Riemannian manifolds with Einstein-like metrics

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RIEMANNIAN MANIFOLDS WITH EINSTEIN-LIKE METRICS

by

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A thesis presented for the degree of

Doctor of Philosophy

at the

University of Durham

June 1985

Department of Mathematical Sciences

University of Durham.

dedicated to

my wife Zawiah

my son Mohd Yazid

my daughter Jawahir

CONTENTS

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Riemannian Manifolds With Einstein-like Metrics

Abstract

In this thesis, we investigate properties of manifolds with Riemannian metrics which satisfy conditions more general than those of Einstein metrics, including the latter as special cases.
The Einstein condition is well known for being the Euler- Lagrange equation of a variational problem.
There is not a great deal of difference between such metrics and metrics with Ricci tensor parallel for the latter are locally Riemannian products of the former.
More general classes of metrics considered include Ricci- Codazzi and Ricci cyclic parallel. Both of these are of constant scalar curvature.

Our study is divided into three parts. We begin with certain metrics in 4-dimensions and conclude our results with three theorems, the first of which is equivalent to a result of Kasner [Ka1] while the second and part of the third is known to Derdzinski [De1,2].

Next we construct the metrics mentioned above on spheres of odd dimension.The construction is similar to Jensen's [Je1] but more direct and is due essentially to Gray and Vanhecke [GV]. In this way we obtain ,beside the standard metric, the second Einstein metric of Jensen. As for the Ricci- Codazzi metrics, they are essentially Einstein, but the Ricci cyclic parallel metrics seem to form a larger class.

Finally,we consider subalgebras of the exceptional Lie algebra g2. Making use of computer programmes in 'reduce' we compute all the corresponding metrics on the quotient spaces associated with G2.

Acknowledgments

The work presented here was carried out between Oct 82 and June 85 in the University of Durham under the supervision of Professor T.J.Willmore.
It was Prof. Willmore who first introduced me the subject of Riemannian Geometry and I am most grateful for his patience in teaching me ,his broad knowledge in the subject and for his unique character which has made my stay in Durham pleasant.
I would also like to thank Dr.L.M.Woodward for his help during the preparation of this thesis and Mr.N.Hall who introduced me the computer programmes in 'reduce'.

Finally I would like to acknowledge financial support from the University Kebangsaan Malaysia, Jabatan Perkhidmatan Awam and Overseas Research Awards for making this period of study possible.

0.1 Preliminaries

Let M be a smooth Riemannian manifold of dimension n, and let g be the metric tensor of M. We recall that the torsion and the curvature operators of a connection D are defined by:

$$T(X,Y) = D_X Y - D_Y X - [X,Y],$$

$$R(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]},$$

where [X,Y] denotes the bracket of two vector fields. We also recall that R(X,Y)Z at point p depends only upon the values of X,Y and Z at p.
We denote the contravariant components of g by g^{ij} and 'raise and lower suffixes' in the usual way; the summation convention is followed.

The Riemannian connection is the unique connection with vanishing torsion tensor for which the covariant derivative of the metric tensor is zero.
We compute the expression of the Christoffel symbols C^k_{ij} in a local coordinate system. The computation gives a proof of the existence and uniqueness of the Riemannian connection.
Since the connection has no torsion, $C^k_{ij} = C^k_{ji}$. Moreover

$$D_k g_{ij} = d_k g_{ij} - C^h_{ki} g_{jh} - C^h_{kj} g_{ih} = 0,$$

$$D_i g_{jk} = d_i g_{jk} - C^h_{ik} g_{jh} - C^h_{ij} g_{kh} = 0,$$

$$D_j g_{ik} = d_j g_{ik} - C^h_{jk} g_{ih} - C^h_{ji} g_{kh} = 0,$$

where D_i denotes the i-th component of the derivative.

Taking the sum of the last two equalities minus the first, we obtain:

$$C^h_{ij} = g^{kh} \{ d_i g_{kj} + d_j g_{ki} - d_k g_{ij} \} / 2.$$

For the curvature tensor corresponding to g, the local components satisfy

$$R^h_{kij} = d_i C^h_{jk} - d_j C^h_{ik} + C^h_{im} C^m_{jk} - C^h_{jm} C^m_{ik}.$$

We now consider the 4-covariant tensor

$R(X,Y,Z,W)=g(X,R(Z,W)Y)$ with components $R_{hkij} = R^m_{kij} g_{hm}$.

The properties of R_{ijkh} and the Bianchi identities are well known:

$$R_{ijkh} = -R_{ijhk} = -R_{jikh} = R_{khij},$$

$R_{ijkh} + R_{ikhj} + R_{ihjk} = 0$ and $D_m R_{ijkh} + D_k R_{ijhm} + D_h R_{ijmk} = 0.$

A Riemann curvature tensor is a complicated object. By viewing the curvature tensor as a function on the Grassmanian of tangent 2-planes we define sectional curvature of a 2-plane P, spanned by an orthonormal basis $\{X,Y\}$, by

$$K(P) = g(R(X,Y)X,Y).$$

It is well known that constant sectional curvature is a very strict condition on the metric g. In fact for a complete Riemannian manifold with constant sectional curvature the universal cover of M is isometric either to an n-sphere or to the flat Euclidean n-space or to a hyperbolic n-space.

By taking a trace of the Riemannian curvature tensor we get the Ricci curvature which is a symmetric 2-tensor in view of the first Bianchi identity. Its components are $R_{ij} = R^{k}_{ikj}$. The scalar curvature is obtained by taking the trace once more, namely $R = R_{ij} g^{ij}$.

Constant scalar curvature is known to be a weak constraint on the metric. On a given manifold M, many Riemannian metrics have constant scalar curvature. For example, all homogeneous Riemannian manifolds are of constant scalar curvature.
In fact Yamabe conjectured that "on a compact manifold, in each conformal class of metrics, there is a metric of constant scalar curvature."

We now return to the Ricci curvature. It also seems to be very broad. For example, Willmore [Wil] in 1956 questioned the existence of a compact manifold with positive definite metric of zero Ricci curvature which is non-flat. In 1977 Yau [Ya] gave an example of such a manifold. But a homogeneous metric with Ricci curvature zero has sectional curvature zero. See Ziller's paper in [WiH].

0.2 Review of Einstein metrics.

We now consider some constraints on Ricci curvature. For example consider the case where the Ricci curvature is proportional to the metric. This condition is known as the Einstein condition. We notice here that the Ricci curvature and the metric g on a manifold of dimension n depend on the same number $n(n+1)/2$ of real parameters. So the equations for metrics satisfying the Einstein condition looks promising since it involves the same number of unknowns as equations. Unfortunately the situation is more complicated as we shall see.

Definition.

An Einstein metric is a metric for which the Ricci tensor and the metric are proportional:

1) $$R_{ij}(p)=f(p)g_{ij}(p).$$

Contracting this equality, we obtain $f(p)=R(p)/n$, which is a constant when $n>2$. Indeed, if we multiply the second Bianchi identity by g^{jm}, we obtain:

$$D^j R_{ijkh} + D_k R_{ih} - D_h R_{ik} = 0,$$

which multiplied by g^{ih} results in $D_k R = 2D^i R_{ik}$. But contracting the covariant derivative of 1) gives $D_k R = nD^i R_{ik}$. Hence when n is different from 2, the scalar curvature R must be constant.

Einstein came to the condition named after him because in his theory of relativity, he proposed that the field equations for the interaction of gravitation and other fields take the form

$$\mathrm{Ricci} - Rg/2 = T$$

where g is the Lorentzian metric of space-time and T is the energy-momentum tensor which is zero in the absence of other fields. This condition is the Euler-Lagrange equation of a variational problem.Namely,if we consider the total scalar curvature,i.e, the integration over M of the scalar curvature $R(g)$ with respect to the volume element induced by the metric g, then the Einstein tensor $\mathrm{Ricci} - Rg/2$ appears as its gradient. Critical points would have to have zero scalar curvature by evaluating the trace.One can remedy this by normalizing the total volume to be 1. The new Euler- Lagrange equation of the constrained functional is now

$\mathrm{Ricci} - Rg/2 = kg$ for some real number k.

See for examples [Pa] and [Mu].In fact this problem was studied by Hilbert in 1915.

It is well known that all 2 or 3-dimensional Einstein spaces are precisely spaces of constant sectional curvature. Thus Einstein metrics not of constant curvature are of dimension at least four.

Examples of Einstein metrics are :

spaces of constant sectional curvature , compact rank one symmetric spaces, i.e, real, complex and quarternionic projective spaces and also the Cayley plane, compact Lie groups with their bi-invariant metrics, and many other non-symmetric spaces such as $SO(pq)/SO(p)\times SO(q)$.

It has been suggested [Be] p. 165 that an interesting class of manifolds consists of those Einstein manifolds that satisfy the additional condition

$$\hat{R}(X,Y) = |R|^2 g(X,Y)/n$$

where

$$\hat{R}(X,Y) = \sum_{i,j,k} R(e_i, e_j, e_k, X)R(e_i, e_j, e_k, Y).$$

We called such manifolds super-Einstein manifolds when $n>4$. Use of the Bianchi identity shows that when $n>4$, a super-Einstein manifold has the property that $|R|^2$ is constant. However, the condition for $\hat{R}$ is satisfied automatically in an Einstein manifold when $n=4$, but in that case $|R|^2$ is not constant in general. For this reason a 4-dimensional Einstein manifold is called super-Einstein when $|R|^2$ is constant.

For example every irreducible symmetric space is a super-Einstein manifold. This condition has been considered by Carpenter, Gray and Willmore. See [GW] and [CGW].

One of the weak points of the Einstein condition is that the product of two Einstein manifolds is not necessarily Einstein. In fact the product of two Einstein metrics is Einstein if and only if the Einstein constants of the two metrics are the same. This defect can be overcome by considering Riemannian metrics with parallel Ricci tensor fields for we have " any Riemannian manifold with parallel Ricci tensor is covered by a Riemannian product of Einstein manifolds." See Theorem 2.1 of [Gr2] p. 262.

0.3 Review of Ricci-Codazzi metrics.

A wider class of metrics is obtained by imposing the condition that the Ricci tensor must satisfy the equation

1) $$D_h R_{ik} - D_k R_{ih} = 0.$$

An alternative definition is to ask for the condition

$$D^j R_{ijkh} = 0,$$

since for all Riemannian metrics we have the identity

$$D^j R_{ijkh} = D_h R_{ik} - D_k R_{ih}.$$

Such metrics are said to have harmonic curvature. It was shown by Bourguignon [Bo3], that this condition is equivalent to insisting that the curvature operator, viewed as a vector valued 2-form, shall be closed.

It will be recalled from classical surface theory, that the second fundamental form (a_{ij}) of an immersed surface in E^3 satisfies the Codazzi equation

2) $$D_h a_{ik} - D_k a_{ih} = 0.$$

Due to the similarity of 1) and 2), we shall say that a metric which satisfies 1) is Ricci-Codazzi.

We emphasize that the condition of harmonicity of the curvature is a third order differential condition on the metric.

This family of metrics includes Einstein metrics in which case equation 1) is trivially satisfied.

It is also clear that every Riemannian manifold with parallel Ricci tensor has harmonic curvature.

This family of metrics also includes conformally flat metrics with constant scalar curvature, for all conformally flat Riemannian manifolds of dimension greater than three and of constant scalar curvature are Ricci- Codazzi. See Theorem 5.1 of [Gr2].

A particular example of a metric of this type is given by:

$$M=\{(x_1,\dots,x_n) \in R^n ;\ x_n > 0\} \text{ with metric}$$

$$ds^2 = x_n^{4/(n-2)} \{dx_1^2 + \dots + dx_n^2\},$$

which has non-parallel Ricci tensor and zero scalar curvature.

Derdzinski [De1,2] has given examples of such metrics on compact manifolds containing the sphere S^1 as a direct factor.

They are not Ricci parallel nor conformally flat manifolds with constant scalar curvature. And this answers in the negative the question raised by Bourguignon [Bo4] whether or not the only metrics with harmonic curvature on compact manifolds are necessarily Ricci parallel.

It is also known that a compact oriented 4-dimensional Riemannian manifold with non-vanishing signature and harmonic curvature is Einstein ([Bo1] p.32) while any compact Riemannian manifold with harmonic curvature and non- negative sectional curvature is in fact Ricci parallel. See Theorem 11.3 of [Gr2]. Furthermore any compact Kahler manifold is covered by a product of Kahler- Einstein manifolds. See [Bo4].

All Ricci-Codazzi metrics are of constant scalar curvature. In fact for an arbitrary local frame field $\{e_1,...,e_n\}$ and a vector field X on M, we have

$$XR = \sum_i D_X R(e_i, e_i) = \sum_i D_{e_i} R(X, e_i).$$

On the other hand it follows from the second Bianchi identity that

$$XR = 2 \sum_i D_{e_i} R(X, e_i)$$

which is valid for any Riemannian manifold. Hence $XR=0$, i.e, the scalar curvature is constant.

This class of metrics has been studied by A.Gray [Gr2] , U.Simon [Si], Bourguignon [Bo3,4], A.Derdzinski [De1,2], C.Shen [DS] and others.

0.4 Review of Ricci cyclic parallel metrics.

Now let G be a Lie transformation group acting on M. Recall that a differential operator $\mathcal{D}$ is said to be invariant with respect to the group G (G-invariant) if $\mathcal{D}(f o g)=(\mathcal{D} f) o g$ holds for any smooth function f and any element g of the group G, where g is understood to denote the action of g on M.

A well known theorem of Lichnerowicz on the algebra of invariant differential operators states that the algebra of such operators on a globally symmetric Riemannian space is commutative, see [Hel] p.396 or [Wi2] p.226. As a consequence another class of metrics called Ricci cyclic parallel metrics was studied by Sumitomo in an attempt to answer the question "to what extent is a Riemannian homogeneous space satisfying the commutativity condition of the algebra of invariant differential operators close to being a symmetric space?." See [Su] and [Si].
One such condition is that the metric shall be Ricci cyclic parallel, namely

$$D_i R_{jk} + D_j R_{ki} + D_k R_{ij} = 0.$$

However the geometric meaning of this condition remains obscure.

An example of such a metric is obtained by considering S^3 as a submanifold of the quaternions. The metric is

$$ds^2 = a dQ_I^2 + b dQ_J^2 + c dQ_K^2 \quad \text{where} \quad dQ_V(X) = \langle X, VN\rangle \quad ; \ V=I,J,K$$

with at least two of the coefficients a, b, c equal, where N denotes the unit outward normal to the sphere. This example gives non-parallel Ricci tensor which satisfies the Ricci cyclic parallel condition.

Theorem 11.4 of [Gr2] states that: a compact Riemannian manifold with non-positive sectional curvature which is Ricci cyclic parallel is Ricci parallel.

This class of metrics is also of constant scalar curvature for at each point of M we have

$$XR = \sum_i D_X R(e_i, e_i) = -2 \sum_i D_{e_i} R(X, e_i).$$

Comparing this with the relation from the preceding section which holds for any Riemannian manifold gives XR=0.

0.5 <u>Relations between the curvature conditions</u>.

We have [Gr2]

$$E \subset RP = RC \cap RCP \begin{array}{l} \subset RC \\ \subset RCP \end{array} \subset RC \cup RCP \subset C$$

with strict relations all over, where E, RP, RC, RCP and C denote respectively spaces satisfying the condition of Einstein, Ricci parallel, Ricci- Codazzi, Ricci cyclic parallel and constant scalar curvature.
In particular the second equality says :metrics which satisfy both Ricci cyclic parallel and Ricci- Codazzi conditions must be Ricci parallel.

Another generalisation of Einstein metrics has been considered by Patterson. In his paper [Pa] p.355 a certain metric of type E_m is defined and he immediately obtained that E_1 is the same as an Einstein metric. We shall not however pursue this concept which is due essentially to Lovelock [Lo] for this generalization differs from what we have considered in the sense that E_1 does not imply E_m for m greater than 1.

In the following three chapters the summary of results are given at the beginning of each chapter.

Chapter1. <u>On Certain Riemannian Metrics In 4-Dimensions</u>.

There are many examples of 4-dimensional Einstein metrics. Famous ones are, example of Schwarzchild of zero scalar curvature which was later generalized by Kottler [Pe1] pp. 79-80 and the example of Kasner [Ka1,2] which is a generalization of De' Sitter's.

In this chapter we study certain metrics in four dimensions and conditions are found for such metrics to be Einstein.Conditions are also found for those metrics to be Ricci-Codazzi and Ricci cyclic parallel.

Since the metrics are irreducible we clearly have that metrics with Ricci tensor parallel are essentially Einstein.

Results on Einstein metrics are summarised in Theorem1 p. 24. These results are equivalent to Kasner's. See [Ka1].

Results previously obtained by Derdzinski [De1,2] for 4-dimensional Ricci- Codazzi metrics here appear in Theorem 2 pp. 31-32 as a special case. Part two of Theorem 3 p. 37 gives a simple example of such metrics.

We have shown that for a special class of 4-dimensional metrics, Ricci cyclic parallel and Einstein are equivalent. In general it seems that for such a metric under consideration the Ricci cyclic parallel condition seems to be more restrictive than the Ricci-Codazzi condition.

1.1 Preparation.

The object is to find the most general form of a 4-dimensional Riemannian metric of the form :

$$ds^2 = dt^2 + e^{2u(1)} dx_1^2 + e^{2u(2)} dx_2^2 + e^{2u(3)} dx_3^2$$

where u(1),u(2),u(3) are functions of the single variable t such that one of the following conditions is satisfied:

a) the space is Einstein ,i.e , the Ricci curvature is proportional to the metric,

b) the space is Ricci cyclic parallel,i.e,

$$D_i R_{jk} + D_j R_{ki} + D_k R_{ij} = 0,$$

c) the space has harmonic curvature which is equivalent to saying that the space is Ricci-Codazzi ,i.e ,$D_i R_{jk} = D_k R_{ji}$.

As preparation we express the given metric in terms of an orthonormal coframe and compute the connection matrix,the curvature matrix and the Ricci tensor in terms of this coframe .We also compute the covariant derivative of the Ricci tensor in order to study the Ricci cyclic parallel and Ricci-Codazzi conditions.

Let

$$1.1 \qquad Q_0 = dt \quad ; \quad Q_i = e^{u(i)} dx_i \qquad ; \quad i = 1,2,3.$$

Then we have

$$1.2 \quad dQ_0 = 0 \ ; \ dQ_i = \dot{u}(i) e^{u(i)} dt \wedge dx_i = \dot{u}(i) Q_0 \wedge Q_i \quad ; \quad i = 1,2,3.$$

The first Cartan structural equation with zero torsion for the Riemannian connection is

1.3 $$dQ_i + \sum_{j=0}^{3} w_{ji} \wedge Q_j = 0 \qquad ; \; i=0,1,2,3,$$

with

1.4 $$w_{ij} + w_{ji} = 0 .$$

We now write

1.5 $$w_{ij} = \sum_{k=0}^{3} A_{ijk} Q_k$$

which defines the functions A_{ijk} with the property $A_{ijk} = -A_{jik}$. We put i=0 into equations 1.3 to get

$$\{A_{100} Q_0 \wedge Q_1 + A_{102} Q_2 \wedge Q_1 + A_{103} Q_3 \wedge Q_1\} + \{A_{200} Q_0 \wedge Q_2 + A_{201} Q_1 \wedge Q_2 + A_{203} Q_3 \wedge Q_2\}$$

$$+ \{A_{300} Q_0 \wedge Q_3 + A_{301} Q_1 \wedge Q_3 + A_{302} Q_2 \wedge Q_3\} = 0.$$

From this we see that

$$A_{i00} = 0 \qquad \text{for all } i=1,2,3,$$

$$A_{j0i} = A_{i0j} \quad ; \quad i,j=1,2,3 \text{ and } i \neq j.$$

Now take i=1 to get

$$dQ_1 + w_{01} \wedge Q_0 + w_{21} \wedge Q_2 + w_{31} \wedge Q_3 = 0$$

which, by the second formula of 1.2 and 1.5, becomes

$$\{\dot{u}(1) - A_{011}\} Q_0 \wedge Q_1 + (A_{210} - A_{012}) Q_0 \wedge Q_2 + (A_{310} - A_{013}) Q_0 \wedge Q_3$$

$$+ (A_{312} - A_{213}) Q_2 \wedge Q_3 + A_{211} Q_1 \wedge Q_2 + A_{311} Q_1 \wedge Q_3 = 0.$$

From this we see that

$$A_{011} = \dot{u}(1) \quad ;$$

$$A_{210} = A_{012} \quad ; \quad A_{310} = A_{013} \quad ; \quad A_{312} = A_{213} \quad ;$$

$$A_{211} = A_{311} = 0 \; .$$

Similar calculations with $i=2$ and $i=3$ give

$$A_{022} = \dot{u}(2) \quad ; \quad A_{033} = \dot{u}(3) \; ,$$

$$A_{120} = A_{021} \; , \; A_{320} = A_{023} \quad ; \quad A_{130} = A_{031} \; , \; A_{230} = A_{032} \; ,$$

$$A_{321} = A_{123} \quad ; \quad A_{231} = A_{132} \; ,$$

$$A_{122} = A_{322} = 0 \quad ; \quad A_{133} = A_{233} = 0 \; .$$

These can be summarised as follows:

1.6 $\quad A_{0ii} = \dot{u}(i) \quad ; \; i=1,2,3,$

1.7 $\quad A_{jki} = A_{ikj} \quad ; \; i,j=0,1,2,3$ and i,j,k distinct,

and all other components are zero.

We now show that all the A's are zero except for A_{0ii}.

We have

$$A_{jki} = A_{ikj} = -A_{kij} = -A_{jik} = A_{ijk} = A_{kji} = -A_{jki} \; ,$$

using the identity 1.7 together with the skew-symmetry property of A with respect to the first two indices. From 1.5 we therefore obtain the connection matrix w given by

1.8 $\quad w_{i0} = -\dot{u}(i)\, Q_i \qquad$ for $i = 1,2,3,$

1.9 $\quad w_{ij} = 0 \qquad$ for $i,j = 1,2,3.$

The curvature matrix is computed using the second structural equation of Cartan, namely

1.10 $$m_{ij} = dw_{ij} - \sum_{k=0}^{3} w_{ik} \wedge w_{kj} \quad \text{for } i,j=0,1,2,3.$$

For $i=0$, we clearly get

$$m_{0j} = dw_{0j} = \{\ddot{u}(j) + \dot{u}(j)^2\} Q_0 \wedge Q_j \quad \text{for } j = 1,2,3,$$

where the last equality follows from 1.2 and 1.8. For i different from j we have similarly

$$m_{ij} = -w_{i0} \wedge w_{0j} = \dot{u}(i)\dot{u}(j) Q_i \wedge Q_j \quad i,j=1,2,3.$$

Note that

1.11 $$2m_{ij} = - \sum_{p,q} R_{ijpq} Q_p \wedge Q_q .$$

We conclude that the only nonzero components of the curvature tensor are :

1.12 $$K_{0i} = R_{0i0i} = - \ddot{u}(i) - \dot{u}(i)^2 \quad \text{for } i = 1,2,3,$$

1.13 $$K_{ij} = R_{ijij} = - \dot{u}(i)\dot{u}(j) \ ; \ i,j=1,2,3 \text{ and } i \neq j.$$

The Ricci tensor can now be computed and we obtain the nonzero components:

1.14 $$R_{00} = - \sum_{i=1}^{3} \ddot{u}(i) - \sum_{i=1}^{3} \dot{u}(i)^2 ,$$

1.15 $$R_{ii} = - \ddot{u}(i) - \dot{u}(i)\{ \sum_{j=1}^{3} \dot{u}(j)\} \quad \text{for } i=1,2,3.$$

The scalar curvature is

1.16 $$R = - 2\{ \sum \ddot{u}(i) + \sum \dot{u}(i)^2 + \sum_{i \neq j} \dot{u}(i)\dot{u}(j)\}.$$

1.2 Einstein Metrics

For metrics of the form

$$2.1 \quad ds^2 = dt^2 + e^{2u(1)} dx_1^2 + e^{2u(2)} dx_2^2 + e^{2u(3)} dx_3^2$$

where $u(i)$ are functions of the single variable t, we have computed the components of the Ricci tensor and we can now study the possibilities of the metrics being Einstein.

The Einstein equations are

$$2.2 \qquad \sum \ddot{u}(i) + \sum \dot{u}(i)^2 = -u/4$$

$$2.3 \qquad \ddot{u}(i) + \dot{u}(i) \sum \dot{u}(j) = -u/4 \quad \text{for } i = 1,2,3$$

where u is the constant scalar curvature.

Summing 2.3 for all $i = 1,2,3$ gives

$$2.4 \qquad \sum \ddot{u}(i) + \{ \sum \dot{u}(i) \}^2 = 3v/4$$

where $v=-u$.

This, together with 2.2, implies

$$2.5 \qquad \dot{u}(1)\dot{u}(2)+\dot{u}(2)\dot{u}(3)+\dot{u}(3)\dot{u}(1) = v/4.$$

We now solve the differential equations 2.2 and 2.3 assuming that the constant v is positive. We write $3v/4 = A^2$ and denote $\sum \dot{u}(i)$ by a. Equation 2.4 now gives

$$2.6 \qquad \dot{a} + a^2 = A^2.$$

Clearly the gradient of each solution curve is negative in the region $|a|>|A|$ while in the region $|a|<|A|$ it is positive.

We rewrite this as

$$\frac{da}{A^2 - a^2} = dt \ , \text{ provided that } A^2 \neq a^2 .$$

i) For $|a|<|A|$, i.e, $A^2-a^2 > 0$ we write $a=A\tanh\#$ with $da=A\operatorname{sech}^2(\#)d\#$ to obtain $\#=A(t-k)$ for some constant k.

We notice that the gradient is $A^2-a^2 = A^2 \operatorname{sech}^2\# > 0$.

