Instability in SU(2) gauge field theory

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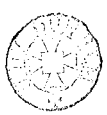
A dissertation submitted for the degree of

Doctor of Philosophy



INSTABILITY IN SU(2) GAUGE FIELD THEORY

PhD. THESIS - ALEXANDER TODD GILLESPIE

ABSTRACT

Classical solutions to SU(2) gauge theory with a static charge source or with wave-like behavior are examined. In both cases gauge rotations cause instability. A quantum mechanical model with a local gauge symmetry is constructed. The quantum numbers of the model are constrained by the local symmetry. The S-matrix elements of SU(2) gauge theory are analysed in terms of angular orientation in gauge space. Most S-matrix elements are found to vanish in a way that indicates that most states are unstable. This result is due to the ambiguity in the time evolution of the states inherent in the local symmetry and it indicates that the gauge must be fixed in the path integral for a well defined dynamical evolution. When the gauge is fixed the result reduces to the conservation of quantised isospin.

PREFACE

The subject of this thesis is the Quantum Field Theory of a form of instability which occurs in non-Abelian gauge theories. It is found in SU(2) theory and should also occur in theories with higher gauge groups. In order to be precise we restrict attention to SU(2).

The work started in an attempt to understand simple classical solutions either with a charge source or with wave-like behaviour. Rotations in gauge space are fundamental to the stability of the solutions, and are at least as important to the theory as angular momentum is to the hydrogen atom.

The first section is a brief account of the relevance of classical equations to quantum theories and introduces notation and conventions. In the second section, the problem of classical solutions for static charge sources is examined. The instability of Abelian solutions is found to be due to gauge rotations. The contents of this section have been published[1].

The third section deals with simple waves. Here again an instability due to rotations in gauge space is found. To gain insight into these rotations, a quantum mechanical model is constructed in the fourth section. This eliminates the complications of dealing with a field theory. In the fifth section a method of dealing with

these rotations in the full theory is developed. Many of the states appear to be unstable as a consequence of the local symmetry. This is due to the ambiguity in the time development of the theory inherent in the local symmetry and which also gives rise to a discontinuity in the form of the S-matrix elements. In order to obtain a well defined dynamical evolution the gauge must be fixed, in which case the analysis in terms of the remaining global symmetry generates the global quantum numbers.

BASIC CONVENTIONS

The metric tensor for Minkowski space is

$$g_{\mu\nu} = Diag\,(1,-1,-1,-1)$$

Unless otherwise stated, all repeated indices are summed over. Since SU(2) gauge theory is non-linear, we have rescaled the fields so that the coupling constant is unity. The other fundamental constants are taken to be

$$\hbar = 1 = c$$

For typographical reasons, we will distinguish between functions and operators by placing the latter inside curved brackets, e.g. $(\underset{\sim}{\Phi})$ is an operator whereas $\underset{\sim}{\Phi}$ is a function.

In certain cases it will be convenient to use a non-covariant analysis. A tilde below a character indicates that it has three spacial components, i.e.

$$\underset{\sim}{x} = (x_1, x_2, x_3)$$

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1.1 CLASSICAL EQUATIONS IN THE REAL WORLD

Under certain conditions, classical equations of motion are accepted as being good approximations to real physical systems. This is one possible justification for examining them. Unfortunately classical equations possess difficulties for the description of bound states such as the hydrogen atom. A more sophisticated approach is that due to Feynman [2]. This is the path-integral formulation of quantum mechanics and it may be developed to include quantum field theory. The path integral formulation asserts that the classical solutions to the equations of motion dominate the time development of quantum theories over the period of time from the distant past to the distant future. The time interval involved must be much longer than any effective parameter of time in the theory.

A slight modification may have to be made to this. Some account must also be taken of quantum numbers. These can be important if they determine whether a system is stable like the ground state of the hydrogen atom, or unstable, like the analogous state of positronium. If the classical solutions of SU(2) gauge field theory show signs of instability it may be fruitful to examine the quantum effects of the symmetries.

1.2 CLASSICAL EQUATIONS IN QUANTUM MECHANICS

The role of classical equations of motion in quantum mechanics has been discussed at length for the case of the simple harmonic oscillator [2]. This example prepares the ground for the path-integral formulation of quantum field theories and many concise accounts of it are found in the literature on gauge theories [3,4,5]. The results are summarised below.

1.2 a) Operators and States

The system can be described by a position operator

$$(Q)$$

As positions are real numbers

$$(Q) = (Q)^{\dagger} \tag{1.1}$$

Momentum is described by the operator

$$(P) = (P)^{\dagger} \tag{1.2}$$

These operators are functions of time and obey the canonical commutation relationship:

$$[(P(t)), (Q(t))] = -i \tag{1.3}$$

The operator $(Q(t))$ is self adjoint and its eigenvalues are not degenerate so its eigenvectors must be orthogonal. We may standardise the normalisation of these eigenvectors to give a complete orthonormal basis of the Hilbert space of the theory. We denote an eigenvector of $(Q(t))$ by

$$|Q,t\rangle : (Q(t))\,|Q,t\rangle = Q\,|Q,t\rangle \qquad (1.4)$$

These eigenvalues are continuous so a sum over all eigenstates is expressed as an integral and the orthonormalisation of the states is expressed in terms of the δ-function, i.e.

and
$$\left.\begin{aligned} &\langle Q_a,t\,|\,Q_b,t\rangle = \delta(Q_a - Q_b) \\ &\int dQ_a\,\langle Q_a,t\,|\,Q_b,t\rangle = 1 \end{aligned}\right\} \qquad (1.5)$$

Energy is described by the Hamiltonian operator

$$(H) = \tfrac{1}{2}(P)^2 + \tfrac{1}{2}\omega^2(Q)^2 \qquad (1.6)$$

The time development of these operators is determined by

$$\left.\begin{aligned} [(H),(Q)] &= -i\,(\dot{Q}) \\ [(H),(P)] &= -i\,(\dot{P}) \\ [(H),(H)] &= -i\,(\dot{H}) = 0 \end{aligned}\right\} \qquad (1.7)$$

1.2 b) The Path Integral

As the above operators are dependent on time, we are implicitly working in the Heisenberg picture. We may remove the time dependence to the states of the theory by working in the Schrödinger picture. The two pictures are related by

$$(O(t))_H = e^{i(H)t}\,(O)_S\,e^{-i(H)t} \qquad (1.8)$$

and

$$|\alpha\rangle_H = e^{i(H)t}\,|\alpha(t)\rangle_S \qquad (1.9)$$

The eigenstate of the Schrödinger representation corresponding to the state $|Q,t\rangle$ is the state

$$|Q\rangle : \quad |Q\rangle = e^{-i(H)t}|Q,t\rangle \tag{1.10}$$

The general form of a matrix element is

$$\begin{aligned}
&{}_H\langle\alpha''|\alpha'\rangle \\
&= {}_S\langle\alpha''(t'')|e^{-i(H)(t''-t')}|\alpha'(t')\rangle_S \\
&= \int dQ''\,dQ'\,{}_S\langle\alpha''(t'')|Q''\rangle\langle Q''|e^{-i(H)(t''-t')}|Q'\rangle \\
&\qquad \times \langle Q'|\alpha'(t')\rangle_S
\end{aligned} \tag{1.11}$$

The time dependence of the theory is determined by the general matrix element

$$\begin{aligned}
&\langle Q''|e^{-i(H)(t''-t')}|Q'\rangle \\
&= \langle Q'',t''|Q',t'\rangle
\end{aligned} \tag{1.12}$$

Let $\tau = t''-t'$. For infinitesimal τ this matrix element becomes

$$\begin{aligned}
&\langle Q'',t''|Q',t'\rangle \\
&\simeq \langle Q''|1-\tfrac{i}{2}[(P)^2+\omega^2(Q)^2]\tau|Q'\rangle
\end{aligned} \tag{1.13}$$

Because of the commutation relations we may use the basis provided by the momentum operator $(P(t))$ to write this as

$$\hat{N}_1\int dP\, e^{iP[Q''-Q']}e^{-\frac{i}{2}\tau[P^2+\omega^2Q'^2]} \tag{1.13a}$$

Evaluating this expression by steepest descents gives

$$\hat{N}_2\, e^{iL([Q''-Q']/\tau,\,Q')\tau} + O[\tau^2] \tag{1.13b}$$

$\hat{N}_1$ and $\hat{N}_2$ are constants and L is the classical action, i.e.

$$L(\dot{Q}, Q) = \tfrac{1}{2}\dot{Q}^2 - \tfrac{1}{2}\omega^2 Q^2 \tag{1.14}$$

For finite-time intervals let

$$\left.\begin{aligned} t^{(0)} &= t' \\ t^{(k)} &= t' + k\tau \\ t^{(n)} &= t' + n\tau = t'' \end{aligned}\right\} \tag{1.15}$$

Then

$$\begin{aligned} &\langle Q'', t'' | Q', t' \rangle \\ &= \int \Pi_{k=0}^{n} dQ^{(k)} \langle Q'', t'' | Q^{(n)}, t^{(n)} \rangle \\ &\quad \times \langle Q^{(n)}, t^{(n)} | Q^{(n-1)}, t^{(n-1)} \rangle \times \cdots \times \langle Q^{(0)}, t^{(0)} | Q', t' \rangle \end{aligned} \tag{1.16}$$

We note

$$\langle Q'', t'' | Q^{(n)}, t^{(n)} \rangle = \delta[Q'' - Q^{(n)}] \tag{1.17}$$

and

$$\langle Q^{(0)}, t^{(0)} | Q', t' \rangle = \delta[Q^{(0)} - Q'] \tag{1.18}$$

As n goes to infinity, $\langle Q^{(k+1)}, t^{(k+1)} | Q^{(k)}, t^{(k)} \rangle$ goes to

$$\hat{N}_2 \exp\{i L(\tfrac{1}{\tau}[Q^{(k+1)} - Q^{(k)}], Q^{(k)})\tau\} \tag{1.19}$$

as before [(1.13b)].

As we take this limit we may rewrite eqn.(1.16) as

$$\langle Q'', t'' \mid Q', t' \rangle$$

$$= \hat{N} \int \prod_{t=t'}^{t''} dQ(t) \exp\left\{ \int_{t'}^{t''} d\tau \, L(\dot{Q}(\tau), Q(\tau)) \right\} \times \delta[Q'' - Q(t'')] \, \delta[Q(t') - Q'] \tag{1.20}$$

The constant $\hat{N}$ is formally infinite.

1.2 c) Sources

There are at least two reasons for adding a source term to the functional integral. One is relevant to the construction of the states of the theory and the other relates to perturbative expansions of more complicated systems.

The position operator obeys the equation of motion:

$$(\ddot{Q}) + \omega^2 (Q) = 0 \tag{1.21}$$

This equation implies that it may be written in the form

$$(Q(t)) = (\alpha)\, e^{-i\omega t} + (\alpha)^\dagger e^{i\omega t} \tag{1.22}$$

where (α) is independent of t .

Thus

$$[(H),(\alpha)^\dagger] = \omega\,(\alpha)^\dagger \tag{1.23}$$

For some energy eigenstate

$$|\psi\rangle : \quad (H)\,|\psi\rangle = E_\psi\,|\psi\rangle$$

$$(H)[(\alpha)^\dagger]^n\,|\psi\rangle = [E_\psi + n\omega][(\alpha)^\dagger]^n|\psi\rangle \tag{1.24}$$

(α) and $(\alpha)^\dagger$ are ladder operators for the energy eigenstates. Note that

$$[(\alpha)^\dagger]^n = \left\{\frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\; e^{-i\omega t}(Q(t))\right\}^n \tag{1.25}$$

The energy eigenstates are constructed from the lowest energy eigenstate by the repeated action of $(Q(t))$, i.e.

$$|\psi\rangle = \int dt_1 \cdots dt_m \; f(t_1 \cdots t_m) \times (Q(t_1)) \cdots (Q(t_m)) |0\rangle \tag{1.26}$$

where

$$\left.\begin{aligned} (H)|0\rangle &= E_0|0\rangle \\ (H)|\psi\rangle &= E_\psi|\psi\rangle \\ E_\psi &\geq E_0 \\ \text{and} \quad t_1 &\geqslant t_2 \geqslant \cdots \geqslant t_m \end{aligned}\right\} \tag{1.27}$$

Since all states can be written as a summation over energy eigenstates, all states can be written in the form:

$$|\alpha\rangle = \int dt_1 \cdots dt_m \; f_\alpha(t_1 \cdots t_m) \times (Q(t_1)) \cdots Q(t_m)|0\rangle \tag{1.28}$$

for some choice of weighting function .

The matrix element

$$\langle \alpha''(t'')|\alpha'(t')\rangle_S$$

can be constructed from elements of the form

$$\langle 0''|T\{(Q(t_1)) \cdots (Q(t_m))\}|0'\rangle$$

These elements can in turn be constructed by adding a source term to the Lagrangian.

Let

$$L' = \tfrac{1}{2}\{\dot{Q}^2 - \omega^2 Q^2\} + s(t)Q$$

Then

$$\langle 0'' | T\{(Q(t_1)) \cdots (Q(t_n))\} | 0' \rangle$$
$$= (-i)^n \frac{\delta^n Z}{\delta s(t_1) \cdots \delta s(t_n)} \Big|_{s=0} \tag{1.29}$$

where

$$Z = Z(s) = \langle 0'' | 0' \rangle$$

calculated for the Lagrangian L' .

The vacuum-to-vacuum matrix element for the Lagrangian L' gives us every matrix element for the Lagrangian L .

The second use of a source term is as a means of calculating orders in perturbation theory. The same term

$$\langle 0'' | T\{(Q(t_1)) \cdots (Q(t_n))\} | 0' \rangle$$

occurs if the Hamiltonian is perturbed to

$$(H') = \tfrac{1}{2}\{(P)^2 + \omega^2 (Q)^2\} + \lambda(V) \tag{1.30}$$

where (V) is a polynominal in (Q) .

In either of these two cases the paths of interest for $Z(s)$ correspond to solutions to the classical equations of motion in the presence of an arbitrary source, i.e. to

$$\ddot{Q} + \omega^2 Q = s(t) \tag{1.31}$$

1.3 CLASSICAL EQUATIONS IN SU(2) GAUGE FIELD THEORY

SU(2) gauge field theory differs from simple quantum mechanical systems in two distinct ways. The first difference is that it is a multi-component field theory. In the Lagrangian there are many operators and these depend on both space and time.

The second difference is that the various operators possess a local symmetry. This may be expressed by the action of a unitary operator, defined at each point of space-time, which acts on the field operators in a specific way but which does not change the Hamiltonian. The interpretation of this symmetry is not entirely clear.

In the following few pages we shall outline the path-integral formulation of SU(2) gauge field theory[3,4,5]. This demonstrates the relevance of the classical solutions to the theory. Problems of stability will arise later which can be analysed in the path integral formulation and so it is sensible to be precise about the framework which we will later use.

1.3 a) The Classical Lagrangian Formulation

The Lagrangian density is

$$\mathcal{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \tag{1.32}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu$$
$$a,b,c \in (1,2,3)$$
$$\mu,\nu \in (0,1,2,3) \tag{1.33}$$

The action is

$$\int d^4x \, \mathcal{L}$$

and the equations of motion may be deduced by requiring the action to be stationary with respect to variations in the gauge fields $\{A^a_\mu\}$. This leads to the equations of motion

$$\partial_\mu F^{a\mu\nu} + \epsilon^{abc} A^b_\mu F^{c\mu\nu} = 0 \qquad (1.34)$$

1.3 b) The Matrix Representation of the Gauge Symmetry

Let

$$T^a = -\frac{i}{2}\sigma^a$$

where σ^a are the Pauli matrices.

We may now define the matrices

$$\left.\begin{aligned} A_\mu &= A^a_\mu T^a \\ F_{\mu\nu} &= F^a_{\mu\nu} T^a \end{aligned}\right\} \qquad (1.35)$$

and

Let

$$G = \exp\{\omega^a T^a\}$$

where $\{\omega^a\}$ is a set of three real functions of space and time.

Then

$$\left.\begin{aligned} G^\dagger &= G^{-1} \\ \mathrm{Det}\, G &= 1 \end{aligned}\right\} \qquad (1.36)$$

and

Under the transformation

$$A_\mu \rightarrow G^{-1} A_\mu G + G^{-1} \partial_\mu G \tag{1.37}$$

we have

$$F_{\mu\nu} \rightarrow G^{-1} F_{\mu\nu} G \tag{1.38}$$

Since

$$\mathcal{L} = Tr\{ F_{\mu\nu} F^{\mu\nu} \}$$

we have

$$\mathcal{L} \rightarrow \mathcal{L} \tag{1.39}$$

The theory is invariant under such a gauge transformation.

1.3 c) The Hamiltonian Formulation

The connection between the Lagrangian and Hamiltonian formulations is made easily if we use the gauge symmetry to transform A_0 to zero.

In this case we may define the conjugate momenta

$$E^a_j = \frac{\delta L}{\delta \partial_0 A^a_j} = \partial_0 A^a_j \tag{1.40}$$

Thus the Hamiltonian density is

$$\mathcal{H} = E^a_j \partial_0 A^a_j - \mathcal{L}$$

$$= \frac{1}{4}\{ -2 E^a_j E^{aj} + F^a_{ij} F^{a\,ij} \} \tag{1.41}$$

1.3 d) Operators and States

The general form of quantised SU(2) gauge field theory can be described in terms of twelve operator counterparts of the gauge field. The operators are $\{(A_\mu^a)\}$.

