Modelling glacial erosional landform development

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MODELLING GLACIAL EROSIONAL LANDFORM DEVELOPMENT

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ABSTRACT

Glacial erosional systems exhibit a complex, highly scale-dependent phenomenology. Some aspects of modelling the development of glacial erosional landforms in response to glacial erosional processes acting over a wide range of scales are considered.

The physics of ice at the glacier sole is discussed. A simple ice-water mixture theory is proposed. A method for finding the solution of the equations of motion of ice at the glacier sole based on the finite element velocities-pressure formulation is shown, which includes novel formulations for the sliding boundary condition, compression of ice and flow of water between ice and bedrock.

These finite element formulations are used to simulate flows at the ice-rock interface. The use of the Laplace equation in simulating uni-axial flow is also considered, and further simulations are carried out using this equation.

The results from these finite element simulations are used to consider erosional processes occurring at the glacier bed. The processes of abrasion are considered, and previous models are shown to be physically inconsistent. Cavitation, transiency and heterogeneity are shown to have an effect on clast-bed contact forces, and the local viscosity of ice is identified as being a further controlling variable on abrasion. These results are used to consider the likely development of hummocks of bedrock.

A mass-balance analysis of basal debris is carried out and shown to have an important effect on erosional patterns.

The equations describing the movement of a surface normal to itself are considered. Various solution techniques for these equations are tested, and requirements for the persistence of form under lowering are given.

The modelling strategy used in this thesis is a nested hierarchy, with the various hierarchical levels corresponding to different scales. The effect of this hierarchisation on the modelling is discussed with respect to the generic properties of the systems, explanation and testability.

"You know, some people call me lazy, but I just like to take my time."

Big Joe Turner

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PREFACE

This thesis is concerned with modelling the development of glacial erosional landforms. In general, the modelling strategy used is that of the direct modelling of the physics of erosion and of erosional systems.

Such a strategy requires understanding of the physics of ice and rock, formulation of equations based on these considerations, and the solution of these equations by often complex numerical techniques.

A three-level hierarchisation is used in this thesis. On the lowest level the processes occurring on the scale of centimetres that act to cause abrasion are considered. The processes that mould the medium-scale topography on the scale of metres to tens of metres, and which include aggregate statements about the processes occurring on the lowest level, form the intermediate level of the hierarchy. Processes occurring on the highest level, which act to produce glacier-wide landforms, include aggregate statements from the lower levels of the hierarchy.

In Chapter 1 a review of glacial erosional geomorphology is undertaken, with an emphasis on modelling procedures appropriate to the processes outlined. Following Evans (1972), a distinction is drawn between specific and non-specific analysis of landforms.

Chapter 2 is an outline of the constraints imposed on glacial erosional landform development from purely geometrical considerations. Requirements for steady state landforms are analysed in some detail, and novel formulations for the solution of the equations of landform development due to the author and to Farmer (personal communication) are given.

In Chapter 3 the physics of ice and glaciers are considered. Attention is focussed on the rheology of ice found at glacier beds, and moisture flow effects are also considered.

Chapter 4 considers the numerical solution of the equations of glacier motion by the finite element velocities-pressure formulation. Novel formulations for the sliding boundary condition and ice-water mixtures are given.

Chapter 5 describes a number of simulations of glacial flow situations. It is found that use of a non-linear rheology in the analysis of basal flow, instead of a linear rheology as is often used, can have a profound effect, the least of which is that of scale dependence.

Chapter 6 uses the results of Chapter 5 to consider the process of abrasion. The analyses of Boulton (1974; 1975) and Hallet (1979; 1981) are shown to be based on physically inconsistent models; however the mechanisms these authors propose are probably of sub-glacial importance. Further mechanisms acting to enhance abrasion are explored, including those arising as a result of cavitation on the lee-side of sub-glacial clasts, transiency and heterogeneity beneath the clast.

Chapter 7 considers the processes that mould medium-scale landforms, and finds that they are complex, and perhaps so complex that hierarchisation procedures may not be practicable.

Chapter 8 considers tool-supply by using a mass-balance analysis of tool production and convection. It is shown that even simple models can produce complex erosion patterns.

A more theoretical study of hierarchisation procedures and their influence on testability is undertaken in Chapter 9.

Chapter 10 provides a discussion and summary of the major results of the thesis.

Appendix 1 provides information on the mathematical notation used in the thesis, while Appendix 2 provides a brief introduction to flow in porous media, of relevance to moisture flow within the ice and for flow under the clast. The figures are all in Volume 2. It should be noted that axis notation adopts the convention of defining a dimensionless quantity, e.g. Distance/m.

CHAPTER 1

GLACIAL EROSIONAL GEOMORPHOLOGY

1.1 The science of geomorphology

The word 'geomorphology' suggests the study of the earth's shape. This is true to a certain extent; however, it cannot be said that geomorphologists seek to explain why the earth is an oblate spheroid. Geomorphology is a study of processes and phenomena occurring at a much smaller scale both in time and in space. Its subject is the surface of the earth. This study includes not only the shape of the earth's surface, though this is the pivot of the discipline, but also the influence of the local geology, hydrology, pedology, climatology and ecology on the earth's surface.

On the broad scale the most potent direct geomorphic agents are mechanical processes. For this reason, landscapes are classified according to the nature of the medium that has applied the morphogenetic force. Four morphogenetic mediums exist in substantial quantities on the earth's surface; air, water, rock and ice. This thesis is concerned with the last.

Large localised volumes of ice on the earth's surface are called glaciers. For this reason it is customary to refer to landscapes that are currently glaciated as glacial landscapes, and those that have been as 'glacierised landscapes'. Such areas exhibit characteristic landforms (Sugden and John, 1976). These landforms are separated into those formed by the action of glacial erosional processes (removal of the underlying rock) and those formed by the processes of glacial deposition (deposition of rock fragments by the glacier). This distinction is not always clear-cut.

1.2 Glacial erosion

In this thesis a definition of glacial erosional activity rather more restricted than usual is adopted. Glacial erosion is defined as removal of rock not previously deposited by the glacier. Glacial deposition is defined as release of rock by the glacier when it loses its competence to carry that rock.

The definition of erosional activity being used in this thesis has been so framed as to exclude re-entrainment of debris deposited by the glacier. Since the force required to separate the rock fragment from its parent material is several orders of magnitude larger than the force required to entrain debris, or to entrain loose debris produced by some other process, it follows that processes that cause entrainment may not cause erosion. Indeed, from the point of view of glacial process, re-entrainment and deposition are both aspects of the transportive capacity of a glacier, and in this respect distinct from erosion.

A further consequence of the disparity between the forces required to erode debris and the forces required to re-entrain debris is that, in general, areas where erosional activity is occurring are swept clear of debris by the glacier. For this reason, a clear distinction may often be made between landscapes whose forms are determined by glacial erosional activity and those landscapes determined by glacial depositional activity.

Yet another consequence of glacial entrainment requiring considerably less power expenditure than glacial erosional activity is that very often deposition is, in the main, the last act of a dying glacier; in other words, a glacial depositional landscape is a snapshot of a glacier in its death-throes. Therefore, the fact that a glacier has deposited at a particular point should not be taken as evidence that the depositional activity was representative of steady-state activity.

Glaciers can also erode fluvio-glacial deposits and weathered rock. Erosion of the former can be regarded as re-entrainment as the strength of these deposits is not great. Weathering can produce rocks of widely varying strength, ranging from the very soft and loose (e.g. soils) to rock which has only been slightly weathered and is therefore of the same hardness as the unweathered rock. This fact means that there is not a rigid distinction between erosion and entrainment.

The sub-systems shaping the surface of the earth, that is the local hydrology, pedology, climatology and ecology, are, for glaciated areas, somewhat simpler than those affecting other geomorphic activities. Soils do not have sufficient time to form under actively eroding glaciers. The ecology is probably restricted to a few bacteria, if anything. The micro-climate at the ice-rock interface is very constant (Battle, 1960; Vivian, 1970). Since the study is restricted to glacial erosional activities, geology enters into the problem exogeneously in general, though it is believed that glacial erosional activity does affect joint patterns in the bedrock (Sugden and John, 1976).

Of the list of sub-disciplines related to geomorphology given at the beginning of the preceding section, this leaves hydrology as the only aspect not considered. In this sense, hydrology means how the shape of the earth affects the passage of water (in whatever phase) across it and through it. This is indeed the crux of glacial erosional studies; how does the shape of the bedrock affect the flow, and in consequence, the erosional activity of the glacier? An equivalent statement is that the geometry of the earth's surface is of dominating interest.

1.3 Introduction to glaciers

A glacier is defined as a flowing body of ice on the land. This definition, then, specifically excludes floating bodies of ice such as the Ross Ice Shelf or the Arctic ice. Further classification of glaciers can be effected by several different sets of criteria.

The first distinction usually drawn is on the basis of the geometry of the ice mass. On the one hand there are ice sheets, which tend to the circular in plan view. Their areal extent is predominantly constrained by the plastic properties of ice and the glacier mass balance. Valley glaciers, on the other hand, are ribbon-like in plan view, having a length that is appreciably greater than their width. The longitudinal extent of valley glaciers is controlled by the mass balance gradient and the plastic properties of the ice, whereas the lateral extent of these glaciers is controlled by the relief of the valley walls. An exception to this rule is the cirque glacier, which tends to the circular in plan view, but is, nonetheless, a valley glacier.

Another distinction commonly drawn is on the basis of the thermal regime of the glacier. A glacier may be at its melting point more or less everywhere. In this instance the glacier is classified as being temperate. Alternatively, the glacier may be predominately below its melting point, in which case it is termed cold.

The motion of glaciers is separated, somewhat arbitrarily, into two components, internal deformation, or shearing, and slip over the bedrock. The internal deformation occurs because ice creeps in response to applied deviatoric stresses. Slip along the bed occurs when the ice is not frozen to the bed, that is when the basal ice is at its pressure melting point.

It can thus be seen that the thermal regime of a glacier has an important influence on its dynamics. If the glacier is cold, then it will be frozen to the bedrock, and sliding can take no part in the glacier's motion. If, on the other hand, the glacier is temperate, then sliding can and does become a substantial part of the motion of the glacier.

Thermal regime also influences creeping of the glacier ice because the viscosity of ice is temperature-dependent (Hobbs, 1974). The relationship is an Arrhenius-type relationship (viscosity is exponentially related to the negative inverse of temperature). Therefore, the viscosity of ice increases as temperature decreases. To put this another way, the ice gets 'harder' as the glacier gets colder.

It can be seen that in general one would expect cold glaciers to flow more slowly than temperate glaciers, as not only do they not possess a sliding component to their motion, but also the ice is stiffer.

Thermal regime also affects the erosive potential of a glacier. As will be seen, many glacial erosional processes require that the glacier be sliding over its bed. Thus, a cold glacier will not have the same erosive potential as a temperate glacier.

1.4 Specific and generic landscape geometries

The landscapes produced as a result of glacial erosional activity are scale-specific, that is their properties vary with the scale of viewing. This apparently contradicts the views of Linton (1963), so a word of explanation is required here. Linton noted that on a scale of metres and on a scale of hundreds of metres, landscapes were asymmetric with respect to the direction of ice flow, and in this sense only he suggested that glacial landscapes were scaleless. If, however, all scales are considered, this is not true. If it were, a glacial surface could not be polished, and it would for example, be common to see small hummocks on the backs of large hummocks.

Because landscapes are scale-specific, one can identify certain units of landscape, for example glacial steps, glacial hummocks, glacial valleys. If the landscape were scaleless, no such meaningful 'individuals' could be identified. Evans (1972) points out that there are two complementary ways of looking at landscapes, the specific and the non-specific.

Specific geomorphology concentrates on identifying units of landscape; for example valleys, barrier bars, dunes, whaleback features. This is the more traditional form of geomorphology. It is of interest because it is natural to human beings to identify and explain patterns. This indeed is the strength of specific geomorphology, in that it seeks to explain phenomena readily observable to humans. The weakness of this approach is that by identifying units on a particular scale, the relationships between both the processes and forms occurring on the scale in question and other scales tend to become obscured.

Generic geomorphology, on the other hand, concentrates on the landscape as a whole, and in particular, the way the properties of the landscape vary with scale. The strength of this approach is that it seeks to examine the relationship between processes occurring at different scales; the weakness is that predictions regarding units are not made.

Generic geomorphology is certainly less well-developed as a discipline than specific geomorphology, and for this reason this thesis tends to concentrate on issues in specific glacial erosional geomorphology. However, some attention is paid to interactions between processes occurring at different scales, in particular in Chapter 9.

1.5 Specific glacial erosional geomorphology

Specific geomorphology is concerned with identifying and explaining the genesis of units of landscape. In general, these units are identified by the pattern-recognising capabilities of the human eye and brain. Whether this necessarily means that these patterns in the landscape necessarily reflect some kind of organisational process within the morphogenetic process is an open question; nor should it be assumed that these morphogenetic patterns will be readily discernible.

The question of context is important in defining the geomorphic unit. It must always be borne in mind that the unit exists because of scale-dependence. If a landscape is scaleless, then no patterns can exist. Examples of scaleless, patternless landforms may be found in Mandelbrot (1982). A consequence of this is that a unit exists because of changes in the scale-dependent phenomenology of the landscape at scales immediately smaller and immediately larger than those of the unit.

The smallest units of glacial erosional phenomena are striation marks (Sugden and John, 1976). The relevance of scale context is immediately apparent here, because if glacial landscapes were not scaleless (in this instance the flatness caused by polishing), then striation marks, would, ipso facto, not be the smallest units. Striation marks are grooves, generally of the order of a millimetre wide and deep, which can extend centimetres or even metres in the direction of the ice flow. Striation marks are caused by the gouging actions of tools being dragged across the bedrock by the moving glacier. A related phenomenon is that of chatter-marks. These are crescentic gouges perpendicular to the ice flow caused by the stick-slip motion of a clast across the bedrock (Gilbert, 1906).

A phenomenon occurring on the scale of metres to kilometres is that of glacial hummocks. These are given various names such as roches moutonnées, whaleback forms, or crag and tail features depending on their size and exact shape (Sugden and John, 1976).

Glacial hummocks tend to be longer longitudinally (with respect to the generalised glacier flow direction) than they are across. Their heights are less than their widths by factors of between approximately two and ten. In general, they have a gentler slope on the upstream side than on the downstream side. This phenomenon is manifested to an extreme in the case of roches moutonnées, where the downstream side is often near vertical with a shattered surface, in contrast to the smooth surface generally exhibited by glacial hummocks. It is relatively uncommon for a hummock to have a smaller hummock on its back. To a certain extent, this is a geometrical problem, because hummocks smaller than about a metre do not tend to exist. Therefore, a hummock has to be relatively large before the geometry of the problem permits it to have a smaller hummock on its back.

A more general observation is that smaller hummocks tend to exist in groups, whereas larger hummocks tend to be on their own. Again, this may be to a certain extent a problem of geometrical context (Evans, personal communication) as, within a glacial valley, there is not sufficient room for a field of large hummocks to develop.

The shape and size of smaller hummocks is very often constrained by jointing of the parent rocks (Addison, 1981; Rastas and Seppälä, 1981). The troughs, where the rock has been eroded away the most, follow the pre-existent jointing. Jointing dipping sub-parallel to the generalised glacier flow direction also has a significant influence on the slope of the upstream face of the hummock.

The shattered lee-sides of roches moutonnées indicate that here a different erosional process to that eroding the upstream and lateral faces of the hummock has occurred. It would seem reasonable to assume that the process causing this shattering only occurs once the hummock has been formed, as hummocks often exist without this down-stream asymmetry. However, it should be noted that the smaller the hummock, the more likely is it to have a shattered lee-side, and it is readily possible that these smaller hummocks are caused by a different set of processes than those causing the larger hummocks. This possibility becomes more conceivable when the point made above regarding hummock size and population is born in mind.

A further point worthy of consideration is whether hummocks are stable phenomena or whether they are transient phenomena. By stable form is meant a form that tends to persist through time; in this context, time is equivalent to generalised lowering of the landscape. This implies that the form retains more or less the same size and shape, though by this definition the form is permitted to migrate upstream or downstream. In principle, therefore, any particular hummock could be traced back to a pre-existent heterogeneity in the landscape.

A transient form, on the other hand, is one that tends to wax and wane with the generalised lowering of the landscape. For such a form to be called transient, one might expect this waxing and waning to occur during of the order of ten hummock heights of generalised lowering or less.

Since glaciers can abrade at the rate of mm/a (Østrem, 1975) and a hummock is minimally of the order of a metre tall, complete extinction of a hummock would take a minimum of one thousand years. Observations under active glaciers have not been going on long enough to confirm or deny that hummocks are transient phenomena. However no hummock has been reported to be abraded away.

More circumstantial evidence coming from the characteristics of hummock populations suggests that they are stable phenomena. This is because within a given group of hummocks, shapes and sizes tend to be rather similar. If hummocks were transient phenomena, then it might be expected that their populations would show a wide range of variation. This homogeneity is not conclusive evidence because it is conceivable that the waxing and waning phases of a hummock might be very brief compared with the 'full' phase.

Glacial erosional forms often associated with hummocks, but somewhat smaller than them, are glacial grooves. When occurring in association with hummocks, they are found at the base of the flanks of the hummock, and extend more or less along the whole length of the hummock. The size of the groove appears to be related to the size of the hummock, possibly because of geometrical constraints. A hummock a couple of metres or so high might be expected to have associated grooves ten centimetres or so wide. Larger hummocks have been seen to have grooves of over a metre in width. In general, these grooves do not seem to have grooves within them, though the giant Kelleys Island grooves (Goldthwait, 1979) are a noticeable exception.

The long profiles of glaciers exhibit characteristic forms. Compared with river valleys, glacial troughs are over-deepened, that is their upstream reaches are lower than those of a hypothetical equivalent river valley. Moreover, the long profiles exhibit asymmetric undulations on the scale of hectometres or kilometres known as glacial steps. These steps have steeper slopes on the side of the glacier facing downstream than on the side facing upstream. The question as to whether these forms are transient or stable phenomena is not so important as in the case of glacial hummocks, because the amplitude of these undulations is often comparable in size to the total lowering of the land surface by the glacier.

The final unit of glacial erosional geomorphology to be considered here is the glacial trough cross-section. This is, of course, the 'U-shaped valley', so called in contradistinction to the V-shaped valley typical of fluvial erosional activity. The U-shaped valley is not a valley in the same sense as a river valley, but rather it is a channel, and if a comparison has to be drawn, it should be compared with the shape of a river channel (which tends to be U-shaped).

Even loosely speaking the trough is not U-shaped in reality (Sugden and John, 1976). When the trough is symmetrical, which it frequently is not, it can be characterised by a generalised parabola (i.e. a curve of the form $y = x^n$, where n is not necessarily equal to 2) (Svensson, 1959; Graf, 1970). Variation in the value of n permits the shape of the trough to vary from a 'V' to a near rectangle. It appears to be a general rule (but by no means universally true) that glacial valleys that have undergone the least erosion tend to be more V-shaped compared with those that have undergone more erosion.

There is no definitive evidence as to whether the troughs reach a steady state form. However, since the trend appears to be from V-shaped to U-shaped, and it is very rare to find a trough with a completely flat bottom, it is reasonable to infer that some kind of a steady state form is reached.

A glacial erosional unit of particular interest is the cirque, which is a rounded hollow with a lip. These can exist in isolation, or form the head of a glacial valley. Their most distinctive feature is the headwall, that is the highest point of the valley. This headwall tends to migrate uphill by mass-wasting and plucking. This migration apparently stops as soon as the headwall gets close to either the top or a ridge of the mountain into which the cirque is incised (White, 1970).

1.6 The generic phenomenology of glacial erosional landforms

Generic phenomenology is the description of a phenomenon as a whole. Since it avoids identifying specific units, which can be given a name, generic phenomenology tends towards adjectival description; for example, such words as "hilly" and "undulating" may be used to describe the generic aspects of a landscape. Attempts to classify the landscape further must rely upon quantitative description.

Evans (1972) has discussed general geomorphometry, and proposed various generic measures of a landscape. These mostly relate to statistical distributions of such parameters as slope, curvature, aspect, etc.

Another important aspect of general geomorphometry is the scale dependence of landform. Thus, a landscape composed of many hillocks of roughly the same size might well have similar distributions of some parameters as a landscape composed of lots of hills and hillocks of different sizes all superimposed upon one another, yet it would appear markedly different to the observer. In the former case, all the variation would be at or around the scale of the hillocks, while in the latter case the range of scales over which there was appreciable variation would be much larger.

Quantitative measures of the generic phenomenology of glacial erosional landforms require numerical data. The altitude matrix approach of Evans (1972) which was followed by Gill (1982, unpublished) in his simulations of glacier mass-balance would seem a viable approach. However, the altitude matrices produced by these workers generally use a grid of 100m, which, as will be seen, is rather too broad for generic measures of geomorphometry. Data at a sufficiently fine resolution do not appear to be available, and for this reason no quantitative attempts to model the generic evolution of a landscape are made in this thesis.

1.7 The processes of glacial erosion

Glacial erosion has been defined previously as the removal of rock fragments from the lithified parent material by the action of the glacier. Many processes effect the removal of such fragments, and these processes act over a wide variety of scales.

A certain dichotomy exists in geomorphology between the concepts of process and form. That the two are linked is central to geomorphology; however, this leads to problems in categorisation. Forms can be defined as forms per se, in which case very little light is cast on their genesis by their system of nomenclature. Alternatively, they can be defined in terms of the processes which are thought to induce them, in which case it is possible that forms which appear to be very different could be given the same name. Either of these approaches is equally valid.

Exactly the same problem arises with respect to the categorisation of process. It has been customary in glacial erosional geomorphology to define processes in terms of the forms they are supposed to produce. Because glacial forms are scale-dependent this can mean that processes which are mechanically indistinguishable are given different names depending on the size of the phenomena they produce. An obvious case in point is that of erosion arising from a tool dragged by moving glacier ice being variously described as scratching, gouging or crushing depending on the size of the tool. On the other hand, defining a process purely mechanically does not inform as to the predominant scale at which this process is acting.

Since this thesis is mainly concerned with process, in general process-oriented definitions will be used. However, for completeness and for cross-reference, two sets of categorisations of glacial processes are given below, one form-oriented, the other process-oriented.

1.8 A categorisation of glacial erosional processes based on process

(a) Abrasion is failure of the bed due to stressing caused by an individual clast. This definition includes the small scale grinding process formally classified as abrasion. In addition, it includes such processes as striating, chatter-marking, the shearing-off of asperities of whatever size by boulders and possibly polishing. In general, because sub-glacial clasts are rarely larger than a few centimetres in size this process is a small-scale process. Currently, the most controversial aspect of abrasion is how the force between clast and bed should be calculated. This topic is discussed in more detail in Chapter 6.

(b) Dynamic stressing failure of the bed caused by stress configurations in the bedrock attributable to the fact that the ice is flowing. Essentially, the process occurs because of the contrast between the high normal and tangential forces imposed on the upstream side of a bump, and the low, hydrostatic stress field on the lee-side of the bump. The resultant of the forces is a moment about the base of the lee-bump which is sufficient to induce fracturing. This process differs from the previous process because it could be caused by a glacier bearing no clasts. This process was first proposed as a geomorphic agent by Boulton (1974) and further discussed by Morland and Morris (1977).

Morland and Morris's investigation suggests that this process requires the incipient formation of a cavity on the lee-side of a glacial hummock for it to be a potent geomorphic agent. For this reason most of its work is done on the small to medium scale, because glacier dynamics are such as to preclude the formation of cavities on the lee-sides of large hummocks (see Chapter 3).

(c) Pressure release is fracturing of the bedrock due to changes in the ambient stress field caused by removal of material by the glacier. This process tends to act on the large scale because large changes in the over-burden pressure are required to induce stressing sufficient to fracture the rock.

(d) Corrosion is chemical attack on the bed.

(e) Meltwater erosion is erosion of the bed caused by the action of meltwater flowing underneath the glacier. This erosion is believed to be caused by blasting of the channel walls by the entrained debris, which possesses a high silt fraction. It is not known, however, which fraction of the debris entrained by water does the most work.

(f) Thermal stressing is erosion caused by thermal fluctuations. A certain element of this wear may be actual fatigue directly attributable to these fluctuations, but it is held that the major component of thermal erosion is freeze-thaw action, which will occur in those parts of the ice-rock interface subjected to thermal fluctuations. This requires that the ice-rock interface be near the pressure melting point.

1.9 Definition of glacial erosional processes based on form

The traditional approach to the categorisation of glacial erosional processes has its basis in the forms these processes are believed to produce. Since forms, by their very nature, manifest on a particular scale, these processes tend, therefore, to be defined on a particular scale.

(i) Polishing is that process which produces the small scale smoothness associated with glacial erosional landforms. Since glacial surfaces are generally smooth, this implies that polishing is a significant component of glacial erosion.

(ii) Striations and chatter marks are glacial erosional forms caused by clasts being dragged along the glacier bed by the moving ice. These processes are unlikely to be major components of glacial erosion. Striae have been observed to form under modern glaciers (Boulton,1974), and are often associated with glaciated landscapes, e.g. Gray (1982).

(iii) Plucking is that process which induces, for example, the rough lee-sides found on roches moutonnées. Shattering tends to occur on the lee-sides of smaller hummocks which are those most likely to have downstream cavities (see Chapter 3). Thus plucking may represent the development of surfaces in a glacial environment which are not subject to abrasion. This correlates with the observation that plucking occurs on large, steep, valley steps (Sugden and John, 1976).

(iv) Meltwater erosion is the process responsible for pot-holes sometimes observed in glacially affected surfaces. Some authors believe it to be responsible for glacial grooving, and it may be responsible for some polishing.

(v) Pressure release is believed to be responsible for some of the jointing often found in glacial troughs. Pressure release occurs when the glacier excavates a trough and releases the overburden pressure. However, further pressure release occurs when the glacier wastes away, and it is not clear how much of the jointing seen today is contemporaneous with the period of active glacial erosion and how much has occurred subsequently.

1.10 Rock-rock wear processes

In this section the wear processes induced when rock is slid over rock are considered. The discussion of wear processes here is not intended to be comprehensive. The study of wear is still in its infancy, and, as such, is in a state of flux. The studies of Riley will be summarised, and discussed in light of the simple models of Rabinowicz.

Riley (1982, unpublished) appears to have been the only recent worker to have studied rock on rock wear. His experiment consisted of grinding a cylinder of fine sandstone (grain-size

= 0.5mm) round a circular path on another block of sandstone. He simulated sub-glacial conditions by conducting the experiments at around freezing point, sliding the cylinder at around 10m/annum and letting the normal stress be about 1bar. A major difference between his experimental conditions and typical sub-glacial conditions was the absence of water.

Riley's carefully executed experiments showed two to three orders of magnitude of scatter in the measured wear. Wear appeared to be related to distance slid by the tool by an equation of the form

$$w = w_{\infty}(1 - e^{-d/b}) \quad 1.1$$

where w is the wear, w_{∞} is the final wear, d is the distance slid and b is a constant. The final wear value, attained when the interface became choked by debris, was variable. Riley attempted to relate the initial wear rate, which again was very variable, to fracture toughness and to the penetration hardness. He found a fairly strong negative relationship between the initial wear rate and the fracture toughness, but no apparent relationship between the initial wear rate and the penetration hardness. The constant b was found to be of the order of 1m, meaning that half the total wear was carried out within less than a metre of distance slid.

Riley also tried wearing ice cylinders on rock, which produced no discernible wear, and gypsum on gypsum, where, after a brief period, wear ceased.

Rabinowicz (1965) postulated four wear mechanisms.

(i) Adhesive wear, which occurs when smooth surfaces are slid over one another, and particles from one surface adhere to the other surface due to chemical bonds of one kind or another.

(ii) Corrosion, or chemical attack.

(iii) Abrasive wear, which relies on one surface being harder than the other. Then asperities in the hard surface can indent the soft surface and plough grooves in the other.

(iv) Surface fatigue wear, which occurs after repeated stressing and destressing. It can only occur in the absence of other types of wear, since they remove material before it can be fatigued.

Adhesive wear occurs because surfaces exhibit attractive forces. Adhesive wear is known to be important in metals (Rabinowicz, 1965), but it is not clear how important adhesive forces are in rocks. The only generalisation that can really be made is that since the chemistry of rocks is rather more varied than the chemistry of metals, processes are likely to be more complex.

For adhesive wear to occur, there must be a weakness within the material such that failure occurs there rather than at the interface. These weaknesses are unpredictable, so a probability model is developed by Rabinowicz. He obtains an equation of the form

$$w = kNd/3p \qquad 1.2$$

where w is the wear, N is the normal stress, d is the distance slid, p is the penetration hardness and k is the probability that failure will occur within the material rather than at the original interface. The factor of one third arises as a result of assumptions about the geometry of the contact.

Rabinowicz also presented a dynamic equilibrium model for the transfer of adhered particles between surfaces. He showed that if the transfer probabilities are independent of the physical environment, then the weight of adherent particles at equilibrium is also independent of the physical environment. At equilibrium, then, net erosion has ceased, and the amount of material that has been removed is independent of the physical environment. In principle, tools can trace out long paths on the glacier bed. All else being equal, it is possible that if adhesive wear were a dominant mechanism, equilibrium could be achieved and net removal of material from the bed could cease. It should be noted that this model predicts wear of a hard bed by a soft tool as well as the other way round.

Riley explains his results for the wear of gypsum on gypsum in terms of an adhesive wear model. Since gypsum is a highly deliquescent salt, one would expect it to exhibit strong surface forces.

The process termed by engineers as abrasion requires a hard abrasive which indents into the abraded material. As the abrasive is swept along it ploughs out material in its way. This, of course, is how striations are formed. Rabinowicz derives the following wear equation:-

$$w = \overline{\tan\theta}\ Nd/\ p \qquad 1.3$$

where θ is the angle which the side of indenter, modelled as a cone, makes with the horizontal and the other symbols have the same meaning as in equation 1.2. It should be noted that this equation and the preceding one predict exactly the same form of relationship for adhesive and abrasive wear.

On a gross scale, Riley's tool and bed were of the same hardness, so no abrasive wear should have occurred. However, given the heterogeneity of Riley's sandstone, it is easy to envisage a large, firmly held grain gouging out or crushing other grains. Crushing would produce smaller fragments, perhaps of the silt size so characteristic of glaciers. Smaller fragment should also be produced by adhesive wear between individual grains. It should be noted that Riley used a sandstone noted for its isotropy and homogeneity, and thus his results should, relatively speaking, be readily interpretable.

Mechanical analysis of the situation, though feasible, presents certain difficulties. The surfaces need to be accurately described, and rheologies have to be provided for all the constituent minerals. These rheologies will have to cover a wide range of conditions. It is easy to envisage a numerical analysis of the situation demanding large computing resources. To permit extrapolation of the results, a large number of numerical experiments would need to be executed; at present this is feasible, though somewhat impractical.

Boulton (1974; 1975) has observed the sub-glacial action of abrading tools. He confirmed that the ice does indeed rub tools over the base, and he noted that these tools carry debris with them that damps out the sharpness of the asperities. Riley's experiments may over-estimate the choking effect of debris. Water, even in small quantities, can flush away debris (Rabinowicz, op. cit.). Riley's experimental conditions, with two flat surfaces rubbing against one another, favour the build up of debris. In general, this kind of situation would not be found under glaciers, as both bed and tool are irregular. This circumstance also makes the attainment of adhesive equilibrium unlikely.

Riley's results do not agree with the simple Rabinowicz models in that abrasion decreased with distance slid. Riley did not, however, systematically vary the normal loading, or the sliding velocity, and thus the dependence of wear on this variable remains unknown for glacial abrasion.

Let us assume, however, a Rabinowicz type relationship i.e.

$$w = R_1 N d$$

where R_1 is a constant.

Differentiating, we obtain

$$a \propto Nv \qquad 1.4$$

where a is the abrasion rate and v is the sliding velocity.

Following Riley and differentiating 1.1 we obtain

$$a \propto v e^{-d/b} \qquad 1.5$$

and we thus obtain the relationship from 1.4 and 1.5

$$a = RvNe^{-d/b} \qquad 1.6$$

where R is a constant.

We define the wear coefficient γ by

$$\gamma = Re^{-d/b} \qquad 1.7$$

giving, from 1.6,

$$a = \gamma N v \qquad 1.8$$

The effective normal stress is defined by

$$AN = F_s - (1 - A_c)p_w$$

where F_s is the force normal to the bed imposed on the top of the clast, p_w is the water pressure underneath the clast, A_c is the area of contact between clast and bed and A is the apparent area of contact between clast and bed. This equation can be divided by A to give

$$N = \sigma_s - (1 - \varphi)p_w \qquad 1.9$$

where σ_s is the stress directed normal to the bed on the clast and φ is the effective contact area coefficient.

It can be seen from equation 1.9 that if the mean stress applied to the top of the clast is equal to the pressure beneath the clast, wear will occur, especially when the effective contact area is high. Hallet (1979; 1981) makes the assertion that wear will only occur when there is a difference between the pressures above and below the clast. However, the presence of any contact area will lead to friction, and our general experience is that friction leads to wear.

No sub-glacial observations of area of contact have been made. For a large clast with no debris between it and the bed (Riley's two-body problem), the contact area coefficient will be very low, as contact will be restricted to a few asperities. When there is a lot of debris between the clast and the bed

(Riley's three-body problem), contact area coefficients will be higher, especially when it is noted that Riley found ... clay-sized minerals in the debris. Whether these small particles would occur under sub-glacial clasts is not known, because, as Riley points out, they may well be washed out by the action of sub-glacial water.

Since Riley attributes some of the decline in the abrasion rate to the build up of debris, there must be some kind of relationship between the effective contact area coefficient ϕ and the wear coefficient γ. Its precise nature has not been determined.

The values of the stress applied by the ice to the clast and the water-pressure underneath the clast evidently also affect the abrasion process. These unknowns are discussed in detail in Chapter 6. To model them requires the determination of the ice-flow about an obstruction. The physics of ice are considered in detail in Chapter 3, and the solution of the equations of ice-motion in Chapters 4 and 5.

1.11 The mechanical properties of rocks

It is impossible in a short section such as this to give anything but the briefest sketch of the mechanical properties of rocks. Thus the aim of this section is to summarise areas of knowledge and ignorance pertinent to glacial erosional studies.

The properties of rock are functions of the spatial and temporal scales of viewing. Thus rock can be considered as to be elastic, plastic and viscous or some combination of these. Viscosity and its attendant phenomenon of ductility are associated with high pressures and temperatures (Jaeger and

Cook, 1976). These conditions do not occur at the surface of the earth, except transiently, so rocks will be treated as elastic brittle materials. It is not valid to treat rocks as homogeneous at all scales of viewing, nor is it valid to treat them as continua. Natural rocks are pervaded by crack and joint systems which may range in spacing from the order of metres to the microscopic (Jaeger and Cook, 1976). Though the width of these joints may be negligible in relation to the spacing in between them they allow discontinuities of strain within the rock which makes consideration of the rock as a continuum, strictly speaking, invalid.

However, to model the response of rock to applied stress it is convenient to treat it as a continuum. Preferably the properties of this continuum should reflect the influence of jointing in some manner. To obtain these properties either an empirical approach or a synthetic approach can be adopted. It is not difficult to see that measurements on large blocks of rocks of the size of hundreds of metres is fraught with difficulties. Also, care must be taken to obviate any influence on the properties of the rocks from the size of the mineral grains.

As with analysing the wear situation, it is possible to consider numerical experiments involving models of joint structure. In the extension of this to a continuum model similar problems would arise. A large number of numerical experiments would have to be performed, and the problem of deriving pertinent statistics to describe the joint structures would arise.

Any synthetic model that attempts to describe the behaviour of jointed rock must use some assumption about the response of the joint to stressing. As Morland (1974) says, "The actual mechanics of joint slip or shear, and compaction or dilation, are complex, depending on the mineral filling of the joint layer and gouge generated by local crushing and sliding, and on the presence of pore fluid". There is no reason to assume that the stress-strain relationship will be simple rather than complex.

Any model that represented the behaviour of a jointed medium as an elastic continuum might be expected to represent it as an anisotropic, non-linear material, which would cause computational difficulties yet without any real guarantee of correspondence with reality.

In erosional studies it is obviously of importance to be able to predict when the rock is going to fail, and also important to predict the planes along which failure is going to occur, to be able to predict morphological development. The actual mechanics of crack development are not really understood, and while there are several failure criteria, it cannot be said they represent experimental results very well (Jaeger and Cook, 1976). For these reasons no specific predictions of morphological development caused by failure can be, or are, made in this thesis.

The situation is not quite irretrievable however. It is possible to predict the lines along which jointing (distinguished here from faulting in the same sense as Price (1966)) are likely to occur. These are, of course, perpendicular to the direction of greatest tension. The actual orientation of joints is obviously of importance in determining morphological development.

It will also be seen to be of interest, in the following two sections, to be able to predict joint spacing. While at present it is not possible to be able to predict absolute joint spacing, it does appear possible, at least on an order of magnitude calculation basis, to be able to predict relative joint spacing. Price (1966) suggested that joint spacing is proportional to the strain energy, and presented some results from the field which appear to support this hypothesis.

The importance of Price's notion lies not in being able to predict the joint spacing in different lithologies as Price did, but in being able to predict, given a constant lithology, the relative joint spacing of different erosional mechanisms.

1.12 Plucking

Plucking has been defined in section 8 of this chapter as failure of the bedrock due to stressing caused by the dynamic glacier ice. While this section will be devoted predominantly to this mechanism, consideration will also be given to other mechanisms of lee-side shattering and entrainment, as these correspond to the morphological definition of plucking.

Plucking was originally thus named because it was believed that ice frozen to the bedrock actually plucked lumps of rock out of the unjointed parent material. However, the strength of the ice-rock bond is an order of magnitude lower than the tensional strengths of most rocks (Hobbs, 1974; Jaeger and Cook, 1976; Glen and Lewis, 1961), so this process can be rejected as a mechanism of shattering, though not as a mechanism for entraining partially or totally loosened blocks.

The other mechanism invoked to explain lee-side shattering is freeze-thaw. This is not a failure process, but a joint exploitation mechanism. The actual jointing, which conveniently happens to be of the right spacing, is supposed to have been provided by some other mechanism. Freeze-thaw has also been invoked to explain erosion at cirque head-walls.

The freeze-thaw mechanism implies reasonably frequent temperature variations. Measurements by Battle (1960) inside a bergschrund and Vivian (1970) in a cavity to the lee of a roche moutonnée indicate that air temperatures are nearly constant; also, with Vivian's observation it is possible to argue that the act of measurement affected the result, because it connected the sub-glacial cavity to the atmosphere by subterranean shafts, so increasing the variance. Since no data are available on the rate of erosion attributable to freeze-thaw action, it will not be incorporated in any model.

Sugden and John (1976) invoke several mechanisms that joint rock so as to make it susceptible to freeze-thaw action. Two of them correspond to processes defined mechanically in this thesis, plucking and pressure release. Both of these processes will be discussed shortly. They also hypothesise that rock can be broken up by ancient, deep, chemical weathering and also by periglacial action prior to the glaciation. At present, however, not enough is known about these models to incorporate them in any self-contained morphogenetic models.

There remains to discuss the process of dynamic stressing failure as defined in Section 8 of this chapter. THe original suggestion that the flow of ice over a bedrock hummock could induce failure in the hummock was made by Boulton (1974, 1975) and by Morland and Morris (1977). Using idealised models for the rheology of ice and the interaction of ice with the bed they calculated the traction exerted by the moving ice on the hummock. Using these tractions as boundary conditions for a linear elastic problem, they were able to calculate the stresses in the hummock. In certain circumstances they found that the stresses induced were sufficient to cause failure. Failure is favoured at the moment of separation of the ice from the rock on the lee-side of the hummock, when the pressure in the cavity is atmospheric. These investigators did not allow the ice to exert any tangential traction on the bed. It is likely that any tangential traction would increase the moment of the hump around a pivot at the base of the lee-side, and thus increase the stressing.

A problem is that there are no data on the relative importance of lee-side shattering and abrasion as agents of erosion though the importance of plucking as a source of tools is considered in Chapter 8. It could be that apparent lee-side shattering is a reflection of the fact that this is an area where little or no abrasion occurs owing to the infrequent contact between ice and bed.

1.13 Pressure release

Pressure release was defined in section 8 as failure of the bed due to stress concentrations caused by morphological change of the bedrock in directions perpendicular to the (generalised) ice flow. Though this is a fairly general definition, it is, as its name suggests, meant to refer to the failure of rock caused by unloading due to removal of the overlying rock by erosion.

It is possible to envisage many different stress histories for the landscape, all of which will be reflected in the state of stress of the rock. Suppose that initially a flat surface of land is lifted to form a plateau. The process of uplift is not reflected in the state of stress of the rock. The climate at this time is non-glacial, so a river cuts down into the land surface, removing rock and reducing the overburden pressure on the underlying rock. The initial state of stress in the rock is hydrostatic . As the overlying rock is removed, the overburden will become dependent on horizontal position as well as vertical position, and the state of stress will no longer be hydrostatic. This state of stress may be sufficient to induce failure of the rock. This is the mechanism referred to as pressure release, though the actual hydrostatic pressure plays no real part in causing failure.

The situation will change if a glacier comes to occupy the valley. Firstly, the presence of the glacier will increase the overburden pressure. A self-contained morphogenetic theory can be developed if it is assumed that the glacier can only remove material that has been broken up by pressure release. Then material can be removed as it is broken up, and the kind of valley cross-section that evolves can be observed. Unfortunately, it is not quite that simple. One can assume that the rate of entrainment of debris is very much greater than the rate of elastic response to changes in the stress field, so that all the loosened blocks are removed immediately. This assumption is obviously rather unrealistic, but if it is not made, then one has to assume a stress-strain behaviour for jointed rocks, and also some kind of relationship between the physical environment and the rate of entrainment. Incorporating the latter assumption means the theory begins to lose its self-containment, because the dynamics of the glacier now have to be taken into account.

If it could be shown that the effect of these assumptions was insignificant, and that the state of stress left by the pre-glacial tectonic and erosional history was similarly not of importance, then this model would be a self-contained morphogenetic theory. If one supposes a glacier trough to be a semi-circle at the bottom of the valley sides, then this is the shape that minimises the curvature of the line joining the two valley sides. Shapes that avoid high curvature are those shapes that are needed to avoid stress concentrations in the rock.

In the preceding section it was stated that pressure release was one of the mechanisms invoked to explain the shattering of rock so that it was susceptible to the effect of freeze-thaw. It is not difficult to see that the efficiency of the pressure-release mechanism as such a preparative agent depends on the spacing of the jointing associated with the mechanism. If it is on a scale an order of magnitude less than the wavelength of the bedrock surface, then it will likely be of importance in the erosion of these hummocks. If it is on a scale comparable with the wavelength of the bedrock, then the jointing could be invoked as an explanation for why the bedrock has a particular wavelength. If the jointing is considerably larger than the wavelength of the bedrock, then it cannot be invoked as a primary erosive agent. As was stated in the section on the mechanics of rock, it is not possible to predict the absolute spacing of joints, though it is possible to compute, for a given lithology, the effect of different stress fields on the joint spacing (see section 11). Thus, it should be possible to compute the relative joint spacing arising as a result of pressure release and dynamic stressing failure.

1.14 Modelling the development of glacial erosional landforms

Modelling the development of landforms is an attempt to elucidate the manner in which landforms are transformed by the processes of erosion and depostion. For glacial processes, these are in the main affected by the local mechanical situation, which is in turn affected by the landform. Modelling the development of landforms therefore requires assessment of the erosion and deposition rates based from the mechanical situation produced by a given landform and climate, computing the resulting change in the landform, and then recalculating the erosion rates for the new landform.

Such a strategy requires understanding of the physics of ice and glaciers. These matters are considered in Chapter 3. It is found that the physical behaviour of flowing ice may be described by sets of partial differential equations. Techniques for the solutions of these are discussed in Chapter 4, while the results from some actual solutions are presented in Chapter 5.

The first local erosion theory in glacial erosional studies appears to have been proposed by Boulton(1974; 1975). This theory is discussed in Chapter 6. Boulton related the abrasion rate to the local mechanical situation, and computed erosion rates as a function of position. This procedure was carried out for medium-scaled landforms (e.g. hummocks in the bedrock) and also for large scale landforms (e.g. glacial troughs). Boulton suggested that his relationship defining the erosion rate was independent of scale. However, in the derivation of his erosion equation, he assumes that the glacier bed is flat. While this is true on the small scale, this is not true for the development of glacial troughs, where the bed is often quite rough.

Hallet (1979; 1981) developed a local erosion theory using a different model for the relationship between the local erosion rate and the mechanical situation. This is also discussed in Chapter 6. He also investigated the effect of such factors as bed roughness and clast concentration on average lowering rates for heterogeneous landforms.

Neither of these two models are based on consistent physical formulations, and they are therefore not used in this thesis. Instead, in this thesis, an implicit hierarchisation is made. In Chapter 6, abrasion theories are considered, in an attempt to produce a relationship between the local erosion rate and the local mechanical situation. In Chapter 7, these considerations are applied to try and predict erosion patterns around sub-glacial hummocks, and to determine what the overall lowering rate of a rough landform might be. In Chapter 8, consideration is made of how large scale variations in the erosion rate might affect large scale landform development.

One assumption made in this hierarchisation is that erosion consists of a continuous component and a series of discrete events. Whether a process is regarded as continuous or discrete depends on the scale of viewing. Thus, 'plucking' is a discrete process on the scale of centimetres, but is a continuous process on the scale of kilometres. Erosion is always regarded as being normal to the bed, since gravitational forces do not play a direct role in glacial erosion. This normality assumption may not be strictly true for pressure release. The consequences of the normality assumption are examined in Chapter 2.

The ultimate aim of these studies is to produce an as yet unrealised comprehensive theory of glacial erosion. Modelling the development of glacial erosional landforms requires landscapes at two different times and a set of processes which vary in time and space according to a set of endogeneous and exogeneous parameters. A comprehensive theory of glacial erosion is one that relates the two landforms and the endogeneous parameters given the exogeneous parameters. Alternatively, a comprehensive theory could, given the two landforms and the endogeneous relationships, infer the exogeneous parameters. Testing of theory would require knowledge of the two landforms, the endogeneous relationships and the exogeneous parameters.

There is no guarantee that such a comprehensive theory can exist. Some consideration of this is given in Chapter 9.

CHAPTER 2

THE GEOMETRY OF GLACIAL EROSIONAL SURFACES

2.1 Introduction

It was suggested in Chapter 1 that the geometry of glacial erosional landforms is of fundamental importance in determining subsequent glacial erosional rates. In this chapter the implications of some aspects of glacial erosional geometry are considered briefly, and the geometrical controls on glacial erosional landform development are considered in some detail.

2.2 Inferences about the relative importance of erosional processes from the geometry of glacial erosional landforms.

If a glacially eroded surface is large enough to provide a statistically stable sample, and if there is no volumetrically significant trend in the relative importance of different glacial erosional processes near deglaciation (since most such surfaces are observed subsequent to glaciation), then the relative areas affected by different erosional processes will reflect the relative erosional rates of different processes.

Glacially abraded surfaces exhibit polishing and scratching. Figure 2.1 shows the possible development of such surfaces. The model of abrasion is a continuum of polishing accompanied by a discrete series of scratches. We may then investigate the relative volumetric importance of scratching and polishing. In Figure 2.1 the distance AB is a unit width and BC the polishing effected in unit time. As a simplification we assume all scratches to be of the same size. We then postulate that a certain number of scratches occur per unit time. The heavy line in the two diagrams represents the steady-state (in some statistical sense). The steady-state time for the case of 3 scratches per unit width per unit time is heavily nicked, probably more so than for natural abraded surfaces (in the absence of any numerical data), while the case for 1 scratch per unit width per unit time is possibly more representative of glacially abraded surfaces. These considerations suggest that polishing is a more significant erosional process than scratching.

Similar criteria apply to the relative areas of abraded and shattered surfaces. While care has to be exercised in determining how much original volume a shattered surface represents, the fact that abraded surfaces seem to exist in larger quantity than shattered surfaces in general indicates (again in the absence of numerical data) that abrasion is the dominant process, unless abrasion of shattered surfaces is, relatively, a rapid process. The formation of convexities by abrasion which cannot sustain the stress imposed on them by the flowing ice will result in abrasion-stimulated shattering.

2.3 Derivation of the abrasion equation

Gravitational forces are much smaller than forces imposed by flowing ice. For this reason abrasion is always considered to act normal to the surface under consideration. In this section we establish a relationship between the abrasion rate a and the rate of change of height $\frac{\partial h}{\partial t}$.

Consider an element of surface dS (see Figure 2.2). The volume abraded in time dt will be adSdt. The change in height is given by the volume removed divided by the projected area of dS in the plane orthogonal to the height, which we call the xy plane. The projected area is dxdy, ignoring second-order effects.

Thus, the change in height dh is given by

$$dh = -adt\,\frac{dS}{dxdy} \qquad 2.1$$

The surface area is given by

$$dS = \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}\,dxdy \qquad 2.2$$

(Wardle, 1965).

By substituting 2.2 into 2.1 and rearranging, we obtain what will henceforth be called the abrasion equation i.e.

$$\frac{\partial h}{\partial t} = -a\sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} \qquad 2.3$$

For two dimensions, $\frac{\partial h}{\partial y} = 0$ and the following simplification, which may be found in Scheidegger (1970), is obtained

$$\frac{\partial h}{\partial t} = -a\sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2} \qquad 2.4$$

If 2.3 is differentiated with respect to x, the following expression for the rate of change of slope with time is obtained for constant abrasion:

$$\frac{\partial\left(\frac{\partial h}{\partial x}\right)}{\partial t} = \frac{-a\,\frac{\partial h}{\partial x}\,\frac{\partial^2 h}{\partial x^2}}{\sqrt{\left(1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right)}} \qquad 2.5$$

A similar expression can be obtained by differentiating with respect to y.

Table 2.1 shows how slope will change with time for different combinations of slope and rate of change of slope under constant abrasion.

Table 2.1

Dependence of change of slope with time on the local geometry

		$\frac{\partial h}{\partial x}$		
		+	0	-
$\frac{\partial^2 h}{\partial x^2}$	+	-	0	+
	0	0	0	0
	-	+	0	-

Thus, for $\frac{\partial^2 h}{\partial x^2} < 0$ (e.g. a ridge) the slope will increase, while for $\frac{\partial^2 h}{\partial x^2} > 0$ (e.g. a trough) the slope will decrease. A ridge will therefore narrow, while a trough will broaden. For $\frac{\partial h}{\partial x} = 0$ or $\frac{\partial^2 h}{\partial x^2} = 0$, the slope at a point will not change.

A landform can be regarded as being in steady state if it persists under generalised lowering. This persistence may be accompanied by the migration of forms. The condition for non-migratory steady state may be obtained by setting the local lowering rate equal to the overall lowering† rate ℓ

$$a\sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} = \ell$$

For migratory steady state the condition for steady state may be obtained by taking the total derivative of the overall

† Note that a positive lowering corresponds to a negative change in height.

lowering rate i.e.

$$\ell = a\sqrt{1 + (\frac{\partial h}{\partial x})^2 + (\frac{\partial h}{\partial y})^2} - \underset{\sim}{m}.\underset{\sim}{\nabla}h$$

where $\underset{\sim}{m}$ is the migration velocity. Thus, for steady state, the abrasion rate must vary with geometry according to

$$a = \frac{\ell + \underset{\sim}{m}.\underset{\sim}{\nabla}h}{\sqrt{(1 + (\frac{\partial h}{\partial x})^2 + (\frac{\partial h}{\partial y})^2)}} \qquad 2.6$$

Rearrangement of this equation leads to the following partial differential equation:

$$(\frac{\partial h}{\partial y})^2(a^2 - m_y^2) + (\frac{\partial h}{\partial x})^2(a^2 - m_x^2) - 2m_y(\ell + m_x(\frac{\partial h}{\partial x}))\frac{\partial h}{\partial y}$$

$$+ a^2 - \ell^2 - 2\ell m_x \frac{\partial h}{\partial x} = 0 \qquad 2.7$$

Setting $m_x = m_y = 0$ and rearranging, we obtain

$$(\frac{\partial h}{\partial y})^2 + (\frac{\partial h}{\partial x})^2 = \frac{\ell^2}{a^2} - 1 \qquad 2.8$$

This equation is similar to the eikonal equation (Sommerfeld, 1954) which is of great importance in optics.

A necessary condition for a summit is $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = 0$. By setting $\frac{\partial h}{\partial x} = 0$ in 2.7, and solving the resulting quadratic in $\frac{\partial h}{\partial y}$, the following expression for $\frac{\partial h}{\partial y}$ is obtained:

$$\frac{\partial h}{\partial y} = (m_y\ell \pm \sqrt{m_y^2\ell^2 - (a^2 - \ell^2)(a^2 - m_y^2)})/(a^2 - m_y^2) \qquad 2.9$$

Thus, when $\frac{\partial h}{\partial x} = 0$, $\frac{\partial h}{\partial y}$ can only be equal to zero when $a = \ell$ (if $a = m_y$, the solution becomes singular). The condition $a = \ell$ only admits a flat plane.

No analytic solutions for either 2.7 or 2.8 have been obtained apart from a plane at arbitrary slope. However, consideration of 2.8 shows that the maximum slope obtainable under constant abrasion in any one direction is given by

$$\frac{a^2}{\ell^2} - 1 \quad .$$

Some numerical solutions have been obtained. These were produced by defining h as a function of x along a boundary. By using central difference approximations to $\frac{\partial h}{\partial x}$ except at the boundaries where forward or backward differences were used as required (Ames, 1977), solutions for $\frac{\partial h}{\partial y}$ could be obtained by solving equation 2.7. Along the lower boundary h was set equal to $\cosh(x)$ with $-1 \leq x \leq 1$. Solutions were not always guaranteed, and were very difficult to obtain for a greater range of x.

Figure 2.3 shows the case of $m_x = m_y = 0$, and $\ell = 2$, $a = 1$ dimensionless units. The resulting surface is curved, but nearly planar. By 2.8 we expect the maximum value of $\frac{\partial h}{\partial y}$ to occur when $\frac{\partial h}{\partial x} = 0$, i.e. when $x = 0$, and $\frac{\partial h}{\partial y}$ to be less elsewhere. This means that the hyperbolic cosine shape should be damped towards the top, though the narrow bounds to the existence of the solution do not make this obvious.

Figure 2.4 shows the case with $a = 1$, $\ell = 2$, $m_y = -0{\cdot}5$, $m_x = 0$. The contours appear to be pulled along the y axis, resulting in flatter slopes compared with the case of no migration.

Figure 2.5 shows the height contours for $a = 1$, $\ell = 2$, $m_x = m_y = -0{\cdot}5$. Here the contours are distorted to the right compared with the case of no migration. The disturbances in the top right-hand corner almost certainly are a consequence of numerical effects.

By retaining the same values for a and ℓ but by setting $m_x = m_y = -0{\cdot}9$ the contour plot shown in Figure 2.6 is obtained. Here it is also evident that an instability has set in, and also that the contours are even more distorted from the case of no migration.

These results show that curved surfaces may exist at least locally under constant abrasion, and that migrating surfaces are flatter than non-migrating surfaces. What would happen physically when a high slope in one direction caused an imaginary slope in the orthogonal direction is not clear: a ridge or trench where the slope becomes undefined is one possibility.

Equation 2.9 is also the condition for a steady-state landform in two dimensions. This equation only admits a constant slope as a solution.

