Group actions on to-grgupoids and crossed complexes, and the homotopy groups of orbit spaces

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GROUP ACTIONS ON ω-GROUPOIDS AND CROSSED COMPLEXES, AND THE HOMOTOPY GROUPS OF ORBIT SPACES

by

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ABSTRACT

This thesis is concerned with some algebraic and topological aspects of group actions on groupoids, ω-groupoids and crossed complexes. One of our main aims is to obtain information on the homotopy groups of orbit spaces.

Let A be a groupoid, ω-groupoid or crossed complex with an action of a group G. The algebraic part of the thesis concentrates on the orbit objects which are universal for G-morphisms into objects with trivial action. Algebraic descriptions are given for orbit groupoids and crossed complexes.

Topological considerations arise as follows. We consider the fundamental groupoid of a space in dimension one, and the homotopy crossed complex of a filtered space in higher dimensions. When the space is equipped with a suitable G-action there is an action induced on the algebraic invariant. We prove that, under suitable conditions, the fundamental groupoid or homotopy crossed complex of the orbit space is the orbit object of the corresponding invariant of the space. In these cases the algebraic descriptions of orbit objects give information on certain relative homotopy groups of the orbit space.

Finally we consider spaces equipped with a cover by subspaces, and various related groupoids. An application of G-groupoids is given to presentations of groups of homeomorphisms.

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CONTENTS.

CHAPTER ONE. INTRODUCTION

In [10] R.Brown and P.J.Higgins define categories of ω-groupoids and crossed complexes, and in a companion paper [11] these are used to give a higher dimensional Seifert - van Kampen theorem. The main part of this thesis is devoted to a consideration of group actions in these categories and some of their topological consequences.

The categories of ω-groupoids $\mathcal{G}$, and crossed complexes $\mathcal{C}$, are both higher dimensional generalisations of the category of groupoids. Both are defined algebraically, but the motivation for the definitions comes from homotopy theory. An important result in the Brown - Higgins theory is that $\mathcal{G}$ and $\mathcal{C}$ are equivalent. This enables Brown and Higgins to prove their higher dimensional Seifert - van Kampen theorem - the Union Theorem of [11] - in $\mathcal{G}$, while using the equivalent statement in $\mathcal{C}$ for the applications. This illustrates what is perhaps a more general guideline. That is, topological problems are often more easily considered in the category of ω-groupoids, while algebraic ones are more fruitfully studied using crossed complexes. Although we have chosen $\mathcal{C}$ as the setting for our main topological theorem, there is some evidence to support this guideline in the work presented here, as we shall indicate.

Our motivation for considering group actions in $\mathcal{G}$ and $\mathcal{C}$ was some work of M.A.Armstrong [2,3] and F.Rhodes [19] on the fundamental group of an orbit space, and a desire to generalise these results to the higher dimensional homotopy groups.

Since the Brown - Higgins theory has been successful in extending the Seifert - van Kampen theorem it is natural to use their approach. In order to see how to proceed in higher dimensions the results in dimension one are first reformulated using fundamental groupoids rather than groups. The reformulated results are more natural and indicate the form of theorems in higher dimensions.

The layout of this thesis is as follows. Chapter two gives the background results concerning the algebra of groupoids, ω-groupoids and crossed complexes. For the most part the groupoid results are well-known and the proofs are omitted. Quotient crossed complexes are defined and studied in some detail as these are required later, but we have omitted the study of quotient ω-groupoids as the details appear rather complicated and they are not used in later chapters. We also give the motivating examples of these objects. Let $\underset{\sim}{X} : X^0 \subseteq X^1 \subseteq \ldots \subseteq X^n \ldots$ be a filtered topological space such that loops in X^0 are contractible in X^1. We give the definitions, due to Brown and Higgins, of a homotopy ω-groupoid $\rho\underset{\sim}{X}$ and a homotopy crossed complex $\pi\underset{\sim}{X}$. This crossed complex is essentially a classical object - in the case where X^0 is a point it is a "homotopy system" of J.H.C.Whitehead. We conclude the chapter with a description of the Brown - Higgins Union Theorem for ω-groupoids.

In chapter three we consider some of the algebraic aspects of group actions. Let A be a groupoid, ω-groupoid or crossed complex with a right action of a group G. Motivated by the definition of orbit space in topology, we define an orbit object $A/[G]$ by a universal property. A description of the orbit object is given in the cases of groupoids and crossed complexes. We prove that, in these cases, $A/[G]$ is isomorphic to a quotient of the semi-direct product $A \tilde{\times} G$ which is defined in a natural way. Using these descriptions some properties of the orbit objects are considered.

Chapter four is topological and contains the main result of the thesis, which generalises the work of Armstrong and Rhodes to higher dimensions. We begin by describing their results on the fundamental group and then give the groupoid formulation. It is shown that under suitable conditions the fundamental groupoid of an orbit space is simply the orbit object (as defined in chapter three) of the fundamental groupoid of the space. Initially Armstrong's results refered to simplicial actions on simplicial complexes, which is the first case we consider. Since then, however, Armstrong has made considerable progress in reducing the conditions under which his results hold. Some consideration is also given here to the more general actions.

For the higher dimensional cases the fundamental groupoid of a pair $X^0 \subseteq X$ is replaced by the homotopy crossed complex of a CW-complex with natural filtration $\underset{\sim}{X} : X^0 \subseteq X^1 \subseteq \ldots \subseteq X^n \ldots$. We consider filtration-preserving actions so there is an action induced on $\pi\underset{\sim}{X}$. Given this however, the result we prove has the same type as the groupoid case. Namely, the homotopy crossed complex of the orbit space is isomorphic to the orbit crossed complex of πX. Finally some corollaries and examples are given.

In chapter five we consider a topological space X with a cover $\mathcal{U}$ by subspaces. In this situation G.Segal [21] defined the classifying space $B\mathcal{U}$ of the cover. We begin by describing this construction. Razak also considered this situation in his thesis [18] and we prove, in section two, a modified form of a conjecture of Razak's.

There is a natural map of $B\mathcal{U}$ onto X which induces a morphism of fundamental groupoids. We study the nature of this morphism and, using the groupoid mapping cylinder of R.H.Crowell and N.Smythe [13] , we relate it to the one-dimensional Brown-Higgins union theorem.

Finally group actions are introduced into this situation. Considering actions which preserve the cover we show how our previous work relates to that of A.M.Macbeath [17] and R.Swan [24] on presentations of groups. In fact, the connection is made via the semi-direct product construction defined in chapter three. We also indicate a conection, again via the semi-direct product, with the Bass - Serre theory of groups acting on trees [22].

The main results of chapters three and four were presented by Professor P.J.Higgins at the 1981 Conference on Category Theory at ~~Oberwollfach~~ Gummersbach and will appear in the proceedings of that conference.

CHAPTER TWO. GROUPOIDS, ω-GROUPOIDS AND CROSSED COMPLEXES

Much of this thesis is concerned with various algebraic objects which are perhaps little known and appreciated. For this reason we devote this chapter to a brief survey of groupoids and some of their generalisations and applications. This survey is necessarily incomplete - we concentrate on those aspects of the theory which are required later. With minor exceptions this material is not new. The basic references are [14] for groupoids and [10] for ω-groupoids and crossed complexes.

1. GROUPOIDS

A groupoid is a small category all of whose morphisms are invertible. To introduce our notation we make the following definition.

2.1 DEFINITION. A groupoid A consists of a set A_0 of vertices, a set A_1 of edges (or arrows) and two maps $\partial^0, \partial^1 : A_1 \to A_0$. For $a \in A_1$, $\partial^0 a$ is called the initial vertex and $\partial^1 a$ the final vertex of a. For each $v \in A_0$ there is a distinguished edge εv called the identity at v with initial and final vertex v. There is a partial composition $+$ on A_1 : $a+b$ is defined if and only if $\partial^1 a = \partial^0 b$ and in this case $\partial^0(a+b) = \partial^0 a$ and $\partial^1(a+b) = \partial^1 b$. This data satisfies the following axioms.

(i) (Associativity). $(a+b)+c = a+(b+c)$ whenever both sides are defined.

(ii) (Identity). Let $a \in A_1$ have initial vertex v

and final vertex w. Then $a + \varepsilon w = a = \varepsilon v + a$.

(iii) (Inverse). For each $a \in A_1$ there exists an edge $-a$ with $\partial^{\alpha}(-a) = \partial^{1-\alpha}a$ for $\alpha = 0,1$ and such that $a - a = \varepsilon\partial^0 a$, $-a + a = \varepsilon\partial^1 a$.

A morphism of groupoids $\theta : A \to B$ is a pair of maps $\theta_i : A_i \to B_i$ for $i = 0,1$ preserving all the structure.

□

We denote the category of groupoids by $\mathfrak{G}$pd. By abuse of notation A_1 is sometimes referred to as a groupoid with vertex set A_0. The set of arrows from v to w (i.e. with initial vertex v and final vertex w) is denoted $A_1(v,w)$. Clearly $A_1(v) = A_1(v,v)$ is a group with respect to $+$. It is called the vertex group of A at v. In particular, a groupoid with a single vertex is a group and the category of groups is embedded in $\mathfrak{G}$pd.

A is said to be connected if $A_1(v,w) \neq \emptyset$ for all $v,w \in A_0$, and ia totally disconnected if $A_1(v,w) = \emptyset$ for $v \neq w$. We also refer to the components of a groupoid with obvious meaning. A is said to be simply connected if $A_1(v,w)$ has at most one element for all $v,w \in A_0$. A connected and simply connected groupoid is called a tree groupoid. The simplest non-trivial example of a tree groupoid is the unit interval groupoid with two vertices 0,1 and two non-trivial edges γ, γ^{-1} where γ is an arrow from 0 to1.

The notion of a subgroupoid is an obvious one. A subgroupoid C of A is wide if $C_0 = A_0$. C is a full subgroupoid if $v,w \in C_0$ implies $C_1(v,w) = A_1(v,w)$.

There are some immediate differences between the theories of groups and groupoids. For example, if $\theta : A \to B$ is a groupoid morphism the image $\theta(A)$ need not be a subgroupoid of B. This is illustrated by the morphism of the unit interval groupoid to the integers (considered as a groupoid with one vertex) which sends γ to the generator. For reasons such as this we divide groupoid morphisms $\theta : A \to B$ into various classes. θ is called <u>vertex injective</u> if $\theta_0 ; A_0 \to B_0$ is injective, <u>piecewise injective</u> if each induced map $\theta(v,w) : A_1(v,w) \to B_1(\theta_0 v, \theta_0 w)$ is injective, and <u>group injective</u> if each $\theta(v) = \theta(v,v)$ is injective. Similar definitions apply with "surjective" replacing "injective".

A subgroupoid N of A is <u>normal</u> if it is wide and $n \in N_1(v)$, $a \in A_1(v,w)$ implies $-a + n + a \in N_1(w)$. The <u>kernel</u> of $\theta : A \to B$ is the set of elements of A which map to identity elements of B. It is easily seen to be a normal subgroupoid of A.

<u>2.2 DEFINITION.</u> Let N be a normal subgroupoid of A. The <u>quotient groupoid</u> A/N and natural projection $\tau : A \to A/N$ are defined by the universal property that any morphism $\theta : A \to B$ with $\ker\theta \supseteq N$ factors uniquely through A/N.

□

The quotient groupoid can be described as follows. Define equivalence relations $\sim$ on A_i $(i=0,1)$ by $v \sim w$ if $N_1(v,w)$ is non-empty (for i=0) and $a \sim b$ if $a = n + b + m$ for some $m,n \in N_1$ (for i=1). Let $(A/N)_i = A_i/\sim$ with the class of a denoted $\bar{a}$. The composition in A/N is given as follows:

$\overline{a} + \overline{b}$ is defined if and only if there exist $a_1 \in \overline{a}$, $b_1 \in \overline{b}$ such that $a_1 + b_1$ is defined in A, and in this case $\overline{a} + \overline{b} = \overline{(a_1+b_1)}$. The proof that $\tau: A \to A/N$, $a \mapsto \overline{a}$ satisfies the universal property of definition 2.2 is given in [14].

2.3 PROPOSITION. Let $\theta : A \to B$ be a groupoid morphism. The following statements are equivalent.

(i) The induced morphism $A/\ker\theta \to B$ is an isomorphism.

(ii) θ is vertex surjective and piecewise surjective.

(iii) θ is surjective and for $v, w \in A_0$, $\theta v = \theta w$ implies $(\ker\theta)_1(v,w) \neq \emptyset$.

Proof. [14; Proposition 25, page 88].

□

The morphisms characterised by proposition 2.3 are called quotient morphisms.

If N is a normal subgroupoid of A then clearly $N_1(v)$ is a normal subgroup of $A_1(v)$ for any $v \in N_0 = A_0$. If C is an arbitrary subgroupoid of A its normal closure $N = \overline{C}$ is the smallest normal subgroupoid of A containing C. If A and C are both connected and $v \in C_0$, then $N_1(v)$ is the (group theoretic) normal closure of $C_1(v)$ in $A_1(v)$. More generally we have the following technical lemma which is required later.

2.4 LEMMA. Let A be a connected groupoid, N a subgroupoid containing the vertex $*$. The vertex group of the normal closure $\overline{N}$ at $*$, $\overline{N} \cap A_1(*)$, is generated by

$$\bigcup_{v \in V} \{ -a + N_1(v) + a \mid a \in A_1(v,*) \}$$

where V is a set of vertices of N containing $*$ and at least one vertex from each component of N.

Proof. By [14; page 95] $\overline{N}$ is generated by N and its conjugates $-a+n+a$ $(a \in A_1,\ n \in N_1)$. Let $x \in \overline{N} \cap A_1(*)$. Then x can be expressed as a word $x = x_1 + \dots + x_s$ where $x_i \in N$ or $x_i = -a_i + n_i + a_i$. By inserting identity elements if necessary we may assume that the word has the following form:

$$x = n_1 + (-a_1+m_1+a_1) + \dots + n_r + (-a_r+m_r+a_r) + n_{r+1},$$

where $n_i, m_i \in N_1$, $a_i \in A_1$. Since $\partial^0(-a_i+m_i+a_i) = \partial^1(-a_i+m_i+a_i)$ it follows that $n_* = n_1 + \dots + n_{r+1}$ is a well-defined element of $N_1(*)$. Define $\hat{n}_i = n_1 + \dots + n_i$, for $i=1,\dots,n+1$ (so $n_* = \hat{n}_{r+1}$). Then

$$\begin{aligned} x &= (\hat{n}_1-a_1+m_1+a_1-\hat{n}_1) + (\hat{n}_2-a_2+m_2+a_2-\hat{n}_2) + \dots \\ &\quad + (\hat{n}_r-a_r+m_r+a_r-\hat{n}_r) + n_* \\ &= \hat{m}_1 + \hat{m}_2 + \dots + \hat{m}_r + n_*, \text{ where } \hat{m}_i = \hat{n}_i-a_i+m_i+a_i-\hat{n}_i. \end{aligned}$$

Consider $\hat{m}_i$. There is an element v_i of V in the component of N containing $\partial^0 m_i = \partial^1 m_i$. Let $\ell_i \in N_1(v_i, \partial^0 m_i)$. Then

$$\begin{aligned} \hat{m}_i &= (\hat{n}_i-a_i-\ell_i) + (\ell_i+m_i-\ell_i) + (\ell_i+a_i-\hat{n}_i) \\ &= -c_i + p_i + c_i \quad \text{where } c_i \in A_1(v_i,*),\ p_i \in N_1(v_i). \end{aligned}$$

So finally, $x = (-c_1+p_1+c_1) + \dots + (-c_r+p_r+c_r) + n_*$
which expresses x in terms of the generators.

□

In view of the existence of various classes of groupoid morphisms it is not immediately obvious how to define the notion of "exactness". Our choice is the following.

<u>2.5 DEFINITION.</u> The sequence $\dots \to A \xrightarrow{\theta} B \xrightarrow{\phi} C \to \dots$ in $\mathcal{G}$pd is <u>exact at B</u> if (i) $\ker\phi = \mathrm{im}\theta$ and (ii) $\theta : A \to \mathrm{im}\theta$ is a quotient morphism.

□

Note that condition (i) of 2.5 implies that $\mathrm{im}\theta$ is a subgroupoid of B. Let E_X denote the trivial subgroupoid with vertex set X. (The only morphisms of E_X are identities). If N is a normal subgroupoid of A, then

$$E_{A_0} \longrightarrow N \hookrightarrow A \xrightarrow{\tau} A/N \longrightarrow E_*$$

is a short exact sequence, where * is the singleton set.

We end this section by describing some groupoids which arise in topology. Let $X^0 \subseteq X$ be a pair of topological spaces. The <u>fundamental groupoid</u> of X relative to X^0, $\pi_1(X,X^0)$, has vertex set X^0 and edges the homotopy classes, relative to end points, of paths in X with ends in X^0. Addition is induced by the usual addition of paths. The groupoid $\pi_1(X) = \pi_1(X,X)$ is called the <u>full</u> fundamental groupoid of X.

There is a modification of this. Suppose now that all loops in X^0 are contractible in X. Then there is a groupoid $\rho_1(X,X^0)$ with vertex set $\pi_0(X^0)$ and edges the relative homotopy classes of paths in X with ends in X^0. If $\pi_0(X^0) = X^0$ (for example, if X^0 is discrete) then $\rho_1(X,X^0) \cong \pi_1(X,X^0)$. The groupoid $\rho_1(X,X^0)$ has been studied by Brown and Higgins [9,11] and Razak [18].

One of the early uses of groupoids in topology was a Seifert - van Kampen theorem for the fundamental groupoid which avoided connectivity assumptions. See, for example, Brown [6; §8.4]. Groupoids appear to be of use because the fundamental groupoid is a better algebraic model of the topology than the fundamental group. This point is demonstrated clearly in our consideration in §4.2 of the fundamental groupoid of an orbit space.

In addition, groupoids admit generalisations to higher dimensions and by modelling arguments used for the fundamental groupoid new information can sometimes be obtained about higher homotopy groups. For example, Brown and Higgins [10,11] have obtained a Seifert - van Kampen type theorem which gives information in higher dimensions. This, however, is the subject of the next section.

2. ω-GROUPOIDS AND CROSSED COMPLEXES

The detailed definition of an ω-groupoid is lengthy and technically complicated [10; pp235 - 238]. Essentially though an ω-groupoid $A = \{A_n \mid n \geqslant 0\}$ is a cubical complex with certain extra "degeneracies" $\Gamma_i : A_{n-1} \to A_n$ $(i=1,\dots,n-1)$ such that the pair (A_n, A_{n-1}) has n "compatible" groupoid structures $+_j$ $(j=1,\dots,n)$ with identities given by ε_j and initial and final maps given by ∂_j^0 and ∂_j^1 respectively. The extra degeneracies are called connections. The length of the definition in [10] is due to the relations which the connections must satisfy, and making the word "compatible" precise. Since our notation is the same as [10] we refer the reader there for details. A morphism of ω-groupoids is a morphism of the underlying cubical complexes which preserves all the additional structure. The resulting category is denoted by $\mathcal{G}$.

By forgetting everything above dimension n we obtain the definition of an n-tuple groupoid, a category $\mathcal{G}_n$ of such objects and a truncation functor $tr^n : \mathcal{G} \to \mathcal{G}_n$. The category $\mathcal{G}_1$ is isomorphic to $\mathcal{G}pd$.

The motivating example of an ω-groupoid is also given in [10]. Let $\underset{\sim}{X} : X^0 \subseteq X^1 \subseteq \dots \subseteq X^n \subseteq \dots$ be a filtered space. Let $\underset{\sim}{I}^n$ denote the n-cube with standard filtration, and let $\rho_n \underset{\sim}{X}$ be the set of filter homotopy classes of filtered maps $\underset{\sim}{I}^n \to \underset{\sim}{X}$. In [11] Brown and Higgins provethe non-trivial fact that under the assumption that loops in X^0 are contractible in X^1 , $\rho\underset{\sim}{X} = \{\rho_n\underset{\sim}{X} \mid n \geqslant 0\}$ is an ω-groupoid. A filtered space satisfying this condition (that loops in X^0 are contractible in X^1) is said to be $\underline{J_0}$-filtered, and $\rho\underset{\sim}{X}$ is called the homotopy ω-groupoid of $\underset{\sim}{X}$.

There is another generalisation of groupoids given by Brown and Higgins in [10]. It is the notion of a crossed complex which we now define.

2.6 DEFINITION. A crossed complex C is a sequence

$$\cdots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \underset{\partial^1}{\overset{\partial^0}{\rightrightarrows}} C_0$$

satisfying the following axioms.

1) (C_1, C_0) is a groupoid with initial and final maps ∂^0 , ∂^1 .

2) For $n \geqslant 2$, $C_n = \{C_n(v) \mid v \in C_0\}$ is a family of groups, which are Abelian for $n \geqslant 3$. (I.e. C_n is a totally disconnected groupoid with vertex set C_0).

3) C_1 acts on C_n $(n \geqslant 1)$ on the right — $x \in C_1(v,w)$ induces an isomorphism $C_n(v) \to C_n(w)$ denoted $a \mapsto a^x$.

4) $\partial : C_n \to C_{n-1}$ $(n \geqslant 2)$ is a morphism of groupoids over C_0 which preserves the action. (C_1 acts on $\{C_1(v) \mid v \geqslant C_0\}$ by conjugation — $y^x = -x + y + x$).

5) $\partial\partial = 0 : C_n \to C_{n-2}$ for $n \geqslant 3$.

6) ∂C_2 acts trivially on C_n for $n \geqslant 3$ and acts by

conjugation on C_2 — $a^{\partial b} = -b + a + b$, for $a, b \in C_2(v)$. □

The term "crossed complex" is used because for $v \in C_0$, $\partial : C_2(v) \to C_1(v)$ is a crossed module as defined by Whitehead. (See [5] for an exposition of the theory of crossed modules).

A morphism of crossed complexes $\theta : C \to C'$ is a family of maps $\theta_n : C_n \to C'_n$ such that for $n \geqslant 1$ (θ_n, θ_0) is a groupoid morphism respecting the action of C_1 and the boundary morphisms ∂. The resulting category is denoted by $\mathcal{C}$.

As for ω-groupoids, ignoring everything above dimension n gives a definition of n-truncated crossed complex, a resulting category $\mathcal{C}_n$ and a truncation functor $tr^n : \mathcal{C} \to \mathcal{C}_n$.

There is the notion of a <u>module over a groupoid</u>. The axioms (1) — (3) of definition 2.6 can be used to define C_n as a module over (C_1, C_0).

Important examples once again come from topology. Let $\underset{\sim}{X} : X^0 \subseteq X^1 \subseteq \dots \subseteq X^n \subseteq \dots$ be a filtered space where now we suppose $\pi_0(X^0) = X^0$. The <u>homotopy crossed complex</u> $C = \pi\underset{\sim}{X}$ of $\underset{\sim}{X}$ is defined as follows. (C_1, C_0) is the fundamental groupoid $\pi_1(X^1, X^0)$ and for $n \geqslant 2$, $v \in C_0$, $C_n(v) = \pi_n(X^n, X^{n-1}, v)$, the usual relative homotopy group. The action of C_1 on C_n is the standard one and $\partial : C_n \to C_{n-1}$ comes from the homotopy sequence of the triple (X^n, X^{n-1}, X^{n-2}).

It is a surprising fact that the categories $\mathcal{G}$ and $\mathcal{C}$ are equivalent. Let A be an ω-groupoid. An element $a \in A_n$ is <u>totally degenerate</u> at $v \in A_0$ if $a = (\varepsilon_1)^n v$. (Brown and Higgins use the term "concentrated at v"). The <u>associated crossed complex</u> $C = \gamma A$ is defined as follows. $(C_1, C_0) = (A_1, A_0)$ and for $n \geqslant 2$, $v \in C_0$
$C_n(v) = \{a \in A_n \mid \partial_i^\alpha a = \varepsilon_1^{n-1} v$, for all $(\alpha, i) \neq (0,1)\}$. So $C_n(v)$ is the set of n-cubes of A all of whose faces except the (0,1)-face are totally degenerate at v. Addition in C_n is induced by $+_i$ $(i \geqslant 2)$ and is independent of i. The boundary morphism $\partial : C_n \to C_{n-1}$ is induced by ∂_1^0, and the action of C_1 is given by

$$a^x = -_n (\varepsilon_1^{n-1} x) +_n a +_n (\varepsilon_1^{n-1} x).$$

In [10] it is proved that $A \mapsto \gamma A$ defines functors $\gamma : \mathcal{G} \to \mathcal{C}$ and $\gamma : \mathcal{G}_n \to \mathcal{C}_n$ $(n \geqslant 0)$ each of which is an equivalence. Furthermore, if $\underset{\sim}{X}$ is a filtered space with $\pi_0 X^0 = X^0$ then $\gamma\rho\underset{\sim}{X}$ is naturally isomorphic to $\pi\underset{\sim}{X}$ [10].

It is interesting to note that there are known to be four other categories non-trivially equivalent to these. They are the categories of simplicial T-complexes, cubical T-complexes, ∞-groupoids and poly-T-complexes. See [10,12] for references to the definitions of these categories and the various equivalences.

In an ω-groupoid A there are compositions $+_i$ in n "different directions" in dimension n. A "word" in A_n is therefore a multi-dimensional composable array $[a_{(p)}]$ — see [10]. An element $a \in A_n$ $(n \geqslant 1)$ is thin if it can be written as the composite of an array where each entry is either of the form $\varepsilon_j y$ or $\bar{i}\,\bar{j} \ldots \bar{\ell}\Gamma_m y$. Thin elements play a special role in the theory of ω-groupoids. The simplicial and cubical T-complexes mentioned above are complexes with thin elements which satisfy three simple axioms. The equivalence between ω-groupoids and cubical T-complexes says that all the ω-groupoid structure may be recovered from knowledge of the thin elements and the three axioms.

We now consider normal subobjects and quotient objects in $\mathcal{G}$ and $\mathcal{C}$. In [16] Howie gives definitions in $\mathcal{G}_2$, the category of double groupoids. A subdouble groupoid N of A is normal if (i) $N_0 = A_0$ and (ii) whenever $n \in N_2$, $a \in A_2$ are such that $\partial_i^1 a \in N_1$ and $n' = \bar{i}a +_i n +_i a$ is defined in A_2 (for i=1 or 2) then $n' \in N_2$. Howie noted that condition (ii) for i=1 is equivalent to condition (ii) for i=2, and also that (N_1, N_0) is a normal subgroupoid of (A_1, A_0).

If $\theta : A \to B$ is a morphism in $\mathcal{G}_2$ its kernel $K = \ker\theta$ is defined as follows. $K_0 = A_0$ and for n=1,2, K_n is the set of elements a of A_n such that $\theta(a)$ is totally degenerate in B. The kernel of θ is clearly normal in A.

The elements of the quotient double groupoid A/N can be described as equivalence classes $A_i/\sim$ (i=0,1,2). For i=0,1 the equivalence relations are those for quotient groupoids given in the previous section. For i=2,

$$a \sim b \quad \text{if} \quad a = \begin{bmatrix} n_1 & t & n_2 \\ t & b & t \\ n_2 & t & n_4 \end{bmatrix} \qquad \begin{array}{l} \ulcorner\!\rightarrow 2 \\ \downarrow 1 \end{array}$$

where $n_i \in N_2$ and the t's are thin elements.

