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CHEBYSHEV SERIES APPROXIMATION ON COMPLEX DOMAINS

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ABSTRACT

This thesis is an account of work carried out at the Department of Mathematics, Durham University, between October 1979 and September 1982. A method of approximating functions in regions of the complex plane is given. Although it is not, in general, a near minimax approximation it is shown that it can give good results. A review of approximation in the complex plane is given in Chapter 1. Chapter 2 contains the basic properties of Chebyshev polynomials and the Chebyshev series, together with methods for calculating the coefficients in the series. The maximum error, over a complex domain, of a truncated Chebyshev series is investigated in Chapter 3 and Chapter 4 shows how the Bessel functions of the first and second kinds of integer order could be approximated over the entire complex plane. Numerical calculations were performed on the NUMAC IBM370.168 computer.

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INTRODUCTION TO COMPLEX APPROXIMATION

The theory of best minimax polynomial approximations is well developed for real valued functions, see Davis (1963), but less well developed for complex valued functions. Algorithms, such as the Remez algorithm for approximation on closed intervals and the iterative scheme described by Ellacott and Williams (1976) for closed regions of the complex plane, do exist to calculate the required polynomials. Opfer (1978) introduced an algorithm based on Kolmogorov's Criterion and Barrodale (1977) has investigated the solution of the best polynomial approximation problem for a number of different norms and suggested that the ℓ_∞ norm,

$$\|g(z)\|_\infty = \max_{1 \leq t \leq M} |g(z_t)|,$$

be replaced by the star norm ℓ_* , defined by

$$\|g(z)\|_* = \max_{1 \leq t \leq M} (\max(|\operatorname{Re}(g(z_t))|, |\operatorname{Im}(g(z_t))|)),$$

thus changing the non-linear problem of finding best ℓ_∞ approximations to the linear one of finding best ℓ_* polynomials. Glashoff and Roleff (1981) reformulate the Chebyshev approximation problem in the complex plane as a problem of linear optimisation in the presence of infinitely many constraints, and show that algorithms exist for its solution.

However, these algorithms are all relatively complex and expensive in computation time, and require the recomputation of all the coefficients when the degree of the approximation is changed. So in many circumstances other approximations, which are easier to produce and nearly as accurate, are used in preference. An approximation, which has a maximum error less than ten times the maximum error of the minimax approximation, over the same region and

of the same degree, is called a near-best or near-minimax approximation.

Near-best polynomial approximations have been found for a restricted set of regions of the complex plane. Geddes and Mason (1975) showed that the truncated Taylor series and the Fourier projection, interpolation at the $(n+1)$ th roots of unity, are near-minimax approximations on the unit disc. The truncated Chebyshev series was shown to be a near-best approximation for a family of ellipses by Geddes (1978), and Mason (1981) showed that an interpolating projection existed that was near-minimax on a circular annulus.

For more arbitrary regions of the complex plane, no such results exist. On closed intervals of the real line, Chebyshev series approximation has been extensively used, Luke (1975, 1977), Clenshaw (1962), Coleman (1980) and the special function section of the NAG subroutine library, because by truncating a Chebyshev series for a function on the interval $[-1, 1]$ we obtain, in many cases, a close approximation to the minimax polynomial of the same degree. The error in truncating a rapidly convergent Chebyshev series $\sum_{k=c}^{\infty} a_k T_k(X)$ after $(n+1)$ terms is dominated by the first neglected term; this, being proportional to $T_{n+1}(X)$, has the equioscillation property characteristic of the error in the minimax polynomial approximation of degree n .

For a domain D of the complex plane we can ask what monic polynomial of degree n will minimise

$$\text{Max}_{Z \in D} |Z^n + a_{n-1} Z^{n-1} + \dots + a_1 Z + a_0|$$

The polynomial $\mathcal{J}_n^D(Z)$ which satisfies this condition is called the Chebyshev polynomial of degree n for the domain D . The success of Chebyshev expansions of the real axis suggests that by truncating an expansion of the form

$$f(z) = \sum_{k=0}^{\infty} b_k \mathcal{J}_k^D(z) \tag{1.1}$$

we could hope to obtain a near-best polynomial approximation for a function f in the domain D . When D is the real interval $[-1,1]$, \mathcal{T}_n^D is 2^{1-n} times the classical Chebyshev polynomial T_n , whereas if D is the unit disc $\mathcal{T}_n^D(z) = z^n$, in which case the right hand side of (1.1) is simply the Taylor series for f . For other domains the properties of the polynomials $\mathcal{T}_n^D(z)$ are not in general known. Explicit expressions for a few polynomials $\mathcal{T}_n^D(z)$ of low degree have been found, Geiger and Opfer (1977), and algorithms exist Elliot (1978) and Opfer (1978) which will produce numerical values of the coefficients for a given domain D and degree n , but without a better understanding of the properties of the polynomials $\mathcal{T}_n^D(z)$ it is not feasible to obtain the coefficients of the expansion (1.1).

Therefore another expansion similar to (1.1), whose coefficients can be more rapidly calculated, would be preferable even if it did not converge as fast as the series in (1.1).

One possibility would be a series in terms of the Faber polynomials related to the compact set of the complex plane on which the function being expanded would be evaluated. Curtiss (1971) gives a good introduction to Faber polynomials and the Faber series, including a study of the convergence of the Faber series. Ellacott (1981a) gives algorithms for calculating the Faber polynomials and the Faber coefficients and for evaluating the Faber series. Also, Ellacott (1981b) has given algorithms for producing rational approximations using the Faber transform.

Expanding in a series of Faber polynomials would seem preferable to finding the Chebyshev polynomials for a particular region, since they are all related to the function $\phi(w)$ which maps the complement of the region conformally onto the exterior of the unit circle. If however, a known set of orthogonal polynomials was chosen, it would reduce the amount of computation time required and the coefficients could be calculated in

the same manner as the expansion of functions of a real variable. The resulting series would probably converge slower than the Faber series of the region as it would not be "tailored" to that specific region. The ease of producing such a series could be the deciding factor, particularly if the series converges quickly enough so that the extra accuracy could be achieved by truncating the series after slightly more coefficients. For these reasons the Chebyshev series has been chosen in this thesis for investigation as an approximation over complex regions.

CHEBYSHEV POLYNOMIALS AND CHEBYSHEV SERIES§1 BASIC PROPERTIES OF CHEBYSHEV POLYNOMIALS

In this section we shall give a brief summary of the properties of Chebyshev polynomials of the first kind. For further details the reader is referred to Rivlin (1974).

The Chebyshev polynomial of the first kind is defined by

$$T_N(z) = \cos(N\Theta) \quad (2.1)$$

where

$$z = \cos \Theta.$$

Normally Θ is taken to be real so that $z \in [-1,1]$, but in the following chapters we shall allow z to take any value in bounded regions of the complex plane. The variable x will be used when the argument is restricted to $[-1,1]$. A closely related set of polynomials, the shifted Chebyshev polynomials of the first kind, are defined as follows

$$T_N^*(z) = T_N(2z-1) \quad (2.2)$$

The properties of these polynomials can easily be derived from those we shall give for $T_N(z)$.

The following recurrence relationship can be obtained from the connection with trigonometric functions:

$$T_{N+1}(z) = 2zT_N(z) - T_{N-1}(z), \quad N \geq 1 \quad (2.3)$$

with $T_0(z) = 1$, $T_1(z) = z$

It is clear from (2.3) that $T_N(z)$ is a polynomial of degree N , with real coefficients and for $N \geq 1$ the leading coefficient is 2^{N-1} . Moreover, if N is even then $T_N(z)$ is an even function and if N is odd then $T_N(z)$

is an odd function. An alternative expression for $T_N(z)$, which is more useful in some circumstances than (2.1), is obtained by solving the second order recurrence equations given in (2.3). They yield

$$T_N(z) = \frac{1}{2} \left\{ (z + \sqrt{z^2 - 1})^N + (z - \sqrt{z^2 - 1})^N \right\}, \quad (2.4)$$

where the branch of $\sqrt{z^2 - 1}$ to be chosen is determined by the conditions

$$\left| z + \sqrt{z^2 - 1} \right| > 1 \quad \text{for } z \notin [-1, 1]$$

and

$$\text{Arg}(\sqrt{z^2 - 1}) = \pi/2 \quad \text{for } z \in [-1, 1]$$

As $\cos N\theta$ has N zeros in $[0, \pi)$ given by

$$\theta_k = \frac{(2k-1)\pi}{2N}, \quad k = 1, 2, \dots, N \quad (2.5)$$

all the zeros $\chi_k^{(N)}$ of $T_N(z)$ lie in the interval $[-1, 1]$ and they are symmetrically distributed about the origin.

This property will now be used to obtain a bound on the absolute value of $T_N(z)$ on a sector D of the unit circle in which

$$\left| \text{Arg } \zeta \right| \leq \theta \leq \pi/2$$

and we shall show that the bound is ~~obtained~~ *obtained*. Since the Chebyshev polynomials have real coefficients it is obvious that

$$\left| T_N(z) \right| = \left| T_N(\bar{z}) \right|,$$

where \bar{z} is the complex conjugate of z . Therefore it is only necessary to consider the sector $0 \leq \arg \zeta \leq \theta$. Furthermore the Maximum Modulus Theorem shows that the maximum occurs on the boundary, i.e. on the radius $\zeta = r \exp(i\theta)$, $0 \leq r \leq 1$, or on the arc $\zeta = \exp(i\theta)$, $0 \leq \theta \leq \theta$.

The zeros of the even polynomial $T_{2m}(\zeta)$ may be labelled as $\pm \chi_k$ ($k = 1, 2, \dots, m$). So we can write

$$T_{2m}(\xi) = 2^{2m-1} \xi \prod_{k=1}^m (\xi^2 - \chi_k^2)$$

and therefore on the unit circle

$$|T_{2m}(e^{i\theta})|^2 = 2^{4m-2} \prod_{k=1}^m (1 + \chi_k^2 - 2\chi_k^2 \cos 2\theta).$$

Clearly, for $0 < \theta < \pi/2$, this is a monotonically increasing function of θ . The maximum value on the arc therefore occurs at $\xi = \text{Exp}(i\theta)$ and as $T_{2m}(1) = 1$ we have

$$|T_{2m}(e^{i\theta})| \geq 1 \text{ for } 0 \leq \theta \leq \pi/2.$$

On the ray $\xi = r e^{i\theta}$,

$$|T_{2m}(r e^{i\theta})| = 2^{4m-2} \prod_{k=1}^m (r^2 + \chi_k^2 - 2r\chi_k \cos 2\theta).$$

If $\pi/4 \leq \theta \leq \pi/2$ each term in the product is positive and monotonically increasing with r . On the other hand, if $0 \leq \theta \leq \pi/4$ the k^{th} term in the product decreases, from the value χ_k^2 when $r = 0$, to a positive minimum at $r = \chi_k (\cos 2\theta)^{1/2}$ and thereafter increases with r . It follows that the product takes its maximum value at $r = 0$ or $r = 1$.

Since

$$|T_{2m}(0)| = 1 \leq |T_{2m}(e^{i\theta})|,$$

we have, for the sector D , the bound

$$|T_{2m}(\xi)| \leq |T_{2m}(e^{i\theta})|.$$

The Chebyshev polynomial $T_{2m+1}(\xi)$ has a zero at the origin and may be written as

$$T_{2m+1}(\xi) = \xi 2^{2m} \prod_{k=1}^m (\xi^2 - \gamma_k^2)$$

where $\pm \gamma_k$ ($k = 1, 2, \dots, m$) are the other zeros. Arguments analogous to those used above show that once again the Maximum modulus on the sector D is attained at the point $\xi = \text{Exp}(i\theta)$.

We have therefore shown that on the sector $D = \{\xi; |\xi| \leq 1, |\text{Arg } \xi| \leq \theta\}$, where $0 \leq \theta \leq \pi/2$,

$$|T_N(\xi)| \leq |T_N(e^{i\Theta})| \quad (2.6)$$

The Chebyshev polynomial of degree N may be written as

$$T_N(\xi) = \frac{1}{2}(W^N + W^{-N}),$$

where

$$W(\xi) = \xi + (\xi^2 - 1)^{\frac{1}{2}}$$

Therefore, on the sector D,

$$|T_N(\xi)| \leq \frac{1}{2}(e^N + e^{-N}), \quad (2.7)$$

where

$$\rho = |e^{i\Theta} + (e^{2i\Theta} - 1)^{\frac{1}{2}}|$$

It is also useful to note that since $\rho \geq 1$, for $0 \leq \Theta \leq \pi/2$,

$$|T_{N+k}(\xi)| \leq \frac{1}{2}(e^N + e^{-N})\rho^k \quad (2.8)$$

in the sector D.

The relationship with trigonometric functions is used to prove two of the most important properties of Chebyshev polynomials, namely their orthogonality over $[-1,1]$ with respect to the weight function $(1-x^2)^{-\frac{1}{2}}$, and the discrete orthogonality. Specifically,

$$\int_{-1}^1 T_j(x) T_k(x) (1-x^2)^{-\frac{1}{2}} dx = \begin{cases} \pi & j = k = 0 \\ 0 & j \neq k \\ \pi/2 & j = k \neq 0 \end{cases} \quad (2.9)$$

and if $\theta_k = k \frac{\pi}{N}$, $k = 0, 1, \dots, N$

and $y_k = \cos \theta_k$, $N > 0$, $S \leq N$ and $k = 0, 1, \dots, N$

$$\sum_{k=0}^N T_r(y_k) T_s(y_k) = \begin{cases} N, & r = 2pN + S, S = 0 \text{ or } N \\ 0, & r \neq 2pN \pm S \\ N/2, & r = 2pN \pm S, S \neq 0 \text{ or } N \end{cases} \quad (2.10)$$

for $p = 0, 1, \dots$

where the double prime on the summation indicates that the first and last terms are halved.

