SELF-DUAL VORTICES IN NONABELIAN GAUGE THEORIES

by

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Dedicated to my parents, Alan and Jill Sophia Burns, with thanks for the support and encouragement that has made this work possible.

CONTENTS Page

ABSTRACT

Using the Atiyah-Ward construction, we examine the solutions of the self-dual Yang-Mills equations for an SU(2) gauge theory, dimensionally reduced from $\mathbb{R}^4$ to $\mathbb{R}^2$. There are two main reasons for doing this:

(i) To provide a large class of relatively simple examples which elucidate how non-singularity and physical field configurations are related to the parameterization of the Atiyah-Ward construction.

(ii) To construct analogues, for pure non-abelian gauge theories, of the superconducting vortex solutions of the abelian Higgs model, in the hope that these will provide the dominant field configurations describing the QCD vacuum.

First, Bäcklund transformations are used to construct axially symmetric solutions, and the analogues of the 't Hooft instantons. These results are then generalised, within the twistor theoretic framework of the Atiyah-Ward construction, to produce an infinite dimensional parameter space of complex non-singular solutions in each of the Atiyah-Ward ansätze. The field configurations are expressible as unitary group integrals occurring in lattice gauge theories - this leads to a simple proof of non-singularity, and a convenient means of calculating properties of the field configurations using strong and weak coupling expansions. The structure of the field configurations is further elucidated using symmetry arguments and numerical computations. Finally, suggestions are made as to how these solutions may play a role in the QCD confinement mechanism.

DECLARATION

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The work for this thesis was carried out at the Department of Mathematical Sciences, University of Durham, Durham, England, during the academic years 1981-1984. This thesis has not been submitted for any other degree.

Sections (1.1), (1.2), (1.3), (2.1) and (5.1) are introductory, and are not claimed as original. All other sections are claimed as original, unless otherwise indicated in the text. The proof of Theorem (2.3.3) was indicated to me by S Rouhani, and Theorem (3.1.1) was proved in collaboration with A D F Puaca.

Parts of sections (2.2), (2.3), and all of Chapter 4 have been published in J Phys A: Math Gen 17 (1984) 689-707. The remaining parts of sections (2.2) and (2.3), and Chapter 5, are in preparation as a Durham preprint.

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CHAPTER 1

INTRODUCTION AND MOTIVATION

1.1 Non-Abelian Gauge Theories

This thesis is concerned with an exploration of certain aspects of the mathematical structure of non-abelian gauge theories (Yang & Mills 1954, Shaw 1955), the quantum field theories currently understood to describe the strong and electroweak interactions of elementary particles, and widely believed to provide (at least a fundamental part of) a framework in which these interactions, together possibly with gravity, arise as the low energy limit of a single unified theory. For a review see Abers & Lee 1973.

Gauge theories are physically motivated by the requirement that they be locally invariant under a fixed, generally non-abelian Lie group G of internal symmetries, in exactly the same way that electromagnetism is invariant under local changes of phase of particle wave functions, and in much the same way that Einstein's theory of general relativity is invariant under local Lorentz transformations. These theories are mathematically attractive in that they are based on the differential geometry of fibre bundles with structure group G in a manner analagous to the way in which general relativity is based on the differential geometry of (pseudo)-

Riemannian manifolds. From this point of view, gauge theories fulfill, to some extent, Einstein's vision that all laws of physics should be ultimately expressible in geometric terms.

Explicitly, a gauge potential is a 4-vector function $A_\mu(x)$ of space-time, taking values in the Lie algebra L(G) of G. This defines a connection of a fibre bundle over space-time with structure group G, and with covariant derivative:

$$D_\mu = \partial_\mu + A_\mu \tag{1.1.1}$$

The gauge field is the curvature tensor:

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{1.1.2}$$

and two gauge potentials define equivalent connections if and only if they are equivalent up to a gauge transformation:

$$A_\mu \to A'_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g \tag{1.1.3}$$

where g(x) is a smooth G-valued function of space-time. This implies that $F_{\mu\nu}$ transforms under the adjoint representation of G, $F_{\mu\nu} \to F'_{\mu\nu} = g^{-1} F_{\mu\nu} g$.

In the absence of other matter fields, ie for a pure gauge theory, the Lagrangian density is proportional to the norm squared of the curvature tensor:

$$L = -\frac{1}{4}||F||^2 = -\frac{1}{4}<F^{\mu\nu}, F_{\mu\nu}> \tag{1.1.4}$$

where <,> denotes the Killing form on L(G). In the classical theory, we are interested in gauge potentials which are the extrema of the action functional $S = \int d^4xL$. The Euler-Lagrange equations for this variational problem are the Yang-Mills equations:

$$D_\mu F^{\mu\nu} = 0 \tag{1.1.5}$$

A similar set of equations, which follow automatically from the definition (1.1.2), are the Bianchi identities:

$$D^*_\mu F^{\mu\nu} = 0 \tag{1.1.6}$$

where ${}^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$ is the Hodge dual of $F^{\mu\nu}$.

Lagrangians describing the interaction of gauge fields with matter fields are constructed from the principal of minimal coupling: simply take a standard (ungauged) Lagrangian of interacting bosonic and fermionic matter fields transforming under certain linear representations of G, replace all space-time derivatives by covariant derivatives (1.1.1), and add the Lagrangian (1.1.4) to provide a kinetic term for the gauge potentials.

Example 1

The gauge theory of the strong interactions, Quantum Chromodynamics (QCD), consists simply of an SU(N) (N=3) gauge potential (the 'gluon' field), interacting with (Dirac) fermionic quarks in the fundamental representation of the colour group SU(N):

$$L = -\frac{1}{4} \| \mathbb{F}_{\mu\nu} \|^2 + \bar{\underset{\sim}{\Psi}} . \not{D} \underset{\sim}{\Psi} \qquad (1.1.7)$$

$$\not{D} \underset{\sim}{\Psi} = \gamma^{\mu} D_{\mu} \Psi = \gamma^{\mu} (\partial_{\mu} + A_{\mu}) \underset{\sim}{\Psi}$$

$$\underset{\sim}{\Psi} \to g \underset{\sim}{\Psi} \qquad \text{under gauge transformations.}$$

This theory is unique in the respect that it is the only known four dimensional theory of the strong interactions which has the experimentally required property of asymptotic freedom ie the running coupling constant tends to zero as the cut-off parameter defining the regularised quantum field theory is taken to its limiting value (ultra-violet cut-off $\Lambda \to \infty$, or, equivalently, lattice spacing $a \to 0$ in the lattice regularised theory). This implies that, at short distances, quarks behave as though they are quasi-free, in agreement with the parton model. The asymptotic freedom property allows the application of perturbation theory to high energy hadronic processes, where it predicts small logarithmic corrections to the scaling predictions of the parton model. These predictions are consistent with observation, but unfortunately, perturbative QCD seems unable to make predictions that are so strikingly accurate that they definitely establish QCD as the undisputed theory of the strong interactions.

Example 2

Another important example of an interacting gauge-matter theory is that of a non-abelian gauge field interaction with a bosonic Higgs field taking values in the Lie algebra of G, and hence transforming under the adjoint representation of the gauge group:

$$L = -\frac{1}{4}\,||F_{\mu\nu}||^2 + \tfrac{1}{2}\,||D_\mu \Phi||^2 - V(\Phi) \qquad (1.1.8)$$

$$D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi]$$

$$\Phi \to g^{-1}\Phi g \qquad \text{under gauge transformations}$$

where the potential V is invariant under gauge transformations. In the case that $V(\Phi)$ has a manifold of degenerate minima acted upon transitively by G, these theories constitute the bosonic sectors of Grand Unified Theories (GUTs) responsible for the high energy symmetry breakdown via the Higgs mechanism (Higgs 1964, 1966, Englert & Brout 1964, Guralnik, Hagen & Kibble 1964).

For example, if G = SU(N), and a minimum Φ_o of $V(\Phi)$ has eigenvalues $\lambda_1,\ldots,\lambda_r$ with degeneracies $N_1,\ldots,N_r$ respectively then the gauge group G undergoes spontaneous symmetry breaking to the subgroup of G leaving Φ_o invariant, ie

$$SU(N) \rightarrow SU(N_1)\times\ldots\times SU(N_r)\times U(1)^{r-1} \tag{1.1.9}$$

eg when $N=5$, $N_1=3$, $N_2=2$, we have the high energy sector of the minimal Grand Unified Theory $SU(5) \rightarrow SU(3)\times SU(2)\times U(1)$.

An important property shared (for somewhat different reasons) by the above two examples is that the particle-spectrum obtained from a naïve perturbative expansion of the defining Lagrangian does not correspond to the low energy physical particle spectrum. For example, the low energy particle spectrum of the strong interaction does not consist of a quark-gluon plasma, as suggested by the Lagrangian (1.1.7) - quarks and gluons are confined in colour singlet states of baryons, mesons, and possibly glueballs etc. Also, as explained in the following section, the classical equations of motion of the Lagrangian (1.1.8) possess finite energy magnetic monopole soliton solutions. If Grand Unified Theories provide a correct description of nature, these magnetic monopoles should provide an important contribution the particle content of the early universe - for example, they have large gravitational mass ($\sim 10^{17}$ GeV) and, when coupled to fermions they have the remarkable property of catalysing baryon decay at approximately strong interaction rates (Callan 1982, Rubakov 1982).

The above examples illustrate the point that probably the most important problem of quantum field theories, particularly asymptotically free quantum field theories, is the determination of the physical particle spectrum from the defining Lagrangian. Until this problem is solved for the relatively simple examples above, it is hard to imagine how the particle spectrum of, say, N=8 supergravity will be determined. Clearly, to solve these problems, we have to go beyond the constraints of perturbation theory - the next section reviews the main approaches to non-perturbative quantum field theory that have been employed to date.

1.2 Non-Perturbative Methods of Quantum Field Theory

There is only one technique of non-perturbative field theory that has so far got anywhere near to making experimentally testable predictions, and that is the technique of Monte Carlo simulations (Binder 1979) of lattice regularised field theory (Wilson 1974). Despite early optimism in the calculation of hadron masses (Hamber & Parisi 1981, Marinari et al 1981), even these techniques have not yet achieved sufficiently high statistics to make reliable predictions. Moreover, even if the quantitative results can be made reliably accurate, these techniques will give us little qualitative insight into the underlying physical processes, in particular into what are the dominant field configurations.

A more ambitious program in this direction was initiated by Polyakov in 1975; he suggested that quantum field theories could be approximated semi-classically by calculating the Gaussian fluctuations around classical solutions (called instantons) of the euclidean space field equations. Classical solutions of the Minkowski space field equations are also important for the rather different reason that, in some theories, they describe topologically stable finite energy soliton-like objects which should provide part of the non-perturbative particle spectrum of the full quantum theory.

We give below brief descriptions of these three topics.

(a) Lattice Gauge Theories

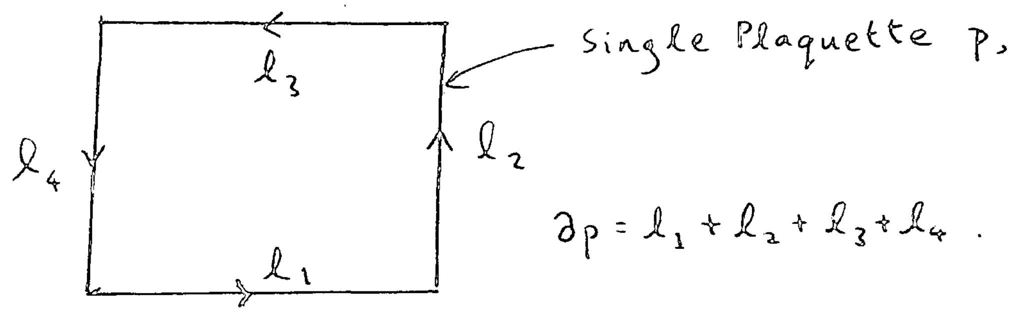
Wilson's formulation of lattice gauge theories (LGT's) exploits in a natural way the geometric interpretation of gauge theories. The fundamental geometric object is the path ordered exponential:

$$U(C) = P \exp \int_C d^4x^\mu A_\mu(x)$$

- which is a function from paths C in space-time to elements U(C) of the gauge group G; U(C) describes the parallel transport of internal symmetry vectors along the curve C. Wilson's idea is to describe gauge field configurations on a lattice by assigning a gauge group element to every elementary path element, or 'link' of the lattice. A path C in the lattice is simply a sequence of consecutive links $\ell_1, \ldots, \ell_n$ say, and the path ordered exponential along C is given by:

$$U(C) = U_{\ell_n} \ldots U_{\ell_2} U_{\ell_1}$$

The Wilson action of a given field configuration is defined in terms of the trace of the path ordered exponentials around elementary loops (ie the boundaries of elementary squares, or plaquettes), as follows:



$$S = \frac{-1}{2g^2} \sum_{\text{Plaquettes}} \mathrm{Tr}\left[\prod_{\partial P} U + \text{h.c.}\right] \tag{1.2.1}$$

where

$$\prod_{\partial P} U = U_{\ell_1} U_{\ell_2} U_{\ell_3} U_{\ell_4},$$

and g is the gauge coupling constant.

The quantum field theory is defined by an equivalent statistical mechanical partition function:

$$Z = \int \prod_{\text{Links}} dU_\ell \, e^{-S[U_\ell]}$$

where the integrals are performed with respect to the Haar measure on G, and the parameter $\beta = 1/g^2$ plays the role of inverse temperature.

Remarks

(1) It is important to note that g^2 is not an absolute constant, but a function of the lattice spacing a. Typically, physical correlation lengths of Green's functions are given by:

$$\ell = n(g^2)a \qquad (1.2.2)$$

where $n(g^2)$ is a dimensionless function of g^2 giving the correlation length in terms of numbers of unit lattice spacings. So, if g were constant, all physical correlation lengths would collapse to zero with the lattice spacing, in the continuum limit $a \to 0$. Instead, a chosen correlation length (or other suitable dimensional physical quantity) is held fixed at its observed value (dimensional transmutation), and then equ (1.2.2) defines g^2 implicitly as a function of a, giving rise to the Callan-Symanzik renormalisation group equations. As a consequence of (1.2.2), we have:

$$n(g^2(a)) \to \infty \qquad \text{as } a \to 0$$

So, in the continuum limit $a \to 0$, lattice Green's functions must be correlated over an infinite number of unit lattice spacings, ie $g^2 \to g_c^2$ where the lattice statistical mechanical theory has a 2nd order phase transition at $\beta_c = 1/g_c^2$. Thus, continuum field theories are defined at 2nd order phase transition points of lattice field theories.

(2) The trace function in (1.2.1) is not the unique possible choice - the universality property of critical phenomena suggests that an identical continuum limit is obtained by replacing it by any function $\chi : G \to \mathbb{C}$

satisfying the same symmetry property as trace:

$$\chi(V^{-1}UV) = \chi(U) \qquad VU, V\varepsilon G$$

ie χ is a class function on G. For a compact group, the Peter-Weyl theorem implies that this can be written uniquely as a linear sum of irreducible characters of G, so the most general possible action is given by:

$$S = - \sum_{\text{Plaquettes}} \sum_r \beta_r (\chi_r [\prod_{\partial P} U] + \text{h.c.}) \tag{1.2.3}$$

where χ_r are the irreducible characters of G, and β_r are associated inverse couplings. (Note: Trace is the character of the fundamental representation). The study of these generalised action (or mixed action) lattice gauge theories is not purely academic. For example, the phase structure of SU(N) LGT's is very much elucidated by studying the behaviour in the $\beta_F\beta_A$ phase plane, where β_F, β_A are the inverse couplings associated with the fundamental and adjoint representations of SU(N) (see eg Drouffe 1982, Caneschi, Halliday & Schwimmer 1982).

(3) Lattice chiral models are defined similarly, as follows:

Given a compact group G, physical states are defined by assigning a group element U_x to every lattice site x, and the (generalised) action is given by:

$$S = - \sum_{\text{links}(xy)} \sum_{r} \beta_r (\chi_r [U_x U_y^{-1}] + \chi_r [U_y U_x^{-1}]) \qquad (1.2.4)$$

Note the global GxG invariance:

$$U_x \to V_L U_x V_R^{-1}, \qquad (V_L, V_R) \in G \times G$$

This class of models is rather general; with a suitable choice of G and suitable restrictions on the action, all the classical spin systems (eg Ising, Potts, Clock & Heisenberg models) can be obtained as special cases.

There is some evidence that the behaviour of d=4 lattice gauge theories is somewhat analagous to that of the corresponding d=2 lattice chiral models; in particular, for a non abelian compact simple group, both are asymptotically free, and Monte Carlo simulations suggest that their phase diagrams have similar structures. Also, the d=1/d=2 lattice chiral/gauge model is trivial in the sense that its partition function factorises into a product of single-site/link partition functions:

$$Z = \int_G dU \exp(-\sum_r \beta_r (\chi_r (U) + \chi_r (U^{-1}))) \qquad (1.2.5)$$

(The theories also reduce to this single integral in the mean field approximation $d \to \infty$.) The integrals

(1.2.5) (for G=U(N)) will reappear in chapters 2 and 5, albeit in a rather different context.

(b) The Semiclassical Approximation

For reviews and references, see Coleman 1977, and Zinn-Justin 1981, 1982.

In quantum field theory, we are largely interested in the evaluation of Green's functions, defined typically by euclidean functional integrals of the form:

$$I = \int [D\Phi] e^{-S(\phi)/g^2} F(\Phi) \tag{1.2.6}$$

Finite dimensional integrals of this form can be evaluated asymptotically as $g^2 \to 0$ using the saddle point approximation. For example, if we have a one dimensional integral of the form (1.2.6), and $S(\Phi)$ has minima $\Phi^{(i)}$, then approximating $S(\Phi)$ by quadratic expansions about $\Phi^{(i)}$ in the neighbourhoods of $\Phi^{(i)}$ leads to an approximation for (1.2.6) as a sum of Gaussian integrals centred about the minima $\Phi^{(i)}$, and we obtain Laplace's result:

$$I \sim \sum_i F^{(i)} e^{-S^{(i)}/g^2} \left[\frac{\pi g^2}{2S''(\Phi^{(i)})}\right]^{\frac{1}{2}}, \quad \text{as } g^2 \to 0 \tag{1.2.7}$$

where

$$F^{(i)} = F(\Phi^{(i)}) \text{ and } S^{(i)} = S(\Phi^{(i)}).$$

This simple picture is complicated by the following two facts:

(1) If $S(\Phi)$, $F(\Phi)$ are analytic in Φ, a better approximation may be obtained by deforming the integration contour to pass through the paths of steepest descent through all saddle points of $S(\Phi)$ in the complex Φ-plane - this is the well known method of steepest descents.

(2) For finite dimensional integrals over $|R^n$, provided the minima of $S(\Phi)$ are isolated, equ (1.2.7) becomes:

$$I \sim \sum_i F^{(i)} e^{-S^{(i)}/g^2} \left[\frac{\pi g^2}{2\det C^{(i)}}\right]^{n/2}, \quad \text{as } g^2 \to 0 \tag{1.2.8}$$

where $C^{(i)}$ is the nxn matrix of second derivatives of S at Φ_i:

$$C^{(i)}_{k\ell} = \left.\frac{\partial^2 S}{\partial\Phi_k \, \partial\Phi_\ell}\right|_{\Phi=\Phi^{(i)}}$$

More generally, the minima of S may occur on a k-dimensional submanifold M of $|R^n$, in which case the sum in (1.2.8) is replaced by an integral over M, and the determinants are replaced by $\det' C^{(i)}$ defined as the products of non-zero eigenvalues of $C^{(i)}$.

These results are expected to generalise, in some sense, to the case of infinite dimensional functional integrals of the form (1.2.6), since one way of defining the latter is as a limit of finite dimensional integrals.

In this case, the saddle points of the action are simply the finite action solutions of the classical equations of motion in euclidean space - these are called <u>instantons</u>. The determinants must also be replaced by suitably regularised functional determinants of the integral operators with kernels:

$$C_o(x,y) = \left.\frac{\delta^2 S}{\delta\Phi(x)\delta\Phi(y)}\right|_{\Phi=\Phi_o}$$

So, to perform a semiclassical approximation of a quantum field theory, at least three non-trivial problems must be solved:

(i) Determine the moduli space M of all instantons.
(ii) Calculate the functional determinants of Gaussian fluctuations about points of M.
(iii) Determine the integration measure on M.

It is not too surprising that this program has been carried out for only a very limited number of models. In most applications, step(i) is simplified first by restricting attention to those instantons of smallest non-zero action, and then by assuming that the other dominant instanton configurations can be approximated in some sense as superpositions of these. This is called the instanton gas approximation, and its validity is rather questionable.

Another source of difficulty is the question of what type of classical solution we expect to contribute to equ (1.2.8). The analogy of the steepest descent approximation for one dimensional integrals strongly suggests that finite action complex saddle points should be just as important as real saddle points, though the deformation of integration contours in infinite dimensional complex configuration space rather defies ordinary geometric intuition. Some authors have also considered the possibility that infinite action solutions (or 'merons') contribute to the semi-classical approximation, despite the fact that the factor e^{-s/g^2} in equ (1.2.8) suggests that these give zero contribution.

(c) Topologically Stable Extended Objects

In this section, we consider solutions of classical field equations in Minkowski space-time $\mathbb{R}^{d,1}$. For a review, see Goddard & Olive 1978. We are particularly interested in stable finite energy solutions with localised energy density - these are called 'extended objects'. If finite energy solutions exist, their stability can often be guaranteed using topological arguments - a continuous time development of a classical field defines a homotopy equivalence of the field configurations at different times, so the homotopy classes of certain associated maps must be conserved.

The prototypical example for us is that of a scalar field thoery with a (possibly gauged) symmetry group G which is spontaneously broken down to a subgroup H, ie

$$L = -\frac{1}{4} F_{\mu\nu}.F^{\mu\nu} + \tfrac{1}{2}D_{\mu}\Phi.D^{\mu}\Phi - V(\Phi) \qquad (1.2.9)$$

where Φ transforms under some linear representation of G, and $V(g\Phi) = V(\Phi)$, $\forall g \in G$.

We assume the conventions that V has absolute minimum value zero, and we define the <u>vacuum manifold</u>

$$M = \{\Phi;\ V(\Phi) = 0\}$$

which we assume to be acted upon transitively by G with isotropy group H, so that M is topologically equivalent to the homogeneous space G/H.

Now, finite energy solutions are expected to satisfy the boundary conditions:

$$\left.\begin{array}{ll}\text{(i)} \quad D_\mu\Phi(\underset{\sim}{x},t) \to 0 & \\ & \text{as } \underset{\sim}{x}\to\infty \\ \text{(ii)} \quad V(\Phi(\underset{\sim}{x},t)) \to 0 & \end{array}\right\} \qquad (1.2.10)$$

ie the scalar field must approach the <u>Higgs vacuum</u> sufficiently quickly at spatial infinity. Condition (ii) implies that the asymptotic Higgs field takes values in the vacuum manifold, hence there exists a map $\Phi_\infty : S^{d-1} \to M$ defined by:

$$\Phi_\infty(\hat{n}) = \lim_{r\to\infty}\Phi(r\hat{n}), \qquad \forall\hat{n}\in S^{d-1}$$

This map determines a homotopy class:

$$q(\Phi)\in\pi_{d-1}(M) \cong \pi_{d-1}(G/H)$$

and, since $\Phi(x,t)$ evolves continuously with time, $q(\Phi)$ is conserved; it is called the <u>topological charge</u> of Φ.

Another important class of models displaying topologically conserved quantities is that of the non-linear sigma models, where a 'free' scalar field is constrained to take values in a compact Riemannian manifold M:

$$L = \tfrac{1}{2}D^\mu\sigma D_\mu\sigma\ , \qquad \sigma\in M$$

In this case, we impose the trivial boundary conditions

$$\sigma(\underset{\sim}{x},t) \to \text{constant} \qquad \text{as } \underset{\sim}{x}\to\infty$$

Hence, the field σ extends to a map on the one point compactificatin of $|R^d$, $\sigma : S^d \to M$, and the homotopy class of this map defines a conserved topological charge:

$$q(\sigma) \varepsilon \pi_d(M)$$

<u>Note</u>. Strictly speaking, the above <u>relative</u> homotopy groups $\pi_n(X)$ should be replaced by <u>absolute</u> homotopy classes $\tilde{\pi}_n(X)$. However, in all applications of interest to us, this distinction is not important.

Example 1: Vortices

The first examples of topologically stable extended objects in relativistic field theories were provided by Nielsen & Olesen in 1973. They considered the Abelian Higgs model, consisting of a U(1) gauge field, A_μ interacting with a complex scalar Higgs field ϕ:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}(D_\mu\phi)^*(D^\mu\phi)-V(\phi) \qquad (1.2.11)$$

$$D_\mu\phi = (\partial_\mu+ieA_\mu)\phi, \qquad V(\phi) = \frac{\lambda^2}{8}(\phi\phi^*-a^2)^2$$

In this case, $M = \{\phi;|\phi|=a\} \cong S^1$, so, in two spatial dimensions, d=2, topological charges are elements of -

$$\pi_1(S^1) \cong \mathbb{Z}$$

ie the topological charge is given by the integer winding number of ϕ as it sweeps round a large circle at infinity; this is called the vorticity of the field configuration. Writing $\phi_\infty = ae^{i\chi}$ for the asymptotic Higgs field, we have:

$$q(\phi) = \frac{1}{2\pi i}[\chi] = \frac{1}{2\pi i}[\ln\phi] \qquad (1.2.12)$$

where $[\cdot]$ denotes the change in going round a large circle at infinity.

The presence of the gauge field implies that more is true; condition (i) of the Higgs vacuum equs (1.2.10) implies that, at sufficiently large distances:

$$A_\mu = \frac{i}{e} \frac{\partial_\mu \phi}{\phi} = \frac{i}{e} \partial_\mu \ln \phi \qquad (1.2.13)$$

Hence, the total magnetic flux $\Phi(R)$ through a large disc of radius R is given by:

$$\Phi(R) = \int_{|x| \leq R} B dS = \int_{|x| = R} A_\mu d\ell^\mu \qquad \text{(by Stokes' theorem)}$$

$$\sim \frac{i}{e} [\ln \phi] \qquad \text{as } R \to \infty \text{, by (1.2.13)}$$

Hence, using (1.2.12), the total magnetic flux Φ in the $x_1 x_2$-plane is given by:

$$\Phi = \frac{2\pi}{e} q(\phi) \qquad (1.2.14)$$

ie <u>the total magnetic flux is quantized in integer multiples of $2\pi/e$.</u>

Now, let C be any simple closed curve which does not pass through any zeros of ϕ. Then we can define the vorticity around C as the Poincare index of the 2-dimensional vector field ϕ, and this reduces to the above definition of vorticity around a circle at infinity. It is therefore natural to identify vortex positions with the zeros of ϕ. It is found that, for vortices of sufficiently large separation, the modulus of the Higgs field differs appreciably from its asymptotic value a only in neighbourhoods of radius $(\lambda a)^{-1}$ around vortex positions and the magnetic field differs appreciably from zero only in neighbourhoods of radius $(ea)^{-1}$ around vortex positions.

It is important to note that the abelian Higgs model is in fact a relativistic version of the Ginzburg-Landau macroscopic theory of superconductivity. The density of Cooper pairs is determined by the Higgs field ϕ, and the Meissner effect (ie the expulsion of a weak applied magnetic field from a superconductor, apart from a small penetration depth) arises as a consequence of the effective mass acquired by the photon via the Higgs mechanism.)

In certain (ie type II) superconductors, a strong applied magnetic field can penetrate the superconductor by the formation of quantized magnetic flux tubes whose interiors are in the normal (ie non-superconducting) phase. These correspond exactly to the above vortex solutions extended along lines in three spatial dimensions.

The above superconducting magnetic vortex lines were proposed by Nielsen and Olesen as field theoretic models for dual strings, which at the time were thought to describe low energy hadronic physics. Since then, various analogues of the above model have been proposed by several different authors (eg 't Hooft 1978, 1979, 1981, Nielsen & Olesen 1979) wherein confinement is conjectured to occur in QCD through the formation of a superconducting ground state. These are mostly based on the observation that, in the superconducting Higgs phase of the abelian Higgs model, we have permanent magnetic confinement - all magnetic flux is squeezed into thin tubes which can only terminate in a magnetic monopole or anti-monopole. Thus, monopole-anti-monopole pairs are joined by magnetic flux tubes, and their energy must vary linearly with separation. 't Hooft uses the analogy of duality transformations in abelian lattice gauge theory to suggest that QCD is in an 'electric confining' phase, which is, in some sense, dual to a 'magnetic confining' Higgs phase. Unfortunately, none of these models has been entirely successful in explaining confinement - the main difficulty seems to be the identification of the correct degrees of freedom in a <u>pure gauge theory</u> (ie <u>without</u> Higgs fields) required to give the desired dynamics.

For completeness, let us note that vortices tend to occur more generally in theories where a gauge group G is spontaneously broken to a <u>discrete</u> subgroup H. In this case, we have $G/H \cong \tilde{G}/\tilde{H}$, where $\tilde{H}$ is the pullback of H in the universal covering group $\tilde{G}$ of G; standard covering space theory then tells us that the vortex charges are elements of -

$$\pi_1(\tilde{G}/\tilde{H}) \cong \pi_0(\tilde{H}) \cong \tilde{H}$$

<u>Example 2: Magnetic Monopoles</u>

The possibility of the existence of magnetic monopoles was first noticed by Dirac (1931). The main observation is that a gauge potential A_μ need not be single valued; we could define $A_\mu \equiv A^i_\mu$ on patches U^i of space-time, provided we impose the consistency condition:

A^i_μ is gauge equivalent to A^j_μ on $U^i \cap U^j$ whenever $U^i \cap U^j$ is non-empty, ie for a U(1) gauge theory:

$$A^i_\mu = A^j_\mu - \partial_\mu \chi^{ij} \qquad \text{on } U^i \cap U^j \tag{1.2.15}$$

where $\exp(-ie\chi^{ij})$ is a single-valued function on $U^i \cap U^j$ ie

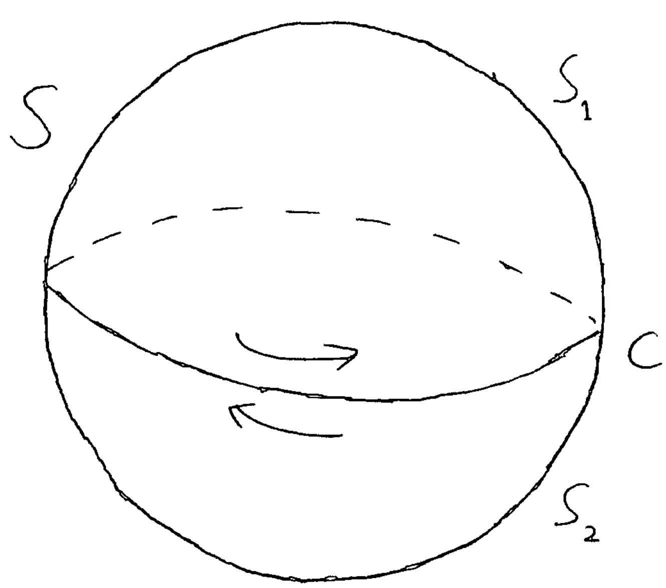
$$\frac{e}{2\pi}[\chi^{ij}]_c \in \mathbb{Z} \tag{1.2.16}$$

where $[\cdot]_C$ denotes the change in χ around a closed curve C.

Condition (1.2.15) simply means that we are treating the gauge potential in its proper geometric setting, as a U(1) -connection on a U(1)-bundle over space-time. Condition (1.2.16) reflects the compactness of the gauge group - it is necessary if the gauge group acts on a complex scalar or spinor field, such as a Schrödinger wave function.

Now, consider a 2-sphere S embedded in 3-space $\mathbb{R}^3$, and cover S with upper and lower hemispheres S_1, S_2 intersecting in the equator C of S

ie



let

$$A_\mu = \begin{cases} A^1_\mu & \text{on } S_1 \\ A^2_\mu & \text{on } S_2 \end{cases}$$

and

$$A^2_\mu = A^1_\mu - \partial_\mu \chi \qquad \text{on } S_1 \cap S_2$$

Then, using Stokes' theorem, the magnetic charge g enclosed by S is given by:

$$g = \int_S \underset{\sim}{B} \cdot d\underset{\sim}{S} = \int_{S_1} d\underset{\sim}{S} \cdot \underset{\sim}{B} + \int_{S_2} d\underset{\sim}{S} \cdot \underset{\sim}{B}$$

$$= \int_C d\ell^k A^1_k - \int_C d\ell^k A^2_k = \int_C d\ell^k \partial_k \chi$$

$$= [\chi]_C = \frac{2\pi n}{e}, \qquad n \in \mathbb{Z}, \quad \text{by (1.2.16)}$$

$$\therefore \quad \frac{ge}{2\pi} \in \mathbb{Z} \qquad \text{ie } \frac{gq}{2\pi h} \in \mathbb{Z} \qquad (1.2.17)$$

the Dirac quantization condition for magnetic charge.

This analysis has been extended to a non-abelian gauge group H by Wu and Yang (1975). In this case, the gauge transformation relating A^1 and A^2 on C defines a map $h: C \to H$, and magnetic charges are classified topologically by the corresponding homotopy classes $[h]\epsilon\pi_1(H)$. Topologically, a non-trivial magnetic charge measures the non-trivial twisting of the H-bundle when restricted to the 2-sphere S. As a consequence, the gauge field <u>must</u> have at least one singularity in the interior of S, since all bundles on contractable spaces are trivial. This unpleasant feature can be avoided if H is the residual gauge symmetry group of a larger spontaneously broken group G, as in (1.2.9) ('t Hooft 1974, Polyakov 1974). In this case, singularities may be replaced by points at which H is not well-defined (eg at zeros of the Higgs field Φ), and in certain cases it is possible to prove the existence of smooth solutions of the equations of motion which asymptotically have non-zero H-magnetic charge. The magnetic charge is in fact identical to the topological charge of Higgs field, $g = q(\Phi)\epsilon\pi_2(G/H)$; this is a consequence of the homotopy exact sequence for the fibration $G \xrightarrow{H} G/H$, which implies, for a simply connected Lie group G:

$$\pi_2(G/H) \cong \pi_1(H)$$

The simplest example occurs when an SU(2) gauge group is broken to U(1) by an adjoint representation (ie isovector) Higgs field $\underset{\sim}{\Phi}$, with:

$$V(\underset{\sim}{\phi}) = \frac{\lambda^2}{4}(\underset{\sim}{\phi}^2 - a^2)^2, \quad \underset{\sim}{\phi} \in |R^3$$

In this case, the vacuum manifold is a 2-sphere so magnetic charges are classified by $\pi_2(S^2) \cong Z$, ie by the winding number of Φ_∞ is it maps the 2-sphere at spatial infinity to a 2-sphere in isospin space. As remarked previously, the general Georgi-Glashow model (1.1.8) with gauge group G and an adjoint representation Higgs field also possesses magnetic monopoles. For example, with symmetry breaking pattern (1.1.9), magnetic charges are classified topologically by:

$$\pi_1\left[SU(N_1)x\ldots xSU(N_r)xU(1)^{r-1}\right] \cong \mathbb{Z}^{r-1}$$

Example 3: Yang-Mills Instantons

Yang-Mills instantons are finite action solutions of the Yang-Mills equations (1.1.5) on euclidean 4-space $|R^4$. The finite action constraint suggests that the gauge potential be 'pure gauge' at infinity

$$\text{ie} \quad A_\mu \sim g^{-1}\partial_\mu g \qquad \text{as } x_\mu \to \infty$$

and any such gauge transformation g(x) defines a map from the 3-sphere at spatial intinity to the gauge group G.

It is easily seen that the homotopy class of this map is gauge invariant, and this defines the <u>instanton charge</u>:

$$q \in \pi_3(G) \cong \mathbb{Z}$$

- for any non-abelian compact simple Lie group G. Note that any finite action solution is extendable to the one-point compactification S^4 of $\mathbb{R}^4$, and the instanton charge can be expressed as an integral over S^4 of the curvature tensor and its dual (Coleman 1977):

$$32\pi^2 q = \int_{S^4} d^4x \langle F_{\mu\nu}, {}^*F_{\mu\nu} \rangle \qquad (1.2.18)$$

1.3 Bogomol'nyi Equations and Dimensional Reduction

An important phenomenon is known to occur in appropriate physical limits of certain classical field theories, and that is the existence of 1st order differential equations, called Bogomol'nyi equations which imply the 2nd order static Euler-Lagrange equations of the theory (Bogomol'nyi 1976).

As a rule, solutions of these 1st order equations are absolute minima (rather than just saddle points) of the energy functional, in distinct topological sectors of the theory.

The prototype examples are provided by the self-dual and anti-self-dual Yang-Mills equations on euclidean 4-space $\mathbb{R}^4$ (Belavin et al 1975).

$$F_{\mu\nu} = \pm {}^*F_{\mu\nu} \tag{1.3.1}$$

These are 1st order equations in the gauge potential A_μ, and, together with the kinematically necessary Bianchi identities (1.1.6), they clearly imply the 2nd order Yang-Mills equations (1.1.5). Finite action solutions of (1.3.1) are absolute minima of the action functional on each of the distinct instanton charge sectors of the theory. For:

$$\frac{1}{4g^2}\int ||F \pm {*F}||^2 = \frac{1}{4g^2}\left(\int ||F||^2 \pm 2\int \langle F, {*F}\rangle + \int ||{*F}||^2\right)$$

$$= 2\left(S \pm \frac{8\pi^2 q}{g^2}\right) \qquad \text{by } (1.2.18)$$

$$\Rightarrow \quad S \geq \frac{8\pi^2 |q|}{g^2}$$

with equality if and only if (1.3.1) is satisfied. Solutions of the self-duality equations must be instantons of positive topological charge, and those of the anti-self-duality equations are anti-instantons, of negative topological charge.

The first step in constructing all instanton solutions of (1.3.1) was taken by Atiyah and Ward in 1977. Using the twistor space construction of Penrose, they showed that there is a one-one correspondence between solutions of the self-duality equations on $\mathbb{R}^4$ (resp S^4), and certain holomorphic vector bundles on $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ (resp $\mathbb{C}P^3$), giving rise to a sequence of distinct ansätze $a_1, a_2 \ldots$ describing self-dual fields. In fact, the Atiyah-Ward construction can be generalised to provide a construction for all (anti)-self-dual gauge potentials on any (anti)-self-dual Riemannian 4-manifold M^4 (ie one for which the Weyl conformal tensor is (anti-)-self-dual). Both the self-duality <u>and</u> anti-self-duality equations can be solved if $W \equiv 0$ ie if M^4

is conformally flat. (Atiyah et al 1978).

The Atiyah-Ward construction actually produces solutions of the self-duality equations which are complex and singular; extra constraints have to be imposed to guarantee reality and non-singularity - the determination of necessary and sufficient conditions for non-singularity is in fact a highly non-trivial unsolved problem. As a result, the Atiyah-Ward construction is not well suited to the construction of instantons. This problem was solved by Atiyah, Drinfeld, Hitchin & Manin (ADHM, 1978) by first noting a result of Serre - that holomorphic bundles on $\mathbb{C}P^3$ are necessarily algebraic - and then using modern algebraic geometric techniques of Horrocks & Barth to construct the required algebraic bundles.

Example 1: Nielsen-Olesen Vortices

Consider the abelian Higgs model, with energy functional:

$$E = \int d^2x\left(\tfrac{1}{2}|D_i\phi|^2 + \tfrac{1}{2}B^2 + \frac{\lambda^2}{8}(|\phi|^2 - a^2)^2\right) \qquad (1.3.2)$$

where $D_i = \partial_i - ieA_i$. In the special case $\lambda^2/e^2 = 1$, this can be written as follows:

$$E = \int d^2x\left[\tfrac{1}{2}|(D_1 \pm iD_2)\phi|^2 + \tfrac{1}{2}(B \pm \tfrac{e}{2}(\phi\bar{\phi} - a^2))^2\right.$$

$$\left.\pm \frac{eBa^2}{2} \mp \frac{i}{2}\varepsilon_{ij}\partial_i(\bar{\phi}D_j\phi)\right]$$

So, if the fields are asymptotically in the Higgs vacuum, the surface term vanishes by Stokes' theorem, and we are left with:

$$E - \tfrac{1}{2}\int d^2x\left(|(D_1 \pm iD_2)\phi|^2 + |B \pm \tfrac{e}{2}(\phi\bar{\phi} - a^2)|^2\right)$$

$$= \pm\frac{ea^2\Phi}{2}, \qquad \Phi = \text{Total Magnetic Flux}$$

$$= \pm\pi a^2 q, \qquad \text{by (1.2.14)}$$

$$\therefore \quad E \geq \pi a^2 |q| \tag{1.3.3}$$

with equality if and only if:

$$(D_1 \pm iD_2)\phi = 0 \tag{1.3.4}$$

$$B \pm \frac{e}{2}(\phi\bar{\phi} - a^2) = 0$$

So, we have obtained two sets of Bogomol'nyi equations for the abelian Higgs model, whose solutions are respectively vortices and anti-vortices saturating the inequality (1.3.3). These equations can be further simplified following Jacobs & Rebbi (1979). Without loss of generality, absorb e into the definition of the gauge potential, and set a=1 by a rescaling of x_i. Then, defining complex co-ordinates $\omega = x_1+ix_2$, $\bar{\omega} = x_1-ix_2$, the first of equs (1.3.4) becomes:

$$(\bar{\partial} - i\bar{A})\phi = (\partial + iA)\bar{\phi} = 0 \qquad (1.3.5)$$

$$\partial\bar{A} - \bar{\partial}A + \frac{i}{4}(\phi\bar{\phi} - 1) = 0$$

Impose Lorentz gauge $\partial\bar{A} + \bar{\partial}A = 0$, so that we can express A in terms of a real superpotential ψ:

$$A = i\partial\psi, \qquad \bar{A} = -i\bar{\partial}\psi$$

Define $f = e^{-\psi}\phi$. Then (1.3.5) is equivalent to:

$$\bar{\partial}f = \partial\bar{f} = 0$$

$$\partial\bar{\partial}\psi = \frac{1}{8}(e^{2\psi}f\bar{f} - 1)$$

The first of these simply states that f is an <u>analytic</u> function of ω; the second can be further simplified by defining:

$$e^{2\chi} = f\bar{f}e^{2\psi} = \chi = \psi + \log|f|$$

giving us the equation:

$$\partial\bar{\partial}\chi = \frac{1}{8}(e^{2\chi} - 1) \qquad (1.3.6)$$

(Note. Strictly speaking, (1.3.6) has δ-function sources at the zeros of f ie at the vortex locations.)

Despite its apparent simplicity, equ (1.3.6) has so far resisted any attempts at an exact solution - not even the axially symmetric charge one vortex is known in closed form. However, there is an existence theorem, due to Taubes (1980), which states that the Bogomol'nyi equs (1.3.3) possess real analytic static multi-vortex solutions for any finite number N of vortices located at arbitrary points in the x_1x_2-plane. This result can be understood intuitively by noticing that the parameter λ^2/e^2 measures the relative strengths of the Higgs attraction and magnetic repulsion between vortices - Jacobs & Rebbi have verified numerically that two vortices attract each other for $\lambda^2/e^2 < 1$, and repel each other for $\lambda^2/e^2 > 1$, at all separations. The intermediate case $\lambda^2/e^2 = 1$ is a sort of non-interacting limit - the forces on vortices exactly balance each other, so that multi-vortex configurations can exist in static equilibrium, as stated above.