Now if $\theta : A \to B$ is a morphism of ω-groupoids, its kernel can be defined to be the set of elements of A whose image is totally degenerate. If we then define normal sub-ω-groupoid in an analogous way to Howie's definition in dimension two, it is clear that the kernel of θ is normal in A. However the equivalence relations used to define the quotient object would then require complicated multi-dimensional arrays.

The following alternative approach may prove more manageable. For the morphism $\theta : A \to B$ define a "kernel system" K as follows. $K_0 = A_0$ and $K_n = \{K_n^i \mid i=1,\ldots,n\}$ where $K_n^i = \{a \in A_n \mid \theta a = \varepsilon_i b \text{ , for some } b \in B_{n-1}\}$. Now to define quotient ω-groupoids we could perhaps use "normal systems" modelled on kernel systems and then use only one-dimensional arrays in defining the required equivalence relations. However the details still appear complicated — we do not know, for example, the minimal axioms for a normal system.

The situation for crossed complexes is considerably simpler, as we now describe.

<u>2.7 DEFINITION.</u> A sub-crossed complex N of C is <u>normal</u> if

(i) (N_1, N_0) is a normal subgroupoid of (C_1, C_0) ,

(ii) $n^x \in N_r(w)$ for all $n \in N_r(v)$, $x \in C_1(v,w)$, $v, w \in C_0$ and $r \geqslant 2$, and

(iii) $a^m - a \in N_r(v)$ for all $a \in C_r(v)$, $m \in N_1(v)$, $v \in C_0$ and $r \geqslant 2$.

□

This definition is a direct generalisation of the definition of normal sub-crossed module (over groups) given in [5].

<u>2.8 DEFINITION.</u> Let $\theta : C \to D$ be a morphism in $\mathcal{C}$. The <u>kernel</u> K of θ is the sub-crossed complex of C given by

(i) $K_0 = C_0$,

(ii) $K_1 = \{x \in C_1 \mid \theta x = \varepsilon w \text{ for some } w \in D_0\}$, and

(iii) for $r \geqslant 2$, $v \in K_0$ $K_r(v) = \{c \in C_r(v) \mid \theta c$ is the identity of $D_r(\theta_0 v)\}$.

□

It is easy to see that the kernel of θ is normal in C. We define the quotient crossed complex by a universal property in an analogous manner to the groupoid case (definition 2.2).

<u>2.9 DEFINITION.</u> Let N be a normal sub-crossed complex of C. The <u>quotient crossed complex</u> C/N and natural projection $\tau : C \to C/N$ are defined by the following universal property.

Let $\theta : C \to D$ be a morphism in $\mathcal{C}$ with $\ker\theta \supseteq N$. Then there is a unique morphism $\theta^* : C/N \to D$ such that $\theta^* \circ \tau = \theta$.

Symbolically, $$\begin{array}{ccc} C & \xrightarrow{\tau} & C/N \\ & {\scriptstyle\theta}\searrow \quad \swarrow{\scriptstyle\theta^*} & \\ & D & \end{array}$$

$\square$

The following description of C/N involves equivalence relations in each dimension $r \geqslant 0$. For $r=0,1$ the equivalence relations are those for quotient groupoids given previously and for $r \geqslant 2$, $a \sim c$ if $a = c^n + m$ for some $n \in N_1$, $m \in N_r$. Let $(C/N)_r = C_r/\sim$ for $r \geqslant 0$, where the equivalence classes are denoted $\langle c \rangle$ in all dimensions. The partition of C_r $(r \geqslant 2)$ into disjoint sets $C_r(v)$, $v \in C_0$ induces a corresponding partition of $(C/N)_r$ into sets $(C/N)_r(\langle v \rangle)$. Define a composition on $(C/N)_r(\langle v \rangle)$ by $\langle a \rangle + \langle c \rangle = \langle a^n + c \rangle$ where $n \in N_1$ and $a^n + c$ is defined in C_r.

<u>2.10 THEOREM</u>. C/N as defined above, with projection $\tau : c \mapsto \langle c \rangle$, is the quotient crossed complex.

<u>Proof</u>. The proof consists of three lemmas.

<u>2.11 LEMMA</u>. $(C/N)_r(\langle v \rangle)$ with composition defined above is a well-defined group and is Abelian for $r \geqslant 3$.

<u>Proof</u>. We first show that the definition of addition is independent of the choices made. So let $a \sim a'$, $c \sim c'$ and let $n_i \in N_1$, $m_i \in N_r$ $(i=0,1)$ be such that $a' = a^{n_1} + m_1$, $c' = c^{n_2} + m_2$. Suppose $n, n' \in N_1$ are such that $a^n + c$ and $(a')^{n'} + c'$ are well defined in C_r. Then

$(a')^{n'} + c' = a^{(n_1 + n')} + m_1^{n'} + c^{n_2} + m_2$.

Now $N_r(v)$ is normal in $C_r(v)$. (For $r \geqslant 3$ the groups are Abelian and normality for r=2 is easy to establish). Hence $m_1^{n'} + c^{n_2} + m_2 = c^{n_2} + m_3$ for some $m_3 \in N_r$. Therefore

$$(a')^{n'} + c' = a^{(n_1+n')} + c^{n_2} + m_3$$
$$= (a^n + c)^{(-n+n_1+n')} - c^{(-n+n_1+n')} + c^{n_2} + m_3 .$$

Since $(-n + n_1 + n') \in N_1$ it follows that $-c^{(-n+n_1+n')} + c^{n_2} \in N_r$ by condition (iii) of definition 2.7. Therefore $(a')^{n'} + c' \sim a^n + c$, so addition is well-defined. It is easy to check that the group structure is inherited.

For $r \geqslant 3$,
$$\begin{aligned} \langle a\rangle + \langle c\rangle &= \langle a^n + c\rangle \\ &= \langle c + a^n\rangle \\ &= \langle (c + a^n)^{-n}\rangle \\ &= \langle c^{-n} + a\rangle \\ &= \langle c\rangle + \langle a\rangle . \end{aligned}$$

This completes the proof.

□

2.12 LEMMA. The crossed complex structure on C induces a crossed complex structure on C/N.

Proof. $\partial : (C/N)_r \to (C/N)_{r-1}$ is given by $\partial\langle c\rangle = \langle\partial c\rangle$, and the action of $(C/N)_1$ is given by $\langle c\rangle^{\langle x\rangle} = \langle c^{n+x}\rangle$ where $n \in N_1$ is such that c^{n+x} is ~~well~~-defined in C_r. (The existence of such an element n follows from the definitions of the equivalence relations).

We must show that conditions (1) — (6) of definition 2.6 hold. (1) is proved in [14], and lemma 2.11 proves (2). That the action is well-defined is proved in a similar way to lemma 2.11.

(4) $$\begin{aligned} \partial(\langle a\rangle + \langle c\rangle) &= \partial\langle a^n + c\rangle \\ &= \langle(\partial a)^n + \partial c\rangle \\ &= \langle\partial a\rangle + \langle\partial c\rangle \\ &= \partial\langle a\rangle + \partial\langle c\rangle , \end{aligned}$$

so $\partial : (C/N)_r \to (C/N)_{r-1}$ is a well-defined morphism of groupoids.

(5) follows immediately from the same law in C.

(6) Let $c \in C_2(v)$, $a \in C_r(w)$ where $v \sim w$. Choose $n \in N_1(v,w)$. Then $\langle a\rangle^{\partial\langle c\rangle} = \langle a^{n+\partial c}\rangle$

$$= \begin{cases} \langle -c+a^n+c\rangle = -\langle c\rangle+\langle a\rangle+\langle c\rangle & \text{for } r=2 \\ \langle a^n\rangle = \langle a\rangle & \text{for } r \geqslant 3. \end{cases}$$

□

<u>2.13 LEMMA.</u> $\tau : C \to C/N$, $c \mapsto \langle c\rangle$ is a $\mathcal{C}$-morphism satisfying the universal property of definition 2.9.

<u>Proof.</u> That τ is a crossed complex morphism is clear from lemma 2.12. Let $\theta : C \to D$ be a morphism in $\mathcal{C}$ such that $N \subseteq \ker\theta$. Define $\theta^* : C/N \to D$ by $\theta^*\langle c\rangle = \theta c$. We must show that θ^* is well-defined. For dimensions 0,1 see [14; Proposition 24].

For dimension $r \geqslant 2$, let $c \sim c'$ in C_r. I.e. $c' = c^n + m$ for some $n \in N_1$, $m \in N_r$. Then $\theta c' = (\theta c)^{\theta n} + \theta m = \theta c$ since $N \subseteq \ker\theta$.

Therefore θ^* is well-defined. It is clearly the unique morphism such that $\theta^* \circ \tau = \theta$.

□

This completes the proof of theorem 2.10.

□

<u>2.14. COROLLARY.</u> For $r \geqslant 2$, $v \in C_0$ $(C/N)_r(\langle v \rangle) \cong C_r(v)/N_r(v)$.

<u>Proof.</u> We have noted above that $N_r(v)$ is normal in $C_r(v)$. Define

$$\psi : C_r(v)/N_r(v) \longrightarrow (C/N)_r(\langle v \rangle) \quad , \quad N_r(v)+c \longmapsto \langle c \rangle .$$

The definition is clearly independent of the choice of c, and gives a well-defined morphism of groups.

To show ψ is injective, let $\psi(N_r(v)+c) = \langle 0 \rangle$, where 0 is the identity of $C_r(v)$. Then $c \sim 0$ in C_r so $c = 0 + m$, where $m \in N_r(v)$. Hence $N_r(v)+c = N_r(v)$, proving ψ is injective.

To show ψ is surjective, let $\langle a \rangle \in (C/N)_r(\langle v \rangle)$. Then $a \in C_r(w)$ where $w \sim v$. Choose $n \in N_1(w,v)$. Then $a^n \in C_r(v)$ and $\psi : N_r(v) + a^n \mapsto \langle a^n \rangle = \langle a \rangle$, proving ψ is surjective.

□

The above corollary indicates that our description of the quotient crossed complex is a generalisation of the description of a quotient crossed module (over groups), given in [5;p9]. We now describe the normal closure of a sub-crossed complex.

2.15.PROPOSITION. Let B be a sub-crossed complex of C, and let $N \subseteq C$ be the following.

(i) (N_1, N_0) is the groupoid normal closure of (B_1, B_0) in (C_1, C_0).

(ii) For $r \geqslant 2$, $N_r = \{N_r(v) \mid v \in N_0\}$ is generated as a (totally disconnected) groupoid by

(a) elements b^x, for $b \in B_r$, $x \in C_1$, and

(b) elements $c^{p+x} - c^x$ for $c \in C_r$, $x \in C_1$, $p \in B_1$.

Then N is the normal closure of B in C. (I.e. N is the smallest normal sub-crossed complex of C containing B).

Proof. Note that for $v \in N_0$, $r \geqslant 2$ $N_r(v)$ is generated as a group by the elements of type (a) and (b) where $b \in B_r(w)$, $c \in C_r(w)$, $p \in B_1(w)$ and $x \in C_1(w,v)$.

We first show that N is a sub-crossed complex of C — i.e. $\partial N_r \subseteq N_{r-1}$ and N_1 acts on N_r. Let $r = 2$. Then $\partial(b^x) = -x + \partial b + x$ which is an element of N_1 since (N_1, N_0) is the normal closure of (B_1, B_0). Also $\partial(c^{p+x} - c^x) = -x - p + \partial c + p - \partial c + x$. Now $\partial c + p - \partial c \in N_1$ by normality, hence $-p + \partial c + p - \partial c = n \in N_1$. Therefore $\partial(c^{p+x} - c^x) = -x + n + x \in N_1$ so $\partial N_2 \subseteq N_1$. Now let $r \geqslant 3$.

Then $\partial(b^x) = (\partial b)^x$ and $\partial(c^{p+x} - c^x) = (\partial c)^{p+x} - (\partial c)^x$ so $\partial N_r \subseteq N_{r-1}$.

It is clear that there is an induced action of N_1 on N_r, so N is a sub-crossed complex of C. By its very construction N is normal and contains B.

It remains to show that N is the smallest such sub-crossed complex, so let M be any normal sub-crossed complex of C containing B. Then $(N_1, N_0) \subseteq (M_1, M_0)$. For $r \geqslant 2$ let c, b, x and p be as above. Then we have (i) $b \in M_r$ so $b^x \in M_r$ since M is normal, and (ii) $p \in M_1$ so $c^p - c \in M_r$ by normality and hence $c^{p+x} - c^x \in M_r$. Therefore $M \subseteq N$ which completes the proof.

□

Finally, we describe the Union Theorem of Brown and Higgins [11]. This is the generalisation of the Seifert - van Kampen theorem mentioned previously which contains information in all dimensions. It was in order to formulate and prove this theorem that much of the theory of ω-groupoids was developed.

<u>2.16 DEFINITION</u>. A filtered space $\underset{\sim}{X} : X^0 \subseteq X^1 \subseteq \ldots \subseteq X^n \subseteq \ldots \subseteq X$ is said to be <u>homotopy full</u> if for all $n > 0$ the induced map $\pi_0 X^0 \to \pi_0 X^n$ is surjective and for $r > n > 0$ and $v \in X^0$ $\pi_n(X^r, X^n, v) = 0$.

□

<u>2.17 THEOREM</u>. (Union Theorem). Let $\underset{\sim}{X}$ be a J_0-filtered space and let $\mathcal{U} = \{ U_\lambda \mid \lambda \in \Lambda \}$ be a cover of X such that the interiors of the U_λ cover X. For $\nu \in \Lambda^n$ set $U_\nu = U_{\nu_1} \cap \ldots \cap U_{\nu_n}$ and let $\underset{\sim}{U}_\nu$ be U_ν with filtration induced from $\underset{\sim}{X}$.

Suppose (i) for $n=1,2$ and all $\nu \in \Lambda^n$, $\underset{\sim}{U}_\nu$ is J_0-filtered, and (ii) for all finite n and all $\nu \in \Lambda^n$, $\underset{\sim}{U}_\nu$ is homotopy full. Then

$$\bigsqcup_{\nu \in \Lambda^2} \rho \underset{\sim}{U}_\nu \overset{a}{\underset{b}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} \rho \underset{\sim}{U}_\lambda \xrightarrow{c} \rho \underset{\sim}{X}$$

is a coequaliser diagram in $\mathcal{G}$ where $\bigsqcup$ denotes coproduct (= disjoint union) , a and b are induced by the inclusions $U_\lambda \cap U_\mu \hookrightarrow U_\lambda$ and $U_\lambda \cap U_\mu \hookrightarrow U_\mu$ respectively, and c is induced by the inclusions $U_\lambda \hookrightarrow X$.

□

The usefulness of the Union Theorem lies in the fact that if $\pi_0 X^0 = X^0$ then $\gamma\rho\underset{\sim}{X}$ is naturally isomorphic to $\pi\underset{\sim}{X}$. Hence applying the equivalence $\gamma : \mathcal{G} \to \mathcal{C}$ gives a coequaliser diagram of homotopy crossed complexes, which has many classical results as corollaries (see [10,11]). The fact that the <u>proof</u> of the Union Theorem takes place in $\mathcal{G}$ illustrates the point made in chapter one that it is often easier to prove topological results in $\mathcal{G}$ rather than $\mathcal{C}$.

The truncation functor $tr^m : \mathcal{G} \to \mathcal{G}_m$ has a right (and a left) adjoint [10], and so preserves colimits (and limits). Hence under the hypotheses of 2.17 we also obtain certain coequalisers in the categories $\mathcal{G}_m$, $m \geqslant 0$. For $m = 1,2$ the homotopy fullness condition (ii) may be weakened — see [18,9] for the minimal conditions.

CHAPTER THREE. G-GROUPOIDS AND GENERALISATIONS

If X is a topological space with group action there is an action induced on the (full) fundamental groupoid $\pi_1(X)$. Similarly if $\underset{\sim}{X}$ is a J_0-filtered space with a filtration preserving group action then the homotopy ω-groupoid $\rho\underset{\sim}{X}$ and homotopy crossed complex $\pi\underset{\sim}{X}$ have induced actions. In this way topological considerations lead us to consider algebraic properties of groupoids, ω-groupoids and crossed complexes with group actions. This is the subject of this chapter.

1. THE GROUPOID CASE

3.1 DEFINITION. Let G be a fixed group. A G-groupoid A is a groupoid together with a right action of G. The automorphism of A corresponding to $g \in G$ is denoted $a \mapsto a^g$. A morphism of G-groupoids or G-morphism is a groupoid morphism which preserves the action.

□

This defines the category of G-groupoids for a fixed group G. Allowing variation of the group would give a more general category of "groupoids with group actions". In this setting a morphism from the G-groupoid A to the H-groupoid B is a groupoid morphism $\theta : A \to B$ and a group morphism $\alpha : G \to H$ such that $\theta(a^g) = (\theta a)^{\alpha g}$. We shall rarely require such generality and for the most part restrict our attention to the category of G-groupoids for fixed G.

Throughout this section A will denote a G-groupoid. Motivated by the notion of an orbit space in topology we seek an analogous definition in the algebraic setting. Simple examples show that the set of orbits of A need not inherit a groupoid structure from A. For example, consider $\mathbb{Z}_2$ acting non-trivially on $\mathbb{Z}$ considered as a groupoid with a single vertex. In the topological context the orbit space is universal for G-maps into a G-space with trivial action. We use this as the definition for groupoids.

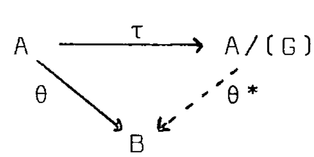
<u>3.2 DEFINITION.</u> The <u>orbit groupoid</u> $A/(G)$ and canonical projection $\tau : A \to A/(G)$ are defined by the following universal prpoerty :

(i) G acts trivially on $A/(G)$ and τ is a G-morphism.

(ii) Let B be any groupoid with trivial G-action and $\theta : A \to B$ a G-morphism. Then there exists a unique morphism $\theta^*: A/(G) \to B$ such that $\theta^*\tau = \theta$.

□

Symbolically we illustrate (ii) by the diagram



A standard argument shows that, if it exists, $A/(G)$ is unique up to (unique) isomorphism. Existence may also be established on "general nonsense" grounds. $A/(G)$ is the colimit in $\mathcal{G}$pd of the diagram whose unique object is A and whose morphisms are the automorphisms of A induced by all the elements of G. The fact that $\mathcal{G}$pd is co-complete [14; Theorem 3, p70] establishes the existence of $A/(G)$.

The notation $A/(G)$ is used to distinguish orbit groupoids from quotient groupoids. (In [15] $A//G$ is used for the orbit groupoid).

The above existence proof for $A/(G)$ is not very useful — we would like to have a description of the orbit groupoid. The defect noted previously that the sets of orbits of A do not necessarily inherit a groupoid structure can be remedied as follows.

Let $\bar{x}$ denote the orbit of $x \in A_1$ and define a relation $\sim$ on the set of orbits by $\bar{x} \sim \bar{y}$ if $x = x_1 + x_2 + \dots + x_n$, $y = y_1 + y_2 + \dots + y_m$ in A_1 and there exist group elements $g_1, \dots, g_n, h_1, \dots, h_m$ such that

$$x_1^{g_1} + \dots + x_n^{g_n} = y_1^{h_1} + \dots + y_m^{h_m} .$$

It is easy to see that $\sim$ is an equivalence relation.

Let $\langle x\rangle$ denote the equivalence class of $\bar{x}$ and B_1 the set of such classes. Also let $\langle v\rangle$ denote the orbit of $v \in A_0$ and B_0 the set of orbits. Then $B = (B_1, B_0)$ inherits a graph structure from A. If $\langle x\rangle, \langle y\rangle \in B_1$ are such that $\partial^0\langle y\rangle = \partial^1\langle x\rangle$ then there exists $g \in G$ such that $\partial^0 y^g = \partial^1 x$ in A_1 and we define $\langle x\rangle + \langle y\rangle = \langle x + y^g\rangle$. It is now routine (but tedious!) to check that B is thus a well-defined groupoid and has the universal property characterising A/(G). These computations are omitted since this description of the orbit groupoid is not easy to work with (and its generalisation to higher dimensions becomes unmanageable). Instead, we give a more elegant and useful description via the semi-direct product construction $A \tilde{\times} G$.

In [7] Brown considers the case of a groupoid G acting on a groupoid A "via a morphism $\omega : A \to G_0$", and he gives a definition of the semi-direct product $A \tilde{\times} G$. Brown attributes the definition to Frölich and calls it the split extension (although later in [8] the terminology semi-direct product is prefered). Our situation is a special case of that considered byBrown and (when modified for right actions) the definition in [7] reduces to the following.

3.3 DEFINITION. The semi-direct product groupoid $A \tilde{\times} G$ has vertex set A_0 , edge set $A_1 \times G$ with initial and final maps $\partial^0(a,g) = \partial^0 a$, $\partial^1(a,g) = \partial^1 a^g$. If $\partial^1 a^g = \partial^0 b$ then composition is given by

$$(a,g) + (b,h) = (a + b^{g^{-1}} , gh) .$$

□

Note that the expression $\partial^1 a^g$ is not ambiguous since $\partial^1(a^g) = (\partial^1 a)^g$.

<u>3.4 LEMMA.</u> $A \tilde{\times} G$ is a well-defined groupoid with identity elements $\varepsilon v = (\varepsilon v, 1)$ and inverses $-(a,g) = (-a^g, g^{-1})$.

<u>Proof.</u> Firstly $\partial^0\big((a,g) + (b,h)\big) = \partial^0 a = \partial^0(a,g)$ and $\partial^1\big((a,g) + (b,h)\big) = \big(\partial^1 b^{g^{-1}}\big)^{gh} = \partial^1 b^h = \partial^1(b,h)$.

Associativity of composition follows from associativity in A and G.

If $\partial^0 a = v$, $\partial^1 a^g = w$ then

$(\varepsilon v, 1) + (a,g) = (\varepsilon v + a, g) = (a,g)$, and

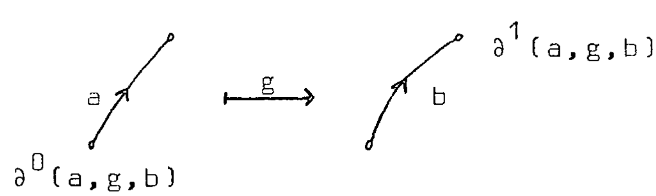
$(a,g) + (\varepsilon w, 1) = (a + \varepsilon w^{g^{-1}}, g) = (a,g)$.

Finally,

$$\begin{aligned}(a,g) - (a,g) &= (a,g) + (-a^g, g^{-1})\\ &= (a-a, 1)\\ &= \varepsilon\partial^0(a,g), \text{ and}\\ -(a,g) + (a,g) &= (-a^g, g^{-1}) + (a,g)\\ &= (-a^g + a^g, 1)\\ &= \varepsilon\partial^1(a,g).\end{aligned}$$

□

It may be helpful to think of the edges of $A \tilde{\times} G$ as triples (a,g,b) where $a^g = b$ in A_1. The diagram below then symbolises an edge, and it can be seen that the definition of addition in $A \tilde{\times} G$ is completely forced.



Define $i : A \to A \tilde{\times} G$ to be the identity on vertices and $a \mapsto (a,1)$ on edges. Also define $p : A \tilde{\times} G \to G$ by $(a,g) \mapsto g$. In the more general case where G is a groupoid Brown proves that i is an isomorphism onto $\ker p$ and p is a groupoid fibration. (See, for example, [14; p98] for the definition of a fibration). Recall that p is piecewise surjective if, for $g \in G$ and $v, w \in A_0$, there exists $(a,g) \in (A \tilde{\times} G)_1(v,w)$ such that $p(a,g) = g$. Therefore p is piecewise surjective if and only if A is connected. The characterisation of quotient morphisms as the vertex surjective, piecewise surjective morphisms (proposition 2.3) gives the following.

3.5 LEMMA. $E_{A_0} \longrightarrow A \xrightarrow{i} A \tilde{\times} G \xrightarrow{p} G \longrightarrow E_*$ is an exact sequence of groupoids if and only if A is connected.

□

This sequence and the exact sequence for a quotient groupoid provide our basic examples of short exact sequences of groupoids.

We now consider some properties of $A \tilde{\times} G$, beginning with the description of the orbit groupoid $A/(G)$.

3.6 THEOREM. Let N be the normal subgroupoid of $A \tilde{\times} G$ generated by elements of the form $(\varepsilon v, g)$ for $v \in A_0$, $g \in G$. Then

$$A/(G) \cong (A \tilde{\times} G)/N .$$

Proof. We verify the universal property of definition 3.2. Let $\pi : A \tilde{\times} G \to (A \tilde{\times} G)/N$, $(a,g) \mapsto \langle a,g \rangle$ be the quotient morphism, and let $\tau : A \to (A \tilde{\times} G)/N$ be the composite $\tau = \pi \circ i$. Hence $\tau(a) = \langle a,1 \rangle$.

Now G acts on $A \tilde{\times} G$ by $(a,g)^h = (a^h, h^{-1}gh)$ such that i and π are G-morphisms. Since

$$(a^h, h^{-1}gh) = (\varepsilon v, h^{-1}) + (a,g) + (\varepsilon w, h)$$

(where $v = \partial^o a^h$, $w = \partial^1 a^g$) the action on $(A \tilde{\times} G)/N$ is trivial.

This verifies the first part of the universal property. For the second part let B be a trivial G-groupoid (i.e. the action is trivial) and let $\theta : A \to B$ be a G-morphism.

Extend θ to $\psi : A \tilde{\times} G \to B$ by defining $\psi(a,g) = \theta a$. Since the action on B is trivial and θ is a G-morphism ψ is well-defined. Now $\psi(\varepsilon v, g) = \theta \varepsilon v$ which is an identity of B , so by the universal property of quotient groupoids (definition 2.2) there is a unique morphism

$$\theta^* : (A \tilde{\times} G)/N \longrightarrow B \quad , \quad \text{such that} \quad \theta^* \circ \pi = \psi .$$

Therefore we have the following commutative diagram in which θ^* is unique for the right hand triangle.

$$\begin{array}{ccccc} A & \xrightarrow{i} & A \tilde{\times} G & \xrightarrow{\pi} & (A \tilde{\times} G)/N \\ & \searrow^{\theta} & \downarrow \psi & \swarrow_{\theta^*} & \\ & & B & & \end{array}$$

To complete the proof we are required to show that θ^* is unique for the outer triangle. So let $\xi : (A \tilde{\times} G)/N \longrightarrow B$ be such that $\xi \circ \tau = \theta^*$.

Since $(a,g) = (a,1) + (\varepsilon\partial^1 a,\ g)$ we have

$$\xi\langle a,g\rangle = \xi\langle a,1\rangle = \theta a = \psi(a,g) \ .$$

That is $\theta^* \circ \pi = \xi$, so by the uniqueness of θ^* in the right hand triangle we have $\xi = \theta^*$ as required.