We require these operators to be self adjoint and to depend on the four coordinates of space and time, i.e.

$$(A_\mu^a) = (A_\mu^a)^\dagger = (A_\mu^a(\underset{\sim}{x}, t)) \tag{1.42}$$

In addition we require the twelve operators to commute at any instant of time, i.e.

$$[(A_\mu^a(\underset{\sim}{x}, t)), (A_\nu^b(\underset{\sim}{y}, t))] = 0 \tag{1.43}$$

The operators at any given instant of time, t say, have simultaneous eigenfunctions which we denote by

$$|\{A\}, t\rangle :$$

$$(A_\mu^a(\underset{\sim}{x}, t))\,|\{A\}, t\rangle$$

$$= A_\mu^a(\underset{\sim}{x})\,|\{A\}, t\rangle \tag{1.44}$$

The set $\{A\}$ is a set of twelve functions of space corresponding to the eigenvalues of the gauge field operators at that particular instant of time. By analogy with the derivatives of functions we may define the operator derivatives of the operator (A_μ^a), i.e.

$$(\partial_\mu A_\nu^a) = \lim_{\delta x \to 0} \{[(A_\nu^a(x + \tfrac{1}{2}\delta x)) - (A_\nu^a(x - \tfrac{1}{2}\delta x))] \div \delta x^\mu\} \tag{1.45}$$

From these operators we may construct the field strength operators

$$(F^a_{\mu\nu}) = (\partial_\mu A^a_\nu) - (\partial_\nu A^a_\mu) + \epsilon^{abc}(A^b_\mu)(A^c_\nu) \tag{1.46}$$

1.3 e) The Path Integral

To be explicit it is convenient to use the gauge invariance of the theory to transform $(A^a_0(\underset{\sim}{x},t))$ to zero.

The theory is then described by nine operators at each point of space time, $\{(A^a_j(\underset{\sim}{x},t))\}$. The eigenfunctions are then

$$|\{\underset{\sim}{A}\},t\rangle :$$

$$(A^a_j(\underset{\sim}{x},t))|\{\underset{\sim}{A}\},t\rangle = A^a_j(\underset{\sim}{x})|\{\underset{\sim}{A}\},t\rangle \tag{1.47}$$

and

$$\langle\{\hat{\underset{\sim}{A}}\},t|\{\underset{\sim}{A}\},t\rangle = \pi_{\underset{\sim}{x},j,a}\,\delta(\hat{A}^a_j(\underset{\sim}{x}) - A^a_j(\underset{\sim}{x})) \tag{1.48}$$

The general matrix element becomes

$$\langle\{\underset{\sim}{A}''\},t''|\{\underset{\sim}{A}'\},t'\rangle$$

For infinitesimal intervals this becomes

$$N_1 \int \Pi_{\underset{\sim}{x},j,a}\, d E_j^a(\underset{\sim}{x}) \exp\{ i E_j^a(\underset{\sim}{x})[A_j''^a(\underset{\sim}{x}) - A_j'^a(\underset{\sim}{x})]\} \times (1 - i\tau H\{\underset{\sim}{E}, \underset{\sim}{A}\}) \tag{1.49}$$

where N_1 is a constant, formally infinite, and $H\{\underset{\sim}{E}, \underset{\sim}{A}\}$ is the classical energy.

Evaluating the integral by steepest descents gives

$$N_2 \exp\{ i\tau \int d^3\underset{\sim}{x}\, \mathcal{L}(\tfrac{1}{\tau}[A_j''^a - A_j'^a], \partial_j A_k'^a, A_j'^a)\} \tag{1.50}$$

We may integrate this up to the form of the finite-time interval matrix element.

Let
$$t^{(0)} = t'$$
$$t^{(k)} = t' + k\tau$$
$$t^{(n)} = t' + n\tau = t''$$

Then

$$\langle \{\underset{\sim}{A}''\}, t'' | \{\underset{\sim}{A}'\}, t' \rangle$$
$$= \int \prod_{k=0}^{n} \Pi_{\underset{\sim}{x},j,a}\, d A_j^{(k)a}(\underset{\sim}{x})$$
$$\times \langle \{\underset{\sim}{A}''\}, t'' | \{\underset{\sim}{A}^{(n)}\}, t^{(n)} \rangle$$
$$\times \langle \{\underset{\sim}{A}^{(n)}\}, t^{(n)} | \{\underset{\sim}{A}^{(n-1)}\}, t^{(n-1)} \rangle$$
$$\times \cdots \times \langle \{\underset{\sim}{A}^{(0)}\}, t^{(0)} | \{\underset{\sim}{A}'\}, t' \rangle$$

$$\langle \{\underset{\sim}{A}^{(0)}\}, t^{(0)} | \{\underset{\sim}{A}'\}, t' \rangle$$
$$= \Pi_{\underset{\sim}{y},j,b}\, \delta(A_j^{(0)b}(\underset{\sim}{y}) - A_j'^b(\underset{\sim}{y}))$$

$$\langle \{\underset{\sim}{A}''\}, t'' | \{\underset{\sim}{A}^{(n)}\}, t^{(n)} \rangle \tag{1.51}$$
$$= \Pi_{\underset{\sim}{y},j,b}\, \delta(A_j''^b(\underset{\sim}{y}) - A_j'^b(\underset{\sim}{y}))$$

As n goes to infinity,

$$\langle \{\underset{\sim}{A}^{(k+1)}\}, t^{(k+1)} | \{\underset{\sim}{A}^{(k)}\}, t^{(k)} \rangle$$

$$= N_2 \exp\{ i\tau \int d^3\underset{\sim}{x}\, \mathcal{L}\left(\tfrac{1}{\tau}[A_j^{(k+1)a}(\underset{\sim}{x}) - A_j^{(k)a}(\underset{\sim}{x})], \partial_j A_\ell^{(k)a}(\underset{\sim}{x}), A_j^{(k)a}(\underset{\sim}{x})\right)\} \tag{1.52}$$

where N_2 is a constant, formally infinite.

As we take this limit we may rewrite eqn. (1.51) as

$$\langle \{\underset{\sim}{A}''\}, t' | \{\underset{\sim}{A}'\}, t' \rangle$$

$$= N \int \prod_{t=t'}^{t''} \pi_{\underset{\sim}{x}, j, a}\, dA_j^a(\underset{\sim}{x}, t)$$
$$\times \exp\{ i \int_{t'}^{t''} d\tau \int d^3y\, \mathcal{L}(\partial_\mu A_\ell^b(y,\tau), A_\ell^b(y,\tau)\}$$
$$\times \delta[A_j''^a(\underset{\sim}{x}) - A_j^a(\underset{\sim}{x}, t'')]\, \delta[A_j^a(\underset{\sim}{x}, t') - A_j'^a(\underset{\sim}{x})] \tag{1.53}$$

where N is a formally infinite constant.

$$= N \int \prod_{t=t'}^{t''} \pi_{\underset{\sim}{x}, \mu, a}\, dA_\mu^a(\underset{\sim}{x}, t)$$
$$\times \exp\{ i \int_{t'}^{t''} d\tau \int d^3y\, \mathcal{L}(\partial_\nu A_\lambda^a(y,\tau), A_\nu^b(y,\tau))\}$$
$$\times \delta[A_0^a(\underset{\sim}{x}, t)]\, \delta[A_j''^a(\underset{\sim}{x}) - A_j^a(\underset{\sim}{x}, t'')]\, \delta[A_j^a(\underset{\sim}{x}, t') - A_j'^a(\underset{\sim}{x})] \tag{1.54}$$

This form of the matrix element can be related to the general matrix element independent of gauge. This manipulation is well known and will be used later.

The method of steepest descents implies that the dominating functions in this integral are those which obey

the classical equations of motion:

$$\partial_\mu F^{a\mu\nu} + \epsilon^{abc} A_\mu^b F^{c\mu\nu} = 0 \qquad (1.55)$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu \qquad (1.56)$$

We may wish to include source fields in order to generate additional matrix elements. In this case the equation of motion becomes

$$\partial_\mu F^{a\mu\nu} + \epsilon^{abc} A_\mu^b F^{c\mu\nu} = J^{a\nu} \qquad (1.57)$$

1.3 f) Gauge Fixing

The gauge fixing term , $\pi_{\underset{\sim}{x},a,t}\,\delta[A_0^a(\underset{\sim}{x},t)]$ occurs naturally here since the connection between the Lagrangian and Hamiltonian is made in terms of the conjugate moment of the theory. This obscures the fact that a general gauge fixing term may be added to gauge independent path integral to avoid the complication of integrating over gauge equivalent paths. There are difficulties in fixing the gauge absolutely[6]. If this can be done, gauge invariance may require it to include a further term resembling a Fermion field — the Faddeev-Popov ghost field[7] .

The important aspect for the latter sections (especially Sections 4 and 5) is the effect of the gauge invariance of the theory on the end-points of the path integral.

1.4 CLASSICAL EQUATIONS AND INSTABILITY

Two examples of instability in SU(2) gauge theory will be described. These are illustrated by two sets of classical solutions to the equations of motion. The first set involves solutions with a static charge source. These solutions, which are generalisations of the Abelian solution, are energetically unstable and can always be perturbed to give a solution of lower energy.

The second set of solutions behave like waves. These are derived as generalisations of the Abelian wave solutions and indicate that the eigenfunctions of the field operators corresponding to the Abelian wave solutions are quantum mechanically unstable. These solutions are equivalent to those already found by Coleman [8].

In both cases the origin of the instability lies in rotation in gauge space. The resolution of the problem seems to involve the relationship of the quantum numbers of the theory with the local gauge symmetry.

2. CHARGE

One of the first problems in electromagnetism is to determine the field due to a static charge source. This is so fundamental to our understanding of electromagnetism that it should be important to examine the corresponding problem for classical SU(2) gauge theory.

The equations which we need to solve, in matrix notation, are

$$D_\mu F^{\mu\nu} = J^\nu$$

where $$J^\nu = \delta^{\nu 0} \rho \qquad (2.1)$$

$$\rho = \rho^a(\underline{x})\, T^a$$

Apart from the insight into the classical version of the theory, these solutions are relevant to the path-integral formulation of the quantum theory for states with no magnetic field. Zero magnetic field means that we need only consider the operator (A_0) which requires only a charge source in the exponential term of the path integral.

2.1 THE PROBLEM

In the equations (2.1) let us take the very simple case of

$$\rho(\underset{\sim}{x}) = \delta^3(\underset{\sim}{x})\, T' \tag{2.2}$$

The equations can then be solved by

$$A_0 = \frac{1}{4\pi r}\, T' \qquad A_i = 0 \tag{2.3}$$

This is a simple form of the Abelian solution to SU(2) gauge theory. It would also be possible to take a number of δ-function sources, all orientated so as to lie in the T' direction of gauge space. This would have a superposition of the Abelian solutions from the individual δ-function sources as a solution.

The equations (2.1) allow for the existence of point-like charged objects interacting via a $1/r^2$-force law. In SU(3) theory the corresponding objects could look like unconfined quarks interacting through a $1/r^2$-force law in addition to the conventional $1/r^2$-force due to their electromagnetic charges. This is so unrealistic that it has prompted the formulation of a generalisation of gauge theories in which these solutions no longer exist [9]. Rather than take such a large step, we shall remain in SU(2) theory and examine the stability of the Abelian solutions.

2.2 STABILITY OF ABELIAN SOLUTIONS

2.2 a) Strong δ-Function Source

The stability of the Abelian solution for a strong δ-function source has already been examined[10]. For sources equal to a δ-function source times a large constant it has been found that certain small variations about the Abelian solution grow exponentially with time. The constant describes the strength of the source in comparison to that of the gauge coupling (here taken equal to unity). Abelian solutions for strong δ-function sources must be unstable.

2.2 b) More General Sources

It is possible to find the solutions from the analogy with electrostatics. Let us set

$$J^{\nu} = \delta^{\nu 0} \sigma(\underset{\sim}{x})\, T' \tag{2.4}$$

in equations (2.1).

The Abelian solutions have the general form:

$$\left.\begin{aligned} A_0 &= \psi\, T' \\ A_i &= 0 \\ -\nabla^2 \psi &= \sigma \end{aligned}\right\} \tag{2.5}$$

where

Apart from these it is possible to find other solutions which have lower energy and have field strengths which in comparison are screened[11-18,33]. Previous alternative solutions have limitations. Instability might be demonstrated for

strong sources[13] or the lower energy solution may be expressed perturbatively in terms of the source strength[18]. Unless there is a three-current source in addition to the charge source of equations (2.1), the magnetic field does not vanish[15,16,17]; though this does not hold if a discontinuous transformation has been introduced[15,16].

2.3 GENERAL PROOF OF INSTABILITY

In this section a general proof that the Abelian solutions of equations (2.1) are unstable will be developed. It is valid for all continuous charge sources which do not depend on time. Many discontinuous sources can be treated as the limiting forms of series of continuous sources and the proof is valid for each member of such series. A related treatment of a configuration of δ - function sources has also been developed [19].

Solutions to equations (2.1) are related to solutions of the equations

$$\left.\begin{aligned} \partial_\mu F^{\mu\nu} &= J^\nu \\ J^\nu &= \delta^{\nu 0}\, \sigma(\underset{\sim}{x})\, T' \end{aligned}\right\} \qquad (2.6)$$

The two sets of solutions are related by a gauge transformation which is independent of time. In general this transformation may have a topological singularity [20]. The Abelian solution to (2.6) is given in eqn. (2.5). New solutions to (2.6) may be generated by turning on a small additional external current J_ν^{E}, and then turning it off again. These new solutions have lower energy and have screened field strengths. In the extreme case there is a new solution with zero energy and complete screening. The proof that the Abelian solutions are unstable with respect to these small perturbations is independent of the strength of the large source.

2.3 a) General Form of the Perturbation

We shall seek a solution to equations (2.6) which preserves some of the properties of the Abelian solution (equations (2.5)).

The new solution has the form

$$\left.\begin{aligned} A_0 &= g^{-1}\phi g \\ A_i &= g^{-1}\partial_i g \\ \text{where}\quad g &= g(\underset{\sim}{x}) \in SU(2) \\ \phi &= \phi^a(\underset{\sim}{x})\, T^a \\ -\nabla^2\phi &= \sigma(\underset{\sim}{x})\, g(\underset{\sim}{x})\, T^1\, g^{-1}(\underset{\sim}{x}) \end{aligned}\right\} \quad (2.7)$$

These fields obey the equation of motion

$$\left.\begin{aligned} D^\mu F_{\mu\nu} &= \delta_{\nu 0}\,\sigma(\underset{\sim}{x})\, T^1 + J_\nu{}^E \\ \text{where}\quad J_\nu{}^E &= g^{-1}\left[\partial_\nu\phi, \phi\right] g \end{aligned}\right\} \quad (2.8)$$

Since $\phi = \phi(\underset{\sim}{x})$, $J_0{}^E = 0$

To make $J_\nu{}^E$ arbitrarily small, we must restrict the magnitude of the component of $\partial_\nu\phi$ which is perpendicular to ϕ . This may be done by making g vary slowly with $\underset{\sim}{x}$. In the limit where g is constant we obtain the solution (2.5) globally transformed.

An alternative method is to make g vary rapidly with $\underset{\sim}{x}$. As we will see in an explicit example, this allows us to make J_ν^E arbitrarily small.

2.3 b) A Specific Perturbation

Let us choose $g(\underset{\sim}{x})$ to be a rotation about the T^3 direction

$$g(\underset{\sim}{x}) = I\cos\alpha/2 + 2T^3\sin\alpha/2 \qquad (2.9)$$

where

$$\alpha = \alpha(\underset{\sim}{x})$$

Equations (2.7) define ϕ

$$-\nabla^2\phi = \sigma(\underset{\sim}{x})\left[T^1\cos\alpha + T^2\sin\alpha\right] \qquad (2.10)$$

This can be solved by

$$\left.\begin{aligned} \phi &= \phi^1 T^1 + \phi^2 T^2 \\ \text{where}\quad -\nabla^2\phi^1 &= \sigma\cos\alpha \\ \text{and}\quad -\nabla^2\phi^2 &= \sigma\sin\alpha \end{aligned}\right\} \qquad (2.11)$$

For a given $\alpha(\underset{\sim}{x})$ these equations may be solved using Green's functions. The external current becomes

$$J_\mu^E = T^3\left[\phi^2\partial_\mu\phi^1 - \phi^1\partial_\mu\phi^2\right] \qquad (2.12)$$

To give a limit which is symmetric and has continuous fields we take

$$\begin{aligned} \alpha = \alpha(v) &= \beta\sin^2 v \quad \text{if}\quad 2n\pi \le v < (2n+1)\pi \\ &= -\beta\sin^2 v \quad \text{if}\quad (2n+1)\pi \le v < 2(n+1)\pi \end{aligned} \qquad (2.13)$$

where $v = k^3 r^3$; $r^2 = x_1^2 + x_2^2 + x_3^2$

and $0 \le \beta \le \pi$

The limit in which J_ν^E goes to zero is that in which k goes to infinity.

2.3 c) Fields and Interaction Terms

The Vector Potential

$$A_i = g^{-1}\partial_i g = \partial_i \alpha\, T^3 \tag{2.14}$$

The Scalar Potential

$$A_0 = g^{-1}[\phi^1 T^1 + \phi^2 T^2]\, g \tag{2.15}$$

where $-\nabla^2 \phi^1 = \sigma \cos\alpha$

and $-\nabla^2 \phi^2 = \sigma \sin\alpha$

We may Fourier decompose $\sin\alpha$:

$$\sin\alpha(v) = \sum_{n=0}^{\infty} S_n \sin[(2n+1)v] \tag{2.16}$$

where

$$\begin{aligned} S_n &= \frac{1}{\pi}\int_0^{2\pi} dv \sin[(2n+1)v]\sin\alpha \\ &= \frac{1}{\pi} \times \sin\beta/2 \int_0^{2\pi} d\mu \sin[(n+1/2)\mu] \\ &\qquad \times \cos(\beta/2 \cos\mu) \\ &+ \frac{1}{\pi} \times \cos\beta/2 \int_0^{2\pi} d\mu \sin[(n+1/2)\mu] \\ &\qquad \times \sin(\beta/2 \cos\mu) \end{aligned} \tag{2.17}$$

$$\int \partial_i \phi^2 \, dS_i = \int \nabla^2 \phi^2 \, d^3\underset{\sim}{x}$$

$$= \sum_{n=0}^{\infty} S_n \int \sigma \sin\left[(2n+1) k^3 r^3\right] d^3\underset{\sim}{x}$$

$$= \sum_{n=0}^{\infty} S_n \int_{r_{min}}^{r_{max}} r^2 \, dr \, \hat{\sigma}(r) \times \sin\left[(2n+1) k^3 r^3\right] \qquad (2.18)$$

where

$$\hat{\sigma}(r) = \int_{\theta_{min}}^{\theta_{max}} d\theta \int_{\psi_{min}}^{\psi_{max}} \sigma(r, \theta, \psi) \sin\theta$$

If σ is non-singular so must be $\hat{\sigma}(r)$. For large k in this case

$$\int_{r_{min}}^{r_{max}} r^2 \, dr \, \hat{\sigma}(r) \sin\left[(2n+1) k^3 r^3\right]$$

goes to zero (proof in Appendix 1), i.e.

$$\int \partial_i \phi^2 \, dS_i \to 0 \quad \text{as} \quad k \to \infty \qquad (2.19)$$

This can only occur if $\phi^2 \to 0$ everywhere (proof in Appendix 2).