Example 2. 't Hooft-Polyakov Monopoles

Consider again the Georgi-Glashow model consisting of an SU(2) gauge theory spontaneously broken to U(1) by an adjoint representation Higgs field Φ. The residual U(1) gauge symmetry is picked out by the direction of Φ in isospin space, so the total U(1) magnetic charge in a volume V is given by:

$$g = \frac{1}{a}\int_{\partial V} \Phi.B_k dS^k = \frac{1}{a}\int_V D_k\Phi.B d^3x$$

-using Stokes' theorem and the equations of motion. Hence the energy functional is given by:

$$E = \tfrac{1}{2}\int d^3x(||B_k||^2 + ||D_k\Phi||^2 + V(\Phi))$$

$$= \pm ag \mp \tfrac{1}{2}\int d^3x(||B_k \mp D_k\Phi||^2 + V(\Phi))$$

ie $\quad E \geqq a|g|$

with equality if and only if:

(i) $B_k = \pm D_k\Phi$ (1.3.7)

(ii) $V(\Phi) \equiv o$ (1.3.8)

ie the Bogomol'nyi equations (1.3.7) imply the 2nd order equations of motion if the Lagrangian is in the <u>Prasad-Sommerfield limit</u> of vanishing Higgs potential, $\lambda \to 0$, but still maintaining the non-trivial boundary conditions $||\Phi||^2 \to a^2$ as $\underset{\sim}{x} \to \infty$. (Prasad & Sommerfield 1975)

Again, there is an existence theorem (Jaffe & Taubes 1980, Taubes 1981) which establishes the existence of sufficiently widely separated static multi-monopole (resp anti-monopole) solutions of (1.3.7). This result also has a physically intuitive interpretation; for $\lambda > 0$, the residual Higgs field has a finite mass, so the Higgs attraction is only short range, and therefore cannot overcome the long range magnetic repulsion between vortices at sufficiently large separation. However, in the Prasad-Sommerfield limit $\lambda \to 0$, the residual Higgs field becomes massless, and hence long range, and this does indeed exactly balance the magnetic repulsion.

Dimensional Reduction

'Dimensional reduction' is the name given to the study of field theories on some Riemannian manifold M which are invariant under some group S of isometries of M. This process generally leads to the construction of a more complicated field theory on the lower dimensional space of orbits of S, M/S; it has been used mainly to construct complicated physically realistic models in four dimensions starting from relatively simple models in higher dimensions, and to study the effects of imposing certain symmetry constraints on solutions of classical field equations (Forgacs & Manton 1980, Chapline & Manton 1981).

The simplest example occurs when a pure gauge theory is reduced from $\mathbb{R}^{4+N}$ to $\mathbb{R}^4$ by requiring the fields to be independent of the extra N dimensions. The Lagrangian density is:

$$L = \frac{1}{4}F_{ab}.F^{ab}$$

where the indices run over ordinary space-time indices μ,ν, and over the extra indices i,j. We impose the constraint $\partial_i \equiv 0$, and define $\Phi_i \equiv A_i$; this implies:

$$F_{\mu i} = \partial_\mu \Phi_i + [A_\mu, \Phi_i] = D_\mu \Phi_i$$

$$F_{ij} = [\Phi_i, \Phi_j]$$

$$\Rightarrow L = \frac{1}{4}F_{\mu\nu}.F^{\mu\nu} + \tfrac{1}{2}D_\mu\Phi_i.D^\mu\Phi_i + \tfrac{1}{2}[\Phi_i,\Phi_j].[\Phi_i,\Phi_j]$$

ie we are left with an interacting gauge theory in ordinary space-time with N adjoint Higgs fields and a non-trivial quartic potential:

$$V(\Phi_i) = \tfrac{1}{2}||[\Phi_i,\Phi_j]||^2$$

For example, we may regard the Georgi-Glashow model as the dimensional reduction of a pure gauge theory from $\mathbb{R}^{4,1}$ to $\mathbb{R}^{3,1}$, with the 4th component of the gauge potential becoming an effective adjoint Higgs field $A_4 \equiv \Phi$

Moreover, for <u>static</u> solutions ($\partial_o \equiv 0$) of the field equations with $A_o \equiv 0$, the Bogomol'nyi equations (1.3.7) are precisely the dimensional reduction of the self-dual Yang-Mills equations (1.3.1) in the four euclidean spatial dimensions (Manton 1978). As a result the techniques developed to solve the self-duality equations have been applied to the Bogomol'nyi equations for magnetic monopoles. It turns out that the original Atiyah-Ward construction is much better suited to the construction of self-dual monopoles than it is for instantons - for an SU(2) gauge group, a complete (4n-1)-parameter family of separated charge n monopole solutions can be obtained in the nth. Atiyah-Ward ansatz (Ward 1981 a,b,c, Prasad 1981, Corrigan & Goddard 1981). Problems still remain however - there is still no general proof that these solutions are non-singular, and it is not known how the 4n-1 parameters of the Corrigan-Goddard ansatz are related to the structure of the physical field configurations. The ADHM construction has also been generalised to give a construction for monopoles for an arbitrary gauge group (Nahm 1981, 1982). This construction guarantees real, non-singular solutions - it is however difficult to implement in practice, involving the solution of systems of 1st order non-linear ordinary differential equations (see eg Bowman et al 1983), and it is unclear whether or not the solutions thus obtained coincide with those obtained from the Atiyah-Ward construction.

Returning to the topic of dimensional reduction, Forgacs & Manton considered the more general situation of a pure gauge theory with gauge group G defined on a space whose extra dimensions form a compact homogeneous manifold, ie $M = \mathbb{R}^d \times S/R$, where S is a compact group of isometries, and R is the isotropy group. Note that for a gauge theory, invariance under S means that S-transformations can be removed by gauge transformations. Forgacs & Manton showed that the isotropy group R has an embedding $\tilde{R}$ in G, and that the resulting gauge symmetry on $\mathbb{R}^d$ is simply the centraliser $C_G(\tilde{R})$ of $\tilde{R}$ in G ie the subgroup of elements of G which commute with $\tilde{R}$. The other components of the gauge field become effective Higgs fields. If, moreover, the embedding $R \to \tilde{R} \leq G$ extends to an embedding $S \to \tilde{S} \leq G$, then the gauge symmetry $C_G(\tilde{R})$ is spontaneously broken to $C_G(\tilde{S})$ on $\mathbb{R}^d$.

eg Take $G = SU(2)$ on $M = \mathbb{R}^2 \times S^2$, where S^2 is realized as the coset space $SU(2)/U(1)$. We can embed the isotropy group onto a maximal torus $\tilde{R} \cong U(1)$ in G, and this extends naturally to an embedding $S \to \tilde{S} = G$. Therefore:

$$C_G(\tilde{R}) \cong U(1), \qquad C_G(\tilde{S}) \cong 1$$

ie we have an effective abelian gauge theory on $\mathbb{R}^2$ which is spontaneously broken to the identity subgroup.

The effective Lagrangian is in fact that of the Bogomol'nyi limit $\lambda=1$ of the abelian Higgs model (1.3.2), and the self-duality equations on $\mathbb{R}^2 \times S^2$ reduce to the Bogomol'nyi equations (1.3.4) for vortices (Ward 1982). Unfortunately however, the Weyl tensor on $\mathbb{R}^2 \times S^2$ is neither self-dual nor anti-self-dual, so the general construction of Atiyah, Hitchin & Singer cannot be applied to give a geometric construction of separated multi-vortices.

Statement of Aims

This work was begun shortly after the proof of Taubes' existence theorems for multi-vortex and multi-monopole solutions, and the geometric construction of monopole solutions in the Atiyah-Ward ansätze. The original aim was to investigate whether or not vortices in the Bogomol'nyi limit of the abelian Higgs model could be constructed in a similar manner to that of self-dual monopoles. For the reasons mentioned above, no progress was made on this problem - the self-duality equations on $\mathbb{R}^2 \times S^2$ are not twistor-solvable, so there is no (known) geometric construction of multi-vortices, despite the purely analytic existence theorems for such solutions.

However, the existence of non-trivial structure in the dimensional reduction $\mathbb{R}^2 \times S^2 \rightarrow \mathbb{R}^2$ of the self-dual Yang-Mills equations suggests that the dimensional reduction $|R^4 \to |R^2$ of the self-duality equations might lead to some interesting solutions analagous in some sense to Nielsen-Olesen vortices. Indeed, following the construction of instanton solutions in four dimensions, and monopole solutions in three dimensions, it is natural to ask what happens in lower dimensions. Such solutions should be relatively simpler than the higher dimensional cases, so they should provide a new class of examples which shed some light on outstanding problems of the Atiyah-Ward construction such as how non-singularity and the structure of physical field configurations are related to the parameterization of the Atiyah-Ward ansätze.

Further motivation is provided by a recent paper of Corrigan & Goddard (1984), where it is established that the ADHMN construction gives rise to some sort of 'reciprocity' between self-dual Yang-Mills systems in 4+0 and 0+4 dimensions for instantons, and in 3+1 and 1+3 dimensions for monopoles. It is further conjectured that this reciprocity is most fully realized for self-dual Yang-Mills systems in 2+2 dimensions, which should be in some sense self-reciprocal. This is precisely the system that we propose to study, albeit within the Atiyah-Ward formalism rather than the ADHMN formalism.

Thus, our primary goal is to study the mathematical structure of the solutions of the self-duality equations, dimensionally reduced from $\mathbb{R}^4$ to $\mathbb{R}^2$. As a more ambitious long-term goal, we are bearing in mind the fact that several authors have proposed an essentially two dimensional confinement mechanism for QCD, arising from the conjectured dominance of superconducting vortex-like structures in the QCD vacuum. We are therefore seeking analogues for pure non-abelian gauge theories of the Nielsen-Olesen vortices of the abelian Higgs model, in the hope that, via the semi-classical approximation, these will provide the dynamical mechanism needed to justify these still rather vague ideas on confinement.

1.4 Real Singular Solutions of the Two Dimensional Self-Duality Equations - A First Attempt

In complexified co-ordinates:

$$y = (x_1 + ix_2)/\sqrt{2} \qquad z = (x_3 - ix_4)/\sqrt{2}$$

$$\bar{y} = (x_1 - ix_2)/\sqrt{2} \qquad \bar{z} = (x_3 + ix_4)/\sqrt{2}$$

the self-duality equations (1.3.1) take the particularly simple form (Yang 1977):

$$F_{yz} = F_{\bar{y}\bar{z}} = F_{y\bar{y}} + F_{z\bar{z}} = 0 \qquad (1.4.1)$$

Let us seek solutions of (1.4.1) for an SU(2) gauge group which are x_3- and x_4-independent. With the convention that the gauge potentials are hermitian, make the ansatz:

$$A_y = \tfrac{1}{2}\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \qquad A_{\bar{y}} = \tfrac{1}{2}\begin{pmatrix} \bar{A} & 0 \\ 0 & -\bar{A} \end{pmatrix}$$

$$A_z = \begin{pmatrix} 0 & \bar{\beta} \\ \bar{\gamma} & 0 \end{pmatrix} \qquad A_{\bar{z}} = \begin{pmatrix} 0 & \gamma \\ \beta & 0 \end{pmatrix} \qquad (1.4.2)$$

where all fields are independent of x_3 and x_4, ie $\partial_z \equiv \partial_{\bar{z}} \equiv 0$. Then, writing $\partial = \partial_y$, $\partial = \partial_{\bar{y}}$, (1.4.1) is equivalent to:

$$\partial\bar{A} - \bar{\partial}A + 2i(\beta\bar{\beta} - \gamma\bar{\gamma}) = 0 \qquad (1.4.3)$$

$$(\partial + iA)\bar{\beta} = (\bar{\partial} - i\bar{A})\beta = 0$$

$$(\partial - iA)\bar{\gamma} = (\bar{\partial} + i\bar{A})\gamma = 0$$

These equations are remarkably similar to the Bogomol'nyi equations for Nielsen-Olesen vortices - they simply replace the constant symmetry breaking mass parameter with another covariantly constant Higgs field, with opposite conventions for covariant derivatives:

$$D_k\beta = (\partial_k - iA_k)\beta, \quad D_k\gamma = (\partial_k + iA_k)\gamma$$

Equs (1.4.3) can be simplified in a similar manner to equs (1.3.5). Impose Lorentz gauge on the abelian part A of the gauge field, $\partial A + \bar{\partial}A = 0$, so there exists a real field ψ such that:

$$A = i\partial\psi, \qquad \bar{A} = -i\bar{\partial}\psi$$

Define:

$$f = \beta e^{-\psi}, \qquad \bar{f} = \bar{\beta}e^{-\psi}$$

$$g = \gamma e^{\psi}, \qquad \bar{g} = \bar{\gamma}e^{\psi}$$

Then equs (1.4.3) are equivalent to:

$$\text{(i)} \quad \partial\bar{f} = \bar{\partial}f = 0 \qquad (1.4.4)$$

$$\text{(ii)} \quad \partial\bar{g} = \bar{\partial}g = 0$$

$$\text{(iii)} \quad \partial\bar{\partial}\psi = f\bar{f}e^{2\psi} - g\bar{g}e^{-2\psi}$$

(i) & (ii) simply mean that f and g must be analytic in y. In order to further simplify equ (iii), we need to consider two distinct cases:

<u>Case (i). One of f,g $\equiv$ 0</u>

Without loss of generality, suppose g $\equiv$ 0, and define the field χ by:

$$e^{2\chi} = f\bar{f}e^{2\psi} = \chi = \psi + \log|f|$$

Then equ (iii) is equivalent to:

$$\partial\bar{\partial}\chi = e^{2\chi} - \Sigma 2\pi\delta \qquad (1.4.5)$$

ie a <u>Liouville equation</u>, with δ-function sources at the zeros of f. Note that the 'Higgs fields' β,γ are given by:

$$\beta = \frac{f}{|f|}e^{-\chi}, \qquad \gamma = 0$$

The Liouville equation is well known to be completely integrable; its solutions are related by Bäcklund transformations to solutions of the Laplace equation, giving the general real solution:

$$e^{2\chi} = \frac{2F'(\omega)F'(\bar{\omega})}{|F(\omega)+F(\bar{\omega})|^2} \tag{1.4.6}$$

where F is an arbitrary analytic function of $\omega = x_1 + ix_2$.

<u>Case (ii) $f \not\equiv 0$, $g \not\equiv 0$</u>

In this case, we can write equ (iii) as follows:

$$\partial\bar{\partial}\psi = \sqrt{f\bar{f}g\bar{g}}\left[\sqrt{\frac{f\bar{f}}{g\bar{g}}}\, e^{2\psi} - \sqrt{\frac{g\bar{g}}{f\bar{f}}}\, e^{-2\psi}\right]$$

So, defining the field χ by:

$$e^{2\chi} = \sqrt{\frac{f\bar{f}}{g\bar{g}}}\, e^{2\chi} = \chi = \psi + \tfrac{1}{2}\log\frac{|f|}{|g|}$$

we see that equ (iii) is equivalent to

$$\partial\bar{\partial}\chi = 2h\bar{h}\sinh 2\chi - \Sigma\, 2\pi\delta \tag{1.4.7}$$

where $h=\sqrt{fg}$, and the δ-function sources occur at the zeros and poles of f and g. In the special case $h \equiv 1$ ie $f \equiv g^{-1}$, equ (1.4.6) reduces to the <u>sinh-Gordon equation</u>, which is also known to be completely integrable - it has an auto-Bäckland transformation which can be used to generate hierarchies of solutions.

In this case, the 'Higgs fields' are given by -

$$\beta = \frac{f}{|f|}.h\bar{h}e^{\chi}, \qquad \gamma = \frac{g}{|g|}.h\bar{h}e^{-\chi}$$

Thus, our original ansatz (1.4.2) has been reduced to two completely integrable equations, both of which are similar to, though not equivalent to, the vortex equation (1.3.4). These results were also obtained by Saclioglu (1981 a,b). He first used the CFTW ansatz (Corrigan & Fairlie 1977, 't Hooft (unpublished), Wilcek 1977) to obtain the solution, given in our notation by -

$$\beta = \frac{i\bar{y}}{r^2 \log r^2}, \qquad \gamma = 0, \qquad \psi = 2\log(\log r)$$

This can also be obtained from the solution of the Liouville equation (1.4.6), with the choice $F(\omega) = \log\omega$. Note that this solution has rather nasty singularities at the origin r=0 and on the circle r=1. In fact, Saclioglu verified that <u>any</u> solution of the Liouville equation gives rise to a self-dual gauge field with singular curves. He then went on to consider the solution obtained from the radial sinh-Gordon equation - in this case, line singularities are avoided, but the solution still has a singularity at the origin, and infinite total action.

This material is included to illustrate a point that will emerge more clearly in the next chapter - that, modulo some reasonable assumptions on boundedness of the fields, all non-vacuum real solutions of the two dimensional self-duality equations are singular. So, in order to construct non-trivial non-singular solutions, we shall have to consider strictly complex gauge fields. Also, we clearly need a more sophisticated approach than the simple ansatz (1.4.2); fortunately, such an approach is provided by the Atiyah-Ward construction.

CHAPTER 2

AXIALLY SYMMETRIC SELF-DUAL VORTICES

2.1 The Atiyah-Ward Construction

In this section, we review the most basic details of the Atiyah-Ward construction, and establish notation, and some preliminary results. Further details may be found in Corrigan et al (1978), Corrigan and Goddard (1981), Prasad (1981) and Prasad and Rossi (1980).

Throughout, we work in complexified euclidean space-time $\mathbb{C}^4$, and we consider complexified gauge fields taking values in $sl(N,\mathbb{C})$, the complexified Lie algebra of $SU(N)$. We denote the Killing form on $sl(N,\mathbb{C})$ by:

$$\begin{aligned} \langle A,B\rangle &= -2\mathrm{Tr}AB \\ ||A||^2 &= \langle A,A\rangle \end{aligned} \qquad A,B\,\varepsilon\, sl(N,\mathbb{C})$$

and we use the convention for gauge transformations:

$$^{G}A_\mu = G^{-1}A_\mu G+G^{-1}\partial_\mu G,\ A_\mu\,\varepsilon\, sl(N,\mathbb{C}),\ G\,\varepsilon\, SL(N,\mathbb{C})$$

We define Yang-variables for $x_\mu\,\varepsilon\,\mathbb{C}^4$:

$$y = \frac{1}{\sqrt{2}} (x_1+ix_2) \qquad \bar{y} = \frac{1}{\sqrt{2}}(x_1-ix_2)$$

$$z = \frac{1}{\sqrt{2}} (x_3-ix_4) \qquad \bar{z} = \frac{1}{\sqrt{2}}(x_3+ix_4)$$

with respect to which the self-duality equations become (Yang 1977):

$$\begin{aligned} &F_{yz} = F_{\bar{y}\bar{z}} = 0 \\ &F_{y\bar{y}} + F_{z\bar{z}} = 0 \end{aligned} \qquad (2.1.1)$$

By (2.1.1)(a), A_y, A_z are pure gauge for fixed $\bar{y}$, $\bar{z}$, and $A_{\bar{y}}$, $A_{\bar{z}}$ are pure gauge for fixed y, z; hence there exist two matrix functions -

$$D,\bar{D} : \mathbb{C}^4 \to SL(N,\mathbb{C})$$

called <u>generating matrices</u>, such that:

$$A_y = D^{-1}D_{,y} \qquad A_z = D^{-1}D_{,z}$$

$$A_{\bar{y}} = \bar{D}^{-1}D_{,\bar{y}} \qquad A_{\bar{z}} = \bar{D}^{-1}\bar{D}_{,\bar{z}}$$

Gauge transformations $A_\mu \to {}^GA_\mu$ induce transformations -

$$D \to DG, \qquad \bar{D} \to \bar{D}G$$

and for fixed $A_y, A_{\bar{y}}, A_z, A_{\bar{z}}$, the matrices $D, \bar{D}$ are determined up to a transformation:

$$D \to \bar{V}(\bar{y},\bar{z})D, \qquad \bar{D} \to V(y,z)\bar{D}$$

where $V,\bar{V}$ are arbitrary $SL(N,\mathbb{C})$-valued functions of the variables indicated.

Now define $J = D\bar{D}^{-1}$. This is clearly a gauge-invariant $SL(N,\mathbb{C})$-valued function, which transforms under V-transformations as:

$$J \to \bar{V}(\bar{y},\bar{z})JV^{-1}(y,z)$$

Also, equ (2.1.1)(b) is equivalent to a chiral model like equation for J:

$$\begin{aligned} &(J_{,\bar{y}}J^{-1})_{,y}+(J_{,\bar{z}}J^{-1})_{,z} = 0 \\ \Leftrightarrow \quad &(J^{-1}J_{,y})_{,\bar{y}}+(J^{-1}J_{,z})_{,\bar{z}} = 0 \end{aligned} \qquad (2.1.2)$$

We now specialize to $SL(2,\mathbb{C})$ gauge fields. Using the gauge freedom for the D-matrices, we may choose the matrices D, $\bar{D}$ to be lower and upper triangular respectively:

$$D = \frac{1}{\sqrt{\phi}}\begin{pmatrix} 1 & 0 \\ \rho & \phi \end{pmatrix}, \qquad \bar{D} = \frac{1}{\sqrt{\phi}}\begin{pmatrix} \phi & -\bar{\rho} \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad J = \frac{1}{\phi}\begin{pmatrix} 1 & \bar{\rho} \\ \rho & \phi^2+\rho\bar{\rho} \end{pmatrix}$$

This is called <u>Yang's R-gauge</u>. $\phi, \rho, \bar{\rho}$ are independent complex valued functions of $y, z, \bar{y}, \bar{z}$. Reality requires:

$$\phi \doteq \phi^*, \qquad \bar{\rho} \doteq \rho^* \tag{2.1.3}$$

where $\doteq$ means equal on $R^4 \subset C^4$.

In R-gauge, the potentials are given by:

$$A_y = \frac{-1}{2\phi}\begin{pmatrix} \phi,_y & 0 \\ 2\rho,_y & -\phi,_y \end{pmatrix}, \quad A_z = \frac{-1}{2\phi}\begin{pmatrix} \phi,_z & 0 \\ 2\rho,_z & -\phi,_z \end{pmatrix}$$

$$A_{\bar{y}} = \frac{1}{2\phi}\begin{pmatrix} \phi,_{\bar{y}} & 2\bar{\rho},_{\bar{y}} \\ 0 & -\phi,_{\bar{y}} \end{pmatrix}, \quad A_{\bar{z}} = \frac{1}{2\phi}\begin{pmatrix} \phi,_{\bar{z}} & 2\bar{\rho},_z \\ 0 & -\phi,_{\bar{z}} \end{pmatrix}$$

and equations (2.1.1)(b), (2.1.2) are equivalent to <u>Yang's R-gauge equations</u>:

$$(\partial_y\partial_{\bar{y}}+\partial_z\partial_{\bar{z}})\log\phi+\frac{1}{\phi^2}(\rho,_y\bar{\rho},_{\bar{y}}+\rho,_z\bar{\rho},_{\bar{z}}) = 0$$

$$(\rho,_y/\phi^2),_{\bar{y}}+(\rho,_z/\phi^2),_{\bar{z}} = 0 \tag{2.1.4}$$

$$(\bar{\rho},_{\bar{y}}/\phi^2),_y+(\bar{\rho},_{\bar{z}}/\phi^2),_z = 0$$

2.1.1 <u>Theorem (Corrigan, Fairlie, Goddard & Yates 1978)</u>

(i) Let $(\phi, \rho, \bar{\rho})$ be a solution of (2.1.4). Then so is $(\phi^I, \rho^I, \bar{\rho}^I)$ where:

$$\phi = \frac{\phi}{\phi^2+\rho\bar{\rho}}, \quad \rho^I = \frac{\bar{\rho}}{\phi^2+\rho\bar{\rho}}, \quad \bar{\rho}^I = \frac{\rho}{\phi^2+\rho\bar{\rho}}$$

and the corresponding potentials are gauge equivalent.

(ii) Let $(\phi, \rho, \bar{\rho})$ be a solution of (2.1.4). Then so is $(\phi^B, \rho^B, \bar{\rho}^B)$ where:

$$\phi^B = 1/\phi$$

$$\rho^B{}_{,y} = -\bar{\rho}_{,z}/\phi^2 \qquad \rho^B{}_{,z} = \bar{\rho}_{,\bar{y}}/\phi^2$$

$$\bar{\rho}^B{}_{,\bar{y}} = \rho_{,z}/\phi^2 \qquad \bar{\rho}^B{}_{,z} = -\rho_{,y}/\phi^2$$

(iii) A solution of (2.1.4) is given by:

$$\rho_{,y} = \phi_{,z} \qquad \rho_{,z} = -\phi_{,\bar{y}}$$

$$\bar{\rho}_{,\bar{y}} = \phi_{,z} \qquad \bar{\rho}_{,\bar{z}} = -\phi_{,y}$$

where $(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})\phi = 0$

<u>Notes</u>

(1) Solution (iii) is just the CFTW ansatz, and constitutes the first of the Atiyah-Ward ansätze a_1.

(2) The transformations B,I are Bäcklund transformations.

They are separately involutive ie $B^2 = I^2 =$ identity. However, the transformation BI is non-involutive and gives rise to a sequence of distinct ansätze

$$a_1 \xrightarrow{BI} a_2 \xrightarrow{BI} a_3 \xrightarrow{BI} \cdots\cdots \xrightarrow{BI} a_n \xrightarrow{BI} \cdots\cdots$$

called the <u>Atiyah-Ward ansätze</u>.

(iv) The BI-transformations can be integrated explicitly as follows:

Suppose we have a sequence of functions $\Delta_k(x)$ of length $2n + 1$, $-n \leq k \leq n$, which satisfy the Cauchy-Riemann like equations:

$$\partial_y \Delta_k = -\partial_{\bar{z}} \Delta_{k+1}\,, \qquad \partial_z \Delta_k = \partial_{\bar{y}} \Delta_{k+1} \tag{2.1.5}$$

$$\Rightarrow \qquad (\partial_y \partial_{\bar{y}} + \partial_z \partial_{\bar{z}}) \Delta_k = 0, \ \forall k$$

We call (Δ_k) a <u>Δ-chain</u>, and (2.1.5) the <u>Δ-chain equations</u>. Define the <u>fundamental nxn matrix</u> $D^{(n)} = (\Delta_{j-i})$

ie $$D^{(n)} = \begin{pmatrix} \Delta_o & \Delta_1 & \Delta_2 & \cdots & \Delta_{n-1} \\ \Delta_{-1} & \Delta_o & \Delta_1 & & \vdots \\ \Delta_{-2} & \Delta_{-1} & \Delta_o & & \vdots \\ \vdots & & & \ddots & \vdots \\ \Delta_{-n+1} & \cdots & \cdots & \cdots & \Delta_o \end{pmatrix}$$

Then, a solution in the a_n ansatz, in Yang's R-gauge, is given by:

$$\phi_n = \det D^{(n)}/\det D^{(n-1)}$$

$$\rho_n = \frac{(-1)^n}{\det D^{(n-1)}} \begin{vmatrix} \Delta_{-1} & \Delta_o & \Delta_1 & \cdots & \Delta_{n-2} \\ \Delta_{-2} & \Delta_{-1} & \Delta_o & & \vdots \\ \Delta_{-3} & \Delta_{-2} & \Delta_{-1} & & \vdots \\ \vdots & & & \ddots & \vdots \\ \Delta_{-n} & \cdots & \cdots & \cdots & \Delta_{-1} \end{vmatrix}$$

$$\bar{\rho}_n = \frac{(-1)^{n-1}}{\det D^{(n-1)}} \begin{vmatrix} \Delta_1 & \Delta_2 & \Delta_3 & \cdots & \Delta_n \\ \Delta_o & \Delta_1 & \Delta_2 & & \vdots \\ \Delta_{-1} & \Delta_o & \Delta_1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ \Delta_{-n+2} & \cdots & \cdots & \cdots & \Delta_1 \end{vmatrix} \qquad (2.1.6)$$

Note that non-singularity is guaranteed if $\det D^{(n)}$ is non-vanishing throughout space-time.

Now, in the construction of monopole solutions, the reduction to $|R^3$ was performed by demanding that the Δ-chain take the form:

$$\Delta_k(x) = e^{iax_4}\, \tilde{\Delta}_k(x_1,x_2,x_3) \qquad (2.1.7)$$

$$\Rightarrow \qquad \phi = e^{iax_4}\tilde{\phi}, \quad \rho = e^{iax_4}\tilde{\rho}, \quad \bar{\rho} = e^{iax_4}\tilde{\bar{\rho}}$$

where $\partial_4\tilde{\phi} = \partial_4\tilde{\rho} = \partial_4\tilde{\bar{\rho}} = 0$

In this case, the fact that Δ_k satisfies the 4-dimensional Laplace equation implies that $\tilde{\Delta}_k$ satisfies the 3-dimensional Helmholtz equation.

$$\nabla^2\tilde{\Delta}_k = a^2\tilde{\Delta}_k$$

The following result guarantees that, provided the conditions of reality and non-singularity are satisfied, then solutions of the form (2.1.7), in the a_n ansatz, describe magnetic monopole configurations of charge n.

Superposition Theorem (Prasad 1981)

Suppose that (2.1.7) is satisfied for $(\phi_k, \rho_k, \bar{\rho}_k)$ in the ansatz a_k, k=1, 2, ...

Then in the nth ansatz:

(i) $$||\Phi||^2 = ||A_4||^2 = a^2 - \sum_{k=1}^{n} \nabla^2 \ln\phi_k$$

$\Rightarrow$ $$||\Phi||^2 = a^2 - \nabla^2 \ln \det D^{(n)}$$

(ii) Energy density

$$\xi = +\tfrac{1}{2}\nabla^2||\Phi||^2 = -\tfrac{1}{2}\nabla^2\nabla^2 \ln \det D^{(n)}$$

(iii) Total Energy, $E_n = 4\pi a n$

2.2 Dimensional Reduction to $\mathbb{R}^2$

Let us now consider the dimensional reduction of a pure SU(2) Yang-Mills gauge theory from $\mathbb{R}^4$ to $\mathbb{R}^2$, requiring the theory to be translation invariant in the extra two dimensions.

Write $A_3 = \Phi_1$, $A_4 = \Phi_2$, and impose the condition $\partial_3 \equiv \partial_4 \equiv 0$. Then we have:

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2], \quad F_{34} = [\Phi_1, \Phi_2]$$

$$F_{13} = D_1 \Phi_1 \qquad , \quad F_{14} = D_1 \Phi_2$$

$$F_{23} = D_2 \Phi_1 \qquad , \quad F_{24} = D_2 \Phi_2$$

Hence the Yang-Mills Lagrangian becomes:

$$L = \tfrac{1}{2}||B||^2 + \tfrac{1}{2}||D_i \Phi_1||^2 + \tfrac{1}{2}||D_i \Phi_2||^2 + \tfrac{1}{2}||[\Phi_1, \Phi_2]||^2$$

where $B = F_{12}$, $i = 1,2$. So, we have an SU(2) gauge field A_i interacting with two adjoint Higgs fields Φ_1, Φ_2 with an extra interaction term:

$$V(\Phi_1, \Phi_2) = \tfrac{1}{2}||[\Phi_1, \Phi_2]||^2$$

This model has also been considered by Nielsen and Olesen (1973), and Lohe (1977). We shall in fact construct solutions with the non-trivial monopole-like boundary conditions:

$$||\Phi_1||^2 \to a^2, \quad ||\Phi_2||^2 \to b^2 \text{ as } (x_1,x_2) \to \infty$$

so we should regard the potential V as the Bogomol'nyi limit $\lambda \to 0$ of the potential:

$$V'(\Phi_1,\Phi_2) = \tfrac{1}{2}||[\Phi_1,\Phi_2]||^2 + \frac{\lambda}{4}(||\Phi||^2 - c^2)^2$$

where $||\Phi||^2 = ||\Phi_1||^2 + ||\Phi_2||^2$, $c^2 = a^2+b^2 \neq 0$, $(c>0)$.

The self-duality and anti-self-duality equations reduce to Bogomol'nyi-like equations for this model:

$$B = \pm[\Phi_1,\Phi_2]$$

$$D_1\Phi_1 \pm D_2\Phi_2 = 0 \qquad (2.2.1)$$

$$D_1\Phi_2 \mp D_2\Phi_1 = 0$$

So, we shall seek solutions of equations (2.2.1) which are essentially the 2-dimensional analogues of self-dual monopoles. The reduction to $\mathbb{R}^2$ is performed in exact analogy with the monopole situation, by requiring that:

$$\Delta_k(x) = e^{i(ax_3+bx_4)}\tilde{\Delta}_k(x_1,x_2)$$

$$\Rightarrow \quad \phi = e^{i(ax_3+bx_4)}\tilde{\phi}, \quad \rho = e^{i(ax_3+bx_4)}\tilde{\rho}, \tag{2.2.2}$$

$$\bar{\rho} = e^{i(ax_3+bx_4)}\tilde{\bar{\rho}}$$

where $\partial_3\tilde{\phi} = \partial_4\tilde{\phi} = 0$ etc

Prasad's Superposition Theorem has some immediate corollaries for the ansatz (2.2.2).

2.2.1 <u>Corollary</u>

Suppose equ (2.2.2) is satisfied. Then the norms of the Higgs fields Φ_1,Φ_2, and the energy density are given, in the nth ansatz by:

$$||\Phi_1||^2 = a^2-\nabla^2\ln\det D^{(n)}, \quad ||\Phi_2||^2 = b^2-\nabla^2\ln\det D^{(n)}$$

$$\Rightarrow \quad ||\Phi||^2 = c^2-2\nabla^2\ln\det D^{(n)}$$

$$\xi = +\tfrac{1}{2}\nabla^2||\Phi_1||^2 = +\tfrac{1}{2}\nabla^2||\Phi_2||^2 = -\tfrac{1}{2}\nabla^2\nabla^2\ln\det D^{(n)}$$

Note that we can replace (a,b) by any 2-vector of length c, by performing an appropriate rotation in the x_3x_4 plane.

Since each Δ_k satisfies the 4-dimensional Laplace equation, equ (2.2.2) implies that each $\tilde{\Delta}_k$ satisfies the 2-dimensional Helmholtz equation:

$$\nabla^2 \tilde{\Delta}_k = c^2 \tilde{\Delta}_k$$

So, if $c>0$, we can expand each $\tilde{\Delta}_k$ in cylindrical Bessel functions:

$$\tilde{\Delta}_k = \sum_{\ell=0}^{\infty} (A_{k,\ell} I_\ell(cr) + B_{k,\ell} K_\ell(cr)) e^{i\ell\theta}$$

where (r,θ) are cylindrical polar co-ordinates in the $x_1 x_2$-plane.

For the most part, we shall consider solutions where each $\tilde{\Delta}_k$ is non-singular. We then have:

$$\tilde{\Delta}_k = \sum_{\ell=-\infty}^{\infty} A_{k,\ell} I_\ell(cr) e^{i\ell\theta} \qquad (2.2.3)$$

$$\sim \frac{e^{cr}}{\sqrt{cr}} \left(f_k(\theta) + \sum_{\ell=1}^{\infty} \frac{f_{k,\ell}(\theta)}{r^\ell}\right), \text{ as } r \to \infty$$

from the asymptotic expansion of $I_\ell(x)$ (Appendix A, equ (A.5)).

2.2.2 <u>Corollary</u>

Suppose the following hypotheses hold:

(i) Equ (2.2.2) is satisfied.

(ii) Each $\tilde{\Delta}_k$ is non-singular ie equ (2.2.3) is satisfied.

(iii) $\mathrm{Det}\,\mathcal{D}^{(n)} \neq 0 \quad \forall x \in \mathbb{R}^2$, so the gauge field configuration is non-singular.

Then, in the nth ansatz:

$$||\Phi_1||^2 \sim a^2 - \frac{nc}{r}, \quad ||\Phi_2||^2 \sim b^2 - \frac{nc}{r}$$

$$\Rightarrow \quad ||\Phi||^2 \sim c^2\left(1 - \frac{2n}{cr}\right) \text{ as } r \to \infty \tag{2.2.4}$$

$$\xi \sim -nc/2r^3 \text{ as } r \to \infty$$

Also, if E(R) is the total energy (or total action) in a disc of radius R centred at the origin, then -

$$E(R) \sim \frac{n\pi c}{R} \to 0 \text{ as } R \to \infty \tag{2.2.5}$$

Hence the total energy, or total action in the x_1x_2-plane is zero.

If, moreover, the associated gauge field is <u>real</u>, then -

$$||\Phi||^2 \equiv c^2 \text{ and } \xi \equiv 0$$

Hence, any solutions satisfying hypotheses (i)-(iii) are either strictly complex, or the Higgs vacuum.

Proof

Hypotheses (i), (ii), (iii) imply, using the asymptotic expansion of equ (2.2.3):

$$\det D^{(n)} \sim e^{in(ax_3+bx_4)} \frac{e^{ncr}}{(cr)^{n/2}} \frac{\delta^{(n)}(\theta)}{r^{\ell}}, \text{ as } r\to\infty$$

where $\delta^{(n)}(\theta)$ is a non-vanishing, non-singular function of θ only, and $\ell \geqq 0$. Hence:

$$\ln \det D^{(n)} \sim ncr + O(\ln r)$$

$$\Rightarrow \quad \nabla^2 \ln \det D^{(n)} \sim \frac{nc}{r} \qquad (\because \nabla^2 \ln r = 0,\ r > 0)$$

$$\Rightarrow \quad \nabla^2\nabla^2 \ln \det D^{(n)} \sim \frac{nc}{r^2}, \text{ as } r\to\infty$$

and equs (2.2.4) follow from Corollary (2.2.1). Also:

$$E(R) = \int_{|x|\leqq R} d^2x\,\xi = \tfrac{1}{2}\int_{|x|\leqq R} d^2x\,\nabla^2 \|\Phi_1\|^2$$

$$= \tfrac{1}{2}\int_{|x|=R} d\underset{\sim}{n}\cdot\underset{\sim}{\nabla} \|\Phi_1\|^2$$

$$\sim n\pi c/R, \text{ as } R\to\infty, \text{ by } (2.2.4).$$

Finally, for real gauge fields, $\xi \geqq 0$ everywhere, since the Killing form on the compact group SU(2) is positive definite. Hence -

$$\int d^2x\, \xi(x) = 0 \quad \Rightarrow \quad \xi \equiv 0 \text{ on } |R^2$$

and

$$\xi = +\frac{1}{4} \nabla^2 ||\Phi||^2, \quad ||\Phi||^2 \to c^2 \text{ as } r \to \infty$$

$$\Rightarrow \quad ||\Phi||^2 \equiv c^2 \text{ on } |R^2$$

QED

In fact, equs (2.2.4) tell us that, for $c > 0$, ξ is negative at sufficiently large distances. This again implies immediately that the associated gauge field is strictly complex. All solutions satisfying hypotheses (i) - (iii) of Corollary (2.2.2) will turn out to be soliton-like enhancements of positive energy density immersed in a sea of negative energy density, in such a way that the total energy integrates to zero. The existence of such solutions was first pointed out by Dolan (1978), who coined the term "voidon".

In addition to the rather strong non-singularity conditions (ii) and (iii), Corollary (2.2.2) supposes that the given self-dual solution arises from one of the a_n ansätze described in theorem (2.1.1) (or, equivalently, that the transition function of its associated holomorphic vector bundle over $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ is equivalent to one which is upper triangular - see Chapter 5.1). This is known to be true for instantons and monopoles, but still requires proof for solutions satisfying our boundary conditions. It is therefore of interest to know what can be said about an arbitrary smooth solution of equs (2.2.1). Some information is provided by the next result.

2.2.3 Theorem (Lohe 1977, Saclioglu 1981)

Let $(A_1, A_2, \Phi_1, \Phi_2)$ be a solution of equs (2.2.1), which is smooth on $\mathbb{R}^2$. Then its total action can be written as a contour integral:

$$S = \pm \lim_{R \to \infty} \oint_{|x|=R} d\ell_j \langle \Phi_2, D_j \Phi_1 \rangle \qquad (2.2.6)$$

Proof

The energy density is given by:

$$\xi = \tfrac{1}{2}(||B||^2 + ||[\Phi_1, \Phi_2]||^2 + ||D_i \Phi_1||^2 + ||D_i \Phi_2||^2)$$

$$= \tfrac{1}{2}(||B \mp [\Phi_1, \Phi_2]||^2 + ||D_1 \Phi_1 \pm D_2 \Phi_2||^2 + ||D_2 \Phi_1 \mp D_1 \Phi_2||^2)$$

$$\pm(\langle B, [\Phi_1, \Phi_2]\rangle - \langle D_1 \Phi_1, D_2 \Phi_2\rangle + \langle D_2 \Phi_1, D_1 \Phi_2\rangle)$$

So if, say, the field is self-dual:

$$\xi = \langle B, [\Phi_1, \Phi_2]\rangle - \langle D_1 \Phi_1, D_2 \Phi_2\rangle + \langle D_2 \Phi_1, D_1 \Phi_2\rangle$$

Expanding this in terms of A_1, A_2, Φ_1, Φ_2, it can be written as a sum of three groups of terms.

(i) $\xi = \langle \partial_2 \Phi_1, \partial_1 \Phi_2 \rangle - \langle \partial_1 \Phi_1, \partial_2 \Phi_2 \rangle$

$+ \langle \partial_1 A_2 - \theta_2 A_1, [\Phi_1, \Phi_2] \rangle$

(ii) $- \langle \partial_1 \Phi_1, [A_2, \Phi_2] \rangle - \langle [A_1, \Phi_1], \partial_2 \Phi_2 \rangle$

$+ \langle \partial_2 \Phi_1, [A_1, \Phi_2] \rangle + \langle [A_2, \Phi_1], \partial_1 \Phi_2 \rangle$

(iii) $+ \langle [A_1, A_2], [\Phi_1, \Phi_2] \rangle - \langle [A_1, \Phi_1], [A_2, \Phi_2] \rangle$

$+ \langle [A_2, \Phi_1], [A_1, \Phi_2] \rangle$

<u>Group (iii)</u>. Note the following identity (which is a consequence of the cyclic property of trace).

(*) $\langle A, [B,C] \rangle = \langle C, [A,B] \rangle = \langle B, [C,A] \rangle$

This, together with the Jacobi identity, gives:

$$\langle [A_1, A_2], [\Phi_1, \Phi_2] \rangle = \langle \Phi_1, [\Phi_2, [A_1, A_2]] \rangle$$

$$= \langle \Phi_1, [[\Phi_2, A_1], A_2] + [A_1, [\Phi_2, A_2]] \rangle \qquad \text{(Jacobi)}$$

$$= \langle [\Phi_2, A_1], [A_2, \Phi_1] \rangle + \langle [\Phi_2, A_2], [\Phi_1, A_1] \rangle \quad (\text{by}(*))$$

$$= \langle [A_1, \Phi_1], [A_2, \Phi_2] \rangle - \langle [A_1, \Phi_2], [A_2, \Phi_2] \rangle$$

Hence, group (iii) vanishes.

Group (i) $= \partial_1\langle\Phi_2,\partial_2\Phi_1\rangle - \partial_2\langle\Phi_2,\partial_1\Phi_1\rangle$ (**)

Group (ii). Rearranging, using (*), this equals:

$$\langle\partial_1 A_2 - \partial_2 A_1, [\Phi_1,\Phi_2]\rangle$$

$$+ \langle A_2,[\partial_1\Phi_1,\Phi_2]\rangle + \langle A_2,[\Phi_1,\partial_1\Phi_2]\rangle$$

$$- \langle A_2,[\Phi_1,\partial_2\Phi_2]\rangle - \langle A_1,[\partial_2\Phi_1,\Phi_2]\rangle$$

$$= \partial_1\langle A_2,[\Phi_1,\Phi_2]\rangle - \partial_2\langle A_1,[\Phi_1,\Phi_2]\rangle$$

$$= \partial_1\langle\Phi_2,[A_2,\Phi_1]\rangle - \partial_2\langle\Phi_2,[A_1,\Phi_1]\rangle \qquad (***)$$

Adding (**) and (***) we obtain:

$$\xi = \partial_1\langle\Phi_2,D_2\Phi_1\rangle - \partial_2\langle\Phi_2,D_1\Phi_1\rangle$$

$$= \varepsilon_{ij}\partial_i\langle\Phi_2,D_j\Phi_1\rangle \qquad (2.2.7)$$

Hence (2.2.6) is an immediate consequence of Stokes' theorem, since, by hypothesis, ξ has no singularities on $|R^2$. QED

2.2.4 Corollary

Let (A_1,A_2,Φ_1,Φ_2) be any real, smooth, finite action solution of equs (2.2.1), for any compact gauge group G, satisfying the boundary conditions:

$$||\Phi_1||^2 \to a^2, \quad ||\Phi_2||^2 \to b^2 \quad \text{as } r\to\infty$$

Then it is the vacuum, ie

$$\xi \equiv 0, \quad ||\phi_1||^2 \equiv a^2, \quad ||\Phi_2||^2 \equiv b^2$$

Proof

The finite action constraint, together with the positive definiteness of the Killing form for a compact group, forces the solution to approach the Higgs vacuum sufficiently quickly at infinity. In particular:

$$||D_i\Phi_j||^2 = 0\left(\frac{1}{r^2}\right) \Rightarrow ||D_i\Phi_j|| = 0\left(\frac{1}{r}\right) \quad \text{as } r\to\infty$$

Also, (2.2.6) gives us:

$$|S| \leq \lim_{R\to\infty} \oint_{|x|=R} d\ell_j ||\Phi_2|| \; ||D_j\Phi_1||$$

$$\to 0 \qquad\qquad \text{as } R\to\infty$$

from the boundary conditions on $||\Phi_i||$ and $||D_i\Phi_j||$.