□

<u>3.7 PROPOSITION</u>. (i) Suppose G acts freely on A (by which we mean that no non-trivial element of G fixes a vertex of A). Then $\tau : A \to A/(G)$ is group injective.

(ii) Suppose G is generated by those of its elements which stabilise some vertex of A. Then A is a tree groupoid implies that $A/(G)$ is also a tree groupoid.

<u>Proof</u>. We use the description of $A/(G)$ given in the previous theorem.

(i) A vertex element of N is of the form $-(a,g) + (\varepsilon v,h) + (a,g)$ where h stabilises $v = \partial^0 a$. Since G acts freely $h=1$. Therefore N is a tree groupoid and

$A \tilde{\times} G \to (A \tilde{\times} G)/N$ is group injective. But $i : A \to A \tilde{\times} G$ is injective so the composite $\pi \circ i$ is group injective.

(ii) Assume A is a tree groupoid and G is generated by its elements which fix some vertex of A. Let $\langle a,g\rangle \in (A \tilde{\times} G)/N$ have $\partial^0\langle a,g\rangle = \langle v\rangle = \partial^1\langle a,g\rangle$. Then $(a,g) \in (A \tilde{\times} G)_1(v^{g_1}, v^{g_2})$ for some $g_1, g_2 \in G$. Hence $a \in A_1(w, w^h)$ where $w = v^{g_1}$ and $h = g_1^{-1}g_2g^{-1}$. By hypothesis $h = h_1h_2 \ldots h_n$ where h_i stabilises some vertex, v_i say.

Since A is connected we may choose $b_i \in A_1(w, v_i)$ for $i=1,\ldots,n$. Consider $c = b_1 - b_1^{h_1} + b_2 - b_2^{h_2} + \ldots + b_n - b_n^{h_n}$. Since h_i stabilises $\partial^1 b_i$ and $\partial^0 b_i^{h_i} = w^{h_1 \ldots h_i} = \partial^0 b_{i+1}$, c is well-defined. Now $\partial^0 c = w$ and $\partial^1 c = w^{h_1 \ldots h_n} = w^h$, so c and a have the same endpoints. But A is a tree groupoid so $a = c$. Therefore

$$(a,g) = \left(b_1 - b_1^{h_1} + \ldots + b_n - b_n^{h_n}, g\right)$$
$$= (b_1 - b_1^{h_1}, 1) + \ldots + (b_n - b_n^{h_n}, 1) + (\varepsilon w^h, g) .$$

Now $(b_i - b_i^{h_i}, 1) = (b_i, 1) + (\varepsilon\partial^1 b_i, h_i^{-1}) - (b_i, 1) + (\varepsilon\partial^0 b_i, h_i)$ which is an element of N. Therefore $(a,g) \in N$ so $\langle a,g\rangle$ is an identity edge and hence $(A \tilde{\times} G)/N$ is a tree groupoid.

□

<u>3.8 PROPOSITION</u>. Let B be a full subgroupoid of the G-groupoid A, with vertex set B_0 which is a union of orbits (i.e. G-stable) and contains every vertex of A which has non-trivial stabiliser. Then $B/(G)$ is a full subgroupoid of $A/(G)$.

Proof. First note that B is G-stable so $B/(G)$ is well-defined. Again we use the description of the orbit groupoid given in theorem 3.6. That is we identify $A/(G)$ with $(A \tilde{\times} G)/N$ and $B/(G)$ with $(B \tilde{\times} G)/M$ where M is generated (as a normal subgroupoid of $B \tilde{\times} G$) by elements of the form $(\varepsilon v, g)$ for $v \in B_0$.

Let $j : B \to A$ be the embedding. Then $j_* : B \tilde{\times} G \longrightarrow A \tilde{\times} G$, $(b,g) \longmapsto (jb,g)$ is an embedding. Let $\theta : B \tilde{\times} G \longrightarrow (A \tilde{\times} G)/N$ be the composite

$$\theta : B \tilde{\times} G \xrightarrow{j_*} A \tilde{\times} G \xrightarrow{\pi} (A \tilde{\times} G)/N .$$

Since $M \subseteq \ker\theta$ there is an induced morphism

$$\theta^* : \left(B \tilde{\times} G\right)/M \longrightarrow \left(A \tilde{\times} G\right)/N .$$

To prove θ^* is an embedding let $x \in N_1$ have vertices in B_0. It is sufficient to show $x \in M$. Write x in the form $x = x_1 + x_2 + \dots + x_n$ where each x_i is of the form $(\varepsilon v, g)$ or a conjugate. We may assume that the word $x_1+x_2+\dots+x_n$ is minimal in the sense that no intermediate vertex lies in B_0.

If $x_1 = (\varepsilon v, g)$, then $\partial^1 x_1 = \varepsilon v^g \in B_0$, so by minimality $x = x_1 \in M$.

If x_1 is a conjugate, $x_1 = (y,h) + (\varepsilon v, g) - (y,h)$, then $\partial^1 x_1 = \partial^0 x = \partial^0 x_1 \in B_0$, so again by minimality $x = x_1$. However in this case for x to be well-defined we require $v^g = v$. The hypotheses now imply $v \in B_0$ so $x_1 \in M$.

Hence $(B \tilde{\times} G)/M$ is a subgroupoid of $(A \tilde{\times} G)/N$.

To prove the fullness condition first note that $B \tilde{\times} G$ is a full subgroupoid of $A \tilde{\times} G$. For if $v,w \in B_0$ and $(a,g) \in (A \tilde{\times} G)_1(v,w)$ then $a \in A_1(v,w^{g^{-1}})$ which is an edge of B since $w^{g^{-1}} \in B$ and B is full in A.

Let the quotient morphism $\pi : A \tilde{\times} G \to (A \tilde{\times} G)/N$ be denoted $(a,g) \mapsto \langle a,g \rangle$, and let $\langle x \rangle = \langle a,g \rangle$ have vertices in $(B \tilde{\times} G)/M$. That is $\partial^0 x$ and $\partial^1 x$ are N-equivalent to vertices u,v say in B_0. Hence there exist $n_1 \in N_1(u,\partial^0 x)$, $n_2 \in N_1(\partial^1 x, v)$. Let $y = n_1 + x + n_2$. Then y has vertices in B_0 and $\langle x \rangle = \langle y \rangle$ in $(A \tilde{\times} G)/N$. Since $B \tilde{\times} G$ is full in $A \tilde{\times} G$ we have $y \in B \tilde{\times} G$. Therefore $\langle y \rangle = \langle x \rangle \in (B \tilde{\times} G)/M$.

This proves that $(B \tilde{\times} G)/M$ is full in $(A \tilde{\times} G)/N$.

□

It has been pointed out by Higgins that the hypotheses of 3.8 may be weakened to require B_0 to be G-stable and meet each component of the fixed point set of every non-trivial group element (see [15]).

3.9 PROPOSITION. Let N be a G-stable normal subgroupoid of the G-groupoid A, and let H be a normal subgroup of G which acts trivially on A. Then

(i) G/H acts naturally on A/N ,

(ii) $N \tilde{\times} H$ (= $N \times H$) is a normal subgroupoid of $A \tilde{\times} G$,

and (iii) $(A/N) \tilde{\times} (G/H) \cong (A \tilde{\times} G)/(N \tilde{\times} H)$.

Proof. Let $\tau : A \to A/N$, $a \mapsto \langle a \rangle$ and $G \to G/H$, $g \mapsto Hg$ be the natural morphisms.

(i) The induced action of G/H on A/N is $\langle a \rangle^{Hg} = \langle a^g \rangle$. To see that this is well-defined, let $b = n+a+m$ with $n,m \in N$ and $k = hg$, $h \in H$. Then $b^k = (n + a + m)^{hg} = n^g + a^g + m^g$ since H acts trivially. Now N is G-stable so $n^g, m^g \in N$. Therefore $\langle b^k \rangle = \langle a^g \rangle$ so the action is well-defined.

(ii) $N \tilde{\times} H$ is clearly a subgroupoid of $A \tilde{\times} G$. For normality, let $(a,g) \in A \tilde{\times} G$ and $(n,h) \in (N \tilde{\times} H)_1(\partial^0 a)$. Then

$$\begin{aligned} -(a,g) + (n,h) + (a,g) &= (-a^g + n^g + a^{h^{-1}g},\ g^{-1}hg) \\ &= (-a^g + n^g + a^g,\ g^{-1}hg) \ , \end{aligned}$$

since H acts trivially. Now $g^{-1}hg \in H$, and $(-a+n+a)^g \in N$ since N is normal and G-stable. Therefore $-(a,g) + (n,h) + (a,g) \in N \tilde{\times} H$, thus proving normality.

(iii) Let $\theta : A \tilde{\times} G \to (A/N) \tilde{\times} (G/H)$, $(a,g) \mapsto \big(\langle a \rangle, Hg\big)$ be the natural morphism. θ is vertex surjective and we show it is piecewise surjective.

Let $v,w \in A_0$, and consider $(\alpha, Hg) \in \big((A/N) \tilde{\times} (G/H)\big)(\langle v \rangle, \langle w \rangle)$. Then $\alpha \in (A/N)(\langle v \rangle, \langle w^{g^{-1}} \rangle)$. Since $\tau : A \to A/N$ is a quotient morphism, there exists $a \in A_1(v, w^{g^{-1}})$ such that $\alpha = \langle a \rangle$. Now $(a,g) \in \big(A \tilde{\times} G\big)_1(v,w)$ and $\theta(a,g) = (\alpha, Hg)$. Therefore θ is piecewise surjective and hence by proposition 2.3 a quotient morphism.

Clearly $N \tilde{\times} H \subseteq \ker\theta$ so there exists a unique morphism

$$\theta^* : (A \tilde{\times} G)/(N \tilde{\times} H) \longrightarrow (A/N) \tilde{\times} (G/H)$$

such that $\theta^* \circ \pi = \theta$ where $\pi : A \tilde{\times} G \to (A \tilde{\times} G)/(N \tilde{\times} H)$ is the quotient morphism. θ^* is clearly surjective.

To show θ^* is injective, suppose $\theta^*\langle a,g\rangle = \theta^*\langle b,k\rangle$. Then $\langle a\rangle = \langle b\rangle$ so $b = n + a + m$ for some $n,m \in N$, and $Hg = Hk$ so $g = hk$ for some $h \in H$. Then

$$(b,k) = (n+a+m,\ hg) = (n,h) + (a,g) + (m^{g^{-1}},1)\ .$$

Therefore $\langle a,g\rangle = \langle b,k\rangle$ so θ^* is injective, which completes the proof.

□

The following result shows that the action of G on A may be factored into an action of a normal subgroup H followed by an action of G/H on the groupoid $A/(H)$. Armstrong uses the analogous topological result in the proof of the main theorem in [3].

3.10 PROPOSITION. Let A be a G-groupoid and H a normal subgroup of G. Then G/H acts naturally on $A/(H)$ and

$$(A/(H))/[G/H] \cong A/(G)\ .$$

Proof. Let $\tau : A \longrightarrow A/(H)$ be the natural morphism, and define a (G/H)-action on $A/(H)$ by $(\tau a)^{Hg} = \tau(a^g)$. This is independent of the choice of $g \in Hg$, for if $g' = hg$ where $h \in H$ then $\tau(a^{g'}) = \tau(a^{gh}) = \tau(a^g)$ since τ is an H-morphism and $A/(H)$ is H-trivial. The action is clearly well-defined.

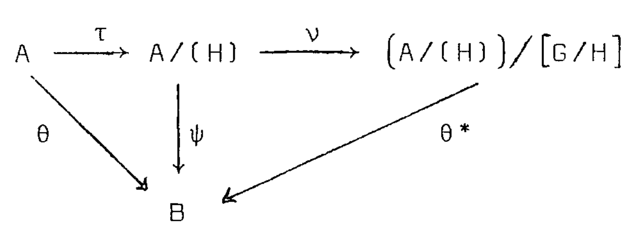
Let $\nu : A/(H) \longrightarrow (A/(H))/[G/H]$ be the natural morphism. To prove the proposition we show that the composite

$$\pi : A \xrightarrow{\tau} A/(H) \xrightarrow{\nu} (A/(H))/[G/H]$$

has the universal property characterising $A/(G)$ (definition 3.2). Define a G-action on $(A/(H))/[G/H]$ by $(\pi a)^g = \pi(a^g)$, so π is automatically a G-morphism. This action is trivial because

$$\pi(a^g) = \nu\left((\tau a)^{Hg}\right) = \nu(\tau a) = \pi(a).$$

This verifies the first part of 3.2. For the second, let $\theta : A \to B$ be a G-morphism where B is G-trivial. Then B is H-trivial and θ is an H-morphism so there is a unique morphism $\psi : A/(H) \longrightarrow B$ such that $\psi \circ \tau = \theta$. Now B is also (G/H)-trivial and ψ is a (G/H)-morphism. Therefore ψ induces a unique θ^* such that $\theta^* \circ \nu = \psi$. The diagram for this is



Finally it is easy to check that θ^* is unique for the outer triangle.

□

Our interest in the semi-direct product is largely in its use for describing the orbit groupoid. However, it appears to be an interesting construction in its own right. For example, the following result, which is due to Brown and Danesh-Naruie [8], shows that any group extension by G comes from a semi-direct product groupoid.

3.11 PROPOSITION. If A is a connected G-groupoid and $* \in A_0$ there is an exact sequence of groups

$$0 \longrightarrow A_1(*) \longrightarrow (A \tilde{\times} G)_1(*) \longrightarrow G \longrightarrow 0 \qquad (\dagger).$$

Conversely, any extension of groups $0 \longrightarrow B \longrightarrow \Sigma \twoheadrightarrow G \longrightarrow 0$ is isomorphic to $(\dagger)$ for some connected groupoid A.

□

Finally in this section we note a connection with the theory of groups acting on trees developed by H.Bass and J.-P.Serre [22]. Let Γ be an abstract connected graph on which the group G acts without reversing an edge. Then G acts on $\pi_1\Gamma$, the free groupoid generated by Γ, and we can form the semi-direct product $\pi_1\Gamma \tilde{\times} G$. In this situation Bass and Serre define a graph of groups $(\mathfrak{G}, \Gamma/G)$ over the quotient graph Γ/G, and its fundamental group $\pi_1(\mathfrak{G}, \Gamma/G, *)$. The Bass - Serre group is non-trivially isomorphic to the vertex group of $\pi_1\Gamma \tilde{\times} G$. Since $\pi_1(\mathfrak{G}, \Gamma/G, *)$ is an extension by

G, this isomorphism may be regarded as an example of proposition 3.11. The connection between semi-direct products and other groups, including the Bass - Serre group, is explored more fully in chapter five.

2. THE ω-GROUPOID AND CROSSED COMPLEX CASES

In this section some of the results of the previous section are generalised to the higher dimensional categories $\mathcal{G}$ and $\mathcal{C}$. Here A will be used to denote an ω-groupoid or crossed complex (as indicated) but always with a right G-action.

For the crossed complex A there is an action of A_1 on A_n $(n \geqslant 2)$ for which the standard notation is $a \mapsto a^x$. In this case therefore the action of the group will be denoted $a \mapsto a \circ g$.

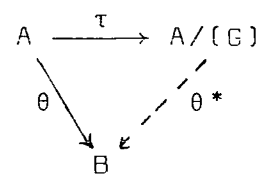
The definition of the orbit object A/(G) in the categories $\mathcal{G}$ and $\mathcal{C}$ is analogous to the groupoid case. We state it here for convenience.

3.12 DEFINITION. Let A be a G-ω-groupoid (respectively G-crossed complex). The orbit ω-groupoid (resp. orbit crossed complex) A/(G) and canonical projection $\tau : A \longrightarrow A/(G)$ are defined by the following universal property.

(i) A/(G) has trivial G-action and τ is a G-morphism.

(ii) Let B be any ω-groupoid (resp. crossed complex) with a trivial G-action and let $\theta : A \to B$ be any G-morphism. Then there is a unique morphism $\theta^* : A/(G) \longrightarrow B$ such that $\theta^* \circ \tau = \theta$.

Symbolically,



□

The above definitions clearly also make sense in the categories $\mathcal{G}_n$ and $\mathcal{C}_n$. Recall that $\tau : A \to A/(G)$ is the colimit of a certain diagram in the appropriate category and that the truncation functor $tr^n : \mathcal{G} \longrightarrow \mathcal{G}_n$ has a right adjoint $cosk^n : \mathcal{G}_n \to \mathcal{G}$ (and similarly for $\mathcal{C}$ and $\mathcal{C}_n$). The truncation functors therefore preserve colimits which proves the following result.

<u>3.13 LEMMA</u>. Let A be a G-ω-groupoid or G-crossed complex. Then

$$tr^n\big(A/(G)\big) \cong \big(tr^n A\big)/(G) .$$

□

The lemma shows that any description of $A/(G)$ in $\mathcal{G}$ or $\mathcal{C}$ automatically gives descriptions in the categories $\mathcal{G}_n$ or $\mathcal{C}_n$ $(n \geqslant 1)$ respectively. The work of the previous section may therefore be used as a basis for inductive descriptions of $A/(G)$.

Recall that the groupoid $A/(G)$ could be defined by using an equivalence relation on the set of orbits of A. By defining different equivalence relations in each dimension this process can be used to define the double groupoid $A/(G)$. The computations involved are complicated and largely uninformative, and there is little hope of repeating this procedure in higher dimensions. For these reasons we make no further mention of this approach but instead generalise the semi-direct product description of the orbit groupoid.

We concentrate mainly on the crossed complex case because quotient objects are easy to handle and the definition of the semi-direct product turns out to be surprisingly simple. As was pointed out in chapter two, a definition of normal sub-ω-groupoid would be complicated and so an analogue of theorem 3.6 would be hard to find. However we do give a definition of $A \tilde{\times} G$ for ω-groupoids because it involves an interesting use of the skeleton functor and because it explains the simplicity of the definition for crossed complexes.

To motivate the definition recall that the edges of the groupoid $A \tilde{\times} G$ could be regarded as triples (x,g,y) where $x^g = y$ in A_1. An obvious analogue in dimension two is to consider arrays

$$\xi = \begin{pmatrix} a_1 & g_1 & a_2 \\ g_2 & & g_3 \\ a_3 & g_4 & a_4 \end{pmatrix} \qquad \begin{array}{l} \longrightarrow 2 \\ \downarrow 1 \end{array}$$

where $a_i \in A_2$, $g_i \in G$ $(i=1,\ldots,4)$ and $a_1^{g_1} = a_2$, $a_1^{g_2} = a_3$ etc. The faces of ξ are given by $\partial_1^0 \xi = (\partial_1^0 a_1, g_1, \partial_1^0 a_2)$ etc. It turns out that, provided the additional condition $g_1 g_3 = g_2 g_4$ is imposed, the set of such arrays form a well-defined double groupoid. The connection is given by

$$\Gamma(x,g,y) = \begin{pmatrix} \Gamma x & g & \varepsilon_1 y \\ g & & 1 \\ \varepsilon_2 y & 1 & \Gamma y \end{pmatrix}$$

and compositions by

$$\begin{pmatrix} a_1 & g_1 & a_2 \\ g_2 & & g_3 \\ a_3 & g_4 & a_4 \end{pmatrix} \underset{2}{+} \begin{pmatrix} b_1 & h_1 & b_2 \\ h_2 & & h_3 \\ b_3 & h_4 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 \underset{2}{+} b_1^{g_1^{-1}} & g_1 h_1 & a_2^{h_2} \underset{2}{+} b_2 \\ g_2 & & h_3 \\ a_3 \underset{2}{+} b_3^{g_4^{-1}} & g_4 h_4 & a_4^{h_4} \underset{2}{+} b_4 \end{pmatrix}$$

and similarly for $\underset{1}{+}$. This double groupoid is $A \tilde{\times} G$.

Notice that the array ξ may be thought of as a pair $(a, \underset{\sim}{g})$, where $a = a_1 \in A_2$ and $\underset{\sim}{g} \in (skG)_2$. Here $sk = sk^1 : \mathcal{G}_1 \to \mathcal{G}$ is the skeleton functor defined in detail in [10], and G is regarded as a groupoid with a single vertex. This observation indicates how to define $A \tilde{\times} G$ in $\mathcal{G}$ as follows.

The set of n-cubes $(n \geqslant 0)$ of $A \tilde{\times} G$ is $A_n \times (skG)_n$ with degeneracies and connections given as products of those in A and skG. The faces of $(a, \underset{\sim}{\theta}) \in (A \tilde{\times} G)_n$ are given by $\partial_i^0 (a, \underset{\sim}{\theta}) = (\partial_i^0 a, \partial_i^0 \underset{\sim}{\theta})$ and $\partial_i^1 (a, \underset{\sim}{\theta}) = \left((\partial_i^1 a)^{v_i \underset{\sim}{\theta}}, \partial_i^1 \underset{\sim}{\theta} \right)$

where $v_i\underset{\sim}{\theta} = \partial_1^0 \ldots \hat{i} \ldots \partial_n^0 \underset{\sim}{\theta}$. (Cf. $u_i\underset{\sim}{\theta} = \partial_1^1 \ldots \hat{i} \ldots \partial_n^1 \underset{\sim}{\theta}$ defined in [10]). Finally, compositions are given by

$$(a, \underset{\sim}{\theta}) +_i (b, \underset{\sim}{\psi}) = \left(a +_i b^{(v_i\underset{\sim}{\theta})^{-1}}, \underset{\sim}{\theta} +_i \underset{\sim}{\psi}\right).$$

The reason that the edges $v_i x$ (beginning at $\partial_1^0 \ldots \partial_n^0 x$) arise in the definitions rather than the edges $u_i x$ (terminating at $\partial_1^1 \ldots \partial_n^1 x$) which are used extensively in [10] is presumably because we have chosen to use right rather than left actions.

Checking that the above definitions satisfy all the axioms for an ω-groupoid is a lengthy process which is omitted since we shall not use this construction.

Recall that in an ω-groupoid an element of dimension n is totally degenerate if it is of the form $\varepsilon_1^n v$ for some vertex v. The elements in dimension n of the associated crossed complex are those all of whose faces except the (0,1)-face are totally degenerate. Let

$$\underset{\sim}{g} = \begin{pmatrix} & g_1 & \\ g_2 & & g_3 \\ & g_4 & \end{pmatrix} \in (skG)_2$$

and suppose $\underset{\sim}{g} \in \gamma(skG)$, the associated crossed complex. Then $g_2 = g_3 = g_4 = 1$, but $g_1 g_3 = g_2 g_4$ by hypothesis so $g_1 = 1$ also, and $\underset{\sim}{g}$ itself is totally degenerate. An inductive argument shows that this is also true in all higher dimensions. Therefore in $\gamma(A \tilde{\times} G)$ the presence of group elements disappears in dimensions greater than one. This leads to the following simple definition of $A \tilde{\times} G$ in the category $\mathcal{C}$.

3.14 DEFINITION. Let A be a G-crossed complex. The semi-direct product crossed complex $A \tilde{\times} G$ is defined as follows.

$$(A \tilde{\times} G)_n = \begin{cases} A_1 \times G & \text{for } n=1 \\ A_n & \text{for } n \neq 1 . \end{cases}$$

The boundaries, identities and compositions are those of A for $n > 1$ and those of the groupoid $A \tilde{\times} G$ for $n=1$ (see definition 3.3), except $\partial : (A \tilde{\times} G)_2 \longrightarrow (A \tilde{\times} G)_1$ which is given by $\partial a = (\partial a, 1)$.

The action of $(A \tilde{\times} G)_1$ on $(A \tilde{\times} G)_n$ is given by

$$a^{(x,g)} = (a^x)\cdot g = (a\cdot g)^{(x\cdot g)} .$$

$\square$

3.15 LEMMA. $A \tilde{\times} G$ as given above is a well-defined crossed complex.

Proof. We must verify the axioms (1)–(6) of definition 2.6. (1) is given in lemma 3.4 and (2) is trivially true.

(3) Let $(x,g) \in (A \tilde{\times} G)_1(v,w)$ and $a \in (A \tilde{\times} G)_n(v)$ for $n \geqslant 2$. Then $a^{(x,g)} = (a^x)\cdot g \in (A \tilde{\times} G)_n(w)$. If $(y,h) \in (A \tilde{\times} G)_1(w,z)$ then

$$a^{((x,g)+(y,h))} = \left(a^{x+y\cdot g^{-1}}\right)\cdot(gh)$$
$$= \left((a\cdot g)^{(x\cdot g+y)}\right)\cdot h$$
$$= \left(a^{(x,g)}\right)^{(y,h)} .$$

Hence the action of $(A \tilde{\times} G)_1$ on $(A \tilde{\times} G)_n$ is well-defined.

(4) For $n \geqslant 3$ the fact that $\partial : (A \tilde{\times} G)_n \longrightarrow (A \tilde{\times} G)_{n-1}$ preserves the action follows from the fact that A is a G-crossed complex. For $n = 2$,

$$\partial\left(a^{(x,g)}\right) = \partial\left((a^x)\cdot g\right)$$
$$= \left((\partial a^x)\cdot g,\ 1\right)$$
$$= \left((-x+\partial a+x)\cdot g,\ 1\right)$$
$$= -(x,g) + (\partial a,1) + (x,g).$$

(5) follows automatically from A.

(6) Let $a \in (A \tilde{\times} G)_2$. Then $\partial a = (\partial a,1)$ acts trivially on $(A \tilde{\times} G)_n$ for $n \geqslant 3$, and if $b \in (A \tilde{\times} G)_2$ we have

$$b^{\partial a} = b^{(\partial a,1)} = (b^{\partial a})\cdot 1 = -a + b + a \text{ , as required.}$$

□

The main result of this section is the following generalisation of theorem 3.6.

<u>3.16 THEOREM</u>. Let A be a G-crossed complex. Then

$$A/(G) \cong (A \tilde{\times} G)/N$$

where N is the normal sub-crossed complex of $A \tilde{\times} G$ generated by elements of the form $(\varepsilon v, g) \in (A \tilde{\times} G)_1$ for $v \in A_0$, $g \in G$.

Remark. At first sight it may seem remarkable that N is generated by elements in dimension one. However from the description of normal closure for subcrossed complexes (proposition 2.15) it follows that N does contain non-trivial elements in higher dimensions. Namely elements of the form $a^{nx} - a^{x}$ where $a \in A_n$, $n \in N_1$ and $x \in (A \tilde{\times} G)_1$.

Proof. Once again the proof consists of showing that the crossed complex $(A \tilde{\times} G)/N$ has the required universal property (definition 3.11). It is similar to the dimension one case (theorem 3.6).

Let $i : A \longrightarrow A \tilde{\times} G$ be the embedding $i_n = id$ $(n \neq 1)$, $i_1 : a \longmapsto (a,1)$, and let $\pi : A \tilde{\times} G \longrightarrow (A \tilde{\times} G)/N$, $x \longmapsto \langle x \rangle$, be the natural projection. We let $\tau : A \longrightarrow (A \tilde{\times} G)/N$ denote the composite

$$A \xrightarrow{\ i\ } A \tilde{\times} G \xrightarrow{\ \pi\ } (A \tilde{\times} G)/N.$$

There is an induced action of G on $A \tilde{\times} G$ which is simply the action of G on A except in dimension one where it is given by $(a,g)\cdot h = (a \cdot h, h^{-1}gh)$. As in the groupoid case $i : A \longrightarrow A \tilde{\times} G$ is a G-morphism.