Now

$$\int d^3\underset{\sim}{x} \left[\partial_i \phi^2\right]^2 = -\int d^3\underset{\sim}{x} \, \phi^2 \nabla^2 \phi^2 \qquad (2.20)$$

also $|\nabla^2 \phi^2| \leq \sigma$

i.e. $|\nabla^2 \phi^2|$ is bounded.

Thus $\int d^3\underset{\sim}{x}\ (\partial_i \phi^2)^2 \to 0$

as $k \to \infty$

i.e. $\partial_i \phi^2 \to 0$ as $k \to \infty$ (2.21)

Next let us Fourier decompose $\cos\alpha$:

$$\cos\alpha(\nu) = C + \sum_{n=1}^{\infty} C_n \cos 2n\nu \tag{2.22}$$

where

$$C = C(\beta) = \frac{1}{\pi}\int_0^{\pi} d\nu \cos\alpha(\nu)$$

$$= \frac{1}{\pi}\cos\beta/2 \int_0^{\pi} d\nu \cos(\beta/2 \cos\nu)$$

$$= \frac{1}{2}\cos\beta/2\,[J_0(\beta/2) + J_0(-\beta/2)]$$

$$= \cos\beta/2\ J_0(\beta/2) \tag{2.23}$$

and

$$C_n = C_n(\beta) = \frac{1}{\pi}\int_0^{\pi} d\nu \cos 2n\nu \cos\alpha$$

$$= \frac{1}{\pi}\cos\beta/2 \int_0^{2\pi} d\mu \cos(\beta/2\cos\mu)\cos n\mu$$

$$+ \frac{1}{\pi}\sin\beta/2 \int_0^{2\pi} d\mu \sin(\beta/2\cos\mu)\cos n\mu \tag{2.24}$$

If n is even,

$$C_n(\beta) = \cos\beta/2 \{ J_n(\beta/2) + J_n(-\beta/2)\}(i)^n$$

$$= 2(i)^n \cos\beta/2 \; J_n(\beta/2) \qquad (2.25)$$

If n is odd

$$C_n(\beta) = \sin\beta/2\{J_n(\beta/2) - J_n(-\beta/2)\}(i)^{n-1}$$

$$= 2(i)^{n-1} \sin\beta/2 \; J_n(\beta/2) \qquad (2.26)$$

J_n are Bessel functions.

For the same reason as before (Appendix 1)

$$\int_{r_{min}}^{r_{max}} r^2 dr \; \hat{\sigma}(r) \cos(2\pi k r^3) \to 0 \text{ as } k \to \infty$$

Thus $$-\int \partial_i \phi' \, dS_i \to c(\beta) \int \sigma \, d^3\underset{\sim}{x} \qquad (2.27)$$

The Abelian solution obeys

$$-\nabla^2 \psi = \sigma$$

Thus

$$\int (\partial_i \phi' - c\,\partial_i \psi) \, dS_i \to 0$$

$$\text{as} \quad k \to \infty \qquad (2.28)$$

If we define $\chi = \phi' - c\psi$ then

$\chi \to 0$ and $\partial_i \chi \to 0$ as $k \to \infty$

for the same reason that

$$\phi^2 \to 0 \quad \text{and} \quad \partial_i \phi^2 \to 0 \quad \text{as} \quad k \to \infty$$

Thus $$\phi' \to c\,\psi$$

and $$\partial_i \phi' \to c\,\partial_i \psi$$

as $$k \to \infty \tag{2.29}$$

We may write the scalar potential as

$$A_0 = g^{-1}[\phi' T' + \phi^2 T^2]\, g$$

where $$\phi^2 \to 0 \; ; \; \partial_i \phi^2 \to 0 \tag{2.30}$$
$$\phi' \to c\psi \; ; \; \partial_i \phi' \to c\,\partial_i \psi$$
$$\text{as} \quad k \to \infty$$

If

$$\beta = 0 \, , \; c(\beta) = 1$$

if

$$\beta \sim 0 \, , \; c(\beta) \sim \frac{1}{\pi} \int_0^{\pi} d\nu \left[1 - \tfrac{1}{2}\alpha^2\right] \tag{2.31}$$
$$\sim 1 - \frac{3}{16}\beta^2$$

if

$$\beta = \pi \, , \; c(\beta) = 0$$

The External Current

$$J_\mu^E = g^{-1}[\partial_\mu \phi, \phi] g$$

$$= T^3 \{ \phi^2 \partial_\mu \phi^1 - \phi^1 \partial_\mu \phi^2 \} \qquad (2.32)$$

Substituting for $\phi^1, \partial_\mu \phi^1, \phi^2, \partial_\mu \phi^2$ gives

$$J_\mu^E \to 0 \quad \text{as} \quad k \to 0 \qquad (2.33)$$

The Magnetic Field

Since $$A_i = g^{-1} \partial_i g$$

$$F_{ij} = 0 \qquad (2.34)$$

The Electric Field

$$F_{io} = g^{-1}[\partial_i \phi^1 T^1 + \partial_i \phi^2 T^2] g$$

$$\to C \partial_i \psi \, g^{-1} T^1 g \quad \text{as} \quad k \to \infty \qquad (2.35)$$

The magnitude of the electric field is reduced by the factor $C(\beta)$ with respect to the Abelian case

$$1 \geq C \geq 0$$

for $$0 \leq \beta \leq \pi \qquad (2.36)$$

The Scalar Interaction

This has the form

$$H_{1S} = -2\int d^3\underset{\sim}{x}\ Tr\{A_0[J_0 + J_0^E]\}$$

$$= -2\int d^3\underset{\sim}{x}\ Tr\{g^{-1}[\phi' T' + \phi^2 T^2] g \sigma T'\}$$

$$= \int d^3\underset{\sim}{x}\ \{\phi' \sigma \cos\alpha + \phi^2 \sigma \sin\alpha\}$$

$$\to c\int \psi\, \sigma \cos\alpha \quad \text{as} \quad k \to \infty \qquad (2.37)$$

H_{1S} may also be written as

$$H_{1S} = -\int d^3\underset{\sim}{x}\ \{\phi' \nabla^2 \phi' + \phi^2 \nabla^2 \phi^2\}$$

$$= \int d^3\underset{\sim}{x}\ \{[\partial_i \phi']^2 + [\partial_i \phi^2]^2\}$$

$$= 2 H_E \qquad (2.38)$$

where $H_E = -\int d^3\underset{\sim}{x}\ Tr\{[F_{oi}]^2\}$

i.e. H_E is the energy associated with the electric field.

The Vector Interaction

$$H_{1V} = -2\int d^3\underset{\sim}{x}\ Tr\{A_i J_i^E\}$$

$$= \int d^3\underset{\sim}{x}\ \{\partial_i \alpha [\phi^2 \partial_i \phi' - \phi' \partial_i \phi^2]\}$$

$$= -\int d^3\underset{\sim}{x}\, \alpha\ \{\phi^2 \nabla^2 \phi' - \phi' \nabla^2 \phi^2\}$$

$$\to -c\int d^3\underset{\sim}{x}\ \psi \alpha \sin\alpha \quad \text{as} \quad k \to \infty \qquad (2.39)$$

The Hamiltonian

The total energy of the system is

$$H = -\int d^3\underset{\sim}{x}\ Tr\ \{ F_{oi} F_{oi} + \tfrac{1}{2} F_{ij} F_{ij} \}$$

$$\to \tfrac{1}{2} c^2 \int d^3\underset{\sim}{x}\ (\partial_i \psi)^2$$

as $k \to \infty$ (2.40)

from eqns. (2.34) and (2.35)

Now the energy of the Abelian solution is

$$H_{abel} = \tfrac{1}{2} \int d^3\underset{\sim}{x}\ (\partial_i \psi)^2$$

Thus as $k \to \infty$

$$H \to c^2 H_{abel}$$

If $\beta = 0$ $\quad H = H_{abel}$

$\beta \sim 0$ $\quad H \sim [1 - \frac{3}{5}\beta^2] H_{abel}$ (2.44)

$\beta = \pi$ $\quad H = 0$

from eqn. (2.31)

2.3 d) Interpretation

In the limit where $k \to \infty$ the solution with $\beta = 0$ is identical to the Abelian solution. If β is small the new solution has an energy close to that of the Abelian solution, but less than it. We may conclude from this that the Abelian solution is not clasically stable. The solutions with lowest energy are those with $\beta = \pi$. In this case the electric field produced by the source disappears. The energy of this completely screened solution is zero.

There is one aspect of the problem of which we have not taken account. In concluding that the Abelian solutions are unstable, we have assumed that no conservation law prevents them from decaying. A spinning top is an example of a system which, when analysed in terms of energy, looks unstable but is unable to decay to a state of lower energy because of the conservation of angular momentum. It has been suggested that similar considerations apply to SU(2) [21]. As k increases the perturbation which we have introduced causes a rotation in gauge space with a frequency which likewise increases. This could violate the conservation of some form of spin in gauge space. This question of gauge spin will be examined in sections 4 and 5.

3. WAVES

Having discussed charge in SU(2) gauge theory, we must next discuss waves. The simplest form of wave solutions are the Abelian waves. These solutions to the classical SU(2) equations of motion correspond to dominant paths in the path integral which lie in an Abelian subspace of the Hilbert space of the gauge field operators. If there is a single energy eigenstate in this subspace, then part of the spectrum of SU(2) gauge theory mimic that of electromagnetism.

To gain greater insight we will introduce the additional gauge degrees of freedom. The relevant solutions are found to be those already discovered by Coleman[8]. These correspond to dominant paths in the path integral which lie in an extended subspace containing the Abelian subspace. These paths allow states in the Abelian subspace to decay to states in the extended subspace outside the Abelian subspace. No state in the Abelian subspace is stable against this form of decay.

3.1 ABELIAN WAVES

3.1 a) The Abelian Subspace

The Abelian subspace is defined here as the space spanned by eigenstates of the gauge field operators such that

$$(A^a_\mu(\underset{\sim}{x}, t))\,|\{A\}, t\rangle$$

$$= \delta_{\mu 1}\,\delta^{a3} A_1^3(\underset{\sim}{x})\,|\{A\}, t\rangle \tag{3.1}$$

The operator equations of motion for the subspace reduce to

$$(\partial_1\,\partial_0\,A_1^3) = 0 \tag{3.2}$$

$$(\partial_0^2 A_1^3 - \partial_2^2 A_1^3 - \partial_3^2 A_1^3) = 0 \tag{3.3}$$

$$(\partial_1\,\partial_2\,A_1^3) = 0 \tag{3.4}$$

$$(\partial_1\,\partial_3\,A_1^3) = 0 \tag{3.5}$$

These have the solution

$$(A_1^3) = \int_0^\infty d\omega \int d^3\underset{\sim}{k} \{\alpha(\underset{\sim}{k}, \omega)\, e^{-i[\omega t + \underset{\sim}{k}\cdot\underset{\sim}{x}]} + \alpha^+(\underset{\sim}{k}, \omega)\, e^{i[\omega t + \underset{\sim}{k}\cdot\underset{\sim}{x}]}\}$$
$$\times\, \delta[k_1]\,\delta[\omega^2 - \underset{\sim}{k}^2] \tag{3.6}$$

We denote this subspace by the set of states $\{|\Delta\rangle\}$.

3.1 b) Matrix Elements

Let us restrict our attention to this Abelian subspace of the theory, and consider S-matrix elements between states in the subspace. These elements are

$$\langle o'' | e^{-i(H)t} | o' \rangle$$

We may evaluate these by the path-integral method. The contributions from the various paths are dominated by those paths corresponding to solutions of the full classical equations of motion. Let us consider paths which remain in the Abelian subspace. For these paths

$$(H) = \frac{1}{2}\int d^3\underset{\sim}{x}\left\{(\partial_0 A_1^3)^2 + (\partial_2 A_1^3)^2 + (\partial_3 A_1^3)^2\right\} \tag{3.7}$$

The classical action for these paths is

$$\int L\,dt = \frac{1}{2}\int d^4x\left\{[\partial_0 A_1^3]^2 - [\partial_2 A_1^3]^2 - [\partial_3 A_1^3]^2\right\} \tag{3.8}$$

The dominating contributions, found from varying the action are paths described by $A_1^3(\underset{\sim}{x}, t)$ which obey the same equations as gauge field operators but in terms of functions, not operators , i.e.

$$\partial_1 \partial_0 A_1^3 = 0 \tag{3.9}$$

$$\partial_0^2 A_1^3 - \partial_2^2 A_1^3 - \partial_3^2 A_1^3 = 0 \tag{3.10}$$

$$\partial_1 \partial_2 A_1^3 = 0 \tag{3.11}$$

$$\partial_1 \partial_3 A_1^3 = 0 \tag{3.12}$$

It is apparent that if the only paths which contribute to S-matrix elements between states of the subspace themselves remain in the subspace, then the classical limit of the time evolution of the subspace is the same as for plane electromagnetic waves.

3.1 c) Energy Eigenstates

To gain insight into the true time development of the Abelian subspace we go back to the operator equations (3.2) to (3.4).

Suppose there is an energy eigenstate in the Abelian subspace, i.e.

$$|\psi\rangle : (H)\,|\psi\rangle = E_\psi |\psi\rangle ; \quad |\psi\rangle \in \{|0\rangle\} \tag{3.13}$$

Now

$$[(H), (A_1^3)] = -i\,(\partial_0 A_1^3) \tag{3.14}$$

So

$$(H)(\alpha(\underset{\sim}{k}, \omega))^\dagger |\psi\rangle = (E_\psi + \omega)(\alpha(\underset{\sim}{k}, \omega))^\dagger |\psi\rangle \tag{3.15}$$

and

$$(H)(\alpha(\underset{\sim}{k}, \omega))|\psi\rangle = (E_\psi - \omega)(\alpha(\underset{\sim}{k}, \omega))|\psi\rangle \tag{3.16}$$

Since energy is positive definite its eigenvalues must have a lower bound. The subspace must contain a minimum energy eigenstate if it contains an eigenstate at all. Call this lowest state

$$|0\rangle_0$$

The operators $\alpha(\underset{\sim}{k},\omega)^{\dagger}$ and $\alpha(\underset{\sim}{k},\omega)$ are creation and annihilation operators for the energy eigenstates of the Abelian subspace. They have the same form as those for plane polarised photons in electromagnetism. The state $|0\rangle_D$ is then the vacuum of the Abelian subspace.

3.1 d) Conclusions

The dominant paths in the path integral which lie in the Abelian subspace are described by the same equations as plane polarised electromagnetic waves. In addition the operator equations of motion on the subspace have solutions whose Fourier components may act as creation and annihilation operators as in electromagnetism. It follows that if there is an energy eigenstate in the Abelian subspace SU(2) gauge theory should contain the spectrum of plane polarised photons in electromagnetism.

3.2 NON-ABELIAN WAVES

It is possible to find more general wave-like solutions to the free equations of motion by expanding the form of the Abelian waves to take account of all three gauge degrees of freedom. These solutions correspond to those found by Coleman[8], and are seen to be the dominating paths in a second subspace of the theory. This subspace contains the Abelian subspace defined in Section 3.1.

The dominating paths allow transitions between states of the Abelian subspace and states of the new subspace outside the Abelian subspace, i.e. they allow states of the Abelian subspace to decay into states outside it.

3.2 a) The Expanded Subspace

The expanded subspace is spanned by eigenstates of the gauge field operators such that

$$(A_\mu^a(\underset{\sim}{x}, t))\,|\{A\}, t\rangle$$

$$= \delta_{\mu 1}\, A_1^a(\underset{\sim}{x})\,|\{A\}, t\rangle \tag{3.17}$$

The operator equations of motion on the subspace reduce to

$$(\partial_1 \partial_0 A_1^a) + \epsilon^{abc}(A_1^b)(\partial_0 A_1^c) = 0 \tag{3.18}$$

$$(\partial_0^2 A_1^a - \partial_2^2 A_1^a - \partial_3^2 A_1^a) = 0 \tag{3.19}$$

$$(\partial_1 \partial_2 A_1^a) + \epsilon^{abc}(A_1^b)(\partial_2 A_1^c) = 0 \tag{3.20}$$

$$(\partial_1 \partial_3 A_1^a) + \epsilon^{abc}(A_1^b)(\partial_3 A_1^c) = 0 \tag{3.21}$$

3.2 b) Classical Solutions

In the path-integral formulation we may once again consider paths which remain in the subspace. The dominant paths are then described by functions which obey the same equations as the operators on the subspace, i.e.

$$\partial_1 \partial_0 A_1^a + \epsilon^{abc} A_1^b \partial_0 A_1^c = 0 \tag{3.22}$$

$$\partial_0^2 A_1^a - \partial_2^2 A_1^a - \partial_3^2 A_1^a = 0 \tag{3.23}$$

$$\partial_1 \partial_2 A_1^a + \epsilon^{abc} A_1^b \partial_2 A_1^c = 0 \tag{3.24}$$

$$\partial_1 \partial_3 A_1^a + \epsilon^{abc} A_1^b \partial_3 A_1^c = 0 \tag{3.25}$$

In order to solve these equations we first transform A_1 to zero. The equations now become

$$\partial_1^2 A_0^a = 0 \tag{3.26}$$

$$\begin{aligned} &\partial_0 \partial_1 A_0^a + \epsilon^{abc} A_0^b \partial_1 A_0^c \\ &- \partial_2 \partial_1 A_2^a - \epsilon^{abc} A_2^b \partial_1 A_2^c \\ &- \partial_3 \partial_1 A_3^a - \epsilon^{abc} A_3^b \partial_1 A_3^c = 0 \end{aligned} \tag{3.27}$$

$$\partial_1^2 A_2^a = 0 \tag{3.28}$$

$$\partial_1^2 A_3^a = 0 \tag{3.29}$$

where

$$A_\mu^a = g^{-1} \partial_\mu g \quad \text{for } \mu \in (0, 2, 3) \tag{3.30}$$

[A_1^a is now zero].