Hence $S = 0$, and the conclusion follows as in Corollary (2.2.2).

QED

Thus, we have proved conclusively, that in order to construct non-trivial non-singular solutions of equs (2.2.1) it is necessary to drop the reality constraint. This will be done in the following, where we shall construct a wealth of non-singular complex solutions, with some properties rather analogous to those of the real Nielsen-Olesen vortices of the Abelian Higgs model.

Effective U(1) Theory

In this section, the R-gauge equations are rewritten in a form where the vortex interpretation of the forthcoming solutions becomes apparent. It is useful at this stage to note two facts which will emerge from the twistor theoretic treatment of Chapter 5.

Fact 1. (Corollary of Theorem (5.2.3)). Suppose that $(\phi,\rho,\bar{\rho})$, $(\phi',\rho',\bar{\rho}')$ are solutions of the self-duality equations obtained from one of the Atiyah-Ward ansätze, after reducing from $\mathbb{R}^4$ to $\mathbb{R}^2$ by the imposition of equ (2.2.2). Then these solutions are gauge equivalent if and only if their corresponding Δ-chains are equivalent up to a scale transformation

ie $\quad \Delta_k'(x) = \lambda\Delta_k(x), \quad \forall k, \ \lambda\in\mathbb{C}\setminus 0$

Such a transofmation clearly leaves $(\phi,\rho,\bar{\rho})$ unchanged (Cf equ (2.1.6)); hence, after reduction from $|R^4$ to $|R^2$, the R-gauge functions $(\phi,\rho,\bar{\rho})$ are gauge invariant scalars. Hence, any new fields defined in terms of $(\phi,\rho,\bar{\rho})$ will also be gauge invariant.

Fact 2. We shall see in Chapter 5 (Theorem (5.3.1)) that it is possible to construct a large family of manifestly non-singular solutions which satisfy the condition -

$$\Delta_k(x) = e^{i(ax_3+bx_4)}\tilde{\Delta}_k(x_1,x_2)$$

$$\text{where } \tilde{\Delta}_{-k} = \tilde{\Delta}_k{}^*, \quad \forall k \tag{2.2.8}$$

This condition implies that -

$$\det D^{(n)} = e^{in(ax_3+bx_4)}\det\tilde{D}^{(n)} \tag{2.2.9}$$

where $\tilde{D}^{(n)} = (\tilde{\Delta}_{j-i})$ is a hermitian nxn matrix, and hence has real determinant. Non-singularity is a consequence of the additional result:

$$\det\tilde{D}^{(n)} > 0, \qquad \forall x\varepsilon|R^2 \tag{2.2.10}$$

Also, substituting (2.2.8) into equs (2.1.6) yields:

$$\tilde{\phi}_n = \frac{\det\tilde{D}^{(n)}}{\det\tilde{D}^{(n-1)}} \tag{2.2.11}$$

$$\tilde{\rho}_n = (-1)^n \frac{\det \tilde{E}^{(n)*}}{\det \tilde{D}^{(n-1)}}$$

$$\tilde{\bar{\rho}}_n = (-1)^{n-1} \frac{\det \tilde{E}^{(n)}}{\det \tilde{D}^{(n-1)}}$$

where

$$\tilde{E}^{(n)} = \begin{pmatrix} \tilde{\Delta}_1 & \tilde{\Delta}_2 & \tilde{\Delta}_3 \cdots & \tilde{\Delta}_n \\ \tilde{\Delta}_o & \tilde{\Delta}_1 & \tilde{\Delta}_2 & \vdots \\ \tilde{\Delta}_{-1} & \tilde{\Delta}_o & \tilde{\Delta}_1 & \vdots \\ \vdots & & \ddots & \vdots \\ \tilde{\Delta}_{-n+2} & \cdots & \cdots & \tilde{\Delta}_1 \end{pmatrix}$$

and these imply -

$$\text{(i)} \quad \tilde{\phi}_n > 0 \qquad \forall x \in \mathbb{R}^2$$

$$\text{(ii)} \quad \tilde{\bar{\rho}}_n = -\tilde{\rho}_n^* \tag{2.2.12}$$

Note that (2.2.12) (ii) is distinctly different from the reality condition (2.1.3).

In terms of $(\tilde{\phi}, \tilde{\rho}, \tilde{\bar{\rho}})$, the R-gauge equations become (writing $m^2 = c^2/2$):

$$\partial_y \partial_{\bar{y}} \log \tilde{\phi} + \frac{1}{\tilde{\phi}^2} (\tilde{\rho},_y \tilde{\bar{\rho}},_{\bar{y}} - m^2 \tilde{\rho}\tilde{\bar{\rho}}) = 0 \tag{2.2.13}$$

$$(\tilde{\rho}_{,y}/\tilde{\phi}^2)_{,\bar{y}} + m^2(\tilde{\rho}/\tilde{\phi}^2) = 0$$

$$(\tilde{\bar{\rho}}_{,\bar{y}}/\tilde{\phi}^2)_{,y} + m^2(\tilde{\bar{\rho}}/\tilde{\phi}^2) = 0$$

Now, using (2.2.9) (i), we can write

$\tilde{\phi} = e^{\chi}$, χ real valued

in terms of which equs (2.2.13) become (dropping the tildes on ρ, $\bar{\rho}$):

$$\partial_y \partial_{\bar{y}} \chi + e^{-2\chi}(\rho_{,y}\bar{\rho}_{,\bar{y}} - m^2 \rho\bar{\rho}) = 0 \qquad (2.2.14)$$

$$\rho_{,y\bar{y}} + m^2\rho = 2(\partial_{\bar{y}}\chi)\rho_{,y}$$

$$\bar{\rho}_{,y\bar{y}} + m^2\bar{\rho} = 2(\partial_y\chi)\bar{\rho}_{,\bar{y}}$$

and these could be regarded as equations for a rather unconventional field theory involving a real Liouville like scalar field χ interacting with a complex scalar field ρ.

A yet more suggestive way of writing these equations is provided by defining the following fields (making use of equs (2.2.12) (ii)):

$$\Psi = e^{-\chi}\tilde{\rho} = \tilde{\rho}/\tilde{\phi}\ , \quad \Psi^* = -e^{-\chi}\tilde{\bar{\rho}} = -\tilde{\bar{\rho}}/\tilde{\phi} \qquad (2.2.15)$$

$$A_y^{ab} = i\partial_y \chi = i\,\partial_y \tilde{\phi}/\tilde{\phi}, \qquad A_{\bar{y}}^{ab} = -i\,\partial_{\bar{y}}\chi = -i\,\partial_{\bar{y}}\tilde{\phi}/\tilde{\phi}$$

$$B^{ab} = \partial_1 A_2^{ab} - \partial_2 A_1^{ab} = -\nabla^2\chi$$

so that A_y^{ab}, $A_{\bar{y}}^{ab}$ represent an 'effective abelian' gauge field such that $\partial_y A_{\bar{y}}^{ab} + \partial_{\bar{y}} A_y^{ab} = 0$ (Lorentz gauge).

Then equs (2.2.14) are equivalent to:

$$-\frac{B^{ab}}{2} = (D_y^{ab}\Psi)(D_y^{ab}\Psi)^* - m^2\Psi\Psi^* \tag{2.2.16}$$

$$D_y^{ab} D_{\bar{y}}^{ab}\Psi + m^2\Psi = 0$$

where $D_y^{ab} = \partial_y - iA_y^{ab}$, $D_{\bar{y}}^{ab} = \partial_{\bar{y}} - iA_{\bar{y}}^{ab}$, so it is natural to regard these as Bogomol'nyi like equations for an unconventional U(1) gauge theory interacting with a complex scalar Higgs field . We shall see that, in the nth ansatz, ψ satisfies the expected boundary conditions for a vortex of charge n.

Note that, in Cartesian co-ordinates, equs (2.2.16) take the manifestly covariant form:

$$-B^{ab} = (D_i^{ab}\Psi)(D_i^{ab}\Psi)^* + i\,\varepsilon_{ij}(D_i^{ab}\Psi)(D_j^{ab}\Psi)^* - c^2\Psi\Psi^* \tag{2.2.17}$$

$$D_i^{ab} D_i^{ab}\Psi + iB^{ab}\Psi + c^2\Psi = 0$$

Finally, let us note how the conventional SU(2) gauge potentials are related to the effective abelian fields. From the expressions for the gauge potentials in Yang's R-gauge we have:

$$A_y = \frac{i}{2}\begin{pmatrix} A_y^{ab} & 0 \\ -2D_y^{ab}\psi & -A_y^{ab} \end{pmatrix}$$

$$A_{\bar{y}} = \frac{i}{2}\begin{pmatrix} A_{\bar{y}}^{ab} & -2(D_y^{ab}\psi)^* \\ 0 & -A_{\bar{y}}^{ab} \end{pmatrix} \qquad (2.2.18)$$

$$A_z = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2) = -\frac{i\gamma}{2\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 2\psi & -1 \end{pmatrix}$$

$$A_{\bar{z}} = \frac{1}{\sqrt{2}}(\Phi_1 - i\Phi_2) = \frac{i\bar{\gamma}}{2\sqrt{2}}\begin{pmatrix} 1 & -2\psi^* \\ 0 & -1 \end{pmatrix} \qquad (2.2.19)$$

Using equ (2.2.19), we can relate $||\Phi||^2$ and ξ to the effective Higgs field as follows:

$$||\Phi||^2 = -2\mathrm{Tr}(\Phi_1^2 + \Phi_2^2) = c^2(2\psi\psi^* - 1) \qquad (2.2.20)$$

$$\Rightarrow \qquad \xi = -\frac{1}{4}\nabla^2||\Phi||^2 = -\frac{c^2}{2}\nabla^2(\psi\psi^*)$$

By (2.2.20), fields satisfying the hypotheses of Corollary (2.2.2) must have:

$$|\Psi| \sim 1 - \frac{N}{2cr} \to 1, \text{ as } r\to\infty \qquad (2.2.21)$$

2.3 Non-Singular Axially Symmetric N-Vortices

The non-singular axially symmetric N-vortex solutions are constructed in a manner analogous to Prasad's construction of the axially symmetric N-monopole solutions (Prasad 1981, Prasad & Rossi 1980). We find that, as is also the case of monopoles, for $N>1$, the energy density is concentrated in an annulus whose radius increases as N increases. In contrast, however, the construction of the axially symmetric N-vortices is much simpler than that for monopoles; the N-vortex solution will be obtained simply by applying the BI-transformations N times to the a_1 ansatz for the single vortex.

Recall that the BPS monopole is constructed in the a_1 ansatz by first defining:

$$\phi_1 = e^{iax_4}\tilde{\Delta}_o(x_1,x_2,x_3), \qquad \nabla^2\tilde{\Delta}_o = a^2\tilde{\Delta}_o$$

To obtain a spherically symmetric, non-singular field configuration, $\tilde{\Delta}_o$ is then chosen to be the non-vanishing spherically symmetric solution of the 3-dimensional Helmholtz equation, namely:

$$\tilde{\Delta}_o = \sinh ar/r$$

Similarly, to construct the 1-vortex solution, we define, in the a_1 ansatz:

$$\phi_1 = e^{i(ax_3+bx_4)}\tilde{\Delta}_o(x_1,x_2), \qquad \nabla^2\tilde{\Delta}_o = c^2\tilde{\Delta}_o \qquad (2.3.1)$$

and choose $\tilde{\Delta}_o$ to be the non-vanishing axially symmetric solution of the 2-dimensional Helmholtz equation, namely:

$$\tilde{\Delta}_o = I_o(cr) \qquad (2.3.2)$$

To obtain the corresponding gauge field configuration, let us write down the CFTW ansatz with the choice (2.3.1):

$$A_1 = \frac{i}{2}\begin{pmatrix} \partial_2 \ln\tilde{\Delta}_o & a+ib \\ -a+ib & -\partial_2\ln\tilde{\Delta}_o \end{pmatrix} \qquad (2.3.3)$$

$$A_2 = -\frac{i}{2}\begin{pmatrix} \partial_1 \ln\tilde{\Delta}_o & -b+ia \\ b+ia & -\partial_1\ln\tilde{\Delta}_o \end{pmatrix}$$

$$A_3 = -\frac{i}{2}\begin{pmatrix} ib & (\partial_2 - i\partial_1)\ln\tilde{\Delta}_o \\ (\partial_2+i\partial_1)\ln\tilde{\Delta}_o & -ib \end{pmatrix}$$

$$A_4 = -\frac{i}{2}\begin{pmatrix} -ia & (\partial_1 - i\partial_2)\ln\tilde{\Delta}_o \\ (\partial_1+i\partial_2)\ln\tilde{\Delta}_o & +ia \end{pmatrix}$$

Hence (2.3.2) gives us the field configuration:

$$A_1 = \frac{i}{2}\begin{pmatrix} c\,\dfrac{I_1(cr)}{I_o(cr)}\dfrac{x_2}{r} & a+ib \\ -a+ib & -c\,\dfrac{I_1(cr)}{I_o(cr)}\dfrac{x_2}{r} \end{pmatrix} \qquad (2.3.4)$$

$$A_2 = -\frac{i}{2}\begin{pmatrix} c\, I_1(cr)\, \frac{x_1}{r} & -b+ia \\ b+ia & -c\, \frac{I_1(cr)}{I_o(cr)}\, \frac{x_1}{r} \end{pmatrix}$$

$$A_3 = \frac{1}{2}\begin{pmatrix} b & -c\, \frac{I_1(cr)}{I_o(cr)}\, \frac{x_1-ix_2}{r} \\ c\, \frac{I_1(cr)}{I_o(cr)}\, \frac{x_1+ix_2}{r} & -b \end{pmatrix}$$

$$A_4 = -\frac{1}{2}\begin{pmatrix} a & ic\, \frac{I_1(cr)}{I_o(cr)}\, \frac{x_1-ix_2}{r} \\ ic\, \frac{I_1(cr)}{I_o(cr)}\, \frac{x_1+ix_2}{r} & -a \end{pmatrix}$$

Using Prasad's Superposition Theorem with:

$$\det D^{(1)} = e^{i(ax_3+bx_4)}\, I_o(cr)$$

we find:

$$||\Phi_1||^2 = \frac{c^2 I_1(cr)^2}{I_o(cr)^2} - b^2, \qquad ||\Phi_2||^2 = \frac{c^2 I_1(cr)^2}{I_o(cr)^2} - a^2 \tag{2.3.5}$$

$$\Rightarrow \quad ||\Phi||^2 = c^2\left[2\,\frac{I_1(cr)^2-1}{I_o(cr)^2}\right]$$

$$E(R) = 2\pi c^2(cR)\,\frac{I_1(cR)}{I_o(cR)}\left[1-\frac{1}{cR}\,\frac{I_1(cR)}{I_o(cR)} - \frac{I_1(cR)^2}{I_o(cR)^2}\right]$$

Note, from (2.3.5), that $||\Phi||^2$ increases monotonically from $-c^2$ to c^2 as r increases from zero to infinity. This is in contrast with the BPS monopole, where the minimum of the norm of the Higgs field is zero. However, we shall see that the effective complex scalar Higgs field Ψ has the expected behaviour for a charge 1 vortex.

Integration of the Bäcklund Transformations

Recall that we are reducing from $\mathbb{R}^4$ to $\mathbb{R}^2$ by demanding that the Δ-chain take the form of equ (2.2.2).

$$\Delta_k(x) = e^{i(\gamma z+\bar{\gamma}\bar{z})/\sqrt{2}}\tilde{\Delta}_k, \qquad \gamma = a+ib, \qquad \bar{\gamma} = a-ib$$

Hence, in terms of $\tilde{\Delta}_k$, the Δ-chain equs (2.1.5) read:

$$\partial_y\tilde{\Delta}_k = \frac{-i\bar{\gamma}}{\sqrt{2}}\tilde{\Delta}_{k+1}, \qquad \partial_{\bar{y}}\tilde{\Delta}_{k+1} = \frac{i\gamma}{\sqrt{2}}\tilde{\Delta}_k \qquad (2.3.6)$$

In terms of cylindrical polar co-ordinates

$y = \frac{r}{\sqrt{2}} e^{i\theta}$, $\bar{y} = \frac{r}{\sqrt{2}} e^{-i\theta}$, we have:

$$\partial_y = \frac{e^{-i\theta}}{\sqrt{2}}(\partial_r - \frac{i}{r}\partial_\theta), \quad \partial_{\bar{y}} = \frac{e^{i\theta}}{\sqrt{2}}(\partial_r + \frac{i}{r}\partial_\theta)$$

Hence, in (r, θ) co-ordinates, the Δ-chain equs read:

$$(\partial_r - \frac{i}{r}\partial_\theta)\tilde{\Delta}_k = -i\bar{\gamma}e^{i\theta}\tilde{\Delta}_{k+1}$$
$$(\partial_r + \frac{i}{r}\partial_\theta)\tilde{\Delta}_{k+1} = i\gamma e^{-i\theta}\tilde{\Delta}_k \qquad (2.3.7)$$

These equations are solved in complete generality as follows:

2.3.1 Theorem. Equs (2.3.7) are solved by:

$$\tilde{\Delta}_k(r,\theta) = (\frac{i\gamma}{c})^k \sum_{\ell=-\infty}^{\infty} (\alpha_{\ell+k} I_\ell(cr) + \beta_{\ell+k}(-1)^\ell K_\ell(cr))e^{i\ell\theta} \qquad (2.3.8)$$

where α_ℓ, β_ℓ are arbitrary complex parameters.

Proof

Since each $\tilde{\Delta}_k$ satisfies the Helmholtz equation, it has an expansion in Bessel functions:

$$\tilde{\Delta}_k(r,\theta) = \sum_{\ell=-\infty}^{\infty} \left[A_{k,\ell} I_\ell(cr) + B_{k,\ell}(-1)^\ell K_\ell(cr) \right] e^{i\ell\theta}$$

Claim

Equs (2.3.7) are satisfied if and only if:

$$A_{k+1,\ell} = \frac{i\gamma}{c} A_{k,\ell+j}, \qquad B_{k+1,\ell} = \frac{i\gamma}{c} B_{k,\ell+1} \qquad (*)$$

For,

$$(\partial_r \mp \frac{i}{r}\partial_\theta)\tilde{\Delta}_k$$

$$= c \sum_{\ell=-\infty}^{\infty} \Big[A_{k,\ell}(I'_\ell(cr) \pm \frac{\ell}{cr} I_\ell(cr)) + B_{k,\ell}(-1)^\ell$$

$$(K'_\ell(cr) \pm \frac{\ell}{cr} K_\ell(cr))\Big] \; e^{i\ell\theta}$$

$$= c \sum_{\ell=-\infty}^{\infty} \Big| \; A_{k,\ell} I_{\ell\mp 1}(cr) + B_{k,\ell}(-1)^{\ell\mp 1} K_{\ell\mp 1}(cr) \Big| \; e^{i\ell\theta}$$

from Appendix A (equ (A.8)).

Hence equ (2.3.7) (ii) is equivalent to:

$$\sum_{\ell=-\infty}^{\infty} \Big[A_{k+1,\ell} I_{\ell+1}(cr) + B_{k+1,\ell}(-1)^{\ell+1} K_{\ell+1}(cr)\Big] \; e^{i\ell\theta}$$

$$= \frac{i\gamma}{c} \sum_{\ell=-\infty}^{\infty} \Big[A_{k,\ell+1} I_{\ell+1}(cr) + B_{k,\ell+1}(-1)^{\ell+1} K_{\ell+1}(cr) \Big] e^{i\ell\theta}$$

and this is equivalent to (*), as required.

Similarly, (2.3.7) (i) $\Leftrightarrow$ (*).

Finally, write:

$$A_{o,\ell} = \alpha_\ell, \qquad B_{o,\ell} = \beta_\ell$$

Then the recurrence relations (*) are solved inductively by:

$$A_{k,\ell} = \left(\frac{i\gamma}{c}\right)^k \alpha_{\ell+k}, \qquad B_{k,\ell} = \left(\frac{i\gamma}{c}\right)^k \beta_{\ell+k}$$

QED

and the result follows.

2.3.2 Corollary

Integration of the BI-transformations on the axially symmetric 1-vortex yields the Δ-chain:

$$\tilde{\Delta}_k = \xi^k I_k(cr), \qquad \xi = \frac{i\gamma}{c} e^{-i\theta} \varepsilon U(1) \qquad (2.3.9)$$

Proof

This follows immediately from the choice $\alpha_o=1$, $\alpha_\ell=0(\ell \neq 1)$, and $\beta_\ell=0\ \forall \ell$, in equ (2.3.8), using $I_k \equiv I_{-k}, k\varepsilon Z$.

QED

We can now complete the description of the a_1 ansatz for the axially symmetric 1-vortex.

We have:

$$\tilde{\phi} = I_o(cr)$$

$$\tilde{\rho} = -\xi^{-1} I_1(cr) = \frac{i\bar{\gamma}}{c} e^{i\theta} I_1(cr)$$

Hence, in the 'effective abelian' picture:

$$A_r^{ab} = 0, \quad A_\theta^{ab} = -c\,\frac{I_1(cr)}{I_o(cr)} \qquad (2.3.10)$$

$$\Rightarrow \quad B^{ab} = -\nabla^2 \ln I_o(cr) = -c^2\left(1-\frac{I_1(cr)^2}{I_o(cr)^2}\right)$$

and

$$\Psi = \frac{i\bar{\gamma}}{c}\, e^{i\theta}\, \frac{I_1(cr)}{I_o(cr)}$$

Hence, Ψ has a unique zero of order 1 at the origin, $|\Psi| \to 1$ monotonically as $r \to \infty$, and Ψ has unit winding number at infinity. So, Ψ satisfies the boundary conditions for a unit vortex, as claimed. Note however that there is an infinite negative effective abelian magnetic flux - this point will be returned to later.

We can now use the Δ-chain (2.3.9) to form solutions in any of the Atiyah-Ward ansätze a_N - these will constitute the axially symmetric N-vortex solutions. eg N=2. In the 2nd ansatz a_2, the above Δ-chain gives:

$$\det\tilde{D}^{(2)} = I_o(cr)^2 - I_1(cr)^2 > 0$$

$$\tilde{\phi} = \frac{I_o^2 - I_1^2}{I_o}$$

$$\tilde{\rho} = \left(\frac{i\bar{\gamma}}{c}\right)^2 e^{2i\theta} \frac{I_1^2 - I_o I_2}{I_o}$$

where $I_k \equiv I_k(cr)$. In this case, non-singularity is automatic since $I_o(x) > I_1(x)$, $\forall x \in \mathbb{R}$.

Hence:

$$\Psi = \left(\frac{i\bar{\gamma}}{c}\right)^2 e^{2i\theta} \frac{I_1^2 - I_o I_2}{I_o^2 - I_1^2}$$

$$\Rightarrow \quad |\Psi| \sim \begin{cases} (cr)^2/8 & r \to 0 \\ 1 & r \to \infty \end{cases}$$

using equs (A.3), (A.5) (Appendix A). So, the boundary conditions for a charge 2 vortex are satisfied.

After some algebra, we also obtain:

$$||\Phi||^2 = c^2 \left[1 - \frac{8 I_o I_1}{r^2 (I_o^2 - I_1^2)} (r(I_o^2 - I_1^2) - I_o I_1)\right] \qquad (2.3.11)$$

$$B^{ab} = c^2 \left[1 - \frac{I_1^2}{I_o^2} - \frac{4 I_o I_1}{r^2 (I_o^2 - I_1^2)} (r(I_o^2 - I_1^2) - I_o I_1)\right]$$

The N-vortex Solution

Substituting the Δ-chain (2.3.9) into equs (2.2.11), and using equ (B.1) (Appendix B) to remove the factors of ξ from the determinants, we obtain, in the Nth ansatz:

$$\tilde{\phi}^{(N)} = \det\tilde{D}^{(N)}/\det\tilde{D}^{(N-1)} \tag{2.3.12}$$

$$\tilde{\rho}^{(N)} = \frac{(-1)^N}{\det\tilde{D}^{(N-1)}}\,\xi^{-N} \begin{vmatrix} I_1 & I_2 & I_3 \cdots & I_N \\ I_o & I_1 & I_2 & \vdots \\ I_{-1} & I_o & I_1 & \vdots \\ \vdots & & \ddots & \vdots \\ I_{-(N-2)} & \cdots & \cdots & I_1 \end{vmatrix}$$

$$\tilde{\bar{\rho}}^{(N)} = \frac{(-1)^{N-1}}{\det\tilde{D}^{(N-1)}}\,\xi^{N} \begin{vmatrix} I_1 & I_2 & I_3 \cdots & I_N \\ I_o & I_1 & I_2 & \vdots \\ I_{-1} & I_o & I_1 & \vdots \\ \vdots & & \ddots & \vdots \\ I_{-(N-2)} & \cdots & \cdots & I_1 \end{vmatrix}$$

where $I_k \equiv I_k(cr)$, and:

$$\det \tilde{D}^{(N)} = \begin{vmatrix} I_o & I_1 & I_2 & \cdots & I_{N-1} \\ I_1 & I_o & I_1 & & \vdots \\ I_2 & I_1 & I_o & & \vdots \\ \vdots & & & \ddots & \vdots \\ I_{N-1} & \cdots & \cdots & \cdots & I_o \end{vmatrix} \qquad (2.3.13)$$

- a persymmetric matrix of Bessel functions.

Numerical Study of the Axisymmetric N-Vortex

Equ (2.3.11) shows that a direct analytic calculation of the gauge invariant quantities, $||\Phi||^2$, ξ is very unwieldy, even in the case N=2. It is therefore necessary to resort to a numerical calculation. This has been performed as follows:

(1) Calculate $\det\tilde{D}^{(N)}$ of equ (2.3.13) on a sufficiently fine grid of points.

(2) Calculate $||\Phi||^2$ and ξ using the expressions in Corollary (2.2.1), with the Laplacian approximated by central differences:

$$\nabla^2 \approx \frac{1}{(\delta x)^2}\left[\delta_x^2 - \frac{\delta_x^4}{12} + \frac{\delta_x^6}{90} - \frac{\delta_x^8}{560}\right]$$

$$+ \frac{1}{(\delta y)^2}\left[\delta_y^2 - \frac{\delta_y^4}{12} + \frac{\delta_y^6}{90} - \frac{\delta_y^8}{560}\right]$$

The accuracy of the algorithms used has been checked by comparing with the known analytic results of equs (2.3.5) and (2.3.11).

In Figure 1, $||\Phi||^2(r)$, $E(R)$ and $\xi(r)$ are plotted for the axially symmetric N-vortex (N=1 to 5) with the characteristic length c^{-1} set equal to unity. Note -

(i) $||\Phi||^2(r)$ increases monotonically from $-c^2$ at the origin to $+c^2$ at infinity, and the profile becomes increasingly spread out as N increases; in fact, $||\Phi||^2(r)$ crosses the zero axis at $r \cong 2N/c$. This agrees very well with the asymptotic formula (2.2.4):

$$||\Phi||^2 \sim c^2\left(1 - \frac{2N}{cr}\right), \quad r \to \infty$$

$$\Rightarrow \quad ||\Phi||^2 = 0 \text{ at } r \cong \frac{2N}{c}$$

(ii) For N>1, the energy density $\xi(r)$ vanishes at the origin, and the region of positive energy density is concentrated in an annulus, inside the region $r \lesssim 2N/c$, where $||\Phi||^2 < 0$. $\xi(r)$ attains its maximum at $r \cong N/c$, for large N.

Hence, the N-vortex solution differs appreciably from the vacuum in a disc of radius 2N units of characteristic length centred at the origin; it is natural to identify this as the 'core' region of the N-vortex, in analogy with the axially symmetric 'monopole core' of Prasad and Rossi.

In the remainder of this section, we shall prove that the axisymmetric N-vortex solutions are non-singular and we shall verify some of the above behaviour for general N by examining the asymptotic behaviour of the fields as $r \to 0$ and $r \to \infty$. The non-singularity proof uses the technique of Prasad and Rossi for proving non-singularity of the axisymmetric monopoles in a neighbourhood of the x_1x_2-plane.

2.3.3 Theorem

For the axially symmetric N-vortex solution (2.3.12), we have:

$$\det\tilde{D}^{(N)} > 0, \quad \forall x \in \mathbb{R}^2$$

Hence the solution is non-singular.

Proof

Using the integral representation for Bessel functions (Appendix A):

$$I_k(z) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{ik\theta} e^{z\cos\theta}$$

we find, from (2.3.13):

$$\det\tilde{D}^{(N)} = \int_0^{2\pi} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi}\, e^{cr(\cos\theta_1 + \ldots + \cos\theta_N)}\, W(\theta_1, \ldots, \theta_N)$$

$$= \frac{1}{N!} \sum_{\sigma\in S_N} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi}\, e^{cr(\cos\theta_1 + \ldots + \cos\theta_N)}\, W(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(N)})$$

where

$$W(\theta_1, \ldots, \theta_N) = \begin{vmatrix} 1 & e^{i\theta} & e^{2i\theta_2} & \cdots & e^{i(N-1)\theta_1} \\ e^{-i\theta_2} & 1 & e^{i\theta_2} & & \vdots \\ e^{-2i\theta_3} & e^{-i\theta_3} & 1 & & \vdots \\ \vdots & & & & \vdots \\ e^{-(N-1)i\theta_N} & \cdots & \cdots & \cdots & 1 \end{vmatrix}$$

Hence, using the Weyl identity (Appendix B, equ (B.3))

$$\sum_{\sigma\in S_N} W(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(N)}) = \prod_{i<j}^{N} 4\sin^2\left(\frac{\theta_i - \theta_j}{2}\right) \qquad (2.3.14)$$

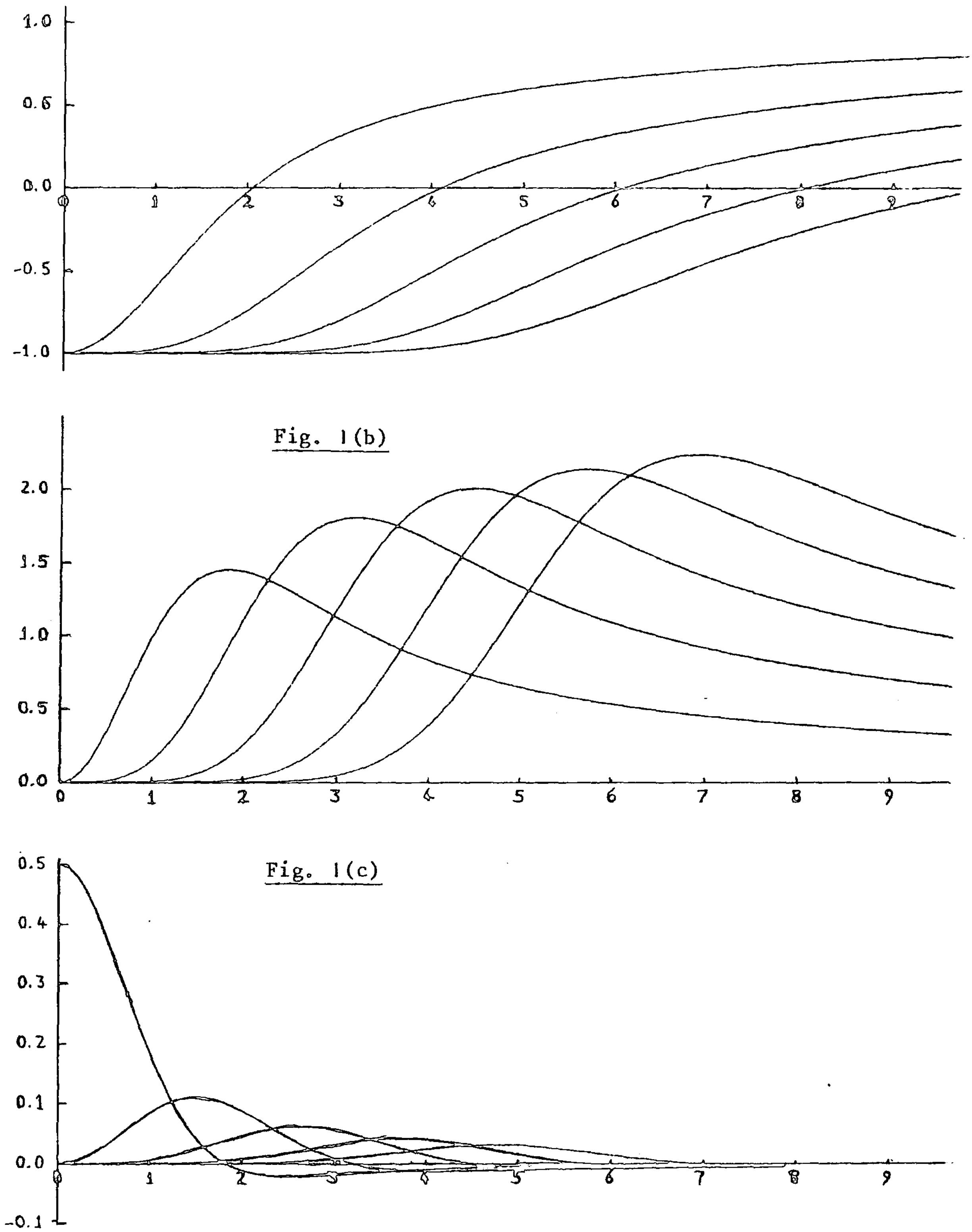
we have:

$$\det\tilde{D}^{(N)} = \frac{1}{N!} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi} \prod_{i<j}^{N} 4\sin^2\left(\frac{\theta_i - \theta_j}{2}\right) \qquad (2.3.15)$$

$$e^{cr(\cos\theta_1 + \ldots + \cos\theta_N)}$$

and the result follows, snce the integrand is positive.

Fig. 1(a)

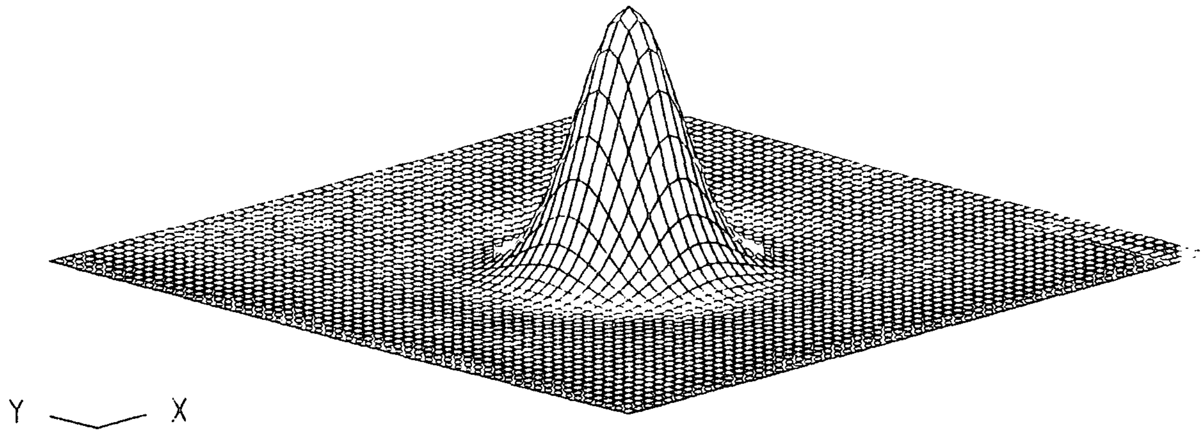


(a) $\|\Phi\|^2$ for axially symmetric N-string, N = 1 to 5.

(b) E(R) for axially symmetric N-string, N = 1 to 5.

(c) Energy density $\mathcal{E}$ of axially symmetric N-string, N = 1 to 5.

N = 1



N = 2

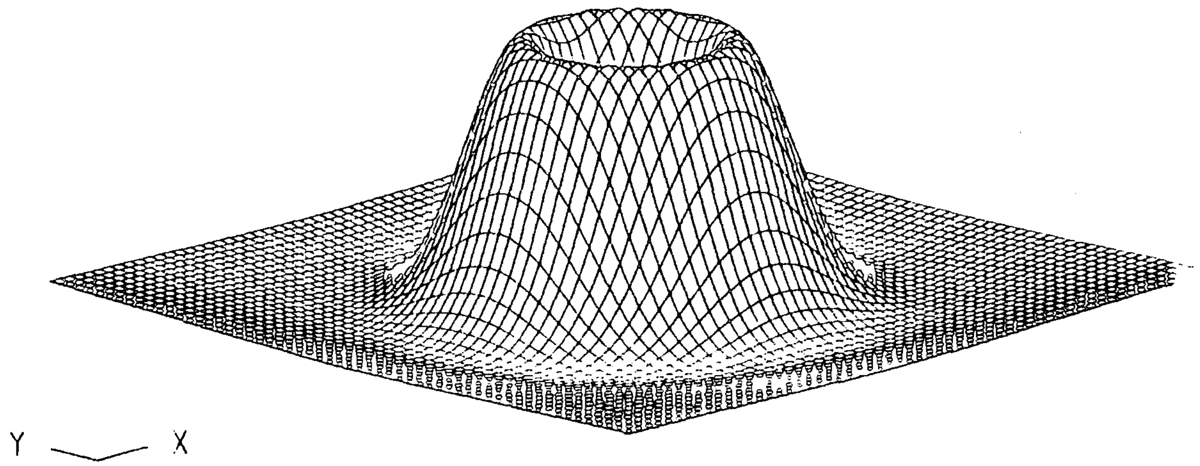


Fig. 1(d). Energy density of axially symmetric 1- and 2-strings.

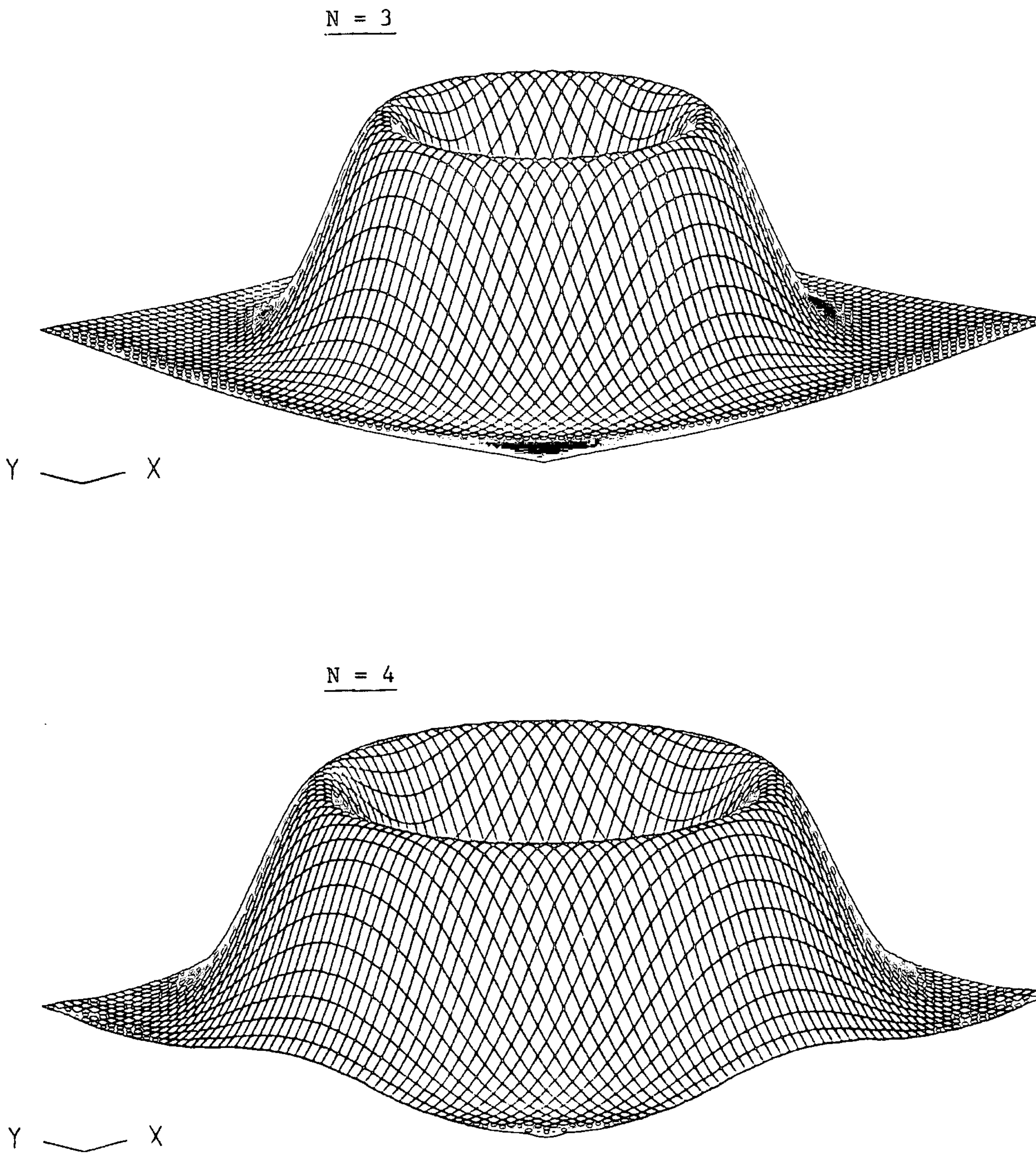


Fig. 1(e). Energy density of axially symmetric 3- and 4-strings.

Relation to Single-Link Lattice Gauge Theory and Asymptotic Behaviour of the Fields

An alternative proof of non-singularity, together with a very compact expression for $\det\tilde{D}^{(N)}$, can be obtained by exploiting a correspondence between Töplitz determinants and U(N) group integrals, explained in Appendix B, Theorem B (whose proof is virtually identical to that of theorem (2.3.3) above).

Throughout this section, we set c=1.

2.3.4 Theorem

For the axially symmetric N-vortex solution,

$$\det\tilde{D}^{(N)} = \int_{U(N)} dU\ e^{\frac{r}{2}\mathrm{Tr}(U+U^{+})} \tag{2.3.16}$$

where dU is the Haar measure on U(N).

Proof

Again use the integral representation for I_k:

$$I_k(r) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{ik\theta} e^{r\cos\theta} = \oint_{U(1)} \frac{dz}{z}.z^k e^{\frac{r}{2}(z+z^*)}$$

This implies, from Theorem B:

$$\det\tilde{D}^{(N)} = \det(I_{\ell-k}(r))$$

$$= \int_{U(N)} dU.\det(e^{\frac{r}{2}(U+U^+)})$$

$$= \int_{U(N)} dU.e^{\frac{r}{2}\mathrm{Tr}(U+U^+)}$$

QED

Remarks

(1) Non-singularity is an immediate consequence of (2.3.16), since the integrand is positive. In fact, it is clear from (2.3.16) that $\det\tilde{D}^{(N)}(r)$ is monotonically increasing from 1 at r=0 to infinity as $r\to\infty$.