It is also of interest to consider the variation of a with position required to maintain a given slope under lowering. The two-dimensional version of 2.6 is

$$a(x) = \frac{\ell + m_x \frac{\partial h}{\partial x}}{\sqrt{1 + (\frac{\partial h}{\partial x})^2}} \qquad 2.10$$

By selecting $h = \sin x$ and assuming a unit lowering rate the variation of a with x and m_x may be seen.

Figure 2.7 shows a plot of a with $m_x = 0$, $\ell/2$ and ℓ. For $m_x = 0$ the abrasion rate is symmetric about the crest, with the maximum values found at the crest. For $m_x = 0{\cdot}5$ the abrasion rate varies slightly on the upstream side and shows high variation on the downstream side. For $m_x = 1{\cdot}0$ the abrasion rate is almost constant over much of the upstream side but varies significantly on the downstream side. Thus nearly constant abrasion can produce a constant form on one side of a hummock if lowering is accompanied by migration.

2.4 The solution of the abrasion equation

In this section we restrict our attention to the two-dimensional abrasion equation, i.e. equation 2.4. The abrasion equation is first order and of second degree. It will now be shown that it is hyperbolic.

By squaring and rearranging 2.4, the following equation is obtained:

$$\left(\frac{\partial h}{\partial t}\right)^2 - a^2\left(\frac{\partial h}{\partial x}\right)^2 = a^2 \qquad 2.11$$

In addition,

$$dh = \frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial t}\,dt \qquad 2.12$$

The characteristic equation is found by following the procedure in Ames (1977), and is found to be defined by

$$\frac{\partial h}{\partial t}\,dt + a^2\,\frac{\partial h}{\partial x}\,dh = 0 \qquad 2.13$$

or

$$\left.\frac{dx}{dt}\right|_c = +a\,\frac{\partial h}{\partial x}\,/\Lambda \qquad 2.14$$

where $\Lambda = \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2}$.

For this first order system there is one characteristic direction, meaning that the equation is hyperbolic (Ames, 1977). The changes of h with x and with t may, along a characteristic, be found by substituting 2.13 in 2.12 and eliminating dt or dx. We obtain

$$\left.\frac{dh}{dt}\right|_c = -a/\Lambda \tag{2.15}$$

$$\left.\frac{dh}{dx}\right|_c = -1/\frac{\partial h}{\partial x} \tag{2.16}$$

Differentiating 2.11 with respect to x to obtain

$$\frac{\partial^2 h}{\partial x \partial t} = -((\frac{\partial h}{\partial x})^2 a \frac{\partial a}{\partial x} + a^2 \frac{\partial h}{\partial x} \frac{\partial^2 h}{\partial x^2} + a \frac{\partial a}{\partial x})/\frac{\partial h}{\partial t} \tag{2.17}$$

and noting that

$$d(\frac{\partial h}{\partial x}) = \frac{\partial^2 h}{\partial x \partial t} dt + \frac{\partial^2 h}{\partial x^2} dx \tag{2.18}$$

by substituting 2.14 and 2.17 in 2.18 we obtain

$$\left.\frac{\partial(\frac{\partial h}{\partial x})}{\partial t}\right|_c = -\frac{\partial a}{\partial x}\Lambda \tag{2.19}$$

When the abrasion rate is invariant horizontally, we obtain

$$\left.\frac{\partial(\frac{\partial h}{\partial x})}{\partial t}\right|_c = 0$$

For this case we can integrate 2.14 and 2.15 to obtain the position x of a characteristic and the value of h along the characteristic:

$$h = \frac{at}{\Lambda} + h_o$$

$$x = \frac{-a \frac{\partial h}{\partial x} t}{\Lambda} + x_o$$

where h_o, x_o are the initial values. The characteristic is a straight line. In the case of $\frac{\partial a}{\partial x} \neq 0$ solutions may be found by the simultaneous integration (usually numerical) of 2.14, 2.15 and 2.19 (see Appendix 1).

Also, by 2.16 the change of h with x along a characteristic is constant under constant abrasion. This line is the projection of the solutions on the characteristic on the hx plane. Since, by 2.16, this line is always normal to the slope, it may be regarded as a surface point trajectory. It also shows that solutions to the abrasion equation for constant abrasion may be constructed geometrically like ray diagrams in media of constant refractive index.

Scheidegger (1970) derives the same characteristics (without deriving 2.19) but suggests that the method of characteristics may only be used along an infinitely long slope. This was not found to be the case in the present studies; indeed, since one of the properties of characteristics is that discontinuities may propagate along them (Ames, 1977), it is hard to see that an infinitely long slope is required.

A test of any algorithm to solve the abrasion equation is to abrade a cylinder. The normals are radial lines, and the shape must be maintained but with a constantly changing radius. All the algorithms mentioned in this chapter were tested in this manner and found to give good results.

Figure 2.8 shows a 'ray' construction for a slope and plateau with a discontinuous slope being abraded. It may be seen that the trajectories cross, which implies that the characteristics have crossed in this region ODBE. This is due to the solution becoming multi-valued, which is a typical feature of hyperbolic equations.

If one considers the solution line ABDEBC it may be seen that there is a cusp catastrophe (Poston and Stewart, 1978) in the surface of the solution. The actual physical solution is given from considerations of mass balance by the line ABC. (A similar consideration is used to obtain the concentration of the shock front in the displacement of immiscible fluids (Dake, 1979), another example of a non-physical multi-valued solution).

The method of characteristics may also be used to determine what will happen to a convexity under constant abrasion. Figure 2.9 shows such a shape defined by a function $h = \exp(-x^2)$. Again, a region where the solution is multi-valued appears, and the convexity becomes cusp-shaped.

The abrasion equation may also be solved by finite difference methods. Scheidegger (1970) uses an explicit backward difference method to solve 2.4 and to obtain a solution for the problem shown in Figure 2.8. His solution differs in that he predicts the singularity to become smoothed (Scheidegger, 1970, Fig. 80).

Equation 2.4 was solved by the author using an explicit central-difference method (Ames, 1977). Figure 2.10 shows a V-shape valley (a simple model for a river valley) being rounded under constant abrasion. It was found that too large time steps resulted in instabilities being propagated in the solutions.

Figure 2.11 shows the solutions obtained for the bell shape. It should be noted that the solution is very poor and this is associated with the formation of a cusp. This is because as the solution becomes multi-valued, the finite difference algorithm is no longer adequate. In fact, the region of instability corresponds with the multi-valued region of the solution obtained by the method of characteristics (see Figure 2.9).

An implicit scheme for the solution of 2.4 was programmed. This resulted in no instability for the abrasion of the V-shape, but the instability associated with the formation of a cusp remained.

A further method of solving the abrasion equation is by parameterising it+. Let the position of any surface point be given by the vector $\underset{\sim}{r} = (x,h)$. The motion of any point on the surface is always normal to the surface i.e.

$$\dot{\underset{\sim}{r}} = a\underset{\sim}{\nu} \qquad 2.20$$

where $\underset{\sim}{\nu}$ is the unit normal vector. This is equivalent to

$$\dot{\underset{\sim}{r}} = a\underset{\approx}{R}\underset{\sim}{\tau} \qquad 2.21$$

where $\underset{\approx}{R}$ is a rotation matrix given by

$$\underset{\approx}{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad 2.22$$

+I am indebted to C. L. Farmer for constructing this formulation. It will be referred to as the Farmer formulation.

and $\underset{\sim}{\tau}$ is given by

$$\underset{\sim}{\tau} = \begin{bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial h}{\partial s} \end{bmatrix}$$

where s is arc length; or more generally by

$$\left\{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2\right\}^{\frac{1}{2}} \tau = \begin{bmatrix} \frac{\partial x}{\partial z} \\ \frac{\partial h}{\partial z} \end{bmatrix} \qquad 2.23$$

where z is a parameter (Wardle, 1965).

Substitution of 2.23 and 2.22 into 2.21 gives

$$\frac{\partial x}{\partial t} = \frac{-a}{\sqrt{\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}} \frac{\partial h}{\partial z} \qquad 2.24a$$

$$\frac{\partial h}{\partial t} = \frac{a}{\sqrt{\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}} \frac{\partial x}{\partial z} \qquad 2.24b$$

These coupled equations have been solved by finite difference methods. This algorithm is somewhat distinct from that used by Scheidegger (1970) in that the motion of a surface point is solved for rather than the change in height at a particular horizontal displacement.

An explicit finite difference algorithm was constructed using a central difference approximation. This algorithm was used to determine the effect of constant abrasion on a V-shaped valley.

Figure 2.12 shows a plot of the result when the surface is defined by 11 points. As a result of using central differences, the slope at the valley bottom was initially zero. As may be seen, rounding of the valley bottom occurs. However, some of the trajectories are curved, which is not what is expected from the results of the analysis of the characteristics.

Figure 2.13 shows the results obtained when the number of mesh points was increased to 91. The same rounding of the valley bottom occurs, but the trajectories plotted are straight lines. The trajectories were thus curved in the coarse mesh case as a result of numerical error. This point is discussed in more detail below.

Figure 2.14 shows the result of using the Farmer formulation on the bell shape (cf. Figure 2.11). As may be seen, the production of a cusp is predicted. The region of multiple solution is different from that obtained by the method of characteristics because trajectories are reflected at singular points in the Farmer formulation.

Figure 2.15 shows the results of setting the abrasion functions to be dependent on position in such a way as to permit a landscape (the bell shape) to persist under generalised lowering. In order to obtain successful results, a very short time step had to be used (the surfaces are not plotted every timestep). Since abrasion is a function of position, the trajectories are curved. It may be seen that trajectories from a large area collapse into a very small area. Thus, any form which might be persistent on a flat surface under abrasion will, on a convexity, eventually be extinguished by being compressed onto a smaller and smaller area. The curvature of the trajectory on the lee-side is a result of numerical error.

Figure 2.16 shows what happens to a persistent migratory form. The same compression of the trajectories is seen. In addition an example is provided, at the left hand corner, of what may happen as a result of numerical error. Under steady state, trajectories should never cross, but merely become asymptotic. If numerical error permits crossing, then a spurious region where the solution becomes multi-valued occurs. This is the cause of the instability in the lower left-hand corner.

The abrasion equation is time-reversible; that is any of the plots may be turned upside down and the results re-interpreted. For example, Figure 2.15 could represent the persistence of a groove under steady-state. In this case, the characteristics emanate from a very small area, and therefore, a large surface area is defined by what was existing in a very small area. Any small deviation is magnified by the action of abrasion.

This point is of importance when a concave cusp is abraded. At the cusp, there is no information on the slopes of the surface. The finite difference algorithms implicitly assume a slope at this point and thus predict a surface. Figures 2.17 and 2.18 show the results of using the method of characteristics to solve the development of a V-shape under constant abrasion. In Figure 2.17 only the characteristic directions at the bottom of the valley slopes are defined. The apparent predicted solution is a flat valley bottom, but this is only because the plotting algorithm draws a straight-line between neighbouring points. In fact, in the whole of the triangular region ABC the solution is undefined, and the valley bottom could be of many different shapes.

Figure 2.18 shows the results of deliberately defining many characteristic directions at the cusp. The plotting algorithm requires knowledge of characteristic neighbours, so if the characteristic directions are set to be smoothly varying with respect to this neighbourhood relationship, the rounded valley bottom seen in this figure is obtained.

If, however, the characteristic directions are set to lie randomly between those for the valley slopes, the result obtained in Figure 2.19 is achieved. Here there are large areas where the solution is multi-valued, and the valley bottom is predicted to have very jagged edges.

In nature, a true cusp would probably not form. However, these results do indicate a high sensitivity to initial conditions in areas of expansion.

Earlier in this chapter it was shown that convex cusps can appear under abrasion. This is unlikely in reality, essentially because we expect higher erosion rates in regions of higher curvature.

The parametric definition of the curvature of a line in a plane is (Wardle, 1965)

$$\kappa = \pm \left| \frac{\frac{\partial h}{\partial z}\frac{\partial^2 x}{\partial z^2} - \frac{\partial x}{\partial z}\frac{\partial^2 h}{\partial z^2}}{\left(\left(\frac{\partial h}{\partial z}\right)^2 + \left(\frac{\partial h}{\partial x}\right)^2 \right)^{3/2}} \right|$$

the sign being chosen according to convention. If the abrasion rate is a function of the curvature, then we see that the process now has a diffusive component.

If we use the Farmer formulation with abrasion rate enhanced by convexity using a linear dependence of the form

$$a = a_o(1 + b\kappa) \qquad 2.25$$

where a_o and b are constants, in the Scheidegger problem (see below) sharp edges becomes rounded (see Figure 2.20). This diffusion produces a result similar to Scheidegger's, suggesting that Scheidegger's prediction of bevelling is a consequence of numerical diffusion, i.e. a spurious diffusive component added to the solution by the error associated with the spatial and temporal discretisation (Ames, 1977). Numerical diffusion is often also called numerical dispersion, although the two are rather separate phenomena (Pinder, personal communication).

Numerical diffusion may also be considered by returning to Figures 2.12 and 2.13, where it was shown that a fine mesh was required to produce straight trajectories. If the fine mesh calculation is repeated but with a dependence of abrasion rate on curvature such that concavities inhibit abrasion, then the result shown in Figure 2.21 is obtained. Here the abrasion function was of the form $a = 0{\cdot}1(1 - 0{\cdot}25\kappa)$. Five hundred timesteps of dimensionless length 0·05 were taken, and surfaces were plotted every hundred timesteps. It may be seen that some of the trajectories are curved, as in Figure 2.12 (the coarse mesh constant abrasion calculation). This suggests that the results obtained using a coarse mesh were affected by numerical diffusion.

Figure 2.22 shows the result of causing the abrasion rate to be enhanced by concavities. This is, in effect, anti-diffusion. A coarse mesh using 21 points was used, and a relationship between abrasion and curvature of the form $a = 0{\cdot}1(1 + 0{\cdot}08\kappa)$ was used. This relationship keeps the valley bottom more pointed; thus the anti-diffusion artificially introduced counters the diffusion arising from the numerical formulation.

Since the region formed from expansion of the cusp is not strictly defined, then it follows that the surface defined by the use of numerical methods is an artefact of the numerical method. Thus, the rounding produced by both the finite difference methods is a result of numerical diffusion. However, any diffusive component in abrasion (caused, for example, by a dependence on curvature) will result in the surface being defined.

As has been stated, these equations were solved using an explicit method. In general, when using explicit methods, there is an upper limit to the time-step which may be used, related to the spatial step length, above which an instability is established. Scheidegger (1970) quotes a stability criterion for the finite difference solution of 2.4; however, it is not clear whether the analysis he implicitly uses is applicable to non-linear equations. Certainly, as reported above, too large a timestep can result in instability for this formulation.

No instability was found for the explicit solution of the Farmer formulation of the abrasion equation (Equations 2.24), and very much larger time steps can thus be used with this method. This does not mean that the use of large time steps will produce the same accuracy as the use of short time steps. Stability conditions do not, of course, apply to the method of characteristics.

When the abrasion rate is made to depend upon curvature, the equations have a diffusive component. In this case, we would expect the maximum time step compatible with stability to be related to the mesh size (Ames, 1977). This was found to be the case; instability here is manifested as a spurious crossing of trajectories, as seen for example in Figure 2.16. The case of anti-diffusion (Figure 2.22) usually results in unconditional instability (Ames, 1977). This was found to be the general rule with the Farmer formulation. However, in the calculation shown in Figure 2.22, the amount of anti-diffusion was presumably less than the diffusion introduced by the numerical scheme.

The method of characteristics cannot be used in general for problems with diffusion, as integrating the second derivatives along a characteristic involves the knowledge of third and higher order derivatives along the characteristics.

In principle, the regions where the solution becomes multi-valued could be eliminated by identifying where characteristics crossed. This, however, would require a complex algorithm for pattern recognition, which has not been attempted.

In conclusion, the method of characteristics has been identified as being the most accurate method for the solution of the abrasion equation, as it is free from numerical diffusion. In those cases where the method of characteristics cannot be used, the Farmer formulation is superior to the Scheidegger formulation.

2.5 The formation of U-shaped valleys

The valleys excavated by glaciers have rounded bottoms compared with their supposed original form, which is a V-shaped river valley. The calculations performed in the previous section indicate that this will occur quite naturally under constant abrasion (subject of course to the proviso that the original shape does not possess a true singular point).

A calculation using a Gaussian abrasion function, where the maximum abrasion rate is found at the valley centre, and the abrasion declines according to $a \propto e^{-x^2}$ has been performed (see Figure 2.23). Again, a U-shape is formed, and a further note of realism is introduced by $\frac{\partial^2 h}{\partial x^2}$ becoming negative on the trough sides.

Figure 2.24 shows the effect of having a harder substratum the surface of which lies beneath the bottom of the V-shaped valley. The effect of the substratum is to round the valley bottom further. It should be noted that the trajectories are 'refracted' at the geological boundary.

Simple abrasion patterns will also produce steady-state forms that are remarkably like U-shaped valleys. Let us suppose that the abrasion function is defined by $a = \frac{c}{h}$ where c is a constant. Then, by 2.10

$$\frac{c}{h} = \frac{\ell}{\sqrt{1 + (\frac{\partial h}{\partial x})^2}}$$

After rearrangement, this leads to the equation

$$x = \int \frac{dh}{\sqrt{\frac{\ell^2 h^2}{c^2} - 1}}$$

which has the solution (Abramowitz and Stegun, 1965)

$$h = \frac{c}{\ell} \cosh \left(\frac{\ell x}{c}\right)$$

A plot of this equation is shown in Figure 2.25. As may be seen, it bears a fair resemblance to a glacial trough.

If one posits (no justification is given here) that concavities lead to enhanced abrasion, and uses a relationship of the form of 2.25, i.e.

$$a = a_o(1 + b\varkappa)$$

then, by using 2.10 the condition for steady state is

$$a_o \left[1 + \frac{b \frac{\partial^2 h}{\partial x^2}}{\Lambda^3} \right] = \frac{\ell}{\Lambda}$$

(since $\varkappa = \frac{\partial^2 h}{\partial x^2} / \Lambda^3$ for $z = x$).

Upon rearrangement, we obtain the following set of ordinary differential equations

$$\frac{\partial h'}{\partial x} + \Lambda^2 (a_o \Lambda - \ell)/b = 0$$

$$h' - \frac{\partial h}{\partial x} = 0 \quad .$$

These equations have been solved by simultaneous integration of h' and h in the region $-10 \leq x < 10$ with a step length of 0·1, $a_o = 1$, $\ell = 4$ and $b = 0 \cdot 1$ to $0 \cdot 8$ with intervals of 0·1. Figure 2.26 shows the results. A trough shape is predicted. As the dependence on curvature increases, the trough becomes more and more V-shaped, and the sensitivity of the shape to curvature decreases.

The simple models used to obtain the 'troughs' in Figures 2.25 and 2.26 have not been given any physical justification, the motivation for their presentation being that many simple models can produce U-shaped valleys. Probably all that is required for a steady state trough is that the abrasion decrease either with distance from the valley centre or with height. The corollary of this is that in the testing of a relationship between abrasion and parameters defined by the glacier flow, the formation of a U-shaped valley is a necessary but not sufficient condition for the validation of the relationship.

2.6 The geometry of sub-glacial hummocks

Sub-glacial hummocks are considered in detail in Chapter 7. However, it is worth re-iterating some of the points already made in this chapter, as well as introducing some new results.

The curved surfaces shown in Figures 2.3 to 2.6 were of concavities, but the results apply equally to convexities. Figures 2.3 to 2.6 may be so interpreted by regarding the contour values as depths and noting migration velocties are of opposite sign.

The steady state solution for a height dependent abrasion function in the previous section (Figure 2.25) could apply equally to hummocks. In this case, abrasion would decline with depth. The width to depth ratio may be altered by changing the ratio $\frac{\ell}{c}$.

Figure 2.26 may also be regarded as providing models of hummocks if it is turned upside-down and supposed, on this scale at least, that convexities lead to increased abrasion. These results suggest that if abrasion rates are controlled by local curvature, and steady states do exist, then the dependence on curvature must be low since high dependence leads to the rather untypical Λ shapes under steady state.

Some calculations have been carried out to represent the influence of abrasion rate on weakness in a rock. The initial surface was set to be flat. The abrasion rate was defined by

$$a = (0{\cdot}5 + 0{\cdot}5e^{-x^2})$$

Figure 2.27 shows the results of this calculation using the Farmer formulation. The width of the section is 8 dimensionless units, and the origin is in the centre. The upper surfaces remain flat, and a cusp has formed. The curvature in the trough is always of the same sign.

A similar calculation was performed with the abrasion rate defined by

$$a = (0{\cdot}5 + 0{\cdot}5e^{-x^2})(1 + b\kappa).$$

Various values of b were used to reduce abrasion in concave regions and enhance abrasion in convex regions. This should at least delay the formation of a cusp; however it was not found possible to prevent such a phenomenon occurring at some stage.

These results suggest that enhanced erosion in areas of weakness tends to produce cusps if the original surface was flat. It can thus be tentatively suggested that weakness induced by jointing is not a sufficient cause for the shapes of glacial hummocks.

2.7 Geometry and lowering rates

It has been seen in the preceding sections that a dependence of erosion on curvature can have a significant influence on landscape evolution. Instead of following the synthetic procedure of establishing erosional behaviour at the lowest scale and proceeding to make predictions at larger and larger scales, one could follow an empirical procedure.

In this procedure, one would postulate that the abrasion rate at any point was defined by the geometry at all scales. Thus if the geometry is represented by n parameters $G_1, \ldots, G_i, \ldots, G_n$, then one could postulate the abrasion rate to be a function defined by

$$a = \sum_{i=1}^{n} \int w_i G_i \, d\phi$$

where ϕ represents scale and w_i is a weighting function dependent on ϕ.

The scale-dependent geometrical parameters would be obtained by smoothing the landscape over larger and larger areas, and the weighting functions would be matching parameters.

The motivation for this approach is that the flow of glaciers is affected by geometry at all scales; if the approach were to be useful, the weighting functions would have to be very scale-dependent, in order to illuminate the scale-dependence of glacial processes. Whether a unique answer would be obtained using this approach is an open question.

CHAPTER 3

THE PHYSICS OF ICE AND GLACIERS

3.1 Introduction

The discussion in the preceding chapters has indicated that in order to model the development of glacial erosional landforms, the ice motion at the glacier sole must be known. The determination of the ice motion is usually effected by solving the equations of glacier motion. In this chapter the establishment of these equations is discussed. Techniques for the solution of these equations are considered in Chapter 4.

The equations describing the dynamics and thermodynamics of a system fall into two classes, balance laws and constitutive relationships. The former are generic relationships independent of the properties of the body under consideration, while the latter depend on the properties of the body.

In this chapter the balance laws and constitutive relationships required to specify completely the equations of ice-motion will be considered. These equations require boundary conditions, which will also be discussed. The discussion will concentrate on steady solutions.

It will become apparent that the appropriateness of a constitutive relationship depends on the scale under consideration. This notion is not made any more precise in this chapter, though some consideration of the effects of scale on modelling strategies is given in Chapter 9.

3.2 Glacier Ice

Glacier ice is a five-component system. Its constituents are water, ice crystals, air, dissolved ions and rock debris. While the multi-component nature of glacier ice obviously leads to many discontinuities in the property fields, it is customary to treat ice as a continuum, and the interactions of the five components as being properties of the continuum. These interactions between components manifest themselves in complex behaviour of the continuum. The reversible transmutability of ice and water is the interaction that appears to induce much of the complexity.

This thesis is concerned with temperate glaciers. These are defined as glaciers composed substantially of temperate ice, which is in turn defined as ice at its pressure melting point. Owing to its granular nature and the fact that, to be in equilibrium at its pressure melting point, an ice surface must be in contact with water, temperate ice contains inter-granular water. In the transient thermodynamic situations found at the glacier bed significant fluctuations in temperature are found, even in temperate glaciers (Robin, 1976). However, sub-freezing point temperatures do not mean that all the water in the glacier is frozen; quite apart from super-cooling, a transient phenomenon, it has been shown that water can exist at grain boundaries at temperatures down to -10^{o}C (Hobbs, 1974).

The discussion of ice will ignore the effect of chemical impurities on the thermodynamic properties of ice. Impurities affect the rheology of ice mono-crystals. Jones and Glen (1969) have shown that some salts soften ice, and Riley et al. (1978) have shown that others turn ice mono-crystals into a work-hardening material. Salt solutions also affect the melting point of ice. Even if the effect of a given concentration of salt on a property were understood, dealing with the actual thermodynamics of salt solutions would be a very complex problem.

The actual physical behaviour of most of the glacier ice components can be described fairly simply if the influence of solutes is ignored. Both air and water can be regarded as viscousless fluids. Water is incompressible and exhibits significant surface tension effects. Air can be regarded as a perfect gas. Rock within the ice is effectively undeformable and unbreakable when it is not in contact with other rocks.

The complications in the physics of the glacier ice components arise from the rheology of the ice crystals and the reversible transmutability of ice and water. The rheology of the ice crystals is complicated, while the transmutability of ice and water, which is a simple matter when the dynamics of the matter are not considered, becomes a very complex matter once the dynamics are considered. These problems are discussed in sections 6 and 7.

3.3 Balance Laws

Hutter (1983) states five balance laws which must be satisfied for a body in motion. These are momentum balance, angular momentum balance, mass balance, energy balance and entropy balance. The first four state that the appropriate quantity is conserved, while the latter requires that the rate of entropy production never be negative.

The equation of momentum balance is

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i - \rho v_i \frac{\partial v_i}{\partial x_i} = 0$$

where $\underset{\sim}{x}$ is the co-ordinate system, $\underset{\approx}{\sigma}$ is the stress tensor, ρ is the density of the glacier ice, $\underset{\sim}{f}$ is a body force and $\underset{\sim}{v}$ is the glacier ice velocity. It is found for ice that the third term is negligible owing to the small velocities of ice flow, so the approximate form of the momentum balance equation used in glaciology is

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i = 0 \qquad 3.1$$

In the absence of body moments, it can be shown (Hutter, 1983) that the requirement for angular momentum balance is that the stress tensor $\underset{\approx}{\sigma}$ is symmetric i.e.

$$\sigma_{ij} = \sigma_{ji}$$

For an incompressible body the mass balance relationship is (Hutter, 1983)

$$\frac{\partial v_i}{\partial x_i} \equiv \dot{\varepsilon}_{ii} \equiv \dot{\varepsilon}_v = 0 \qquad 3.2$$

where $\dot{\varepsilon}$ is the strain-rate tensor defined by

$$\dot{\varepsilon}_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

and $\dot{\varepsilon}_v$ is known as the volumetric strain-rate.

Ice and water are generally regarded as being incompressible fluids. However, a mixture of ice and water is not incompressible, because the phase change involves a density change.

The density ρ^{iw} of an ice-water mixture is given by

$$\rho^{iw} = \theta\rho^{w} + (1 - \theta)\rho^{i} \qquad 3.3$$

where θ is the volume fraction of water, also called the moisture content, ρ^{w} is the density of water and ρ^{i} is the density of ice, both taken at the melting point.+

In certain cases discussed in section 7 of this chapter it is suggested that ice under pressure will melt internally, and the water so produced will be expelled. Two mass balance equations are required here, for the ice-water mixture and for the water itself.

For the mixture the relationship is

$$\rho^{iw}\dot{\varepsilon}_{v}^{iw} = - \rho^{w}Q \qquad 3.4$$

where Q is the volume rate of production of water per unit volume (the negative sign arises because compression is conventionally negative).

For the water, the mass balance relationship is, for small θ,

$$Q - \frac{\partial\theta}{\partial t} - \underset{\sim}{\nabla}.(\theta\underset{\sim}{v}^{iw}) + \underset{\sim}{\nabla}.(\underset{\sim}{v}^{w}) = 0 \qquad 3.5$$

where $\underset{\sim}{v}^{w}$ is the water Darcy velocity relative to the ice.

+ In this chapter superscripts will refer to phase: i for ice, w for water and iw for the ice-water mixture.

The first term represents the rate of production of water by ice melting, the third term represents the change of moisture content at a point due to convection of ice, while the fourth term represents the flow of water through the ice. By expanding the third term and substituting equations 3.3 and 3.4 into 3.5, we obtain

$$-\dot{\varepsilon}^{iw}_{v}(1 - \theta)\frac{\rho^{i}}{\rho^{w}} - \frac{\partial\theta}{\partial t} - \underset{\sim}{v}^{iw}\cdot\underset{\sim}{\nabla}\theta + \underset{\sim}{\nabla}\cdot\underset{\sim}{v}^{w} = 0 \qquad 3.6$$

It is assumed that the flow is steady and that the moisture content is constant then this equation simplifies to

$$-\dot{\varepsilon}^{iw}_{v} + \frac{\rho^{w}}{(1-\theta)\rho^{i}}\underset{\sim}{\nabla}\cdot\underset{\sim}{v}^{w} = 0 \qquad 3.7$$

The general statement for energy balance is similar to that for the mass-balance for the water†:

$$-QL\rho^{w} + E - \underset{\sim}{\nabla}\cdot(\underset{\sim}{v}^{iw}c^{i}\ T) - \underset{\sim}{\nabla}\cdot(\underset{\sim}{v}^{w}c^{w}T) + \underset{\sim}{\nabla}\cdot\underset{\sim}{q} = 0 \qquad 3.8$$

where L is the latent heat of fusion of ice, E is the shear heating, c is the specific heat capacity, T is the temperature, and $\underset{\sim}{q}$ is the heat flow in the ice water mixture: again the moisture content is assumed to be small.

The shear-heating is given by

$$E = \dot{\varepsilon}_{ij}\cdot\sigma_{ij}$$

The heat expended in melting or freezing is given by

$$QL\rho^{w}$$

which accounts for the first term in equation 3.8. The third

† Fowler (1984) derives a similar equation and gives a fuller discussion of ice-water mixture theory.

term arises from the convection of heat by the ice-water mixture, the fourth term by convection of heat by the water flowing through the ice water mixture, the fifth term by conduction of heat through the ice-water mixture.

When the ice is temperate, it can sometimes be assumed (Hutter, 1983) that the temperature gradients are negligible. In this case conduction and convection are not important in the heat transfer process, and the heat balance is maintained by the melting and freezing of water. In this case, the heat and mass-balance equations may be combined to give

$$\dot{\varepsilon}_v^{iw} + \frac{E}{\rho^{iw} L} = 0 \qquad 3.9$$

The entropy balance requires that the rate of entropy production be greater than or equal to zero. This requirement usually acts as a constraint on the constitutive relationships, and where it is required it will be indicated.

3.4 Constitutive relationships

(a) Rheological relationship

The rheological relationship relates the stress and strain-rate tensors,though these relationships can involve other variables.

Ice is generally regarded as a viscous fluid. Strictly speaking, fluids are regarded as being isotropic. The most general relationships that can exist between the stress tensor $\underset{\approx}{\sigma}$ and the strain rate tensor $\underset{\approx}{\dot{\varepsilon}}$ are (Hutter, 1983;

Eringen, 1967)

$$\dot{\varepsilon}_{ij} = A_{\varepsilon}\delta_{ij} + B_{\varepsilon}\sigma'_{ij} + C_{\varepsilon}\sigma'_{ik}\sigma'_{kj} \qquad 3.10a$$

$$\sigma_{ij} = -(\pi + A_{\sigma})\delta_{ij} + B_{\sigma}\dot{\varepsilon}_{ij} + C_{\sigma}\dot{\varepsilon}_{ik}\dot{\varepsilon}_{kj} \qquad 3.10b$$

where

$$\sigma'_{ij} = \sigma_{ij} - p\delta_{ij}$$

and δ_{ij} is the Kronecker delta, $\underset{\approx}{\sigma'}$ is the stress deviator tensor, π is the thermodynamic pressure, p is the hydrostatic pressure defined by

$$p = \tfrac{1}{3}\sigma_{ii}$$

and A_{ε}, B_{ε}, C_{ε}, A_{σ}, B_{σ}, C_{σ} are functions of such factors as the ice temperature, the moisture content, the ice fabric, and are also functions of the first, second and third invariants of the stress tensor (equation 3.10a) or the strain-rate tensor (equation 3.10b).

The second invariant of a tensor $\underset{\approx}{\omega}$ is defined as

$$\tfrac{1}{2}\omega_{ij}\omega_{ij} ,$$

the effective value of a tensor as the square root of the second invariant and the third invariant as

$$\frac{1}{6}\omega_{ik}\omega_{kj}\omega_{ji}$$

(Hutter, 1983).

It should be noted that straining can occur in directions in which there are no stresses because of the third term on the right-hand side of 3.10a. The influence of this effect is considered in Chapters 6 and 7 where it is called the 'third term effect'.

It is conventionally assumed in glaciology (Hutter, 1983) that the third-term effect cannot arise, and that the stress-strain rate relationship is not a function of the third invariant.

If the ice is compressible then the following relationship may be obtained from equation 3.10b

$$\sigma_{ij} = (-\underline{\pi} + \lambda\dot{\varepsilon}_v)\delta_{ij} + 2\mu\dot{\varepsilon}_{ij} \qquad 3.12$$

where $\lambda = -A_\sigma/\dot{\varepsilon}_v$

and $2\mu = B_\sigma$

The function μ is the viscosity of the ice, and the function λ is known as the dilatational viscosity of the ice.

It can be shown (Eringen, 1967) that if

$$3\lambda + 2\mu = 0$$

then $\pi = p$

This is known as the Stokes condition. It is a requirement of positive entropy production that

$$\mu \geq 0$$

$$3\lambda + 2\mu \geq 0$$

For incompressible flow, the necessity for distinguishing between the hydrostatic and thermodynamic pressures is lost, and 3.12 becomes

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\dot{\varepsilon}_{ij} \qquad 3.13$$

The dependence of the viscosities on the ice properties is considered in the following sections.

Insertion of 3.12 or 3.13 into the momentum balance equation produces the Navier-Stokes equations. These may be simplified for certain flows.

Plane strain flow occurs when no flow or straining occurs in one direction, say the z direction. In this case, according to 3.12 or 3.13, $\sigma_{xz} = \sigma_{yz} = 0$, and $\sigma_{zz} = -p$. The momentum balance equations then become

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho g_x = 0 \qquad 3.14a$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho g_y = 0 \qquad 3.14b$$

$$\frac{\partial \sigma_{zz}}{\partial z} = 0 \qquad 3.14c$$

In the case of uni-axial flow, flow occurs in one direction only, say the z direction, and no change in flow is allowed along the z direction. Flow is assumed to be incompressible. In this case $\dot{\varepsilon}_{xx} = \dot{\varepsilon}_{yy} = \dot{\varepsilon}_{xy} = 0$,

$$\dot{\varepsilon}_{xz} = \tfrac{1}{2} \frac{\partial v_z}{\partial x}$$

$$\dot{\varepsilon}_{yz} = \tfrac{1}{2} \frac{\partial v_y}{\partial z}$$

Substitution of these relationships into the Navier-Stokes equations produces:

$$\frac{\partial p}{\partial x} + \rho g_x = 0 \qquad 3.15a$$

$$\frac{\partial p}{\partial y} + \rho g_y = 0 \tag{3.15b}$$

$$\nabla^T.(\mu\nabla v_z) + \rho g_z = 0 \tag{3.15c}$$

(b) Heat flow

Heat flow is defined by

$$\underset{\sim}{q} = k\underset{\sim}{\nabla}T$$

where k is the thermal conductivity of the ice-water mixture.

(c) Moisture flow

Empirical observations appear to be limited to the qualitative observations of Carol (1947). Hutter (1983) suggests a diffusion-type law. However, the motivation for including the moisture term in this thesis was the observation of Carol, which suggests that at the base of the glacier the flow of moisture may be controlled by a Darcy type flow law (see Appendix 2) i.e.

$$\underset{\sim}{v}^w - \varkappa\underset{\sim}{\nabla}p^w = 0 \tag{3.16}$$

where $\varkappa$ is the permeability.

If 3.16 is substituted into 3.7, we obtain

$$\dot{\varepsilon}_v^{iw} - \frac{\rho^w}{(1-\theta)\rho^{iw}}\varkappa\nabla^2 p = 0 \tag{3.17}$$

or

$$\dot{\varepsilon}_v^{iw} - \chi\nabla^2 p = 0 \tag{3.18}$$

where χ is the diffusivity.

The subject is considered in section 7.

3.5 Boundary conditions

Small-scale boundary conditions for temperate ice involve either an ice-gas boundary or an ice-water boundary. Since both of these fluids may be regarded as being viscousless+ in comparison with ice, the usual boundary condition specified is that the tangential traction T_τ is zero and the normal traction T_ν is equal to the pressure in the gas or in the water.

At the boundary the ice must be at its pressure melting point. The melting point is related to the normal traction acting on the boundary by the Clausius-Clapeyron relationship (Hutter, 1983) i.e. $\Delta T_m = - C\Delta p$ where T_m is the melting point of ice and C is the Clausius constant.

When viewed on the larger scale, the glacier ice is an ice-water mixture. In this case, water may flow out of the mixture at the boundary. In this thesis we follow Nye (1969) in assuming that the flow of the water $\underset{\sim}{q}^w$ lying in between the ice and bedrock is governed by a Couette type flow i.e.

$$\underset{\sim}{q}^w \propto \underset{\sim}{\nabla} p^w \qquad 3.19$$

If the water pressure is assumed to be equal to the ice pressure (as is, in fact, tacitly assumed in the mixture theory above (see Fowler (1984)), this means that there is an inconsistency in the boundary condition, as the water pressure should be equal to the normal stress acting on the ice surface. No resolution of this inconsistency is offered here, the justification for the inconsistency being that it allows a simple model for water flow within the ice to be developed. The alternative is to assume that the glacier sole is impervious to water flow.

A similar relationship to 3.19 is used for water flow

+i.e. perfect fluids with zero viscosity between clast and bed-rock.

On the larger scale, the ice-rock boundary is smoothed, and the drag imposed on the ice when it flows round bed-rock obstructions not included in the more general description is regarded as a frictional component of the glacier motion. This is represented as a boundary condition relating for example the generalised velocity tangential to the smoothed bed, the tangential and normal tractions and the water pressure beneath the glacier. This boundary condition, often called the sliding law, is discussed in sections 10 and 11.

3.6 The rheology of glacier ice

Ice crystals are anisotropic. There is one plane, the basal plane, in which ice deforms very much more readily than any other. This deformation is known as easy glide. Ice in easy glide exhibits work softening, that is a creep rate increasing with total strain. Higher stresses soften the ice. Deformation in other planes is called hard glide. Stress levels about ten times greater than those for easy glide are needed to initiate creep. In these directions ice crystals exhibit work hardening (Hobbs, 1974).

When polycrystalline ice with randomly orientated crystals is subjected to a constant stress, three phases of behaviour are exhibited. The primary or transient creep is decelerating, the secondary creep occurs at a constant rate, while the tertiary creep is reaccelerating. Deformation is non-recoverable (Hobbs, 1974).

Polycrystalline ice is much harder than ice crystals deforming in the basal plane; in fact its hardness is more theorder of that for hard glide. Polycrystalline ice, like crystals undergoing hard glide deformation, exhibits work-hardening (Hobbs, 1974).

If the ice crystals have any preferred orientation, the glacier ice's continuum properties will be anisotropic. Lile (1978) has attempted to quantify this. He calculated a correction factor for the viscosity of ice based on the volume fraction of ice crystals orientated in any particular direction. The agreement between his results and his data is very good, particularly considering he used a linear flow law. However his method does depend on a uniform distribution of crystals with particular orientations within the actual ice sample. If a set of crystals orientated along a particular plane lie adjacent to one another on this same plane this will represent a line of weaknesses in the glacier ice that corresponds to a 'fault'. Expression of this phenomenon when the glacier ice is being considered as a continuum is a problem. The difficulty lies in devising a measure that represents the effect of non-uniform orientation distributions within an ice sample. If too small a sample is taken then the scale is not sufficiently great for the effects of individual crystals to be negligible. If, however, too large a sample is taken the problem of the possible occurrence of a line of weakness reappears.

Since snow crystals settle in no preferred direction, stationary ice formed from snow by firnification is isotropic. Anisotropy occurs after the ice has been worked. The actual processes of anisotropification are not well understood. It is an observed fact that as ice is worked, more and more of the basal planes become oriented with the direction of maximum shear (Budd, 1972). This can be explained by several, not mutually exclusive, processes. Rotation is one, annihilation of unfavourably orientated crystals by fracture is another; in very high energy environments one might expect melting. Recrystallisation in an orientation favourable to shear is a fourth mechanism.

The discussion this far has been limited to processes occurring within the crystals. Much of glacier flow can be attributed to processes occurring at grain boundaries: indeed it is obvious that there must be some kind of boundary adjustment to allow for the fact that the ice-crystals are anisotropic and not all orientated in the same direction. Grain boundary processes are characteristic of (relatively) high-energy environments, and are at least partly responsible for the apparent variation of the activation energy for the creep of glacier ice with temperature (Barnes et al. 1971).

Boundary adjustment processes include slipping along the grain boundary, pressure melting and melting due to strain energy (classified as a boundary process because melting must occur at a boundary) (Hobbs, 1974). Duval (1977) has shown that the viscosity of ice is a monotone decreasing function of the moisture content. It is not difficult to envisage grain boundary water as a lubricant: the more water there is, the less coupled the grains will be. It cannot be assumed that the ice is always fully saturated; its voids may be occupied by air and water vapour. Lack of saturation will increase the number of surfaces, and increase the importance of surface tension effects, which will tend to couple the grains together.

The influence of absolute grain size on the rheology of ice has been investigated by Baker (1978), Baker and Gerberich (1979) and Duval and LeGac (1980). All these experiments were performed at temperatures substantially below the melting point, the maximum temperature being -7^{o}C. Baker (1978) found that the viscosity of ice was a non-monotone function of grain size, but his results may be suspect in that in some cases the experiments were not sustained for a sufficiently long period for the ice to enter into the secondary creep phase (see discussion at the end of Baker's paper). Duval and LeGac found that whereas the secondary creep rate was not influenced by the grain size, the primary creep rate was, a result compatible with Baker's. It is difficult to asses how relevant these results are to the deformation of temperate ice. In a high energy environment, where grain boundary processes dominate, smaller-grained ice might be more fluid, since it has more grain boundary per unit volume. It is characteristic of tertiary-creeping ice for it to have smaller-grained ice.

Reaccelerating, or tertiary creep is characteristic of ice in a high energy environment. Its appearance is not dictated by microscopic ageing processes in the way the transition from primary to secondary creep is. Rather, its appearance is a manifestation of the fact that the fabric of the glacier ice changes as work is done on it. Tertiary creep is characterised by recrystallisation in preferred directions leading to increased anisotropy of the ice sample, grain boundary movement and production of melt-water by pressure melting and shear heating (Hobbs, 1974). Note that anisotropification by itself results in an entropy decrease, and that the required global increase is due to the action of associated processes. The appearance of tertiary creep in laboratory experiments is partly a consequence of the fact that most analogue models are samples with randomly orientated crystals, and thus will maximise the anisotropification effect.

The demonstration by Duval (1977) that the viscosity of ice was dependent on the moisture content may be partly a reflection of the dependence of the viscosity on the structure of the ice: indeed, it is possible that in certain situations the influence of the two effects will not be separable.

The influence of rock debris on the rheology of glacier ice is potentially a subject of some importance given that the volume concentration of debris can reach 40% in the basal layers (Boulton, 1974). Baker (1978) and Baker and Gerberich (1979) have investigated the effect of debris on ice, and Watts (1974) has conducted a theoretical analysis. Baker and Gerberich (1979) considered ice where the inclusions were of a size comparable to the crystals. They found an increasing fraction of inclusions increased the fluidity of ice; however, owing to the effect inclusions had on the crystallisation process, they could not control the crystal sizes.

When the inclusions are of a size comparable to the ice crystals, it is difficult to reason in a general manner as to how the presence of debris will influence the glacier ice rheology. In high energy environments, where grain boundary processes dominate, if a major deformation process is boundary slipping, debris might be expected not to make much of a difference, as the difference between the rheologies of the ice and the debris would be unimportant. Conversely, in a high energy environment it could be argued that it would enhance pressure melting. In low energy environments, they might be expected to increase the rigidity of the ice, as the internal rheologies of the constituents would be the dominant influence on deformation. This is not the result obtained by Baker.

3.7 Dynamics of the ice-water transition

Since this thesis is concerned with temperate glaciers, that is glaciers which are at their pressure melting point, the dynamics of the ice-water transition are of some importance. While, strictly speaking, the ice-water transition is reversible, in practice it is irreversible, as the operations of freezing and melting have to be carried out at an infinitely low speed to maintain reversibility.

In this section two points will be considered. The first one is the subject of regelation theory, that is theory describing the flux of ice past obstacles by melting on the upstream side of the obstacle where the melting point is lower due to the increased pressure and the water flowing round the obstacle under the action of the pressure gradient to refreeze on the other side. The second point is the importance of the internal melting process.

The phenomenon of regelation has been a contentious matter since the middle of the last century (Watts, 1974), and theory to describe it has been re-invented several times.

An analysis of the motion of a wire through the ice by regelation by Nye (1967) has shown that the velocity of motion will be proportional to the driving force. However agreement between this prediction and the results as reported by Nunn and Rowell (1967), Townsend and Vickery (1967) and Drake and Shreve (1973) is poor. The theory over-predicts velocities by factors of up to one thousand. Drake and Shreve attribute this poor agreement to the operation of several mechanisms not considered in the analysis carried out by Nye. These include accumulation of impurities on the lee-side of the obstacle that alter the melting point; an additional super-cooling on the lee-side needed to produce a finite rate of freezing, and the fact that the lee-pressure has a lower limit provided by the triple point of water.

Representations of the regelative mechanism in glaciological models, ignoring as they do the additional processes postulated by Drake and Shreve, cannot in general be said to correspond very closely with reality.

The process of bulk internal melting, as described by Robin (1976) and by Goodman et al. (1979) has many consequences of importance. It means that glacier ice cannot be treated as being incompressible; that the assumption of thermodynamic reversibility is no longer viable and that the assumption that the ice is never frozen to the bed in temperate glaciers cannot be sustained. It is a consequence of the fact that ice is not a continuum, or to put it another way, a consequence of the finite size of ice crystals.

Consider a sample of saturated temperate ice. Quite apart from the boundaries of the actual sample, there are many internal ice-water interfaces owing to the crystalline nature of glacier ice. Now, the Clausius relationship above does not in fact refer to the hydrostatic pressure but to the normal stress component on the interface. In a sample of glacier ice the normal stress on the crystal boundary will be a function of the orientation of the boundary. Changes in the stress field will cause ice to melt or water to freeze. Since the ice-water transformation is not volume-preserving, changes in the stress field applied to saturated glacier ice will cause changes in volume of the sample; in other words the incompressibility assumption is not strictly valid. However, the compressibility due to the phase transition ($\frac{1}{V}\frac{dV}{dp}$ where V is the volume) is the same order as that of water, i.e. very small.

In actual fact the compression will not be reversible. This is because it is extremely unlikely that the ice will recover its fabric on the reverse cycle. This means that the total surface area (and hence the total surface energy which is a function of the surface curvature and thus of the grain shape) will alter. The energy freed will go towards keeping water melted. Another energy-absorbing process will be deformation of the ice-crystals (Goodman et al. 1979).

If the voids are connected the moisture will, on comparison of the saturated ice sample, flow through the ice from regions of high pressure to regions of low pressure. Robin (1976) used this fact to suggest a possible mechanism for the stick-slip process of glacier sliding observed by Vivian (1970). As the saturated glacier ice approaches a bump, the ice will become compressed and some will turn to water. The water is supposed to be expelled from the ice by action of the pressure and to flow, sub-glacially, away from the region. This process is irreversible as it leads to the collapse of the capillaries within the ice, and also leads to loss of heat from the ice. Robin (1976) postulates that this could cause the temporary freezing of ice to the bed. The expulsion of water melted internally could also be an important source of sub-glacial water.

In the solution of the equations for the pressure and temperature fields within the body of the ice the temperature is not coupled to the stress by the Clausius relationship, indeed it cannot be because no boundaries are defined, and thus no melting point can be defined. It has been found during attempts to solve the regelation problems for certain geometries success is not guaranteed (Nye, 1967; Morris, 1976; 1979). It is necessary to invoke at least one of three mechanisms (Morris, 1979):-

(i) Internal melting and freezing.

(ii) Formation of water cavities.

(iii) Formation of gas cavities.

Utilising the first option would involve dropping the incompressibility assumption for ice. In its irreversible form this relationship would violate the continuum assumption as this assumes the ice consists of an infinite number of infinitesimally small crystals.

Another inconsistency in the Nye formulation of the regelation problem is that he assumes (explicitly) the process is steady-state, whereas it is, in fact, a Carnot cycle (Weertman, 1979). This will lead to residual melting. In principle, the extra dissipation arising as a result of this effect could be allowed for, and the driving force established from energy balance considerations. As it is, the assumptions of thermodynamic reversibility and that consistent steady-state models can be established are not valid.

3.8 Modelling the rheology of ice

In the preceding sections, the complexity of the response of ice to applied stresses has been described and discussed. In this section we discuss the possibility of modelling these processes and discuss the rheological models used in the glacier ice flow modelling described in Chapters 4 and 5.

The following factors have been identified in section 6 as having an empirically observed effect on the rheology of poly-crystalline ice. They are ice fabric, moisture content, strain-history, anisotropy and the presence of inclusions of foreign material.

If fabric did not change with time, then modelling the influence of fabric would be simply a matter of adjusting the viscosity so that it was appropriate for the ice-sample under consideration. However, fabric does evolve with time; ice crystal sizes and shapes change with time. Modelling this would require knowledge of how the ice fabric changed in association with all the other unknowns. In addition, the ice would now be a memory fluid, that is its response would be governed by its past history, and modelling would have to be effected by integrations over the path of any ice particle. While this is not an enormous difficulty in transient modelling, the steady-state solution of ice-flow problems would be changed from an elliptic problem to a mixed elliptic-hyperbolic problem and the dangers of ill-posedness could very well arise (Ames, 1977).

Similar considerations apply to modelling the effect of the strain history on the rheology of the ice. At the base of glaciers, where ice is being subject to rapidly fluctuating stresses and strains, this could make analysis quite difficult.

Rational continuum mechanics does not permit the existence of anisotropic fluids (Eringen, 1967), and the description of viscous anisotropic behaviour properly comes under the properties of liquid crystals. Thus, any attempt to model anisotropy and anisotropification will require the use of a flow law based on different principles to those of an isotropic fluid (Hutter, 1983).

The influence of inclusions on glacier ice rheology has been investigated using inclusions of roughly the same size as the ice-crystals in the ice sample (see section 6). A theoretical investigation into the effect of inclusions on Newtonian regelating ice has been carried out by Watts (1974). Since this model of ice assumes the ice to be non-crystalline, the empirical and theoretical studies are not really comparable. The use of a linear rheology and the use of the suspect regelation theory make the applicability of Watts' results to real situations questionable. One conclusion though is worth mentioning; that when regelation is very efficient, that is when the particles are small (but still larger than the crystal size) the viscosity of the ice is actually reduced by the addition of particles. The role of the clasts as stress concentrators must also increase the plasticity of the ice owing to its non-linear rheology.

Any modelling of the influence of clasts requires the tracking of clast concentrations within the glacier, which, as with the fabric evolution equations, changes the nature of the equation from that of an elliptic towards that of a hyperbolic equation.

If it could be assumed that the moisture content of the glacier ice did not vary significantly with time, then the viscosity could again be adjusted so as to account for the influence of water within the ice.

However, if moisture transport can occur the question then arises as to whether the dominant mechanism of moisture transfer in ice is by saturated flow or by turbulent pipe flow. If the latter situation is the case then the solution of the moisture transfer equation becomes a macroscopically stochastic process, because the appearance of moulins and smaller pipes cannot be predicted exactly. Their size and shape can vary rapidly, and in general lend transiency to the pore-water situation. Engelhardt et al. (1978) suggest that the pipe flow systems and the pore-water systems are effectively uncoupled owing to the extremely slow flow of water within the pore system. If this is the case, then the dependence of the ice viscosity on moisture content can be ignored, because the moisture content is only a reflection of the fabric of the ice.

If, on the other hand, porous flow is of importance, then the moisture content is of importance for the solution of the glacier flow, because it couples the Navier-Stokes equations, the mass balance equation and the heat flow equation. The deviatoric stress field will probably affect the voids ratio, and thus the hydraulic conductivity. If water is flowing fast through the pores, it could enlarge them, affecting the hydraulic conductivity which, in turn governs the rate of flow of the moisture. This would affect the heat transfer and the fabric, which would manifest itself in changes in viscosity and so in the rate of heat production.

In the bulk of the glacier, the evidence suggests that porous flow is negligible (Engelhardt, 1978). Against this must be set the famous observation of Carol (1947), who stated that ice on the upstream side of a roche moutonnée had water "exuding from countless tiny capillaries" and that this ice had "the consistency of cheese". (It is not clear whether Carol was referring to a soft cheese such as Camembert or a harder cheese such as Gruyère.) Ice of this consistency occurs in situations which could crucially determine the clast-bed contact force and thus abrasion (see Chapter 6).

If ice is significantly compressible, then the question arises as to what the dilatational viscosity is. It is unlikely that the Stokes condition will be maintained and thus the setting of the dilatational viscosity to -2/3 the shear viscosity will not be valid. Indeed, the dilatational viscosity might play the role of providing the dissipation required to melt the ice by compression in one formulation, whereas in other formulations of the physics of compressing ice it might not. There is a possibility of a creep instability in the compression of ice if the shear-heating arising as a result of the compression produces significant quantities of melt-water, which then increases the compression of the ice. Obviously this cannot be a normal state of affairs, as glaciers would then melt under their own weight. This instability would then occur when the ability of the ice to transmit the water was high. It is not clear whether the transport of moisture is governed by a diffusion process, as suggested by Hutter (1983), or by a Darcy flow process, as suggested previously in this chapter, or both, the Darcy process becoming important when the moisture content was high enough to provide inter-connecting pores. The results of Nye and Frank (1973) suggest that this regime of interconnecting voids, which is presumably what Carol (1947) was observing, cannot be a steady situation.

Two simple models of the compressing flow of ice are suggested in equations 3.7 and 3.9. In the absence of a complete theory describing the properties of the ice-water mixture, it is not possible to say how approximate these theories are; they have been included because it is relatively easy to include these equations in the finite element formulation of the equations of glacier flow described in the next chapter. In particular, 3.7 contains no guarantee that the heat balance is maintained, though the presence of a dilatational viscosity ensures dissipation on volume change.

Carol's observation suggested that the water was visibly exuding from the ice. If we take this as indicating a pore velocity of 1 m/day, assume pressure gradients to be 1 bar/m, and assume the porosity to be equal to the moisture content (2%) (Duval, 1977), this indicates a permeability of the order of ten m/bar.a (see Appendix 2).

The third invariant of the deviatoric stress contains information on the stress configuration, e.g. whether the ice is compressing or shearing (Lile, 1978). The nature and probability of debris collisions will be dependent on whether the ice is being compressed, sheared or pulled apart. The moisture content is a function of the compression and possibly also of the state of shear. Evidence on the response of laboratory ice is contradictory. Glen (1958) stated that the rheology is a function of the third invariant, whereas Legac and Duval (1980) imply that it is not. No 'third term' dependence has been reported to occur..