Equation (2.9) is proved by the change of variable $x = \cos \theta$, and it then becomes a standard trigonometric identity. The discrete orthogonality can be proved using the following:

$$(1) \quad T_r(x)T_s(x) = \frac{1}{2} \{ T_{r+s}(x) + T_{|r-s|}(x) \}$$

$$(2) \quad T_r(y_k) = \cos(r k\pi/N)$$

and

$$(3) \quad \sum_{k=0}^{N-1} \cos(j k\pi/N) = \begin{cases} N & j = 2Np \\ 0 & j \neq 2Np \end{cases} \quad p=0,1,\dots \quad (2.11)$$

§2 CHEBYSHEV SERIES

We define the Chebyshev coefficients of a function, $f(z)$, by

$$a_j = \frac{2}{\pi} \int_{-1}^1 f(x)T_j(x) (1-x^2)^{-\frac{1}{2}} dx \quad (2.12)$$

$j = 0, 1, \dots$

We call $\sum_{j=0}^{\infty} a_j T_j(z)$ the Chebyshev series for $f(z)$, the prime on summation indicates that the first term is to be halved. The series may or may not converge.

Definition

The Chebyshev ellipse ξ_ρ , $\rho > 1$ is defined by

$$\xi_\rho = \left\{ \omega \mid \omega = \frac{1}{2} (z+z^{-1}), \quad |z| = \rho \right\} \quad (2.13)$$

For values of the argument lying on a Chebyshev ellipse we have the following lemma.

LEMMA 1

Let $z \in \xi_\rho$, $\rho > 1$, then $\exists w \in \{ w \mid |w| = \rho \}$

such that

$$T_N(z) = \frac{1}{2}(W^N + W^{-N})$$

Proof.

From the definition of $\xi_\rho \exists W \in \{W \mid |W| = \rho\}$

such that

$$z = \frac{1}{2}(W + W^{-1})$$

substituting for z in (2.12) gives

$$\begin{aligned} T_N(z) &= \frac{1}{2} \left[\left(\frac{W+W^{-1}}{2} + \sqrt{\frac{W^2 - 2 + W^{-2}}{4}} \right)^N + \left(\frac{W+W^{-1}}{2} + \sqrt{\frac{W^2 - 2 + W^{-2}}{4}} \right)^{-N} \right] \\ &= \frac{1}{2} \left[\left(\frac{W+W^{-1}}{2} + \frac{(W-W^{-1})}{2} \right)^N + \left(\frac{W+W^{-1}}{2} + \frac{(W-W^{-1})}{2} \right)^{-N} \right] \\ &= \frac{1}{2} (W^N + W^{-N}) \end{aligned} \quad (2.14)$$

We shall now prove a theorem on the convergence of the Chebyshev series of a function.

Theorem 1

Let $f(z)$ be analytic in the interior of ξ_ρ , $\rho > 1$, ~~and not continuous on the boundary~~ but not in the interior of any $\xi_{\rho'}$ with $\rho' > \rho$, and continuous on ξ_ρ

Then

$$f(z) = \sum_{j=0}^{\infty} a_j T_j(z),$$

where a_j is defined by (2.12).

The series converges absolutely and uniformly on any closed set in the interior of ξ_ρ and diverges exterior to ξ_ρ .

Moreover

$$\limsup_{j \rightarrow \infty} |a_j|^{1/j} = \frac{1}{\rho}$$

Proof

For u in the interior of ξ_ρ

$$f(u) = \frac{1}{2\pi i} \int_{\xi_\rho} \frac{f(z) dz}{z-u} \quad (2.15)$$

But since

$$(z-u)T_n(u) = z T_n(u) - \frac{1}{2}(T_{|n-1|}(u) + T_{n+1}(u)),$$

we have

$$\frac{1}{z-u} = \sum_{j=0}^{\infty} \frac{2 T_j(u)}{(z + \sqrt{z^2-1})^j \sqrt{z^2-1}} \quad (2.16)$$

and the series converges absolutely at any u in the interior of \mathcal{E}_ρ .

So substituting (2.16) in (2.15) gives

$$f(u) = \frac{1}{2\pi i} \int_{\mathcal{E}_\rho} \sum_{j=0}^{\infty} \frac{2 T_j(u) f(z)}{(z + \sqrt{z^2-1})^j \sqrt{z^2-1}} dz \quad (2.17)$$

Interchanging integration and summation yields

$$f(u) = \sum_{j=0}^{\infty} a_j T_j(u) \quad (2.18)$$

where
$$a_j = \frac{1}{\pi i} \int_{\mathcal{E}_\rho} \frac{f(z)}{(z + \sqrt{z^2-1})^j \sqrt{z^2-1}} dz \quad (2.19)$$

Now replace u in (2.16) with the real variable $x \in [-1, 1]$.

That gives

$$\frac{1}{z-x} = \sum_{j=0}^{\infty} \frac{2 T_j(x)}{(z + \sqrt{z^2-1})^j \sqrt{z^2-1}}$$

and from the orthogonality of the Chebyshev polynomials we get

$$\frac{2}{\pi} \int_{-1}^1 \frac{T_j(x)}{(z-x)\sqrt{1-x^2}} dx = \frac{2}{\sqrt{z^2-1} (z + \sqrt{z^2-1})^j}$$

So

$$\begin{aligned} a_j &= \frac{1}{2\pi i} \int_{\mathcal{E}_\rho} f(z) \frac{2}{\pi} \int_{-1}^1 \frac{T_j(x)}{(1-x^2)^{1/2}} \frac{1}{(z-x)} dx dz \\ &= \frac{1}{\pi^2 i} \int_{-1}^1 \frac{T_j(x)}{(1-x^2)^{1/2}} \int_{\mathcal{E}_\rho} \frac{f(z)}{(z-x)} dz dx \\ &= \frac{2}{\pi} \int_{-1}^1 \frac{T_j(x) f(x)}{(1-x^2)^{1/2}} dx \end{aligned}$$

To prove the absolute and uniform convergence on closed sets in the

interior of \mathcal{E}_ρ we chose a ρ' such that $1 < \rho' < \rho$. Then $f(z)$ is analytic in

and on $\mathcal{E}_{\rho'}$ and we may write

$$a_j = \frac{1}{\pi^2 i} \int_{\mathcal{E}_{\rho'}} f(w) \int_{-1}^1 \frac{T_j(x)}{(w-x)\sqrt{1-x^2}} dx dw$$

If we let

$$Q_j(w) = \int_{-1}^1 \frac{T_j(x)}{(w-x)(1-x^2)^{1/2}} dx \quad (2.20)$$

Then

$$|a_j| \leq \frac{1}{\pi \alpha} \alpha(\xi_{\rho'}) \max_{w \in \xi_{\rho'}} |f(w)| \max_{w \in \xi_{\rho'}} |Q_j(w)|, \quad (2.21)$$

where $\alpha(\xi_{\rho'})$ is the length of $\xi_{\rho'}$.

To bound $|Q_j(w)|$ we make the change of variable $w = \frac{1}{2}(z+z^{-1})$, where $z = \rho' e^{i\psi}$. This yields

$$\begin{aligned} Q_j(w) &= 2 \int_{-1}^1 \frac{T_j(x)}{(1-x^2)^{1/2}(z+z^{-1}-2x)} dx \\ &= \frac{2}{z} \int_{-1}^1 \frac{T_j(x)}{(1-x^2)^{1/2} \left\{1 - 2\frac{x}{z} + \frac{1}{z^2}\right\}} dx \end{aligned} \quad (2.22)$$

Putting $x = \cos \theta$ in (2.22) gives

$$Q_j(w) = \frac{2}{z} \int_0^\pi \frac{\cos j\theta}{\left(1 - \frac{2\cos\theta}{z} + \frac{1}{z^2}\right)} d\theta.$$

Now

$$\frac{\sin \theta}{z} \left\{ \frac{1}{1 - \frac{2\cos\theta}{z} + \frac{1}{z^2}} \right\} = \sum_{M=1}^{\infty} \frac{\sin(M\theta)}{z^M}.$$

The last series converges uniformly and absolutely for $0 \leq \theta \leq \pi$ and for all $|z| \geq \rho'' > 1$.

$$\text{Hence } Q_j(w) = \sum_{M=1}^{\infty} \frac{2}{z^M} \int_0^\pi \frac{\cos(j\theta)\sin(M\theta)}{\sin\theta} d\theta,$$

but from Gradshteyn and Ryzhik (1965) page 366 we have

$$\int_0^\pi \frac{\cos(n\theta)\sin(m\theta)}{\sin\theta} d\theta = \begin{cases} 0 & m \leq n \\ \pi & m > n \text{ if } m+n \text{ is odd} \\ 0 & m > n \text{ if } m+n \text{ is even} \end{cases}$$

So

$$\begin{aligned} Q_j(w) &= \frac{2\pi}{z^{j+1}} \sum_{k=0}^{\infty} z^{-2k} \\ &= \frac{2\pi}{z^{j+1}(1-z^{-2})} \end{aligned}$$

$$\text{Therefore } \max_{w \in \xi_{\rho'}} |Q_j(w)| = \frac{2\pi (\rho')^{-j}}{(\rho' - \rho'^{-1})} \quad (2.23)$$

Combining (2.21) with (2.23) we get

$$|a_j| \leq K (\rho')^{-j} \quad (2.24)$$

where K is independent of j .

Therefore

$$\lim_{j \rightarrow \infty} \sup |a_j|^{1/j} \leq \frac{1}{\rho'}$$

Since $\rho' < \rho$ is arbitrary, it follows that

$$\lim_{j \rightarrow \infty} \sup |a_j|^{1/j} \leq \frac{1}{\rho}$$

However from (2.14) we have for $w \in \mathbb{E}_{\rho'}$

$$\lim_{j \rightarrow \infty} \sup |T_j(w)|^{1/j} = \rho' < \rho$$

So the series converges uniformly and absolutely inside every $\mathbb{E}_{\rho'}$, $\rho' < \rho$.

Suppose now that

$$\lim_{j \rightarrow \infty} \sup |a_j|^{1/j} < \frac{1}{\rho}$$

Then the series would converge uniformly and absolutely in the interior of a larger ellipse $\mathbb{E}_{\rho''}$, $\rho'' > \rho$, and provide an analytic continuation of $f(z)$ there. This is a contradiction, because by hypothesis \mathbb{E}_{ρ} is the largest ellipse in which $f(z)$ is analytic. So

$$\lim_{j \rightarrow \infty} \sup |a_j|^{1/j} = \rho^{-1} \quad \text{and the series diverges}$$

in the exterior of \mathbb{E}_{ρ} .

§3 NUMERICAL CALCULATION OF CHEBYSHEV COEFFICIENTS

The calculation of the coefficients in (2.12) can be approached in a number of ways. We will give here three methods, the first uses the discrete orthogonality property of Chebyshev polynomials and the other two use differential equations satisfied by the function being expanded.

§3.1 COLLOCATION METHOD

If we define the coefficients $a_j^{(N)}$, $j = 0, 1, \dots, N$ such that the

polynomial $\sum_{j=0}^N a_j^{(N)} T_j(z)$ collocates $f(z)$ at the $(N+1)$ points given by

$$y_k = \cos k\pi/N, \quad k = 0, 1, \dots, N,$$

we can write

$$f(y_k) - \sum_{j=0}^N a_j^{(N)} T_j(y_k) = 0 \quad \text{for } k=0, 1, \dots, N.$$

So

$$\begin{aligned} \sum_{k=0}^N f(y_k) T_r(y_k) &= \sum_{k=0}^N T_r(y_k) \sum_{j=0}^N a_j^{(N)} T_j(y_k) \\ &= \sum_{j=0}^N a_j^{(N)} \sum_{k=0}^N T_r(y_k) T_j(y_k) \\ &= \left(\frac{2}{N}\right)^{-1} a_r^{(N)}, \quad \text{for } r=0, 1, \dots, N \end{aligned}$$

because of the orthogonality property (2.10).

We shall now show the relationship between the coefficients $a_r^{(N)}$ and the Chebyshev coefficients A_r . If the function $f(x)$ has a convergent Chebyshev series on $[-1, 1]$ then we can write

$$\begin{aligned} a_r^{(N)} &= \frac{2}{N} \sum_{k=0}^N \sum_{j=0}^{\infty} a_j T_j(y_k) T_r(y_k) \\ &= \frac{2}{N} \sum_{j=0}^{\infty} a_j \sum_{k=0}^N T_j(y_k) T_r(y_k) \\ &= a_r + \sum_{p=1}^{\infty} (a_{2pN-r} + a_{2pN+r}) \quad r=0, 1, \dots, N. \end{aligned} \quad (2.25)$$

This shows that if the coefficients converge rapidly to zero that $a_r^{(N)}$ is a good approximation to A_r for $r < N$ and $a_N^{(N)}$ is a good approximation to $2A_N$.

The error incurred when using these 'approximate' coefficients

is

$$\begin{aligned} f(z) - \sum_{j=0}^N a_j^{(N)} T_j(z) &= \sum_{j=0}^N a_j T_j(z) - \sum_{j=0}^N a_j^{(N)} T_j(z) + \sum_{j=N+1}^{\infty} a_j T_j(z) \\ &= \sum_{j=0}^{N-1} \sum_{p=1}^{\infty} [a_{2pN-j} + a_{2pN+j}] T_j(z) \\ &\quad + \sum_{p=1}^{\infty} a_{(2p+1)N} T_N(z) + \sum_{k=N+1}^{\infty} a_k T_k(z) \\ &= \sum_{k=N+1}^{\infty} a_k [T_k(z) - T_{|N-(k+N) \bmod 2N|}(z)] \end{aligned} \quad (2.26)$$

If the approximation is to be used in a region such that

$$|T_j(z)| \leq \sigma^j, \quad \sigma \gg 1$$

then given that $f(z)$ has a convergent Chebyshev series we can write the error, caused by truncating the Chebyshev series after $(N+1)$ terms, is bounded by

$$\sum_{k=N+1}^{\infty} |a_k| \sigma^k,$$

and the error

in approximating $f(z)$ by the collocation polynomial of same degree is

$$\text{bounded by } \sum_{k=N+1}^{\infty} |a_k| [\sigma^k + \sigma^N]$$

So the collocation polynomial has a larger error bound, but it is less than twice the bound for the truncated Chebyshev series.