(2) The expression (2.3.16) is, in fact, the single-site/ link partition function for a U(N) lattice chiral/gauge theory, with the radial variable r replacing the inverse coupling (or inverse temperature) β. Hence, 'small r' and 'large r' expansions of (2.3.16) are equivalent respectively to 'strong coupling' and 'weak coupling' expansions of the corresponding lattice gauge theory, and these are obtained by referring to relevant parts of the literature on lattice gauge theories. This is very convenient, since the obvious procedure of inserting power series and asymptotic expansions of Bessel functions into equ (2.3.13) becomes virtually impracticable for $N\gtrsim 4$.

(3) Using the lattice gauge theory analogy, we can also obtain an extremely useful and suggestive expression

for the effective complex scalar Higgs field $\psi^{(N)}$.
From equ (2.3.12), with a slight change in previous notation, we have:

$$\psi^{(N)} = (i\bar{\gamma})^N e^{iN\theta} \frac{\det\tilde{E}^{(N)}}{\det\tilde{D}^{(N)}} \qquad (|\gamma| = 1)$$

where $\det\tilde{E}^{(N)} = \det(I_{\ell-k+1}(r))$.

$\therefore$ Using theorem B:

$$I_{k+1}(r) = \oint_{U(1)} \frac{dz}{z} \cdot z^k . z e^{\frac{r}{2}(z+z^*)}$$

$$\Rightarrow \quad \det\tilde{E}^{(N)} = \int_{U(N)} dU \det(Ue^{\frac{r}{2}(U+U^+)})$$

$$\therefore \quad \det\tilde{E}^{(N)} = \int_{U(N)} dU.\det U.e^{\frac{r}{2}\mathrm{Tr}(U+U^+)} \tag{2.3.17}$$

$$= \frac{1}{N!} \int_o^{2\pi} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi} \prod_{i>j}^{N} 4\sin^2\left(\frac{\theta_i-\theta_j}{2}\right) e^{i(\theta_1+\ldots+\theta_N)}$$

$$e^{r(\cos\theta_1+\ldots+\cos\theta_N)}$$

$$\Rightarrow \quad \psi^{(N)} = (i\bar{\gamma}e^{i\theta})^N \frac{\int_{U(N)} dU.\det U.e^{\frac{r}{2}\mathrm{Tr}(U+U^+)}}{\int_{U(N)} dU.e^{\frac{r}{2}\mathrm{Tr}(U+U^+)}}$$

$$\therefore \quad \Psi^{(N)} = (i\bar{\gamma}e^{i\theta})^N \langle \det U \rangle \tag{2.3.18}$$

where $\langle \cdot \rangle$ denotes the expectation value with respect to the partition function (2.3.16). This immediately tells us that $|\Psi^{(N)}| \leq 1$, $\forall x \in \mathbb{R}^2$.

In fact, from (2.3.17) and (2.3.15), we deduce:

$$\det\tilde{E}^{(N)}(0) = \int_{U(N)} dU.\det U = 0$$

and $\det\tilde{E}^{(N)} \sim \det\tilde{D}^{(N)}$ as $r \to \infty$

$$\Rightarrow \quad \Psi^{(N)}(0) = 0, \quad |\Psi^{(N)}| \to 1 \quad \text{as } r \to \infty,$$

and, from (2.3.18), $\Psi^{(N)}$ has winding number N at infinity, as required for the vortex interpretation.

In fact, regarding r as an inverse temperature 1/T, a nice physical picture emerges. At the vortex centre, $r=0 \Rightarrow T=\infty$, so the probability distribution of $U \in U(N)$ is completely random (wrt the Haar measure), so, $|\Psi| = |\langle \det U \rangle| = 0$.

As $r \to \infty$, $T \to 0$, and the probability distribution becomes frozen around the identity element of U(N), so $|\Psi| = |\langle \det U \rangle| \to 1$ as $r \to \infty$.

Small r Behaviour

Expanding the partition function (2.3.16) in powers of r, we obtain:

$$\det\tilde{D}^{(N)} = \sum_{m,n=0}^{\infty} \frac{1}{m!n!}\left(\frac{r}{2}\right)^{m+n} \int_{U(N)} dU.(\mathrm{Tr}U)^m(\mathrm{Tr}U^+)^n$$

But, the above cumulants vanish unless m=n (cf Appendix B, equ (B.4), hence -

$$\det\tilde{D}^{(N)} = \sum_{n=0}^{\infty} \frac{1}{(n!)^2}\left(\frac{r}{2}\right)^{2n} \int_{U(N)} dU.(\mathrm{Tr}U)^n(\mathrm{Tr}U^+)^n$$

Rules for calculating strong coupling cumulants have been given by Bars and Green (1979), Bars (1980), and Samuel (1980). The results that we need are:

$$\int dU.(\mathrm{Tr}U)^n(\mathrm{Tr}U^+)^n = \begin{cases} n! \quad , & n \leq N \\ (N+1)!-1, & n=N+1 \end{cases}$$

$$\Rightarrow \quad \det\tilde{D}^{(N)} = \sum_{n=0}^{N+1} \frac{1}{n!}\left(\frac{r}{2}\right)^{2n} - \frac{1}{(N+1)!^2}\left(\frac{r}{2}\right)^{2(N+1)} + O(r^{2(N+2)})$$

$$\therefore \quad \det\tilde{D}^{(N)} = e^{(r/2)^2}\left(1-\frac{1}{(N+1)!^2}\left(\frac{r}{2}\right)^{2(N+1)}\right) + O(r^{2(N+2)}) \tag{2.3.19}$$

$$\& \quad \ln\det\tilde{D}^{(N)} = \frac{r^2}{4} - \frac{1}{(N+1)!^2}\left(\frac{r}{2}\right)^{2(N+1)} + O(r^{2(N+2)})$$

Hence, using Corollary (2.2.1), we obtain small r expansions for $||\Phi||^2, \xi$:

$$||\Phi||^2 = -1+\frac{2}{(N!)^2}\left(\frac{r}{2}\right)^{2N}+0(r^{2(N+1)})$$

(2.3.20)

$$\xi = \frac{1}{2(N-1)!^2}\left(\frac{r}{2}\right)^{2(N-1)}+0(r^{2N})$$

in qualitative agreement with the above numerical results.

Let us also examine the small r behaviour of the effective abelian fields. Using (2.3.19), we have:

$$\mathfrak{F} = \frac{\det\tilde{D}^{(N)}}{\det\tilde{D}^{(N-1)}} = 1+\frac{1}{(N!)^2}\left(\frac{r}{2}\right)^{2N}+0(r^{2N+2})$$

$$\Rightarrow \quad \chi = \ln\mathfrak{F} = \frac{1}{(N!)^2}\left(\frac{r}{2}\right)^{2N}+0(r^{2N+2}) \qquad (2.3.21)$$

$$\Rightarrow \quad B^{ab} = -\nabla^2\chi = \frac{-1}{(N-1)!^2}\left(\frac{r}{2}\right)^{2(N-1)}+0(r^{2N})$$

Finally, let us calculate the small r behaviour of $\Psi^{(N)}$. Expanding equ (2.3.17) in powers of r:

$$\det\tilde{E}^{(N)} = \sum_{m,n=0}^{\infty} \frac{1}{m!n!}\left(\frac{r}{2}\right)^{m+n} \int_{U(N)} dU.\det U.(\mathrm{Tr}U)^n(\mathrm{Tr}U^+)^m$$

$$= \sum_{n=0}^{\infty} \frac{1}{(N+n)!n!}\left(\frac{r}{2}\right)^{N+2n} \int_{U(N)} dU.\det U.(\mathrm{Tr}U)^n(\mathrm{Tr}U^+)^{N+n}$$

(from equ (B.4), since detU consists of an ε-tensor contracted with N copies of $U_{\alpha\beta}$).

$$\det\tilde{E}^{(N)} = \frac{1}{N!}\left(\frac{r}{2}\right)^N \int dU.\det U.(\mathrm{Tr}U^+)^N + 0(r^{N+2})$$

But,

$$\int_{U(N)} dU.\det U.(\mathrm{Tr}U^+)^N$$

$$= \int_0^{2\pi} \frac{d\theta}{2\pi} \int_{SU(N)} dV.\det(e^{i\theta}V).(e^{-i\theta}\mathrm{Tr}V^+)^N$$

$$= \int_{SU(N)} dV.(\mathrm{Tr}V^+)^N, \quad (\because \det V=1)$$

$$= \int_{SU(N)} dV.V^+_{\alpha_1\alpha_1}.V^+_{\alpha_2\alpha_2}\ldots V^+_{\alpha_N\alpha_N}$$

$$= \frac{1}{N!}\varepsilon_{\alpha_1\ldots\alpha_N}\ \varepsilon_{\alpha_1\ldots\alpha_N} \qquad \text{(from equ (B.5))}$$

$$=\frac{N!}{N!} = 1$$

$$\therefore \quad \det\tilde{E}^{(N)} = \frac{1}{N!}\left(\frac{r}{2}\right)^N + O(r^{N+2}) \qquad (2.3.22)$$

$$\therefore \quad \psi^{(N)} = (i\bar{\gamma}e^{i\theta})^N \frac{r^N}{2^N . N!} + O(r^{N+2}) \qquad (2.3.23)$$

ie $\psi^{(N)}$ has a zero of order N at the origin, as indeed it must, if it is to describe N vortices superposed at the origin.

Large r Behaviour

Brower and Nauenberg (1981) have considered the leading order weak coupling behaviour of the partition function of a single-link U(N) lattice gauge theory in the presence of an arbitrary matrix source J. Using a saddle point approximation, they find, for non-singular J:

$$Z(J,J^+) = \int_{U(N)} dU . e^{\beta \mathrm{Tr}(UJ+J^+U^+)}$$

$$\sim \exp\left(2\beta \sum_{i=j}^{N} \sqrt{x_i}\right) \prod_{n,m=1}^{N} \left[\frac{\pi}{\beta(\sqrt{x_n}+\sqrt{x_m})}\right]^{\frac{1}{2}}, \text{ as } \beta\to\infty,$$

where $x_1, \ldots, x_n$ are the (non-zero) eigenvalues of J^+J.

This implies, in our case of interest:

$$\det\tilde{D}^{(N)} \sim \text{const}\ \frac{e^{Nr}}{r^{N^2/2}} \quad \text{as } r\to\infty \tag{2.3.24}$$

$$\ln\det\tilde{D}^{(N)} \sim Nr - \frac{N^2}{2}\ln r + \text{const}$$

This result can be understood qualitatively by noting that the saddle point of the integrand of equ (2.3.16) occurs at U=1 - hence the factor e^{Nr}, and that we have to integrate over a neighbourhood of this saddle point in the N^2-dimensional manifold U(N) - hence the factor $r^{-N^2/2}$.

This result tells us nothing new about the asymptotic behaviour of $||\Phi||^2$, ξ and $\Psi^{(N)}$, but it does allow us to determine the asymptotic behaviour of the effective abelian gauge field. We find -

$$\tilde{\phi} \sim e^r r^{-(N-\frac{1}{2})}$$

$$\chi \sim r-(N-\tfrac{1}{2})\ln r \tag{2.3.25}$$

$$\therefore \quad A_r^{ab} \sim 0, \quad A_\theta^{ab} \sim -1 + (N-\tfrac{1}{2})\frac{1}{r}$$

$$\text{and} \quad B^{ab} = -\nabla^2\chi \sim -\frac{1}{r}, \quad \text{as } r\to\infty \tag{2.3.26}$$

We can use these results to calculate the total effective abelian magnetic flux $\Phi^{(N)}(R)$ in a large disc of radius R centred at the origin. We have:

$$\Phi^{(N)}(R) = \int_{|x|\leq R} d^2x\ B^{ab} = -\int_{|x|\leq R} d^2x \nabla^2\chi = \int_{|x|=R} d\underset{\sim}{n}\cdot\nabla\chi$$

$$\sim -2\pi R(1-(N-\tfrac{1}{2})\tfrac{1}{R}),\ \text{as } R\to\infty,\ \text{by } (2.3.25)$$

$$\therefore \quad \Phi^{(N)}(R) \sim 2\pi(N-\tfrac{1}{2})-2\pi R,\ \text{as } R\to\infty \qquad (2.3.27)$$

Hence we have the bizarre result that the axisymmetric N-vortex has infinite negative total magnetic flux, whereas the <u>difference</u> in fluxes of distinct assätze is quantized in the same units as Nielsen-Olesen vortices:

$$\Phi^{(N)}(R) - \Phi^{(M)}(R) \sim 2\pi(N-M),\ \text{as } R\to\infty \qquad (2.3.28)$$

<u>Large N Behaviour</u>

Gross & Witten (1980) evaluated the partition function -

$$Z = \int_{U(N)} dU\ e^{\beta \mathrm{Tr}(U+U^{+})} = \int_{U(N)} dU\ e^{\frac{N}{\lambda}\mathrm{Tr}(U+U^{+})}$$

for fixed $\lambda = N/B$, in the limit $N\to\infty$. They found that, in this limit, the above weak and strong coupling results are <u>exact</u> for $\lambda \leq 2$, $\lambda \geq 2$ respectively, ie $$\lim_{N\to\infty} \frac{\ln Z}{N^2} = \begin{cases} \dfrac{1}{\lambda^2} , & \lambda \geqq 2 \\ \dfrac{2}{\lambda} + \tfrac{1}{2}\ln\dfrac{\lambda}{2} - \dfrac{3}{4}, & \lambda \leqq 2 \end{cases}$$

This abrupt change in analyticity behaviour at $\lambda=2$ gives rise to the Gross-Witten 3rd order phase transition in the large-N limit of lattice QCD_2.

For us, it implies, as $N\to\infty$:

$$\ln \det \mathcal{D}^{(N)} = \begin{cases} \dfrac{r^2}{4} , & r \leqq N \\ Nr - \dfrac{N^2}{2}\ln\dfrac{r}{N} - \dfrac{3N^2}{4} , & r \geqq N \end{cases} \qquad (2.3.29)$$

By Corollary (2.2.1), this gives:

$$||\Phi||^2 = \begin{cases} -1 , & r < N \\ 1 - \dfrac{2N}{r} , & r > N \end{cases} \qquad (2.3.30)$$

Note that $||\Phi||^2$ is continuous, with discontinuous 1st derivative at r=N; $||\Phi||^2$ crosses the zero axis at precisely r=2N, as guessed from the previous numerical results.

Also -

$$\xi = \begin{cases} 0 & r < N \\ -N/2r^3 & r > N \end{cases}$$

and the discontinuity in the 1st derivative of $||\Phi||^2$ implies that the positive energy density is concentrated on a δ-function supported on the circle $r=N$. Explicitly:

$$\xi = \frac{1}{2N}\delta(r-N) - \frac{N}{2r^3} H(r-N) \qquad (2.3.31)$$

where H is the Heaviside function, and the coefficient of the δ-function is obtained from the requirement $\int d^2x\, \xi = 0$. Hence, in the large-N limit, the energy density attains its maximum on a ring of radius N, as conjectured from the numerical results.

Finally, the large N-behaviour of the effective abelian gauge field is given by:

$$\chi = \ln \det \tilde{D}^{(N)} - \ln \det \tilde{D}^{(N-1)}$$

$$= \begin{cases} 0 & , \quad r \leqq N-1 \\ \frac{r^2}{4} - (N-1)r + \frac{N^2}{2}\ln\frac{r}{N} + \frac{3N^2}{4} & , \quad N-1 \leqq r \leqq N \\ r + (N-\frac{1}{2})\ln r - \frac{3}{2}(N-\frac{1}{2}) & , \quad r \geqq N \end{cases}$$

$$\therefore \quad -B^{ab} = \nabla^2\chi = \begin{cases} 0 & r \leqq N-1 \\ 1-\frac{(N-1)}{r} & N-1 \leqq r \leqq N \\ \frac{1}{r} & r \geqq N \end{cases} \tag{2.3.32}$$

(note that this is continuous, with discontinuous 1st derivatives).

Hence, at least in the large N limit, the effective abelian magnetic field also has a ring-like structure concentrated in the region $r \approx N$.

Summary of Results

Since some of the above results are somewhat disjointed, let us now summarize what we have learnt about the axisymmetric N-vortex configurations.

(1) The short distance behaviour (equs (2.3.20)) and the long distance behaviour (equs (2.2.4)) of $||\Phi||^2$ and ξ are consistent, for general N, with the ring-like structure conjectured from numerical results for small N. We have also verified, in the large N limit, our conjectured picture of the 'vortex core' consisting of the region $r \lesssim 2N$ where $||\Phi||^2 < 0$, with the energy density attaining its maximum on the ring $r \approx N$.

(2) The effective complex scalar field $\Psi^{(N)}$ satisfies <u>all</u> the boundary conditions required to describe a Nielsen-Olesen vortex of charge N located at the origin, ie

(i) It has a unique zero of order N at the origin (equ (2.3.23)).

(ii) $|\Psi^{(N)}|$ converges monotonically to unity as $r \to \infty$.

(iii) The change of phase of $\Psi^{(N)}$ around any simple closed path encircling the origin is $2\pi N$. (From equ (2.3.18) and the remarks thereafter.)

The peculiar fact that $||\Phi||^2$ interpolates from $-c^2$ at the origin to $+c^2$ at infinity is a simple consequence of the above behaviour of Ψ, and equ (2.2.20):

$$||\Phi||^2 = c^2(2\Psi\Psi^* - 1)$$

(3) Although the effective abelian magnetic flux does not satisfy the flux quantization law of Nielsen-Olesen vortices, it does consist of a <u>fixed</u> infinite negative flux plus a finite flux obeying the quantization law:

$$\Phi^{(N)}(R) \sim (-2\pi R - \pi) + 2\pi N, \quad \text{as } R \to \infty.$$

It is amusing to note that if we 'divide' two distinct ansätze a_N, a_M $(N > M)$ as follows:

$$\tilde{\phi} = \tilde{\phi}^{(N)} / \tilde{\phi}^{(M)}, \qquad \tilde{\rho} = \tilde{\rho}^{(N)} / \tilde{\rho}^{(M)}, \qquad \bar{\tilde{\rho}} = \bar{\tilde{\rho}}^{(N)} / \bar{\tilde{\rho}}^{(M)}$$

and then form the fields (A_μ^{ab}, Ψ) from $(\tilde{\phi}, \tilde{\rho}, \bar{\tilde{\rho}})$ as previously

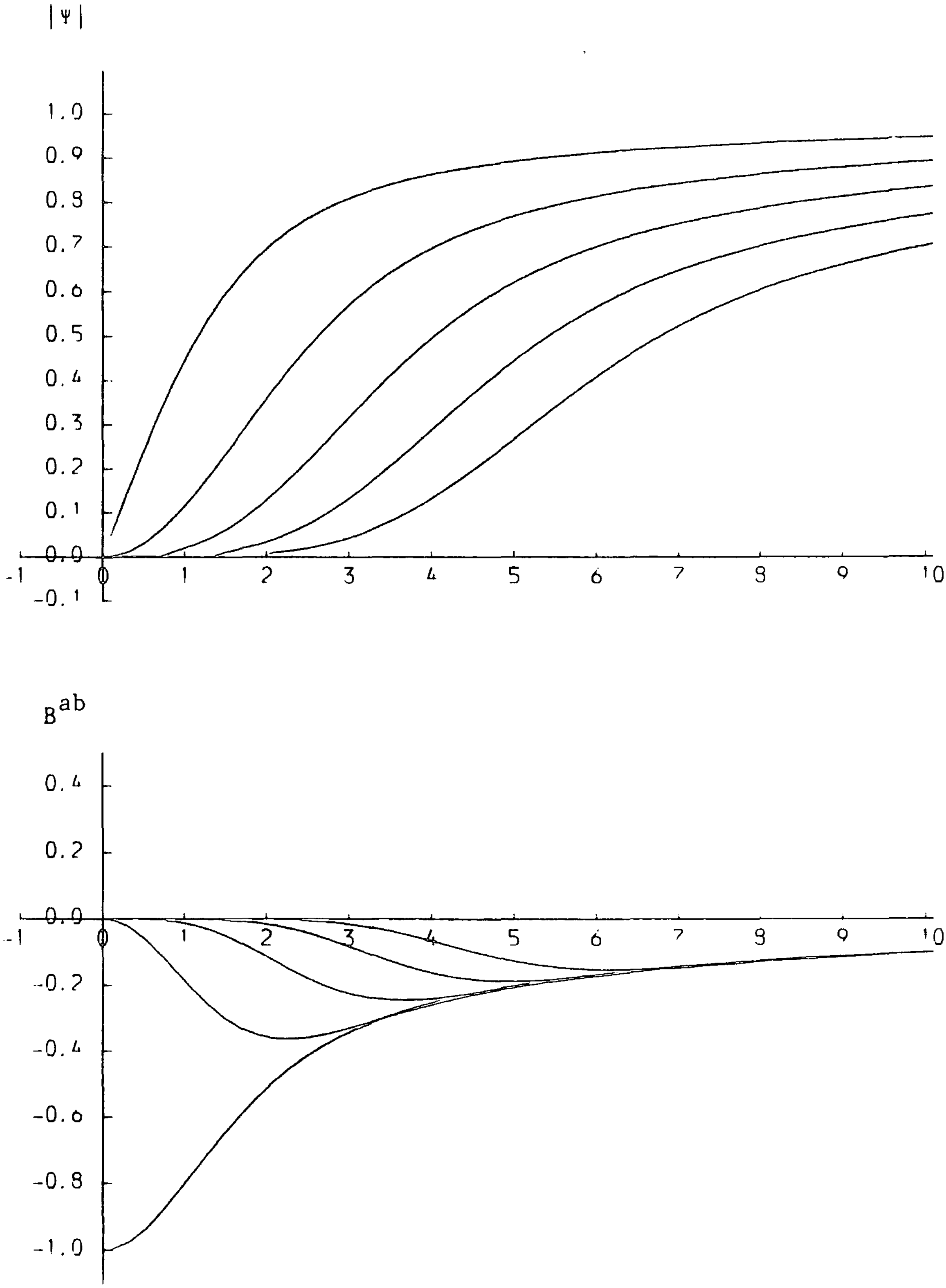
ie $\quad \Psi = \Psi^{(N)} / \Psi^{(M)}$

$$A_\mu^{ab} = A_\mu^{ab(N)} - A_\mu^{ab(M)}$$

then Ψ satisfies the boundary conditions for a Nielsen-Olesen vortex of charge N-M, <u>and</u> the total magnetic flux of A_μ^{ab} is $2\pi(N-M)$. It is an interesting open question whether any of these bear any relation to the Nielsen-Olesen vortices in the Bogomol'nyi limit of the abelian Higgs model.

For completeness, Ψ and B are plotted in Figure 2, for N=1 to 5.

Fig 2



Effective complex scalar Higgs field $|\Psi|$, and effective abelian magnetic field B^{ab} for the axially symmetric N-vortex, N=1 to 5.

2.4 Some Singular Axially Symmetric Solutions

If in theorem (2.3.1), we choose $\alpha_\ell = 0$, and $\beta_\ell = \delta_{\ell,0}$, $\forall \ell \varepsilon Z$, then we obtain the Δ-chain

$$\tilde{\Delta}_k = (-\xi)^k K_k(cr), \qquad \xi = \frac{i\gamma}{c} e^{-i\theta} \tag{2.4.1}$$

Hence, in the a_N ansätze, we obtain solutions analogous to the axisymmetric N-vortices, with $I_k(cr)$ replaced everywhere by $(-1)^k K_k(cr)$. Using equ (B.1), these solutions are:

$$\tilde{\phi}^{(N)} = \det\tilde{D}^{(N)}/\det\tilde{D}^{(N-1)} \tag{2.4.2}$$

$$\tilde{\rho}^{(N)} = \frac{\xi^{-N}}{\det\tilde{D}^{(N-1)}} \begin{vmatrix} K_1 & K_2 & K_3 \cdots & K_N \\ K_o & K_1 & K_2 & \vdots \\ K_{-1} & K_o & K_1 & \vdots \\ \vdots & & \ddots & \vdots \\ K_{-(N-2)} & \cdots & \cdots & K_1 \end{vmatrix}$$

$$\bar{\rho}^{(N)} = \frac{-\xi^{N}}{\det\tilde{D}^{(N-1)}} \begin{vmatrix} K_1 & K_2 & K_3 \cdots & K_N \\ K_o & K_1 & K_2 & \vdots \\ K_{-1} & K_o & K_1 & \vdots \\ \vdots & & \ddots & \vdots \\ K_{-(N-2)} & \cdots & \cdots & K_1 \end{vmatrix}$$

where $K_k \equiv K_k(cr)$, and:

$$\det\tilde{D}^{(N)} = \begin{vmatrix} K_o & K_1 & K_2 \cdots & K_{N-1} \\ K_1 & K_o & K_1 & \vdots \\ K_2 & K_1 & K_o & \vdots \\ \vdots & & \ddots & \vdots \\ K_{N-1} & \cdots & \cdots & K_o \end{vmatrix} \qquad (2.4.3)$$

In the following figure, $||\Phi||^2, \xi$ are plotted for these solutions, with N=1 to 4.

Notice that the asymptotic behaviour of $||\Phi||^2, \xi$ is rather different than that for the non-singular solutions (2.3.12). Using the asymptotic expansions of $K_k(z)$ (equ (A.5)), we have (cf Proof of Corollary (2.2.2)):

$$\det\tilde{D}^{(N)} \sim \frac{e^{-Ncr}}{(cr)^{N/2}} \frac{\delta^{(N)}(\theta)}{r^{\ell}}, \qquad r \to \infty$$

$$\Rightarrow \left.\begin{array}{l} ||\Phi||^2 \sim c^2(1+\frac{2N}{cr}) \\ \\ \xi \sim Nc/2r^3 \end{array}\right\} \text{ as } r \to \infty$$

Hence, in this case, $||\Phi||^2 \to c^2+$, $\xi \to 0+$, as $r \to \infty$.

Also, from equ (A.4), these solutions are singular at r=0. For example, in the case N=1:

$$\tilde{\Delta}_o \sim -\ln cr, \qquad r \to 0$$

$$\Rightarrow \quad ||\Phi||^2 \sim c^2\left(1 - \frac{2}{(cr)^2 \ln(cr)^2}\right), \qquad r \to 0$$

In fact, the short distance behaviour of the N=1 solution is identical to that of Saclioglu's singular solution, since the latter occurs in the 't Hooft ansatz, with the choice $\Delta_o(x) = \ln r$. However, the singularity at finite r is now avoided, since $K_o(cr) > 0$, $\forall r > 0$. This phenomenon carries over to general N, as the next result shows:

2.4.1 Theorem

For the solutions (2.4.2),

$$\det\tilde{D}^{(N)} \neq 0, \qquad \forall r > 0.$$

Hence the solutions are non-singular for $r \neq 0$.

Proof (cf Proof of theorem (2.3.3)).

Use the integral representation (equ (A.7), Watson p 181):

$$K_k(cr) = \tfrac{1}{2}\int_{-\infty}^{\infty} dU.e^{ku}e^{-cr\cosh u}$$

$$\Rightarrow \quad \det\tilde{D}^{(N)} = \frac{1}{2^N}\int_{-\infty}^{\infty} du_1 \ldots du_N\; e^{-cr(\cosh u_1 + \ldots + \cosh u_N)}$$

$$V(u_1, \ldots, u_N)$$

$$= \frac{1}{2^N N!} \sum_{\sigma \varepsilon S_N} \int_{-\infty} du_1 \ldots du_N e^{-cr(\cosh u_1 + \ldots + \cosh u_N)} V(u_{\sigma(1)}, \ldots, u_{\sigma(N)})$$

where

$$V(u_1, \ldots, u_N) = \begin{vmatrix} 1 & e^{u_1} & e^{2u_1} & \cdots & e^{(N-1)u_1} \\ e^{-u_2} & 1 & e^{u_2} & & \vdots \\ e^{-2u_3} & e^{-u_3} & 1 & \ddots & \vdots \\ \vdots & & & \ddots & \vdots \\ e^{-(N-1)u_N} & \cdots & \cdots & \cdots & 1 \end{vmatrix}$$

Weyl's identity (equ (2.3.14)) implies (setting $\theta_k = -iu_k$):

$$\sum_{\sigma \varepsilon S_N} V(u_{\sigma(1)}, \ldots, u_{\sigma(N)}) = (-1)^{\frac{1}{2}N(N-1)} \prod_{i<j}^{N} 4\sinh^2\left(\frac{u_i - u_j}{2}\right)$$

$$\Rightarrow \quad \det \tilde{D}^{(N)} =$$

$$\frac{(-1)^{\frac{1}{2}N(N-1)}}{2^N . N!} \int_{-\infty}^{\infty} du_1 \ldots du_N \prod_{i<j}^{N} 4\sinh^2\left(\frac{u_i - u_j}{2}\right) e^{-cr(\cosh u_1 + \ldots + \cosh u_N)} \qquad (2.4.4)$$

and the result follows, since the integral is positive.

QED

Remark

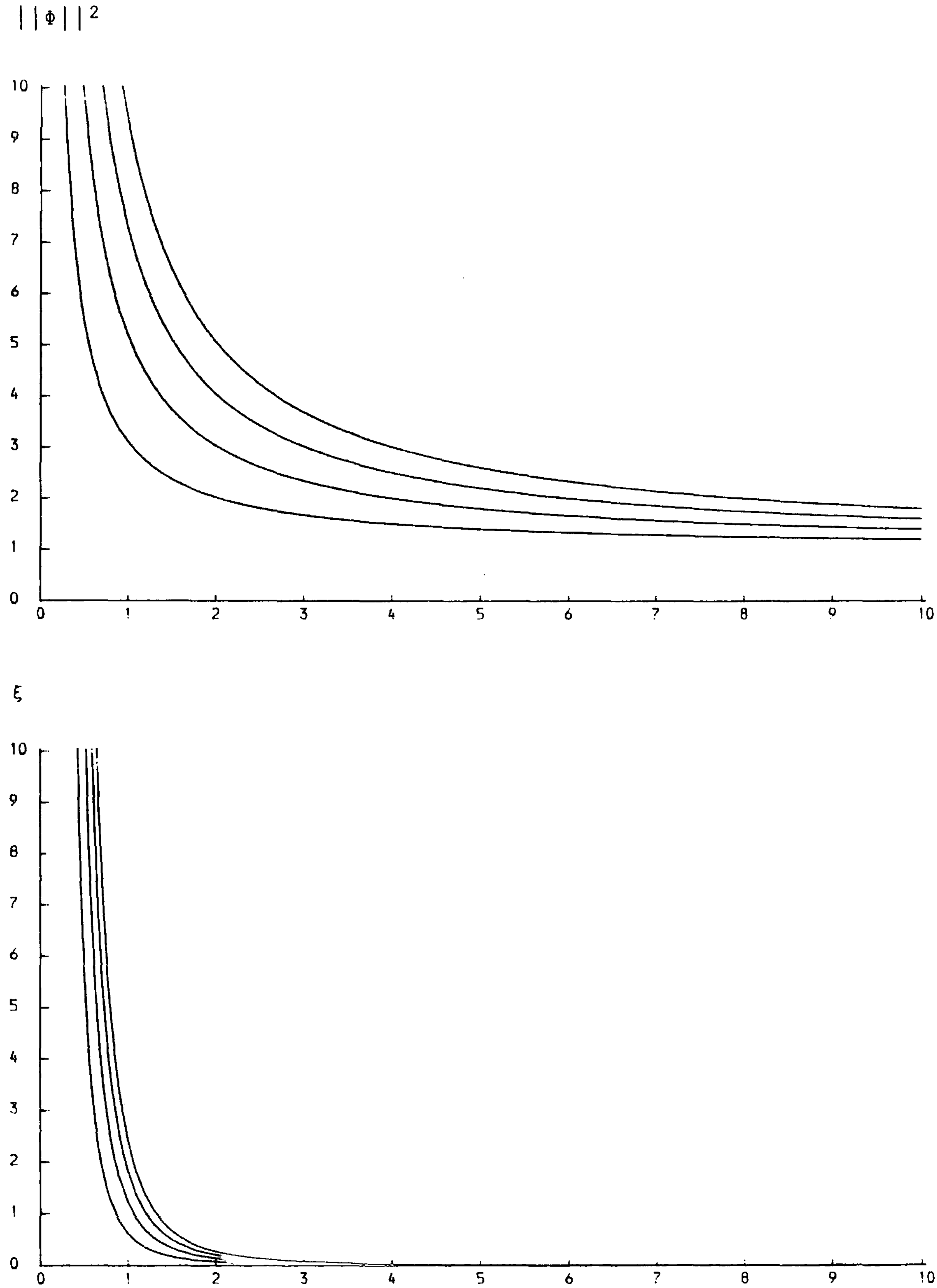
The structure of equ (2.4.4), by analogy with equs (2.3.15), (2.3.16), suggests that the fundamental determinant of the singular solutions (2.4.2) satisfies:

$$\det\tilde{D}^{(N)} \propto \int_{H^+} dR\,.\,e^{-\frac{r}{2}\mathrm{Tr}(R+R^{-1})}$$

where H^+ is the non-compact N^2-dimensional sub-manifold of GL(N,C) consisting of positive hermitian matrices $R = e^H$ (H hermitian). Since any $G \varepsilon GL(N,C)$ can be expressed uniquely as $G = RU$, $R \varepsilon H^+$, $U \varepsilon U(N)$, this seems to indicate some connection between general self-dual solutions in a_N, and group integrals over GL(N,C).

Since we are interested, ultimately, in the construction of solutions which may have some application to the semi-classical approximation, we shall concentrate in the following chapters on the construction of further non-singular solutions.

Fig 3



$||\Phi||^2, \xi$ for the singular solution (2.4.2), N=1 to 4

CHAPTER 3

ONE DIMENSIONAL SOLUTIONS

3.1 Dimensional Reduction to $\mathbb{R}^1$

In this chapter, we address ourselves briefly to the question of what happens if we reduce the self-duality equations a further dimension from $\mathbb{R}^4$ to $\mathbb{R}^1$. In this case, we have a third effective Higgs field $\Phi_3 = A_2$, and the Bogomol'nyi equations (2.2.1) become:

$$\begin{aligned} D\Phi_1 &= [\Phi_2, \Phi_3] \\ D\Phi_2 &= [\Phi_3, \Phi_1] \\ D\Phi_3 &= [\Phi_1, \Phi_2] \end{aligned} \tag{3.1.1}$$

where $D = \partial + [A, \cdot]$, $\partial = \partial_1$, $A = A_1$.

The $d = 1$ gauge field A can be removed by a gauge transformation to Coulomb gauge, giving us the equations:

$$\partial\Phi_i = \tfrac{1}{2}\varepsilon_{ijk}[\Phi_j, \Phi_k] \tag{3.1.2}$$

These are Nahm's equations (Nahm 1981, Corrigan & Goddard 1984), which occur in the extension of the ADHM construction to include self-dual monopoles. Let us briefly examine the behaviour of solutions of these equations. Note that, in the SU(2) case, (Φ_1, Φ_2, Φ_3) forms a triad of 3-vectors; it is useful to bear this analogy in mind in the general case, and define the "volume element":

$$V = \langle \Phi_1, [\Phi_2, \Phi_3] \rangle \tag{3.1.3}$$

Then, using equ (3.1.2), and the cyclic property of the Killing form, we obtain the following equations:

eg $$\frac{d}{dx}\langle \Phi_1, \Phi_2 \rangle = \langle \Phi_1, \frac{d\Phi_2}{dx} \rangle + \langle \frac{d\Phi_1}{dx}, \Phi_2 \rangle$$

$$= \langle \Phi_1, [\Phi_3, \Phi_1] \rangle + \langle [\Phi_2, \Phi_3], \Phi_2 \rangle$$

$$= \langle \Phi_3, [\Phi_1, \Phi_1] \rangle + \langle [\Phi_2, \Phi_2], \Phi_3 \rangle$$

$$= 0 \quad \text{etc}$$

and $$\frac{d}{dx}\langle \Phi_1, \Phi_1 \rangle = 2\langle \Phi_1, \frac{d\Phi_1}{dx} \rangle$$

$$= 2\langle \Phi_1, [\Phi_2, \Phi_3] \rangle$$

$$= 2V \quad \text{etc}$$

ie $$\frac{d}{dx}\langle \Phi_i, \Phi_j \rangle = 2\delta_{ij} V \tag{3.1.4}$$

$$\Rightarrow \quad \langle \Phi_i, \frac{d\Phi_i}{dx} \rangle = V, \qquad i=1,2,3$$

Similarly:

$$\frac{dV}{dx} = \langle \frac{d\Phi_1}{dx}, [\Phi_2,\Phi_3] \rangle + \langle \Phi_1, [\frac{d\Phi_2}{dx},\Phi_3] \rangle + \langle \Phi_1, [\Phi_2,\frac{d\Phi_3}{dx}] \rangle$$

$$= \langle \frac{d\Phi_1}{dx}, [\Phi_2,\Phi_3] \rangle + \langle \frac{d\Phi_2}{dx}, [\Phi_3,\Phi_1] \rangle + \langle \frac{d\Phi_3}{dx}, [\Phi_1,\Phi_2] \rangle$$

$$= \langle \frac{d\Phi_1}{dx}, \frac{d\Phi_1}{dx} \rangle + \langle \frac{d\Phi_2}{dx}, \frac{d\Phi_2}{dx} \rangle + \langle \frac{d\Phi_3}{dx}, \frac{d\Phi_3}{dx} \rangle$$

$$\frac{dV}{dx} = \sum_{i=1}^{3} \left|\left|\frac{d\Phi_i}{dx}\right|\right|^2 \qquad (3.1.5)$$

This implies $dV/dx \geq 0$ when Φ_1, Φ_2, Φ_3 are in the real Lie algebra of a compact group, since the Killing form is then positive definite. Equs (3.1.4), (3.1.5) give us the further equation:

$$\frac{d^2}{dx^2} ||\Phi_i||^2 = \sum_{k=1}^{3} \left|\left|\frac{d\Phi_k}{dx}\right|\right|^2, \qquad i=1,2,3 \qquad (3.1.6)$$

Now, since equs (3.1.2) are a special case of equs (2.2.1), we know from Theorem (2.2.3) that they can have no non-trivial non-singular real solutions. We can in fact improve on this result, in this case, by showing that any real non-trivial solution of (3.1.2) is bounded below by a function with a pole singularity.

3.1.1 Theorem

Let (Φ_1, Φ_2, Φ_3) be a real solution of equs (3.1.2) for a compact gauge group. Suppose that, for a fixed point $x_o \in \mathbb{R}$, we have:

$$V(x_o) = V_o \neq 0 \Rightarrow A_i^2 = ||\Phi_i(x)||^2 \neq 0, \forall i$$

Let $A = \max A_i$. Then the following inequalities are satisfied:

$$V(x) \geq \frac{V_o}{\left[1 - \frac{V_o}{A^2}(x-x_o)\right]^3} \tag{3.1.7}$$

$$||\Phi_i||^2 \geq A_i^2 - A^2 + \frac{A^2}{\left[1 - \frac{V_o}{A^2}(x-x_o)\right]^2}, \quad i=1,2,3 \tag{3.1.8}$$

Proof. Step 1

Note, from (3.1.4), that:

$$V(x_o) \neq 0 \Rightarrow \Phi_i(x_o) \neq 0, \left.\frac{d\Phi_i}{dx}\right|_{x=x_o} \neq 0, \forall i$$

$$\Rightarrow \left.\frac{dV}{dx}\right|_{x=x_o} > 0, \qquad \text{by (3.1.5)}$$

Hence, by continuity, $\Phi_i \neq 0$, $d\Phi_i/dx \neq 0$ and $dV/dx > 0$ in some neighbourhood of x_o; in particular, we are allowed to divide by $||\Phi_i||^2$ in this neighbourhood.

Step 2

Use the Cauchy-Schwarz inequality, together with (3.1.4), (3.1.5) to obtain a differential inequality for V. (This is valid, since the Killing form is positive definite).

We have, for each i=1,2,3, by Cauchy-Schwarz:

$$<\Phi_i,\frac{d\Phi_i}{dx}>^2 \leqq ||\Phi_i||^2 \ ||\frac{d\Phi_i}{dx}||^2$$

$$\Rightarrow \quad ||\frac{d\Phi_i}{dx}||^2 \geqq \frac{<\Phi_i,\frac{d\Phi_i}{dx}>}{||\Phi_i||^2}$$

$$= \frac{V^2}{||\Phi_i||^2}, \text{ by } (3.1.4)$$

$$\therefore \quad \frac{dV}{dx} \geqq \sum_{i=1}^{3} \frac{V^2}{||\Phi_i||^2}, \text{ by } (3.1.5) \qquad (3.1.9)$$

Step 3

Solve (3.1.4) in a neighbourhood of x_o in terms of the function:

$$W(x) = \int_{x_o}^{x} dx'V(x') \Leftrightarrow \frac{dW}{dx} = V(x),\ W(x_o) = 0$$

Then, if $||\phi_i(x_o)||^2 = A_i^2$, (3.1.4) is solved by:

$$||\Phi_i(x)||^2 = A_i^2 + 2W(x) \tag{3.1.10}$$

and equ (3.1.9) becomes:

$$\frac{d^2W}{dx^2} \geq \sum_{i=1}^{3} \frac{1}{(A_i^2+2W)}\left(\frac{dW}{dx}\right)^2 \geq \frac{3}{(A^2+2W)}\left(\frac{dW}{dx}\right)^2 \tag{3.1.11}$$

where $A = \max A_i$.

Step 4

Finally, let $W_1 = \frac{A^2}{2}+W$. Then (3.1.11) implies:

$$\frac{d^2W_1}{dx^2} \geq \frac{3}{2W_1}\left(\frac{dW_1}{dx}\right)^2, \quad W_1(x_o) = \frac{A^2}{2}, \quad \left.\frac{dW}{dx}\right|_{x=x_o} = V_o \tag{3.1.12}$$

Since $W_1 > 0$ in a neighbourhood of x_o, we may write this as:

$$W_1\frac{d^2W_1}{dx^2} - \frac{3}{2}\left(\frac{dW_1}{dx}\right)^2 \geq 0$$

$$\Leftrightarrow \quad -W_1^{5/2}\frac{d^2}{dx^2}\left(W_1^{-\frac{1}{2}}\right) \geq 0$$

$$\therefore \quad \frac{d^2(W_1^{-\frac{1}{2}})}{dx^2} \leq 0 \qquad \text{(provided } W_1 > 0\text{)} \tag{3.1.13}$$

Integrate (3.1.13) twice, using the initial conditions in (3.1.12). We obtain, after the first integration:

$$\frac{d}{dx}(W_1^{-\frac{1}{2}}) + \frac{\sqrt{2}}{A^2} V_o \leq 0 \qquad (3.1.14)$$

and, after the second integration:

$$W_1^{-\frac{1}{2}} - \frac{\sqrt{2}}{A} + \frac{\sqrt{2}}{A} V_o(x-x_o) \leq 0$$

$$\Rightarrow \qquad W_1(x) \geq \frac{A^2}{2} \frac{1}{\left|\frac{V_o(x-x_o)-1}{A^2}\right|^2} \qquad (3.1.15)$$

Equ (3.1.7) is now an immediate consequence of (3.1.14), and equ (3.1.8) follows from equ (3.1.10):

$$||\Phi_i||^2 = A_i^2 - A^2 + 2W_1 \qquad \underline{\text{QED}}$$

Hence, each $||\Phi_i||$ is bounded below by a function with a simple pole. Indeed, it is easily seen that any pole singularity of a solution of (3.1.2) must be a simple pole singularity. This is in fact an essential feature of Nahm's construction - monopole solutions are constructed from normalizable solutions of the dimensionally reduced Dirac equation in the presence of solutions of equ (3.1.2), and the pole singularities ensure that there are just the right number of normalizable solutions.