Recall that for the quotient crossed complex $\big((A \tilde{\times} G)/N\big)_r = (A \tilde{\times} G)/\sim$ where the equivalence relations $\sim$ are defined in chapter two. To show that there is an induced action on $(A \tilde{\times} G)/N$ we are required to prove that the action on $A \tilde{\times} G$ preserves these equivalence relations. The cases $r=0,1$ are considered in the proof of proposition 3.6. For $r \geqslant 2$, $a \sim b$ if $a = b^n + m$ for some $n \in N_1$, $m \in N_r$. In this case for $g \in G$ we have

$$a \cdot g = (b^n + m) \cdot g = (b \cdot g)^{n \cdot g} + m \cdot g.$$

It is therefore sufficient to show that N is G-stable (for then $n \cdot g \in N_1$, $m \cdot g \in N_r$ so $a \cdot g \sim b \cdot g$).

Theorem 3.6 shows that (N_1, N_0) is G-stable. For $r \geqslant 2$ N_r is generated by elements of the form $a^{nx} - a^x$ where $a \in A_r$, $n \in N_1$ and $x \in (A \tilde{\times} G)_1$. Now $(a^{nx} - a^x) \cdot g =$ $(a \cdot g)^{(n \cdot g)(x \cdot g)} - (a \cdot g)$ which is also a generator of N_r, so N_r is G-stable.

Hence there is an induced action on $(A \tilde{\times} G)/N$ given by $\langle a \rangle \cdot g = \langle a \cdot g \rangle$, and π is a G-morphism. Thus $\tau : A \to (A \tilde{\times} G)/N$ is also a G-morphism.

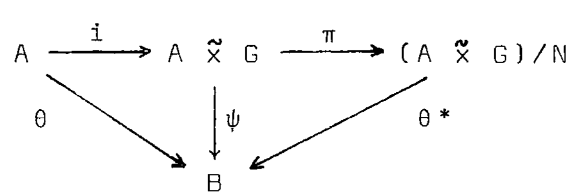
To show that $(A \tilde{\times} G)/N$ is G-trivial we must prove that $a \sim a \cdot g$ for $a \in (A \tilde{\times} G)_r$. Again the cases $r=0,1$ are given in the proof of theorem 3.6, and for $r \geqslant 2$ $a \cdot g = a^{(\varepsilon v, g)}$ where $a \in A_r(v)$ so $a \sim a \cdot g$.

This verifies the first part of the universal property.

For the second part, let B be a G-crossed complex with trivial action and $\theta : A \to B$ a G-morphism.

Define $\psi : A \tilde{\times} G \to B$ by $\psi_r = \theta_r$ for $r \neq 1$ and $\psi_1 : (a,g) \longmapsto \theta_1 a$. Since the action on B is trivial ψ is a well-defined morphism. Now $\psi_1(\varepsilon v, g) = \theta_1 \varepsilon v$ is an identity of B so $N \subseteq \ker\psi$. By the universal property of quotient crossed complexes (definition 2.9) there is a unique morphism $\theta^* : (A \tilde{\times} G)/N \to B$ such that $\theta^* \circ \pi = \theta$.

Hence the following diagram commutes and θ^* is unique for the right hand triangle.



We require θ^* to be unique for the outer triangle, so let $\eta : (A \tilde{\times} G)/N \longrightarrow B$ be such that $\eta \circ \tau = \theta$. I.e. $\eta_1\langle x,1\rangle = \theta_1(x)$ and $\eta_r\langle a\rangle = \theta_r(a)$ for $r \neq 1$.

Now $(x,g) = (x,1) + (\varepsilon\partial^1 x, g)$ so $\langle x,g\rangle = \langle x,1\rangle$. Therefore $\eta_1\langle x,g\rangle = \theta_1(x)$ so $\eta \circ \pi = \psi$. The uniqueness of θ^* for the right hand triangle in the above diagram now gives $\eta = \theta^*$ which completes the proof.

□

<u>3.17 COROLLARY.</u> For $r \geq 2$ and $v \in A_0$

$$\left(A/(G)\right)_r(\langle v \rangle) \cong A_r(v)/P$$

where P is the subgroup of $A_r(v)$ generated by elements of the form $a^m - a$ for $a \in A_r(v)$ and $m = -x + n + x$, $x \in (A \tilde{\times} G)_1(w,v)$, $n \in N_1(w)$.

<u>Proof.</u> By the previous theorem and corollary 2.14 of chapter 2 we have the isomorphism

$$\left(A/(G)\right)_r(\langle v \rangle) \cong \left(A \tilde{\times} G\right)_r(v)/\overline{N}_r(v)$$

where $\overline{N}$ is the normal closure of N. Now $(A \tilde{\times} G)_r = A_r$ for $r \geq 2$, and by proposition 2.15, $\overline{N}_r(v)$ is generated as a group by elements $c^{n+x} - c^x$ where $c \in A_r(w)$, $x \in (A \tilde{\times} G)_1(w,v)$ and $n \in N_1(v)$. Let $a = c^x \in A_r(v)$. Then $c^{n+x} - c^x = a^m - a$, as above.

□

<u>3.18 COROLLARY.</u> If G acts freely on A (i.e. no non-trivial element of G fixes a vertex of A) then

$$\left(A/(G)\right)_r(\langle v \rangle) \cong A_r(v) \quad , \quad \text{for } v \in A_0 \text{ , } r \geq 2.$$

<u>Proof.</u> This follows from corollary 3.15 since if G acts freely on A then (N_1, N_0) is a tree groupoid so m is the identity element of $N_1(v)$ and hence P is the trivial subgroup.

□

Finally we consider the effect of a group action on the fundamental groupoid of a crossed complex. The definition of this groupoid, given by Brown and Higgins in [12], is once again motivated by topology. Let C be any crossed complex. Then ∂C_2 is a normal, totally disconnected subgroupoid of C_1, and the <u>fundamental groupoid</u> of C is defined to be the quotient $C_1/\partial C_2$.

The motivation for this definition is as follows. Let $\underset{\sim}{X} : X^0 \subseteq X^1 \subseteq \ldots \subseteq X^n \subseteq \ldots$ be a filtered space such that $\pi_0 X^0 = X^0$, and let $C = \pi \underset{\sim}{X}$. Suppose further that $\underset{\sim}{X}$ is homotopy full (see definition 2.16). Then a standard use of exact sequences shows that $\pi_1(\underset{\sim}{X}, *) \cong \pi_1(X^1, *)/\partial\pi_2(X^2, X^1, *)$ for any $* \in X^0$. Therefore for homotopy full filtrations $\pi_1 C \cong \pi_1 \underset{\sim}{X}$.

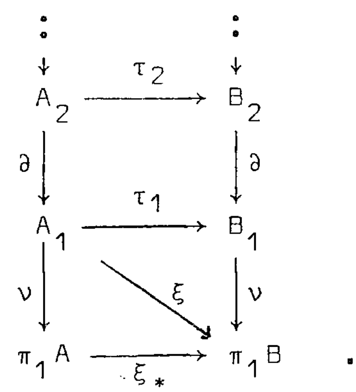
<u>3.19 THEOREM</u>. Let A be a G-crossed complex. Then

$$\pi_1\big(A/(G)\big) \cong (\pi_1 A)/(G).$$

<u>Proof</u>. We show that the canonical morphism $A \longrightarrow A/(G)$ in $\mathcal{C}$ induces a groupoid morphism $\pi_1 A \longrightarrow \pi_1(A/(G))$ which has the universal property characterising orbit groupoids.

Let $B = A/(G)$ and let both the quotient morphisms $A_1 \to \pi_1 A$ and $B_1 \to \pi_1 B$ in $\mathcal{G}pd$ be denoted by ν. If $\tau : A \to A \tilde{\times} G \to B$ is the natural morphism in $\mathcal{C}$ then the composite $\xi : A_1 \xrightarrow{\tau_1} B_1 \xrightarrow{\nu} \pi_1 B$ is a well-defined groupoid morphism.

Let $a \in A_2$. Then $\xi \partial a = \nu\tau_1\partial(a) = \nu\partial(\tau_2 a)$, which is an identity of $\pi_1 B$. Therefore $\partial A_2 \subseteq \ker\xi$ so ξ induces a morphism $\xi_* : \pi_1 A \to \pi_1 B$. The following diagram illustrates the situation.



We verify the universal property of definition 3.2 for $\xi_* : \pi_1 A \longrightarrow \pi_1 B$.

(i) Clearly $\pi_1 B$ is G-trivial (since B is), and it is readily checked that ξ_* is a G-morphism.

(ii) Let C be any G-trivial groupoid and $\theta : \pi_1 A \longrightarrow C$ any G-morphism. Then $\psi = \theta\nu : A_1 \longrightarrow C$ is a G-morphism, and hence induces a unique morphism $\psi_* : B_1 \longrightarrow C$ such that $\psi_* \circ \tau_1 = \psi$.

Let $a \in A_2$. Then $\tau_2(a) \in B_2$, and every element of B_2 is of this form. Therefore

$$\psi_* \partial(\tau_2 a) = \psi_* \tau_1 \partial a = \psi \partial a = \theta\nu\partial a \quad ,$$

which is an identity of C. Hence $\partial B_2 \subseteq \ker\psi_*$ so ψ_*

induces a unique morphism $\theta^* : \pi_1 B \longrightarrow A$ such that $\theta^* \circ \nu = \psi_*$. It is readily checked that $\theta^* \xi_* = \theta$, and it only remains to show that θ^* is the unique such morphism. So let $\eta : \pi_1 B \to C$ be such that $\eta \xi_* = \theta$.

I.e. $\eta \xi_* \nu = \theta \nu = \psi$

$\therefore \quad \eta \xi = \psi$

$\therefore \quad \eta \nu \tau_1 = \psi$

$\therefore \quad \eta \nu = \psi_*$ (by the universal property of ψ_*)

$\therefore \quad \eta = \theta^*$ (by the universal property of ν).

This completes the proof.

□

CHAPTER FOUR. THE HOMOTOPY GROUPS OF ORBIT SPACES

Let X be a topological space with right G-action, and let X/G be the orbit space. This chapter concerns the question : what can be said about the homotopy groups of X/G? We show that under certain circumstances we can give good algebraic models for the topology of group actions.

Armstrong [2,3] and Rhodes [19] have considered the case of the fundamental group. We begin by describing their results which we present in a slightly modified form. In particular, since both authors considered left actions some of their definitions need to be adjusted to fit our setting.

In section two these results are generalised to fundamental groupoids. Some work is required for this modest generalisation, but mostly this concerns the algebra of G-groupoids considered in the previous chapter. The pay-off is that the results are more natural and this indicates how to proceed in higher dimensions.

The higher dimensional result given in section three deals with the case where X is a CW-complex with a cellular group action (satisfying anadditional condition). The orbit space has a filtration induced from the standard filtration on X and we describe the corresponding homotopy crossed complex. As a consequence information on certain relative homotopy groups of X/G is obtained.

1. THE WORK OF ARMSTRONG AND RHODES

In [2] Armstrong considers a simplicial complex K on which G acts simplicially. Let $X = |K|$ be its polyhedron. Armstrong considers the following conditions.

Condition 1 : If x is a 1-simplex of K with vertices v and w then no group element maps v to w.

Condition 2 : If two n-simplexes of K have vertices $v_1,\dots,v_n,a$ and $v_1,\dots,v_n,b$ respectively, where a and b are in the same G-orbit, then there exists a group element g such that $v_i^g = v_i$ for $i=1,\dots,n$ and $a^g = b$.

Define a simplicial complex K/G to have vertices the orbits of vertices of K, and such that the orbits $\overline{v}_0,\dots,\overline{v}_n$ span a simplex of K/G if and only if $v_0,\dots,v_n$ span a simplex of K. (That is there exist representatives v_i of the orbit $\overline{v}_i$ with this property).

4.1 THEOREM. (Armstrong). If conditions 1 and 2 above hold then $X/G = |K|/G$ is homeomorphic to $|K/G|$. Furthermore, the action of G on the second derived complex $K^{(2)}$ always satisfies conditions 1 and 2.

Proof. [2; Theorems 1 and 2].

□

We also consider the following condition.

Condition 3 : Let H be a subgroup of G, $(v_0,\dots,v_n)$ and $(v_0^{h_0},\dots,v_n^{h_n})$ simplexes of K where $h_i \in H$ for $i=0,\dots,n$. Then there exists an element h of H such that $v_i^h = v_i^{h_i}$ for $i=0,\dots,n$.

4.2 LEMMA. Condition 3 for $H = G$ is equivalent to conditions 1 and 2.

Proof. Assume condition 3 for $H = G$. Suppose the vertices v, v^g span a simplex of K. Then (v,v) and (v,v^g) are both simplexes. Therefore there exists $h \in G$ such that $v^h = v$ and $v^h = v^g$. Hence $v = v^g$ so (v,v^g) is not a 1-simplex of K, so condition 1 holds. Condition 2 is simply a special case of condition 3.

Conversely, assume conditions 1 and 2, and let $(v_0,\dots,v_n)$ and $(v_0^{g_0},\dots,v_n^{g_n})$ be simplexes of K where $g_i \in G$ for $i=0,\dots,n$. Condition 1 implies that these simplexes have the same dimension (i.e. they have the same number of distinct vertices). We prove condition 3 by induction on n, the case $n=0$ being trivial.

Now $(v_0,\dots,v_{n-1})$ and $(v_0^{g_0},\dots,v_{n-1}^{g_{n-1}})$ are simplexes so by the inductive hypothesis there exists $g \in G$ such that $v_i^g = v_i^{g_i}$, for $i = 0,\dots,n-1$. We now apply condition 2 to the simplexes $(v_0,\dots,v_n)^g = (v_0^g,\dots,v_n^g)$ and

$(v_0^{g_0},\ldots,v_n^{g_n})$. Since $(v_0^{g},\ldots,v_n^{g}) = (v_0^{g_0},\ldots,v_{n-1}^{g_{n-1}},v_n^{g})$ it follows by condition 2 that there is an element $h \in G$ such that $v_i^{g_i h} = v_i^{g_i}$ for $i = 0,\ldots,n-1$ and $v_n^{gh} = v_n^{g_n}$. Therefore $gh \in G$ satisfies $v_i^{gh} = v_i^{g_i}$ for $i = 0,\ldots,n$, which completes the inductive step.

□

Bredon [4; p116] calls a G-simplicial complex regular if it satisfies condition 3 for every subgroup H of G. He also proves the following analogue of theorem 4.1.

4.3 THEOREM (Bredon). If K is a regular G-simplicial complex then $|K|/G$ is homeomorphic to $|K/G|$. Furthermore, the second derived complex is always regular.

□

To prove topological results about X/G we may assume either regularity or conditions 1 and 2, and work with the simplicial complex K/G. In particular, for the fundamental group $\pi_1(X/G,*)$ we may work instead with the combinatorially defined edge-path group $\pi_1(K/G,*)$. Using this approach Armstrong proves the following in [2].

4.4 THEOREM (Armstrong). If X is connected and simply connected then

$$\pi_1(X/G,*) \cong G/H$$

where H is the normal subgroup generated by elements with a non-empty fixed point set.

□

Later, in [3], Armstrong generalises this theorem to the case where G acts discontinuously on a path-connected, locally compact metric space X, and more recently (private communication) to discontinuous actions on a path-connected Hausdorff space. Armstrong also has a recently published version [M.A.Armstrong, Calculating the fundamental group of an orbit space, Proc. Amer. Math. Soc. 84 (1982), 267-271] which includes continuous as well as discontinuous group actions.

Rhodes' approach is somewhat different. Given a topological space X (not necessarily a polyhedron) with G-action he defines a group $\sigma = \sigma(X,*,G)$ for $* \in X$. It is an extension of $\pi_1(X,*)$ by G. To define σ, consider pairs (p,g) where $g \in G$ and p is a path in X from $*$ to $*^{g^{-1}}$. Define an equivalence relation on the set of such pairs by $(p,g) \sim (p',g')$ if $g = g'$ and p is homotopic to p' relative to end points. The elements of σ are the equivalence classes, which are denoted $[p,g]$, with composition

$$[p,g] + [q,h] = [p + q^{g^{-1}}, gh] .$$

<u>4.5 PROPOSITION</u> (Rhodes). There is an exact sequence of groups

$$0 \longrightarrow \pi_1(X,*) \xrightarrow{i} \sigma(X,*,G) \xrightarrow{\phi} G \longrightarrow 0$$

where $i : [p] \mapsto [p,1]$, and $\phi : [p,g] \longmapsto g$.

□

4.6 THEOREM (Rhodes). If G acts simplicially on the simplicial complex K and $X = |K|$ is its polyhedron, then

$$\pi_1(X/G,*) \cong \sigma/\sigma'$$

where σ' is the subgroup generated by elements of the form $[p - p^g, g^{-1}]$, (and of course where g fixes the end point of the path p).

□

This result includes theorem 4.4 for if $\pi_1(X,*) = 0$ then $\sigma \cong G$ and $\sigma' \cong H$. The group σ acts on the universal cover $\tilde{X}$ of X so that $\tilde{X}/\sigma$ and X/G are homeomorphic. Theorem 4.6 may be proved using Armstrong's proof of theorem 4.4 for this action on $\tilde{X}$.

2. THE FUNDAMENTAL GROUPOID OF AN ORBIT SPACE

Rhodes' group $\sigma(X,*,G)$ may appear at first sight to be an unnatural construction. However it is simply the vertex group at $*$ of the semi-direct product groupoid $\pi_1(X) \tilde{\times} G$, where $\pi_1(X)$ is the full fundamental groupoid of X. So Rhodes is really using groupoids. More precisely we have the following.

4.7 PROPOSITION. The exact sequence

$$0 \longrightarrow \pi_1(X,*) \xrightarrow{i} \sigma(X,*,G) \xrightarrow{\phi} G \longrightarrow 0$$

of proposition 4.5 is the vertex sequence at $* \in X$ of the sequence

$$E_X \longrightarrow \pi_1(X) \xrightarrow{i} \pi_1(X) \tilde{\times} G \xrightarrow{p} G \longrightarrow E_*$$

given in lemma 3.5.

<u>Proof</u>. The elements of $\pi_1(X) \tilde{\times} G$ are $([p],g)$ where $[p]$ is the homotopy class, relative to end points, of a path p in X. Now $\partial^0([p],g) = * = \partial^1([p],g)$ implies that p runs from $*$ to $*g^{-1}$, so $(\pi_1(X) \tilde{\times} G)(*)$ is Rhodes' group $\sigma(X,*,G)$. It is clear that the groupoid morphisms induce those of proposition4.5.

□

It has been noted by Higgins that the Abelian condition in theorem 2 of [19] is unnecessary. The groups $\sigma(X,x_0,G)$ and $\sigma(X,x_1,G)$ are, in general, isomorphic if and only if x_0 and x_1 lie in the same component of the groupoid $\pi_1(X) \tilde{\times} G$. This is clearly true in the case where $x_1 = x_0{}^g$ since $(\varepsilon x_0,g)$ has initial vertex x_0 and final vertex x_1, where εx_0 is the class of the identity path at x_0. The mistake in Rhodes' example [19; p638] is that $G_1 = \{e, g_1g_2\}$ and not $\{e\}$ as stated.

It is worth noting here that if Γ is an abstract G-graph then there is an obvious way to define $\sigma = \sigma(\Gamma,v,G)$ combinatorially for $v \in \Gamma_0$ using edge-paths. Suppose the action does not reverse edges of Γ. Then Rhodes' group is isomorphic to the Bass - Serre group $\pi_1(\mathcal{G}, \Gamma/G, *)$ defined in [22]. This follows from proposition 4.7 and the remark in chapter three that $\pi_1(\mathfrak{G}, \Gamma/G, *)$ is isomorphic to the vertex group of $\pi_1\Gamma \tilde{\times} G$. (The isomorphism $\sigma \cong \pi_1(\mathfrak{G}, \Gamma/G, v)$ may also be established using the Bass - Serre structure theorem. One constructs a tree $\hat{\Gamma}$, the universal cover of Γ, and an action of σ on $\hat{\Gamma}$ so that $\hat{\Gamma}/\sigma \cong \Gamma/G$). The connection between semi-direct products and the Bass—Serre theory is considered in chapter five.

We now return to simplicial actions on a simplicial complex K. The edge-path fundamental group has a groupoid analogue $\pi_1(K,K^0)$ defined as follows. It has vertex set the 0-skeleton K^0 and generators the 1-simplexes. For every 2-simplex (v_0,v_1,v_2) there is a relation $(v_0,v_1) + (v_1,v_2) = (v_0,v_2)$. The following is a groupoid version of a standard result on the fundamental group mentioned earlier, from which it can be deduced.

4.8 PROPOSITION. Let $X = |K|$ be the polyhedron of a simplicial complex K, and $X^0 = |K^0|$. Then $\pi_1(X,X^0) \cong \pi_1(K,K^0)$.

□

4.9 THEOREM. Let G act simplicially on the simplicial complex K, and let $X = |K|$. Then

$$\pi_1(X,X^0)/(G) \cong \pi_1(X/G,X^0/G) \ .$$

Proof. By theorem 4.3 (or theorem 4.1) we may assume K is regular (or satisfies conditions 1 and 2). It is therefore sufficient to show

$$\pi_1(K,K^0)/(G) \cong \pi_1(K/G,K^0/G).$$

Since K is regular (or satisfies conditions 1 and 2) the n-simplexes of K/G are just orbits of n-simplexes of K. The natural projection $\tau : K \to K/G$ induces a morphism

$$\tau_* : \pi_1(K,K^0) \longrightarrow \pi_1(K/G,K^0/G) ,$$

and we verify the universal property of definition 3.2 for τ_*.

Clearly $\pi_1(K/G,K^0/G)$ is G-trivial and τ_* is a G-morphism. Let A be any G-trivial groupoid and $\theta : \pi_1(K,K^0) \longrightarrow A$ any G-morphism. Let $x \in K^1/G$ and choose some $\hat{x} \in K^1$ covering x (i.e. $\tau\hat{x} = x$). Define $\theta^*(x) = \theta(\hat{x})$. The definition is independent of the chioce of $\hat{x}$ for if $\tau(y) = x$ then $y^g = \hat{x}$ for some $g \in G$ so

$$\theta(\hat{x}) = \theta(y^g) = (\theta y)^g = \theta(y).$$

Also if σ is a 2-simplex of K/G which gives rise to a relation $x + y = z$ in $\pi_1(K/G,K^0/G)$ then a lift $\hat{\sigma}$ of σ gives rise to the relation $\hat{x} + \hat{y} = \hat{z}$ in $\pi_1(K,K^0)$. Therefore

$$\theta^*(x+y) = \theta(\hat{x}) + \theta(\hat{y}) = \theta(\hat{z}) = \theta^*(z).$$

Hence θ^* preserves the relations of $\pi_1(K/G,K^0/G)$ and defines a morphism

$$\theta^* : \pi_1(K/G,K^0/G) \longrightarrow A .$$

Clearly $\theta^* \circ \tau_* = \theta$ and θ^* is the unique such morphism.

□

4.10 COROLLARY. In theorem 4.9 we may replace X^0 with Y^0 which is any G-stable set of vertices containing all the vertices with non-trivial stabilisers.

Proof. This follows from the corresponding algebraic result (proposition 3.8).

□

Theorem 4.9 together with the algebraic description of the orbitgroupoid (theorem 3.6) gives an isomorphism

$$\pi_1(X/G,\ X^0/G) \cong \left(\pi_1(X,X^0) \tilde{\times} G\right)/N ,$$

where N is the normal closure of the subgroupoid, M say, generated by elements of the form $(\varepsilon v, g)$ for $v \in X^0$, $g \in G$. (Here εv denotes the class of the identity path at v). In fact M consists entirely of such elements.

Choose a vertex $*$. We have seen that Rhodes' group $\sigma = \sigma(X,*,G)$ is precisely the vertex group at $*$ of $\pi_1(X,X^0) \tilde{\times} G$. We now show that the vertex group of N at $*$ is Rhodes' group σ', and hence we recapture Rhodes' theorem 4.6. To establish this fact recall lemma 2.4

which gives generators for the vertex group of a normal closure of a subgroupoid. This shows that $N_1(*)$ is generated by elements of the form

$$n = (\alpha, g) + (\varepsilon v, h) - (\alpha, g) \quad ,$$

where h stabilises v. Now $n = (\alpha - \alpha^k,\ k^{-1})$ where $k = gh^{-1}g^{-1}$ stabilises $\partial^1\alpha$. Hence $N_1(*)$ is precisely Rhodes' group σ' as claimed.

□

We believe that the result is more natural in the groupoid setting. The content of theorem 4.9 is really that, under the given hypotheses, the algebra of the fundamental groupoid accurately models the topology. It also indicates how to proceed in higher dimensions — the fundamental groupoid is replaced by the homotopy crossed complex of a suitable filtered space but the form of the result is the same.

We now consider more general actions, in particular the discontinuous actions of Armstrong [3]. We begin with a few remarks concerning arbitrary G-spaces. For any G-space X there is a diagram

$$\begin{array}{ccc}
\pi_1(X) & \xrightarrow{\ i\ } & \pi_1(X) \tilde{\times} G \\
\Big\downarrow p_* & & \Big\downarrow \phi \\
\pi_1(X/G) & & \pi_1(X)/(G) \cong \left(\pi_1(X) \tilde{\times} G\right)/N
\end{array}$$

where $\pi_1(X)$ is the full fundamental groupoid, p_* is induced by $p : X \to X/G$, N is given by theorem 3.6 and $\phi : (\alpha, g) \longmapsto \langle\alpha, g\rangle$ is the quotient morphism.

Define $q : \pi_1(X) \tilde{\times} G \longrightarrow \pi_1(X/G)$ by $q(\alpha, g) = p_*(\alpha)$. Since the action on $\pi_1(X/G)$ is trivial q is a well-defined morphism and clearly $N \subseteq \ker q$. Therefore, by the universal property of quotient morphisms (definition 2.2), there is a morphism

$$q_* : \left(\pi_1(X) \tilde{\times} G\right)/N \longrightarrow \pi_1(X/G)$$

induced by q. The purpose of this section is to discover when q_* is an isomorphism. (Theorem 4.9 says q_* is an isomorphism for simplicial actions). It will always be assumed that the action is discrete in the sense that the orbit of a point in X is a discrete subspace of X.

<u>4.11 DEFINITION</u>. A map $p : X \to Y$ has <u>path lifting up to homotopy</u> if, given a path $\gamma : I \to Y$ and a point $x \in p^{-1}\gamma(0)$, there is a path $\tilde{\gamma} : I \to X$ such that $\tilde{\gamma}(0) = x$ and $p \circ \tilde{\gamma}$ is homotopic, relative to end points, to γ.

□

<u>4.12 LEMMA</u>. If X is a G-space and $p : X \to X/G$ has path lifting up to homotopy, then

$$q : \pi_1(X) \tilde{\times} G \longrightarrow \pi_1(X/G)$$

defined above is a quotient morphism.

Proof. We use the characterisation of quotient morphisms as the vertex surjective, piecewise surjective morphisms (proposition 2.3).