§3.2 LANCZOS' τ -METHOD

If a function satisfies a differential equation of order q , which has only polynomial coefficients, then substituting a polynomial of degree n for the function will leave a residual polynomial $R_N(z)$ of degree N say. If the boundary conditions are met, only $(n+1-q)$ coefficients in the approximating polynomial are independent. So in constructing $R_N(z)$ we may impose $(n+1-q)$ conditions upon its coefficients to give a unique approximation of degree n .

The conditions chosen by Lanczos (1938) are that the residual polynomial be a linear combination of m Chebyshev polynomials, where $m = N+q-n$. For example if $m = 1$ then the polynomial would be $\tau T_N(z)$, while if $m = 2$ it would be $\tau_1 T_N(z) + \tau_2 T_{N-1}(z)$. The values of the coefficients τ are automatically determined by solving the resultant differential equation. When τ is small we may expect the polynomial to be a good approximation to the function.

§3.3 CLENSHAW'S METHOD

This method was introduced by Clenshaw (1957) for functions satisfying

a linear differential equation with polynomial coefficients. By substituting separate Chebyshev series for the function and each of its derivatives into the differential equation and then equating coefficients of $T_j(z)$, it is possible to produce recurrence relations for the coefficients.

To equate coefficients we must be able to write the coefficients of $Z^p g(z)$, $p=0,1,\dots$, in terms of the coefficients of $g(z)$. To solve the subsequent relations we need to be able to relate the coefficients of $g'(z)$ to those for $g(z)$. If we define $C_R(g(z))$ to be the R^{th} Chebyshev coefficient of $g(z)$ for $R > 0$, and twice this for $R=0$, then the required relations are:

$$C_R(z^p g(z)) = \frac{1}{2^p} \sum_{s=0}^p \binom{p}{s} C_{|R-p+2s|}(g(z)) \quad (2.27)$$

and

$$2RC_R(g'(z)) = \{ C_{|R-1|}(g(z)) - C_{R+1}(g(z)) \} \quad (2.28)$$

The differential equation yields an infinite system of equations in the unknown coefficients and a number of initial or boundary conditions, which can be solved directly using (2.28). However, it is possible to use (2.28) repeatedly to eliminate the coefficients of the derivatives and then solve a system of equations in the coefficients of the function only. For example suppose $f(z)$ satisfies the O.D.E.

$$zf''(z) - f'(z) + z^2 f(z) = 0, \quad (2.29)$$

with the conditions

$$f(0) = 1, f(1) = 0,$$

then if we let

$$\begin{aligned} C_R(f''(z)) &= a''_R \\ C_R(f'(z)) &= a'_R \\ C_R(f(z)) &= a_R \end{aligned}$$

and equate coefficients of $T_R(z)$ in (2.29) we get

$$\frac{1}{2} [a''_{R-1} + a''_{R+1}] - a'_R + \frac{1}{4} [a_{R-2} + 2a_R + a_{R+2}] = 0 \quad (2.30)$$

$R = 0, 1, \dots$

Subtracting the $(R+2)^{\text{th}}$ equation from R^{th} and using (2.28) to eliminate the coefficients of $f''(z)$ gives

$$R a'_{R+(R+2)} a'_{R+2-(R+1)} a_{R+1} + \frac{1}{4} [a_{R-2} + a_R - a_{R+2} - a_{R+4}] = 0 \quad (2.31)$$

$R = 0, 1, \dots$

which can be rearranged to give

$$a'_R + a'_{R+2} - 4a_{R+1} + \frac{1}{4(R+1)} [a_{R-2} + a_R - a_{R+2} - a_{R+4}] = 0 \quad (2.32)$$

$R = 0, 1, \dots$

The coefficients from the expansion of first derivative can similarly be eliminated to yield the recurrence relations

$$2(R-1)a_{R+1} + 2(R+5)a_{R+3} + \frac{1}{4} \left\{ \frac{a_R - a_{R+4}}{(R+1)(R+3)} + \frac{a_{R-2}}{(R+1)} + \frac{a_{R+6}}{(R+3)} - \frac{2(R+2)a_{R+2}}{(R+1)(R+3)} \right\} = 0 \quad (2.33)$$

$R = 0, 1, \dots$

The boundary conditions yield the equations

$$\sum_{j=0}^{\infty} (-1)^j a_{2j} = 1 = f(0)$$

$$\sum_{j=0}^{\infty} a_j = 0 = f(1) \quad (2.34)$$

To solve systems of equations similar to (2.33) with conditions similar to (2.34) we find solutions from (2.33) using backward recurrence from the equation having a_{N-1} as its lowest order coefficient, with a_N and higher order coefficients given arbitrary values, for various values of N . A linear combination of a sufficient number of these solutions can be taken to satisfy the equations (2.34) and any equations of (2.33) not used in the backward recurrence. For instance in the above example we could use backward recurrence from $R = N+1$ to $R = 2$ to find an initial independent solution of the recurrence formulae and this would

Leave equations $R = 0$ and 1 to be satisfied as well as the two equations in (2.34). Therefore we would need a linear combination of four independent solutions. However in many cases we shall have only one or two outstanding conditions to satisfy.

§3.4 CHOICE OF METHOD

The methods require quite different information about a function to be known. The first requires the evaluation of the function at $(N+1)$ points. The accuracy of the coefficients depends on the accuracy of these function evaluations and therefore, this method is useful only if ^{the} evaluation method used is inferior to the Chebyshev series either in the speed of calculation or in the region over which it can be used. The other methods require the knowledge of a differential equation satisfied by the function and a small number of values of the function or its derivatives. If these values are known exactly, then the accuracy of the coefficients calculated using Clenshaw's method is restricted only by round-off errors if N is very large. The coefficients given in later chapters were calculated using Clenshaw's method. Clenshaw's and Lanczos' methods are further compared by Fox (1962).

stability?

CHAPTER 3

COMPARISON OF BEST POLYNOMIAL APPROXIMATION WITH TRUNCATED CHEBYSHEV SERIES OF SAME DEGREE

§1 THE RELATIVE FACTOR

Chebyshev series approximation has been used extensively for continuous functions of a real variable. Schonfelder (1978, 1980), Schonfelder et al (1980), Razaz & Schonfelder (1980) and Shepherd & Laframboise (1981) are just a few examples. The error in this type of approximation, in comparison to the error in the best polynomial approximation of the same degree, was investigated by Powell (1967). He showed that on the interval $[-1,1]$ the maximum norm of the error caused by using a truncated Chebyshev series of degree N was within a relative factor ν_N , the N 'th Lebesgue constant, of the norm of the minimax error. This can be written as

$$\|e_N(z)\|_{\mathcal{I}} \leq (1 + \nu_N) \|E_N^{\mathcal{I}}(z)\|_{\mathcal{I}} \quad (3.1)$$

where

$E_N^{\mathcal{I}}(z)$ is the error at z of the best polynomial approximation on $[-1,1]$ of order N to the function $f(z)$, and $e_N(z)$ is the error at z of the truncated Chebyshev series of $f(z)$. The norm is defined by

$\|f(z)\|_{\mathcal{I}} = \max_{z \in [-1,1]} |f(z)|$ and the Lebesgue constant by

$$\nu_N = \frac{1}{(2N+1)} + \frac{2}{\pi} \sum_{t=1}^N \frac{1}{t} \tan\left(\frac{\pi t}{2N+1}\right)$$

This result was extended by Geddes (1978) to Chebyshev ellipses, with the space of functions restricted to those functions which are analytic within the ellipse and continuous onto the boundary. In this chapter a bound on the relative factor will be obtained when the interval

$[-1,1]$ is replaced by any closed, bounded domain D , which encloses the interval $[-1,1]$.

Γ_D is defined to be the boundary of D . The set of points in the interior of ξ_ρ will be denoted $I(\xi_\rho)$ and the closure of $I(\xi_\rho)$ as $\bar{I}(\xi_\rho)$. We shall now define ξ_R to be the smallest Chebyshev ellipse which contains D .

So

$$\bar{I}(\xi_\rho) \supset D \quad \text{if and only if} \quad \rho \geq R.$$

The space of functions analytic within D and continuous onto the boundary Γ_D will be denoted $\bar{A}(D)$. We shall assume all functions to be approximated belong to this space. The norm $\| \cdot \|_D$ is defined as follows:

$$\|F(z)\|_D = \max_{z \in \Gamma_D} |F(z)|. \quad (3.2)$$

For a given function $f(z)$ and a region D the error in the best approximation of order N is

$$E_N^D(z) = f(z) - \sum_{j=0}^N b_j T_j(z) \quad (3.3)$$

where the coefficients b_j ($j=0,1,\dots,N$) are the solution of the minimisation problem

$$\text{Min}_{a_0, a_1, \dots, a_N} \left\| f(z) - \sum_{j=0}^N a_j T_j(z) \right\|_D \quad (3.4)$$

This minimisation problem has a unique solution, see Davis (1963). The error in the truncated Chebyshev series of degree N will be denoted by

$$e_N(z) = f(z) - \sum_{j=0}^N c_j T_j(z). \quad (3.5)$$

From (2.12), the coefficients c_j , ($j=0,1,\dots,N$) above are given by

$$c_j = \frac{2}{\pi} \int_{-1}^1 \frac{f(t) T_j(t)}{(1-t^2)^{1/2}} dt \quad (3.6)$$

As $f(z)$ belongs to $\bar{A}(D)$, but not necessarily to $\bar{A}(\bar{I}(\xi_R))$ the series may not converge for all $z \in \bar{I}(\xi_R)$. Subtracting equation (3.5) from (3.3) yields

$$E_N^D(z) - e_N(z) = \sum_{j=0}^{N-1} (c_j - b_j) T_j(z) \quad (3.7)$$

which is a polynomial of degree N . From (3.5) and (3.6) it can be shown that

$$(c_j - b_j) = \frac{2}{\pi} \int_{-1}^1 \frac{E_N^D(t) T_j(t)}{(1-t^2)^{1/2}} dt$$

Substituting the above into (3.7) and taking the norm over D of both sides we see that

$$\begin{aligned} \left\| E_N^D(z) - e_N(z) \right\|_D &= \frac{2}{\pi} \max_{z \in \bar{D}} \left| \sum_{j=0}^{N-1} \int_{-1}^1 \frac{E_N^D(t) T_j(t)}{(1-t^2)^{1/2}} dt T_j(z) \right| \\ &\leq \frac{2}{\pi} \max_{z \in \bar{E}_R} \left| \sum_{j=0}^{N-1} \int_{-1}^1 \frac{E_N^D(t) T_j(t)}{(1-t^2)^{1/2}} dt T_j(z) \right| \end{aligned} \quad (3.8)$$

The inequality stems from the maximum modulus principle. We shall assume $R > 1$, since $R = 1$ has already been dealt with by Powell.

§2 UPPER BOUNDS ON THE RELATIVE FACTOR

We shall now find upper bounds for the right hand side of (3.8) by obtaining bounds on the integral independent of z . By elementary manipulation of summation, integration and modulus signs we have

$$\left| \sum_{j=0}^{N-1} \int_{-1}^1 \frac{E_N^D(t) T_j(t) T_j(z)}{(1-t^2)^{1/2}} dt \right| \leq \|E_N^D(z)\|_I \int_{-1}^1 \left| \sum_{j=0}^{N-1} \frac{T_j(t) T_j(z)}{(1-t^2)^{1/2}} \right| dt \quad (3.9)$$

It will now be shown that if $z \in \bar{E}_R$ and $N > 0$ then

$$\int_{-1}^1 \left| \sum_{j=0}^{N-1} \frac{T_j(t) T_j(z)}{(1-t^2)^{1/2}} \right| dt \leq \frac{R^{N-1} (R+1) (1+R^{-(2N+1)})}{(1-R^{-1})^2} \quad (3.10)$$

By definition

$$T_j(z) = \cos(j \cos^{-1}(z)),$$

and as $z \in \bar{E}_R$

$$z = \frac{1}{2} (R e^{i\theta} + R^{-1} e^{-i\theta}), \quad \theta \in (-\pi, \pi] \quad (3.11)$$

This means that

$$\cos^{-1}(z) = -\theta + i\sigma,$$

where $\sigma = \ln R > 0$.