In the studies of ice-flow described in Chapters 4 and 5, a power law relationship for the rheology of ice will be used. In most cases equation 3.13 is used to establish the relationship between the stress and the strain-rate. The viscosity is related to the effective strain rate by a relationship of the form

$$\mu = A \, \dot{\varepsilon}^{\frac{1-n}{n}} \qquad 3.20$$

The constant n is empirically determined and is usually accepted (Hutter, 1983) to be 3. A is a constant related to the constant in Glen's Law. This was mistakenly set at

1.815 bar.a$^{\frac{1}{3}}$ (twice the value for the Glen flow relationship cited in Paterson (1981)) in the modelling. No account is taken of many of the complexities outlined above; in particular, no account is taken of the third invariant, and no third-term effects can arise.

In most of the modelling incompressibility was assumed; otherwise the volumetric strain-rate was defined either by 3.7 or 3.9. The density of ice was set at 910 kg/m^3, and the latent heat of fusion at 3.3×10^5 J/kg. The dilatational viscosity was set arbitrarily, as no measurements of it have been made.

Other flow laws have been suggested (Hutter, 1983). Many of these suggest a dependence of the second invariant of the strain-rate on a polynomial of the second invariant of the deviatoric stress. However, it is not possible to use these to define a viscosity of ice, which was required by the solution method used in Chapters 4 and 5.

The discussion above has suggested that the use of a Glen rheology will not necessarily produce accurate representations of glacial flow situations, and that moisture flow may be of crucial importance. However, the Glen relationship was used because of its widespread acceptance, and also to see how important use of a non-linear rheology was when modelling glacial flow situations.

Regelation is not modelled in the thesis. The primary motivation for this was that it is not always possible to avoid ill-posed problems (see section 7). A justification for ignoring regelation is given by Fowler (1979) who suggests that it is not significant for obstacles much larger than a centimetre.

3.9 Some aspects of glacier hydrology and cavitation

The purpose of this section is not to give an account of the complex and contentious subject of glacier hydrology. There are many excellent papers on the subject, ranging from discussion of percolation through the ice (Nye and Mae, 1972; Shreve, 1972; Nye and Frank, 1973), conditions for the formation of intra-glacial and sub-glacial channels (Weertman, 1972; Röthlisberger, 1972; Nye, 1976), observations of heads in boreholes (Hodge, 1974; Engelhardt, 1978; Engelhardt et al. 1978) and a discussion of the relationship between the head in a borehole and the sub-glacial water-pressure (Engelhardt, 1978). The main purpose of this section is to discuss the ideas proposed by Engelhardt (1978) and to try to relate them to the more detailed situation at the base of the glacier with special regard to erosional processes.

Glacier ice flowing over an obstacle can separate on the lee-side, leaving a cavity. Cavitation occurs when the normal stress on the lee-side of the obstacle is reduced beneath a certain critical pressure. This pressure is usually the triple point of water (approximately zero bars), though if there is a supply of water other than that supplied by stoss-side melting the pressure of this water will be the cavity pressure.

The minimum normal stress on the lee-side is governed by the generalised ice velocity and viscosity and the geometry. The greater the velocity and viscosity, the lower the stress, while the greater the obstacle the smaller the stress

(Watts, 1974, though see Chapter 6).

The contents of and the pressure within this cavity are a matter of some contention, certainly in theory, if not in practice. That cavities exist is beyond doubt (Vivian and Bocquet, 1973; Boulton et al., 1979), which are filled with air, if only because of their connection with a system of underground tunnels. Lliboutry (1979) suggests that four situations can exist on the lee-side of an obstacle.

(a) No cavity.

(b) A cavity filled with stagnant ice or water which is hydrostatically stressed.

(c) A cavity filled with water vapour or air.

(d) A cavity filled with water that is connected by channels to other cavities.

The cavity pressure can thus be a function of global conditions and not local conditions. It is evident that the different stress situations caused by these different conditions in the cavity will affect both abrasion and plucking. These conditions also affect the sliding of the glacier over the bedrock, and it is a matter of some importance to know which, if any, of these situations predominates.

It is a tenet of nearly every model of the ice-rock interface that the ice is separated from the rock by a thin layer of water (Boulton, 1975; Hallet, 1979; Kamb, 1970; Lliboutry, 1979; Weertman, 1979). In these models, the water cannot support shear stresses, so the state of stress in the water is hydrostatic and imposed by the ice. Water flow occurs under the action of pressure gradients. The water layer is likely to be only a few microns thick (Nye, 1969) and flow through it is, therefore, likely to be slow. Nye (1976) and Weertman (1972) have considered the problem of flow in channels at the base of the glacier. The existence of the channels is due to continued melting due to the contact with the flowing water, without which the channel would close under the action of the stress. Röthlisberger (1972) concluded that intra-glacial channels were stable, and that the bigger they were the more stable they were. Weertman (1972) investigated the stability of channels within the ice and their ability to drain the areas between channels. He concluded that sub-glacial channels were stable but that the stress field associated with them would in general seal them off from the sub-glacial sheet of water. This was not a rigid conclusion, but one dependent on the flow law of the ice and on the stress environment. Weertman also concluded that channels incised into the bed-rock would be collectors of water of sub-glacial origin.

Measurements of bore-hole heads (Hodge, 1974; Engelhardt, et al. 1978) indicate that they fluctuate rapidly, when they are connected to the sub-glacial drainage system. Inspection of Hodge's results suggest that the average height of water in connected bore-holes is two-thirds of the glacier height. If the bore-hole is not connected to the sub-glacial system, then the height of water in it indicates that the basal water pressure is equal to the ice-overburden pressure (Engelhardt ., 1978).

The conclusion to be drawn from these observations is that there are two sub-glacial hydraulic regimes. On the one hand there is the unconnected hydraulic regime, which includes (a), (b) and (c) of the above scheme, and the connected hydraulic regime, which includes (c) and (d). Whether a cavity is connected to the main drainage appears to be a matter of chance; Lliboutry (1979) postulates a cyclic scheme of connection and isolation. The pressure in connected sub-glacial channels is dependent on the water input to the glacier.

The results from both Hodge's and Engelhardt's investigations suggest very strongly that bore-holes act as attractors for sub-glacial channels. If we imagine channels to be of the order of metres wide, then connection of the spatially infrequent bore-holes should be uncommon in a broad valley glacier. Also, once a bore-hole is connected, it remains connected. The exact reason for this stability is not known. The fact that the bore-hole is a pressure sink may be important. If this is so, then it is likely that cavities will be connected, because they are also capable of acting as pressure sinks.

It is not known how many cavities within a glacier are connected to the main drainage system, and how many are isolated; with present theory it is not possible to predict whether a cavity is connected to a channel or not. Nor is it possible to predict the pressure within the cavity if it is connected. Thus the pressure in sub-glacial cavities has to be taken as a random variable with unknown pressure distribution, with a lower limit provided by the triple point pressure of water, and an average upper limit provided by the overburden pressure of the ice.

3.10 Sliding laws

The concept of the sliding law, or the friction law, as Lliboutry (1979) terms it, has been introduced in section 5 of this chapter. Many formulations of such laws have been attempted (Fowler, 1979; Kamb, 1970; Lliboutry, 1968, 1979; Morland, 1976a, 1976b; Nye, 1969, 1970; Weertman, 1957, 1964). For various reasons, mostly of a methodological nature, none of these laws have been experimentally confirmed. The main purpose of this section is not to give individual critiques of the various sliding laws, but rather to discuss the general principles by which these laws are formulated in the context of what has been said in preceding sections about the physical processes occurring in ice and glaciers.

All analyses of the sub-glacial situation are based on some kind of model of the ice-rock interface. The ice in all the above models is assumed to be at the pressure-melting point, and in all of these models except for Morland (1976b) the ice-rock interface is not allowed to sustain any tangential stresses. In between the ice and the rock there is generally supposed to be a thin layer of water. This layer is allowed, in some of the theories, to thicken appreciably on the lee-side of obstacles (Lliboutry, 1968, 1979). Water flow can take place, and the process of regelation described in section 4 is incorporated in all these models. The rheology of ice is usually modelled either as a Newtonian (linear) viscous fluid or as a Glen fluid (i.e. a fluid using the Glen constitutive relationship). The lee-side cavities, if they exist, are usually supposed to be full of stagnant, hydrostatically stressed water or containing air at atmospheric pressure. Lliboutry (1979) allows the pressure to be an externally determined variable.

It should be apparent from the discussion in the preceding sections that the physical model of the ice-rock interface presented by these authors is not a particularly realistic one. Observations by Kamb and La Chapelle (1964) and Boulton (1974) at the base of temperate glaciers show that the ice found at the base of a temperate glacier is quite distinctive in its properties. Kamb and La Chapelle termed this layer of ice the regelation layer. It contains grains of ice very much smaller than those in the ice above it, many ice bubbles, and substantial proportions of debris; Boulton (1974) states that the volume concentration of debris in ice can reach 40%. This layer was observed by Kamb and La Chapelle to be between 0mm thick and 29mm thick. Boulton (1974) has observed the layer to be up to 100mm thick. Typically, this layer is thinner on the upstream side of obstacles and thicker on the down-stream side.

In general, observations of the ice indicate that it is at the pressure-melting point. There is a theoretical difficulty here because any observation necessitates altering the stress field in the vicinity of the ice. It should be apparent from discussions in preceding sections that the properties and thermodynamic state of ice, especially when it is at its melting point, are very sensitive to changes in the stress field. However, Kamb and La Chapelle (1964) did observe that when the pressure was released from the ice it did immediately freeze to the bedrock. This could be confirmation of Robin's (1976) "stick-slip" mechanism.

In all the models except Morland (1976b) it is assumed that the ice exerts no tangential stress on the rock. This is because the ice is assumed to carry no debris. This assumption is obviously highly unrealistic and contrary to any hypothesis that the glacier can abrade. Morland (1976b) uses two different traction models in his formulation; a Coulomb model, where the traction is proportional to the normal stress, and a frictional resistance model, where the traction is a function of the sliding velocity. Both of these models are ad hoc. What little experimental evidence there is on stresses and velocities at the base of a glacier presents rather a confused picture (Boulton et al., 1979).

Without the benefit of empirical evidence, it is difficult even to begin to speculate on the nature of the traction relationship. A Coulomb criterion is probably sufficient to describe the behaviour of individual clasts, but to extrapolate this behaviour to an assemblage of clasts carried by a viscous medium is not justified (as indeed Morland states). A relationship between the stress and the velocity is intuitively appealing. The processes of ice flow around rocks will be discussed further below, and these apply equally well to flow around clasts that are retarded by frictional drag. Indeed, one way of dealing with the clast problem would be to regard clasts as an additional roughness component.

Ice at the base of glaciers consists of regelation ice, which is young ice and ice which, because it has been near the bed, has been heavily worked. Its stress and strain history is one of transiency in both magnitude and direction. In view of what has been said in the preceding sections, any assumption that the rheology of ice at the base of a glacier can be described adequately using a Glen rheology should be treated with a certain amount of caution.

All of the above models model the regelation phenomenon using the simple theory. Glacier slip, as originally proposed by Weertman (1957) and every subsequent author, is thus a combination of regelation and the increased viscous deformation. Since the speed of the regelation phenomenon is governed by the heat flux available to melt the ice, and this heat flux is in turn governed by the temperature gradient across the obstacle, regelation will be, according to the simple theory, a very efficient mechanism for flowing round small obstacles (Weertman, 1957). Large obstacles, on the other hand, enhance the rate of plastic creep. This is because the rate of strain is a function of stress deviators, and the bigger the obstacle, the larger the volume in which it increases the stress deviation. The fact of the stress deviator enhancement is independent of the flow law. However, if the flow law is a power function of the stress deviator invariants as in Glen's Law, the plastic flow will become more important. It should be noted that owing to the dependence of the flow law on the stress deviator invariant, any globally imposed non-hydrostatic stress field will have an effect on the relative importance of enhanced plastic flow.

Weertman (1957) used the fact that regelative flow is efficient around small obstacles while plastic flow is efficient around large obstacles to introduce the concept of the controlling obstacle size. This was the obstacle size where the two processes were as efficient as one another at flowing round it the obstacle.

Weertman's model can be criticised on a number of grounds. His approach to calculating stress fields is ad hoc to the point of error in his 1957 paper, a mistake which he rectified to a certain extent in his 1964 paper (Weertman, 1964). Most restrictive, however, is his model of the bedrock morphology, which consisted of cubes of rock protruding from the bed. Nye (1969; 1970) and Kamb (1970) produced formulations of a sliding law where the bed morphology was represented by a Fourier series; the only restriction on the morphology was that slopes had to be small. They replaced the concept of a controlling obstacle size with a controlling wavelength. Because the only way they could obtain analytical, rather than numerical, solutions to the problem was by assuming a linear flow law, the controlling wavelength is a function of the arbitrarily chosen viscosity. For a stress situation fairly typical of a glacier base, Nye (1969) gives a controlling wavelength of the order of decimetres. Nye (1969) also points out that not only will the controlling wavelength be of importance, but also the distribution of wavelengths adjacent to the controlling wavelength.

Fowler (1979) analysed the glacier sliding problem in general terms. He found that if a power law rheology were used, the controlling wavelength was of the order of 1mm, appreciably less than Nye's figure. It is fair to say that 'convex' rugosities do not exist on this scale on abraded surfaces, and for this reason Fowler neglects the influence of regelation in his subsequent analysis. By assuming the standard model for the ice-rock interface (ignoring, of course, regelation), he uses variational methods to calculate upper and lower bounds for the drag on the base. He derived a sliding law of the same form as Weertman viz.

$$u_b = \frac{\sigma}{\nu^{n+1}} \left(\frac{\tau_b}{R}\right)^n$$

where u_b is the sliding velocity, τ_b the basal shear stress, and σ, ν, R are parameters describing the roughness and n is the constant in the Glen strain-stress relationship 3.20.

The question of whether the regelative process can be ignored is of some importance. Fowler (1979), discussing the regelation ice observed by Kamb and La Chapelle (1964), Kamb (1970) and Boulton (1974) says "it is not clear whether these observations contradict (my own) theory, but in any case it must be remembered that the neglect of hydrological theories may be of profound significance in this respect".

The hydrological effects Fowler refers to have already been touched on in section 7. Fowler is calling upon influx of water from any source to the low pressure areas on the lee-side of obstacles to produce regelation ice. Internal bulk-melting is one such process. Another process, as evinced in a study of ice-sliding by Budd et al. (1979), is shear melting of the ice when it is being worked heavily. That regelation ice is only due to surficial pressure melting is not proven, and Souchez and Lorrain (1978) produce chemical evidence to indicate that it is due to internal melting.

So far the discussion has only touched on what Nye (1969) has identified as the central problems in the formulation of a sliding law; the problem of describing the geometry of the rock surface and the problem of describing the geometry of the glacier base. When there is no cavitation these problems are, of course, the same. Nye (1969) and Kamb (1970) found that, with a linear flow law, all that was needed to describe the bedrock was the product of the auto-correlation of the amplitude and the mean square of the amplitude. This is not true for all flow laws.

Owing to the presence of cavities,the general problem of finding the position of the glacier base to the lee of an obstacle would appear to be a free-surface problem, and consequently is not usually soluble by analytical means. Lliboutry(1979) avoids this problem by not treating it as a free surface problem; he assumes that flow with cavitation is roughly the same on the downstream side of the bump as it would be on the pure sine profile having a maximum slope t, the value of t being such that the minimum value of the normal pressure is N, the autonomously determined pressure in the cavity. This means that the roofs of cavities are supposed to have a constant slope for a given stress situation.

Sub-glacial observations by Vivian (1980) contradict both Lliboutry's model and the supposition that the problem is a strict free-surface. Vivian's observations indicate that the place where the ice re-touches the bed is always the same, and that changes in cavitation manifest themselves as changes in where the ice leaves the bedrock. This observation is rather difficult to explain.

3.11 Two phenomenological sliding laws

The purpose of this section is to discuss, briefly, the work of Reynaud (1973) and of Bindschadler (1983). Both of these authors produced simple, phenomenologically derived relationships that can be used as the boundary condition for the base of the glacier. Synthesising boundary conditions for the base of the glacier, is, of course, the object of all those workers who have formulated sliding models (see the preceding section), but the applicability of their results is hampered by relating morphological concepts of roughness to notions of roughness obtained by consideration of process. The work of Bindschadler and of Reynaud, while not free of arbitrary assumptions, suggests that the gross nature of the sliding law may be very much simpler than considerations of the processes indicate. Why this might be so is considered briefly at the end of this section.

Reynaud (1973) considered the flow of ice through a cross-section of the Athabasca Glacier using data provided by Raymond (1971). He assumed a steady state with all flow parallel to the valley sides. In this case the Navier-Stokes equations reduce to the Laplace equations. He used Glen's Law to decribe the stress-strain relationship of the ice, and for the sliding law he adopted a Coulomb condition, viz.

$$\sigma_b = \eta N$$

where σ_b is the basal shear stress, η the coefficient of friction and N is the difference between the ice over-burden and the water pressure. The pressure difference is estimated using the following relationship

$$N = K[(\rho_I h_I - \rho_w h_w) + (\rho_w - \rho_I)H]$$

where K is a constant somewhat intuitively derived (Lliboutry, 1968) related to the geometry of the sub-glacial streams, ρ_I, ρ_w are the densities of ice and water respectively, h_I is the thickness of the ice, h_w is the piezometric height corresponding to the water channels and H is the difference in height between the point under consideration and the height of the point where the water cavities link with the main water channel. The coefficient of friction is not given but derived from the condition that the shear stress is zero along the surface and the middle of the (assumed) symmetric channel.

Reynaud ends up with one variable parameter, the piezometric height with which to fit the data. He achieves a remarkably good fit by using a piezometric height of two-thirds the glacier height. This height is typical of the average heights found by Hodge in bore-holes (Hodge, 1974).

Whether Reynaud's success is serendipitous is a moot point. The undetermined constant K, which is supposed to represent the sub-glacial hydraulic geometry, provides an extra matching parameter. However, as Engelhardt (1978) has argued, the heights of water in bore-holes are as much a reflection of the sub-glacial channel patterns as the sub-glacial water-pressure. Why a Coulomb criterion should reproduce the velocity distribution so well, and why the coefficient of friction should be constant over the bed are not obvious. One implication must be that the water pressure is of crucial importance in determining the way the ice and the bed interact, so much so that the actual shape of the bed ceases to be important.

Bindschadler (1983) analysed field data from the long profile of the Variegated Glacier. He calculated the basal shear stress by assuming it to be proportional to the drop in normal stress across bedrock humps, and he calculated the water pressure by assuming all flow was contained in sub-glacial channels incised into the ice using Röthlisberger's (1972) theory. He found that the basal velocity was strongly correlated with what he termed an "index of bed separation". This was defined as the basal shear stress divided by the effective normal pressure (equal to the ice overburden pressure minus the water pressure).

Bindschadler's results and Reynaud's results do not, at first sight, appear to correspond. Part of the reason may be that they come from different section types. However, they do both contain the result that the shear stresses are related to the effective normal stress, though they are calculated in different ways.

Both Reynaud and Bindschadler suggest that glacier sliding is intimately connected with the difference between the ice over-burden pressure and the water-pressure in the connected regime. If we assume that cavities are attractors for sub-glacial channels, as was argued in section 9, then it is possible that the majority of cavity pressures are controlled by the fluctuations in the pressures of the connected regime. The cavity pressure will affect the sliding in two ways.

Firstly, the drag due to sliding consists of two components: an increased pressure on the upstream side and a suction on the downstream side. The cavity pressure provides an upper bound to the suction; in other words drag does not increase as fast with velocity in the cavitated regime as it does in the uncavitated regime.

Secondly, the cavity pressure controls the degree of cavitation. For a given velocity, the higher the cavity pressure, the larger the obstacle that can have a cavity on its lee-side.

3.12 Conclusions on sliding

The conclusions of this chapter must all be rather pessimistic. The rheology of glacier ice, especially near the ice-rock interface, is still far from being understood. Sliding laws based on consideration of small-scale processes have not been very successful, and no-one has succeeeded in explaining Reynaud's or Bindschadler's results from a theoretical stance, which means that we cannot, in all confidence, extend their relationships to other glaciers. In later chapters in this thesis a Weertman relationship is used, i.e.

$$v_\tau = ST_\tau \qquad 3.21$$

where v_τ is the tangential velocity and S is the bed smoothness, and T_τ the tangential traction. The justification for using it is that every investigator apart from Reynaud (1973) has suggested that the tangential velocity and traction are related such that they increase with one another.

CHAPTER 4

SOLUTION OF THE EQUATIONS OF GLACIER MOTION

4.1 Introduction

The equations of glacier motion have been presented in Chapter 3. These equations describe the slow creeping flow of a material which may be either compressible or incompressible.

In general, analytical solutions of these equations, which describe glacier flow, do not exist except when the domain of interest is of a simple shape. If, in addition, a non-linear relationship exists between the stress and strain rates, the class of problems for which solutions can be found by analytic means becomes even more restricted. The equations of glacier motion do have one considerable simplification compared with the full Navier-Stokes equations because the convection of momentum can be ignored owing to its small size relative to the viscous forces. This not only leads to a simplification of the formulation of the equations, but also avoids considerable numerical difficulties.

Since analytic solutions are in general unobtainable, numerical solutions which require, in practice, the use of a digital computer have to be used. The glacier-flow equations are, of course, a set of partial differential equations. Two methods are currently widely used to solve partial differential equations, the finite difference model and the finite element model.

The finite difference method has been used in other parts of this thesis, for example to solve the abrasion equation (see Chapter 2) and the eikonal equation (see also Chapter 2).

For the equations of glacier motion the finite element method has been used.

Both the finite difference method and the finite element method rely on the principle of discretisation. This is when the domain of interest is treated as a set of discrete sub-domains.

The essential difference between the finite difference methods and the finite element methods is that in the former the derivatives are approximated by piece-wise functions (usually piece-wise linear), but in the latter the equations are formulated such that the discretisation error is minimised over a discrete sub-domain. In certain cases this can lead to exactly the same numerical formulation (Zienkiewicz, 1977[+]). Accepted wisdom is that for a comparable grid (sub-domain) size, the finite element method is the more accurate of the two but that solutions are more expensive to obtain.

The finite element method was chosen to solve the equations of motion in this study partly because of its success in handling the elliptic problems posed by the steady-state glacier flow equations, but mainly because of the method's ability to deal with geometrically complex domains and boundaries. The novel basal boundary condition formulation described below would have been rather more difficult to express in terms of finite differences.

Zienkiewicz describes three finite element formulations for solving the incompressible Navier-Stokes equations. These are

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[+]All subsequent references to Zienkiewicz made in this chapter are to this book.

the penalty function formulation, the streamline formulation and the velocities-pressure formulation.

The penalty function formulation, which has been used by Iken (1981) to solve the equations of glacier motion, relies on the identity between the equations describing slow viscous flow and the equations describing the response of an incompressible elastic medium (Zienkiewicz). When a material is incompressible, it has Poisson's ratio of 0.5. If this value is inserted into the standard equations of elasticity (Zienkiewicz) then these equations become singular. The penalty function avoids this problem by setting the value of the Poisson's ratio to very nearly one half. Unless certain precautions are taken, which are described in Zienkiewicz, the near singularity of the matrix can result in the solution becoming very ill-conditioned.

The streamline formulation, which has been applied to glacier flow by Hooke et al. (1979), depends on defining a function ϕ such that $\frac{\partial \phi}{\partial x} = - v_y$ and $\frac{\partial \phi}{\partial y} = v_x$, where v_x is the velocity in the x direction and v_y is the velocity in the y direction. This formulation has proved difficult to extend to three dimensions. The full formulation is presented in Zienkiewicz. A problem with the streamline method is that because the second derivatives of the streamline function enter into the equations as well as the derivatives, the derivatives have to be solved for (Zienkiewicz). Traction boundary conditions are not straightforward to incorporate.

The third formulation for the solution of the Navier-Stokes equations is the velocities-pressure formulation. This has not been used so far in glacial flow problems. This formulation solves directly for the velocities and the hydrostatic pressure. The incompressibility condition enters as a further constraint with the pressure serving as a Lagrangian multiplier (Zienkiewicz). There are difficulties with this formulation which mean that the pressure solution is never as accurate as the velocity solution. The formulation also leads to semi-definite matrices, which require special linear equation solvers to deal with them and are prone to ill-conditioning.

Originally it was hoped that three dimensional modelling of glacier flow problems would be practicable. For this reason, the streamline formulation was rejected. The velocities-pressure formulation was chosen in preference to the penalty function formulation because the pressure enters naturally into the former formulation. This was felt to be an advantage should phase changes from ice to water be modelled.

6.2 Weighted Residual Methods

The principles of the finite element method have been described by many authors, e.g. Zienkiewicz, Huyakorn and Pinder (1983). At its most basic level, the finite element method is a way of formulating the numerical solution of differential equations such that the error introduced by their discretisation (a necessary step for numerical solution) is minimised. The finite element method is, therefore, a method of weighted residuals.

Consider a set of differential equations

$$L(\hat{u}) - \hat{f} = 0 \qquad 4.1$$

where $\hat{u}$ is the unknown variable vector, L is a linear operator and $\hat{f}$ is a constant vector. In order for the equation to be solvable by numerical means, the continuous variable $\hat{u}$ has to be approximated by a piecewise variable u, where

$$u = \sum_{i=1}^{n} N_i u_i = \underset{\sim}{N}\underset{\sim}{u} , \qquad 4.2a$$

similarly:

$$f = \sum_{i=1}^{n} N_i f_i = \underset{\sim}{N}\underset{\sim}{f} \qquad 4.2b$$

where N_i is an interpolation function and u_i, f_i are values of u, f at a particular point or node i. $\underset{\sim}{N}$, $\underset{\sim}{u}$ and $\underset{\sim}{f}$ are $1 \times n$, $n \times 1$ and $n \times 1$ vectors of N_i, u_i and f_i respectively. n is the number of points from which interpolation is made. The interpolation function is generally called the trial function (Huyakorn and Pinder, 1983).

Thus, the trial functions can be regarded as interpolation functions for the unknown values of the variables.

Equation 4.1 may be approximated by substituting in 4.2a, 4.2b to obtain

$$L(u) - f = \xi \qquad 4.3$$

where ξ is the discretisation error.

The method of weighted residuals seeks to determine the unknown u_i such that the error ξ is minimised. This is done by ensuring that the average error over the sub-domain is zero. This condition is specified by

$$\int_R W\xi dR = 0 \qquad 4.4$$

where W is an arbitrary weighting function and R is the region of interest (i.e. the sub-domain of interpolation). The form of these functions is not specified by the formulation, and they can therefore be selected on the basis of their ability to improve the solution or for their computational convenience.

The weighting functions W in this study were chosen to be identical to the trial functions N. This choice of weighting functions is known as the Galerkin method. This method has been shown (Zienkiewicz) to produce very accurate results. Another advantage of this method is that the resulting set of equations turn out to be symmetrical, which produces considerable computational economy.

The weighted residual method minimises the discretisation error over a domain. For computational convenience, the domain is of a simple shape, and the number of solution points are kept small. This means that the domain of interest in the grand solution has to be represented by many of these sub-domains or finite elements. In order to ensure that the solution is compatible, the points of interest belong, in general, to several of the sub-domains over which the discretisation error is minimised, and a set of simultaneous equations describing the whole of the domain of interest are generated. To see how this is effected, we must consider the weighted residual statements again.

For the sake of illustration, let us consider a one-dimensional example. We consider the nodes 1 and 2, at which the unknowns are u_1 and u_2. The discretisation error is given by (from 4.2 and 4.3)

$$\sum_{i=1}^{2} L(N_i u_i) - \sum_{i=1}^{2} N_i f_i = \xi$$

the summation being taken outside of the operator because it is linear. If the matrices are written explicitly, and noting that f is interpolated similarly to u, we obtain the following expression for the discretisation error.

$$[LN_1, LN_2]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + [N_1, N_2]\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \xi \qquad 4.5$$

Using 4.4 to minimise the discretisation error, and using the Galerkin approach of setting the weighting function W equal to the trial function N, the following relationship is obtained from equation 4.5:

$$\int_R \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} [LN_1, LN_2]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dR + \int_R \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} [N_1, N_2]\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} dR = 0$$

which may be recast as the matrix equation.

$$\underset{\approx}{K}\underset{\sim}{u} + \underset{\approx}{H}\underset{\sim}{f} = 0$$

where the element matrices are defined by

$$K_{ij} = \int_R N_i L N_j dR$$

$$H_{ij} = \int_R N_i N_j dR$$

Thus, for the two unknowns, u_1 and u_2 two simultaneous linear equations have been established which may be solved to obtain u after the specification of boundary conditions (see below).

Certain constraints exist on the nature of the trial functions N. An obvious one is that if the nodal unknowns for any sub-domain or element are equal, then the interpolated value must be equal to the nodal values. This places the following requirement on the trial functions:

$$\sum_{i=1}^{n} N_i = 1$$

Also, at a nodal point, the interpolated value must be equal to the nodal value at that point, meaning that all the other nodal contributions must be zero.

For the two-noded one dimensional element considered above, these requirements taken together mean that the trial functions $\tilde{N}$ must be linear, and thus the variation of the solution u within the element must also be linear.

In general, it is to be expected that the unknown will vary with position in a manner more complex than linear. There are two ways to get round this problem. The first is to increase the order of the trial functions so as to make their variation potentially more complex. In the one-dimensional case this could be effected by putting in an extra node. Thus, each trial function would have to equal 1 at the node to which it belonged, and 0 at the other two nodes, imposing a quadratic form on the trial function. This process would result in the creation of a full three by three matrix, which would then have to be inverted.

An alternative approach is to use three nodes, but to consider the one-dimensional domain as being modelled using two, connected, two-noded elements. In this case the middle node has to satisfy minimisation of the discretisation error for two elements simultaneously. If the element equations are derived, then it appears that the system is over-determined, in that there are four equations and three unknowns. These equations are

$$\begin{bmatrix} K^1_{11} & K^1_{12} \\ K^1_{21} & K^1_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} H^1_1 \\ H^1_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} K^2_{22} & K^2_{23} \\ K^2_{32} & K^2_{33} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} H^2_2 \\ H^2_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where superscripts denote the element, and may be combined into the grand system of equations

$$\begin{bmatrix} K^1_{11} & K^1_{12} & 0 \\ K^1_{11} & K^1_{22}+K^2_{22} & K^2_{23} \\ 0 & K^2_{32} & K^2_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} H^1_1 \\ H^1_2+H^2_2 \\ H^3_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

to form a solvable set of equations. It should be noted that this set of equations possess a band-structure, i.e. all the matrix elements greater than a certain distance from the diagonal are zero. Matrices with band structures allow efficiencies to be made in their solution procedures (Zienkiewicz), and, in general, finite element grids may be designed so as to minimise the bandwidth of the matrix, that is the distance of the zero entry from the diagonal.

Thus, where two lower-order elements have been used instead of one, the resulting grand solution matrix is of the same order as that obtained by using one higher order element, but it possesses a structure that makes its solution more economical. The drawback is that it can only represent the unknown varying in a piecewise linear fashion, rather than the quadratic fashion permitted by the higher-order element.

The grand assembly process produces a singular matrix, that is an infinite number of solutions can satisfy it. In order to produce a solution, boundary conditions have to be put in. These boundary conditions are of two types, Dirichlet and Neumann conditions. Both of these affect the vector $\underset{\sim}{f}$.

Neumann boundary conditions define the values of the derivatives of the nodal variables. If, for example, the heat conduction equation were being solved and it was desired to specify a non-conducting boundary, this would be effected by setting the temperature gradient at this boundary to zero. If, on the other hand, it was known that a certain boundary was at a fixed temperature, this could be effected by forcing the nodal variables at the boundary to take on that value. This is a Dirichlet boundary condition.

4.3 Solution of the Uni-axial Flow Equations

It has been shown (Chapter 3) that for the case of uni-axial flow the steady-state Navier-Stokes equations reduce to the Laplace equation viz.

$$\frac{\partial p}{\partial x} + \rho g_x = 0 \tag{4.6a}$$

$$\frac{\partial p}{\partial y} + \rho g_y = 0 \tag{4.6b}$$

$$\nabla^T.(\mu \nabla v_z) + \rho g_z = 0 \tag{4.6c}$$

where p is hydrostatic pressure, ρ is the density of the ice, $\underset{\sim}{g}$ is the acceleration due to gravity, x, y, z is the co-ordinate system with z in the flow direction, ρ is the viscosity of the ice, and v_z is the velocity in the flow direction. Since pressure does not enter into equation 4.6c, the equations are no longer coupled and 4.6c may be solved separately. This equation may be solved approximately using the method of weighted residuals with the following formulation.

The approximate value v_z of the solution within any element is given by

$$v_z = \sum_{i=1}^{n} N_i v_{zi}$$

and the value of the derivative and second derivative by the equations

$$\frac{\partial v_z}{\partial x} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial x} v_{zi} \quad , \quad \frac{\partial v_z}{\partial y} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial y} v_{zi}$$

and

$$\frac{\partial^2 v_z}{\partial x^2} = \sum_{i=1}^{n} \frac{\partial^2 N_i}{\partial x^2} v_{zi} \quad , \quad \frac{\partial^2 v_z}{\partial y^2} = \sum_{i=1}^{n} \frac{\partial^2 N_i}{\partial y^2} v_{zi}$$

To minimise the discretisation error using the Galerkin method, the following condition must hold:

$$\int_R \underset{\sim}{N}^T \frac{\partial \left[\mu \left(\frac{\partial \underset{\sim}{N}}{\partial x}\right)\right]}{\partial x} dR \; v_z$$

$$+ \int_R \underset{\sim}{N}^T \frac{\partial \left[\mu \left(\frac{\partial \underset{\sim}{N}}{\partial y}\right)\right]}{\partial y} dR \; v_z + \int_R \underset{\sim}{N}^T \rho \underset{\sim}{N} \, dR g_z = 0$$

As it stands, this set of equations is asymmetric. The equations may be made symmetric by use of Green's theorem (Zienkiewicz, 1977), and the following expression is obtained.

$$\int_R \frac{\partial \underset{\sim}{N}}{\partial x}^T \mu \frac{\partial \underset{\sim}{N}}{\partial x} dR \; v_z + \int_R \frac{\partial \underset{\sim}{N}}{\partial y}^T \mu \frac{\partial \underset{\sim}{N}}{\partial y} dR \; v_z$$

$$+ \int_\Gamma \underset{\sim}{N}^T t_z d\Gamma + \int_R \underset{\sim}{N}^T \rho \underset{\sim}{N} g_z dR = 0 \qquad 4.7$$

where Γ is the surface and t_z is the boundary traction. Before Green's theorem was used, second derivatives appeared in the equations, and this would have required continuity across the elements of the first derivative (Zienkiewicz, 1977). This would, in turn, have increased the number of equations to be solved as both the nodal variable and its derivative would have had to be solved. Use of Green's theorem means that the highest order of derivative now appearing is one, meaning that only the nodal unknowns have to be solved for. Also, the viscosities need no longer be continuous from element to element, and if they vary within an element their derivatives need not be evaluated.

The integration over the element area to obtain the nodal stiffnesses can be obtained by analytic means for simple elements. However, when quadratic or higher order trial functions are used, numerical integration is used. Details may be found in Zienkiewicz.

In the case of a Dirichlet boundary condition (i.e. a prescribed velocity) the equation for that particular nodal variable is replaced by the equation

$$1.v_z = v'_{zi} \tag{4.8}$$

where v'_{zi} is the prescribed velocity. This makes the coefficient matrix asymmetric, so to rectify this, the column containing the prescribed variable is set to zero and the entry that was there is multiplied by the prescribed value and subtracted from the right hand side, for each row other than the row containing the prescribed variable.

In glaciological studies a bed-sliding boundary condition of the form

$$v_{zb} = St_{zb} \tag{4.9}$$

is often found, where v_{zb} is the sliding velocity, t_{zb} is the tangential traction at the ice-rock interface in the direction of sliding and S is the smoothness of the bed. This is termed the Weertman boundary condition, after Weertman (1957).

The weighted residual statement is

$$\int_\Gamma \underset{\sim}{N}^T t_{zb} d\Gamma - \int_\Gamma \underset{\sim}{N}^T \frac{\underset{\sim}{N}}{S} d\Gamma \, v_{zb} = 0$$

The first term in this equation corresponds to the third term in 4.7, and thus

$$\int \underset{\sim}{N}^T \frac{\underset{\sim}{N}}{S} d\Gamma \, v_{zb}$$

may be substituted into 4.7 to simulate the basal sliding relationship.

This boundary condition is known as a mixed boundary condition because it contains both Dirichlet-type terms and Neumann-type terms. This formulation is that shown in Zienkiewicz for heat radiation at a boundary.

4.4 Solution of the Navier-Stokes equations for plane strain flow

The equations describing the plane strain steady flow of an incompressible viscous fluid are (Chapter 3)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho g_x = 0 \qquad 4.10a$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho g_y = 0 \qquad 4.10b$$

where $\sigma_{ij} = 2\mu\dot{\varepsilon}_{ij} - p\delta_{ij} = 2\mu.\frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] - p\delta_{ij}$

and $\dot{\varepsilon}_v = 0$ where $\dot{\varepsilon}_v = \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy}$ 4.10c

Here there are three nodal unknowns, the x and y velocities and the pressure. These equations have been formulated in terms of the strain rates rather than the velocities. If the velocities and pressures are interpolated within the element in the normal way i.e.

$$\hat{\underset{\sim}{v}} = \underset{\sim}{N}\underset{\sim}{v}$$

$$\hat{p} = \underset{\sim}{L}\underset{\sim}{p}$$

where $\underset{\sim}{L}$ is the trial function for the pressure.

the strain rates are then approximated by

$$\dot{\underset{\sim}{\varepsilon}} = B\underset{\sim}{v}$$

where

$$\dot{\underset{\sim}{\varepsilon}} = \begin{bmatrix} \dot{\varepsilon}_{xx} \\ \dot{\varepsilon}_{yy} \\ 2\dot{\varepsilon}_{xy} \end{bmatrix}$$

and

$$\underset{\approx}{B} = \begin{bmatrix} \frac{\partial N}{\partial x} & 0 \\ 0 & \frac{\partial N}{\partial y} \\ \frac{\partial N}{\partial y} & \frac{\partial N}{\partial z} \end{bmatrix}$$

for each node of the element.

Let $\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$, $\underset{\approx}{D} = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$, $\underset{\sim}{m} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

The weighted residual statements of the momentum balance euqations are

$$\int_R \underset{\sim}{N}^T \frac{\partial \sigma_{xx}}{\partial x}\, dR + \int_R \underset{\sim}{N}^T \frac{\partial \sigma_{xy}}{\partial y}\, dR + \int_R \underset{\sim}{N}^T \rho g_x dR = 0$$

$$\int_R \underset{\sim}{N}^T \frac{\partial \sigma_{yx}}{\partial x}\, dR + \int_R \underset{\sim}{N}^T \frac{\partial \sigma_{yy}}{\partial y}\, dR + \int_R \underset{\sim}{N}^T \rho g_y dR = 0$$

Use of Green's Theorem (Zienkiewicz, 1977) results in

$$\int_R \frac{\partial \underset{\sim}{N}^T}{\partial x} \sigma_{xx} dR + \int_R \frac{\partial \underset{\sim}{N}^T}{\partial y} \sigma_{xy} dR + \int_R \underset{\sim}{N}^T \rho g_x dR + \int_\Gamma \underset{\sim}{N}^T \underset{\sim}{N} t_x d\Gamma = 0$$

$$\int_R \frac{\partial \underset{\sim}{N}^T}{\partial x} \sigma_{xy} dR + \int_R \frac{\partial \underset{\sim}{N}^T}{\partial y} \sigma_{yy} dR + \int_R \underset{\sim}{N}^T \rho g_y dR + \int_\Gamma \underset{\sim}{N}^T \underset{\sim}{N} t_y d\Gamma = 0$$

where $\underset{\sim}{t}$ is the surface traction on the boundary. These equations may be expressed in matrix form

$$\int_R \underset{\approx}{B}^T \underset{\approx}{D} \underset{\approx}{B} dR \underset{\sim}{v} - \int_R \underset{\approx}{B}^T \underset{\sim}{m} L dR p + \int_R \underset{\sim}{N}^T \rho g dR + \int_\Gamma \underset{\sim}{N}^T \underset{\sim}{N} \underset{\sim}{t} d\Gamma = 0 \qquad 4.11$$

In addition, these equations have to satisfy the continuity equation

$$\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} = 0$$

Now $[\dot{\varepsilon}_{xx}, \dot{\varepsilon}_{yy}, 0] = m^T Bv$ and thus the approximate continuity equation is

$$\underset{\sim}{m}^T \underset{\approx}{B} \underset{\sim}{v} = \xi$$

where ξ is the discretisation error.

If we chose $\underset{\sim}{L}$, the pressure interpolation function, as the weighting function, the weighted residual statement is

$$- \int \underset{\sim}{L}^T \underset{\sim}{m}^T \underset{\approx}{B} \underset{\sim}{v} = 0 \qquad 4.12$$

Thus, for each node, the Navier-Stokes equations become

$$\begin{bmatrix} K_{xx} & K_{xy} & J_x^T \\ K_{yx} & K_{yy} & J_y^T \\ J_x & J_y & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ p \end{bmatrix} + \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = 0$$

where

$$\begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} = \int_R \underset{\approx}{B}^T \underset{\approx}{D}\underset{\approx}{B} dR$$

$$[J_x \quad J_y] = - \int_R \underset{\sim}{L}^T \underset{\sim}{m}^T \underset{\approx}{B} dR$$

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \int_R \underset{\sim}{N}^T \rho \underset{\approx}{g} dR + \int_\Gamma \underset{\sim}{N}^T \underset{\sim}{N} \underset{\sim}{t} d\Gamma$$

For an element of n nodes, the structure of B given above is repeated n times columnwise, as is N, meaning that the elements K_{ij} of the matrices are in fact $n \times n$ sub-matrices. If the pressure is interpolated at ℓ nodes per element then the elements J_i are $\ell \times m$ matrices. The elements f_i of the vector $\underset{\sim}{f}$ are $m \times 1$ vectors.

The interpolation function for the strain rate is thus used as the weighting function for the momentum equations, and the interpolation function for the pressure is used as the weighting function for the continuity equation so as to preserve symmetry. It should be noted that the diagonal entry for the pressure equation is zero. This makes the resulting set of equations semi-definite (Zienkiewicz), and means that special techniques have to be used for the solution of the resulting set of equations.

Dirichlet conditions are inserted in exactly the same manner as for uni-axial flow. Neumann conditions usually come in the form of normal and tangential stresses at the surface of the domain of interest. To find the nodal force caused by these stresses, the stresses have to be resolved into x and y direction forces by use of the Cauchy formula (Eringen, 1967), i.e.

$$t = \sigma_b \nu_b$$

where σ_b is the state of stress at the boundary and ν_b is the normal to the boundary, and these forces have to be integrated along the boundary.

It is often found in glaciological studies that there is a Dirichlet condition imposed on the normal velocity at a boundary while there is no such condition on the tangential velocity. This boundary condition can be inserted by a rotation of the co-ordinate axes at a particular node such that one of the axes is tangential to the surface and the other normal. Then, the unknowns correspond to the tangential and normal velocities.

A formulation preserving matrix symmetry can be established as follows. Let the normal and tangential velocities w_n, w_t be related to the x and y direction velocities by the following relationship

$$W = Rv \qquad 4.13$$

where

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

where α is the slope the boundary makes with the x axis.

Because $R \cdot R^T = I$ where I is the unit matrix

$$v = R^T w$$

$$\hat{v} = NR^T w$$

and $$\dot{\varepsilon} = BR^T w$$

If we select the weighting function to be RN^T and use Green's Theorem, the weighted residual statements 4.11 become

$$\int_R BB^T DBR^T dR w + \int_R BB^T mL dR p + \int_R BN \rho g dR + \int_\Gamma RN^T . NR^T t d\Gamma = 0 \qquad 4.14$$

where the vectors v and t are now expressed in the rotated co-ordinate frame. This formulation is given by Zienkiewicz (1977).

Insertion of a Weertman bed-sliding condition may be effected if the rotated co-ordinate system is used. Then, the normal velocity is zero, and the tangential velocity is related to the boundary tangential stress by a relationship of the form of equation 4.9.

The weighted residual statement of this relationship is

$$\int_\Gamma RN^T NR^T t d\Gamma = \int_\Gamma RN^T SNR^T w d\Gamma \qquad 4.15$$

where

$$t = \begin{bmatrix} t_n \\ t_t \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$$

This may be inserted into equation 4.14 by replacing the appropriate fourth term in equations by the right-hand side of equation 4.15. Since $w_n = 0$ is specified by a Dirichlet condition, insertion of this does not force t_n to be zero.

This is a novel formulation. It should be noted that this is a weighted residual formulation using the velocities-pressure formulation and does not contradict Hutter's proof (1983) that a sliding law formulation may not be derived for the streamline formulation using variational principles.

The boundary conditions for the Navier-Stokes equation need to be specified with some care. For two-dimensional flow at least three velocities must be specified, one of which must be orthogonal to the other two. Otherwise, the matrix would be singular because three rigid-body motions are possible, translation in each of the two directions and rotation.

If all the boundary conditions are specified velocities, then the pressure must be specified at one point and one point only. Criteria for the best selection of this point are given by Jackson (1982). If Dirichlet conditions are specified then extreme caution must be taken to avoid a violation of the incompressibility condition. In practice, when using curved geometries, round-off error makes it impossible to respect this condition (Jackson, op. cit.) and the use of Dirichlet conditions has been kept to a minimum in this study.

If a Weertman boundary condition is used, then care has to be taken over the specification of the boundary conditions at the end of the Weertman section. If a node marks the boundary from a Weertman section to a Neumann section of the boundary, then no contribution from the Neumann section should be made to the force vector.

4.5 Solution of the non-linear equations

In many instances in glaciology it is found that the material properties of the ice or the ice-rock interface are not constant, but are dependent on the stress or velocity environment. In this case the viscosity or smoothness are not known. One way of solving this problem is to make an initial guess at the unknown material parameter and then solve the equations. The values of the solution are then used to calculate the element material properties, and a new solution is obtained. This process is continued until convergence is obtained, for example

$$\frac{\langle \underset{\sim}{u}^{n+1} \cdot \underset{\sim}{u}^{n+1} \rangle - \langle \underset{\sim}{u}^{n} \cdot \underset{\sim}{u}^{n} \rangle}{\langle \underset{\sim}{u}^{n} \cdot \underset{\sim}{u}^{n} \rangle} \leq \zeta \qquad 4.16$$

where $\langle \underset{\sim}{a} \cdot \underset{\sim}{a} \rangle$ is the inner-product of the vector $\underset{\sim}{a}$, $\underset{\sim}{u}^{n+1}$ is the new solution, $\underset{\sim}{u}^{n}$ is the previous solution and ζ is the convergence criterion.

This method was tried but it was found that convergence was not obtained.

A faster method of converging (in the sense that convergence is obtained after fewer iterations) is the Newton-Raphson scheme.

Let $\underset{\sim}{\psi} = \underset{\approx}{L}\underset{\sim}{u} + \underset{\sim}{b} = 0$ where $\underset{\sim}{\psi}$, $\underset{\sim}{u}$ and $\underset{\sim}{b}$ are $m \times 1$

vectors, $\underset{\sim}{u}$ is the unknown and $\underset{\approx}{L}$ is an $m \times m$ matrix dependent on $\underset{\sim}{u}$.

Now

$$\frac{\partial \underset{\sim}{\phi}}{\partial \underset{\sim}{u}} = \frac{\partial \underset{\approx}{L}\underset{\sim}{u}}{\partial \underset{\sim}{u}}$$

If, at the nth iteration, the estimate of $\underset{\approx}{L}$ is $\underset{\approx}{L}^n$ and $\underset{\sim}{u}$ is $\underset{\sim}{u}^n$, then an improved solution may be obtained using the Newton-Raphson formula following Zienkiewicz (1977)

$$\underset{\approx}{L}^{n+1}\underset{\sim}{u}^{n+1} + \underset{\sim}{b} \approx \underset{\approx}{L}^n\underset{\sim}{u}^n + \underset{\sim}{b} + \frac{\partial \underset{\approx}{L}\underset{\sim}{u}}{\partial \underset{\sim}{u}} \Delta \underset{\sim}{u}^n$$

where $\Delta \underset{\sim}{u}^n$ is a correction vector.

Since we wish $\underset{\approx}{L}^{n+1}\underset{\sim}{u}^{n+1} + \underset{\sim}{b} = 0$ and we define $\underset{\sim}{r}^n = \underset{\approx}{L}^n\underset{\sim}{u}^n + \underset{\sim}{b}$ we find that

$$\frac{\partial \underset{\approx}{L}\underset{\sim}{u}}{\partial \underset{\sim}{u}} \Delta \underset{\sim}{u}^n = - \underset{\sim}{r}^n$$

and this equation system may be solved for Δu^n.

We then set

$$\underset{\sim}{u}^{n+1} = \underset{\sim}{u}^n + \Delta \underset{\sim}{u}^n$$

and proceed with the iteration until condition 4.16 is satisfied.

$\underset{\approx}{L}\underset{\sim}{u}$ is an $m \times 1$ vector so $\frac{\partial(\underset{\approx}{L}\underset{\sim}{u})}{\partial \underset{\sim}{u}}$ is an $m \times m$ matrix. Consider the weighted residual statement 4.11, and note that

$$\underset{\approx}{L}\underset{\sim}{u} \equiv \int \underset{\approx}{B}^T . \underset{\approx}{D} . \underset{\sim}{\varepsilon} dR + \int \underset{\approx}{B}^T . \underset{\sim}{m} . \underset{\sim}{L} dRp$$

Noting also $\frac{\partial \dot{\underset{\sim}{\varepsilon}}}{\partial \underset{\sim}{v}} = \underset{\approx}{B}$ then, by differentiating with respect to $\underset{\sim}{v}$ and p we obtain

$$\frac{\partial \underset{\approx}{L}\underset{\sim}{u}}{\partial \underset{\sim}{u}} = \int_R \underset{\approx}{B}^T . \underset{\sim}{m} . \underset{\sim}{L} dR + \int_R \underset{\approx}{B}^T \frac{\partial \underset{\approx}{D}\dot{\underset{\sim}{\varepsilon}}}{\partial \dot{\underset{\sim}{\varepsilon}}} . \underset{\approx}{B} dR$$

Now

$$\underset{\approx}{D}\dot{\underset{\sim}{\varepsilon}} = \begin{bmatrix} 2\mu\dot{\varepsilon}_{xx} \\ 2\mu\dot{\varepsilon}_{yy} \\ 2\mu\dot{\varepsilon}_{xy} \end{bmatrix}$$

$$\underset{\approx}{D}_\Gamma = \frac{\partial \underset{\approx}{D}\dot{\underset{\sim}{\varepsilon}}}{\partial \dot{\underset{\sim}{\varepsilon}}} = 2 \begin{bmatrix} \mu + \dot{\varepsilon}_{xx}\frac{\partial\mu}{\partial\dot{\varepsilon}_{xx}} & \dot{\varepsilon}_{xx}\frac{\partial\mu}{\partial\dot{\varepsilon}_{yy}} & \dot{\varepsilon}_{xx}\frac{\partial\mu}{\partial\dot{\varepsilon}_{xy}} \\ \dot{\varepsilon}_{yy}\frac{\partial\mu}{\partial\dot{\varepsilon}_{xx}} & \mu + \dot{\varepsilon}_{yy}\frac{\partial\mu}{\partial\dot{\varepsilon}_{yy}} & \dot{\varepsilon}_{yy}\frac{\partial\mu}{\partial\dot{\varepsilon}_{xy}} \\ \dot{\varepsilon}_{xy}\frac{\partial\mu}{\partial\dot{\varepsilon}_{xx}} & \dot{\varepsilon}_{xy}\frac{\partial\mu}{\partial\dot{\varepsilon}_{yy}} & \mu + \dot{\varepsilon}_{xy}\frac{\partial\mu}{\partial\dot{\varepsilon}_{xy}} \end{bmatrix}$$

This matrix is asymmetric, meaning the grand solution matrix is no longer symmetric.

A similar procedure is followed for the Weertman boundary condition where the roughness is dependent on the sliding velocity. In this formulation the roughness was assumed independent of the local viscosity for both plane-strain flow and for uni-axial flow.

Convergence was achieved using the Newton-Raphson method, but it was not guaranteed. Success depended on setting the initial, first guess viscosities to values higher than were expected in the solution. Using a velocity dependent smoothness in the Weertman boundary condition caused convergence to fail.

4.6 Solutions for compressing flow

In Chapter 3 it was shown that ice could compress for at least three reasons. These were melting of the ice due to shear heating, melting of ice due to pressure changes and possible compressing due to rheological effects.

Finite formulations for describing the first two processes were developed. It was shown in Chapter 3 that the mass balance condition for internal melting could be described by the following expression when the flow was steady:

$$\dot{\varepsilon}_v - \chi\nabla^2 p = 0$$

where χ is the diffusivity which is a function of the permeability $\varkappa$.

As can be seen, the pressure now enters into the continuity equation through a diffusive term. This expression can be transformed using Green's theorem, and the resulting weighted residual statement for the continuity equation is

$$\int \underset{\sim}{L}^T \underset{\sim}{m}^T \underset{\approx}{B} dR \underset{\sim}{v} + \int_R \frac{\partial \underset{\sim}{L}}{\partial x}^T \chi \frac{\partial \underset{\sim}{L}}{\partial x} dRp + \int_R \frac{\partial \underset{\sim}{L}}{\partial y}^T \chi \frac{\partial \underset{\sim}{L}}{\partial y} dRp + \int_\Gamma \underset{\sim}{L}^T q d\Gamma = 0 \qquad 4.17$$

where $q = \varkappa \frac{dp}{d\nu}$ where ν is the normal to the boundary.

The momentum balance equations now have to be altered because the dilatational viscosity now has an effect (see Chapter 3). The viscosity matrix D (see equation 4.11) becomes

$$\begin{bmatrix} 2\mu + \lambda & \lambda & 0 \\ \lambda & 2\mu + \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

where λ is the dilatational viscosity.

The boundary conditions for the moisture flow take two forms. The first is a no-flow boundary or a boundary where the flow rate was specified. This is defined by defining q in the fourth term of 4.17. The other type of boundary was of water flow along the bed. This was assumed to be a Couette flow governed by Laplace's equation, suitable for an ideal fluid (Landau and Lifshitz, 1959), i.e. a flow where the discharge of water was assumed to be directly proportional to the pressure gradient in the sub-glacial water-film. The weighted residual stratement becomes, after use of Green's Theorem:

$$\int_{\Gamma} \frac{\partial \underset{\sim}{L}}{\partial \tau}^{T} C \frac{\partial \underset{\sim}{L}}{\partial \tau} \, d\Gamma - \int_{\Gamma} \underset{\sim}{L} q \, d\Gamma = 0$$

where τ is the direction tangential to the boundary, C is the conductivity of the water film and the second term is identical to the fourth term of equation 4.17.

If the bed is assumed to have been rotated into normal and tangential components, then the implementation of the equations is very easy.

In the actual finite element formulation, the water-pressure in the sub-glacial water film has been assumed to be equal to the pressure of the ice at the boundary of the ice. This is probably not true, but since the purpose of the solution was to provide an indication of the importance of along bed flow to the movement of ice, this approximation was felt to be adequate.

In the case of compressing flow around a clast, a route for water exists under the clast. Again Couette flow was assumed, and the same weighted residual statement as for the along bed flow was used, though no sources or sinks were permitted.