Equations (3.26), (3.28) and (3.30) imply that

$$A_\mu^a = -x_1 \alpha_\mu^a + \beta_\mu^a \tag{3.31}$$

where

$$\left.\begin{aligned} &\partial_1 \alpha_\mu^a = 0 = \alpha_1^a \\ &\text{and} \\ &\partial_1 \beta_\mu^a = 0 = \beta_\mu^a \end{aligned}\right\} \tag{3.32}$$

Let

$$g|_{x_1=0} = h \tag{3.33}$$

then

$$\beta_\mu = \beta_\mu^a T^a = h^{-1} \partial_\mu h \tag{3.34}$$

[from eqn. (3.30)].

Equation (3.34) implies that we may gauge transform β_μ to zero without loss of generality, i.e.

$$A_\mu^a = g^{-1}\partial_\mu g \quad \text{for } \mu \in (0,2,3) \tag{3.35}$$

$$\left.\begin{aligned} A_\mu^a &= -x_1 \alpha_\mu^a \\ &\text{where} \\ \partial_1 \alpha_\mu^a &= 0 = \alpha_1^a \end{aligned}\right\} \tag{3.36}$$

Eqn.(3.35) implies that

$$\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0$$
$$\text{for } \mu, \nu \in (0,2,3) \tag{3.37}$$

Substituting from eqn.(3.36) gives

$$-x_1[\partial_\mu \alpha_\nu^a - \partial_\nu \alpha_\mu^a]$$
$$+x_1^2 \epsilon^{abc} \alpha_\mu^b \alpha_\nu^c = 0 \tag{3.38}$$

for $\mu, \nu \in (0,1,2,3)$ now,

i.e.

$$\epsilon^{abc} \alpha_\mu^b \alpha_\nu^c = 0 \tag{3.39}$$

and

$$\partial_\mu \alpha_\nu^a - \partial_\nu \alpha_\mu^a = 0 \tag{3.40}$$

Eqn.(3.39) implies that

$$\alpha_\mu{}^a = \alpha_\mu n^a \tag{3.41}$$

where

$$n^a n^a = 1 \tag{3.42}$$

Substituting into eqn.(3.40) gives

$$[\partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu] n^a + [\partial_\mu n^a \alpha_\nu - \partial_\nu n^a \alpha_\mu] = 0 \tag{3.43}$$

i.e.

$$\partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu = 0 \tag{3.44}$$

and

$$\partial_\mu n^a \alpha_\nu - \partial_\nu n^a \alpha_\mu = 0 \tag{3.45}$$

Eqn.(3.44) implies that

$$\alpha_\nu = \partial_\nu \alpha \tag{3.46}$$

and eqn.(3.45) implies that

$$\partial_\mu n^a = \Lambda^a \alpha_\mu \tag{3.47}$$

Substituting eqns.(3.41),(3.46) and (3.47) into eqn.(3.36) gives

$$A_\mu^a = -x_1 \, n^a \, \partial_\mu \alpha \tag{3.48}$$

where

$$\left.\begin{aligned} &\partial_\mu n^a = \Lambda^a \partial_\mu \alpha \\ &\text{and} \\ &\partial_1 \alpha = 0 \end{aligned}\right\} \tag{3.49}$$

Eqn.(3.42) implies that

$$n^a \Lambda^a = 0 \tag{3.50}$$

Substituting eqn.(3.48) into (3.27) gives

$$\begin{aligned} &[\partial_0^2 \alpha - \partial_2^2 \alpha - \partial_3^2 \alpha] \, n^a \\ &\quad + [\{\partial_0 \alpha\}^2 - \{\partial_2 \alpha\}^2 - \{\partial_3 \alpha\}^2] \, \Lambda^a = 0 \end{aligned} \tag{3.51}$$

Using eqn.(3.50) gives

$$\partial_0^2 \alpha - \partial_2^2 \alpha - \partial_3^2 \alpha = 0 \tag{3.52}$$

and

$$[\partial_0 \alpha]^2 - [\partial_2 \alpha]^2 - [\partial_3 \alpha]^2 = 0 \tag{3.53}$$

Using eqn.(3.49) gives

$$\partial_\mu \partial^\mu \alpha = 0 \tag{3.54}$$

and

$$[\partial_\mu \alpha][\partial^\mu \alpha] = 0 \tag{3.55}$$

Eqn.(3.54) implies that

$$\alpha(x) = \int d^4p\, e^{ip\cdot x}\, \tilde{\alpha}(p)\, \delta(p^2) \tag{3.56}$$

Note that this means that

$$[\alpha(x)]^2 = \int d^4p\, d^4q\, e^{i(p+q)\cdot x} \times \tilde{\alpha}(p)\, \tilde{\alpha}(q)\, \delta(p^2)\, \delta(q^2)$$

$$= \int d^4p\, d^4k\, e^{ik\cdot x}\, \delta(p^2) \times \delta(k^2 - 2p\cdot k)\, \tilde{\alpha}(p)\, \tilde{\alpha}(k-p)$$

$$= \int d^4k\, e^{ik\cdot x} \left\{ \int d^4p\, \delta(p^2) \times \delta(k^2 - 2p\cdot k)\, \tilde{\alpha}(p)\, \tilde{\alpha}(k-p) \right\} \tag{3.57}$$

Substituting eqn.(3.56) into eqn.(3.55) gives

$$0 = \int d^4p\, d^4q\; p\cdot q\; e^{i(p+q)\cdot x} \times \tilde{\alpha}(p)\tilde{\alpha}(q)\,\delta(p^2)\,\delta(q^2)$$

$$= \int d^4p\, d^4k\; p\cdot k\; e^{ik\cdot x}\tilde{\alpha}(p)\tilde{\alpha}(k-p) \times \delta(p^2)\,\delta(k^2-2p\cdot k)$$

$$= \int d^4k\; e^{ik\cdot x}\tfrac{1}{2}k^2\Big\{\int d^4p\;\delta(p^2) \times \delta(k^2-2p\cdot k)\tilde{\alpha}(p)\tilde{\alpha}(k-p)\Big\} \qquad (3.58)$$

Fourier decomposing eqn.(3.58) gives

$$k^2\int d^4p\;\delta(p^2)\,\delta(k^2-2p\cdot k) \times \tilde{\alpha}(p)\tilde{\alpha}(k-p) = 0 \qquad (3.59)$$

Comparing eqns.(3.57) and (3.59) indicates that the only values of k which contribute to $[\alpha(x)]^2$ through the term

$$\int d^4p\;\delta(p^2)\,\delta(k^2-2p\cdot k)\,\tilde{\alpha}(p)\tilde{\alpha}(k-p)$$

are those for which $k^2 = 0$, i.e.

$$2p \cdot k = 0 \tag{3.60}$$

$$\Rightarrow p = \lambda k \tag{3.61}$$

The simultaneous solutions to both (3.54) and (3.55) are characterised by a specific direction of propagation. We chose this to be the 3-direction. In this case, we may substitute the solution of (3.54) and (3.55) into eqn. (3.48) to give

$$A_\mu^a = -x_1 n^a \partial_\mu \alpha(\tau) \tag{3.62}$$

where

$$\left.\begin{aligned} \tau &= t - x_3 \\ \text{and} \\ n^a &= n^a(\tau) \end{aligned}\right\} \tag{3.63}$$

This has the same form as the solutions already found by Coleman [8].

These solutions may be gauge transformed to the gauge where only A_1^a are non-zero. In this gauge

$$A_1 = A_1^a T^a = g \partial_1 g^{-1}$$

$$g = g(x_1, \tau) : g^{-1} \partial_\tau g$$

$$= -x_1 \alpha(\tau) n^a(\tau) T^a \tag{3.64}$$

3.2 c) Halpern Copies

Let us consider two solutions of the form given by eqn.(3.64). The first is

$$A_\mu = \delta_{\mu 1}\, g\, \partial_1 g^{-1}$$

where $g = g(x_1, \tau) : g^{-1}\partial_\tau g = -x_1 \alpha^a(\tau) T^a$ (3.65)

The second is

$$\hat{A}_\mu = \delta_{\mu 1}\, \hat{g}\, \partial_1 \hat{g}^{-1}$$

where $\hat{g} = \hat{g}(x_1, \tau) : \hat{g}^{-1}\partial_\tau \hat{g} = -x_1 \hat{\alpha}^a(\tau) T^a$ (3.66)

and

$$\hat{\alpha}^a(\tau) T^a = k(\tau)\, \alpha^a(\tau)\, T^a\, k^{-1}(\tau) \qquad (3.67)$$

for some gauge transformation $k(\tau)$.

Eqn.(3.65) gives

$$F_{01} = g\, \alpha^a(\tau)\, T^a\, g^{-1} = -F_{31} \qquad (3.68)$$

and eqn.(3.66) gives

$$\hat{F}_{01} = \hat{g}\, \hat{\alpha}^a(\tau)\, T^a\, \hat{g}^{-1} = \hat{g}\, k\, \alpha^a(\tau)\, T^a\, k^{-1} \hat{g}^{-1}$$

$$= -\hat{F}_{31} \qquad (3.69)$$

It is clear from eqns. (3.68) and (3.69) that the field strengths of the two solutions may be gauge transformed into one another.

The gauge fields themselves cannot in general be transformed into one another. To see this we note that it is possible to gauge transform the first solution to the form

$$B_\mu = \delta_{\mu 0} B_0 + \delta_{\mu 3} B_3 = h^{-1} A_\mu h + h^{-1} \partial_\mu h$$

where

$$B_0 = - B_3 = -x_1 k \alpha^a T^a k^{-1} + k \partial_z k^{-1}$$

if (3.70)

$$h = g k^{-1}$$

We may transform the second solution to the form

$$\hat{B}_\mu = \delta_{\mu 0} \hat{B}_0 + \delta_{\mu 3} \hat{B}_3 = \hat{g}^{-1} \hat{A}_\mu g + \hat{g}^{-1} \partial_\mu \hat{g}$$

where

$$\hat{B}_0 = - \hat{B}_3 = - x_1 \hat{\alpha}^a T^a$$
$$= - x_1 k \alpha^a T^a k^{-1} \qquad (3.71)$$

The x_1 and z dependance of β_μ and $\hat{\beta}_\mu$ implies that they are not gauge equivalent unless $\partial_z k = 0$ or $k \alpha^a T^a k^{-1} = \alpha^a T^a$. If β_μ and $\hat{\beta}_\mu$ are not gauge equivalent, the two solutions are Halpern copies (ref. 22). In general, Halpern copies have field stengths which are identical or can be made so by a gauge transformation, but which arise due to gauge fields which are not gauge equivalent.

3.2 d) Decay of Abelian States

Let us choose a particular example of two Halpern copies. The first solution is

$$\left.\begin{aligned} &A_\mu = \delta_{\mu 1} A_1 \\ &\text{where} \\ &A_1{}^a = \delta^{a3} \beta(\tau) \end{aligned}\right\} \qquad (3.72)$$

i.e.

$$\left.\begin{aligned} &g^{-1}\partial_\tau g = -x_1 \alpha^a(\tau) T^a \\ &\text{where} \\ &\alpha^a(\tau) = \delta^{a3}\partial_\tau \beta \end{aligned}\right\} \qquad (3.73)$$

This is an Abelian solution corresponding to a dominant path in the Abelian subspace.

The second solution is

$$\left.\begin{aligned} &\hat{A}_\mu = \delta_{\mu 1}\, \hat{g}\, \partial_1 \hat{g}^{-1} \\ &\text{where} \\ &\hat{g} = \hat{g}(x_1, \tau) : \hat{g}^{-1}\partial_\tau \hat{g} = -x_1 k(\tau)\alpha^a T^a k(\tau) \end{aligned}\right\} \qquad (3.74)$$

and where $k(\tau)$ is a gauge transformation such that

$$\lim_{t\to-\infty} k = I \; ; \; \lim_{t\to-\infty} \partial_0 k = 0 \qquad (3.75)$$

It is clear that as $t \to -\infty$ the two solutions converge. The first solution is a dominant path in

the Abelian subspace, whereas the second solution is a dominant path in the extended subspace starting, at $t = -\infty$, in the Abelian subspace.

3.3 CONCLUSIONS

It is clear from Section (3.2d) that the dominant paths in the extended subspace correspond to Halpern copies of the Abelian solutions propagating along defined rays in space. Each such Abelian solution corresponds to a dominant path in the Abelian subspace, and corresponding to each such path there are an infinite number of Halpern copies which approach it asymptotically as $t \to -\infty$ but which diverge into the extended subspace at later times.

These Halpern copies provide paths for the decay of states in the Abelian subspace. In the quantum theory we would expect these paths to provide mixing between states in the Abelian subspace and other subspaces of the same form but set at different orientations in gauge space.

4. QUANTUM NUMBERS IN A LOCAL GAUGE THEORY
— A QUANTUM MECHANICAL EXAMPLE —

The classical solutions describing static charges and plane waves are seen [Sections 2 and 3] to be complicated by the effects of rotations in gauge space. It seems sensible then to examine the quantum numbers associated with such rotations and this is simplified if a quantum mechanical case is examined. A more general approach will be developed in Section 5.

In this section we will construct a quantum mechanical model which possesses a local [i.e. time-dependent] symmetry. This avoids the complications due to a field theory. The techniques used in the three-component harmonic oscillator provide the basis for analysing the model. The gauge symmetry is found to give a constraint on the quantum numbers. This is to be expected from the Lagrangian which is required to include a term resembling a Lagrange undetermined multiplier.

4.1 THREE COMPONENT HARMONIC OSCILLATOR

This system is simple and straightforward but has the additional factor of spin which is not present in the one-component theory. The Hamiltonian is

$$(H) = \tfrac{1}{2}\{(\pi^a)^2 + \omega^2(\Phi^a)^2\} \tag{4.1}$$

This corresponds to a classical action of

$$L = \tfrac{1}{2}\{\dot{\Phi}^a\dot{\Phi}^a - \omega^2\Phi^a\Phi^a\} \tag{4.2}$$

The operators (π^a) and (Φ^a) are time-dependent, self-adjoint and act in some Hilbert space.

In the classical three-component harmonic oscillator we obtain the equations of motion by requiring the variation of the action with respect to the fields to be zero. In the quantum theory we obtain the equations of motion by imposing commutation relations on the operators. These are

$$\left.\begin{aligned}&[(\pi^a(t)),(\Phi^b(t))] = -i\delta^{ab}\\ &[(\pi^a(t)),(\pi^b(t))] = 0 = [(\Phi^a(t)),(\Phi^b(t))]\end{aligned}\right\} \tag{4.3}$$

and

$$\left.\begin{aligned}&[(H),(\Phi^a)] = -i(\partial_0\Phi^a)\\ &[(H),(\pi^a)] = -i(\partial_0\pi^a)\end{aligned}\right\} \tag{4.4}$$

Equations (4.3) and (4.4) lead to

$$\left.\begin{aligned}(\pi^a) &= (\partial_0 \Phi^a) \\ (\partial_0^2 \Phi^a) &= -\omega^2 (\Phi^a)\end{aligned}\right\} \tag{4.5}$$

Up to this point the only change from the one-component theory is the additional index on the operators.

Spin can now be introduced. Define

$$(S^a) = \epsilon^{abc} (\Phi^b)(\pi^c) \tag{4.6}$$

Equations (4.4) and (4.5) imply that

$$[(H), (S^a)] = -i(\partial_0 S^a) = 0 \tag{4.7}$$

In addition, equation (5.3) allows us to write

$$[(S^a), (S^b)] = -\frac{i}{2} \epsilon^{abc} (S^c) \tag{4.8}$$

Defining

$$(S)^2 = (S^a)(S^a) \tag{4.9}$$

we conclude that

$$\left.\begin{aligned}[(S)^2, (S^a)] &= 0 \\ [(S)^2, (H)] &= 0\end{aligned}\right\} \tag{4.10}$$

Since (π^a) and (Φ^a) are self adjoint so are (H), (S^a) and $(S)^2$. Equations (4.7), (4.8) and (4.10) imply that we may choose (H), (S^3) and $(S)^2$ to have simultaneous eigenfunctions.

We may also introduce the ladder operators

$$(S^{\pm}) = (S^1) \pm i(S^2) \tag{4.11}$$

By considering the effects of these operators on the simultaneous eigenfunctions we may deduce that the eigenvalues of $(S)^2$ and (S^3) are discrete[23]. We may write these eigenfunctions as

$$|k, \sigma, \sigma^3\rangle$$

where

$$\left.\begin{aligned} (H)|k, \sigma, \sigma^3\rangle &= E_{k,\sigma}|k, \sigma, \sigma^3\rangle \\ (S)^2|k, \sigma, \sigma^3\rangle &= \sigma(\sigma+1)|k, \sigma, \sigma^3\rangle \\ (S^3)|k, \sigma, \sigma^3\rangle &= \sigma^3|k, \sigma, \sigma^3\rangle \end{aligned}\right\} \tag{4.12}$$

The labels σ and σ^3 are both either integral or half-integral, and

$$-\sigma \leq \sigma^3 \leq \sigma$$

4.1 a) Fourier Decomposition

We may Fourier decompose $(\mathcal{E}^a)$:

$$(\mathcal{E}^a) = (\alpha^a)\, e^{-i\omega t} + (\alpha^a)^\dagger\, e^{i\omega t} \tag{4.13}$$

Equations (4.3) and (4.5) then imply

$$[(\alpha^a), (\alpha^b)^\dagger] = \frac{1}{2\omega}\, \delta^{ab} \tag{4.14}$$

The operators (α^a) and $(\alpha^a)^\dagger$ allow us to construct a Fock space. We expect to build energy eigenstates by the action of $(\alpha^a)^\dagger$ on the vacuum.