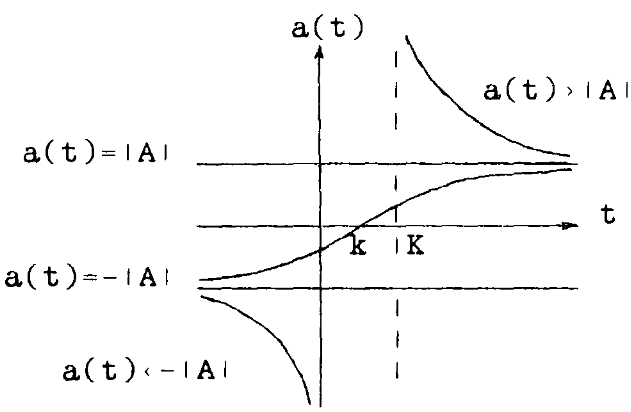
ii) For $|a|>|A|$, i.e, $A^2-a^2 < 0$ we write $a=A\coth\#$ with $da=-A\operatorname{cosech}^2(\#)d\#$ to obtain $\#=A(t-K)$ for some constant K.

In this case the gradient is $A^2-a^2 = -A^2 \operatorname{cosech}^2\# < 0$.

Hence

$$a = \begin{cases} A\tanh A(t-k) \ , & |a|<|A| \\ A\coth A(t-K) \ , & |a|>|A| , \ t\neq K. \end{cases} \tag{2.7}$$

Now observe that $a=\pm A$ is also a solution of the equation. There appear to be a possibility that there may be some t for which $a(t)^2 = A^2$ although this does not hold for all t. However, a general uniqueness theorem for differential equations guarantees that this cannot happen, and we have in fact obtained all solutions of equation 2.6.



Multiplying the first, the second and the third equations of 2.3 by $\dot{u}(2)\dot{u}(3)$, $\dot{u}(3)\dot{u}(1)$ and $\dot{u}(1)\dot{u}(2)$, respectively, and summing yields

$$2.8 \qquad \frac{d}{dt}\{\dot{u}(1)\dot{u}(2)\dot{u}(3)\}+3\dot{u}(1)\dot{u}(2)\dot{u}(3)\sum \dot{u}(i) = \frac{v}{4}\sum_{i\neq j} \dot{u}(i)\dot{u}(j).$$

Denote $\dot{u}(1)\dot{u}(2)\dot{u}(3)$ by b and use relations 2.5 and the first part of 2.7 (i.e for $|a|<|A|$) to get

$$2.9 \qquad \dot{b} + 3bA \tanh A(t-k) = A^4/9.$$

Recall that the integrating factor of the differential equation 2.9 is $\cosh^3 A(t-k)$ .On multiplying both sides of this equation by this integrating factor and then integrating with respect to t ,we obtain

$$b\cosh^3 A(t-k) = \frac{A^3}{27} \sinh^3 A(t-k) + \frac{A^3}{9} \sinh A(t-k) + B$$

from which we finally get

$$b= \frac{A^3}{27}\tanh^3 A(t-k)+ \frac{A^3}{9}\tanh A(t-k)\operatorname{sech}^2 A(t-k)+ B\operatorname{sech}^3 A(t-k)$$

where B is constant of integration. We also use : $\sinh 3x=3\sinh(x) +4\sinh^3 x$ and $\cosh 3x=4\cosh^3 x -3\cosh(x)$. From equations 2.4,2.5,2.8 and the relation $3v=4A^2$ we see that $\dot{u}(1)$,$\dot{u}(2)$ and $\dot{u}(3)$ are roots of the cubic equation

2.10

$$y^3-y^2 A\tanh At+y\frac{A^2}{3} -\{\frac{A^3}{27} \tanh^3 At+ \frac{A^3}{9}\tanh(At)\operatorname{sech}^2 At+ B\operatorname{sech}^3 At\}=0.$$

(Notice that we can carry out the transformation $t'=t-k$ and then work in terms of t'.)

Similarly for $|a|>|A|$ we obtain that $\dot{u}(i)$ are roots of

$$y^3-y^2 A\coth At+y\frac{A^2}{3}-\left\{\frac{A^3}{27}\coth^3 At-\frac{A^3}{9}\coth At.\operatorname{cosech}^2 At+C\operatorname{cosech}^3 At\right\}=0,$$

for non-zero t.

We remember that in obtaining the above equations ,we have assumed that the scalar curvature is negative. When it is positive we obtain that $\dot{u}(i)$ are roots of

2.10′

$$y^3-y^2 A\cot(At)-y\frac{A^2}{3}-\left\{\frac{A^3}{27}\cot^3 At-\frac{A^3}{9}\cot(At)\operatorname{cosec}^2 At+D\operatorname{cosec}^3 At\right\}=0.$$

The next case to consider is when the scalar curvature is zero.In this case the differential equation 2.6 becomes

2.6′′ $$\dot{a}+a^2=0$$

with the general solution $a = 1/(t-k)$,whence the differential equation 2.9 is reduced to

2.9′′ $$\dot{b}+3b/(t-k)=0.$$

It is quite clear that the general solution of this equation is given by $b = E(t-k)^{-3}$ for some constants E and k.

Thus in the case of zero scalar curvature, $\dot{u}(i)$ are roots of

2.10′′ $$y^3-y^2t^{-1}-Et^{-3}=0 .$$

Again we have carried out the transformation $t'=t-k$.

The only case left is the case when $a = A \neq 0$.

We have from 2.2 and 2.4,

$$\dot{u}(1)^2 + \dot{u}(2)^2 + \dot{u}(3)^2 = A^2/3 , \tag{2.2'''}$$

$$\dot{u}(1) + \dot{u}(2) + \dot{u}(3) = A. \tag{2.4'''}$$

We have the identity

$$(\dot{u}(1)-A/3)^2 + (\dot{u}(2)-A/3)^2 + (\dot{u}(3)-A/3)^2$$

$$= \Sigma\, \dot{u}(i)^2 - 2A(\Sigma\, \dot{u}(i))/3 + A^2/3.$$

Using 2.2''' and 2.4''' we see that

$$(\dot{u}(1)-A/3)^2 + (\dot{u}(2)-A/3)^2 + (\dot{u}(3)-A/3)^2 = 0$$

and hence

$$\dot{u}(1)=\dot{u}(2)=\dot{u}(3)=A/3.$$

We have seen that the study of Einstein metrics on a four dimensional Riemannian manifold with metrics of the form

$$ds^2 = dt^2 + e^{2u(1)} dx_1^2 + e^{2u(2)} dx_2^2 + e^{2u(3)} dx_3^2$$

where u(1), u(2), u(3) are functions of the single variable t, is reduced to the study of solutions of the following cubic equations:

$$(y - A/3)^3 = 0 , \tag{2.11}$$

$$y^3 - y^2 t^{-1} - Et^{-3} = 0 , \tag{2.12}$$

2.13

$$y^3 - y^2 A\tanh At + y\frac{A^2}{3} - \left\{\frac{A^3}{27}\tanh^3 At + \frac{A^3}{9}\tanh(At)\,\mathrm{sech}^2 At + B\,\mathrm{sech}^3 At\right\} = 0,$$

$$y^3 - y^2 A\coth At + y\frac{A^2}{3} - \left\{\frac{A^3}{27}\coth^3 At - \frac{A^3}{9}\coth(At)\,\mathrm{cosech}^2 At + C\,\mathrm{cosech}^3 At\right\} = 0,$$

2.14

$$y^3 - y^2 A\cot(At) - y\frac{A^2}{3} - \left\{\frac{A^3}{27}\cot^3 At - \frac{A^3}{9}\cot(At)\,\mathrm{cosec}^2 At + D\,\mathrm{cosec}^3 At\right\} = 0.$$

The cubic equations 2.12, 2.13 and 2.14 can be reduced to the following:

2.12′ $3x^3 - xt^{-2} - Lt^{-3} = 0 \quad ; \quad x + t^{-1}/3 = y,$

2.13′ $3x^3 + xA^2\,\mathrm{sech}^2 At - M\,\mathrm{sech}^3 At = 0 \quad ; \quad x + A(\tanh At)/3 = y,$

$3x^3 - xA^2\,\mathrm{cosech}^2 At - N\,\mathrm{cosech}^3 At = 0 \quad ; \quad x + A(\coth At)/3 = y,$

2.14′ $3x - xA\,\mathrm{cosec}\, At - P\,\mathrm{cosec}\, At = 0 \quad ; \quad x + A(\cot At)/3 = y,$

where L, M, N, P are all constants.

Using a familiar method of solving cubic equations, it is not difficult to prove that the solutions of 2.12′, 2.13′ and 2.14′ are respectively given by:

$y = A(i)/t$, $A(\tanh At)/3 - B(i)\,\mathrm{sech} At$, $A(\coth At)/3 + C(i)\,\mathrm{cosech} At$ and $y = A(\cot At)/3 + D(i)\,\mathrm{cosec} At$.

Moreover, in view of 2.13 and 2.14, the constants B(i), C(i) and D(i) must satisfy the relations:

$$\sum V(i) = 0\ , \quad \sum_{i\neq j} V(i)V(j) = -A^2/3 \text{ for } V = B, C, D.$$

We prove the following theorem which is equivalent to a result of Kasner [Ka1].

Theorem_1

Metrics of the form 2.1 could only be Einstein if and only if one of the following conditions holds:

1) $u(i) = tA/3 + b(i)$ where A and b(i) are constants. In this case the sectional curvature is constant $-A^2$.

2) $u(i) = A(i)\log t$ with the relations $\sum A(i) = \sum A(i)^2 = 1$. This is the case for zero scalar curvature.

3) a) $\dot{u}(i) = A(\tanh At)/3 + B(i)\operatorname{sech} At$ with the relations $\sum B(i) = 0$ and that $\sum B(i)^2 = 2A^2/3$. This is the case when the scalar curvature is negative and it takes value $-4A^2/3$.

b) $\exp\{2u(i)\} = \sinh^{2/3}(At)\tanh^{2C(i)/A}(At/2)$ with the relations $\sum C(i) = 0$ and that $\sum C(i)^2 = 2A^2/3$. This is the case when the scalar curvature is negative and it takes value $-4A^2/3$.

4) $\exp\{2u(i)\} = \sin^{2/3}(At)\tan^{2D(i)/A}(At/2)$ with the relations $\sum D(i) = 0$ and that $\sum D(i)^2 = 2A/3$. This is the case when the scalar curvature is positive and it takes value $4A^2/3$.

Proof

1) Clearly in this case $\dot{u}(i) = A/3$; $i = 1,2,3$. They are roots of the cubic 2.11. More over from 1.12 and 1.13 we see that the sectional curvatures are all constant and equal to $-A^2$.

2) When $u(i) = A(i)\log t$, i.e. $\dot{u}(i) = A(i)/t$, we have

$\Sigma \dot{u}(i) = \Sigma A(i)t^{-1}$ and $\Sigma \dot{u}(i)\dot{u}(j) = \Sigma A(i)A(j)t^{-2}$.

In order to satisfy 2.12(which corresponds to the case of zero scalar curvature) we must have $\Sigma A(i) = 1$ and

$\Sigma A(i)A(j) = 0$, which imply $\Sigma A(i) = \Sigma A(i)^2 = 1$.

It is not difficult to see that this is impossible if exactly one of the coefficients is zero.

3)b) For the case $\exp\{2u(i)\} = \sinh^{2/3}(At)\tanh^{2C(i)/A}(At/2)$, it is not difficult to get

$$\dot{u}(i) = \frac{A}{3}\coth(At) + C(i)\operatorname{cosech}(At).$$

We have also seen that $\Sigma C(i)C(j) = -A^2/3$. Multiplying this negative constant by four we see from 2.5 that the scalar curvature is $-4A^2/3$.

3)a), 4) These are proved by the similar method above.

<u>Examples</u>.

In case 2) we can take for example, $A(i)$ are just cyclic permutations of $\{2/3, 2/3, -1/3\}$. In this case the sectional curvatures given by 1.12 and 1.13 give

$$K_{0i} = A(i)\{1-A(i)\}t^{-2} \ ; \ K_{ij} = -A(i)A(j)t^{-2} \qquad \text{for } i=1,2,3.$$

Thus this example gives Ricci flat nonflat spaces. The second example when $A(i)$ are cyclic permutations of $\{1,0,0\}$ would give flat spaces.

In case 3) we can take for example, $B(i)$ are cyclic permutation of $\{2,-1,-1\}$ with $A=3$. Other simple example of $B(i)$ are given by cyclic permutations of $\{-2,1,1\}$ with the same $A=3$.

1.3 Calculations for Ricci-Codazzi and Ricci cyclic parallel.

In order to be able to consider the Ricci-Codazzi and the Ricci cyclic parallel conditions we compute the covariant derivative of the Ricci tensor using the usual formula:

$$\sum R_{ij,s}Q_s = dR_{ij} - \sum w_{ip}R_{pj} - \sum w_{jp}R_{ip} .$$

In our case the metric is diagonalized and the formula reduces to

$$\sum R_{ij,s}Q_s = dR_{ij} - w_{ij}(R_{jj} - R_{ii}) .$$

Making use of the equations 1.19 ,1.10 ,1.15 and 1.16 we obtain

$$\sum R_{00,s}Q_s = dR_{00} = -\{ \sum \dddot{u}(i) + 2\sum \ddot{u}(i)\dot{u}(i) \} Q_0 ,$$

$$\sum R_{ii,s}Q_s = dR_{ii} = -\{ \dddot{u}(i) + \ddot{u}(i)\sum \dot{u}(j) + \dot{u}(i)\sum \ddot{u}(j) \} Q_0 ,$$

$$\sum R_{0i,s}Q_s = -(R_{ii} - R_{00})w_{0i} = \dot{u}(i)\{ R_{00} - R_{ii} \} Q_i \quad \text{for } i = 1,2,3,$$

$$\sum R_{ij,s}Q_s = 0 \quad \text{for } i,j=1,2,3 \text{ and } i \neq j.$$

Thus the only nonvanishing of $D_i R_{jk}$ are given by:

3.1 $$D_0 R_{00} = \dot{R}_{00} = -\sum \dddot{u}(i) - 2\sum \ddot{u}(i)\dot{u}(i)$$

3.2 $$D_0 R_{ii} = \dot{R}_{ii} = -\dddot{u}(i) - \ddot{u}(i)\sum \dot{u}(j) - \dot{u}(i)\sum \ddot{u}(j)$$

3.3 $$D_i R_{0i} = \dot{u}(i)\{ R_{00} - R_{ii} \} \quad \text{for } i=1,2,3.$$

We now consider the case of Ricci-Codazzi, namely

$$D_i R_{jk} = D_k R_{ji} \qquad \text{for all} \quad i = 0,1,2,3.$$

Clearly the identity is automatically satisfied for the case $i = k$ .We also see from 3.1 ,3.2 and 3.3 that the identity is trivial for the cases when i,j,k are distinct.So we have only to consider whether or not $D_i R_{ij}$ is equal to $D_j R_{ii}$. Again this is trivial when $i=j$. Hence the only cases to be considered are:

$$D_i R_{00} = D_0 R_{0i} \quad \text{for} \quad i = 1,2,3,$$

$$D_0 R_{11} = D_1 R_{10} \; ; \; D_2 R_{11} = D_1 R_{12} \; ; \; D_3 R_{11} = D_1 R_{13} ,$$

$$D_0 R_{22} = D_2 R_{20} \; ; \; D_1 R_{22} = D_2 R_{21} \; ; \; D_3 R_{22} = D_2 R_{23} ,$$

$$D_0 R_{33} = D_3 R_{30} \; ; \; D_1 R_{33} = D_3 R_{31} \; ; \; D_2 R_{33} = D_3 R_{32} .$$

All these cases are trivial except for the three conditions,

$$D_0 R_{ii} = D_i R_{i0} \quad ; \quad i = 1,2,3.$$

We therefore have the following result.

Consider metrics of the form

$$ds^2 = dt^2 + \Sigma e^{2u(i)} (dx_i)^2$$

where $u(i)$ are functions of the single variable t .
Necessary and sufficient conditions for Ricci-Codazzi are

$$\dot{R}_{ii} = \dot{u}(i)(R_{00} - R_{ii}) \qquad \text{for} \quad i = 1,2,3.$$

(If the space is Einstein then the above condition is automatically satisfied.) On substituting the values of the Ricci tensors the conditions become:

3.4 $$\dddot{u}(i)+ \ddot{u}(i)\{\sum\dot{u}(j)+\dot{u}(i)\}+ \dot{u}(i)^2\sum\dot{u}(j)- \dot{u}(i)\sum\dot{u}(j)^2 =0$$

for $i = 1,2,3$.

We now consider the condition for Ricci cyclic parallel, namely

$$D_i R_{jk} + D_j R_{ki} + D_k R_{ij} = 0 \qquad \text{for all } i,j,k=0,1,2,3.$$

Since the only nonzero components of the derivative are

$$D_0 R_{00} , \; D_0 R_{ii} , \; D_i R_{0i} ,$$

we have only to consider

$$D_0 R_{00} = 0 \quad \text{and} \quad D_0 R_{ii} + D_i R_{i0} + D_i R_{0i} = 0 \quad \text{for } i=1,2,3.$$

We have therefore obtained the following result.

Necessary and sufficient conditions for the space to be Ricci cyclic parallel are :

R_{00} is a constant and $\dot{R}_{ii} + 2\dot{u}(i)\{R_{00} - R_{ii}\} =0$ for $i=1,2,3$.

These conditions are equivalent to

3.5 $$\sum \dddot{u}(j) + 2 \sum \ddot{u}(j)\dot{u}(j) = 0$$

3.6 $$\dddot{u}(i)+ \ddot{u}(i)\{2\dot{u}(i)+\dot{u}(j)+\dot{u}(k)\}+ 2\dot{u}(i)\{\dot{u}(j)^2+\dot{u}(k)^2\}$$

$$- 2\dot{u}(i)^2\{\dot{u}(j)+\dot{u}(k)\}+ 3\dot{u}(i)\{\ddot{u}(j)+\ddot{u}(k)\} = 0,$$

where i,j,k in 3.6 are cyclic permutations of 1,2,3.

We have been unable to tackle both problems of Ricci-Codazzi and Ricci cyclic parallel in general because of the nature of the differential equations.

1.4 Special cases

We now simplify our problem considerably by considering the case when $u(1)=u(2)=u(3)=u$, say.
In this case the solutions of the Einstein equations 2.2 and 2.3, which now reduce to

$$3\ddot{u}(t) + 3\dot{u}(t)^2 = v/4 \quad \text{and} \quad \ddot{u}(t) + 3\dot{u}(t)^2 = v/4,$$

must be $\ddot{u}(t) = 0$.

The Ricci cyclic parallel conditions 3.5 and 3.6 are now just

$$\dddot{u}(t) + 2\ddot{u}(t)\dot{u}(t) = 0 \quad \text{and} \quad \dddot{u}(t) + 10\ddot{u}(t)\dot{u}(t) = 0$$

which clearly imply $\ddot{u}(t) = 0$. Thus we have:

Ricci cyclic parallel spaces with metrics of the form

$$ds^2 = dt^2 + e^{2u(t)}\{ dx_1^2 + dx_2^2 + dx_3^2 \}$$

are necessarily Einstein spaces of constant curvature. The spaces are flat when u is just a constant function and we shall always exclude this trivial case from our discussion. The proof is immediate from part 1) of theorem 1.

Lastly the Ricci- Codazzi conditions 3.4 reduce to

$$\dddot{u}(t)+4\ddot{u}(t)\dot{u}(t) = 0, \text{ i.e. } \ddot{u}(t)+2\dot{u}(t)^2 = k ,$$

where k is a constant.
Clearly the solution $\ddot{u} = 0$ of the Einstein equations satisfy the differential equation above. In general we have to consider three cases :

case(1): $k=0$. In this case it is quite easy to see that $u(t)=\frac{1}{2}\log|t-c|$ is a solution of $\ddot{u}(t)+2\dot{u}(t)^2=0$. This solution does not come from a Ricci parallel metric. See equation 3.1.

case(2): k is positive, say $k=K/2$.

We write

$$2e^{2u}=p \text{ ,i.e, } 2u=\log(p/2) \text{ ; } 2\dot{u}=\dot{p}/p \text{ ; } 2\ddot{u}=\ddot{p}/p-4\dot{u}^2 .$$

These together with $\ddot{u}(t)+2\dot{u}(t)^2=K/2$ give $\ddot{p}-pK=0$ with a solution

$$p(t) = a\exp\{tK^{1/2}\} + b\exp\{-tK^{1/2}\}$$

where a and b are constants. So in this case we have

$$e^{2u(t)} = A\exp\{tK^{1/2}\} + B\exp\{-tK^{1/2}\}$$

where A and B are nonnegative arbitrary constants not both equal to zero.

case(3): k is negative.We obtain a similar result as above.

From the three cases above we obtain the following theorem which is known to Derdzinski [De1,2].

Theorem 2

Metrics of the form

$$ds^2 = dt^2 + e^{2u(t)}\{dx_1^2 + dx_2^2 + dx_3^2\}$$

are Ricci-Codazzi if the function e^{2u} is given by:

1) $e^{2u(t)} = t$,

2) $e^{2u(t)} = A\exp(tK^{1/2}) + B\exp(-tK^{1/2})$,

and the one which comes from the Einstein space $\ddot{u}(t)=0$.

The fact that this particular metric is not a good candidate for the study of Ricci cyclic parallel metrics can be seen by the following more general results. Derdzinski[De1] has considered the following construction.

Let (M,h^M) and (N,h^N) be Riemannian manifolds and F is a positive function on M. Define the F-warped product $M\times_F N$ of M and N to be the Riemannian manifold $(M\times N, h^M \times_F h^N)$ with

$$(h^M \times_F h^N)_{(x,y)}(U+X, V+Y) = h^M_x(U,V) + F(x)h^N_y(X,Y) ,$$

where U,V and X,Y are respectively tangent vectors of M and N at points x and y.
The local coordinate expressions for some geometric quantities when M is of 1-dimension are given as follows: Let I be an interval of real numbers, considered with its standard metric, F a positive smooth function on I and (N,h) an (n-1)-dimensional Riemannian manifold. Denoting by g the F-warped product metric of $I\times_F N$ and by R its Ricci tensor, and letting the indices i,j,k run through 1,..,n-1 we get for a given chart t=x(0),x(1),...,x(n-1) of $I\times_F N$ with $g_{00} = 1$, $g_{0i} = 0$ and $g_{ij} = Fh_{ij}$, the non-zero components of the Ricci tensor and its derivative, namely

$$R_{00} = \frac{1-n}{4}\{2\ddot{q} + (\dot{q})^2\},$$

$$R_{ij} = -p_{ij} - \frac{e^q}{4}\{2\ddot{q} + (n-1)(\dot{q})^2\}h_{ij},$$

and

$$D_0 R_{00} = \frac{1-n}{2}\{\dddot{q} + \ddot{q}\dot{q}\},$$

$$D_0 R_{ij} = -\dot{q}\,p_{ij} - \frac{e^q}{2}\{\dddot{q} + (n-1)\ddot{q}\dot{q}\}h_{ij},$$

$$D_i R_{0j} = -\dot{q}\,p_{ij}/2 + \frac{2-n}{4}e^q\ddot{q}\dot{q}\,h_{ij},$$

$$D_k R_{ij} = D'_k p_{ij},$$

where $q=\log F$ and D', p denote the Riemannian connection and Ricci tensor of (N,h), respectively, while the components of h, p and Dp are considered with respect to the chart $x(1),\ldots,x(n-1)$ of N.

If F is non-constant and $n \geq 3$, then $I \times_F N$ has harmonic curvature if and only if (N,h) is an Einstein space and the positive function $f = F^{n/4}$ on I satisfies the ordinary differential equation

$$\ddot{f} - \frac{nk}{4(n-1)} f^{1-4/n} = bf$$

for some real number b, k being the constant scalar curvature of N.

In our case we have $n=4$ and $k=0$. Futhermore $f=F=e^{2u}$ so that $\ddot{f} = bf$ becomes $\ddot{u} + 2\dot{u}^2 = b$ which agrees with our result for the case of Ricci-Codazzi.

The case of the Ricci cyclic parallel spaces which are the F-warped products of the form $I \times_F N^{n-1}$ is as follows:

The conditions $D_0 R_{00} = 0$, $D_0 R_{ij} + D_i R_{j0} + D_j R_{0i} = 0$, respectively implies

$$\dddot{q}(t) + \ddot{q}(t)\dot{q}(t) = 0,$$

$$\dot{q}(t)p_{ij} = \frac{2-n}{4} e^{q} \dot{q}(t)\ddot{q}(t)h_{ij} - e^{q}\{\dddot{q}(t) + (n-1)\ddot{q}(t)\dot{q}(t)\}h_{ij}/4$$

and these two together give

$$p_{ij} = \frac{2-n}{2} \ddot{q}(t)e^{q} h_{ij}$$

which means N is an Einstein space.

We therefore have

$$\frac{2-n}{2} \ddot{q}(t)e^{q(t)} = \frac{S}{n-1}$$

where S is the scalar curvature of N. Thus a necessary condition for the warped product $I \times_F N$ to be Ricci cyclic parallel is

$$\ddot{q}(t) + \frac{2S}{(n-1)(n-2)} e^{-q(t)} = 0 \quad ; \quad n \geqslant 3$$

and

$$\dddot{q}(t) + \ddot{q}(t)\dot{q}(t) = 0.$$

But these are exactly the conditions for the Ricci tensor of the warped product $I \times_F N$ to be parallel. See Derdzinski [De2] p.147. So we see that the warped product of this type is not a good candidate for studying the Ricci cyclic parallel case. In fact in our case we have S=0 and hence $\ddot{q}(t) = 0$ which is the solution of the Einstein equations.

The next simplest case to consider is when metrics take the form

$$ds^2 = dt^2 + \sum (t-k)^{2a(i)} dx_i^2$$

where a(i) and k are just constants. In this special case we have

4.1 $\quad u(i)=a(i)\log(t-k);\ \dot{u}(i)= \frac{a(i)}{t-k};\ \ddot{u}(i)=-\frac{a(i)}{(t-k)^2};\ \dddot{u}(i)= \frac{2a(i)}{(t-k)^3}$

and the Einstein spaces correspond to part (2) of theorem1 with the relation $\sum a(i) = \sum a(i)^2 = 1$.

Moreover we have seen that the scalar curvature is zero.

In the case of Ricci cyclic parallel it is not difficult to see that the space is necessarily Einstein for condition 3.5 implies that $\sum a(i) = \sum a(i)^2$.