Shear-heating is very straightforward to describe using the velocities-pressures formulation. The continuity condition now becomes

$$\dot{\varepsilon}_v - \dot{\varepsilon}_{vm} = 0$$

where $\dot{\varepsilon}_{vm}$ is the rate of volumetric straining due to the melting of the ice. This rate of volumetric straining depends on how quickly the water is expelled. If it is assumed that the moisture content is steady, then the volumetric strain rate is related to the rate of heating by the relationship

$$\dot{\varepsilon}_{vm} = \frac{\sigma_{ij}\dot{\varepsilon}_{ij}}{H_F \rho}$$

where H_F is the latent heat of fusion of ice and the tensor summation convention is used. The weighted residual statement for the modified continuity condition is now

$$\int_R \underset{\sim}{L}^T \underset{\sim}{m}^T \underset{\approx}{B} dR \underset{\sim}{v} - \int_R \underset{\sim}{L}^T \dot{\varepsilon}_{vm} dR = 0$$

Since the volumetric strain rate is dependent on the motion in a non-linear manner, the set of equations becomes non-linear and an iterative method of obtaining the solution has to be used. This was done by direct iteration, i.e. the motion was solved for assuming no compression of the ice, the shear heating was calculated, the right-hand side modified and the process was continued. Since these calculations were done using a linear rheology, the grand stiffness[+] matrix remained unchanged. Advantage could thus be made of the ability to solve several systems of the equations with the same coefficient matrix by saving the factorised form of the matrix.

4.7 Problems with the velocities-pressure formulation

A problem with the velocities-pressure formulation is that unless special care is taken the pressure solution can exhibit large, spurious, oscillations (Zienkiewicz). This occurs because it is possible to set up a set of equations using the velocities-pressures formulation to solve the Navier-Stokes equations which, while completely specifying the velocity field, do not specify the pressure field. This problem is discussed by Jackson and Cliffe (1980). A partial way of getting round this problem is to interpolate the pressure using a lower order polynomial than is used for the velocities. This does not completely get round the problem, as specification of too many Dirichlet boundary conditions can still cause spurious pressure solutions to occur. This problem may also arise in other formulations which give rise to zeros on the diagonal of the grand solution matrix.

[+] In finite element terminology the solution matrix is often termed the 'stiffness matrix'.

4.8 Computer program for the finite element solution of the Navier-Stokes equations

A computer program has been written in FORTRAN IV to solve the Navier-Stokes equations using a finite element velocities-pressure formulation. The program consists of nearly ten thousand cards, and has the ability to solve uni-axial and plane strain flow problems.

Input is by a keyword system, giving flexibility in the data input order and robustness against data errors. Output comes in two forms, line printer and a data-set for use with a graphics post-processor written in FORTRAN 77 and using GHOST-80 (Sutherland and Prior, 1983) sub-routines.

After the data has been input, checked and undergone initial processing, the 'force vector' has been computed (i.e. the terms on the right-hand side arising as a result of the boundary conditions, body forces and internal melting) the element stiffness matrices are computed and stored. The grand stiffness matrix is then assembled row by row. This prevents the whole of the stiffness matrix from ever needing to be in store at one time.

Once the row of the stiffness matrix has been computed, the row is passed to the linear equation solver, where it is reduced. These factors are then stored or written out to disc until the back-substitution phase occurs, when the solution is obtained.

If the problem is linear, then a certain amount of processing of the results occurs so as to obtain stresses and strain rates, and the results are written to the line printer and to the graphics data set if requested.

If the problem is non-linear, then the solution is used to calculate the solution dependent material parameters, and the residual at each node is calculated by reassembling each row of the grand stiffness matrix and multiplying it by the new solution vector. The Jacobian matrix is then computed and the correction vector is obtained by the linear equation solver. The correction vector is added to the original solution vector, and convergence of the solution is checked. If convergence has been achieved, then the steps described in the preceding paragraph are carried out; otherwise the residual vector is found again, and the same processes are gone through again until convergence is obtained or the maximum number of iterations is reached. At each iteration a report on the accuracy of the solution is given.

As has been stated above, the matrices formed by the velocities pressures formulation are semi-definite, and in consequence require a special solution technique to obtain solutions. When the calculations were carried out, only one linear equation solver, the Harwell Subroutine Library MA32 package (Hopper, 1981) was available to solve these equations. This package is not able to take account of any symmetry in the matrix, and thus this part of the saving introduced by using the Galerkin formulation was lost. However, a saving was obtained by only having to compute half the matrix, which is a considerable portion of the cost of the computation in linear problems.

The elements used in the two-dimensional calculations were isoparametric quadratic triangles (Zienkiewicz). This means that the elements are triangular in shape, and that the sides of the triangles are defined by quadratic functions. The sides may therefore be curved in shape, which permits more accurate modelling of curved boundaries. The triangles are defined by six nodes; three at the vertices, and three along the edges. The closer the edge nodes are to the mid-point of the side, the more accurate the solution is (Zienkiewicz) and this condition has been adhered to as far as is possible. Because there are three nodes along each edge, the trial function for the velocities, which has to go through the velocity values at each of the nodes, is quadratic. This increases the accuracy of the velocity solution compared with linear interpolation over half the grid size, but results in greater connectivity of the nodes which in turn results in greater expense in both assembling the elements and solving the set of linear equations. Because the velocities-pressure formulation requires that the pressure be interpolated to a lower order than the velocities, the pressures are interpolated linearly and therefore only solved for at the vertex nodes. This in turn means that the continuity condition is only imposed at the corner nodes. Quadratic and linear interpolation functions over triangles are given in Zienkiewicz. Pressure was also interpolated linearly in the compressing flow formulation. Quadratic interpolation of the velocities over quadratic isoparametric triangles was used in the uni-axial flow formulation.

Triangles were selected because of their ability to represent complex geometrical shapes with greater facility than rectangles. In a three dimensional formulation, cubic quadratic elements must be used because of the problems in envisaging combinations of tetrahedrons. Interpolation formulae for these elements are also given in Zienkiewicz.

In non-linear problems element properties (e.g. viscosities) vary over the element. However, it has been found more accurate to calculate the property at the element mid-point and use this over the whole element (G. Pinder, personal communication).

4.9 The testing of the finite element program

The program was tested by using the 2 element model shown in Figure 4.1. Simple shear was specified by setting the velocities along the bottom edge to zero and by specifying shearing functions along the other edges. Accurate solutions were obtained for linear and Glen rheologies.

When Dirichlet boundary conditions were specified all around the boundary the correct velocity solution was obtained for a linear rheology, but the pressure solution was inaccurate due to it adopting a higher mode of variation (Jackson and Cliffe, 1980). When this model was run with a Glen rheology the influence of the poor pressure solution caused non-convergence of the solution.

The Weertman boundary condition was checked for pure shear using traction boundary conditions and found to give exact answers.

The program was tested to simulate flow under gravity down an infinite inclined plane. The solution to this is given by Carson (1971).

$$v_x = \frac{\rho g_x}{2\mu} z(2h - z), \qquad v_z = 0$$

$$p = \rho g_z(h - z)$$

the co-ordinate system xz is arranged such that x lies down the line of maximum slope, z is the normal to the plane, h is the height of the free surface, v_x, v_z are the velocities in the subscript direction, g_x, g_z are the accelerations due to gravity in these directions, p is the pressure and the pressure at the free surfaces is zero. The stress field $\underset{\approx}{\sigma}$ is given by

$$\underset{\approx}{\sigma} = \begin{bmatrix} \rho g_z(h - z) & \rho g_x(h - z) \\ \rho g_x(h - z) & \rho g_z(h - z) \end{bmatrix} \qquad 4.18$$

Since the variation in velocities is quadratic and in pressure linear, a two element model representing a rectangle (Figure 4.1) should represent the flow. A zero velocity boundary condition was placed along the bottom of the section, and the appropriate stresses were put along the edges according to equation 4.18.

The finite element solution was found to agree exactly with the analytical solution to 4 decimal places (the number printed).

The velocity profile for a Glen rheology is given by Carson (1971):

$$v_{xs} - v_x = \frac{2A\rho^n g^n \sin^n (h-z)^{n+1}}{n + 1}$$

where v_{xs} is the surface velocity and A and n define the stress strain relationship proposed by Glen (1955) (see Chapter 3) viz.

$$\dot{\varepsilon}_{xz} = A\tau_{xz}^{n}$$

A value of n = 3 was selected.

Test runs were carried out using a 9 × 9 node grid and a 17 × 17 node rectangular grid built up from the 3 × 3 node grid shown in Figure 4.1.

Plots of $\log(v_{xs} - v_x)$ against $\log(h - z)$ are shown in Figures 4.2 and 4.3. It should be noted that the surface velocities are different and that the slope of the line for the 9 × 9 grid is 1.96 and for the 17 × 17 grid is 2.88 rather than the 4.0 predicted by analysis. It was noted that convergence was very slow for this problem, which is often found when Neumann boundary conditions are used (Zienkiewicz), and thus convergence may not have been attained. Strictly speaking the surface viscosity is infinite and the model therefore underestimates viscosity gradients.

The velocity distribution of a fluid between two co-axial infinitely long rotating cylinder was also determined. The solution for this is given by Landau and Lifshitz (1959).

$$v_c = \frac{\Omega_o R_o^2}{R_o^2 - R_I^2} r - \frac{\Omega_o R_o^2 R_I^2}{R_o^2 - R_I^2} \frac{1}{r} \qquad 4.19$$

where Ω_o is the angular velocity of the outer cylinder, the inner cylinder is stationary, R_o is the radius of the outer cylinder, R_I is the radius of the inner cylinder, r is the distance from the centre and v_c is the circumferential velocity.

A 5 × 5 grid was set up with $R_I = 1$, $R_O = 2$, and $\Omega_O = \frac{1}{2}$. A segment subtending 10° at the centre was used.

Traction boundary conditions with a pressure of 0.0 were specified. Figure 4.4 shows the circumferential velocity plotted against $r - \frac{1}{r}$. Equation 4.19 predicts this to be linear. A good fit to a straight line was obtained (r = 0.999). The pressure solution showed slight errors.

Figure 4.4 shows the analytical solution and the solution computed using the finite element program using a five by five grid. This shows that even a coarse grid gives accurate solutions.

The uni-axial flow formulation was tested for pure shear using linear and Glen rheologies, with and without Weertman boundary conditions using Dirichlet boundary conditions. The results were found to be exact. No facilities for Neumann boundary conditions or gravity-driven flow were programmed, so these were not tested.

The testing of the compressing flow formulation was hampered by lack of analytical solutions for the case $\nabla^2 p \neq 0$.

Validation was effected having the two element configuration (Figure 4.1) represent a constant pressure gradient. This was done by setting traction boundary conditions that gave a uniform pressure gradient and checking that the computed boundary moisture fluxes were correct. They were found to be exact.

An analytical solution for the water flow along the one-dimensional bed could not be found. However the uncoupled variant of the formulation, representing under-clast flow, was tested under a uniform pressure gradient and found to give exact answers.

The shear-heating formulation was tested for pure-shear with a two-element model (Figure 4.1) and found to give exact results.

The testing program showed that accurate answers could be obtained for linear rheologies and Glen rheologies where the variations in viscosities were not severe for both the plane-strain formulation and the uni-axial formulation. When there were large variations in the viscosity produced as a consequence of using a Glen rheology, it was found that the finite element formulation tended to under-estimate the variations in the viscosity.

The fact that sufficiently demanding analytical solutions could not be obtained for the compressing flow formulation means that the results from computations using this formulation should be treated with some caution. However, the results described in the next chapter indicate that results are what might be expected, and it was concluded that the formulation was probably correct.

4.10 Lagrangian multiplier formulation for the Weertman bed sliding condition and for inclusion motion

An attempt was made to extend the velocities-pressures formulation to model the effect of inclusions within ice. Inclusions are stones, and are therefore rigid. The surface of the stone may be regarded as smooth, in which case the tangential traction imposed by the ice is zero; alternatively, the surface can be regarded as rough, in which case the traction can be regarded as proportional to the tangential velocity (Watts, 1974).

The motion of the clast has three unknowns in plane strain flow; the x and y direction velocities and the angular velocity. Since the formulation is steady state, no acceleration is allowed. This means that the sum of the forces must be zero, and that the total moment exerted on the inclusion must also be zero.

The velocity of any point on the surface of the inclusion is related to the angular velocity ω by the equation

$$\begin{bmatrix} v_{xs} \\ v_{ys} \end{bmatrix} = \begin{bmatrix} 1 & 0 & r\sin\beta \\ 0 & 1 & -r\cos\beta \end{bmatrix} \begin{bmatrix} v_{xc} \\ v_{yc} \\ \omega \end{bmatrix} \qquad 4.20a$$

where v_{xs}, v_{ys} are the velocities on the surface, v_{xc}, v_{yc} are the velocities of the centre of mass of the inclusion, r is the distance of the surface point from the centre of mass of the inclusions and β is the angle the line joining the centre of mass to the surface point makes with the x-axis.

This equation may be put in matrix form:

$$\underset{\sim}{v} = \underset{\approx}{A}\, \underset{\sim}{c} \qquad 4.20b$$

It is convenient to take the axis about which the clast rotates as the centre of mass of the inclusion. These velocities can be rotated into a set of velocities normal and tangential to the interface by using equation 4.13.

The conditions for the ice velocity at the interface, assuming no separation, are that the velocity of the ice normal to the interface is equal to the velocity of the clast surface in the same direction at this point, and that the difference between the velocities of the ice and the clast surface tangential to the surface is equal to the roughness of the interface multiplied by the tangential stress at the interface. Two extra equations have been introduced at the expense of introducing four new variables; the clast x and y velocities, the clast angular velocity and the tangential traction at the clast-ice interface. To obtain a solvable set of equations, a further unknown is introduced; the normal traction at the ice rock interface.

To solve the equations an element is constructed containing an arbitrary odd number of nodes whose geometry is defined by piecewise quadratic interpolation. For each node the following set of equations are constructed. Eight equations have to be solved: the x and y momentum balances for each surface point, the incompressibility condition for the ice at the surface point, the conditions for the surface ice velocities, the x and y momentum balances for the inclusion, and the angular momentum balance for the inclusion. Using the notation from equation 4.4 we find that these eight equations can be represented by four matrix equations:

$$\underset{\approx}{D}\underset{\approx}{B}\underset{\sim}{v} + \underset{\sim}{m}\underset{\sim}{L}p + \underset{\approx}{R}^T\underset{\sim}{s} = 0 \qquad 4.21a$$

$$\underset{\sim}{m}^T\underset{\approx}{B}\underset{\sim}{v} = 0 \qquad 4.21b$$

$$\underset{\approx}{R}\underset{\sim}{v} + \underset{\approx}{S}\underset{\sim}{s} - \underset{\approx}{R}\underset{\approx}{A}\underset{\sim}{c} = 0 \qquad 4.21c$$

$$-\underset{\approx}{A}^T\underset{\approx}{R}^T\underset{\sim}{s} = 0 \qquad 4.21d$$

where $\underset{\sim}{s} = \begin{bmatrix} s_n \\ s_t \end{bmatrix}$ = the normal and tangential tractions at the the surface, $\underset{\approx}{S} = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}$ where S is the smoothness, and $\underset{\approx}{A}$ is defined in equations 4.20.

Equation 4.21a represents the momentum balance, 4.21b the continuity equation, 4.21c the equations defining the normal velocity and the tangential velocity, and 4.21d the momentum balance for the inclusion.

The origin of the first and second terms in equation 4.21a have been explained previously. $\underset{\sim}{s}$ is the surface traction which has to be rotated into x and y co-ordinates. The origin of 4.21b has been explained previously.

Equation 4.21c is framed in terms of normal and tangential velocities. Thus, the x and y velocities at the surface have to be rotated into their components normal and tangential to the surface. The normal velocity is equal to the velocity of the clast at the surface. The difference between the tangential velocity of the ice and the inclusion velocity in this direction is equal to the smoothness S multiplied by the tangential traction s_t. If S is zero, there is no slip. If S is very large, s_t becomes very small, and the boundary condition tends to perfect slip.

Noting that $\underset{\sim}{t} = \underset{\approx}{R}^T \underset{\sim}{s}$, substituting this into 4.21d and multiplying out, the following set of equations is obtained:

$$t_x = 0$$

$$t_y = 0$$

$$t_x r\sin\alpha - t_y r\cos\alpha = 0$$

These equations are for a node i, and for an element modelling an inclusion with n nodes the equations are

$$\sum_{i=1}^{n} t_{xi} = 0$$

$$\sum_{i=1}^{n} t_{yi} = 0$$

$$\sum_{i=1}^{n} (t_{xi} r_i \sin\alpha_i - t_{yi} r_i \cos\alpha_i) = 0$$

where the subscript i refers to the ith node.

These equations respectively define the momentum balance on the inclusion in the x and y directions and the total couple on the clast to be zero.

The tractions enter as Lagrangian multipliers (Zienkiewicz 1977) in the velocity constraint equations and the inclusion velocities $\underset{\sim}{c}$ as Lagrangian multipliers in the inclusion momentum balance equations.

Weighted residual statements are made as follows:

$$\int_R \underset{\approx}{B}^T \underset{\approx}{D} \underset{\approx}{B} \underset{\sim}{v} dR + \int_R \underset{\approx}{B}^T \underset{\sim}{m} \underset{\sim}{L} p dR + \int_R \underset{\approx}{N}^T \underset{\approx}{R}^T \underset{\sim}{J} \underset{\sim}{s} \, d\Gamma = 0 \qquad 4.22a$$

$$\int_R \underset{\sim}{L}^T \underset{\sim}{m}^T \underset{\approx}{B} \underset{\sim}{v} dR = 0 \qquad 4.22b$$

$$\int_\Gamma \underset{\sim}{J}^T \underset{\approx}{R} \underset{\sim}{N} \underset{\sim}{v} d\Gamma + \int_\Gamma \underset{\approx}{J}^T \underset{\approx}{S} \underset{\sim}{J} \underset{\sim}{s} d\Gamma - \int_\Gamma \underset{\sim}{J}^T \underset{\approx}{R} \underset{\approx}{A} \underset{\sim}{c} d\Gamma = 0 \qquad 4.22c$$

$$-\int_\Gamma \underset{\approx}{A}^T \underset{\approx}{R}^T \underset{\sim}{J} \underset{\sim}{s} d\Gamma = 0 \qquad 4.22d$$

where $\underset{\sim}{J}$ is the interpolation function for the surface stresses. Because the inclusion velocities are point values they require no interpolation functions.

This element set of equations can be assembled into the

Navier-Stokes equations in the normal fashion.

Tests with this formulation proved very unsuccessful. The programming was checked to determine that the coefficients being formed were correct. This having been confirmed, the robustness of the formulation was checked by simplifying it. In section 4 a method of inserting the Weertman boundary condition was given. The method outlined above for solving clast transport problems can be simplified to give an alternative formulation for the same boundary condition.

These equations are exactly the same as equations 4.22 but now $\underset{\sim}{c} \equiv 0$.

The order of the interpolation functions for the normal and tangential tractions can be chosen arbitrarily. Two orders were tried: quadratic and linear. With quadratic interpolation the results obtained were poor. With linear interpolation, the following sets of results were obtained with a two element model (Figure 4.1). The basal boundary condition was a Weertman bed with a smoothness of 1.0. The boundary conditions on the other sides were either all of the von Neumann type or of the Dirichlet type. In each case they were designed to produce simple shear with the shear stress being one arbitrary unit, thus producing a sliding velocity of one. Four computations were performed. The results are given in Table 4.1 along with the true results.

In the cases where the pressure was set to zero, it can be seen that the velocities, pressure and the tangential traction computed are correct, but that the computed normal traction shows a spurious mode; i.e. instead of the result being one, the normal traction takes on a higher component of

Variable	Traction boundary conditions				Dirichlet boundary conditions			
	Ambient Pressure = 0		Ambient Pressure = 1		Ambient Pressure = 0		Ambient Pressure = 1	
	True	Computed	True	Computed	True	Computed	True	Computed
Node 1 Pressure	0.0	0.0	1.0	0.713	0.0	0.0*	1.0	1.0*
Node 1 Tangential Pressure	1.0	1.0	1.0	0.803	1.0	1.0	1.0	0.898
Node 1 Normal Traction	0.0	1.0	-1.0	0.0	0.0	1.0	-1.0	0.237
Node 7 Pressure	0.0	0.0	1.0	0.742	0.0	0.0	1.0	-0.0485
Node 7 Tangential Traction	1.0	1.0	1.0	1.20	1.0	1.0	1.0	1.11
Node 7 Normal Traction	0.0	-1.0	-1.0	-2.0	0.0	-1.0	-1.0	-0.804
Node 3 Pressure	0.0	0.0	1.0	1.14	0.0	0.0	1.0	-0.152
Node 9 Pressure	0.0	0.0	1.0	1.11	0.0	0.0	1.0	1.07
Velocity solution	Exact		Sliding velocities poor. Rest approximately correct		Exact		Sliding velocities poor, others correct because mainly defined by boundary conditions	

* Prescribed pressure

Table 4.1

Results from use of the Lagrangian multipliers formulation for the Weertman boundary condition variation, and only has an average value of zero. When the pressure is set to one, the normal traction shows the same oscillation about the correct value. However, all the other solutions show deterioration, presumably because the spurious mode of the normal traction is now interfering with them in some undetermined way. The tangential traction does seem to show a spurious mode, because the average value is close to the true average value, but the same is not true for the pressure solution.

When the order of interpolation is set to one, with the quadratic nodes being used the result is that the mid-side nodes are not constrained by the correct boundary condition. When this Weertman boundary condition was run with the grids described in Chapter 5, where there is flow around a curved boundary, the results obtained were very poor. This was not surprising in view of the spurious traction modes, the lack of proper constraint and the probable ill-conditioning of the resulting set of equations due to their lack of diagonal dominance.

The poor results obtained with the higher order of interpolation were almost certainly due to spurious traction modes being introduced.

Though the causes of these spurious modes were not investigated in detail, it is likely, in view of the similar nature of the equations, that these modes arise in exactly the same way as do the spurious modes of the pressure solution described above. The remedy of dropping the order of interpolation does not seem to work.

While this formulation is not the best way of solving the simple Weertman condition, it is needed for the inclusion transport problem because the extra variables defining the clast velocity have to be solved for. Thus, currently, there is no finite element method for solving the equations of motion of inclusions carried by shearing ice.

4.11 Conclusions

A finite element program has been written to simulate the steady-state flow of glacial ice with a variety of rheologies. Novel formulations have been introduced for the Weertman boundary condition in basal sliding, compressing flow, water flow along the bed and the transport of inclusion within ice. The Weertman boundary condition has been found to be reliable. The compressing flow formulation and bed water flow formulation have not been rigorously tested because of the lack of available analytical solutions. The clast transport formulation was found to be unusable as a result of numerical problems.

It was found that when modelling situations with a strain-rate dependent viscosity, large numbers of elements had to be used in order to obtain answers that were not particularly accurate. Thus, in flow simulations where a Glen rheology is used, the results should be interpreted with a certain amount of caution.

CHAPTER 5

THE FINITE ELEMENT ANALYSIS OF SOME GLACIAL FLOWS AT ICE-ROCK INTERFACES

5.1 Introduction

The principles of finite element analysis have been described in the previous chapter. In this chapter we discuss the application of the finite element method to certain flow situations at the glacier sole.

Owing to the large demands made on computing resources, the number of calculations performed had to be kept to a minimum, and it was not therefore possible to perform as many simulations as was desired. Typically, a solution of a plane-strain flow problem with a Glen rheology took around 60s processing time on a CRAY 1 computer. For uni-axial flow such a problem took of the order of 10s.

Sections 5.2 to 5.9 discuss the results of the plane-strain flow analysis, while the uni-axial flow modelling is discussed in section 5.10.

5.2 Calculations carried for plane-strain flow

Two basic meshes were used, one for flow over a semi-cylindrical ridge (Mesh C) and one for flow over a truncated sine ridge (Mesh S), the sine ridge being defined between 0 and π. For reasons enlarged upon below, the boundaries for these meshes had to be at large distances from the obstruction. For this reason a nested approach was used. The elements distal from the ridge were given large dimensions, while closer to the ridge, where it was anticipated that gradients in the velocity and pressure fields would be greater, the elements were smaller. Figure 5.1 shows the nesting approach for Mesh C, while Figures 5.2 and 5.3 show details from Mesh C and Mesh S respectively close to the obstruction.

The mesh design was arrived at after a certain amount of experimentation. Because of the ninety degree corner in Mesh C more elements were required in this mesh than in Mesh S. Mesh C incorporated 769 nodes and 352 six-noded iso-parametric triangular elements. With the velocities interpolated quadratically over the elements and their pressures linearly, this gave a total of 1747 unknowns. Mesh S used 555 nodes and 252 of the same elements and had a total of 1262 unknowns.

The maximum dimensions of Mesh C were 21 nodes in the y direction and 43 nodes in the x directions. The surface of the semi-cylinder was described by 25 nodes.

Mesh S had dimensions 15 nodes by 45 nodes. The surface of the sine ridge was described by 17 nodes.

The mesh sizes are dimensionless, and thus all lengths are defined in terms of arbitrary length units (alu). The regular triangles close to the semi-cylindrical ridge (Figure 5.2) have length 1 alu, and the vertical element boundaries in Mesh S (Figure 5.3) are separated by 1 alu.

In both cases the overall physical dimensions of the grids were 6000 alu in the x-direction and 3000 alu in the y-direction. The radius of the semi-cylinder was 4 alu. The height of the sine ridge was 4 alu, and its wavelength 16 alu.

For both meshes the problem solved was the passage of the obstruction through a medium filling a half-plane. Table 5.1 lists the calculations performed. Because of a slight gridding error, the Mesh C obstacle is not quite a semi-cylinder: however, it is unlikely that this affected results significantly.

Table 5.1 Plane strain calculations

No	Mesh[1]	Rheology[2]	Bed[3] Smoothness alu/bar.a	Diffusivity[4] alu^2/bar.a	Cavity[5] Pressure Bar	Distal Velocity alu/a
1	C	L				10
2	C	L				50
3	C	L				100
4	C	L				200
5	C	L				400
6	C	L			0	10
7	C	L			0	400
8	C	L		1		10
9	C	L		1		50
10	C	L		1		100
11	C	L		1		200
12	C	L		1		400
13	C	L		10		10
14	C	L		10		50
15	C	L		10		100
16	C	L		10		200
17	C	L		10		400
18	C	L	10			10
19	C	L	50			50
20	C	L	100			100
21	C	L	200			200
22	C	L	400			400
23	C	G				10
24	C	G				50
25	C	G				200
26	C	G				400
27	C	G			0	10
28	C	G			0	400
29	C	G		1		10
30	C	G		1		50
31	C	G		1		100
32	C	G		1		200
33	C	G		1		400
34	C	G	10			10
35	C	L		S		10
36	C	L		S		50
37	C	L		S		100
38	C	L		S		200
39	C	L		S		400
40	S	L				10
41	S	L				50
42	S	L				100
43	S	L				200
44	S	L				400
45	S	G				10
46	S	G				50
47	S	G				100
48	S	G				200
49	S	G				400

Table 5.1 Plane strain calculations (continued)

1. C - Semi-cylindrical ridge
   S - Truncated sine ridge

2. L - Linear rheology
   G - Glen rheology

3. No smoothness shown indicates perfect slip

4. No diffusivity shown indicates incompressible rheology
   S - indicates shear-heating formulation

5. No cavity pressure shown indicates no cavity

Boundary conditions were set as follows. Along the bed (see Figure 5.4) the normal velocity was set to zero, except when a cavitation boundary condition was specified on the lee-side of the obstacle. In this case, the velocities were undefined, and the normal traction was set to the desired cavity pressure and the tangential traction to zero. The ice was fixed to the leading edge of the half-cylinder, and also to the trailing edge for non-cavitated problems.

Along the distal edges of the mesh the normal traction (the "distal pressure") was set at 10 bars, corresponding to an approximate glacier depth of 110m. For perfect slip problems the tangential traction along these edges was set at zero, while for a rough bed the tangential traction was set at 1 bar.

The velocity of the obstruction (the "distal velocity") through the medium was set at nodes where the traction boundaries met the glacier bed. These could be set arbitrarily for non-shearing flow. For shearing flow, these velocities had to be set in conjunction with the bed smoothness such that at the boundary the relationship

$$v_\tau = ST_\tau$$

where v_τ is the sliding velocity, S is the bed smoothness and T_τ is the tangential traction. The shearing stress was, in all cases, 1 bar.

When there is an obstruction and the ice is in motion the boundary conditions at infinity and at the obstruction are not consistent. There is a trivial reason which applies to both shearing and non-shearing flows which is that the boundaries are not strictly at infinity, but only at a very large distance. There is a rather more subtle inconsistency which applies only to the non-shearing flows.

This inconsistency is touched upon by Watts (1974) in his work on the flow of regelating ice past obstacles, and expounded upon more fully by severalwriters on fluid dynamics, for example Landau and Lifshitz (1959). At large distances from the obstacle the convective momentum terms begin to dominate the viscous terms. Since the fluid is supposed to be infinite in extent, this domination will occur sooner or later, whatever the Reynolds' number of the flow. The domination of the fluid motion by convective terms means that compatible boundary conditions do not exist for flows both at infinity and at the cylinder surface. However, while this problem does mean that the flow in the whole region cannot be solved for, the flow in the regions close to the obstruction, which is the area of interest, can be solved for a linear rheology, as here the viscous terms dominate the flow.

The consequence of this incompatibility is that the flows in the regions distal from the obstruction are not solved accurately in these finite element analyses. It might be argued that these poor solutions in the distal region of the flow could be due to ill-conditioning caused by the element shapes. However, the flows in the distal regions for the shearing flow with the rough basal boundary were very accurately computed, meaning that the inaccuracy in the distal flows observed in the non-shearing flow cannot be the result of this form of ill-conditioning.

In analyses performed using a Glen rheology, it was noted that the flow solutions in the distal regions were rather better than those obtained using the linear rheology. The reason for this is presumably related to the fact that in this instance the viscosities tend to infinity at infinity, meaning that the momentum terms cannot dominate the viscous terms as they do for a linear rheology.

It could be argued that since using infinite boundaries causes problems, then it might be better to use some other boundary conditions. This however, is by no means easy. A problem with the velocities-pressure formulation is that if too many Dirichlet boundary conditions are used, the pressure solution exhibits spurious oscillations (see Chapter 4). Thus, while it is possible to design a set of Dirichlet boundary conditions that do not globally violate mass-balance, these can result in the pressure solution being unreliable. Thus, traction boundary conditions had to be set. The only way to set consistent traction boundary conditions for incompressible flow is to set them at infinity. Thus, quasi-infinite boundaries have been used in these calculations.

Initial calculations for a Glen rheology with a very tight convergence criterion indicated that the Newton-Raphson scheme did not allow the solution to converge very fast. The optimum over-relaxation factor (Zienkiewicz, 1977) was found to be 1.6. Because steady-state problems were being solved, high accuracy was not essential. Thus, in order to save upon the large demands of computing resources made by the finite element analysis, a loose convergence criterion was set. Convergence of the velocity solution was defined as the inner-product of the updated velocity solution with itself not differing from that of the previous interaction by more than 5%. Since this is a constraint on global error it is possible for individual nodal values not to have converged by this amount. However, in incompressible flow disturbances propagate very rapidly, meaning that a gross error in one particular value would not get diffused out by the system; in other words if one nodal value has an error, it is probable that other nodal values will be in error.

Because a lower order of pressure interpolation than for the velocity interpolation is required in the velocities-pressure formulation (see Chapter 4), the pressure solution tends to be less accurate than the velocity solution. A consequence of this is that the convergence of the pressure solution is slower than that of the velocity solution. In consequence, the convergence criterion of the pressure solution was set at 10%.

It was also found that the velocity solution reached the velocity convergence criterion of 5% before the pressure solution reached its convergence criterion of 10%, and that by the time the pressure solution had converged to within 5% the velocity solution had generally converged to between 0.1% and 1.0%. Another way of looking at this is to say that an accurate velocity solution is required before the pressure solution can converge.

In general the computed points show a certain amount of scatter. This is due to round-off error and in the case of the non-linear rheologies, incomplete (though satisfactory) convergence. If it is assumed that these errors are normally distributed, regression analysis may be used.

For linear rheologies a viscosity of 1 bar.a was used, a typical value for glaciers. For non-linear rheologies, the viscosity μ is defined by

$$\mu = 1.815\varepsilon^{-\frac{2}{3}}$$

where ε is the square root of the second strain rate invariant (see Chapter 3) measured in a^{-1} and μ is measured in bar.a.

The diffusivity χ has been defined in Chapter 3 (equations 3.17, 3.18) by the relationship

$$\chi = \frac{\rho^{iw}\varkappa}{(1 - \theta)\rho^{i}}$$

If we assume $\rho^{iw}/(1 - \theta)\rho^{i}$ to be $O(1)$ then $\chi \approx \varkappa$. In Chapter 3 $\varkappa$ was estimated to be of the order of $10 m^2$/bar.a. Thus, the diffusivity χ is also of the order of $10 m^2$/bar.a. Now if 1 alu = 1 m, χ is of the order of 10 alu^2/bar.a, while if 1 alu = 1 mm, χ is of the order of 10^{-5}alu^2/bar.a.

In computations performed using a compressible rheology, the diffusivity has been set at a constant value throughout the mesh. This is not physically realistic, as was explained in Chapter 3. However, if was felt to be best to investigate the properties of a uniform compressing fluid initially.

For fluids with a linear rheology the dilatational viscosity was set so as not to violate the Stokes condition. Some calculations were performed where the dilatational viscosity was set to zero (for want of any better value, see Chapter 3). These calculations did not produce significant changes in the results. For the compressing Glen rheology the dilatational viscosity was set to zero.

Bed smoothnesses have units of alu/bar.a. No empirical values exist: however T_τ is $O(1)$ bar, v_τ is $O(100)$ m/a, giving $S = O(100)$ m/bar.a on the large scale.

In section 3 to 9 the results from the plane strain computations are reported. For uncavitated flow, the problem is symmetric. Results from the linear problems respected this symmetry, and for non-linear flow this symmetry was respected approximately. Since this thesis is concerned with glacial erosion, the flows of ice around cavity roofs have not been reported, but the flows in the contacted (eroding) upstream areas have been.

Owing to data-handling problems it is not possible to report the results from the shear-heating formulation in detail. However, it is worth noting that for distal velocities of 10 and 50 alu/a, convergence of this non-linear problem was obtained, while for distal velocities of 100 alu/a and above, convergence was not obtained. This may be indicative of a compressional shear instability (see Chapter 3), and would bear further investigation.

Regression analysis has been used. Calculations were performed using subroutines GØ2BAF, GØ2CAF AND GØ2CGF from the NAG Library (NAG, 1981). Significance testing using the t-test was carried out by following procedures outlined in Walpole and Myers (1978). Unless stated otherwise, a two-tail 99% confidence limit has been used.

5.3 Velocity Distributions

When considering glacial abrasion, the most important aspect of velocity distributions are the slip velocities at the ice-rock interface. In this section these slip velocities are reported. For the semi-cylindrical ridge the slip velocity v_τ has been regressed, linearly and logarithmically, against the sine of the angle γ subtended at the centre of the base of the semi-cylinder.

In addition, some arrow plots of velocities have been produced to show more generalised flow patterns. The horizontal and vertical scales of these plots are equal. Arrows originate at the nodes, point in the direction of flow and have lengths proportional to the speed at that point.

Table 5.2 shows the results of the regression for v_τ against sin γ .

The linear regression implies a relationship of the form

$$v_\tau = C + D\sin\gamma \qquad 5.1$$

where C and D are constants, and the logarithmic regression a relationship of the form

$$v_\tau = A\sin^m\gamma \qquad 5.2$$

where A and m are constants.

These regressions were done to find the broad patterns of relationships so that they could be used in further analysis. Thus, in certain cases it was found that although the correlation was very strong, the relationship was obviously more complex, and no attempt has been made to fit more complex models or to explain these higher order deviations.

In general m was found to vary slightly with the distal velocity v_d. The coefficient A was found to vary with v_d, and A has therefore been regressed on v_d. The results are given in Table 5.3. Table 5.4 gives the inferred relationships between v_τ, v_d and γ using an average value of m.

Figure 5.5 shows a velocity vector plot for a linear incompressible rheology with a distal velocity of 100 alu/a. A plot showing a larger region is given in Figure 5.6. These show that the constraint of no normal velocity has been satisfied, as do all the subsequent vector plots.

Immediately in front of the leading edge there is a relatively stagnant region, which is also found immediately behind the trailing edge. The velocity vectors have a significant upwards component at distances of 30 alu upstream of the obstruction.

Figure 5.7 shows the regressions of v_τ against $\sin\gamma$. In

Table 5.2 Regression of v_τ against $\sin\gamma$ for the semi-cylindrical ridge

No	R[1]	S[2]	x[3]	P_c[4]	V_d[5]	Linear regression[6]			Logarithmic regression[7]			
						r	C	D	r	A	m	Δm
1	L				10	1·00	0·001	1·43	1·00	1·43	1·01	0·02
2	L				50	1·00	0·003	7·15	1·00	7·15	1·01	0·02
3	L				100	1·00	0·006	14·3	1·00	14·3	1·01	0·02
4	L				200	1·00	0·012	28·6	1·00	28·6	1·01	0·02
5	L				400	1·00	0·025	57·2	1·00	57·2	1·01	0·02
6	L			0	10	-0·948	1·41	- 12·8	0·985	11·0	1·28	0·22
7	L			0	400	0·998	-2·79	76·2	1·00	74·1	1·15	0·02
8	L		1		10	0·981	-0·02	1·39	0·999	1·37	1·05	0·04
9	L		1		50	0·981	-0·12	6·95	0·999	6·85	1·05	0·04
10	L		1		100	0·981	-0·24	13·9	0·999	13·7	1·05	0·04
11	L		1		200	0·981	-0·48	27·8	0·999	27·4	1·05	0·04
12	L		1		400	0·981	-0·95	55·6	0·999	54·8	1·05	0·04
13	L		10		10	0·982	0·03	1·40	0·999	1·38	1·06	0·05
14	L		10		50	0·982	-0·13	7·00	0·999	6·88	1·06	0·05
15	L		10		100	0·982	-0·25	14·0	0·999	13·3	1·06	0·05
16	L		10		200	0·982	-0·50	27·9	0·999	27·5	1·06	0·05
17	L		10		400	0·982	-1·00	55·8	0·999	55·0	1·06	0·05
18	L	10			10	0·998	-0·50	11·1	0·999	10·7	1·14	0·04
19	L	50			50	0·999	-0·72	35·6	0·999	35·5	1·07	0·02
20	L	100			100	0·998	-1·00	61·9	1·00	60·3	1·06	0·02
21	L	200			200	1·00	-1·47	105	1·00	105	1·05	0·02
22	L	400			400	1·00	-2·14	169	1·00	170	1·05	0·02
23	G				10	0·974	-1·06	8·95	0·994	7·89	1·41	0·15
24	G				50	0·976	-5·18	44·7	0·994	38·9	1·38	0·15
25	G				200	0·977	-19·8	175	0·994	154·9	1·36	0·14
26	G				400	0·979	-36·7	342	0·995	302	1·34	0·14
27	G			0	10	0·987	-0·904	10·1	0·998	9·27	1·32	0·08
28	G		1		10	0·984	-0·702	6·49	0·998	5·98	1·47	0·10
29	G		1		50	0·985	-3·93	36·5	0·998	33·9	1·46	0·09
30	G		1		100	0·985	-7·69	74·1	0·998	67·6	1·43	0·09
31	G		1		200	0·987	-15·4	152	0·998	141	1·41	0·04
32	G		1		400	0·988	-29·5	305	0·998	282	1·38	0·04
34	G	10			10	0.965	- 0.429	2.91	0.995	2.52	1.62	0·16

Table 5.2. (continued)

1. L - Linear rheology
   G - Glen rheology

2. S = Bed smoothness in alu/bar.a
   no smoothness shown indicates perfect slip

3. χ = Diffusivity in alu^2/bar.a
   No diffusivity shown indicates incompressibility

4. P_c = Cavity pressure in bar
   No cavity pressure shown indicates no cavity

5. Distal velocity in alu/a

6. Relationship of the form
   $v_\tau = C + D \sin\gamma$ with correlation r

7. Relationship of the form
   $v_\tau = A \sin^m\gamma$ with correlation r
   Δm = 99% confidence limits for m using t-test

Table 5.3 Regression of A against v_d for v_τ

Rheology	Rough Bed	Diffusivity $alu^2/bar.a$	r	E	ℓ	$\Delta\ell$
L	X	0	1·000	0·143	1·00	0·0
L	X	1	1·00	0·139	1·00	0·0
L	X	10	1·00	0·140	1·00	0·0
L	√	0	1·00	1·88	0·754	0·040
G	X	0	1·00	0·809	0·990	0·033
G	X	1	1·00	1·09	1·05	0·05

These figures are for a regression relationship of the form

$$A = E v_d^{\ell}$$

where v_d is the distal velocity and A is the regression coefficient from the relationship

$$v_\tau = A \sin^m \gamma \qquad \text{(see Table 5.2)}$$

r is the correlation coefficient.

$\Delta\ell$ is the 99% confidence limit for ℓ using the t-test.

Table 5.4 Regression relationship between v_τ, v_d and $\sin\gamma$.

1. Linear incompressible rheology

$$v_\tau = 0{\cdot}143\ v_d \sin\gamma$$

2. Linear compressible rheology, diffusivity = 1·0 alu^2/bar.a

$$v_\tau = 0{\cdot}139\ v_d \sin^{1{\cdot}05}\gamma$$

3. Linear compressible rheology, diffusivity = 10·0 alu^2/bar.a

$$v_\tau = 0{\cdot}140\ v_d \sin^{1{\cdot}06}\gamma$$

4. Linear incompressible rheology, rough bed

$$v_\tau = 1{\cdot}88\ v_d^{0{\cdot}754} \sin^{1{\cdot}07}\gamma$$

5. Glen incompressible rheology

$$v_\tau = 0{\cdot}809\ v_d^{0{\cdot}99} \sin^{1{\cdot}37}\gamma$$

6. Glen compressible rheology, diffusivity = 1·0 alu^2/bar.a

$$v_\tau = 0{\cdot}551\ v_d^{1{\cdot}05} \sin^{1{\cdot}43}\gamma$$

both linear and logarithmic cases the relationships are very strong. The exponent m is not significantly different from 1.0 at the 99% confidence level. v_τ scales exactly with v_d. The crestal velocity $v_c = 0.143v_d$, thus when $v_d = 400$ alu/a, $v_c = 57.2$ alu/a.

Using cavitation boundary conditions on the lee-side, a cavity pressure of 0.0 bar, and a distal velocity of 400 alu/a gives the relationship between v_τ and $\sin\gamma$ shown in Figure 5.8. The exponent m is 1.15, significantly different from 1.0. The crestal velocity here was computed to be 76.2 alu/a.

The cavity was found to be opening and the higher crestal velocity relative to the uncavitated case is partly a function of this extension and partly a function of the fact that the cavity provides a bound on the suction retarding the ice, meaning the obstruction provides less drag.

With a lee-side cavity and $v_d = 10$ alu/a, the cavity was found to collapse. The direction of flow on the (generalised) upstream side of the cylinder is reversed, and velocities (Figure 5.9) were an order of magnitude higher than for the uncavitated case, the crestal velocity being 15.2 alu/a as compared with 1.43 alu/a respectively.

The exponent m was found to be 1.22, significantly different from 1.0.

For a compressing rheology with a diffusivity of 1 alu^2/bar.a Figure (5.10) it is again found that the relationship between v_τ and $\sin\gamma$ is nearly linear. The exponent m is, however, at 1.06, significantly different from 1.0. The slip velocity v_τ scales exactly with the distal velocity v_d. Crestal velocities were only slightly lower than for the incompressible case.

A velocity vector plot with a diffusivity of 10 alu^2/bar.a is given in Figure 5.11. Upstream velocities away from the obstruction have a smaller vertical component than in the incompressible case owing to the compression of the ice. The exponent m is 1.05 (Figure 5.12) but significantly different from 1.0. The slip velocity scales exactly with the distal velocity. Crestal velocities are identical for those when the diffusivity was 1 alu^2/bar.a.

Figure 5.13 gives a velocity vector plot for shearing rough bed flow with an incompressible linear rheology, a bed smoothness of 100 alu/bar.a and a distal velocity of 100 alu/a. The regression shows that there is a nearly linear relationship between v_τ and $\sin\gamma$ (Figures 5.14, 5.15). The exponent m varies from 1.14 when the distal velocity was 10 alu/a and the smoothness 10 alu/bar.a to 1.05 when the distal velocity was 400 alu/a and the smoothness 400 alu/bar.a. In all cases the exponents were significantly different from zero. The coefficient A in the logarithmic regression scaled with the distal velocity raised to 0.754, which is significantly different from 1.0. Crestal velocities varied from 10.9 alu/a when the distal velocity was 10 alu/a to 168 aul/a when the distal velocity was 400 alu/a.

The reason for the high crestal velocities as compared with the non-shearing flow is the fact of the shearing. This velocity ratio decreases with the smoothness raised to 0.754 as well, as conditions approach perfect slip.

Figure 5.16 shows a velocity vector plot for an incompressible Glen rheology. The slip velocities show a strongly and significantly non-linear relationship (Figures 5.17, 5.18).

The exponent m lies between 1.41 and 1.34 and shows a slight decline with distal velocities. Crestal velocities are much higher than for the linear case, the ratio being 6.3 for a distal velocity of 10 alu/a and 5.9 for a distal velocity of 400 alu/a. The coefficient A scales nearly linearly with the distal velocity, the relationship not being significantly different from 1.0. The exponent m regresses significantly with distal velocity.

A velocity vector plot for the cavitated case with the cavity pressure equal to 0 bar is shown in Figure 5.19. The exponent m in the logarithmic regression between v_τ and $\sin\gamma$ is 1.32 (Figure 5.20), significantly different from 1.0. The crestal velocity is slightly higher (9.55 alu/a) than for the uncavitated case (8.98 alu/a).

Figure 5.21 shows a velocity vector plot for a compressing Glen rheology with a diffusivity of 1 alu^2/bar.a. The velocities have a smaller vertical component than for the incompressible case owing to the compression.

The logarithmic regression of v_τ against $\sin\gamma$ (Figure 5.22) gives exponents ranging between 1.47 and 1.38 all significantly different from 1.0, slightly higher than for the incompressible case. The exponent shows a monotone decrease with distal velocity. The crestal velocities are substantially lower than for the incompressible case, the ratios ranging from 0.71 for a distal velcoity of 10 alu/a to 0.88 for a distal velocity of 400 alu/a.

The coefficient A scales with the distal velocity raised to 1.05. This relationship is not significantly different from being linear.

One calculation was carried out for shearing flow with an incompressible Glen rheology and a smoothness of 10 alu/bar.a.

The logarithmic regression (Figure 5.23) gives an exponent of 1.62, significantly different from 1.0. The crestal velocity is 2.78 alu/a, very much lower than for the corresponding non-shearing flow (8.98 alu/a). This is the opposite trend to that of the linear rheology.

A velocity vector plot for flow over the truncated sine ridge is given in Figure 5.24. It should be noted that in this case, since the trailing edge and leading edge corner angles were not 90^{o}, the ice was not fixed here, and the corners have been smoothed. The relative shallowness of the slope means that there is not such a significant stagnation zone, and shear strains are somewhat lower.

It was not found possible to characterise the slip velocity distributions using simple regression models. Figure 5.25 shows the slip velocities normalised to the crestal velocity for flows, using a linear incompressible rheology, for which the slip velocities scaled exactly with the distal velocity, and also for a Glen rheology, for distal velocities of 10, 50, 100, 200 and 400 alu/a. Point velocities for the Glen rheology were regressed against the distal velocity. The correlation coefficients were all greater than 0.99. Figure 5.26 shows a plot of the regression exponent ℓ in the relationship

$$v_\tau = E v_d^{\ell}$$

with 99% confidence limits against position. In all cases the regression exponent ℓ is significantly different from 1.0. The value of ℓ ranges from 0.92 at the leading edge to 0.88 at the crest.

To summarise, for linear rheologies v_τ is proportional to

$\sin\gamma$, the same result as obtained by Watts (1974) for flow around a sphere. For non-linear rheologies v_τ is proportional to $\sin\gamma$ raised to some power, which is greater than 1. There is some indication that this power changes with the distal velocity. Compressing rheologies do not significantly affect basal velocity distributions. For flow around a truncated sine ridge, velocity distributions are affected by the distal velocity.

5.4 Pressure Distributions

In this section the relationships between the hydrostatic pressure in the ice, position and distal velocity are considered. The hydrostatic pressure is defined as

$$p = \tfrac{1}{3}\sigma_{ii}$$

For plane strain flow, $p = \sigma_{zz}$, and thus $p = \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy})$. As in the previous section we concentrate on distributions at the ice-rock interface. Regressions of the increase in pressure p' above the distal pressure p_d are regressed linearly and logarithmically against $\cos\gamma$. For flow around a sphere Watts (1974) found that p' was directly proportional to $\cos\gamma$. In addition some pressure contours in the region around the obstructions have been produced (e.g. Figure 5.27). These contours are exact representations of the solution. In straight-sided elements the contours are straight lines, while in distorted elements the contours are conic sections (see Chapter 4).

Table 5.5 shows the regressions of p' against $\cos\gamma$. The models chosen were

$$p' = C + D\cos\gamma$$

and $$p' = A\cos^m\gamma$$

where C, D, A and m are constants. Because the pressure was

TABLE 5.5

Regression of p' against cosγ for the semi-cylindrical ridge

						Linear Regression[6]			Logarithmic Regression[7]			
No	R[1]	S[2]	χ[3]	P_c[4]	V_d[5]	r	C	D	r	A	m	Δm
1	L				10	0•948	-0•008	0•84	0•916	0•9	0•952	0•646
2	L				50	0•948	-0•038	4•20	0•916	4•5	0•952	0•646
3	L				100	0•948	-0•075	8•4	0•916	8•9	0•952	0•646
4	L				200	0•948	-0•151	16•8	0•916	17•8	0•952	0•646
5	L				400	0•948	-0•301	33•6	0•916	35•5	0•952	0•646
7	L			0	400	0•799	-12•0	44•7	0•977	32•5	1•06	0•78
8	L		1		10	0•999	-0•015	0•56	0•998	0•56	1•14	0•13
9	L		1		50	0•999	-0•060	2•8	0•998	2•8	1•14	0•13
10	L		1		100	0•999	-0•120	5•6	0•998	5•6	1•14	0•13
11	L		1		200	0•999	-0•241	11•2	0•998	11•2	1•14	0•13
12	L		1		400	0•999	-0•482	22•3	0•998	22•4	1•14	0•13
13	L		10		10	1•00	-0•043	0•14	1•00	0•14	1•08	0•04
14	L		10		50	1•00	-0•0107	0•68	1•00	0•68	1•08	0•04
15	L		10		100	1•00	-0•0214	1•36	1•00	1•35	1•08	0•04
16	L		10		200	1•00	-0•0428	2•72	1•00	2•70	1•08	0•04
17	L		10		400	1•00	-0•0856	5•43	1•00	5•40	1•08	0•04
18	L	10			10	0•945	-0•312	5•79	0•944	5•09	0•493	0•582
19	L	50			50	0•948	-1•68	20•2	0•922	17•5	0•611	0•862
20	L	100			100	0•948	-3•07	35•5	0•922	30•7	0•627	0•888
21	L	200			200	0•972	-1•96	57•2	0•943	52•4	0•640	0•910
22	L	400			400	0•984	2•04	86•8	0•973	91•2	1•02	0•56
23	G				10	0•928	16•3	131	0•912	148	0•776	0•803
24	G				50	0•930	27•4	226	0•913	257	0•785	0•808
25	G				200	0•932	42•5	365	0•914	411	0•806	0•821
26	G				400	0•933	52•1	468	0•915	525	0•826	0•836
28	G			0	400	0•930	71•1	440	0•922	574	1•33	1•27
29	G		1		10	0 999	0•230	11•7	0•999	11•7	0•960	0•078
30	G		1		50	0•999	1•35	51•0	0•999	58•3	0•943	0•082
31	G		1		100	0•998	2•64	91•44	0•998	94•0	0•928	0•119
32	G		1		200	0•997	5•44	162	0•998	166	0•915	0•138
33	G		1		400	0•994	17•0	235	0•999	243	0•731	0•075
34	G	10			10	0•899	8•73	88•2	0•883	97•6	0•815	0•994

Table 5.5. (continued)

1. L - Linear rheology
   G - Glen rheology

2. S = Bed smoothness in alu/bar.a
   no smoothness shown indicates perfect slip

3. χ = Diffusivity in alu^2/bar.a
   No diffusivity shown indicates incompressibility

4. P_c = Cavity pressure in bar
   No cavity pressure shown indicates no cavity

5. Distal velocity in alu/a

6. Relationship of the form
   $p' = C + D \cos\gamma$ with correlation r

7. Relationship of the form
   $p' = A \cos^m \gamma$ with correlation r

   Δm = 99% confidence limits for m using t-test interpolated at a lower order than the velocities, there are fewer sample points. The pressure solution shows more scatter than the velocity solution. In many of the cases the deviations from the regression line exhibit a periodicity in that they alternate between being positive and negative. This may be a manifestation of a spurious pressure mode in the solution (see Chapter 4), in which case the smoothing introduced by the regression will remove some of the effect of this spurious mode.

In all of the cases but two the coefficients of correlation between p' and $\cos\gamma$ and $\log_{10}p'$ and $\log_{10}\cos\gamma$ are greater than 0.9, indicating a strong relationship between p' and $\cos\gamma$. Table 5.6 gives the regression relationships between A and v_d and Table 5.7 gives the inferred relationships between p, v_d and $\cos\gamma$. Figure 5.27 shows pressure contours around a semi-cylindrical ridge for a fluid with a distal velocity of 100 alu/a and an incompressible linear rheology. As would be expected, the highest pressures are at the leading edge, and are symmetric about the half cylinder.

The regression of p' against $\cos\gamma$ is shown in Figure 5.28 In the logarithmic plot there are only five points owing to zero and negative values. Correlation coefficients are high, probably because of the outliers. The exponent m is computed at 0.952 with confidence limits of 0.646, in other words the relationship is not significantly different from being linear. The pressure solution scales exactly with the velocity. With regard to the possible oscillations in the pressure solution, this is more consistent with a spurious mode occurring in the pressure solution than with a manifestation of ill-conditioning, where we would expect errors to change with the distal velocity.

Table 5.6

Regression of A against v_d for p'

Rheology	Rough Bed	Diffusivity alu² /bar.a	r	E	ℓ	$\Delta\ell$
L	X	0	1·00	0·072	1·00	0·00
L	X	1	1·00	0·056	1·00	0·00
L	X	10	1·00	0·027	1·00	0·00
L	√	0	0·985	1·25	0·754	0·450
a	X	0	1·00	67·25	0·342	0·013
a	X	1	0·994	1·95	0·829	0·301

These figures are for a regression relationship of the form

$$A = Ev_d^{\ell}$$

where v_d is the distal velocity and A is the regression coefficient from the relationship

$$p' = A \cos^m \gamma$$

r is the correlation coefficient.

$\Delta\ell$ is the 99% confidence limit using the t-test.