The left hand side of (3.10) can now be written as

$$\int_0^\pi \left| \sum_{j=0}^{N-1} \cos j\psi \cos j(-\theta + i\sigma) \right| d\psi. \quad (3.12)$$

Equation (3.12) can be rearranged using property 1 of (2.11) and

$$\sum_{j=0}^{N-1} \cos j\phi = \frac{\frac{1}{2} \sin(N + \frac{1}{2})\phi}{\sin \frac{1}{2}\phi}, \quad (3.13)$$

to give

$$\frac{1}{4} \int_0^\pi \left| \frac{\sin(N + \frac{1}{2})(\psi - \theta + i\sigma)}{\sin \frac{1}{2}(\psi - \theta + i\sigma)} + \frac{\sin(N + \frac{1}{2})(\psi + \theta - i\sigma)}{\sin \frac{1}{2}(\psi + \theta - i\sigma)} \right| d\psi \quad (3.14)$$

$$= \frac{R^N}{2} \int_0^\pi \left(\frac{U_N(\theta, R, \psi)}{U_0(\theta, R, \psi)} \right)^2 d\psi \quad (3.15)$$

where

$$U_j(\theta, R, \psi) = \left\{ \begin{aligned} &\cos^2 j\psi (1 + 2R^{-2(j+1)} \cos 2(j+1)\theta + R^{-4(j+1)}) \\ &- 2 \cos j\psi \cos(j+1)\psi (R^{-1}(1 + R^{-2(j+2)}) \cos \theta + R^{-2(j+1)} \cos 2(j+1)\theta) \\ &+ \cos^2(j+1)\psi (R^{-2}(1 + 2R^{-2j} \cos j\theta + R^{-4j})) \end{aligned} \right\}_{j=0, N}.$$

Equation (3.15) was obtained by writing each of the sines in (3.14) in terms of exponentials and taking the modulus. $U_j(\theta, R, \psi)$ are even, 2π -periodic functions of θ therefore only values of θ between 0 and π are of interest. The inequality (3.10) comes from noting that

$$U_N(\theta, R, \psi) \leq \left\{ (1 + R^{-2(N+1)}) \cos N\psi + R^{-1}(1 + R^{-2N}) \cos(N+1)\psi \right\}^2$$

and

$$U_0(\theta, R, \psi) \geq (1 - R^{-1})^4.$$

Therefore

$$\frac{R^N}{2} \int_0^\pi \left(\frac{U_N(\theta, R, \psi)}{U_0(\theta, R, \psi)} \right)^2 d\psi \leq \frac{R^N}{2(1-R^{-1})^2} \int_0^\pi \left| \frac{(1+R^{-2(N+1)}) \cos N\psi + R^{-1}(1+R^{-2N}) \cos(N+1)\psi}{1-R^{-1}} \right|^2 d\psi \quad (3.16)$$

$$\leq \frac{R^N}{2(1-R^{-1})^2} \left\{ (1+R^{-2(N+1)}) \int_0^\pi |\cos N\psi| d\psi + R^{-1}(1+R^{-2N}) \int_0^\pi |\cos(N+1)\psi| d\psi \right\}$$

$$= \frac{R(1+R^{-2})(1+R^{-(2N+1)})}{(1-R^{-1})^2}, \quad N > 0. \quad (3.17)$$

We have now obtained the bound given in (3.10), but there is an upper bound on (3.14), which is better than (3.17) for $R < 6$, in terms of an elliptic integral. We first note that

$$\frac{1}{4} \int_0^\pi \left| \frac{\sin(N+\frac{1}{2})(\psi-\theta+i\sigma)}{\sin\frac{\alpha}{2}(\psi-\theta+i\sigma)} + \frac{\sin(N+\frac{1}{2})(\psi+\theta-i\sigma)}{\sin\frac{\alpha}{2}(\psi+\theta-i\sigma)} \right| d\psi$$

$$\leq \frac{1}{2} \int_0^\pi \left| \frac{\sin(N+\frac{1}{2})(\psi+i\sigma)}{\sin\frac{\alpha}{2}(\psi+i\sigma)} \right| d\psi \quad (3.18)$$

$$= \frac{R^N}{2} \left(\frac{1+R^{-2(N+1)}}{1+R^{-2}} \right)^{\frac{1}{2}} \int_0^\pi \left(\frac{1-L \cos(2N+1)\psi}{1-M \cos(\psi)} \right)^{\frac{1}{2}} d\psi,$$

where

$$L = \left\{ \cosh(2N+1)\sigma \right\}^{-1} \quad (3.19)$$

and $M = \left\{ \cosh \sigma \right\}^{-1}$

To obtain the upper bounds in terms of an elliptic integral we shall now prove the following Lemma.

LEMMA 2

For all $x, y \in [0, 1)$ and k a positive integer,

$$I_k(x, y) = \int_0^\pi \left(\frac{1-x \cos(2k+1)\psi}{1-y \cos \psi} \right)^{\frac{1}{2}} d\psi \leq \int_0^\pi \frac{d\psi}{(1-y \cos \psi)^{\frac{1}{2}}} \quad (3.20)$$

Proof

We shall show that the derivative of I_k with respect to x is non-positive

$$\frac{\partial}{\partial x} I_k(x, y) = -\frac{1}{2} \int_0^\pi \frac{\cos(2k+1)\psi d\psi}{(1-y \cos \psi)^{\frac{1}{2}} (1-x \cos(2k+1)\psi)^{\frac{1}{2}}} \quad (3.21)$$

Since $0 \leq x < 1$, we have that

$$\cos(2k+1)\psi \leq \frac{\cos(2k+1)\psi}{(1-x \cos(2k+1)\psi)^{\frac{1}{2}}}, \quad 0 \leq \psi \leq \pi.$$

So

$$\begin{aligned}
 \frac{\partial}{\partial x} I_k(x,y) &\leq - \frac{1}{2} \int_0^\pi \frac{\cos(2k+1)\psi}{(1-y\cos\psi)^{3/2}} d\psi \\
 &= \frac{1}{(1+y)^{3/2}} \int_0^\pi \frac{\cos(2k+1)2\psi'}{(1-2y\sin^2\psi'/(1+y))^{3/2}} d\psi' \\
 &= \frac{(-1)^{2k+1}}{(1+y)^{3/2}} \frac{\pi}{2} \sum_{j=2k+1}^\infty \frac{(2j-1)!! (y/a)^j (1+y)^{-j}}{(j+2k+1)! (j-2k-1)!} \\
 &\leq 0
 \end{aligned}$$

and the lemma is proved.

Since L and M in (3.19) are both positive and less than one we have

$$\begin{aligned}
 \int_0^\pi \left| \sum_{j=0}^{N-1} \cos j\psi \cos j(-\theta+is) \right| d\psi &\leq \frac{R^N (1+R^{2(2N+1)})^{1/2}}{2 (1+R^{-2})^{1/2}} \int_0^\pi \frac{d\psi}{(1-M\cos\psi)} \\
 &= \frac{R^N (1+R^{-2(2N+1)})^{1/2}}{(1+R^{-2})} \int_0^{\pi/2} \frac{d\psi}{(1-k^2\sin^2\psi)^{1/2}} \quad (3.23)
 \end{aligned}$$

where $k^2 = (\cosh(\sigma/a))^{-2} = 4R(1+R)^{-2}$.

From (3.10) and (3.23) we have,

$$\left\| E_N^D - e_N(z) \right\|_D \leq \frac{2}{\pi} \left\| E_N^D(z) \right\|_{\mathcal{I}} R^N A,$$

where $A = \min \left(\frac{(1+R^{-1})(1+R^{-(2N+1)})}{(1-R^{-1})^2}, \frac{(1+R^{-2(2N+1)})^{1/2}}{(1+R^{-1})} \int_0^{\pi/2} \frac{d\psi}{(1-k^2\sin^2\psi)^{1/2}} \right)$

Therefore, we have

$$\begin{aligned}
 \left\| e_N(z) \right\|_D &\leq \left\| E_N^D(z) \right\|_D + \frac{2}{\pi} \left\| E_N^D(z) \right\|_{\mathcal{I}} R^N A \\
 &\leq \left(1 + \frac{2}{\pi} R^N A \frac{\left\| E_N^D(z) \right\|_{\mathcal{I}}}{\left\| E_N^D(z) \right\|_D} \right) \left\| E_N^D(z) \right\|_D
 \end{aligned}$$

So we have an upper bound on the Relative factor of

$$\frac{2R^N A}{\pi} \frac{\left\| E_N^D(z) \right\|_{\mathcal{I}}}{\left\| E_N^D(z) \right\|_D} \quad (3.24)$$

In (3.24) $\left\| E_N^D(z) \right\|_{\mathcal{I}} \leq \left\| E_N^D(z) \right\|_D$ and in many cases $\left\| E_N^D(z) \right\|_{\mathcal{I}} \ll \left\| E_N^D(z) \right\|_D$. So we can say

$$\left\| e_N(z) \right\|_D \leq \left(1 + \frac{2R^N A}{\pi} \right) \left\| E_N^D(z) \right\|_D$$

§3 NUMERICAL COMPARISON OF BOUNDS

The integral given by (3.12) was calculated numerically using a

Gauss-Konrad integration routine from the NAG library, for various values of Θ , σ and N . Some of these results are produced in Table 3.1. From these results it can be seen that for $N\sigma$ greater than 1 the maximum value, as Θ varies, appears to be reached at or near $\Theta = \pi/2$. This is in contrast to the case $\sigma = 0$, where the maximum is obtained at $\Theta = 0$, for all N .

The maximum values, obtained above for each value of N and σ , are compared with the bounds (3.10) and (3.23) in Table 3.2. It can be seen that (3.23) is a closer bound than (3.10) in all the cases given here, but (3.10) improves and (3.23) worsens as σ increases. The elliptic integral in (3.23) was calculated using a NAG special function routine.

4 LOWER BOUND ON THE RELATIVE FACTOR FOR $J_0(z)$ OVER D

Given

$$D = \{ z \mid |z| \leq 8, \quad 0 \leq \text{Arg } z \leq \pi/2 \}$$

&

$$\gamma = 8 e^{i\pi/4}$$

we define

$$A_N(z) = J_0(z) - \sum_{k=0}^{N'} a_{2k} T_{2k}(z/\gamma)$$

$$C_N^\infty(z) = J_0(z) - \sum_{k=0}^{N'} c_{2k}^\infty (z/\gamma)^{2k}$$

and

$$C_N^*(z) = J_0(z) - \sum_{k=0}^{N'} c_{2k}^* (z/\gamma)^{2k},$$

the coefficients are given by

$$a_{2k} = \frac{2}{\pi} \int_{-1}^1 \frac{J_0(\gamma t) T_{2k}(t) dt}{(1-t^2)^{1/2}} \quad \Leftrightarrow \quad J_0(\gamma t) \sim \sum_{k=0}^{\infty} a_{2k} T_{2k}(t) \quad \text{where } t \in [-1, 1]$$

and

$$c_{2k}^\infty \quad \text{and} \quad c_{2k}^*, \quad k = 0, 1, \dots, N$$

$$\| C_N^\infty \|_\infty \quad \text{and} \quad \| C_N^* \|_*$$

$\Rightarrow J_0(z) \sim \sum_{k=0}^{\infty} a_{2k} T_{2k}(z/\gamma)$
minimise
respectively over D .

Then from Barrodale (1977) it can be shown that

$$\sqrt{2} \frac{\|A_N\|_{\infty}}{\|C_N^*\|_{\infty}} \geq \frac{\|A_N\|_{\infty}}{\|C_N^{\infty}\|_{\infty}} \geq \frac{1}{\sqrt{2}} \frac{\|A_N\|_{\infty}}{\|C_N^{\#}\|_{\infty}}$$

So by calculating the best polynomial approximation in the star norm and the maximum error, in the star norm, of the truncated Chebyshev series, we can obtain upper and lower bounds on the relative factor. Numerical results for $\mathcal{J}_0(z)$ on D for $N = 5(15)15$ are given below.

N	Lower Bound	Upper Bound
5	5.41	10.82
10	36.55	73.09
15	231.08	462.16

Although the results show that the truncated Chebyshev series approximation to $\mathcal{J}_0(z)$ is not a near-best approximation on D , the lost accuracy can be retrieved by only slightly increasing the degree of the approximation.

TABLE 3.1

	$\ominus \sigma$	0.05	0.1	0.2	0.3	0.4
N=5	0	3.14	3.33	4.17	5.85	8.81
	$\pi/6$	2.71	3.01	4.13	6.10	9.36
	$\pi/3$	2.71	3.05	4.29	6.42	9.89
	$\pi/2$	2.71	3.06	4.33	6.51	10.30
N=10	0	3.78	4.73	9.90	2.44(+1)	6.32(+1)
	$\pi/6$	3.40	4.81	1.11(+1)	2.72(+1)	6.91(+1)
	$\pi/3$	3.51	5.05	1.17(+1)	2.89(+1)	7.31(+1)
	$\pi/2$	3.68	5.19	1.19(+1)	2.93(+1)	7.43(+1)
N=20	0	5.33	1.12(+1)	7.07(+1)	4.89(+2)	3.45(+3)
	$\pi/6$	5.62	1.32(+1)	8.22(+1)	5.49(+2)	3.77(+3)
	$\pi/3$	5.95	1.39(+1)	8.67(+1)	5.66(+2)	3.99(+3)
	$\pi/2$	6.02	1.40(+1)	8.80(+1)	5.74(+2)	4.05(+3)

TABLE 3.2

	σ	Num.Int.	(3.10)	(3.23)
N=5	0.05	3.14	1.66(+3)	3.86
	0.10	3.33	4.62(+2)	4.00
	0.20	4.33	1.67(+2)	5.56
	0.30	6.51	1.2(+2)	8.51
	0.40	10.30	3.8(+2)	13.40
N=10	0.05	3.78	1.83(+3)	4.54
	0.10	5.19	6.42(+2)	6.23
	0.20	1.19(+1)	4.15(+2)	1.51(+1)
	0.30	2.93(+1)	5.22(+2)	3.82(+1)
	0.40	7.43(+1)	8.39(+2)	9.90(+1)
N=20	0.05	6.02	2.52(+3)	7.13
	0.10	1.40(+1)	1.59(+3)	1.70(+1)
	0.20	8.80(+1)	3.03(+3)	9.70(+1)
	0.30	5.74(+2)	1.05(+3)	7.65(+2)
	0.40	4.05(+3)	4.59(+4)	5.40(+3)

CHAPTER 4

APPROXIMATION OF $J_N(z)$ AND $Y_N(z)$ OVER THE ENTIRE COMPLEX PLANE

§1 INTRODUCTION TO BESSEL FUNCTION APPROXIMATION

For the Bessel function $J_N(z)$ it is possible in principle to use the Taylor expansion within a disc $|z| \leq R$ and an asymptotic expansion for $|z| > R$. This approach suffers from cancellation errors and the need for an increasingly large number of terms in the Taylor series as $|z|$ increases, as well as from the limitation on the accuracy achievable from an asymptotic expansion for a given value of R .

