matrix I of (5.6) does not appear and we reduce to $\hat{F}_\pm = \frac{1}{\sqrt{2}}(\psi_\pm+\bar{\psi}_\mp)$. There are just four fermion states , and the degeneracy of a given bosonic level E_b is as follows .

1) $E_b = 0$. States ψ, $\frac{1}{\sqrt{2}}(1+\bar{\psi}\psi)$ are annihilated .
   States $\bar{\psi}$, $\frac{1}{\sqrt{2}}(1-\bar{\psi}\psi)$ are degenerate at zero .
2) $E_b > 0$. All four states are degenerate and leave E_b unchanged .

It will be seen that the energy-level structure (in terms of values , not degeneracy) is precisely the same as for the n = 1 case with just fermions . As stated earlier , for n = 1 the two generalisations are identical , and the apparent introduction of additional fermionic degrees of freedom is fictitious . Only for $n \geq 2$ do the differences between the two sorts of generalisation manifest themselves .

We may generalise this structure to the n-site case without difficulty . We now have

$$\hat{F}_{\pm} = \frac{1}{2^{n-1}\sqrt{2n}} \left\{ (\psi_{\pm} + \bar{\psi}_{\mp})^a I^b \ldots I^n + \text{cyclic perms.} \right\}$$

5.8)
$$\begin{cases} (\psi_+ + \bar{\psi}_-)^j = (\sigma_3 \otimes \sigma_3) \otimes \ldots \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \otimes \ldots \otimes (1 \otimes 1) = (\psi_- + \bar{\psi}_+)^{T\, j} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j^{th} \text{ place}) \\ I^j = (1 \otimes 1) \otimes \ldots \otimes I \otimes \ldots \otimes (1 \otimes 1) \end{cases}$$

with 4^n fermionic states . These affect a bosonic level E_b as follows .

1) $E_b = 0$. Such a state becomes $\frac{1}{2} . 4^n$ -fold degenerate .

2) $E_b > 0$. This becomes split into the following levels -

$$\frac{E_b}{\lambda^{n-1}} , \frac{E_b}{\lambda^{n-2}} \ldots , E_b , \ldots E_b . \lambda^{n-2} , E_b . \lambda^{n-1}$$

with degeneracies such that levels $E_b . \lambda^{j-2^{n-1}}$ and $\frac{E_b}{\lambda^{j-2^{n-1}}}$ both have degeneracy $4 . {}^{2^{n-1}}C_{j-1}$

E.g. for n = 4 we have

level	$\frac{E_b}{\lambda^3}$	$\frac{E_b}{\lambda^2}$	$\frac{E_b}{\lambda}$	E_b	$E_b.\lambda$	$E_b.\lambda^2$	$E_b.\lambda^3$
degen.	4	24	60	80	60	24	4

The fermion number operator in this general case is

$$\hat{f} = (\psi_-\psi_+ + \bar{\psi}_-\bar{\psi}_+)^a 1^b \ldots 1^n + \text{cyclic perms.}$$

The state of lowest fermion number , $\bar{\psi} \otimes \bar{\psi} \otimes \ldots \otimes \bar{\psi}$, is always one of those which leave E_b unchanged , and the Schrodinger equation for such a state is always the direct analogy of (5.7) . As a general rule , the states with very high or low levels are those which are "rougher" in terms of distribution of the fermion content - in other words , states in which the fermion number is the same from site to site (such as $\bar{\psi} \otimes \ldots \otimes \bar{\psi}$)

do not alter the bosonic value much if at all , whereas the states with rapid alternations of ψ's and $\bar{\psi}$'s from site to site tend to give the extremes .

This is important because of the following . It will be noted that as we increase the number of sites , the value of the lowest energy level (which we suppose to be strictly positive , i.e. we are considering a spontaneously broken model) tends to decrease as $1/2^n$. Because of the greater number of degrees of freedom , one also expects the lowest level to increase with n linearly (cf. harmonic oscillator with n degrees of freedom for which $E_n = n\hbar\ (\omega + \frac{1}{2})$) , but this increase is clearly not sufficient to outweigh the power-law decrease . Thus in the $n \to \infty$ limit the lowest energy-level , even in a spontaneously broken model , tends down to zero . We have not had time to explore the consequences of this for field theory , but we expect that this is related to the observation that the lowest-lying states are those with rapid fluctuation of fermion content . It is possible that we are seeing in this phenomenon the analogy in quantum mechanics of the species-doubling which causes difficulties with lattice calculations in field theories with fermionic degrees of freedom [40] . If this is the case , then we expect to find some mechanism operating in field theories such as entropy which would suppress such violently oscillating states .

We have constructed in this chapter two essentially different generalisations of the single site supersymmetric quantum mechanics treated in chapters 2 and 3 . We have shown above that they coincide for n = 1 but differ markedly for cases

with larger n . The first , which is the more obvious , leads to a structure which is the same in essence as the original case in that the multiplicity of fermion variables with sites produces no new features . The second , which incorporates anti-fermions as well as fermions , produces a nontrivial change to the bosonic spectrum - the fermionic interactions split any bosonic level into a sequence of levels with regular ratios spacing them . This second model has many interesting features , particularly as regards the behaviour for a large number of sites and thus the approach to field theory .

Finally we wish to point out that there are other ways of generalising (5.6) which coincide up to and including n = 3 but can differ for higher cases (5.8) is the simplest such generalisation , in which the matrix I is used in $\hat{F}_+$ for all sites other than the one at which the active part of the operator is located . It could be argued that this choice induces fermionic transitions which have a certain non-local character which may be unacceptable . Specifically , creation or annihilation of fermions induces (through the form of the I-matrix chosen) transitions of the form $1 \leftrightarrow \bar{\psi}\psi$ at sites which in the physical arrangement of the sites may be well separated . To avoid this we could adopt instead the following generalisation :

$$5.9) \qquad \hat{F}_+ = N \sum_{\text{cyclic}} \left(1^q \cdots 1^{j-2} I^{j-1} (\psi_+ + \bar{\psi}_-)^j I^{j+1} 1^{j+2} \cdots 1^n \right) \quad (N \text{ normalises})$$

where 1 is the normal unit matrix to the appropriate tensor power , and labels are taken to be given modulo n . This new generalisation is easily seen to induce transitions $1 \leftrightarrow \bar{\psi}\psi$ only at sites adjoining the active site and thus acts in a more local manner . We have not investigated the properties

of such a model in detail yet (mainly because differences only start to appear for $n \geq 4$) but it is probable that the effect will be that the spectrum will be less changed by the inclusion of the fermions than in the case (5.8) - perhaps a bosonic level E_b will only ever be split into three , at $E_b \cdot \lambda$, $\frac{E_b}{\lambda}$, and E_b (before , the highest power of λ appearing was λ^{n-1} and there were (n-1) I-matrices in the products) .

An advantage of this more varied generalisation is that it allows us to consider various geometric arrangements of the sites . The form of (5.9) implicitly assumes that the sites are arranged round a circle so that each site has exactly two neighbours . Alternatively we may arrange the sites along a line by not identifying the two ends . Clearly by varying the locations of where 1's and I's are inserted we are allowing different nearest-neighbour structures . In general different such structures will display different patterns of energy-splitting for a given bosonic level . So , for example , the original proposal (5.8) assumes that every site is adjacent to every other (in the sense of being neighbours for the fermionic interaction - recall we have not assumed specific forms for the bosonic operators $\hat{B}_{\pm}$. A more complicated example is the following . Arrange n sites in an nxn array numbered as (for n = 4)

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

and let

5.10) $$\hat{F}_+ = N \sum_{\text{cyclic}} \left(1^1 \ldots I^{i-4} 1^{i-3} 1^{i-2} I^{i-1} (\psi_+ + \psi_-)^i I^{i+1} 1^{i+2} 1^{i+3} I^{i+4} \ldots 1^n \right)$$

where site-labels are taken modulo n , so that for example $I^{19} = I^{3}$. Then if we let n tend to infinity while retaining this structure we can end up with a field theory based on a torus instead of a circle - or , by not identifying opposite sides we may work on a plane instead of a torus . Evidently , a large number of possibilities may be extracted from a scheme such as this .