Table 5.7

Regression relationships between p', v_d and $\cos\gamma$

1. Linear incompressible rheology

$$p' = 0{\cdot}072\ v_d \cos\gamma$$

2. Linear compressible rheology, diffusivity = 1.0 alu²/bar.a

$$p' = 0{\cdot}056\ v_d \cos\gamma$$

3. Linear compressible rheology, diffusivity = 10.0 alu²/bar.a

$$p' = 0{\cdot}027\ v_d \cos\gamma$$

4. Glen incompressible rheology

$$p' = 67{\cdot}25\ v_d^{0\cdot342}\ \cos^{0\cdot80}\gamma$$

For a distal velocity of 400 alu/a the pressure increase at the leading edge is 29.1 bar.

There are seven points on the linear plot and five on the logarithmic plot. The plots suggest the linear model to be the best.

With a cavitation boundary condition and a distal velocity of 400 alu/a, there are only five points plotted on the logarithmic relationship (see Figure 5.29) as there is a negative outlier. The correlation coefficient for the linear relationship is low (0.799), because it includes the outliers. The pressure deviation at the leading edge is 32.1 bar, 10% higher than for the uncavitated case, presumably because the ice is extending.

The plots suggest that the linear model is certainly adequate towards the leading edge (where $\cos\gamma$ is high). The logarithmic plot is heavily influenced by an outlier.

Figure 5.30 shows the regression of p' against $\cos\gamma$ for a linear compressible rheology, a distal velocity of 400 alu/a and a diffusivity of 1 $alu^2/bar.a$. The most noticeable feature is the low scatter of the points compared with the incompressible case. This is because the pressure diffusion term $(\chi\nabla^2 p)$ makes the solution matrix definite, and thus spurious pressure modes will not occur (see Chapter 4). Correlation is high (0.999) for both the regressions. The exponent m in the logarithmic regression is 1.14, with confidence limits of $\pm$ 0.13. The pressure deviation at the leading edge is 22.1 bar for a distal velocity of 400 alu/a, 25% lower than for the corresponding incompressible case. The pressure solution scales exactly with the distal velocity.

The fit to both plots is good. Given the linear nature of the physical system it is probably best to assume a linear relationship between p' and $\cos\gamma$ despite there being a 99% certainty that it is significantly non-linear.

Figure 5.31 shows a pressure contour plot for a linear compressible rheology with a diffusivity of 10 alu^2/bar.a and a distal velocity of 400 alu/a. The pressure contours are rather smoother than for the incompressible case, reflecting the increased accuracy of the solution, and show a more radial pattern. The contouring algorithm has failed to plot in certain elements; these have been manually inserted.

Figure 5.32 shows the regression of p' against $\cos\gamma$. The correlation coefficient is 1.00 in both cases. The exponent m is 1.08 with confidence limits of $\pm$ 0.04. The pressure solution scales exactly with the distal velocity. At a distal velocity of 400 alu/a, the pressure increase at the leading edge is 10.8 bar, approximately $\frac{1}{2.04}$ that for when the diffusivity was 1 alu^2/bar.a. Note that $2.04 \approx 10^{\frac{1}{3}}$.

It is probably best to accept a linear relationship between p' and $\cos\gamma$, for the same reasons as advanced for the case when the diffusivity was 1 alu^2/bar.a.

Figure 5.33 shows the pressure contours for a linear incompressible fluid with a rough bed of smoothness 100 alu/bar.a and a distal velocity of 100 alu/a. These contours show a similar pattern to the perfect slip case.

Figures 5.34 and 5.35 show the regressions of p' against $\cos\gamma$ for rough beds when the distal velocities are 10 alu/a and 400 alu/a and the smoothnesses 10 alu/bar.a and 400 alu/bar.a respectively. The presence of negative deviations on the upstream side means that only five points have been used in the logarithmic regression except for when the distal velocity is equal to 400 alu/a, and the bed is correspondingly smoother. The correlation coefficients are generally greater for the linear regression then for the logarithmic regression. The exponent m varies from 0.49 to 1.02 but is never significantly different from 1.0. The coefficient A scales with the distal velocity raised to the 0.754 with confidence limits of $\pm$ 0.451 meaning that m is not significantly different from 1. The leading edge pressure increment for a distal velocity of 400 alu/a. is 85.2 bar, nearly thrice as much as that for the incompressible case.

For the case of the distal velocity being 10 alu/a, it is difficult to decide from the plots which model is more suitable. The linear plot suggests a non-linear relationship, but since there are only five points on the logarithmic plot, it cannot be taken to be conclusive.

For the case of the distal velocity being 400 alu/a, where the bed is correspondingly smoother, the linear model seems satisfactory on the plot.

There does seem to be a trend of the exponent on the smoothness, the relationship becoming more linear as the smoothness increases. The regression was not found to be significant.

Figure 5.36 shows a pressure contour plot for an incompressible Glen rheology for a distal velocity of 200 alu/a. The contours are rather less smooth than for the linear case (Figure 5.27). There is apparently a point of maximum pressure at $\gamma = 30^{\circ}$. This is a reflection of the oscillations in the pressure solution. The true maximum pressure is at the leading edge of the obstacle (see Figures 5.37, 5.38 - the same pattern occurs for all the distal velocities) but the contouring algorithm has failed to plot here. The contours are more closely concentrated towards the crest of the ridge.

This concentration is due to the fact that the contours no longer show a linear relationship with $\cos\gamma$. The results of the regression analysis give values for m of around 0.81 with confidence limits of around $\pm$ 0.81, meaning that the relationship between p' and $\cos\gamma$ is not significantly different from being linear. The coefficient A scales with v_d raised to the power 0.34. This exponent has confidence limits of $\pm$ 0.013, meaning the relationship is significantly non-linear. The leading-edge pressure increment for a distal velocity of 400 alu/a is 483 bar, 16.4 times as much as for the corresponding non-linear case.

A multiple logarithmic linear regression was carried out, with p' as the dependent variable and v_d and $\cos\gamma$ as the independent variables. For 24 variables and 2 independent variables there are 21 degrees of freedom (Walpole and Myers, 1978). The results are shown in Table 5.8. This shows that the relationship between p' and $\cos\gamma$ is significantly non-linear at the 98% level.

Figure 5.39 shows pressure contour plots for cavitated flow, at a distal velocity of 400 alu. The pressure contours are very irregular and the contouring algorithm has repeatedly failed to plot.

The low correlation coefficients for the regression (Figure 5.40) suggest that the solution is rather poor. The outlier distorts the logarithmic plot. The linear plot suggests a near-linear relationship, which is unexpected given the the results from the uncavitated case; the outlier-influenced logarithmic plot gives m=1.33, which is not significantly different from 1.0.

Table 5.8

Multiple linear logarithmic regression of p' on v_d, $\cos\gamma$ for an incompressible Glen flow

Model relationship: $p' = A\, v_d^{\ell} \cos^{m}\gamma$

Correlation matrix:

	p'	v_d	$\cos\gamma$
p'	1.0	0.747	0.610
		1.0	0.0
			1.0

Coefficient of multiple correlation = 0.963

$A = 68.47$

	$\Delta(99\%)$	$\Delta(98\%)$
$\ell = 0.338$	$\pm$ 0.075	$\pm$ 0.067
$m = 0.798$	$\pm$ 0.214	$\pm$ 0.191

$\Delta(x\%)$ = Confidence limits at x% level

Figure 5.41 shows pressure contours for a compressible Glen rheology with the diffusivity equal to 1 alu^2/bar.a and a distal velocity of 400 alu/a. These contours are smoother and more radial than for the incompressible case.

The regression of p' against $\cos\gamma$ (Figure 5.42) gives high correlation coefficients. The coefficient m varies between 0.96 for a distal velocity of 10 alu/a and 0.73 for a distal velocity of 400 alu/a. The latter relationship is significantly non-linear.

The regression of A against v_d gives an exponent of 0.83 with confidence limits of $\pm$ 0.30, meaning that the relationship is not significantly non-linear. The maximum pressure increment at the leading edge is 246 bar for a distal velocity of 400 alu/a, half of that for the incompressible case.

The plots show that outliers severely distort the logarithmic regression, and that the linear model is best. A logarithmic regression of the coefficient D against the distal velocity produced a relationship of the form

$$D \propto v_d^{0.821 \pm 0.451}$$

This is comparable with that obtained by regressing the logarithmic model against v_d.

Figure 5.43 shows the regression of p' against $\cos\gamma$ for the shearing flow of the fluid with an incompressible Glen rheology and a rough bed. The distal velocity was 10 alu/a and the smoothness 10 alu/bar.a. The points show a high degree of scatter. The exponent m in the regression relationship is 0.82 with confidence limits of ± 0.99, meaning that the relationship is not significantly non-linear.

The high degree of scattering of the points makes model fitting difficult; it could even be argued that the relationship between p' and $\cos\gamma$ is non-monotonic. The relationship does appear to be slightly non-linear.

It proved difficult to find a relationship between the pressure deviation and position for flow around the truncated sine ridge. The pressures normalised to the maximum pressure are shown in Figure 5.44. The pressures scaled exactly with the distal velocity for the linear case. As the distal velocity increases the behaviour of p' for the Glen rheology case becomes more like that of the linear case.

Regressions were carried out for p' at each point against v_d (see Table 5.9). The node numbers are shown in

Table 5.9

Regression of nodal pressure increase against distal velocity

Node	r	k	Δk	E
1	0.994	0.630	0.225	13.4
2	0.996	0.678	0.197	28.4
3	0.993	0.612	0.237	22.44
4	0.994	0.632	0.217	15.24

Figure 5.3. The correlation coefficient r is very high. The parameters k and E appear in the regression relationship

$$p' = E v_d^k \;.$$

k is significantly different from 1.0.

Figure 5.45 shows pressure contours for a calculation using a Glen rheology. These appear somewhat smoother than those observed for the flow over a half-cylinder. This may only be an artefact of the fact that fewer elements have been used.

Table 5.10 shows the maximum pressures recorded for flow over a sine ridge and flow over a semi-cylindrical ridge.

Table 5.10

Maximum pressure increments

Rheology	Distal Velocity alu/a	Sine Ridge p'max/bar	Semi-cylindrical Ridge p'max/bar
Linear	400	50	29.4
Glen	10	147	136
Glen	50	364	234
Glen	200	1040	378
Glen	400	1790	483

The fact that the point of maximum pressure can potentially lie some away behind the leading edge is of some importance as will be seen in Chapter 6. The question arises as to whether the fact that the point of maximum pressure is not at the leading edge is a real effect or a manifestation of a spurious pressure mode. The fact that the contours are smooth suggests that a spurious mode may not be present, or if it is present, not manifesting itself significantly. The flow over the sine ridge is a rather easier problem for the finite element program to solve than is the flow over the half-cylinder, because this latter flow has a 90° corner. We might thus expect the solution to be more accurate for the flow over the truncated sine ridge.

The conclusion used in Chapter 6 is that the fact that the point of maximum pressure lies behind the trailing edge may well be a true representation of the solution.

To summarise, for flow of a fluid with a linear rheology over a half-cylinder, it is probably best to accept that p' is proportional to $\cos\gamma$, the result obtained by Watts (1974) for flow around a sphere. For non-linear rheologies, there is fairly strong evidence that p' is proportional to some power of $\cos\gamma$, and that this power is less than 1. Both these relationships imply that p' is at its greatest at the leading edge. For linear rheologies p' is proportional to v_d, while for non-linear rheologies p' is proportional to v_d^{ℓ}, where $0 < \ell < 1$.

The same relationship with the distal velocity is noted for flow over a truncated sine ridge; here, however, there is a certain amount of evidence to believe that the point of maximum pressure lies someway behind the leading edge. Use of a compressing rheology causes maximum pressures to be decreased. Use of a Glen rheology increases the maximum pressure on an obstacle. Maximum pressures recorded on the upstream side of a sine ridge are far greater than those on a semi-cylindrical ridge, despite the sine ridge being less rough (in the sense of height/length) than the semi-cylindrical ridge.

5.5 Volumetric strain rates

The volumetric strain rate $\dot{\varepsilon}_v (\equiv \dot{\varepsilon}_{ii})$ is a measure of the compression or dilation of the ice. For incompressible fluids, the volumetric strain rate is zero. When ice is compressing, the volumetric strain rate is negative, and when the ice is dilating, the volumetric strain rate is positive.

In this section the volumetric strain rates observed in compressing flow are reported. At each element centroid $\dot{\varepsilon}_v$ was computed and these numbers have been written on to the finite element grid. Figure 5.46 shows such a plot for a linear rheology, a diffusivity of 1 alu^2/bar.a and a distal velocity of 400 alu/a. It was found that $\dot{\varepsilon}_v$ scaled exactly with distal velocity. It can be seen that $\dot{\varepsilon}_v$ is at a maximum near the ice-ridge interface, and shows a complex pattern of behaviour along the interface.

Figure 5.47 shows a similar plot for a linear rheology, a diffusivity of 10 alu^2/bar.a and a distal velocity of 200 alu/a. Volumetric strain rates for this rheology scaled exactly with the distal velocity. These results show a similar pattern to those obtained with the lower diffusivity. The element volumetric strain rates were correlated and found to be related by a relationship of the form

$$\dot{\varepsilon}_{vk10} = 5.20\ \dot{\varepsilon}_{vk1}^{\ 0.41} \qquad (r = 0.96)$$

where $\dot{\varepsilon}_{vk10}$ is the volumetric strain rate for a diffusivity of 10 alu^2/bar.a and $\dot{\varepsilon}_{vk1}$ that for a diffusivity of 1 alu^2/bar.a.

The confidence limits for the exponent are $\pm$ 0.09, meaning that the relationship is significantly non-linear. Why the relationship should take this form is not obvious.

Figures 5.48 and 5.49 show $\dot{\varepsilon}_v$ for a Glen rheology, with v_d = 10 alu/a and 400 alu/a respectively. These strain rates are an order of magnitude higher than those obtained using a linear rheology, because of the greater pressures found in the former case. The same pattern of high compression near the ridge and close to the leading edge is observed. The element volumetric strain rates have been regressed logarithmically against the distal velocity, implying a relationship of the form $\dot{\varepsilon}_v \propto v_d^m$. The correlation coefficients were all greater than 0.95. Figure 5.50 plots the regression exponents m against position. None of these exponents were significantly different from 1.0, but a pattern exists where the exponents are most linear near the ridge and near the crest of the ridge; in other words the straining patterns change less with velocity away from the ridge.

It is of interest to determine the total compression occurring from the leading edge to the crest. The total volumetric strain is given by

$$\varepsilon_v = \int \dot{\varepsilon}_v dt = \int \frac{\dot{\varepsilon}_v}{v_\tau} d\tau$$

since

$$v_\tau = \frac{d\tau}{dt} \qquad 5.3$$

where v_τ is the tangential velocity and τ is the tangential distance.

This integral has been evaluated approximately by use of the formula

$$\int \frac{\dot{\varepsilon}_v}{v_\tau} d\tau \approx \Sigma \frac{\dot{\varepsilon}_v^i}{v_\tau^i} \Delta s_i \quad ,$$

and where $\dot{\varepsilon}_v^i$ is the interface element volumetric strain rate, v_τ^i is the computed tangential velocity at the middle of the corresponding element side and Δs_i is the length of the element side.

Now, the definition of $\dot{\varepsilon}_v$ is

$$\dot{\varepsilon}_v = \frac{1}{V}\frac{dV}{dt}$$

where V is the volume.

Using 5.3 this gives

$$\ell n \frac{V_\tau}{V_o} = \int_o^\tau \frac{\dot{\varepsilon}_v}{v_\tau} d\tau'$$

where V_o is the original volume and V_τ the volume at τ'. The total compression may be computed. The concentration factor Γ is given by V_o/V_τ.

Table 5.11 gives the concentration factor at the ridge and the logarithmic regression component of Γ against γ, γ being proportional to τ. For linear rheologies Γ is independent of velocity, but the relationship between Γ and γ changes with the diffusivity. Also, as would be expected, an increase in the diffusivity causes an increase in the concentration. For Glen rheologies, Γ decreases with the distal velocity. The correlation coefficient of the relationship is -0.96 and the regression relationship is

$$\Gamma = 2.15 \times 10^4 \times v_d^{-1.02}$$

In addition, the pattern of concentration changes. For a low distal velocity the relationship between Γ and γ is nearly parabolic, while for high distal velocity it is nearly linear.

Table 5.11

Regression of concentration factor

Γ against γ

Rheology	Diffusivity alu²/bar.a	Distal Velocity alu/a	Correlation Coefficient	m	Γ at ridge
Linear	1	400	0·973	0·104	1·549
Linear	10	200	1·00	0·487	5·89
Glen	1	10	0·994	1·92	1380
Glen	1	50	0·994	1·79	763
Glen	1	100	0·995	1·49	201
Glen	1	200	0·994	1·33	105
Glen	1	400	0·994	1·02	32·6

To summarise, volumetric strain rates vary with velocities and are higher for Glen rheologies with these boundary conditions. Concentration factors depend on the rheology, diffusivity, and for non-linear rheologies, the distal velocity.

5.6 Water flow in compressible flow

The formulation for compressible flow is based on the assumption that water within the ice flows under the action of pressure gradients. Thus, moisture flow lines may be elucidated by inspecting the pressure contours (Figures 5.31, 5.41). These show that the water flows from the upstream side, over the obstacle and is absorbed in the downstream side, where $\nabla^2 p$ is positive. This is a similar process to regelation. It should be noted that the water flow lines are not in the same direction as the ice flow lines.

5.7 Viscosity distributions

For a Glen rheology the viscosity of the ice μ is defined by

$$\mu = A\dot{\varepsilon}^{-\frac{2}{3}} \qquad 5.4$$

where $\dot{\varepsilon}$ is the effective strain rate and A is a constant (see Chapter 3).

Element centroid viscosity values for flow over obstacles have been plotted onto diagrams in a similar fashion to those displaying element volumetric strain rates.

Figures 5.51 and 5.52 show the viscosity variations for an incompressible Glen rheology for distal velocities of 10 alu/a and 400 alu/a respectively. Obvious features are that the ice is much harder near the leading edge of the ridge than it is at the crest. The maximum to minimum viscosity ratios are 6.8 for a distal velocity of 10 alu/a and 6.5 for the distal velocity of 400 alu/a.

The element viscosities were regressed against distal velocity to obtain a power law relationship of the form $\mu \propto v_d^n$. The correlation coefficients were all greater than 0.98 and exponent n was found to vary slightly, having a mean value of -0.663 with standard duration 0.079, and was always significantly different from 1.0. This value of the exponent is of course consistent with the viscosity relationship 5.4 and the fact observed in section 3 that slip velocities scaled with distal velocity.

Figure 5.53 shows a plot of viscosities for the cavitated case with a distal velocity of 10 alu/a. The ice is slightly softer than for the uncavitated case (Figure 5.51) because the cavity is opening and the ice therefore extending.

Figures 5.54 and 5.55 show element centroid viscosities for compressing Glen rheologies with diffusivities of 1 alu^2/bar.a and distal velocities of 10 alu/a and 400 alu/a respectively. The actual viscosities are not comparable with those for the Glen rheologies since the slip velocities are different; however, the same pattern of finding the harder ice at the leading edge was found. The maximum to minimum interfacial viscosity ratio is 4.7 for a distal velocity of 10 alu/a and 5.4 for a distal velocity of 400 alu/a. The trend is different from the case for incompressible rheologies.

Centroid viscosities were regressed against distal velocities to establish the exponent n in a relationship

$$\mu \propto v_d^n$$

The correlation coefficient was always greater than 0.98. The exponent n had a mean value of 0.684 with standard deviation 0.017 and was always significantly different from unity.

Figure 5.56 shows the element centroid viscosities for rough bed flow, a distal velocity of 10 alu/a and a bed smoothness of 10 alu/bar.a. The ice is somewhat stiffer than for the smooth bed case because slip velocities are lower. The largest viscosities are again found at the leading edge. The maximum to minimum interfacial viscosity ratio is 7.7, higher than for the case of perfect slip.

Figure 5.57 shows the viscosity for the case of the flow over the truncated sine ridge for a distal velocity of 10 alu/a, while Figure 5.58 shows the same for a distal velocity of 400 alu/a.

Again, the ice is more viscous at the leading edge. However, the maximum to minimum viscosity ratio is rather lower than for the case of flow around the half-cylinder, being 1.60 when the distal velocity is 10 alu/a and 1.45 when the distal velocity is 400 alu/a. Viscosities increase rapidly vertically.

The centroid viscosities have been regressed against the distal velocity to establish the exponent in a relationship form

$$\mu \propto v_d^n$$

Figure 5.59 shows a plot of n against position. A slight trend of decreasing dependence on distal velocity may be observed from edge to crest.

In all the calculations viscosities in the distal regions were observed to be very much higher. For the flow around the half-cylinder the distal viscosities were around 10^4bar.a in non-shearing flow. This is not surprising as the viscosity should tend to infinity at infinite distances. For flow around the sine ridge the distal viscosities were 10^5bar.a, an order of magnitude higher than for the flow around the half-cylinder.

For the shearing flow, the distal viscosities were of the order 10^3bar.a. The value expected, given pure shear in the distal regions of the flow and a flow law constant of 1.815 bar.$a^{\frac{1}{3}}$, is 24 bar.a. Thus, the solution obtained in the distal regions of the flow is somewhat inaccurate.

To summarise, viscosity can show a marked variation around obstacles. This variation becomes less marked as the obstacle becomes shallower. Viscosity decreases as velocity increases.

5.8 Computed tractions at the ice-rock interface

It is of fundamental importance to know the tractions at the ice rock interface. For the case of perfect slip, the tangential traction T_τ is, by definition, zero, while for a rough bed the tangential traction is equal to the slip velocity v_τ divided by the smoothness S .

The normal traction T_ν is equal to the normal stress $\sigma_{\nu\nu}$. This is (see Chapter 3) given by

$$\sigma_{\nu\nu} = (- p + \lambda\dot{\varepsilon}_v)\delta_{ij} + 2\mu\dot{\varepsilon}_{\nu\nu}$$

For incompressible flow, $\dot{\varepsilon}_{\nu\nu} = - \dot{\varepsilon}_{\tau\tau} = - \frac{\partial v_\tau}{\partial \tau}$, while for compressible flow $\dot{\varepsilon}_{\nu\nu} + \dot{\varepsilon}_{\tau\tau} = \dot{\varepsilon}_V = \chi \nabla^2 p$.

In this section the quality of the regression models established in the preceding chapters is discussed in order to produce predictive equations for the normal stress. As will be seen in Chapter 6, a conservative hypothesis is that relationships, especially the pressure relationship, are linear. The approach therefore will be to adopt a conservative assumption of linearity, and only reject it if this seems inescapable.

Computed slip velocities at the interface

For the linear rheologies with smooth beds the tangential velocity distributions are represented well by the linear regression lines. The constants in these regressions are all small compared with the slope, and it can therefore be assumed that

$$v_\tau \propto \sin\gamma$$

In addition, the tangential velocities all scale with velocity, giving the inferred relations in Table 5.4.

For the non-linear rheologies it was found that the logarithmic models gave the best fit, although in many cases there was a slight higher order variation. The exponent in the logarithmic regression varied slightly. For the incompressible case the regression of the exponent m on v_d was significant, but it was not for the compressible case. In both cases the mean value was taken in inferring the generic relationship. This produced a relationship of the form

$$v_\tau = A \sin^m\gamma$$

where $m = 1.37$ for incompressible flow and 1.43 for compressible flow. The regression of A on v_d in both cases was not significantly different from being linear, and the linear relationship was therefore accepted (Table 5.4). For rough bed flow the logarithmic relationship provided the best fits. However, for the linear rheology the exponent varied. A mean value of 1.07 has been taken.

The regression of A on v_d gives the inferred relationship in Table 5.4. For the case of the Glen rheology one run at 10 alu/a gave an exponent of 1.62.

The cavitated flows produced greater errors, but did not produce greatly different patterns from their uncavitated counterparts, apart from the case when the cavity was collapsing.

The tangential strain-rate was computed by differentiating v_τ with respect to τ, where $\tau = 4\pi \sin\gamma$ in alu.

No analytical function was obtained for slip velocities over the sine ridge.

Computed ice pressures at the interface

The fewer data points and their greater scatter compared with slip velocities made the pressure increase more difficult to characterise than the slip velocities.

For the linear rheologies and the compressing Glen rheologies, the conclusion was that p' was proportional to $\cos\gamma$. Regression of the regression coefficient for this relationship against v_d produced the relationship shown in Table 5.7.

For the incompressible Glen rheology and the rough bed flow neither the linear nor the logarithmic relationship gave a good fit. The pattern of fluctuations around the regression line for the linear incompressible rheology (Figure 5.28) suggested that there might be spurious modes in the pressure solution. The occurrence of these modes is a consequence of mesh geometry (Jackson and Cliffe, 1980), and thus would arise equally for the non-linear case. Fitting of a higher order model was not felt to be justified in view of the uncertainity surrounding the pressure calculations.

However, we note that the logarithmic model gives an exponent of less than one, and that this model over predicts the decay of the pressure increment from the leading edge. Since linearity is a conservative model for the purposes of explaining glacial abrasional phenomena (see Chapter 6), then adopting this logarithmic regression model is a conservative approach.

The simulations modelling perfect slip may be used to model clasts sliding over a flat bed at an arbitrary velocity, since the slip velocity of the ice over the flat bed does not affect the flow. This is not the case for the rough bed flow, and it is therefore not so important for the purposes of this thesis to determine whether there is non-linearity in the relationship between p' and $\cos\gamma$. However, it may be cautiously suggested that there is, and that this non-linearity increases as the smoothness decreases for a linear rheology. The limited evidence from the rough bed Glen rheology calculation suggests that there is a non-linear relationship.

It is noted that p' increased with $v_d^{0.75}$, though this relationship was not significantly different from being linear. However, the slip velocities scale with the distal velocities to the same power, meaning that the magnitude of p' and the slip velocities increase directly with one another. This idea is intuitively appealing, given the linear nature of the flow system and on this basis the relationship between p' and v_d is cautiously accepted.

Inferred relationships between p', $\cos\gamma$ and v_d are shown in Table 5.7. Linear relationships between p' and $\cos\gamma$ are accepted for all the rheologies apart from the incompressible Glen flow. No attempt has been made to characterise the rough bed flows.

<u>Volumetric strain rates at the interface</u>

The volumetric strain rate $\dot{\varepsilon}_v$ is given by

$$\dot{\varepsilon}_v = \chi\nabla^2 p = \chi\nabla^2 p' \quad \text{since} \quad p = p' + p_d$$

where p_d is the constant distal pressure.

Noting that for compressible flow the pressure contours are nearly radial, this relationship may be approximated by

$$\dot{\varepsilon}_v \approx \chi\frac{\partial^2 p'}{\partial\tau^2}$$

Given the analytic relationships between p' and $\cos\gamma$ of the form $p' = A\cos\gamma$ where A is a constant for compressing flow, and noting that $\tau = 4\gamma$ (since the radius of the obstacle was 4 alu) the following relationship for $\dot{\varepsilon}_v$ at the ice-rock interface is obtained:

$$\dot{\varepsilon}_v = -\frac{A\cos\gamma}{16}$$

The values computed from this relationship and that at the corresponding element centroid are given in Table 5.12.

For the linear rheologies the numbers are of the same order. In view of all the approximations made and of the fact that the volumetric strain rates are centroid values this is a

TABLE 5.12

Comparison of interfacial and centroid volumetric strain rates

	Linear rheology $\chi = 1 \cdot 0$ alu^2/bar.a		Linear rheology $\chi = 10 \cdot 0$ alu^2/bar.a		Glen rheology $\chi = 1 \cdot 0$ alu^2/bar.a	
τ	Element	Semi-Analytic	Element	Semi-Analytic	Element	Semi-Analytic
0•524	-0•0415	-0•0338	-0•150	-0•165	-1•20	-0•827
1•571	-0•0288	-0•0315	-0•166	-0•154	-2•26	-0•769
2•618	-0•0290	-0•0270	-0•151	-0•132	-1•60	-0•676
3•665	-0•0290	-0•0207	-0•129	-0•101	-1•62	-0•515
4•712	-0•0735	-0•0130	-0•140	-0•0636	-2•80	-0•325
5•670	+0•0015	-0•0045	-0•0109	-0•0217	-0•276	-0•111

TABLE 5.13

Comparison of interfacial and centroid viscosities for perfect slip

	Incompressible Rheology		Compressible Rheology	
τ	Element	Semi-Analytic	Element	Semi-Analytic
0•524	1•57	1•52	1•35	1•45
1•571	0•77	1•23	0•67	1•23
2•618	0•72	1•22	0•69	1•24
3•665	0•51	1•37	0•54	1•38
4•712	0•23	1•76	0•29	1•81
5•760	0•37	3•54	0•46	3•63

satisfactory cross-check of consistency for the relatively untested (see Chapter 4) compressing flow formulation. For the Glen compressing flow the match is worse. Why this is so is not clear.

Viscosities at the interface

For perfect slip, viscosities may be calculated by using the relationships (see Chapter 3)

$$\dot{\varepsilon}_{\nu\nu} = \dot{\varepsilon}_{V} - \dot{\varepsilon}_{\tau\tau}$$

$$\dot{\varepsilon} = (\tfrac{1}{2}(\dot{\varepsilon}^{2}_{\nu\nu} + \dot{\varepsilon}^{2}_{\tau\tau}))^{\frac{1}{2}} \quad \text{(since} \quad \dot{\varepsilon}_{\tau\nu} = 0)$$

and $\mu = A\,\dot{\varepsilon}^{-\frac{2}{3}}$

Table 5.13 shows a comparison of element centroid viscosities and interfacial viscosities computed using the analytical relationships for strain-rates.

As can be seen, the semi-analytic method generally predicts stiffer ice, but in general the match is within a factor of two or three, except at the very crest. Here, symmetry predicts zero strain rates and thus infinite viscosities. The rise in viscosity is captured by the semi-analytic method. It is not clear whether the rise in viscosity observed in the finite element calculations is a consequence of the fact that the centroid of the last element is more removed from the interface than the preceding element, or whether it is reflecting the lower strain-rates. It should be noted that although the viscosity tends to infinity, the stress deviator does not. At the crest, symmetry demands that $\nabla^2 p = 0$, which means that for the compressible and incompressible flow that $\dot{\varepsilon} = \dot{\varepsilon}_{\tau\tau}$.

We therefore find that the stress deviator $\sigma'_{\tau\tau} = 2\mu\varepsilon_{\tau\tau} = 2A\dot{\varepsilon}^{\frac{1}{3}}_{\tau\tau} = 0$ at the crest. It can be shown (Eringen,

1967) that

$$\pi - p = \dot{\varepsilon}_v(\lambda + \frac{2\mu}{3})$$

where π is the thermodynamic pressure and λ the dilatational viscosity. If the Stokes condition is not respected then the velocities-pressure formulation solves for the thermodynamic pressure when it should really solve for the hydrostatic pressure. For the compressing Glen flow, λ was set at zero, and the difference between π and p is given by

$$\pi - p = \frac{2\dot{\varepsilon}_v\mu}{3}$$

Inspection of Tables 5.12 and 5.13 shows that for a distal velocity of 10 alu/a the error was of the order 0.01 bar, insignificant compared with the distal pressure of 10 bar.

Normal stresses at the interface

The normal stress given by

$$\sigma_{\nu\nu} = - p + \lambda\dot{\varepsilon}_v + 2\mu\dot{\varepsilon}_{\nu\nu}$$

may now be computed at the ice-rock interface. Figures 5.60 to 5.64 show plots of the ice pressure and the normal traction against $\cos\gamma$ (equivalent to distance from the leading edge) for all the five rheologies which could be characterised.

For the linear rheologies the normal traction is a linear function of $\cos\gamma$. For the non-linear rheologies this is not the case. It should be noted that for the incompressible rheology the deviatoric stress is insignificant compared with the hydrostatic stress, and that for the compressible case the non-linearity of the relationship between $\sigma_{\nu\nu}$ and $\cos\gamma$

arises because of the non-linear behaviour of the deviatoric stress.

The fact that the deviatoric stress is insignificant for the incompressible case is a result of there being a very fluid layer at the ice-rock interface relative to the more distal regions. This is also the explanation for the high stresses observed relative to the linear case. This flow configuration is unlikely to occur in real glaciers, where straining will not be so concentrated.

The total force acting downwards on the upstream-side of the clast is given by

$$\int \sigma_{\nu\nu} dx \quad ,$$

where x is the horizontal distance from the leading edge. It should be noted that for a given leading edge traction, this force is greater for the non-linear rheologies than for the linear rheologies.

Since it was not found possible to characterise the velocity and ice pressure distributions around the truncated sine ridge simply, the analysis carried out for the flow around the half-cylinder cannot be repeated. However, it is worth performing an order of magnitude calculation to see whether the point of maximum pressure observed behind the leading edge is reflected by there being a similarly positioned point of maximum normal stress. For a distal velocity of 10 alu/a the interfacial viscosities were of the order of 0.6 bar.a. The tangential strain rate never exceeds $\approx 0.3\ a^{-1}$. This gives an upper estimate to the deviatoric normal stress of 0.2 bar, which is insignificant compared with the pressure increment at the point of maximum pressure, which is 147 bar. Thus, the normal stress may be approximated by the ice pressure, and the point of maximum water pressure does lie downstream of the leading edge.

Tangential stresses at the ice-rock interface

For perfect slip the tangential stress at the interface is zero, but for rough beds it is not. It is important to note that the flow exerts a clockwise couple on the obstruction. This couple will act to increase the stress fields in the obstruction.

For a distal velocity of 10 alu/a and a smoothness of 10 alu/bar.a the tangential traction exerted on the crest of the obstruction was 1.09 bar, and for a distal velocity of 400 alu/a and a smoothness of 400 alu/bar.a the traction was 0.42 bar. The applicability of these results to glacial situations awaits the determination of values for the smoothness.

For the Glen rheology the distal tangential traction was 0.278 bar at a distal velocity of 10 alu/a and a smoothness of 10 alu/bar.a.

5.9 Scaling

The mesh geometry has been measured in alu. These results may therefore be applied for obstacles of any dimension provided the correct scaling factors are used. Define ω as the scaling factor in alu/m. Let the subscript p denote the value measured in the m-bar-a system, and the subscript a the value measured in the alu-bar-a system. Then, by dimensional analysis,

$$h_p = h_a/\omega \qquad 5.3a$$

where h is a height

$$v_p = v_a/\omega \qquad 5.3b$$

$$S_p = S_a/\omega \qquad 5.3c$$

$$\chi_p = \chi_a/\omega^2 \qquad 5.3d$$

$$p_a = p_p \qquad 5.3c$$

Pressure does not change with scale because it is measured in bar .

Consider the case in which the physical velocity v_p and the mesh obstacle height h_a are kept constant. Then by 5.3a and 5.3b the quantities v_a/ω and $h_p\omega$ are kept constant. Thus, as ω increases (i.e. the mesh becomes smaller), h_p decreases and v_a increases.

Thus, a decrease in physical height by a factor of 40 is equivalent to increasing the distal velocity measured in arbitrary length units by a factor of 40.

Thus, the results obtained for a series of computations with differing distal velocity measured in arbitrary length units for constant height measured in arbitrary length units apply equally to a series of computations with constant physical velocity and differing physical height. For a linear rheology, since $p' \propto v_d$, $p' \propto h^{-1}$, while for a Glen rheology $p' \propto v_d^{\frac{1}{3}}$, meaning that $p' \propto h^{-\frac{1}{3}}$.

For compressible rheologies, the analysis is more complicated. If we follow the procedure of keeping h_a and v_p fixed we find from 5.3 that the physical smoothness and diffusivity also change. For example, the calculations carried out for a diffusivity of 1 alu^2/bar.a with distal velocities of 10 alu/a and 400 alu/a and an obstacle height of 4 alu could equally well refer to calculations carried out at a distal velocity of 10 m/a, but with obstructions of height 4 m and 10 cm respectively. By 5.3d the diffusivity in the former case would be 1 m^2/bar.a, while in the latter case it would be $\frac{1}{1600}$ m^2/bar.a, i.e. very much lower than that estimated in Chapter 3.

The results from the calculations using linear compressible rheologies showed that the pressure increment decreased as the diffusivity increased for constant distal velocity. Let us suppose that $p' \propto \chi^{-m}$.

Consider an experiment 1 where the pressure increment p_1' at an arbitrary point is computed for diffusivity χ_1 and obstacle height h_1.

This is equivalent to an experiment 2 with diffusivity $\chi_2 = \chi_1/\omega^2$ and obstacle height $h_2 = h_1/\omega$ where $p_2' = p_1'$.

If we assume that $p' \propto \chi^{-m}$, then the p_3' for an experiment 3 with diffusivity $\chi_3 = \chi_1$ and obstacle height $h_3 = h_1/\omega$ may be predicted since $\chi_3 = \chi_1$ and $\chi_2 = \chi_1/\omega^2$ then $p_3'/p_2' = p_3'/p_1' = (\chi_3/\chi_2)^{-m} = \omega^{-2m}$. Now $\frac{h_3}{h_1} = \omega^{-1}$ and therefore $\frac{p_3'}{p_1'} = \left(\frac{h_3}{h_1}\right)^m$.

It was noted in section 5 that as χ increased tenfold, p' decreased by a factor of $\approx 10^{\frac{1}{3}}$ suggesting that $m = \frac{1}{3}$. Thus, p' increases with h, whereas for the case of purely viscous flow it decreased. This is somewhat similar to the combined effects of regelation and plastic flow (see Chapter 3) though in this case the increase of p' with h occurs for flow occurring by both methods.

The simple power law relationship between p' and χ cannot be universally valid since it predicts p' to be infinite for the incompressible case. However, since p' decreases with h for an incompressible case there must for example be a value of diffusivity where p' is independent of h , and thus in general scale-dependency will be sensitive to the diffusivity.

Similar considerations apply to the bed smoothness. By 5.3c, as ω increases, that is the mesh size decreases, the physical smoothness changes in inverse proportion. Thus, the results obtained for a series of increasing distal velocities and identically increasing smoothness are equivalent to maintaining the distal velocity and smoothness and varying the height. Thus, since $p' \propto v_d^{0.75}$, we find that $p' \propto h^{-0.75}$, a different relationship from the case of non-shearing flow and perfect slip. Since the crestal tangential traction declined with v_d and S it will also be less for smaller obstacles.

The flows around the truncated sine ridge using an incompressible Glen rheology showed significant variations in their patterns as the distal velocity changed. This is equivalent to saying that for different scales and a constant distal velocity the flow patterns change with scale.

The same is true of flow around the semi-cylindrical ridge, where there was a statistically significant dependence of the relationship between v_τ and $\cos\gamma$. Thus, we expect strain-rate patterns to be scale dependent for non-linear rheologies.

In conclusion, different material properties can alter the scaling behaviour of the flow system. How the different scaling properties interact with each other has not yet been determined. The uni-axial flow modelling described in the remainder of this chapter investigates some of these properties.

None of the results outlined in this chapter are strictly comparable with other reported analytic and numerical works apart from the observation of reverse flow upon cavity collapse, which was also observed by Iken (1980). Another 'predictable' feature has the maximum pressure in the upstream edge of the half-cylinder increasing with (approximately) the cube root of the velocity (Weertman, 1957). Less expected was the fact that for the truncated sine ridge the exponent was near $\frac{2}{3}$, and that pressure and velocity distribution showed significant changes with distal velocity and thus scale.

A feature that was probably not significant was the very high stresses computed compared with sub-glacial observations (Boulton et al., 1979). Although the ice was stiffer than glacier ice by a factor of two (see Chapter 3), the high pressures are consequences of the low straining in the distal regions and would not occur under glaciers.

5.10 Uni-axial flow modelling

The uni-axial flow formulation obtains solutions for fluid flow in a plane orthogonal to the direction of glacier flow. The basic assumptions of the formulation are that the ice flows in one direction only, and that it is incompressible. These two assumptions taken together mean that there can be no changes in velocity field along the direction of the velocity field.

When these assumptions are put into the Navier-Stokes equations, these equations reduce to the Laplace equation (see Chapter 3) i.e. the equation describing heat flow or pressure behaviour of fluids in a porous medium. It is possible to solve these equations using the finite element method (see Chapter 4). In addition, it is straightforward to incorporate a basal boundary condition of a Weertman type.

The obvious case where the assumptions implicit in the uni-axial flow formulation are least violated is in the case of a cross-section of a valley glacier. On this broad scale, the incompressibility conditon is unlikely to be seriously violated by bulk internal melting, and longitudinal velocity gradients can be small compared with velocity gradients orthogonal to the principal direction of flow. Studies using simulation models have been carried out by Reynaud (1973).

Similar studies were not carried out using the finite element program described in Chapter 4 principally because the results of small scale studies (reported in this chapter and discussed in Chapters 6 and 7) indicate that a satisfactory relationship between abrasion rate and the basal velocities and stresses at a scale where hummocks have been smoothed does not exist. The studies reported below were part of the attempt to establish a satisfactory medium-scale erosion law and to try and explain the genesis, shape and size of small and medium scale glacial erosional landforms.

The problem studied was flow over sub-glacial hummocks. Thus, finite element grids representing such landforms were set up. The grid used is shown in Figure 5.65. As can be seen, the cross sections of three identical hummocks are represented. The cross-sections of the hummocks are described by a sine wave in the range 0 to π .

A serious objection to this model of the flow around hummocks is that the assumptions necessary to the applicability of the uni-axial flow formulation are severely violated. Significant flows orthogonal to the principal direction of ice flow do exist, and, these lead, as a consequence of the incompressibility condition, to the assumption of there being small longitudinal velocity gradients being also violated. However, if it is supposed that these hummocks are longitudinally symmetric, that the cross-sections shown in Figure 5.65 represent the mid points and that the edge effects do not distort the flow around the central hummock compared with the case where there is an infinite row of hummocks, then, by symmetry, all the velocity vectors must be longitudinal and the longitudinal strain rates must be zero. In this case, the uni-axial flow assumption is a fair one.

In the absence of a practical method of solving the full three dimensional equations of ice flow, the approximation of uni-axial flow is the best that can be done if we wish to investigate the reasons why the cross-sections of roches moutonnées are shaped as they are. Since, as will be seen in the discussion in Chapter 7, these results are used in a qualitative way only, the approximation does help us to understand sub-glacial processes a little further.

In all the computations described in this section, ice has been modelled as possessing a Glen rheology. A global convergence criterion of 0·01% was set. This is a lot more stringent than that used for the plane strain formulation. This was done because if the 1% criterion used for the plane strain formulation was used in the uniaxial flow formulation the results could be seen to be non-physical.

The reason why the uni-axial flow formulation requires a more stringent convergence criterion than the plane strain formulation is a consequence of the different nature of the two equations. For plane strain flow, significant deviations from the true solution affect neighbouring areas significantly. This is not true for the Laplace equation. This means that unless a tight convergence criterion is set for uni-axial flow, significant local deviations of the flow will appear in the solution. In practice the solutions converged to within 0.1 or 0.3% after the maximum number of iterations (15). To avoid excessive computing costs, these solutions were accepted after inspection to ensure physical reasonableness.

In this modelling a linear Weertman boundary condition was used i.e.

$$V_\tau = ST_\tau$$

where V_τ is the slip velocity, S is the smoothness and T_τ is the tangential traction at the interface. It is argued in Chapter 6 that this may not be the best basal boundary criterion for the representation of the flow of dirty ice over polished bedrock. The implications of this are considered later.

The boundary conditions were specified velocities around the top and side edges and a Weertman boundary condition with a linear stress-velocity relationship along the bottom edge. There is not the same need to specify boundary conditions at infinity as there is with the plane strain flow formulation as the incompressibility condition cannot be violated by poor choice of the boundary conditions. The velocity along the

Dirichlet boundaries has been set at a constant value. This means that along the Dirichlet boundaries the strain rate in the direction parallel to the boundary is zero. This produces an inconsistency where the Dirichlet boundary meets the Weertman boundary. However, around the middle hummock, the velocity gradients are approximately horizontal, and the boundary effect is fairly small. Gravity has not been included in the problem.

In most of the calculations the bed smoothness has been set at a constant value along the bed. In one set of calculations the smoothness varies with height. Three sets of runs were performed. The first set used a hummock with height 2 alu and trough to trough distance 8 alu. The distance from the upper boundary to the bottom of the trough 18 alu. Smoothness was spatially constant. This set of runs is called the wide-hummocked case. The other two sets of runs used a mesh geometry with half the width, i.e. a distance of trough to trough of 4 alu, but with the same vertical dimension. In one case (called the narrow hummocked case) the smoothness was constant in space; in the smooth-crested case the smoothness varied with height. The minimum value was set at the bottom of the trough, and changed with height h according to $S = \frac{1}{4-h}$, reaching infinity at the crest. In fact, element side mid-point values have been taken. This means that the minimum smoothness was multiplied by 1.62 along the lower part of the hummock and by 13.1 along the upper part of the hummock.

For the wide hummocked configuration forty nine calculations were carried out. These were for values of the smoothness ranging from 0.002 alu/bar.a unit to 100 alu/bar.a, and for velocities ranging from 10 alu/a to 400 alu/a (see Table 5.14).

Table 5.14

Uni-axial flow calculations

(a) Wide hummocked case

Five distal velocities (10,50,100,200,400 alu/a) ×
ten smoothnesses (0·002,0·01,0·02,0·1,0·2,1,2,10,20,100 alu/bar.a)

Missing calculation : v_d = 100 alu/a, s = 0·002 alu/bar.a

(b) Narrow hummocked case

Three distal velocities (10,100,400 alu/a) ×
five smoothnesses (0·002,0·01,0·02,0·1,1 alu/bar.a)

No missing calculations

(c) Smooth-crested case

Three distal velocities (10,100,400 alu/a) ×
five smoothnesses (0·002,0·01,0·02,0·1,1 alu/bar.a)

Missing calculation: v_d = 100 alu/a, s = 0·002 alu/bar.a

For the narrow hummocked configurations 15 calculations covering a range of five bed smoothnesses from 0.02 alu/bar.a and three boundary velocities of 10 alu/a to 400 alu/a were performed (see Table 5.14).

For the smooth-crested configuration 14 calculations were carried out using the same parameter ranges as for the previous configuration (see Table 5.14).

The longitudinal velocities computed are shown on contour plots (Figures 5.66 - 5.79). The contours are those of the computed velocity solution. The velocity solution was interpolated quadratically in each element. In undistorted elements the contours are conic sections, while in the distorted elements the contours are conics mapped onto quadratic surfaces.

Since the velocity solution is represented by a piecewise quadratic function, the velocity gradient (i.e. the strain rates) are not guaranteed to be continuous. If the solution is accurate the contours will be smooth, while regions of poorer solution will be represented by breaks in the contour slopes at element boundaries. Examples of this occurring can be seen in the troughs in Figures 5.76 - 5.79.

One feature of the calculations that has not been explained is that an oscillation in the computed velocities with a frequency of two nodes was observed. At the very edge of the solution, this resulted in there being negative velocities at some nodes. The effect of this on the velocity contours can be seen in several of the plots of velocity contours, e.g. Figure 5.66. Detailed inspection of printed results showed that these oscillations do still exist at the bed in the middle of the section, but they are very much subdued compared with those at the edge. These oscillations are probably a consequence of the inconsistent boundary conditions where the Dirichlet boundary meets the Weertman boundary.

In general the velocity contours are affected by the hummock, and tend to follow it in a damped fashion. Figures 5.66 and 5.71 show a series of calculations for increasing smoothness. For the very rough beds, the velocity contours are packed very closely near the crest, and the spacing increasing in the troughs, meaning that strain rates are highest (and viscosities lowest) at the crest. As the bed becomes smoother, the slip velocity increases and some of the contours are no longer plotted.

Figures 5.67 and Figures 5.73 to 5.75 show plots for a constant smoothness but increasing distal velocity. It may be seen that as the boundary velocity increases the velocity contours become packed closer and closer together, increasing strain-rates and decreasing viscosity.

Figures 5.76 to 5.79 show velocity contours for the case where the wavelength of the hummocks is halved. A notable feature is that variation in strain rates from crest to trough is less marked than for the wider hummocks: in other words the flow is less choked for narrower troughs, a counter-intuitive observation. It is possible that as the troughs become narrower still, the flow within the troughs will become choked. The flow solution in the troughs is, however, poorer than for the wide hummocks, as indicated by the breaks in slopes of the velocity contours. The likely effect of this is to reduce velocity gradients because of numerical diffusion.

Figure 5.79 shows a velocity contour plot for the case of smoothness increasing with height. The contours in the troughs are rather less parallel to the boundary than for the other cases because the ice is slipping rather more easily over the crests.

The results from the calculations have been processed so as to provide some indication of the variation in abrasion rate across the hummock.

As a first approximation let us assume that the smoothness S is inversely proportional to clast concentration and to viscosity. For convenience, the coefficient of proportionality is included with the clast concentration. This quantity is called the clast concentration factor c which increases with clast concentration. Thus,

$$S = \frac{1}{\mu c} \qquad 5.4$$

Two models of abrasion rate are considered here (see Chapter 6 for a justification).

Model A states the abrasion rate A is given by

$$A_a = R_a \mu v c \qquad 5.5a$$

where R_a is a wear constant, and v is the slip velocity; and model B defines another wear constant R_b which is used in the relationship

$$A_b = R_b v c \qquad 5.5b$$

Such a form might arise if abrasion were controlled by contact area effects.

Now, since the viscosity varies across the hummock, while the smoothness S was either constant or constrained in a particular way, the clast concentration factor will vary according to

$$C_t = \frac{1}{\mu_t S_t} \quad , \quad C_c = \frac{1}{\mu_c S_c} \qquad 5.6$$

where the subscript t refers to the trough and the subscript c refers to the crest. This convention will be followed in the remainder of the chapter. The clast concentration is thus varying in an uncontrolled fashion in these computations. In order to make results more comparable with one another, instead of considering the actual abrasion rates as given by equations 5.5, we consider the abrasion rate per unit clast concentration, B_a and B_b.

Using 5.5 and 5.6 and setting $R_a = R_b = 1$ (since the values of these constants have not been determined empirically or theoretically), we obtain

$$B_{ac} = \mu_c v_c \qquad 5.7a$$

$$B_{at} = \mu_t v_t \qquad 5.7b$$

$$B_{bc} = v_c \qquad 5.7c$$

$$B_{bt} = v_t \qquad 5.7d$$

Before considering the results in detail, it is necessary to consider scaling. The meshes are again in arbitrary length units. We restate the appropriate equations 5.3 i.e.

$$h_p = h_u/\omega \qquad 5.8a$$

$$v_p = v_u/\omega \qquad 5.8b$$

$$S_p = S_u/\omega \qquad 5.8c$$

but using the subscript u to represent unspecified length units. The case of interest will be varying ω while keeping v_p and S_p constant. Thus, we keep the ratios v_u/ω, S_u/ω constant and the product $h_p\omega$ constant since h_u is invariant. Thus, as ω (alu/m) increases, the scale decreases and the v_u and S_u corresponding to the constant physical values increase.

The results for the three sets of runs have been plotted on contour maps, with the distal velocity and the smoothness as the independent variables. Plots (Figures 5.80 to 5.106) are of the crest, trough and ratio of crest to trough values of slip velocity, viscosity and the product of viscosity and velocity.

At the back of the thesis may be found a transparency, which when superimposed over the plots, shows scaling behaviour. If a line is followed in the direction of the arrows, scale is increased for constant smoothness and boundary velocity. These lines were obtained by use of Equations 5.8.

For the wide hummock case, these contour maps come from 5 points in the velocity plane by 11 points in the smoothness plane, while for the other sets of calculations the results were plotted from a 3 × 5 grid. In certain cases the contouring algorithm (the Butland method (NAG, 1981)) has produced crossing of contours, because the contouring algorithm can imply a higher order variation than exists. However, since the purpose of these studies is to illustrate generic properties of the system rather than detailed predictions, it was felt that this method of displaying the results was the most direct. A total of two missing values existed. These were filled in by interpolation using regression relationships with high (> 0.98) correlation coefficients.

Figures 5.80 to 5.88 shows the slip velocities. Crest velocities (Figures 5.80 to 5.82) increase with distal velocity and with smoothness, though at a lower rate than direct proportionality. The surface in logarithmic space is planar apart from when smoothnesses are high for the constant smoothness case. For the case of the smooth crest, the relationship is more complicated but shows a similar variation. The contours are not parallel to the scaling lines, and as scale increases for a given velocity and smoothness, crestal velocities decrease. Crestal velocities are highest for the smooth-crested case, and lowest for the wide hummock case (Figures 5.80 to 5.82), apart from the behaviour being slightly more irregular for the smooth-crested hummocks. Trough velocities decrease along the scaling lines, and are highest for the smooth crested case and mostly lowest for the wide hummocks.

Velocity ratios (Figures 5.86 to 5.88) exhibit a much higher order of variation. They are very high for the smooth crested case (Figure 5.88), and are on average greater for the wide hummocks than for the narrow hummocks.

The contours are not parallel to the scaling lines for rough beds, but become sub-parallel as the smoothness increases. For the cases of the smoothness not changing spatially, the velocity ratio decreases with an increase in scale. For the spatially varying smoothness the ratio is not a monotone function of scale.

Crest viscosities (Figures 5.89 to 5.91) decrease with increasing distal velocity, and increase slowly with smoothness, the rate increasing for very smooth beds. Crest viscosities are approximately equal for the narrower hummock cases when smoothness is low but are generally lower than for the wide hummock case. Crest viscosities increase markedly with scale.

Trough viscosities exhibit similar patterns (Figures 5.92 to 5.94). There is some anomalous behaviour for the wide hummocked case. This is where viscosities are high, and the values may be inaccurate because of exponentiation errors arising when the viscosity is computed. Trough viscosities are markedly higher for the wide hummocked case than for the narrow hummocked case and the smooth-crested case, the latter two being of the same order. As scale increases, trough viscosities decrease. This is consistent with the fact that the wide hummocks, and therefore wide troughs, have higher viscosities.

Viscosity ratios (Figures 5.95 to 5.97) show complex patterns of variation, though part of this may be a result of the fact that the quantity is a ratio of two exponentiated numbers. For the two narrow-hummocked cases the viscosity ratio is not so strongly dependent on velocity as on smoothness, but does increase with scale. Given the uncertainty inherent in the calculation of the ratio, it may be that the contours are parallel to the scaling lines. For the smooth crested case the viscosity ratio is greater than one for high smoothnesses, while for the narrow-hummocked case it lies between 0.1 and 0.5. The variation is much more marked for the wide hummocked case, where the viscosity ratio varies between 0.01 and being greater than one. For very low smoothnesses the viscosity ratio does show a marked scaling effect, the ratio decreasing with scale.

Figures 5.98 to 5.106 display the velocity-viscosity products. Crestal values (Figures 5.98 to 5.100) show a slight decrease with velocity but increase markedly with smoothness. As scale increases, the product decreases. The wide hummocked case and the narrow-hummocked case exhibit similar values, while the smooth-crested case shows values nearly an order of magnitude higher.

The trough viscosities (Figures 5.101 to 5.103) show a broadly similar spatial pattern. The higher order variations are probably a consequence of exponentiation error. Values for the smooth crested case and the wide hummocked case exhibit similar values for high velocity, while those for the narrow hummocked case are half an order of magnitude lower. Values decrease slowly with scale.