Summing up equation 3.6 for i=1,2,3 and using 3.5 gives

$$\dot{u}(1)\dot{u}(2)+\dot{u}(2)\dot{u}(3)+\dot{u}(3)\dot{u}(1) = c$$

for a constant c. But then we must have

4.2 $\quad a(1)a(2)+a(2)a(3)+a(3)a(1) = 0.$

Now it is clear that the relation $\sum a(i)=\sum a(i)^2=1$ is essential.

For the case of Ricci-Codazzi ,we substitute $\dot{u}(i),\ddot{u}(i)$ and $\dddot{u}(i)$ from 4.1 into equation 3.4 to get

4.3 $$a(i)\{2 -(\Sigma a(j)+a(i)) +a(i)\Sigma a(j) -\Sigma a(j)^2\} = 0 \; ; \; i=1,2,3.$$

If all a(i) are different from zero we must then have the relation

4.4 $$d = 2 - c + a(i)(c - 1) \qquad \text{for all } i = 1,2,3$$

where c,d denote $\Sigma\, a(i)$, $\Sigma\, a(i)^2$, respectively. Summing for $i = 1,2,3$ in 4.4 gives

4.5 $$3d = 6 -4c + c^2.$$

On the other hand we also have

4.6 $$\Sigma\, \dddot{u}(i) + \Sigma\, \ddot{u}(i)\, \Sigma\, \dot{u}(i) + \Sigma\, \ddot{u}(i)\dot{u}(i) = 0$$

obtained by summing up the three Ricci-Codazzi conditions of 3.4 .This together with 4.1 would imply

4.7 $$d = 2c - c^2 .$$

Eliminating d from equations 4.5 and 4.7 would give

$$2c^2 -5c +3 = (2c-3)(c-1) =0.$$

When c=1, d is equal to 1 and when c = 3/2 ,d is equal to 3/4 .The first case is none other than the Einstein solution while the second one does not even give Ricci parallel. For example we take a(i) to be {1/2,1/2,1/2}. In fact this is the only solution for the second case:

for from $\Sigma a_i =3/2$ and $\Sigma a_i^2 =3/4$, we easily see that

$$\Sigma(a_i -1/2)^2 = \Sigma a_i^2 +3/4 -\Sigma a_i =0.$$

If however only one of the coefficients say a(1) is nonzero then condition 4.3 reduces to a(1)=1. This is just a flat space.

Now suppose a(k)=0 but a(i), a(j) are different from zero. From 4.4 we obtain

$$2d = 4 - 3c + c^2 \tag{4.8}$$

On eliminating d from 4.7 and 4.8 we get c=1 or 4/3. When c=1, d is equal to 1 and when c =4/3, d is equal to 8/9. But the only solution of the first case is either a(i)=0 and a(j)=1 or the other way round (which is absurd), while for the second case we must have a(i)=a(j)=2/3. We can now conclude :

Theorem 3

There are metrics of the form

$$ds^2 = dt^2 + \sum (t-k)^{2a(i)} dx_i^2 ,$$

which are Ricci-Codazzi but not Einstein. These are given by:

1) a(i) = 1/2 for all i=1,2,3.

2) {a(1), a(2), a(3)} are just permutations of {2/3, 2/3, 0}.

Note that the first part is similar to part 1 of Theorem2.

Chapter 2. Metrics On Certain Odd Dimensional Spheres.

In this chapter we shall use the Cartan structure equations to compute curvature. This method is similar to Jensen's but more direct. See [Je1], [Gr1] and [Ne].
We explain our notation in 1) while in 2) we consider a naturally defined 2-parameter family of metrics on the spheres S^{2n+1} and the curvature is then computed. The same is done in 3) for the 4-parameter family of metrics on S^{4n+3}.
In 4) and 5) we discuss the possibilities of those metrics being Einstein, Ricci- Codazzi or Ricci cyclic parallel on those spheres.
In particular we obtain the Einstein metric of Jensen [Je1] beside the standard one.

We conclude our investigation in Theorem4 p. 66, Theorem5 p. 68 and Theorem6 p. 70. From these theorems it seems that on a homogeneous space the Ricci-Codazzi condition is more strict than the Ricci cyclic parallel condition.

The non-associativity of Cayley numbers prevent us from generalizing the method to S^{8n+7} to obtain the third Einstein metric of Bourguignon and Karcher [BK].

2.1 NOTATION

Let M be an n-dimensional Riemannian manifold with metric $\bar{ds}^2$. In a neighbourhood of each point let $\bar{Q}_1, \ldots, \bar{Q}_n$ be 1-forms which orthonormalize the metric,

i.e $$\bar{ds}^2 = \bar{Q}_1^2 + \ldots + \bar{Q}_n^2 .$$

The connection forms $\bar{w}$'s are the unique solutions of the structural equations of Cartan

1.1 $$d\bar{Q}_i = \sum_j \bar{w}_{ij} \wedge \bar{Q}_j \qquad ; \quad i=1,\ldots,n$$

with

1.2 $$\bar{w}_{ij} + \bar{w}_{ji} = 0 \qquad ; \quad i,j=1,\ldots,n.$$

The curvature forms $\bar{m}_{ij}$ are given by the relation

1.3 $$d\bar{w}_{ij} = \bar{m}_{ij} + \sum_k \bar{w}_{ik} \wedge \bar{w}_{kj} \qquad ; \quad i,j = 1,\ldots,n$$

where our connection and curvature forms $\bar{w}_{ij}$, $\bar{m}_{ij}$ correspond to the ω_i^j, Ω_i^j, respectively of Kobayashi and Nomizu [KN1].

The components of the curvature tensor (relative to $\bar{Q}_1, \ldots, \bar{Q}_n$) are given by

1.4 $$2\bar{m}_{ij} = - \sum_{k,l} \bar{R}_{ijkl} \bar{Q}_k \wedge \bar{Q}_l .$$

The sectional curvature ,Ricci curvature and scalar curvature are respectively given by

1.5 $\bar{K}_{ij} = \bar{R}_{ijij}$, $\bar{R}_{ij} = \sum_k \bar{R}_{ikjk}$, $\bar{R} = \sum_i \bar{R}_{ii}$.

We also recall that the manifold M is of constant sectional curvature k if

1.6 $\bar{m}_{ij} = -k\bar{Q}_i \wedge \bar{Q}_j$;i,j = 1,...,n.

See [KN1] p.204.

Let $\bar{D}$ be the Riemannian connection of an n-dimensional unit sphere S^n and let $\{E_1,...,E_n\}$ be the local orthonormal frame field dual to $\{\bar{Q}_1,...,\bar{Q}_n\}$ .Then the connection $\bar{D}$ and the connection forms $\bar{w}_{ij}$ are related by the formula

1.7 $\bar{w}_{ij}(X) = \langle \bar{D}_X E_i , E_j \rangle$ for all vector fields X on S^n.

We regard S^n as the unit sphere in R^{n+1} and let N be a globally defined unit normal vector field to S^n .Thus if $\{u_1,...,u_{n+1}\}$ denote the natural coordinate functions on R^{n+1} , we have

1.8 $N = \sum_i u_i \partial/\partial u_i$

for

$$\sum_{i=1}^{n+1} (u_i)^2 = 1 \quad \text{on} \quad S^n .$$

Let D be the Riemannian connection of R^{n+1}, the formulas of Gauss and Weingarten are

$$D_X Y = \bar{D}_X Y + h(X,Y)N \; ; \; X,Y \text{ are tangent vector fields to } S^n$$

$$D_X N = - A_N(X) + D^{\perp}_X N$$

where h is the second fundamental form, while A_N and $D^{\perp}_X N$ are the tangential and normal components. The second equation reduces to

1.9 $$D_X N = X$$

for from $\langle N,N \rangle = 1$ we get $\langle D_X N,N \rangle = 0$ and hence $\langle D^{\perp}_X N,N \rangle = 0$.

Since $D^{\perp}_X N$ is is a scalar multiple of N, we must have $D^{\perp}_X N=0$ at each point p of S^n. Moreover we have $A_N = -Id$ for our sphere is of radius 1. See Kobayashi and Nomizu [KN2] p. 30. This can also be shown by a straight forward calculation using the fact that the Riemannian connection of Euclidean space is flat .

2.2 METRICS ON SPHERES S^{2n+1}

Let $d\bar{s}^2$ denote the standard metric on S^{2n+1} with sectional curvature 1 and we regard S^{2n+1} as the unit sphere in R^{2n+2}. R^{2n+2} has a naturally defined almost complex structure I, (i.e, at every point, I is an endomorphism of the tangent space such that $I^2 = -Id$ where Id is the identity transformation) which is compatible with the metric in the sense that $\langle IX,IY\rangle=\langle X,Y\rangle$ for all vector fields X,Y. We then have a globally defined tangent vector field IN to S^{2n+1} for $\langle IN,N\rangle=-\langle N,IN\rangle$ implies IN is perpendicular to N.

For each point p in S^{2n+1}, let $\{E_1,\ldots,E_{2n+1}\}$ be an orthonormal frame field defined in a neighbourhood of p such that

$$2.1 \qquad E_{2n+1} = IN$$

and

$$2.2 \qquad IE_i = E_{i+n} \quad ; \quad IE_{i+n} = -E_i \qquad \text{for } i=1,\ldots,n.$$

Denote by $\{\bar{Q}_1,\ldots,\bar{Q}_{2n},\bar{Q}_I\}$ the 1-forms dual to $\{E_1,\ldots,E_{2n},IN\}$ and we write

$$2.3 \qquad \bar{Q}_{I(i)} = \bar{Q}_{i+n} \quad ; \quad \bar{Q}_{I(i+n)} = -\bar{Q}_i \qquad \text{for } i=1,\ldots,n.$$

We therefore have $d\bar{s}^2 = \bar{Q}_1^2 + \ldots + \bar{Q}_n^2 + \bar{Q}_{I(1)}^2 + \ldots + \bar{Q}_{I(n)}^2 + \bar{Q}_I^2$.

Denote by $\bar{w}_{ij}$ and $\bar{m}_{ij}$ the connection and the curvature forms of $d\bar{s}^2$ relative to the frame field $\{E_1, \ldots, E_{2n}, IN\}$.

Lemma.

The connection forms $\bar{w}_{ij}$ satisfy

2.4 $\quad \bar{w}_{iI} = \bar{Q}_{I(i)} \quad ; \quad \bar{w}_{I(i)I} = -\bar{Q}_i \quad ; \quad i=1,\ldots,n,$

2.5 $\quad \bar{w}_{I(i)I(j)} = \bar{w}_{ij} \quad ; \quad \bar{w}_{I(i)j} = -\bar{w}_{iI(j)} \quad ; i,j=1,\ldots,n.$

Proof

Let $\bar{D}$ and D denote the Riemannian connections of S^{2n+1} and R^{2n+2} respectively. Then $\bar{D}$ and $\bar{w}_{ij}$ are related by the formula

2.6 $\quad \langle \bar{D}_X E_i, E_j \rangle = \bar{w}_{ij}(X)$ for all vector field X on S^{2n+1}, $i,j=1,\ldots,2n+1.$

We have

$$\bar{w}_{iI}(X) = \langle \bar{D}_X E_i, IN \rangle = \langle D_X E_i, IN \rangle = -\langle E_i, D_X IN \rangle$$

$$= -\langle E_i, IX \rangle = \langle IE_i, X \rangle = \bar{Q}_{I(i)}(X) \; ;$$

$$\bar{w}_{I(i)I}(X) = \langle \bar{D}_X IE_i, IN \rangle = \langle D_X IE_i, IN \rangle = -\langle IE_i, D_X IN \rangle$$

$$= -\langle IE_i, IX\rangle = -\langle E_i, X\rangle = -\bar{Q}_i(X) \quad ;$$

$$\bar{w}_{I(i)I(j)}(X) = \langle \bar{D}_X IE_i, IE_j\rangle = \langle D_X IE_i, IE_j\rangle = \langle D_X E_i, E_j\rangle$$

$$= \langle \bar{D}_X E_i, E_j\rangle = \bar{w}_{ij}(X) \quad ;$$

$$\bar{w}_{I(i)j}(X) = \langle \bar{D}_X IE_i, E_j\rangle = \langle D_X IE_i, E_j\rangle = \langle ID_X E_i, E_j\rangle$$

$$= -\langle D_X E_i, IE_j\rangle = \langle E_i, D_X IE_j\rangle = \bar{w}_{I(j)i}(X) \; .$$

In the calculation above we used the fact that the almost complex structure I of R^{2n+2} is parallel.

Now consider metrics on S^{2n+1} of the form

$$ds^2 = A^2 \{ \bar{Q}_1^2 + .. + \bar{Q}_n^2 + \bar{Q}_{I(1)}^2 + .. + \bar{Q}_{I(n)}^2 \} + a^2 \bar{Q}_I^2$$

where A and a are non-zero constants.

If we write

2.7 $$Q_i = A \bar{Q}_i \quad ; \ i=1,..,2n$$

and

2.8 $$Q_I = a \bar{Q}_I$$

then

$$ds^2 = Q_1^2 + .. + Q_{2n}^2 + Q_I^2 .$$

Let w_{ij} and m_{ij} $(1 \leqslant i,j \leqslant 2n+1)$ be the corresponding connection and curvature forms respectively. We have the relations between $w, \bar{w}$ and $\bar{Q}$ as follows :

2.9 $$w_{ij} = w_{I(i)I(j)} = \bar{w}_{ij} = \bar{w}_{I(i)I(j)} \quad ; 1 \leqslant i,j \leqslant 2n, j \neq I(i),$$

2.10 $$w_{iI(i)} = \bar{w}_{iI(i)} + A^{-2} (a^2 - A^2) \bar{Q}_I \quad ; 1 \leqslant i \leqslant 2n,$$

2.11 $$w_{iI} = A^{-1} a \bar{w}_{iI} = A^{-1} a \bar{Q}_{I(i)} \quad ; 1 \leqslant i \leqslant 2n.$$

Proof

$$dQ_I = \sum_j w_{Ij} \wedge Q_j = A \sum w_{Ij} \wedge \bar{Q}_j . \tag{2.12}$$

On the other hand

$$dQ_I = d(a\,\bar{Q}_I) = a\,d\bar{Q}_I = a \sum_j \bar{w}_{Ij} \wedge \bar{Q}_j . \tag{2.13}$$

Comparing 2.12 and 2.13 we have

$$w_{iI} = A^{-1} a\bar{w}_{iI} = A^{-1} a\bar{Q}_{I(i)}, \tag{2.14}$$

where the last equality is obtained from the first equality of 2.4 .This proves 2.11.

$$dQ_i = \sum_j w_{ij} \wedge Q_j + w_{iI(i)} \wedge Q_{I(i)} + w_{iI} \wedge Q_I$$

$$= A \sum_j w_{ij} \wedge \bar{Q}_j + A\, w_{iI(i)} \wedge \bar{Q}_{I(i)} + A^{-1} a^2 \bar{Q}_{I(i)} \wedge \bar{Q}_I . \tag{2.15}$$

On the other hand

$$dQ_i = d(A\,\bar{Q}_i) = A\,d\bar{Q}_i$$

$$= A \sum_j \bar{w}_{ij} \wedge \bar{Q}_j + A\,\bar{w}_{iI(i)} \wedge \bar{Q}_{I(i)} + A\,\bar{w}_{iI} \wedge \bar{Q}_I$$

$$= A \sum_j \bar{w}_{ij} \wedge \bar{Q}_j + A\,\bar{w}_{iI(i)} \wedge \bar{Q}_{I(i)} + A\,\bar{Q}_{I(i)} \wedge \bar{Q}_I . \tag{2.16}$$

Using the uniqueness of the solution of Cartan structure equation we get

$$w_{ij} = \bar{w}_{ij} \quad \text{for } j \neq I(i), \tag{2.17}$$

which satisfy 2.9 . Furthermore

equations 2.15, 2.16 and 2.17 would give

$$\bar{Q}_{I(i)} \wedge \{-A\, w_{iI(i)} + A^{-1} a^{2} \bar{Q}_{I}\} = \bar{Q}_{I(i)} \wedge \{-A\, \bar{w}_{iI(i)} + A\, \bar{Q}_{I}\}$$

from which we obtain

$$w_{iI(i)} = \bar{w}_{iI(i)} + A^{-2}(a^{2} - A^{2})\,\bar{Q}_{I}$$

and completes the proof.

We now compute the curvature forms m_{ij} using identities 2.9, 2.10 and 2.11. For $j \neq I(i)$

$$m_{ij} = dw_{ij} - \sum_{k} w_{ik} \wedge w_{kj} - w_{iI(i)} \wedge w_{I(i)j} - w_{iI(j)} \wedge w_{I(j)j} - w_{iI} \wedge w_{Ij}$$

$$= d\bar{w}_{ij} - \sum_{k} \bar{w}_{ik} \wedge \bar{w}_{kj} - \{\bar{w}_{iI(i)} + A^{-2}(a^{2} - A^{2})\,\bar{Q}_{I}\} \wedge \bar{w}_{I(i)j}$$

$$+ \bar{w}_{iI(j)} \wedge \{\bar{w}_{jI(j)} + A^{-2}(a^{2} - A^{2})\,\bar{Q}_{I}\} + A^{-2} a^{2} \bar{Q}_{I(i)} \wedge \bar{Q}_{I(j)}$$

2.18 $$= \{d\bar{w}_{ij} - \sum \bar{w}_{ik} \wedge \bar{w}_{kj} - \bar{w}_{iI(i)} \wedge \bar{w}_{I(i)j} - \bar{w}_{iI(j)} \wedge \bar{w}_{I(j)j} - \bar{w}_{iI} \wedge \bar{w}_{Ij}\}$$

$$+ A^{-2}(a^{2} - A^{2})\bar{Q}_{I(i)} \wedge \bar{Q}_{I(j)} - A^{-2}(a^{2} - A^{2})\bar{Q}_{I} \wedge \{\bar{w}_{I(i)j} + \bar{w}_{iI(j)}\}$$

but the last term is zero by the second equality of 2.5, hence

2.19 $$m_{ij} = \bar{m}_{ij} + A^{-2}(a^{2} - A^{2})\,\bar{Q}_{I(i)} \wedge \bar{Q}_{I(j)} \quad ; 1 \leq i, j \leq 2n;\ j \neq I(i).$$

We now use the fact that the metric $d\bar{s}^2$ is of constant sectional curvature 1, i.e,

$$\bar{m}_{ij} = -\bar{Q}_i \wedge \bar{Q}_j \quad , \ 1 \leqslant i, j \leqslant 2n+1$$

and we finally obtain

2.20 $$m_{ij} = -A^{-2} Q_i \wedge Q_j - (A^2 - a^2) A^{-4} Q_{I(i)} \wedge Q_{I(j)}$$

by using the identity 2.7 .Similar method of calculation would give

2.21 $$m_{iI(i)} = -A^{-4}(4A^2 - 3a^2) Q_i \wedge Q_{I(i)} - \sum_{k \neq i} A^{-4} (A^2 - a^2) Q_k \wedge Q_{I(k)}$$

{ notice here that k does not take values i or I(i)}

2.22 $$m_{iI} = - a^2 A^{-4} Q_i \wedge Q_I \quad , \ i=1,..,2n.$$

From three identities 2.20,2.21 and 2.22 ,we obtain the components of the curvature tensor together with sectional curvatures and Ricci tensors :

$$R_{ijI(i)I(j)} = (A^2 - a^2) A^{-4}/2 \ ; \ 1 \leqslant i,j \leqslant n \ ; \ j \neq i, I(i),$$

$$R_{iI(i)kI(k)} = (A^2 - a^2) A^{-4}/2 \ ; \ 1 \leqslant i,j \leqslant n \ ; \ i \neq k,$$

$$K_{ij} = A^{-2} \qquad ; \ 1 \leqslant i,j \leqslant 2n \ ; \ j \neq i, I(i),$$

$$K_{iI(i)} = (4A^2 - 3a^2)\,A^{-4} \quad ; \quad K_{iI} = a^2 A^{-4} \quad ;$$

$$R_{II} = 2na^2 A^{-4} \quad ; \quad R_{ii} = 2((n+1)A^2 - a^2)A^{-4}\,,$$

$$i = 1, \ldots, 2n.$$

The scalar curvature of the metric ds^2 is given by

$$R = 2n((2n+2)A^2 - a^2)\,A^{-4}$$

2.3 METRICS ON S^{4n+3}

We generalize the method used on S^{2n+1}. Let $d\bar{s}^2$ be the standard metric on S^{4n+3} with constant sectional curvature 1 and we regard S^{4n+3} as the unit sphere in the right quaternionic vector space H^{n+1}. Let I,J and K denote the transformation on H^{n+1} which are left multiplication by the quaternions i,j and k, respectively. Then I,J and K are quaternionic linear on H^{n+1}. Thus if N is the unit normal vector field to S^{4n+3} we have three tangent vector fields IN,JN and KN to S^{4n+3} which are globally defined. For each point p of S^{4n+3}, let $\{E_1, \ldots, E_{4n+3}\}$ be an orthonormal frame field defined in a neighbourhood of p such that

3.1 $\quad IE_i = E_{n+i} \; ; \; JE_i = E_{2n+i} \; ; \; KE_i = E_{3n+i}, \quad i=1,\ldots,n$

and

3.2 $\quad E_{4n+1} = IN \; ; \; E_{4n+2} = JN \; ; \; E_{4n+3} = KN.$

Writing

3.3 $\quad E_{n+i} = E_{I(i)} \; ; \; E_{2n+i} = E_{J(i)} \; ; \; E_{3n+i} = E_{K(i)},$

we clearly have the following relations

3.4 $\quad IE_{I(i)} = JE_{J(i)} = KE_{K(i)} = -E_i, \quad i=1,\ldots,n.$

Now let

$$\{\bar{Q}_1,..,\bar{Q}_n,\bar{Q}_{I(1)},..,\bar{Q}_{I(n)},..,\bar{Q}_{J(1)},..,\bar{Q}_{J(n)},\bar{Q}_{K(1)},..,\bar{Q}_{K(n)},\bar{Q}_I,\bar{Q}_J,\bar{Q}_K\}$$

be the 1-forms dual to $\{E_1,...,E_{4n+3}\}$.

We set

3.5 $V(V(i))=-i$ and $\bar{Q}_{-i} = -\bar{Q}_i$ for all $V=I,J,K$ and all $i=1,..,n$.

So we have the metric

$$d\bar{s}^2 = \sum_i \bar{Q}_i^2 + \sum_i \bar{Q}_{I(i)}^2 + \sum_i \bar{Q}_{J(i)}^2 + \sum_i \bar{Q}_{K(i)}^2 + \bar{Q}_I^2 + \bar{Q}_J^2 + \bar{Q}_K^2 .$$

Following the notation of the preceding section we let $\bar{w}_{ij}$ and $\bar{m}_{ij}$ be the connection and curvature forms of the metric $d\bar{s}^2$ relative to the frame field $\{E_1,...,E_{4n+3}\}$.

Lemma

3.6 $\bar{w}_{iV} = \bar{Q}_{V(i)}$ for all $V=I,J,K$, $l=1,..,n$,

3.7 $\bar{w}_{ij} = \bar{w}_{V(i)V(j)}$ for all $V=I,J,K$, $1\leqslant i,j\leqslant n$,

3.8 $\bar{w}_{V(i)V} = -\bar{Q}_i$; $\bar{w}_{U(i)V} = -\bar{Q}_{W(i)}$; $\bar{w}_{U(i)W} = \bar{Q}_{V(i)}$,

where U,V,W are cyclic permutations of I,J,K.

3.9 $\bar{w}_{iU(j)} = -\bar{w}_{U(i)j} = -\bar{w}_{V(i)W(j)} = \bar{w}_{W(i)V(j)}$, $1\leqslant i,j\leqslant n$,

3.10 $\bar{w}_{IJ} = -\bar{Q}_K$; $\bar{w}_{JK} = -\bar{Q}_I$; $\bar{w}_{KI} = -\bar{Q}_J$.

We prove 3.9 .

The first equality is the second equality in 2.5 while

$$\bar{w}_{V(i)W(j)}(X) = \langle \bar{D}_X VE_i , WE_j \rangle = -\langle \bar{D}_X E_i , VWE_j \rangle = -\bar{w}_{iU(j)}(X)$$

which proves the second part of 3.9.

From above we also get

$$\langle \bar{D}_X E_i , VWE_j \rangle = \langle \bar{D}_X WE_i , WVWE_j \rangle = \langle \bar{D}_X WE_i , VE_j \rangle = \bar{w}_{W(i)V(j)}(X)$$

and completes the proof.

All the others can be proved similarly.

Recall that we have the metric of constant sectional curvature 1

$$d\bar{s}^2 = \sum_i \bar{Q}_i^2 + \sum_i \bar{Q}_{I(i)}^2 + \sum_i \bar{Q}_{J(i)}^2 + \sum \bar{Q}_{K(i)}^2 + \bar{Q}_I^2 + \bar{Q}_J^2 + \bar{Q}_K^2 .$$

Now consider metrics of the form

$$ds^2 = A^2 d\bar{s}^2 + (a(I)^2 - A^2)\bar{Q}_I^2 + (a(J)^2 - A^2)\bar{Q}_J^2 + (a(K)^2 - A^2)\bar{Q}_K^2$$

which clearly equivalent to

$$A^2 \sum_i (\bar{Q}_i^2 + \bar{Q}_{I(i)}^2 + \bar{Q}_{J(i)}^2 + \bar{Q}_{K(i)}^2) + a(I)^2 \bar{Q}_I^2 + a(J)^2 \bar{Q}_J^2 + a(K)^2 \bar{Q}_K^2$$

where $A, a(I), a(J), a(K)$ are constants. These metrics are globally defined for at each point $p \in S^{4n+3}$ (represented as the quotient spaces $Sp(n+1)/Sp(n)$) ,the tangent space decomposes into the direct sum of a 4n-dimensional subspace and a 3-dimensional subspace, each invariant under the linear isotropy representation of $Sp(n)$. Furthermore the action on the three dimensional subspace is trivial.