3.2 The Complex Soliton Solution

We shall now construct a necessarily complex non-singular solution of equ (3.1.2) for an SU(2) gauge group. The construction is exactly analagous to that of the axisymmetric 1-vortex. We reduce to $|R^1$ by demanding:

$$\Delta_k(x) = e^{i(mx_2+ax_3+bx_4)}\tilde{\Delta}_k(x) \qquad (3.2.1)$$

$\Rightarrow$ $\tilde{\Delta}_k$ must satisfy the d=1 Helmholtz equation

$$\frac{d^2}{dx^2}\tilde{\Delta}_k = \lambda^2\tilde{\Delta}_k, \quad \lambda^2 = m^2+c^2 = m^2+a^2+b^2 \qquad (3.2.2)$$

Prasad's superposition theorem implies, for solutions satisfying (3.2.1):

$$||\Phi_1||^2 = ||A_3||^2 = a^2 - \frac{d^2}{dx^2}\ln\det\tilde{D}^{(N)} \qquad (3.2.3)$$

$$||\Phi_2||^2 = ||A_4||^2 = b^2 - \frac{d^2}{dx^2}\ln\det\tilde{D}^{(N)}$$

$$||\Phi_3||^2 = ||A_2||^2 = m^2 - \frac{d^2}{dx^2}\ln\det\tilde{D}^{(N)}$$

$$\Rightarrow \quad ||\Phi||^2 \underset{def}{=} ||\Phi_1||^2 + ||\Phi_2||^2 + ||\Phi_3||^2$$

$$= \lambda^2 - 3\frac{d^2}{dx^2}\ln\det\tilde{D}^{(N)}$$

$$\xi = -\tfrac{1}{2}\frac{d^4}{dx^4}\ln\det\tilde{D}^{(N)}$$

In the a_1 ansatz, in order to guarantee non-singularity, choose $\tilde{\Delta}_o$ to be the non-vanishing solution of equ (3.2.2), which is symmetric under $x \to -x$, namely:

$$\tilde{\Delta}_o(x) = 2\cosh\lambda x = e^{\lambda x} + e^{-\lambda x} \qquad (3.2.4)$$

We can use equ (2.3.3) to write down the corresponding gauge field configuration; this is simplified if we use the global SO(3) invariance of equs (3.1.2):

$$\Phi_i \to R_{ij}\Phi_j, \qquad R\varepsilon SO(3)$$

to choose $a=b=0$, $\lambda=m\neq 0$:

$$A = A_1 = -\frac{m}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (3.2.5)$$

$$\Phi_1 = A_3 = -\frac{m}{2}\begin{pmatrix} 0 & 1+\tanh mx \\ 1-\tanh mx & 0 \end{pmatrix}$$

$$\Phi_2 = A_4 = -\frac{im}{2}\begin{pmatrix} 0 & \tanh mx+1 \\ \tanh mx-1 & 0 \end{pmatrix}$$

$$\Phi_3 = A_2 = -\frac{i}{2}\begin{pmatrix} m\tanh mx & 0 \\ 0 & -m\tanh mx \end{pmatrix}$$

A is removed by applying a gauge transformation such that:

$$\frac{dg}{dx} = gA$$

ie $$g(x) = \begin{pmatrix} e^{-mx/2} & 0 \\ 0 & e^{mx/2} \end{pmatrix}$$

after which we obtain the simplified expressions:

$$\Phi_1 = -\frac{m}{2}\,\text{sech}mx.\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\frac{m}{2}\,\text{sech}mx.\sigma_1 \qquad (3.2.6)$$

$$\Phi_2 = \frac{m}{2}\,\text{sech}mx.\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{m}{2}\,\text{sech}mx.\sigma_2$$

$$\Phi_3 = -\frac{im}{2}\tanh mx.\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\frac{im}{2}\tanh mx.\sigma_3$$

and we recognize the appearance of functions which are familiar from the profiles of sine-Gordon and ϕ^4-theory solitons. We call this solution the 'complex soliton' or 'complex wall' solution.

Using equ (3.2.3), we have:

$$||\Phi||^2 = m^2(1 - 3\,\text{sech}^2 mx) \qquad (3.2.7)$$

$$\xi = m^4\text{sech}^2 mx(3\,\text{sech}^2 mx - 2)$$

and the total energy E(R) in an interval (-R,R) is given by:

$$E(R) = 2m^3 \operatorname{sech}^2 mR . \tanh mR \qquad (3.2.8)$$

$$\sim 2m^3 e^{-2mR}, \qquad R \to \infty$$

so, $E(R) \to 0$ as $R \to \infty$, as in the case of complex vortices - the "complex soliton" is another example of a "voidon" (Dolan 1978).

These functions are plotted in Figure 4. Note that the fields approach their asymptotic values much more quickly than vortices, since there is an exponential decay rather than a power law decay to the Higgs vacuum.

For completeness, let us compute the full Δ-chain generated by the complex soliton solution. After reduction to $|R^1$, the Δ-chain equs (2.3.6) read:

$$\left(\frac{d}{dx} + m\right)\tilde{\Delta}_k = -i\gamma\overline{\tilde{\Delta}}_{k+1} \qquad (3.2.9)$$

$$\left(\frac{d}{dx} - m\right)\tilde{\Delta}_{k+1} = i\gamma\tilde{\Delta}_k$$

Also, since each $\tilde{\Delta}_k$ satisfies the Helmholtz equ (3.2.2), we may write:

$$\tilde{\Delta}_k(x) = \alpha_k e^{\lambda x} + \beta_k e^{-\lambda x}$$

and equs (3.2.9) are then equivalent to the recurrence relations:

$$\alpha_{k+1} = \frac{i\gamma}{(\lambda-m)}\alpha_k, \qquad \beta_{k+1} = \frac{-i\gamma}{(\lambda+m)}\beta_k$$

$$\Leftrightarrow \quad \alpha_k = \frac{(i\gamma)^k}{(\lambda-m)^k}\alpha_o, \qquad \beta_k = \frac{(-i\gamma)^k}{(\lambda+m)^k}\beta_o$$

So, the general solution of equs (3.2.9) is given by:

$$\tilde{\Delta}_k = (i\gamma)^k\left(\alpha_o\frac{e^{\lambda x}}{(\lambda-m)^k} + (-1)^k\beta_o\frac{e^{-\lambda x}}{(\lambda+m)^k}\right) \qquad (3.2.10)$$

For the complex soliton solution, $\alpha_o=\beta_o=1$, so its corresponding Δ-chain is:

$$\tilde{\Delta}_k = (i\gamma)^k\left[\frac{e^{\lambda x}}{(\lambda-m)^k} + (-1)^k\frac{e^{-\lambda x}}{(\lambda+m)^k}\right] \qquad (3.2.11)$$

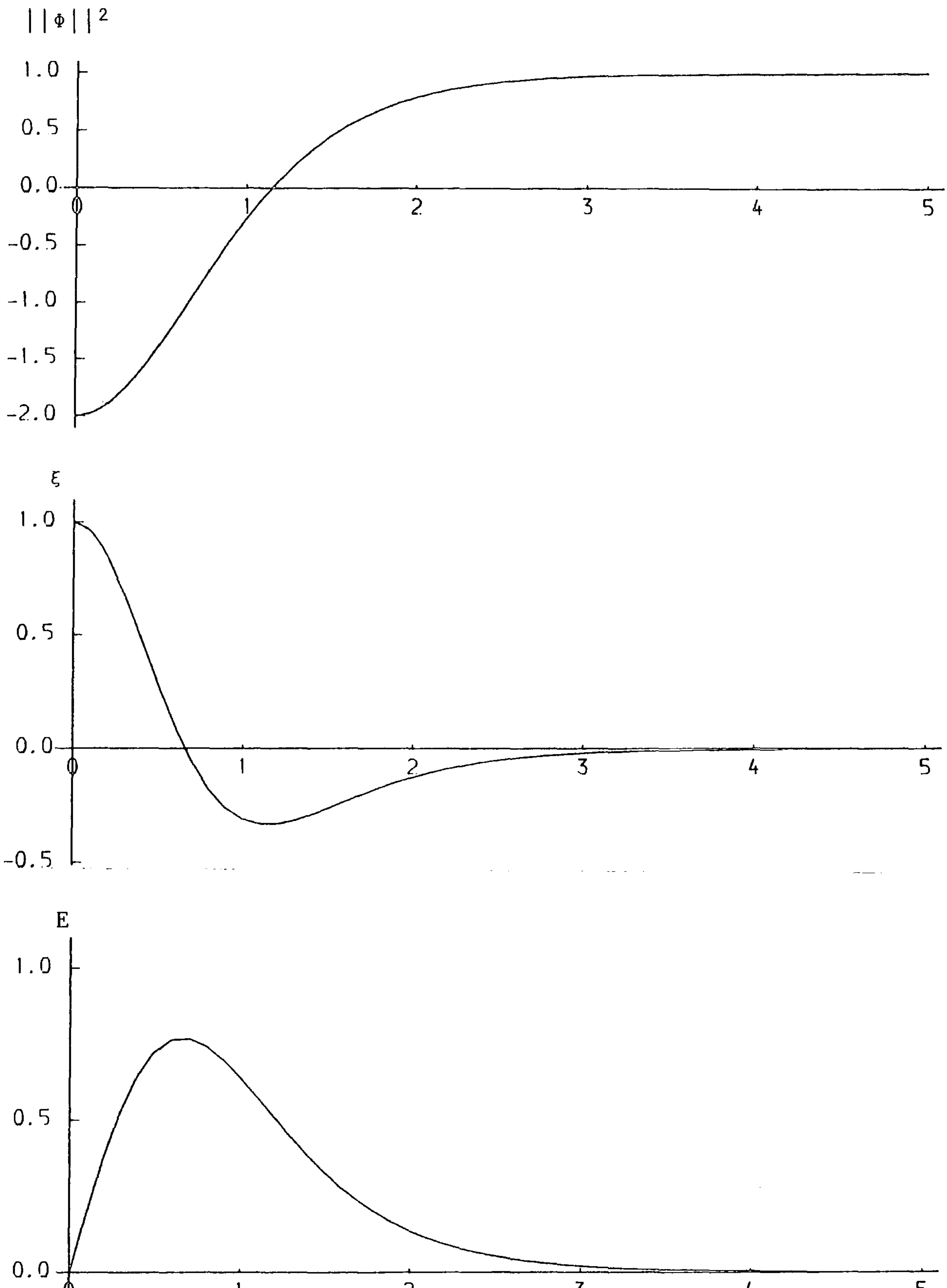
This Δ-chain does not generate any new solutions in the higher ansätze, since it is easily seen that:

$$\det\tilde{D}^{(2)} = \text{constant}$$

$$\det\tilde{D}^{(N)} = 0, \qquad \forall N \geq 3$$

This is not too surprising, since we are solving the first order system (3.1.2), which has a finite number of degrees of freedom.

Fig 4



$||\Phi||^2(x)$, $\xi(x)$, and $E(x)$ for the complex soliton solution.

CHAPTER 4

SEPARATED VORTEX SOLUTIONS I: BÄCKLUND TRANSFORMATIONS APPROACH

4.1 Separated Charge 2 Solutions

This chapter represents a first attempt at the construction of finitely separated vortex solutions. A more general and more complete approach is adopted in Chapter 5. Recall, from Chapter 2, that the behaviour of the axially symmetric N-vortex solutions was found to be more or less analogous to that of the axially symmetric N-monopole solutions. In contrast we shall find that, at least for large values of the separation parameter, the behaviour of the separated vortex solutions is remarkably different to that of the separated monopole solutions.

Again, the actual construction of the solutions is much simpler than in the monopole case. In analogy with the 't Hooft ansatz for separated instantons, we simply linearly superpose the functions Δ_o for axially symmetric 1-vortices in the 1st ansatz, and then integrate up to the Nth ansatz to ensure correct asymptotic behaviour of the fields. The remarkable thing is that, unlike the cases of instantons and monopoles, this procedure does not give rise to uncontrollable singularities in the higher ansätze.

For example, in the N=2 case, we take:

$$\phi_1 = e^{i(ax_3+bx_4)}\tilde{\Delta}_o(x_1,x_2)$$

where $\tilde{\Delta}_o$ is chosen to be a linear sum of two $\tilde{\Delta}_o$'s for axially symmetric 1-vortices situated at the points $(h_1,k_1),(h_2,k_2)$ say,

$$\text{ie} \quad \tilde{\Delta}_o = \alpha I_o(cr_1) + \beta I_o(cr_2) \qquad (4.1.1)$$

where α, β are positive real constants, equivalent up to a common scale factor, and:

$$r_1 = \left[(x_1-h_1)^2+(x_2-k_1)^2\right]^{\frac{1}{2}},$$

$$r_2 = \left[(x_1-h_2)^2+(x_2-k_2)^2\right]^{\frac{1}{2}}$$

Since the Δ-chain equs are linear, integration of the Bäcklund transformations simply gives us a linear superposition of Δ-chains of the type equ (2.3.9):

$$\tilde{\Delta}_k = \alpha\xi_1^k I_k(cr_1) + \beta\xi_2^k I_k(cr_2) \qquad (4.1.2)$$

where

$$\xi_1 = \frac{i\gamma r_1}{\sqrt{2}c(y-y_1)}, \quad \xi_2 = \frac{i\gamma r_2}{\sqrt{2}c(y-y_2)} \quad \varepsilon U(1)$$

and

$$y_1 = \frac{1}{\sqrt{2}}(h_1+ik_1), \qquad y_2 = \frac{1}{\sqrt{2}}(h_2+ik_2)$$

Hence, working in the a_2 ansatz, the fundamental determinant is given by:

$$\det\tilde{D}^{(2)} = \begin{vmatrix} \tilde{\Delta}_o & \tilde{\Delta}_1 \\ \tilde{\Delta}_{-1} & \tilde{\Delta}_o \end{vmatrix}$$

$$= \alpha^2\left[I_o(cr_1)^2-I_1(cr_1)^2\right] + \beta^2\left[I_o(cr_2)^2-I_1(cr_2)^2\right] \qquad (4.1.3)$$

$$+ \alpha\beta\left[2I_o(cr_1)I_o(cr_2) - (\xi_2\xi_1^{-1}+\xi_1\xi_2^{-1})I_1(cr_1)I_1(cr_2)\right]$$

In this case, it is possible to give a "bare hands" proof of non-singularity. Since $\xi_1,\xi_2\varepsilon U(1)$, we have:

$$\xi_2\xi_1^{-1} + \xi_1\xi_2^{-1} = \xi_1\bar{\xi}_2 + \xi_2\bar{\xi}_1$$

$$= 2\mathrm{Re}\,\xi_1\bar{\xi}_2$$

$$= 2\mathrm{Re}\,\xi_1\xi_2^{-1}$$

and $\xi_1\xi_2^{-1}\varepsilon U(1) \Rightarrow |\mathrm{Re}\,\xi_1\xi_2^{-1}| \leq 1$

Hence, from (4.1.3)

$$\det\tilde{D}^{(2)} = \alpha^2[I_o(cr_1)^2-I_1(cr_1)^2] + \beta^2[I_o(cr_2)^2-I_1(cr_2)^2]$$

$$+ 2\alpha\beta[I_o(cr_1)I_o(cr_2) - Re(\xi_1\xi_2^{-1})I_1(cr_1)I_1(cr_2)]$$

$$\geq \alpha^2[I_o(cr_1)^2-I_1(cr_1)^2] + \beta^2[I_o(cr_2)^2-I_1(cr_2)^2]$$

$$+ 2\alpha\beta[I_o(cr_1)I_o(cr_2)-I_1(cr_1)I_1(cr_2)]$$

$$> o, \quad \forall x\in\mathbb{R}^2$$

since $I_o(r) > I_1(r) \geq o, \quad \forall r\in\mathbb{R}$

Hence the charge 2 solution is non-singular.

Numerical Study

Without loss of generality, let us consider displacements along the x_1-axis, centred at the origin, ie let:

$$r_1 = \sqrt{(x_1+h)^2+x_2^2}, \qquad r_2 = \sqrt{(x_1-h)^2+x_2^2} \qquad (4.1.4)$$

Then $Re\xi_1\bar{\xi}_2 = \frac{r^2-h^2}{r_1r_2}$, so we have:

$$\det\tilde{D}^{(2)} = \alpha^2[I_o(cr_1)^2-I_1(cr_1)^2] + \beta^2[I_o(cr_2)^2-I_1(cr_2)^2]$$

$$+ 2\alpha\beta[I_o(cr_1)I_o(cr_2) - \frac{(r^2-h^2)}{r_1r_2}I_1(cr_1)I_1(cr_2)]$$

$$(4.1.5)$$

This solution has been studied numerically for various values of the separation parameter h, with characteristic length $c^{-1}=1$. As in Chapter 2, this was achieved by evaluating equ (4.1.5) and its 1st and 2nd Laplacians on a sufficiently fine grid of points. By virtue of equ (2.2.20), we can identify the vortex locations with the points where $||\Phi||^2$ attains its absolute minimum value of -1. This occurs at the points $(x_1,x_2) = (\pm h_{phys},0)$; we call h_{phys} the physical separation parameter.

We might expect intuitively that $h_{phys} \sim h$; however, this is certainly <u>not</u> the case for large values of h. $||\Phi||^2$ and ξ are plotted along the x_1 axis in figure 5(a) for h=2.5, 5.0, 7.5, 10.0 and in figure 5(b) for h=20, 30, 40, 50, with $\alpha = \beta = 1$. We find numerically:

$$h_{phys} \sim h \qquad \text{for } 0 \leq h \lesssim 0.6$$

and for large values of h we have the table:

h	h_{phys}	ξ_{max}	ξ_{min}
1	0.9	0.16	-0.24
3	1.5	0.342	-0.167
10	2.0	0.687	-0.257
30	2.5	0.875	-0.296
50	2.7	0.925	-0.309
80	2.9	0.951	-0.316

So, for large h, $\underline{h}_{phys}$ <u>varies extremely slowly with h</u>.

Also, note that the energy density profile of an isolated 1-vortex (Fig 1) has:

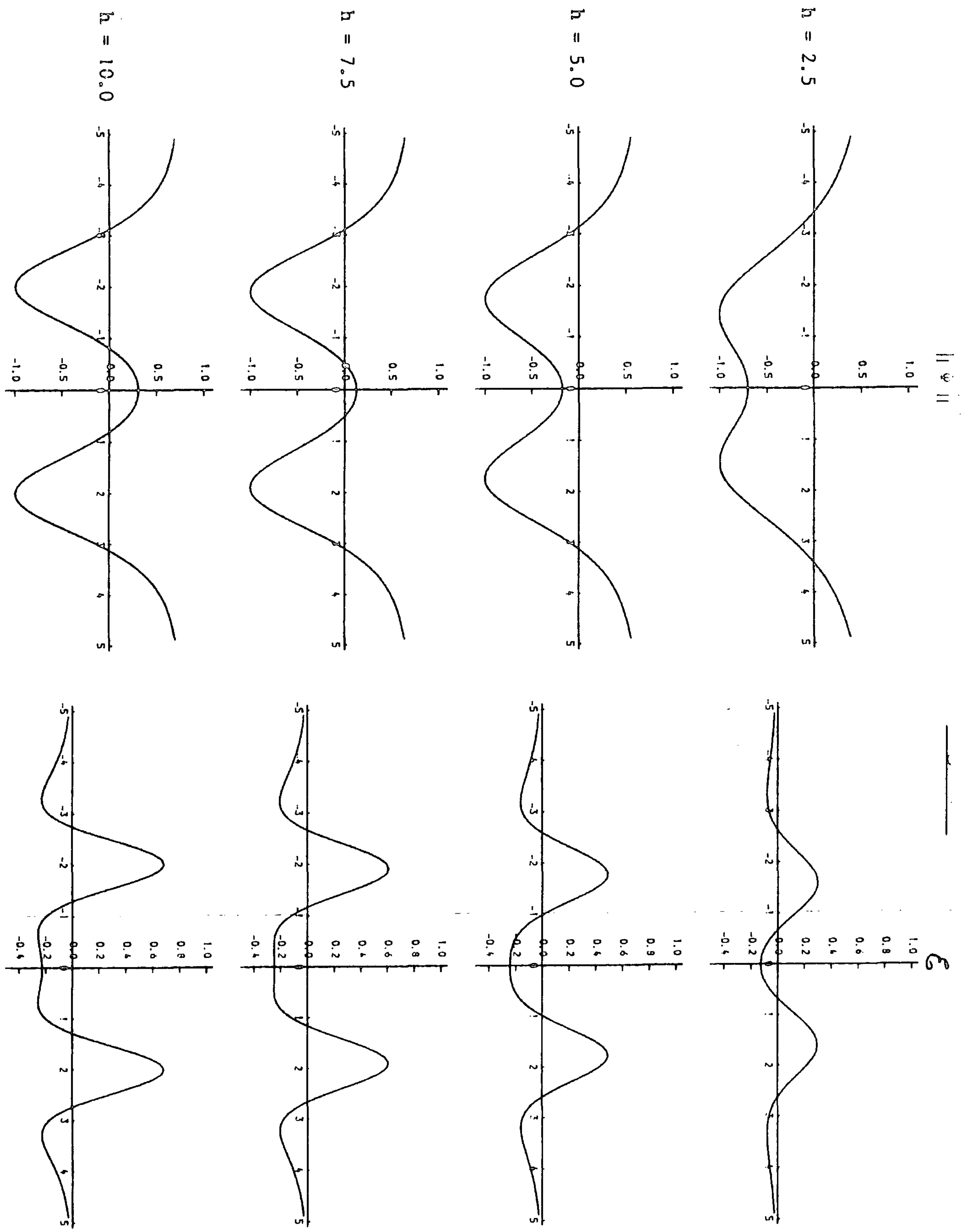
$$\xi_{max} = 0.5, \qquad \xi_{min} \cong -0.02$$

So, the energy density of the separated solution is clearly not converging to that of two isolated 1-vortices; it is much more strongly peaked along the x_1-axis.

Variation of the parameter β/α does not alter this behaviour significantly; for sufficiently large values of h it simply has the effect of translating the profiles along the x_1-axis, with the actual separation of the vortices remaining unchanged.

The vortex profiles along the x_2-axis are even more surprising. Whereas the separation along the x_1-axis varies very slowly with h, there is an 'elongation' along the x_2-axis, roughly of the same order as h. This behaviour is clearly shown in the contour plots of Figures 6(a) and 6(b), where the energy density is plotted between $-4 \leq x_1, x_2 \leq 4$ and $-4 \leq x_1 \leq 4$, $-8 \leq x_2 \leq 8$ respectively, for various values of the separation parameter. The corresponding surface plots are shown in Figures 7(a) and 7(b).

Thus we reach the conclusion that, at least within this class of separated solutions, it is impossible to approach the energy-density and Higgs field profiles of two isolated 1-vortices at large separation. In fact, comparison with the plots of the complex soliton in Figure 4 seems to suggest that the energy density profile of the separated 2-vortex is approaching that of two complex solitons.


h = 2.5
h = 5.0
h = 7.5
h = 10.0
||ψ||
ω

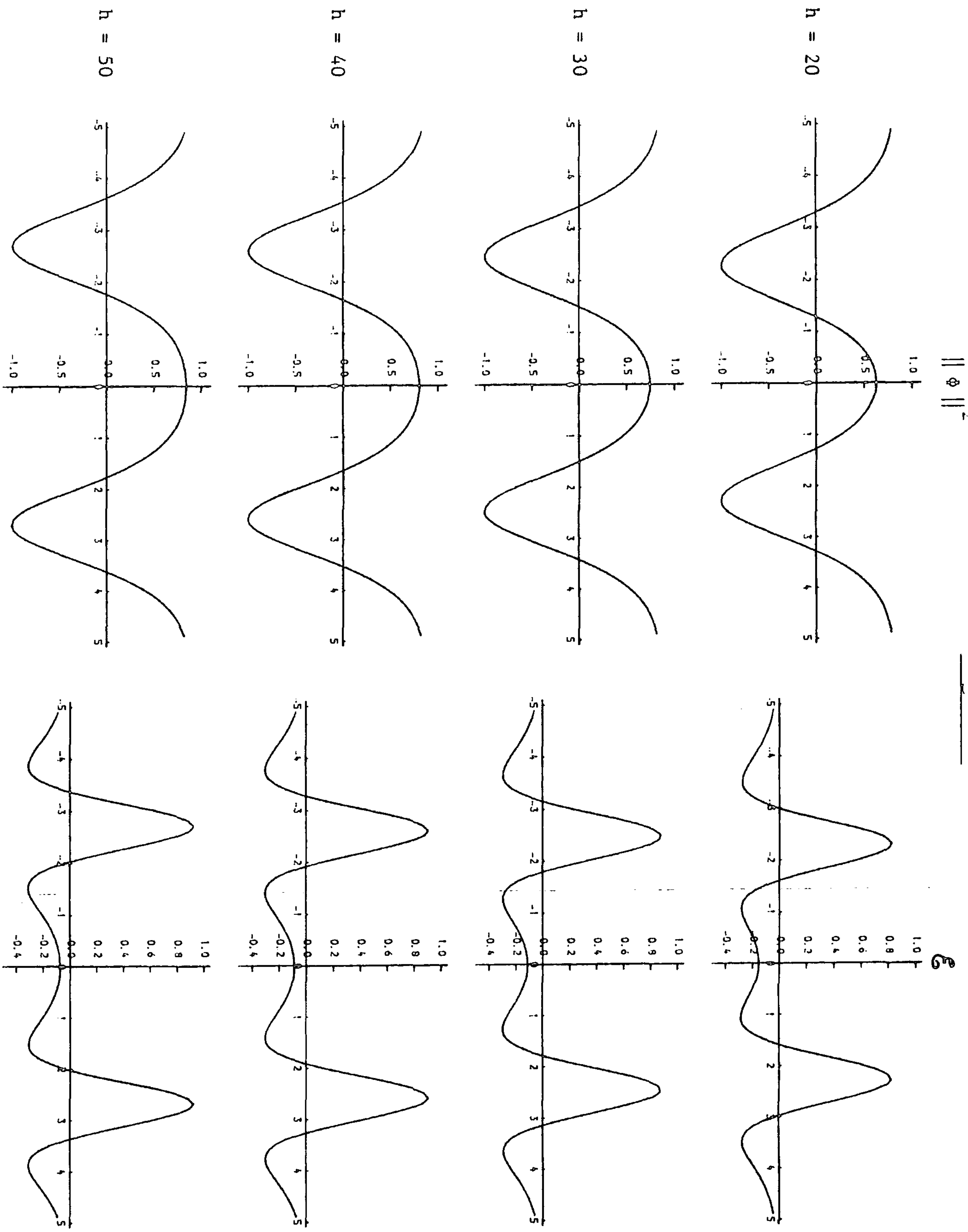
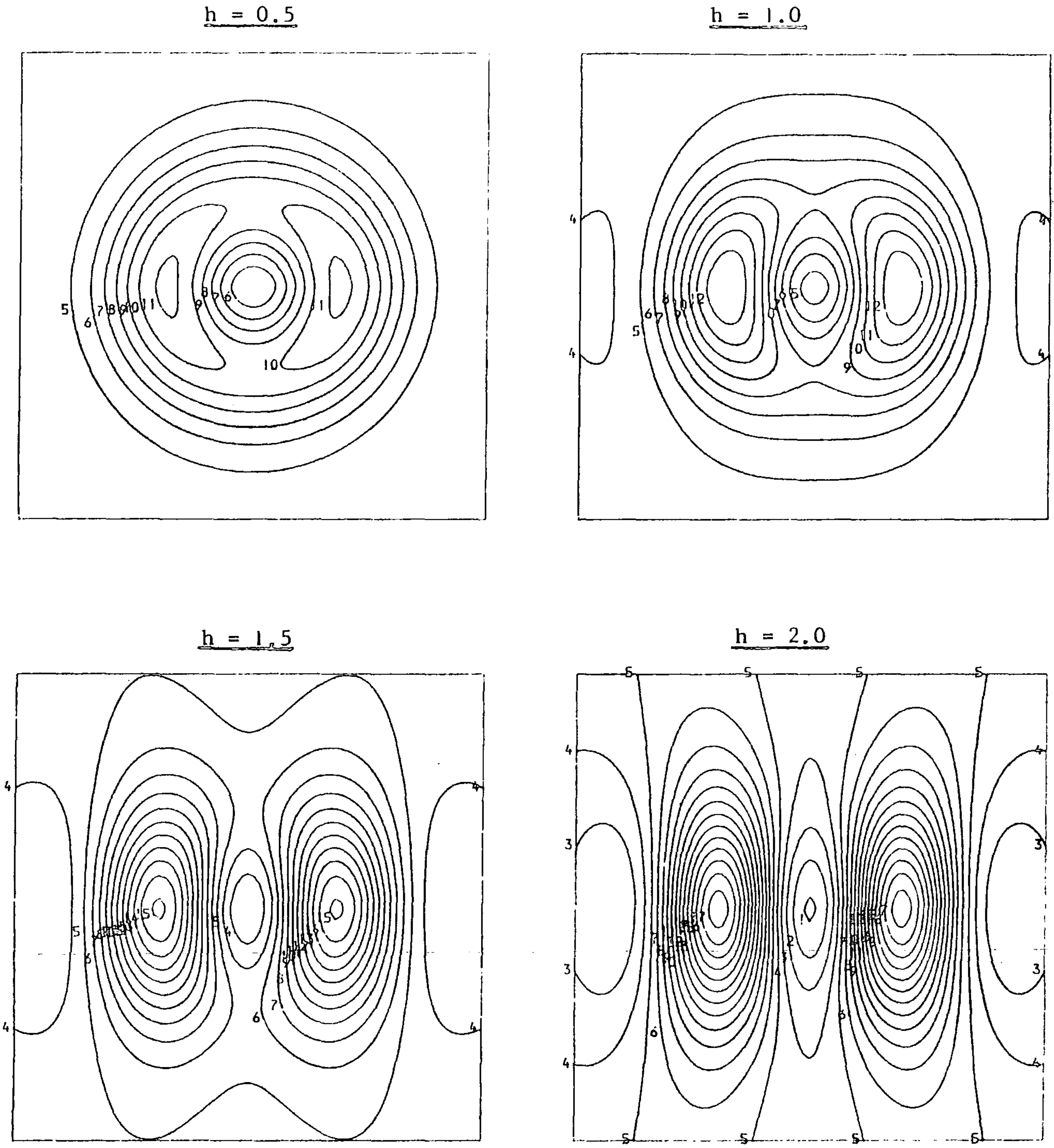

h = 20
h = 30
h = 40
h = 50
$\|\Phi\|^2$

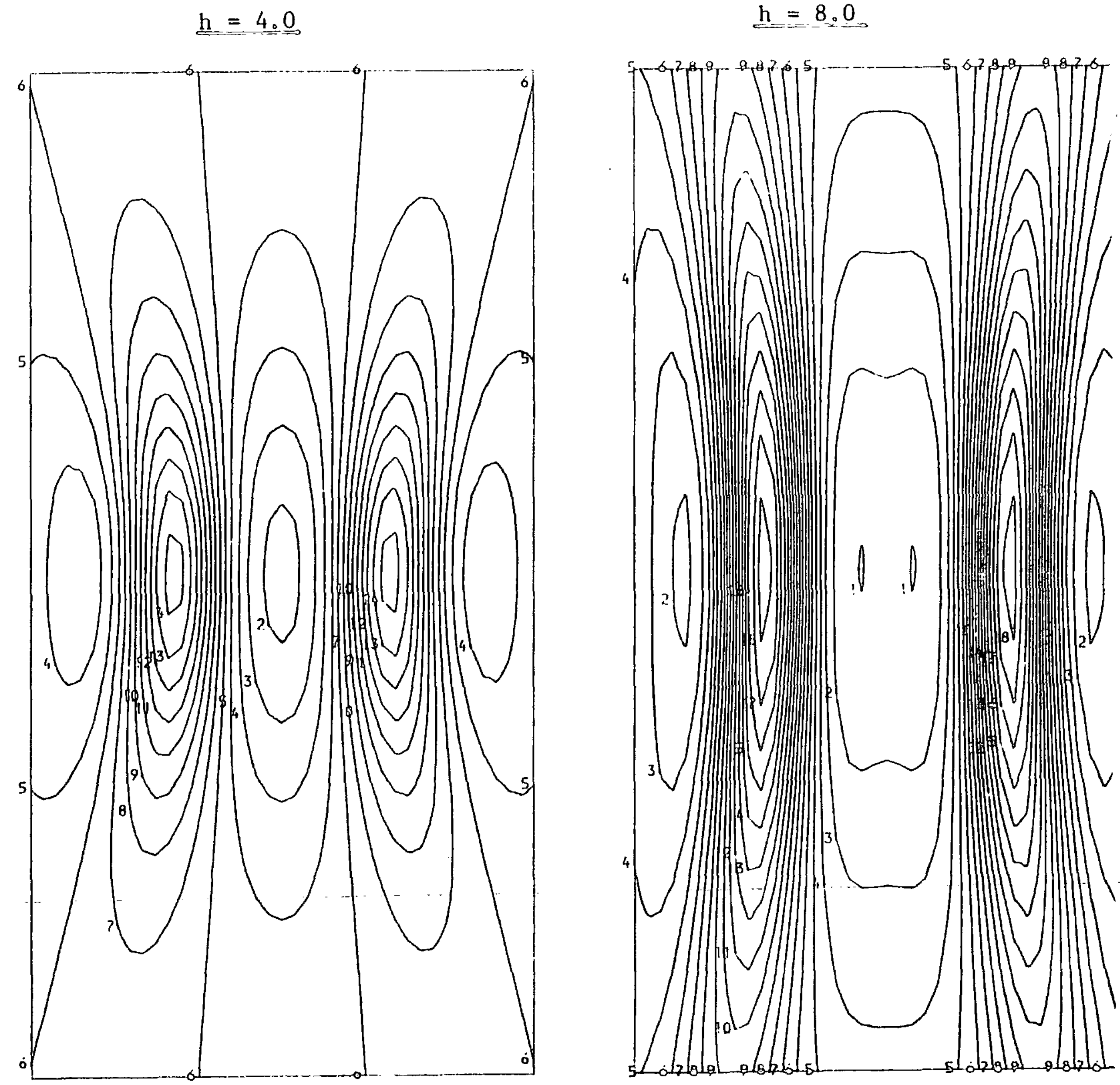
Fig 6(a)



Contour Plots of Energy density for the separated 2-string at h = 0.5,1.0, 1.5,2.0.

<u>Contour Key:</u> At Contour n, $\mathcal{E} = -0.1 + 0.02n$.

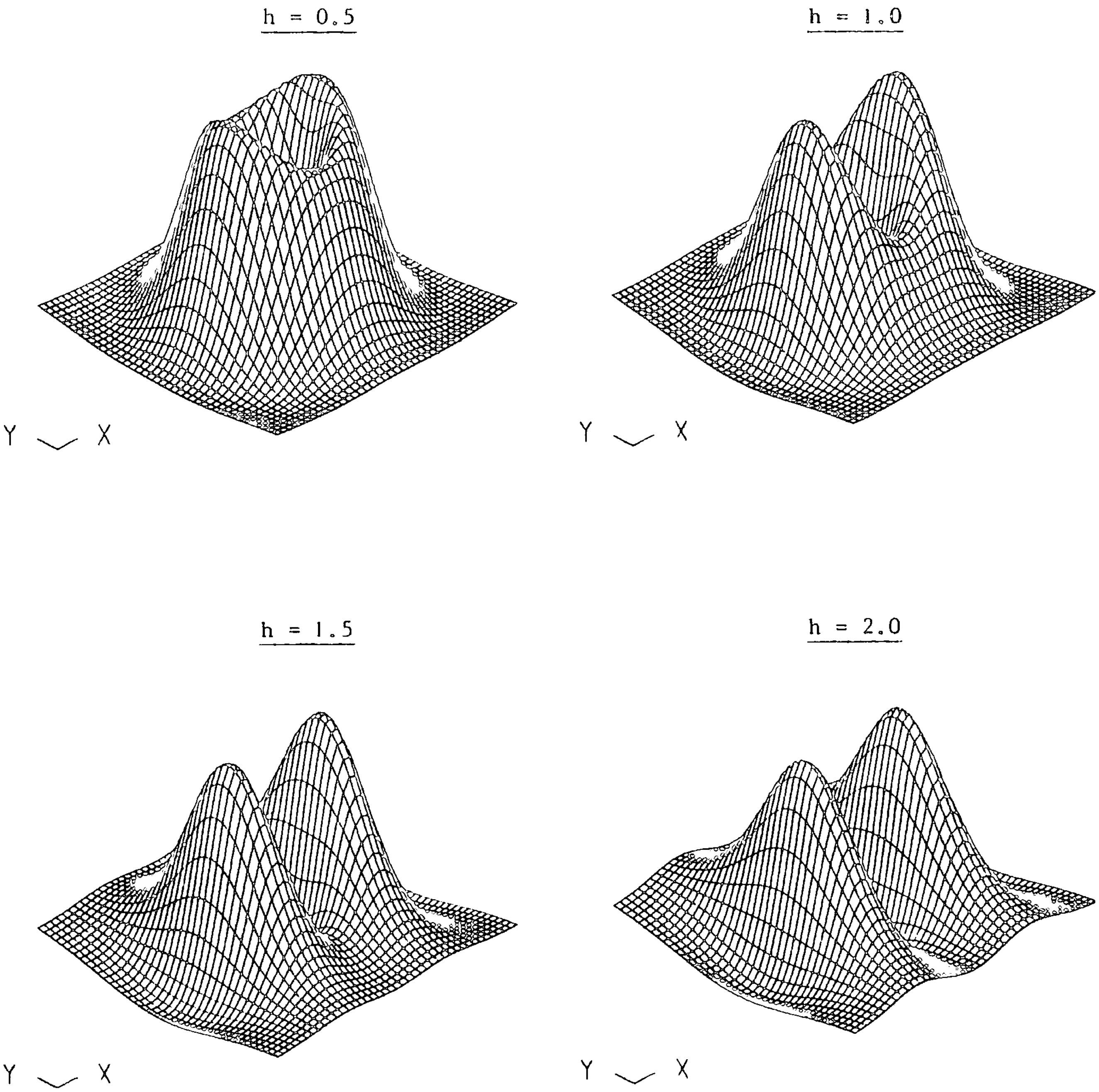
Fig 6(b)



Contour Plots of energy density for the separated 2-string h = 4.0 and 8.0

Contour Key: At contour n, $\mathcal{E} = -0.3 + 0.05n$.

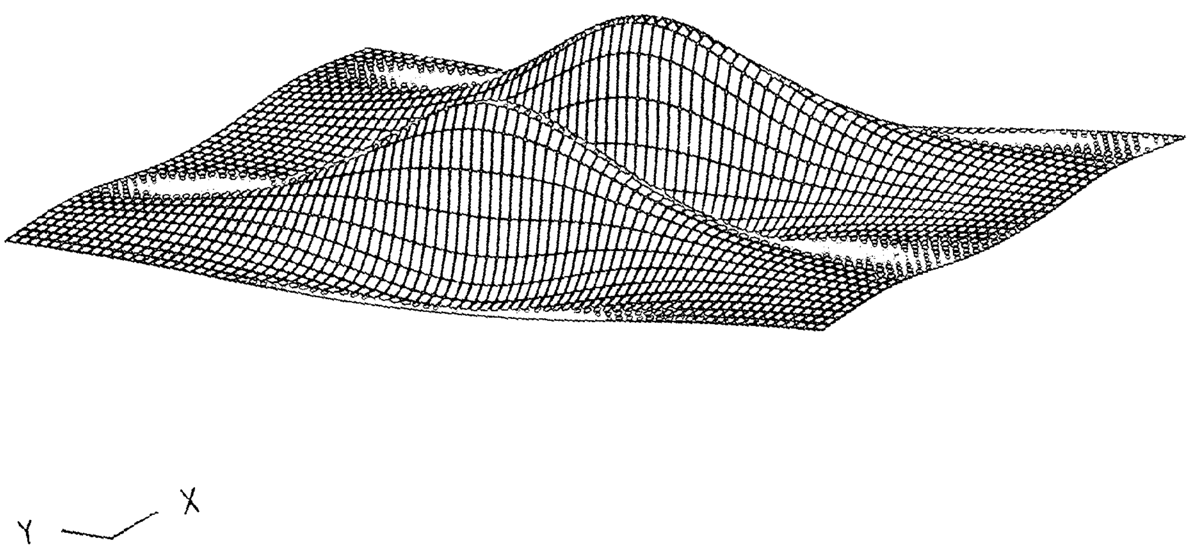
Fig 7(a)



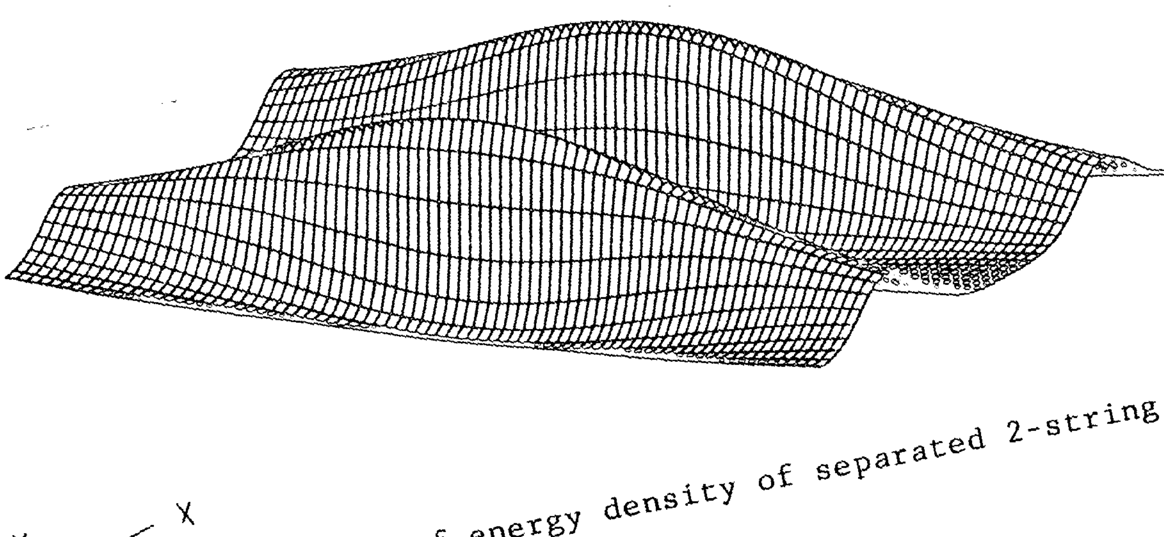
Surface Plots of energy density of separated 2-string
h=0.5, 1.0, 1.5, 2.0.

Fig 7(b)

h = 4.0



h = 8.0



Surface Plots of energy density of separated 2-string h=4.0, 8.0.

Asymptotic Behaviour as $h\to\infty$

The above numerical results are so counter-intuitive that we clearly need some analytical results to test their validity. To this end, let us perform an asymptotic expansion of equ (4.1.5) as $h\to\infty$, for fixed $x,y \ll h$.

Explicitly, we shall assume that:

$$x,y \lesssim O(\log h) \Rightarrow e^x = O(h) \text{ etc},$$

and ignore terms of order x^2/h^2, y^2/h^2. Thus we have:

$$r_1 \sim h\left(1 + \frac{x}{h} + \frac{y^2}{h^2}\right), \qquad r_2 \sim h\left(1 - \frac{x}{h} + \frac{y^2}{h^2}\right) \qquad (4.1.6)$$

$$\Rightarrow \quad \frac{1}{r_1} \sim \frac{1}{h}\left(1 - \frac{x}{h}\right), \qquad \frac{1}{r_2} \sim \frac{1}{h}\left(1 + \frac{x}{h}\right)$$

$$\Rightarrow \quad \frac{-(r^2-h^2)}{r_1 r_2} \sim 1$$

Inserting the asymptotic expansions (A.5) into equ (4.1.5), we obtain:

$$\det\tilde{D}^{(2)} \sim \alpha^2 \frac{e^{2r_1}}{2\pi r_1^2} + \beta^2 \frac{e^{2r_2}}{2\pi r_2^2} + 2\alpha\beta \frac{e^{r_1+r_2}}{2\pi\sqrt{r_1 r_2}}\left(2 - \frac{1}{4r_1} - \frac{1}{4r_2}\right)$$

and, using equs (4.1.6) this becomes:

$$\det\tilde{D}^{(2)} \sim \frac{e^{2h}}{2\pi h^2} e^{y^2/h}(\alpha^2 e^{2x} + \beta^2 e^{-2x}) + \frac{\alpha\beta}{\pi} \frac{e^{2h}}{h}\left(2 - \frac{1}{2h}\right)$$

$$\sim e^{2h} \frac{e^{y^2/h}}{2\pi h^2} (\alpha^2 e^{2x} + \beta^2 e^{-2x} + 4\alpha\beta h) \qquad (4.1.7)$$

If we reparameterize α,β as follows:

$$\alpha = \delta e^{-x_o}, \quad \beta = \delta e^{x_o} \Rightarrow x_o = \tfrac{1}{2}\ln\beta/\alpha$$

then equ (4.1.7) becomes:

$$\det\tilde{D}^{(2)} \sim 2\delta^2 \frac{e^{2h}}{\pi h^2} e^{y^2/h}\left[\sinh^2(x-x_o) + h\right] \qquad (4.1.8)$$

So, for large h, varying β/α merely translates the profiles a distance x_o along the x-axis, as discovered previously from numerical calculations. So, without loss of generality, we may set $x_o=0$ to obtain:

$$\ln\det\tilde{D}^{(2)} \sim \frac{y^2}{h} + \ln(\sinh^2 x + h) + \text{constant}$$

$$\Rightarrow \quad \nabla^2 \ln\det\tilde{D}^{(2)} \sim 2h. \frac{1+2\sinh^2 x}{(h+\sinh^2 x)^2}$$

$$\therefore \quad ||\Phi||^2 \sim 1 - 4h \cdot \frac{1+2\sinh^2 x}{(h+\sinh^2 x)^2} \tag{4.1.9}$$

So, for $x,y \lesssim O(\log h)$, $||\Phi||^2$ is independent of y. This clearly gives the 'elongation' behaviour along the y-axis which was observed previously.