Figures 5.104 to 5.106 show the viscosity-velocity product ratios. For the spatially constant smoothness cases marked dependence on the distal velocity and the smoothness is exhibited, the product ratio increasing with both. In consequence, a marked dependence on scale exists for these two. The range of values for the wide hummocked case is much greater (≈ 0.02 to ≈ 1.0) than for the narrow-hummocked case (≈ 0.5 to ≈ 1.1). The smooth crested case exhibits a very complex pattern with fairly strong scale dependency in most regions and with values (≈ 1 to ≈ 6) rather greater than for the other two cases.

It is of interest to compare these solutions with the analytical solution for an infinitely wide flat bed. The velocity at a height h, v_h, is defined as is the smoothness S. The constant velocity gradient is given by

$$\frac{\partial v}{\partial y} = \frac{v_h - v_s}{h}$$

where s is the slip velocity and y is the vertical co-ordinate. The strain rate $\dot{\varepsilon}_{yz} = \frac{1}{2}\frac{\partial v}{\partial y}$, where z is the flow direction. Since the flow is pure shear, the effective strain rate $\dot{\varepsilon} = \dot{\varepsilon}_{yz}$, the viscosity $\mu = A\dot{\varepsilon}_{yz}^{-\frac{2}{3}}$, and the shear stress $\sigma_{yz} = 2\mu = 2A\left(\frac{v_h - v_s}{2h}\right)^{\frac{1}{3}}$ where A is the constant in the rheological relationship (see Chapter 3). The tangential traction in the z direction $T_z = v_s/S$, and also equals the basal shear stress.

By equating the traction and the shear stress expressions, we obtain

$$\frac{v_s}{S} = 2A\left(\frac{v_h - v_s}{h}\right)^{\frac{1}{3}}$$

Rearranging this produces the following cubic for v_s

$$v_s^3 + \frac{4A^3S^3}{h}v_s - \frac{4A^3S^3}{h}v_h = 0$$

This equation has one real root (Abramowitz and Stegun,1965) given by

$$v_s = AS\sqrt[3]{\frac{4}{h}}\left\{\left[\frac{v_h}{2} + \left(\frac{4A^3S^3}{27h} + \frac{v_h^2}{4}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \left[\frac{v_h}{2} - \left(\frac{4A^3S^3}{27h} + \frac{v_h^2}{4}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}\right\}$$

As may be seen, there is a complex relationship between the slip velocity and the other parameters even for this geometrically simple problem. The sliding velocity increases non-linearly with the smoothness and the boundary velocity, as was found with the hummocky bed. The height h is equivalent to a scale factor; thus as h increases, while the velocity and smoothness are maintained, the slip velocity decreases. This too happened with the hummocky bed.

Of specific geomorphological interest are the slip velocities and the viscosity-velocity product, which have been posited (Equations 5.7) to be indices of the abrasion rate divided by the clast concentration. These parameters have both been shown to vary with the distal velocity, the smoothness and therefore scale. More importantly, their ratios at trough and crest change with distal velocity, smoothness and scale; these relationships are often complex. The implications of these results for glacial hummock development are discussed in Chapter 7.

It would also be of interest to interpret these results in terms of an average lowering rate for a given basal shear stress-sliding velocity situation at the glacier base. However, this is fraught with difficulties. The first is the definition of sliding velocity. This term is often used rather loosely in glaciology. In small scale studies an observation means the actual slip velocity at the ice-rock interface, while in the large scale modelling of ice-sheets it refers to a discharge through an area of unit width and arbitrary height, meaning therefore that cold glaciers can slide (Morland and Smith, 1984). All that is usually said about this height is that it is small compared with the thickness of the glacier.

Similarly difficult to define is the basal tangential traction. While traction is defined in units of force/area in the modelling of ice sheets, to be consistent with the definition of sliding velocity the traction must be defined in terms of the mean shear stress over the height through which the discharge is being computed in order to obtain a sliding velocity. Thus, to define abrasion rates in terms of generalised sliding velocities and shear stresses begs the question of what is meant by the sliding velocities and the shear stresses.

In conclusion, these results indicate a marked sensitivity of results to sub-glacial conditions, hummock geometry and smoothness distributions in space. These conclusions apply specifically to a Weertman basal boundary condition where the tangential traction increases with the velocity. In Chapter 6 it is suggested that a Coulomb relationship may be more applicable to the ice-rock interface, where the tangential traction is related to the normal traction. Since clast concentration is an uncontrolled variable in these experiments, the results could be re-interpreted so that the extra drag due to velocity could be attributed to a different clast concentration. In this case the quantities μv and v still operate as indices of abrasion per unit clast concentration.

5.11 Conclusions on finite element modelling

Numerical solutions of partial differential equations are inevitably accompanied by a certain amount of noise, and part of the art of numerical analysis is the interpretation of results. The essential issue is that of the credibility of the results. The testing of the finite element program described in Chapter 4 indicated that it worked well in a number of simple test-cases, except for the problem of gravity-driven flow of an infinite medium, where it was suggested that the slow convergence properties of that particular system may have meant that the solution quoted was not the converged solution.

For the case of the flow over the semi-cylindrical ridge, the drag increased with the velocity to the power of one third, an 'expected' result. This suggests that good solutions were obtained for these cases. These solutions were observed to converge more swiftly than the inclined plane problems, which is another reason for accepting the results. For the truncated sine ridge, the drag increased with the distal velocity raised to the power 0.6, which is not the expected result. However, convergence was again rapid for these calculations, suggesting that accurate solutions were obtained; for this shape, therefore, the expected drag-velocity relationship does not hold, presumably because straining patterns change significantly with the distal velocity, as was observed.

In general, the smoothness of the velocity contours representing the solutions obtained from the uni-axial flow modelling suggested that good solutions were obtained, even though convergence was rather slow.

The finite element method does, therefore, seem to be a viable method of solving the flows at the ice-rock interface, provided that sufficient care is taken in interpreting the results. Obviously there are many more situations that could profitably be investigated.

CHAPTER 6

ICE-CLAST-BED PROCESSES

6.1 Introduction

Abrasion has been defined in Chapter 1 as the erosion caused by the forcing of rock against bedrock by the action of the flowing ice. In this chapter, the process is considered in some detail.

The model of the clast-bed-ice-water configuration used in this thesis will now be outlined. It is assumed, unless otherwise stated, that the ice is at its pressure-melting point. On the broad scale, this is a reasonable assumption, as the ice needs to slide over the bedrock in order to drag the abrading clasts, and in order to slide, it needs to be at its pressure melting point.

The clast is assumed to be in contact with the bedrock (Figure 6.1), and enveloped by a mixture of water and gas. Ice cannot adhere directly to the clast unless it is frozen to the clast. In general, the lower surface of the clast and the bed-rock will not be conforming. The geometry of the contact will be discussed in more detail in the next section. The ice is separated from the bedrock and the clast by a thin film of water or a cavity filled with gas or water.

Since the ice is nowhere in contact with the clast, the forces on the clast are not determined directly by the stresses in the ice, but rather by how these stresses are transmitted through the fluids surrounding the clast.

The first person to attempt to calculate the forces acting on a clast was McCall (1960). He reached the conclusion that the maximum clast-bed contact force was 1 bar, the yield stress of ice, because at stresses greater than this, the ice would be squeezed under the clast. McCall's model is wrong because he ignores the presence of the water-film surrounding the clast. In a hydrostatic environment the stress level in this film will be the same as in the ice, and the water will thus oppose the movement of the ice, whatever the rheology of the ice (McCall's assumption of a perfectly plastic rheology is poor, given the discussion in Chapter 3). The validity of McCall's model in non-hydrostatic stress environments is discussed in section 6.3.

The next person to attempt to calculate the force-balance on an abrading clast was Boulton (1974; 1975). In this model, he assumes that the normal traction at the base of the glacier is transmitted directly to the clast, and that the clast transmits this force to the bed and the water between the clast and the bed. The water pressure between the clast and the bed is allowed to take on an autonomously determined value between zero and the pressure at the base of the ice. In general, Boulton's model is not physically consistent. If the configuration is in steady state, and the pressure in the water layer between clast and bed is not equal to the pressure in the ice, then the force balance at the base of the glacier no longer exists. If the situation is dynamic, and the water has been rapidly drained, then the model is more consistent, but only an approximation, as the ice would have a downwards velocity component which would affect the stress fields around the clast, meaning that the normal traction exerted on the clast through the water by the ice would no longer be hydrostatic. To summarise, Boulton's model cannot represent a steady-state, and for transient situations it is an approximate model.

The first worker to use a model in which the clast was enveloped by pressurised water was Hallet (1979). Hallet implicitly uses the correct calculation for the force pressing the clast towards the bed (Watts, 1974), i.e.

$$\oint p_w \nu_z dS \qquad 6.1$$

where p_w is the water pressure, ν_z is the vertical component of the normal to the clast's surface, and the integral is a surface integral taken over the surface S of the clast. The pressurised water in between the clast and the bed acts to separate the clast and the bed. Hallet assumes this force to be

$$\int p_w dU$$

where U is the under-clast area. In fact, as was pointed out in Chapter 1, the force is

$$\int (1 - \varphi) p_w dU \qquad 6.2$$

where φ is coefficient of effective contact area, that is the proportion of clast-bed contact area which can bear a load. Thus, Hallet's calculation is only really correct when the effective contact area coefficient is very small. The details of the clast-bed contact are discussed in the next section.

Consider the case when the projections of the pressure contours above the clast onto the under-clast area exactly follow the pressure contours under the clast (this is the result predicted by Watts' (1974) analysis).

Then, by the definition of the surface integral,

$$\oint p_w \nu_z dS = \int p_w dU \qquad 6.3$$

If we assume that the effective contact area coefficient φ is constant over the bed, then by subtracting the downwards force 6.3 from the upwards force 6.2 we obtain the clast-bed contact force

$$\varphi \int p_w dU \qquad 6.4$$

The clast-bed contact stress is obtained by differentiating 6.4 with respect to U:

$$\varphi p_w \qquad 6.5$$

for constant φ.

This stress will result in friction between the clast and the bed, and it is a general rule that where there is friction there will be wear.

6.2 The clast-bed contact

No direct measurements of the effective area of contact between the clast and the bed appear to have been made. Implicit in Hallet's work is that it is very low, consisting solely of a few asperities per clast. In this case φ will indeed be low, and under-clast water pressure will suppress much abrasion. However, Boulton (1975) and Boulton et al. (1979) have stressed, both in words and in diagrams, that the area between the clast and the bed is frequently occupied by debris ploughed up and along by the advancing clast. Boulton (1974) suggests that this debris will increase the area of contact.

Obviously this is true to a certain extent, but how much is not clear. Also, it is not clear what portion of the load will be borne by the asperities touching the bed, and what portion by the accumulation of debris.

The ploughing action of the clast will act so as to pack the debris into the voids between the clast and the bed, but it is not apparent that this packing action will be of sufficient strength to cause the bulk of the load of the clast to be borne by the debris.

To analyse this matter in more detail, in the absence of observations, some assumptions about the stress-strain behaviour of the debris must be made. This will depend on the grain size distribution of the under-clast debris. If an appreciable fraction of its particles are sand-size, then it is unlikely that the packing action of the tangential frictional forces will consolidate the debris. Curvature of the glacier bed will not be conducive to packing.

If an appreciable fraction of the debris underneath the clast is silt or clay, as Riley (1982, unpublished) suggests, then the under-clast debris may well possess cohesion. If the debris contains air as well as water then capillary action will tend to consolidate the grains. In addition, if the surface of the bedrock is rough on the small scale, capillary action will tend to pull the debris and the bedrock together. Another way of putting this is to say that rough surfaces will get abraded faster.

If the debris is frozen together, then this will serve to increase its rigidity and increase contact forces.

As has been pointed out in Chapter 2 the fact that glacier surfaces are highly polished rather than being rough in the direction transverse to ice flow suggests that the bulk of glacier abrasional activity consists of small scale events rather than deep scoring and grooving.

This may be a consequence of several factors. Firstly, in general tools may not be pressed against the bed with any great force. Thus, any indenting asperity which might cause wear would not penetrate the bed very deeply, and in consequence not produce deep striae.

Secondly, the size of the asperities which do the bulk of the work could be limited. This would be due to the fact that the larger and sharper an asperity is, the more likely it is to get crushed, leaving the duller, blunt asperities to do the bulk of the work. Equally possible is that much of the polishing work is done by the smaller debris being ploughed along by the clasts. If the presence of such debris increases the effective contact area as has been suggested above, then it may very well play an important role in the abrasion process.

If indeed the contact area is an important factor in determining the abrasive potency of a clast, it is of importance to be able to determine how this might vary through a clast's travel.

Boulton (1978, unpublished) has stated an important principle. This is that if the clast is of equal hardness (without being too precise as to what exactly hardness is) then it is reasonable to expect that the clast loses material as quickly by abrasion as does the bedrock.

Now the above discussion regarding contact areas has implied a two stage model of glacial abrasion. Firstly, large clasts are crushed to produce much smaller grains; grains of this size are also produced when striations and chatter marks are created. This debris is then dragged by the moving clast to polish the glacial surface. In this two stage model it is then assumed that the material produced by the abrasive action of the fines is lost.

It has already been argued that the geomorphological evidence suggests that striating and chatter-marking cannot be the processes of glacial erosion that produce most of the excavation of a surface. However, if production of the smaller debris necessary to increase contact areas were solely due to these mechanisms, and the smaller debris was destroyed once it had carried out abrasion, then one would expect the glacial surface to be half as rough as that produced by a surface being abraded by striation and chatter marking alone, as the polishing action of the fines is assumed to flatten a surface. This result still holds if the fines produced by the destruction of the clast are taken into account, provided that it is assumed that the action of the fines is equally directed towards the bedrock and to the clast.

A test of the adequacy of this two stage model is whether a glacially abraded surface is half as rough as it would be if it were created by striations: in other words is the average width of the area between striae equal to the striation width?

The answer to this question must be that in general it is rather greater. However, many questions are begged, e.g.

how much the original striation size varies; nor is it clear whether there is one wear process, or whether there are two wear processes, one acting on the large scale producing striae, the other smaller process causing polishing.

Perhaps the most important factor that is ignored by the two stage model is that it is most probable that there are more than two stages of breakdown. To construct a mathematical model of this process would be speculative at present, but it is worth considering the factors that would affect the model, and exactly what the model might be able to predict.

Firstly, a relationship is required between the size of an abrading fragment and its daughter product. It is most likely that this relationship will not be a constant ratio, but will vary with size. Thus, large clasts, which typically have a few asperities, will 'see' the bed as being very smooth, and produce fragments typically of the same order of size as the asperities. Smaller fragments, which might 'see' the small scale perhaps microscopic roughness of the bed, will tend to produce fragments more nearly their own size.

Fragments of whatever size will have a certain tendency to leave the system. Some will tend to get left behind the ploughing clast; others will tend to get washed out by the water flowing under the glacier. The latter process will certainly cause smaller particles to become entrained in the flow of water in preference to the larger particle: in consequence the fragments that will tend to get left behind are far more likely to be the larger ones that cannot be entrained. There may well be a lower limit to the particle size that can remain under a clast, unless some sub-clast areas are isolated from the water flow.

All other things being equal, it is possible that a dynamic equilibrium of clasts being created and destroyed could be established. Each particular size range would be created at a certain rate and destroyed by crushing or removed by the processes mentioned above.

The action of the clast ploughing this debris will tend to cause movements of the sub-clast debris relative to the bed. The possibility of any particular grain touching the bed will be a function of not only its size but the size distribution of all the debris under the clast, as will the average and peak contact forces experienced by these grains. The arguments put forward in this section have suggested that the presence of debris under the clast may affect the abrasional activity of glaciers. Riley's work (1982, unpublished), reported in Chapter 1, suggests that as the debris concentration increases, the abrasional potency of the clast declines. This, however, does not invalidate the arguments put forward in this section. Firstly, as the fine debris concentration increases, the clast-bed contact stress will also increase because the effective area of contact will increase. Secondly, the action of flushing water, as Riley points out, may serve to stop the choking effect of the debris on abrasion by washing out the smaller debris, though this will prevent the build up of the effective contact area.

Thirdly, the observations of Boulton et al.(1979) suggest that many clasts plough debris. Thus, while the abrasional potency of fresh surfaces may be higher, the larger number of worn surfaces actually performing the abrasive work may mean that worn surfaces do a substantial proportion, if not the bulk, of the abrasion carried out underneath a glacier.

6.3 The Geometry of the ice-clast contact

Provided there are no sources or sinks of water underneath the clast the pressure distribution underneath the clast is governed by the pressures at the edge of the clast, where the ice meets the bed or there is a cavity. The pressure distribution at the edge of the clast-bed interface is determined by the flow of ice around the clast and the generalised stress environment at that point of the glacier bed. In this section the flow of ice and water around the top of the clast are considered in more detail.

Consider the regelating flow of Newtonian ice past a sphere. A solution for this problem has been given by Watts (1974). Watts suggests that this solution is also applicable to the case of a hemispherical knob resting on an infinite flat bed. Watts' reason for suggesting this is that since the flow around the sphere is symmetrical, then the same analysis must apply to the hemispherical knob in the plane.

However, this assumption of symmetry is not justified. There are two reasons for this. The first reason is that there is, for temperate ice, a water film between the flat part of the glacier bed and the ice that is not considered in the analysis for flow around the sphere. This water film provides an extra drainage path for the water melted on the upstream side of the knob which is not included in Watts' analysis. Secondly, in general, water will be able to flow underneath the clast, as has already been pointed out in the preceding section. Again, this extra drainage path is not considered in Watts' analysis.

We may use Watts' analysis to consider the water pressure distribution in the water-film between the flat part of the bed and the ice. Watts' solution is given in terms of spherical co-ordinates r, θ, ϕ with velocity components v_r, v_θ, v_ϕ.

In spherical co-ordinates, r is the distance from the origin (the centre of the sphere), θ is the angle made with the line in which the direction of generalised ice-flow is occurring and ϕ is the rotation about the direction of generalised ice-flow.

Watts gives the following relationships:

$$v_r = U\left(1 - \frac{a^3}{r(a^2 + a_*^2)}\right)\cos\theta \qquad 6.6a$$

$$v_\theta = -U\left(1 - \frac{a^3}{2r(a^2 + a_*^2)}\right)\sin\theta \qquad 6.6b$$

$$v_\phi = 0 \qquad 6.6c$$

where U is the generalised flow velocity, a is the radius of the sphere, a_* is $\frac{3\mu C}{L\rho_s}$, where μ is the viscosity of the ice, C is the Clausius constant, L is the volumetric latent heat of ice and ρ_s is the thermal resistivity of the system.

Now, in spherical co-ordinates

$$\sigma_{rr} = - p + 2\mu \frac{\partial v_r}{\partial r} \qquad 6.7a$$

$$\sigma_{\theta\theta} = - p + \frac{2\mu}{r}\left(\frac{\partial v_\theta}{\partial\theta} + v_r\right) \qquad 6.7b$$

$$\sigma_{\phi\phi} = - p + \frac{2\mu}{r}\left(v_r + v_\theta \cot\theta + \operatorname{cosec}\theta \frac{\partial v_\phi}{\partial\phi}\right) \qquad 6.7c$$

where σ is the stress and p the local pressure (Landau and Lifshitz, 1959).

Watts' solution for the pressure field is

$$p = - \frac{a^3 \mu U \cos\theta}{r^2(a^2 + a_*^2)} + p_\infty \qquad 6.8$$

where p_∞ is the pressure at an infinite distance away (referred to as the distal pressure).

By differentiating the velocity expressions 6.6 to obtain the strain rates and substituting these in the stress relationships together with the pressure relationship 6.9, we obtain the following relationships for the stress field:

$$\sigma_{rr} = - p_\infty - \frac{2\mu a^3 U}{r^2(a^2 + a_*^2)} \cos\theta \qquad 6.9a$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = - p_\infty + \frac{\mu a^3 U}{r^2(a^2 + a_*^2)} \cos\theta \qquad 6.9b$$

The normal traction exerted on the flat part of the glacier bed is equal to $-\sigma_{\theta\theta}$, and is equal to the water pressure. Now, the water pressure at this point must be equal to the normal traction from considerations of force balance. Along the bed $\theta = 0$, and just upstream from the sphere $r \approx a$ giving the water pressure as $p_\infty - a\mu U/(a^2 + a_*^2)$.

The water pressure at the leading edge of the clast (equal to σ_{rr} by force balance) may be obtained from 6.9a, and is found to be

$$\frac{2a\mu U}{a^2 + a_*^2} + p_\infty \quad .$$

Thus, at the very leading edge of the clast Watts' analysis predicts a discontinuity in the pressure in the water-film. This is not surprising because Watts ignored the influence of the film over the flat part of the bed.

The idea that the point of maximum stress occurs at the leading edge of the clast is intuitively appealing, so it is worth considering qualitatively what might happen if this situation were to arise. Naturally the situation would be highly unstable. Water would flow very rapidly from the leading edge of the clast upstream under the bed, and the steady-state required by Watts' analysis would be lost. One way the system could avoid the situation would be to form a cavity on the stoss-side of the knob, which would act so as to mollify the discontinuity. However, there would still be high pressure gradients in the water film.

Stoss-side cavities have been observed to occur under glaciers (Boulton et al., 1979). Analysis of the shape of the clast is a free-surface problem, and it is unlikely that this problem could be solved by analytical means.

Another possible mechanism for avoiding the discontinuity in the water pressure is for the ice to creep under the tool, forming a finger of ice under the clast: this has also been observed to occur by Boulton et al. (1979). The finger will lengthen the drainage path, and so mollify the singularity. A further way in which the system might act to mollify the singularity is for the point of maximum water pressure, predicted by Watts' analysis to reside at the leading edge of the clast, to move away from the leading edge of clast.

These three mechanisms are considered in this section. In the next section their influence on the clast-bed contact force is considered, along with some other possible system configurations that might affect the clast-bed contact force.

Let us consider in more detail the formation of a stoss-side cavity. Watts' analysis only applies for hemi-spherical knobs, and it is worth considering whether this is a special case, or whether similar considerations might apply to other geometries.

To begin with, consider the case where the angle between the ice-clast contact and the ice-bed contact is more than 90°. When the ice is stationary, this configuration is stable. Immediately the ice begins to move, the stress begins to change. If there is perfect slip along the ice-bed boundary and the ice-clast boundary, then the shearing stress along these interfaces must be zero. Thus, the directions of the principal stresses must be at 45° to these zero shear stresses by the definition of principal stresses (Carson, 1971). Since the principal stresses can exist in one direction only if there is not to be a discontinuity in the stress field, this situation is only possible if the angle subtended at the edge is 90°, or if the edge is mollified into a continuous form. It is unlikely that the configuration with a sharp edge of 90° is stable, so it is therefore probable that the mollification of the shape into the continuously differentiable form will occur.

As was mentioned previously, the break in the slope of the ice boundary produces a break in the water pressure when Watts' analysis is applied to hemispherical knobs.

The principal stresses σ_{11}, σ_{22} are given (see Chapter 3) by

$$\sigma_{11} = -\pi + \lambda\dot{\varepsilon}_v + 2\mu\dot{\varepsilon}_{11}$$

$$\sigma_{22} = -\pi + \lambda\dot{\varepsilon}_v + 2\mu\dot{\varepsilon}_{22}$$

where π is the thermodynamic pressure, λ is the dilatational viscosity, μ the shear viscosity, $\dot{\varepsilon}_v$ the volumetric strain rate and $\dot{\varepsilon}_{11}$, $\dot{\varepsilon}_{22}$ the longitudinal strain-rates. If the water pressure is not to be discontinuous around the 90° corner, then the following relationship must hold: $\sigma_{11} = \sigma_{22}$, which means that

$$\dot{\varepsilon}_{11} = \dot{\varepsilon}_{22}$$

In an incompressible fluid, this can only occur if no straining is occurring, because of the continuity condition $\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} = 0$ (see Chapter 3). In a compressible fluid strain-rate equality can be satisfied when distortion of the medium is occurring.

If the incompressible ice is flowing, and if the point of maximum pressure were at the leading edge, then this would create a force imbalance at the leading edge that would act to push the ice at the leading edge of the clast away from the clast, so creating a stoss-side cavity. A diagram of the ice-clast contact geometries considered is shown in Figure 6.2.

Further analysis of a free-surface problem is required to determine the shape of the stoss-side cavity and the new water-pressure distribution in the film. However, either the point of maximum water pressure will no longer be localised at the leading edge, because the whole cavity is now the area of maximum pressure; or alternatively the point of maximum water pressure may move away from the point where the stoss-side thin water film becomes very narrow. This possibility is considered in more detail below.

If the original angle between the ice surface enveloping the clast and the bed is acute, then the arguments presented above preventing this being a stable situation still apply, and the discontinuity in the slope will be mollified. Since the original angle was acute, it is possible if not likely that the edge will soften into a wedge. It is possible that this wedge may inveigle its way under the clast, stopping abrasion and possibly floating the tool. This is not dissimilar to what McCall predicted might happen, but in this analysis the influence of the water film enveloping the clast is also considered.

If flotation were to occur, this might well occur by the tool rotating about a tool-bed contact. Thus, the ice need not initially support the whole weight of the tool, but only provide a sufficient moment to raise the tool. Because the rotation would be occurring about an axis distant from the wedge of ice, only a small moment would be required to initiate rotation.

Opposing the rotation caused by wedging is a force acting over the top of the clast. This is unlikely to be any greater than the pressure in the wedge, which will be at the leading edge. The propensity of the wedge to extend parallel to the bed and perpendicular to the bed by viscous flow has not been computed; nor is it known whether the wedge is likely to grow as a result of regelation.

The further the finger has established itself under the clast, the greater is the couple that it can exert; thus at some point the process of wedge growth may be unstoppable. Countering this is the fact that the longer the wedge grows, the larger the couple that can be exerted on it, which may cause the wedge or finger to get broken off. This situation has been illustrated by Boulton et al. (1979). If the fracture plane slopes so that the angle between the ice-clast contact and the ice-bed contact is obtuse, then a stoss-side cavity may form and the wedge become stagnant.

Another factor inhibiting wedge growth is that there may be a generalised component of melting producing a flow normal to the bed. This could cause any wedge to get melted away if the melting component increased suddenly. This might happen if the tool slid onto the upstream side of a sub-glacial hummock.

Since the tip of the wedge cannot exhibit a sharp edge, the edge must be rounded, which will provide a limit to the smallest gap into which the wedge may inveigle itself. This will depend on the largest curvature that can be borne by the flowing ice; this will depend on a number of factors including the crystalline structure of the ice at that point. Also, since rigid ice can bear an arbitrarily sharp angle, it is probable that the slower the ice is moving, the sharper the edge may be.

Rotation of the tool to allow lifting will also be favoured if there is a large overhang of the clast over the bed on the stoss side, as this permits the edge a large contact area over which to exert a couple.

If there is a cavity on the lee-side of the clast, then the finger may tend to roll the clast rather than lift it. If there is no cavity, then the process of separation must be lifting. This lifting will be assisted by the suction occurring on the lee-side of the clast. If the clast is sufficiently rotund, lifting rather than rolling it will be the only way of separating the clast from the bed.

Another mechanism causing separation of the tool from the bed might be the freezing of the water layer beneath the clast, onto which more water is frozen. This might occur on the lee-side of glacial hummocks if the ice surface is not separated from the bed-rock, and thus reduce the erosive potential of glacial action in these areas.

These discussions about the process of wedging have been rather qualitative, but they do suggest that the process may not be uncommon. Indeed, given the instability of any sharp edge, it is quite possible that the initial angle of the ice-clast contact with the ice-bed contact determines whether fingering will occur or not; if the angle is acute, it will occur, if it is not, then a stoss-side cavity will be formed. If a finger gets broken off, then depending on the angle of the fracture surface, either the growth of the finger will start again, or a stoss-side cavity will form.

The third way of avoiding the discontinuity in the pressure field mentioned at the beginning of this section is for the point of maximum pressure to move away from the leading edge of the clast. The justification for this occurring is that in general, the drainage path under the clast may be more transmissible than the path around the upper surface of the clast because of its greater thickness, and the transmissibility of the drainage path in the zone between ice and bed will be greater than the path over the clast because of its greater area. Thus, to make the total transmissibility of all the paths equal, the point of maximum pressure must move away from the clast edge so that the paths involving the 'short circuits' next to the bed also have some resistance from the paths adjacent to the clast. This argument depends upon there being a source of water at the point of maximum pressure and is only a possibility; it is not clear whether the dynamics or thermodynamics of ice permit the mechanism to operate.

Cavities on the lee-sides of clasts have been observed frequently (Boulton et al., 1979). Since the motion of the ice is acting to separate the ice-surface on the lee-side of clasts from the clast, it is unlikely that ice will wedge its way under the clast from this end.

Cavitation occurs when the normal traction at the ice surface becomes less than the pressure. If the rate of water supply to the cavity is low, then a lower bound to the cavity pressure is given by the triple-point pressure of water, which is effectively zero (Hobbs, 1974). If the rate of water supply is appreciable, then the water pressure in the cavity may take on arbitrary values dictated by the influx and efflux rates of water to and from the cavity. The cavity pressure may range between the triple point pressure and the ambient pressure.

6.4 Clast-bed Contact Forces

In this section, the effects of the factors discussed in the two preceding sections on the clast-bed contact forces are discussed. Let us consider a clast in a coordinate system xyz where the x and y axes are in the plane of the bed and the z axis is perpendicular to these two. If we assume that the generalised ice flow is in the direction of x, then there is no need to consider the y axis further, because symmetry will mean that there can be no total force acting in the y direction.

The downward force acting on the clast is given by

$$\oint p_w \nu_z dS \qquad 6.10$$

and the downstream force acting on the clast is

$$\oint p_w \nu_x dS \qquad 6.11$$

where p_w is the water pressure, ν_x, ν_z are components of the clast surface normal, and the integral is taken over the surface S (Watts, 1974). The water pressure enters into the equations because it is assumed that the ice is everywhere temperate. The water pressure at any point must be equal to the normal stress at the ice surface for momentum balance.

Inspection of the above equations shows that the forces exerted on the clasts are determined by the water pressure above the clast, the slope of the clast and the area over which the force acts.

All other things being equal, a loose rule is that a flowing fluid will exert the maximum stress on that part of the obstacle which presents a surface most perpendicular to the generalised flow direction. This will occur when ν_x is greatest and ν_z is therefore smallest, that is when the force exerted on the clast is the greatest, the vertical component of this force is the smallest.

Furthermore, this point of maximum slope often occurs at the leading edge of the clast, where the water films above and below the clast meet and are therefore coupled. This causes the maximum water pressure underneath the clast to be the same as that above the clast, and therefore inhibit abrasion.

The upward force exerted on the clast by the action of the water film underneath the clast has been shown (equation 6.2) to be

$$\int p_w(1 - \varphi)dU \qquad 6.12$$

Abrasion is favoured when the difference in the values predicted by equations 6.10 and 6.12 is greatest.

It is instructive to consider the force exerted by the clast on the bed for Newtonian regelating flow past a half-sphere. If we accept Watts' analysis, despite the problems discussed in the previous section, then the vertical forces due to the motion of the ice on the upstream and downstream sides cancel each other out, meaning that the total force given by 6.10 is

$$\pi a^2 p_\infty$$

where π has its usual meaning.

The force exerted by the under-clast film given by 6.12 is

$$p_{\infty}(1 - \varphi)\pi a^2$$

meaning that the total clast-bed contact force is $\varphi\pi a^2$. Thus, when the effective contact area coefficient is low, abrasion will be inhibited.

When the ice conforms to the clast at all points, then the projected area of the ice clast contact area is the same as the area between the clast and the bed, which will reduce the abrasional potential of the clast when the effective contact area coefficient is low. However, when cavities exist on the lee or stoss sides of the clast, then the projected area of contact will not in general be the same as the area of clast-bed contact.

If the line defining the separation of the cavity from the flanks is horizontal, then given that there is no pressure variation in the cavity, the separation line defines an isobar of the water-pressure in the film. In this case the projected area of the contacted area on the clast top is the same as the area under the clast which is bounded by ice and not cavity. In this case, if the contact area is low, the upward and downward forces will be equalised, and abrasional activity will be low.

If, for a lee-side cavity, the line of separation of the cavity from the flanks slopes downstream from the point of separation at the roof, then the projected area of downwards action where the pressure is greater than the cavity pressure will be less than the the equivalent area under the clast, reducing the contact force. If the line of separation along the flanks of the cavity slopes upstream from the point of separation of the cavity roof, then the projected area of the clast-ice contact will be greater than the area under the clast, increasing the clast-bed contact force.

These arguments may be considered from a slightly different point of view by looking at Figure 6.3. This is a cross-sectional view of ice flowing over an obstruction, the plane of the drawing being in the plane of generalised motion. The point s represents the point of separation of the cavity roof from the clast, and the lines su, sb, sd represent projections onto the plane of the figure of the possible lines of separation of the cavity flanks from the clast.

The graph beneath the figure represents the water pressure beneath the clast. Also, the line p_u b' represents a possible pressure configuration above the clast. The pressure is assumed to decrease linearly with distance from the leading edge, the result predicted for a sphere by Watts (1974). The force exerted on the top of the clast is the integral over the surface of the clast of the pressure component in the vertical direction, which is the triangle $p_u p_d b'$. Let us assume the coefficient of effective contact is negligibly small. Then, the force pointing upwards on the clast is given by the triangle $p_u p_d x'$ where x' is the point where the cavity separates from the clast at the glacier bed. If x' is coincident with b', then the net force will be zero. If x' is coincident with d' then the net force acting on the clast will be upwards. If x' is coincident with u', then the net force acting on the clast will be downwards, and abrasion can occur. It should be noted that the lower the cavity pressure, the greater the clast-bed contact force. Thus, clasts with gas filled cavities on the lee-side can potentially exert the greatest contact forces.

This argument is only suggestive of a lee-side cavitation mechanism for enhancing abrasion. The assumptions about the pressure distributions may be wrong, because the dynamics and thermodynamics of the problem may alter the pressure distribution above the clast such as to minimise the gain provided by the increase in area over which the ice can act on the top surface of the clast. The answer lies in the full three-dimensional solution of the free-surface problem, which is not currently practicable (see Chapter 4).

Boulton et al. (1979) do not report on the precise geometry of the cavities existing on the lee-sides of clasts, so it is not possible to say whether the sub-glacial evidence favours this mechanism occurring. It is probable that if the ice is extending in a direction in the plane of the bed but perpendicular to the generalised ice motion, then the action of the ice flow will be to pull the ice away from the flanks of the clast. This situation will be found on the upstream faces of sub-glacial hummocks, thus enhancing abrasion on these faces.

In the description of the above mechanism, it was assumed that the point of maximum pressure was at the leading edge of the clast. This need not be so. Hallet (1979) pointed out that if the point of maximum water pressure were not at the leading edge of the clast, then the tool-bed contact force would be increased. This may be seen by considering Figures 6.4 - 5. These are similar to Figure 6.3, but here it is assumed that the point of maximum pressure resides some way above the leading edge of the clast. Two situations are considered; one where the point of minimum pressure is at the trailing edge of the clast, the other a symmetrical pressure configuration, where the point of minimum pressure is at the opposite point on the clast surface to the point of maximum pressure. In the latter case, if it is assumed that the water pressure underneath the clast declines linearly, the existence of the point of minimum pressure causes the total force on the surface of the sphere to be exactly balanced by the force acting upwards (retaining the negligible contact area assumption). In the former case, the greatest positive pressure deviation is more than the greatest minimum pressure deviation, and thus the force acting downwards on the clast is greater than the force acting upwards.

Hallet's mechanism for the point of maximum pressure being above the leading edge of the clast is a component of ice flow towards the bed caused by melting of the ice. While Hallet's analysis, based on Watts' analysis, cannot be correct, the discussion in the previous section has suggested that we are not in a position to predict what will happen at the leading edge of the clast. Ignoring wedging, which would severely reduce abrasional activity, two mutually inclusive possibilities were suggested.

Firstly, a stoss-side cavity may open up. If this opens up such that the point of maximum pressure is at the point of cavity closure, then the whole cavity will be at this maximum pressure, and the abrasional advantage will be lost.

Secondly, the point of maximum pressure may move up even further away from the leading edge of the clast, in order to equalise the hydraulic resistivity of all the paths. This could affect the total downwards force acting on the tool.

If the concept of the equalisation of hydraulic resistivity is a valid one in that it does not violate the other balance conditions in the ice, then there is no need for there to be an ice flow normal to the bed for the point of maximum pressure to lie away from the leading edge; this situation may occur without water flow effects occurring, e.g. the flow over the sine ridge. For longitudinally symmetric obstacles, the point of minimum water pressure will be so placed as to produce no net increase in the downwards force on the clast. If the clast is asymmetrical, this equalisation might not occur. When there is a cavity, the symmetry is destroyed (Figure 6.6). The cavity pressure must be less than the ambient pressure by an amount less than the pressure difference between the maximum pressure and the ambient pressure because the cavity pressure, minimally the triple point of water, provides a lower limit to the minimum water pressure. Thus, the total downwards force on the clast will be increased. The more transmissible the drainage path under the clast, the further the point of maximum pressure will move away from the leading edge if the concept of equalisation of hydraulic resistance is valid. Here , the clast-bed contact force will be enhanced most when there is little debris under the clast.

It is possible that no stable position exists for the point of maximum water presure. However, provided that at some point it is not either at the leading edge of the clast or at the point of separation of the stoss-side cavity, then during these periods abrasion will be enhanced.

When there is debris underneath the clast, this will act to impede the flow. If the debris is sufficiently fine, and tightly packed, then the flow through the debris layer will be controlled by a Darcy type flow (see Appendix 2).

For the sake of illustration, let us consider a clast whose contact with the bed forms a square. Let us suppose the dimensionless pressure at the upstream face is equal to 1, and the downstream dimensionless pressure equal to 0. Furthermore, let us suppose that the pressure gradient along the sides is constant, and that these pressure contours extend over the top of the clast.

The total force acting downwards on the clast is therefore 0.5. If there is a negligible contact area between the clast and the bed, and the impedance to flow is constant under the clast, then the total force acting upwards on the clast will also be 0.5.

Now consider the situation when there is variable resistance to the flow of water under the clast, caused, for example, by variability in the particle size distribution of the debris.

First of all consider the case when the transmissibility is lowest at the upstream edge, and increases towards the down-stream edge. This could be caused by the silt-size fraction being concentrated at the upstream edge of the clast. If the pressure distribution around the clast remains unchanged in decreasing linearly with distance then the pressure contours will take on the form shown in Figure 6.7. These contours were obtained by numerical solution of the Laplace equation (see Chapter 3) using finite differences (Ames, 1977).

It can be seen that the pressure contours bow towards the downstream end, meaning that the total force exerted upwards on the clast is now greater than 0.5.

If on the other hand, the transmissibility decreased towards the down-stream edge, which is possibly more likely as the action of the flowing water will be to pack the fines towards this end, then the situation is reversed, and the pressure contours bow towards the upstream end (Figure 6.8). In this case, the force acting upwards on the clast will be less than the downwards force provided by the action of ice on top of the clast.

Because it is postulated that the transmissibility is varying underneath the clast, this means that the coefficient of effective contact area must also be varying. If it is supposed that this coefficient is related to the porosity of the material, and we also note that the transmissibility of the area under the clast will be directly related to the permeability, we can use the empirical relationship (Dake, 1979) $\theta = A \log(Bk)$ where θ is the porosity, k the permeability and A and B are constants.

In the under-clast flow model just described, the actual values of the permeability do not enter the equation, because Dirichlet conditions have been prescribed at the boundary, and it is only the relative values of the permeability that are of importance. In the calculations about to be described, the range of permeability values set ranged from 1 to 9. If we assume, rather arbitrarily, that these correspond to porosities of 0.5 and 0.95, and also assume that the coefficient of effective contact area is equal to one minus the porosity, then if a logarithmic model as above is assumed, it is possible to calculate the variation of the effective contact coefficient in space.

A series of calculations have been carried out. In each case, the pressures were as described above, and the upstream permeability was set at 5. The permeability was assumed to vary linearly downstream, but not transversely. The total force acting upwards on the clast ignoring any contact area effects was computed, also the total force using the relationship between effective contact area and permeability. The results are presented in Table 6.1.

Table 6.1

The effect of varying under-clast transmissibility on clast-bed contact forces

Downstream Transmissibility	Downstream Contact Area Coefficient	% Force Increment* Ignoring Contact Area Effects	% Force Increment Including Contact Area Effects
1	0.500	10.0	25.2
2	0.642	6.21	22.2
3	0.725	3.6	20.0
4	0.784	1.6	18.4
5	0.830	0.0	17.0
6	0.867	-1.2	16.0
7	0.899	-2.4	15.2
8	0.926	-3.2	14.4
9	0.956	-4.0	13.8

Upstream transmissibility = 5, upstream coefficient of effective contact area = 0.830

* - indicates upwards force

It can be seen that when the permeability is decreasing downstream, the force acting upwards on the clast is less than the force acting downwards; thus these conditions will favour abrasion. When the effective contact area effects are not considered for the case of the steepest permeability gradient, the resulting clast-bed contact stress is approximately 10% of the overburden pressure. Thus, under a glacier 330m thick, where the overburden stress is 30 bar, the clast bed contact stress would be 3 bar. If the contact area effects are considered in the stress calculation, then the bed-clast contact stress is 25% of the overburden, i.e. 7.5 bar.

As the permeability variation decreases, the clast-bed contact forces decrease. When the permeability is constant, there is no clast-bed contact stress if the contact area effects are ignored. When the permeability increases downstream, the upward force is greater than the downward force. If, however, contact area effects are considered, then there still is a compressive clast-bed contact stress.

Obviously the model described above is based on some rather arbitrary assumptions, but it does suggest that certain variations in the distribution of choked debris will affect the clast-bed contact force, unless the coupling of the ice dynamics with the under-clast transmissibility distribution is such as always to preclude this. This possibility cannot be assessed at the moment.

Another point worth noting is that the greater the clast-bed contact force, the less likely the clast is to be lifted, which would result in the loss of much of the choked debris. Also, the greater the contact force, the more tightly packed the debris will become, so increasing the coefficient of effective contact area.

Non-linearities in the rheology of the ice may have an influence on the clast-bed contact forces. In Chapter 5 it was shown that the excess water pressure p_w around the curved surface of a semi-cylinder over which ice with a Glen rheology was flowing was related to the angle θ subtended at the middle of the base of the cylinder by a relationship of the form

$$p_w = \cos^m\theta$$

Also the horizontal distance x from the centre of the cylinder is given by

$$x = r\cos\theta$$

For convenience consider the case $r = 1$. Let us assume that the cylinder is of unit width and ignore the fact that this violates the plane strain assumption made in the calculation.

We thus get

$$p_w = x^m$$

which when integrated gives the force acting on the upstream side of the cylinder $\frac{1}{m+1}$.

In order to calculate the upward force provided by the pressure some assumptions must be made about the pressure field, at the clast-bed interface. If it is assumed that the pressure is declining linearly with distance under the cylinder then the upwards force is $\frac{1}{2}$. If it is further assumed that the pressure contours lie perpendicular to the flow direction on the curved surface, and that therefore the contours on the vertical edges of the clast are not vertical, we may use the downward force calculated above, and find that the total force acting on the clast is $\frac{1-m}{m+1}$: thus the direction of the force depends on whether m is less than or greater than 1.

If it is assumed that the contours on the vertical faces are vertical, then the force may be computed by assuming a constant transmissibility under the clast and solving the Laplace equation (see Figure 6.9). For m = 0.8 unit length, the upward force is 0.531 which is less than the downward force of 0.555 by 0.024.

Where no cavity exists, the upstream and downstream normal force imbalance will cancel out, giving no total clast-bed contact force. When a cavity is present, the plane-strain calculations give a maximum value of m of 1.33. This gives a downward force of 4/7, and, repeating the underclast pressure distribution calculation (Figure 6.10) we obtain an upward force of 0.459, meaning that the total upward force is 0.030.

The initial nature of the results obtained for cavitated flow with a Glen rheology suggests that this rheology is not conducive to abrasion. However, the situation represented in the modelling was a transient situation because the cavity was opening, and it is possible that with a steady flow m might return to being less than 1, the result obtained in the non-cavitated case, and abrasion will be enhanced. Moreover, the value of m=1.33 is outlier-influenced, and the true value is probably nearer 1.0 (see Chapter 5).

The results from the flow over a truncated sine ridge indicated that the point of maximum pressure might lie behind the leading edge. In this case, if this situation persists when a cavity opens, abrasion will be enhanced.

The smaller the clast, the greater the maximum pressure for a given distal velocity (see Chapter 5). This will have two effects. Firstly it will increase the possibility of cavitation, and thus enhanced abrasion.

Secondly, all these abrasive effects described above increase with the maximum pressure value. Thus, smaller clasts will have greater abrasional potential. The same increase in abrasion occurs with increases in the ice-clast relative velocity. The maximum pressure is also related to the local viscosity of the ice (e.g. Watts' (1974) analysis), which will affect abrasion. The effect of this possible dependence of the normal force on these effects is examined in the next section.

The sub-glacial environment is characterised by transiency and it is therefore likely that the water pressures will vary in the area around the clast.

If a diffusive medium is disturbed, then these disturbances permeate through the system. The higher the diffusion (related to the transmissibility, see Appendix 2) of the system, the more rapidly these changes are diffused. Consider now the general case that the diffusivity of the drainage path above the clast is different to that beneath the clast.

Let us suppose to begin with that the area under the clast bears no debris, and is more diffusive than the drainage path above the clast. Then, if the ambient pressure drops, the pressure under the clast will fall more rapidly than the pressure above the clast, and so, before the system re-equilibrates, abrasion will be enhanced. If the ambient pressure increases, then abrasion will be reduced.

If, on the other hand, the under-clast area is choked with debris, and is less diffusive than the drainage path over the clast, abrasion will be enhanced during periods of increasing pressure, and diminished during periods of decreasing pressure. Thus, transiency may play a role in abrasion.

Robin (1976) has suggested that the process of internal melting of ice in regions of increased pressure upstream could lead to the ice freezing onto the obstacle on the downstream side. If this mechanism applied to clasts, then they could get frozen to the ice on their downstream sides. The ice velocity here has a downwards component, which would increase bed-contact forces. This increase in forces would only produce abrasion if the ice were not frozen to the bed as well.

Transiency will also affect the proportion of the glacier load borne by the sub-glacial water film away from the clasts and that borne by the water film around the clasts. During periods of declining pressure in the sub-glacial water-film, the load borne by the clasts will increase, and the lower the clast concentration, the larger the increase will be.

Hallet's mechanism for moving the point of maximum water pressure away from the leading edge relies on the ice moving towards the clast because it is being melted at the bed. Provided that Hallet's mechanism does operate (see above) then this may be generalised to include cases where the ice is moving towards the bed because of compression of the ice due to shear heating, bulk internal melting or rheological effects arising from dependence of the stress-strain relationship on third term effects.

To conclude, the two generally accepted abrasion models of Boulton and Hallet have been shown to be based on inconsistent physical models. Nevertheless, the processes they describe may still be operative. In addition it is possible to postulate several other effects relying on the presence of debris underneath the clast, cavities on the lee-side of the clast, transiency and additional rheological effects to explain the abrasive action of clasts. The relative magnitude of these effects cannot be currently assessed, nor has it been definitely proved that these effects can exist. Further research is needed into rock-rock abrasion and the rheology of basal ice, and larger computers will have to be made available for longer periods to solve the equations of flow around clasts.

6.5 Friction and Abrasion by Individual Clasts

In the preceding section the movement of clasts over the ice-rock interface was discussed. It was pointed out that whereas the traction exerted on the clast tangential to bed was due to viscous drag, at least three factors could be affecting the traction exerted on the clast normal to the bed. These factors were viscous drag, the geometry of the clast coupled with the ambient pressure and the geometry of the ice-clast contact. In view of the complexity of and the uncertainty surrounding these effects, it is not yet possible to compute with any certainty the clast-bed contact force.

It is normal to assume a Coulomb friction criterion in abrasional studies, e.g. Boulton (1975), Hallet (1981), that is that the tangential traction T_τ exerted on the particle and the normal traction T_ν exerted on the clast are related by

$$T_\tau = \eta T_\nu \qquad 6.13$$

where η is the coefficient of friction. The abrasion rate $\dot{A}$ is assumed to be of the form (see Chapter 1)

$$\dot{A} = RT_\nu v_c \qquad 6.14$$

where R is a wear constant and v_c is the clast velocity.

Since the tangential traction is primarily a consequence of viscous drag, it is reasonable to assume the tangential traction is related to the particle velocity and the ice sliding velocity v_i by a relationship of the form

$$T_\tau = k_{1\tau}(v_i - v_c) \qquad 6.15$$

where $k_{1\tau}$ is a function depending on the clast geometry and, in some way, the second invariant of the strain rate (because the viscosity of the ice is related to the second invariant). The exact nature of the relationship between the tangential traction and the second invariant will depend on how much contribution to the local second invariant is made by the bulk flow of the ice. The function $k_{1\tau}$ will also depend directly on the relative velocity of the ice and the clast. This is because when the relative velocity exceeds a certain critical value determined by the ambient pressure, the clast geometry and the second invariant of the strain rate, the ice will begin to separate from the lee-side of the particle. Once this begins to occur, the suction component of the viscous drag will begin to be lost, and, if everything else were equal, the tangential traction would tend asymptotically towards roughly half the original value.

The normal traction can be described by a relationship of the form

$$T_\nu = k_{1\nu}(v_i - v_c) + k_{2\nu}\varphi p_\infty + k_{3\nu}v_n + k_{4\nu} \qquad (6.16)$$

where $k_{1\nu}$, $k_{2\nu}$, $k_{3\nu}$ and $k_{4\nu}$ are functions, φ is the coefficient of effective contact area, p_∞ is the distant pressure and v_n is the generalised velocity of the ice normal to the bed. The dependence of the functions $k_{1\nu}$ to $k_{4\nu}$ on the basal environment is discussed below. The various terms of the expression represent contributions by differing processes to the traction.

The first term represents the contribution made by ice flow parallel to the bed. In the absence of cavitation, this will depend on asymmetry and heterogeneity beneath the clast. It is likely to be larger once cavitation has occurred. The function $k_{1\nu}$ is therefore dependent on the relative velocity of the ice and the particle. It is dependent on this for another reason as well, because the influence the ice-clast contact geometry has on the normal traction is directly dependent on the pressures in the basal water layer enveloping the clast, which are non-linear functions of the relative velocity of the ice and the clast. These pressures are also dependent on the rheology of the ice, so the function $k_{1\nu}$ will also be dependent on the second invariant of the strain rate and the permeability of the ice.

The second term in equation 6.16 represents the difference between the downward force applied to the tool by the layer of water enveloping it and the upward force provided by this same layer. The difference is due to the fact that there is a contact area between the clast and the bed over which the water cannot act. The function $k_{2\nu}$ is therefore a function of clast geometry. The third term represents the contribution to the normal forces made by the mechanisms proposed by Hallet (1979; 1981), i.e. basal melting and longitudinal stretching of the ice. The function $k_{3\nu}$ will therefore be dependent on the second invariant of the flow, the ice permeability and the ice-clast contact geometry, which is in turn dependent on the relative velocity of the ice and the tool. The fourth term is to represent possible compressional effects, that is the possibility that the ice contracts when it shears. Since this is also a viscous drag effect, the function $k_{4\nu}$ will depend on the second invariant of the strain rate and the geometry of the ice-particle contact.

In order to obtain the sliding velocity of the clast, the tangential and normal tractions (6.15, 6.16) are related through the Coulomb friction criterion (6.13). The clast velocity is defined by the following expression

$$v_c = v_i - \eta(k_{2\nu}\varphi p_\infty + k_{3\nu}v_n + k_{4\nu})/(k_{1\tau} - \eta k_{1\nu}) \qquad 6.17$$

though of course several of the functions on the right-hand side are dependent on the relative velocity of the clast and the ice.

If, for the sake of argument, a linear model is assumed (i.e. these functions are independent of the relative velocity of the ice and particle) then it can be seen that the clast velocity is not directly proportional to the ice velocity, but different from it by an amount depending on the distal pressure, the clast geometry and the ice rheology. If the clast velocity is computed to be less than zero, then lodgement will occur; if the denominator in this equation is negative, as it will be if $k_{1\tau} < \mu k_{1\nu}$, then this means that the tangential stress is only sufficient to move the particle if upward suction is occurring, and again the particle will be lodged.

It should be noted that if the functions $k_{2\nu}$, $k_{3\nu}$, $k_{4\nu}$ are all zero, then the particle will move with the speed of the ice: that is if normal forces are solely a function of the relative velocity of the ice and the clast, then the clast must float.

When $k_{1\tau} - \eta k_{1\nu} = 0$, then the clast velocity is not defined.

The tangential and normal tractions may be obtained by substituting for the clast velocity (6.17) in equations 6.15 and 6.16. The following relationships are obtained:

$$T_\tau = k_{1\tau}\eta(k_{2\nu}\varphi p_\infty + k_{3\nu}v_n + k_{4\nu})/(k_{1\tau} - \eta k_{1\nu}) \qquad 6.18$$

$$T_\nu = k_{1\nu}\eta(k_{2\nu}\varphi p_\infty + k_{3\nu}v_n + k_{4\nu})/(k_{1\tau} - \eta k_{1\nu}) + k_{2\nu}\eta p_\infty + k_{3\nu}v_n + k_{4\nu} \qquad 6.19$$

It can be seen that, using the linear model, the tractions are independent of the ice sliding velocity. Since the traction imposed on the clast must be equated by a corresponding force on the ice, this result indicates that the basal boundary condition for dirty ice sliding over a smooth bed should not include a velocity dependent term if the linear model is valid.

The abrasion rate may be obtained by substituting for the normal traction (6.19) and the clast velocity (6.17) in equation (6.14). The following relationship is obtained:

$$\dot{A} = R(\varphi k_{2\nu}p_{\infty}+k_{3\nu}v_n+k_{4\nu})\left[\frac{\eta k_1}{k_{1\tau}-\eta k_{1\nu}} + 1\right]\left[v_i - \frac{\eta(k_{2\nu}\varphi p_{\infty}+k_{3\nu}v_n+k_{4\nu})}{k_{1\tau} - \eta k_{1\nu}}\right] \quad 6.20$$

Thus, the abrasion rate increases linearly with the ice velocity, but is quadratic in the distal pressure, and the terms representing the contribution of normal velocity and compression. These quadratic terms all have a negative coefficient, meaning that as the contribution from any of these terms increases, the abrasion rate will first increase, reach a maximum and then decrease. Figure 6.11 shows a contour plot of abrasion rate against the ice sliding velocity and the distal pressure, with the $k_{3\nu}$ and $k_{4\nu}$ terms ignored.

In view of the uncertainty surrounding the relative importance of these terms, it is not possible to say much further about this expression. However, two factors are worthy of discussion; the occurrence of cavitation, and how the presence of many clasts at the ice-rock interface will affect the abrasion law.