Chapter 6 - Conclusions

It remains now to make some concluding remarks . These will be of a general nature , as specific comments concerning the various methods have been made at the appropriate points of the text .

We set out initially to attempt to understand the nature and extent of the spontaneous breakdown of supersymmetry , and towards this end have considered a number of quantum-mechanical models , both broken and unbroken . In super-symmetric quantum-mechanical models , the criterion for deciding whether or not the symmetry is broken is particularly simple (although in field theories it is not , so we seek some alternative characterisation) - one looks at the trial wave-function $\psi \sim e^{-v(q)}$ (where v(q) is the superpotential) and sees if such a state is normalisable . If it is , then it is a state of zero energy and supersymmetry is unbroken ; otherwise it is broken . Normally (and always in the text above) this corresponds to checking only the asymptotic behaviour of v(q) for large $|q|$, but see below for other possibilities . So with our cases the parity of the highest power of q in v discriminates - even unbroken and odd broken . Although this is very convenient for ease of decision , it leads to the following problem . Normally to begin to study a broken symmetry we first consider the unbroken case and then switch on a breaking term which may be taken arbitrarily small in order to study the effect . Here we cannot do this , as the difference between broken and unbroken models is contained in the dominant behaviour of v(q) .

Thus it makes no sense , for example , to start with the (unbroken) model $v = \frac{1}{2}\omega q^2$ and then switch on a breaking term $v = \frac{1}{3}\varepsilon q^3$ - however small ε is , the new potential , the bosonic potential , and the full potential used in the text are all completely altered . Thus , our choice of whether or not supersymmetry is to be broken is a fundamental one to be made at the start of the calculation , not an optional one which can be left open at first .

The investigations of this thesis have not shed much light on the mechanism by which supersymmetry is broken dynamically . By now a variety of models of supersymmetric quantum mechanics have been studied by numerous people , and in some ways the question becomes more open as more is discovered about the models - see [29] for a summary of some of the cases . The work of chapter 2 indicates that the mechanism must somehow blend perturbative and nonperturbative effects rather than being linked to just one , although clearly in certain asymptotic regions one or the other may dominate . The work in chapter 3 shows that , at least in strong-coupling regions , the relative sizes of effects termed "perturbative" and "nonperturbative" may be very similar , and so to neglect either is hazardous . The mixed wave-function method developed and explained in chapter 2 provides answers which are numerically good (presumably because it blends perturbative and non-perturbative ideas) , but it does not provide a great deal of physical insight into the breaking mechanism . In the past it has been suggested that the existence of instanton solutions could act as a trigger to break the symmetry , but this now seems unlikely - we have seen that instantons are present in

some unbroken theories , both in the formalism using the bosonic potential V_b and the one we use , with the full potential V_+ . Similarly there are broken models with no perturbative breaking and without instantons - an example is the superpotential $v = \frac{1}{3}aq^3$. We can see from the earlier comments in this chapter why instantons are insufficient - supersymmetry breakdown is governed by the leading behaviour of the superpotential , while the existence of instantons is decided by the specific form of the non-leading terms . So for example the two superpotentials $v = \frac{1}{4}q^4 - \frac{1}{2}q^2$, $v = \frac{1}{4}q^4 - q$ both lead to unbroken theories , but the former has instantons (in both V_b and V_+) whereas the latter does not . In [29] it is suggested that the necessary condition is slightly more complicated : instantons in the V_b problem will signal supersymmetry breaking unless the phase transition when the fermionic part of V_+ is turned on is of first order (so that the expectation value of the coordinate changes discontinuously) . This criterion (which is derived empirically rather than physically or theoretically) is satisfactory for the models so far considered , but as yet it is too early to assess its general value . At first sight it seems once again to be probing non-leading terms , but this in fact is not clear as the leading power of the fermionic part of the interaction has the same parity as that of the superpotential , so the criterion may be adequate . We feel that the question is still very much open .

One point which arises from chapter 2 concerning single-site problems is the following . In the introduction it was stressed that for a supersymmetric model to be physically plausible , it would have to be broken in some way in order to split up the

particle supermultiplets containing bosons and fermions of equal mass . However , the broken models discussed here do not do that - the sole qualitative effect of the breaking is to lift the lowest energy-level from zero and render it degenerate . All excited states of an unbroken model come in degenerate pairs (one bosonic , one fermionic) with a nondegenerate (bosonic) vacuum ; all states without exception of a broken model come in degenerate pairs ! This is rather disappointing , and questions the use of the word broken .The situation in the models constructed in chapter 5 is a little more encouraging - using the more elaborate models splits up a single bosonic level into several levels of different energies . However , each of the new levels is still degenerate and contains an equal number of states with odd and even total fermion number , so the problem is not yet fully resolved here . However , it is known that for supersymmetric Yang-Mills models , certain types of explicit breaking can be introduced (without spoiling the finiteness of the models) which give each particle a different mass , subject to an overall sum-rule [41] . We therefore suggest that the rather rigid nature of energy-levels in the broken cases here is an artifact of the small number of degrees of freedom , and need not persist in field theory .

As our measure of the extent of supersymmetry breaking we have used the value of ground-state energy . As explained before , this has the advantage of being readily calculable in many (approximate) ways , using both Hamiltonian and Lagrangian techniques . It is particularly suitable for cases in which we know in advance that the theory is broken and wish only to assess the extent of this . It is not suited to situations in

which we wish to test whether a given model is unbroken or softly broken , since numerical estimates have intrinsic inaccuracies , and to decide whether a small answer is "really" zero or else positive , with no additional evidence , is clearly impossible . The problem here is that very few techniques provide answers which are guaranteed to be lower bounds - either the answer is known to be an upper bound (e.g. most variational methods) or else it is ambiguous (e.g. zero-point estimates or instanton calculations) . Procedures which do guarantee lower bounds , such as the method of [7] , tend to converge more slowly than standard upper-bound methods ; for problems of the type considered here , we already have a lower bound - zero - and so may need to expend considerable energy to improve this in a softly-broken case . It would therefore be advantageous to have a general tool for establishing whether or not breaking occurs , before attempting to assess its extent . In quantum mechanics this is straightforward ; in general , some functional methods have been established [15] based on an original idea in [43] , but they are extremely difficult to apply in most cases .

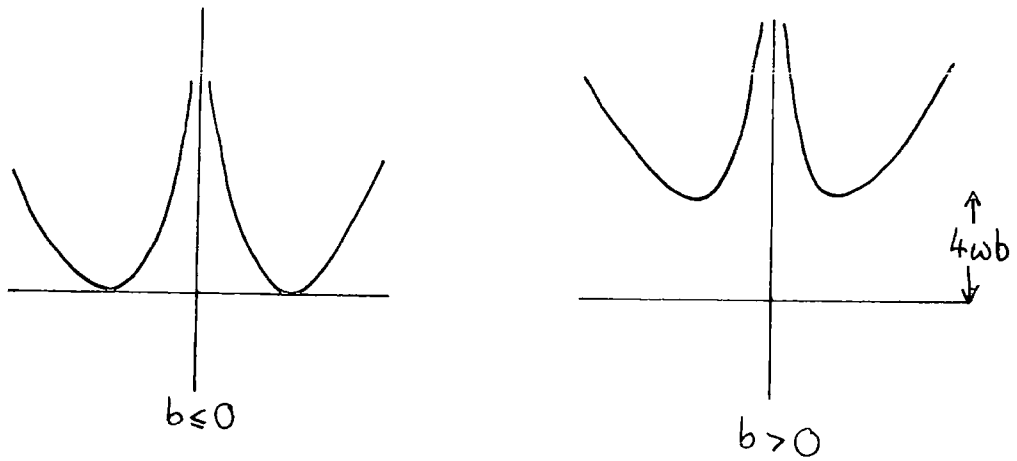
As a final general point concerning these models , we return to a question raised in the introduction - what happens if we allow the superpotential to contain non-polynomial terms in an attempt to duplicate the effects generated by renormalisation ? As a specific model to analyse we take a family of super-potentials

$$W = \tfrac{1}{2}\omega q^2 + \tfrac{1}{2} b \ln q^2$$

parametrised by b . This gives bosonic and full potentials

$$\begin{cases} V_b = \left(\omega q + \dfrac{b}{q}\right)^2 \\ V_+ = \omega^2 q^2 + (2b-1)\omega + \dfrac{b(b+1)}{q^2} \end{cases}$$