Also, analysis of (4.1.9) gives:

$$||\Phi||^2_{min} \sim -1 \text{ at } \sinh^2 x \sim h, \text{ as } h\to\infty$$

$$\Rightarrow \quad \sinh^2 h_{phys} \sim h, \text{ as } h\to\infty$$

$$\therefore \quad h_{phys} \sim \log 2\sqrt{h}, \text{ as } h\to\infty \tag{4.1.10}$$

This is in good agreement with the above numerical results. To obtain some idea of how slow the growth of h_{phys} is, note that $h = 5000 \Rightarrow h_{phys} \sim 4.95$.

4.2 Separated Charge N Solutions

The obvious ansatz for separated N-vortex configurations which generalizes that of the separated 2-vortex of equ (4.1.2) is to choose:

$$\tilde{\Delta}_o = \sum_{\ell=1}^{N} \alpha_\ell I_o(cr_\ell), \qquad \alpha_\ell > 0 \tag{4.2.1}$$

$$r_\ell = \left[(x_1 - x_1^{(\ell)})^2 + (x_2 - x_2^{(\ell)})^2\right]^{\frac{1}{2}}$$

where $(x_1^{(\ell)}, x_2^{(\ell)})$ are N fixed points in the x_1x_2-plane, and then integrate this up to the Nth ansatz.

This gives us the Δ-chain:

$$\tilde{\Delta}_k = \sum_{\ell=1}^{N} \alpha_\ell \xi_\ell{}^k I_k(cr_\ell), \qquad \alpha_\ell > 0 \tag{4.2.2}$$

where

$$\xi_\ell = \frac{i\gamma r_\ell}{\sqrt{2}c(y - y^{(\ell)})} \quad \varepsilon U(1)$$

and

$$y^{(\ell)} = \frac{1}{\sqrt{2}}(x_1^{(\ell)} + ix_2^{(\ell)})$$

The proof that this gives us a non-singular solution is a combination of the above proofs of non-singularity of the separated 2-vortex and axially symmetric N-vortex solutions. First note that the term-by-term expansion of $\det\tilde{D}^{(N)}$ can be arranged into sums of complex conjugate pairs:

$(\xi+\xi^{-1})$ x Product of Bessel functions

where ξ is a product of $\xi_\ell^{\pm 1}$, $\ell=1,\dots,N$; hence $\xi \varepsilon U(1)$.

So, $\xi+\xi^{-1} = \xi+\bar{\xi} = 2\mathrm{Re}\,\xi$, and this is bounded by ± 2. Hence we deduce that it is sufficient to prove:

$$\delta^{(N)} = \begin{vmatrix} I_o(r_1) & I_1(r_1) & I_2(r_1) & \cdots & I_{N-1}(r_1) \\ I_1(r_2) & I_o(r_2) & I_1(r_2) & & \vdots \\ I_2(r_3) & I_1(r_3) & I_o(r_3) & \ddots & \vdots \\ \vdots & & & \ddots & \vdots \\ I_{N-1}(r_N) & \cdots & \cdots & \cdots & I_o(r_N) \end{vmatrix}$$

$$\Rightarrow \quad \delta^{(N)} > 0 \qquad \forall r_1,\dots,r_N \varepsilon \mathbb{R}$$

<u>Proof</u> (cf Thm (2.3.3))

Using the integral representation (A.7) for Bessel functions, and the Weyl identity (B.3):

$$\delta^{(N)} = \frac{1}{N!} \sum_{\sigma\varepsilon S_N} \int_o^{2\pi} \frac{d\theta_1}{2\pi} \frac{d\theta_N}{2\pi}\, e^{(r_1\cos\theta_1+\dots+r_N\cos\theta_N)}$$

$$W(\theta_{\sigma(1)},\dots,\theta_{\sigma(N)})$$

$$= \frac{1}{N!} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi} e^{(r_1 \cos\theta_1 + \ldots + r_N \cos\theta_N)}$$

$$\prod_{i<j}^{N} 4\sin^2\left(\frac{\theta_i - \theta_j}{2}\right) > 0$$

since integrand is positive. <u>QED</u>

<u>Remark</u>

We have in fact constructed a much larger family of non-singular solutions than we originally intended; the above proof actually goes through for <u>any</u> ansatz a_k, not only k=N. For example, equ (4.2.1) clearly gives a non-singular solution in the a_1-ansatz, which we might interpret as a 'distorted' 1-vortex solution. In Chapter 5, this family of solutions will be extended much further.

CHAPTER 5

SEPARATED VORTEX SOLUTIONS II: TWISTOR SPACE APPROACH

In this chapter, we construct an infinite dimensional parameter space of complex vortex solutions, strictly containing those found in Chapter 2 and Chapter 4.

Recall that, in Chapter 2, Bäcklund transformations were applied to a non-singular axially symmetric solution in the a_1 ansatz to construct non-singular axially symmetric solutions in the a_n ansätze ($n \geq 2$). Separated solutions were constructed in Chapter 4 by applying Bäcklund transformations to linearly superposed solutions in the a_1 ansatz.

The point of view here is somewhat different; we shall make more direct use of the twistor space formalism underlying the Atiyah-Ward construction. Explicitly, we shall write down an ansatz for the transition functions of the holomorphic vector bundles over $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ corresponding to non-singular self-dual gauge fields on $\mathbb{R}^2$ via the Atiyah-Ward correspondence. This ansatz may be regarded as being analagous to (though much simpler than) the Corrigan-Goddard ansatz for monopoles (Corrigan & Goddard 1981). As in the case of the finitely separated monopole solutions, this approach serves to illustrate the power of twistor space methods over Bäcklund transformation methods.

It is hoped that the relative simplicity of these solutions will help to shed some light on outstanding problems for the monopole solutions. For example, there is still no proof of non-singularity of the Corrigan-Goddard ansatz for finitely separated monopoles, even in the charge 2 case, and the only proof of non-singularity of the general axially symmetric case relies on sophisticated algebraic geometric techniques related to the ADHMN construction (Hitchin 1983). Also, it is not at all clear how the parameters in the Corrigan-Goddard ansatz are related to the physical parameters of the monopoles, for example their positions. Both of these problems are greatly simplified for our class of complex vortex solutions.

5.1 Review of Twistor Space

Twistor space was introduced by Penrose (1967) as a convenient device for discussing the conformal geometry of 4-dimensional Minkowski Space. It is in fact more properly related to the conformal geometry of 4-dimensional complex euclidean space, $\mathbb{C}^4$; results on real euclidean space $\mathbb{R}^4$ and real Minkowski space $\mathbb{R}^{3,1}$ are obtained by restriction to appropriate real subspaces.

A null vector $x_\mu \varepsilon \mathbb{C}^4$ satisfies $x_\mu x_\mu = 0$.

A null 2-plane Z in $\mathbb{C}^4$ is one such that every tangent vector is null.

Given a 2-plane Z in $\mathbb{C}^4$, choose two linearly independent tangent vectors u_μ, v_μ. These define a non-zero 2-form -

$$u \wedge v \varepsilon \Lambda^2 \mathbb{C}^4 \cong \mathbb{C}^6$$

- which is uniquely determined up to complex scalar multiplication, so it defines a unique point in $\mathbb{C}P^5$.

A plane is (anti)-self-dual if a representative of its class of 2-forms is (anti)-self-dual.

An α-plane is a self-dual null plane.

A β-plane is an anti-self-dual null plane.

The construction of twistor space begins with the identification of points $x_\mu \varepsilon \mathbb{C}^4$ with 2x2 complex matrices as follows:

$$x = x_4 + ix.\sigma \qquad \Rightarrow \qquad \det x = x_\mu x_\mu \qquad (5.1.1)$$

This defines an isomorphism $\mathbb{C}^4 \cong S \otimes S'$, where S is the space of complex 2-spinors equipped with the symplectic form $\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

A (projective) twistor has homogeneous co-ordinates given by a pair of complex 2-spinors -

$$Z^\alpha = [\omega, \pi] \varepsilon \mathbb{C}P^3, \qquad (\omega, \pi) \varepsilon S \otimes S' \setminus 0$$

The (projective) line at infinity is defined by -

$$I_\infty = \{[\omega, \pi] ; \quad \pi = 0\} \cong \mathbb{C}P^1$$

Points of $\mathbb{C}P^3 \setminus I_\infty$ are related to points $x_\mu \varepsilon \mathbb{C}^4$ via the equation -

$$\omega = x\pi, \qquad \pi \neq 0 \qquad (5.1.2)$$

It is easily seen that -

(a) For fixed $x \in \mathbb{C}^4$, the set of all $[\omega, \pi]$ satisfying (5.1.2) form a projective line in $\mathbb{CP}^3 \setminus I_\infty$. Conversely, any line in $\mathbb{CP}^3 \setminus I_\infty$ arises from a unique point x.

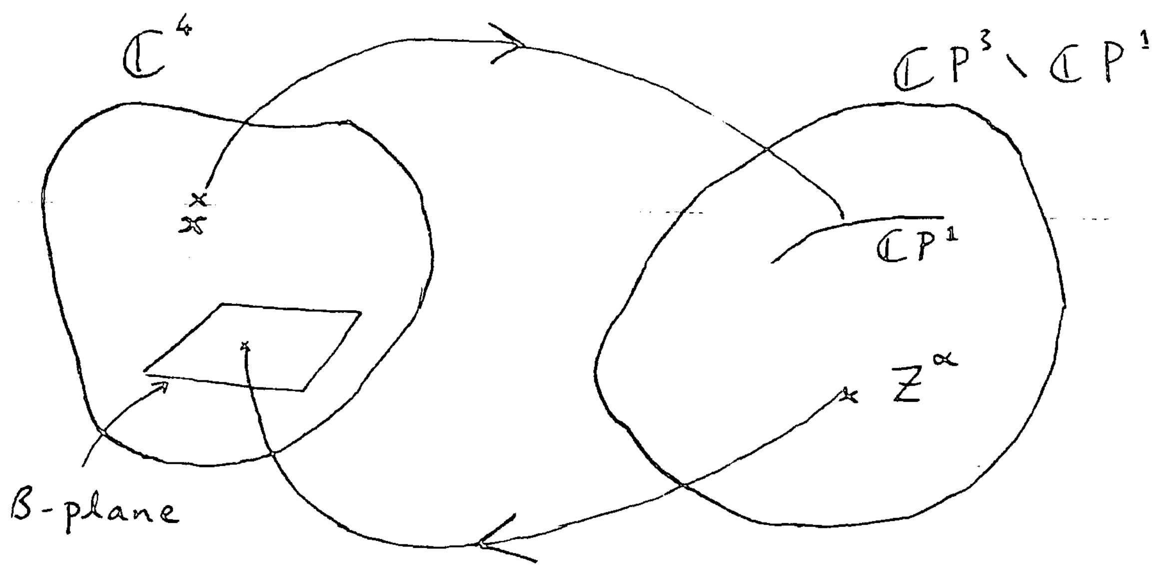
(b) For fixed $[\omega, \pi] \in \mathbb{CP}^3 \setminus \mathbb{CP}^1$, the set of all x satisfying (5.1.2) forms a β-plane in $\mathbb{C}^4$, and conversely.

Hence (5.1.2) defines 1-1 correspondences -

$$\{\text{Points in } \mathbb{C}^4\} \leftrightarrow \{\text{Projective lines in } \mathbb{CP}^3 \setminus \mathbb{CP}^1\} \tag{5.1.3}$$

$$\{\beta\text{-plane in } \mathbb{C}^4\} \leftrightarrow \{\text{Points in } \mathbb{CP}^3 \setminus \mathbb{CP}^1\} \tag{5.1.4}$$

ie



(5.1.3) implies that $\mathbb{C}^4$ can be embedded in the moduli space of <u>all</u> projective lines in $\mathbb{C}P^3$, which, using Plücker co-ordinates, can be identified with a quadratic variety in $\mathbb{C}P^5$, the Klein quadric Q_5 (Atiyah 1979). Q_5 is thus identified as the conformal compactification of $\mathbb{C}^4$, with "lines through infinity in $\mathbb{C}P^3$ corresponding to "points at null infinity" in the extension of $\mathbb{C}^4$. The conformal compactifications $\mathbb{R}^4 \to S^4$, $\mathbb{R}^{3,1} \to \tilde{M}^4$ (compact Minkowski Space) are obtained by restriction to appropriate real subspaces.

In fact, restricting to $x \in \mathbb{R}^4$, equation (5.1.2) defines a bundle projection map -

$$\mathbb{C}P^3 \setminus \mathbb{C}P^1 \to \mathbb{R}^4 \tag{5.1.5}$$

with typical fibre $\mathbb{C}P^1$, and this extends, as above, to the Penrose fibration

$$\mathbb{C}P^3 \xrightarrow{\mathbb{C}P^1} S^4 \tag{5.1.6}$$

The Atiyah-Ward correspondence between self-dual gauge fields on $\mathbb{R}^4$ and holomorphic vector bundles on $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ can either be obtained using (5.1.4) (Ward 1977, 1982b), or using the fibration (5.1.5) to pull back bundles (Atiyah 1979). We shall give a statement of the main theorem, referring to the above for proofs, and a brief description of the construction of the holomorphic vector bundle and the reconstruction of its appropriate gauge field (Corrigan et al 1978).

5.1.1 Lemma

A gauge connection A_μ on $\mathbb{C}^4$ is self-dual if and only if its restriction to any β-plane is flat.

5.1.2 Theorem (Atiyah-Ward Correspondence)

There is a 1-1 correspondence between -

(i) Self-dual $GL(n,\mathbb{C})$-connections on $\mathbb{C}^4$.

(ii) Holomorphic rank n vector bundles over $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ which are trivial when restricted to projective lines.

Note

We obtain self-dual G-connections for subgroups $G \subseteq GL(n,\mathbb{C})$ by imposing certain restrictions on the bundle E eg for self-dual $SL(n,\mathbb{C})$-connections it is required that the determinantal line bundle, detE, be trivial.

The flatness of restrictions of self-dual connections to β-planes is simply a consequence of the equations

$$F_{yz} = F_{\bar{y}\bar{z}} = 0$$

-since it is always possible to choose co-ordinates where $\partial/\partial y$ and $\partial/\partial z$ are tangent to a given β-plane. This in turn implies that the restriction of D_μ to a β-plane is integrable ie that parallel transport of a column vector function $\Psi : Z \to \mathbb{C}^n$ via the equation

$$v^\mu D_\mu \Psi = 0, \qquad \forall v^\mu \text{tangent to } Z$$

is independent of the path chosen on Z.

(Explicitly, we can define -

$$G_Z(x_o,x) = \text{Pexp} \int_{x_o}^{x} A_\mu(x')dx'^\mu; \quad x,x_o \varepsilon Z$$

where the path-ordered integral is independent of the path in Z from x_o to x, and then -

$$v^\mu(\partial_\mu + A_\mu)\Psi_Z = 0, \qquad \forall v^\mu \text{tangent to } Z$$

$$\Leftrightarrow \qquad \Psi_Z(x) = G_Z^{-1}(x,x_o)\Psi_Z(x_o).)$$

Hence, given a β-plane $Z\varepsilon\mathbb{CP}^3\setminus\mathbb{CP}^1$ (by (5.1.4)), we can define the fibre over Z by -

$$E_Z = \{\Psi;\ v^\mu D_\mu \Psi = 0,\ \forall v^\mu \text{tangent to } Z\} \cong \mathbb{C}^n$$

and this defines a vector bundle $E \to \mathbb{CP}^3\setminus\mathbb{CP}^1$ satisfying the requirements of (ii).

We next need to set up co-ordinates and transition functions for this vector bundle. Note first that

$\mathbb{C}P^3 \setminus \mathbb{C}P^1$ is covered by the two standard affine pieces -

$$U_1 = \{[\omega, \pi];\ \pi_1 \neq 0\},\ U_2 = \{[\omega, \pi];\ \pi_2 \neq 0\}$$

Next, given a β-plane Z, choose -

$$x_z^{(1)} = \begin{pmatrix} \omega_1/\pi_1 & 0 \\ \omega_2/\pi_1 & 0 \end{pmatrix} \quad , \pi_1 \neq 0$$

$$x_z^{(2)} = \begin{pmatrix} 0 & \omega_1/\pi_2 \\ 0 & \omega_2/\pi_2 \end{pmatrix} \quad , \pi_2 \neq 0$$

Then $\Psi_z(x_z^{(1)})$ are co-ordinates for E over U_1

and $\Psi_z(x_z^{(2)})$ are co-ordinates for E over U_2

and the co-ordinates on the two patches are related on the overlap $U_1 \cap U_2$ by -

$$\Psi_z(x_z^{(1)}) = G_z(x_z^{(1)}, x_z^{(2)})\Psi_z(x_z^{(2)})$$

Hence, very conveniently, E is locally trivial on U_1 and U_2, and its isomorphism class is specified by the single transition function $G : U_1 \cap U_2 \to GL(n, \mathbb{C})$ given by -

$$G(\omega, \pi) = G_z(x_z^{(1)}, x_z^{(2)})$$

$G(\omega,\pi)$ is called the <u>patching function</u>, and it is determined up to bundle equivalence transformations -

$$G \rightarrow \tilde{G} = \Lambda_1 G \Lambda_2 \qquad (5.1.7)$$

where $\Lambda_i : U_i \rightarrow GL(n,\mathbb{C})$ $(i=1,2)$ are holomorphic maps on the standard affine pieces U_1, U_2. Note that bundle equivalence transformations contain gauge transformations of the associated self-dual connections.

We must also see how to reconstruct $A_\mu(x)$ from $G(\omega,\pi)$. Given $x \varepsilon \mathbb{C}^4$, let $\hat{x}$ be the corresponding projective line in $\mathbb{C}P^3 \setminus \mathbb{C}P^1$. Since $E|\hat{x}$ is trivial, we can "split" G as follows:

$$G(x\pi,\pi) = H(x,\zeta)K(x,\zeta)^{-1}$$

where $\zeta = \pi_1/\pi_2 \varepsilon \mathbb{C}_\infty$, the Riemann sphere, and:

<u>$H(x,\zeta)$ is analytic on U_1 (ie for $\zeta \neq 0$)</u>

$K(x,\zeta)$ is analytic on U_2 (ie for $\zeta \neq \infty$)

These are determined up to gauge transformations:

$$H(x,\zeta) \rightarrow H(x,\zeta)\Gamma(x), \quad K(x,\zeta) \rightarrow K(x,\zeta)\Gamma(x)$$

To reconstruct A_μ, we have to solve

$$A_\mu dx^\mu = G_z^{-1}\partial_\mu G_z dx^\mu, \qquad (dx)\pi = 0$$

But -

$$A_\mu dx^\mu = A_{\alpha\beta}dx^{\alpha\beta} = (A_{11}-\zeta A_{12})dx_{11}+(A_{21}-\zeta A_{22})dx_{21}$$

So, we obtain (for i=1,2):

$$A_{i1}-\zeta A_{i2} = H^{-1}\left(\frac{\partial}{\partial x_{i1}} - \zeta\frac{\partial}{\partial x_{i2}}\right)H = K^{-1}\left(\frac{\partial}{\partial x_{i1}} - \zeta\frac{\partial}{\partial x_{i2}}\right)K$$

<u>Reconstruction of the Atiyah Ward Ansätze</u>

Let us define a convenient set of co-ordinates of twistor space. We have:

$$x = i\begin{pmatrix} x_3-ix_4 & x_1-ix_2 \\ x_1+ix_2 & -x_3-ix_4 \end{pmatrix} = \sqrt{2}i\begin{pmatrix} z & \bar{y} \\ y & -\bar{z} \end{pmatrix}$$

$$\therefore \quad \omega = x\pi$$

$$\Rightarrow \quad \omega_1 = i\left[(x_3-ix_4)\pi_1+(x_1-ix_2)\pi_2\right]$$

$$\omega_2 = i\left[(x_1+ix_2)\pi_1-(x_3+ix_4)\pi_2\right]$$

Define co-ordinates (μ,ν,ζ) of $U_1\cap U_2$ by:

$$2\mu = \frac{\omega_1}{\pi_1}, \qquad 2\nu = \frac{\omega_2}{\pi_2}, \qquad \zeta = \frac{\pi_1}{\pi_2} \tag{5.1.8}$$

$$\Rightarrow \quad \mu = \frac{i}{2}\left[\frac{(x_1-ix_2)+(x_3-ix_4)}{\zeta}\right] = \frac{i}{\sqrt{2}}\frac{(\bar{y}+z)}{\zeta}$$

$$\nu = \frac{i}{2}\left[(x_1+ix_2)\zeta-(x_3+ix_4)\right] = \frac{i}{\sqrt{2}}(y\zeta-\bar{z})$$

In terms of these, inhomogeneous co-ordinates on U_1, U_2 are given respectively by:

$$U_1 : \left(\frac{\omega_1}{\pi_1}, \frac{\omega_2}{\pi_1}, \frac{\pi_2}{\pi_1}\right) = (2\mu,\ 2\nu\zeta^{-1}, \zeta^{-1}) \tag{5.1.9}$$

$$U_2 : \left(\frac{\omega_1}{\pi_2}, \frac{\omega_2}{\pi_2}, \frac{\pi_1}{\pi_2}\right) = (2\mu\zeta, 2\nu, \zeta)$$

In general, the splitting procedure described above is very difficult to implement. However, it has been done explicitly by Corrigan et al (1978) for $SL(2,\mathbb{C})$ patching functions which are equivalent to an upper triangular patching function of the form:

$$G(\mu,\nu,\zeta) = \begin{pmatrix} \zeta^n & \Delta(\mu,\nu,\zeta) \\ 0 & \zeta^{-n} \end{pmatrix} \tag{5.1.10}$$

where Δ is an arbitrary holomorphic function of μ,ν,ζ.

The condition that the patching function

$G(x\pi,\pi)$ depends on x,ζ only through the variables μ,ν,ζ implies -

$$D_i G = 0, \text{ where } D_i = \frac{\partial}{\partial x_{i1}} - \zeta\frac{\partial}{\partial x_{i2}}$$

$$\Rightarrow \quad D_i \Delta = 0, \text{ if } G \text{ takes the form (5.1.10)}$$

$$\Rightarrow \quad \left(\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)\Delta = \left(\frac{\partial}{\partial z} - \zeta\frac{\partial}{\partial y}\right)\Delta = 0 \qquad (5.1.11)$$

Now suppose that $\Delta(\mu,\nu,\zeta) = \Delta(x;\zeta)$ has Laurent expansion:

$$\Delta(x;\zeta) = \sum_{k=-\infty}^{\infty} \Delta_k(x)\zeta^{-k}$$

$$\Rightarrow \quad \Delta_k(x) = \frac{1}{2\pi i}\oint\frac{d\zeta}{\zeta}\,\zeta^k\Delta(x,\zeta) \qquad (5.1.12)$$

Then (5.1.11) is precisely equivalent to the condition that the Laurent coefficients $\Delta_k(x)$ satisfy the Δ-chain equations (2.1.5) - this is in fact the natural origin of the Δ-chain equations. We therefore call $\Delta(x,\zeta)$ the generating function of the Δ-chain $(\Delta_k(x))$. After performing the splitting procedure, Corrigan et al found that (5.1.10) gave rise to self-dual gauge fields, given in Yang's R-gauge by equations (2.1.6).

5.2 Dimensional Reduction in Twistor Space

Recall that, in Chapter 2.2, we guaranteed x_3- and x_4-translational invariance of a self-dual $SL(2,\mathbb{C})$ gauge potential by requiring that its Δ-chain satisfy equ (2.2.2). This implies that the generating function $\Delta(x;\zeta)$ takes the form:

$$\Delta(x;\zeta) = e^{i(ax_3+bx_4)}\tilde{\Delta}(x_1,x_2;\zeta) \qquad (5.2.1)$$

But, from (5.1.8):

$$\begin{aligned} \mu-\nu &= ix_3 + \frac{i}{\sqrt{2}}\left(\frac{\bar{y}}{\zeta} - y\zeta\right) \\ \mu+\nu &= x_4 + \frac{i}{\sqrt{2}}\left(\frac{\bar{y}}{\bar{\zeta}} + y\zeta\right) \end{aligned} \qquad (5.2.2)$$

Hence, equ (5.2.1) is equivalent to -

$$\Delta(\mu,\nu,\zeta) = e^{a(\mu-\nu)+ib(\mu+\nu)}f(\mu,\nu,\zeta)$$

where, keeping x_1,x_2 fixed:

$$\partial_3 f = \partial_4 f = 0 = \partial f/\partial(\mu-\nu) = \partial f/\partial(\mu+\nu) = 0$$

$\Rightarrow$ f is a function of ζ only.

Hence we have proved:

5.2.1 Proposition

Dimensional reduction from $\mathbb{R}^4$ to $\mathbb{R}^2$ via the imposition of equ (2.2.2) is implemented, in the twistor space construction, by a choice of generating function of the form:

$$\Delta(\mu,\nu,\zeta) = e^{a(\mu-\nu)+ib(\mu+\nu)}f(\zeta) = e^{\gamma\mu-\bar{\gamma}\nu}f(\zeta) \tag{5.2.3}$$

where f is an arbitrary function which is analytic in some annular neighbourhood of $\{0 < |\zeta| < \infty\}$

QED

Let us now determine the patching functions of previously constructed solutions.

(a) The Axisymmetric N-Vortex Solutions

Since all non-singular axially symmetric N-vortices are obtained from the same Δ-chain, equ (2.3.9), their associated patching functions must have the same generating function. From equ (2.3.9):

$$\Delta(x;\zeta) = e^{i(ax_3+bx_4)} \sum_{k=-\infty}^{\infty} \left(\frac{\xi}{\zeta}\right)^k I_k(cr), \quad \xi = \sqrt{2}\frac{i\gamma\bar{y}}{cr}$$

This is simplified using the generating function for Bessel functions (equ (A.6)):

$$\Delta(x;\zeta) = \exp\left(\frac{i}{\sqrt{2}}(\gamma z+\bar{\gamma}\bar{z})\right)\exp\left(\frac{cr}{2}\left(\frac{\xi}{\zeta}+\frac{\zeta}{\xi}\right)\right)$$

$$= \exp\left(\frac{i}{\sqrt{2}}\left(\gamma\left(z+\frac{\bar{y}}{\zeta}\right) + \bar{\gamma}(\bar{z} - y\zeta)\right)\right), \text{ by definition of } \xi$$

$$= e^{\gamma\mu-\bar{\gamma}\nu} \qquad , \quad \text{by (5.1.8)}$$

$$\therefore \quad \Delta(\mu,\nu,\zeta) = e^{a(\mu-\nu)+ib(\mu+\nu)} \tag{5.2.4}$$

So, comparing with equ (5.2.3), we see that the axially symmetric N-vortices have the simplest possible patching function, corresponding to the choice $f(\zeta) \equiv 1$.

For completeness, let us include the translational degrees of freedom ie let us calculate the patching function of an axially symmetric N-vortex situated at $(x_1^{(1)}, x_2^{(1)}) \in \mathbb{R}^2$.

Under a translation in $\mathbb{R}^2$:

$$y \to y - y^{(1)}, \qquad y^{(1)} = \frac{1}{\sqrt{2}}(x_1^{(1)} + ix_2^{(1)})$$

we have:

$$\mu \to \mu - \frac{i}{\sqrt{2}}\frac{\bar{y}^{(1)}}{\zeta}, \qquad \nu \to \nu - \frac{i}{\sqrt{2}} y^{(1)}\zeta$$

$$\therefore \quad e^{\gamma\mu-\bar{\gamma}\nu} \to e^{\gamma\mu-\bar{\gamma}\nu}\exp\left(\frac{i}{\sqrt{2}}\left(\bar{\gamma}y^{(1)}\zeta - \frac{\gamma\bar{y}^{(1)}}{\zeta}\right)\right)$$

Hence, the patching function for the linearly translated axially symmetric N-vortex is defined by:

$$f(\zeta) = \exp\left(\frac{i}{\sqrt{2}}\left(\bar{\gamma} y^{(1)}\zeta - \frac{\gamma \bar{y}^{(1)}}{\zeta}\right)\right) \qquad (5.2.5)$$

(b) <u>The Separated N-Vortex Solutions (Chapter 4.2)</u>

The Δ-chain of equ (4.2.2) is just the linear superposition of the Δ-chains of linearly translated axially symmetric vortices. Hence, the patching function is defined by:

$$f(\zeta) = \sum_{\ell=1}^{N} \alpha_\ell \exp\left(\frac{i}{\sqrt{2}}\left(\gamma y^{(\ell)}\zeta - \frac{\gamma \bar{y}^{(\ell)}}{\zeta}\right)\right) \qquad (5.2.6)$$

where

$$\alpha_\ell > 0, \quad y^{(\ell)} = \frac{1}{\sqrt{2}}(x_1^{(\ell)} + i x_2^{(\ell)})$$

We can now use the calculation of equ (5.2.4) to relate the general solution of the Δ-chain equs (equ (2.3.8), Theorem (2.3.1)) with equ (5.2.3).

5.2.2 <u>Proposition</u>

Suppose the generating function satisfies equ (5.2.3), where $f(\zeta)$ has Laurent expansion.

$$f(\zeta) = \sum_{\ell=-\infty}^{\infty} f_\ell \zeta^{-\ell}$$

Then the associated Δ-chain takes the form of equ (2.3.8), with:

$$\alpha_\ell = \left(\frac{i\gamma}{c}\right)^{-\ell}\bar{f}_\ell, \quad \beta_\ell = 0, \quad \forall \ell \varepsilon \mathbb{Z}$$

<u>Proof</u>

From equ (5.2.4), we have:

$$e^{a(\mu-\nu)+ib(\mu+\nu)} = e^{i(ax_3+bx_4)} \sum_{\ell=-\infty}^{\infty} \xi^\ell I_\ell(cr)\zeta^{-\ell}$$

$$\Rightarrow \quad \Delta(\mu,\nu,\zeta) = e^{i(ax_3+bx_4)}\left(\sum_{\ell=-\infty}^{\infty} \xi^\ell I_\ell \zeta^{-\ell}\right)\left(\sum_{m=-\infty}^{\infty} f_m \zeta^{-m}\right)$$

$$= e^{i(ax_3+bx_4)} \sum_{k=-\infty}^{\infty} \left(\sum_{\ell+m=k} f_m \xi^\ell I_\ell\right)\zeta^{-k}$$

$$\Rightarrow \quad \tilde{\Delta}_k = \sum_{\ell+m=k} f_m \xi^\ell I_\ell$$

$$= \left(\frac{i\gamma}{c}\right)^k \sum_{\ell=-\infty}^{\infty} \left(\frac{i\gamma}{c}\right)^{-(k+\ell)} f_{k+\ell} I_\ell(cr)e^{i\ell\theta}$$

and, comparing with equ (2.3.8), the result follows.

<u>QED</u>

As a consequence, any solution derived from this framework automatically satisfies the non-singularity condition of equ (2.2.3), and hence has the asymptotic behaviour of equs (2.2.4), (2.2.5), Corollary (2.2.2).

The singular solutions of Chapter 2.4 do not seem to fit naturally into the twistor space construction.

Bundle Equivalence Transformations

Let $G(\mu,\nu,\zeta)$ be a patching function satisfying equ (5.2.3), ie

$$G(\mu,\nu,\zeta) = \begin{pmatrix} \zeta^n & e^{\gamma\mu-\bar{\gamma}\nu}f(\zeta) \\ 0 & \zeta^{-n} \end{pmatrix}$$

This is bundle equivalent (written $G \cong \tilde{G}$) to patching functions of the form:

$$\tilde{G} = \Lambda_1 G \Lambda_2, \quad \Lambda_i : U_i \to SL(2,\mathbb{C}) \qquad (*)$$

where Λ_1, Λ_2 are holomorphic, ie in terms of the inhomogeneous co-ordinates of equ (5.1.9):

$$\Lambda_1 = \Lambda_1(\mu, \nu\zeta^{-1}, \zeta^{-1}), \qquad \Lambda_2 = \Lambda_2(\mu\zeta, \nu, \zeta)$$

where the functions are analytic in the variables indicated.

Hence we can transfer the μ,ν-dependence to the diagonal entries of G, using a bundle equivalence transformation with:

$$\Lambda_1 = \begin{pmatrix} e^{-\gamma\mu} & 0 \\ 0 & e^{\gamma\mu} \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} e^{-\bar{\gamma}\nu} & 0 \\ 0 & e^{\bar{\gamma}\nu} \end{pmatrix}$$

$$\Rightarrow \quad \tilde{G}(\mu,\nu,\zeta) = \begin{pmatrix} e^{-(\gamma\mu-\bar{\gamma}\nu)}\zeta^{n} & f(\zeta) \\ 0 & e^{(\gamma\mu-\bar{\gamma}\nu)}\zeta^{-n} \end{pmatrix} \qquad (5.2.7)$$

We now ask the following question:

Given two patching functions of the form (5.2.7), defined respectively by analytic functions $f(\zeta)$, $\tilde{f}(\zeta)$, what are the necessary and sufficient conditions on f and $\tilde{f}$ which ensure that the two patching functions are bundle equivalent?

By (*) we must have:

$$\begin{pmatrix} a_\infty & b_\infty \\ c_\infty & d_\infty \end{pmatrix} \begin{pmatrix} e^{-(\gamma\mu-\bar{\gamma}\nu)}\zeta^{n} & \tilde{f}(\zeta) \\ 0 & e^{(\gamma\mu-\bar{\gamma}\nu)}\zeta^{-n} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-(\gamma\mu-\bar{\gamma}\nu)}\zeta^{n} & f(\zeta) \\ 0 & e^{(\gamma\mu-\bar{\gamma}\nu)}\zeta^{-n} \end{pmatrix} \begin{pmatrix} a_o & b_o \\ c_o & d_o \end{pmatrix}$$

where a_o, b_o, c_o, d_o are analytic on $U_2(|\zeta| < 1+\varepsilon)$
and $a_\infty, b_\infty, c_\infty, d_\infty$ are analytic on $U_1(|\zeta| > 1-\varepsilon)$

This implies:

$$\begin{pmatrix} a_\infty e^{-(\gamma\mu-\bar{\gamma}\nu)}\zeta^n & a_\infty\tilde{f}+b_\infty e^{(\gamma\mu-\bar{\gamma}\nu)}\zeta^{-n} \\ c_\infty e^{-(\gamma\mu-\bar{\gamma}\nu)}\zeta^n & c_\infty\tilde{f}+d_\infty e^{(\gamma\mu-\bar{\gamma}\nu)}\zeta^{-n} \end{pmatrix}$$

$$= \begin{pmatrix} c_o f+a_o e^{-(\gamma\mu-\bar{\gamma}\nu)}\zeta^n & d_o f+b_o e^{-(\gamma\mu-\bar{\gamma}\nu)}\zeta^n \\ c_o e^{(\gamma\mu-\bar{\gamma}\nu)}\zeta^{-n} & d_o e^{(\gamma\mu-\bar{\gamma}\nu)}\zeta^{-n} \end{pmatrix}$$

which, in turn, implies:

(1) $(a_\infty-a_o)e^{-(\gamma\mu-\bar{\gamma}\nu)}\zeta^n = c_o f$

(2) $(d_o-d_\infty)e^{(\gamma\mu-\bar{\gamma}\nu)}\zeta^{-n} = c_\infty\tilde{f}$

(3) $a_\infty\tilde{f}-d_o f = b_o e^{-(\gamma\mu-\bar{\gamma}\nu)}\zeta^n - b_\infty e^{(\gamma\mu-\bar{\gamma}\nu)}\zeta^{-n}$

(4) $c_o = c_\infty e^{-2(\gamma\mu-\bar{\gamma}\nu)}\zeta^{2n}$

Since the exponential term in (4) has essential singularities both at $\zeta=0$ and $\zeta=\infty$, the analyticity constraints on c_o, c_∞ require:

$$c_o \equiv c_\infty \equiv 0$$

Hence, again using the analyticity constraints:

(1) $\Rightarrow a_o = a_\infty \Rightarrow a_o = a_\infty = \text{constant}$

(2) $\Rightarrow d_o = d_\infty \Rightarrow d_o = d_\infty = \text{constant}$

Hence (3) implies:

$$\tilde{f}(\zeta) = kf(\zeta) + \beta_0 e^{-(\gamma\mu-\bar{\gamma}\nu)}\zeta^n + \beta_\infty e^{(\gamma\mu-\bar{\gamma}\nu)}\zeta^{-n}$$

where $k\varepsilon\mathbb{C}$, and β_0, β_∞ are analytic on U_1, U_2 respectively,

ie β_0 is an analytic function of $(\mu\zeta, \nu, \zeta)$

β_∞ is an analytic function of $(\mu, \nu\zeta^{-1}, \zeta^{-1})$

But $e^{\pm(\gamma\mu-\bar{\gamma}\nu)}$ has essential singularities both at $\zeta=0$ and $\zeta=\infty$. So the condition that $\tilde{f}$ depends on ζ only forces:

$$\beta_0 \equiv \beta_\infty \equiv 0$$

Hence we have proved:

5.2.3 <u>Theorem</u>

Two patching functions of the form described in Proposition (5.2.1), defined respectively by analytic functions $f(\zeta)$, $\tilde{f}(\zeta)$, are bundle equivalent if and only if:

$$\tilde{f}(\zeta) = kf(\zeta), \qquad k\varepsilon\mathbb{C}\setminus 0$$

<u>QED</u>

As a Corollary, we obtain Fact 1, quoted in the section on effective abelian fields in Chapter 2.2.

5.3 Patching Functions for Non-Singular Solutions

The question to which we address ourselves in this section is, what choice of $f(\zeta)$ in equ (5.2.3) guarantees us non-singular solutions?

Recall from equ (5.2.5) that the axially symmetric 1-vortex solution situated at the point $(x_1^{(1)}, x_2^{(1)}) \in \mathbb{R}^2$ is given, in the 1st ansatz, by:

$$f(\zeta) = \exp(\alpha\zeta + \frac{\bar{\alpha}}{\zeta}), \qquad \alpha = \frac{i\bar{\gamma}}{\sqrt{2}}(x_1^{(1)} + ix_2^{(1)}) \qquad (5.3.1)$$

Let us now make two observations:

(1) Experience with self-dual monopoles (Ward 1981, Corrigan & Goddard 1981) suggests that non-singularity is guaranteed if $\Delta(\mu,\nu,\zeta)$ has no zeros on $U_1 \cap U_2$. (There is still, however, no general proof of this statement). Imposing this condition, in our case, means that f may be written as an exponential:

$$f(\zeta) = \exp P(\zeta) \qquad (5.3.2)$$

where P is analytic in some annular neighbourhood of $\{0 < |\zeta| < \infty\}$

(2) Hence, a natural generalization of equ (5.3.1) is provided by choosing $f(\zeta)$ of the form (5.3.2), with, in the Nth ansatz:

$$P(\zeta) = \alpha_0 + \sum_{k=1}^{N} \left(\alpha_k \zeta^k + \frac{\bar{\alpha}_k}{\zeta^k}\right), \quad \alpha_0 \varepsilon \mathbb{R}, \ \alpha_k \varepsilon \mathbb{C} (k \geq 1) \quad (5.3.3)$$

Remarks

(1) Equ (5.3.3) is equivalent to requiring that P be a homogeneous polynomial of degree N on CP^1, satisfying the extra condition:

$$\overline{P(\zeta)} = P(\bar{\zeta}^{-1}) \qquad (5.3.4)$$

This is similar to, though not the same as, the condition that P be a hermitian polynomial on $\mathbb{C}P^1$. We shall see later that it gives rise to manifestly non-singular solutions.

(2) Note that $P(\zeta)$ in equ (5.3.3) has 2N+1 real parameters. However, α_0 may be removed by a scale transformation, leaving 2N free parameters, precisely the number required to describe the positions of N finitely separated vortices. (This interpretation is made with some reservations, however, since the forthcoming proof of non-singularity does <u>not</u> require the constraint deg $P \leq N$).