Equations (4.4) and (4.13) imply that

$$\left.\begin{aligned} [(H), (\alpha^a)] &= -\omega\, (\alpha^a) \\ [(H), (\alpha^a)^\dagger] &= \omega\, (\alpha^a)^\dagger \end{aligned}\right\} \tag{4.15}$$

This implies that

$$\begin{aligned} (H)(\alpha^a)^\dagger |0\rangle &= \omega\, (\alpha^a)^\dagger |0\rangle + (\alpha^a)^\dagger (H)|0\rangle \\ &= (\omega + E_0)(\alpha^a)^\dagger |0\rangle \end{aligned} \tag{4.16}$$

where $|0\rangle$ is the vacuum state and has energy E_0 .

4.1 b) Spin in the Path Integral

We may gain more insight by rewriting the Hamiltonian in terms of spin (see ref. 24 and 25).

Set

$$\left.\begin{aligned} (\Phi^a) &= (\Phi)(n^a) \\ \text{where} \quad & \\ (n^a)(n^a) &= I \end{aligned}\right\} \tag{4.17}$$

(Φ) and (n^a) have simultaneous eigenvalues which are obtained from the eigenvalues of (Φ^a).

Then

$$(S)^2 = (\Phi)^4 (\partial_0 n^a)(\partial_0 n^a) \tag{4.18}$$

Let

$$(\pi) = (\partial_0 \Phi) \tag{4.19}$$

From equations (4.2), (4.18) and (4.19)

$$(H) = \tfrac{1}{2}\left\{(\pi)^2 + \omega^2 (\Phi)^2 + (S)^2/(\Phi)^2\right\} \tag{4.20}$$

The path-integral formulation for S-matrix elements between states of the form $|k, s, s^3\rangle$ gives stationary action for paths which obey the equation of motion

$$\partial_0^2 \Phi = -\omega^2 \Phi + s(s+1)/\Phi^3 \qquad (4.21)$$

This is in agreement with the equations of motion for Φ^a.

This discussion of spin is quite general. It is a consequence of introducing three components into the harmonic oscillator. There would be no distinction drawn here between the quantisation of angular momentum or of isospin.

4.2 THE Q-MODEL

There are three major differences between SU(2) gauge theory and the three-component harmonic oscillator. SU(2) gauge theory is non-linear, possesses a local symmetry and has operators dependent on all four space-time variables. In this section we construct a model which enables us to examine the first two of these three differences. By limiting dependence of operators to the time variable alone, we can examine the quantum behaviour of the theory through canonical quantisation. We shall call this model the Q-model because of the similarity of certain spin operators to the charge operator of other systems.

We shall examine first the structure of this model in a global form before making the symmetry local and so creating the Q-model. The Q-model has the property that it is equivalent to spacially invariant SU(2) gauge theory.

4.2 a) The Global Model

Classical Definition

The classical system of interest here has the Lagrangian

$$\left.\begin{aligned} L &= -\tfrac{1}{4}\, G^{a}_{\mu\nu}\, G^{a\mu\nu} \\ G^{a}_{\mu\nu} &: \quad G^{a}_{\mu\nu} = -\, G^{a}_{\mu\nu} \\ \text{and} \quad & \\ G^{a}_{oi} &= \partial_o B^{a}_{i} \\ G^{a}_{ij} &= \epsilon^{abc}\, B^{b}_{i}\, B^{c}_{j} \end{aligned}\right\} \qquad (4.22)$$

There is a deliberate notational similarity to SU(2) gauge theory. At present we shall ignore this and examine the model as defined by eqn.(4.22).

Classical Solutions

Eqn.(4.22) gives rise to the classical equation of motion

$$\partial_o G^{a}_{oj} = \epsilon^{abc}\, B_i^{\,b}\, G^{c}_{ij} \qquad (4.23)$$

The theory has an O(3) x SU(2) global symmetry with corresponding constants of motion.

The quantities

$$Q^{a} = \epsilon^{abc}\, B^{b}_{i}\, G^{c}_{oi} \qquad (4.24)$$

will be particularly important. From eqns.(4.22) and (4.23) we see that

$$\partial_0 Q^a = 0 \tag{4.25}$$

The triplet of constants $\{Q^a\}$ we shall call Q-spin.

We may find a solution to eqn.(4.23) by using the ansatz

$$B^a_i = \epsilon^a{}_i{}^\rho \beta^\rho \tag{4.26}$$

Substituting eqn.(4.26) into eqn.(4.22) gives

$$\left.\begin{aligned} G^a_{0i} &= \epsilon^a{}_i{}^\rho \partial_0 \beta^\rho \\ G^a_{ij} &= \epsilon_{ij}{}^\rho \beta^a \beta^\rho \end{aligned}\right\} \tag{4.27}$$

Substituting eqn.(4.27) into eqn.(4.23) gives

$$\epsilon^a{}_i{}^\rho \partial_0^2 \beta^\rho = - \epsilon^a{}_i{}^\rho \beta^\rho \beta^\tau \beta^\tau \tag{4.28}$$

or

$$\partial_0^2 \beta^\rho = - \beta^\rho [\beta]^2 \tag{4.29}$$

Eqn.(4.29) is the equation of motion for an anharmonic three-component oscillator.

Substituting eqn.(4.26) into eqn.(4.24) gives

$$\begin{aligned} Q^a &= \epsilon^{abc}\epsilon^{b}{}_{i}{}^{p}\epsilon^{c}{}_{i}{}^{q}\beta^p\partial_0\beta^q \\ &= \epsilon^{apq}\beta^p\partial_0\beta^q \end{aligned} \tag{4.30}$$

We may further simplify eqn.(4.29) by writing

$$\left.\begin{aligned} &\beta^p = \gamma n^p \\ &\text{where} \\ &n^p n^p = 1 \end{aligned}\right\} \tag{4.31}$$

Substituting eqn.(4.31) into eqn.(4.29) gives

$$\begin{aligned} &\partial_0^2\gamma\, n^p + 2\,\partial_0\gamma\,\partial_0 n^p \\ &\qquad + \gamma\,\partial_0^2 n^p = -\gamma^3 n^p \end{aligned} \tag{4.32}$$

i.e. $$\partial_0^2\gamma + \gamma\, n^p\partial_0^2 n^p = -\gamma^3 \tag{4.33}$$

and $$\epsilon^{abc}\{2\,\partial_0\gamma\, n^p\partial_0 n^q + \gamma\, n^p\partial_0^2 n^q\} = 0 \tag{4.34}$$

Eqn.(4.34) is equivalent to eqn.(4.25) with this ansatz.

Substituting eqn.(4.31) into eqn.(4.30) gives

$$\epsilon^{\alpha\beta\gamma} n^\beta \partial_0 n^\gamma = Q^\alpha/\gamma^2 \tag{4.35}$$

Since eqn.(4.31) implies that

$$n^\gamma \partial_0 n^\gamma = 0 \tag{4.36}$$

eqn.(4.35) implies that

$$\partial_0 n^\alpha = \epsilon^{\alpha\rho\sigma} Q^\alpha n^\rho/\gamma^2 \tag{4.37}$$

Thus

$$\begin{aligned} n^\sigma \partial_0^2 n^\sigma &= \epsilon^{\alpha\rho\sigma} n^\sigma \partial_0 n^\rho Q^\alpha/\gamma^2 \\ &= -Q^\alpha Q^\alpha/\gamma^4 \end{aligned} \tag{4.38}$$

[using eqn.(4.35).]

Substituting into eqn.(4.33) gives

$$\left.\begin{aligned} &\partial_0^2 \gamma = q^2/\gamma^2 - \gamma^3 \\ &\text{where} \\ &q^2 = Q^\alpha Q^\alpha \end{aligned}\right\} \tag{4.39}$$

q is also a constant of the motion.

The classical Hamiltonian corresponding to eqn. (4.22) is

$$H = \tfrac{1}{2}\left[G^{a}_{oi}\right]^2 + \tfrac{1}{4}\left[G^{a}_{ij}\right]^2 \qquad (4.40)$$

Substituting eqns.(4.22),(4.26) and (4.31) into eqn.(4.40) gives

$$E = (\partial_o \gamma)^2 + q^2/\gamma^2 + \tfrac{1}{2}\gamma^4 \qquad (4.41)$$

where $E \equiv H$ is the energy of a solution. Naturally, for a solution,

$$\partial_o E = 0 \qquad (4.42)$$

Thus solutions which obey the ansatz, eqn.(4.26) are characterised by energy and Q-spin. They may be written as

$$\left.\begin{aligned} & B^{a}_{i} = \epsilon^{a}{}_{i}{}^{\rho}\,\beta^{\rho} \\ & \text{where} \\ & \beta^{\rho} = \gamma\, n^{\rho} \;:\; n^{\rho} n^{\rho} = 1 \\ & \text{and} \\ & t = \int d\gamma \left\{ E - \tfrac{1}{2}\gamma^4 - q^2\gamma^{-2} \right\}^{-1/2} \\ & \epsilon^{a\rho\varrho}\, n^{\rho}\, \partial_o\, n^{\varrho} = Q^{a}/\gamma^2 \\ & \text{with} \\ & Q^{a} Q^{a} = q^2 \;;\; \partial_o Q^{a} = 0 \end{aligned}\right\} \qquad (4.43)$$

For simplicity we could choose $Q^{a} = \delta^{a3} q$.

Quantum Mechanical Form

The Hamiltonian operator of the quantum mechanical form of the global model is

$$\left.\begin{aligned}(H) &= \tfrac{1}{4}\{2(D^a_i)(D^a_i) + (G^a_{ij})(G^a_{ij})\} \\ \text{where} \\ (G^a_{ij}) &= \epsilon^{abc}(B^b_i)(B^c_j)\end{aligned}\right\} \qquad (4.44)$$

The canonical quantisation conditions are

$$\left.\begin{aligned}[(D^a_i(t)),(B^b_j(t))] &= -i\delta^{ab}\delta_{ij} \\ [(D^a_i(t)),(D^b_j(t))] &= 0 \\ [(B^a_i(t)),(B^b_j(t))] &= 0\end{aligned}\right\} \qquad (4.45)$$

The time development of the theory is determined by

$$\left.\begin{aligned}&[(H),(B^a_j)] = -i(\partial_0 B^a_j) \\ &\text{and} \\ &[(H),(D^a_j)] = -i(\partial_0 D^a_j)\end{aligned}\right\} \qquad (4.46)$$

These lead to the operator equations of motion

$$\left.\begin{aligned}&(D^a_i) = (\partial_0 B^a_i) \\ &\text{and} \\ &(\partial_0 B^a_i) = (B^a_j)(B^b_i)(B^b_j) \\ &\qquad\qquad -(B^a_i)(B^b_j)(B^b_j)\end{aligned}\right\} \qquad (4.47)$$

The global model has quantum numbers associated with the full O(3) x SU(2) symmetry. In particular the operators for Q-spin are

$$(Q^a) = \epsilon^{abc}(B^b_j)(D^c_j) \tag{4.48}$$

Eqns.(4.47) imply

$$[(H),(Q^a)] = -i(\partial_0 Q^a) = 0 \tag{4.49}$$

Eqns.(4.45) imply

$$[(Q^a),(Q^b)] = -\tfrac{i}{2}\epsilon^{abc}(Q^c) \tag{4.50}$$

If we define $(Q)^2 = (Q^a)(Q^a)$ then equations (4.49) and (4.50) imply that

$$[(Q)^2,(Q^a)] = 0$$
and
$$[(Q)^2,(H)] = 0 \tag{4.51}$$

In exact analogy with the three-component harmonic oscillator we may arrange for $(H),(Q)^2$ and (Q^3) to have simultaneous eigenfunctions . We shall denote these by

$$|k, q, q^3\rangle$$

where

$$(H)\,|k, q, q^3\rangle = E_{k,q}\,|k, q, q^3\rangle$$

$$(Q)^2\,|k, q, q^3\rangle = q(q+1)\,|k, q, q^3\rangle \qquad (4.52)$$

$$(Q^3)\,|k, q, q^3\rangle = q^3\,|k, q, q^3\rangle$$

With

$$-q \leq q^3 \leq q \qquad (4.53)$$

The labels q and q^3 are again both integral or half-integral.

In this model Q-spin is a manifestation of the global SU(2) symmetry in the same way that the angular momentum is a manifestation of the rotational invariance of the three-dimensional harmonic oscillator or of the hydrogen atom. It should similarly play an important part in our understanding of this system.

4.2 b) The Q-Model

In order to make the global symmetry of section 4.2a) into a local symmetry, we must introduce quantities which have indeterminate time development into the Lagrangian. These quantities have some of the properties of Lagrange multipliers. There are three of these, which we shall call B_0^a and introduce by changing the definition of G_{oi}^a in eqns.(4.22).

The Q-model has the classical Lagrangian

$$L = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}$$

where $G^a_{\mu\nu}$ is antisymmetric in μ and ν

$$G^a_{oi} = \partial_o B^a_i + \epsilon^{abc} B^b_o B^c_i$$

and (4.54)

$$G^a_{ij} = \epsilon^{abc} B^b_i B^c_j$$

In the matrix notation

$$B_\mu = B_\mu{}^a T^a$$

The local gauge symmetry of the Q-model can be expressed by the transformations

$$\left.\begin{aligned} B_i &\to g^{-1} B_i g \\ B_o &\to g^{-1} B_o g + g^{-1} \partial_o g \end{aligned}\right\} \qquad (4.55)$$

where $g = g(t) \in SU(2)$.

These transformations do not alter the classical Lagrangian.

The Hamiltonian operator is

$$(H) = \frac{1}{4}\{ 2 (D^a_i)(D^a_i) + (G^a_{ij})(G^a_{ij}) \} \qquad (4.56)$$

The absence of conjugate momenta for the B_o^a and gauge transformation of eqns.(4.55) cause difficulties for canonical quantisation. These are eliminated if we choose the gauge where $B_o^a = 0$.

In this gauge the canonical quantisation relations are

$$\left.\begin{aligned} [(D_i^a(t)), (B_j^b(t))] &= -i\,\delta^{ab}\delta_{ij} \\ [(D_i^a(t)), (D_j^b(t))] &= 0 \\ [(B_i^a(t)), (B_j^b(t))] &= 0 \end{aligned}\right\} \qquad (4.57)$$

These are equivalent to eqns.(4.45).

Time dependence of the operators are determined by eqns.(4.46) leading to the equations of motion (4.47). In this gauge we may define Q-spin by

$$(Q^a) = \epsilon^{abc} (B^b_j)(D^c_j) \qquad (4.58)$$

which is identical to eqn.(4.48). The arguments used in Section 4.2a) imply that we may choose to describe the energy eigenstates as

$$|k, q, q^3\rangle$$

which obeys eqns.(4.52.)

This provides a valid description of the quantum version of the Q-model in the gauge where $(B_0^a) = 0$. As the physical content of the theory is independent of the choice of gauge, it is at first sight identical to the physical content of the global model of Section 4.2a). The difference is due to the classical equations of motion which arise from variations in the classical action caused by variations in the B_0^a .

In operator form this equation of motion is

$$\epsilon^{abc}(B_j{}^b)(\partial_0 B_j^c) = 0$$

$$\text{i.e. } (Q^a) = 0 \qquad (4.59)$$

Thus the energy eigenstates must be of the form

$$|k, q=0, q^3=0\rangle$$

The constraint imposed when the global SU(2) invariance is made into a local SU(2) invariance is exemplified when classical solutions are sought which obey the ansatz given in eqn.(4.26). The solutions for the global symmetry correspond to solutions to a three-dimensional anharmonic oscillator [eqn. (4.29)] with Q-spin corresponding to angular momentum [eqn. (4.30)]. For the local symmetry the equations of motion can be made identical to eqn.(4.29) but with angular momentum, i.e. Q-spin, limited to zero.

Since the Q-model is identical to spacially invariant SU(2) gauge theory, a corresponding constraint should be expected in the general form of SU(2) gauge theory.

4.2 c) Interpretation

The Q-model is one of the simplest systems with a local SU(2) symmetry. Because the operators depend only on time, we can quantise by imposing canonical commutation relations (eqns.(5.24)) in the gauge where the eigenvalues of (B_0^a) are zero.

Once this gauge has been chosen it is possible to define the quantum operator for Q-spin. This is quantised for the same reasons that spin in the three-component harmonic oscillator is quantised and is a manifestation of the global SU(2) invariance. The additional complication that this global symmetry is one aspect of a local symmetry means that the eigenvalues of Q-spin are zero (eqns.(4.59)).

The Q-model has been chosen to be equivalent to SU(2) gauge theory with gauge fields independent of space. In other theories spacial independence is taken to imply that we are dealing with a system in its rest frame[26]. For SU(2) gauge theory the question of its rest frame is complicated by the fundamental Lorentz covariance.

However, the Q-model does describe the large coupling limit of SU(2) gauge theory. If we reintroduce the coupling constant by rescaling the gauge fields and take the limit where size of the terms in the coupling constant swamps the spacial variations, we obtain the spacially invariant theory. The derivative with respect to time must be retained in order that the conjugate momenta may still be defined.

The constraint imposed when the global SU(2) invariance is made into a local SU(2) invariance is exemplified when classical solutions are sought which obey the ansatz given in eqn.(4.26). The solutions for the global symmetry correspond to those for a 3-dimensional anharmonic oscillator [eqn.(4.29)] with Q-spin corresponding to angular momentum [eqn.(4.30)]. For the local symmetry the equations of motion can be made identical to eqn.(4.29) but with angular momentum, i.e. Q-spin, limited to zero.

Since the Q-model is identical to spacially invariant SU(2) gauge theory, a corresponding constraint should be expected in the general form of SU(2) gauge theory.

5. A SIMPLIFICATION OF SU(2) MATRIX ELEMENTS

In quantum mechanics a global symmetry may be used to relate S-matrix elements, with important consequences for the structure of the theory. Likewise in SU(2) gauge field theory we expect S-matrix elements to be related by the gauge symmetry.

In Section 5.1 the general form of the S-matrix elements is obtained. There is little new in this and the section is included only for completeness. In Section 5.2 the equality of certain S-matrix elements is deduced as a consequence of the gauge symmetry. This is a more powerful result than for a global symmetry as could be expected.