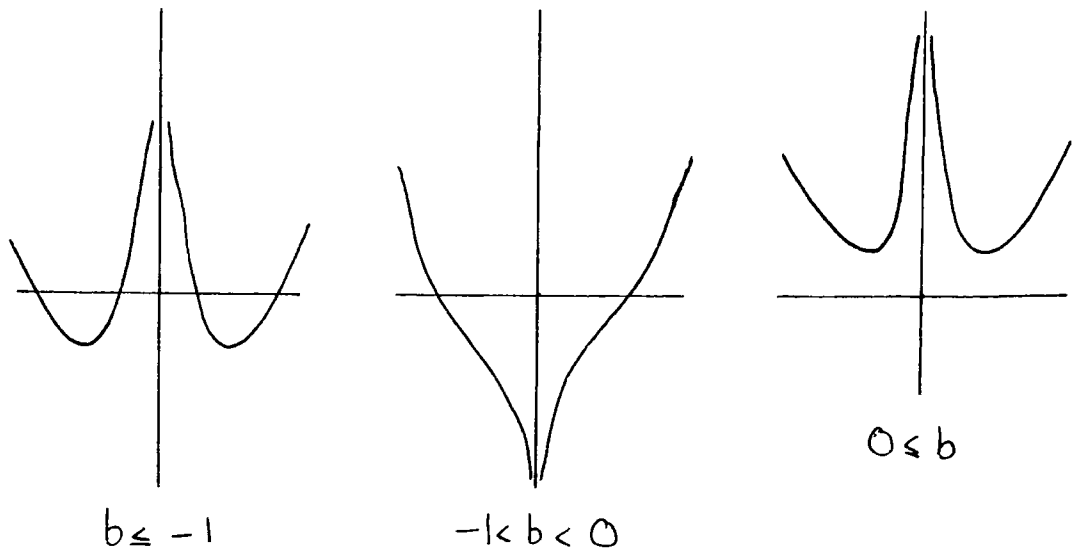
FIG. 6.1



a) V_b

$$v = \tfrac{1}{2}\omega q^2 + \tfrac{1}{2} b \ln q^2$$

b) V_+



Note $V_+(q_{min}) < V_b(q_{min})$ in all ranges.

This family of models is currently under investigation by W.J.Zakrzewski and myself , and few definite results are yet available . However , preliminary results indicate that the behaviour of the models is in many ways very odd . Figs. 6.1a,b show the various generic forms of V_b and V_+ , and we shall begin by analysing the expected behaviour of the models according to conventional wisdom . Consider fig. 6.1a . For $b > 0$ the models appear to be broken by perturbative effects , since the classical minimum is above zero . For $b < 0$ the classical minimum is at zero , so the models are not broken perturbatively , but because of the degeneracy we can expect to find tunnelling solutions which could break the symmetry nonperturbatively - the discussion in [38] follows precisely these lines . Now look directly at the superpotential . Since for large $|q|$, $v \sim q^2$ the usual argument would conclude that the models are always unbroken , regardless of the value of b . However , careful application of this argument requires that we look more closely at the normalisability of the trial wave-function $\psi \sim e^{-v}$. This requires

$$\int \psi^2 dq = \int e^{-2v} dq = \int |q|^{-2b} e^{-\omega q^2} dq < \infty$$

$$\Longleftrightarrow \quad -\infty < b \leq \frac{1}{4}$$

- the critical behaviour of ψ is now at ~~that~~ the origin , since we have fixed the long-range behaviour appropriately . Thus for $b \leq \frac{1}{4}$ the models are not broken , and for $b > \frac{1}{4}$ they are broken . This discontinuous change in the behaviour following the continuous change of the parameter b provides another surprise , and contradicts several prior expectations (see e.g. [15] , [43] , or remarks earlier in this chapter) . Notice that the arguments based on the use of V_b gave the wrong

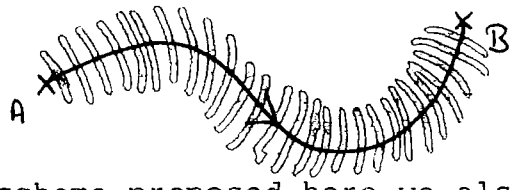
conclusions for some ranges of b . For $0 \leq b \leq \frac{1}{4}$, the wave-function is peaked at the origin , so we are most likely to find the particle in the middle of the potential barrier . For $-1 < b < 0$, the wave- function vanishes at the origin , so we are least likely to find the particle in the potential well . Only for $b \leq -1$ does the shape of the wave-function accord with prior expectations . One other surprise comes in the region $\frac{1}{8} < b \leq \frac{1}{4}$.Here the classical minimum of the Schrodinger potential for the problem is above zero , yet the energy of the true ground-state is exactly zero - somehow the quantum fluctuations manage to reduce the ground level to a lower value than the classical one .

This investigation is far from complete , but the above results show that the inclusion of non-standard terms in the superpotential may lead to some very strange effects . Since this is done in order to imitate terms arising in the effective potential from renormalisation in field theories , we believe that further study is amply justified .

We turn now to the extended semiclassical calculations of chapters 3 and 4 . These use solutions of the classical equations of motion other than the usual (instanton) ones , patched onto constant solutions at either end , the idea being to allow for a spread in the value of the energy of the system from its classical value . The fact that there are such general solutions in many (and presumably all) models has been known for some time , but the practical use of such solutions has been uncertain and their physical relevance questioned many times on several grounds , one of which is

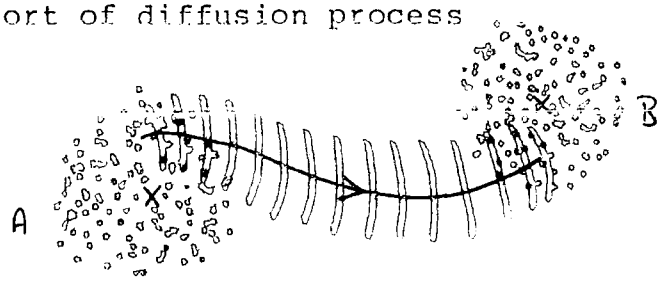
that they are oscillatory and thus have infinite action if allowed to fill the whole of space . Therefore it is satisfying to have found a possible use , as otherwise their existence in so many theories would be rather mysterious . The specific problem of infinite action is avoided in the method here by simply not using the entire solution , but only a part of one period in a localised region of space (the dilute gas approximation is assumed to cover multiple oscillations) . In order to have a specific functional form to insert into the functional integral , we are then required to use paths which are only piecewise solutions of the equations of motion and which have jumps on curves of measure zero relative to the whole space . These jumps are used only for calculational purposes , and the particle should be imagined as slowly meandering to the starting-point of the general solution under the influence of quantum-mechanical fluctuations . A helpful picture suggested by A.D.Burns in the course of a discussion is the following . Classically we think of a particle moving from point A to point B along a definite path ;

In the usual semiclassical calculations we then allow for Gaussian fluctuations orthogonal to this path ;



With the scheme proposed here we also allow for fluctuations

around the initial and final points , presumably governed by some sort of diffusion process

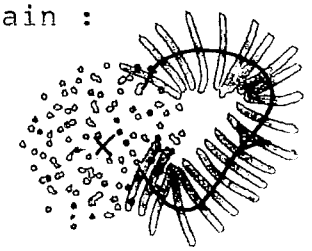


Clearly now different classical paths are allowed , since they need link only the vicinities of A and B rather than the points A and B .