Clearly q is vertex surjective — the vertex map of q is $p : X \to X/G$. Let $x_i \in X$ and $p(x_i) = \bar{x}_i$ for $i=1,2$, and let $\beta \in \pi_1(X/G)(\bar{x}_1, \bar{x}_2)$. By hypothesis β has a representative path γ which has a lift $\tilde{\gamma}$ where $\tilde{\gamma}(0) = x_1$. Let α be the class of $\tilde{\gamma}$. Then $p_* \partial^1 \alpha = p\tilde{\gamma}(1) = \bar{x}_2$, so there is a group element g such that $\partial^1 \alpha^g = x_2$. Hence $(\alpha, g) \in \left(\pi_1(X) \tilde{\times} G\right)_1(x_1, x_2)$ is such that $q(\alpha, g) = \beta$. Therefore q is piecewise surjective which completes the proof.

□

4.13 DEFINITION. Let X be a G-space. The action is discontinuous if

(i) the stabiliser of each point of X is finite,

and (ii) each $x \in X$ has a neighbourhood U such that if $x^g \neq x$ then $U \cap U^g \neq \emptyset$.

□

In [3] Armstrong generalises his theorem 4.4 to discontinuous actions on path-connected, locally compact metric spaces. Note that if the action is also fixed point free then $q_* : \left(\pi_1(X) \tilde{\times} G\right)/N \to \pi_1(X/G)$ is an isomorphism. To see this note that $p : X \to X/G$ is a covering projection in this case ([23; theorem 7, page 87]), so we can lift homotopies from X/G to X. A standard type of covering space argument shows that $\ker q$ is contained in (and hence equal to) N and so q_* is an isomorphism. The isomorphism also holds under Armstrong's hypothesis as we now show.

4.14 THEOREM. Let X be a path-connected, locally compact metric space and let G act discontinuously. Suppose either

(i) X is simply connected, or

(ii) X is locally connected and semi-locally simply connected.

Then

$$\pi_1(X)/(G) \cong \pi_1(X/G).$$

Proof. (i) We use Armstrong's trick of factoring the action into an action of H followed by a free action of G/H on X/H, where H is the normal subgroup of G generated by the elements with fixed points.

Armstrong shows that X/H is simply connected, and proposition 3.7(ii) shows that $\pi_1(X)/(H)$ is simply connected. Hence $\pi_1(X)/(H) \cong \pi_1(X/H)$. The result now follows from proposition 3.10 and the fact that q_* is an isomorphism for free actions.

(ii) The hypotheses on X ensure that it has a path-connected, simply connected covering space $\tilde{X}$. Also $\tilde{X}$ is locally compact and metric (M.A.Armstrong — private communication).

We can take $\tilde{X} = \text{costar}_{x_0}\pi_1(X)$ for some point x_0 of X, with a suitable topology. (See, for example, W.S.Massey [Algebraic Topology: An Introduction, Graduate Texts in Mathematics 56, Springer 1977]. In fact Massey uses $\text{star}_{x_0}\pi_1(X)$, but for our purposes the costar is more convenient).

Let $\tilde{G} = (\pi_1(X) \tilde{\times} G)(x_0)$. Then $\tilde{G}$ acts on $\tilde{X}$ by $\alpha^{(\beta,g)} = \alpha^g + \beta^g$, and $\tilde{X}/\tilde{G}$ is homeomorphic to X/G (see Armstrong [3] and Rhodes [19]). In fact this action is discontinuous and satisfies the hypotheses of part (i). Hence

$$\pi_1(X/G) \cong \pi_1(\tilde{X})/(\tilde{G}).$$

Finally we show $\pi_1(\tilde{X})/(\tilde{G}) \cong \pi_1(X)/(G)$. First note that the groupoids have the same vertex set. $\pi_1(\tilde{X})/(\tilde{G})$ has vertex group isomorphic to $\tilde{G}/\tilde{H}$ where $\tilde{H}$ is the normal subgroup generated by elements with fixed points. But this is precisely the group $N(x_0)$ where N is the normal subgroupoid of $\pi_1(X) \tilde{\times} G$ considered above. Hence the vertex groups of $\pi_1(\tilde{X})/(\tilde{G})$ and $\pi_1(X)/(G)$ are also isomorphic. Since both groupoids are connected, this completes the proof.

□

We have now proved that $q_* : \pi_1(X)/(G) \to \pi_1(X/G)$ is an isomorphism in some special cases. In each case it was assumed that the space X was reasonably nice, together with some form of discontinuity assumption on the action. That some form of discreteness condition is necessary is readily seen — consider Armstrong's example of the reals acting freely on the real line by addition. However the full weight of the discontinuity assumption (definition 4.13) is not strictly necessary. The following example indicates that the finite stabilisers assumption is not necessary.

4.15 EXAMPLE. We first consider the standard action of the integers on the real line by addition modified by adding two fixed points — one at $+\infty$, and one at $-\infty$.

More precisely, let the generator of $G = \mathbb{Z}$ act on $X = [0,1]$ by

$$t \mapsto \begin{cases} 2t & \text{if } t \leqslant 1/3 \\ \frac{1}{2}(1+t) & \text{if } t > 1/3. \end{cases}$$

Let $X^0 = \{0,1\}$. Then $\pi_1(X,X^0)$ is the unit interval groupoid $\mathcal{J}$ and the induced action is trivial since the above map is homotopic, relative to X^0, to the identity. Hence the orbit groupoid is also the unit interval groupoid.

The action restricted to the open interval $(0,1)$ is equivalent to the standard action of $\mathbb{Z}$ on $\mathbb{R}$. A fundamental domain for this restricted action is $[1/3,2/3]$ and the orbit space is a circle.

Hence X/G consists of a circle C (the image of $[1/3.2/3]$) together with two points which, by a slight abuse of notation, we denote 0 and 1. The topology on the orbit space consists of the usual open sets on the circle C together with $C \cup \{0,1\}$, $V_0 = C \cup \{0\}$ and $V_1 = C \cup \{1\}$.

We use the Seifert - van Kampen theorem to show that X/G is simply connected. X/G is the union of the open sets V_0 and V_1 whose intersection is the circle C which is path-connected.

For $i=0,1$, V_i is contractible, and hence is path-connected and simply connected. Define a map

$$H : V_i \times I \longrightarrow V_i , \quad \text{by}$$

$$(x,t) \longmapsto \begin{cases} i & \text{if } t=1 \\ x & \text{if } t<1. \end{cases}$$

H is clearly continuous and is therefore the required homotopy from the identity map on V_i to the constant map at the point i.

The Seifert - van Kampen theorem now implies that X/G is simply connected.

Hence $\pi_1(X/G, X^0/G)$ is the unit interval groupoid and the induced morphism $q_*: \pi_1(X,X^0)/(G) \longrightarrow \pi_1(X/G,X^0/G)$ is an isomorphism.

In this example the action is not discontinuous since the points 0 and 1 of X have infinite stabilisers. The second condition of definition 4.13 is satisfied however.

Now let $Y = X/G = C \cup \{0,1\}$ and let $\mathbb{Z}_2$ act on Y by interchanging 0 and 1, and rotating the circle C by π. The groupoid $\pi_1(Y,Y^0)$ is the unit interval groupoid, and the induced $\mathbb{Z}_2$ action sends the generator to its inverse. Hence $\pi_1(Y,Y^0)/[\mathbb{Z}_2]$ is isomorphic to $\mathbb{Z}_2$.

The new orbit space $Y/\mathbb{Z}_2$ consists of a circle (the image of C) with a single additional point (the image of Y^0). The only open set containing this additional point is the whole space. A similar argument to the one above shows that $Y/\mathbb{Z}_2$ is path-connected and simply connected. Therefore $\pi_1(Y/\mathbb{Z}_2, Y^0/\mathbb{Z}_2)$ is the trivial group and we obtain an example where q_* fails to be an isomorphism.

Note that in this case the condition (ii) of definition 4.13 is not satisfied. Indeed Y is not even Hausdorff.

We noted in section one that Armstrong has recently generalised his theorem 4.4 to discontinuous actions on a connected, simply connected Hausdorff space X. The groupoid result (that q_* is an isomorphism) can be deduced from this as in theorem 4.14. To prove that $q_* : \pi_1(X)/(G) \longrightarrow \pi_1(X/G)$ is an isomorphism in the non-simply connected case some local conditions need to be assumed.

It would be interesting to know the most general conditions on the space and the action under which q_* is an isomorphism.

3. THE HIGHER DIMENSIONAL CASE

To prove analogous results to those of the previous section in higher dimensions we use the algebraic machinery of crossed complexes. We are led therefore to consider filtered spaces $\underset{\sim}{X} : X^0 \subseteq X^1 \subseteq \dots \subseteq X^n \subseteq \dots \subseteq X$ with a filtration preserving action and such that $\pi_0 X^0 = X^0$ (or more generally the filtration is J_0). We restrict attention to CW-complexes.

4.16 DEFINITION. A G-CW-complex is a CW-complex X on which G acts by cellular homeomorphisms satisfying the following condition.

(*) If a group element maps a closed cell to itself it does so by the identity homeomorphism.

□

In the case where G is finite this definition is due to Bredon [Equivariant Cohomology Theories, Springer Lecture Notes in Math. 34 (1967)]. By a CW-complex we mean the space together with the characteristic maps of the cells. Bredon notes that the characteristic maps "may be chosen" equivariantly, and he appears to assume this. This point is examined in more detail below.

The concept has been extended to topological groups G independently by S.Illman [Equivariant Algebraic Topology, Ph.D. Thesis, Princeton (1972)] and T.Matumoto [On G-CW-complexes and a Theorem of J.H.C.Whitehead, J.Fac.Sci.Univ.

Tokyo, Sect.1A 18 (1971), 363 - 374]. These more general spaces are no longer "classical" CW-complexes. Illman used these complexes to consider equivariant cohomology theory, and more recently S.Waner [Trans.Amer.Math.Soc., 258 (1980), 351 - 368, 369 - 384, 385 - 405] has used them to develop an equivariant homotopy theory.

Our definition is the case where G is discrete. We always take the standard filtration on the G-CW-complex X. Then $\underset{\sim}{X}$ is J_0-filtered and homotopy full [11]. This situation includes that of regular G-simplicial complexes as the following lemma shows.

4.17 LEMMA. Let K be a G-simplicial complex satisfying condition (1) of the previous section (page 58). Then its polyhedron $X = |K|$ is a G-CW-complex.

Proof. It is well-known that a polyhedron is a CW-complex and that simplicial maps induce cellular maps. (See J.P.May [Simplicial Objects in Algebraic Topology, Van Nostrand Mathematical Studies 11 (1967)], for example).

Suppose K satisfies condition (1). Let σ be the realisation of the n-simplex $(v_0,\dots,v_n)$ and suppose $\sigma^g = \sigma$. If there is a vertex v_i $(0 \leqslant i \leqslant n)$ such that $v_i^g \neq v_i$ then (v_i, v_i^g) is a 1-simplex, which contradicts condition (1). Hence $v_i^g = v_i$ for $i=0,\dots,n$, so g maps σ to itself by the identity homeomorphism.

□

Let X be a G-CW-complex. We do not assume that the characteristic maps are preserved by the action. However, the next result shows that X is G-homeomorphic to a G-CW-complex where the characteristic maps are preserved by the action. We call this a "re-characterisation", and it justifies Bredon's remark that the characteristic maps "may be chosen" equivariantly [4; Chapter II, §1].

<u>4.18 PROPOSITION.</u> Let X be a G-CW-complex. Then there is a G-CW-complex Y and a cellular G-homeomorphism $k : Y \to X$ such that the action on Y preserves the characteristic maps.

<u>Proof.</u> We use induction to define Y and k together. Let $X^0 = \{x_\lambda \mid \lambda \in \Lambda_0\}$. There is an induced action on the index set Λ_0 given by $\lambda^g = \mu$ if $x_\lambda{}^g = x_\mu$. Choose one index from each orbit — the letter ξ will denote a chosen index throughout. Let $Y^0 = \{y_\lambda \mid \lambda \in \Lambda_0\}$ be a set in bijective correspondence with X^0, the bijection being $k_0 : Y^0 \to X^0$, $y_\lambda \mapsto x_\xi{}^g$ where $\lambda = \xi^g$ (and ξ is a chosen index). This begins the induction.

Suppose now that Y^{n-1} and $k_{n-1} : Y^{n-1} \longrightarrow X^{n-1}$ have been defined such that k_{n-1} is a cellular G-homeomorphism and the action on Y^{n-1} commutes with the characteristic maps. Let $X^n - X^{n-1} = \{e_\lambda \mid \lambda \in \Lambda_n\}$, a family of open n-cells. Also let $h_\lambda : (I^n, \mathring{I}^n) \longrightarrow (\bar{e}_\lambda, \mathring{e}_\lambda)$ be the characteristic map of the cell e_λ, where $\mathring{e}_\lambda = \bar{e}_\lambda - e_\lambda$. We sometimes denote $h_\lambda | \mathring{I}^n$ by ∂h_λ.

As for $n=0$, there is an action induced on Λ_n, and we choose a set of representatives of the orbits. Let $A = \coprod_{\lambda\in\Lambda_n} I^n_\lambda$ be Λ_n disjoint copies of I^n and let $B = \coprod_{\lambda\in\Lambda_n} \dot{I}^n_\lambda$. Define

$$\psi_\lambda : \dot{I}^n_\lambda \longrightarrow Y^{n-1} \qquad \text{by} \qquad \psi_\lambda = k^{-1}_{n-1} \circ g_* \circ \partial h_\xi$$

where ξ is the representative of the orbit containing λ, $g \in G$ is such that $\lambda^g = \xi$ and g_* is the homeomorphism of X induced by g. This definition is independent of the choice of g, for if g' also maps λ to ξ then $g_*|\bar{e}_\lambda = g'_*|\bar{e}_\lambda$ by condition (*) of the definition (4.16) of a G-CW-complex.

$\psi = \coprod_{\lambda\in\Lambda_n} \psi_n$ is a map $B \to Y^{n-1}$. Define Y^n to be the adjunction space $Y^n = Y^{n-1} \sqcup_\psi A$. Then there is a pushout of spaces

$$\begin{array}{ccc} B & \hookrightarrow & A \\ {\scriptstyle\psi}\downarrow & & \downarrow \\ Y^{n-1} & \longrightarrow & Y^n \end{array} \qquad (*)$$

We use the universal property of this pushout to extend k_{n-1} to k_n. Define

$$\theta_\lambda : I^n_\lambda \longrightarrow X^n \qquad \text{by} \qquad \theta_\lambda = g_* \circ h_\xi$$

where ξ is a representative index and $\xi^g = \lambda$. Again this is independent of the choice of g and

$\theta_\lambda | \dot{I}^n_\lambda = g_* \circ \partial h_\xi = k_{n-1} \circ \psi_\lambda.$ Hence

$$\theta = \bigsqcup_{\lambda \in \Lambda_n} \theta_\lambda : A \longrightarrow X^n$$

is such that $\theta | B = k_{n-1} \circ \psi$. The universal property of (*) now provides a unique map $k_n : Y^n \to X^n$ extending k_{n-1}.

Clearly Y^n is constructed so that the action commutes with the characteristic maps. We now show that k_n is a cellular G-homeomorphism.

A point of A is (t,λ) where $t \in I^n$, $\lambda \in \Lambda_n$. G acts on A by $(t,\lambda)^g = (t,\lambda^g)$ and $\theta : (t,\lambda) \mapsto g_* h_\xi(t)$. Let $\ell \in G$. Then

$$\begin{aligned} \theta\left((t,\lambda)^\ell\right) &= \theta(t,\lambda^\ell) \\ &= (g\ell)_* h_\xi(t) \qquad \text{since} \quad \xi^{g\ell} = \lambda^\ell \\ &= \ell_* g_* h_\xi(t) \\ &= \left(\theta(t,\lambda)\right)^\ell. \end{aligned}$$

Therefore θ is a G-map. Now $\theta | B$ is a homeomorphism, and by the inductive hypothesis k_{n-1} is a cellular G-homeomorphism. Hence k_n is a cellular G-homeomorphism which completes the inductive step.

Let Y be the colimit of the system $Y^0 \subseteq Y^1 \subseteq \ldots \subseteq Y^n \subseteq \ldots$ and let $k : Y \to X$ be induced by the maps $Y^n \xrightarrow{k_n} X^n \hookrightarrow X$. A standard limit argument completes the proof.

□

It is easy to see that the equivalence relation on Y given by the group action is a cellular equivalence in the sense of A.T.Lundell and S.Weingram [The Topology of CW-complexes, Van Nostrand (1969), page 32]. Hence Y/G is canonically a CW-complex and the projection $\tau : Y \to Y/G$ is a cellular map. If $\bar{\lambda}$ denotes the orbit of $\lambda \in \Lambda_n$ then the orbit of e_λ is an n-cell of Y/G which can be denoted by $e_{\bar{\lambda}}$. Its characteristic map is $h_{\bar{\lambda}} = \tau \circ h_\xi$ where ξ is the representative of $\bar{\lambda}$. In fact the projection of any characteristic map of Y is a characteristic map of Y/G.

The G-homeomorphism $k : Y \to X$ induces a homeomorphism $\hat{k} : Y/G \longrightarrow X/G$ which exhibits X/G as a CW-complex.

Let $\pi\underset{\sim}{X}$ denote the homotopy crossed complex of $\underset{\sim}{X}$ (with standard filtration). In order to prove our main theorem we need an algebraic description of $\pi\underset{\sim}{X}$ which we describe below. This description (theorem 4.19) essentially goes back to J.H.C.Whitehead and follows from the Union Theorem — Whitehead considered the case over a single vertex.

The cell structure on X may be considered as (defining) a "presentation" of $\pi\underset{\sim}{X}$ in some sense. (Indeed, for double groupoids Howie [16] uses CW-complexes to define the notion of a presentation). The re-characterisation $k : Y \to X$ may be viewed algebraically as giving a new presentation — one on which the group acts. These remarks will be made more precise later.

We now describe $\pi\underset{\sim}{Z}$ for an arbitrary CW-complex Z. Let 0 be the base point of I^n, and assume the characteristic maps of Z map 0 to the 0-skeleton Z^0. Let e_λ be an n-cell of Z, for $n \geqslant 1$, with characteristic map $f_\lambda : (I^n, \dot{I}^n, 0) \longrightarrow (Z^n, Z^{n-1}, Z^0)$. This determines an element c_λ of $\pi_n(Z^n, Z^{n-1}) = \{\pi_n(Z^n, Z^{n-1}, v) \mid v \in Z^0\}$. Let $\partial_1^0 I^n = \{(t_1, \ldots, t_n) \in I^n \mid t_1 = 0\}$ and $J^{n-1} = \overline{(\dot{I}^n - \partial_1^0 I^n)}$. Then c_λ has a representative $\hat{f}_\lambda : (I^n, \dot{I}^n, J^{n-1}) \longrightarrow (Z^n, Z^{n-1}, Z^0)$. (I.e. $f_\lambda \simeq \hat{f}_\lambda$). $\partial_1^0 I^n$ is homeomorphic to I^{n-1}. Let $d\hat{f}_\lambda = \hat{f}_\lambda | \partial_1^0 I^n$. So $d\hat{f}_\lambda$ may be considered as a map $(I^{n-1}, \dot{I}^{n-1}) \rightarrow (Z^{n-1}, Z^0)$, and hence determines an element dc_λ of $\pi_{n-1}(Z^{n-1}) = \{\pi_{n-1}(Z^{n-1}, v) \mid v \in Z^0\}$. By regarding $d\hat{f}_\lambda$ as a map $(I^{n-1}, \dot{I}^{n-1}, J^{n-1}) \longrightarrow (Z^{n-1}, Z^{n-2}, Z^0)$ it represents an element ∂c_λ of $\pi_{n-1}(Z^{n-1}, Z^{n-2})$.

<u>4.19 THEOREM</u>. Let Z be a CW-complex and let c_λ, dc_λ and ∂c_λ be as defined above. Then :

(i) For $n > 2$, $\pi_n(Z^n, Z^{n-1})$ is a free $\pi_1(Z^{n-1}, Z^0)$-module on the set $\{c_\lambda \mid \lambda \in \Lambda_n\}$. The morphism $\partial : \pi_n(Z^n, Z^{n-1}) \rightarrow \pi_{n-1}(Z^{n-1}, Z^{n-2})$ is induced by $c_\lambda \mapsto \partial c_\lambda$.

(ii) For $n=2$, $\pi_2(Z^2, Z^1)$ is the free crossed $\pi_1(Z^1, Z^0)$-module on $d : \{c_\lambda \mid \lambda \in \Lambda_2\} \longrightarrow \{\pi_1(Z^1, v) \mid v \in Z^0\} \subseteq \pi_1(Z^1, Z^0)$ given by $c_\lambda \mapsto dc_\lambda$.

<u>Proof</u>. [11; Example 4 of §7, page 33].

□

Definitions of free module and free crossed module over a groupoid are given in [12]. They are simply extensions to the groupoid case of the usual definitions over groups. In [12] a notion of free crossed complex is defined and it is noted that $\pi \underset{\sim}{Z}$ is free in this sense. (There are, of course, many forgetful functors with domain $\mathcal{C}$ which have left adjoints. Thus there are several possible definitions of "free" crossed complex, but the one given in [12] seems the most useful. In her thesis [5], Bolton considered various types of "freeness" for crossed modules over groups).

We now have all the machinery required to prove the following theorem which is the main result of this chapter.

4,20 THEOREM. Let $\underset{\sim}{X}$ be a G-CW-complex with standard filtration. Then

$$(\pi \underset{\sim}{X})/(G) \cong \pi(\underset{\sim}{X}/G).$$

Proof. By proposition 4.18 we may assume that the characteristic maps of X are equivariant, and hence X/G is canonically a CW-complex. Let $\tau : X \to X/G$ be the natural projection. We verify the universal property of definition 3.12 for the induced map $\tau_* : \pi \underset{\sim}{X} \longrightarrow \pi(\underset{\sim}{X}/G)$.

It is clear that G acts trivially on $\pi(\underset{\sim}{X}/G)$ and that τ_* is a G-morphism.

Let A be a crossed complex with trivial G-action and let

$\theta : \pi X \longrightarrow A$ be a G-morphism. We use the freeness of $\pi(X/G)$ to define the required morphism $\theta^* : \pi(X/G) \longrightarrow A$.

$(X^1/G, X^0/G)$ is a (topological) graph so $\pi_1(X^1/G, X^0/G)$ is the free groupoid with vertex set X^0/G and generators the 1-cells of X/G. Define

$$(\theta_1^*, \theta_0^*) : \pi_1(X^1/G, X^0/G) \longrightarrow A \quad \text{by} \quad x_{\bar{\lambda}} \longmapsto \theta x_\lambda$$

where for $i=0,1$ $x_{\bar{\lambda}}$ is an i-cell of X/G and x_λ is an i-cell of X such that $\tau x_\lambda = x_{\bar{\lambda}}$. The definition is independent of the choice of x_λ, for if $\tau x_\mu = x_{\bar{\lambda}}$ then there exists $g \in G$ such that $x_\lambda{}^g = x_\mu$ and hence $\theta x_\mu = \theta x_\lambda$.

Since $\pi_1(X^1/G, X^0/G)$ is free, $(\theta_1^*, \theta_0^*) = tr^1\theta^*$ is a well-defined groupoid morphism, and since $\pi_1(X^1, X^0)$ is also free it is clear that $tr^1\theta^*$ is the unique morphism such that $tr^1\theta^* \circ tr^1\tau_* = tr^1\theta$.

We now extend this to a morphism of crossed modules by defining $\theta_2^* : \pi_2(X^2/G, X^1/G) \longrightarrow A_2$. By theorem 4.19(ii) applied to X/G it is sufficient to define θ_2^* on the family $\{a_\mu\}$ where a_μ is determined by the characteristic map of a 2-cell of X/G. That is, a_μ is the class of $\tau \circ f_\lambda$ for some (indeed any) $\lambda \in \Lambda_2$ where μ is the orbit of λ, and f_λ is a characteristic map of X. We define

$$\theta_2^* : \pi_2(X^2/G, X^1/G) \longrightarrow A_2 \quad \text{by} \quad a_\mu \longmapsto \theta_2 c_\lambda$$

where $c_\lambda \in \pi_2(X^2, X^1)$ is the element determined by f_λ. As in dimension one this is independent of the chioce of λ. This extends $tr^1\theta^*$ to a morphism of crossed modules

$$tr^2\theta^* = (\theta_2^*, \theta_1^*, \theta_0^*) : tr^2\pi(\underset{\sim}{X}/G) \longrightarrow tr^2 A$$

provided $\partial\theta_2^* a_\mu = \theta_1^* da_\mu$ in A_1, where da_μ is defined by $\tau \circ \partial f_\lambda$. But

$$\partial\theta_2^* a_\mu = \partial\theta_2 c_\lambda = \theta_1[\partial f_\lambda] = \theta_1^* \tau_*[\partial f_\lambda] = \theta_1^* da_\mu$$

as required. To show $tr^2\theta^* \circ tr^2\tau_* = tr^2\theta$ we again appeal to theorem 4.19(ii) but this time applied to X. By this result we need only check the above condition on $\{c_\lambda \mid \lambda \in \Lambda_2\}$. It is clear that $\theta_2^*\tau_* c_\lambda = \theta_2 c_\lambda$ by the independence of the choice of λ in the definition of θ_2^*. It is also obvious that $tr^2\theta^*$ is the unique morphism with this property.

We now use an inductive argument. For $n \geqslant 3$, suppose that

$$tr^{n-1}\theta^* : tr^{n-1}\pi(\underset{\sim}{X}/G) \longrightarrow tr^{n-1}A$$

has been defined in $\mathcal{C}_n$ and is the unique morphism such that $tr^{n-1}\theta^* \circ tr^{n-1}\tau_* = tr^{n-1}\theta$. To extend to $tr^n\theta^*$ we use theorem 4.19(i) to define

$$\theta_n^* : \pi_n(X^n/G, X^{n-1}/G) \longrightarrow A_n \quad \text{by} \quad a_\mu \longmapsto \theta_n c_\lambda$$

as in dimension two. We need to verify that $\theta_{n-1}^* \partial = \partial \theta_n^*$ for $tr^n \theta^*$ to be well-defined. There are two cases.

Case (i) : $n > 3$. In this case $\pi_1(X^{n-1}/G, X^0/G) = \pi_1(X^{n-2}/G, X^0/G)$ so $\pi_n(X^n/G, X^{n-1}/G)$ and $\pi_{n-1}(X^{n-1}/G, X^{n-2}/G)$ are modules over the same groupoid. Therefore it is sufficient to check $\theta_{n-1}^* \partial = \partial \theta_n^*$ on the free generators. For these elements we have

$$\partial \theta_n^* a_\mu = \partial \theta_n c_\lambda = \theta_n \partial c_\lambda = \theta^*_{n-1} \tau_* \partial c_\lambda = \theta_{n-1}^* \partial \tau_* c_\lambda = \theta_{n-1}^* \partial a_\mu$$

as required.