The equality of these S-matrix elements allows for simplification. When analysed in terms of angular dependance in gauge space most S-matrix elements seem to vanish. This is demonstrated in Section 5.3. The states with non-zero S-matrix elements have a simplified form which has a similar structure in gauge space as the hydrogen atom's s=0 states in real space. This effect is due to the ambiguity in the time evolution of the states inherent in the local symmetry and indicates that the gauge must be fixed in path integral calculations in order to obtain a well defined dynamical evolution.

In section 5.4 the effect of the global symmetry is examined and in section 5.5 a comparison is made with

electromagnetism.

The results are summarised in Section 5.6

5.1 GENERAL FORM OF S-MATRIX ELEMENTS

We may write the general S-matrix element in the gauge where $\{(A_0^a)\}$ is zero [cf. eqn.(1.54)]

$$\begin{aligned}
&\langle \{\underset{\sim}{A}''\}, t'' | \{\underset{\sim}{A}'\}, t' \rangle \\
&= N_1 \int \prod_{t=t'}^{t''} \Pi_{\underset{\sim}{x},\mu,a} \, dA_\mu^a(\underset{\sim}{x}, t) \exp[i S\{A\}] \\
&\times \delta[A_0^a(\underset{\sim}{x}, t)] \, \delta[A_j''^a(\underset{\sim}{x}) - A_j^a(\underset{\sim}{x}, t'')] \\
&\times \Pi_j \delta[A_j'^a(\underset{\sim}{x}) - A_j^a(\underset{\sim}{x}, t')]
\end{aligned} \tag{5.1}$$

Let us choose some gauge transformation

$$\left.\begin{aligned}
h(\underset{\sim}{x}, t) : \; h(\underset{\sim}{x}, t'') &= I = h(\underset{\sim}{x}, t') \\
\partial_0 h(\underset{\sim}{x}, t'') &= 0 = \partial_0 h(\underset{\sim}{x}, t')
\end{aligned}\right\} \tag{5.2}$$

and define

$$\begin{aligned}
B_\mu^{(h)} &= B_\mu^{(h)a} T^a \\
&= h^{-1} B_\mu h + h^{-1} \partial_\mu h \\
&= A_\mu
\end{aligned} \tag{5.3}$$

Eqn.(5.2) implies that

$$\left.\begin{aligned} B_\mu^{(h)}(\underset{\sim}{x}, t'') &= B_\mu(\underset{\sim}{x}, t'') \\ \text{and} \\ B_\mu^{(h)}(\underset{\sim}{x}, t') &= B_\mu(\underset{\sim}{x}, t') \end{aligned}\right\} \tag{5.4}$$

Substituting eqns. (5.3) and (5.4) into eqn.(5.1) gives

$$\begin{aligned} &\langle \{\underset{\sim}{A}''\}, t'' | \{A'\}, t' \rangle \\ &= N_1 \int \prod_{t=t'}^{t''} \prod_{\underset{\sim}{x},\mu,a} dB_\mu^{(h)a}(\underset{\sim}{x},t) \exp\left[i S\{B^{(h)}\}\right] \\ &\quad \times \delta\left[B_0^a(\underset{\sim}{x}, t'')\right] \delta\left[B_0^a(\underset{\sim}{x}, t')\right] \\ &\quad \times \prod_j \delta\left[A_j''^a(\underset{\sim}{x}) - B_j^a(\underset{\sim}{x}, t'')\right] \delta\left[B_j^a(\underset{\sim}{x}, t') - A_j'^a(\underset{\sim}{x})\right] \\ &\quad \times \prod_{\tau>t'}^{<t''} \prod_{\underset{\sim}{y},b} \delta\left[B_0^{(h)b}(\underset{\sim}{y}, \tau)\right] \end{aligned} \tag{5.5}$$

Now the integral over $\pi^a d B_\mu^{(h)a}$ is $\int \pi^a d B_\mu^a$ ..
as the Jacobian of the transformation is unity and the classical action is also invariant so we may write

$$\langle \{\underset{\sim}{A}''\}, t'' | \{\underset{\sim}{A}'\}, t' \rangle$$

$$= N_1 \int \prod_{t=t'}^{t''} \pi_{\underset{\sim}{x},\mu,a} \, d B_\mu^a(\underset{\sim}{x}, t) \exp[i S\{B\}]$$

$$\times \delta[A_\mu''^a(\underset{\sim}{x}) - B_\mu^a(\underset{\sim}{x}, t'')]$$

$$\times \delta[A_\mu'^a(\underset{\sim}{x}) - B_\mu^a(\underset{\sim}{x}, t')]$$

$$\times \prod_{\tau>t'}^{<t''} \pi_{y,b} \, \delta[B_o^{(h)b}(y, \tau)] \qquad (5.6)$$

Since $A_o'^a(\underset{\sim}{x}) = 0 = A_o^{a\,''}(\underset{\sim}{x})$

Let us define Δ^{-1} as

$$\Delta^{-1} = \int \prod_{t>t'}^{<t''} \prod_{\underset{\sim}{x},a} d\omega_g^a(\underset{\sim}{x}, t) \, \delta[B_o^{(g)a}(\underset{\sim}{x}, t)] \qquad (5.7)$$

Where

$$g(\underset{\sim}{x}, t) = \exp\{\omega_g^a(\underset{\sim}{x}, t) \, T^a\}$$

We may write $B_0^{(g)}$ as

$$B_0^{(g)} = (kg)^{-1}\,\partial_0\,(kg)$$

for some gauge transformation $k(\underset{\sim}{x}, t)$.

In addition, as integrating over $d\omega_g^a$ integrates over the entire gauge volume, it is irrelevant whether we integrate over $d\omega_g^a$ or $d\omega^a_{(k,g)}$.

Thus we may rewrite eqn.(5.7) as

$$\Delta^{-1} = \int \prod_{t>t'}^{<t''} \pi_{\underset{\sim}{x},a}\, d\omega_g^a(\underset{\sim}{x}, t)\, \delta[\{g^{-1}\partial_0 g\}^a] \qquad (5.8)$$

Δ^{-1} is now seen to be independent of B_0^a .

We may also note that

$$\int \prod_{t>t'}^{<t''} \pi_{y,b}\, d\omega_h^b \,\langle \{A''\}, t'' | \{A'\}, t'\rangle$$

$$= K \langle \{A''\}, t'' | \{A'\}, t'\rangle \qquad (5.9)$$

since the matrix element is independent of the integration. Substituting eqn.(5.9) into (5.6) we obtain

$$\langle \{\underset{\sim}{A}''\}, t'' | \{\underset{\sim}{A}'\}, t' \rangle$$

$$= K^{-1} N_1 \int \prod_{t=t'}^{t''} \pi_{\underset{\sim}{x},a,\mu} \, dB^a_\mu(\underset{\sim}{x},t) \, \exp[iS\{B\}]$$

$$\times \delta[A''^a_\mu(\underset{\sim}{x}) - B^a_\mu(\underset{\sim}{x},t'')] \, \delta[A'^a_\mu(\underset{\sim}{x}) - B^a_\mu(\underset{\sim}{x},t')]$$

$$\times \int \prod_{\tau>t'}^{<t''} \pi_{y,b} \, d\omega_h{}^b(y,\tau) \, \delta[B^{(h)}{}^b_o(y,\tau)]$$

$$= K^{-1}\Delta^{-1} N_1 \int \prod_{t=t'}^{t''} \prod_{\underset{\sim}{x},\mu,a} dA^a_\mu(\underset{\sim}{x},t) \, \exp[iS\{A\}]$$

$$\times \delta[A''^a_\mu(\underset{\sim}{x}) - A^a_\mu(\underset{\sim}{x},t'')]$$

$$\times \delta[A'^a_\mu(\underset{\sim}{x}) - A^a_\mu(\underset{\sim}{x},t')] \qquad (5.10)$$

[using eqn.(5.3)].

As K and Δ are constants they may be absorbed into the normalisation so that we may write

$$\langle \{\underset{\sim}{A}''\}, t'' | \{\underset{\sim}{A}'\}, t' \rangle$$

$$= \hat{N} \int \prod_{t=t'}^{t''} \pi_{\underset{\sim}{x},\mu,a} \, dA^a_\mu(\underset{\sim}{x},t) \, \exp[iS\{A\}] \qquad (5.11a)$$

$$\times \delta[A''^a_\mu(\underset{\sim}{x}) - A^a_\mu(\underset{\sim}{x},t'')]$$

$$\times \delta[A'^a_\mu(\underset{\sim}{x}) - A^a_\mu(\underset{\sim}{x},t')]$$

We may generalise this by defining

$$\langle \{A''\}, t'' | \{A'\}, t' \rangle$$

$$= \hat{N} \int \prod_{t=t'}^{t''} \prod_{\underset{\sim}{x},\mu,a} d A_\mu^a(\underset{\sim}{x}, t) \exp[i S\{A\}]$$

$$\times \delta[A_\mu''^a(\underset{\sim}{x}) - A_\mu^a(\underset{\sim}{x}, t'')]$$

$$\times \delta[A_\mu'^a(\underset{\sim}{x}) - A_\mu^a(\underset{\sim}{x}, t')] \qquad (5.11b)$$

Eqn.(5.11b) is the general gauge independent form of the S-matrix elements. As this form contains an implicit sum over all gauges, it is not useful for conventional path-integral calculations.

5.2 EQUALITY OF S-MATRIX ELEMENTS

Let us choose some general gauge transformation and define

$$B_\mu^{(g)} = B_\mu^{(g)a} T^a$$
$$= g^{-1} B_\mu g + g^{-1} \partial_\mu g$$
$$= A_\mu \qquad (5.12)$$

Substituting into eqn. (5.11) gives

$$\langle \{A''\}, t'' | \{A'\}, t' \rangle$$
$$= \hat{N} \int \prod_{t=t'}^{t''} \Pi_{\underset{\sim}{x},\mu,a} \, d B_\mu^{(g)a}(\underset{\sim}{x}, t) \exp [i S\{B^{(g)}\}]$$
$$\times \delta [A''^a_\mu(\underset{\sim}{x}) - B_\mu^{(g)a}(\underset{\sim}{x}, t'')]$$
$$\times \delta [A'^a_\mu(\underset{\sim}{x}) - B_\mu^{(g)a}(\underset{\sim}{x}, t')]$$

(5.13)

Once again we note that the Jacobian of the gauge transformation is unity and that the action is gauge invariant, so eqn.(5.13) gives

$$\langle \{A''\}, t'' | \{A'\}, t' \rangle$$

$$= \hat{N} \int \prod_{t=t'}^{t''} \Pi_{\underset{\sim}{x},\mu,a} \, d B^a_\mu(\underset{\sim}{x}, t) \, \exp[i S\{B\}]$$

$$\times \delta[A''^a_\mu(\underset{\sim}{x}) - B^{(g)a}_\mu(\underset{\sim}{x}, t'')]$$

$$\times \delta[A'^a_\mu(\underset{\sim}{x}) - B^{(g)a}_\mu(\underset{\sim}{x}, t')] \qquad (5.14)$$

Let

$$\left.\begin{aligned} A''_\mu(\underset{\sim}{x}) &= \{g^{-1}\hat{A}''_\mu(\underset{\sim}{x})g + g^{-1}\partial_\mu g\}|_{t=t''} \\ \text{and} \\ A'_\mu(\underset{\sim}{x}) &= \{g^{-1}\hat{A}'_\mu(\underset{\sim}{x})g + g^{-1}\partial_\mu g\}|_{t=t''} \end{aligned}\right\} \qquad (5.15)$$

Now

$$\pi^a \delta[A''^a_\mu(\underset{\sim}{x}) - B^{(g)a}_\mu(\underset{\sim}{x}, t'')]$$

$$= \pi^a \delta[\{g^{-1}(\underset{\sim}{x}, t'')[\hat{A}''^a_\mu(\underset{\sim}{x}) - B^a_\mu(\underset{\sim}{x}, t'')]g(\underset{\sim}{x}, t'')\}^a] \qquad (5.16)$$

The right-hand side of eqn.(5.16) is zero unless $\hat{A}''^a_\mu(\underset{\sim}{x})$ equals $B^a_\mu(\underset{\sim}{x}, t'')$, i.e.

$$\pi^a \delta[A''^a_\mu(\underset{\sim}{x}) - B^{(g)a}_\mu(\underset{\sim}{x}, t'')]$$

$$= \pi^a \delta[\hat{A}''^a_\mu(\underset{\sim}{x}) - B^a_\mu(\underset{\sim}{x}, t'')] \qquad (5.17)$$

Substituting eqn.(5.17) into (5.14) gives

$$\langle \{A''\}, t'' | \{A', t'\} \rangle$$

$$= \hat{N} \int \prod_{t=t'}^{t''} \pi_{\underset{\sim}{x},\mu,a} \, dB^a_\mu(\underset{\sim}{x}, t) \exp[i S\{B\}]$$

$$\times \delta[\hat{A}''^a_\mu(\underset{\sim}{x}) - B^a_\mu(\underset{\sim}{x}, t'')] \, \delta[\hat{A}'^a_\mu(\underset{\sim}{x}) - B^a_\mu(\underset{\sim}{x}, t')]$$

$$= \hat{N} \int \prod_{t=t'}^{t''} \pi_{\underset{\sim}{x},\mu,a} \, dA^a_\mu(\underset{\sim}{x}, t) \exp[i S\{A\}]$$

$$\times \delta[\hat{A}''^a_\mu(\underset{\sim}{x}) - A^a_\mu(\underset{\sim}{x}, t'')] \, \delta[\hat{A}'^a_\mu(\underset{\sim}{x}) - A^a_\mu(\underset{\sim}{x}, t')] \qquad (5.18)$$

Comparing eqn.(5.18) with eqn.(5.11) gives

$$\langle \{A''\}, t'' | \{A'\}, t' \rangle$$

$$= \langle \{\hat{A}''\}, t'' | \{\hat{A}'\}, t' \rangle \qquad (5.19)$$

where the eigenvalues of the initial and final eigenstates are noted by eqn.(5. 15).

5.3 SIMPLIFICATION OF NON-ZERO S-MATRIX ELEMENTS

Let us choose a function $g(\underset{\sim}{x},t)$ such that

$$\partial_0 g|_{t=t'} = \partial_0 g|_{t=t''} = 0 \tag{5.20}$$

In this case the effect of $g(\underset{\sim}{x},t)$ at $t=t'$ and $t=t''$ may be expressed in terms of functions of space, i.e.

$$g(\underset{\sim}{x},t') = \exp[\alpha' T_3]\exp[\beta' T_2]\exp[\gamma' T_3]$$

and (5.21)

$$g(\underset{\sim}{x},t'') = \exp[\alpha'' T_3]\exp[\beta'' T_2]\exp[\gamma'' T_3]$$

Following eqns. (5.15) and (5.19) we may define the states

$$|\{A''\},\alpha'',\beta'',\gamma'';t''\rangle = |\{\hat{A}''\},t''\rangle \tag{5.22}$$

and

$$|\{A'\},\alpha',\beta',\gamma';t'\rangle = |\{\hat{A}'\},t'\rangle \tag{5.23}$$

The functions $\alpha'',\alpha',\beta'',\beta',\gamma''$ and γ' are defined by $g(\underset{\sim}{x},t)$ which obeys eqn (5.20) but is otherwise arbitrary. Thus we may treat these functions as arbitrary functions of space. These functions define spherical rotations so we may decompose states in eqns. (5.22) and (5.23) as typically

$$|\{A'\}, \alpha', \beta', \gamma'; t'\rangle$$
$$= \Pi_{\underset{\sim}{x}} \sum_{\ell'(\underset{\sim}{x}), m'(\underset{\sim}{x}), n'(\underset{\sim}{x})} D^{\ell'}_{m'n'}(\alpha'(\underset{\sim}{x}), \beta'(\underset{\sim}{x}), \gamma'(\underset{\sim}{x}))$$
$$\times |\{A'\}, \ell', m', n'; t'\rangle$$

(5.24)

where $\ell'(\underset{\sim}{x}), m'(\underset{\sim}{x})$ and $n'(\underset{\sim}{x})$ are integer valued functions of space and each coefficient D^{ℓ}_{mn} at each point in space is a matrix element of the rotation operator between eigenstates of spin (ref. 35).

Inverting eqn. (5.24) gives

$$|\{A'\}, \ell', m', n'; t'\rangle$$
$$= \int \Pi_{\underset{\sim}{x}} d\alpha'(\underset{\sim}{x})\, d[\cos\beta'(\underset{\sim}{x})]\, d\gamma'(\underset{\sim}{x}) \left[\frac{\ell'(\underset{\sim}{x})+1}{8\pi}\right]$$
$$\times D^{*\,\ell'(\underset{\sim}{x})}_{m'(\underset{\sim}{x})\, n'(\underset{\sim}{x})}(\alpha'(\underset{\sim}{x}), \beta'(\underset{\sim}{x}), \gamma'(\underset{\sim}{x}))$$
$$\times |\{A'\}, \alpha', \beta', \gamma'; t'\rangle$$

(5.25)

where the integral is over

$$0 \le \alpha'(\underset{\sim}{x}) < 2\pi; \quad 0 \le \beta'(\underset{\sim}{x}) < \pi; \quad 0 \le \gamma'(\underset{\sim}{x}) < 2\pi$$
$$\forall\ \alpha'(\underset{\sim}{x}), \beta'(\underset{\sim}{x}), \gamma'(\underset{\sim}{x})$$

This form of transformation is quite common. A similar device is used in the discussion of vacuum

tunnelling (refs. 27,28 & 29). The state $|\{A'\}, \ell', m', n'; t'\rangle$ is the weighted sum over all states with eigenvalues related to $\{A'\}$ by a gauge transformation obeying eqn. (5.20).