Cavitation occurs when the normal stress on the glacier sole goes below the triple point of water, which is effectively at zero pressure. As a rough approximation, it can be said that this normal stress is related to the tangential traction imparted on the clast by the flowing ice. If we suppose that the maximum normal stress at the ice-water contact is directly proportional to the tangential traction imparted on the clast by the flowing ice, then the condition for cavitation may be defined by using an index C, i.e.

$$CT_{\tau} > p_{\infty}$$

If equation 6.18 is substituted into this condition, then the following relationship is obtained after some rearrangement.

$$\frac{k_{1\tau}\eta k_{2\nu}\varphi}{k_{1\tau} - \eta k_{1\nu}} + \frac{k_{1\tau}\eta k_{3\nu}v_n}{(k_{1\tau} - \eta k_{1\nu})p_{\infty}} + \frac{k_{1\tau}\eta k_{4\nu}}{(k_{1\tau} - \eta k_{1\nu})p_{\infty}} > \frac{1}{C} \qquad 6.21$$

Thus, it can be seen that as the distal pressure declines, cavitation is more likely to occur. Another noticeable feature is that the ice velocity does not enter directly into the expression, though it does enter indirectly through the functions $k_{1\tau}$, $k_{1\nu}$, $k_{3\nu}$, $k_{4\nu}$. These functions are all dependent on the ice rheology, and it is quite likely that they depend in similar ways. This means that the first term will be independent of the rheology, whereas an increase in the hardness of the ice will increase the magnitude of the second and third terms. Thus, as the ice gets harder, cavitation will become more likely. The function $k_{1\nu}$, which describes the influence of the geometry of the ice-clast contact, is strongly dependent on the relative velocity of the ice and the clast. When conditions are insufficient to produce cavitation, this function will take on a value that is approximately zero, and will only increase once cavitation starts.

There is nothing in this formulation to prohibit the existence of more than one solution for the particle velocity. Since the functions $k_{1\tau}$ and $k_{1\nu}$ take on different values depending on the degree of cavitation, it is entirely possible that several solutions for the particle velocity exist. This in turn means that the tractions and the abrasion rate may take on several different values for the same set of sub-glacial conditions, all corresponding to different degrees of cavitation. The multi-valuedness of the abrasion rate means that it is possible that a clast could take on different values of abrasive power within a very short distance. This is obviously one possible mechanism for chatter-marking.

One of the aims of this investigation was to try and determine an abrasion relationship for ice. Equation 6.20 forms such a relationship, but the unknown functions $k_{1\tau}$, $k_{1\nu}$, $k_{2\nu}$, $k_{3\nu}$, $k_{4\nu}$ prevent it being used predictively.

If we lump together the terms $k_{3\nu}v_n$ and $k_{4\nu}$ into a new term

$$D\mu = k_{3\nu}v_n + k_{4\nu}$$

where D is a function and μ is the viscosity, let

$$E\mu = k_{1\tau} - \eta k_{2\nu}$$

$$F\mu = k_{1\nu}$$

then 6.20 becomes

$$\dot{A} = R(p_\infty \varphi k_{2\nu} + D\mu)\left(\frac{F}{E} - 1\right)\left(v_i - \frac{k_{2\nu}\varphi p_\infty}{E\mu} + \frac{D}{E}\right) \qquad 6.22$$

We note that the abrasion rate is almost linear in the viscosity if the functions D, E and F are independent of the viscosity; i.e. if we assume that the forces exerted on the clast by the flowing ice are proportional to the viscosity.

As a first approximation to 6.22 we can assume the abrasion rate is defined by

$$\dot{A} = Vv_i\mu$$

where V is assumed to be a constant and for dirty ice

$$\dot{A} = Wv_i\mu c \qquad 6.23$$

where W is assumed to be a constant and c is the clast concentration by volume at the glacier sole.

The question of how assemblages of clasts behave is a very complex one, as the behaviour of one will affect the behaviour of the others because of the coupling provided by the Navier-Stokes equations. There are also difficulties in extending the formulation describing the behaviour of individual particles to the case of many clasts.

The latter difficulties arise because the sliding velocity of the ice and distal pressure are fictional quantities which refer to values that would be taken up at infinity if there were no other clasts in the system. Since there are other clasts in the system it is not quite clear what is meant by either of these values. Also, the formulation is inconsistent in that in computing the viscous drag terms it assumes that there is no overall shear in the ice flow, while the term allowing for third term effects implies that there is. However, even though the sliding velocity and ambient pressure have not been rigorously defined, we shall continue to use them since they do still have a certain meaning.

The question of the coupling of flows around individual particles is a complex one. Not only do the actual flowlines distort one another, but also the extra drag provided at the bed by the particles will influence the stress-field, which will in consequence affect the viscosity.

The uni-axial flow modelling described in Chapter 5 used a sliding law that was velocity-dependent. This is now recognised to be ill-chosen in view of the discussion above. However, one important result emerging from those calculations was that the abrasion rate (using 6.23) did not change linearly with the clast concentration. This was due to the fact that as clast concentrations got higher, the additional roughness provided by these induced additional shearing in their immediate vicinity. Also, as the boundary velocity increased, the additional shearing induced caused the viscosity at the glacier sole to decrease.

These calculations indicate the kinds of effects that might be expected. Whether the choice of an inappropriate basal boundary condition has an important effect is not known; however, that additional roughness will affect the drag or the abrasion in increased proportions does seem likely in view of the reduction of viscosity at the glacier sole.

In Chapter 1 it was decided to assume a linear relationship between the abrasion rate and the normal stress. If this is not true, then equation 6.20 will adopt a more complicated form. In particular, one might expect abrasion to be limited at low stresses, in particular those arising as a result of contact area effects, where the point stresses may not be very high at all. However, contact area effects will still affect the frictional force: for this reason we might expect abrasion to decline under high overburdens, as suggested by Boulton (1974).

CHAPTER 7

MEDIUM SCALE PROCESSES AND HIERARCHISATION

7.1 Introduction

In this chapter we consider medium-scale processes, that is processes occurring around and as a result of hummocks of sizes ranging from hundreds of centimetres to hundreds of metres. Two aspects of medium-scale processes are discussed. The first aspect is how the variations in the parameters defining the flow field act together to mould the hummocks. The second aspect is related to the fact that the scale may be smoothed to model the erosional properties of whole glaciers. Thus, the effect of hummocks on large scale erosion patterns is considered.

In this chapter, cavitation refers to the formation of cavities behind clasts. When it refers to the formation of cavities behind hummocks, this will be made clear.

7.2 Hummock Development

In Chapter 2 it was argued that abrasion may play a significant role in the formation of glacial erosional landforms. In Chapter 6 the factors controlling the abrasive potency of a clast were analysed. In the following sections the variations of these factors around hummocks are analysed in order to try and form a view as to how abrasion might vary around a hummock.

7.3 Velocity fields and clast concentration variation around hummocks

Velocity fields and clast concentrations have to be considered together because clast concentrations are governed by the velocity fields.

Clasts are carried within the basal debris layer, which generally has distinctive characteristics (see Chapter 3). The discharge of clasts Q through a unit width is given by

$$Q = v_{\tau} b C \qquad 7.1$$

where v_{τ} is the ice slip velocity, b is the thickness of the basal debris layer and C is the volume concentration of the clasts. We assume for simplicity in this analysis that clasts actually in contact with the bedrock are transported at the same speed as the ice. We consider longitudinal straining only and ignore clast concentration changes arising as a result of erosion and deposition.

There are two mass balances to be considered: that of the clasts and that of the ice. Along any section Q must remain constant. If the ice is not melting, then the quantity $v_{\tau}b$ must also remain constant. This will remain true while b remains greater than the clast height. Under these conditions the clast concentration C will remain constant. All else being equal the abrasion rate is then controlled by clast discharge, and is thus proportional to the velocity.

If the thickness of the basal debris layer b is less than the clast height, then the clast discharge is given by

$$Q = v_{\tau} D \qquad 7.2$$

where D is the area concentration of the clasts at the glacier sole. The clast discharge must remain constant, and thus the quantity $v_{\tau}D$ is invariant. Under these conditions, all else being equal, the abrasion rate will remain constant along the line of flow.

It is probable that in certain instances a combination of the two situations described above will occur. Thus, the b will decrease along a section where v_τ and therefore the abrasion rate are increasing, until it is less than the clast height, at which point the abrasion rate will cease to increase.

Melting can occur at the very glacier sole, in which case the ice can have a normal velocity prescribed, and also by internal melting, in which case a volumetric strain-rate may also be prescribed.

The loss in volume per unit width of ice by melting at the glacier sole is given by

$$\int \frac{v_\nu}{v_\tau}\, d\tau \qquad 7.3$$

where ν, τ represents a co-ordinate system normal and tangential to the bed.

The volumetric strain-rate $\dot{\varepsilon}_V$ is defined as $\frac{1}{V}\frac{dV}{dt}$ where V is volume and t is time. The proportional loss in volume due to internal melting is thus given by

$$V/V_o = \exp\left(\int \frac{\dot{\varepsilon}_V}{v_\tau}\, d\tau\right) \qquad 7.4$$

The former process has been analysed by Nye (1969; 1970). Consider the plane strain flow of a regelating Newtonian fluid around a shallow sine wave defined by

$$h = A \sin kx .$$

Nye finds the velocity solution to be

$$v_x = U \qquad 7.5a$$

$$v_h = \frac{UAk_*^2 \; k \cos kx}{k_*^2 + k^2} \qquad 7.5b$$

where U is the generalised velocity, $\underset{\sim}{v}$ is the velocity at the ice rock interface, and

$$k_*^2 = L/4CK\mu$$

where L is the latent heat of ice, C is the Clausius-Clapeyron constant, K is the thermal conductivity of ice and μ is the (constant) viscosity of the ice.

Now

$$v_\tau = v_x \cos\alpha + v_h \sin\alpha \qquad 7.6a$$

$$v_\nu = -v_x \sin\alpha + v_h \cos\alpha \qquad 7.6b$$

and $\tan\alpha = \frac{dh}{dx}$.

The total melting M is obtained substituting equations 7.5 into 7.6, the results into 7.4 and noting that

$d\tau = \sqrt{1 + (\frac{\partial h}{\partial x})^2}\, dx$.

We suppose that the thickness of the basal debris layer increases with scale, and that a standard shape is being considered. These may be prescribed by setting $A = \frac{1}{k}$ (for example) and $b = \frac{1}{k}$ (for example).

The ratio of melting M to the thickness of the debris layer for a standard hummock shape of wavelength $\frac{1}{k}$ is given by setting $A = \frac{1}{k}$ into 7.5, substituting these into 7.6 to obtain the interfacial velocities, substituting these into 7.4 and dividing by $b(= \frac{1}{k})$ to obtain

$$\frac{M}{b} = \int \frac{k^2 \cos kx \sqrt{1 + \cos^2 kx}}{k_*^2 + k^2 + k_*^2 \cos^2 kx} dx$$

When k is very large (i.e. a small obstacle), the right-hand side becomes

$$\int \cos kx \sqrt{1 + \cos^2 kx}\, dx$$

while when k is very small (a large obstacle) this tends to zero. Thus, using Nye's regelation theory, we find that clast concentration is scale-dependent, and that clast concentration will increase towards the top of the crest.

Similar conclusions were reached with the calculations performed with a compressible non-linear rheology described in Chapter Five.

The situation is more complicated with three-dimensional flows. For ice suffering no volume loss, the clast concentration effects depend on whether the thickness of the basal transport layer becomes smaller than the height of the clasts.

Flow tends to be around hummocks rather than over them (Nye, 1983 unpublished). The basal debris layer should thus tend to be thicker around the sides of hummocks than on the crests.

If there are melting effects, then the clasts will become concentrated. If we assume that concentration occurs mostly at the leading edge, because maximum pressures and maximum melting occurs here, then subsequent diversion of flow around the hummocks will result in concentration around the flanks of hummocks, as observed by Boulton (1974; 1975).

The finite element analysis using the incompressible Glen rheology with a smooth bed showed that slip velocity patterns were dependent on the general flow velocity and scale. Straining patterns are therefore dependent on these parameters meaning that abrasion patterns will be scale-dependent whether the clast flux patterns are described by 7.1 or by 7.2.

The problem is further complicated by the fact that the presence of clasts will result in friction at the glacier bed increasing. This effect is not considered in Nye's analysis. However, if there is no melting, and the basal debris layer remains larger than the clast height, then the finite element analyses for sliding with friction are a valid model. These suggest that velocities will increase to the crest in a similar manner as to when there is perfect slip, and thereafter decline.

The uni-axial flow analyses described in Chapter 5 indicated that flow was choked in troughs. Thus, even if clast concentrations are higher in the troughs, as observed by Boulton (1974), the abrasion rate may not increase in proportion.

The analysis given in Chapter 6 suggested that the tractions imposed on a sliding clast by the moving ice were independent of ice velocity. Thus, all else being equal, the propensity for cavity formation on the lee-side of a clast will not change with velocity.

The basal debris layer is often called the regelation layer because of the presence of regelation ice (see Chapter 3). This is often assumed to be formed as a result of freezing on the lee-sides of obstacles. However, it is equally possible that it is formed as a result of straining around clasts. If the former case is true then the average thickness of the basal debris layer decreases with obstacle size (Nye, 1970); if the latter case is true the situation becomes very complicated to analyse. We now consider these effects in more detail.

If the flow were exactly as Nye (1969; 1970) describes it, then if the clasts are all in contact with the bed at the crest of an obstacle, they should all be lifted off the bed by regelation immediately after the crest and, if all the obstacles are of the same size, not retouch the bed until the next crest is reached (Nye, personal communication). Thus, the clasts will all be at the interface of the regelation layer and the true glacier ice. This is not observed. This may be because:

(a) Inhomogeneities in the ice give the clasts a quasi-random component of motion within the basal debris layer. Whether this component of motion would have any trend is not known.

(b) Regelation on the lee-side of hummocks is not a significant process and regelation arises as a result of pressure melting around clasts. Any thickening of the basal debris layer is at least partly a result of tangential straining.

It should be noted that if (a) is true, because clasts are almost always found with regelation ice then either the mean velocity of the clasts is always toward the glacier sole, or else the clasts create regelation ice with them as they move away from the sole. If the basal debris layer primarily arises as a result of straining around clasts it is not clear how its thickness relates to obstacle dimension.

If there is a cavity on the lee-side of the hummock, then the water melted on the upstream side of the hummock may not all refreeze. This would imply that progressive concentration down-glacier should occur if many cavities are present, unless the clasts have a random, diffusive component to their motion which causes them to migrate away from the glacier sole.

In conclusion, when the ice is undergoing no melting, the straining of ice will thin the debris layer when tangential velocities are increasing, and thicken it when they are decreasing, but keep the concentration constant, in which case the abrasion rate will be dependent on the velocities, all else being equal. If the basal debris layer is stretched so much that no more clasts may be brought to the surface, the abrasion rate will be independent of velocity, all else being equal. If there is melting, concentration will occur from the leading edge to the crest . Three dimensional flows will cause this concentrated ice to be diverted around the flanks of hummocks. Slip velocities generally increase away from the leading edge until the longitudinal mid-point is reached, whereafter they decline. The propensity to cavitation is independent of velocity.

Straining patterns and thus clast concentration patterns are expected to be scale dependent.

7.4 Viscosity variations around hummocks

The results from the finite element analysis indicated that ice flowing over and around obstacles exhibited viscosity variations which arise as a result of variations in strain rates. In plane strain flow, these variations depended on how steep the obstacle was and to a certain extent on scale and velocity. The viscosity variations from leading edge to crest were of the order of a divisor of five for a steep obstacle, and of the order of a divisor of one and a half for a shallower obstacle. Viscosities increased from the leading edge to the trailing edge.

In the uni-axial flow calculations, where flow was inhibited by the presence of roughness at the glacier bed, variations in viscosity around hummocks of orders of magnitude were computed, maximum values being observed in troughs.

The analysis presented in Chapter 6 suggested that the abrasion rate increases with the viscosity of the ice as a result of the larger forces exerted on the clast by the stiffer ice. In addition, the propensity to cavitation, which may have a bearing on the abrasion rate, increases with the viscosity. Thus, all else being equal, we should expect maximum abrasion rates at the leading edges, trailing edges and lower flanks of hummocks.

7.5 Water pressure variations around hummocks

In Chapter 6 it was shown that a quadratic relationship existed between the abrasion rate and the water pressure when considering the abrasive action of a single clast. This quadratic relationship arose as a consequence of the dependence of the abrasion rate upon the normal stress between clast and bed for two reasons. Firstly, the simple model proposed in Chapter 1 suggested that the abrasion rate increased in direct proportion with the normal stress. Secondly, as the normal stress increases, the frictional drag imparted upon the clast by the bed increases, reducing the clast velocity for a given ice velocity.

The water pressure affects the clast-bed contact stress through contact area effects. As pointed out in Chapter 6, it is not clear whether the average stress increase imposed by this mechanism will produce sufficient local stresses to cause abrasion to occur. However, increases in water pressure will result in increases in the frictional resistance when effective contact areas are high, and thus high water pressures will slow down clasts.

However, when considering the passage of dirty ice over bedrock, the abrasion rate is not determined by the clast velocity but by the clast discharge over a given area. By mass balance (ignoring comminution) this must remain constant if clasts are not being brought to the glacier sole by straining and by melting. Thus, the quadratic dependency of abrasion rate on the normal stress no longer exists, and the relationship is simply the empirical relationship between the abrasion rate and the normal stress. It must be stressed that this consideration only applies to the elucidation of relative abrasion rates along a transect where clast discharge is kept constant.

Thus, any direct dependence of abrasion rate upon water pressure variations depends on contact area effects having a direct effect on abrasion, which has not been unequivocally established. If there is a connection, then abrasion will be highest where water pressures are highest, i.e. towards the leading edges of hummocks. Minimum abrasion rates will be towards the trailing edges of hummocks if no cavity exists behind the hummock, or at the cavity boundary if such a cavity does exist.

If cavitation does have an influence on clast-bed contact forces, as argued in the previous chapter, then low water pressure will have the effect of reducing abrasion because high water pressures suppress cavitation.

7.6 Variations in other factors influencing abrasion rate around hummocks

In Chapter 6 it was suggested that enhanced clast-bed contact forces could arise from Hallet's mechanisms (Hallet, (1979;1981)) and as a result of other rheological effects. The relationship between the abrasion rate and these quantities is quadratic for individual particles, and linear for dirty ice along a particular flow-line (see the previous section).

Hallet (1979;1981) postulated two mechanisms for enhanced clast-bed contact forces. These were a component of the generalised ice-flow existing normal to the bed due to melting of the ice at the bed, and straining of the ice parallel to the bed, i.e. longitudinal or transverse straining.

Hallet gives two reasons as to why there might be melting at the glacier bed. The first is pressure melting in regions of high pressure, as suggested by numerous authors, e.g. Weertman (1957). It was argued in Chapter 3 that pressure melting may not be as important a mechanism as is suggested by many theories: this view is supported by Fowler (1979). In addition, as obstacles get larger, pressure melting becomes progressively less important (Nye,1969). Thus, Hallet's pressure melting effect has a scale dependency, being significant only for small obstacles.

Where pressure melting is significant, we expect it to be very closely correlated with pressure: thus maximum abrasion rates should be at or near the leading edges of hummocks, all else being equal. If no cavity exists on the downstream side of a hummock, regelation will prevent the pressure melting mechanism from operating.

Hallet also pointed out that melting arising from geothermal heating will affect abrasion. If, as a first approximation, we assume that the geothermal heat flux is strictly in the opposite direction to gravity, and align the lowering vector with the gravity vector, then it is possible to compute how abrasion will vary with slope if we follow Hallet in assuming that the clast-bed contact force is proportional to the melting velocity.

The component of heat flow normal to the bed q_n is given by

$$q_n = q/(1 + h_x^2 + h_y^2) = qc$$

where h is height, the subscripts x,y denote differentiation with respect to these axes, q is the geothermal heat flux and c is the direction cosine of the surface normal.

If we assume for the sake of argument that the abrasion rate is numerically equal to the normal heat flux, and note that

$$h_t = -a/c$$

where the subscript t denotes differentiation with respect to time and a is the abrasion rate (this relationship is obtained in Chpater 2) we find that

$$h_t = -q$$

in other words if geothermal melting were the sole cause of abrasion, any surface form would be stable under abrasion.

Hallet also suggested that extension of the ice will increase clast-bed contact forces. In general, extension will be found on the upstream faces of hummocks, enhancing clast-bed contact forces in these areas, while the opposite effect will occur on the downstream faces.

A generalisation of Hallet's mechanisms proposed in the previous chapter is that volume-loss within the ice will increase clast-bed contact forces. This volume loss was attributed to internal bulk-melting and frictional heating.

In many ways internal bulk-melting is similar to regelation. Thus, we expect the melting to occur on the upstream side, and refreezing to occur on the downstream side, although the probable dependence of permeability on moisture content will mean that refreezing is likely to take place onto the glacier sole rather than within the ice on the lee-side of obstacles as was modelled in the finite element calculations reported in Chapter 5. These calculations showed that hydrodynamic effects caused internal bulk-melting to be more efficient as a means of flowing around smaller obstacles; this is also what would be expected from thermodynamic considerations, which were not included in the formulation used in Chapter 5. Thus, we would expect the volume loss, and thus the abrasion, to be greatest around the upstream sides of obstacles and around smalller obstacles.

Shear melting will occur where straining is greatest. For the calculations using non-linear rheologies reported in Chapter 5 this occurred at the leading and trailing edges in plane-strain flows, and at hummock crests for uni-axial flows; thus, a somewhat confused picture is presented.

The exact influence of any third-term effects awaits the determination of the existence and nature of these effects. However, it is reasonable to suppose that they will be at their maximum where straining is at a maximum, and should thus be expected to exert their maximum influence on abrasion similarly to frictional melting.

The influence of all these factors on cavitation is unknown; it might be expected that high clast-bed forces suppress it.

7.7 Variations in transiency around hummocks

Fluctuations in the basal water pressure have been identified in Chapter 6 as a possible mechanism for enhancing abrasion rates. The sub-glacial environment is highly transient (see Chapter 3).

Some of this transiency arises as a result of stick-slip motion. This will cause pressure fluctuations which are likely to be related in magnitude to the absolute magnitude of the pressure deviation from the overburden, i.e. at the leading and trailing edges.

Where there are cavities behind hummocks, fluctuating pressures in the cavity and opening and closing of the cavity might be expected to produce the maximum transiency in the water pressure at points nearest the separation and grounding lines.

7.8 Variations in abrasion patterns around hummocks

If we ignore the influence of cavities existing behind hummocks, then the likely variations in abrasion rates across longitudinal sections of hummocks are rather difficult to characterise. Velocity and clast concentrations suggest that the maximum abrasion rates will be found near the longitudinal midpoint. Viscosity variations suggest maximum abrasion rates at the leading and trailing edges. The influence of the the other factors is difficult to assess because of uncertainties in how the mechanisms operate, but it appears that they will in general produce maximum abrasion away from the crest.

If the abrasion rate is significantly affected by cavitation, then we note that the maximum propensity to cavitation will be found near the trailing edge of the hummock where pressures are low and viscosities high. The point of minimum propensity to cavitation will most probably be found somewhere near the leading edge, but it depends on the details of the viscosity variation and the pressure variation. The propensity to cavitation may be approximately characterised by an index

$$\mu(p_o - p)$$

where μ is the viscosity, p is the local pressure at the ice-rock interface and p_o the ice overburden pressure. The greater this value, the higher the cavitation propensity. The finite element analyses described in Chapter 5 suggested that variations in $p_o - p$ far exceeded those in μ: however, it was not clear whether these large fluctuations arose as a result of the fact the flow past the obstacle was of that of an infinite volume of fluid. For this reason we cannot place the position of minimum propensity to cavitate at the point of maximum water pressure.

If a cavity exists behind an obstacle, then the lowest pressures will be found at the crest: here, however the ice is softer. It is not therefore possible to predict the exact position of maximum propensity to cavitation. However, for shallow hummocks the viscosity variations along the hummocks were rather subdued, and the position of maximum propensity to cavitate will be found where the pressure is least, i.e. at the crest.

In the absence of cavities behind hummocks, transient water pressures arising from stick-slip motion will produce maximum abrasion rates at the leading and trailing edges, while the influence of cavities behind hummocks gives maximum enhancement to abrasion rates on the crest and at the grounding line.

The plane-strain finite element calculations indicated that all these effects are dependent on scale and velocity.

Transverse variations in abrasion patterns have one simplification relative to longitudinal profiles in that variations in the steady-state pressure are likely to be very much subdued.

The finite element analysis of uni-axial flow utilised two simple abrasion models: one where the abrasion rate per unit of clast concentration was dependent on the clast velocity, while the second suggested that the abrasion rate per unit of clast concentration was related to the product of viscosity and velocity. All the discussion in Chapter 6 suggests that there may well be a strong dependence on the ice viscosity.

If we accept that the abrasion rate is affected by the ice viscosity, then either the results of the uni-axial flow modelling can be interpreted in terms of the actual abrasion rate, in which case, since the clast concentration was controlled by the viscosity in these numerical experiments the abrasion rate at a point is affected by the sliding velocity (see Chapter 5); alternatively we may interpret the results in terms of the abrasion rate per unit clast concentration, and ignore the fact that the results from the smooth-crested calculations indicated a strong dependence of flows upon clast concentration variations.

In either case the absolute and relative patterns are dependent on velocity, smoothness, geometry, clast concentration patterns and scale in frequently non-linear and often non-monotonic patterns. This is equally true for the model of abrasion where the abrasion rate is independent of viscosity.

In view of the uncertainties associated with the correct choice of parameters describing the basal ice (see Chapter 3) and the difficulties in interpreting the modelling results it is prudent to refrain from further analysis except to say that complexity and scale-dependency are to be expected. However, one other fairly firm conclusion that can be drawn relates to viscosity variation from the crest to the trough. For the case of the flat bed, no transverse viscosity variation is expected. The viscosity variation for the wide hummocks was rather greater than that for the narrow hummocks. As the hummocks become wider and wider, the situation becomes more akin to the flat bed. On this basis we might expect a non-monotonic relationship between the viscosity ratio and the hummock width for a given height.

In conclusion, abrasion rates are likely to behave in a complex and scale-dependent manner along and across hummocks. If cavitation is a significant control on abrasion, then we might expect abrasion to increase from the leading edge to the trailing edge.

7.9 Hummock evolution

In this section we consider how hummocks might evolve, the possibility of persistent forms evolving and the influence of lee-side shattering processes on hummock evolution.

In many cases hummock shapes, especially those on the small scale, are associated with the presence of joints (e.g. Rastas and Seppälä, 1981). However, it cannot always be unequivocally argued that these joints existed before the hummock was formed; nor has it been unequivocally demonstrated that areas close to joints are necessarily less resistant to abrasion. In addition, the geometrical evidence cited in Chapter 2 suggests that if rock weakness were the sole cause of hummock development transverse to the direction of glacier flow, then it is likely that a cusp would develop, which is untypical of transverse glacial erosional forms: indeed, in view of the tendency of convexities to collapse into cusps, the absence of such features is a phenomenon requiring explanation.

Boulton (1974) has suggested that hummocks will act to stream debris around themselves, thus increasing abrasion rates around the flanks. All else being equal, any convexity will grow. In view of the complex behaviour of the parameters controlling abrasion rates, it is not possible to confirm or deny Boulton's suggestion. We might expect the clast concentration ratio from trough to crest to increase with the hummock height; however there is an upper limit to the clast concentration ratio, imposed by the fact that clasts have finite sizes; thus the ice cannot be compressed infinitely, which may in turn place a control on hummock development. Other constraints will be imposed by the flow-related influences on abrasion rate discussed previously.

Another aspect of the problem of hummock evolution is whether hummocks may be persistent under lowering; this includes persistence associated with migration.

Hummocks are unlikely to migrate in directions transverse to the generalised flow direction because it is to be expected that flow fields are transversely symmetric about the hummock. Persistence therefore requires lower abrasion on the steeper flanks; this would arise as a result of choking of flow, if hummocks are developing in groups, though increases in viscosity in the troughs will doubtless affect this conclusion. If a hummock is developing in isolation, then we might expect transverse variations in the flow fields to be more subdued, and clast concentration effects to be more important, although the coupling of the flow and the clast concentrations will undoubtedly affect matters.

The uni-axial flow modelling indicates that the relationship between crest and trough flow fields is also affected by the generalised velocity, smoothness, scale and geometry. It might therefore be possible that for a particular geometry, size and average flow field the crest and trough lowering rates will be constant. This, then, is an explanation for the scale-dependency of glacial erosional bedforms. However, this does not prove that such a form is resistant to perturbation, nor does it prove that there is only one set of these parameters that results in stability.

If the width of the hummock is controlled significantly by jointing, then a stable form for a given average flow field would be geometry dependent. Again, there might be more than one stable form.

These arguments cannot be considered in isolation from the generalised lowering of the landform. If the geometry of the hummocks affects the absolute lowering rate (for example the crestal viscosity-velocity product differs between the narrow and wide hummocks for the same distal velocity and smoothness), then that particular geometry which produces the lowest overall lowering rate will be the most stable in time, although not necessarily when considered in the lowering space.

The mechanisms for halting hummock growth have depended to a large extent on other hummocks checking their growth. It is therefore possible that a lone hummock may well continue to grow indefinitely (this is not to suggest that all such hummocks are formed in this manner). If groups of hummocks do grow together this begs the question of why the hummocks should all begin to grow together. One explanation for this might be a parent heterogeneity (e.g. jointing) whose influence persists through time because of the persistence of hummocks under lowering.

If persistent forms can evolve, then it is of interest to know how much generalised lowering has to be affected before this can be attained. However, all that may currently be said is that the stronger the initial contrasts between crest and trough lowering rates the smaller the generalised lowering that need be effected for persistent forms to evolve.

Many of the above considerations regarding the development and persistence of transverse hummock profiles apply to the longitudinal profiles. However, there are four fundamental differences:

(a) Flow choking is less likely to play a role in inhibiting abrasion at the bases of hummocks.

(b) Persistence with migration does not require extreme variations in abrasion rate across one side; thus for example near-constant abrasion could be occurring on the upstream side.

(c) Lee-side shattering may influence the development of hummock form.

(d) Pressure variations along the hummock may significantly affect abrasion.

The lesser effect of flow choking means that hummocks are more likely to grow indefinitely. If, however, cavitation does play a role in enhancing abrasion then abrasion should be enhanced on the down-stream side but diminished on the upstream side, meaning that stable hummocks will migrate upstream. Migration, if it exists, can offer important information about abrasion patterns. The existence of shattered lee-side faces on hummocks almost certainly arises as a result of the existence of cavities behind the hummocks (see Chapter 1). Such shattering might be a continuous process or might equally occur very occasionally, removing large blocks of bedrock.

If it is a continuous process, no migration is occurring and the hummock is a stable form, then the vertical face must remain constant in position (i.e. no erosion) while the vertical lowering rate immediately downstream of the shattered face (where no abrasion is likely to occur as the ice will rarely contact this area) must equal the lowering rate produced as a result of abrasion. If the hummock is migrating upstream then this constraint must also apply, while the vertical face must retreat at the migration velocity. These conditions are rather unlikely. If the hummock is migrating downstream, then plucking must produce the same vertical lowering rate of the vertical shattered face, while the tangent of the slope of the shattered face must equal the ratio of migration velocity to the lowering rate. Again this is rather unlikely. We thus conclude either

(a) shattering is not a continuous process,

(b) steady-state forms do not exist.

If block removal is an infrequent but volumetrically significant process, then we might expect to see horizontal shattered surfaces raised above the base of the hummock. This is observed (Boulton, personal communication). If migration were in the downstream direction, then it might be expected that such shattered surfaces might be observed fairly close to the upstream edge, while if migration were upstream, these surfaces would be shortened by the action of abrasion. Evidence on this is rather unclear, and depends on how much of a hummock is removed by a catastrophic failure.

Either of these shattering mechanisms requires the moulding and maintenance of hummocks by the action of abrasion. Thus, the rate of erosion by shattering will be strongly related to the abrasion rate. This has important consequences for glacier-wide erosion patterns as will be seen in the next chapter.

It is possible that hummock formation is a consequence of large block removal and subsequent smoothing of sharp faces by, in the main, abrasion. If this is the case, then in hummocky areas, such large block removal must be the main form of erosion. If this were the case, then we would expect glacial surfaces to exhibit much more shattering, unless such shattering only occurs under deep glaciers, and the abrasion only represents a patina of erosion occurring during the retreat phase of a glacier.

To summarise, it is not possible with the evidence to hand to conclude whether hummocks are stable forms and, if they are, whether they migrate. The modelling evidence suggests that this is possible, and that stability might be related to scale.

7.10 <u>Construction of large scale erosional relationships</u>

If it were possible correctly to compute erosion and plucking rates then it would be possible to further compute the large scale lowering rate of the landform. The question arises as to whether, given a statistical description of a landform in a particular region, it would be possible to use this to calculate the basal flow field at the bottom of a glacier defined by a given mass-balance distribution, and from this obtain information about the overall lowering rate of a landform and the evolution of the parameters describing the smaller-scale features of the landform.

The ultimate confirmation or denial of this hypothesis cannot, at present, be made. However in view of the sensitivity of the local abrasion rates to local conditions, in particular clast concentration patterns, which may be profoundly influenced by the local geometry and are also strongly coupled with the flow fields, and given the difficulty in defining suitable average velocities and stresses (see Chapter 5), the possibility of being able to construct large scale relationships must be viewed with some scepticism. Some of the philosophical implications of this are discussed in Chapter 9.

It is worth commenting on the erosional theories of Nye and Martin (1968), Budd et al. (1979) and Metcalfe (1979). Nye and Martin made statements about the production of stagnant, non-abrading zones of ice. These zones arise as a result of a perfect-plasticity assumption for the ice rheology. Such an assumption will enhance the production of stagnant zones. Also, as has been discussed above, the slip velocity is only one of several controlling factors on abrasion.

Budd et al. (1979) produced a relationship between erosion and the ice flow based on the results of sliding ice over a rough granite plate. Here, rock fragments were presumably plucked by the sliding ice. The application of the relationship derived by these workers to other situations is made difficult by (i) the surface was rough on the small scale (i.e. O(mm)) and (ii) the effect of large scale roughness cannot be accounted for because the experiments were carried out over a flat plate.

Metcalfe (1979) suggested that the large-scale rate of erosion might be related to the power expenditure at the base of a glacier. While this idea is intuitively appealing, the complexity of the processes causing erosion makes it impossible at this time to justify such a relationship by appeal to smaller scale processes.

It is, however, possible to make some qualitative statements about broad-scale variation in clast-bed contact forces and abrasion rates. High overburdens will increase clast-bed contact forces when contact area effects are important. A hummocky bed will increase average clast-bed contact forces through the Hallet mechanisms and the other mechanisms discussed in this chapter and the preceding chapter. High overburdens will inhibit cavitation, meaning that if cavitation is an important control on clast-bed contact forces, high overburdens will reduce clast-bed contact forces, exactly the opposite effect to that produced by contact area effects.

The longitudinal variation of erosion rates on the broad scale depends upon how the concentration of abrading clasts varies along a particular stream-line. Two extreme situations are possible to envisage. On the one hand exchange of clasts between the active fraction at the glacier sole and the non-abrading fraction within the basal debris layer might be so limited that the concentrations of the two clast fractions become uncoupled. This is the case assumed for flow along individual hummocks, where the presence of a clast-bed contact force makes it unlikely that a clast will be re-incorporated into the ice. In this case, the abrasion rate is linearly related to the clast-bed contact force (see the arguments presented earlier in this chapter).

The extreme opposite of this situation occurs when re-incorporation of active clasts into the basal debris layer is so frequent that the active clast concentration and the clast concentration in the basal debris layer become equal. This situation is likely to occur on the broad scale, where the effects of straining in the regelation layer rooving cavities behind hummocks and regelation are likely to mix clasts very well.

If we ignore deposition and comminution, then along any streamline the active clast concentration will remain constant. In this case, the relationship between the abrasion rate and the normal force will adopt the quadratic form of equation 6.20, because the active clast concentration is not, in this case, varying with the active clast velocity, while the average active clast velocity is affected by frictional retardation. Thus, we see that the functional form of the abrasional relationship is profoundly influenced by exchange processes between the basal debris layer and the active debris layer. More sub-glacial observations are required to determine the significance of these exchange processes.

Thus, longitudinal variation of abrasion rates may follow either the linear model or the quadratic model, depending on the degree of mixing. Comparisons between glaciers must be made on the basis of the active clast discharge; how this relates to the clast discharge in the basal debris layer depends on the mixing process.

CHAPTER 8

VARIATIONS IN CLAST CONCENTRATION ON A GLACIER-WIDE SCALE

The concentration of clasts in the glacier sole has an important effect on the rate of abrasion produced by the dirty ice sliding over the glacier bed. Clast concentrations vary at the glacier bed on a number of scales. The variations that might be expected around a sub-glacial hummock have been discussed in Chapter 7 and have been advanced as a possible factor contributing to the development of these shapes.

On a broader scale, clast concentrations are affected by the input of rock into the system, either from direct action of glacial erosion or by material falling into the glacier. A point made by Boulton (1978; unpublished) is that as these tools abrade away the bed, they themselves are worn down. Boulton further made the point that if the tools are of the same hardness as the bed then it should be expected that the loss of volume from the tool should equal the loss of volume from the glacier bed.

This idea has been examined in Chapters 2 and 6. There it was suggested that the process of tool comminution may not be a simple one-stage process, and that the products of abrasion may themselves play a significant part in the abrasion process. This suggestion was advanced on two grounds. Firstly, the geomorphological evidence on the finest scale (i.e. polishing) suggests that a substantial amount of the abrasion may be carried out by the smallest particles which are, in all probability, the products of abrasion. Secondly, a kind of natural selection mechanism may be in operation. The hardness of bedrock underlying glaciers will vary in general. Harder rocks incorporated into the clast population will, because of their hardness, survive longer in the sub-glacial environment and therefore carry out more abrasion than clasts from the softer constituents of the bedrock.

In this section a model is established which describes the changes in clast population characteristics taking into account all the variables that are believed to affect the formation and destruction of clasts and, in consequence, the erosion of bedrock. This model is framed at a scale above that of smaller hummocks, i.e. these hummocks are smoothed in the description of the glacer bed. The governing equations are given, and the assumption and empirical relationships required are noted. In light of what has been discussed in the preceding chapters, it will be seen that it is not yet possible to establish such a model in a sufficiently rigorous form to make confident predictions about the development of glacially affected landforms. A numerical scheme for a very simple model is derived, and some results using this model are produced.

The basic assumption of the model is that the clast population can be characterised by two parameters, the size of the clast and the hardness of the clast. The size of the clast enters because there are many reasons to suggest that the size of the clast plays an important role in determining the stress imposed on the clast by the action of ice flowing past it (see Chapter 6).

The evidence presented from the finite element calculations described in Chapter 5 suggests that the shape of a clast has an important influence on the stresses that can be imparted on it by the action of the flowing ice. This is obviously important. However, as a first approximation, it will be assumed that these differences can be resolved into a typical value for a particular size of clast. If the sub-glacial system were linear or approximately linear, this approximation might be used with some confidence. However, the glacial system is very non-linear, and, in consequence, it should always be borne in mind that this assumption that clast shape is unimportant may mask some important effects.

It is assumed that the abrasion rate is some function of the sliding velocity, the basal stress field and the clast concentration. These variables are all coupled by the equations of glacier motion. This coupling has been discussed in preceding chapters. The clast concentration enters into the abrasion equation not directly but as the apparent contact area of particle per unit area of glacier sole. This is obviously dependent on the clast concentration, but there are a number of other factors that need to be considered. This apparent contact area is not related to the coefficient of effective contact area, φ (see Chapter 6).

As will be seen, the model for the variation of clast concentration with position is based on the principle of conservation of mass. Clast concentrations are computed by dividing the total volume of clasts by the volume of ice that contains them. The clast concentration is not necessarily equal to the contacted area, but it is assumed to be so in this analysis.

As Boulton (see above) pointed out, wear is a two-surface phenomenon and thus we should expect both surfaces to be worn. Against this must be put the arguments that in order to abrade, a harder surface must indent a softer surface. However, as discussed in Chapter 1, the hetereogeneity of many rocks found in the natural environment and the existence of worn clasts suggests that wear is a two-surface phenomenon.

In this chapter a parameter H, called the hardness, is defined for a material such that

$$w_i / w_j = H_j / H_i \qquad 8.1$$

where the subscripts i and j refer to surfaces i and j and w refers to the amount of wear under some standard test condition. It is further assumed that given two experiments which define $\frac{w_i}{w_j}$ and $\frac{w_i}{w_k}$, where k refers to a surface k, that

$$\frac{w_j}{w_k} = \frac{w_i}{w_k} \Big/ \frac{w_i}{w_j} \quad .$$

We adopt the following definitions:

The vector $\underset{\sim}{\phi}$ defines the size and shape of a clast.

The vector $\underset{\sim}{\alpha}$ defines the physical properties of a clast or surface.

$A(\underset{\sim}{\phi})$ defines the abrasive potency of a clast of particular size and shape (see Chapter 6).

G is a function dependent on the local physical environment.

$D(\underset{\sim}{\phi},\underset{\sim}{\alpha})$ is the volume concentration of the clast fraction defined by $\underset{\sim}{\phi}$ and $\underset{\sim}{\alpha}$

v is the clast velocity, which may well be a function of $\underset{\sim}{\phi}$, $\underset{\sim}{\alpha}$ and correlated with G

b is the thickness of the basal debris layer in the glacier

$Q(\underset{\sim}{\phi},\underset{\sim}{\alpha},\underset{\sim}{\phi}_m,\underset{\sim}{\alpha}_m)$ is the probability that mother clasts characterised by $\underset{\sim}{\phi}_m,\underset{\sim}{\alpha}_m$ will, under the action of abrasion become clasts (where a 'clast' may be very small indeed) characterised by $\underset{\sim}{\phi}$ and $\underset{\sim}{\alpha}$.

$P(\underset{\sim}{\phi},\underset{\sim}{\alpha},s)$ represents the probability that bedrock at a point s on the surface becomes a clast defined by $\underset{\sim}{\phi}$, $\underset{\sim}{\alpha}$ as a result of 'plucking' which in itself arises as a result of modification of bedrock forms by abrasion into shapes favouring the process of 'plucking' (see Chapter 7).

$R(\underset{\sim}{\phi},\underset{\sim}{\alpha},s)$ represents the probability that bedrock at a point s on the surface becomes a clast defined by $\underset{\sim}{\phi}$, $\underset{\sim}{\alpha}$ as a result of grinding.

$S(\underset{\sim}{\phi},\underset{\sim}{\alpha},s)$ represents the input rate of clasts characterisable by $\underset{\sim}{\phi}$, $\underset{\sim}{\alpha}$ at a point s as a result of processes independent of the abrasion rate, for example some forms of 'plucking', some of the debris arising from extra-glacial sources.

Since Q, P and R are probability functions, we may impose certain limitations on their forms:

$$\iint Q(\underset{\sim}{\psi},\underset{\sim}{\alpha},\underset{\sim}{\psi}_m,\underset{\sim}{\alpha}_m)d\underset{\sim}{\psi}\ d\underset{\sim}{\alpha} = 1 \qquad 8.2$$

$$\iint P(\underset{\sim}{\psi},\underset{\sim}{\alpha},s) + R(\underset{\sim}{\psi},\underset{\sim}{\alpha},s)d\underset{\sim}{\psi}\ d\underset{\sim}{\alpha} = 1 \qquad 8.3$$

We suppose that the abrasion rate $\frac{d^2a}{d\underset{\sim}{\psi}d\underset{\sim}{\alpha}}$ attributable to any clast function $\underset{\sim}{\psi}$, $\underset{\sim}{\alpha}$ is given by

$$\frac{\partial^2 a}{\partial\underset{\sim}{\psi}\partial\underset{\sim}{\alpha}} = A(\psi)GF(H(\underset{\sim}{\alpha})/H(\underset{\sim}{\alpha}(s)))\ D(\underset{\sim}{\psi},\underset{\sim}{\alpha})v \qquad 8.4$$

(see Chapter 7) where $\underset{\sim}{\alpha}(s)$ is the vector defining the physical properties of the surface being eroded and F is a function.

The rate of destruction $\dot{V}_d$ of the clast function is defined by

$$\dot{V}_d(\underset{\sim}{\psi},\underset{\sim}{\alpha}) = -A(\underset{\sim}{\psi})GF(H(\underset{\sim}{\alpha}(s))/H(\underset{\sim}{\alpha}))D(\underset{\sim}{\psi},\underset{\sim}{\alpha})v \qquad 8.5$$

Now, considering 8.1, replacing i by the fraction $\underset{\sim}{\psi}$, $\underset{\sim}{\alpha}$ and j by the surface point s, we obtain

$$w_{\underset{\sim}{\psi}\underset{\sim}{\alpha}}/w_s = H_s/H_{\underset{\sim}{\psi}\underset{\sim}{\alpha}} \qquad 8.6$$

Noting that $w_{\underset{\sim}{\psi}\underset{\sim}{\alpha}} = \dot{V}_d(\underset{\sim}{\psi},\underset{\sim}{\alpha})\delta t$, $w_s = \frac{\partial^2 a}{\partial\underset{\sim}{\psi}\partial\underset{\sim}{\alpha}}\delta t$ and substituting these relationships 8.4 and 8.5 into 8.6, we obtain the functional equation

$$\frac{F(H(\underset{\sim}{\alpha})/H(\underset{\sim}{\alpha}(s))}{F(H(\underset{\sim}{\alpha}(s)/H(\underset{\sim}{\alpha}))} = \frac{H(\underset{\sim}{\alpha})}{H(\underset{\sim}{\alpha}(s))} \qquad 8.7$$

A simple solution to this is

$$F(\beta) = \sqrt{\beta} \qquad 8.8$$

where β is any argument.

The total abrasion at any point is given by substituting 8.8 into 8.4 and integrating over $\underset{\sim}{\phi}$, $\underset{\sim}{\alpha}$ i.e.

$$a = \iint A(\underset{\sim}{\phi}) G \sqrt{\frac{H(\underset{\sim}{\alpha})}{H(\underset{\sim}{\alpha}(s))}} D(\underset{\sim}{\phi},\underset{\sim}{\alpha}) v \, d\underset{\sim}{\phi} d\underset{\sim}{\alpha} \qquad 8.9$$

The total rate of creation of clasts from the bedrock as a result of abrasion is given by

$$\dot{V}_a(\underset{\sim}{\phi},\underset{\sim}{\alpha}) = (R(\underset{\sim}{\phi},\underset{\sim}{\alpha},s) + P(\underset{\sim}{\phi},\underset{\sim}{\alpha},s))a \qquad 8.10$$

while the rate of creation of clasts by comminution of other clasts is given by

$$\dot{V}_c(\underset{\sim}{\phi},\underset{\sim}{\alpha}) = - \iint Q(\underset{\sim}{\phi},\underset{\sim}{\alpha},\underset{\sim}{\phi}_m,\underset{\sim}{\alpha}_m) \dot{V}_d(\underset{\sim}{\phi}_m,\underset{\sim}{\alpha}_m) d\underset{\sim}{\phi}_m d\underset{\sim}{\alpha}_m \qquad 8.11$$

The mass balance equation for clasts is given by

$$\dot{V}(\underset{\sim}{\phi},\underset{\sim}{\alpha}) - \frac{\partial(vbD(\underset{\sim}{\phi},\underset{\sim}{\alpha}))}{\partial s} = \frac{\partial(bD(\underset{\sim}{\phi},\underset{\sim}{\alpha}))}{\partial t} \qquad 8.12$$

where $\frac{\partial(bD(\underset{\sim}{\phi},\underset{\sim}{\alpha}))}{\partial t}$ is the change in volume of the fraction $\underset{\sim}{\phi}$, $\underset{\sim}{\alpha}$ through time and

$$\dot{V}(\underset{\sim}{\phi},\underset{\sim}{\alpha}) = - \dot{V}_d(\underset{\sim}{\phi},\underset{\sim}{\alpha}) + \dot{V}_a(\underset{\sim}{\phi},\underset{\sim}{\alpha}) + \dot{V}_c(\underset{\sim}{\phi},\underset{\sim}{\alpha}) + S(\underset{\sim}{\phi},\underset{\sim}{\alpha}) \qquad 8.13$$

The model ignores deposition.

Equation 8.12 defines a set of equations in real $\underset{\sim}{\phi}$ and $\underset{\sim}{\alpha}$ spaces, i.e. an undenumerably infinite number of equations. In general, numerical solutions will have to be obtained, which thus requires discretisation of the $\underset{\sim}{\phi}$ and $\underset{\sim}{\alpha}$ spaces. It is convenient to do this by defining a set of n species each defined by a specific $\underset{\sim}{\phi}$ and $\underset{\sim}{\alpha}$. Each clast fraction $\underset{\sim}{\phi}$, $\underset{\sim}{\alpha}$ is thus represented by a species i. Replacing the arguments $\underset{\sim}{\phi}$, $\underset{\sim}{\alpha}$ by the subscript i, the clast fraction $\underset{\sim}{\phi}_m, \underset{\sim}{\alpha}_m$ by j and also using the subscript k to represent a species, letting the subscript s represent the surface and restricting our attention to one dimension we obtain the following n equations by substituting 8.5, 8.9, 8.10 and 8.11 into 8.13 and the result into 8.12:

$$- A_i G \sqrt{\frac{H_s}{H_i}} D_i v + \sum_{j=1}^{n} Q_{ij} A_j G \sqrt{\frac{H_s}{H_j}} D_j v$$
$$+ (R_i(s) + P_i(s) \sum_{k=1}^{n} A_k G \sqrt{\frac{H_k}{H_s}} D_j v + S_i(s)$$
$$= \frac{\partial(vbD_i)}{\partial s} + \frac{\partial(bD_i)}{\partial t} \qquad 8.14$$

For simplicity we assume that v and b are constant in space (see Chapter 7 for discussion) and that the process is in steady-state. These equations may be solved by setting the clast concentrations at a point and integrating (usually numerically) along s.

The abrasion rate a is given by

$$\sum_{k=1}^{n} A_k G \sqrt{\frac{H_k}{H_s}} D_j$$

and, if it is assumed that extra-glacial clast input is negligible, the total lowering rate ℓ is given by

$$\ell = a + \sum_{i=1}^{k} S_i(s)$$

A simple solution may be obtained by adopting a two species model. We define $\underset{\sim}{A} = (A,0)$, $\underset{\sim}{H} = (H,H)$, $\underset{\sim}{P} + \underset{\sim}{R} = (P_1, 1-P_1)$, $\underset{\sim}{S} = (S_1, S_2)$

$$\underset{\approx}{Q} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad H_s = H$$

and the volume concentrations at the upstream end of the section $\underset{\sim}{D}_o = (D_o, 0)$. This model defines the breakdown of an abrading species 1 into a non-abrading species 2. The destruction of bedrock by abrasion and abrasion-associated plucking provides contributions to both the species. The following two equations are obtained from 8.14

$$-AGD_1v + P_1AGD_1v + S_1 = vb\,\frac{dD_1}{ds} \qquad 8.15a$$

$$AGD_1v + (1-P_1)AGD_1v + S_2 = vb\,\frac{\partial D_2}{\partial s} \qquad 8.15b$$

The solutions for these equations are

$$D_1 = \frac{S_1(1-e^{-\frac{T_1 s}{b}})}{vT_1} + D_o e^{-\frac{T_1 s}{b}} \qquad 8.16a$$

$$D_2 = s\left(\frac{U_1 S_1}{T_1} + S_2\right) + \frac{U_1 T_1 v}{b}\left(D_o - \frac{S_1}{vT_1}\right)\left(1 - e^{-\frac{T_1 s}{b}}\right) \qquad 8.16b$$

where $T_1 = AG(1 - P_1)$, $U_1 = AG(2 - P_1)$.

For $S = 0$, i.e. no tool supply without abrasion, these simplify to

$$D_1 = D_o e^{-\frac{T_1 s}{b}} \qquad 8.17a$$

$$D_2 = \frac{U_1T_1vD_o}{b}\left(1 - e^{-\frac{T_1 s}{b}}\right) \qquad 8.17b$$

Thus, as $s \to \infty$, $D_1 \to 0$ and $D_2 \to U_1T_1vD_o/b$, a constant value. For $S_1 \neq 0$, D_1 tends to a constant value, representing a steady-state between destruction and creation, while D_2 tends to a linear increase with s.

The abrasion rate is proportional to D_1, and thus declines with distance if $S_1 = 0$, or reaches a steady state if $S_1 \neq 0$. The lowering rate behaves in the same fashion.

TABLE 8.1

Parameter definitions for 3 species model

$\underset{\sim}{H} = (1, 1\cdot5, 1)$

$\underset{\sim}{A} = (1, 1\cdot5, 0)$

$\underset{\sim}{P} = (0\cdot5, 0, 0)$

$\underset{\sim}{R} = (0, 0\cdot25, 0\cdot25)$

$\underset{\sim}{S} = 0$

$$\underset{\approx}{Q} = \begin{bmatrix} 0 & 0\cdot5 & 0\cdot5 \\ 0 & 0 & 1\cdot0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\underset{\sim}{D}_o = [0\cdot5, 0, 0]$

$H_s = 1\cdot0$

$v = 1$

$b = 1$

$G = 1$

$\Delta s = 0\cdot1$

A three species model was established using the parameter definitions in Table 8.1. This represents a situation with no tool supply independent of the abrasion rate. Species 1 is comminuted equally into a harder species, Species 2, representing perhaps the harder minerals composing species 1 and into a non-abrading Species 3. Species 2 is also comminuted into the non-abrading Species 3 (Species 3 here might consist of two mineralogically distinct species; there is, however, no need to distinguish them in the model). Plucking dependent on the abrasion rate replenishes Species 1 only, while attrition of bedrock replenishes Species 2 and 3. The upstream input consists of Species 1 only. Figure 8.1 shows the abrasion rate, lowering rate, and species concentrations in dimensionless units against distance.

The abrasion rate and lowering rate are equal. Initially Species 1 declines in concentration with distance, and then begins to show a concentration increase. This is because the abrasion rate shows an increase, as a result of the creation of the harder, abrading Species 2. This species shows an increase in concentration with distance, the rise in concentration apparently being linear. The final wear product, Species 3, shows an increase with distance which is tending towards linear.

A four species model has been established using the parameter definitions given in Table 8.2. No tool supply independent of the abrasion rate exists. The bedrock consists of two rock types, one harder than the other. The section consists of a length of hard rock, a length of soft rock and a further length of hard rock.

TABLE 8.2

Parameter definitions for four species model with no abrasion rate independent plucking, two bedrock types, α and β.

$\underset{\sim}{H} = (5,5,1,1)$

$\underset{\sim}{A} = (1,0,1,0)$

$\underset{\sim}{P}_{\alpha} = (0{\cdot}3,0,0,0)$ $\quad$ $\underset{\sim}{P}_{\beta} = (0,0,0{\cdot}3,0)$

$\underset{\sim}{P}_{\beta} = (0,0{\cdot}7,0,0)$ $\quad$ $\underset{\sim}{R}_{\beta} = (0,0,0{\cdot}7,0)$

$S_{\alpha} = S_{\beta} = 0$

$$\underset{\approx}{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$H_{\alpha} = 5{\cdot}0$, $\quad$ $H_{\beta} = 1{\cdot}0$

Bedrock α : $0 \leq s < 1{\cdot}0$, $\quad 2{\cdot}0 \leq s \leq 2{\cdot}9$

Bedrock β : $1{\cdot}0 \leq s < 2{\cdot}0$

$v = 1$

$b = 1$

$G = 1$

$\Delta s = 0{\cdot}1$

The abrading Species 1 comminutes into non-abrading Species 2, while the softer abrading Species 3 comminutes into the softer non-abrading 4. Erosion of the hard rock section results in replenishment of Species 1 and 2, while erosion of the soft rock replenishes Species 3 and 4. Thus, the hard bed-rock and Species 1 and 2 form a group of hard rocks, while the soft bedrock and Species 3 and 4 represent a group of soft rocks. Upstream input is of Species 1. Figure 8.2 gives the results.