Case (ii) : $n=3$. Here $\pi_2(X^2/G, X^1/G)$ is a crossed module over the groupoid $\pi_1(X^1/G, X^0/G)$ and $\pi_3(X^3/G, X^2/G)$ is a module over the groupoid

$$\pi_1(X^2/G,\ X^0/G) \cong \pi_1(X^1/G,\ X^0/G) \big/ \partial \pi_2(X^2/G,\ X^1/G).$$

Let $\beta : \pi_1(X^1/G, X^0/G) \longrightarrow \pi_1(X^2/G, X^0/G)$ be the quotient morphism. We also use β to denote $A_1 \longrightarrow A_1/\partial A_2$, the corresponding morphism for A.

Since $\theta_1^*\left(\partial \pi_2(X^2/G, X^1/G)\right) \subseteq \partial A_2$, θ_1^* induces a morphism $\psi : \pi_1(X^2/G, X^0/G) \longrightarrow A_1/\partial A_2$. The situation is illustrated by the following commutative diagram.

$$\begin{array}{ccc}
\pi_2(X^2/G, X^1/G) & \xrightarrow{\theta_2^*} & A_2 \\
\partial\big\downarrow & & \big\downarrow\partial \\
\pi_1(X^1/G, X^0/G) & \xrightarrow{\theta_1^*} & A_1 \\
\beta\big\downarrow & & \big\downarrow\beta \\
\pi_1(X^2/G, X^0/G) & \xrightarrow{\psi} & A_1/\partial A_2
\end{array}$$

As a (totally disconnected) groupoid $\pi_3(X^3/G, X^2/G)$ is generated by elements $a_\mu{}^x$ where $x \in \pi_1(X^2/G, X^0/G)$ and, as usual, a_μ is determined by a characteristic map. Now

$$\begin{aligned}
\partial\theta_3^*\left(a_\mu{}^x\right) &= \partial\left((\theta_3^* a_\mu)^{\psi x}\right) \\
&= \left(\partial\theta_3^* a_\mu\right)^{\theta_1^*\bar{x}} && \text{for some } \bar{x} \text{ such that } \beta\bar{x} = x \\
&= \left(\theta_2^*\partial a_\mu\right)^{\theta_1^*\bar{x}} \\
&= \theta_2^*\left(\partial a_\mu{}^{\bar{x}}\right) \\
&= \theta_2^*\partial\left(a_\mu{}^x\right) && \text{as required.}
\end{aligned}$$

Hence in all cases $(n \geqslant 3)$ we obtain a well-defined morphism $tr^n\theta^* : tr^n\pi(\underset{\sim}{X}/G) \longrightarrow tr^n A$ which is easily seen to be the unique morphism such that $tr^n\theta^* \circ tr^n\tau_* = tr^n\theta$. This completes the inductive step, and hence we obtain the required morphism $\theta^* : \pi(\underset{\sim}{X}/G) \longrightarrow A$.

□

<u>Remarks</u>. (1) We noted earlier that the cell structure on X may be viewed as (defining) a presentation of $\pi\underset{\sim}{X}$ in some sense. The cells and their boundaries give generators and relations as described by theorem 4.19. The content of the re-characterisation (proposition 4.18) is then that $\pi\underset{\sim}{X}$ has a presentation (given by Y) such that the group acts <u>on the</u>

presentation. Viewed in this way, theorem 4.20 says that the orbit crossed complex $(\pi\underset{\sim}{X})/(G)$ has a presentation obtained from the presentation of $\pi\underset{\sim}{X}$ by selecting a generator from each orbit of generators and similarly for relations (when interpreted appropriately).

(2). The proof of the isomorphism of theorem 4.20 crucially depends on liftability of maps $I^n \to X/G$ to X. This was achieved by making the fairly strong assumptions that X and X/G are CW-complexes which meant we only had to lift characteristic maps. We have been unable to establish the isomorphism under more general hypotheses due to this strong liftability requirement.

(3). Recall theorem 3.19 which gives an isomorphism $\pi_1\big(A/(G)\big) \cong \big(\pi_1 A\big)/(G)$ for the G-crossed complex A, and note that if $A = \pi\underset{\sim}{Z}$ for a CW-complex Z then $\pi_1(Z,Z^0) = \pi_1 A$. These results together with theorem 4.20 give an isomorphism $\pi_1(X/G, X^0/G) \cong \pi_1(X,X^0)/(G)$. Hence we recapture theorem 4.9 but in the more general context of CW-complexes.

4.21 COROLLARY. Let X be a G-CW-complex, $v \in X^0$ and $n \geqslant 2$. Then

$$\pi_n(X^n/G, X^{n-1}/G, \tau v) \cong \pi_n(X^n, X^{n-1}, v)/N$$

where N is the subgroup generated by elements of the form $(a\circ g)^{(x\circ g\ -\ x)} - a$, for $a \in \pi_n(X^n, X^{n-1}, v)$, $x \in \pi_1(X^1)(v,w)$

and $g \in G$ stabilising $w \in X^0$.

Proof. From corollary 3.17 we have that N is generated by elements of the form $a^m - a$ where $a \in \pi_n(X^n, X^{n-1}, v)$ and $m = -y+n+y$, $y \in (\pi_1(X^1, X^0) \tilde{\times} G)(w,v)$ and $n = (\varepsilon w, h)$ where h stabilises $w \in X^0$. Now y may be written $y = (z,k)$ where $k \in \pi_1(X^1)(w, v^{k^{-1}})$. Then

$$\begin{aligned} m &= (-z\cdot k,\ k^{-1}) + (\varepsilon w, h) + (z,k) \\ &= (-z\cdot k + z\cdot h^{-1}k,\ k^{-1}hk) \\ &= (x - x\cdot g^{-1},\ g) \qquad \text{where} \quad x = -z\cdot k \quad \text{and} \quad g = k^{-1}hk. \end{aligned}$$

Therefore

$$\begin{aligned} a^m - a &= a^{(x - x\cdot g^{-1}, g)} - a \\ &= (a\cdot g)^{(a\cdot g - x)} - a \quad \text{, as required.} \end{aligned}$$

□

In the case where G acts freely the corollary gives $\pi_n(X^n/G, X^{n-1}/G, \tau v) \cong \pi_n(X^n, X^{n-1}, v)$. This is a standard result since, in this case, $\tau : X \longrightarrow X/G$ is a generalised cover in the sense of S.T.Hu [Homotopy Theory, Academic Press, 1959].

It is perhaps unfortunate that the main theorem does not yield results about the absolute groups — at least, not directly. It may be that this is intrinsic to the problem. For example, in [20] Rhodes generalises his group σ to higher dimensions but simple (two dimensional) examples show that the analogue of theorem 4.6 is false in higher dimensions. In dimension two, however, we can glean a little more information, because in this case the morphism $\partial : \pi_2(Z^2, Z^1) \longrightarrow \pi_1(Z^1, Z^0)$ which appears in $\pi \underset{\sim}{Z}$ is part of the homotopy sequence for the pair (Z^2, Z^1). In particular we have the following.

<u>4.22 COROLLARY.</u> Let X be a G-CW-complex such that $\pi_2(X^2, v) = 0$, for $v \in X^0$. Then $\partial : \pi_2(X^2, X^1, v) \longrightarrow \pi_1(X^1, v)$ is injective and

$$\pi_2(X^2/G, \tau v) \cong \big(\pi_2(X^2, X^1, v) \cap K\big)/N$$

where $\partial K = \overline{K} = \ker\big(\tau_* : \pi_1(X^1, v) \longrightarrow \pi_1(X^1/G, \tau v)\big)$ and N is defined in the previous corollary.

<u>Proof.</u> From the homotopy exact sequence for (X^2, X^1) it is immediate that ∂ is injective. Combining this with the sequence for $(X^2/G, X^1/G)$ gives the following diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \pi_2(X^2/G, \tau v) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 \longrightarrow & N & \longrightarrow & \pi_2(X^2, X^1, v) & \xrightarrow{\tau_*} & \pi_2(X^2/G, X^1/G, \tau v) & \longrightarrow & 0 \\
 & & & \downarrow \partial & & \downarrow \partial & & \\
0 \longrightarrow & \overline{K} & \longrightarrow & \pi_1(X^1, v) & \xrightarrow{\tau_*} & \pi_1(X^1/G, \tau v) & & \\
 & & & \downarrow & & \downarrow & & \\
 & & & \pi_1(X^2, v) & & \pi_1(X^2/G, \tau v) & &
\end{array}$$

The exactness of the first row follows from corollary 4.21.

Hence $\pi_2(X^2/G,\tau v) \quad \ker\big(\partial : \pi_2(X^2/G, X^1/G, \tau v) \longrightarrow \pi_1(X^1/G,\tau v)\big)$.

Simple diagram chasing now proves the result.

$\square$

We complete this chapter by illustrating theorem 4.20 with a few examples.

<u>4.23 EXAMPLE.</u> Let $X = S^n = \{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} \mid \Sigma x_i^{\,2} = 1\}$, $n \geq 2$ and let $G = \mathbb{Z}_2$ act by reflection in the equatorial hyperplane. I.e. the generator $g \in G$ maps $(x_0,\ldots,x_n)$ to $(x_0,\ldots,x_{n-1},-x_n)$. Consider the following cell structure on X. $X^0 = X^1 = \ldots = X^{n-2} = * = (1,0,\ldots,0)$, $X^{n-1} = \{\underline{x} \in S^n \mid x_n = 0\}$, and $X^n - X^{n-1} = \sigma_+ \cup \sigma_-$ where $\sigma_\pm = \{\underline{x} \in S^n \mid x_n \gtrless 0\}$. So X has four cells, $X = * \cup e_{n-1} \cup \sigma_+ \cup \sigma_-$.

Let $A = \pi \underset{\sim}{X}$. Then $A_0 = *$, $A_1 = \ldots = A_{n-2} = 0$, $A_{n-1} = \mathbb{Z}$ and $A_n = \mathbb{Z} \times \mathbb{Z}$. $\partial : A_n \longrightarrow A_{n-1}$ projects both factors of A_n onto A_{n-1}, and G acts on A_n by interchanging the factors.

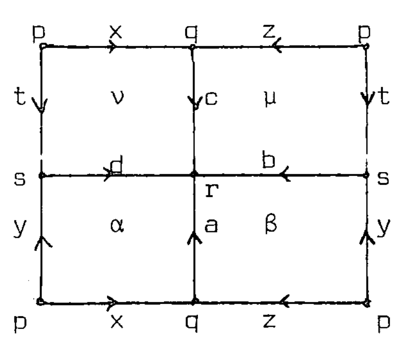
Let $B = A/(G)$. Then $B_r = A_r$ for $r < n$, $B_n = \mathbb{Z}$ and $\partial : B_n \longrightarrow B_{n-1}$ is an isomorphism. The orbit space $\underset{\sim}{X}/G$ is an n-disc with standard cell decomposition into three cells. Clearly $\pi(\underset{\sim}{X}/G) \cong B$.

Although this example is essentially trivial, it does illustrate two points. The first is the point made previously that the algebra is an accurate model for the topology. Secondly, it illustrates that theorem 4.20 is false for arbitrary cell structures. For suppose we give $X = S^n$ its standard cell decomposition with just two cells. Then $A = \pi \underset{\sim}{X}$ has only one non-trivial group, $A_n = \mathbb{Z}$. The reflection homeomorphism has degree -1 so $\mathbb{Z}_2$ acts on A_n non-trivially. Therefore $B = A/(G)$ has the group $\mathbb{Z}_2$ in dimension n, whereas $\pi(\underset{\sim}{X}/G)$ is trivial. (Here, of course, $\underset{\sim}{X}/G$ does not have the structure of a CW-complex).

The antipodal action on S^n is almost as easy. In this case though we require two r-cells for each $r=0,1,\ldots,n$ so the algebra is more complicated.

4.24 EXAMPLE. We now consider the example given by Rhodes in [20; §9] which shows that the analogue of his theorem 4.6 does not hold in higher dimensions.

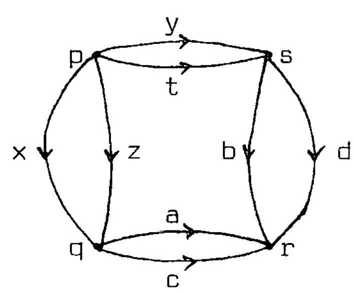
Let X be the torus, obtained from I^2 by identifying opposite sides. $G = \langle g \mid g^4 = 1 \rangle$ acts on I^2 by (anticlockwise) rotation and induces an action on the torus. Let $\underset{\sim}{X}$ be the torus with the following cell structure : it has four 0-cells, eight 1-cells and four 2-cells.



The diagram gives the following description of $A = \pi\underset{\sim}{X}$ (where, by a slight abuse of notation, we use the same letter to denote a cell and the corresponding generator in A).

$A_0 = \{p, q, r, s\}$.

A_1 is the free groupoid on the following graph.



A_2 is the free crossed A_1-module on generators α, β, μ, ν with $\partial\alpha = x+a-d-y \in A_1(p)$, $\partial\beta = -z+y+b-a \in A_1(q)$, $\partial\mu = -b-t+z+c \in A_1(r)$, and $\partial\nu = d-c-x+t \in A_1(s)$.

The generator g of G acts on A as follows.

A_0 : $p \circ g = p$, $q \circ g = s$, $r \circ g = r$, $s \circ g = q$.

A_1 : $x \circ g = y$, $y \circ g = z$, $z \circ g = t$, $t \circ g = x$,
$a \circ g = b$, $b \circ g = c$, $c \circ g = d$, $d \circ g = a$.

A_2 : $\alpha \circ g = \beta^{-z}$, $\beta \circ g = \mu^{-b}$, $\mu \circ g = \nu^{d}$, $\nu \circ g = \alpha^{x}$.

We restrict to the vertex p which is fixed by G. A maximal tree in the above graph has three edges so $A_1(p) \cong F_5$, the free group on five generators. $A_2(p)$ is the free crossed $A_1(p)$-module on four generators, namely $\overline{\alpha} = \alpha$, $\overline{\beta} = \beta^{-z}$, $\overline{\mu} = \mu^{-c-z}$, $\overline{\nu} = \nu^{-t}$. (The element $\overline{\theta}$ is θ "transposed" to the vertex p by a path in $\partial\theta$). The boundary morphism $\partial : A_2(p) \longrightarrow A_1(p)$ is induced by $\partial\overline{\beta} = z + \partial\beta - z$, etc.

$A_2(p)$ has a presentation, due to Whitehead, with generators $(\overline{\theta}, w)$ for $\overline{\theta} \in \{\overline{\alpha}, \overline{\beta}, \overline{\mu}, \overline{\nu}\}$ and $w \in A_1(p)$, and relations

$$-(\overline{\theta}, w) + (\overline{\phi}, w') + (\overline{\theta}, w) = (\overline{\phi},\ w' - w + \partial\theta + w).$$

$A_1(p)$ acts on $A_2(p)$ by $(\overline{\theta}, w)^{w'} = (\overline{\theta},\ w + w')$ and the boundary morphism is given by $\partial(\overline{\theta}, w) = -w + \partial\overline{\theta} + w$.

Let $B = A/(G)$. By corollary 3.17 there is an isomorphism

$$B_2(<p>) \cong A_2(p)/N$$

where N is generated by elements of the form

$$\left((\overline{\theta}, w)\circ h\right)^{(w'\circ h\ -\ w')} - (\overline{\theta}, w),$$

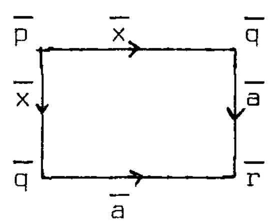
for $w' \in A_1(p, v)$ and $h \in G$ stabilising v.

Since p is fixed by every element of G, taking w' to be the identity at p shows that the elements $(\overline{\theta}, w)\circ h - (\overline{\theta}, w)$ lie in N. Also $A_1(p)$ is easily seen to be generated by the elements $w'\circ h - w'$, so the elements $(\overline{\theta}\circ h, 1) - (\overline{\theta}, w)$

lie in N (where h depends on w). Combining this with the element above (for w=1) we see that $(\overline{\theta},w) - (\overline{\theta},1)$ lies in N.

Therefore a presentation of $B_2(<p>)$ is obtained from that of $A_2(p)$ by adding the relations $(\overline{\theta},1) = (\overline{\theta}\circ h,1)$ and $(\overline{\theta},1) = (\overline{\theta},w)$. The generators of $A_2(p)$ are all identified, and the relations of $A_2(p)$ are trivial. Hence $B_2(<p>) \cong \mathbb{Z}$.

The orbit space X/G is S^2 with the following cell structure. (We use $\overline{x}$ to denote the orbit of x etc.).



Since X^1/G is contractible, $\pi_2(X^2/G, X^1/G, \overline{p}) \cong \pi_2(S^2,*) \cong \mathbb{Z}$, as required.

This example shows that, even for fairly simple actions, the algebra can become quite complicated, largely due to the complicated structure of the second relative homotopy group as a crossed module.

We have been able to establish the isomorphism $\pi(\underset{\sim}{X})/(G) \cong \pi(\underset{\sim}{X}/G)$ only in the case where X is a G-CW-complex. In view of Armstrong's success in reducing the conditions required for the groupoid result we feel that the isomorphism for crossed complexes probably holds more generally (although much stronger liftability conditions are required in this case). It would be interesting to know the most general conditions under which theorem 4.20 holds.

CHAPTER FIVE. RAZAK'S CONJECTURE AND RELATED RESULTS

This chapter concerns topological spaces X with a cover $\mathcal{U} = \{X_\lambda \mid \lambda \in \Lambda\}$ by subspaces, and various associated groupoids. In particular we consider fundamental groupoids of the classifying space $B\mathcal{U}$ of the cover, defined by Segal [21].

Section one gives the definition of the classifying space and some background results. There are natural projections of the classifying space onto X and onto the nerve of the cover. In his thesis [18] Razak made a conjecture concerning the morphism of fundamental groupoids induced by the projection of $B\mathcal{U}$ onto the nerve. In section two we prove a modified form of the conjecture.

The morphism $\pi_1 B\mathcal{U} \to \pi_1 X$ induced by the projection of $B\mathcal{U}$ onto X was also considered by Razak. In particular, he proved that the morphism is a homotopy equivalence of groupoids in the case where the interiors of the members of $\mathcal{U}$ cover X. In section three we indicate that this morphism is closely related to the one-dimensional union theorem via a groupoid construction of Crowell and Smythe [13].

Several authors have considered presentations of groups of homeomorphisms of a space with a fundamental domain. For example, see Macbeath [17], Swan [24], and Abels [1]. Section four applies the modified Razak conjecture, proved in section two, and the notion of the semi-direct product groupoid to this situation.

Finally, we conclude with some general remarks concerning possible future developments of the theory.

1. THE CLASSIFYING SPACE $B\mathcal{U}$

In this section we give some constructions and results required later. For the most part this material is well-known or is due to Razak [18]. We follow the notation and approach of [18] which is the basic reference for this section.

Let $\mathcal{U} = \{X_\lambda \mid \lambda \in \Lambda\}$ be a cover of the space X by subspaces. We begin by constructing a space $B\mathcal{U}$ defined by Segal [21]. First give the index set Λ a well-ordering, and let $\Lambda^{(n)}$ denote the set $\{(\lambda_1, \ldots, \lambda_n) \in \Lambda^n \mid \lambda_1 \leqslant \ldots \leqslant \lambda_n\}$. Define a simplicial space $N\mathcal{U}$ by

$$(N\mathcal{U})_n = \coprod_{\nu \in \Lambda^{(n+1)}} X_\nu$$

where $X_\nu = X_{\lambda_0} \cap \ldots \cap X_{\lambda_n}$. An element of $(N\mathcal{U})_n$ is $(x, \lambda_0, \ldots, \lambda_n)$ where $x \in X_\nu$, $\nu = (\lambda_0, \ldots, \lambda_n)$. The face and degeneracy maps are defined in the usual way. The <u>classifying space</u> $B\mathcal{U}$ of the cover $\mathcal{U}$ is the geometric realisation of $N\mathcal{U}$, given by

$$B\mathcal{U} = \left(\coprod_{n \geqslant 0} (N\mathcal{U})_n \times \Delta^n\right) \Big/ \sim$$

where Δ^n is the standard n-simplex (in $\mathbb{R}^{n+1}$) and $\sim$ is the usual equivalence relation. $B\mathcal{U}$ has a natural filtration by skeleta :

$$B\mathcal{U} : B\mathcal{U}^0 \subseteq B\mathcal{U}^1 \subseteq \ldots \subseteq B\mathcal{U}^n \subseteq \ldots \quad .$$

There is a natural projection $p : B\mathcal{U} \longrightarrow X$ induced by the map $N\mathcal{U} \to X$, $(x,\lambda_0,\ldots,\lambda_n) \mapsto x$. In [21] Segal proved that p is a homotopy equivalence in the case where $\mathcal{U}$ is a numerable cover, and Razak showed that the induced morphism $p_* : \pi_1 B\mathcal{U} \to \pi_1 X$ is a homotopy equivalence (of groupoids) in the case where the interiors of the X_λ's cover X. In the proof of his result Razak established the following [18; Chapter 2, diagram 2.1, page 70].

<u>5.1 LEMMA</u>. The following diagram is a pushout in $\mathcal{G}pd$.

$$\begin{array}{ccc}
\bigsqcup_{\nu\in\Lambda^{(n+1)}} \pi_1(X_\nu \times \partial\Delta^n) & \xrightarrow{k_n} & \pi_1(B\mathcal{U}^{n-1}) \\
\big\downarrow j_n & & \big\downarrow J_n \\
\bigsqcup_{\nu\in\Lambda^{(n+1)}} \pi_1(X_\nu \times \Delta^n) & \xrightarrow{K_n} & \pi_1(B\mathcal{U}^n)
\end{array}$$

where j_n, J_n are induced by inclusions, and k_n, K_n are inducedby the identification map $\bigsqcup_{n\geqslant 0} (N\mathcal{U})_n \times \Delta^n \to B\mathcal{U}$. □

In this chapter we are usually concerned with the full subgroupoids $\pi_1(B\mathcal{U}^n, B\mathcal{U}^0)$ of $\pi_1(B\mathcal{U}^n)$. It is convenient, therefore, to have a version of lemma 5.1 in terms of these groupoids.

5.2 COROLLARY. The following diagram is a pushout in $\mathcal{G}$pd, where $(\Delta^n)^0$ is the set of vertices (0-skeleton) of Δ^n.

$$\begin{array}{ccc} \coprod_{\lambda\in\Lambda^{(n+1)}} \pi_1(X_\nu \times \partial\Delta^n, X_\nu \times (\Delta^n)^0) & \xrightarrow{k_n} & \pi_1(B\mathcal{U}^{n-1}, B\mathcal{U}^0) \\ \downarrow j_n & & \downarrow J_n \\ \coprod_{\lambda\in\Lambda^{(n+1)}} \pi_1(X_\nu \times \Delta^n, X_\nu \times (\Delta^n)^0) & \xrightarrow{K_n} & \pi_1(B\mathcal{U}^n, B\mathcal{U}^0). \end{array}$$

Proof. By [14; Theorem 2] each groupoid in the diagram is a retract of the corresponding groupoid in 5.1. The corollary now follows from Brown'sresult [6; 6.6.7] that a retract of a pushout is a pushout.

□

Razak defines certain elements $\phi_{\lambda\mu}(x) \in \pi_1(B\mathcal{U}^1, B\mathcal{U}^0)$ for $x \in X_\lambda \cap X_\mu$, $(\lambda,\mu) \in \Lambda^{(2)}$. $\phi_{\lambda\mu}(x)$ is the class of the map $I \longrightarrow B\mathcal{U}^1$ given by $t \longmapsto K_1(x,\lambda,\mu,t)$ and has initial vertex (x,λ), final vertex (x,μ). We extend the definition to all $(\lambda,\mu) \in \Lambda^2$ by defining $\phi_{\mu\lambda}(x) = -\phi_{\lambda\mu}(x)$ for $\lambda \leqslant \mu$. The element $J_2\phi_{\lambda\mu}(x) \in \pi_1(B\mathcal{U}^2, B\mathcal{U}^0)$ will generally be denoted $\phi_{\lambda\mu}(x)$ also.

5.3 LEMMA. $\pi_1(B\mathcal{U}^1, B\mathcal{U}^0)$ is generated by $J_1\pi_1(B\mathcal{U}^0)$ and the elements $\phi_{\lambda\mu}(x)$ for $x \in X_{\lambda\mu} = X_\lambda \cap X_\mu$.

Proof. From corollary 5.2 with n=1 it follows that $\pi_1(B\mathcal{U}^1, B\mathcal{U}^0)$ is generated by the images of J_1 and K_1. Identify $\pi_1(X_\nu \times \Delta^1, X_\nu \times \partial\Delta^1)$ with $\pi_1(X_\nu) \times \mathcal{I}$ where $\mathcal{I}$ is the unit interval groupoid generated by $\gamma \in \mathcal{I}_1(0,1)$. Then

$\pi_1(X_\nu) \times \mathcal{J}$ is generated by elements of the form $(\alpha, 0)$ and $(\varepsilon x, \gamma)$ where $\alpha \in \pi_1(X_\nu)$ and $x \in X_\nu$. To emphasise the role of $\nu = (\lambda, \mu) \in \Lambda^2$ we shall usually write $(\alpha, 0)$ as $(\alpha, \nu, 0)$ or $(\alpha, \lambda, \mu, 0)$ and similarly for $(\varepsilon x, \gamma)$.

Under the above identification K_1 maps $(\varepsilon x, \nu, \gamma)$ to $\phi_{\lambda\mu}(x)$ and $(\alpha, \nu, 0)$ to $i_\lambda(\alpha)$ where $i_\lambda : \pi_1 X_\lambda \hookrightarrow \pi_1(B\mathcal{U}^0) \xrightarrow{J_1} \pi_1(B\mathcal{U}^1, B\mathcal{U}^0)$. Hence $i_\lambda(\alpha) \in J_1\pi_1(B\mathcal{U}^0)$ so the images of J_1 and K_1 are precisely the stated generators.

□

Razak showed [18; p74] that for $x \in X_{\lambda\mu\eta} = X_\lambda \cap X_\mu \cap X_\eta$ the relation $\phi_{\lambda\mu}(x) + \phi_{\mu\eta}(x) = \phi_{\lambda\eta}(x)$ holds in $\pi_1(B\mathcal{U}^2, B\mathcal{U}^0)$ but not in $\pi_1(B\mathcal{U}^1, B\mathcal{U}^0)$. Indeed these relations are the essential difference between the two groupoids as the next result shows.

<u>5.4 PROPOSITION.</u> $J_2 : \pi_1(B\mathcal{U}^1, B\mathcal{U}^0) \longrightarrow \pi_1(B\mathcal{U}^2, B\mathcal{U}^0)$ is a quotient morphism whose kernel is the normal subgroupoid R generated by elements of the form $\phi_{\lambda\mu}(x) + \phi_{\mu\eta}(x) + \phi_{\eta\lambda}(x)$ for $x \in X_{\lambda\mu\eta}$.

<u>Outline proof.</u> We shall omit some of the details since they are routine and similar arguments are given in greater detail later.

J_2 is the identity on vertices. From the pushout of corollary 5.2 with n=2, and the fact that Δ^2 is simply connected, it follows that J_2 is surjective. Hence it is piecewise surjective and therefore a quotient morphism by proposition 2.3.