The state $|\{A''\}, \ell'', m'', n''; t''\rangle$ is constructed analogously so that

$$
\begin{aligned}
&|\{A''\}, \ell'', m'', n''; t''\rangle \\
&\quad = \int \Pi_{\underset{\sim}{x}} \left[\frac{\ell''(\underset{\sim}{x})+1}{8\pi}\right] d\alpha''(\underset{\sim}{x})\, d[\cos\beta''(\underset{\sim}{x})]\, d\gamma''(\underset{\sim}{x}) \\
&\qquad \times D^{*\ell''(\underset{\sim}{x})}_{m''(\underset{\sim}{x})\, n''(\underset{\sim}{x})}(\alpha''(\underset{\sim}{x}), \beta''(\underset{\sim}{x}), \gamma''(\underset{\sim}{x})) \\
&\qquad \times |\{A''\}, \alpha'', \beta'', \gamma''; t''\rangle
\end{aligned}
\tag{5.26}
$$

Forming the S-matrix elements of the states in eqns. (5.25) and (5.26) gives

$$
\begin{aligned}
&\langle\{A''\}, \ell'', m'', n''; t''|\{A'\}, \ell', m', n'; t'\rangle \\
&\quad = \int \Pi_{\underset{\sim}{x}}\, d\alpha''(\underset{\sim}{x})\, d[\cos\beta''(\underset{\sim}{x})]\, d\gamma''(\underset{\sim}{x}) \\
&\qquad \times d\alpha'(\underset{\sim}{x})\, d[\cos\beta'(\underset{\sim}{x})]\, d\gamma'(\underset{\sim}{x}) \\
&\qquad \times \left[\frac{\ell''(\underset{\sim}{x})+1}{8\pi}\right]\left[\frac{\ell'(\underset{\sim}{x})+1}{8\pi}\right] D^{\ell''}_{m''n''}(\alpha'', \beta'', \gamma'') \\
&\qquad \times D^{*\ell'}_{m'n'}(\alpha', \beta', \gamma') \\
&\qquad \times \langle\{A''\}, \alpha'', \beta'', \gamma''; t''|\{A'\}, \alpha', \beta', \gamma'; t'\rangle
\end{aligned}
\tag{5.27}
$$

Substituting eqns. (5.22) and (5.23) into eqn. (5.18) gives

$$\langle \{A''\}, \alpha'', \beta'', \gamma''; t'' | \{A'\}, \alpha', \beta', \gamma'; t' \rangle = \langle \{A''\}; t'' | \{A'\}, t' \rangle \tag{5.28}$$

Substituting eqn. (5.28) into eqn. (5.27) gives

$$\begin{aligned} &\langle \{A''\}, \ell'', m'', n''; t'' | \{A'\}, \ell', m', n'; t' \rangle \\ &= \langle \{A''\}; t'' | \{A'\}; t' \rangle \\ &\quad \times \int \Pi_{\underset{\sim}{x}} d\alpha''(\underset{\sim}{x})\, d[\cos\beta''(\underset{\sim}{x})]\, d\gamma''(\underset{\sim}{x}) \left[\frac{\ell''(\underset{\sim}{x})+1}{8\pi}\right] \\ &\quad \times d\alpha'(\underset{\sim}{x})\, d[\cos\beta'(\underset{\sim}{x})]\, d\gamma'(\underset{\sim}{x}) \left[\frac{\ell'(\underset{\sim}{x})+1}{8\pi}\right] \\ &\quad \times D^{\ell''}_{m''n''}(\alpha'', \beta'', \gamma'')\, D^{*\ell'}_{m'n'}(\alpha', \beta', \gamma') \end{aligned} \tag{5.29}$$

$= 0$ unless $\ell''(\underset{\sim}{x}) = \ell'(\underset{\sim}{x}) = m''(\underset{\sim}{x}) = m'(\underset{\sim}{x}) = n''(\underset{\sim}{x}) = n'(\underset{\sim}{x}) = 0$

by orthogonality and using $D^{0}_{00} = 1$

The states $\{|\{A'\}, \ell', m', n'; t'\rangle\}$ form a complete basis for the Hilbert space. According to eqn. (5.29) the only states with non-zero S-matrix elements are superpositions of the states

$$\{|\{A'\}, \ell' = 0, m' = 0, n' = 0; t'\rangle\}$$

We relabel these states in general as

$$|\{A\}, 0, t\rangle = |\{A\}, \ell=0, m=0, n=0; t\rangle$$
$$= \int \Pi_{\underset{\sim}{x}} d\alpha(\underset{\sim}{x})\, d[\cos\beta(\underset{\sim}{x})]\, d\gamma(\underset{\sim}{x})$$
$$\times \left[\frac{\{\ell(\underset{\sim}{x})=0\}+1}{8\pi}\right]$$
$$\times |\{A\}, \alpha, \beta, \gamma; t\rangle \qquad (5.30)$$

where

$$(A_\mu{}^a(\underset{\sim}{x}, t))\,|\{A\}, \alpha, \beta, \gamma; t\rangle$$
$$= [g^{-1} A_\mu{}^b(\underset{\sim}{x})\, T^b g + g^{-1}\partial_\mu g]^a\,|_t$$
$$\times |\{A\}, \alpha, \beta, \gamma; t\rangle \qquad (5.31)$$

and

$$g|_t = \exp[\alpha T_3]\ \exp[\beta T_2]\ \exp[\gamma T_3]$$
$$\partial_0 g|_t = 0 \qquad (5.32)$$

At first sight eqn. (5.29) may seem to violate S-matrix unitarity. This is resolved in Appendix 3 where this is seen to be a consequence of the inclusion of the gauge degrees of freedom in the Lagrangian.

If the only valid states of the theory were to be of the form given in eqn. (5.30) there would be serious consequences. It is apparent that these states are a gauge invariant superposition of states related by gauge transformations. If the entire system of operators and

states is gauge rotated then only the operators would change. It follows that the expectation values of both the field operators and of gauge covariant operators between states of the form given in eqn. (5.30) would be zero.

The flaw in this result is manifest in the discontinuity in eqn. (5.29) as t'' goes to t' which is a consequence of the similar discontinuity in eqn. (5.19). These discontinuities are due to the ambiguity in the dynamical evolution of the states due to the local symmetry. This ambiguity must be removed in order to obtain a well defined theory.

In Appendix 3 the gauge degrees of freedom were removed from the Lagrangian by a change of variable. In SU(2) gauge theory the gauge degrees of freedom may be removed either by adding a gauge-fixing term to the Lagrangian or by placing a restriction on the physical states (refs. 3,4 & 5).

5.4 EFFECTS OF GLOBAL SYMMETRY

If the arbitrary dynamical evolution is removed by eliminating the gauge degrees of freedom then only the global symmetry remains. The general form of a global rotation is:-

$$g = \exp[\alpha T_3] \exp[\beta T_2] \exp[\gamma T_3] \qquad (5.33)$$

The coefficients α, β & γ are now constants. Following eqn. (5.22),

let
$$|\{A\}, \alpha, \beta, \gamma; t\rangle = (G(\alpha,\beta,\gamma))|\{A\}; t\rangle \quad \forall\ |\{A\}; t\rangle \qquad (5.34)$$
$(G(\alpha,\beta,\gamma))$ is the rotation operator.

Eqns. (5.15) and (5.22) now give

$$\langle\{A''\}; t''|\{A'\}; t'\rangle = \langle\{A''\}, \alpha, \beta, \gamma; t''|\{A'\}, \alpha, \beta, \gamma; t'\rangle \qquad (5.35)$$

or
$$\langle\{A''\}|e^{-i(H)(t''-t')}|\{A'\}\rangle = \langle\{A''\}|(G)^{\dagger} e^{-i(H)(t''-t')}(G)|\{A\}\rangle \qquad (5.36)$$

or
$$(G)^{\dagger}(H)(G) = (H)$$

or
$$[(H),(G)] = 0 \qquad (5.37)$$

As expected the rotation operator commutes with the Hamiltonian.

Following eqn. (5.24) let us write

$$|\{A\}, \alpha, \beta, \gamma; t\rangle = \Sigma_{\ell,m,n} D^{\ell}_{mn}(\alpha,\beta,\gamma) \times |\{A\}, \ell, m, n; t\rangle \qquad (5.38)$$

Inverting eqn. (5.38) gives

$$
\begin{aligned}
&|\{A\}, \ell, m, n; t\rangle \\
&\quad = \int_0^{2\pi} d\alpha \int_0^{\pi} d[\cos\beta] \int_0^{2\pi} d\gamma \\
&\qquad \times D^{*\ell}_{mn}(\alpha,\beta,\gamma)\left[\frac{\ell+1}{8\pi}\right] \\
&\qquad \times |\{A\}, \alpha, \beta, \gamma; t\rangle \\
&\quad = \int_0^{2\pi} d\alpha \int_0^{\pi} d[\cos\beta] \int_0^{2\pi} d\gamma \\
&\qquad \times D^{*\ell}_{mn}(\alpha,\beta,\gamma)\left[\frac{\ell+1}{8\pi}\right] \\
&\qquad \times (G(\alpha,\beta,\gamma))\,|\{A\}; t\rangle \\
&\quad = (P^{\ell}_{mn})\,|\{A\}; t\rangle
\end{aligned}
\tag{5.39}
$$

where

$$
\begin{aligned}
(P^{\ell}_{mn}) = \int_0^{2\pi} d\alpha \int_0^{\pi} d[\cos\beta] \int_0^{2\pi} d\gamma \left[\frac{\ell+1}{8\pi}\right] \\
\times D^{*\ell}_{mn}(\alpha,\beta,\gamma)\,(G(\alpha,\beta,\gamma))
\end{aligned}
\tag{5.40}
$$

Substituting (5.40) into (5.37) gives

$$
[(H), (P^{\ell}_{mn})] \tag{5.41}
$$

Thus

$$
(H)(P^{\ell}_{mn})\,|\{A\}; t\rangle = (P^{\ell}_{mn})(H)\,|\{A\}; t\rangle \tag{5.42}
$$

The labels l, m and n do not change with time and so must be associated with constants of the motion.

Consider an eigenstate of the isospin operator associated with the global symmetry, $|\{\psi\}, l', n'\rangle$, which has total isospin $\sqrt{[l'(l'+1)]}$ with isospin n' along T_3.

$$
\begin{aligned}
&(P^{l}{}_{m\,n})\,|\{\psi\}, l', n'\rangle \\
&\quad = \int_0^{2\pi} d\alpha \int_0^{\pi} d[\cos\beta] \int_0^{2\pi} d\gamma \\
&\qquad \times \left[\frac{l+1}{8\pi}\right] D^{*l}{}_{m\,n}(\alpha,\beta,\gamma) \\
&\qquad \times \Sigma_{m'} D^{l'}{}_{m'n'}(\alpha,\beta,\gamma)\,|\{\psi\}, l', m'\rangle \\
&\quad = \Sigma_{m'} \delta^{l,l'} \delta_{m,m'} \delta_{n,n'}\,|\{\psi\}, l', m'\rangle \\
&\quad = \delta^{l,l'} \delta_{n,n'}\,|\{\psi\}, l, m\rangle
\end{aligned}
\tag{5.43}
$$

Now $|\{A\}; t\rangle$ may be decomposed in terms of the eigenstates of the isospin operator. From eqn. (5.43) it follows that $|\{A\}, l, m, n\rangle$ is a superposition of states with total isospin $\sqrt{[l(l+1)]}$ and isospin m along T_3 and is constructed from components of $|\{A\}; t\rangle$ with total isospin $\sqrt{[l(l+1)]}$ with isospin n along T_3.

i.e.

$$
\begin{aligned}
&\langle \{\psi\}, l', m' |\, \{A\}, l, m, n; t\rangle \\
&\qquad = \langle \{\psi\}, l, n |\, \{A\}; t\rangle\, \delta^{l,l'} \delta_{m,m'}
\end{aligned}
$$

Thus $|\{A\}, l, m, n\rangle$ must itself be an eigenstate of isospin

5.5 COMPARISON WITH ELECTROMAGNETISM

Similar consequences to those in Sections 5.1 to 5.3 apply to electromagnetism, but with different consequences. The same general arguments which lead to eqn. (5.19) still apply but now there are no gauge indices and the gauge rotations belong to U(1).

The appropriate elements of U(1) are [cf. eqns. (5.20) and (5.21)]

$$g(\underset{\sim}{x}, t): \partial_0 g|_{t=t'} = \partial_0 g|_{t=t''} = 0 \tag{5.44}$$

$$\begin{aligned} g(\underset{\sim}{x}, t') &= e^{i\omega'(\underset{\sim}{x})} \\ g(\underset{\sim}{x}, t'') &= e^{i\omega''(\underset{\sim}{x})} \end{aligned} \tag{5.45}$$

The eigenstates of the gauge field operators are $|\{A\}, t\rangle$:

$$\begin{aligned} (A_\mu(\underset{\sim}{x}, t))\,|\{A\}, t\rangle \\ = A_\mu(\underset{\sim}{x})\,|\{A\}, t\rangle \end{aligned} \tag{5.46}$$

These may be relabelled in terms of the gauge

rotations [cf. eqns.(5.22) and (5.23)] as

$$|\{A\}, \omega, t\rangle : \quad (A_\mu(\underset{\sim}{x}, t))\,|\{A\}, \omega, t\rangle$$
$$= \hat{A}_\mu(\underset{\sim}{x})\,|\{A\}, \omega, t\rangle \qquad (5.47)$$

where

$$\left.\begin{aligned} i\,\hat{A}_\mu(\underset{\sim}{x}) &= i\,A_\mu(\underset{\sim}{x}) + g^{-1}\partial_\mu g\,|_t \\ \text{i.e.} \\ \hat{A}_\mu(\underset{\sim}{x}) &= A_\mu(\underset{\sim}{x}) + \partial_\mu \omega(\underset{\sim}{x}) \end{aligned}\right\} \qquad (5.48)$$

Eqns.(5.47) and (5.48) give the general form of the end-points with gauge rotations obeying eqns.(5.44) and (5.45).

In analogy with eqn.(5.25) we may define the states

$$|\{A\}, k, t\rangle$$
$$= \tilde{\Lambda}^{-1}\int \Pi_{\underset{\sim}{x}}\, d\omega(\underset{\sim}{x})\, e^{i\,k(\underset{\sim}{x})\,\omega(\underset{\sim}{x})}\,|\{A\}, \omega, t\rangle \qquad (5.49)$$

The gauge symmetry implies that

$$\langle \{A''\}, \omega'', t'' | \{A'\}, \omega', t' \rangle$$

$$= \langle \{A''\}, t'' | \{A'\}, t' \rangle \qquad (5.50)$$

Eqns.(5.49) and (5.50) imply [cf. eqn.(5.29)]

$$\langle \{A''\}, k'', t'' | \{A'\}, k', t' \rangle$$

$$= \tilde{\Lambda}^{-2} \langle \{A''\}, t'' | \{A'\}, t' \rangle$$

$$\times \int \Pi_{\underset{\sim}{x}} d\omega''(\underset{\sim}{x}) d\omega'(\underset{\sim}{x}) \exp i\{k'(\underset{\sim}{x})\omega'(\underset{\sim}{x}) - k''(\underset{\sim}{x})\omega''(\underset{\sim}{x})\} \qquad (5.51)$$

The only non-zero S-matrix elements occur between states of the form

$$|\{A\}, k=0, t\rangle = \tilde{\Lambda}^{-1} \int \Pi_{\underset{\sim}{x}} d\omega(\underset{\sim}{x}) |\{A\}, \omega, t\rangle \qquad (5.52)$$

States of this form are independent of gauge transformations whose time derivatives at time t vanish. By fixing the gauge we may examine the time

development of the individual states which are summed over in eqn. (5.52). Summing over these states does not invalidate the results of computations in a specific gauge. Since the non-zero S-matrix elements occur between states of the form given in eqn. (5.52) , the energy eigenstates must be independent of gauge transformations whose time derivatives vanish at time t . Equation (5.50) exhibits a similar discontinuity as t'' goes to t' to that in eqn. (5.19).

A critical difference between electromagnetism and SU(2) gauge theory is that when the gauge is fixed in electromagnetism there is no remaining global symmetry for the gauge fields. This corresponds to the observation that SU(2) gauge fields carry isospin, whereas photons are uncharged. A second difference is that since the electric and magnetic fields are gauge invariant their matrix elements between states of the form given in eqn. (5.52) are not forced to be zero.

5.6 SUMMARY

In this section we have seen that the SU(2) symmetry relates S-matrix elements of the theory. This allows a simplification of the S-matrix elements. Most of these seem to vanish in a way that suggests that most of the states are fundamentally unstable. The matrix elements of the gauge field operators and of gauge covariant operators between the remaining states vanish.

The origin of this result is the ambiguity in the time evolution of the states due to the local symmetry. This ambiguity gives rise to a discontinuity and must be removed in order to obtain a well defined dynamical evolution, leaving the global symmetry and associted quantum numbers.

Similar considerations apply to electromagnetism but here the matrix elements of the electric and magnetic fields are not forced to vanish when the dynamical evolution is not properly defined and there is no remaining global symmetry of the electromagnetic gauge fields once the gauge is fixed.

6. CONCLUSIONS

The classical solutions to SU(2) gauge theory for static charges and for plane waves are associated with instabilities due to rotations in gauge space. The quantum numbers for these rotations can be examined in a special quantum mechanical model. The local nature of the gauge symmetry in the model gives rise to a constraint on the quantum numbers.

In the full theory stable states seem to be constrained to be gauge invariant due to the way the gauge symmetry relates S-matrix elements. This would force matrix elements of gauge field operators and of gauge covariant operators to vanish. This result is due to the ambiguity in the dynamical evolution of the theory inherent in the local symmetry which causes a discontinuity. When the ambiguity is removed, the global symmetry remains and is associated with its quantum numbers.

Although a corresponding analysis can be carried out for electromagnetism the matrix elements do not vanish in a corresponding way and there is no remaining global symmetry of the gauge field operators once the ambiguity in dynamical evolution is removed.

ACKNOWLEDGEMENTS

I would like to thank D.B. Fairlie and E. Corrigan for their encouragement and stimulating remarks. I am grateful in addition to the Science Research Council for its financial support.