The effect of this seems to be to increase the domain of validity of the semiclassical calculation in terms of accuracy of answers . The normal (i.e. instanton + dilute gas) calculation is only reliable in the weak-coupling limit - this does not include the CP^{n-1} case because of the effects of the conformal invariance . The references of [6] show that including complex-valued solutions of the equations of motion (i.e. complex saddles of the action) gives the same answer as the dilute gas method in weak-coupling cases , and is more reliable in strong-coupling regions . The solutions used in these methods involve elliptic functions , coming from direct integration of the equations of motion . It seems possible that our method , using general (real) elliptic solutions with relaxed boundary conditions , phase-space integrals and the dilute gas method , is in fact picking up the same effects as the complex complex saddle-point method . If so , the conceptual advantages of the method described here are that it is more easy to visualise the behaviour of the classical solutions and it avoids the use of complexified variables .

In this thesis we have only used this method in cases where

there is a double- well structure , and thus a classical degeneracy in the problem . However , because of the jumps (or diffusion regions) one can envisage performing a similar calculation in situations where there is only a single global minimum , perhaps with a subsidiary local minimum elsewhere as in calculations concerning the decay of a metastable vacuum state . Examples of such a potential may be found in chapter 2 as cases of spontaneously broken supersymmetry . In such a situation one can imagine using a solution which jumps out from the minimum , moves on a classical path for a time , and then jumps back again :



The secondary (false) minimum will act to reduce the ground-state energy from that calculated using just the true minimum , and we can hope to evaluate this reduction through an extended semiclassical calculation using solutions such as the above . With the normal approach there is no solution other than the trivial one which has the correct boundary conditions , and with the usual approach we cannot determine this splitting . With our approach there is no essential difference between this calculation and the one treated in the text .

In chapter 5 we have proposed and started to investigate a generalisation of the one-site models of chapters 2 and 3 to an arbitrary finite number of sites which may if desired be arranged in different geometrical configurations . This is another area which has great potential for further study ,

particularly as regards the limiting case of $n \to \infty$, i.e. the approach to field theory . The effect on bosonic eigenstates is to split up the degeneracy of a single level in a well-defined manner ; however with some of the cases considered the value of the lowest energy level for a spontaneously broken model $\sim n/\lambda^n$ as $n \to \infty$. The consequences of this for field theory are unclear . It is possible that these very low-lying states , which also exhibit rapid alternations in their fermion content , are suppressed in field theories by some sort of entropy factor , and thus the levels which seek to vanish in the $n \to \infty$ limit are in fact absent . Alternatively the problem may be related to our choice of the nearest-neighbour structure , and so choosing a more realistic one might solve it . Whichever the case , the properties of these models have an intrinsic interest , and further study would be worthwhile .

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Appendix 2.1 - Constructing the supersymmetric quantum-mechanical Lagrangian

We briefly review the procedure for constructing the most general such theory in superspace . This treatment may be found in [18] and we include it purely for convenience .

The superspace is given by (t,θ,θ^*) where t is a normal (i.e. commuting) variable and θ,θ^* are Grassmannian (anticommuting) . Thus they satisfy

$$\{\theta,\theta^*\} = \{\theta,\theta\} = \{\theta^*,\theta^*\} = [\theta,t] = [\theta^*,t] = 0$$

Infinitesimal supersymmetry transformations (which are Grassmann-even translations in superspace) are

$$t \to t' = t - i(\theta^*\varepsilon - \varepsilon^*\theta) ,$$
$$\theta \to \theta' = \theta + \varepsilon \quad , \quad \theta^* \to \theta^{*\prime} = \theta^* + \varepsilon^*$$

where $\varepsilon, \varepsilon^*$ are constant anticommuting parameters . Finite transformations are generated by

$$G = \exp\, i(\varepsilon^* Q^* + Q\varepsilon)$$

where Q,Q^* are the supersymmetry generators . These transform all quantities as

$$A \to A' = GAG^*$$

which infinitesimally gives

$$\delta A = i\,[\varepsilon^* Q^* + Q\varepsilon , A]$$

To reproduce the previous transformations of t,θ,θ^* we require

$$Q = i\partial_\theta - \theta^*\partial_t \quad , \quad Q^* = -i\partial_{\theta^*} + \theta\partial_t$$

so that $\{Q,Q^*\} = 2i\partial_t = 2H \; ; \; [Q,H] = [Q^*,H] = 0$

Covariant derivatives for these transformations are given by

$$D_\theta = \partial_\theta - i\theta^*\partial_t \quad , \quad D_{\theta^*} = \partial_{\theta^*} - i\theta\partial_t$$

We now consider a superfield which is real and transforms as a scalar : $\phi^*(t,\theta,\theta^*) = \phi(t,\theta,\theta^*)$

$$\phi(t,\theta,\theta^*) \to \phi'(t',\theta',\theta^{*\prime}) = \phi(t,\theta,\theta^*)$$

Expanding this in powers of θ, θ^* gives the following terms which we use to define the fields $q(t)$, $\psi(t)$, $\psi^*(t)$, $D(t)$,

$$\phi(t,\theta,\theta^*) = q(t) + i\theta\psi(t) - i\psi^*(t)\theta^* + \theta^*\theta\, D(t)$$

$$\rightarrow \delta\phi = \delta q(t) + i\theta\,\delta\psi(t) - i\,\delta\psi^*(t)\,\theta^* + \theta^*\theta\,\delta D(t)$$

But $\delta\phi = i[\varepsilon^* Q^* + Q\varepsilon, \phi]$ and so

$$\begin{cases} i\delta q(t) = \varepsilon^*\psi^*(t) - \psi(t)\varepsilon \ ; \\ \delta\psi(t) = \varepsilon^*\dot{q}(t) - i\varepsilon^* D(t) \ ; \quad \delta\psi^*(t) = \varepsilon\dot{q}(t) + i\varepsilon D(t) \\ \delta D(t) = \varepsilon\dot{\psi}(t) + \dot{\psi}^*(t)\varepsilon^* = \dfrac{d}{dt}\left(\varepsilon\psi + \psi^*\varepsilon^*\right) \end{cases}$$

The action of covariant derivatives on ϕ is

$$D_\theta\phi = i\psi - \theta^* D - i\theta^*\dot{q} + \theta^*\theta\dot{\psi} \quad \text{etc..}$$

The most general action in superspace, invariant under supersymmetry transformations is

$$S = \int dt\, d\theta^*\, d\theta \left\{ \tfrac{1}{2} |D_\theta\phi|^2 - f(\phi) \right\}$$

where $f(\phi)$ is some polynomial in ϕ (choosing a polynomial here is conventional because of limitations in field theory concerning renormalisability , but it is not clear that this restriction should still apply in quantum-mechanical models) . We now apply the integration rules for anticommuting variables [13] ,

$$\int \theta\, d\theta = \int \theta^*\, d\theta^* = 1 \quad , \quad \int d\theta = \int d\theta^* = 0$$

to see that only the coefficient of $\theta^*\theta$ in $\frac{1}{2}|D_\theta\phi|^2 - f(\phi)$ will contribute to the action :

$$|D_\theta\phi|^2 = \theta^*\theta\left(\dot{q}^2 + D^2 + i(\psi^*\dot{\psi} - \dot{\psi}^*\psi)\right) + \text{lower-order terms}$$

$$f(\phi) = \theta^*\theta\left(-D\,v'(q) - \tfrac{1}{2}[\psi^*, \psi]\,v''(q)\right) + \cdots$$

where $v(q) = f(\phi)|_{\theta=\theta^*=0}$

Integrating over θ, θ^* we find

$$S = \int dt \left[\tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) + \tfrac{1}{2}D^2 - D\,v'(q) + \tfrac{1}{2}[\psi^*, \psi]\,v''(q)\right]$$

In the quantum formulation of the theory we will functionally integrate over all fields $q(t)$, $\psi(t)$, $\psi^*(t)$, $D(t)$. Notice that D enters only quadratically and without derivatives , so this functional integration may be immediately performed to give an

effective action

$$S' = \int dt \left[\tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \tfrac{1}{2}v'^2 - \tfrac{1}{2}[\psi^*,\psi]v'' \right]$$

which appears in the generating functional

$$Z = \int [dq][d\psi^*][d\psi]\, e^{iS'}$$

With the identification of v(q) as the superpotential , we see that this effective action is the one used as a starting-point in the text .