Since there is no input of Species 3 and 4 over the initial stretch, the behaviour predicted by equations 8.17 is followed. The abrasion rate increases upon entering the soft stretch, and the rate of comminution of Species 1 decreases, resulting in the rate of creation of Species 2 decreasing. The concentrations of Species 3 and 4 increase.

Upon re-entering the hard-bed stretch, the abrasion rate drops: however the abrasion rate is higher than it would have been had the soft stretch not been there, which is represented by the point Z. This is because Species 1 has not been as severely comminuted by the soft rock as it would have been by the hard rock. The concentration of Species 2 rises at a faster rate. The concentration of Species 3 drops because there is no replenishment, while the concentration of Species 4 increases at a declining rate owing to the comminution of Species 3.

A similar model with additional replenishment independent of the abrasion rate has been established which is otherwise similar to the previous model apart from the fact that the harder group of rocks is only twice as hard as the softer group of rocks, and that there is no upstream tool input.

TABLE 8.3

Parameter definitions for four species model with some plucking independent of abrasion rate, bedrocks α and β.

$\underset{\sim}{H} = (2,2,1,1)$

$\underset{\sim}{A} = (1,0,1,0)$

$\underset{\sim}{P}_\alpha = (0\cdot3,0,0,0) \qquad \underset{\sim}{P}_\beta = (0,0,0\cdot3,0)$

$\underset{\sim}{R}_\alpha = (0,0\cdot7,0,0) \qquad \underset{\sim}{R}_\beta = (0,0,0,0\cdot7)$

$\underset{\sim}{S}_\alpha = (0\cdot7,0,0,0) \qquad \underset{\sim}{S}_\beta = (0,0,0\cdot7,0)$

$$\underset{\approx}{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$H_\alpha = 2\cdot0, \qquad H_\beta = 1\cdot0$

Bedrock α : $0 \leq s < 1\cdot0, \quad 2\cdot0 \leq s \leq 2\cdot9$

Bedrock β : $1\cdot0 \leq s < 2\cdot0$

$v = 1$

$b = 1$

$G = 1$

$\Delta s = 0\cdot1$

The abrasion rate starts off at zero, and increases with the increase in concentration of Species 1. Along the soft section, the abrasion rate is enhanced, but decreases after the end of the soft section. Again, the point Z represents the abrasion rate which would be occurring if the soft section had not been present. In this case, the computed abrasion rate is lower because the harder Species 1 has not been replenished over the soft section.

The lowering rate is a constant amount higher than the abrasion rate. The concentration of Species 2 rises continually. Species 3 increases in concentration over the soft rock section, and is comminuted over the hard rock section. Species 4 increases in concentration over the soft section as a result of bedrock attrition and comminution of Species 3, and increases in concentration over the hard section for the latter reason.

These models, with all their simplifications, show that complex large scale patterns in clast species concentrations and in erosion may be established as a result of mass-balance considerations. Of particular note are the results given by the four species models. The soft section would, if concentration were ignored, be eroded symmetrically about its mid-point. However, when tool supply is considered, the erosion pattern is asymmetrical. This may be one causative agent of the asymmetry of large scale glacial steps if they are affected by lithology.

These results also show that whether significant tool supply arises independently of the abrasion rate is of importance in determining whether erosion increases or decreases down glacier. It certainly must occur under large ice sheets, or they would never be able to abrade; indeed one of the reasons why large ice sheets do not sometimes erode under their centres (White, 1972) may be the low tool supply.

The mass-balance models presented in this chapter obviously require coupling with the solution of the glacier motion. The complexity of this task has been illustrated in Chapter 7.

CHAPTER 9

HIERARCHICAL MODELS OF THE DEVELOPMENT OF GLACIAL EROSIONAL LANDFORMS

9.1 Hierarchical modelling of glacial systems

In preceding chapters the heterogeneity of glacial ice down to the very smallest scales has been described and emphasised. In general, as has been outlined in Chapter 4, numerical methods have to be used to solve the equations governing the flow of glaciers.

It is of interest to determine the extent of the computing resources required to solve these equations if a model of a whole glacier was established which accounted directly for all the narrow-scale heterogeneity. Let us consider a valley glacier 10km long, 500m wide and 100m thick. This contains a volume of 500 000 000m^3 of ice. If we supposed, on average, that the small-scale heterogeneity of ice could be described by cubic meshes of 1cm size in the bottom five metres of the ice, 10cm in the next 15m and 2m in the remaining 80m, then the total number of mesh points required for 1m^2 of glacier surface would be $5 \times 100^3 + 15 \times 10^3 + 30 \times 2^3$, which equals more than five million for each column. There are 10 000 × 500 such columns, giving a total number of mesh points of more than 25 million million. This is more than 1000 million times the number of mesh points used in the finite element calculations described in Chapter 5, which filled up one of the larger computers currently available in the United Kingdom.

These solutions were obtained by using direct methods to solve the resulting systems of linear equations, a necessity in most cases because these equations described incompressible flow (see Chapter 4). Solution time using direct methods increases with the third power of the number of unknowns (Forsythe and Moler, 1967), so the execution time would have increased according to $(10^9)^3$, i.e. 10^{27}. Since these calculations were taking of the order of a minute, a calculation for the whole glacier would take 10^{21} years, i.e. approximately 10^{12} times the current age of the universe. Thus, it can be concluded that calculations involving detailed description of the whole glacier are not practicable given the current state of the arts of computer hardware manufacture and numerical analysis.

The usual way out of this difficulty is to use hierarchical models. In these models, numerical or laboratory experiments are carried out on fine-scale models of the ice, and it is either hoped or assumed that these calculations will produce constitutive relationships for the aggregate. For example, a numerical analyst might be interested in determining an overall viscosity for an ice sample which has varying viscosity within it. The analyst might set up some kind of numerical model which described the variations of the viscosity within the ice sample, and perform various calculations with different stress configurations. The analyst could then compute the average stress and strain rates for each run, and from these derive a viscosity for the aggregate.

The advantage of hierarchical modelling strategies for glacial systems is obvious when it is noted that the bulk of the mesh-points for the whole-glacier solution are concentrated at the base of the glacier. Hierarchisation could take several stages.

Firstly, the heterogeneity of basal ice could be considered by taking mesh-points of the order of one centimeter in size. This would be sufficient to describe the effect of clasts, larger bubbles and the different properties of the regelation layer. The numerical experiments so derived would derive constitutive relationships for the aggregate which described the relationships between stress and strain fields, the properties of the ice such as its moisture content, temperature, and also friction laws for the dirty ice/glacier bed interface. If it is assumed that 8000 mesh points are the current practical limit for computing, then this permits an aggregate cube with sides of length 20cm (as the cube root of 8000 is 20).

The next step in the hierarchisation procedure is to use the aggregated constitutive relationship for the twenty centimetre samples to be used in the further set of numerical experiments that would generate sets of results for cubes of ice of volume $20^3 \times (20cm)^3$, i.e. 4m blocks. These constitutive relationships would depend, to a greater or lesser extent, on all the properties of the samples as described above, as well as heterogeneities introduced on the scale in between 20cm and 4m. The obvious example in this case for basal ice is the presence of bedrock obstructions.

The next stage of the hierarchisation would be done using blocks of the size 20 × 4m = 80m. This would take account of all the heterogeneity occurring at scales below 4m through the aggregated constitutive relationships, and also heterogeneity at larger scales, for example larger bed-rock obstructions.

The hierarchisation, if faithfully carried out, would result in blocks with aggregated constitutive relationships of size 1600m, 32km, etc. Obviously for the valley glacier described above, the hierarchisation need not extend this far in the vertical and transverse directions, and aggregates with a greater longitudinal dimension could be generated.

The success of the hierarchisation procedure depends on how accurately the aggregate constitutive relationships define the properties of the aggregate. To illustrate this point, let us consider a specific example.

Suppose we wished to derive an aggregate constitutive relationship that accounts for the effect of temperature variations within a sample of ice. The average temperature of the aggregate is simply the volume-weighted average of all the temperatures of the elements of the aggregate. We would now consider each possible combination of temperatures of these elements that gives the average aggregate temperature being considered. Obviously, since temperature is continuous, it is not possible to consider every possible permutation as there would be an infinite number of them. Rather, the temperature field would be discretised at a sufficiently small interval to capture all the variation. (This begs the question of how small the discretisation interval need be, which will be returned to later.)

For each permutation of element states, the analyst computes or measures the aggregate property of interest, in this case the aggregate viscosity. The results may indicate that the aggregate is insensitive to the element state permutations, or, conversely, that it is very sensitive.

If the results are insensitive, then the hierarchisation procedure is well-justified, since small scale heterogeneities do not affect the aggregate result. The penalty for this success is that information is lost about the arrangement of the element states. This information may be needed for other purposes.

If, on the other hand, the derived values of the aggregate property are extremely sensitive to the arrangement of the element states, then the hierarchisation procedure is not well-justified. This sensitivity may manifest itself in two different ways.

Firstly, the distribution of the aggregate values for the various element states may not be very peaked; in other words, lack of information about the arrangement of the element states would mean that the derived aggregate properties have to be treated as a random variable whose values are controlled by the probability distribution obtained by the series of numerical or laboratory experiments.

Secondly, the mapping of the element states onto the aggregate value may not be smooth. For example, if two experiments were done to derive aggregate values where only one of the element states differed in value, and then a further experiment was done where an intermediate value of the element state were chosen, then the hierarchisation procedure could be regarded as smoothly mapping if the aggregate value so derived were in between the aggregate values derived for the previous two experiments. However, there is no guarantee that this would occur, in which case interpolation between aggregate values would be a dangerous procedure. There may be some kind of relationship between the two ways sensitivity manifests itself, but its precise nature, if it exists is by no means obvious.

If the aggregate value were known by other means and the hierarchisation were smoothly mapping, the arrangement of the element states could be determined. If the hierarchisation were not smoothly mapping, then small errors in the measured aggregate value could lead to large errors in the derived element states.

The apparent sensitivity of the hierarchisation procedure is conditioned by the measuring scale used for the aggregate value. If it is fairly coarse, then a not particularly peaked distribution would be seen as a sharply peaked distribution, and the hierarchisation thus regarded as well-justified. The setting of measurement scales is essentially an arbitrary procedure, and is usually governed by the purposes for which the aggregate value is going to be used. This requirement can be determined more precisely if the aggregate value is going to be used in a further hierarchisation. If the higher-level hierarchisation is insensitive, then it does not really matter how coarse the measuring scale for the lower-level hierarchisation is, as the variation will get lost in the second hierarchisation. If, on the other hand, the higher-level hierarchisation is sensitive then too coarse a measuring scale for the lower-order hierarchisation will create, over the two hierarchisations, a spurious insensitivity that is an artefact of the hierarchisation procedures used.

When a system is moderately sensitive, it is possible to improve the hierarchisation by including more parameters defining the element states. If, however, the hierarchisation is highly sensitive, then the number of parameters required to define the aggregate state will become so large that the original point of the hierarchisation, that of practicable computing, will be lost. This effect becomes especially important in the modelling of transient problems, where equations describing the temporal evolution of the additional parameters would also be required.

So far the discussion of hierarchisation procedures has been restricted to variations of internal parameters, e.g. the viscosity. However, exactly the same arguments apply to the boundary conditions. Experiments on sliding with cavities, for example, could be carried out for a variety of average pressures, which would affect the processes of cavity formation. Experiments could also be carried out with the same average pressure at the boundary but with a whole variety of pressure distributions around the boundary, and it might be found that the derived sliding relationship for the bed was either sensitive or insensitive.

The situation is complicated by the fact that in a higher-level model using the aggregate values, the aggregates only communicate using the average values, or at least, a reduced set of parameters describing the boundary conditions. This is not important if the hierarchisation procedure is insensitive with respect to the boundary conditions. If, however, the procedure is sensitive then the communication between aggregates through their boundary conditions is reduced, and the behaviour of the numerical system composed of the aggregates will be damped with respect to the physical system.

A consequence of this is that the propagation of effects through the glacier will also be damped. If the glacier were a highly sensitive system, then small perturbations would be propagated through the glacier causing fluctuations in other parts of the glacier. If the hierarchisation acts so as to give a spurious damping to the glacial system because of either coarse aggregate measuring scales or insufficient parameters describing the element states then the fluctuation distribution computed using this hierarchisation scheme will be altered.

The application of the foregoing considerations to the modelling of glacial systems and in particular glacial erosional systems raises two questions; firstly are these systems sensitive, and secondly what is the importance of the result?

On the broad scale it is unlikely that glacial systems are sensitive. This conclusion can be reached on several grounds. Firstly, on the broad scale, glaciers and ice-caps are rather sluggish and smoothly varying. Variations in the bedrock topography only produce damped variations on the glacier surface (Nye, 1959). These phenomena are partly a consequence of the very low Reynolds numbers of glacier flow. This means that the flow on the broad scale, and all other scales, is laminar. Laminar flow systems tend to be fairly insensitive to perturbation, unlike turbulent flows, which are extremely sensitive to perturbations over a wide range of scales. Another indication of the insensitivity of glaciers on the broad scale is their amenability to numerical modelling and the ability of modellers to derive phenomenological basal sliding relationships (see Chapter 3).

On narrower scales however, it seems that glacial systems are rather more sensitive. Detailed observations at the glacier bed (Vivian and Bocquet, 1973; Boulton et al., 1979) indicate that transiency is characteristic of these scales. One of the aims of the numerical modelling described in Chapter 5 was to gain some idea of the sensitivity of smaller scale glacier systems. This attempt was defeated by numerical problems. Numerical problems can be indicative of a sensitive physical system, since a sensitive physical system is very often reflected in a sensitive numerical system. Unfortunately, it does not follow

(i) that a sensitive numerical system indicates a sensitive physical system or

(ii) that any aggregate value probability distributions obtained using the numerical system would be the aggregate value probability distribution of the physical system;

this demonstrates the importance of laboratory and field work in determining system properties.

If it is accepted that smaller-scale glacial systems, especially at the glacier-bed interface, are sensitive it is natural to enquire as to what causes this sensitivity. The heterogeneity of the material composing the basal ice, described in Chapter 3, is one causative factor; temperature fluctuations related to basal freezing are another. An important externally determined variable is the water pressure, which can affect glacier motion on the very broadest scale.

9.2 Testability and hierachical systems

The question of what effect the sensitivity of a glacial system has on the modelling process depends on what is being asked of the glacier model. This latter question is usually of the form 'How closely do the predictions of the model agree with the data?' This question is intimately connected to how scale-specific the data are. Thus, if a system is very insensitive, in other words its properties do not change with the scale of viewing, then data measured over a particular scale are extrapolable over a wider range of scales. Thus, for example, a basal boundary condition measured over a scale of a few metres could be applied to the glacier as a whole if the glacier system were sufficiently insensitive. If the system is highly sensitive, then it is also highly scale specific, and relationships derived on the basis of measurements at one particular scale are not applicable to any other scale.

These issues are central to the modelling of the development of glacial erosional landforms, and in particular, the testing of these models. In Chapter 1 it was suggested that glacial erosional modelling deals with three variables; the present landscape, a landscape at some other time, and the glacier dynamics. Ideally, these variables form a closed system, and given any two of them, the unknown could be inferred. A corollary is that given all three, the model should be testable. It will be shown below that with hierarchical models this testability is not guaranteed; in other words several models may give the correct answer.

The essence of the argument with glacial erosional systems is firstly, hierarchisations are required to have the possibility of computing testable propositions and secondly, because two distinct hierarchisation processes are required, the models so generated are unlikely to be testable unless the glacial system is rather less sensitive than has been suggested in this thesis.

Let us assume that the dominant process of glacial erosion is glacial abrasion. While this may not be strictly true, the arguments about to be put forward remain valid provided that the dominant erosional process is one that acts over scales of less than a few metres. Then, following Boulton (1974) or Hallet (1979; 1981) we may derive an abrasion model, i.e. a relationship between the abrasion rate due to a clast and parameters describing the local physical and possibly chemical environment. We would then seek to validate the law. This might be done by seeing what shape of landform might be produced as the result of a clast, acting according to the derived abrasion law acting, on the bedrock.

To validate the law we would need to observe some portion of the glacier-bed over a period of several years. We might assume that the glacier-bed was in a steady state; then we could obtain the velocity distributions locally and see whether the predicted abrasion rates agreed with those required to maintain a steady state.

If the glacial system is highly insensitive, i.e. any fluctuation on whatever scale is very rapidly damped by the glacier, this procedure is valid. Extreme insensitivity of the glacier will lead to near-constancy of conditions at all scales. This criterion would also apply to the effect of measurement; the effect of a measurement carried out on an extremely insensitive system will rapidly be smoothed out.

If on the other hand, the glacial system is highly sensitive, then the velocity and stress fields in the area of observation will fluctuate. Attempts to measure the change of these properties will be bedevilled by what Kamb (1978, unpublished) termed the 'sub-glacial uncertainty principle' i.e. any measurement will affect the object being measured. The more sensitive the sub-glacial system is, the more serious will the measurement effect be, yet the more measurements will be required.

To be able to make predictions when systems are insensitive requires correct calculations of fluctuation fields. These calculation techniques may, if we are lucky, be able to take some account of measurement interference. Correct calculation of the local fluctuation field can only be obtained if they are made within their larger context. Thus, in order to establish the fluctuation field in the area under observation, the calculations must be done over a sufficiently large area that the fluctuation field is correctly predicted by the numerical model. This argument is of course circular, since the fluctuation field is not known, so the sufficiently large area is not known. However, the more sensitive the glacial system, the larger the area required will be. In addition, the context of this area is not known. Initially, at least, the whole glacier has to be modelled in order to determine the fluctuation field for the area under consideration.

The success of this procedure again depends upon the sensitivity of the glacial dynamic system. If the system is fairly insensitive, then a hierarchical model of the glacier could be established where the flows distant from the area of observations were represented by higher level hierarchical models. If, on the other hand, the glacier system is highly sensitive, then this procedure will not produce the correct fluctuations in the velocity and stress fields around the hummock. The only way of avoiding this problem would be either to increase the number of parameters transferring information across the hierarchical levels, or to decrease the coarseness of the hierarchical system. In the limit the original problem arises, that is unpracticably long times in which to compute the glacier flow at a sufficiently fine scale.

Another problem is that even if we do match the observed fluctuation field in the area of interest and we are confident that measurement effects have not distorted the results significantly, we are not sure whether the match is due to coincidence or not. What is required here is a formalism that permits us to express the degree of confidence in our results. Presumably this would be based on statistical theory in some way, but beyond that it is not clear how this formalism should operate.

The above arguments show that the testability of propositions generated at a low level in the hierarchy when hierarchical models have been used is by no means guaranteed. However, propositions generated at a high level can be tested. For example, a basal sliding relationship may be created using hierarchical techniques, used in a numerical model of a glacier, and found to be valid. A problem arises when two or more hierarchical schemes are used in conjunction with one another.

Glacial erosional landforms exhibit a scale-dependent phenomenology. This may only be a consequence of the scale-dependence of glacial flow systems, but it is also likely that it is a consequence of the scale dependence of wear laws, i.e. laws that relate the erosion rate of the surface to the ambient physical and chemical conditions. Wear laws at any scale may be derived either phenomenologically or by synthesis, i.e. by the use of hierarchical modelling techniques. An example of this is found in Hallet (1981) who derived relationships between the lowering rate of a landform being subjected to glacial abrasion, the roughness of the landform and the debris concentration at the sole of the glacier. All the comments previously made about hierarchisation procedures apply to the hierarchisation of wear laws and these hierarchisations, which would permit investigation of the influence of bed roughness and debris concentration like Hallet, plus other variables such as debris constitution, temperature, the stress and velocity fields etc., might exhibit sensitivity in exactly the way the glacial dynamic systems might exhibit sensitivity.

If the hierarchisations of the wear models do exhibit sensitivity then additional parameters could be introduced to transfer information from one level of a hierarchy to another. These might or might not be the parameters that are used to transfer information across the glacier dynamical system hierarchy.

If the glacial dynamical system is sensitive to these parameters, then the introduction of these extra parameters is a bonus, because they improve the hierarchisation. However, the fact that they were not included is quite likely to mean that the dynamical system is insensitive to these erosion related parameters. Thus, when the whole glacier model is run, attempts to determine the values of these erosion-dependent parameters will be fraught with uncertainty because of an effect mentioned previously, that a well-justified hierarchisation necessarily loses information about the element states.

Let us suppose that a would-be modeller of glacial erosional landforms has constructed hierarchical models of the glacier dynamical system and the erosion system. This modeller has been able to reconstruct a previous glacial landscape by some method which does not concern us, knows the current landscape, and knows the climatic history of the region in the intervening period, also by means which do not concern us. From this climatic record the modeller is able to reconstruct the glacier dynamics and so, by using the wear law, is able to reconstruct the broad scale evolution of the landform. The question is can the modeller thereby claim that the model of the abrasion process used in the erosional system hierarchical model has been validated?

In view of all the foregoing discussion, the answer must be that the case is not proven without a great deal more knowledge about the sensitivity of the hierarchisation procedures. If the abrasion process is controlled by variables whose fluctuations are not adequately represented by the hierarchisation of the glacial dynamical system, then the answer is no.

However, we are still left with the fact that the development of the glacial landform has been matched. If happenstance is assumed not to be the answer, then the conclusion must be that the precise nature of the wear law does not matter, and that the way the glacier erodes on the broad scale is independent of narrow-scale properties. In other words, the way in which the glacier erodes is dependent on the way the systems combine, that is the scale-dependent amplifying and damping properties of the systems and the way they interact at different scales. Validation of this proposition might be effected by putting in a totally different wear law and seeing whether the results of the exercise were different.

What has happened is that the modeller has set out to test the proposition that his abrasion model is correct and found that it is untestable. The fact that a model framed at the finest scale is untestable however is a very useful result, because it forces us to create a different proposition, pitched, in effect, at a different scale. This proposition is that the way components of the glacial system at a particular scale interact means the form of the wear law is not crucial in determining certain phenomena of glacially eroded landforms.

At which particular scale this occurs has not yet been determined. The fact that it occurred at a particular scale would be a consequence of the scale-dependent damping and amplifying properties of the glacier dynamical and erosional systems, in other words which hierarchisations were respectively insensitive and sensitive. This is a very general explanation of why the phenomenology of glacial eroded landforms is scale dependent, and further reiteration of the point made in Chapters 1 and 7 that explaining the size of glacial landforms is perhaps more important than explaining their shapes.

This chapter is almost entirely qualitative. This is because no quantitative formalism exists for expressing the concepts discussed. The fact that sensitivity is an expression of the information that is available about the elements of a system suggests that the sensitivity is closely related to information and entropy. Whether the entropy of a hierarchisation distribution is a sufficient expression of the sensitivity is an open question.

Furthermore, the discussion suggests that the notion of testability may best be expressed not on a binary scale of testable or not testable, but rather as a real number. Thus, a certain narrow scale notion may be testable within certain limits. For example, we might feel that a wide variety of narrow scale wear laws might give rise to the same broad scale results, but it is extremely unlikely that all do. However, it is not known how many do, and without this knowledge it is not possible to establish a degree of testability. We may be able to resolve this difficulty to a certain extent by using Occam's Razor, but there is no guarantee that the Razor will be able to separate all the theories.

The most important conclusion that can be reached as a result of all these arguments is that testing a model involves considerations at other scales than the scale at which the model is pitched. Processes occurring at other scales affect the scale being considered. (The essence of the reductionist method is isolation of the object of interest from processes occurring at other scales.) The extent to which these processes at other scales are important is governed by the hierarchical sensitivity of the system. Without due consideration of these effects, it is not possible to determine whether a particular model being proposed is testable.

9.3 The nature of the abrasion equation and its effect on modelling the development of glacial erosional landforms

In this section we enquire more deeply into the nature of the abrasion equation, and see how its nature affects our ability to carry out the glacial erosional programme (see Chapter 1) and thus to test wear models.

The abrasion equation is a hyperbolic equation, and is thus capable of producing multi-valued solutions. These were shown in Chapter 2 to present serious numerical difficulties. These multi-valued solutions also present a more profound difficulty with regard to the programme of glacial erosional landform modelling. This is discussed below.

Let us assume to begin with that we have sufficient computing resources and sufficient time to model glaciers to any degree of fineness we so wish. For the sake of discussion, consider the evolution of the cross-section of a valley from the original V-shape carved out by the river to a U-shape arising as a result of glacial action. Because the hypothetical model of the glacier is able to encompass any degree of fineness required, it is possible to model the evolution of the landscape at all scales, from the gross shape of the valley to small grooves. (Since this discussion is restricted to the properties of the abrasion equation, striae, which violate an assumption made in formulating the abrasion equation, cannot really be modelled.)

Thus, with the hypothetical model, it should be possible to start off with the existing landscape and predict the position of every roche moutonnée and other sub-glacial forms. However, once the practicalities of computing begin to restrict the fineness of scale that can be modelled, problems begin to arise. It may be supposed that hummocks of a small size so affect the flow of the glacier around them that the resulting erosion acts to increase their size, albeit up to a certain limit, yet any heterogeneity in the landform that has a horizontal dimension below the mesh size will not be described by the mesh, and never grow to a hummock of appreciable size.

In practice hierarchical strategies have to be adopted in the modelling of glacial dynamical systems. It is not clear whether, by use of such modelling strategies, the potential of the combined dynamical and erosional hierarchical models to predict the growth of landforms at particular scales is enhanced or decreased.

In order to answer this question the mechanisms by which hierarchical models can create or destroy bed roughness must be considered. At the very highest level of the hierarchy, roughness is created by different basal elements of the glacier model having different lowering rates. Thus, the smallest-scale roughness can only have a wavelength of twice the element size. This dimension will depend on the hierarchisation strategy used, which will in turn depend on what the modeller has felt to be a natural hierarchisation procedure and on the constraints imposed by the computing system used by the modeller.

If the elements of the hierarchical structure are of an appreciable size, they will have a basal boundary condition that is dependent on the topography of the bed in some way. Either this may be assumed to be unchanging in time, in which case no extra roughness at scales below the highest level can be introduced, or the wear law may contain some terms which describe the evolution of roughness under particular conditions.

Now, describing the shapes of arbitrary surfaces is, in general, only possible by using a large quantity of numbers. For example, the heights at arbitrary grids may be used, or the surface may be described by use of Fourier analysis or some other series expansion method; in general, though, it is not straightforward to describe a surface by only a few numbers.

When using hierarchical models, where the aim is to reduce the transfer of information across the levels of the hierarchy by reducing the number of parameters, this becomes a problem. This is because as soon as the number of parameters being used is decreased, the number of wavelengths (which describe roughness) is reduced. This will certainly filter out smaller wavelength features, which might have grown to become bigger features, within the hierarchical level under consideration.

In addition, a problem arises when a feature grows so big that it can only be described at the next highest level of the hierarchy. The details of the bedform at the lower level may be represented by an arbitrary number of parameters, which have been selected on the grounds of system hierarchical sensitivity and the constraints of the computing system. The parameters representing the lower scale bed-forms could for example be a selected number of mesh points, or perhaps the coefficients of a Fourier series. In the former case, transfer of roughness up the scale is possible, because tilting of the bed could be recognised, this tilting perhaps being part of a larger-scale roughness. However, the representation of smaller roughnesses would not be recognised by the rather few grid points representing only a smooth version of the bed. Obviously smaller-scale roughness can be included in some other way in the glacial dynamical system, but if its evolution is linked with the evolution of larger-scale roughness, then some representation of the evolution of smaller-scale roughness in the lower level of the hierarchy must be included in the higher level description of the bed-form.

The evolution of roughness in glacial systems at different scales does appear to be coupled; for example, smaller hummocks do not tend to lie on the backs of larger hummocks. Thus, uncoupling the evolution of the lower level small-scale roughness from that of the larger, to reduce these linkages and save computing time, could lead to incorrect predictions about the variation of roughness with scale.

Another way of representing the lower level topography would be to use a Fourier series. By selecting a few wavelengths and allowing the amplitudes to evolve with the lowering of the landscape, the influence and evolution of the smaller wavelength roughnesses could be modelled, albeit somewhat approximately. However, this would predetermine the largest roughness that could be described at the lower level, and possibly prevent the transfer of roughness from a lower level to a higher level.

It can, therefore, be seen that hierarchical modelling of glacial erosional landforms forward in time can lead to predictions about the distribution of roughness with scale that are artefacts of the hierarchisation procedure. Whether this arises as a result of the procedures suggested above, or whether this is a general result applicable to all hierarchisation procedures is an open question.

Another factor comes into play when one is trying to reconstruct a pre-existent landform. This is the fact that erosional systems destroy information about themselves; in other words they can be described as forgetful systems. One reason for this is the hyperbolic nature of the abrasion equation. As was described in Chapter 2, it is possible for the solution to become multi-valued, and this includes situations where parts of the landform 'neck off'. An areal example appears in the evolution of cliffs with the formation of pillars from heads. This situation rarely occurs in the glacial erosional case. Instead, what happens is that a part of the exposed landform collapses into an area of zero size. All the smaller scale information that this part of the landform contained, i.e. the smaller scale roughness, is lost, and cannot be recreated.

In practice, as was also suggested in Chapter 2, the formation of such cusps may not occur all that frequently, if at all. In that chapter, a phenomenological model was proposed where enhanced erosion of regions of high curvature occurred. This model took on the form of the diffusion equation. Now, diffusive systems are highly forgetful systems, i.e. many significantly different initial states can give rise to very similar final states. For example, material given an arbitrary initial temperature field will eventually attain the same temperature throughout, and at this point it is not possible to compute its initial temperature distribution. In actual fact, the attainment of equilibrium takes an infinite time, but coarseness in measurement and in calculation results in the elapsed time at which two different initial states become indistinguishable being finite.

The same principle applies to landforms. If the way erosion varies with topography is equivalent to a diffusion process, different landscapes at an initial time can give rise to the same final landscape. This is, of course, more or less what W. M. Davis (1909) was saying.

In Chapter 2 it was suggested that landscape modelling could be carried out using scale-dependent diffusion coefficients with positive or negative values. This can present a numerical problem, because any attempt to solve the diffusion equation with a negative diffusivity (this most commonly occurs when trying to integrate the diffusion equation backwards in time) leads to extreme numerical instability in a very short time (Ames, 1977). This is partly a consequence of the 'forgetfulness' of the diffusion equation.

A final point worth making concerns landscapes that have reached steady-state. Since a steady-state is unchanging through time, the steady-state has no memory, and contains no information about its past. Thus it contains no information about how recently it reached steady state, and in the reconstruction of landforms it is not possible to specify the (backward-counted) time at which the landform should start diverging from steady state. Though, strictly speaking, a steady-state is never reached, measurement error and computational error may prevent us from distinguishing one near-steady-state from another.

In conclusion then,

(i) the use of hierarchical strategies may prevent the proper description of the evolution of smaller scale-roughness. If these smaller-scale roughnesses couple with other factors, this may lead to errors in the prediction of landscape evolution;

(ii) erosional systems are naturally forgetful, and this limits the accuracy to which previous landscapes can be modelled.

No consideration has been given to the problem of how accurately the dynamics of the glaciers through time might be deduced given an initial and final landform. Firstly, there is no limiting result stating that given a topography and a climate there can be one and only one arrangement of the glacier and all its dynamic and thermodynamic states.

Secondly, there is no guarantee that there is only one path between two landforms. In view of what has been said about the forgetfulness of erosional landforms this is also unlikely. Thus, it is unlikely that there is a unique way of inferring the glacier dynamics that have occurred to change one landscape into another, nor, in consequence, is it likely that the climatic history of an area can be inferred from the erosional history of one glacier.

These concepts are similar to the equifinality concept (Chorley and Kennedy, 1971). However, they explain why it may not be possible, by increasing the fineness at which effects are measured, to produce progressively better approximations to the current landscape and thus infer progressively more about, for example, previous landscapes. Another way of saying this is that equifinality is a scale-dependent concept, and thus meaningless when referenced without a scale.

9.4 Generic modelling of glacial erosional landscape evolution

The preceding sections have concentrated on the possibility of modelling the evolution of specific descriptions of landscape. The conclusion reached was that if hierarchical modelling strategies are used, the propositions and conclusions made may well be untestable. However, this does not necessarily apply to generic description of the landscape.

As was outlined in Chapter 2, generic descriptions sacrifice description of individual units to describe the landscape as a totality. Of course, if it were possible to describe the specific evolution of a landscape then there would be no need to have generic models.

Hierarchisation can be applied to a landscape, and the scale dependence of its properties therefore determined. A generic approach to landscape evolution would then investigate the scale dependence of the landform and try to relate it to the scale dependence of the glacial systems.

How successful this approach might be is obviously an open question. Similarly open is the question of whether such an approach might be testable. No attempt to resolve these issues will be made here, except to repeat a suggestion that testability might well be best represented on a continuum scale rather than on a binary scale, and that assessing the testability of generic procedures may well be an important part of the formalism of their establishment.

CHAPTER 10

DISCUSSION

10.1 Introduction

While the principal aim of this thesis has been to model the development of glacial erosional landforms, the decision taken at the outset to try and establish a hierarchical model based on considerations of the physical processes acting over all scales has inevitably meant that glacier physics and geomorphological explanation have had to be considered and commented upon. In this final chapter attention is divided between glacier physics, geomorphological explanation and the methodology of modelling the development of glacial erosional landforms.

10.2 The modelling of glacier physics

A major part of the research was the writing of the finite element program described in Chapter 4. This has proved itself to be a viable and instructive method of investigating steady small and medium scale glacier flows.

The major problem with the velocities-pressure formulation is the inaccuracies that sometimes appear in the pressure solution. These were most apparent in the flow over the semi-cylindrical ridge, which is, it should be pointed out, with its 90° corner, a rather difficult problem.

The advantage with the velocities-pressure formulation is that the pressure term is solved for explicitly, and this will almost certainly be of value when proper mixture theories become available. It has been shown that the velocities-pressure formulation is sufficiently flexible to incorporate modelling of such features as rough beds and coupled flow of water within the ice and along the bed, though no successful formulation for modelling clast transport within the ice was found. It is, however, unlikely that the streamlines formulation or the penalty-function formulation would be able to model this phenomenon any more successfully.

The results from the finite element modelling have shown the significant effect modelling a non-linear rheology can have, in particular on the scale-dependence of flows.

The discussion of glacier physics presented perhaps a somewhat personal view of the processes occurring at the base of a glacier. In this section regelation in its 'classical' form was not considered in great detail. This was for several reasons: firstly the experimental evidence that suggests that theory overestimates its importance; secondly the theoretical problems associated with creating well-posed problems; and thirdly, the most important reason, the evidence that intra-glacial melting may be of substantial importance. The finite element calculations using the simple mixture theory suggested that internal compression could alter flows significantly, and a mixture theory capable of reproducing the observations of Carol (1947) must be produced† before we can begin to predict erosional patterns occurring around sub-glacial hummocks. Such a theory would have one considerable numerical advantage in that the ice would be a compressible fluid; against this must be set the fact that convective terms (of moisture and of heat) might transpire to be of importance, which could lead to numerical difficulties due to numerical diffusion (a spurious attenuation of concentration

† Fowler (1984) has produced such a theory modelling Darcy flow.

gradients) or numerical dispersion (a spurious high frequency mode in the solution arising from the different models of the solution propagating with different velocity) (Huyakorn and Pinder 1983).

The finite-element method is easily extended to three dimensions; indeed the programming for this has been carried out, though the computing resources required may not yet be available. Many of the applications of numerical flow analysis (e.g. clast cavity geometry, clast concentration patterns) demand three-dimensional solutions, and only when these can be carried out will we be in a position to be able to make confident quantitative predictions.

The discussion on abrasion in Chapter 6 has shown that the models of Boulton (1974; 1975) and Hallet (1979; 1981) are based on inconsistent formulations. This discussion also suggested that enhanced clast-bed contact forces may arise for a variety of reasons. No detailed physical model was set up because of the reliance on three-dimensional non-linear free surface flows, for which no solutions of glaciological interest are known. Thus, whether these mechanisms are physically plausible is unknown; however, the fundamental asymmetry produced by a cavity makes the arguments compelling to the author at least, and coupled with the importance of lee-side cavitation on hummocks for glacier physics and erosion means that cavitation phenomena must be regarded as of paramount importance in studies of processes occurring at the temperate ice/rock interface.

Many uncertainties in the properties of ice and in particular basal ice still exist. Much more empirical evidence is needed; however obtaining such information either in the laboratory or in the field is by no means a trivial task. It may transpire that many experiments will be performed on numerical models as occurs frequently in the elucidation of the properties of turbulent flows.

10.3 Geomorphological explanation

If any part of this thesis can be said to have arrived at some definite conclusions it is the investigation of the properties of the abrasion equation found in Chapter 2. While the mathematical investigation is of some interest, it contains nothing that would have startled Fresnel (1788-1827): the author has expected to come across work produced a century or so ago giving exactly the same results. Many of the results are probably known to any geomorphologist (especially the 'cause' of U-shaped valleys) who has thought about the geometrical nature of abrasion. However, if the work presented does codify in some way the geometrical consequences of the abrasion equation then it will have been of some use.

The geomorphological explanation given in Chapter 7 does not prove in any rigorous sense why landforms are of the shape they are. Rather, the calculations show how complex and interlinked are processes on the hummock scale. In view of the probable complexity of the constitution and rheology of basal ice, as outlined in Chapter 3, it is not certain that rigorous demonstration can be offered. However, the uni-axial flow modelling in particular did suggest one avenue; that of determining the scale-dependence of glacial flows. These are generic properties rather than specific properties. If the scale-dependence of glacial erosional landforms is related to the scaling properties of glacial flows, then there is a new field of analysis available: that of investigating scale dependence and its sensitivity to geometry, variability in the basal ice, etc. It can further be argued that predicting the scale-dependent phenomenology of landforms is a more efficient test of a theory than predicting actual shapes.

Chapter 8 used a simple model which showed how the constraints imposed by debris mass-balance may be of profound importance in glacial erosional systems.

10.4 Modelling the development of glacial erosional landforms

In Chapter 1 it was stated that a hierarchical approach to landform modelling was to be undertaken. Models were to be constructed at the very smallest scale, and these results generalised to broader and broader scales.

This pattern was followed in Chapter 6 to 8. The processes affecting abrasion were found to be complex, and the difficulties that occurred in constructing statements about the abrasive action of debris-bearing ice made the adoption of simple models inescapable in the discussion of medium-scale erosion patterns given in Chapter 7. A similar problem forced another simple erosion model to be adopted in Chapter 8, which concentrated on broad-scale erosion patterns in relation to debris mass-balance.

The implicit aim in establishing a hierarchical model of glacial erosional processes is to relate ultimately the large scale features of erosional patterns to the properties of the fundamental particles, because even at the lowest level of the hierarchisation considered in this thesis there is a further implied hierarchisation through the properties of crystals down to the constituents of atomic nuclei. Such hierarchisations have been successful because throughout this hierarchisation they have well defined units which have distinct scale boundaries.

The hierarchisation in this thesis assumed, rather tacitly, that the clast and the hummock could similarly be identified as units. Many of the results in this thesis contradict this assumption.

The Concise Oxford Dictionary (Sykes, 1976) has, as its first definition of explanation, 'to make known in detail'. This is often interpreted as the demonstration of the relationships between a system and its elements. Hierarchisation thus fulfils an explicative purpose. If the properties of units can be explained in the properties of sub-units then it is felt that a system is 'explained'. But if no such units or species exist, specific explanation is not possible, and if we are to explain, we are compelled to seek some kind of generic explanation. To illuminate what is meant by 'generic explanation' we must discuss scale-dependence a little further.

A unit can only be identified because of acute changes in the property of a system with scale. It is reasonable to suppose that the explanation for this severe scale-dependence lies in the scaling properties of interlinked systems.

Units may be regarded as extreme examples of scale-dependence. In the general case we might expect more gradual variations in scale-dependence. Explanation may then consist of correlating the scale-dependence of related systems.

The extreme example of a scaleless system is given by a chaotic system, e.g. turbulent flow (Eckmann, 1981). No detailed explanation may be confidently given for any particular detail of the flow, because of the extreme sensitivity of such flows to initial conditions and perturbation. Rather, such flows are explained generically.

In between chaotic systems and strict hierarchies there potentially lie a wide variety of systems exhibiting different patterns of scale-dependence. In view of the above discussion, it might be expected that each system has its own explicative limit, ranging from highly specific for strict hierarchies to completely generic for chaotic systems. (Such a limit is intimately related to the problems pertaining to hierarchisation procedures outlined in Chapter 9.) Whether this limit is quantifiable and how it is related to the scaling properties of any system are questions that must be answered before we can claim to have 'explained' the development of glacial erosional landforms. In addition, ways of determining the scale-dependence of systems without scale-discretisation (hierarchisation) profoundly influencing the result must be found.

APPENDIX 1. MATHEMATICAL ADDENDUM

(a) Numerical integration along arcs has been carried out using a forward-Euler scheme i.e.

$$a_{i+1} = a_i + \left(\frac{da}{dx}\right)_i \Delta x$$

where a_{i+1} is the value of a at $i+1^{th}$ point, a_i the value of a at the i^{th} point, $\left(\frac{da}{dx}\right)_i$ is the estimate of $\frac{da}{dx}$ at the i^{th} point and Δx is the step-length across the arc.

(b) Vectors are indicated by one underlining, e.g. $\underset{\sim}{A}$ while tensors and matrices are indicated by two underlinings e.g. $\underset{\approx}{B}$. A distinction is assumed to be made by the underlining; for example, in Chapter 4, R and $\underset{\approx}{R}$ are completely separate entities. The superscript T implies the vector or matrix transpose. The Einstein tensor summation convention has been used throughout; i.e repeated indices imply summation over those indices; thus, for a second rank, second order tensor $\underset{\approx}{\omega}$:

$$\omega_{ii} = \omega_{11} + \omega_{22}$$

$$\omega_{ij}.\omega_{ij} = \omega^2_{11} + \omega^2_{22} + \omega^2_{12} + \omega^2_{21}$$

(c) The following is a list of symbols followed by the chapter numbers in which they are used. This list does not include cases where symbols are not used after the first expression in which they have been defined. If no chapter number is defined the symbol is used universally.

a	:	Abrasion rate (2,7)
	:	Sphere radius (3)
a_*	:	$3\mu C/L\rho_s$ (3)
a_o	:	Term in curvature-dependent abrasion function (2)
b	:	Decay constant in wear equation (1)
	:	Coefficient in curvature-dependent abrasion function (2)
	:	Thickness of basal debris layer (7,8)
c	:	Coefficient in height-dependent abrasion function (2)
	:	Clast concentration factor (5)
	:	Clast concentration (6)
c^w	:	Specific heat capacity of water (3)
c^{iw}	:	Specific heat capacity of ice-water mixture (3)
$\underset{\sim}{c}$	:	Clast velocity (4)
d	:	Distance slid by clast (1)
$\hat{f}$	:	Finite element approximation to arbitrary force vector (4)
f	:	Arbitrary force (4)
$\underset{\sim}{f}$	:	Body force vector (3)
	:	Finite element force vector (4)
$\underset{\sim}{g}$	:	Acceleration due to gravity
h	:	Height
h_a	:	Height in arbitrary length units (5)
h_p	:	Height in physical units (5)
h_u	:	Height in arbitrary length units (5)
h_o	:	Initial height (2)
k	:	Wear constant (1)
k	:	Coefficient in logarithmic regression (5)
k	:	Wave number (7)

k_*	:	$1/4CK\mu$ (7)
$k_{1\nu}$	:	Clast normal traction function describing effect of ice-clast relative velocity (6)
$k_{2\nu}$	:	Clast normal traction function describing effect of contact area effects (6)
$k_{3\nu}$	:	Clast normal traction function describing effect of normal velocity (6)
$k_{4\nu}$	:	Clast normal traction function describing effect of compressional effects (6)
$k_{1\tau}$	:	Clast tangential traction function (6)
ℓ	:	Lowering rate (2,8)
ℓ	:	Coefficient in logarithmic regression relationship (5)
m	:	Coefficient in logarithmic regression relationship (5)
$\underset{\sim}{m}$	:	Migration velocity (2)
$\underset{\approx}{m}$	:	Matrix defining volumetric strain-rate (4)
n	:	Exponent in Glen model of ice rheology
p	:	Penetration hardness (1)
	:	Hydrostatic pressure (3,4,5,6)
p'	:	Pressure difference (5)
p_w	:	Water pressure under clast (1,6)
p^w	:	Water pressure (3)
$\underset{\sim}{q}$	:	Heat flux (3)
$\underset{\sim}{q}^w$	:	Water flux (3)
r	:	Co-ordinate in cylindrical system (4)
	:	Correlation coefficient (5)
	:	Co-ordinate in spherical system (6)
$\underset{\sim}{r}$	:	Position vector (2)

s	:	Arc length (2)
	:	Position (8)
$\underset{\sim}{s}$	:	Tractions at ice-clast interface (5)
t	:	Time (2)
t_{zb}	:	Basal traction in z direction (5)
$\underset{\sim}{t}$	:	Traction (5)
u	:	Finite element approximation to unknown (5)
$\hat{u}$	:	Unknown (5)
v	:	Clast velocity (1,8)
v_a	:	Velocity in arbitrary length units (5)
v_b	:	Boundary velocity (5)
v_c	:	Crestal velocity (5)
	:	Clast velocity (6)
v_d	:	Distal velocity (5)
v_p	:	Velocity in physical units (5)
v_s	:	Slip velocity (5)
v_t	:	Trough velocity (5)
v_u	:	Velocity in arbitrary length units (5)
v_{xs}	:	Ice velocity in x-direction at ice-clast surface (4)
v_{yc}	:	Clast velocity in y-direction (4)
v_{ys}	:	Ice velocity in y-direction at ice-clast surface (4)
v_{zi}	:	Prescribed velocity for uni-axial flow (4)
v_{zb}	:	Slip velocity for uniaxial flow (4)
$\underset{\sim}{v}$	:	Velocity
$\underset{\sim}{v}^{iw}$	:	Ice-water mixture velocity (3)
$\underset{\sim}{v}^{w}$	:	Water velocity (3)
w	:	Wear (1)
w_i	:	Weighting function for scale-dependency (2)

w_i : Wear of species i (8)

w_s : Wear of surface (8)

$w_{\phi\alpha}$: Wear of species $\phi\alpha$ (8)

w : Final wear (1)

$\underset{\sim}{w}$: Ice velocity at ice-rock interface

x : Co-ordinate in Cartesian system

x_o : Initial position (2)

y : Co-ordinate in Cartesian system

z : Parameter (2)

: Co-ordinate in Cartesian system (3,4)

A : Apparent area of contact between clast and bed (1)

: Coefficient in Glen model of ice rheology (3,4,5)

: Constant in logarithmic regression relationship (5)

: Amplitude (7)

$\dot{A}$: Abrasion rate (6)

A_a : Abrasion rate (5)

A_b : Abrasion rate (5)

A_c : Effective area of contact between clast and bed (1)

A_i : Abrasive potency of species i (8)

$A(\phi)$: Abrasive potency of clast size ϕ (8)

A_ε : Rheology function (3)

A_σ : Rheology function (3)

$\underset{\approx}{A}$: Transformation matrix to obtain clast surface point velocity from clast velocity (5)

B_a : Abrasion rate per unit clast concentration (5)

B_b : Abrasion rate per unit clast concentration (5)

B_{ac} : B_a at hummock crest (5)

B_{at} : B_a in trough (5)

B_{bc} : B_b at crest (5)

B_{ε} : Rheology function (3)

B_{σ} : Rheology function (3)

$\underset{\approx}{B}$: Strain-rate operator matrix (4)

C : Clausius-Clapeyron constant (3)

: Hydraulic conductivity of thin-water film (4)

: Constant in linear regression relationship (5)

: Cavitation condition constant (6)

: Volume concentration of clasts (8)

C_{ε} : Rheology function (3)

C_{σ} : Rheology function (3)

D : Coefficient in linear regression relationship (5)

D : Normal traction function (6)

D : Area concentration of clasts at the glacier sole (7)

D_i : Clast volume concentration of species i (8)

D_o : Initial clast volume concentration (8)

$D(\underset{\sim}{\phi},\underset{\sim}{\alpha})$: Clast volume concentration of species $\underset{\sim}{\phi}$, $\underset{\sim}{\alpha}$ (8)

$\underset{\approx}{D}$: Viscosity matrix (4)

E : Shear heating (3)

: Constant in logarithmic regression relationship (5)

: Traction function (6)

F : Traction function (6)

F_s : Force imposed on top of clast (6)

$F(H(\alpha),H(\alpha(s)))$: Abrasion function dependent on materials (8)

G : Abrasion function dependent on stress environment (8)

G_i : Geometry parameter (2)

H_i : Hardness of species i (8)

H_s : Hardness of bed rock (8)

$H_{\underset{\sim}{\phi}\underset{\sim}{\alpha}}$: Hardness of species $\underset{\sim}{\phi}$, $\underset{\sim}{\alpha}$ (8)

$\underset{\approx}{H}$: Stiffness matrix (4)

$\underset{\approx}{J}$: Stiffness matrix (4)

K : Thermal conductivity of ice (7)

$\underset{\approx}{K}$: Stiffness matrix (4)

L : Latent heat of ice (3)

: Linear operator (4)

$\underset{\sim}{L}$: Interpolation function (4)

M : Melting (7)

N : Normal force (1)

$\underset{\sim}{N}$: Interpolation function (4)

P(s) : Transition probability for plucking of bedrock (8)

$P(\underset{\sim}{\phi},\underset{\sim}{\alpha})$: Transition probability for plucking of bedrock (8)

Q : Volume production of water per unit volume (3)

: Clast discharge (7)

Q : Transition probability for clast comminution (8)

$Q(\underset{\sim}{\phi},\underset{\sim}{\alpha},\underset{\sim}{\phi}_m,\underset{\sim}{\alpha}_m)$: Transition probability for clast comminution (8)

R : Wear constant (7)

: Region (3)

$R(\underset{\sim}{\phi},\underset{\sim}{\alpha})$: Transition probability for comminution of bedrock (8)

R_1 : Wear constant (1)

R_a : Wear constant (5)

R_b : Wear constant (5)

R : Transition probability for comminution of bedrock (8)

$\underset{\approx}{R}$: Rotation matrix (4)

S : Surface area (2)

: Smoothness (3,4,5)

: Clast upper surface (6)

S_a : Smoothness in arbitrary length units (5)

S_p : Smoothness in physical units (5)

S_u : Smoothness in arbitrary length units (5)

$S(\underset{\sim}{\phi},\underset{\sim}{\alpha},s)$: Supply rate of tools independent of abrasion (8)

$S(s)$: Supply rate of tools independent of abrasion (8)

$\underset{\approx}{S}$: Smoothness matrix (4)

T : Temperature (3)

$\underset{\sim}{T}$: Traction

U : Underclast area (6)

: Generalised velocity (6,7)

V : Volume (5,7)

V : Wear constant (6)

V_o : Initial volume (5,7)

V_τ : Volume at position τ (5)

$\dot{V}$: Volume rate of change of clasts (8)

$\dot{V}_a$: Volume rate of creation of clasts from abrasion (8)

$\dot{V}_c$: Volume rate of creation of clasts by comminution (8)

$\dot{V}_d$: Volume rate of distruction of clasts (8)

W : Galerkin weighting functions (4)

: Wear constant (6)

α : Slope of bed (4,7)

$\underset{\sim}{\alpha}$: Physical properties of clast (8)

$\underset{\sim}{\alpha}_m$: Physical properties of mother clast (8)

$\underset{\sim}{\alpha}_s$: Physical properties of bedrock (8)

β : Cylindrical co-ordinate for clast surface (4)

γ : Wear coefficient (1)

: Angle subtended at centre of base of semi-cylinder (5)

δ_{ij} : Kronecker δ ; $i \neq j$, $\delta = 0$; $i = j$, $\delta = 1$.

$\dot{\varepsilon}_v$: Volumetric strain-rate

$\dot{\varepsilon}_{vm}$: Volumetric strain-rate due to melting (3)

$\dot{\underset{\sim}{\varepsilon}}$: Strain-rate vector (4)

$\dot{\underset{\approx}{\varepsilon}}$: Strain-rate tensor

ζ : Convergence criterion (5)

η : Coefficient of friction

θ : Vertex angle of indenting cone (1)

: Moisture content (3,4)

: Co-ordinate in spherical system (6)

$\varkappa$: Curvature (2)

: Permeability (3,4,5)

λ : Dilatational viscosity

μ : Viscosity

μ_c : Viscosity at hummock crest (3)

μ_t : Viscosity at hummock trough (5)

ν : Normal co-ordinate

$\underset{\sim}{\nu}$: Normal vector

ξ : Discretisation error

π : 3.14159

: Thermodynamic pressure (3,5,6)

ρ : Density

ρ^{iw} : Density of ice-water mixture (3)

ρ^{w} : Density of water (3)

ρ_s : Thermal resistivity of ice (6)

$\underset{\approx}{\sigma}$: Stress

$\underset{\approx}{\sigma}'$: Deviatoric stress

σ_s : Stress applied to top of clast (1)

τ : Tangential co-ordinate

$\underset{\sim}{\tau}$: Tangent vector

φ	:	Effective contact area coefficient (1,6)
χ	:	Diffusivity
χ_a	:	Diffusivity in arbitrary length units (3)
χ_p	:	Diffusivity in physical units (5)
ϕ	:	Scale (2)
	:	$\underset{\approx}{L}\underset{\sim}{u} + \underset{\sim}{b}$ (5)
	:	Co-ordinate in spherical system (6)
$\underset{\sim}{\phi}$	:	Clast geometry (8)
$\underset{\sim}{\phi}_m$	:	Mother clast geometry (8)
ω	:	Angular velocity (4)
	:	Scaling factor (5)
Γ	:	Boundary (4)
	:	Clast concentration factor (5)
Λ	:	$\sqrt{(1 + (\frac{\partial h}{\partial x})^2)}$ (2)

APPENDIX 2. FLOW IN POROUS MEDIA

The physical principles of flow in porous media are described in many texts, e.g. Todd (1972), Dake (1979).

The Darcy velocity $\underset{\sim}{u}$ is given by Darcy's Law

$$\underset{\sim}{u} = - k\underset{\sim}{\nabla}p \qquad \text{A2.1}$$

where k is the permeability (in general a tensor) μ is the viscosity and p is the presssure. (It should be noted that the exact definition of permeability varies between disciplines.) Darcy's Law is a statement of momentum balance.

Mass balance requires

$$\underset{\sim}{\nabla}.\underset{\sim}{u} = - \theta c \frac{\partial p}{\partial t} \qquad \text{A2.2}$$

where θ is the porosity and c is the compressibility.

Combination of A2.1 and A2.2 gives

$$\frac{\partial p}{\partial t} = \frac{k}{\theta c} \nabla^2 p = \chi \nabla^2 p \qquad \text{A2.3}$$

where χ is the diffusivity.

The transmissibility is defined by kA where A is the area of flow. The pore velocity $\underset{\sim}{v}$ is defined by $\underset{\sim}{v} = \underset{\sim}{u}/\theta$ and is the actual fluid velocity within the pores.

In Chapter 3 the equation

$$\dot{\varepsilon}_V = \chi \nabla^2 p$$

where $\dot{\varepsilon}_V$ is the volumetric strain-rate of the ice-water mixture is derived. χ is called the diffusivity here because of the similarity of this equation to the diffusion equation A2.3.

Similar equations apply to viscous flow between two plates (Landau and Lifshitz, 1959), and for convenience porous media flow terms are extended to this case for flow under clasts where there is little debris (see Chapter 6).

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