We have noted that Razak's work shows that $R \subseteq \ker J_2$, so there is an induced morphism

$$J_2^* : \pi_1(B\mathcal{U}^1, B\mathcal{U}^0)/R \longrightarrow \pi_1(B\mathcal{U}^2, B\mathcal{U}^0).$$

The pushout of corollary 5.2 (again with n=2) can now be used to construct the inverse isomorphism to J_2^*.

□

Finally in this section we recall some facts about the nerve of the cover $\mathcal{U}$. The <u>nerve</u> $K = K\mathcal{U}$ is an (abstract) simplicial complex whose n-simplexes are (n+1)-tuples $(\lambda_0, \ldots, \lambda_n)$ where $X_{\lambda_0 \ldots \lambda_n} = X_{\lambda_0} \cap \ldots \cap X_{\lambda_n} \neq \emptyset$.

By proposition 4.8 the fundamental groupoid $\pi_1(K, K^0)$ (or $\pi_1(K^2, K^0)$) has vertex set K^0, generators $[\lambda, \mu]$ where $X_{\lambda\mu} \neq \emptyset$ and relations $[\lambda, \mu] + [\mu, \eta] = [\lambda, \eta]$ where $X_{\lambda\mu\eta} \neq \emptyset$.

Also the map $N\mathcal{U} \to K$, $(x, \lambda_0, \ldots, \lambda_n) \mapsto (\lambda_0, \ldots, \lambda_n)$ induces a morphism $\pi_1(B\mathcal{U}^2, B\mathcal{U}^0) \to \pi_1(K, K^0)$, where $\pi_1(K, K^0)$ is identified with $\pi_1(|K|, |K^0|)$. But this is the subject of the next section.

2. RAZAK'S CONJECTURE

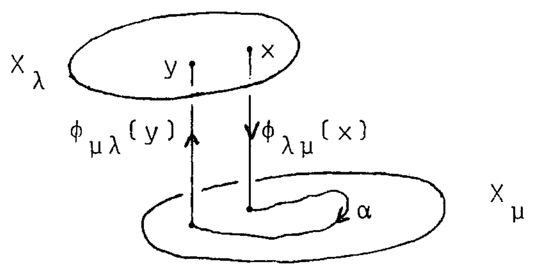
Let $\mathcal{U} = \{X_\lambda \mid \lambda \in \Lambda\}$ be a cover of the space X such that the interiors of the sets X_λ cover X. There is a simplicial map $f : N\mathcal{U} \to K$, given by $(x, \lambda_0, \ldots, \lambda_n) \mapsto (\lambda_0, \ldots, \lambda_n)$. Its realisation $B\mathcal{U} \to |K|$ which we also denote by f induces a groupoid morphism $f_* : \pi_1(B\mathcal{U}^2, B\mathcal{U}^0) \to \pi_1(K, K^0)$. Here $\pi_1(K, K^0)$ is the edge-path fundamental groupoid which is isomorphic to $\pi_1(|K|, |K^0|)$. Razak shows that f_* is surjective — a generator of $\pi_1(K, K^0)$ is $[\lambda, \mu]$ where $X_\lambda \cap X_\mu \neq \emptyset$, and $f_* \phi_{\lambda\mu}(x) = [\lambda, \mu]$ for any $x \in X_\lambda \cap X_\mu$. He makes the following conjecture [18; p89].

5.5 CONJECTURE (Razak). Let $N = \ker f_*$. Then N is generated as a normal subgroupoid of $\pi_1 X$ by the elements represented by a path in at least one of the sets $X_\lambda \cup X_\mu$. □

First note that the conjecture is not well-posed as it stands. N cannot be regarded as a subgroupoid of $\pi_1(X)$ as the vertex set is not correct. Note also that f_* is a quotient morphism if and only if each subset X_λ is path connected. This follows from the characterisation of quotient morphisms given in proposition 2.3. We make this additional assumption. Razak did not assume this, and he appears to use a weaker notion of exactness to ours (definition 2.5). The reason for makingthe assumption is that we then have at our disposal the universal property of quotient morphisms (definition 2.2). The following result may be regarded as a modified form of the conjecture.

5.6 THEOREM. Suppose the sets $X_\lambda \in \mathcal{U}$ are path-connected. Then $N = \ker f_*$ is generated as a normal subgroupoid of $\pi_1(B\mathcal{U}^2, B\mathcal{U}^0)$ by $\pi_1(B\mathcal{U}^0)$ and elements of the form $\phi_{\lambda\mu}(x) + i_\mu(\alpha) + \phi_{\mu\lambda}(y)$, where $x,y \in X_{\lambda\mu}$, $\alpha \in \pi_1(X_\mu)(x,y)$ and $i_\mu : \pi_1 X_\mu \hookrightarrow \pi_1(B\mathcal{U}^0) \xrightarrow{J_2 J_1} \pi_1(B\mathcal{U}^2, B\mathcal{U}^0)$.

Note. In passing to $\pi_1(B\mathcal{U}^2, B\mathcal{U}^0)/N$, the elements of $\pi_1(B\mathcal{U}^0)$ have the effect of killing the homotopy which comes from the X_λ's, and the other generators of N identify $\phi_{\lambda\mu}(x)$ with $\phi_{\lambda\mu}(y)$ for $x,y \in X_{\lambda\mu}$. A picture of these elements is the following.



Proof. Let M be the normal subgroupoid generated by $\pi_1(B\mathcal{U}^0)$ and the elements $\phi_{\lambda\mu}(x) + i_\mu(\alpha) + \phi_{\mu\lambda}(y)$. Since f_* is a quotient morphism it will be sufficient to establish the isomorphism

$$\pi_1(B\mathcal{U}^2, B\mathcal{U}^0)/M \cong \pi_1(K, K^0).$$

We first show $M \subseteq N = \ker f_*$. If $\alpha \in \pi_1(X_\mu)$ then clearly $f_* i_\mu(\alpha) = \varepsilon\mu$. Also $f_*\phi_{\lambda\mu}(x) = [\lambda, \mu]$ so

$$f_*\big(\phi_{\lambda\mu}(x) + i_\mu(\alpha) + \phi_{\mu\lambda}(y)\big) = [\lambda,\mu] + \varepsilon\mu + [\mu,\lambda] = \varepsilon\lambda.$$

Hence $M \subseteq N$, so f_* induces a unique morphism ψ such that the following diagram commutes (where τ is the natural morphism).

$$\begin{array}{ccc} \pi_1(B\mathcal{U}^2,\ B\mathcal{U}^0) & \xrightarrow{\ \tau\ } & \pi_1(B\mathcal{U}^2,\ B\mathcal{U}^0)/M \\ & {\scriptstyle f_*}\searrow \quad \swarrow{\scriptstyle \psi} & \\ & \pi_1(K,\ K^0). & \end{array}$$

We show that ψ is an isomorphism by constructing its inverse θ. Recall that $\pi_1(K,K^0)$ has generators $[\lambda,\mu]$ where $X_{\lambda\mu} \neq \emptyset$ and relations $[\lambda,\mu] + [\mu,\eta] = [\lambda,\eta]$ where $X_{\lambda\mu\eta} \neq \emptyset$. Define θ on vertices by $\theta(\lambda) = \tau(x,\lambda)$ where $x \in X_\lambda$, and on generators by $\theta[\lambda,\mu] = \tau\phi_{\lambda\mu}(x)$ for $x \in X_{\lambda\mu}$. To prove that θ is well-defined we must show (i) that the definition is independent of the choice of the x's and (ii) that θ preserves the relations of $\pi_1(K,K^0)$.

(i) If $x,y \in X_\lambda$ then there is a path in X_λ from x to y so $\tau(x,\lambda) = \tau(y,\lambda)$. If $x,y \in X_{\lambda\mu}$ then there is an element $\alpha \in (\pi_1 X_\mu)(x,y)$ so

$$\begin{aligned} \tau\phi_{\lambda\mu}(x) &= \tau\left(\phi_{\lambda\mu}(x) + i_\mu(\alpha) + \phi_{\mu\lambda}(y)\right) + \tau\phi_{\lambda\mu}(y) \\ &= \tau\phi_{\lambda\mu}(x). \end{aligned}$$

Therefore the definition of θ is independent of the choices made.

(ii) By proposition 5.4 the relation $\phi_{\lambda\mu}(x) + \phi_{\mu\eta}(x) = \phi_{\lambda\eta}(x)$ holds in $\pi_1(B\mathcal{U}^2,\ B\mathcal{U}^0)$ for $x \in X_{\lambda\mu\eta}$. Therefore θ preserves the relations of $\pi_1(K,K^0)$.

Now $\psi\theta[\lambda,\mu] = \psi\tau\phi_{\lambda\mu}(x) = f_*\phi_{\lambda\mu}(x) = [\lambda,\mu]$, so $\psi\theta$ is the identity on $\pi_1(K,K^0)$. To show $\theta\psi$ is also the identity recall lemma 5.3 and proposition 5.4 which imply that $\pi_1(B\mathcal{U}^2, B\mathcal{U}^0)$ is generated by $i_\lambda(\alpha)$ for $\alpha \in \pi_1(X_\lambda)$ and $\phi_{\lambda\mu}(x)$ for $x \in X_{\lambda\mu}$. Now

$$\theta f_* i_\lambda(\alpha) = \theta\varepsilon\lambda = \varepsilon\tau(x,\lambda) = \tau i_\lambda(\alpha) \text{ , and}$$

$$\theta f_*\phi_{\lambda\mu}(x) = \theta[\lambda,\mu] = \upsilon\phi_{\lambda\mu}(x).$$

Therefore $\theta \circ f_* = \tau$. That is $\theta\psi\tau = \tau$, so $\theta\psi$ is the identity on $\pi_1(B\mathcal{U}^2, B\mathcal{U}^0)/M$. Thus θ and ψ are inverse isomorphisms, which completes the proof.

□

To explain why theorem 5.6 is a modification of Razak's conjecture recall that there is a natural map $p : B\mathcal{U} \longrightarrow X$ which induces $p_* : \pi_1(B\mathcal{U}^2, B\mathcal{U}^0) \longrightarrow \pi_1(X)$. Now $\ker f_*$ is generated by elements of the form $i_\lambda(\alpha) + \phi_{\lambda\mu}(x) + i_\mu(\beta) + \phi_{\mu\lambda}(y)$. Under p_* such an element maps to $\alpha+\beta \in \pi_1(X)$, which has a representative in $X_\lambda \cup X_\mu$. It should be noted that the elements of $\pi_1(X)$ arising in this way have end points in the same X_λ. Conversely, any element of $\pi_1(X)$ with a representative in some $X_\lambda \cup X_\mu$ is the projection under p_* of an element of $\ker f_*$ provided it has end points in the same X_λ.

3. CONNECTIONS WITH THE 1-DIMENSIONAL UNION THEOREM AND GROUPOID MAPPING CYLINDER

Let X be a space with a cover $\mathcal{U} = \{X_\lambda \mid \lambda \in \Lambda\}$ and let $B\mathcal{U}$ denote the classifying space. In this section we consider the morphism $p_* : \pi_1(B\mathcal{U}^1, B\mathcal{U}^0) \longrightarrow \pi_1(X)$ induced by the projection $p : B\mathcal{U} \longrightarrow X$ defined in section one. We prove that p_* is a quotient morphism and give its kernel. This result is very close to the 1-dimensional union theorem — its proof uses the union theorem. The connection between the two results is given by the groupoid mapping cylinder construction of Crowell and Smythe [13].

The following result follows from the one-dimensional union theorem (see [9], [18]).

5.7 THEOREM. Let $\mathcal{U} = \{X_\lambda \mid \lambda \in \Lambda\}$ be a cover of X such that the interiors of the elements of $\mathcal{U}$ cover X. Then the diagram

$$\bigsqcup_{\nu \in \Lambda^2} \pi_1(X_\nu) \overset{a}{\underset{b}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} \pi_1(X_\lambda) \xrightarrow{c} \pi_1(X)$$

is a coequaliser in $\mathcal{G}pd$, where a and b are induced by the inclusions $X_\nu = X_{\lambda\mu} \hookrightarrow X_\lambda$, $X_\nu \hookrightarrow X_\mu$ respectively, and c is induced by the inclusions $X_\lambda \hookrightarrow X$.

□

The main result of this section is the following. We now assume that the interiors of the members of $\mathcal{U}$ cover X.

5.8 THEOREM. The morphism $p_* : \pi_1(B\mathcal{U}^1, B\mathcal{U}^0) \longrightarrow \pi_1(X)$ defined previously is a quotient morphism whose kernel is the normal subgroupoid N generated by the elements $\phi_{\lambda\mu}(x)$ for $x \in X_{\lambda\mu}$, $(\lambda,\mu) \in \Lambda^2$.

Proof. First note that using the pushout of corollary 5.2 with $n=1$, p_* can be defined by the diagram

$$\begin{array}{ccc}
\bigsqcup_{\nu\in\Lambda^2} \pi_1(X_\nu \times \partial\Delta^1) & \xrightarrow{k_1} & \pi_1(B\mathcal{U}^0) \\
\downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle f} \\
\bigsqcup_{\nu\in\Lambda^2} \pi_1(X_\nu \times \Delta^1, X_\nu \times \partial\Delta^1) & \xrightarrow[g]{} & \pi_1(X)
\end{array}$$

where f is induced by inclusions $X_\lambda \hookrightarrow X$, and with the identification $\pi_1(X_\nu \times \Delta^1, X_\nu \times \partial\Delta^1) \cong \pi_1(X_\nu) \times \mathcal{J}$ (see lemma 5.3), g is defined by $g(\varepsilon x,\nu,\gamma) = \varepsilon x$, $g(\alpha,\nu,0) = \alpha$. (Strictly, $g(\alpha,\nu,0) = ca(\alpha) = cb(\alpha)$, where a,b,c are given in theorem 5.7).

Let N be as stated. Then $N \subseteq \ker p_*$ since $p_*\phi_{\lambda\mu}(x) = g(\varepsilon x,\lambda,\mu,0) = \varepsilon x$.

The next step is to prove that p_* is a quotient morphism. We use the characterisation of quotient morphisms given in proposition 2.3. Let $\alpha \in \pi_1(X)(x,y)$ — say $\alpha = [\ell]$ for some path ℓ in X. Now ℓ has a subdivision $\ell = \ell_1 + \ldots + \ell_n$ such that ℓ_i maps the subinterval $[t_i, t_{i+1}]$ into X_{λ_i} for some $\lambda_i \in \Lambda$. Define

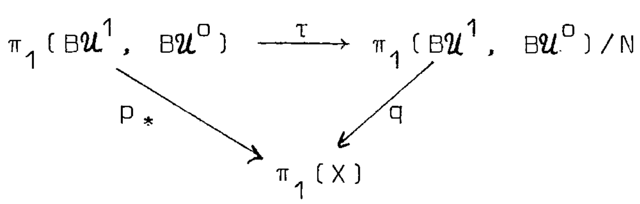
$\hat{\alpha} \in \pi_1(B\mathcal{U}^1, B\mathcal{U}^0)$ by

$$\hat{\alpha} = [\ell_1] + \phi_{\lambda_1\lambda_2}(\partial^1\ell_1) + [\ell_2] + \dots + \phi_{\lambda_{n-1}\lambda_n}(\partial^1\ell_{n-1}) + [\ell_n].$$

Now $p_*(\hat{\alpha}) = [\ell_1] + \dots + [\ell_n] = \alpha$, so p_* is surjective.

Suppose $v, w \in B\mathcal{U}^0$ are such that $p_*v = p_*w$. Then $v = (x,\lambda)$ and $w = (x,\mu)$ for some $x \in X_{\lambda\mu}$. However $\phi_{\lambda\mu}(x) \in N_1(v,w) \subseteq (\ker p_*)(v,w)$ so v and w lie in the same component of the kernel. Therefore, by proposition 2.3, p_* is a quotient morphism.

Since $N \subseteq \ker p_*$ there is a unique morphism q such that the following diagram commutes (where τ is the natural projection).



To show that q is an isomorphism (i.e. $\ker p_* = N$) we use theorem 5.7 to construct its inverse. Define

$$r : \bigsqcup_{\lambda \in \Lambda} \pi_1(X_\lambda) \longrightarrow \pi_1(B\mathcal{U}^1, B\mathcal{U}^0)/N$$

by $r(\alpha,\lambda) = \tau i_\lambda(\alpha)$ (where $i_\lambda : \pi_1(X_\lambda) \hookrightarrow \pi_1(B\mathcal{U}^0) \xrightarrow{J_1} \pi_1(B\mathcal{U}^1, B\mathcal{U}^0)$). Let $(\alpha,\lambda,\mu) \in \pi_1(X_{\lambda\mu})$. Then $ra(\alpha,\lambda,\mu) = r(\alpha,\lambda) = \tau i_\lambda(\alpha)$, and similarly $rb(\alpha,\lambda,\mu) = \tau i_\mu(\alpha)$. Now

Razak notes [18; p77] that $(\alpha,\lambda) = \phi_{\lambda\mu}(\partial^0\alpha) + (\alpha,\mu) + \phi_{\mu\lambda}(\partial^1\alpha)$. Hence $\tau i_\lambda(\alpha) = \tau i_\mu(\alpha)$ so $ra = rb$. Therefore, by theorem 5.7, r induces a morphism

$$r^* : \pi_1(X) \longrightarrow \pi_1(B\mathcal{U}^1, B\mathcal{U}^0)/N$$

such that $r^*c = r$. We claim that q and r^* are inverse isomorphisms.

First note that for $(\alpha,\lambda) \in \pi_1(X_\lambda)$

$$qr(\alpha,\lambda) = q\tau i_\lambda(\alpha) = p_* i_\lambda(\alpha) = \alpha = c(\alpha,\lambda).$$

So $qr = c$ and hence $qr^*c = c$. Therefore $qr^* = id_{\pi_1(X)}$.

Also $r^*p_*i_\lambda(\alpha) = r^*(\alpha) = r(\alpha,\lambda) = \tau i_\lambda(\alpha)$, and
$r^*p_*\phi_{\lambda\mu}(x) = r^*\varepsilon x = \tau\phi_{\lambda\mu}(x)$.

By lemma 5.3 the elements of the form $i_\lambda(\alpha)$ for $\alpha \in \pi_1(X_\lambda)$, and $\phi_{\lambda\mu}(x)$ for $x \in X_{\lambda\mu}$ generate $\pi_1(B\mathcal{U}^1, B\mathcal{U}^0)$. Hence $r^*p_* = \tau$ so $r^*q = id$.

Therefore p and r^* are inverse isomorphisms, completing the proof.

□

<u>5.9 COROLLARY</u> (Razak). The morphism $p'_* : \pi_1(B\mathcal{U}^2, B\mathcal{U}^0) \to \pi_1(X)$ induced by $p : B\mathcal{U} \longrightarrow X$ is an equivalence of groupoids.

Proof. There is a commutative diagram

$$\begin{array}{ccc} \pi_1(B\mathcal{U}^1, B\mathcal{U}^0) & \xrightarrow{J_2} & \pi_1(B\mathcal{U}^2, B\mathcal{U}^0) \\ & {\scriptstyle p_*}\searrow \quad \swarrow{\scriptstyle p'_*} & \\ & \pi_1(X) & \end{array}$$

where J_2 and p_* are quotient morphisms. It follows that p'_* is a quotient morphism whose kernel is the normal subgroupoid generated by the elements $\phi_{\lambda\mu}(x)$, for $x \in X_{\lambda\mu}$. Hence the components of $\pi_1(B\mathcal{U}^2, B\mathcal{U}^0)$ and $\pi_1(X)$ are in bijective correspondence (given by p'_*). Now the kernel of p'_* is simply connected since the relation $\phi_{\lambda\mu}(x) + \phi_{\mu\eta}(x) + \phi_{\eta\lambda}(x)$ holds in $\pi_1(B\mathcal{U}^2, B\mathcal{U}^0)$. Therefore the vertex groups of the corresponding components of $\pi_1(B\mathcal{U}^2, B\mathcal{U}^0)$ and $\pi_1(X)$ are isomorphic. A result of Higgins [14; Corollary 2, p47] completes the proof.

□

Note that in general the kernel of $p_* : \pi_1(B\mathcal{U}^1, B\mathcal{U}^0) \longrightarrow \pi_1(X)$ is not simply connected since it contains the non-trivial loop $\phi_{\lambda\mu}(x) + \phi_{\mu\eta}(x) + \phi_{\eta\lambda}(x)$ where $x \in X_{\lambda\mu\eta}$. We feel that the above route to corollary 5.9 is slightly simpler than Razak's proof. By passing to the quotient groupoids the choices involved in defining a morphism $\pi_1(X) \longrightarrow \pi_1(B\mathcal{U}^2, B\mathcal{U}^0)$ can be avoided. This is, of course, largely a matter of presentation — the ideas behind our approach are the same as Razak's.

We now investigate the connection between theorems 5.7 and 5.8.

We show that $\pi_1(B\mathcal{U}^1, B\mathcal{U}^0)$ is essentially a mapping cylinder groupoid as defined by Crowell and Smythe in [13]. The following generator-relation description of the mapping cylinder construction is somewhat different to that given in [13], and is introduced to avoid the consideration of equivalence relations on groupoids which are not congruences.

Let $\Gamma = (\Gamma_1, \Gamma_0)$ be a directed graph. We adopt our notation for groupoids and let $\partial^0 x$, $\partial^1 x$ denote the initial and final vertices respectively of the edge $x \in \Gamma_1$. A <u>Γ-diagram of groupoids</u> $(\mathcal{A}, \Gamma)$ consists of a family $\mathcal{A} = \{A^v \mid v \in \Gamma_0\}$ of groupoids, and a family $\{\theta_x : A^{\partial^0 x} \longrightarrow A^{\partial^1 x} \mid x \in \Gamma_1\}$ of groupoid morphisms. (More formally, a Γ-diagram of groupoids is a functor $F(\Gamma) \longrightarrow \mathcal{G}pd$, where $F(\Gamma)$ denotes the free category on Γ).

<u>5.10 DEFINITION</u>. The <u>mapping cylinder groupoid</u> $m = m(\mathcal{A}, \Gamma)$ of a Γ-diagram of groupoids is the groupoid with vertex set $\bigsqcup_{v \in \Gamma_0} (A^v)_0$, generators

(i) the elements of A^v for $v \in \Gamma_0$, and

(ii) elements $\gamma_x(w) \in m_1\big((w, \partial^0 x), (\theta_x w, \partial^1 x)\big)$ for $w \in \big(A^{\partial^0 x}\big)_0$,

and defining relations

(a) the relations of the groupoids A^v, $v \in \Gamma_0$, and

(b) $\theta_x(a) = -\gamma_x(\partial^0 a) + a + \gamma_x(\partial^1 a)$ for $a \in A^{\partial^0 x}$, $x \in \Gamma_1$.

□

Presentations of groupoids have been considered by Higgins — see [14], for example. According to Higgins' work relations are ordered pairs of groupoid words, so our relations of type (b) should be written $\left(\theta_x(a),\ -\gamma_x(\partial^0 a) + a + \gamma_x(\partial^1 a)\right)$. In this case the initial and final vertices of the words are equal, which is why we can write the relations in the more familiar form.

One can verify directly that this description gives a groupoid isomorphic to that of Crowell and Smythe. We omit this technical argument since the reason the mapping cylinder has been introduced here is the following result, due to Crowell and Smythe, which we prove directly.

<u>5.11 THEOREM</u>. Let $(\mathcal{A}, \Gamma)$ be a diagram of groupoids and $\varinjlim(\mathcal{A}, \Gamma)$ be its colimit in $\mathcal{G}$pd. Then

$$\varinjlim(\mathcal{A}, \Gamma) \cong m(\mathcal{A}, \Gamma)/P$$

where P is the normal subgroupoid of $m(\mathcal{A}, \Gamma)$ generated by the elements $\gamma_x(w)$ for $w \in \left(A^{\partial^0 x}\right)_0$, $x \in \Gamma_1$.

<u>Note</u>. In the terminology of Crowell and Smythe P is the normal closure in $m(\mathcal{A}, \Gamma)$ of the subgroupoid they called the <u>core</u>.

<u>Proof</u>. From the presentation of $m(\mathcal{A}, \Gamma)$ we see that the quotient $m(\mathcal{A}, \Gamma)/P$ has a presentation with generators

the elements of A^v for $v \in \Gamma_0$, and relations $(\theta_x(a), a)$ for $a \in A^{\partial^0 x}$, $x \in \Gamma_1$ together with the relations of A^v for $v \in \Gamma_0$.

We use this presentation to show that m/P has the universal property characterising $\underrightarrow{\lim}(\mathcal{A}, \Gamma)$. Let Δ be the elements of A^v considered as a generating graph for m/P and let $\tau : \Delta \longrightarrow m/P$ be the natural morphism.

Let $\psi^v : A^v \longrightarrow B$, $v \in \Gamma_0$, be a family of groupoid morphisms commuting with the morphisms $\theta^x : A^{\partial^0 x} \longrightarrow A^{\partial^1 x}$. Then $\psi = \{\psi^v\} : \Delta \longrightarrow B$ is a graph map under which the relations of m/P hold. Hence there is a unique morphism $\psi_* : m/P \longrightarrow B$ such that $\psi_* \circ \tau = \psi$. This is precisely the universal property characterising $\underrightarrow{\lim}(\mathcal{A}, \Gamma)$.

□

Now consider the diagram

$$\bigsqcup_{\nu \in \Lambda^2} \pi_1(X_\nu) \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} \pi_1(X_\lambda)$$

given in theorem 5.7. Let Γ be the 1-skeleton of the nerve $K\mathcal{U}$ of the cover $\mathcal{U} = \{X_\lambda \mid \lambda \in \Lambda\}$, subdivided once. Then Γ is a graph with vertex set $\Lambda \cup \Lambda^{(2)}$, and with edges connecting $(\lambda, \mu) \in \Lambda^{(2)}$ to each of the vertices $\lambda, \mu \in \Lambda$. Directing the edges from (λ, μ) to λ and from (λ, μ) to μ, we see that the above diagram is a Γ-diagram of groupoids. We denote it by (π, Γ).

<u>5.12 PROPOSITION.</u> $\pi_1(B\mathcal{U}^1, B\mathcal{U}^0)$ is isomorphic to the full subgroupoid of $m(\pi,\Gamma)$, the mapping cylinder of (π,Γ), on the vertex set $\bigsqcup_{\lambda\in\Lambda} X_\lambda$.