Appendix 2.2 - Details for the mixed wave-function method

This should be read in conjunction with fig. 2.6 . We show in detail the calculation for the case

$$\psi_0(q) = (1 + a_0(q-q_0)) \exp -\tfrac{1}{2}\lambda_0(q-q_0)^2$$

Let $\varepsilon = q - q_0$, then

$$\begin{aligned}\hat{H}\psi_0 = E\psi_0 &= E(1+a_0\varepsilon)\exp(-\tfrac{1}{2}\lambda_0\varepsilon^2) \\ &= \tfrac{1}{2}\left(-\frac{d^2\psi_0}{dq^2} + (L^2q^4 - 2LMq^2 - 2Lq + M^2)\psi_0\right) \\ &= \tfrac{1}{2}\left(-\lambda_0 - 3\lambda_0 a_0\varepsilon + \lambda_0^2\varepsilon^2 + \lambda_0^2 a_0\varepsilon^3 + (L^2q^4 - 2LMq^2 - 2Lq + M^2)(1+a_0\varepsilon)\right)\exp(-\tfrac{1}{2}\lambda_0\varepsilon^2)\end{aligned}$$

Replacing q by $q_0 + \varepsilon$ and comparing quadratic and quartic powers in ε yields

$$\begin{cases} \lambda_0 = \sqrt{6L^2q_0^2 - 2LM} \\ a_0 = -1/4q_0 \end{cases}$$

Around the complex stationary points $q = q_\pm$ we perform a similar procedure to get wave-functions $\psi_\pm$ say , and then construct the representative wave-function $\tilde{\psi}$ around Re q_+ by

$$\tilde{\psi}(Q) = \tfrac{1}{2}(\psi_+(Q_+) + \psi_-(Q_-))$$

where Re Q_+ = Re Q_- = Q and $Q_\pm$ lie on the lines $L_\pm$ of fig. 2.6 .

In our case this gives

$$\tilde{\psi}(Q) = (1 + \tilde{a}(Q - \mathrm{Re}\, q_+))\exp\left(-\tfrac{1}{2}\tilde{\lambda}(Q - \mathrm{Re}\, q_+)^2\right)$$

where

$$\begin{cases} \tilde{\lambda} = \sqrt{16L^2q_+^2 - 2LM}\ / \cos^2\chi \\ \tilde{a} = -\mathrm{Re}\,(1/4q_+\cos\chi) \end{cases}$$

These expressions for $\lambda_0, \tilde{\lambda}, a_0, \tilde{a}$ require a little modification when $\mu < \mu_c$, but this is straightforward .

For versions of the method in which these parameters were optimised in a minimisation routine , the above values were used as first guesses .

Next we show the differences between calculated and optimised values of the parameters for the case $\mu = 0$ (as in the text , we fix L = 0.3 for purposes of illustration) . Method (a)

fixes $a_0 = \tilde{a} = 0$, while method (b) allows them to vary .

Method	Parameter λ_0	$\tilde{\lambda}$	a_0	$\tilde{a}$	α	E
(a) calc.	0.87	1.16	-	-	1.19	0.2133
opt.	1.24	1.25	-	-	1.15	0.1367
(b) calc.	0.87	1.16	-0.21	0.11	1.31	0.1303
opt.	1.23	0.96	-0.06	0.33	1.10	0.1265

Compare the strict bounds which give $0.1245 < E_0 < 0.1266$. It will be seen that none of the parameters is altered drastically by the optimisation process (i.e. by orders of magnitude) . For the region near $\mu = \mu_c$, where the pair of complex saddles becomes degenerate and $\tilde{\lambda} \to 0$, we find that $\tilde{\lambda}$ is changed by a large amount - this is to be expected as the optimisation process was intended to circumvent the problems arising from this .

Appendix 3.1 - Instanton contribution for supersymmetric quantum mechanics

As mentioned in the text , there are (at least) two ways of performing this calculation , as in [27] or [17] . In [3] we gave details for the second of these - for variety we here show working for the method of [27] . The motivation and background for the calculation for the calculation are given in this reference , so we will only mention that we seek to express an amplitude (given by a functional integral) in the form

$$\langle \varphi_2, T \,|\, \varphi_1, -T \rangle = N e^{-2E_q T/\hbar} \sinh 2.\Delta E.T/\hbar$$

and to this level of approximation the lowest energy level is $E_q - \Delta E$. N is a normalisation factor . The superpotential for this problem is

$$U = \tfrac{1}{4} L q^4 - \tfrac{1}{2} M q^2 = \tfrac{1}{4} (L^{1/4} q)^4 - \tfrac{1}{2} \mu (L^{1/4} q)^2$$

giving the Schrodinger potential

$$V_+ = L^{1/2} \left((L^{1/4} q)^6 - 2\mu (L^{1/4} q)^4 + (\mu^2 - 3)(L^{1/4} q)^2 + \mu \right)$$

We are interested in the case $\mu > -\sqrt{3}$, for which there are two degenerate minima of V_+ located at

$$q^2 = \alpha^2 = \frac{1}{L^{1/2}} \left(\frac{2\mu + \sqrt{\mu^2+9}}{3} \right)$$

We must solve

$$\tfrac{1}{2} m \dot{q}^2 = \frac{\hbar^2}{2m} (V_+(q) - V_{min})$$

$$V_{min} = \frac{L^{1/2}}{27} \left(2\mu^3 - 27\mu - 2(\mu^2+9)\sqrt{\mu^2+9} \right)$$

The solution is

$$\bar{q}(t) = \frac{\alpha . \tanh \omega(t-t_0)}{\sqrt{1+\beta . \operatorname{sech}^2 \omega(t-t_0)}}$$ - as in the text we set $t_0 = 0$

where $\begin{cases} \omega^2 = \dfrac{L^{3/2}\hbar^2}{m^2} \alpha^2 \sqrt{\mu^2+9} \\ \beta = 9 L \alpha^4 \,/\, 2(9 - \mu^2 + \mu\sqrt{\mu^2+9}) \end{cases}$

Now $L[q] = \tfrac{1}{2} m \dot{q}^2 + \frac{\hbar^2}{2m} V_+(q) = m\dot{q}^2 + \frac{\hbar^2}{2m} V_{min} \equiv m\dot{q}^2 + E_{min}$

and so the action of the instanton is

$$S[\bar{q}] = 2.E_{min}.T + \frac{m\alpha^2\omega}{4\beta} \left\{ 2\beta - 1 + \frac{4\beta+1}{\sqrt{\beta(1+\beta)}} \ln\left(\sqrt{\beta} + \sqrt{1+\beta}\right) \right\}$$

$$\equiv 2.E_{min} T + m J$$

The functional integral we wish to perform is

$$\int [dq] \exp\left[-\frac{1}{\hbar}\int_{-T}^{T} dt \left(\frac{m\dot{q}^2}{2} + \frac{\hbar^2}{2m} V_+(q)\right)\right] \quad \text{subject to} \quad q(\pm T) = \pm\alpha$$

which by a change of variable $y(t) = q(t) - \bar{q}(t)$ becomes

$$e^{-S[\bar{q}]/\hbar} \int [dy] \exp\left[-\frac{1}{\hbar}\int_{-T}^{T} dt \left(\frac{m\dot{y}^2}{2} + \frac{\hbar^2\sqrt{\Omega}}{2m}(L^{1/4}y)^2\left(15(L^{1/4}\bar{q})^4 - 12\mu(L^{1/4}\bar{q})^2 + \mu^2 - 3\right) + \text{higher powers which we discard}\right)\right]$$

subject to $y(\pm T) = 0$

As explained in [27] we perform this in two different ways in order to calculate the determinants involved . The first way is via a change of variable

$$z(t) = y(t) - \int_{-T}^{t} \frac{\dot{N}(\tau)}{N(\tau)} y(\tau)\, d\tau \qquad \text{where here } N(t) = \dot{\bar{q}}(t)$$

giving $$I = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{1}{N(T)N(-T)}\right)^{1/2} \left(\int_{-T}^{T} \frac{dt}{N^2(t)}\right)^{-1/2} = \sqrt{\frac{\omega}{\pi\hbar}}$$

and the second uses an expansion in normal modes to give

$$I = \left|\frac{Dy}{DC_n}\right| \prod_n \left(\frac{\pi\omega^2}{E_n^2}\right)^{1/2}$$