APPENDIX 1

Proof that $\int \partial_i \phi^2 \, dS_i \to 0$ as $k \to \infty$

$$\int \partial_i \phi^2 \, dS_i = \sum_{n=1}^{\infty} S_n \int_{r_{min}}^{r_{max}} r^2 \, \hat{\sigma}(r) \sin(2n+1) k^3 r^3 \, dr$$

Let $r^3 = u$, $\hat{\sigma}(r) = \bar{\sigma}(u)$; $\bar{\sigma}(u)$ cannot be singular so

$$\bar{\sigma}(u) = \sum_{l=0}^{\infty} \sigma_l u^l$$

This series will always converge. Thus

$$\int \partial_i \phi^2 \, dS_i = \sum_{n=1}^{\infty} S_n \tfrac{1}{3} \int_{u_{min}}^{u_{max}} du \sum_{l=0}^{\infty} \sigma_l u^l \sin(2n+1) k^3 u$$

Let $m = (2n+1)$

$$\lim_{k \to \infty} \int_0^{\bar{u}} du \, u^l \sin m k^3 u \quad (l \geq 0)$$

$$= \lim_{k \to \infty} (m k^3)^{-l-1} \, \mathrm{Im} \left\{ \int_0^{\infty} dv \, v^l e^{iv} \right\}$$

$$= \lim_{k \to \infty} (m k^3)^{-l-1} \, \mathrm{Im} \left\{ - \int_{i\infty}^{0} dv \, v^l e^{iv} \right\}$$

$$= \lim_{k \to \infty} (m k^3)^{-l-1} \, \mathrm{Im} \left\{ (i)^{l+1} \int_0^{\infty} u^l e^{-u} \, du \right\}$$

$$= \lim_{k \to \infty} (m k^3)^{-l-1} \, l! \sin \tfrac{1}{2} l \pi$$

$$= 0$$

Thus

$$\int \partial_i \phi^2 \, dS_i \to 0 \quad \text{as} \quad k \to \infty$$

APPENDIX 2

Proof that $\phi^2 \to 0$ as $k \to \infty$

Assume that as $k \to \infty$, ϕ^2 develops a discontinuity. This means that $|\nabla^2 \phi^2|$ must increase indefinitely at some point. But $|\nabla^2 \phi^2| \leq |\sigma|$ which is finite. As we take the limit ϕ^2 cannot develop a discontinuity.

Next assume that as $k \to \infty$, ϕ^2 develops a local maximum on some region R. R can be a single point. This maximum cannot be a discontinuity, so there must exist an equipotential surface S around R with $\phi^2|_S = \phi^2|_R - \delta\phi^2$. We can always choose a surface close enough to R that $\phi^2 > \phi^2|_S$ inside S, and that $|\int \partial_i \phi^2 d S_i| > 0$. This contradicts the result of Appendix 2 that $\int \partial_i \phi^2 d S_i \to 0$. Thus ϕ^2 cannot have a local maximum. Equipotentials of ϕ^2 must extend to infinity.

At infinity ϕ^2 is zero. Thus $\phi^2 \to 0$ everywhere.

APPENDIX 3

S-Matrix Unitarity

Equation (5.29) seems to violate S-matrix unitarity. To gain insight into this problem we examine the quantum mechanical form of the CP(1) model[34].

The corresponding classical Lagrangian is

$$L = \tfrac{1}{2}\,[\dot{z}_\alpha - z_\beta^* \dot{z}_\beta z_\alpha][\dot{z}_\alpha - z_\gamma^* \dot{z}_\gamma z_\alpha]^* \qquad \text{(A3.1)}$$

where

$$z_\alpha^* z_\alpha = 1 \quad ; \quad \alpha, \beta, \gamma \in \{1,2\}$$

The quantum theory has operators $(z_\alpha(t))$ and $(z_\alpha(t))^\dagger$. The eigenstates of these operators may be written as $\{|x_1, x_2, y_1, y_2; t\rangle\}$ where

$$(z_\alpha(t))\,|x_1, x_2, y_1, y_2; t\rangle = [x_\alpha + i y_\alpha]\,|x_1, x_2, y_1, y_2; t\rangle$$

and (A3.2)

$$(z_\alpha(t))^\dagger\,|x_1, x_2, y_1, y_2; t\rangle = [x_\alpha - i y_\alpha]\,|x_1, x_2, y_1, y_2; t\rangle$$

The classical Lagrangian is invariant under the gauge transformation:

$$z_\alpha \to \hat{z}_\alpha = e^{i\vartheta} z_\alpha$$
$$z_\alpha^* \to \hat{z}_\alpha^* = e^{-i\vartheta} z_\alpha^* \qquad \text{(A3.3)}$$

where $\vartheta = \vartheta(t)$

To make use of this symmetry in the quantum theory we relabel the eigenstates in terms of the angular coordinate ϑ, i.e.

$$|x_1, x_2, y_1, y_2, \vartheta; t\rangle = |\hat{x}_1, \hat{x}_2, \hat{y}_1, \hat{y}_2; t\rangle \qquad \text{(A3.4)}$$

where

$$\hat{x}_\alpha = x_\alpha \cos\vartheta - y_\alpha \sin\vartheta$$
$$\hat{y}_\alpha = x_\alpha \sin\vartheta + y_\alpha \cos\vartheta$$

Equations (A3.1), (A3.2) and (A.34) imply that

$$\langle x_1'', x_2'', y_1'', y_2'', \vartheta''; t'' | x_1', x_2', y_1', y_2', \vartheta'; t'\rangle$$
$$= \langle \hat{x}_1'', \hat{x}_2'', \hat{y}_1'', \hat{y}_2''; t'' | \hat{x}_1', \hat{x}_2', \hat{y}_1', \hat{y}_2'; t'\rangle$$
$$= N \int \prod_{t=t'}^{t''} d\hat{x}_1(t)\, d\hat{x}_2(t)\, d\hat{y}_1(t)\, d\hat{y}_2(t)$$
$$\times \exp\{i \hat{S}(t'', t')\}$$
$$\times \delta(\hat{x}_1(t'') - \hat{x}_1'')\, \delta(\hat{x}_2(t'') - \hat{x}_2'')\, \delta(\hat{y}_1(t'') - \hat{y}_1'')$$
$$\times \delta(\hat{y}_2(t'') - \hat{y}_2'')\, \delta(\hat{x}_1(t') - \hat{x}_1')\, \delta(\hat{x}_2(t') - \hat{x}_2')$$
$$\times \delta(\hat{y}_1(t') - \hat{y}_1')\, \delta(\hat{y}_2(t') - \hat{y}_2')$$

where

$$\hat{S}(t'', t') = \int_{t'}^{t''} dt\, L(\dot{\hat{z}}_\alpha, \dot{\hat{z}}_\alpha^*, \hat{z}_\alpha, \hat{z}_\alpha^*)$$

$$= N \int \prod_{t=t'}^{t''} dx_1(t)\, dx_2(t)\, dy_1(t)\, dy_2(t)$$
$$\times \exp\{i S(t'', t')\}$$
$$\times \delta(x_1(t'') - x_1'')\, \delta(x_2(t'') - x_2'')\, \delta(y_1(t'') - y_1'')$$
$$\times \delta(y_2(t'') - y_2'')\, \delta(x_1(t') - x_1')\, \delta(x_2(t') - x_2')$$
$$\times \delta(y_1(t') - y_1')\, \delta(y_2(t') - y_2')$$

where

$$\hat{x}_\alpha(t) = x_\alpha(t)\cos\vartheta(t) - y_\alpha(t)\sin\vartheta(t)$$
$$\hat{y}_\alpha(t) = x_\alpha(t)\sin\vartheta(t) + y_\alpha(t)\cos\vartheta(t)$$

and

$$\vartheta(t'') = \vartheta''\,; \quad \vartheta(t') = \vartheta'$$
$$\hat{x}_\alpha'' = x_\alpha''\cos\vartheta'' - y_\alpha''\sin\vartheta''$$
$$\hat{y}_\alpha'' = x_\alpha''\sin\vartheta'' + y_\alpha''\cos\vartheta''$$

and

$$S(t'', t') = \int_{t'}^{t''} dt\; L(\dot{z}_\alpha, \dot{z}_\alpha^*, z_\alpha, z_\alpha^*)$$

$$= \langle x_1'', x_2'', y_1'', y_2''; t'' | x_1', x_2', y_1', y_2'; t' \rangle \qquad \text{(A3.5)}$$

Since ϑ is an angular variable we may write

$$|x_1, x_2, y_1, y_2, \vartheta; t\rangle = \sum_m e^{-im\vartheta} |x_1, x_2, y_1, y_2, m; t\rangle \qquad \text{(A3.6)}$$

Inverting equation (A3.6) gives

$$|x_1, x_2, y_1, y_2, m; t\rangle$$
$$= \frac{1}{2\pi} \int_0^{2\pi} d\vartheta\; e^{im\vartheta} |x_1, x_2, y_1, y_2, \vartheta; t\rangle \qquad \text{(A3.7)}$$

Equations (A3.5) and (A.37) imply that

$$\begin{aligned}
&\langle x_1'', x_2'', y_1'', y_2'', m''; t'' | x_1', x_2', y_1', y_2', m'; t' \rangle \\
&= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} d\theta'' \int_0^{2\pi} d\theta' \exp\{-i[m''\theta'' - m'\theta']\} \\
&\quad \times \langle x_1'', x_2'', y_1'', y_2'', \theta''; t'' | x_1', x_2', y_1', y_2', \theta'; t' \rangle \\
&= \langle x_1'', x_2'', y_1'', y_2''; t'' | x_1', x_2', y_1', y_2'; t' \rangle \\
&\quad \times \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} d\theta'' \int_0^{2\pi} d\theta' \exp\{-i[m''\theta'' - m'\theta']\} \\
&= \langle x_1'', x_2'', y_1'', y_2''; t'' | x_1', x_2', y_1', y_2'; t' \rangle \delta_{m'';0} \delta_{m';0}
\end{aligned} \tag{A3.8}$$

Equation (A3.8) like equation (5.29) seems to imply that the S-matrix violates unitarity. The explanation becomes apparent if we write

$$\begin{aligned}
z_1 &= e^{i(\psi+\chi)} \cos\phi \\
z_1^* &= e^{-i(\psi+\chi)} \cos\phi \\
z_2 &= e^{i(\psi-\chi)} \sin\phi \\
z_2^* &= e^{-i(\psi-\chi)} \sin\phi
\end{aligned} \tag{A3.9}$$

The angular variables ψ, χ and ϕ allow us to relabel the eigenstates so that

$$|x_1, x_2, y_1, y_2; t\rangle = |\psi, \chi, \phi; t\rangle \tag{A3.10}$$

Equations (A3.4),(A.39) and (A3.10) imply that

$$|x_1, x_2, y_1, y_2, \theta; t\rangle = |\psi+\theta, x, \varphi; t\rangle \qquad \text{(A3.11)}$$

Since ψ is an angular variable

$$|\psi, x, \varphi; t\rangle = \sum_n e^{-in\psi} |n, x, \varphi, t\rangle \qquad \text{(A3.12)}$$

and so

$$|n, x, \varphi; t\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\psi\, e^{in\psi} |\psi, x, \varphi; t\rangle \qquad \text{(A3.13)}$$

Equation (A3.7),(A3.11) and (A3.12) imply that

$$\begin{aligned}
&|x_1, x_2, y_1, y_2, m; t\rangle \\
&= \left(\tfrac{1}{2\pi}\right) \int_0^{2\pi} d\theta\, e^{im\theta} |x_1, x_2, y_1, y_2, \theta; t\rangle \\
&= \left(\tfrac{1}{2\pi}\right) \int_0^{2\pi} d\theta\, e^{im\theta} |\psi+\theta, x, \varphi; t\rangle \\
&= \left(\tfrac{1}{2\pi}\right) \int_\psi^{\psi+2\pi} d\hat{\psi}\, e^{im(\hat{\psi}-\psi)} |\hat{\psi}, x, \varphi; t\rangle \\
&= e^{-im\psi} |m, x, \varphi; t\rangle
\end{aligned} \qquad \text{(A3.14)}$$

Substituting (A3.14) into equation (A3.8) gives

$$e^{i[m''\psi'' - m'\psi']} \langle m'', x'', \varphi''; t''|m', x', \varphi'; t'\rangle \qquad \text{(A3.15)}$$

$$= \langle \psi'', x'', \varphi''; t''|\psi', x', \varphi', t'\rangle\, \delta_{m'',0}\, \delta_{m',0}$$

Let

$$|x, \varphi; t\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\psi\, |\psi, x, \varphi; t\rangle \qquad \text{(A3.16)}$$

Equation (A3.15) implies that

$$\langle \psi'', x'', \varphi''; t'' | \psi', x', \varphi'; t'\rangle$$
$$= \langle x'', \varphi''; t'' | x', \varphi'; t'\rangle \qquad \text{(A3.17)}$$

Equation (A3.17) implies that the apparent violation of S-matrix unitarity corresponds to the elimination of the ψ -degree of freedom in the S-matrix elements. Substituting equations (A3.9) into the Lagrangian verifies this by giving

$$L = \frac{1}{2}\{[\dot{\varphi}]^2 + [\dot{x}]^2[1 - \cos 2\varphi]\} \qquad \text{(A3.18)}$$

From this it is obvious that the S-matrix elements must be independent of the ψ -degree of freedom. No information concerning this degree of freedom is preserved as the states evolve and the S-matrix appears to violate unitarity.

The energy eigenstates of this model must have periodic time evolution and so must have non-zero S-matrix elements. They must be composed of states of form $|x, \varphi; t\rangle$. The matrix elements of the

operators ($Z_\alpha(t)$) and ($Z_\alpha(t)$)† between such states are zero, e.g.

$$\langle x'', \varphi'', t'' | (Z_1(t')) | x', \varphi', t' \rangle$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} d\psi'' \int_0^{2\pi} d\psi' \, e^{i(\psi' + x')} \cos\varphi'$$

$$\times \langle \psi'', x'', \varphi''; t'' | \psi', x', \varphi'; t' \rangle$$

from eqn.(A3.16)

$$= e^{ix'} \cos\varphi' \langle x'', \varphi''; t' | x', \varphi'; t' \rangle$$

$$\times \left(\frac{1}{2\pi}\right) \int_0^{2\pi} d\psi' e^{i\psi'} \quad \text{from eqn.(A3.17)}$$

$$= 0 \qquad \text{(A3.19)}$$

The other three operators behave similarly. This means that the Fourier components of these operators cannot be creation and annihilation operators for the quanta of the model.

In SU(2) gauge field theory no information concerning the gauge degrees of freedom is preserved as the states evolve. The loss of this information is the origin of the apparent violation of unitarity. A similar result to equation (A3.19) also holds.

REFERENCES

1) GILLESPIE, A.
J. Phys. G.: Nucl. Phys; 5, N5, 619 (1979)

2) FEYNMAN, R.P. & A.R. HIBBS
'Quantum Mechanics and Path Integrals'
McGraw Hill Book Company (1965)

3) TAYLOR, J.C.
'Gauge Theories of Weak Interactions'
Cambridge University Press (1976)

4) ABERS, E.S. & B.W. LEE
'Gauge Theories'
Phys. Rep. 9C N1 (1973)

5) MARCIANO, W. & H. PAGELS
'Quantum Chromodynamics'
Phys. Rep. 36C N3 (1978)

6) GRIBOV, V.N.
'Instability of Non-Abelian Gauge Theories and the Impossibility of Choosing the Coloumb Gauge'
From: 'Materials for the XII Winter School of the Leningrad Nuclear Research Institute' (1977)

7) FADDEEV, L.D. & V.N. POPOV
Phys. Lett. 25B, N29 (1967)

8) COLEMAN, S.
Phys. Lett. 70B N1, 59 (1977)

9) ADLER, S.
Phys. Rev. D17, 3212 (1978)

10) MANDULA, J.E.
Phys. Lett. 67B, N2, 175 (1977).

11) MAGG, M.
Phys. Lett. 77B, N2, 199 (1978)

12) MAGG, M.
Phys. Lett. 78B, N4, 481 (1978)

13) SIKIVIE, P.
Phys. Rev. D20, 877 (1979)

14) SIKIVIE, P. & N.O. WEISS
Phys. Rev. Lett. 40, N2, 1411 (1978)

15) SIKIVIE, P. & N.O. WEISS
Phys. Rev. D18, 3809 (1978)

16) SIKIVIE, P. & N.O. WEISS
Phys. Rev. D20, 487 (1978)

17) CAHILL, K.
Phys. Rev. Lett 41, 599 (1978)

18) JACKIW, R., L. JACOBS & C. REBBI
Phys. Rev. D20, 474 (1979)

19) CORRIGAN, E.
Phys. Lett. 82B, 407 (1979)

20) KALB, M.
Phys. Rev. D18, 2909 (1978)

21) JACKIW, R. & P. ROSSI
Phys. Rev. D21, 248 (1980)

22) HALPERN, M.B.
Nucl. Phys. B139, 477 (1978)

23) MESSIAH, A.
'Quantum Mechanics VI', p.298
McGraw-Hill Book Company

24) RAJARAMEN, R.
'Some Non-perturbative Semi-classical Methods in Quantum Field Thoery'
Phys. Rep. 21C, N5 (1975)

25) RAJARAMEN, R. & E.J. WEINBERG
Phys. Rev. D11, N10, 2950 (1975)

26) GASIOROWITZ, S.
'Elementary Particle Physics', pp.19,46,55
John Wiley & Sons, Inc. (1966)

27) JACKIW, R. & C. REBBI
Phys. Rev. Lett. 37, N3, 172 (1976)

28) CALLAN, C.G., R.F. DASHEN & D.J. GROSS
Phys. Lett. 63B, N3, 334 (1976)

29) CALLAN, C.G., R.F. DASHEN & D.J. GROSS
Phys. Lett. 66B, N4, 375 (1977)

30) GUPTA, S.N.
Proc. Phys. Soc. A63, 681 (1950)

31) GUPTA, S.N.
Proc. Phys. Soc. A64, 426 (1951)

32) BLEULER, K.

Helv. Phys Acta. 23, 567 (1950)

33) CAMPBELL, W.B.

Phys. Lett. 103B, N6, 455 (1981)

34) DIN, A.M. & W.J. ZAKRZEWSKI

Lett. Al Nuovo Cimento 28, N4, 121 (1980)

35) ROSE, M.E.

'Elementary Theory of Angular Momentum', p 52

John Wiley & Sons, Inc. (1957)