Polynomial approximations for Bessel functions of real argument were obtained by Allen (1954, 1956). Ardill and Moriarty (1977) produced approximations for J_0 and J_1 , by extrapolating Allen's approximations to complex arguments. Because the accuracy of such an approximation decreases rapidly as $\arg z$ is increased, Ardill and Moriarty (1977) divided the first quadrant into three regions in which they used extrapolations of polynomial approximations for J_0 and J_1 , valid on the real axis, for the modified Bessel functions I_0 and I_1 , or for the Kelvin functions ber and bei (see Abramowitz and Stegun (1965), p.379). In view of the nature of these approximations there is no basis for a theoretical appraisal of the error involved. The error can be assessed, as for Allen's approximations for real arguments, only by tabulation. Furthermore, the numerical results given by Ardill and Moriarty (1977) show that the truncation error is substantially greater in some regions than in others; in the authors' own words "...considerable improvement ... could be made by devising a new polynomial approximation for the region given approximately by $60^\circ \leq \arg z \leq 80^\circ$ ". It follows that this approach is not a useful one if the objective is a routine which will calculate $J_N(z)$ to a specified

accuracy. Approximations, based on Chebyshev series, for Bessel functions of real argument have been obtained by Clenshaw (1962), Luke (1975) and Coleman (1980).

Our approach is to divide the complex plane into a number of sectors and, in the absence of an expansion of the form (1.1) we use in each sector an approximation

$$f(z) \approx \sum_{k=0}^n a_k T_k(g(z/\delta)), \quad (4.1)$$

the complex constant δ being chosen so that $g(z/\delta)$ is real on the central ray of the sector, where $g(\omega)$ is a function chosen to improve convergence for a particular function $f(z)$. Our expansions for $J_N(z)$ and the generation of their coefficients are described in Section 2. The study in Section 3, of the truncation error requires an upper-bound for $|T_n(z)|$ on a sector; such a bound was established in Chapter 2. Section 4 describes the coefficient tables and provides a guide to the accuracy achievable. In section 5 expansions for $Y_0(z)$ and $Y_1(z)$ are produced.

§2. EXPANSIONS FOR $J_N(z)$

Throughout this chapter we shall concentrate on values of z in the first quadrant, $0 \leq \arg z \leq \frac{\pi}{2}$, only, since symmetry relations can be used to deduce the values of $J_N(z)$ elsewhere. It is convenient to consider separately two regions of the complex plane, an inner region $|z| \leq R$ and an outer region $|z| > R$, each of which will be mapped into the unit disc. It will be assumed that N is a positive integer, but the extension to non-integral orders is straightforward.

§2.1 The inner region

Clenshaw (1966) tabulated coefficients for expansions of the form

$$J_n(X) = \frac{1}{n!} \left(\frac{X}{2} \right)^n \sum_{r=0}^{\infty} a_{nr} T_{nr}(X/8) \quad (4.2)$$

on the interval $[-8, 8]$, for $n = 0$ and 1 . Coefficients of the corresponding expansions for $n = 2, 3, \dots, 10$ were calculated by Coleman (1980) and are incorporated in a routine which calculates $J_n(X)$ to a specified accuracy.

If x is replaced by the complex variable z the expansion (4.2) still converges. Since $z^{-n} J_n(z)$ is a regular function of z it can be shown (Rivlin (1974) p.143) that, for any $\rho > 1$,

$$|a_{nr}| \leq \frac{2M(\rho)}{\rho^{nr}}$$

where $M(\rho)$ is an upper bound on $|2^{-n} z^{-n} J_n(z)|$ for $|z| = 8 \cdot \rho$

Furthermore

$$|T_{nr}(\zeta)| \leq \frac{1}{2} [(1 + \sqrt{2})^{nr} + (1 - \sqrt{2})^{nr}]$$

for $|\zeta| \leq 1$, the bound being attained when $\zeta = \pm i$. It follows that

$$|a_{nr} T_{nr}(z/8)| \leq \frac{2M(\rho) (1 + \sqrt{2})^{nr}}{\rho^{nr}}$$

and since we may take $\rho > (1 + \sqrt{2})$ the convergence of the series is established. There is nevertheless a substantial deterioration in the rate of convergence as $\arg z$ increases. For example, if the series is truncated to give a polynomial of degree 20, the first neglected term increases by a factor of 1.3×10^8 as $\arg z$ increases from 0° to 90° , while $|z| = 8$.

It was shown in Chapter 2 that in the sector $|\zeta| \leq 1$, $|\arg \zeta| \leq \Theta$

$$|T_n(\zeta)| \leq |T_n(e^{i\Theta})|.$$

The right-hand side of this inequality is a monotonically increasing function of Θ for $0^\circ < \Theta < 90^\circ$ so the smaller the angular range about the real axis in which we use a given number of terms of the expansion (4.2) the less significant the loss of accuracy.

The advantages of a Chebyshev expansion on the real interval $[-1,1]$ may be retained for any particular ray in the complex plane by writing

$$\begin{aligned} J_n(z) &= \frac{1}{n!} \left(\frac{z}{2}\right)^n \sum_{r=0}^{\infty} a_{2r} T_{2r}(z/\gamma) \\ &= \frac{1}{n!} \left(\frac{z}{2}\right)^n V(z/\gamma) \end{aligned} \quad (4.3)$$

where γ is a complex number having the same argument as Z , and such that $0 \leq z/\gamma \leq 1$. The function V in eqn. (4.3) satisfies the differential equation

$$tV'' + (2n+1)V' + \gamma^2 tV = 0 \quad (4.4)$$

where $V(0) = 1$, $V'(0) = 0$ and the prime denotes differentiation with respect to $t = z/\gamma$. We now seek a solution of this equation of the form

$$V(t) = \sum_{r=0}^{\infty} a_{2r} T_{2r}(t) = \sum_{r=0}^{\infty} a_{2r} T_r^*(t) \quad (4.5)$$

Clenshaw's method applied to eqn (4.4) gives the recurrence relations

$$a'_{2r-1} = a'_{2r+1} + 4r a_{2r} \quad (4.6)$$

$$a_{2r-2} = a_{2r+2} - \frac{4r}{\gamma^2} [a'_{2r-1} + a'_{2r+1} + 4n a_{2r}] \quad (4.7)$$

which may be solved as indicated in Chapter 2, starting with

$$a_{2N} = 1, \quad a_{2r} = 0 \quad \text{for } r > N$$

$$a'_{2r+1} = 0 \quad \text{for } r \geq N$$

for some sufficiently large N . The results are normalised using the initial condition $V(0) = 1$.

We have used the method for $|\gamma| = 8$ and $\arg \gamma = 7.5^\circ (15^\circ) 82.5^\circ$, thus providing a separate expansion for each 15° sector in the first quadrant. The accuracy of these expansions is discussed in Section 3.

The generalized hypergeometric function can be defined by the series

$$\begin{aligned}
& {}_pF_q (\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; z) \\
&= \sum_{k=0}^{\infty} \left[\prod_{h=1}^p (\alpha_h)_k z^k / \prod_{h=1}^q (\beta_h)_k k! \right], \\
& (\alpha)_k = \Gamma(\alpha+k) / \Gamma(\alpha),
\end{aligned}$$

$$p \leq q \text{ or } p = q + 1 \text{ and } |z| < 1,$$

where $\Gamma(z)$ is the gamma function. The general theory of the expansion of these functions in a series of the same kind is described in Luke (1969). It is useful in this context, because both Bessel functions and Chebyshev polynomials can be written as hypergeometric functions;

$$\begin{aligned}
T_n(x) &= {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) \\
J_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu+1)} {}_0F_1(\nu+1; -z^2/4)
\end{aligned}$$

In Luke (1977) are recurrence relations equivalent to (4.6) and (4.7), and in Luke (1975) a generalised case of the expansion

$$v(t) = \sum_{j=0}^{\infty} b_j T_j(t^2) \quad (4.8)$$

is given. Initially this appears to be a superior method of expansion, because it can be used not only in its original sector, but also in a sector at right angles to it. The use of symmetry relations means that if, as before, the quadrant is divided into 15° sectors we need only find expansions in the three sectors for which

$$|\arg z| \leq \pi/4.$$

However, the series in (4.8) converge slightly slower than those in (4.5) and therefore the time required to calculate the Bessel function to the required accuracy is increased.

§2.2 The outer region

The Bessel function $J_\lambda(z)$ may be expressed as

$$J_n(z) = \frac{1}{2} [H_n^{(1)}(z) + H_n^{(2)}(z)]$$

in terms of the Hankel functions of the first and second kinds which have the asymptotic forms

$$H_n^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z - \frac{1}{2}n\pi - \frac{1}{4}\pi)}$$

$$H_n^{(2)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z - \frac{1}{2}n\pi - \frac{1}{4}\pi)}$$

as $z \rightarrow \infty$. Any ray in the region $|z| \gg |\gamma|$ may be mapped onto $[0, 1]$ by the transformation $t = \gamma/z$ and the function

$$u^{(a)}(t) = \left(\frac{\pi z}{2}\right)^{1/2} \exp[i(z - \frac{1}{2}n\pi - \frac{1}{4}\pi)] H_n^{(2)}(z)$$

satisfies the differential equation

$$t^2 u'' + 2(t + i\gamma)u' + (\frac{1}{4} - n^2)u = 0 \quad (4.10)$$

where the prime denotes differentiation with respect to the real variable t , and $u^{(a)}(0) = 1$. Similarly extraction of the asymptotically dominant term in $H_n^{(1)}(z)$ yields a function $u^{(1)}(t)$ which satisfies a differential equation differing from (4.10) only in the sign of $i\gamma$.

Clenshaw's method, used to find a solution of the form

$$u^{(i)}(t) = \sum_{r=0}^{\infty} a_r^{(i)} T_r^*(t), \quad i = 1, 2 \quad (4.11)$$

for eqn. (4.10), gives the recurrence relations

$$[r(r+1) + \frac{1}{4} - n^2]a_{r-1} = (\frac{1}{4} - n^2)a_r - \frac{1}{4}(r+1) [7a'_{r+1} + a'_{r+3} + 4(a'_{r+2} + a'_r)] - 8i\gamma(r+1)a_{r+1} \quad (4.12)$$

$$a'_{r-1} = 4r a_r + a'_{r+1} \quad (4.13)$$

When these, and the corresponding equations for the Chebyshev coefficients of $u^{(a)}(t)$, are solved the results may be combined to evaluate the Bessel functions of the first and second kinds, since

$$J_n(z) = (2\pi z)^{-1/2} [u^{(1)}(t) \exp\{i\alpha(z)\} + u^{(2)}(t) \exp\{-i\alpha(z)\}] \quad (4.14)$$

and

$$Y_n(z) = -i(2\pi z)^{-\frac{1}{2}} \left[U^{(n)}(t) E_{\kappa\rho} \{i\alpha(z)\} - U^{(n)}(t) E_{\kappa\rho} \{-i\alpha(z)\} \right] \quad (4.15)$$

where

$$\alpha(z) = z - \frac{1}{2} n\pi - \frac{1}{4} \pi$$

The expansions in (4.11) are particular cases of series discussed by Luke (1977, p.88). In the real case the differential equations (4.10) has only one solution of the required form and the coefficients can be found, as described in Chapter 2, by backward recurrence from an arbitrary starting point and the use of the initial condition. When $\text{Im } \gamma > 0$ there is no longer just one solution, since (4.10) is also satisfied by

$$W(t) = \begin{cases} e^{a i \gamma / t} u(-t) & , t > 0 \\ 0 & , t \leq 0 \end{cases}$$

It follows that any solution of equations (4.12) and (4.13) corresponds to a linear combination of $U(t)$ and $W(t)$. The unwanted solution $W(t)$ may be removed by solving a boundary value problem in the manner described in Chapter 2. The known value of $J_n(\gamma)$ is used in addition to the condition $U(0) = 1$. In practice the extent to which the unwanted solution $W(t)$ enters will depend on the value of $\text{Im } \gamma$. In producing the tables we solved the recurrence systems with two different starting values and decided by comparison of the results whether to solve an initial - or boundary - value problem.