In the limit $T \to \infty$ (which we require) $E_0^2 \to 0$; this zero-mode corresponds to invariance under time-translations . In the finite-time case E_0^2 is non-zero and for large T may be found by boundary perturbation theory -

$$E_0^2 = 4^3 m\omega^3 \alpha^2 (1+\beta)^2 e^{-4\omega T} \Big/ J$$

(It should be mentioned that all these calculations involve some degree of approximation , and the validity of some of the steps is questionable) . The quasi-zero mode is taken care of by the use of the Faddeev-Popov method of collective coordinates which introduces an additional determinant in place of the divergent eigenvalue . The new value of the path-integral is

$$\bar{I} = \Delta_{FP} \left|\frac{Dy}{DC_n}\right| \prod_{E_n^2 \neq E_0^2} \left(\frac{\pi\omega^2}{E_n^2}\right)^{1/2} \cdot \frac{\omega^2}{\hbar}$$

$$= \Delta_{FP} \cdot I \cdot \sqrt{\frac{E_0^2}{\pi\omega^2}} \cdot \frac{\omega^2}{\hbar}$$

where $\Delta_{FP} = \int_{-T}^{T} dt\, \Psi_0 \dot{\bar{q}}$, where Ψ_0 is the eigenfunction corresponding to E_0^2 . For large T $\Psi_0 \propto \dot{\bar{q}}$ and we find

$$\Delta_{FP} = \sqrt{\frac{\hbar J}{\omega^2}}$$

Collecting terms , $\overline{I} = \sqrt{m}\cdot\frac{\omega^2\alpha}{\pi\hbar}\cdot 8(1+\beta)\cdot e^{-2\omega T}$

Thus to this depth of approximation , the kernel we wish to evaluate is $e^{-S[\bar{q}]/\hbar}\cdot 2T\cdot\overline{I}$

$$= \frac{16\alpha\omega^2(1+\beta)\sqrt{m}\,T}{\pi\hbar}\cdot e^{-2(E_{min}+\hbar\omega)T/\hbar}\; e^{-mJ/\hbar}$$

This gives the first term in the expansion for sinh , together with the normalisation factor which physically represents the extent to which the two wells can mix , i.e. the overlap of two wave-functions located in each well . This factor may be found by the methods of [27] to be

$$N = \sqrt{\frac{2\omega}{\pi\hbar}}$$

One may also check that the next term in the power-series for sinh is given by solutions of the form instanton+ (anti-instanton,instanton pair) , at least insofar as the dilute gas approximation may be trusted for this problem (this assumes that all multi-instanton quasi-solutions may be approximated by simple superpositions of single instanton solutions , i.e. that all the instantons concerned are well-separated) . So the kernel is given by

$$N\,e^{-2E_q T/\hbar}\left(\frac{2\cdot\Delta E\cdot T}{\hbar} + \frac{1}{3!}\left(\frac{2\cdot\Delta E\cdot T}{\hbar}\right)^3 + \cdots\right)$$

and hence we find

$$E_q = E_{min} + \hbar\omega$$

$$= \frac{\hbar^2\sqrt{L}}{m}\left\{\frac{1}{54}\left(2\mu^3 - 27\mu - 2(\mu^2+9)\sqrt{\mu^2+9}\right) + \sqrt{\frac{2\mu+\sqrt{\mu^2+9}}{3}}\right\}$$

and $\Delta E = \frac{\hbar}{2T}\cdot\frac{16\alpha\omega^2(1+\beta)\sqrt{m}\,T}{\pi\hbar}\sqrt{\frac{\pi\hbar}{2\omega}}\,e^{-mJ/\hbar}$

$$= \frac{\hbar^2\sqrt{L}}{m}\cdot 8(\mu^2+9)^{3/8}\left(\frac{2\mu+\sqrt{\mu^2+9}}{3}\right)^{5/4}\cdot\frac{1+\beta}{\sqrt{2\pi}}\cdot\exp\left[-\frac{(\mu^2+9)^{1/4}}{4\beta}\left(\frac{2\mu+\sqrt{\mu^2+9}}{3}\right)^{3/2}\left\{2\beta-1+\frac{4\beta+1}{\sqrt{\beta(\beta+1)}}\cdot\ln\left(\sqrt{\beta}+\sqrt{1+\beta}\right)\right\}\right]$$

For $\mu=0$ these become (see next page)

From the general expressions we see that for all values of μ, dimensional parameters produce a harmless overall scale, rather than acting to suppress the instanton contribution.

Expressions for $\mu=0$:

$$E_q = \frac{k^2\sqrt{L}}{m}\left(-1+\sqrt{3}\right)$$

$$\Delta E = \frac{k^2\sqrt{L}}{m}\cdot\frac{36}{\sqrt{2\pi\sqrt{3}}}\, e^{-\frac{3}{2}\ln(2+\sqrt{3})}$$

Appendix 3.2 - Dimensional analysis for instanton calculations

In the text we have written the Euclidean Lagrangian for the anharmonic (quartic) oscillator as

$$L = \tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}\left(q^2 - \tfrac{1}{2}\mu^2\right)^2$$

in which μ^2 is used as a parameter . The most obvious way to alter this to include dimensional quantities explicitly in a dimensionally correct scheme is to rewrite it as

$$L = \frac{m}{2}\dot{q}^2 + \frac{\hbar^2}{2mL^2}\left(q^2 - \tfrac{1}{2}M^2\right)^2$$

where M has the dimensions of a length and L of a length cubed . Then a dimensionless quantity is given by

$$\hat{\mu} = M/L^{1/3}$$

and it may readily be checked that $\hat{\mu}$ enters all expressions in the same way as the original μ . The dimensional scale for energies is $\hbar/mL^{2/3}$.

We now compare this with the standard approach which comes from direct generalisation of the equivalent classical problem ;

$$\hat{H} = \tfrac{1}{2m}\hat{p}^2 - \tfrac{1}{2}m\omega^2\hat{q}^2 + (m^2\Omega^3/2\hbar)\hat{q}^4 + \text{const.}$$

where the constant is arranged to give a classical minimum at zero . ω and Ω are characteristic frequencies , giving respectively the rates of oscillation at the bottom of one of the wells and between the two wells - i.e. the short and long range oscillations . The dimensionless parameter characterising the problem is the ratio between these two frequencies ,

$$\chi = \omega/\Omega$$

where for a weak-coupling problem we expect χ to be large (slow rate of oscillation between wells as opposed to within a well) . Rewriting the above Lagrangian ,

$$L = \frac{m}{2}\dot{q}^2 + \frac{m^2\Omega^3}{2\hbar}\left(q^2 - \frac{\hbar\omega^2}{2m\Omega^3}\right)^2$$

we may identify our original parameters as

$$M = \frac{\omega}{\Omega}\left(\frac{\hbar}{m\Omega}\right)^{1/2} \quad , \quad L = \left(\frac{\hbar}{m\Omega}\right)^{3/2}$$

and so $\chi = \frac{\omega}{\Omega} = \frac{M}{L^{1/3}} = \hat{\mu} = \mu$.

Thus μ is indeed the appropriate dimensionless parameter to choose . In this second formulation the dimensional scale of energy is given by $\hbar\Omega$ (or equivalently $\hbar\omega$, but we are particularly interested in strong-coupling cases near $\omega = 0$ with Ω finite) .