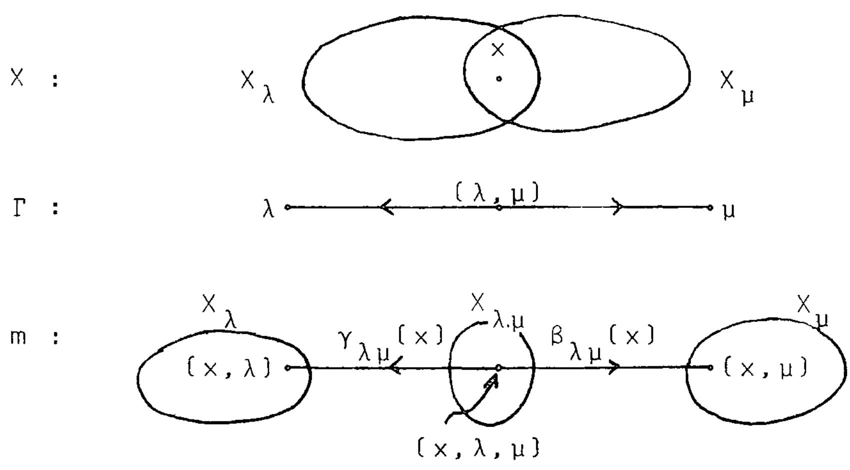
Further, under the isomorphism the normal subgroupoid N of $\pi_1(B\mathcal{U}^1, B\mathcal{U}^0)$ generated by the elements $\phi_{\lambda\mu}(x)$ is isomorphic to the corresponding full subgroupoid P of $m(\pi,\Gamma)$ given in theorem 5.11.

<u>Proof</u>. Let $\hat{m} = \hat{m}(\pi,\Gamma)$ be the full subgroupoid of $m = m(\pi,\Gamma)$ on the vertex set $\bigsqcup_{\lambda\in\Lambda} X_\lambda$, and let $\hat{P}$ be the corresponding full subgroupoid of P.

Now m has generators the elements of $\pi_1(X_\lambda)$, $\lambda\in\Lambda$, the elements of $\pi_1(X_{\lambda\mu})$, $(\lambda,\mu)\in\Lambda^{(2)}$, and elements

$$\gamma_{\lambda\mu}(x)\in m_1\big((x,\lambda,\mu),(x,\lambda)\big)\ ,\quad \beta_{\lambda\mu}(x)\in m_1\big((x,\lambda,\mu),(x,\mu)\big).$$

A diagram for these elements is the following.



The relations of m are those of the $\pi_1(X_\lambda)$ and $\pi_1(X_{\lambda\mu})$, and

$$\text{(i)} \quad (\alpha,\lambda) = -\gamma_{\lambda\mu}(x) + (\alpha,\lambda,\mu) + \gamma_{\lambda\mu}(y)$$

$$\text{(ii)} \quad (\alpha,\mu) = -\beta_{\lambda\mu}(x) + (\alpha,\lambda,\mu) + \beta_{\lambda\mu}(y)$$

for $\alpha \in \pi_1(X_{\lambda\mu})(x,y)$.

The subgroupoid $\hat{m}$ is a retract of m [14; Theorem 2, p47]. For $\alpha \in m_1(v,w)$ a retraction $\rho : m \to \hat{m}$ is given by

$$\rho_1 : \alpha \mapsto \begin{cases} \alpha & \text{if } v,w \in \hat{m}_0 \\ \alpha + \gamma_{\eta\xi}(y) & \text{if } v \in \hat{m}_0,\ w = (y,\eta,\xi) \\ -\gamma_{\lambda\mu}(x) + \alpha & \text{if } v = (x,\lambda,\mu),\ w \in \hat{m}_0 \\ -\gamma_{\lambda\mu}(x) + \alpha + \gamma_{\eta\xi}(y) & \text{if } v = (x,\lambda,\mu),\ w = (y,\eta,\xi). \end{cases}$$

By a result of Higgins [14; Proposition 28, p91] ρ is a quotient morphism with simply connected kernel. Hence adding the relations $\left(\gamma_{\lambda\mu}(x),\ \varepsilon(x,\lambda)\right)$ to those above gives a presentation for $\hat{m}$. This presentation has generators the elements of the $\pi_1(X_\lambda)$'s and elements $\beta_{\lambda\mu}(x) \in \hat{m}_1\left((x,\lambda),(x,\mu)\right)$ The relations are those of the $\pi_1(X_\lambda)$'s and

$$(\alpha,\mu) = -\beta_{\lambda\mu}(x) + (\alpha,\lambda) + \beta_{\lambda\mu}(y) \quad \text{for } \alpha \in \pi_1(X_{\lambda\mu})(x,y).$$

In particular, $\hat{m}$ is itself a mapping cylinder groupoid.

We can now define a morphism $\theta : \hat{m}(\pi,\Gamma) \longrightarrow \pi_1(B\mathcal{U}^1,\ B\mathcal{U}^0)$ by $i_\lambda : \pi_1(X_\lambda) \hookrightarrow \pi_1(B\mathcal{U}^0) \xrightarrow{J_1} \pi_1(B\mathcal{U}^1,\ B\mathcal{U}^0)$ and

$\beta_{\lambda\mu}(x) \mapsto \phi_{\lambda\mu}(x)$. That the relations hold in $\pi_1(B\mathcal{U}^1, B\mathcal{U}^0)$ under θ follows from a remark of Razak [18; p77]. Therefore θ is well-defined, and it easily seen to be an isomorphism by using corollary 5.2 for example.

Finally $\hat{P}$ is generated by the elements $\beta_{\lambda\mu}(x)$ so clearly $\theta(\hat{P}) = N$.

□

5.13 COROLLARY. $\pi_1(B\mathcal{U}^1, B\mathcal{U}^0)/N \cong \varinjlim(\pi,\Gamma)$ where N is the normal subgroupoid defined in 5.8.

Proof. The corollary follows from 5.11 and5.12 since comparing the presentations given in the previous proposition shows $m/P \cong \hat{m}/\hat{P}$.

□

The corollary indicates that the one dimensional union theorem (5.7) and theorem 5.8 are essentially equivalent statements.

4. CONNECTIONS WITH MACBEATH-SWAN THEORY

In [17] Macbeath considers coverings of a G-space X which are the translates, under the action, of a subspace V. Let $\mathcal{U} = \{V^g \mid g \in G\}$ be such a covering, indexed by the elements of G. Macbeath defined a group, which we denote $M = M(\mathcal{U})$, to have generators (g) where $V \cap V^g \neq \emptyset$, $g \in G$ and relations $(g)(h) = (gh)$ where $V \cap V^h \cap V^{gh} \neq \emptyset$. (In fact we have modified the definition to convert from left actions, which Macbeath considered, to right actions).

Define $\phi : M \to G$ by $\phi : (g) \mapsto g$. Macbeath proved the following [17; Theorem 1].

<u>5.14 THEOREM</u>. Suppose V is open in X. Then

(i) ϕ is surjective if X is connected, and

(ii) ϕ is an isomorphism if X is connected and simply connected, and V is path-connected.

□

There is a generalisation of part (ii) of the theorem, due to Swan [24], which deals with non-simply connected X. The statement of Swan's result involves a morphism $\theta : \pi_1(X,*) \longrightarrow M$ which we now describe. Let $*$ be a base point in V, and let $\alpha \in \pi_1(X,*)$. Then α is represented by a loop ℓ which may be subdivided $\ell = \ell_1 + \ldots + \ell_n$ such that each path ℓ_i is contained in V^{g_i} for some $g_i \in G$. We always choose $g_1 = g_n = 1$. Define
$\theta(\alpha) = (g_1 g_2^{-1})(g_2 g_3^{-1}) \ldots (g_{n-1} g_n^{-1})$.

<u>5.15 THEOREM</u> (Swan). Suppose X is path-connected and V is open and path-connected. Then

(i) the above definition of θ is independent of the choices made and gives a well-defined morphism $\theta : \pi_1(X,*) \longrightarrow G$, and

(ii) there is an exact sequence of groups

$$0 \longrightarrow P \hookrightarrow \pi_1(X,*) \xrightarrow{\theta} M \xrightarrow{\phi} G \longrightarrow 0$$

where P is the normal subgroup of $\pi_1(X,*)$ generated by those elements which have representatives of the form $p+\ell-p$

where p is a path with initial point $*$ and ℓ is a loop contained in $V^g \cup V^h$ for some $g,h \in G$.

Proof. (i) [24; Lemmas 1.3 and 1.4]

(ii) [24; Theorem 1.1].

□

We will show that the work of section two together with the notion of semi-direct product groupoid gives a groupoid version of Swan's result. This approach is not new. Razak showed that Swan's theorem was closely related to his conjecture, although the statement of Razak's theorem 3.2 [18; p91] is not quite correct as we shall explain. Also Abels [1] used semi-direct product groupoids to prove a result similar to our next theorem.

Let X be a path-connected G-space and let $\mathcal{U} = \{X_\lambda \mid \lambda \in \Lambda\}$ be a G-invariant cover of X. (That is, for $g \in G$, $\lambda \in \Lambda$ we have $X_\lambda{}^g \in \mathcal{U}$). Then G acts on the nerve $K = K\mathcal{U}$ and on the classifying space $B\mathcal{U}$ of the cover, and induces actions on the corresponding fundamental groupoids.

5.16 THEOREM. Let X be a path-connected G-space and $\mathcal{U} = \{X_\lambda \mid \lambda \in \Lambda\}$ a G-invariant cover by path-connected open sets. The following is a commutative diagram of groupoids with exact rows and column (where N and f_* are given in theorem 5.6).

$$\begin{array}{ccccccccc}
 & & E_{B\mathcal{U}^0} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & N & & & & & & \\
 & & \cap & & & & & & \\
 & & \downarrow & & & & & & \\
E_{B\mathcal{U}^0} & \longrightarrow & \pi_1(B\mathcal{U}^2,\ B\mathcal{U}^0) & \xrightarrow{i} & \pi_1(B\mathcal{U}^2,\ B\mathcal{U}^0)\ \tilde{\times}\ G & \longrightarrow & G & \longrightarrow & E_* \\
 & & \downarrow f_* & & \downarrow f_*\times \mathrm{id} & & \downarrow \mathrm{id} & & \\
E_{K_0} & \longrightarrow & \pi_1(K,\ K^0) & \xrightarrow{j} & \pi_1(K,\ K^0)\ \tilde{\times}\ G & \longrightarrow & G & \longrightarrow & E_* \\
 & & \downarrow & & & & & & \\
 & & E_* & & & & & &
\end{array}$$

<u>Proof</u>. The column is given by theorem 5.6, and the rows by lemma 3.5.

□

<u>5.17 COROLLARY</u>. Under the hypotheses of theorem 5.16 there is an exact sequence of groupoids

$$E_{B\mathcal{U}^0} \longrightarrow N \hookrightarrow \pi_1(B\mathcal{U}^2,\ B\mathcal{U}^0) \xrightarrow{\xi} \pi_1(K,K^0)\ \tilde{\times}\ G \rightarrow G \longrightarrow E_*$$

where $\xi = j \circ f_*$.

<u>Proof</u>. This is an immediate consequence of the theorem since the composition of quotient morphisms is a quotient morphism.

□

We will show that the corollary is a groupoid version of Swan's theorem (5.15). Recall the result of Razak (corollary 5.9) which states that if the cover $\mathcal{U}$ is open the groupoids $\pi_1(B\mathcal{U}^2, B\mathcal{U}^0)$ and $\pi_1(X)$ are equivalent, and hence have isomorphic vertex groups. We also require the following result (cf. Abels [1; Proposition 5.3]).

5.18 PROPOSITION. Let $\mathcal{U} = \{V^g \mid g \in G\}$ be a cover of X. Macbeath's group $M = M(\mathcal{U})$ is isomorphic to the vertex group of $\pi_1(K,K^0) \tilde{\times} G$ at $1 \in K^0$, where 1 is the identity of G, and $K = K\mathcal{U}$ is the nerve of the cover.

Proof. $\pi_1(K,K^0)$ has vertex set G, generators $[g,h]$ where $V^g \cap V^h \neq \emptyset$, and relations $[g,h] + [h,k] = [g,k]$ where $V^g \cap V^h \cap V^k \neq \emptyset$. Let Σ denote the vertex group of $\pi_1(K,K^0) \tilde{\times} G$ at $1 \in K^0 = G$. An element of Σ is $([1,g_1] + [g_1,g_2] + \ldots + [g_n,g_{n+1}],\ g_{n+1}^{-1})$ where $V^{g_i} \cap V^{g_{i+1}} \neq \emptyset$ for $i=0,1,\ldots,n$ (and $g_0=1$). Such an element may be written

$$([1,g_1] + [g_1,g_2]+\ldots+ [g_{n-1},g_n],g_n^{-1}) + ([1,g_{n+1}g_n^{-1}],g_ng_{n+1}^{-1}).$$

An inductive argument now shows that Σ is generated by elements of the form $([1,g],\ g^{-1})$ where $V \cap V^{g^{-1}} \neq \emptyset$ with defining relations

$$([1,g],\ g^{-1}) + ([1,h],\ h^{-1}) = ([1,hg],\ (hg)^{-1})$$

if $V \cap V^g \cap V^{hg} \neq \emptyset$. The map $([1,g],\ g^{-1}) \mapsto (g^{-1})$ therefore defines an isomorphism of Σ onto M.

□

5.19 PROPOSITION. Let $\mathcal{U} = \{V^g \mid g \in G\}$ be an open cover of X where V is path-connected. Let $* \in V$ be a base point. The vertex sequence

$$0 \longrightarrow N(*,1) \longrightarrow \pi_1(B\mathcal{U}^2, B\mathcal{U}^0)(*,1) \longrightarrow \left(\pi_1(K,K^0) \tilde{\times} G\right)(1) \longrightarrow G \longrightarrow 0$$

of the sequence of corollary 5.17 is isomorphic to Swan's exact sequence (theorem 5.15(ii)).

Proof. Let $p : B\mathcal{U}^2 \longrightarrow X$ be the projection defined in section one. By corollary 5.9 there is an isomorphism $p_* : \pi_1(B\mathcal{U}^2, B\mathcal{U}^0)(*,1) \longrightarrow \pi_1(X,*)$ and clearly $p_*\left(N(*,1)\right) = P$. The previous proposition gives an isomorphism $q : \left(\pi_1(K,K^0) \tilde{\times} G\right)(1) = \Sigma \longrightarrow M$. Hence we have a diagram whose vertical morphisms are isomorphisms.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & N(*,1) & \hookrightarrow & \pi_1(B\mathcal{U}^2, B\mathcal{U}^0)(*,1) & \xrightarrow{\xi} & \Sigma & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow p_* & & \downarrow p_* & & \downarrow q & & \downarrow id & & \\
0 & \longrightarrow & P & \hookrightarrow & \pi_1(X,*) & \xrightarrow[\theta]{} & M & \xrightarrow[\phi]{} & G & \longrightarrow & 0
\end{array}$$

It remains to show that the diagram is commutative. Clearly the outside squares commute. For the inside square, let $\alpha \in \pi_1(B\mathcal{U}^2, B\mathcal{U}^0)(*,1)$. Then

$$\alpha = \alpha_0 + \phi_{g_0 g_1}(\partial^1 \alpha_0) + \alpha_1 + \phi_{g_1 g_2}(\partial^1 \alpha_1) + \dots$$

$$+ \alpha_n + \phi_{g_n g_{n+1}}(\partial^1 \alpha_n) + \alpha_{n+1}$$

where for $i = 0,1,\dots,n+1$, $g_i \in G$ (and $g_0 = g_{n+1} = 1$), $\alpha_i \in \pi_1(V^{g_i})$, and $V^{g_i} \cap V^{g_{i+1}} \neq \emptyset$.

Now $$\begin{aligned} q\xi(\alpha) &= q\circ j\circ f_*(\alpha) \\ &= q\circ j\left([g_0,g_1] + \dots + [g_n,g_{n+1}]\right) \\ &= (g_0g_1^{-1})\,(g_1g_2^{-1})\,\dots\,(g_ng_{n+1}^{-1}) \\ &= \theta(\alpha_0 + \dots + \alpha_n) \\ &= \theta p_*(\alpha). \end{aligned}$$

This completes the proof.

□

Theorem 5.16 also implies Abels' theorem 5.14 [1], since $B\mathcal{U} \simeq X$ in the case where $\mathcal{U}$ is G-numerable. Indeed the above results are similar to those in §5 of [1].

We now consider Razak's theorem 3.2 [18; p91]. This states that for $\mathcal{U} = \{V^g \mid g \in G\}$ there is an isomorphism between Macbeath's group M and the fundamental group of $|K|/G$ where $|K|$ is the realisation of K. An example below shows that this is false. The correct result is the following, which is essentially what Razak proves.

<u>5.20 PROPOSITION</u>. If $\mathcal{U} = \{V^g \mid g \in G\}$ is a cover of X, and K is the nerve then

$$\pi_1(K,\ K^0)/(G) \cong M.$$

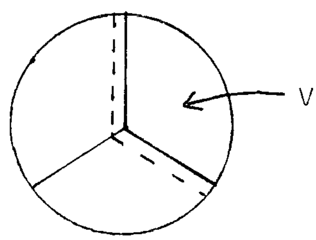
<u>Proof</u>. From theorem 3.6 there is an isomorphism

$$\pi_1(K,\ K^0)/(G) \cong \left(\pi_1(K,K^0)\ \tilde{\times}\ G\right)/N$$

where N is the normal subgroup generated by elements of the form $([g,g], h)$ for $g,h \in G$. Therefore N is a tree groupoid and hence $\pi_1(K,K^0)/(G)$ and $\pi_1(K,K^0) \tilde{\times} G$ have isomorphic vertex groups. Now K^0/G is a single point so $\pi_1(K,K^0)/(G)$ is already a group which is isomorphic to M by proposition 5.18.

□

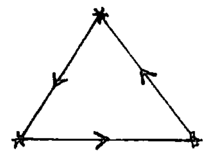
<u>5.21 EXAMPLE</u>. Let $G = \mathbb{Z}_3$ act on an open 2-disc by rotation, and let V be an open neighbourhood of the fundamental domain as indicated.



Then $K = K\mathcal{U}$ is a 2-simplex (together with its faces). K is simply connected so $\pi_1(K,K^0)/(G)$ is isomorphic to $G = \mathbb{Z}_3$, and is isomorphic to M by the previous proposition.

Now $|K|$ is a geometric 2-simplex and G acts by rotation. The orbit space is homeomorphic to a 2-disc. Hence $\pi_1(|K|/G,*) = 0$, so we have a counter-example to Razak's theorem 3.2.

In fact, Razak actually proves $\pi_1(|K/G|, |K^0/G|) \cong M$. In our example $|K/G|$ is the CW-complex with one r-cell for $r=0,1,2$, illustrated below.



Clearly $\pi_1(|K/G|, |K^0/G|) \cong \mathbb{Z}_3 = G$.

The problem is, of course, that $|K|/G$ and $|K/G|$ are not homeomorphic in this case. Conditions under which these spaces are homeomorphic are given in Chapter four.

Finally we note briefly some connections with the Bass-Serre theory of groups acting on graphs [22]. The Bass-Serre structure theorem concerns a group G acting on a graph Γ so that no edge is reversed. The quotient graph Γ/G together with the stabilisers of edges and vertices of Γ give rise to a "graph of groups" $(\mathcal{G}, \Gamma/G)$. This is a diagram of groups over Γ/G subdivided once in the sense of Crowell and Smythe (see section three, this chapter). For a vertex $*$ of Γ/G, Bass and Serre define the fundamental group of the graph of groups, $\pi_1(\mathcal{G}, \Gamma/G, *)$, together with a morphism $\phi : \pi_1(\mathcal{G}, \Gamma/G, *) \longrightarrow G$. The definition of the fundamental group is similar to that of (the vertex group of) the mapping cylinder given in section three.

The Bass-Serre structure theorem states that there is an exact sequence of groups

$$0 \longrightarrow \pi_1(\Gamma, \hat{*}) \longrightarrow \pi_1(\mathcal{G}, \Gamma/G, *) \xrightarrow{\phi} G \longrightarrow 0$$

where $\hat{*}$ is some vertex of Γ in the orbit $*$.

Bass and Serre were primarily interested in the case where Γ is a tree, and the theorem may then be regarded as giving a presentation of G in terms of the stabilisers and certain other elements arising from circuits in Γ/G.

In fact, the fundamental group $\pi_1(\mathcal{G},\Gamma/G,*)$ is isomorphic to the vertex group at $\hat{*}$ of the semi-direct product $\pi_1\Gamma \mathbin{\tilde{\times}} G$. More precisely there is an isomorphism $\psi : \pi_1(\mathcal{G},\Gamma/G,*) \longrightarrow (\pi_1\Gamma \mathbin{\tilde{\times}} G)(*)$ such that the following diagram commutes, where the top row is the vertex sequence of the one given in lemma 3.5.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(\Gamma,\hat{*}) & \longrightarrow & (\pi_1\Gamma \mathbin{\tilde{\times}} G)(*) & \longrightarrow & G & \longrightarrow & 0 \\
 & & \Big\downarrow \mathrm{id} & & \Big\downarrow \psi & & \Big\downarrow \mathrm{id} & & \\
0 & \longrightarrow & \pi_1(\Gamma,\hat{*}) & \longrightarrow & \pi_1(\mathcal{G},\Gamma/G,*) & \xrightarrow{\ \phi\ } & G & \longrightarrow & 0
\end{array}$$

To prove this it is more natural to use groupoids. ψ is defined as a groupoid morphism with domain $\pi_1\Gamma \mathbin{\tilde{\times}} G$ and codomain the fundamental groupoid of $(\mathcal{G},\Gamma/G)$ as defined by Higgins [The fundamental groupoid of a graph of groups; J. London Math. Soc. (2), 13 (1976), 145-149]. The details are long and somewhat uninformative.

A few things emerge from the diagram above. The first is that (a combinatorial version of) Rhodes' group $\sigma(\Gamma,*,G)$ is isomorphic to the Bass-Serre fundamental group. This was noted in chapter four and follows from the diagram above together with proposition 4.7.

Secondly, the earlier parts of this chapter establish a relationship between semi-direct products and the work of Macbeath and Swan. Hence there is a connection between the way presentations of groups are obtained in the Macbeath-Swan and Bass-Serre theories. We are unsure of the precise nature of the link between the two, but it seems fairly strong. It would be interesting to have a clear formulation of this relationship.

5. CONCLUSION

In concluding we make some general remarks concerning possible further developments of this work. There are perhaps two areas where further investigation may prove fruitful.

The first directly concerns the work of this chapter. In exploring the various inter-relations between the Brown-Higgins union theorem, the work of Bass-Serre and Macbeath-Swan it may be possible to formulate a more general result incorporating some of their common features. The data for such a result would involve a G-space X with an invariant cover $\mathcal{U}$ by subspaces. Suppose that a description of $\pi_1 X \tilde{\times} G$ could be given in terms of groupoids $\pi_1 X_\lambda \tilde{\times} G_\lambda$ where the subgroup G_λ stabilises $X_\lambda \in \mathcal{U}$ and similar groupoids for the intersections $X_\lambda \cap X_\mu$. Such a description might involve a mapping cylinder construction. If the action were trivial this situation would reduce to that of the 1-dimensional union theorem, and if the X_λ's were simply connected it would reduce to that of the Macbeath-Swan theory. In addition, taking X as a graph with certain canonical choices for the X_λ might link up with the results of Bass-Serre.

The second area we have in mind is the establishment of the isomorphism $\pi(\underset{\sim}{X}/G) \cong \pi\underset{\sim}{X}/(G)$ of theorem 4.20 under more general hypotheses. For example, some relaxation might be achieved by using the union theorem in the proof rather than the free description of the homotopy crossed complex (theorem 4.19) which is itself a corollary of the union theorem. To obtain a significant relaxation of the hypotheses, however, substantial topological problems need to be resolved. In view of Armstrong's success in reducing the hypotheses for the dimension one result, some progress in dimension two may be possible, for example.

Finally on a more speculative note, we observe the similarity between the statements of theorem 4.20 and the union theorem (2.17). Both take the general form that the functor π from a certain category of J_0-filtered spaces to the category of crossed complexes preserves colimits of a certain type. It would be rewarding to be able to incorporate both into a theorem giving fairly general conditions under which π preserves colimits.

REFERENCES

[1] H.ABELS. Generators and relations for groups of homeomorphisms, Proc. Conf. on Transformation Groups, Univ. Newcastle-upon-Tyne 1976, London Math. Soc. Lecture Notes 26 (1977), 3-20.

[2] M.A.ARMSTRONG. On the fundamental group of an orbit space, Proc. Cambridge Philos. Soc. 61 (1965), 639-646.

[3] M.A.ARMSTRONG. The fundamental group of the orbit space of a discontinuous group, Proc, Cambridge Philos. Soc. 64 (1968), 299-301.

[4] G.E.BREDON. Introduction to Compact Transformation Groups, Academic Press (1972).

[5] C.E.BOLTON. Crossed modules, M.Phil. thesis, Univ. of London (1978).

[6] R.BROWN. Elements of Modern Topology, McGraw Hill (1968).

[7] R.BROWN. Groupoids as coefficients, Proc. London Math. Soc. (3) 25 (1972), 413-426.

[8] R.BROWN and G.DANESH-NARUIE. The fundamental groupoid as a topological groupoid, Proc. Edinburgh Math. Soc. 19 (1974), 237-244.

[9] R.BROWN and P.J.HIGGINS. On the connection between the second relative homotopy groups of some related spaces, Proc. London Math. Soc. (3) 36 (1978), 193-212.

[10] R.BROWN andP.J.HIGGINS. On the algebra of cubes, J. Pure Appl. Algebra 21 (1981), 233-260.

[11] R.BROWN and P.J.HIGGINS. Colimit theorems for relative homotopy groups, J. Pure Appl. Algebra 22 (1981), 11-41.

[12] R.BROWN and P.J.HIGGINS. Crossed complexes and non-Abelian extensions, Proc. 1981 Conf. on Category Theory, Springer Lecture Notes in Math., to appear.

[13] R.H.CROWELL and N.SMYTHE. The subgroup theorem for amalgamated free products, HNN-constructions and colimits, Proc. Second Internat. Conf. on the Theory of Groups, Canbera 1973, Springer Lecture Notes in Math. 372 (1974), 241-280.

[14] P.J.HIGGINS. Notes on Categories and Groupoids, Van Nostrand Reinhold Mathematical Studies 32 (1971).

[15] P.J.HIGGINS and J.TAYLOR. The fundamental groupoid and the homotopy crossed complex of an orbit space, Proc. 1981 Conf. on Category Theory, Springer Lecture Notes in Math., to appear.

[16] J.HOWIE. Topics in the theory of double groupoids, Ph.D. thesis, Univ. of London, (1977).

[17] A.M.MACBEATH. Groups of homeomorphisms of a simply connected space, Ann. of Math. 79 (1964), 473-488.

[18] A.RAZAK. Union theorems for double groupoids and groupoids; some generalisations and applications, Ph.D. thesis, Univ. of Wales, (1976).

[19] F.RHODES. On the fundamental group of a transformation group, Proc. London Math. Soc. (3) 16 (1966), 635-650.

[20] F.RHODES. Homotopy groups of transformation groups, Canadian J. Math. 21 (1969), 1123-1136.

[21] G.SEGAL. Classifying spaces and spectral sequences, Publ. Math. I.H.E.S. 34 (1968), 105-112.

[22] J.-P.SERRE. Arbres,amalgames, SL_2, Astérisque 46, Société mathématique de France, Paris (1977).

[23] E.H.SPANIER. Algebraic Topology, McGraw Hill Series in Higher Mathematics (1966).

[24] R.G.SWAN. Generators and relations for certain special linear groups, Advances in Math. 6 (1971), 1-77.