§3 ERROR BOUNDS

A real - or complex-valued function f of a real variable x may, under suitable conditions, be expressed on the interval $[-1,1]$ as a Chebyshev series

$$f(x) = \sum_{r=0}^{\infty} b_r T_r(x)$$

where

$$b_r = \frac{2}{\pi} \int_{-1}^1 \frac{f(x) T_r(x)}{\sqrt{1-x^2}} dx$$

In particular, if f is differentiable to all orders on $[-1, 1]$ and $|f^{(r)}(x)| \leq Q_r$ on that interval, the work of Elliott (1963) shows that

$$|b_r| \leq \frac{Q_r}{2^{r-1} r!}$$

To apply the result it is convenient to write equation (4.5) as

$$2^n n! z^{-n} J_n(z) = f(x) = \sum_{r=0}^{\infty} a_{ar} T_r(x)$$

where

$$x = 2t^2 - 1 = 2(z/\gamma)^{1/2} - 1$$

Then

$$\begin{aligned} \frac{d^r f}{dx^r} &= 2^n n! \left[\frac{\gamma}{2} \right]^{ar} \left[\frac{1}{z} \frac{d}{dz} \right]^r \left[\frac{J_n(z)}{z^n} \right] \\ &= \frac{(-1)^r 2^n n!}{z^{n+r}} \left[\frac{\gamma}{2} \right]^{ar} J_{n+r}(z) \end{aligned}$$

(See Watson (1944) p.18). Watson (1944) p.49 establishes the bound

$$|J_m(z)| \leq \frac{|z|^m \exp |Im z|}{2^m m!} \quad (4.16)$$

Thus if $|z| \leq |\gamma|$ and $\arg z = \arg \gamma = \phi$ then

$$|a_{ar}| \leq \frac{2 n!}{2^{4r} r! (n+r)!} |\gamma|^{ar} \exp(|\gamma| |\sin \phi|) \quad (4.17)$$

Replacing (4.16) by the inequality

$$|J_m(z)| \leq \frac{|z|^m}{2^m m!} \exp \left(\frac{1}{4} \frac{|z|^2}{m+1} \right)$$

we obtain the alternative bound

$$|a_{ar}| \leq \frac{2 |\gamma|^{ar} n!}{2^{4r} r! (n+r)!} \exp \left(\frac{1}{4} \frac{|\gamma|^2}{(n+r+1)} \right) \quad (4.18)$$

From Luke (1977) p.77 we know that

$$a_{ar} = \frac{2(-1)^r \gamma^{ar} n!}{2^{4r} r! (n+r)!} {}_1F_2 \left(r + \frac{1}{2}; n+r+1, 2r+1; -\gamma^2/4 \right)$$

where ${}_1F_a(a; b, c; z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b)\Gamma(c)z^k}{\Gamma(a)\Gamma(b+k)\Gamma(c+k)k!}$

so $|a_{ar}| \leq \frac{2n!}{2^{4r}} \frac{|\gamma|^{2r}}{r!(n+r)!} \sum_{k=0}^{\infty} \frac{(n+r)!r!(2r+2k)!}{(r+k)!(r+n+k)!(2r+k)!k!} \frac{|\gamma|^{2k}}{16^k}$

The ratio of successive terms in this series is

$$\frac{(2r+2k)(2r+2k-1)}{16(r+k)(r+n+k)(2r+k)k} |\gamma|^2 < \frac{|\gamma|^2}{4(2r+k)k}$$

since $n > -\frac{1}{2}$, and therefore

$$\begin{aligned} |a_r| &< \frac{2n!}{2^{4r}} \frac{|\gamma|^{2r}}{r!(n+r)!} \left[1 + \sum_{k=1}^{\infty} \left\{ \frac{|\gamma|^2}{4(2r+1)} \right\}^k \frac{1}{k!} \right] \\ &= \frac{n! \cdot 2 \cdot |\gamma|^{2r}}{2^{4r} r!(n+r)!} \exp \left[\frac{1}{4} \frac{|\gamma|^2}{(2r+1)} \right] \end{aligned} \tag{4.19}$$

We now have three upper bounds on the coefficients, which differ only in the argument of the exponential. It can easily be seen that (4.19) is a lower bound than (4.18) if $r > n$ and vice versa if $n > r$, whilst (4.17) is a lower upper bound than both if

$$|\sin \phi| < \frac{|\gamma|}{4 \cdot \text{Max}(r+n+1, 2r+1)}$$

Table 4.1 gives some comparisons of these bounds with the values of $|a_{ar}|$ when $n = 1$.

Suppose that the Chebyshev series for $2^n n! z^{-n} T_n(z)$ on the ray $\arg z = \phi$ is truncated to give a polynomial of degree of $2m$ and the resulting approximation is used in the domain

$$D = \left\{ z; |z| \leq |\gamma|, \phi - \theta \leq \arg z \leq \phi + \theta \right\},$$

where θ is given. Then the truncation error is

$$E_{m,n}(z) = \sum_{r=m+1}^{\infty} a_{ar} T_{ar}(z/\gamma)$$

and its magnitude is bounded by

$$2, n! A_{m,n} \sum_{r=m+1}^{\infty} \frac{|\gamma|^{2r}}{4^{ar}} \frac{|T_{ar}(e^{i\theta})|}{r!(n+r)!}$$

with

$$A_{m,n} = \text{Min} \left(\text{Exp}(|\gamma| \sin \phi), \text{Exp} \left(\frac{1}{4} \frac{|\gamma|^2}{2m+3} \right), \text{Exp} \left(\frac{1}{4} \frac{|\gamma|^2}{n+m+2} \right) \right)$$

Since

$$|T_{2r}(e^{i\theta})| \leq \frac{1}{2} (e^{2m+2} + e^{-(2m+2)}) e^{2(r-m-1)}$$

with
$$\rho = |e^{i\theta} + \sqrt{e^{2i\theta} - 1}|$$

and

$$\frac{(m+1)!(n+m+1)!}{(m+1+k)!(n+m+1+k)!} \leq \frac{1}{(m+2)^k (n+m+2)^k}$$

it follows that

$$|E_{m,n}(z)| \leq \frac{n! A_{m,n}}{(m+1)!(n+m+1)!} \cdot \left(\frac{|\gamma|}{4}\right)^{2(m+1)} \frac{(e^{2m+2} + e^{-(2m+2)})}{1 - R_{m,n}}$$

where
$$R_{m,n} = \frac{|\gamma|^2 e^2}{16(m+2)(m+n+2)} \quad (4.20)$$

When $R_{m,n} \ll 1$ the truncation error is dominated by the 1st neglected term. For example, when $|\gamma| = 8$ and $\Theta = 7.5^\circ$, $R_{22,0} \approx 0.014$ and $R_{32,0} \approx 0.007$.

The corresponding bound for the expansion (4.8) is

$$|E_{m,n}^*(z)| \leq \frac{|\gamma|^{2m+2}}{2^{2m+3}} \frac{n! B_{m,n}}{(m+1)!(m+n+1)!} \frac{\{\delta^{m+1} + \delta^{-(m+1)}\}}{1 - P_{m,n}}$$

where

$$\delta = |\text{Exp}(2i\theta) + \sqrt{\text{Exp}(4i\theta) - 1}| \geq 1$$

$$P_{m,n} = \frac{|\gamma|^2 \delta}{16(m+2)(m+n+2)}$$

and

$$B_{m,n} = \text{Exp} \left\{ |\gamma|^4 / (64(m+2)(m+n+2)(m+n+3)) \right\} \quad (4.21)$$

For $\Theta = \pi/24$ the ratio of these bounds is approximately $1:(1.6)^{m+1}$, which indicates why the previous expansion is preferred.

§3.1 Estimation of the required number of terms

We wished to write a program capable of calculating the Bessel function $J_n(z)$, in the inner region, to a user specified accuracy, less than some maximum attainable accuracy. To do this efficiently we require a method

of calculating the smallest number of terms needed in the truncated Chebyshev series. It is assumed that the degree of the truncated series would be at least $2M_0$, where $M_0 + 1 > |\gamma|$, and that $\rho < 4$ and $|\gamma| > 1$.

With these assumptions we have for $M \geq M_0$,

$$\frac{|z/2|^n}{n!} |E_{m,n}(z)| \leq \frac{\text{Exp}(|\gamma|/8)}{(m+1)!} \left(\frac{\rho|\gamma|}{4}\right)^{2(m+1)} \frac{(0.5|\gamma|)^n}{(m+n+1)!} \left(\frac{1 + \rho^{-4|\gamma|}}{1 - \rho^2/16}\right), \quad (4.22)$$

where the left hand side is the error in approximation of $J_n(z)$. Now set

$x = m + 1$ and note that

$$\sqrt{2\pi} \text{Exp}(-x + 1/2x) x^{x+1/2} \geq \Gamma(x+1) \geq \sqrt{2\pi} \text{Exp}(-x) x^{x+1/2} \quad x > 0.$$

We can now write

$$\frac{1}{n!} \left| \frac{z}{2} \right|^n |E_{x-1,n}(z)| \leq \frac{1}{2} \left(\frac{\rho|\gamma|e}{4}\right)^{2x} \frac{1}{(x+n)^{x+n+1/2}} \cdot \frac{K_n}{x^{x+1/2}}$$

where

$$K_n = \frac{e^{|\gamma|/8} (1 + \rho^{-4|\gamma|})}{\pi (1 - \rho^2/16)} \left(\frac{|\gamma|e}{2}\right)^n$$

The bound in the last equation is a monotonic decreasing function of x .

So if the accuracy required is $\frac{1}{2} \times 10^{-N}$, and we find γ such that

$$\frac{1}{2} \times 10^{-N} = \frac{1}{2} \left(\frac{\rho|\gamma|e}{4}\right)^{2\gamma} \frac{K_n}{(\gamma+n)^{\gamma+n+1/2} \gamma^{\gamma+1/2}}$$

then by setting

$$M = [\gamma] + 1 \geq M_0$$

we have

$$\frac{|z/2|^n}{n!} |E_{M,n}(z)| \leq \frac{1}{2} \times 10^{-N}$$

The value of γ can be obtained iteratively using

$$y_{j+1} = \frac{(2y_j - \frac{1}{2} \log_e y_j - (n + \frac{1}{2}) \log_e (n + y_j) + \frac{1}{2} \left\{ \frac{y_j}{n + y_j} \right\} + Nk_1 + k_2 + k_3)}{\left\{ \log_e \{y_j (y_j + n)\} - 2 \log_e \left\{ \frac{\rho|\gamma|}{4} \right\} + \frac{1}{2y_j} + \frac{1}{2(y_j + n)} \right\}} \quad (4.23)$$

where

$$k_1 = -\log_e 10, \quad k_2 = \frac{1}{2} + \frac{|\gamma|}{8} - \log_e \left\{ \frac{\pi (1 - \rho^2/16)}{(1 + \rho^{-4|\gamma|})} \right\}$$

and $k_3 = n + n \log_e (|\delta|/a)$.

Furthermore, if $\gamma > \frac{1}{a}$ and $\gamma_j > (\rho|\delta|/4)$, the iteration converges to γ from above, so one iteration gives an upper bound on γ , which can be used to determine the number of terms required.

Examples:

(1) $|\delta| = 8$ $\rho^2 = 2.03, n = 0$

N	γ_0	γ_1	γ_2
4	10	10.7	10.7
24	20	23.63	
40	28	31.65	

(2) $|\delta| = 2$ $\rho^2 = 1, n = 0$

N	γ_0	γ_1	γ_2
16	10	9.09	9.07

The initial value γ_0 was chosen to be $\{|\delta| + N/2\}$ as this satisfies the condition $\gamma_j > |\delta|$ and uses $N/2$ as an estimate of the variation of the number of coefficients required with N . Substituting values $[\gamma_{i-1}]$ and $[\gamma_i]$ from the first example into (4.22) gives the following results:

$[\gamma_i]$	Bound using $[\gamma_{i-1}]$	Bound using $[\gamma_i]$	Required Accuracy
10	0.2×10^{-3}	0.13×10^{-4}	0.5×10^{-4}
23	0.18×10^{-23}	0.25×10^{-25}	0.5×10^{-24}
31	0.30×10^{-39}	0.24×10^{-41}	0.5×10^{-40}

From these results we can see that $M = [\gamma_i]$ is the smallest integer which reduces the bound on the error to the required level and therefore, we do not have to iterate to convergence. They also show that for M large, adding an extra coefficient gives an extra 2 decimal place accuracy. So although the relative factor of a truncated Chebyshev series to the best polynomial approximation of the same degree may be large, the extra accuracy can be

produced by adding just one or two coefficients to the series. The extra evaluation time is more than offset against the difficulty in calculating best polynomial approximations.

§3.2 Asymptotic estimates for the outer region

The dependence of the rate of convergence in the inner region, on $|\delta|$ is evident from the inequality (4.20). By decreasing $|\delta|$ the rate of convergence in the inner region is increased, but at the expense of a slower rate in the outer region. The bound (4.20) applies to the inner region $|z| \leq |\delta|$ only. For $|z| > |\delta|$ we do not have useful bounds for the coefficients of the expansions (4.11), but their asymptotic forms have been investigated by Miller (1966) and by Luke (1977). The latter reference gives two slightly different asymptotic estimates:

$$Q_r^{(3+\epsilon)/2} \sim \frac{4(-1)^r (\pi/3)^{3/2} (2r^2 \epsilon \delta i)^{1/6} \text{Exp} \{S(2P-3) + R/S\}}{r \Gamma(\frac{1}{2}-n) \Gamma(\frac{1}{2}+n)} \quad (4.24)$$

and

$$Q_r^{(3+\epsilon)/2} \sim \frac{4(-1)^r (\pi/3)^{3/2} (2r^2 \epsilon \delta i)^{1/6} \text{Exp} \{ \frac{2}{3} \epsilon \delta i - 3s + (R + \frac{4\delta^2}{15}) s^{-1} \}}{r \Gamma(\frac{1}{2}-n) \Gamma(\frac{1}{2}+n)} \quad (4.25)$$

where

$$S = (2r^2 \epsilon \delta i)^{1/3}, \quad y = (2r \delta \epsilon i / r)^{1/3}$$

$$P = \frac{1}{2} \left\{ 1 - y / \sinh(y) \right\}, \quad R = n^2 - \frac{1}{9}$$

and $\epsilon = \pm 1$.

Tables 4.2 and 4.3 contain comparisons of our calculated coefficients with these estimates, in every case the estimate (4.24) is better than (4.25), but both are remarkably accurate. The value $|\delta| = 8$ was chosen after experimentation with a number of different values. No compelling reason was found to change the value from that chosen by Clenshaw (1962) for the real case.