For the supersymmetric problem we have

$$L = \frac{m}{2}\dot{q}^2 + \frac{\hbar^2}{2m}\left(L^2 q^6 - 2LMq^4 + (M^2 - 3L)q^2 + M\right)$$

and rewriting this in terms of characteristic frequencies gives

$$L = \frac{m}{2}\dot{q}^2 + \frac{1}{2}m\omega^2 q^2 - \frac{m^2\Omega^3}{2\hbar}q^4 + \frac{m^3\hat{\Omega}^4}{2\hbar^2}q^6 + \text{const.}$$

Then dimensionless ratios are

$$\chi_1 = \frac{\omega}{\Omega} = \sqrt{\mu^2 - 3}\,/\,(2\mu)^{1/3}$$

$$\chi_2 = \frac{\omega}{\hat{\Omega}} = \sqrt{\mu^2 - 3}$$

where here μ is the dimensionless parameter used in the text , $\mu = \frac{M}{\sqrt{L}}$. So , as explained in the text , the location of weak and strong coupling regimes is more obscure in this case , but seems to be qualitatively similar to the quartic problem .

Appendix 3.3 - Calculating amplitudes for non-instanton solutions

There are surprisingly few differences here from calculation of the instanton amplitude , but most of the intermediate steps can only be expressed in the form of integrals to be performed numerically , rather than exactly calculation . Therefore , this appendix should be read in conjunction with appendix 3.1 .

The Lagrangian for the quartic problem (dropping dimensional factors) is

$$L = \tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}\left(q^2 - \tfrac{1}{2}\mu^2\right)^2$$

We take as the classical solution about which to expand

$$\bar{q}(t) \quad \begin{cases} -\sqrt{\mu^2/2} & \cdots\cdots \quad -T < t < T1 \\ q_\varepsilon(t) & \cdots\cdots \quad T1 < t < T2 \\ \sqrt{\mu^2/2} & \cdots\cdots \quad T2 < t < T \end{cases}$$

where $q_\varepsilon(t)$ is any member of the family of solutions (3.1.14) - it satisfies

$$\dot{q}_\varepsilon^2 = \left(q_\varepsilon^2 - \tfrac{1}{2}\mu^2\right)^2 + \varepsilon$$

and the instanton has $\varepsilon = 0$.

The classical action is

$$\bar{S} = \int_{-T}^{T} L[\bar{q}]\,dt = \int_{T1}^{T2} \dot{q}_\varepsilon^2\,dt - \frac{\varepsilon}{2}(T2 - T1)$$

Expanding about the classical solution in the functional integral gives

$$I = \int [dy]\, \exp -\int_{-T}^{T} dt \left(\tfrac{1}{2}\dot{y}^2 + \tfrac{1}{2}\left(6\bar{q}^2 - \mu^2\right)y^2 + \text{higher powers}\right)$$

where $y(t) = q(t) - \bar{q}(t)$

As before this is done in two ways , first by a change of variable to

$$z(t) = y(t) - \int_{-T}^{t} \frac{\dot{N}(\tau)}{N(\tau)}\, y(\tau)\,d\tau$$

Before we had $N(t) = \dot{\bar{q}}(t)$, but here we cannot do this as $N(t)$ must be continuous . To determine $N(t)$ we must return to its

definition [27] by $\ddot{N}(t) = \left(6\bar{q}^2(t) - \mu^2\right) N(t)$

and solve this in the three regions separately . Thus

$$N(t) = \begin{cases} \dot{q}_1 \exp\left[\sqrt{2\mu^2}\,(t-T_1)\right] & \cdots\cdots\ -T < t < T_1 \\ \dot{\bar{q}}(t) & \cdots\cdots\cdots\ T_1 < t < T_2 \\ \dot{q}_2 \exp\left[-\sqrt{2\mu^2}\,(t-T_2)\right] & \cdots\cdots\ T_2 < t < T \end{cases}$$

where $\dot{q}_{1,2} = \dot{\bar{q}}(T_1, T_2)$ - N is now clearly continuous . Then

$$I = \frac{1}{\sqrt{2\pi}} \left(\int_{-T}^{T} dt\ \frac{N(T)N(-T)}{N^2(t)} \right)^{-1/2} \equiv \sqrt{\frac{\sqrt{2\mu^2}}{2\pi}} \cdot \frac{1}{J}$$

$$= \sqrt{\frac{\sqrt{2\mu^2}}{2\pi}} \left[\dot{q}_1 \dot{q}_2\, e^{-\sqrt{2\mu^2}\,(2T-(T_2-T_1))} \left\{ \frac{e^{2\sqrt{2\mu^2}\,(T+T_1)} - 1}{\dot{q}_1^2} + \sqrt{2\mu^2} \int_{T_1}^{T_2} \frac{dt}{\dot{\bar{q}}^2(t)} + \frac{e^{2\sqrt{2\mu^2}\,(T-T_2)} - 1}{\dot{q}_2^2} \right\} \right]^{-1/2}$$

Alternatively we may expand in normal modes to get the same answer as before ,

$$I = \left| \frac{Dy}{Dc_n} \right| \prod_n \left(\frac{\pi\mu^2}{E_n^2} \right)^{1/2}$$

where the eigenfunction Ψ_n corresponding to E_n^2 is normalised by $\int_{-T}^{T} \Psi_n^2(t)\, dt = 1/\mu^2$

In particular we have $\Psi_0(t) \propto N(t)$

Estimating E_0^2 by boundary perturbation theory gives

$$E_0^2 = \frac{2\sqrt{2\mu^2}\ \dot{q}_1 \dot{q}_2\, e^{-\sqrt{2\mu^2}\,(2T-(T_2-T_1))}}{\int_{-T}^{T} N^2(t)\, dt}$$

To cope with the quasi-zero-mode of time translations we introduce a Faddeev-Popov determinant , which to our degree of approximation is

$$\Delta_{FP} = \int_{-T}^{T} \Psi_0(t)\, N(t)\, dt$$

and normalising Ψ_0 correctly gives this as

$$\Delta_{FP} = \frac{1}{\mu} \sqrt{\int_{-T}^{T} N^2(t)\, dt}$$

The full nonperturbative path-integral incorporating this is

$$\bar{I} = \Delta_{FP} \left| \frac{Dy}{Dc_n} \right| \prod_{E_n^2 \neq E_0^2} \left(\frac{\pi\mu^2}{E_n^2} \right)^{1/2} \cdot \mu^2 = \Delta_{FP} \cdot I \cdot \sqrt{\frac{E_0^2}{\pi\mu^2}}\ \cdot \mu^2$$

We may now nearly reconstruct the full amplitude , but first

must fix the normalisation factor which patches multiple contributions together . Physically this represents the extent of overlap between the two wells , as explained in appendix 3.1 , and so has a value independent of the value of $\mathcal{E}$. Calculating it for the instanton , we find it to be

$$N = \sqrt{\frac{\sqrt{2}\mu^2}{\pi}}$$

On reconstruction we find

$$E_0 = \sqrt{\frac{\mu^2}{2}}$$

- i.e. the quadratic energy is independent of $\mathcal{E}$ - this is because it is fixed by the properties of the potential around its minima and does not depend on how we travel from one minimum to the other .

$$\Delta E = \sqrt{\frac{2}{\pi}\sqrt{\frac{\mu^2}{2}}}\ \frac{\sqrt{\dot{q}_1\dot{q}_2}\ e^{\sqrt{\frac{\mu^2}{2}}(T_2 - T_1)}\ e^{-\bar{S}}}{J}$$

- this does depend on $\mathcal{E}$ through $\dot{q}_1$, $\dot{q}_2$, S and J .

In fact for numerical computation we do not directly use $\mathcal{E}$ - instead for each phase-space integral we pick the values of the jumps in q and p as the independent variables and hence derive the appropriate value of $\mathcal{E}$ from $\mathcal{E} = p^2 + \left(q^2 - \frac{1}{2}\mu^2\right)^2$. This also provides a delta-function in the integral since the value of stays fixed along the path $q_{\mathcal{E}}(t)$.