We then write

3.11 $$A\,\bar{Q}_i = Q_i \qquad \text{for } i = 1,\dots,4n$$

and

3.12 $$a(V)\bar{Q}_V = Q_V \qquad \text{for } V = I, J, K.$$

Then the metric ds^2 can be written as

$$ds^2 = Q_1^2 + \dots + Q_{4n}^2 + Q_I^2 + Q_J^2 + Q_K^2 .$$

As before we also denote by w_{ij}, m_{ij} the corresponding connection and curvature forms. Now we find the relations between w, $\bar{w}$ (and $\bar{Q}$) making use of the identities 3.11 and 3.12 .

3.13 $$dQ_I = \sum_i^{4n} w_{Ii} \wedge Q_i + w_{IJ} \wedge Q_J + w_{IK} \wedge Q_K$$

$$= A \sum_i^{4n} w_{Ii} \wedge \bar{Q}_i + a(J)\, w_{IJ} \wedge \bar{Q}_J + a(K)\, w_{IK} \wedge \bar{Q}_K .$$

On the other hand

3.14 $$dQ_I = d(a(I)\,\bar{Q}_I) = a(I)\, d\bar{Q}_I$$

$$= a(I) \sum_i \bar{w}_{Ii} \wedge \bar{Q}_i + a(I)\, \bar{w}_{IJ} \wedge \bar{Q}_J + a(I)\, \bar{w}_{IK} \wedge \bar{Q}_K .$$

We take

3.15 $$w_{Vi} = A^{-1} a(V)\bar{w}_{Vi} = -A^{-1} a(V)\bar{Q}_{V(i)} \quad ; \; V=I,J,K,$$

where the last equality follows from the first equality of 3.6 .We must then have

3.16 $$a(J)w_{IJ}\wedge\bar{Q}_J + a(K)w_{IK}\wedge\bar{Q}_K = a(I)\{\bar{w}_{IJ}\wedge\bar{Q}_J + \bar{w}_{IK}\wedge\bar{Q}_K\}$$

Using 3.10, the right hand side of 3.16 becomes $2a(I)\bar{Q}_J\wedge\bar{Q}_K$.
Unique solutions of 3.16 are now given by

3.17 $$w_{IJ} = \frac{\{a(I)^2+a(J)^2-a(K)^2\}}{a(I)a(J)}\bar{w}_{IJ} = \frac{\{a(K)^2-a(I)^2-a(J)^2\}}{a(I)a(J)}\bar{Q}_K .$$

3.18 $$w_{IK} = \frac{\{a(I)^2+a(K)^2-a(J)^2\}}{a(I)a(K)}\bar{w}_{IK} = \frac{\{a(I)^2+a(K)^2-a(J)^2\}}{a(I)a(K)}\bar{Q}_J .$$

[check: $a(J)w_{IJ}\wedge\bar{Q}_J + a(K)w_{IK}\wedge\bar{Q}_K = \bar{Q}_K\wedge\bar{Q}_J\{a(K)^2-a(I)^2-a(J)^2$
$-(a(I)^2+a(K)^2-a(J)^2)\}/a(I) = 2a(I)\bar{Q}_J\wedge\bar{Q}_K$.]

3.19 $$dQ_i = \sum_{j\neq V(i)}^{4n} w_{ij}\wedge Q_j + \sum_V w_{iV(i)}\wedge Q_{V(i)} + \sum_V w_{iV}\wedge Q_V$$

$$= A\{\sum_{j\neq V(i)}^{4n} w_{ij}\wedge\bar{Q}_j + \sum_V w_{iV(i)}\wedge\bar{Q}_{V(i)}\}$$

$$+ a(I)w_{iI}\wedge\bar{Q}_I + a(J)w_{iJ}\wedge\bar{Q}_J + a(K)w_{iK}\wedge\bar{Q}_K .$$

On the other hand

3.20 $$dQ_i = Ad\bar{Q}_i = A\{\sum_{j\neq V(i)}^{4n}\bar{w}_{ij}\wedge\bar{Q}_j + \sum_V\bar{w}_{iV(i)}\wedge\bar{Q}_{V(i)} + \sum_V\bar{w}_{iV}\wedge\bar{Q}_V\}.$$

Comparing both identities , we can take

$$w_{ij} = \bar{w}_{ij} \quad \text{for} \quad 1 \leq j \leq 4n \quad \text{and} \quad j \neq V(i) \ ; \ V=I,J,K. \tag{3.21}$$

Equating the remaining terms of the equations 3.19 and 3.20 and using identity 3.15 we get

$$w_{iV(i)} = \bar{w}_{iV(i)} + A^{-2}\{a(V)^2 - A^2\}\bar{Q}_V \ ; V=I,J,K; i=1,..,4n. \tag{3.22}$$

Now

$$dQ_{I(i)} = \sum_{j\neq i} w_{I(i)j}\,\hat{}\,Q_j + w_{I(i)i}\,\hat{}\,Q_i + \sum_V w_{I(i)V(i)}\,\hat{}\,Q_{V(i)} + \sum_V w_{I(i)V}\,\hat{}\,Q_V$$

$$= A\sum_{j\neq i} \bar{w}_{I(i)j}\,\hat{}\,\bar{Q}_j + A[\bar{w}_{I(i)i} - ((a(I)^2 - A^2)A^{-2})\bar{Q}_I]\,\hat{}\,\bar{Q}_i + Aw_{I(i)J(i)}\,\hat{}\,\bar{Q}_{J(i)}$$

$$+ Aw_{I(i)K(i)}\,\hat{}\,\bar{Q}_{K(i)} + a(I)^2/_A\,\bar{w}_{I(i)I}\,\hat{}\,\bar{Q}_I + a(J)^2/_A\,\bar{w}_{I(i)J}\,\hat{}\,\bar{Q}_J + a(K)^2/_A\,\bar{w}_{I(i)K}\,\hat{}\,\bar{Q}_K$$

by 3.11,3.12,3.15 and 3.22. On the other hand

$$dQ_{I(i)} = Ad\bar{Q}_{I(i)} = A[\sum_{j\neq i} \bar{w}_{I(i)j}\,\hat{}\,\bar{Q}_j + \bar{w}_{I(i)i}\,\hat{}\,\bar{Q}_i + \sum_V \bar{w}_{I(i)V(i)}\,\hat{}\,\bar{Q}_{V(i)} + \sum_V \bar{w}_{I(i)V}\,\hat{}\,\bar{Q}_V].$$

Cancelling like terms of the above two identities we get

$$- A\bar{Q}_I\,\hat{}\,\bar{Q}_i\{a(I)^2 - A^2\}A^{-2} + A\{w_{I(i)J(i)} - \bar{w}_{I(i)J(i)}\}\,\hat{}\,\bar{Q}_{J(i)}$$

$$+ A\{w_{I(i)K(i)} - \bar{w}_{I(i)K(i)}\}\,\hat{}\,\bar{Q}_{K(i)} + \{a(I)^2 - A^2\}/_A\,\bar{w}_{I(i)I}\,\hat{}\,\bar{Q}_I$$

$$+ \{a(J)^2 - A^2\}/_A\,\bar{w}_{I(i)J}\,\hat{}\,\bar{Q}_J + \{a(K)^2 - A^2\}/_A\,\bar{w}_{I(i)K}\,\hat{}\,\bar{Q}_K = 0.$$

The first and fourth terms cancel by the first identity of 3.8.We later use the second and the third identities of the same equation for the last two terms to obtain

$$\{Aw_{I(i)J(i)} - A\bar{w}_{I(i)J(i)} - A^{-1}(a(K)^2 - A^2)\bar{Q}_K\} \wedge \bar{Q}_{J(i)} +$$

$$\{Aw_{I(i)K(i)} - A\bar{w}_{I(i)K(i)} + A^{-1}(a(J)^2 - A^2)\bar{Q}_J\} \wedge \bar{Q}_{K(i)} = 0.$$

So we have

3.23 $$w_{U(i)V(i)} = \bar{w}_{U(i)V(i)} + A^{-2}\{a(W)^2 - A^2\}\bar{Q}_W,$$

3.24 $$w_{U(i)W(i)} = \bar{w}_{U(i)W(i)} - A^{-2}(a(V)^2 - A^2)\bar{Q}_V,$$

where U,V,W are cyclic permutations of I,J,K.

We now list all the relations that we have obtained

3.25 $$w_{ij} = w_{V(i)V(j)} = \bar{w}_{ij} \quad ; \ i,j=1,\ldots,4n \text{ and } j \neq V(i),$$

3.26 $$w_{iV} = A^{-1}a(V)\bar{w}_{iV} = A^{-1}a(V)\bar{Q}_{V(i)} \quad ,V=I,J,K,i=1,..,4n,$$

3.27 $$w_{iU(i)} = -w_{V(i)W(i)} = \bar{w}_{iU(i)} + A^{-2}\{a(U)^2 - A^2\}\bar{Q}_U,$$

3.28 $$w_{UV} = \frac{\{a(U)^2 + a(V)^2 - a(W)^2\}}{a(U)a(V)}\bar{w}_{UV} = \frac{-\{a(U)^2 + a(V)^2 - a(W)^2\}}{a(U)a(V)}\bar{Q}_W,$$

3.29 $$w_{U(i)V(i)} = \bar{w}_{U(i)V(i)} + A^{-2}\{a(W)^2 - A^2\}\bar{Q}_W.$$

Identities 3.27 to 3.29 are for all cyclic permutations of I,J,K. We now check the relations that we have obtained. Firstly

$$dQ_i = \sum_{k\neq V(i)} w_{ik} \wedge Q_k + \sum_V w_{iV(i)} \wedge Q_{V(i)} + \sum_V w_{iV} \wedge Q_V$$

$$=A \sum_k \bar{w}_{ik} \wedge \bar{Q}_k + A\{\sum_V \bar{w}_{iV(i)} + \frac{a(V)^2 - A^2}{A^2} \bar{Q}_V\} \wedge \bar{Q}_{V(i)} + \sum_V \frac{a(V)^2}{A} \bar{w}_{iV} \wedge \bar{Q}_V$$

$$= A\, d\bar{Q}_i + \sum_V \frac{a(V)^2 - A^2}{A} \bar{Q}_V \wedge \bar{Q}_{V(i)} + \sum_V \frac{a(V)^2 - A^2}{A} \bar{w}_{iV} \wedge \bar{Q}_V .$$

Notice that the last two terms cancel. Secondly

$$dQ_I = \sum_j w_{Ij} \wedge Q_j + \sum_V w_{IV} \wedge Q_V$$

$$= \sum_j a(I)\bar{w}_{Ij} \wedge \bar{Q}_j + \frac{a(I)^2+a(J)^2-a(K)^2}{a(I)} \bar{w}_{IJ} \wedge \bar{Q}_J + \frac{a(I)^2+a(K)^2-a(J)^2}{a(I)} \bar{w}_{IK} \wedge \bar{Q}_K$$

$$=a(I)d\bar{Q}_I + \{\frac{a(I)^2+a(J)^2-a(K)^2}{a(I)} - a(I)\}\bar{w}_{IJ} \wedge \bar{Q}_J + \{\frac{a(I)^2+a(K)^2-a(J)^2}{a(I)} - a(I)\}\bar{w}_{IK} \wedge \bar{Q}_K$$

$$= a(I)\, d\bar{Q}_I - \frac{a(J)^2 - a(K)^2}{a(I)} \bar{Q}_K \wedge \bar{Q}_J + \frac{a(K)^2 - a(J)^2}{a(I)} \bar{Q}_J \wedge \bar{Q}_K$$

which clearly give $a(I)d\bar{Q}_I$ since the last two terms cancel. Lastly

$$dQ_{I(i)} = \sum_{j\neq i} w_{I(i)j} \wedge Q_j + w_{I(i)i} \wedge Q_i + \sum_V w_{I(i)V(i)} \wedge Q_{V(i)} + \sum_V w_{I(i)V} \wedge Q_V$$

$$= A\sum_j \bar{w}_{I(i)j} \wedge \bar{Q}_j + A\{\bar{w}_{I(i)i} - \frac{a(I)^2 - A^2}{A^2}\bar{Q}_I\} \wedge \bar{Q}_i + A\{\bar{w}_{I(i)J(i)} + \frac{a(K)^2 - A^2}{A^2}\bar{Q}_K\} \wedge \bar{Q}_{J(i)}$$

$$+ A\{\bar{w}_{I(i)K(i)} - \frac{a(J)^2 - A^2}{A^2}\bar{Q}_J\} \wedge \bar{Q}_{K(i)} + \frac{a(I)^2}{A}\bar{w}_{I(i)I} \wedge \bar{Q}_I \quad ,$$

$$+ \frac{a(J)^2}{A}\bar{w}_{I(i)J} \wedge \bar{Q}_J + \frac{a(K)^2}{A}\bar{w}_{I(i)K} \wedge \bar{Q}_K$$

$$= Ad\bar{Q}_{I(i)} - \frac{a(I)^2 - A^2}{A}\bar{Q}_I \wedge \bar{Q}_i + \frac{a(K)^2 - A^2}{A}\bar{Q}_K \wedge \bar{Q}_{J(i)} - \frac{a(J)^2 - A^2}{A}\bar{Q}_J \wedge \bar{Q}_{K(i)}$$

$$- \frac{a(I)^2 - A^2}{A}\bar{Q}_i \wedge \bar{Q}_I - \frac{a(J)^2 - A^2}{A}\bar{Q}_{K(i)} \wedge \bar{Q}_J + \frac{a(K)^2 - A^2}{A}\bar{Q}_{J(i)} \wedge \bar{Q}_K$$

$$= Ad\bar{Q}_{I(i)}$$

for the second, third and the fourth terms cancel with the fifth, seventh and the sixth terms, respectively.

We compute the components of the curvature tensor of the metric ds^2 using the second structural equation of Cartan. For each i, let j be different from $I(i), J(i), K(i)$, then

$$m_{ij} = dw_{ij} - \sum_k w_{ik} \wedge w_{kj} - \sum_V w_{iV(i)} \wedge w_{V(i)j} - \sum_V w_{iV(j)} \wedge w_{V(j)j} - \sum_V w_{iV} \wedge w_{Vj}$$

(note here that k is different from $V(i), V(j)$.)

$$= d\bar{w}_{ij} - \sum_k \bar{w}_{ik} \wedge \bar{w}_{kj} - \sum_V \{\bar{w}_{iV(i)} + \frac{a(V)^2 - A^2}{A^2}\bar{Q}_V\} \wedge \bar{w}_{V(i)j}$$

$$- \sum_V \bar{w}_{iV(j)} \wedge \{\bar{w}_{V(j)j} - \frac{a(V)^2 - A^2}{A^2} \bar{Q}_V\} - \sum_V \frac{a(V)^2}{A^2} \bar{w}_{iV} \wedge \bar{w}_{Vj}$$

which then reduces to

$$m_{ij} = \bar{m}_{ij} - \sum_V \frac{a(V)^2 - A^2}{A^2} \bar{Q}_V \wedge \{\bar{w}_{V(i)j} + \bar{w}_{iV(j)}\} + \sum_V \frac{a(V)^2 - A^2}{A^2} \bar{Q}_{V(i)} \wedge \bar{Q}_{V(j)} .$$

The second term vanishes by identity 3.9 while identity 3.11 and the fact that the original metric is of constant sectional curvature 1 ,imply

3.30 $$m_{ij} = -A^{-2} Q_i \wedge Q_j - \sum_V \frac{A^2 - a(V)^2}{A^4} Q_{V(i)} \wedge Q_{V(j)} , 1 \leq i,j \leq 4n, j \neq V(i).$$

Next

$$m_{iI(i)} = d\{\bar{w}_{iI(i)} + \frac{a(I)^2 - A^2}{A^2} \bar{Q}_I\} - \sum_k \bar{w}_{ik} \wedge \bar{w}_{kI(i)} - \sum_V \frac{a(V)^2}{A^2} \bar{w}_{iV} \wedge \bar{w}_{VI(i)}$$

$$- \{\bar{w}_{iJ(i)} + \frac{a(J)^2 - A^2}{A^2} \bar{Q}_J\} \wedge \{\bar{w}_{J(i)I(i)} - \frac{a(K)^2 - A^2}{A^2} \bar{Q}_K\}$$

$$- \{\bar{w}_{iK(i)} + \frac{a(K)^2 - A^2}{A^2} \bar{Q}_K\} \wedge \{\bar{w}_{K(i)I(i)} + \frac{a(J)^2 - A^2}{A^2} \bar{Q}_J\} .$$

Thus

3.31 $$m_{iI(i)} = \bar{m}_{iI(i)} + \frac{a(I)^2 - A^2}{A^2} d\bar{Q}_I - \{ \frac{a(J)^2 - A^2}{A^2} \bar{Q}_J \wedge \bar{w}_{J(i)I(i)}$$

$$+ \frac{a(K)^2-A^2}{A^2}\bar{Q}_K \wedge \bar{w}_{K(i)I(i)} - \frac{a(K)^2-A^2}{A^2}\bar{w}_{iJ(i)} \wedge \bar{Q}_K + \frac{a(J)^2-A^2}{A^2}\bar{w}_{iK(i)} \wedge \bar{Q}_J$$

$$- 2 . \frac{a(J)^2-A^2}{A^2} . \frac{a(K)^2-A^2}{A^2}\bar{Q}_J \wedge \bar{Q}_K \} - \sum_V \frac{a(V)^2-A^2}{A^2}\bar{w}_{iV} \wedge \bar{w}_{VI(i)} .$$

We know that

$$\bar{m}_{iI(i)} = - \bar{Q}_i \wedge \bar{Q}_{I(i)}$$

$$d\bar{Q}_I = \sum_k \bar{w}_{Ik} \wedge \bar{Q}_k + \bar{w}_{Ii} \wedge \bar{Q}_i + \sum_V \bar{w}_{IV(i)} \wedge \bar{Q}_{V(i)} + \sum_V \bar{w}_{IV} \wedge \bar{Q}_V$$

$$= \sum_k \bar{Q}_k \wedge \bar{Q}_{I(k)} + 2\bar{Q}_i \wedge \bar{Q}_{I(i)} + 2\bar{Q}_{J(i)} \wedge \bar{Q}_{K(i)} + 2\bar{Q}_J \wedge \bar{Q}_K$$

by using 3.7,3.8 and 3.10 .The third expression in 3.31 now becomes

$$\frac{a(J)^2-A^2}{A^2}\bar{Q}_J \wedge \bar{Q}_{J(i)I(i)} - \frac{a(K)^2-A^2}{A^2}\bar{w}_{iJ(i)} \wedge \bar{Q}_K - 2 . \frac{a(J)^2-A^2}{A^2} \frac{a(K)^2-A^2}{A^2}\bar{Q}_J \wedge \bar{Q}_K$$

$$+ \frac{a(K)^2-A^2}{A^2}\bar{Q}_K \wedge \bar{w}_{K(i)I(i)} + \frac{a(J)^2-A^2}{A^2}\bar{w}_{iK(i)} \wedge \bar{Q}_J$$

which,after cancelling the first with the fifth by the identity 3.9 and the second with the fourth,simplifies to

$$-2A^{-4}\{a(J)^2-A^2\}\{a(K)^2-A^2\}\bar{Q}_J \wedge \bar{Q}_K .$$

Furthermore making use of the identities 3.6 and 3.8,

$$\sum_V \{a(V)^2 - A^2\}\bar{w}_{iV}\hat{\bar{w}}_{VI(i)} = \{a(I)^2 - A^2\}\bar{w}_{iI}\hat{\bar{w}}_{II(i)} + \{a(J)^2 - A^2\}\bar{w}_{iJ}\hat{\bar{w}}_{JI(i)}$$

$$+ \{a(K)^2 - A^2\}\bar{w}_{iK}\hat{\bar{w}}_{KI(i)}$$

$$= \{A^2 - a(I)^2\}\bar{Q}_i\hat{\bar{Q}}_{I(i)} + \{a(J)^2 + a(K)^2 - 2A^2\}\bar{Q}_{J(i)}\hat{\bar{Q}}_{K(i)} .$$

Finally equation 3.31 reduces to

3.32
$$m_{iI(i)} = -\bar{Q}_i\hat{\bar{Q}}_{I(i)} + A^{-2}\{a(I)^2 - A^2\}\{\sum_{k\neq i}\bar{Q}_k\hat{\bar{Q}}_{I(k)} + 2\bar{Q}_i\hat{\bar{Q}}_{I(i)}$$

$$+ 2\bar{Q}_{J(i)}\hat{\bar{Q}}_{K(i)} + 2\bar{Q}_J\hat{\bar{Q}}_K\} + 2A^{-4}\{a(J)^2 - A^2\}\{a(K)^2 - A^2\}\bar{Q}_J\hat{\bar{Q}}_K$$

$$- A^{-2}\{a(J)^2 + a(K)^2 - 2A^2\}\bar{Q}_{J(i)}\hat{\bar{Q}}_{K(i)} + A^{-2}\{a(I)^2 - A^2\}\bar{Q}_i\hat{\bar{Q}}_{I(i)}$$

$$= -A^{-4}\{4A^2 - 3a(I)^2\}Q_i\hat{Q}_{I(i)} + A^{-4}\{2a(I)^2 - a(J)^2 - a(K)^2\}Q_{J(i)}\hat{Q}_{K(i)}$$

$$-A^{-4}\{A^2 - a(I)^2\}\sum_{k\neq i} Q_k\hat{Q}_{I(k)} + \frac{2\{A^2(a(I)^2 - a(J)^2 - a(K)^2) + a(J)^2a(K)^2\}}{a(J)a(K)A^4} Q_J\hat{Q}_K .$$

Similarly we have

$$m_{iI} = dw_{iI} - \sum_{k\neq i} w_{ik}\hat{w}_{kI} - \sum_V w_{iV(i)}\hat{w}_{V(i)I} - \sum_V w_{iV}\hat{w}_{VI}$$

which after quite a lengthy calculation reduces to

3.33 $$m_{iI} = -A^{-4}a(I)^2 Q_i \wedge Q_I - \frac{a(I)^2a(J)^2 - A\{a(I)^2+a(J)^2-a(K)^2\}}{a(I)a(J)A^4} Q_J \wedge Q_{K(i)}$$

$$- \frac{A\{a(J)^2-a(I)^2-a(K)^2\}-a(I)^2a(K)^2}{a(I)a(K)A^4} Q_K \wedge Q_{J(i)} .$$

Lastly

3.34 $$m_{IJ} = \frac{a(I)^2a(J)^2+A\{a(K)^2-a(I)^2-a(J)^2\}}{a(I)a(J)A^4} \sum_k Q_k \wedge Q_{K(k)}$$

$$+ \frac{3a(K)^4-a(I)^4-a(J)^4+2a(I)^2a(J)^2-2a(I)^2a(K)^2-2a(J)^2a(K)^2}{a(I)^2a(J)^2a(K)^2} Q_I \wedge Q_J .$$

From identities 3.30,3.32,3.33 and 3.34 we see that the sectional curvatures are given by :

3.35 $$K_{ij} = A^{-2} \qquad \text{for } 1\leq i,j\leq 4n \quad \text{and} \quad j\neq V(i) \quad ; \quad V=I,J,K$$

3.36 $$K_{iV} = A^{-4}a(V)^2 \quad , \quad 1\leq i \leq 4n$$

3.37 $$K_{iV(i)} = 4A^{-2} - 3a(V)^2A^{-4} \quad , \quad 1\leq i\leq 4n$$

3.38 $$K_{UV} = \frac{a(U)^4+a(V)^4-3a(W)^4-2\{a(U)^2a(V)^2-a(U)^2a(W)^2-a(V)^2a(W)^2\}}{a(U)^2a(V)^2a(W)^2}$$

where U,V and W in the last identity are cyclic permutations of I,J,K; while the other nonzero components of the curvature tensor are :

3.39 $$R_{ijV(i)V(j)} = R_{iV(i)jV(j)} = \{A^{-2} - a(V)^{2}A^{-4}\}/2,\ 1\leq i,j\leq 4n,\ j\neq V(i)$$

3.40 $$R_{iU(i)V(i)W(i)} = A^{-4}\{a(V)^{2}+a(W)^{2}-2a(U)^{2}\}/2 \quad , \quad i=1,\ldots,4n$$

3.41 $$R_{iU(i)VW} = \frac{\{A^{2}(a(V)^{2}+a(W)^{2}-a(U)^{2}) - a(V)^{2}a(W)^{2}\}}{a(V)a(W)A^{4}} \quad , \quad i=1,\ldots,4n$$

3.42 $$R_{iUW(i)V} = \frac{A^{2}\{a(U)^{2}+a(V)^{2}-a(W)^{2}\} - a(U)^{2}a(V)^{2}}{2a(U)a(V)A^{4}} \quad , \quad i=1,\ldots,4n$$

3.43 $$R_{iUWV(i)} = \frac{A^{2}\{a(U)^{2}+a(W)^{2}-a(V)^{2}\} - a(U)^{2}a(W)^{2}}{2a(U)a(W)A^{4}} \quad , \quad i=1,\ldots,4n.$$

Notice here that the last four identities are for all cyclic permutations of I,J,K .For example :

$$R_{iI(i)JK} + R_{iJKI(i)} + R_{iKI(i)J} = R_{iI(i)JK} - R_{iJI(i)K} - R_{iKJI(i)}$$

$$= \frac{\{A(a(J)^{2}+a(K)^{2}-a(I)^{2}) - a(J)^{2}a(K)^{2}\}}{a(J)a(K)A^{4}} - \frac{\{A(a(J)^{2}+a(K)^{2}-a(I)^{2})-a(J)^{2}a(K)^{2}\}}{2a(J)a(K)A^{4}}$$

$$- \frac{\{A(a(K)^{2}+a(J)^{2}-a(I)^{2})-a(K)^{2}a(J)^{2}\}}{2a(K)a(J)A^{4}}$$

by consecutively using the identities 3.41, 3.42, 3.43 ;and the above clearly vanishes and satisfies the Bianchi identity.