(3) The linear parameter $\alpha_1 \varepsilon \mathbb{C}$ in equ (5.3.3) corresponds to the two translational degrees of freedom; it can be removed by a translation (cf equ (5.2.5)):

$$y \to y' = y - y_o, \qquad \text{where } \alpha_1 = \frac{i\bar{\gamma}y_o}{\sqrt{2}} \tag{5.3.5}$$

Calculation of Δ-chain

For completeness, let us compute the corresponding Δ-chain when f is given by equ (5.3.2), and $P(\zeta)$ is a general function analytic on some annular neighbourhood of $\{0 < |\zeta| < \infty\}$, with Laurent expansion:

$$P(\zeta) = \sum_{\ell=1}^{\infty} (\alpha_\ell \zeta^\ell + \bar{\alpha}_\ell \zeta^{-\ell} + \beta_\ell \zeta^\ell - \bar{\beta}_\ell \zeta^{-\ell}) \tag{5.3.6}$$

$$\Rightarrow \quad \Delta(x;\zeta) = e^{a(\mu-\nu)+ib(\mu+\nu)} e^{P(\zeta)}$$

$$= e^{i(ax_3+bx_4)} \exp\left(\frac{1}{\sqrt{2}i} (\bar{\gamma}y\zeta - \gamma\bar{y}\zeta^{-1}) + \alpha_1 \zeta + \bar{\alpha}_1 \zeta^{-1} + \beta_1 - \bar{\beta}_1 \zeta^{-1} \right)$$

$$\exp\left(\sum_{\ell=2}^{\infty} (\alpha_\ell \zeta^\ell + \bar{\alpha}_\ell \zeta^{-\ell} + \beta_\ell \zeta^\ell - \bar{\beta}_\ell \zeta^{-\ell}) \right)$$

and, applying a translation of the form (5.3.5) to remove α_1, this becomes:

$$\Delta(x;\zeta) = e^{i(ax_3+bx_4)} \exp\left(\frac{1}{\sqrt{2}i} (\bar{\gamma}y'\zeta - \gamma\bar{y}'\zeta^{-1}) + \beta_1 \zeta - \bar{\beta}_1 \zeta^{-1} \right)$$

$$\exp\left(\sum_{\ell=2}^{\infty} (\alpha_\ell \zeta^\ell + \bar{\alpha}_\ell \zeta^{-\ell} + \beta_\ell \zeta^\ell - \bar{\beta}_\ell \zeta^{-\ell} \right)$$

Now, parameterize $(x_1, x_2) \in \mathbb{R}^2$ by cylindrical polar co-ordinates (r_1, ψ_1) defined by:

$$\frac{\overline{\gamma}\gamma'}{\sqrt{2}i} = \frac{cr_1 e^{-i\psi_1}}{2}$$

Similarly, parameterize α_ℓ, β_ℓ as follows:

$$\left.\begin{aligned} \alpha_\ell &= \frac{cr_\ell e^{-i\ell\psi_\ell}}{2}, \qquad \ell \geqq 2 \\ \beta_\ell &= \frac{cs_\ell e^{-i\ell\chi_\ell}}{2}, \qquad \ell \geqq 1 \end{aligned}\right\} \tag{5.3.7}$$

$$\Rightarrow \quad \Delta(x;\zeta) = e^{i(ax_3+bx_4)}$$

$$\exp\left(\frac{cr_1}{2}(e^{-i\psi_1}\zeta + e^{i\psi_1}\zeta^{-1}) + \frac{cs_1}{2}(e^{-i\chi_1}\zeta - e^{i\chi_1}\zeta^{-1})\right)$$

$$\exp\left(\frac{c}{2}\sum_{\ell=2}^{\infty} r_\ell(e^{-i\ell\psi_\ell}\zeta^\ell + e^{i\ell\psi_\ell}\zeta^{-\ell}) + s_\ell(e^{-i\ell\chi_\ell}\zeta^\ell - e^{i\ell\chi_\ell}\zeta^{-\ell})\right)$$

$$\therefore \quad \tilde{\Delta}_k(x) = \frac{1}{2\pi i}\oint \frac{d\zeta}{\zeta}\, \zeta^k \tilde{\Delta}(x;\zeta)$$

$$= \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{ik\theta}\, e^{cr_1\cos(\theta-\psi_1)}\, e^{irs_1\sin(\theta-\chi_1)}$$

$$\exp\left(c\sum_{\ell=2}^{\infty}(r_\ell\cos\ell(\theta-\psi_\ell) + is_\ell\sin\ell(\theta-\chi_\ell))\right)$$

Hence we finally obtain:

$$\tilde{\Delta}_k(x) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{ik\theta} \exp(c \sum_{\ell=1}^{\infty} (r_\ell \cos\ell(\theta-\psi_\ell) + i s_\ell \sin\ell(\theta-\chi_\ell))) \tag{5.3.8}$$

If $P(\zeta)$ also satisfies the constraint (5.3.4), then $\beta_\ell = 0$, $\forall \ell$, so the Δ-chain is given by:

$$\tilde{\Delta}_k(x) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{ik\theta} \exp(c \sum_{\ell=1}^{\infty} r_\ell \cos\ell(\theta-\psi_\ell)) \tag{5.3.9}$$

Note that the integral in equ (5.3.9) is a rather natural generalization of the integral representation for modified Bessel functions equ (A.7) (i), from which the Δ-chain of equ (2.3.9) can be recovered in the axisymmetric case $P(\zeta) \equiv 0$. Note also that the polar co-ordinates (r_1, ψ_1) are related to the original co-ordinates (r, θ) by:

$$r_1 = r, \quad e^{i\psi_1} = \xi = \frac{i\gamma}{c} e^{-i\theta}$$

5.3.1 Theorem

Suppose that, in the Nth Atiyah-Ward ansatz, the generating function is given by:

$$\Delta(\mu, \nu, \zeta) = e^{\gamma\mu - \bar{\gamma}\nu} f(\zeta)$$

where $f(\zeta) = e^{P(\zeta)}$, and $\overline{P(\zeta)} = P(\bar{\zeta}^{-1})$.

Then:

(i) $\tilde{\Delta}_{-k} = \tilde{\Delta}^*_k$, $\quad \forall k \in \mathbb{Z}$

(ii) $\mathrm{Det}\tilde{D}^{(N)}$ is real valued; hence the gauge invariant quantities $||\Phi||^2$, ξ of the associated field configuration are real valued.

(iii) $\mathrm{Det}\tilde{D}^{(N)} > 0$, $\quad \forall x \in \mathbb{R}^2$; hence the associated field configuration is non-singular.

Proof

(i) is an immediate consequence of equ (5.3.9), and it implies that $\tilde{D}^{(N)}$ is hermitian. Hence $\det\tilde{D}^{(N)}$ is real valued, and so are $||\Phi||^2$, ξ, from Corollary (2.2.1).

The non-singularity proof follows the now familiar argument of Theorem (2.3.3). Using equ (5.3.9), we have:

$$\det\tilde{D}^{(N)}$$

$$= \frac{1}{N!} \sum_{\sigma \in S_N} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi} \exp\left(c \sum_{i=1}^{N} \sum_{\ell=1}^{\infty} r_\ell (\cos\ell(\theta_i - \psi_\ell))\right)$$

$$W(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(N)})$$

$$= \frac{1}{N!} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi} \prod_{i<j}^{N} 4\sin^2\left(\frac{\theta_i-\theta_j}{2}\right) \tag{5.3.10}$$

$$\exp\left(c \sum_{\ell=1}^{\infty} r_\ell \left(\sum_{i=1}^{N} \cos\ell(\theta_i-\psi_\ell)\right)\right)$$

> 0, since the integrand is positive.

QED

Remarks

(1) (i) and (ii) justify Fact 2, quoted in the section on effective U(1) fields in Chapter 2.2.

Note that the condition that $||\Phi||^2$, ξ be real valued seems intimately related to non-singularity. If $P(\zeta)$ is a general analytic function, the pure imaginary term in the exponent of equ (5.3.8) causes the non-singularity proof to fail (the integrand of equ (5.3.10) is no longer positive), and it causes $\det\mathfrak{D}^{(N)}$, $||\Phi||^2$ and ξ to be complex valued.

(2) In the above theorem, there is no restriction that P be a polynomial of bounded degree. In fact, the theorem goes through for the even more general choice:

$$f(\zeta) = \alpha_\ell e^{P_\ell(\zeta)} \tag{5.3.11}$$

where $\alpha_\ell > 0$, and $\overline{P_\ell(\zeta)} = P_\ell(\bar{\zeta}^{-1})$, $\forall \ell$

(The separated solutions studied in Chapter 4 are of this type). Since gauge transformations are equivalent only to scale transformations of the generating functions, this means that, in each ansatz, we have constructed an extremely large infinite dimensional parameter space of complex non-singular self-dual gauge fields.

Relation with Generalized Action Lattice Gauge Theory

Consider solutions satisfying the hypotheses of theorem (5.3.1), with c=1. Using equ (5.3.9) the Δ-chain can be rewritten as:

$$\Delta_k(x) = \oint_{u(1)} \frac{dz}{z}.z^k \exp\left(\tfrac{1}{2}\sum_{\ell=1}^{\infty} r_\ell (z^\ell e^{-i\ell\psi_\ell} + z^{-\ell} e^{i\ell\psi_\ell})\right)$$

So, by Theorem B, the fundamental determinant is given by:

$$\det\tilde{D}^{(N)} = \int_{U(N)} dU.\det \exp\left(\tfrac{1}{2}\sum_{\ell=1}^{\infty} r_\ell \left[U^\ell e^{-i\ell\psi_1} + U^{+\ell} e^{i\ell\psi_\ell}\right]\right)$$

$$= \int_{U(N)} dU.\exp\left(\tfrac{1}{2}\sum_{\ell=1}^{\infty} r_\ell \mathrm{Tr}\left[U^\ell e^{-i\ell\psi_\ell} + U^{+\ell} e^{i\ell\psi_\ell}\right]\right) \tag{5.3.12}$$

ie $\det\tilde{D}^{(N)}$ is equal to the partition function for a single-site/link <u>mixed action</u> U(N) lattice chiral/ gauge theory, with the parameters $r_\ell e^{-i\ell\psi_\ell}$ playing the role of complex inverse couplings or temperatures. In this formalism, non-singularity is simply a consequence of the reality of the Hamiltonian of the corresponding lattice gauge theory.

Note that the spatial parameters $r_1 e^{-i\psi_1}$ form the complex inverse temperature associated with the Wilson component of the action, so, as previously noted, the axisymmetric case $r_\ell=0\ \forall \ell\geq 2$ corresponds to the usual Wilson lattice gauge theory. In fact, translation invariance of the Haar measure can be used to remove the angular dependence from the Wilson component of the action in (5.3.12):

$$\det\tilde{D}^{(N)} = \int_{U(N)} dU \exp\left(\frac{r_1}{2} Tr(U+U^+)\right) \qquad (5.3.13)$$

$$\exp\left(\tfrac{1}{2} \sum_{\ell=2} r_\ell Tr\left[U^\ell e^{-i\ell(\psi_\ell-\psi_1)} + U^{+\ell} e^{i\ell(\psi_\ell-\psi_1)}\right]\right)$$

We can also obtain a formula for the effective complex scalar field $\Psi^{(N)}$ analagous to that of the axisymmetric case. Using Theorem B, we have, in the notation of equ (2.2.1):

$$\det\tilde{E}^{(N)} = \int_{U(N)} dU.\det U.\exp\left(\tfrac{1}{2}\sum_\ell r_\ell Tr(U^\ell e^{-i\ell\psi_\ell} + U^{+\ell} e^{i\ell\psi_\ell})\right) \qquad (5.3.14)$$

$$= e^{iN\psi_1} \int dU.\det U.\exp\left(\frac{r_1}{2} Tr(U+U^+)\right)$$

$$\exp\left(\tfrac{1}{2} \sum_{\ell=2}^{\infty} r_\ell Tr(U^\ell e^{-i\ell(\psi_\ell-\psi_1)} + U^{+\ell} e^{i\ell(\psi_\ell-\psi_1)})\right) \qquad (5.3.15)$$

again using translation invariance of the Haar measure.

Hence we obtain:

$$\psi^{(N)} = (-1)^N \frac{\det\tilde{E}^{(N)*}}{\det\tilde{D}^{(N)}} \tag{5.3.16}$$

$$= (-1)^N \langle \det U^+ \rangle$$

$$= (-e^{-i\psi_1})^N \langle \det U^+ \rangle' = (i\bar{\gamma} e^{i\theta})^N \langle \det U^+ \rangle'$$

where $\langle \cdot \rangle$, $\langle \cdot \rangle'$ denote the expectation values with respect to the partition functions in equs (5.3.12), (5.3.13) respectively.

Note that, for <u>fixed</u> r_ℓ ($\ell \geq 2$), we have from equs (5.3.13), (5.3.15):

$$\begin{cases} \det\tilde{D}^{(N)} \sim \displaystyle\int dU \exp\left(\frac{r_1}{2} \mathrm{Tr}(U+U^+)\right) \\ \\ \det\tilde{E}^{(N)} \sim e^{iN\psi_1} \displaystyle\int dU.\det U.\exp\left(\frac{r_1}{2} \mathrm{Tr}(U+U^+)\right), \quad r_1 \to \infty \end{cases}$$

ie as $r_1 \to \infty$, <u>these solutions have the same asymptotic behaviour as the axially symmetric N-vortices</u>.

In particular, $|\Psi^{(N)}| \to 1$, $\Psi^{(N)}$ has winding number N at infinity, and the total magnetic flux obeys the 'quantization law' (2.3.27) as before, so these solutions describe non-singular vortices of charge N.

<u>Aside</u>

Writing $F(r,\psi) = \ln \det\tilde{D}^{(N)}(r,\psi)$ for the free energy of the partition function (5.3.12), and using equ (2.2.20) which relates $||\Phi||^2$ to Ψ, we obtain a remarkable formula for the single-link U(N) mixed action lattice gauge theory:

$$\nabla^2 F = 1 - \langle \det U \rangle \langle \det U^+ \rangle \tag{5.3.17}$$

It is not at all obvious how this formula could be derived directly.

5.4 Study of Field Configurations

We shall now study the physical field configurations of the solutions of Theorem (5.3.1), with the additional constraint that, in the Nth ansatz, P is a homogeneous polynomial of degree $\leq N$. This gives us a total of 2N free parameters, which are expected to describe the positions of N finitely separated vortices. We shall use a combination of symmetry arguments, small parameter expansions, and numerical computations to provide evidence for this hypothesis.

Let us recall how euclidean symmetries are realized in twistor space (Corrigan & Goddard 1981, O'Raifeartaigh, Rouhani and Singh 1981, 1982). O(4) transformations are realized on $x \in M(2,\mathbb{C})$ (of equ (5.1.1)) by $SL(2,\mathbb{C})$ transformations:

$$x \to uxu^{-1}, \qquad u \in SL(2,\mathbb{C})$$

and the corresponding action on twistor space is given by:

$$\omega \to u\omega, \qquad \pi \to u\pi$$

In particular:

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad \zeta \to \frac{a\zeta + b}{c\zeta + d}$$

so $SL(2,\mathbb{C})$ acts as the group of conformal transformations on the Riemann sphere $\mathbb{C}P^1$ spanned by ζ.

Note the actions on real space and twistor space of the following $SL(2,\mathbb{C})$ transformations:

<u>SL(2,ℂ) Transformation</u>	<u>Action on $\mathbb{R}^4$</u>	<u>Action on $\mathbb{C}P^3 \setminus \mathbb{C}P^1$</u>
$\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$	$: y \to e^{-2i\phi} y$, $z \to z$	$: (\mu,\nu,\zeta) \to (\mu,\nu,e^{2i\phi}\zeta)$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$: y \to \bar{y}$, $z \to -\bar{z}$	$: (\mu,\nu,\zeta) \to (\nu,\mu,\zeta^{-1})$
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$: y \to -\bar{y}$, $z \to -\bar{z}$	$: (\mu,\nu,\zeta) \to (\nu,\mu,-\zeta^{-1})$

Hence, if by an appropriate rotation in the x_3x_4-plane, we choose $\gamma=-\bar{\gamma}$, the factor $e^{\gamma\mu-\bar{\gamma}\nu}$ of the generating function is left unchanged by $O(2)$ transformations of the x_1x_2-plane.

Therefore, $O(2)$ transformations of our solutions correspond to transformations of the analytic functions $P(\zeta)$ induced by the following transformations of the Riemann sphere:

Rotations:

$$y \to e^{i\phi} y : \qquad \zeta \to e^{-i\phi}\zeta \tag{5.4.1}$$

Reflections:

$$\begin{aligned} x_1 &\to -x_1 : \qquad & \zeta &\to -\zeta^{-1} \\ x_2 &\to -x_2 : \qquad & \zeta &\to \zeta^{-1} \end{aligned} \tag{5.4.2}$$

Also, $|R^2$-translations of our solutions are given by (cf equ (5.2.5)):

$$y \to y - \beta : \quad P(\zeta) \to P(\zeta) + \alpha\zeta + \frac{\bar{\alpha}}{\zeta}, \quad \alpha = \frac{i}{\sqrt{2}}\bar{\gamma}\beta \tag{5.4.3}$$

Finally, gauge transformations are given by:

$$P(\zeta) \to P(\zeta) + \alpha_o \qquad \alpha_o = \text{constant} \tag{5.4.4}$$

Examples (1) Axial Symmetry

P(ζ) gives rise to an axially symmetric field if and only if:

$$P(\zeta) = P(e^{i\phi}\zeta), \quad \forall\phi \Rightarrow P \equiv \alpha_o = \text{constant}$$

Hence, the only axially symmetric solutions in this formalism are the non-singular axially symmetric vortices studied in Chapter 2.3.

(2) Reflection Symmetry

C_2 symmetry under (i)$x_1 \to -x_1$ or (ii)$x_2 \to -x_2$ requires respectively, in the notation of equ (5.3.3):

(i) $P(\zeta) = P(-\zeta^{-1}) \Rightarrow \alpha_k = (-1)^k \bar{\alpha}_k$

(ii) $P(\zeta) = P(\zeta^{-1}) \Rightarrow \alpha_k = \bar{\alpha}_k$

and these imply respectively:

(i) $P(\zeta) = \sum_{k=1}^{N} \varepsilon_k(\zeta^k + (-1)^k \zeta^{-k}), \quad \varepsilon_k \in \begin{cases} \mathbb{R}, & k \text{ even} \\ i\,\mathbb{R}, & k \text{ odd} \end{cases}$

(ii) $P(\zeta) = \sum_{k=1}^{N} \varepsilon_k(\zeta^k + \zeta^{-k}), \quad \varepsilon_k \in \mathbb{R}$

Hence, symmetry under $D_2 \cong C_2 x C_2$ generated by reflections in both co-ordinate axes requires:

$$P(\zeta) = \sum_k \varepsilon_{2k}(\zeta^{2k} + \zeta^{-2k}), \qquad \varepsilon_{2k} \in \mathbb{R} \tag{5.4.5}$$

ie only even degree terms with real coefficients are allowed.

(3) Reduction of Charge 2 Case

$$P(\zeta) = \alpha_1 \zeta + \bar{\alpha}_1 \zeta^{-1} + \alpha_2 \zeta^2 + \bar{\alpha}_2 \zeta^{-2}$$

Remove $\alpha_1, \bar{\alpha}_1$ by a translation, and remove the phase of α_2 by a rotation. This leaves us with the D_2-symmetric polynomial:

$$P(\zeta) = \varepsilon(\zeta^2 + \zeta^{-2}), \quad \varepsilon \in \mathbb{R}^+ \qquad (5.4.6)$$

Hence, the general charge 2 solution has preferred axes, with respect to which it is D_2-symmetric. So, we expect the vortices to be separated along one of these axes, with ε parameterizing the separation distance.

(4) <u>D_N-Symmetry</u>

Let D_N be the dihedral group of order 2N generated by the rotation and reflection:

$$y \to e^{i2\pi/N} y, \qquad x_2 \to -x_2$$

Clearly, in the Nth assatz, $P(\zeta)$ is invariant under rotation through $2\pi/N$ if and only if:

$$P(\zeta) = \alpha\zeta^N + \bar{\alpha}\zeta^{-N}$$

As above, we can remove the phase of α by a rotation to leave:

$$P(\zeta) = \varepsilon(\zeta^N + \zeta^{-N}), \quad \varepsilon \in \mathbb{R}^+ \qquad (5.4.7)$$

and this is now also invariant under $x_2 \to -x_2 : \zeta \to \zeta^{-1}$, so it is the general D_N-symmetric solution. We expect it to describe N separated vortices situated on the vertices of a regular polygon of N vertices, again with ε parameterizing the separation distance.

More generally, the D_M-symmetric solution, for $M \leq N$ is given by:

$$P(\zeta) = \sum_{k=1}^{Q} \varepsilon_k(\zeta^{Mk} + \zeta^{-Mk}), \qquad \varepsilon_k \in \mathbb{R}$$

where $N = MQ + R, \quad Q,R \in \mathbb{Z}, \quad 0 \leq R < N-1$

<u>Small Parameter Expansions</u>

Let us consider the small ε behaviour of the simplest D_M-symmetric solution in the Nth assatz, $(M \leq N)$, which we parameterize as:

$$P(\zeta) = (\tfrac{\varepsilon}{2})^M(\zeta^M + \zeta^{-M}) \qquad (5.4.8)$$

In order to verify that this gives us a separated N-vortex solution, we must determine the locations of the zeros of the effective complex scalar Higgs field $\Psi^{(N)}$ of equ (5.3.16). Since $0 < \det\tilde{D}^{(N)} < \infty$, $\forall x \in \mathbb{R}^2$, this is equivalent to determining the zeros of $\det\tilde{E}^{(N)}$, where, from equ (5.3.15), writing $(r,\psi) = (r_1,\psi_1)$:

$$\det \tilde{E}^{(N)}$$

$$= e^{iN\psi} \int_{U(N)} dU.\det U.\exp(\frac{r}{2}\,\mathrm{Tr}(U+U^{+}))$$

$$\exp((\frac{\varepsilon}{2})^{M}\,\mathrm{Tr}(U^{M}e^{iM\psi}+U^{+M}e^{-iM\psi}))$$

Expanding this to lowest order in ε gives:

$$\det \tilde{E}^{(N)} = \det \tilde{E}^{(N)}\Big|_{\varepsilon=0}$$

$$+\ e^{iN\psi}(\frac{\varepsilon}{2})^{M} \int_{U(N)} dU.\det U.\mathrm{Tr}(U^{M}e^{iM\psi}+U^{+M}e^{-iM\psi})$$

$$\sum_{k=0}^{\infty} \frac{1}{k!}(\frac{r}{2})^{k}(\mathrm{Tr}(U+U^{+}))^{k}$$

Next, consider the behaviour of this expression for small r; assuming $r=0(\varepsilon)$, and ignoring terms of $0(\varepsilon^{N+1})$, we have, from equs (2.3.22) and (B.4):

$$\det \tilde{E}^{(N)} =$$

$$e^{iN\psi}\Big[\frac{1}{N!}(\frac{r}{2})^{N} + (\frac{\varepsilon}{2})^{M}\frac{e^{-iM\psi}}{(N-M)!}(\frac{r}{2})^{N-M}$$

$$\int_{U(N)} dU.\det U.\mathrm{Tr}(U^{+M})(\mathrm{Tr}U^{+})^{N-M}\Big]$$

But, (cf proof of equ (2.3.22)):

$$\int_{U(N)} dU.\det U.\mathrm{Tr}(U^{+M})(\mathrm{Tr}U^{+})^{N-M}$$

$$= \int_{SU(N)} dV.\mathrm{Tr}(V^{+M})(\mathrm{Tr}V^{+})^{N-M}$$

$$= \int_{SU(N)} dV(V^{+}_{\alpha_1\alpha_2} V^{+}_{\alpha_2\alpha_3} \cdots V^{+}_{\alpha_M\alpha_1})(V^{+}_{\alpha_{M+1}\alpha_{M+1}} \cdots V^{+}_{\alpha_N\alpha_N})$$

$$= \frac{1}{N!}\, \varepsilon_{\alpha_1\alpha_2\cdots\alpha_M\alpha_{M+1}\cdots\alpha_N}\ \varepsilon_{\alpha_2\alpha_3\cdots\alpha_1\alpha_{M+1}\cdots\alpha_N}$$

(from (B.5))

$$= \frac{(-1)^{M+1}}{N!}\ \varepsilon_{\alpha_1\cdots\alpha_N}\ \varepsilon_{\alpha_1\cdots\alpha_N}$$

$$= (-1)^{M+1}$$

So we obtain:

$$\det\hat{E}^{(N)} = \frac{1}{N!}(\frac{r}{2})^N e^{iN\psi} + \frac{(-1)^{M+1}}{(N-M)!}(\frac{\varepsilon}{2})^M (\frac{r}{2})^{N-M} e^{i(N-M)\psi}$$

$$= \frac{1}{2^N N!}\left[z^N + (-1)^{M+1}\frac{N!}{(N-M)!}\varepsilon^M z^{N-M}\right] \qquad (5.4.9)$$

where $z = re^{i\psi}$. Hence:

$$\det\tilde{E}^{(N)} = \frac{(-1)^M}{2^N N!} z^{N-M} \left[(-z)^M - \frac{N!}{(N-M)!} \varepsilon^M \right]$$

$$\therefore \quad \det\tilde{E}^{(N)} = 0 \Leftrightarrow z = 0 \text{ or } (-z)^M = \frac{N!}{(N-M)!} \varepsilon^M$$

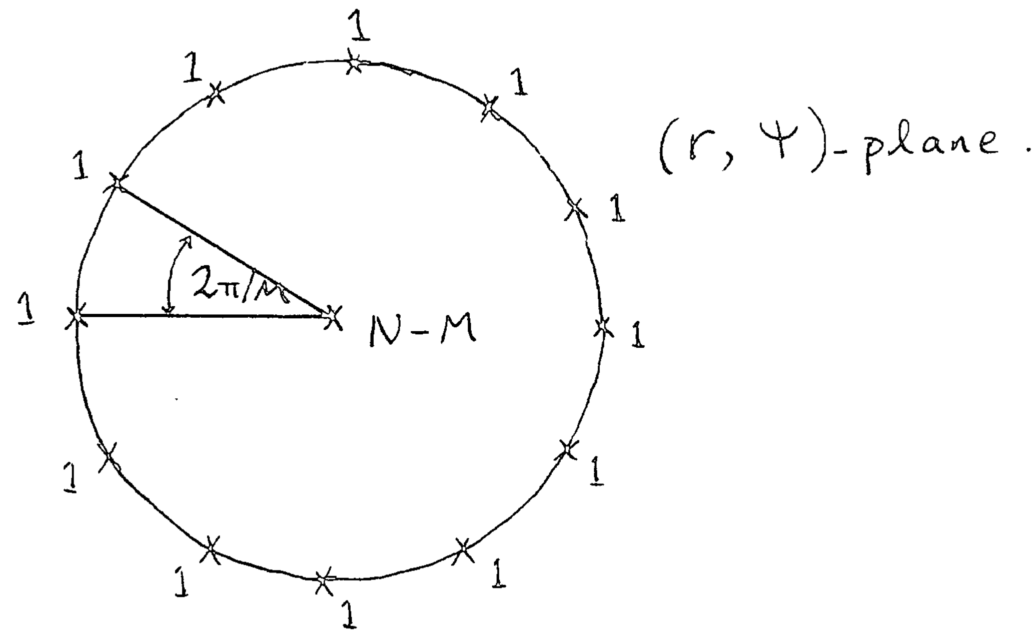
Hence, if $M < N$, there is a zero of order $N-M$ at the origin, and the other zeros are of order 1, at the points (r,ψ) given by:

$$r = \left[\frac{N!}{(N-M)!} \right]^{1/M} \varepsilon \tag{5.4.10}$$

$$\psi = \pi + \frac{2\pi k}{M}, \quad k=0, \ldots , M-1$$

So, as expected from symmetry considerations, we have a vortex of charge $N-M$ at the origin, with M charge 1 vortices sitting on the vertices of a regular M-gon centred at the origin. Moreover, for small ε, the physical separation varies linearly with ε.

i.e.



In the D_N-symmetric case, M=N, the vortex at the origin disappears, and, as conjectured above, we have N single vortices sitting on the vertices of a regular N-gon, of physical radius:

$$r = (N!)^{1/N}\varepsilon \qquad (5.4.11)$$

Finally, we can use equ (5.4.9) to obtain a small parameter expansion for the solution with general $P(\zeta)$, $\deg P \leq N$. Parameterize $P(\zeta)$ as follows:

$$P(\zeta) = \sum_{k=1}^{N} (\varepsilon_k/2)^k (\zeta^k e^{-ik\psi_k} + \zeta^{-k} e^{ik\psi_k})$$

Suppose r and ε_k $(2 \leq k \leq N)$ are all of the same order ε, and work to order ε^N. We have, as above:

$$\det\hat{E}^{(N)} = e^{iN\psi} \int_{U(N)} dU \det U \exp(\tfrac{r}{2}\mathrm{Tr}(U+U^+))$$

$$\exp(\Sigma(\varepsilon_k/2)^k \mathrm{Tr}[U^k e^{-ik(\psi_k-\psi)} + U^{+k} e^{ik(\psi_k-\psi)}])$$

$$= e^{iN\psi}\Big[\frac{1}{N!}(\frac{r}{2})^N + \sum_{k=1}^{N}(\frac{\varepsilon_k}{2})^k(\frac{r}{2})^{N-k}\frac{e^{ik(\psi_k-\psi)}}{(N-k)!}$$

$$\int_{U(N)} dU.\det U.\mathrm{Tr}(U^{+k}).(\mathrm{Tr}U^+)^{N-k}\Big]$$

$$= \frac{1}{2^N N!}\left[r^N e^{iN\psi} + \sum_{k=1}^{N}(-1)^{k+1}\frac{N!}{(N-k)!}(\varepsilon_k e^{i\psi_k})^k r^{N-k} e^{i(N-k)\psi}\right] \tag{5.4.12}$$

Hence we finally obtain, in terms of the complex variable $z = re^{i\psi}$:

$$\det\tilde{E}^{(N)} = \frac{1}{2^N . N!}\left[z^N - \sum_{k=1}^{N}\frac{N!}{(N-k)!}(-\varepsilon_k e^{i\psi_k})^k z^{N-k}\right]$$

ie if $\alpha_k \varepsilon \mathbb{C}$, and $\alpha_k = 0(\varepsilon^k)$, and $r = 0(\varepsilon)$, then:

$$P(\zeta) = \sum_{k=1}^{N}(\bar{\alpha}_k\zeta^k + \alpha_k\zeta^{-k})$$

$$\Rightarrow \quad \det\tilde{E}^{(N)} = \frac{1}{N!}\left[(\tfrac{z}{2})^N - \sum_{k=1}^{N}(-1)^k\frac{N!}{(N-k)!}\alpha_k(\tfrac{z}{2})^{N-k}\right] + 0(\varepsilon^{N+1}) \tag{5.4.13}$$

So, if $P(\zeta)$ is a polynomial of degree $\leqq N$, then for sufficiently small values of its coefficients, and for sufficiently small z, $\det\tilde{E}^{(N)}$ is a polynomial of degree N in the complex variable z. So, by the fundamental theorem of algebra, $\det\tilde{E}^{(N)}$ has precisely N zeros, so the solution describes N separated vortices which are close together.

Moreover, <u>any</u> complex polynomial of degree N can be written in the form of (5.4.12) or (5.4.13), with a suitable choice of parameters ε_k, ψ_k. Thus, given any cluster of N points sufficiently close together in the complex (r,ψ)-plane, there exists a degree N polynomial $P(\zeta)$ which describes N vortices situated approximately at these points.

Numerical Study

The gauge invariant quantities $||\Phi||^2$, ξ are plotted between $-5 \leqq x_1 \leqq 5$, $-5 \leqq x_2 \leqq 5$ for the D_N - symmetric N-vortex solution with N=2,3,4 in Figures 8, 9, 10 respectively, for increasing values of the separation parameter ε, with parameterization (5.4.8):

$$P(\zeta) = (\frac{\varepsilon}{2})^N (\zeta^N + \zeta^{-N})$$

The same method was used as that for the numerical work in Chapter 2 and Chapter 4; $\det\tilde{D}^{(N)}$ was calculated on a fine grid of points by applying numerical integration routines to the integral expression (5.3.9) for the Δ-chain.

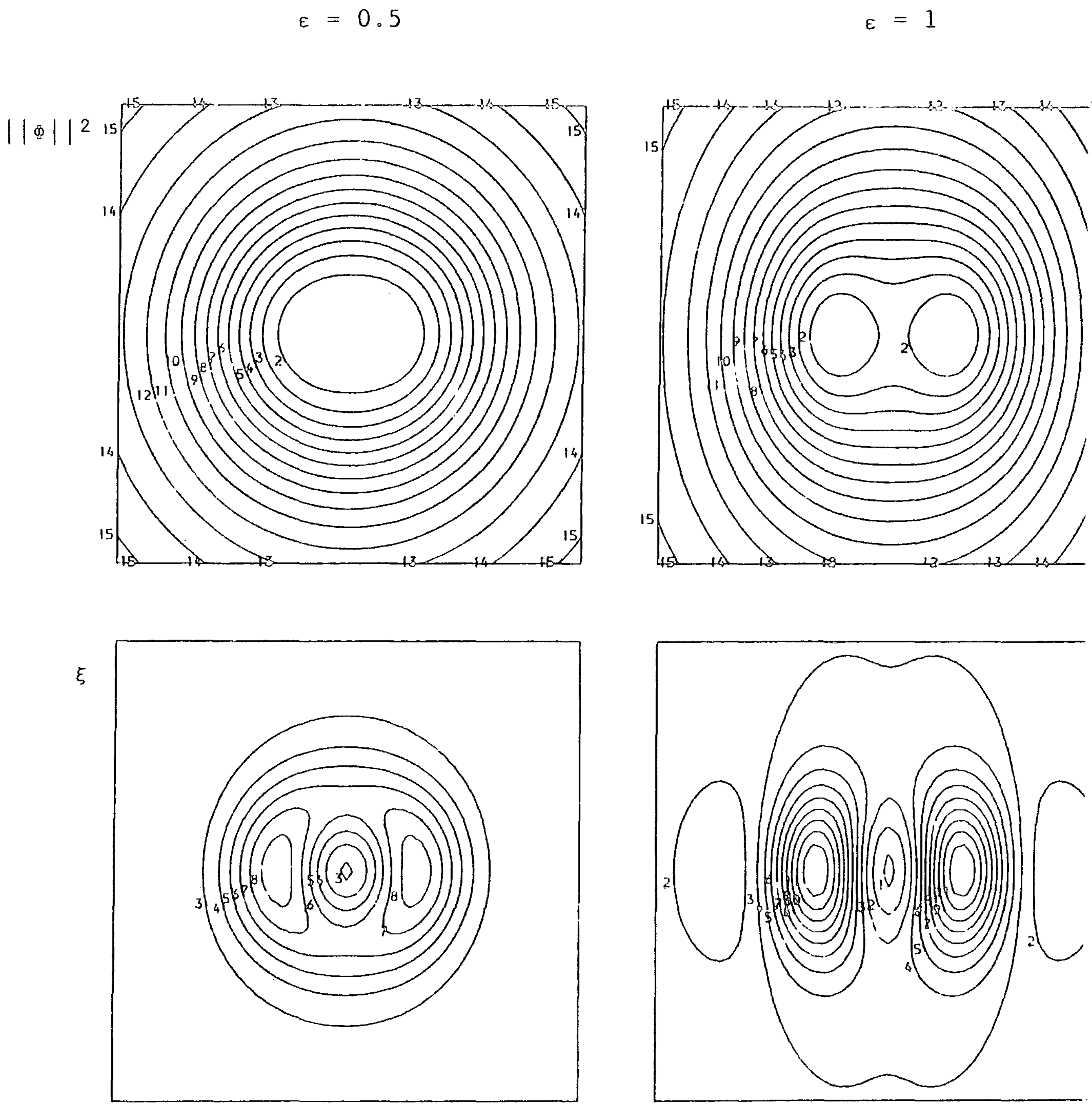
In view of the naturality of the solutions, together with the above nice analysis of the case of small separation, we might expect these solutions to display behaviour analagous to that of the Nielsen-Olesen vortices in the Bogomol'nyi limit of the Abelian Higgs model ie to separate out to an approximate linear superposition of isolated 1-vortices. Unfortunately however, this is not the case - the solutions still exhibit bizarre unexpected behaviour for large values of the separation parameter ($\varepsilon \gtrsim 1$).

The charge 2 separated solution exhibits the same elongation behaviour perpendicular to the axis of separation as that observed for the charge 2 solutions in Chapter 4; however, unlike the solutions of Chapter 4, the physical separation seems to remain the same order of magnitude as the separation parameter ε.

For the higher charge solutions (N=3,4), there is not only a distortion in the shape of the vortices for large separation parameter ($\varepsilon \gtrsim 1$), but also a development of secondary minima of the Higgs field and corresponding maxima of the energy density - for $\varepsilon > 1$ and $N > 2$, the charge N solution seems to have 2N local minima of the Higgs field.

Thus, whereas it is possible to interpret the solutions as N finitely separated vortices for <u>small</u> values of the separation parameter $\varepsilon \lesssim 1$, this interpretation does not seem entirely tenable for $\varepsilon > 1$. Again it seems impossible, within this class of solutions, to approach an approximate linear superposition of isolated vortices at large separation.

Fig 8(a)



Contour Plots of the separated 2-vortex solution, $\varepsilon = 0.5$, $\varepsilon = 1$

Contour Key: $||\Phi||^2_n = -1.0 + 0.05(n-1)$

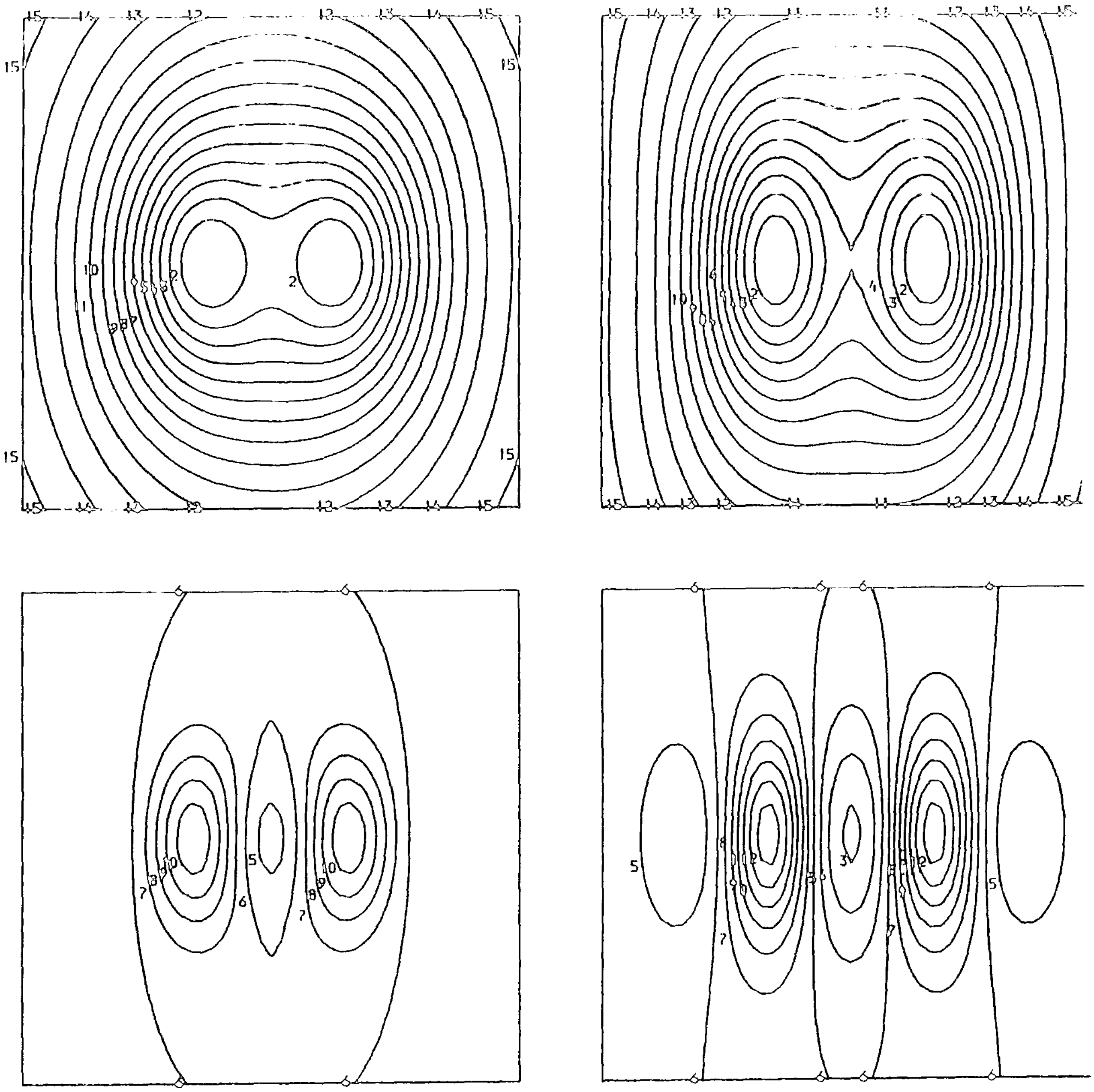
$\xi_n = -0.05 + 0.025(n-1)$

Fig 8(b)

$\varepsilon = 1.5$ $\varepsilon = 2$

$||\Phi||^2$

ξ



Same as Fig 8(a), $\varepsilon = 1.5, 2$

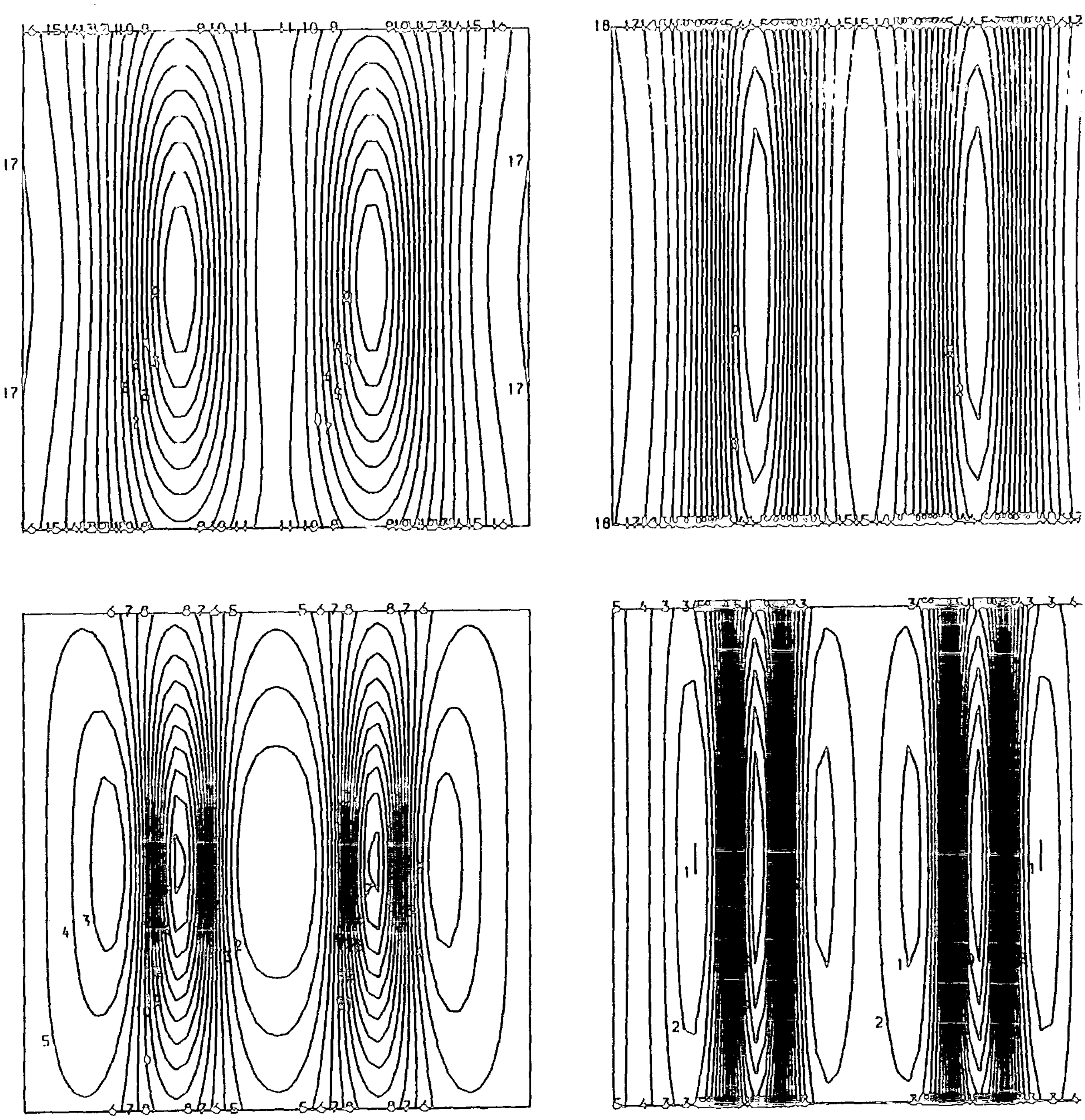
Contour Key: $\xi_n = -0.25 + 0.05(n-1)$

Fig 8(c)

$\varepsilon = 2.5$ $\varepsilon = 3$

$||\phi||^2$

ξ



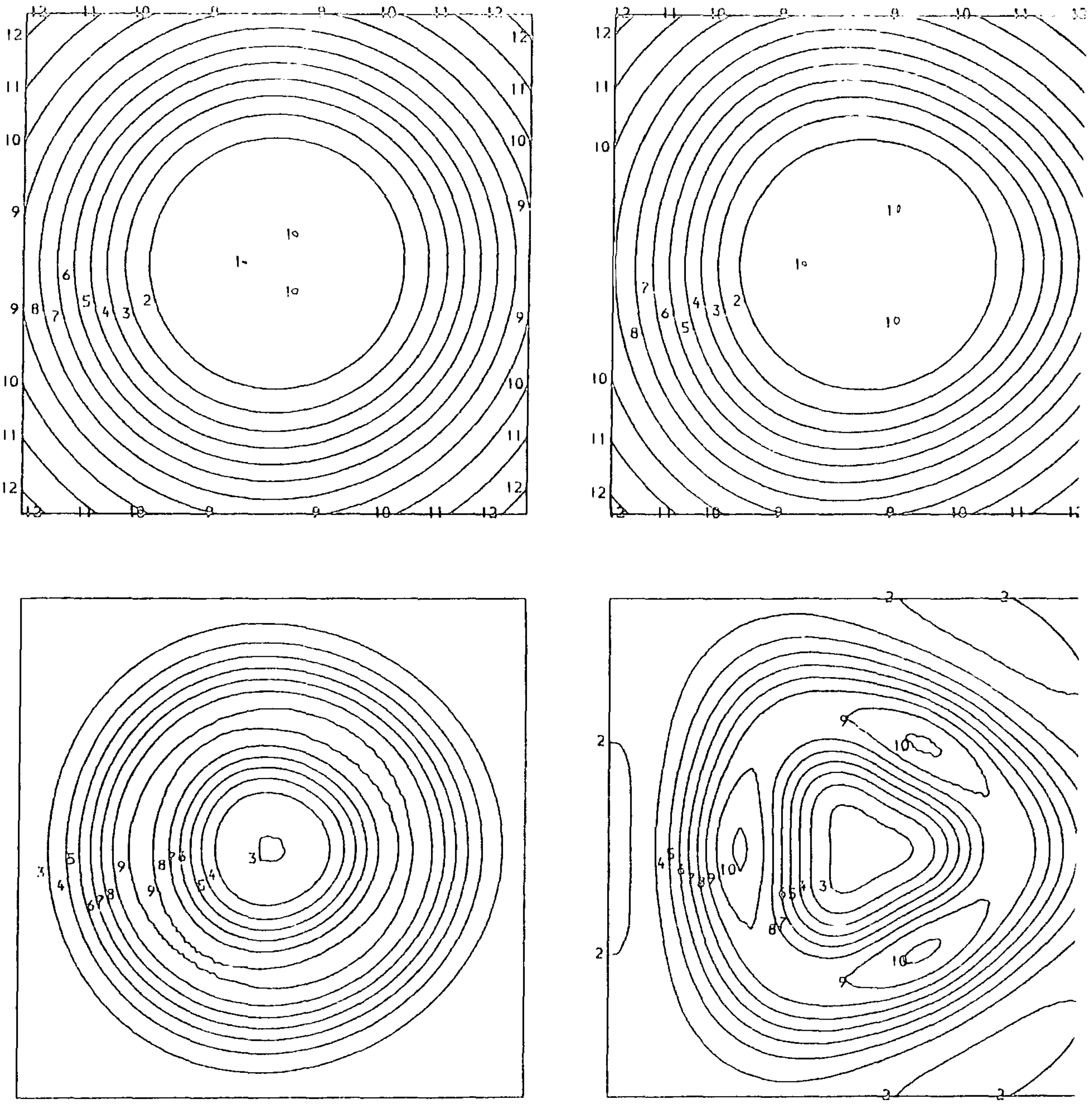
Same as Fig 8(b), $\varepsilon = 2.5, 3$

Fig 9(a)

$\varepsilon = 0.5$ $\qquad\qquad$ $\varepsilon = 1$

$||\Phi||^2$

ξ



Contour Plots of the D_3-symmetric separated 3-vortex solution, $\varepsilon = 0.5, 1$.