§4. The coefficient tables for $J_N(z)$

The coefficients of the Chebyshev expansions (4.5) and (4.11), for $J_0(z)$ and $J_1(z)$ with $|z| = 8$, were calculated in quadruple-precision on an IBM 370 computer, and the results were rounded to double-precision form. For each Bessel function there are six tables, one corresponding to each of the rays $\arg z = 7.5^\circ (15^\circ) 82.5^\circ$. In each table the top set of figures corresponds to the expansion (4.5) and the two lower sets are the coefficients $\alpha_r^{(1)}$ and $\alpha_r^{(2)}$ to be used for $|z| > 8$.

The number of coefficients tabulated in each case, and the number of decimal digits to which they are quoted, were determined by the requirement that a certain accuracy be achievable in the calculation of the corresponding Bessel function in double precision arithmetic. In the inner region the Chebyshev series converges rapidly and the accuracy is limited only by rounding error. Table 4.4 shows the accuracy achievable for $|z| \leq 8$ with the tabulated coefficients. As $\arg z$ increases the low order coefficients increase in magnitude but $\sum_{r=0}^{\infty} (-1)^r \alpha_{2r} = 1$ in all cases; the resulting loss of significant figures reduces the achievable accuracy as shown in the table.

In the outer region the truncation error, rather than the rounding error, is the decisive factor. The truncation error increases as $\arg z$ is increased and we maintain the achievable accuracy (Table 4.5) at a fixed level for $0 < \arg z \leq 75^\circ$ by increasing the number of coefficients retained in each of the series (4.11). For $75^\circ < \arg z \leq 90^\circ$ we have accepted a lower level of accuracy as the convergence of the series has then become very slow. The relative errors quoted in Table 4.5 are not applicable in the vicinity of the zeros of J_0 and J_1 , all of which lie on the real axis. The tabulated coefficients are sufficiently accurate to permit calculations with absolute error no greater than $\frac{1}{2} \times 10^{-14}$ near the zeros.

The coefficients of the Chebyshev expansion (4.5), for $J_0(z)$, $J_1(z)$, ..., $J_{10}(z)$ with $|\lambda| = 8$ are also tabulated in quadruple precision form.

§5 Expansions for $Y_0(z)$ and $Y_1(z)$ for $|z| \leq |\lambda|$

In the outer region the Bessel functions of the second kind can be calculated from the Hankel functions. $Y_0(z)$ and $Y_1(z)$ have singularities at the origin, but as the form of the singularity is known in each case, we can subtract it and expand the remaining auxiliary function, which is analytic.

For $Y_0(z)$ and $Y_1(z)$ we choose

$$V_0(z) = Y_0(z) - \frac{2}{\pi} \log_e \left(\frac{z}{\lambda} \right) \cdot J_0(z) \quad (4.26)$$

and

$$\left(\frac{z}{\lambda} \right) V_1(z) = Y_1(z) + \frac{2}{\pi} \left(\frac{1}{z} - \log_e \left(\frac{z}{\lambda} \right) \right) \cdot J_1(z) \quad (4.27)$$

Putting $z = \lambda t$ we find that these auxiliary functions satisfy the non-homogenous differential equations

$$t^2 V_0'' + V_0' + \lambda^2 t V_0 = \frac{4\lambda}{\pi} J_1(\lambda t)$$

and

$$t^2 V_1'' + 3t V_1' + \lambda^2 t^2 V_1 = \frac{4}{\pi} (1 - J_0(\lambda t) + J_2(\lambda t)) \quad (4.28)$$

where the dash represents differentiation with respect to t . Since we already have the expansions of the Bessel functions of the first kind we can now find the coefficients in the expansions

$$V_i(z) = \sum_{j=0}^{\infty} (b_{i,aj}) \cdot T_{2j}^{(i)}(t) \quad i = 0, 1 \quad (4.29)$$

using Clenshaw's method. The recurrence relations obtained are

$$b_{0,ar} = b_{0,a(r+4)} + \frac{2}{\pi} (a_{1,ar} - a_{1,a(r+4)}) - \frac{4(r+1)}{\lambda^2} (b_{0,a(r+3)} + b_{0,a(r+1)}) \quad r = 0, 1, \dots \quad (4.30)$$

and

$$\begin{aligned}
 b_{1,2r} = & b_{1,2r+4} - b_{1,2r+2} - b_{1,2r} - \frac{16}{\gamma^2 \pi} (a_{0,2r} - a_{0,2r+2}) \\
 & + \frac{4}{\pi} (a_{2,2r-2} + a_{2,2r} - a_{2,2r+2} - a_{2,2r+4}) \pi^{-1} \\
 & - 4/\gamma^2 ((r+1) b'_{1,2r-1} + (2r+1) b'_{1,2r+1} + b'_{1,2r+3}) \quad (4.31) \\
 & r=0,1,\dots
 \end{aligned}$$

where

$a_{i,j}$ is the coefficient of $T_j(t)$ in the expansion for $J_i(z)$ given by (4.5).

As previously we have

$$b'_{i,2r+1} = 2(2r+2) b_{i,2r+2} + b'_{i,2r+3} \quad i=0,1..$$

We now set the coefficients $b_{i,2k}$ to zero for $k > N$ and recur backwards using the known values of $a_{i,j}$. The solution obtained is a solution of the differential equation of the form

$$W_k(z) = V_k(z) + A k! \left(\frac{2}{z}\right)^k J_k(z)$$

The value of A can be found from the initial conditions

$$V_0(0) = \frac{2\chi}{\pi}, \quad V_1(0) = \frac{2\chi-1}{\pi}$$

where

$$\chi = \text{Euler's Constant} = 0.57721\dots$$

To get the value of the coefficients of $V_k(z)$ we can subtract the appropriate multiple of the coefficients for the Bessel function of the first kind from those obtained by recurrence.

The coefficients in the expansion (4.29) are tabulated in quadruple precision form for $i=0$ and 1 . The coefficients for $Y_0(z)$ and $Y_1(z)$ converge to zero at a similar rate to those for $J_0(z)$ and $J_1(z)$.

TABLE 4.1

Values of $|a_{nr}|$ for $J_1(z)$ with $\gamma = 8 \exp(i\phi)$, and upper bounds (4.17), (4.18) and (4.19).

	r	$ a_{nr} $	(4.17)	(4.18)	(4.19)
0	0	3.2(-1)	2.0	6.0(+3)	1.8(+7)
	5	7.3(-3)	2.4(-2)	2.3(-1)	1.0(-1)
	10	7.4(-9)	1.4(-8)	5.5(-8)	3.1(-8)
	20	1.2(-26)	1.8(-26)	3.7(-26)	2.6(-26)
	30	8.2(-49)	1.1(-48)	1.7(-48)	1.4(-48)
$\pi/4$	0	3.3(-1)	5.7	6.0(+3)	1.8(+7)
	5	7.6(-3)	6.7(-2)	2.3(-1)	1.0(-1)
	10	7.6(-9)	4.1(-8)	5.5(-8)	3.1(-8)
	20	1.2(-26)	5.9(-26)	3.7(-26)	2.6(-26)
	30	8.2(-49)	3.0(-48)	1.7(-48)	1.4(-48)
$11\pi/24$	0	6.1(+1)	5.6(+3)	6.0(+3)	1.7(+7)
	5	6.9(-2)	6.6(+1)	2.3(-1)	1.0(-1)
	10	2.7(-8)	4.0(-5)	5.5(-8)	3.1(-8)
	20	2.5(-26)	4.9(-23)	3.7(-26)	2.6(-26)
	30	1.4(-48)	2.9(-45)	1.7(-48)	1.4(-48)

TABLE 4.2

Values of $|a_r^{(1)}|$ for the Hankel function $H_0^1(z)$, and the estimates E_1 and E_2 , obtained from (4.24) and (4.25) respectively.

ϕ	r	$ a_r^{(1)} $	E_1	E_2
7.5°	20	3.39(-22)	3.43(-22)	3.85(-22)
	30	5.11(-29)	5.13(-29)	5.49(-29)
	40	4.49(-35)	4.50(-35)	4.72(-35)
82.5°	20	1.67(-23)	1.71(-23)	1.53(-23)
	30	4.80(-31)	4.88(-31)	4.57(-31)
	40	8.88(-38)	8.99(-38)	8.60(-38)

TABLE 4.3

Values of $|a_r^{(2)}|$ for the Hankel function $H_0^2(z)$, and estimates E_1 and E_2 , obtained from (4.24) and (4.25) respectively.

ϕ	r	$ a_r^{(2)} $	E_1	E_2
-7.5°	20	1.22(-21)	1.21(-21)	1.33(-21)
	30	3.69(-28)	3.64(-28)	3.84(-28)
	40	6.10(-34)	6.07(-34)	6.29(-34)
82.5°	20	1.24(-16)	1.20(-16)	1.08(-16)
	30	8.79(-21)	8.65(-21)	8.14(-21)
	40	1.84(-24)	1.82(-24)	1.75(-24)

TABLE 4.4

Maximum absolute error in using (4.5) with the tabulated coefficients
for $|z| \leq 8$

arg z	n = 0	n = 1
0°-15°	0.5(-14)	0.5(-14)
15°-30°	1.0(-14)	1.0(-14)
30°-45°	0.5(-13)	0.5(-13)
45°-60°	1.0(-13)	2.0(-13)
60°-75°	0.5(-12)	0.5(-12)
75°-90°	0.5(-12)	0.5(-12)

TABLE 4.5

Maximum relative error in using (4.11) with the tabulated coefficients
for $|z| \geq 8$

arg z	n = 0	n = 1
0°-15°	0.25(-14)	0.25(-14)
15°-30°	0.25(-14)	0.25(-14)
30°-45°	0.25(-14)	0.25(-14)
45°-60°	0.5(-14)	0.5(-14)
60°-75°	0.5(-14)	0.5(-14)
75°-90°	0.1(-13)	0.1(-13)

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APPENDIX A

Tables of Coefficients for the Chebyshev expansions

A: For $|z| \leq 8$

$$J_n(z) = \frac{1}{n!} \left(\frac{z}{2}\right)^n \sum_{r=0}^{\infty} a_{nr} T_{nr}(t)$$

with

$$t = \left(\frac{z}{8}\right) \text{Exp}(-i\phi), \quad \phi = 7.5^\circ(15^\circ)82.5^\circ,$$

$$n = 0, 1.$$

B: For $|z| > 8$

$$J_n^{(1)}(z) = \frac{1}{2} [H_n^{(1)}(z) + H_n^{(2)}(z)]$$

$$J_n^{(2)}(z) = \frac{1}{2} [H_n^{(1)}(z) - H_n^{(2)}(z)]$$

where

$$H_n^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \text{Exp}\{i\alpha(z)\} \sum_{r=0}^{\infty} a_r^{(1)} T_r^*(t)$$

and

$$H_n^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \text{Exp}\{-i\alpha(z)\} \sum_{r=0}^{\infty} a_r^{(2)} T_r^*(t)$$

with

$$\alpha(z) = z - \frac{1}{2}n\pi - \frac{1}{4}\pi$$

and

$$t = \left(\frac{8}{z}\right) \text{Exp}(i\phi) \quad \phi = 7.5^\circ(15^\circ)82.5^\circ$$

$$n = 0, 1.$$

COEFFICIENTS FOR $J_0(z)$ $\phi = 7.5$

r	a_{ar}	a_{ar}
0	0.40795869055667140+00	-0.56442646929690940-01
1	0.76307829250131400-01	-0.61302046672716610-01
2	0.29061006595818790+00	-0.20911435561449690+00
3	-0.41239519311709360+00	-0.49959512745050740-01
4	0.14405373906065260+00	0.91255766225367240-01
5	-0.22191368433940320-01	-0.29516142867063310-01
6	0.15821011678673100-02	0.47897681004629970-02
7	-0.59234157455040-05	-0.4786371071377800-03
8	-0.93656954952600-05	0.322257062586540-04
9	0.9800718275080-06	-0.15271062036970-05
10	-0.588490058690-07	0.513703842020-07
11	0.248418431170-08	-0.116593048650-08
12	-0.7916558530-10	0.1300272270-10
13	0.197135290-11	0.21989560-12
14	-0.38986100-13	-0.15719640-13
15	0.612880-15	0.469890-15
16	-0.7490-17	-0.9910-17
17	0.70-19	0.160-18

r	a_r
0	0.19972093278930000+01
1	-0.1516374747527360-02
2	-0.118392601149590-03
3	0.2739226670890-05
4	0.82915499780-07
5	-0.101411905840-07
6	0.2312874930-09
7	0.391970850-10
8	-0.49083610-11
9	0.1266660-12
10	0.3932760-13
11	-0.659150-14
12	0.33700-15
13	0.59950-16
14	-0.15410-16
15	0.1680-17
16	0.60-19
17	-0.50-19
18	0.10-19

r	a_r
0	-0.15204871279421320-01
1	-0.7549109144717700-02
2	0.56461738916240-04
3	0.2978267560210-05
4	-0.158614165510-06
5	-0.9370458410-09
6	0.6623690540-09
7	-0.403132210-10
8	-0.13886480-11
9	0.4976390-12
10	-0.4057150-13
11	-0.129200-14
12	0.78540-15
13	-0.9870-16
14	0.1540-17
15	0.1810-17
16	-0.380-18
17	0.30-19
18	0.00+00

r	a_r
0	0.20012235425941850+01
1	0.474841871038830-03
2	-0.137202226054980-03
3	-0.65744713620-07
4	0.204799529390-06
5	-0.31171482170-08
6	-0.8151343660-09
7	0.511219570-10
8	0.40675200-11
9	-0.7989440-12
10	0.1211370-13
11	0.1083010-13
12	-0.137610-14
13	-0.46380-16
14	0.36710-16
15	-0.4310-17
16	-0.290-18
17	0.180-18
18	-0.20-19

r	a_r
0	0.15617615943316040-01
1	0.7822498531306660-02
2	0.9262723570900-05
3	-0.4420964498870-05
4	0.192215667580-07
5	0.120091037020-07
6	-0.4060054090-09
7	-0.589994200-10
8	0.64160370-11
9	0.1927840-12
10	-0.9655760-13
11	0.615670-14
12	0.101110-14
13	-0.24430-15
14	0.10650-16
15	0.4410-17
16	-0.1000-17
17	0.40-19
18	0.20-19