We then compute the components of the Ricci tensor and write for the nonzero components :

3.44 $$R_{ii} = \sum_V R_{iViV} + \sum_V R_{iV(i)iV(i)} + \sum_j R_{ijij}$$

$$= \sum_V A^{-4} a(V)^2 + \sum_V \{4A^{-2} - 3a(V)^2 A^{-4}\} + (4n-4)A^{-2}$$

$$= (4n+8)A^{-2} - 2A^{-4}\{a(I)^2 + a(J)^2 + a(K)^2\},$$

3.45 $$R_{II} = K_{IJ} + K_{IK} + \sum_i K_{Ii}$$

$$= \frac{2\{a(I)^4 - a(J)^4 - a(K)^4 + 2a(J)^2 a(K)^2\}}{a(I)^2 a(J)^2 a(K)^2} + 4na(I)^2 A^{-4}$$

The other components R_{JJ} and R_{KK} can easily be read off from the second identity above. Lastly the scalar curvature is:

3.46 $$R = 16n(n+2)A^{-2} - 4nA^{-4}(a(I)^2 + a(J)^2 + a(K)^2) - \frac{2\{a(I)^4 + a(J)^4 + a(K)^4 - 2a(I)^2 a(J)^2 - 2a(J)^2 a(K)^2 - 2a(K)^2 a(I)^2\}}{a(I)^2 a(J)^2 a(K)^2}.$$

2.4 Discussion of metrics on S^{2n+1}

We now study the possibilities of the metrics on S^{2n+1} constructed in section two of being Einstein ,Ricci-Codazzi or Ricci cyclic parallel.

We first consider the Einstein case.Since

$$R_{ii} = 2\{(n+1)A^{-2} - a^2 A^{-4}\} \quad \text{and} \quad R_{II} = 2na^2 A^{-4}$$

it is easy to see that the metric is Einstein if and only if $a^2 = A^2$ .This is just the standard metric with constant sectional curvature A^{-2} .

As for the other two cases we first have to compute the components of covariant derivative of the Ricci tensor using the usual formula :

$$\sum_s H_{ij's} Q_s = dH_{ij} + \sum_p H_{ip} w_{pj} + \sum_p H_{pj} w_{pi} \quad \text{for any 2-form } H,$$

where $'$ denotes the covariant derivative.

In our case ,we have

$$\sum_s R_{iI's} Q_s = (R_{II} - R_{ii}) w_{Ii} = aA^{-2} (R_{ii} - R_{II}) Q_{I(i)}$$

where the last equality follows from 2.7 and 2.11.
The other non-trivial equations are

$$\sum_s R_{II(i)'s} Q_s = (R_{ii} - R_{II}) w_{I(i)I} = - aA^{-2} (R_{ii} - R_{II}) Q_i$$

where the last equality follows from the second identity of 2.4 and 2.11. Thus the only nonzero components of the derivative of the Ricci tensor are

4.1 $$R_{iI'I(i)} = - R_{II(i)'i} = 2aA^{-6}(n+1)(A^{2} - a^{2}).$$

Theorem 4

All the metrics constructed in section 2 are Ricci cyclic parallel while the Ricci-Codazzi metrics are necessarily Einstein .

The theorem is immediately proved in view of 4.1.

2.5 Discussion of the metrics on S^{4n+3}

Following the results in section 3 we write down the Einstein equations, namely

5.1 $$\frac{a(U)^4-a(V)^4-a(W)^4+2a(V)^2a(W)^2}{a(U)^2a(V)^2a(W)^2} + 2na(U)^2A^{-4} = c,$$

where U,V,W are cyclic permutations of I,J,K;

5.2 $$(2n+4)A^{-2} - A^{-4}\{a(I)^2+a(J)^2+a(K)^2\} = c.$$

From the first three equations (obtained by cyclic permutations of U,V,W in 5.1) we get

5.3 $$\frac{a(U)^4-a(V)^4+a(V)^2a(W)^2-a(U)^2a(W)^2}{a(U)^2a(V)^2a(W)^2} = nA^{-4}\{a(V)^2-a(U)^2\},$$

{again here U,V,W are cyclic permutations of I,J,K}
which can be simplified to

5.4 $$\{a(I)^2-a(J)^2\}\{a(I)^2+a(J)^2-a(K)^2+nA^{-4}a(I)^2a(J)^2a(K)^2\} = 0,$$

5.5 $$\{a(J)^2-a(K)^2\}\{a(J)^2+a(K)^2-a(I)^2+nA^{-4}a(I)^2a(J)^2a(K)^2\} = 0,$$

5.6 $$\{a(K)^2-a(I)^2\}\{a(K)^2+a(I)^2-a(J)^2+nA^{-4}a(I)^2a(J)^2a(K)^2\} = 0.$$

If $a(I)^2, a(J)^2, a(K)^2$ are all unequal, we must then have

$$a(I)^2+a(J)^2-a(K)^2 + nA^{-4}a(I)^2a(J)^2a(K)^2 = 0,$$

$$a(J)^2+a(K)^2-a(I)^2 + nA^{-4}a(I)^2a(J)^2a(K)^2 = 0,$$

$$a(K)^2+a(I)^2-a(J)^2 + nA^{-4}a(I)^2a(J)^2a(K)^2 = 0,$$

which imply $a(I)^2=a(J)^2=a(K)^2$, contradicting our assumption. If however two of the constants are equal, say $a(I)^2=a(J)^2$, then from 5.6 we see that we must have

$$a(K)^2\{1 + nA^{-4}a(I)^4\} = 0,$$

which is absurd .

So the only other possibility is when all the coefficients are equal, say a^2. In this case the Einstein equations reduce to :

5.7 $$a^{-2} + 2na^2A^{-4} = (2n+4)A^{-2} - 3a^2A^{-4} = c.$$

Theorem 5

There are two Einstein metrics on S^{4n+3} obtained by the construction above. One is the standard metric and the other is of non-constant sectional curvature.

Proof

The Einstein equations 5.7 can be reduced to

$$(A^2 - a^2)\{A^2 - (2n+3)a^2\} = 0 .$$

The metric with $A^2 = a^2$ is just the standard metric with sectional curvature A^{-2}. The other metric with $A^2 = (2n+3)a^2$ is of non-constant sectional curvature taking values $\{(2n+3)a\}^{-2}$, $(2n+3)^{-1}a^{-2}$, $(8n+9)\{(2n+3)a\}^{-2}$ and a^{-2}. Clearly the maximum and minimum sectional curvature are respectively a^{-2}, $\{(2n+3)a\}^{-2}$ and thus this Einstein metric has pinching $(2n+3)^{-2}$. This metric was discovered by Jensen [JE1].

On computing the derivative of the Ricci tensor we obtain

$$\sum_s R_{IJ's}\, Q_s = (R_{II} - R_{JJ})w_{IJ} = \frac{a(K)^2 - a(I)^2 - a(J)^2}{a(I)a(J)a(K)}\{R_{II} - R_{JJ}\}Q_K$$

(in view of 3.28); which then reduces to

$$\sum_s R_{IJ's}\, Q_s = \frac{4\{a(I)^2 - a(J)^2\}\{a(K)^2 - a(I)^2 - a(J)^2\}}{a(I)a(J)a(K)}\{ nA^{-4} - \frac{a(K)^2 - a(I)^2 - a(J)^2}{a(I)^2 a(J)^2 a(K)^2}\}Q_K .$$

Similarly, we write other nontrivial relations :

$$\sum_s R_{iI's}\, Q_s = (R_{ii} - R_{II})w_{iI} =$$

$$\left[\frac{a(J)^4 + a(K)^4 - a(I)^4 - 2a(J)^2 a(K)^2}{a(I)^2 a(J)^2 a(K)^2 A^2} - \frac{2n\{a(I)^2 - A^2\} - 4A^2 + a(I)^2 + a(J)^2 + a(K)^2}{A^6}\right]2a(I)Q_{I(i)}$$

$$\sum_s R_{II(i)'s}\, Q_s = (R_{ii} - R_{II})w_{I(i)I} = a(I)A^{-2}(R_{II} - R_{ii})Q_i$$

From these equations we list the nonzero components :

$$R_{UV'W} = \frac{a(W)^2-a(U)^2-a(V)^2}{a(U)a(V)a(W)}(R_{UU} - R_{VV})$$

where U,V,W are cyclic permutations of I,J,K;

$$R_{iV'V(i)} = a(I)A^{-2}(R_{VV} - R_{ii}) \qquad \text{for } V=I,J,K,$$

$$R_{VV(i)'i} = - R_{iV'V(i)} \qquad \text{for } V=I,J,K.$$

We now study the conditions for Ricci cyclic parallel and Ricci-Codazzi.

<u>Theorem 6</u>

Metrics constructed in section 3 are Ricci-Codazzi if and only if they are Einstein .However the metrics obtained by this construction are Ricci cyclic parallel if and only if at least two of the coefficients $a(I)^2, a(J)^2$ $a(K)^2$ are equal.

<u>Proof</u>

Part one follows quite clearly since we have

$$R_{II(i)'i} = - R_{iI'I(i)} \quad \text{and} \quad R_{iV'V(i)} = a(V)A^{-2}\{R_{VV} - R_{ii}\}.$$

As for the second part we first see that the relation

$$R_{iV'V(i)} + R_{VV(i)'i} + R_{V(i)i'V} = 0 \quad \text{for} \quad V=I,J,K.$$

is automatically satisfied for the third term is zero while the first and the second terms cancelled out each other. The only other condition to consider is:

$$R_{IJ'K} + R_{JK'I} + R_{KI'J} = 0.$$

But the cyclic sum above can be shown to be

$$-\frac{16\{a(I)^2-a(J)^2\}\{a(J)^2-a(K)^2\}\{a(K)^2-a(I)^2\}}{\{a(I)a(J)a(K)\}^3}$$

which completes our proof .

We note that the previous analysis uses the fact that $a(I)^2, a(J)^2, a(K)^2$ and A^2 are constants, but the argument goes through if for example we replace $a(I)^2$ by $-a(I)^2$ on pages 45 and 52. We have therefore also constructed indefinite metrics on S^{2n+1} and S^{4n+3} having the required property.

The corresponding isometry groups for the metrics of Theorem4, Theorem5 and Theorem6 can all be read off from Ziller's paper [Zi].

Chapter 3. Metrics On Homogeneous Manifolds Associated With The Exceptional Group G2.

We explain our notation in the first five sections. In section 6 we describe the set of all G-invariant metrics on G/H, where G is a compact connected Lie group and H a closed subgroup such that G acts effectively on G/H.
Formula for computing the curvature is given in page 84. In section 8 we consider the exceptional Lie algebra g2 with all its subalgebras.

We check our programmes in "reduce" by applying them to S^7 and CP^3 and compare the known results on Einstein metrics. See [Je2] and [Zi].

Finally we consider the possibilities of the quotients spaces associated with G2 admitting the Einstein metric, Ricci cyclic parallel metric and Ricci-Codazzi metric. From this investigation,it seems that in this case the class of Ricci cyclic parallel metrics forms a larger class.

3.1 Reflection in Euclidean space of dimension n

Geometrically, a reflection in E^n is an invertible linear transformation of order 2 leaving pointwise fixed some hyperplane (subspace of codimension one) and sending any vector orthogonal to that hyperplane into its negative. Evidently it is 'orthogonal', preserving the inner product. Let A be a non-zero vector in E^n and let $A^{@} = \{ x \in E^n \mid \langle x,A\rangle = 0 \}$ be its orthogonal complement called the reflecting hyperplane.

The projection of a vector B into the reflecting hyperplane $A^{@}$ is to be B-rA, where a real number r is to be chosen so that B-rA is in $A^{@}$. So we must have

$$0 = \langle B - rA,\ A\rangle = \langle B,A\rangle - r\langle A,A\rangle$$

and hence $r = \langle B,A\rangle / \langle A,A\rangle$.

Clearly then the reflection of B in $A^{@}$ is given by an explicit formula :

$$T_A(B) = B - 2rA = B - 2A\langle B,A\rangle/\langle A,A\rangle .$$

(it sends A to -A and fixes all points in $A^{@}$.)

For orthogonality, we see that

$$\langle T_A(B), T_A(C)\rangle = \langle B-2\frac{\langle A,B\rangle}{\langle A,A\rangle}A\ ,\ C-2\frac{\langle A,C\rangle}{\langle A,A\rangle}A\rangle$$

$$= \langle B,C\rangle - 2.\frac{\langle A,C\rangle}{\langle A,A\rangle}\langle B,A\rangle - 2.\frac{\langle A,B\rangle}{\langle A,A\rangle}\langle A,C\rangle - 4\frac{\langle A,B\rangle\langle A,C\rangle}{\langle A,A\rangle}$$

which is just $\langle B,C\rangle$.

The number $2\langle B,A\rangle/\langle A,A\rangle$ will be abbreviated by (B,A) which is linear only in the first variable. We shall also write E for E^n .

3.2. Root systems

A subset Q of the Euclidean space E is called a root system in E if the following properties are satisfied:

a) Q is finite, spans E and does not contain 0 ;

b) if A is in Q, the only multiple of A in Q other than A itself is -A ;

c) if A is in Q, the reflection T_A leaves Q invariant;

d) if A,B are both in Q then (B,A) is an integer.

For n is less than 3 we can describe the root system Q by simply drawing a picture.
There is only one possibility in case n=1, for in view of b, we must have Q={A,-A}.
There are exactly four possibilities in case n=2 .This is because the property d, limits severely the possible angles occuring between pairs of roots.
Recall that the cosine of the angle # between vectors A,B in E is given by the usual formula $\langle A,B\rangle=|A||B|\cos\#$.
Therefore

$$(B,A)=2\langle B,A\rangle/\langle A,A\rangle= 2\cos\#.|B|/|A| \text{ and } (A,B)(B,A)=4\cos^2\#.$$

Since (A,B) and (B,A) are integers then the last number $4\cos^2\#$ is also integer. Moreover since

$$|\cos^2\#| \leqslant 1 \text{ and } (A,B),(B,A) \text{ have like sign}$$

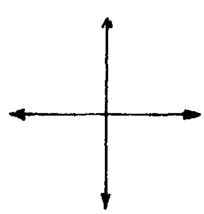
the following are the only ones when $A\neq\pm B$ and $|B|\geqslant|A|$:

(A,B)	(B,A)	Angle #	$\|B\|^2 / \|A\|^2$
0	0	$\pi/2$	arbitrary
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

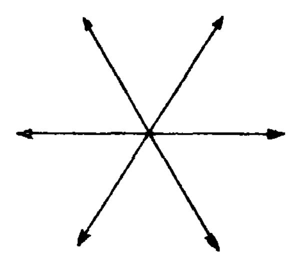
Table 3.2

The above angles and relative lengths are portrayed in figure 4.1 below:

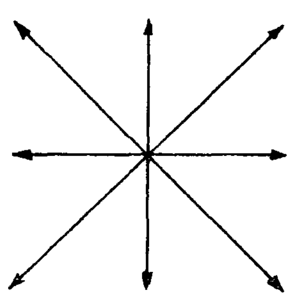
$A_1 \times A_1$



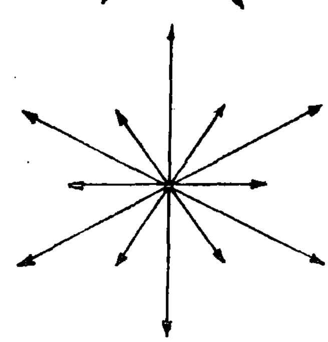
A_2



B_2



G_2



We will consider the fourth diagram in more detail later on.

3.3 Lie Groups

A Lie group G is a group which is at the same time a differentiable manifold such that the map $G\times G\to G$, defined by $(a,b)\mapsto ab^{-1}$, is differentiable.

Left translation by an element g of G is the map $L_g:G\to G$, defined by $L_g(p)=gp, p\in G$.
If a vector field X on G satisfies

$$(dL_g)X = X \quad \text{for all } g \text{ in } G,$$

then X is called a left invariant vector field. Thus if X is left invariant, then it is uniquely determined by $X(e)$, where e is the identity element of G. Conversely a tangent vector X at e gives rise to a left invariant vector field $X(g)=(dL_g)X(e)$.

Similarly, a covariant tensor field B of order r on G is left-invariant if

$$(dL_g)\, B = B.$$

We remark that if $\{X_1,\ldots,X_r\}$ is a basis of smooth left-invariant vector fields, then $B(X_1,\ldots,X_r)$ is constant.

We also have similar properties for right translation by an element g of G denoted by R_g.

We define $Ad_g X = dR_g \circ dL_g^{-1}(X)$. Clearly $Ad_{gh} = Ad_g Ad_h$.

Since for each g, the map $h\mapsto ghg^{-1}$ is an automorphism of g, $Ad_g[X,Y] =[Ad_g X, Ad_g Y]$.

Set ad $=d(\mathrm{Ad})$, i.e, the differential of the adjoint representation ; then $\mathrm{ad}_X Y=[X,Y]$.

Moreover, every Lie group has a left-invariant Riemannian metric, while on a compact connected Lie group a bi-invariant Riemannian metric always exists.

Proposition.

Let $\langle\ ,\ \rangle$ be a left invariant metric on G and let X,Y,Z be left invariant vector fields. Denote by D the corresponding Levi-Civita connection. Then :

i) $D_X Y = \{[X,Y] - (\mathrm{ad}_X)^* Y - (\mathrm{ad}_Y)^* X\}/2$,

where A^* denotes the adjoint of the linear transformation A with respect to $\langle\ ,\ \rangle$,

ii) $\langle R(X,Y)Z,W\rangle = \langle D_X Z, D_Y W\rangle - \langle D_Y Z, D_X W\rangle - \langle D_{[X,Y]} Z, W\rangle$.

Proof.

By left invariance we have

$$0 = X\langle Y,Z\rangle = \langle D_X Y,Z\rangle + \langle Y,D_X Z\rangle,$$

$$0 = Y\langle X,Z\rangle = \langle D_Y X,Z\rangle + \langle X,D_Y Z\rangle,$$

$$0 = Z\langle X,Y\rangle = \langle D_Z X,Y\rangle + \langle X,D_Z Y\rangle.$$

Subtracting the third of these equations from the sum of the first two and using

$$D_V W - D_W V - [V,W] = 0$$

yields

$$2\langle D_X Y,Z\rangle = \langle [X,Y],Z\rangle - \langle Y,[X,Z]\rangle - \langle X,[Y,Z]\rangle$$

from which i) readily follows.

By left invariance ,$X\langle D_Y Z,W\rangle = 0$. Therefore

$$\langle D_X D_Y Z,W\rangle = -\langle D_Y Z,D_X W\rangle$$

$$-\langle D_Y D_X Z,W\rangle = \langle D_X Z,D_Y W\rangle$$

$$-\langle D_{[X,Y]} Z,W\rangle = -\langle D_{[X,Y]} Z,W\rangle .$$

Adding these equations gives ii).

3.4 Lie algebras

A vector space p over a field F, is a Lie algebra if in addition to its vector space structure it possesses a product, that is a map $p \times p \longrightarrow p$ taking the pair (X,Y) to the element $[X,Y]$ of p which has the following properties:

i) $[X,Y]$ is bilinear for all X,Y in p,

ii) $[X,X] = 0$ for all X in p,

iii) $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ for all X,Y,Z in p.

Property iii) is called the Jacobi identity. We note that $[X,[Y,Z]]$ is not necessarily equal to $[[X,Y],Z]$, thus the bracket product is not in general associative. As a simple consequence of properties i) and ii), we have

$0 = [X+Y,X+Y] = [X,X]+ [X,Y]+ [Y,X]+ [Y,Y] = [X,Y]+ [Y,X]$.

Thus $[X,Y]=-[Y,X]$ showing the bracket is anticommutative, assuming F has characteristic$\neq 2$.

Conversely, if the characteristic of F is different from 2, the anticommutativity of bracket implies ii).

Let p be a Lie algebra and let m,h be subspaces of p. Let $[m,h]$ be the subspace of p spanned by elements of the form $[X,Y]$, $X\in m, Y\in h$. If $[m,m]$ is in m then m is called a subalgebra of p.

If $[h,p]$ is contained in h, then h is called an ideal of p.

The derived series of p is the decreasing sequence of ideals

$D^0p, D^1p, \ldots$ of p defined inductively by $D^0p=p, D^{i+1}p=[D^ip, D^ip]$.

The descending central series of p is the decreasing sequence of ideals $C^0 p$, $C^1 p$, .. of p defined inductively by

$$C^0 p = p \ , \ C^{i+1} p = [p, C^i p] \ .$$

Evidently $D^i p \subset C^i p$. The Lie algebra p is <u>abelian</u> if $D^1 p = 0$, <u>nilpotent</u> if $C^i p = 0$ for some i and <u>solvable</u> if $D^i p = 0$ for some i. Every Lie algebra has a unique maximal solvable ideal, called the <u>radical</u> of p. A Lie algebra p is said to be <u>semisimple</u> if its radical is zero. A Lie algebra is said to be <u>simple</u> if it is not abelian and has no non-zero ideal other than p itself.

A subalgebra h of the Lie algebra p is called a <u>Cartan subalgebra</u> if it is nilpotent and it does not contain as an ideal in any larger subalgebra of p,
i.e, if [X,Y] is in h for all Y in h, then X is in h.
The dimension of a Cartan subalgebra h of p is called the rank of p. A theorem states that all such subalgebras have the same dimension.
A real semisimple algebra p is said to be <u>compact</u> if its Cartan-Killing form is negative definite, where the Cartan-Killing form is a symmetric bilinear form on p defined by

$$\mathrm{Kil}(X,Y) = \mathrm{Trace}(\, ad_X \circ ad_Y \,) \quad ; \ X, Y \text{ in } p \ .$$

Furthermore a connected, complex semisimple Lie group G is compact if its Lie algebra is of compact type ([KN2]p.252).
We remember that the Cartan-Killing form is associative, in the sense that Kil([X,Y],Z)=Kil(X,[Y,Z]) since we know that Trace([X,Y]Z)=Trace(X[Y,Z]). See Humphreys [Hu] p.19.

3.5 The root system of a compact semisimple Lie algebra.

Let h be a Cartan subalgebra of the complex Lie algebra g and let a be a linear function on h. Let g_a denote the linear subspace of g, invariant under ad_h, given by

$$g_a = \{ Y \in g \mid [X,Y] = a(X)Y \text{ ; for all } X \in h \}.$$

If g_a is not the zero element, the linear function a (an element of the dual space of h) is called a root of the Lie algebra g with respect to h and such g_a is called a root subspace. The collection of all nonzero roots form a root system and Theorem 4.2 of [Hel] p.141 gives:

i) $g = h + \sum g_a$ summation is over all nonzero roots,

ii) for each nonzero root a, g_a is of complex dimension 1,

iii) the only roots proportional to a are $-a, 0, a$.

It is not difficult to see that a root system of a compact semisimple Lie algebra is a 'root system' in the sense discussed earlier. See for example [Ma] pp. 108-112.

3.6 Riemannian Homogeneous Metrics.

A metric on M is called Riemannian homogeneous if there exists a group G of isometries acting transitively on M. If we fixed a point p of M and let H be the isotropy group (the subgroup of isometries leaving p fixed) then M is diffeomorphic to G/H.

Let G be a compact connected Lie group and H a closed subgroup such that G acts effectively on G/H, i.e, H contains no non-trivial normal subgroup of G.
We denote by g, h the Lie algebra of G and H, by Ad_G the adjoint action of G on g and by ad its derivative, i.e, $ad_X(Y) = [X,Y]$ for all X, Y in g.

Let B be the negative of the Cartan-Killing form of g. We choose a complement m of h in g such that $g = h \oplus m$ and $[h,m]$ (m which always exists since H is compact.
Then m can be identified with the tangent space of G/H at the coset eH where e is the identity element of G. Corollary 3.2 of [KN2] pp. 201-202 gives a one-to-one correspondence between the set of G-invariant Riemannian metrics on G/H and the set of Ad_H-invariant inner products on m.

To describe the set of all G-invariant metrics on G/H, let $m = m_0 + m_1 + .. + m_r$ be the decomposition of m into H-modules where m_0 is the submodule of m on which H acts as identity and $m_1, \dots, m_r$ are irreducible H-modules.

Such a decomposition is not unique if some of the representations of Ad_H on m_i are equivalent to each other. But the subspace m_0 and the numbers $d_i = \dim m_i$ are independent of the chosen decomposition.

We decompose m_0 further into B-orthogonal 1-dimensional subspaces $m_0 = m_{r+1}, \ldots, m_s$; $s-r>1$ to get $m = m_1 + \ldots + m_r + m_{r+1} + \ldots + m_s$. For each decomposition there is the family of Ad_H-invariant diagonal metrics:

$$\langle , \rangle = x_1 B|_{m_1} + \ldots + x_s B|_{m_s} , \quad x_i > 0.$$

Conversely, every Ad_H-invariant inner product on m belongs to the family of Ad_H-invariant diagonal metrics of some decomposition of m. In fact, for a given Ad_H- invariant inner product $\langle , \rangle$ on m, we can diagonalize $\langle , \rangle$ with respect to B to obtain a decomposition of m into eigenspaces of $\langle , \rangle$, which are orthogonal with respect to both B and $\langle , \rangle$. These eigenspaces are Ad_H-invariant and so can be decomposed into irreducible summands which are orthogonal with respect to B and $\langle , \rangle$. Then $\langle , \rangle$ has the form above with respect to this decomposition, where x_i's are the eigenvalues of $\langle , \rangle$ with respect to B.

The Levi-Civita connection of the metric on m is given by

$$D_X Y = \frac{1}{2}[X,Y]_{|m} + U(X,Y)$$

where U is a symmetric 2-form on m determined by

$2\langle U(X,Y),Z\rangle = -\langle X,[Y,Z]\rangle - \langle [X,Z],Y\rangle$; X,Y,Z in m.

See Nomizu [No] p. 52 or [KN2] p. 201.

The curvature tensor is then computed by the formula:

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]_{|m}} Z - [[X,Y]_{|h}, Z].$$

See [No] p. 47.

In suffix notation (without summation convention), we have

$$2C_{hjk} = M_{hjk} - M_{jkh} g_{hh} / g_{kk} - M_{hkj} g_{jj} / g_{kk},$$

$$R_{hpqk} = \Sigma_j C_{pqj} C_{hjk} g_{kk} - \Sigma_m C_{hqm} C_{pmk} g_{kk} - \Sigma_s M_{hps} C_{sqk} g_{kk} - \Sigma_s M_{hps} M_{sqk} g_{kk},$$

where the last sum is over all generators of h and for simplicity, the metric considered is diagonalized.