Contour Key: $||\Phi||^2_n = -1.0 + 0.05(n-1)$

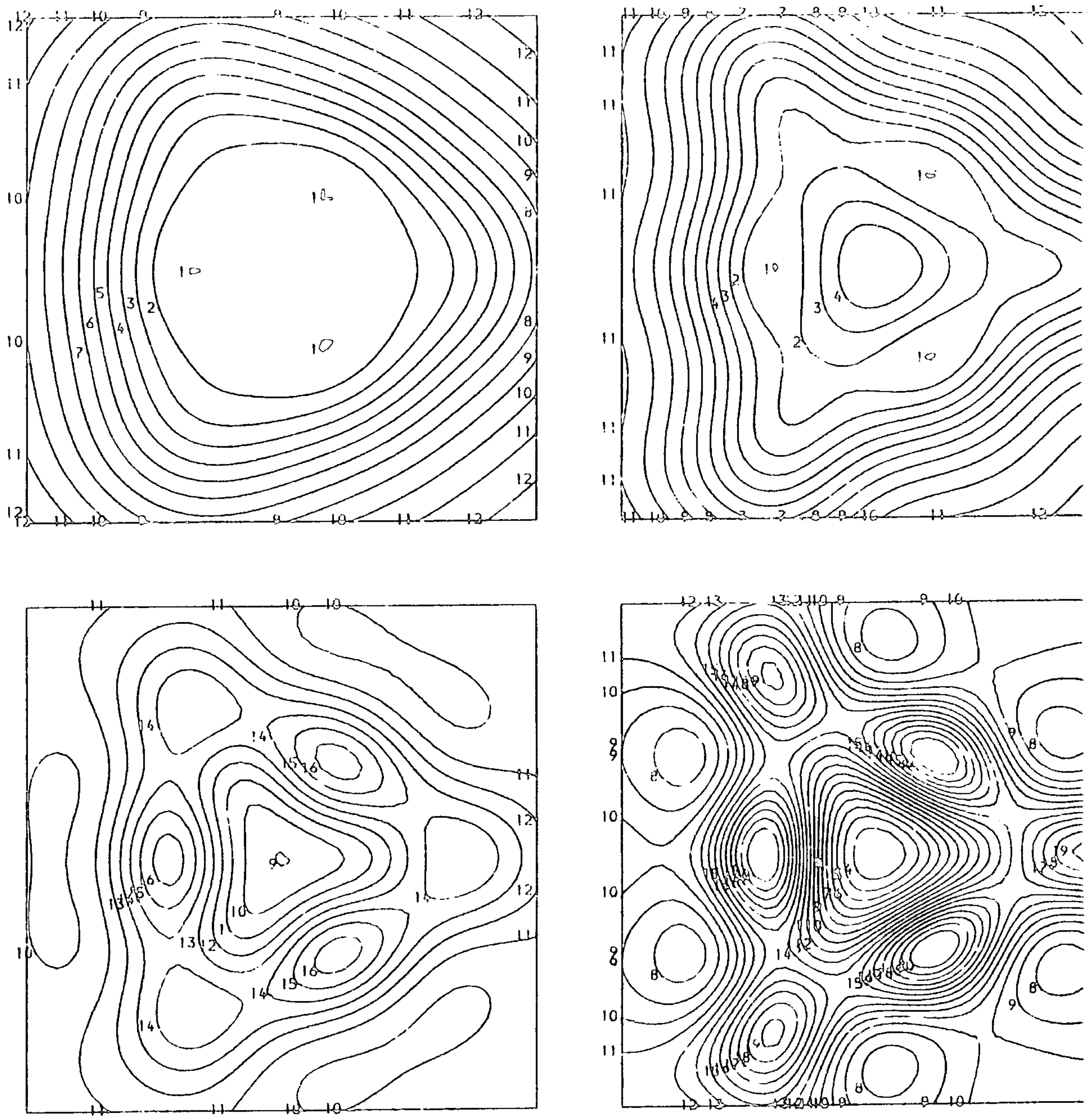
$\xi_n = -0.02 + 0.01(n-1)$

Fig 9(b)

$\varepsilon = 1.5$ $\varepsilon = 2$

$||\Phi||^2$

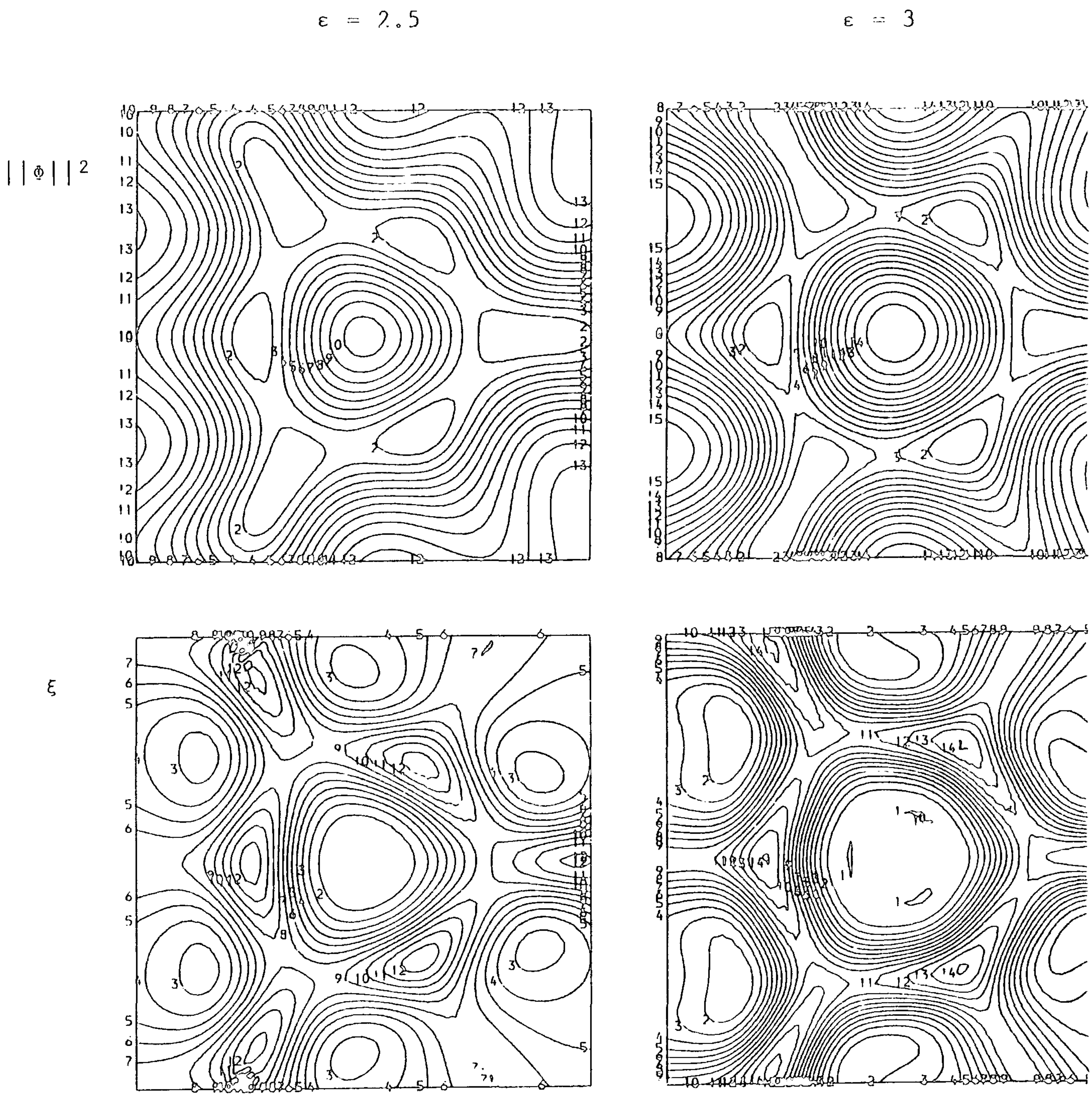
ξ



Same as Fig 9(a), $\varepsilon = 1.5, 2$

Contour Key: $\xi_n = -0.2 + 0.02(n-1)$

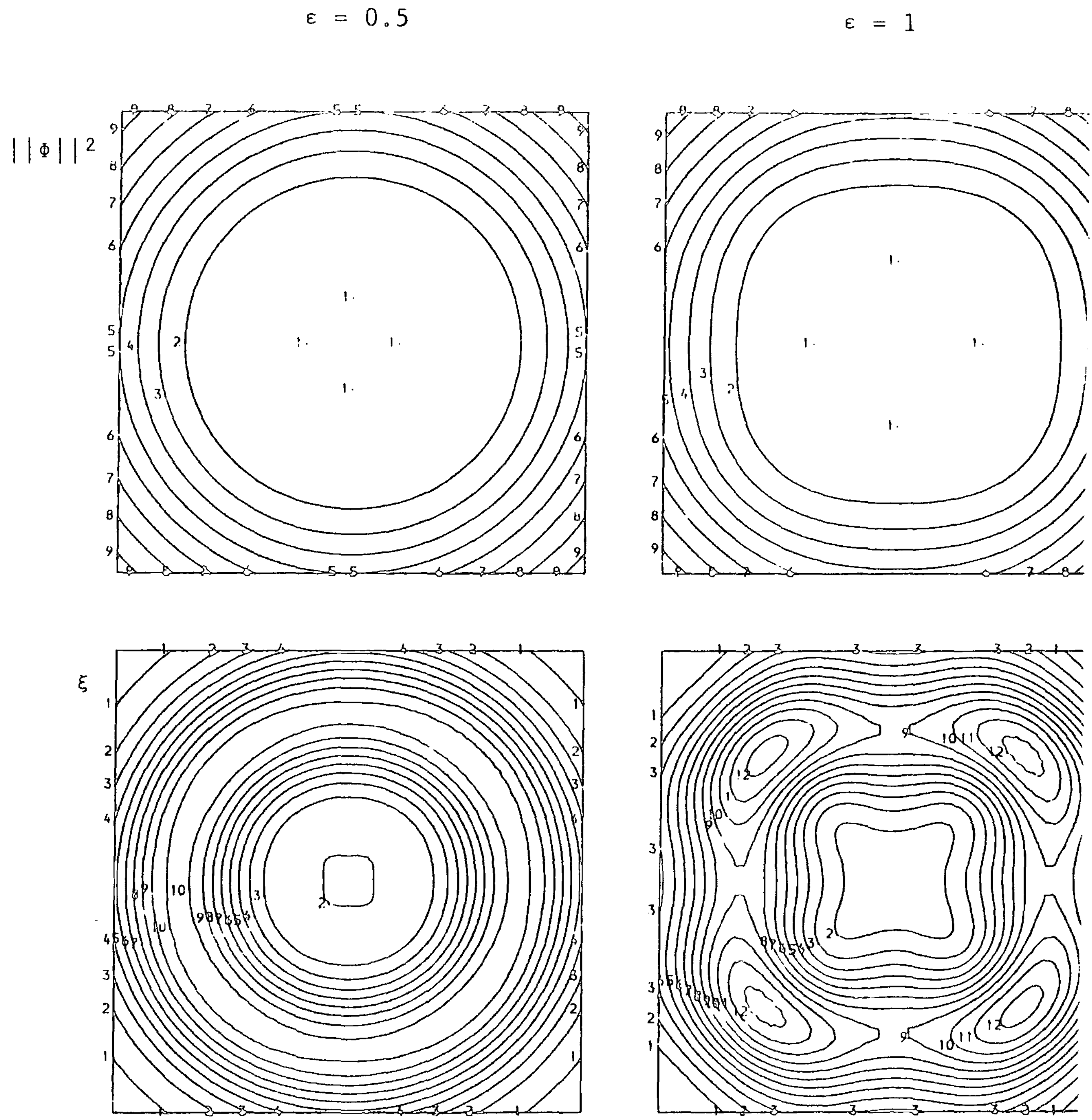
Fig 9(c)



Same as Fig 9(a), ε = 2.5,3

Contour Key: $\xi_n = -0.25 + 0.05(n-1)$

Fig 10(a)



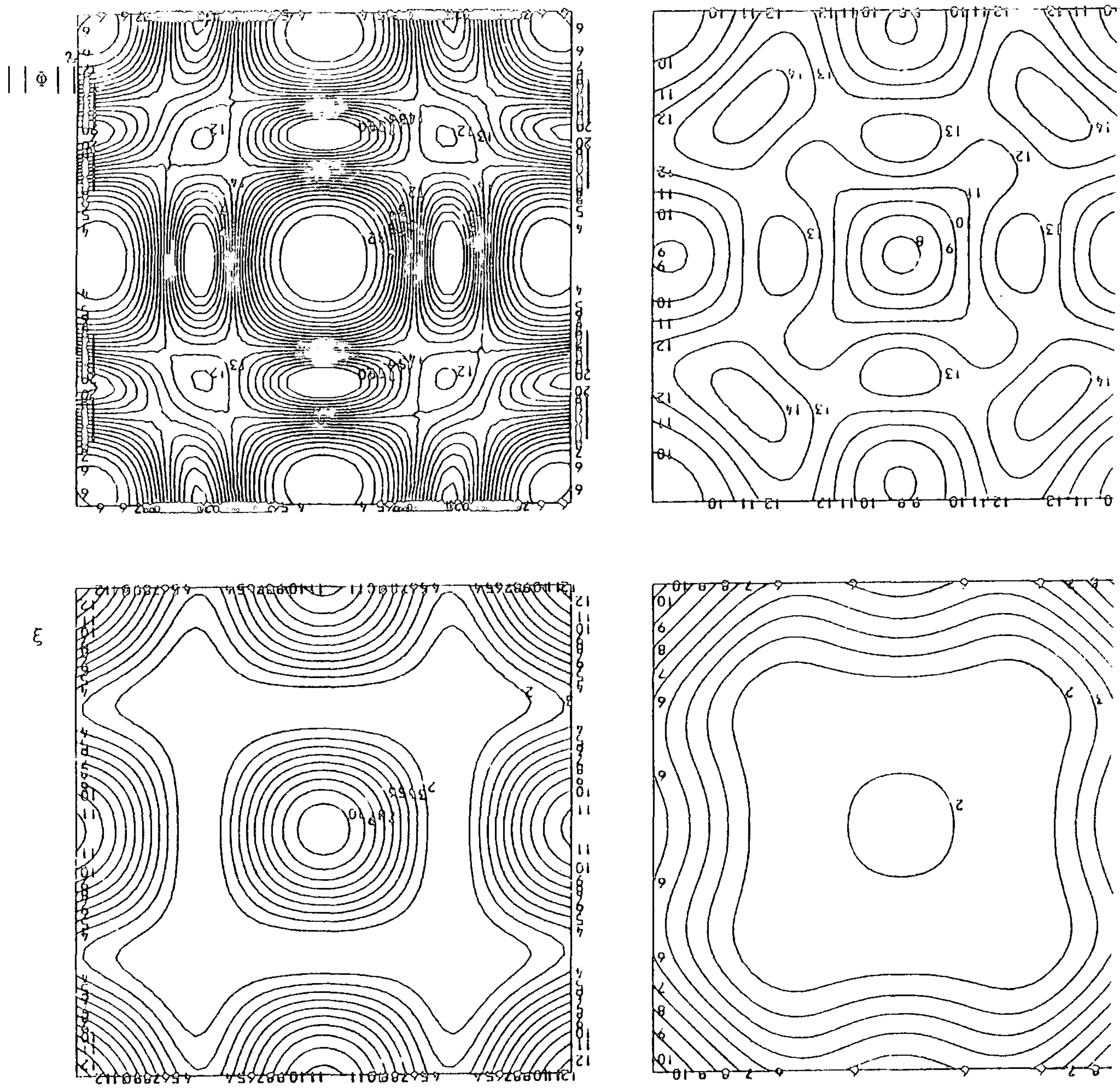
Contour Plots of $||\Phi||^2$, ξ for the D_4 symmetric separated 4-vortex solution, $\varepsilon = 0.5, 1$.

Contour Key: $||\Phi||_n^2 = -1.0 + 0.05(n-1)$

$\xi_n = -0.005 + 0.005(n-1)$

Fig 10(b)

ε = 1.5 ε = 2



Same as Fig 10(a), $\varepsilon = 1.5, 2$

Contour Key: $\xi_n = -0.25 + 0.025(n-1)$

CHAPTER 6

CONCLUSIONS

6.1 Summary of Results

Since our original goal was to find, for pure nonabelian gauge theories, structures analogous to the superconducting vortices of the abelian Higgs model, let us compare the properties of our solutions with those of Nielsen-Olesen vortices. First of all, we have seen that, after dimensional reduction, the R-gauge equations reduce to equations for an unconventional U(1) gauge theory interacting with a massive complex scalar field Ψ involving higher derivative couplings, and with mass of the same order as the characteristic mass c. In the nth Atiyah-Ward ansatz, Ψ satisfies boundary conditions identical to those of the complex scalar field of a Nielsen-Olesen vortex of charge n. For all values of c, $|\Psi|$ is asymptotically equal to unity, and the characteristic length c^{-1} measures the size of regions of space over which $|\Psi|$ differs appreciably from its asymptotic value. Unlike Nielsen-Olesen vortices, which have a short range magnetic field with finite quantized flux, the effective magnetic field of our solutions is the superposition of a long range part with infinite total flux, and a short range part with finite total flux obeying the same quantization law as Nielsen-Olesen vortices.

The non-singularity condition forces the solutions to be strictly complex of zero total action. This in turn, leaves far fewer constraints on the dimension of the parameter space of solutions than is the case for the real charge n multi-vortices in the Bogomol'nyi limit of the abelian Higgs model, which are know to form a 2n-dimensional manifold (Weinberg 1979). In contrast, we have constructed an infinite dimensional parameter space of complex non-singular solutions in each of the Atiyah-Ward ansätze. We can however identify a natural 2n-parameter subfamily of solutions in the nth ansatz (ie those with deg $P(\zeta) \leq n$) which, for small separation parameters (less than the characteristic length c^{-1}), behave like n finitely separated vortices. Unfortunately, numerical evidence indicates that this interpretation breaks down for large separations. This is a little disappointing - it was expected by the author that this 2n-parameter family would display behaviour analagous to that of Nielsen-Olesen vortices in the Bogomol'nyi limit $\lambda = 1$, ie that, for sufficiently large separations, the energy density would be an approximate linear superposition of energy densities of separated vortices.

The class of non-singular solutions constructed in Chapter 5 was obtained using educated guesswork from the monopole construction of Corrigan & Goddard (1981) - it was guessed that one of the conditions guaranteeing non-singularity was the non-vanishing of the patching function. The fact that our solutions do indeed turn out to be non-singular lends some credance to (though it does not prove) the assertion that monopoles obtained from the Corrigan-Goddard ansatz are non-singular. It might be an interesting exercise to take a Fourier transform in the x_3-variable of the Δ-chain equations for monopoles, thus obtaining the Δ-chain equations for vortices, with $c = \sqrt{a^2 + p^2}$ (p = conjugate momentum), and thus express multi-monopoles in some sense as 'Fourier transforms' of multi-vortices. The results of this thesis may then be useful in calculating properties of multi-monopole field configurations, and, perhaps in finding a relatively simple proof of non-singularity.

It is still unclear whether or not the ADHMN construction could be generalized to describe our solutions. In its present formulation, the ADHMN construction yields <u>real</u> non-singular solutions of the self-duality equations, so it would have to undergo a non-trivial modification in order to yield <u>complex</u> non-singular solutions. It may just be that Corrigan-Goddard reciprocity is trivial in 2+2 dimensions in that the only real non-singular solution is the vacuum.

Finally, let us remark that, although we have concentrated here on self-dual vortex solutions of SU(2) gauge theory, such solutions clearly also exist for larger gauge groups G - the simplest possible of such solutions could be obtained from embeddings of SU(2) in G, though the general case is expected to be more complicated than this (cf work on multimonopoles in higher gauge groups (Athorne 1983, Bais & Wilkinson 1979, Ward 1982a). Since the long range behaviour of magnetic fields of self-dual vortices is similar to that of monopoles, it is also expected that the general analysis of nonabelian magnetic charge due to Goddard, Olive & Nuyts (1977) will go through word-for-word in the definition of self-dual vortex charges for higher gauge groups.

6.2 Speculations on Possible Physical Applications

The contribution of complex saddle points of the classical action to the semi-classical approximation of the functional integral has been studied in the context of the quantum mechanical double well oscillator by Richard & Rouet (1981 a,b) and in the context of finite temperature quantum mechanics and field theory by Lapides & Mottola (1982). Both sets of authors find that this approximation procedure is an ideal substitute for the instanton gas approximation. It is therefore natural to suggest that our class of non-singular complex solutions to pure nonabelian gauge theory should provide an important contribution to the semi-classical evaluation of QCD. Indeed, the fact that these solutions are 'voidons', ie they are complex solutions of zero total action, seems absolutely essential if this interpretation is going to be correct. Solutions with <u>finite action</u> on $\mathbb{R}^2$ would have <u>infinite action</u> on $\mathbb{R}^4$, and hence would not contribute to the semi-classical formula (1.2.8) due to the exponential damping factor:

$$e^{-S/g^2h}$$

Moreover, the above exponential damping factor suggests that, if complex saddle points do indeed contribute to the semi-classical evaluation of QCD, then those with zero total action should provide a dominant contribution of roughly the same order as the real perturbative vacuum. This idea was first suggested by Dolan (1978).

Before these ideas can properly be put to the test, two non-trivial problems have to be solved:

(1) The functional determinants of Gaussian fluctuations about our solutions have to be computed. If there exists a reformulation of the ADHMN construction for complex solutions in 2+2 dimensions, then it should prove useful in this problem. (cf corresponding problem for instantons (Corrigan et al 1979, Osborn 1979)).

(2) Since the instanton gas approximation is clearly inapplicable for saddle points of zero total action, the full details have to be worked out of deforming the functional integration contour in complex configuration space, and of performing the integral over the appropriate infinite dimensional moduli space of complex solutions.

Since we are ultimately interested in finding an essentially two dimensional mechanism giving rise to the QCD confinement phenomenon, let us briefly review some of the progress that has already been made in this direction, and make some suggestions as to how the self-dual vortex solutions might add to this progress.

(a) Lattice Gauge Theories

It was shown by Wilson (1974) that a pure lattice gauge theory with static quark sources is confining if the expectation value of the loop correlation function

$$W(c) = \chi(\prod_{c} U_{\ell})$$

satisfies the 'area law':

$$\langle W(c)\rangle \sim e^{-\sigma A} \qquad \text{as } A\to\infty$$

where A is the area of a minimal surface spanning the loop C. (χ = trace for the fundamental (Wilson) coupling). This gives rise to an asymptotically linear interquark potential $V(R) \sim \sigma R$ as R ; σ is called the string tension. Wilson showed moreover that all lattice gauge theories (ie nonabelian and abelian) are confining at sufficiently strong coupling, with $\sigma \sim \ln g^2$ as $g \to \infty$. The confining phase of LGT is in fact analagous to the high temperature disordered phase of the corresponding lattice chiral model, the string tension being analagous to the mass gap = inverse correlation length of the 2-point correlation function:

$$\langle\chi(U_x U_y^{-1})\rangle \sim e^{-|x-y|/\ell}, \qquad |x-y|\to\infty$$

The conventional wisdom is that nonabelian gauge theories will be confining in the continuum limit if there is no 2nd order phase transition separating the confining strong coupling region from the asymptotically free weak coupling region. The difference between nonabelian and abelian LGT's should be that the nonabelian theories remain disordered for all values of the coupling, whereas, at weak coupling, abelian theories should be in an ordered 'Coulomb phase' with a perimeter law for Wilson loops. So, the main question that has to be answered is, what are the excitations that disorder lattice gauge theories? In particular, why are nonabelian theories disordered at weak coupling, whereas abelian theories are not?

These questions have been partially answered by Monte Carlo simulations of SU(N) LGT's and chiral models (see eg Ardill et al 1982, Creutz et al 1983, Caneschi et al 1982, Kogut et al 1981, 1982, 1983). For a fundamental (Wilson) coupling:

$$\beta_F \chi_F(U) = \beta_F \mathrm{Tr}\, U$$

a rapid crossover from weak to strong coupling behaviour is observed for N=2, 3, whereas for $N \geq 4$ there is 1st order phase transition separating the weak and strong coupling regions. This is however not deconfining

- it just gives rise to a discontinuity in the string tension. This phenomenon can be better understood by considering the theory with adjoint coupling.

$$\beta_A \chi_A(U) = \beta_A((\mathrm{Tr}U)(\mathrm{Tr}U^+) - \mathrm{Tr}\,1))$$

In this theory, elements of the centre $\mathbb{Z}_N$ of SU(N) are effectively trivial, so the global group is actually $SU(N)/\mathbb{Z}_N$. Monte Carlo simulations show that there is always a 1st order phase transition in these models due to the destruction of $\mathbb{Z}_N$-monopole condensates and thin $\mathbb{Z}_N$-vortices, which are responsible for confinement in the strong coupling phase. It should be noted that, like the 1st order phase transitions, these topological objects are artifacts of the lattice regularization.

In the mixed theory, with both fundamental and adjoint couplings, the 1st order phase transition point on the β_A-line forms part of a line of 1st order phase transition points extending towards the β_F-line. For N=2,3, this line terminates before it reaches the β_F-line, giving rise to the rapid crossover behaviour of the Wilson model, whilst for $N \geq 4$ it terminates after it has crossed the Wilson line, giving rise to the 1st order phase transition in the Wilson model. Of course, it is still possible to continue round this line for negative β_A, and thus connect the strong coupling region to the weak coupling region.

The lesson to be learned from this is clear - the fluctuations guaranteeing confinement at strong coupling are <u>not</u> responsible for confinement at weak coupling. The 1st order phase transition line is associated with a <u>change</u> of the confinement mechanism, which, at weak coupling, must be due to fluctuations associated with the nonabelian degrees of freedom of the gauge group.

(b) <u>Instantons</u>

The instanton contributions to the functional integral have been calculated exactly for the 2-dimensional $\mathbb{C}P^n$ models (Eichenherr 1978). In the $\mathbb{C}P^1$ case, the k-instanton solutions yield an effective partition function equivalent to that of a classical neutral Coulomb gas of 2k particles at the critical temperature T=1 where the pressure diverges (Berg & Lüscher 1979). This indicates that the dilute gas approximation (DGA) is rather poor for this model (it is in fact divergent). Four dimensional gauge theories are expected to suffer from the same problem, but unfortunately there is no corresponding exact calculation of the semi-classical partition function to test this hypothesis.

Nielsen-Olesen vortices and 't Hooft-Polyakov monopoles are, respectively, the instantons of the 1+1 dimensional abelian Higgs model, and the 2+1 dimensional Georgi-Glashow model. It has been shown (using the DGA) that these give rise to confinement in these models, essentially because of the restoration of apparently broken gauge symmetries due to the instantons' tunnelling between perturbatively degenerate vacua (Coleman 1977, Polyakov 1977). Again, there is no corresponding result for 4-dimensional gauge theories - Yang Mills instantons decay far too rapidly to affect the integral for large Wilson loops (Coleman 1977).

It has been argued, again using the DGA, that Yang-Mills instantons are responsible for the rapid crossover from weak to strong coupling in SU(2) LGT (Callan, Dashen & Gross 1979). However, as remarked earlier, this phenomenon seems more related to the first order phase transition line in the mixed action theory. In fact, studies of mixed action lattice chiral models (Kogut et al 1981, 1983) indicate that even models without instantons have a rapid crossover - it just happens that those models with instantons have a more rapid crossover than those without.

The upshot of all this is that, despite some success in lower dimensional theories, there is very little evidence that instantons play an essential role in the QCD confinement phenomenon. From our point of view, this is not too disappointing - if complex zero action solutions do contribute, they are certainly expected to dominate the instanton contributions, and it is hard to imagine how instantons (or merons) could give rise to an essentially two dimensional confinement mechanism.

(c) The Copenhagen Vacuum

The Copenhagen group has proposed that confinement follows in QCD if the QCD vacuum is equivalent to a statistical mechanical system of random colour magnetic flux (Ambjorn & Olesen 1980, Nielsen & Olesen 1979, Olesen 1982). This picture is inspired by a phenomenon that occurs in solid state physics, namely, the large distance dynamics of a d-dimensional system coupled to a random magentic field is equivalent to that of the (d-2)-dimensional system without the random field. It is then argued, heuristically, that the Wilson correlation function of QCD_4 should behave similarly to that of QCD_2. A more explicit (but none-the-less still heuristic) argument has been given by Nielsen & Olesen, and it goes as follows.

The flux Φ through a domain D is supposed to be random in the sense that:

$$\langle\Phi\rangle_D = 0, \qquad \langle\Phi^2\rangle_D \neq 0$$

and Φ is uncorrelated from domain to domain. One then considers a domain D, of area A, divided into small uncorrelated domains D_i. Then:

$$\langle\Phi^2\rangle_D = \sum_i \langle\Phi^2\rangle_{D_i} \propto A$$

so, using the central limit theorem to approximate the statistical distribution of Φ by a normal distribution:

$$\rho(\Phi) = \frac{1}{\sqrt{\pi\sigma A}} \exp(-\Phi^2/\sigma A)$$

we obtain, for a curve C spanning the domain D:

$$W(C) = \mathrm{Tr}\ P \exp(\oint_C A_\mu dx^\mu) = e^{i\Phi}$$

$$\Rightarrow \quad \langle W(C)\rangle = \langle e^{i\Phi}\rangle_D \sim \int d\Phi_D\, e^{i\Phi_D} \rho(\Phi_D)$$

$$\Rightarrow \quad \langle W(C)\rangle \sim e^{-\sigma A}$$

ie the area law. Perhaps the most severe criticism of this argument has been given by Lüscher, who pointed out that it is by no means obvious why the <u>minimal</u> surface spanned by C should be picked out. However, it has been argued by Olesen that, conversely, if one <u>assumes</u> the area law, then it follows that some additive flux must be independently distributed over the minimal surface spanning a Wilson loop. Thus, an important constraint on any candidate theory of confinement along these lines, is that it must give rise to random fluxes on minimal surfaces, but <u>not</u> on non-minimal ones. Note also that the additivity property of the flux is rather non-trivial for nonabelian theories, but it might follow if the vacuum is dominated by topologically non-trivial objects, such as effective abelian magnetic flux tubes.

Finally, another criticism that could be raised is that the abelian-like flux has not been properly defined, and no dynamical mechanism has been proposed for the formation of magnetic flux tubes. Ambjorn & Olesen have considered the possibility that $\mathbb{Z}_N$ vortices give rise to the desired dynamics. However, from the above remarks on lattice gauge theories, it seems, at least to the author, that topological objects related to the centre of the gauge group are only relevant at strong coupling, and not in the weak coupling continuum limit. Thus it is tempting to make the alternative proposal that, via the semi-classical approximation, <u>self-dual vortices</u> provide the dynamical mechanism for the formation of the Copenhagen vacuum. In this case, the flux is effectively abelian, with gauge group given by a maximal torus in SU(N) singled out by a choice of

R-gauge, and it seems at least plausible that, in analogy with the instantons of the $\mathbb{C}P^n$ models, the suitably deformed semi-classical functional integral is equivalent to the partition function of a random vortex gas, with the required properties for confinement. The main advantage of this approach, if it works, is that is should give a well-defined analytic calculational procedure for making non-perturbative predictions, such as the ratio between the string tension and the Λ-parameter.

Of course, until the difficult problems (1) & (2) have been solved, and hard calculations have been done, this idea must retain the status of a rather bold conjecture. However, it is hoped that the conceptual simplicity and the structural richness of the self-dual vortex solutions will persuade physicists that it is worth the effort to check whether these ideas are correct or not.

APPENDIX A

Properties of Modified Bessel Functions

(Abromowitz & Stegun, Watson)

We shall deal throughout with modified Bessel functions $I_n(z)$, $K_n(z)$ of integer order. Note the identity:

$$I_n(z) = I_{-n}(z), \qquad K_n(z) = K_{-n}(z) \tag{A.1}$$

Power Series Expansions

$$I_n(z) = \left(\frac{z}{2}\right)^n \sum_{K=0}^{\infty} \frac{(\frac{1}{2}z)^{2K}}{k!(n+k)!} \tag{A.2}$$

Small z Behaviour

$$I_n(z) = (\tfrac{1}{2}z)^n \left[\frac{1}{n!} + \frac{z^2}{4(n+1)!} + \frac{z^4}{32(n+2)!} + \dots \right] \tag{A.3}$$

$$K_o(z) \sim -\ln z \tag{A.4}$$

as $z \to 0$

$$K_n(z) \sim \frac{(n-1)!}{2} \left(\frac{z}{2}\right)^{-n}, \quad (n > 0)$$

Asymptotic Expansions ($z \to \infty$):

$$I_n(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left[1 - \frac{(4n^2-1)}{8z} + \dots \right], \quad |\arg z| < \frac{\pi}{2} \tag{A.5}$$

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z}\left[1 + \frac{(4n^2-1)}{8z} + \ldots\right], \quad |\arg z| < \frac{3\pi}{2}$$

Generating Function:

$$e^{\frac{z}{2}(t+\frac{1}{t})} = \sum_{k=-\infty}^{\infty} t^k I_k(z) \qquad (A.6)$$

Integral Representations

$$I_k(z) = \int_0^{2\pi} \frac{d\theta}{2\pi} \cos k\theta . e^{z\cos\theta} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{ik\theta} e^{z\cos\theta} \qquad (A.7)$$

$$K_k(z) = \int_0^{\infty} du . \cosh ku . e^{-z\cosh u} = \int_{-\infty}^{\infty} \frac{du}{2} . e^{ku} e^{-z\cosh u}$$

Derivative Recurrence Relations

If $L_n(z)$ is a linear combination of $I_n(z)$ and $(-1)^n K_n(z)$:

$$L_n'(z) \mp \frac{n}{z} L_n(z) = L_{n\pm 1}(z) \qquad (A.8)$$

APPENDIX B

Töplitz Determinants and U(N) Group Integrals

Given a sequence $(G_k;\ -\infty<k<\infty)$, the associated NxN Töplitz determinant is defined by:

$$\det(G_{\ell-k}) = \begin{vmatrix} G_o & G_1 & G_2 & \cdots & G_{N-1} \\ G_{-1} & G_o & G_1 & & \vdots \\ G_{-2} & G_{-1} & G_o & & \vdots \\ \vdots & & & \ddots & \vdots \\ G_{-N+1} & \cdots & \cdots & \cdots & G_o \end{vmatrix}$$

This is precisely the form of $\det D^{(N)}$ encountered in the text.

A well known elementary property of Töplitz determinants is the following:

$$\det(\lambda^{\ell-k}G_{\ell-k}) = \det G_{\ell-k}, \qquad \lambda\in\mathbb{C} \tag{B.1}$$

Proof

$$\det(\lambda^{\ell-k}G_{\ell-k}) = \begin{vmatrix} G_o & \lambda G_1 & \lambda^2 G_2 & \cdots & \lambda^{N-1}G_{N-1} \\ \lambda^{-1}G_{-1} & G_o & \lambda G_1 & & \vdots \\ \lambda^{-2}G_{-2} & \lambda^{-1}G_{-1} & G_o & & \vdots \\ \vdots & & & \ddots & \vdots \\ \lambda^{-(N-1)}G_{-N+1} & \cdots & \cdots & \cdots & G_o \end{vmatrix}$$

$$= \lambda^{-1}\lambda^{-2}\ldots\lambda^{-(N-1)} \begin{vmatrix} G_o & \lambda G_1 & \lambda^2 G_2 & \cdots & \lambda^{N-1} G_{N-1} \\ G_{-1} & \lambda G_o & \lambda^2 G_1 & & \vdots \\ G_{-2} & \lambda G_{-1} & \lambda^2 G_o & \ddots & \vdots \\ \vdots & & & \ddots & \vdots \\ G_{-N+1} & \cdots & \cdots & \cdots & \lambda^{N-1} G_o \end{vmatrix}$$

and the result follows, after cancelling the factors of λ from the columns.

QED

Given $F: U(N) \to \mathbb{C}$, define its U(N) group integral:

$$I(F) = \int dU F(U)$$

where dU is the Haar measure on U(N), ie the unique measure such that:

$$dU = d(UV) = d(VU), \quad \forall V \varepsilon U(N)$$

and

$$\int_{U(N)} dU = 1$$

Diagonalising $U = T^+ \Lambda T$, $T \varepsilon U(N)$, $\Lambda = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_N})$, the Haar measure is given by (Weyl 1939):

$$dU = dT . d\mu(\theta_i)$$

where

$$d\mu(\theta_i) = \frac{1}{N!} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \Delta(e^{i\theta_i}) \Delta(e^{-i\theta_i})$$

$$= \frac{1}{N!} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \prod_{k<\ell}^{N} 4\sin^2\left(\frac{\theta_k-\theta_\ell}{2}\right)$$

where $\Delta(x_i)$ is the Vandermonde determinant:

$$\Delta(x) = \det(x_k^{\ell-1}) = \prod_{k<\ell}^{N} (x_k - x_\ell)$$

When F is <u>invariant</u> ie $F(V^+UV) = F(U)$, $\forall U, V \in U(N)$, the dT integration is trivial, and we are left with:

$$I(F) = \int d\mu(\theta_i) F(\Lambda)$$

Using these results, we can establish a remarkable connection between Töplitz determinants and U(N) group integrals.

<u>Theorem B (Prasad & Rossi, 1980)</u>

Suppose G_k is defined by:

$$G_k = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{ik\theta} G(e^{i\theta}) = \oint_{U(1)} \frac{dz}{z} . z^k G(z, z^*)$$

where $G : U(1) \to \mathbb{C}$ is defined by a power series:

$$G(z, z^*) = \sum_{1,\ell} g_{k,\ell} z^k (z^*)^\ell$$

Extend this to a map $G : U(N) \to M(N, \mathbb{C})$ defined by:

$$G(U, U^+) = \sum_{k,\ell} g_{k,\ell} U^k (U^+)^\ell$$

Then,

$$\det(G_{\ell-k}) = \int_{U(N)} dU.\det G(U,U^{+})$$

$$= \frac{1}{N!}\int_{o}^{2\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \cdot \prod_{k<\ell}^{N} 4\sin^2\left(\frac{\theta_k-\theta_\ell}{2}\right) \cdot \prod_{i=1}^{N} G(e^{i\theta_i}) \tag{B.2}$$

Proof

Using the integral rperesentation for G_k, we have:

$$\det(G_{\ell-k}) = \int_{o}^{2\pi} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi} . G(e^{i\theta_1}) \cdots G(e^{i\theta_N}) W(\theta_1,\ldots,\theta_N)$$

where

$$W(\theta_1,\ldots,\theta_N) = \det(e^{i(\ell-k)\theta_k})$$

Since θ_i are dummy integration variables, we can rewrite this as:

$$\det(G_{\ell-k}) = \frac{1}{N!} \sum_{\sigma\epsilon S_N} \int_{o}^{2\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \prod_{i=1}^{N} G(e^{i\theta_i}) W(\theta_{\sigma(1)},\ldots,\theta_{\sigma(N)})$$

and the result follows from Weyl's identity:

$$\sum_{\sigma\epsilon S_N} W(\theta_{\sigma(1)},\ldots,\theta_{\sigma(N)}) = \Delta(e^{i\theta_i})\Delta(e^{-i\theta_i})$$

$$= \prod_{i<j}^{N} 4\sin^2\left(\frac{\theta_i-\theta_j}{2}\right) \tag{B.3}$$

QED

In a power series expansion of equ (B.2), we will typically have to evaluate integrals of the form:

$$I_{\alpha_1\beta_1\ldots\alpha_m\beta_m\gamma_1\delta_1\ldots\gamma_n\delta_n} = \int_{U(N)} dU.U_{\alpha_1\beta_1}\ldots U_{\alpha_m\beta_m}U^{+}_{\gamma_1\delta_1}\ldots U^{+}_{\gamma_n\delta_n}.$$

Writing $U = e^{i\theta}V$, $V\varepsilon SU(N)$, we have:

$$I_{\alpha_1\beta_1\ldots\gamma_n\delta_n} = \int_0^{2\pi}\frac{d\theta}{2\pi}e^{i(m-n)\theta}\int_{SU(N)} dV.V_{\alpha_1\beta_1}\ldots V_{\alpha_m\beta_m}V^{+}_{\gamma_1\delta_1}\ldots V^{+}_{\gamma_n\delta_n} \qquad \text{(B.4)}$$

$$= \begin{cases} 0 & , \quad m\neq n \\ \int_{SU(N)} dV.V_{\alpha_1\beta_1}\ldots V_{\alpha_m\beta_m}V^{+}_{\gamma_1\delta_1}\ldots V^{+}_{\gamma_n\delta_n}, & \quad m=n \end{cases}$$

Rules for calculating the latter SU(N) integrals have been given by Creutz, 1977. The most elementary of these is:

$$\int_{SU(N)} dV.V_{\alpha_1\beta_1}\ldots V_{\alpha_N\beta_N} = \frac{1}{N!}\,\varepsilon_{\alpha_1\ldots\alpha_N}\,\varepsilon_{\beta_1\ldots\beta_N} \qquad \text{(B.5)}$$

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