Appendix 5.1 - Propagating fermion states for two sites including anti-fermions

There are 16 independent states , with an original basis at each site chosen to be $(\bar{\psi}, \frac{1}{\sqrt{2}}(1 \pm \bar{\psi}\psi), \psi)$. States which propagate under the Hamiltonian (5.) are

	Fermion number	State
1)	2	$\psi \otimes \psi$
2)	1	$\frac{1}{2}\left[(1-\bar{\psi}\psi)\otimes\psi - \psi\otimes(1-\bar{\psi}\psi)\right]$
3)	1	$\frac{1}{2}\left[(1+\bar{\psi}\psi)\otimes\psi - \psi\otimes(1+\bar{\psi}\psi)\right]$
4)	1	$\sqrt{\frac{\sqrt{5}-1}{8\sqrt{5}}}\left[(1-\bar{\psi}\psi)\otimes\psi + \psi\otimes(1-\bar{\psi}\psi)\right] - \sqrt{\frac{\sqrt{5}+1}{8\sqrt{5}}}\left[(1+\bar{\psi}\psi)\otimes\psi + \psi\otimes(1+\bar{\psi}\psi)\right]$
5)	1	$\sqrt{\frac{\sqrt{5}+1}{8\sqrt{5}}}\left[(1-\bar{\psi}\psi)\otimes\psi + \psi\otimes(1-\bar{\psi}\psi)\right] + \sqrt{\frac{\sqrt{5}-1}{8\sqrt{5}}}\left[(1+\bar{\psi}\psi)\otimes\psi + \psi\otimes(1+\bar{\psi}\psi)\right]$
6)	0	$\frac{1}{16}\left[2\psi\otimes\bar{\psi} + 2\bar{\psi}\otimes\psi + (1+\bar{\psi}\psi)\otimes(1-\bar{\psi}\psi) - (1-\bar{\psi}\psi)\otimes(1+\bar{\psi}\psi)\right]$
7)	0	$\frac{1}{\sqrt{8\sqrt{5}(\sqrt{5}+1)}}\left[-2\psi\otimes\bar{\psi} + 2\bar{\psi}\otimes\psi + (1+\bar{\psi}\psi)\otimes(1-\bar{\psi}\psi) + (1-\bar{\psi}\psi)\otimes(1+\bar{\psi}\psi) - (\sqrt{5}+1)(1-\bar{\psi}\psi)\otimes(1-\bar{\psi}\psi)\right]$
8)	0	$\frac{1}{\sqrt{8\sqrt{5}(\sqrt{5}-1)}}\left[-2\psi\otimes\bar{\psi} + 2\bar{\psi}\otimes\psi + (1+\bar{\psi}\psi)\otimes(1-\bar{\psi}\psi) + (1-\bar{\psi}\psi)\otimes(1+\bar{\psi}\psi) + (\sqrt{5}-1)(1-\bar{\psi}\psi)\otimes(1-\bar{\psi}\psi)\right]$
9)	0	$\frac{1}{16}\left[2\psi\otimes\bar{\psi} + 2\bar{\psi}\otimes\psi - (1+\bar{\psi}\psi)\otimes(1-\bar{\psi}\psi) + (1-\bar{\psi}\psi)\otimes(1+\bar{\psi}\psi)\right]$
10)	0	$\frac{1}{\sqrt{8\sqrt{5}(\sqrt{5}+1)}}\left[2\psi\otimes\bar{\psi} - 2\bar{\psi}\otimes\psi + (1+\bar{\psi}\psi)\otimes(1-\bar{\psi}\psi) + (1-\bar{\psi}\psi)\otimes(1+\bar{\psi}\psi) - (\sqrt{5}+1)(1+\bar{\psi}\psi)\otimes(1+\bar{\psi}\psi)\right]$
11)	0	$\frac{1}{\sqrt{8\sqrt{5}(\sqrt{5}-1)}}\left[2\psi\otimes\bar{\psi} - 2\bar{\psi}\otimes\psi + (1+\bar{\psi}\psi)\otimes(1-\bar{\psi}\psi) + (1-\bar{\psi}\psi)\otimes(1+\bar{\psi}\psi) + (\sqrt{5}-1)(1+\bar{\psi}\psi)\otimes(1+\bar{\psi}\psi)\right]$
12)	-1	$\frac{1}{2}\left[(1+\bar{\psi}\psi)\otimes\bar{\psi} - \bar{\psi}\otimes(1+\bar{\psi}\psi)\right]$
13)	-1	$\frac{1}{2}\left[(1-\bar{\psi}\psi)\otimes\bar{\psi} - \bar{\psi}\otimes(1-\bar{\psi}\psi)\right]$
14)	-1	$\sqrt{\frac{\sqrt{5}-1}{8\sqrt{5}}}\left[(1+\bar{\psi}\psi)\otimes\bar{\psi} + \bar{\psi}\otimes(1+\bar{\psi}\psi)\right] - \sqrt{\frac{\sqrt{5}+1}{8\sqrt{5}}}\left[(1-\bar{\psi}\psi)\otimes\bar{\psi} + \bar{\psi}\otimes(1-\bar{\psi}\psi)\right]$
15)	-1	$\sqrt{\frac{\sqrt{5}+1}{8\sqrt{5}}}\left[(1+\bar{\psi}\psi)\otimes\bar{\psi} + \bar{\psi}\otimes(1+\bar{\psi}\psi)\right] + \sqrt{\frac{\sqrt{5}-1}{8\sqrt{5}}}\left[(1-\bar{\psi}\psi)\otimes\bar{\psi} + \bar{\psi}\otimes(1-\bar{\psi}\psi)\right]$
16)	-2	$\bar{\psi}\otimes\bar{\psi}$

Supersymmetry transformations communicate within the following pairs (recall that $\hat{O}_+$ ($\hat{O}_-$) reduces (increases) charge by one) -

1,2 ; 3,6 ; 4,7 ; 5,8 ; 9,13 ; 10,14 ; 11,15 ; 12,16 .

The additional degeneracy because the Schrodinger equations are identical relates the following groups -

A : 1,3,9,12 ; 4,10 ; 5,11

B : 2,6,13,16 ; 7,14 ; 8,15

If a bosonic level has energy E_b , then

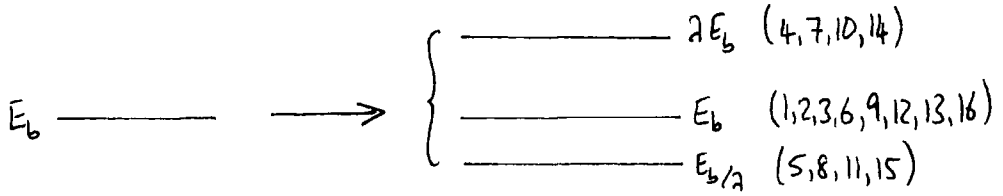
1) $E_b = 0$. All states in B are annihilated by both $\hat{O}_+$ and $\hat{O}_-$, and all states in A have zero energy . Thus the ground-state in this case is 8-fold degenerate . The fermionic content of such a state is

(1x(f=1) + 3x(f=0) + 3x(f=-1) + 1x(f=-2)) ,

so that the quantity Δ defined in [43] is

1x(-1) + 3x(1) + 3x(-1) + 1x(1) = 0 - as expected , for an unbroken model $\Delta = 0$. This quantity Δ may be used in some cases to tell if a particular model is broken or unbroken - in reference [15] it is redefined in terms of a functional integral .

2) $E_b > 0$. The first groups in both A and B are all degenerate with energy E_b , the second groups all degenerate with energy λE_b , the third groups all degenerate with energy E_b/λ , where $\lambda = \left(\frac{\sqrt{5}+1}{2}\right)^2 \approx 2.6$. Thus the fermions split the original single level as follows



Here the fermion content of the three states is

1x(f=1) + 2x(f=0) + 1x(f=-1)

1x(f=2) + 2x(f=1) + 2x(f=0) + 2x(f=-1) + 1x(f=-2)

1x(f=1) + 2x(f=0) + 1x(f=-1)

and so the quantity Δ = 1x(-1) + 2x(1) + 1x(-1) = 0 .

Thus even though a state such as the first of these could represent the ground-state of a spontaneously broken theory , we still have $\Delta = 0$ - this is the ambiguous situation in which knowledge of Δ does not help us to discriminate between broken and unbroken cases .

Thus (unless there is an additional degeneracy from the bosonic sector) in an unbroken theory the ground-state has zero energy and is 8-fold degenerate ; in a spontaneously broken theory it has strictly positive energy and is 4-fold degenerate .