We have written

$$D_{X(i)} X(j) = \Sigma_k C_{ijk} X(k)$$

$$\langle R\{X(i),X(j)\}X(k),X(h)\rangle = R_{ijkh}$$

and M_{ijk} denotes the k-th component of the bracket product $[X(i),X(j)]$.

3.7 S7 and CP3

Let V be a 2n-dimensional vector space over F. Let f be a nondegenerate skew-symmetric form on V given by the matrix

$$s = \begin{vmatrix} 0 & I_n \\ -I_n & 0 \end{vmatrix}.$$

Denote by sp(2n,F), the symplectic algebra, which by definition consists of all endomorphisms x of V satisfying $f(x(u),v)=-f(u,x(v))$. In matrix terms, the condition for

$$x=\begin{vmatrix} m & n \\ p & q \end{vmatrix} \quad ;m,n,p,q \in gl(n,F)$$

to be symplectic is that $sx=-x^t s$, i.e, that $n^t=n, p^t=p, m^t=-q$. A basis of this algebra say sp(2,C) is given by ;

$$e_1 = e_{11} - e_{33} \; ; \; e_2 = e_{22} - e_{44} \; ;$$

$$e_3 = e_{13} \; ; \; e_4 = e_{24} \; ; \; e_5 = e_{14} + e_{23} \; ;$$

$$e_6 = e_{31} \; ; \; e_7 = e_{42} \; ; \; e_8 = e_{32} + e_{41} \; ;$$

$$e_9 = e_{12} - e_{43} \; ; \; e_{10} = e_{21} - e_{34} \, ,$$

where here and in the sequel e_{ij} is the matrix having 1 in the (i,j) position and 0 elsewhere. See for example Humphreys [Hu] page 3. The bracket product is computed using the relation

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$$

which follows from the relation $e_{ij} e_{kl} = \delta_{jk} e_{il}$.

The bracket product is given by table 1 below.

Table 1

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
e_1	0	0	$2e_3$	0	e_5	$-2e_6$	0	e_8	e_9	$-e_{10}$
e_2	0	0	0	$2e_4$	e_5	0	$-2e_7$	$-e_8$	$-e_9$	e_{10}
e_3	$-2e_3$	0	0	0	0	e_1	0	e_9	0	$-e_5$
e_4	0	$-2e_4$	0	0	0	0	e_2	e_{10}	$-e_5$	0
e_5	$-e_5$	$-e_5$	0	0	0	e_{10}	e_9	e_1+e_2	$-2e_3$	$-2e_4$
e_6	$2e_6$	0	$-e_1$	0	$-e_{10}$	0	0	0	e_8	0
e_7	0	$2e_7$	0	$-e_2$	$-e_9$	0	0	0	0	e_8
e_8	e_8	e_8	$-e_9$	$-e_{10}$	$-e_1-e_2$	0	0	0	$2e_7$	$2e_6$
e_9	$-e_9$	e_9	0	e_5	$2e_3$	$-e_8$	0	$-2e_7$	0	e_1-e_2
e_{10}	e_{10}	$-e_{10}$	e_5	0	$2e_4$	0	$-e_8$	$-2e_6$	$-e_1+e_2$	0

The Killing metric is computed and we list the non-zero components:

$$\mathrm{Kil}(1,1)=\mathrm{Kil}(2,2)=\mathrm{Kil}(5,8)=\mathrm{Kil}(9,10) = 12,$$

$$\mathrm{Kil}(3,6)=\mathrm{Kil}(4,7)= 6.$$

Since the algebra needed is compact we now diagonalize the metric and make it negative definite by taking :

$E_1 = ie_1$; $E_2 = ie_2$

$E_3 = i(e_3 + e_6)$; $E_4 = e_3 - e_6$; $E_5 = i(e_4 + e_7)$; $E_6 = e_4 - e_7$

$E_7 = i(e_5 + e_8)$; $E_8 = e_5 - e_8$; $E_9 = i(e_9 + e_{10})$; $E_{10} = e_9 - e_{10}$.

Clearly we have:

$Kil(1,1)=Kil(2,2)=Kil(7,7)=...=Kil(10,10)=-24,$

$Kil(3,3)= \ ... \ =Kil(6,6)=-12.$

The bracket product is now given by table 2 next page.

It is not difficult to see that in this case we have the subalgebra sp(1,C) of sp(2,C) generated by E_1, E_3 and E_4 under which action the quotient sp(2,C)/sp(1,C) is split into :

3-dimensional m_0 generated by E_2, E_5 and E_6 ;

4-dimensional m_1 generated by E_7, E_8, E_9 and E_{10} .

The corresponding homogeneous space is known to be S^7. See for example [Jel]p.599.We now consider metrics of the form :

$$g = -dKil|_{m_1} + h \quad \text{where } h \text{ is an arbitrary metric on } m_0 .$$

Table 2

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
E_1	0	0	$-2E_4$	$2E_3$	0	0	$-E_8$	E_7	$-E_{10}$	E_9
E_2	0	0	0	0	$-2E_6$	$2E_5$	$-E_8$	E_7	E_{10}	$-E_9$
E_3	$2E_4$	0	0	$-2E_1$	0	0	$-E_{10}$	$-E_9$	E_8	E_7
E_4	$-2E_3$	0	$2E_1$	0	0	0	E_9	$-E_{10}$	$-E_7$	E_8
E_5	0	$2E_6$	0	0	0	$-2E_2$	E_{10}	$-E_9$	E_8	$-E_7$
E_6	0	$-2E_5$	0	0	$2E_2$	0	E_9	E_{10}	$-E_7$	$-E_8$
E_7	E_8	E_8	E_{10}	$-E_9$	$-E_{10}$	$-E_9$	0	$-2E_1-2E_2$	$2E_4+2E_6$	$-2E_3+2E_5$
E_8	$-E_7$	$-E_7$	E_9	E_{10}	E_9	$-E_{10}$	$2E_1+2E_2$	0	$-2E_3-2E_5$	$-2E_4+2E_6$
E_9	E_{10}	$-E_{10}$	$-E_8$	E_7	$-E_8$	E_7	$-2E_4-2E_6$	$2E_3+2E_5$	0	$-2E_1+2E_2$
E_{10}	$-E_9$	E_9	$-E_7$	$-E_8$	E_7	E_8	$2E_3-2E_5$	$2E_4-2E_6$	$2E_1-2E_2$	0

i.e, $g(2,2) = 12a$; $g(5,5) = 12b$; $g(6,6) = 12c$

$$g(7,7) = g(8,8) = g(9,9) = g(10,10) = 24d .$$

We now examine the result enclosed in the appendixl.

The Einstein equations are :

$$a^2bc + 2a^2d^2 - 2b^2d^2 + 4bcd^2 - 2c^2d^2 = bcd^2(12az),$$

$$ab^2c + 2b^2d^2 - 2c^2d^2 + 4acd^2 - 2a^2d^2 = acd^2(12bz),$$

$$abc^2 + 2c^2d^2 - 2a^2d^2 + 4abd^2 - 2b^2d^2 = abd^2(12cz),$$

$$-a \quad -b \quad -c \quad +12d = d(24dz).$$

If two of the first three coefficients are equal, say $a=b$, the first two equations would imply $a=b=c$, while if all a,b,c are distinct, subtracting the second by the first equation would give

$$abc + 4d^2(a+b) - 4cd^2 = 0.$$

Similarly we have

$$abc + 4d^2(a+c) - 4bd^2 = 0,$$

$$abc + 4d^2(b+c) - 4ad^2 = 0,$$

which then give

$$3abc + 4d^2(a+b+c) = 0 \text{ ,which is absurd.}$$

So the only possibility left is when a,b,c are all equal, say a. In this case the non-zero components of the Ricci tensor are :

$$R(i,i) = 2 + a^2d^{-2} \qquad \text{for } i=2,5,6,$$

$$R(i,i) = 12 - 3a/d \qquad \text{for } i=7,8,9,10.$$

Eliminating the constant of the Einstein equations would give $a/d = 2$ or $2/5$. The Einstein metric with $a/d = 2$ corresponds to the standard metric on S^7 with constant sectional curvature $1/12a$, i.e, the metric is of constant

sectional curvature 1 when a is taken to be 1/12 .
The second metric with a/d =2/5 has non-constant positive sectional curvature taking values 1/300a, 17/300a and 1/12a. Thus the second Einstein metric has pinching 1/25. This metric was first discovered by Jensen [Je1] pp. 612-613. See also [Je2].

The result enclosed in the appendix also confirms that a Ricci cyclic parallel metric is obtained when at least two of the coefficients are equal while Ricci- Codazzi spaces are necessarily Einstein. Compare our results in chapter 2.

We now consider the projective space CP^3 written as $Sp2/Sp1\times U1$ in which case the splitting of m is given by $m = m_1 + m_2$, where the

2-dimensional m_1 is generated by E_5 and E_6 while the

4-dimensional m_2 is generated by E_7, E_8, E_9 and E_{10}.

Following the preceding procedure we consider metrics with :

$$g(5,5) = g(6,6) = 12a,$$

$$g(7,7) = \ \ldots \ = g(10,10) = 24b\ .$$

From the result enclosed in appendix we have the nonzero components of the Ricci tensor :

$$R_{ii} = (a^2 + 4b^2)b^{-2} \qquad \text{for} \quad i=5,6,$$

$$R_{ii} = 2(6b - a)b^{-1} \qquad \text{for} \quad i=7,8,9,10.$$

The solutions of the Einstein equations are easily found to be $a/b = 2$ or 1. The Einstein metric with $a/b = 2$ is the standard metric on the complex projective space CP^3. On substituting $a=2b$ we found that the sectional curvatures take values $1/6b$ and $1/24b$, i.e, the maximum and the minimum sectional curvatures are respectively $1/6b, 1/24b$. Thus the metric has positive sectional curvature with pinching 1/4 as we might have expected .

For the other Einstein metric the sectional curvature takes values $1/48a$, $5/48a$, $1/6a$ and $1/3a$, i.e, the maximum and the minimum sectional curvatures are $1/3a$, $1/48a$, respectively. Thus the second metric with $a=b$ also has positive sectional curvature but the pinching is 1/16. It can also be shown that this metric is naturally reductive even though such a metric on CP^n is not naturally reductive for n different from 2. This kind of metric was first discovered by Ziller [Zi] p. 358.

It is also clear that that the Ricci-Codazzi metrics obtained by this method are essentially Einstein for we need

$$a^2 - 3ab + 2b^2 = 0 \text{ , i.e, } a = b \text{ or } a = 2b.$$

However the Ricci cyclic parallel metrics seem to be less restrictive since all the metrics considered are automatically Ricci cyclic parallel.

3.8 The Exceptional Lie Algebra g2.

It is known that the 14-dimensional Lie algebra of type g2 is a subalgebra of so(7). The following construction can be found in Humphreys [Hu] pp 103-104.

The two dimensional Cartan subalgebra h of g2 is

$$h = \{ \sum_{i=1}^{3} a_i d_i \text{ such that } \sum a_i = 0 \}$$

where $d_i = e_{i+1,i+1} - e_{i+4,i+4}$. Obviously $\{d_1, d_2\}$ form a basis of h.

Corresponding to the six long roots in g2 we choose certain root vectors $g_{i,-j}$ (i is different from j) of so(7) relative to h as follows :

$$g_{1,-2} = g^t_{2,-1} = e_{23} - e_{65},$$

$$g_{1,-3} = g^t_{3,-1} = e_{24} - e_{75},$$

$$g_{2,-3} = g^t_{3,-2} = e_{34} - e_{76},$$

while for the short roots, we take

$$g_1 = -g^t_{-1} = \sqrt{2}(e_{12} - e_{51}) - (e_{37} - e_{46}),$$

$$g_2 = -g^t_{-2} = \sqrt{2}(e_{13} - e_{61}) + (e_{27} - e_{45}),$$

$$g_3 = -g^t_{-3} = \sqrt{2}(e_{14} - e_{71}) - (e_{26} - e_{35}).$$

We notice here that each of the twelve vectors listed above is a common eigenvector for adh and none of them centralizing h. As for the Lie bracket product we have

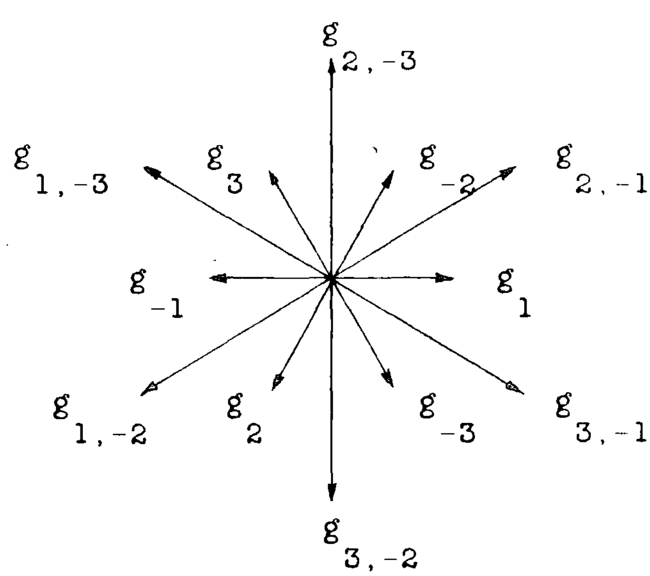
$$[g_{i,-j}, g_{k,-p}] = \delta_{jk} g_{i,-p} - \delta_{ip} g_{k,-j},$$

$$[g_i, g_{-i}] = 3d_i - (d_1 + d_2 + d_3),$$

$$\left.\begin{aligned} [g_k, g_{i,-j}] &= \delta_{ik} g_j \\ [g_{-k}, g_{i,-j}] &= -\delta_{jk} g_{-i} \\ [g_i, g_j] &= +2g_{-k} \\ [g_{-i}, g_{-j}] &= +2g_k \end{aligned}\right\} \text{ i, j, k are cyclic permutations of 1,2,3}$$

$$[g_i, g_{-j}] = 3g_{j,-i}, \quad i \neq j.$$

See table 3 next page for details of the product. The root diagram of g2 is given by



Root diagram of g2

On computing the Killing metric, the non-zero components are given by :

TABLE 3

	$g_{1,-2}$	$g_{3,-1}$	g_2	g_{-3}	$g_{2,-3}$	g_1	$e_7=d_1-d_3$	$e_8=d_2-d_3$	g_{-1}	$g_{3,-2}$	g_3	g_{-2}	$g_{1,-3}$	$g_{2,-1}$
$g_{1,-2}$	0	$-g_{3,-2}$	0	0	$g_{1,-3}$	$-g_2$	$-g_{1,-2}$	$g_{1,-2}$	0	0	0	g_{-1}	0	e_7-e_8
$g_{3,-1}$	$g_{3,-2}$	0	0	0	$-g_{2,-1}$	0	$2g_{3,-1}$	$g_{3,-1}$	g_{-3}	0	$-g_1$	0	$-e_7$	0
g_2	0	0	0	$3g_{3,-2}$	g_3	$-2g_{-3}$	0	g_2	$3g_{1,-2}$	0	$2g_{-1}$	$-e_7+2e_8$	0	g_1
g_{-3}	0	0	$-3g_{3,-2}$	0	$-g_{-2}$	$-3g_{3,-1}$	g_{-3}	g_{-3}	$2g_2$	0	e_7+e_8	$-2g_1$	$-g_{-1}$	0
$g_{2,-3}$	$-g_{1,-3}$	$g_{2,-1}$	$-g_3$	g_{-2}	0	0	$-g_{2,-3}$	$-2g_{2,-3}$	0	e_8	0	0	0	0
g_1	g_2	0	$2g_{-3}$	$3g_{3,-1}$	0	0	g_1	0	$2e_7-e_8$	0	$-2g_{-2}$	$3g_{2,-1}$	g_3	0
e_7	$g_{1,-2}$	$-2g_{3,-1}$	0	$-g_{-3}$	$g_{2,-3}$	$-g_1$	0	0	g_{-1}	$-g_{3,-2}$	g_3	0	$2g_{1,-3}$	$-g_{2,-1}$
e_8	$-g_{1,-2}$	$-g_{3,-1}$	$-g_2$	$-g_{-3}$	$2g_{2,-3}$	0	0	0	0	$-2g_{3,-2}$	g_3	g_{-2}	$g_{1,-3}$	$g_{2,-1}$
g_{-1}	0	$-g_{-3}$	$-3g_{1,-2}$	$-2g_2$	0	$-2e_7+e_8$	$-g_{-1}$	0	0	0	$-3g_{1,-3}$	$2g_3$	0	$-g_{-2}$
$g_{3,-2}$	0	0	0	0	$-e_8$	0	$g_{3,-2}$	$2g_{3,-2}$	0	0	$-g_2$	g_{-3}	$-g_{1,-2}$	$g_{3,-1}$
g_3	0	g_1	$-2g_{-1}$	$-e_7-e_8$	0	$2g_{-2}$	$-g_3$	$-g_3$	$3g_{1,-3}$	g_2	0	$3g_{2,-3}$	0	0
g_{-2}	$-g_{-1}$	0	$-e_7-2e_8$	$2g_1$	0	$-3g_{2,-1}$	0	$-g_{-2}$	$-2g_3$	$-g_{-3}$	$-3g_{2,-3}$	0	0	0
$g_{1,-3}$	0	e_7	0	g_{-1}	0	$-g_3$	$-2g_{1,-3}$	$-g_{1,-3}$	0	$g_{1,-2}$	0	0	0	$-g_{2,-3}$
$g_{2,-1}$	$-e_7+e_8$	0	$-g_1$	0	0	0	$g_{2,-1}$	$-g_{2,-1}$	g_{-2}	$-g_{3,-1}$	0	0	$g_{2,-3}$	0

$$Kil(g_{i,-j}\ ,\ g_{j,-i}) = Kil(d_1 - d_3\ ,\ d_2 - d_3) = 8\ ,$$

$$Kil(d_1 - d_3 , d_1 - d_3) = Kil(d_2 - d_3 , d_2 - d_3) = 16,$$

$$Kil(\ g_i\ ,\ g_{-i}\ ) = -24.$$

Since the semisimple algebra of type g2 is compact and the Killing metric should be negative definite ,we choose a basis of g2 denoted by

$\{g'_j , g'_{-j} , g'_{j,-k} , h'_1 , h'_{-1}\}$;j,k=1,2,3 and j is different from k

where

i) $g'_j = (g_j + g_{-j})/2$ and $g'_{-j} = (g_j - g_{-j})/2i$ for j=1,2,3,

ii) $g'_{k,-j} = (g_{k,-j} + g_{j,-k})/2i$ and $g'_{j,-k} = (g_{k,-j} - g_{j,-k})/2$ for $j>k$,

iii) $h'_1 = (d_1 - d_3 + d_2 - d_3)/2i$ and $h'_{-1} = \{d_1 - d_3 - (d_2 - d_3)\}/2i$.

It is quite easy to see from i) that

$$4Kil(g'_j , g'_j) = 2Kil(g_j , g_{-j}),$$

$$-\ 4Kil(g'_{-j} , g'_{-j}) = -2Kil(g_j , g_{-j}),$$

$$Kil(g'_j , g'_{-j}) = 0$$

which in view of relations earlier on give

$$Kil(g'_j , g'_j) = Kil(g'_{-j} , g'_{-j}) = -12.$$

Similarly ,from ii) and iii) we obtain

$Kil(g'_{i,-j}, g'_{i,-j}) = Kil(h'_{-1}, h'_{-1}) = -4$ and $Kil(h'_{1}, h'_{1}) = -12$

while all other components are zero.

Writing $h'_{1}, h'_{-1}, g'_{1}, g'_{-1}, g'_{2}, g'_{-2}, g'_{3}, g'_{-3}, g'_{1,-2}, g'_{2,-1}$ $g'_{1,-3}, g'_{3,-1}, g'_{2,-3}, g'_{3,-2}$ respectively by $e_1, \ldots, e_{14}$, the Lie product is now given by table4 next page.

The nonabelian subalgebras of g2 as listed in D'Atri and Ziller [DZ] page 60 are :

so(4) (g2 ; su(3) (g2 ; so(3) (g2
u(2)=so(2)⊕bl (so(4) (g2 ; u(2)'=so(2)⊕bl'(so(4) (g2
bl (g2 ; bl'(g2 ; b (so(4) (g2.

Here so(4) = bl ⊕ bl'; bl ≅ bl'≅ so(3) is the splitting of so(4) into simple ideals and b (so(4) is the usual imbedding of so(3) in so(4).The two imbedding of u(2) in g2 and bl,bl' in g2 are not conjugate and the subalgebra so(3) (g2 is maximal .
Futhermore ,su(3),so(4) and maximal so(3) (g2 act irreducibly on their complements.See [Wol] and [Dy].

We can take e_1, e_2, e_7, e_8, e_9 and e_{10} as a basis of so(4) which splits into two ideals, namely e_1, e_7, e_8 and e_2, e_9, e_{10}.

Another basis of so(4) is given by $\{e_3, e_5, e_7, e_{10}, e_{12}, e_{14}\}$.

TABLE 4

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}
e_1	0	0	$-\frac{1}{2}e_4$	$\frac{1}{2}e_3$	$-\frac{1}{2}e_6$	$\frac{1}{2}e_5$	e_8	$-e_7$	0	0	$-\frac{3}{2}e_{12}$	$\frac{3}{2}e_{11}$	$-\frac{3}{2}e_{14}$	$\frac{3}{2}e_{13}$
e_2	0	0	$-\frac{1}{2}e_4$	$\frac{1}{2}e_3$	$\frac{1}{2}e_6$	$-\frac{1}{2}e_5$	0	0	$-e_{10}$	e_9	$-\frac{1}{2}e_{12}$	$\frac{1}{2}e_{11}$	$\frac{1}{2}e_{14}$	$-\frac{1}{2}e_{13}$
e_3	$\frac{1}{2}e_4$	$\frac{1}{2}e_4$	0	$-\frac{1}{2}e_1-\frac{3}{2}e_2$	$e_7-\frac{3}{2}e_{10}$	$-e_8-\frac{3}{2}e_9$	$-e_5-\frac{3}{2}e_{12}$	$e_6-\frac{3}{2}e_{11}$	$\frac{1}{2}e_6$	$\frac{1}{2}e_5$	$\frac{1}{2}e_8$	$\frac{1}{2}e_7$	0	0
e_4	$-\frac{1}{2}e_3$	$-\frac{1}{2}e_3$	$\frac{1}{2}e_1+\frac{3}{2}e_2$	0	$-e_8+\frac{3}{2}e_9$	$-e_7-\frac{3}{2}e_{10}$	$e_6+\frac{3}{2}e_{11}$	$e_5-\frac{3}{2}e_{12}$	$-\frac{1}{2}e_5$	$\frac{1}{2}e_6$	$-\frac{1}{2}e_7$	$\frac{1}{2}e_8$	0	0
e_5	$\frac{1}{2}e_6$	$-\frac{1}{2}e_6$	$-e_7+\frac{3}{2}e_{10}$	$e_8-\frac{3}{2}e_9$	0	$\frac{3}{2}e_2-\frac{1}{2}e_1$	$e_3-\frac{3}{2}e_{14}$	$-e_4-\frac{3}{2}e_{13}$	$\frac{1}{2}e_4$	$-\frac{1}{2}e_3$	0	0	$\frac{1}{2}e_8$	$\frac{1}{2}e_7$
e_6	$-\frac{1}{2}e_5$	$\frac{1}{2}e_5$	$e_8+\frac{3}{2}e_9$	$e_7+\frac{3}{2}e_{10}$	$\frac{1}{2}e_1-\frac{3}{2}e_2$	0	$-e_4+\frac{3}{2}e_{13}$	$-e_3-\frac{3}{2}e_{14}$	$-\frac{1}{2}e_3$	$-\frac{1}{2}e_4$	0	0	$-\frac{1}{2}e_7$	$\frac{1}{2}e_8$
e_7	$-e_8$	0	$e_5+\frac{3}{2}e_{12}$	$-e_6-\frac{3}{2}e_{11}$	$-e_3+\frac{3}{2}e_{14}$	$e_4-\frac{3}{2}e_{13}$	0	e_1	0	0	$\frac{1}{2}e_4$	$-\frac{1}{2}e_3$	$\frac{1}{2}e_6$	$-\frac{1}{2}e_5$
e_8	e_7	0	$-e_6+\frac{3}{2}e_{11}$	$-e_5+\frac{3}{2}e_{12}$	$e_4+\frac{3}{2}e_{13}$	$e_3+\frac{3}{2}e_{14}$	$-e_1$	0	0	0	$-\frac{1}{2}e_3$	$-\frac{1}{2}e_4$	$-\frac{1}{2}e_6$	$-\frac{1}{2}e_6$
e_9	0	e_{10}	$-\frac{1}{2}e_6$	$\frac{1}{2}e_5$	$-\frac{1}{2}e_4$	$\frac{1}{2}e_3$	0	0	0	$-e_2$	$-\frac{1}{2}e_{14}$	$\frac{1}{2}e_{13}$	$-\frac{1}{2}e_{12}$	$\frac{1}{2}e_{11}$
e_{10}	0	$-e_9$	$-\frac{1}{2}e_5$	$-\frac{1}{2}e_6$	$\frac{1}{2}e_3$	$\frac{1}{2}e_4$	0	0	e_2	0	$-\frac{1}{2}e_{13}$	$-\frac{1}{2}e_{14}$	$\frac{1}{2}e_{11}$	$\frac{1}{2}e_{12}$
e_{11}	$\frac{3}{2}e_{12}$	$\frac{1}{2}e_{12}$	$-\frac{1}{2}e_8$	$\frac{1}{2}e_7$	0	0	$-\frac{1}{2}e_4$	$\frac{1}{2}e_3$	$\frac{1}{2}e_{14}$	$\frac{1}{2}e_{13}$	0	$-\frac{1}{2}e_1-\frac{1}{2}e_2$	$-\frac{1}{2}e_{10}$	$-\frac{1}{2}e_9$
e_{12}	$-\frac{3}{2}e_{11}$	$-\frac{1}{2}e_{11}$	$-\frac{1}{2}e_7$	$-\frac{1}{2}e_8$	0	0	$\frac{1}{2}e_3$	$\frac{1}{2}e_4$	$-\frac{1}{2}e_{13}$	$\frac{1}{2}e_{14}$	$\frac{1}{2}e_1+\frac{1}{2}e_2$	0	$\frac{1}{2}e_9$	$-\frac{1}{2}e_{10}$
e_{13}	$\frac{3}{2}e_{14}$	$-\frac{1}{2}e_{14}$	0	0	$-\frac{1}{2}e_8$	$\frac{1}{2}e_7$	$-\frac{1}{2}e_6$	$\frac{1}{2}e_5$	$\frac{1}{2}e_{12}$	$-\frac{1}{2}e_{11}$	$\frac{1}{2}e_{10}$	$-\frac{1}{2}e_9$	0	$-\frac{1}{2}e_1+\frac{1}{2}e_2$
e_{14}	$-\frac{3}{2}e_{13}$	$\frac{1}{2}e_{13}$	0	0	$-\frac{1}{2}e_7$	$-\frac{1}{2}e_8$	$\frac{1}{2}e_5$	$\frac{1}{2}e_6$	$-\frac{1}{2}e_{11}$	$-\frac{1}{2}e_{12}$	$\frac{1}{2}e_9$	$\frac{1}{2}e_{10}$	$\frac{1}{2}e_1-\frac{1}{2}e_2$	0

We now consider the subalgebras of g2 which do not act irreducibly on their complements, namely u(2), u(2)′, bl, bl′ and b.

We have lemmas 2.5.1 and 2.5.2 of [Va] which give a one to one correspondence between the set of all subalgebras of g2 and the set of all connected Lie subgroups of G2. All these subgroups must be compact since a simple Lie group with a noncompact subgroup must itself be noncompact. See [Gi] p.330. Proposition 4.2 of [KN1] p.43 enables us to consider homogeneous spaces G2/W, where W is one of the Lie subgroups.

The components of the curvature tensor are computed using the formula on page 84. Moreover we investigate the possibilities of the homogeneous spaces admitting the Einstein, Ricci- Codazzi or Ricci cyclic parallel metrics. From this investigation it seems that the class of Ricci-Codazzi metrics is more restrictive than the class of Ricci cyclic parallel metrics.

<u>G2/B1</u>

In this case b1 is generated by e_1, e_7 and e_8. The splitting of m is : $m = m_0 + m_1$,where the 3-dimensional m_0 is generated by e_2, e_9 and e_{10} ; 8-dimensional m_1 is generated by $e_3, \ldots, e_6, e_{11}, \ldots, e_{14}$. We consider metrics with

$g(2,2)=4a$, $g(9,9)=4b$, $g(10,10)=4c$;

$g(3,3)=\ldots= g(6,6)=12d$, $g(11,11)=\ldots=g(14,14)=4d$.

The Einstein equations are

$$a^2bc + d^2(a^2 - b^2 - c^2) + 2bcd^2 = 2bcd^2 .4av,$$

$$ab^2c + d^2(b^2 - c^2 - a^2) + 2cad^2 = 2cad^2 .4bv,$$

$$abc^2 + d^2(c^2 - a^2 - b^2) + 2abd^2 = 2abd^2 .4cv,$$

$$-a - b - c + 16d = 8d.4dv.$$

For a=b=c ,the equations above reduce to

$$a^4 + a^2d^2 = 2a^2d^2 .4av,$$

$$-3a + 16d = 8d.4dv,$$

which on eliminating v yields

$$7a^2/d^2 - 16a/d + 4 = (a/d - 2)(7a/d - 2) = 0 .$$

For the first case a=b=c=2d the sectional curvatures take positive and negative values while for the second metric the sectional curvatures are non-negative. They take values

$\{1/224d, 11/224d, 1/16d, 89/672d, 35/32d\}$. Thus the Einstein metric with $a=b=c=2d/7$ has pinching $1/245$.

All Ricci-Codazzi metrics for the case $a=b=c$ are Einstein since we need $7a^2 - 16ad + 4d^2 = 0$.

It is not difficult to show that there are no solutions of Einstein equations in the case when exactly two of the coefficients a, b, c are equal. For example, when $a=c$ the equations reduce to

$$a^3 + 2ad^2 - bd^2 = 2ad.4av,$$

$$a^2 b^2 + b^2 d^2 = 2a^2 d.4bv,$$

$$16d - 2a - b = 8d.4dv.$$

Multiplying the first by b and subtracting the second yields

$b(a-b)(a^2 + 2d^2) = 0$, which is absurd unless $a=b$.

When the coefficients a, b, c are distinct we obtain from the first three equations:

$$(a-b)\{abc + 2d^2(a+b-c)\} = 0,$$

$$(b-c)\{abc + 2d^2(b+c-a)\} = 0,$$

$$(c-a)\{abc + 2d^2(c+a-b)\} = 0,$$

from which we get

$3abc + 2d^2(a+b+c) = 0$, which is absurd.

For the space to be Ricci cyclic parallel we need

$$ab^2+bc^2+ca^2-ac^2-ba^2-cb^2=0,$$

which can be written as

$$(a-b)(b-c)(c-a)=0.$$

Thus, for an arbitrary d, the space is Ricci cyclic parallel if and only if at least two of the coefficients a, b, c are equal. But all such metrics are Ricci parallel.

G2/B1′

In this case b1′ is generated by e_2, e_9 and e_{10}. The splitting of m is given by $m = m_0 + m_1 + m_2$ where the 3-dimensional m_0 is generated by e_1, e_7 and e_8; 4-dimensional m_1 is generated by e_3, e_4, e_5 and e_6; 4-dimensional m_2 is generated by e_{11}, e_{12}, e_{13} and e_{14}. We consider metrics with

$g(1,1)=12a$, $g(7,7)=12b$, $g(8,8)=12c$;

$g(3,3)= \ldots = g(6,6)=12d$; $g(11,11)= \ldots = g(14,14) =4f$.

The Einstein equations for a=b=c and d=f are

$$10a^4d^2 + 2a^2d^4 = 4a^2d^4.12av,$$

$$48a^2d^2 - 15a^3d = 8a^2d^2.12dv,$$

which on eliminating v gives

$$35a^2/d^2 - 48a/d + 4 = 0 \text{ ,i.e , } a/d = \{24 \pm 2(109)^{1/2}\}/35 .$$

In this case (a=b=c and d=f) all the metrics considered are automatically Ricci cyclic parallel, while all Ricci-Codazzi metrics are necessarily Einstein.

It can be shown that there are no solutions of the Einstein equations for d=f unless a=b=c.

As for Ricci cyclic parallel (when $d=f$) we need

$$(a-b)(b-c)(c-a)=0$$

and

$$b^2 +bc +2ad -2ab -bd -cd =0 \text{ and } c^2 +bc +2ad -2ac -bd -cd =0.$$

If $a=b$ then we must have $a=b=c$, the case already considered.

If $b=c$, we need $b^2 +ad -ab -bd =(b-d)(b-a) =0$.

Thus $b=c=d=f$ with arbitrary a gives Ricci cyclic parallel metrics. But there is no Ricci-Codazzi metric for such case.

G2/B

In this case b is generated by e_{10}, e_{12} and e_{14}.

The splitting of m is given by $m = m_1 + m_2 + m_3$, where

5-dimensional m_1 is generated by $e_1, e_2, e_9, e_{11}, e_{13}$,

3-dimensional m_2 is generated by e_3, e_5, e_7,

3-dimensional m_3 is generated by e_4, e_6 and e_8.

We consider metrics with

$Ki(1,1)=12a$, $Ki(2,2)=4a$, $Ki(9,9)=Ki(11,11)=Ki(13,13) = 4a$,

$Ki(3,3)=Ki(5,5)=Ki(7,7)=12b$ and $Ki(4,4)=Ki(6,6)=Ki(8,8)=12c$.

The Einstein equations are

$$a^2 - b^2 - c^2 + 8bc = 4bc.4av,$$

$$18ac^2 + 2ab^2 - 5ca^2 + 5cb^2 - 5c^3 = 4ac^2.12bv,$$

$$- 4ab^2 - 5ca^2 - 5cb^2 + 5c^3 + 24abc = 4abc.12cv.$$

For $b=c$ the equations reduce to

$$a^2 + 6b^2 = 16b^2 av \quad \text{and} \quad -5a^2 + 20ab = 48b^2 av,$$

which has no real solution.

In fact in this case ($b=c$), Ricci-Codazzi conditions

reduce to $4a^2 -10ab +9b^2 =0$, which has no real solutions.

Similar case holds for $a=c$ or $a=b$.

However all metrics (with $b=c$) are Ricci cyclic parallel.

<u>G2/U2</u>

u2 is generated by e_1, e_2, e_7 and e_8. The splitting of of m is given by $m = m_1 + m_2$, where the 8-dimensional m_1 is generated by $e_3, \ldots, e_6, e_{11}, \ldots, e_{14}$; 2-dimensional m_2 is generated by e_9 and e_{10}.

We consider metrics with

$g(3,3) = \ldots = g(6,6) = 12a$; $g(11,11) = \ldots = g(14,14) = 4a$;
$g(9,9) = g(10,10) = 4b$.

The Einstein equations are

$$8a - b = 4a.4av,$$

$$2a^2 + b^2 = 2a^2.4bv,$$

which on eliminating v gives $a/b = 1/2$ or $3/2$.
For the Einstein metric with $a/b=1/2$ the sectional curvature takes both positive and negative values, while for the second metric the sectional curvature is non-negative. It takes values $\{1/96a, 1/32a, 1/16a, 1/12a, 11/96a, 1/4a, 3/8a\}$. Thus the second Einstein metric with $2a=3b$ has pinching $1/36$.

All Ricci-Codazzi metrics are necessarily Einstein for we need

$$4a^2 - 8ab + 3b^2 = 0$$

which gives $a/b = 1/2$ or $3/2$.

However all the metrics considered are automatically Ricci cyclic parallel.

G2/U2′

In this case u2′ is generated by e_1, e_2, e_9 and e_{10}. We have the splitting of m given by $m = m_1 + m_2 + m_3$, where

4-dimensional m_1 is generated by e_3, e_4, e_5 and e_6 ;

2-dimensional m_2 is generated by e_7 and e_8 ;

4-dimensional m_3 is generated by e_{11}, e_{12}, e_{13} and e_{14}.

We consider metrics with

$g(3,3) = \dots = g(6,6) = 12a$; $g(7,7) = g(8,8) = 12b$;
$g(11,11) = \dots = g(14,14) = 4c$.

The Einstein equations are

$$3a^3 - 3ab^2 - 3ac^2 + 24abc - 4b^2c = 4abc.12av,$$

$$2b^2c + 3ab^2 - 3ac^2 - 3a^3 + 8a^2c = 2a^2c.12bv,$$

$$8ab - a^2 - b^2 + c^2 = 4ab.4cv .$$

It is not difficult to show that there is no solution of Einstein equations for a=c. In fact in this case there are no Ricci-Codazzi metrics for we need

$$4a^2-24ab+17b^2=0 \ ; \ 4a^2-24ab+15b^2=0 \ ; \ 4a^2-32ab+17b^2=0,$$

which has no solutions.

Similarly there are no solutions for a=b or b=c. In fact for b=c the above equations reduce to

$$3a^3 + 18ac^2 - 4c^3 = 48a^2c^2v,$$

$$-3a^3 + 8ca^2 + 2c^3 = 24a^2c^2v,$$

$$8ac - a^2 = 16ac^2v.$$

From which we obtain

$$6a^3 - 4c^3 - 24ca^2 + 18ac^2 = 0,$$

$$3a^3 - 4c^3 + 8ca^2 = 0,$$

with no solution in common.

A necessary and sufficient condition for Ricci cyclic parallel metric is a=b=c.

APPENDIX1 : RESULTS ON S^7

```
1$Log Output: MAN3, 16:52:27 Wed Apr 03/85

$RUN ETC:RDC2.LISP SCARDS=*SOURCE*+*MSOURCE* PAR=R=250P
Execution begins   16:52:29
STANDARD LISP  INITIAL CORE ALLOCATION:  FREE CELLS =    83490, BPS =    86016, PDS =     3000.

   REDUCE 2 (Apr-15-79 (MTS Aug-18-80)) ...

   LINELENGTH(90);
   120

   IN S7ABC;

      COMMENT    ---1)This programme will compute the components of the curvature
            tensor of  any given homogeneous manifold. We shall also  compute the
            sectional curvature, Ricci curvature, the covariant derivative of the
            Ricci tensor and the  cyclic sum of this derivative together with the
            Ricci-Codazzi condition;

      COMMENT ---------2)We represent the k-th component of the bracket product
            [e(i),e(j)]  by  M(i,j,k) .
            In  the case of  S(7) obtained from  Sp(2)/Sp(1)  we have seen that
            the splitting of m is given by m = m(0) + m(1) ,where the
            3-dimensional  m(0)  is generated by  e(2),e(5),e(6)  while the
            4-dimensional  m(1)  is generated by  e(7),e(8),e(9) and  e(10).
            From the  table of the bracket products of the elements of sp(2) we
            now list all the  non-zero  components of  M(i,j,k) involving  only
            e(2),e(5),e(6),e(7), e(8),e(9)  and e(10). We only need to list the
            lower triangle elements since the bracket product is skew-symmetric;

     N:=10;

     N := 10

     ARRAY M(N,N,N);

           M( 5, 2, 6):= 2$   M( 6, 2, 5):=-2$   M( 6, 5, 2):= 2$
           M( 7, 2, 8):= 1$   M( 7, 5,10):=-1$   M( 7, 6, 9):=-1$
           M( 8, 2, 7):=-1$   M( 8, 5, 9):= 1$   M( 8, 6,10):=-1$
           M( 8, 7, 2):= 2$   M( 9, 2,10):=-1$   M( 9, 5, 8):=-1$
           M( 9, 6, 7):= 1$   M( 9, 7, 6):=-2$   M( 9, 8, 5):= 2$
           M(10, 2, 9):= 1$   M(10, 5, 7):= 1$   M(10, 6, 8):= 1$
           M(10, 7, 5):=-2$   M(10, 8, 6):=-2$   M(10, 9, 2):=-2$

      COMMENT---3)We now state the skew-symmetry property of the bracket
        product namely, M(i,j,k) = -M(j,i,k)    for all   i,j,k=2,5,6,.,10;

        FOR ROW:=2:10 DO
            FOR COL:=2:ROW-1 DO
                FOR K:=2:10 DO
                IF NOT M(ROW,COL,K)=0 THEN M(COL,ROW,K) := -M(ROW,COL,K);

      COMMENT---4)The metric on m obtained from the Killing metric on Sp(2)
        by the method described in page 83 is given by;
```

```
ARRAY KI(N);

      KI(2) := 12*A  $   KI(5) := 12*B  $  KI(6) := 12*C  $
      KI(7) := 24*D  $   KI(8) := 24*D  $  KI(9) := 24*D  $
      KI(10):= 24*D  $

COMMENT-----5)The inverse metric is then;

ARRAY HI(N);

      HI(2)  :=1/(12*A)$  HI(5)  :=1/(12*B)$  HI(6)  :=1/(12*C)$
      HI(7)  :=1/(24*D)$  HI(8)  :=1/(24*D)$  HI(9)  :=1/(24*D)$
      HI(10) :=1/(24*D)$

 COMMENT------6)We now compute the Christoffel symbols of the Levi-Civita
             connection  given in page 84, making use of the fact that
             the "array"  M is rather sparse. This is  done by single
             pass through the array M;

ARRAY C(N,N,N);

      FOR H:=2:10 DO
          FOR J:=2:10 DO
              FOR K:=2:10 DO
              IF NOT M(H,J,K) = 0 THEN <<
              C(H,J,K) := C(H,J,K) + M(H,J,K)/2 ;
              C(K,H,J) := C(K,H,J) - M(H,J,K)*KI(K)*HI(J)/2 ;
              C(H,K,J) := C(H,K,J) - M(H,J,K)*KI(K)*HI(J)/2 ;
                                        >>;

COMMENT --7)We will now compute the components of the curvature tensor given
  by the formula in p. 84. We first compute the sum of the first three terms.
  Again we exploit  the fact  that both arrays M and C are rather sparse,i.e,
  we make single passes over the arrays accumulating "information" only when
  we have non-zero elements;

COMMENT---8)In order to be able to compute the components of the curvature
  tensor we have seen from  the last term of the last term of the formula
  that we need where now i,k take values either 1,3 or 4. We denote these
  components by MC(i,j,k);
ARRAY MC(N,N,N);

      MC( 7, 1, 8):= 1$    MC( 7, 3,10):= 1$    MC( 7, 4, 9):=-1$
      MC( 8, 1, 7):=-1$    MC( 8, 3, 9):= 1$    MC( 8, 4,10):= 1$
      MC( 9, 1,10):= 1$    MC( 9, 3, 8):=-1$    MC( 9, 4, 7):= 1$
      MC(10, 1, 9):=-1$    MC(10, 3, 7):=-1$    MC(10, 4, 8):=-1$
      MC( 8, 7, 1):= 2$    MC( 9, 7, 4):=-2$    MC( 9, 8, 3):= 2$
      MC(10, 7, 3):= 2$    MC(10, 8, 4):= 2$    MC(10, 9, 1):= 2$

COMMENT----9) We also have the skew-symmetry property ;

          FOR ROW:=1:10 DO
              FOR COL:=1:ROW-1 DO
                  FOR K:=1:10 DO
                  IF NOT MC(ROW,COL,K)=0 THEN
                  MC(COL,ROW,K):=-MC(ROW,COL,K);
```

```
ARRAY R(N,N,N,N);

  FOR P := 2:10 DO <<
     FOR Q := 2:10 DO
         FOR J := 2:10 DO
             IF NOT C(P,Q,J) = 0 THEN <<
                FOR H := 2:10 DO
                    FOR K := 2:10 DO <<
                        IF NOT C(H,J,K) = 0 THEN
                        BEGIN
                          SCALAR TEMP;
                          TEMP := C(P,Q,J) * C(H,J,K) * KI(K);
                          R(H,P,Q,K) := R(H,P,Q,K) + TEMP;
                          R(P,H,Q,K) := R(P,H,Q,K) - TEMP;
                        END;

            IF NOT M(H,K,P) = 0 THEN
            R(H,K,Q,J) := R(H,K,Q,J) - M(H,K,P)*C(P,Q,J)*KI(J);
                                      >>;
                                       >>;
                    >>;

  FOR H:=2:10 DO
      FOR P:=2:10 DO
          FOR L:=1:4 DO
              IF NOT MC(H,P,L) = 0 THEN <<
                 FOR Q:=2:10 DO
                     FOR K:=2:10 DO
                         IF NOT MC(L,Q,K) = 0 THEN <<
                            BEGIN
                              SCALAR TEM;
                              TEM := MC(H,P,L)*MC(L,Q,K)*KI(K);
                              R(H,P,Q,K) := R(H,P,Q,K) - TEM;
                            END;
                                                   >>;
                                        >>;

COMMENT---10)We can now compute the sectional curvature S;

ARRAY S(N,N);

      FOR P:=2:10 DO
          FOR K:=P+1:10 DO
          BEGIN
              S(P,K):=S(K,P):= R(K,P,P,K)*HI(P)*HI(K);
              IF NOT S(P,K)=0 THEN WRITE
              "S(",P, "," ,K, ")=S(",K, "," ,P, ")= ",S(P,K)$
          END;
```

$$S(2,5)=S(5,2)= (A^2 - 2*A*B + 2*A*C + B^2 + 2*B*C - 3*C^2)/(12*A*B*C)$$

$$S(2,6)=S(6,2)= (A^2 + 2*A*B - 2*A*C - 3*B^2 + 2*B*C + C^2)/(12*A*B*C)$$

$$S(2,7)=S(7,2)= A/(48*D^2)$$

$S(2,8)=S(8,2)= A/(48*D^2)$

$S(2,9)=S(9,2)= A/(48*D^2)$

$S(2,10)=S(10,2)= A/(48*D^2)$

$S(5,6)=S(6,5)= (- 3*A^2 + 2*A*B + 2*A*C + B^2 - 2*B*C + C^2)/(12*A*B*C)$

$S(5,7)=S(7,5)= B/(48*D^2)$

$S(5,8)=S(8,5)= B/(48*D^2)$

$S(5,9)=S(9,5)= B/(48*D^2)$

$S(5,10)=S(10,5)= B/(48*D^2)$

$S(6,7)=S(7,6)= C/(48*D^2)$

$S(6,8)=S(8,6)= C/(48*D^2)$

$S(6,9)=S(9,6)= C/(48*D^2)$

$S(6,10)=S(10,6)= C/(48*D^2)$

$S(7,8)=S(8,7)= (- 3*A + 8*D)/(48*D^2)$

$S(7,9)=S(9,7)= (- 3*C + 8*D)/(48*D^2)$

$S(7,10)=S(10,7)= (- 3*B + 8*D)/(48*D^2)$

$S(8,9)=S(9,8)= (- 3*B + 8*D)/(48*D^2)$

$S(8,10)=S(10,8)= (- 3*C + 8*D)/(48*D^2)$

$S(9,10)=S(10,9)= (- 3*A + 8*D)/(48*D^2)$

COMMENT——11)The Ricci tensor is given by;

```
ARRAY RICCI(N,N);

      FOR P:=2:10 DO
          BEGIN
              RICCI(P,P) := (FOR Q:=2:10 SUM S(P,Q))*KI(P) $
              IF NOT RICCI(P,P)=0 THEN WRITE
              "RICCI(",P, ",",P, ")= ",RICCI(P,P)$
          END;

                2            2  2      2  2          2      2  2        2
RICCI(2,2)= (A *B*C + 2*A *D  - 2*B *D  + 4*B*C*D  - 2*C *D )/(B*C*D )

                   2  2      2             2      2  2      2  2        2
RICCI(5,5)= ( - 2*A *D  + A*B *C + 4*A*C*D  + 2*B *D  - 2*C *D )/(A*C*D )

                   2  2        2           2      2  2      2  2        2
RICCI(6,6)= ( - 2*A *D  + A*B*C  + 4*A*B*D  - 2*B *D  + 2*C *D )/(A*B*D )

RICCI(7,7)= ( - A - B - C + 12*D)/D

RICCI(8,8)= ( - A - B - C + 12*D)/D

RICCI(9,9)= ( - A - B - C + 12*D)/D

RICCI(10,10)= ( - A - B - C + 12*D)/D

COMMENT----12)We now compute the covariant derivative of the Ricci tensor;

ARRAY DEL(N,N,N);

      FOR L:=2:10 DO
          FOR J:=2:10 DO
              FOR K:=2:10 DO
              BEGIN
                 DEL(L,J,K):=-C(K,L,J)*RICCI(J,J) -C(K,J,L)*RICCI(L,L);
                 IF NOT DEL(L,J,K)=0 THEN WRITE
                 "DEL(",L, ",",J, ",",K, "):=  ",DEL(L,J,K);
              END;

                   3          3  2     2    2       2    2       2    2      3      2  2
DEL(2,5,6):=  ( - A *B*C - 4*A *D  + A *B*C  - 4*A *B*D  + 8*A *C*D  + A*B *C - A*B *C  +

                    2  2        2  2      3  2      2    2        2  2          2
               4*A*B *D  - 4*A*C *D  + 4*B *D  - 8*B *C*D  + 4*B*C *D )/(A*B*C*D )

               3          3  2    2  2        2    2       2    2      2  2        2  2
DEL(2,6,5):=  (A *B*C + 4*A *D  - A *B *C - 8*A *B*D  + 4*A *C*D  + A*B *C  + 4*A*B *D  -

                   3       2  2      2    2        2  2      3  2          2
               A*B*C  - 4*A*C *D  - 4*B *C*D  + 8*B*C *D  - 4*C *D )/(A*B*C*D )

                     2          2  2      2         2                        2  2         2
DEL(2,7,8):=  ( - 3*A *B*C - 4*A *D  - A*B *C - A*B*C  + 12*A*B*C*D + 4*B *D  - 8*B*C*D

                 2  2          2
               + 4*C *D )/(2*B*C*D )
```

$$\mathrm{DEL}(2,8,7) := (3 \cdot A^{2} \cdot B \cdot C + 4 \cdot A^{2} \cdot D^{2} + A \cdot B^{2} \cdot C + A \cdot B \cdot C^{2} - 12 \cdot A \cdot B \cdot C \cdot D - 4 \cdot B^{2} \cdot D^{2} + 8 \cdot B \cdot C \cdot D^{2} - 4 \cdot C^{2} \cdot D^{2})/(2 \cdot B \cdot C \cdot D^{2})$$

$$\mathrm{DEL}(2,9,10) := (3 \cdot A^{2} \cdot B \cdot C + 4 \cdot A^{2} \cdot D^{2} + A \cdot B^{2} \cdot C + A \cdot B \cdot C^{2} - 12 \cdot A \cdot B \cdot C \cdot D - 4 \cdot B^{2} \cdot D^{2} + 8 \cdot B \cdot C \cdot D^{2} - 4 \cdot C^{2} \cdot D^{2})/(2 \cdot B \cdot C \cdot D^{2})$$

$$\mathrm{DEL}(2,10,9) := (- 3 \cdot A^{2} \cdot B \cdot C - 4 \cdot A^{2} \cdot D^{2} - A \cdot B^{2} \cdot C - A \cdot B \cdot C^{2} + 12 \cdot A \cdot B \cdot C \cdot D + 4 \cdot B^{2} \cdot D^{2} - 8 \cdot B \cdot C \cdot D^{2} + 4 \cdot C^{2} \cdot D^{2})/(2 \cdot B \cdot C \cdot D^{2})$$

$$\mathrm{DEL}(5,2,6) := (- A^{3} \cdot B \cdot C - 4 \cdot A^{3} \cdot D^{2} + A^{2} \cdot B \cdot C^{2} - 4 \cdot A^{2} \cdot B \cdot D^{2} + 8 \cdot A^{2} \cdot C \cdot D^{2} + A \cdot B^{3} \cdot C - A \cdot B^{2} \cdot C^{2} + 4 \cdot A \cdot B^{2} \cdot D^{2} - 4 \cdot A \cdot C^{2} \cdot D^{2} + 4 \cdot B^{3} \cdot D^{2} - 8 \cdot B^{2} \cdot C \cdot D^{2} + 4 \cdot B \cdot C^{2} \cdot D^{2})/(A \cdot B \cdot C \cdot D^{2})$$

$$\mathrm{DEL}(5,6,2) := (A^{2} \cdot B^{2} \cdot C - A^{2} \cdot B \cdot C^{2} - 4 \cdot A^{2} \cdot B \cdot D^{2} + 4 \cdot A^{2} \cdot C \cdot D^{2} - A \cdot B^{3} \cdot C + 8 \cdot A \cdot B^{2} \cdot D^{2} + A \cdot B \cdot C^{3} - 8 \cdot A \cdot C^{2} \cdot D^{2} - 4 \cdot B^{3} \cdot D^{2} - 4 \cdot B^{2} \cdot C \cdot D^{2} + 4 \cdot B \cdot C^{2} \cdot D^{2} + 4 \cdot C^{3} \cdot D^{2})/(A \cdot B \cdot C \cdot D^{2})$$

$$\mathrm{DEL}(5,7,10) := (A^{2} \cdot B \cdot C - 4 \cdot A^{2} \cdot D^{2} + 3 \cdot A \cdot B^{2} \cdot C + A \cdot B \cdot C^{2} - 12 \cdot A \cdot B \cdot C \cdot D + 8 \cdot A \cdot C \cdot D^{2} + 4 \cdot B^{2} \cdot D^{2} - 4 \cdot C^{2} \cdot D^{2})/(2 \cdot A \cdot C \cdot D^{2})$$

$$\mathrm{DEL}(5,8,9) := (- A^{2} \cdot B \cdot C + 4 \cdot A^{2} \cdot D^{2} - 3 \cdot A \cdot B^{2} \cdot C - A \cdot B \cdot C^{2} + 12 \cdot A \cdot B \cdot C \cdot D - 8 \cdot A \cdot C \cdot D^{2} - 4 \cdot B^{2} \cdot D^{2} + 4 \cdot C^{2} \cdot D^{2})/(2 \cdot A \cdot C \cdot D^{2})$$

$$\mathrm{DEL}(5,9,8) := (A^{2} \cdot B \cdot C - 4 \cdot A^{2} \cdot D^{2} + 3 \cdot A \cdot B^{2} \cdot C + A \cdot B \cdot C^{2} - 12 \cdot A \cdot B \cdot C \cdot D + 8 \cdot A \cdot C \cdot D^{2} + 4 \cdot B^{2} \cdot D^{2} - 4 \cdot C^{2} \cdot D^{2})/(2 \cdot A \cdot C \cdot D^{2})$$

$$\mathrm{DEL}(5,10,7) := (- A^{2} \cdot B \cdot C + 4 \cdot A^{2} \cdot D^{2} - 3 \cdot A \cdot B^{2} \cdot C - A \cdot B \cdot C^{2} + 12 \cdot A \cdot B \cdot C \cdot D - 8 \cdot A \cdot C \cdot D^{2} - 4 \cdot B^{2} \cdot D^{2} + 4 \cdot C^{2} \cdot D^{2})/(2 \cdot A \cdot C \cdot D^{2})$$

$$\mathrm{DEL}(6,2,5) := (A^{3} \cdot B \cdot C + 4 \cdot A^{3} \cdot D^{2} - A^{2} \cdot B^{2} \cdot C - 8 \cdot A^{2} \cdot B \cdot D^{2} + 4 \cdot A^{2} \cdot C \cdot D^{2} + A \cdot B^{2} \cdot C^{2} + 4 \cdot A \cdot B^{2} \cdot D^{2} - A \cdot B \cdot C^{3} - 4 \cdot A \cdot C^{2} \cdot D^{2} - 4 \cdot B^{2} \cdot C \cdot D^{2} + 8 \cdot B \cdot C^{2} \cdot D^{2} - 4 \cdot C^{3} \cdot D^{2})/(A \cdot B \cdot C \cdot D^{2})$$

$$\mathrm{DEL}(6,5,2):=(A^{2} B^{2} C-A^{2} B C^{2}-4 A^{2} B D^{2}+4 A^{2} C D^{2}-A B^{3} C+8 A B^{2} D^{2}+A B C^{3}-8 A C^{2} D^{2}-4 B^{3} D^{2}-4 B^{2} C D^{2}+4 B C^{2} D^{2}+4 C^{3} D^{2})/(A B C D^{2})$$

$$\mathrm{DEL}(6,7,9):=(A^{2} B C-4 A^{2} D^{2}+A B^{2} C+3 A B C^{2}-12 A B C D+8 A B D^{2}-4 B^{2} D^{2}+4 C^{2} D^{2})/(2 A B D^{2})$$

$$\mathrm{DEL}(6,8,10):=(A^{2} B C-4 A^{2} D^{2}+A B^{2} C+3 A B C^{2}-12 A B C D+8 A B D^{2}-4 B^{2} D^{2}+4 C^{2} D^{2})/(2 A B D^{2})$$

$$\mathrm{DEL}(6,9,7):=(-A^{2} B C+4 A^{2} D^{2}-A B^{2} C-3 A B C^{2}+12 A B C D-8 A B D^{2}+4 B^{2} D^{2}-4 C^{2} D^{2})/(2 A B D^{2})$$

$$\mathrm{DEL}(6,10,8):=(-A^{2} B C+4 A^{2} D^{2}-A B^{2} C-3 A B C^{2}+12 A B C D-8 A B D^{2}+4 B^{2} D^{2}-4 C^{2} D^{2})/(2 A B D^{2})$$

$$\mathrm{DEL}(7,2,8):=(-3 A^{2} B C-4 A^{2} D^{2}-A B^{2} C-A B C^{2}+12 A B C D+4 B^{2} D^{2}-8 B C D^{2}+4 C^{2} D^{2})/(2 B C D^{2})$$

$$\mathrm{DEL}(7,5,10):=(A^{2} B C-4 A^{2} D^{2}+3 A B^{2} C+A B C^{2}-12 A B C D+8 A C D^{2}+4 B^{2} D^{2}-4 C^{2} D^{2})/(2 A C D^{2})$$

$$\mathrm{DEL}(7,6,9):=(A^{2} B C-4 A^{2} D^{2}+A B^{2} C+3 A B C^{2}-12 A B C D+8 A B D^{2}-4 B^{2} D^{2}+4 C^{2} D^{2})/(2 A B D^{2})$$

$$\mathrm{DEL}(8,2,7):=(3 A^{2} B C+4 A^{2} D^{2}+A B^{2} C+A B C^{2}-12 A B C D-4 B^{2} D^{2}+8 B C D^{2}-4 C^{2} D^{2})/(2 B C D^{2})$$

$$\mathrm{DEL}(8,5,9):=(-A^{2} B C+4 A^{2} D^{2}-3 A B^{2} C-A B C^{2}+12 A B C D-8 A C D^{2}-4 B^{2} D^{2}+4 C^{2} D^{2})/(2 A C D^{2})$$

$$\mathrm{DEL}(8,6,10) := (A^2 \cdot B \cdot C - 4 \cdot A^2 \cdot D^2 + A \cdot B^2 \cdot C + 3 \cdot A \cdot B \cdot C^2 - 12 \cdot A \cdot B \cdot C \cdot D + 8 \cdot A \cdot B \cdot D^2 - 4 \cdot B^2 \cdot D^2 + 4 \cdot C^2 \cdot D^2)/(2 \cdot A \cdot B \cdot D^2)$$

$$\mathrm{DEL}(9,2,10) := (3 \cdot A^2 \cdot B \cdot C + 4 \cdot A^2 \cdot D^2 + A \cdot B^2 \cdot C + A \cdot B \cdot C^2 - 12 \cdot A \cdot B \cdot C \cdot D - 4 \cdot B^2 \cdot D^2 + 8 \cdot B \cdot C \cdot D^2 - 4 \cdot C^2 \cdot D^2)/(2 \cdot B \cdot C \cdot D^2)$$

$$\mathrm{DEL}(9,5,8) := (A^2 \cdot B \cdot C - 4 \cdot A^2 \cdot D^2 + 3 \cdot A \cdot B^2 \cdot C + A \cdot B \cdot C^2 - 12 \cdot A \cdot B \cdot C \cdot D + 8 \cdot A \cdot C \cdot D^2 + 4 \cdot B^2 \cdot D^2 - 4 \cdot C^2 \cdot D^2)/(2 \cdot A \cdot C \cdot D^2)$$

$$\mathrm{DEL}(9,6,7) := (- A^2 \cdot B \cdot C + 4 \cdot A^2 \cdot D^2 - A \cdot B^2 \cdot C - 3 \cdot A \cdot B \cdot C^2 + 12 \cdot A \cdot B \cdot C \cdot D - 8 \cdot A \cdot B \cdot D^2 + 4 \cdot B^2 \cdot D^2 - 4 \cdot C^2 \cdot D^2)/(2 \cdot A \cdot B \cdot D^2)$$

$$\mathrm{DEL}(10,2,9) := (- 3 \cdot A^2 \cdot B \cdot C - 4 \cdot A^2 \cdot D^2 - A \cdot B^2 \cdot C - A \cdot B \cdot C^2 + 12 \cdot A \cdot B \cdot C \cdot D + 4 \cdot B^2 \cdot D^2 - 8 \cdot B \cdot C \cdot D^2 + 4 \cdot C^2 \cdot D^2)/(2 \cdot B \cdot C \cdot D^2)$$

$$\mathrm{DEL}(10,5,7) := (- A^2 \cdot B \cdot C + 4 \cdot A^2 \cdot D^2 - 3 \cdot A \cdot B^2 \cdot C - A \cdot B \cdot C^2 + 12 \cdot A \cdot B \cdot C \cdot D - 8 \cdot A \cdot C \cdot D^2 - 4 \cdot B^2 \cdot D^2 + 4 \cdot C^2 \cdot D^2)/(2 \cdot A \cdot C \cdot D^2)$$

$$\mathrm{DEL}(10,6,8) := (- A^2 \cdot B \cdot C + 4 \cdot A^2 \cdot D^2 - A \cdot B^2 \cdot C - 3 \cdot A \cdot B \cdot C^2 + 12 \cdot A \cdot B \cdot C \cdot D - 8 \cdot A \cdot B \cdot D^2 + 4 \cdot B^2 \cdot D^2 - 4 \cdot C^2 \cdot D^2)/(2 \cdot A \cdot B \cdot D^2)$$

```
COMMENT----13)The nonzero components of RC(i,j,k)=DEL(i,j,k)-DEL(k,j,i) are;

ARRAY RC(N,N,N);

      FOR L:=2:10 DO
          FOR J:=2:10 DO
              FOR K:=L+1:10 DO
              BEGIN
                 RC(L,J,K) := DEL(L,J,K)-DEL(K,J,L);
                 IF NOT RC(L,J,K)=0 THEN WRITE
      "RC(",L, ",", J, "," , K, "):=-RC(",K, "," ,J, "," ,L, "):= ",RC(L,J,K)$
              END;
```

$$RC(2,5,6) := -RC(6,5,2) := (-A^3 B C - 4 A^3 D^2 - A^2 B^2 C + 2 A^2 B C^2 + 4 A^2 C D^2 + 2 A B^3 C - A B^2 C^2 - 4 A B^2 D^2 - A B C^3 + 4 A C^2 D^2 + 8 B^3 D^2 - 4 B^2 C D^2 - 4 C^3 D^2)/(A B C D^2)$$

$$RC(2,6,5) := -RC(5,6,2) := (A^3 B C + 4 A^3 D^2 - 2 A^2 B^2 C + A^2 B C^2 - 4 A^2 B D^2 + A B^3 C + A B^2 C^2 - 4 A B^2 D^2 - 2 A B C^3 + 4 A C^2 D^2 + 4 B^3 D^2 + 4 B C^2 D^2 - 8 C^3 D^2)/(A B C D^2)$$

$$RC(2,7,8) := -RC(8,7,2) := (-3 A^2 B C - 4 A^2 D^2 - A B^2 C - A B C^2 + 12 A B C D + 4 B^2 D^2 - 8 B C D^2 + 4 C^2 D^2)/(2 B C D^2)$$

$$RC(2,8,7) := -RC(7,8,2) := (3 A^2 B C + 4 A^2 D^2 + A B^2 C + A B C^2 - 12 A B C D - 4 B^2 D^2 + 8 B C D^2 - 4 C^2 D^2)/(2 B C D^2)$$

$$RC(2,9,10) := -RC(10,9,2) := (3 A^2 B C + 4 A^2 D^2 + A B^2 C + A B C^2 - 12 A B C D - 4 B^2 D^2 + 8 B C D^2 - 4 C^2 D^2)/(2 B C D^2)$$

$$RC(2,10,9) := -RC(9,10,2) := (-3 A^2 B C - 4 A^2 D^2 - A B^2 C - A B C^2 + 12 A B C D + 4 B^2 D^2 - 8 B C D^2 + 4 C^2 D^2)/(2 B C D^2)$$

$$RC(5,2,6) := -RC(6,2,5) := (-2 A^3 B C - 8 A^3 D^2 + A^2 B^2 C + A^2 B C^2 + 4 A^2 B D^2 + 4 A^2 C D^2 + A B^3 C - 2 A B^2 C^2 + A B C^3 + 4 B^3 D^2 - 4 B^2 C D^2 - 4 B C^2 D^2 + 4 C^3 D^2)/(A B C D^2)$$

$$RC(5,7,10) := -RC(10,7,5) := (A^2 B C - 4 A^2 D^2 + 3 A B^2 C + A B C^2 - 12 A B C D + 8 A C D^2 + 4 B^2 D^2 - 4 C^2 D^2)/(2 A C D^2)$$

$$RC(5,8,9):=-RC(9,8,5):=(-A^2 \cdot B \cdot C+4 \cdot A^2 \cdot D^2-3 \cdot A \cdot B^2 \cdot C-A \cdot B \cdot C^2+12 \cdot A \cdot B \cdot C \cdot D-8 \cdot A \cdot C \cdot D^2-4 \cdot B^2 \cdot D^2+4 \cdot C^2 \cdot D^2)/(2 \cdot A \cdot C \cdot D^2)$$

$$RC(5,9,8):=-RC(8,9,5):=(A^2 \cdot B \cdot C-4 \cdot A^2 \cdot D^2+3 \cdot A \cdot B^2 \cdot C+A \cdot B \cdot C^2-12 \cdot A \cdot B \cdot C \cdot D+8 \cdot A \cdot C \cdot D^2+4 \cdot B^2 \cdot D^2-4 \cdot C^2 \cdot D^2)/(2 \cdot A \cdot C \cdot D^2)$$

$$RC(5,10,7):=-RC(7,10,5):=(-A^2 \cdot B \cdot C+4 \cdot A^2 \cdot D^2-3 \cdot A \cdot B^2 \cdot C-A \cdot B \cdot C^2+12 \cdot A \cdot B \cdot C \cdot D-8 \cdot A \cdot C \cdot D^2-4 \cdot B^2 \cdot D^2+4 \cdot C^2 \cdot D^2)/(2 \cdot A \cdot C \cdot D^2)$$

$$RC(6,7,9):=-RC(9,7,6):=(A^2 \cdot B \cdot C-4 \cdot A^2 \cdot D^2+A \cdot B^2 \cdot C+3 \cdot A \cdot B \cdot C^2-12 \cdot A \cdot B \cdot C \cdot D+8 \cdot A \cdot B \cdot D^2-4 \cdot B^2 \cdot D^2+4 \cdot C^2 \cdot D^2)/(2 \cdot A \cdot B \cdot D^2)$$

$$RC(6,8,10):=-RC(10,8,6):=(A^2 \cdot B \cdot C-4 \cdot A^2 \cdot D^2+A \cdot B^2 \cdot C+3 \cdot A \cdot B \cdot C^2-12 \cdot A \cdot B \cdot C \cdot D+8 \cdot A \cdot B \cdot D^2-4 \cdot B^2 \cdot D^2+4 \cdot C^2 \cdot D^2)/(2 \cdot A \cdot B \cdot D^2)$$

$$RC(6,9,7):=-RC(7,9,6):=(-A^2 \cdot B \cdot C+4 \cdot A^2 \cdot D^2-A \cdot B^2 \cdot C-3 \cdot A \cdot B \cdot C^2+12 \cdot A \cdot B \cdot C \cdot D-8 \cdot A \cdot B \cdot D^2+4 \cdot B^2 \cdot D^2-4 \cdot C^2 \cdot D^2)/(2 \cdot A \cdot B \cdot D^2)$$

$$RC(6,10,8):=-RC(8,10,6):=(-A^2 \cdot B \cdot C+4 \cdot A^2 \cdot D^2-A \cdot B^2 \cdot C-3 \cdot A \cdot B \cdot C^2+12 \cdot A \cdot B \cdot C \cdot D-8 \cdot A \cdot B \cdot D^2+4 \cdot B^2 \cdot D^2-4 \cdot C^2 \cdot D^2)/(2 \cdot A \cdot B \cdot D^2)$$

$$RC(7,2,8):=-RC(8,2,7):=(-3 \cdot A^2 \cdot B \cdot C-4 \cdot A^2 \cdot D^2-A \cdot B^2 \cdot C-A \cdot B \cdot C^2+12 \cdot A \cdot B \cdot C \cdot D+4 \cdot B^2 \cdot D^2-8 \cdot B \cdot C \cdot D^2+4 \cdot C^2 \cdot D^2)/(B \cdot C \cdot D^2)$$

$$RC(7,5,10):=-RC(10,5,7):=(A^2 \cdot B \cdot C-4 \cdot A^2 \cdot D^2+3 \cdot A \cdot B^2 \cdot C+A \cdot B \cdot C^2-12 \cdot A \cdot B \cdot C \cdot D+8 \cdot A \cdot C \cdot D^2+4 \cdot B^2 \cdot D^2-4 \cdot C^2 \cdot D^2)/(A \cdot C \cdot D^2)$$

```
                                   2          2  2       2            2                      2
RC(7,6,9):=-RC(9,6,7):= (A *B*C - 4*A *D  + A*B *C + 3*A*B*C  - 12*A*B*C*D + 8*A*B*D  - 4*

                        2  2      2  2       2
                       B *D  + 4*C *D )/(A*B*D )

                                     2          2  2        2          2                        2
RC(8,5,9):=-RC(9,5,8):= ( - A *B*C + 4*A *D  - 3*A*B *C - A*B*C  + 12*A*B*C*D - 8*A*C*D

                             2  2      2  2        2
                        - 4*B *D  + 4*C *D )/(A*C*D )

                                     2          2  2       2            2                      2
RC(8,6,10):=-RC(10,6,8):= (A *B*C - 4*A *D  + A*B *C + 3*A*B*C  - 12*A*B*C*D + 8*A*B*D  -

                          2  2      2  2       2
                       4*B *D  + 4*C *D )/(A*B*D )

                                       2          2  2       2          2                    2  2
RC(9,2,10):=-RC(10,2,9):= (3*A *B*C + 4*A *D  + A*B *C + A*B*C  - 12*A*B*C*D - 4*B *D  + 8

                             2      2  2       2
                       *B*C*D  - 4*C *D )/(B*C*D )

    COMMENT----14)Finally we check whether the Ricci cyclic parallel conditions are
                satisfied.  We print non-zero components of
                RCP(i,j,k)=DEL(i,j,k)+DEL(j,k,i)+DEL(k,i,j);

    ARRAY RCP(N,N,N);

        FOR L:=2:10 DO
            FOR J:=L:10 DO
                FOR K:=L:10 DO
                BEGIN
                    RCP(L,J,K):= DEL(L,J,K) + DEL(J,K,L) + DEL(K,L,J);
                    IF NOT RCP(L,J,K)=0 THEN WRITE
                    "RCP(",L, ",", J, ",", K, ") := ",RCP(L,J,K)$
                END;

                            2      2        2      2    2        2
RCP(2,5,6) := (16*( - A *B + A *C + A*B  - A*C  - B *C + B*C ))/(A*B*C)

                            2      2        2      2    2        2
RCP(2,6,5) := (16*( - A *B + A *C + A*B  - A*C  - B *C + B*C ))/(A*B*C)

    COMMENT----15)PROGRAMMES IN REDUCE ENDED;

$
SIGNOFF

ENTERING LISP...

*** END OF DATA
Execution terminated   16:55:30  T=129.963  RC=0  $8.14
```

APPENDIX2 : RESULTS ON CP^3

COMMENT——Metrics considered are of the form;

$$g(5,5) := 12*A \ \$ \quad g(6,6) := 12*A \ \$ \quad g(7,7) := 24*B \ \$$$
$$g(8,8) := 24*B \ \$ \quad g(9,9) := 24*B \ \$ \quad g(10,10) := 24*B \ \$$$

COMMENT——The sectional curvature S is given by;

$$S(5,6) := S(6,5) := 1/(3*A)$$

$$S(5,7) := S(7,5) := A/(48*B^2)$$

$$S(5,8) := S(8,5) := A/(48*B^2)$$

$$S(5,9) := S(9,5) := A/(48*B^2)$$

$$S(5,10) := S(10,5) := A/(48*B^2)$$

$$S(6,7) := S(7,6) := A/(48*B^2)$$

$$S(6,8) := S(8,6) := A/(48*B^2)$$

$$S(6,9) := S(9,6) := A/(48*B^2)$$

$$S(6,10) := S(10,6) := A/(48*B^2)$$

$$S(7,8) := S(8,7) := 1/(6*B)$$

$$S(7,9) := S(9,7) := (-3*A + 8*B)/(48*B^2)$$

$$S(7,10) := S(10,7) := (-3*A + 8*B)/(48*B^2)$$

$$S(8,9) := S(9,8) := (-3*A + 8*B)/(48*B^2)$$

$$S(8,10) := S(10,8) := (-3*A + 8*B)/(48*B^2)$$

$$S(9,10) := S(10,9) := 1/(6*B)$$

COMMENT——The Ricci tensor is given by;

$$RICCI(5,5) := (A^{2} + 4*B^{2})/B^{2}$$

$$RICCI(6,6) := (A^{2} + 4*B^{2})/B^{2}$$

$$RICCI(7,7) := (2*(- A + 6*B))/B$$

$$RICCI(8,8) := (2*(- A + 6*B))/B$$

$$RICCI(9,9) := (2*(- A + 6*B))/B$$

$$RICCI(10,10) := (2*(- A + 6*B))/B$$

COMMENT——The non-zero components of the derivative of Ricci tensor are;

$$DEL(5,7,10) := (2*(A^{2} - 3*A*B + 2*B^{2}))/B^{2}$$

$$DEL(5,8,9) := (2*(- A^{2} + 3*A*B - 2*B^{2}))/B^{2}$$

$$DEL(5,9,8) := (2*(A^{2} - 3*A*B + 2*B^{2}))/B^{2}$$

$$DEL(5,10,7) := (2*(- A^{2} + 3*A*B - 2*B^{2}))/B^{2}$$

$$DEL(6,7,9) := (2*(A^{2} - 3*A*B + 2*B^{2}))/B^{2}$$

$$DEL(6,8,10) := (2*(A^{2} - 3*A*B + 2*B^{2}))/B^{2}$$

$$DEL(6,9,7) := (2*(- A^{2} + 3*A*B - 2*B^{2}))/B^{2}$$

$$DEL(6,10,8) := (2*(- A^{2} + 3*A*B - 2*B^{2}))/B^{2}$$

$$DEL(7,5,10) := (2*(A^{2} - 3*A*B + 2*B^{2}))/B^{2}$$

$$DEL(7,6,9) := (2*(A^{2} - 3*A*B + 2*B^{2}))/B^{2}$$

$$DEL(8,5,9) := (2*(- A^{2} + 3*A*B - 2*B^{2}))/B^{2}$$

$$DEL(8,6,10) := (2*(A^{2} - 3*A*B + 2*B^{2}))/B^{2}$$

$$DEL(9,5,8) := (2*(A^2 - 3*A*B + 2*B^2))/B^2$$

$$DEL(9,6,7) := (2*(- A^2 + 3*A*B - 2*B^2))/B^2$$

$$DEL(10,5,7) := (2*(- A^2 + 3*A*B - 2*B^2))/B^2$$

$$DEL(10,6,8) := (2*(- A^2 + 3*A*B - 2*B^2))/B^2$$

COMMENT——The nonzero components of the difference RC(i,j,k)=DEL(i,j,k)−DEL(k,j,i) are;

$$RC(5,7,10) := -RC(10,7,5) := (2*(A^2 - 3*A*B + 2*B^2))/B^2$$

$$RC(5,8,9) := -RC(9,8,5) := (2*(- A^2 + 3*A*B - 2*B^2))/B^2$$

$$RC(5,9,8) := -RC(8,9,5) := (2*(A^2 - 3*A*B + 2*B^2))/B^2$$

$$RC(5,10,7) := -RC(7,10,5) := (2*(- A^2 + 3*A*B - 2*B^2))/B^2$$

$$RC(6,7,9) := -RC(9,7,6) := (2*(A^2 - 3*A*B + 2*B^2))/B^2$$

$$RC(6,8,10) := -RC(10,8,6) := (2*(A^2 - 3*A*B + 2*B^2))/B^2$$

$$RC(6,9,7) := -RC(7,9,6) := (2*(- A^2 + 3*A*B - 2*B^2))/B^2$$

$$RC(6,10,8) := -RC(8,10,6) := (2*(- A^2 + 3*A*B - 2*B^2))/B^2$$

$$RC(7,5,10) := -RC(10,5,7) := (4*(A^2 - 3*A*B + 2*B^2))/B^2$$

$$RC(7,6,9) := -RC(9,6,7) := (4*(A^2 - 3*A*B + 2*B^2))/B^2$$

$$RC(8,5,9) := -RC(9,5,8) := (4*(- A^2 + 3*A*B - 2*B^2))/B^2$$

$$RC(8,6,10) := -RC(10,6,8) := (4*(A^2 - 3*A*B + 2*B^2))/B^2$$

COMMENT——The non-zero components of DEL(i,j,k)+DEL(j,k,i)+DEL(k,i,j) are;

```
$
SIGNOFF
ENTERING LISP...
*** END OF DATA
Execution terminated   20:01:26  T=86.98  RC=0  $5.44
```

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