COEFFICIENTS FOR $J_0(z)$ $\phi = 22.5$

r	a_{2r}	a_{2r}
0	0.196233881365254940+01	0.164861222859825240+00
1	0.14586386625130950+01	-0.12544466638704220+00
2	0.99548777603655730+00	-0.10344229108502020+01
3	-0.69363067919803990+00	-0.59323596240726290+00
4	-0.92103394031436980-01	0.27615145780052250+00
5	-0.54325155286706500-01	-0.12476381839349760-01
6	-0.53126997414577370-02	-0.46544745856727890-02
7	-0.313387287174160-04	0.6362292727826230-03
8	0.342873932912790-04	-0.260135301745500-04
9	-0.22152426452250-05	-0.4599299867230-06
10	0.4779114966140-07	0.8240144194770-07
11	-0.93772468430-09	-0.313460160430-08
12	-0.8555941840-10	0.4049268570-10
13	0.214976920-11	-0.84820190-12
14	-0.13044850-13	-0.44965040-13
15	-0.366120-15	0.802280-15
16	0.13280-16	-0.4640-17
17	-0.180-18	-0.90-19

r	a_r
0	0.19935131054295560+01
1	-0.3319196444255620-02
2	-0.73441443030410-04
3	0.3737120090050-05
4	-0.75032763990-07
5	-0.33930193340-08
6	0.5571257220-09
7	-0.321282990-10
8	-0.1121430-12
9	0.2706120-12
10	-0.3600210-13
11	0.235660-14
12	-0.8620-16
13	-0.5080-16
14	0.8270-17
15	-0.740-18
16	-0.10-19
17	0.20-19
18	-0.40-20

r	a_r
0	-0.13832018383615870-01
1	-0.6814262241695100-02
2	0.192213889059340-03
3	0.313900166510-06
4	-0.140262200500-06
5	0.771099094030-08
6	-0.960963020-10
7	-0.285282290-10
8	0.36843970-11
9	-0.2225950-12
10	-0.516990-14
11	0.332550-14
12	-0.48100-15
13	0.3590-16
14	0.1220-17
15	-0.920-18
16	0.170-18
17	-0.20-19
18	0.00+00

r	a_r
0	0.20053177738260890+01
1	0.2542329296990900-02
2	-0.120170427429640-03
3	-0.3482858517330-05
4	0.143709603890-06
5	0.132218047240-07
6	-0.6381030370-09
7	-0.963341570-10
8	0.74912770-11
9	0.10238170-11
10	-0.15339260-12
11	-0.1047360-13
12	-0.390480-14
13	-0.656660-16
14	-0.97720-16
15	0.12640-16
16	0.1600-17
17	-0.660-18
18	0.390-19
19	0.210-19
20	-0.50-20
21	0.00+00

r	a_r
0	0.14968708481942420-02
1	0.7568320558896080-02
2	-0.807452227954180-04
3	-0.3422098643420-05
4	-0.193818784070-06
5	0.33037100720-08
6	-0.10616440260-08
7	-0.627618810-10
8	-0.96905330-11
9	0.10253030-11
10	-0.10846780-12
11	-0.2418370-13
12	-0.65960-15
13	-0.62850-15
14	-0.42220-16
15	-0.13930-16
16	0.3080-17
17	0.70-19
18	-0.1270-18
19	0.180-19
20	0.20-20
21	-0.10-20

COEFFICIENTS FOR $J_0(z)$ $\phi = 37.5$

P	a_{1P}	a_{2P}
0	0.6711192732258907000D+01	0.83712702818187070D+01
1	0.71500895090149270D+01	0.59935320789535870D+01
2	0.621599845055422610D+01	-0.19486400634296970D+00
3	0.91766948812533030D+00	-0.24065140778222980D+01
4	-0.567773565838961930D+00	-0.31719376085575830D+00
5	-0.52934967577101030D-01	0.92811127584290710D-01
6	-0.10994988091648050D-01	0.53006984777873090D-02
7	-0.3315544108098570D-03	-0.9660000943464270D-03
8	-0.641585382952800D-04	-0.116441403663700D-04
9	-0.192622784860D-07	-0.32792409067350D-05
10	-0.1309846008230D-05	-0.243897004330D-07
11	-0.168160777960D-08	-0.413530518190D-08
12	-0.103711528320D-09	0.70597257860D-10
13	0.2185934690D-11	0.2054503640D-11
14	0.312224550D-13	-0.533025000D-13
15	-0.10573080D-14	-0.3278130D-15
16	-0.11760D-17	0.173790D-16
17	0.2390D-18	-0.410D-19

P	a_{1P}
0	0.19903453514550460D+01
1	-0.48458093189399830D-02
2	-0.15517452292490D-04
3	0.2319112892250D-05
4	-0.1436070455590D-06
5	0.49066235950D-08
6	0.66290330D-11
7	-0.226695340D-10
8	0.23074010D-11
9	-0.2304390D-12
10	0.1115110D-13
11	0.49990D-15
12	-0.22630D-15
13	0.3790D-16
14	-0.4490D-17
15	0.360D-18
16	0.00D+00
17	-0.10D-19
18	0.20D-20

P	a_{1P}
0	-0.11639234118612650D-01
1	-0.5696876791120940D-02
2	0.120646852906100D-03
3	-0.2117642443590D-05
4	-0.179503666700D-07
5	0.57733100550D-08
6	-0.4767936730D-09
7	0.263230620D-10
8	-0.5165050D-12
9	-0.1195350D-12
10	0.2321990D-13
11	-0.271220D-14
12	0.22030D-15
13	-0.6410D-17
14	-0.2030D-17
15	0.560D-18
16	-0.90D-19
17	0.10D-19
18	-0.10D-20

P	a_{1P}	a_{2P}
0	0.20092036666877190D+01	0.13214907754115090D-01
1	0.4541978994061490D-02	0.6745117980415220D-02
2	-0.65149863654560D-04	0.138067792653490D-03
3	-0.5428112934930D-05	0.156405264520D-06
4	-0.119337987910D-06	-0.261936829340D-06
5	0.147555616640D-07	-0.138903793930D-07
6	0.15004313420D-08	0.9350951390D-09
7	-0.650658410D-10	0.1705095680D-09
8	-0.209903820D-10	-0.49717500D-11
9	0.443995550D-12	-0.28152980D-11
10	0.40980060D-12	0.5407140D-13
11	-0.985050D-14	0.6414450D-13
12	-0.1065680D-13	-0.233680D-14
13	0.603730D-15	-0.184940D-14
14	0.328940D-15	0.161120D-15
15	-0.42280D-16	0.58550D-16
16	-0.10070D-16	-0.10910D-16
17	0.2760D-17	-0.1570D-17
18	0.1830D-18	0.6780D-18
19	-0.1600D-18	-0.20D-20
20	0.120D-19	-0.360D-19
21	0.70D-20	0.50D-20
22	-0.20D-20	0.10D-20

COEFFICIENTS FOR $J_0(z)$ $\phi = 52.5$

r	a_r	b_r
0	-0.24615607680072100+02	0.40708184519741680+02
1	-0.13674680433263380+02	0.36750868601784120+02
2	0.42050323926583390+01	0.20367739691527430+02
3	0.586222759766447910+01	0.35548361381913600+01
4	0.13204034634908640+01	-0.61553461742536760+00
5	-0.12011362049486120-01	-0.21071435764786350+00
6	-0.19372490663037710-01	-0.10318073190657240-01
7	-0.1319376588675230-02	0.1086868574307410-02
8	0.328820941236510-04	0.977692403701930-04
9	0.495612000024880-05	0.1008907163270-06
10	0.666210813440-07	-0.1822328826430-06
11	-0.49411940730-08	-0.39106755590-08
12	-0.14323130560-09	0.9609017440-10
13	0.114262290-11	0.386637610-11
14	0.81399050-13	0.872960-15
15	0.429640-15	-0.1367940-14
16	-0.18410-16	-0.12440-16
17	-0.2360-18	0.1940-18

r	a_r	b_r
0	0.19878411243937910+01	-0.87911119172615750-02
1	-0.6038323189307780-02	-0.4281323482052130-02
2	0.41903843695520-04	0.110943187326070-03
3	0.690039947840-06	-0.3292755752140-05
4	-0.92176460190-07	0.98313058650-07
5	-0.65907466430-08	-0.19622250780-08
6	-0.4095719820-09	-0.921720090-10
7	0.223506830-10	0.195075790-10
8	-0.3550580-12	-0.22017030-11
9	-0.2079470-13	0.2069150-12
10	0.1059950-13	-0.1692010-13
11	-0.174970-14	0.110150-14
12	0.22720-15	-0.2790-16
13	-0.2560-16	-0.8190-17
14	0.2480-17	0.2230-17
15	-0.180-18	-0.400-18
16	0.00+00	0.60-19

r	a_r	b_r
0	0.20125582661731310+01	0.10396291901150900-01
1	0.6300905165603860-02	0.5353125376053160-02
2	0.18188557607200-04	0.159575311521380-03
3	-0.3915033136280-05	0.4607235879250-05
4	-0.347060124640-06	-0.90777364410-08
5	-0.146007728380-07	-0.227517457000-07
6	0.12414432150-08	-0.23429196180-08
7	0.3106696790-09	0.173963280-10
8	0.123824040-10	0.402265350-10
9	-0.533111820-11	-0.35132040-11
10	-0.76545750-12	-0.73926430-12
11	0.10923080-12	-0.15778290-12
12	0.3256060-13	0.1758000-13
13	-0.3162300-14	-0.6866310-14
14	-0.1489260-14	-0.648580-15
15	0.151710-15	-0.331950-15
16	0.75620-16	0.39570-16
17	-0.11130-16	0.17440-16
18	-0.4010-17	-0.3270-17
19	0.9840-18	-0.9040-18
20	0.1920-18	0.2980-18
21	-0.890-19	0.360-19
22	-0.430-20	-0.260-19
23	0.760-20	0.60-21
24	-0.70-21	0.210-20
25	-0.50-21	-0.40-21
26	0.20-21	-0.10-21

COEFFICIENTS FOR $J_0(z)$ $\phi = 67.5$

r	$a_r^{(1)}$	$a_r^{(2)}$
0	-0.120232334004003580+03	-0.666910182233263760+02
1	-0.991702227887441290+02	-0.3991478265313556590+02
2	-0.49482551698438320+02	-0.14800978678558560+01
3	-0.128334635332439320+02	0.63351619838680030+01
4	-0.139224145634922310+01	0.23497334840795790+01
5	0.358734426312289920-01	0.36184555175082580+00
6	0.24623296955251350-01	0.25407683584997800-01
7	0.2598902940404090-02	0.273286148968950-03
8	0.1238547420465580-03	-0.864737060665300-04
9	0.1623338187743000-05	-0.681640336862700-05
10	-0.12846188590200-06	-0.23322393287200-06
11	-0.799950011100-08	-0.267182166000-08
12	-0.205515990400-09	0.93893769100-10
13	-0.195512950-11	0.4801536300-11
14	0.3750730-13	0.9592080-13
15	0.152150-14	0.75440-15
16	0.2590-16	-0.890-17

r	$a_r^{(1)}$
0	0.198611136347907940+01
1	-0.6854361987473120-02
2	0.37216072860410-04
3	-0.15826142664400-05
4	0.22986899400-07
5	-0.87193215600-09
6	-0.16493739300-09
7	-0.1799823100-10
8	-0.173446200-11
9	-0.17355600-12
10	-0.16876300-13
11	-0.1641600-14
12	-0.158100-15
13	0.14700-16
14	-0.12600-17
15	0.900-19
16	-0.00+00

r	$a_r^{(1)}$
0	-0.5466993118181470-02
1	-0.2653394957211820-02
2	0.77118451024820-04
3	-0.2849994799580-05
4	0.126423891270-06
5	-0.63725645700-08
6	-0.3487255450-09
7	-0.1975144200-10
8	0.107621800-11
9	-0.4631700-13
10	-0.245100-15
11	-0.478800-15
12	-0.96400-16
13	0.15400-16
14	-0.22500-17
15	0.3200-19
16	-0.400-19

r	$a_r^{(2)}$
0	0.201504999312310760+01
1	0.762973668663711300-02
2	0.1060525890331400-03
3	0.11428837593700-05
4	-0.167423262880-06
5	-0.3253413411400-07
6	-0.400385738000-08
7	-0.22171255900-09
8	0.5217775700-10
9	0.1680145800-10
10	-0.1022039950-11
11	-0.572029300-12
12	-0.130234400-12
13	0.16414400-13
14	-0.95419000-14
15	-0.3688300-15
16	-0.7018000-15
17	0.39900-17
18	-0.569200-16
19	-0.59700-18
20	-0.510500-17
21	0.26600-18
22	-0.488600-18
23	-0.63800-19
24	-0.45800-19
25	0.11900-19
26	-0.3700-20
27	-0.1900-20
28	-0.1600-21
29	0.2600-21
30	-0.3000-22
31	-0.3000-22
32	0.1000-22

r	$a_r^{(2)}$
0	0.6663530963222060-02
1	0.3455325755816240-02
2	0.129764771192460-03
3	0.6569784075860-05
4	0.383152400240-06
5	-0.171132698620-07
6	-0.121108689800-08
7	-0.53541494900-09
8	-0.7736796500-10
9	0.1699555500-11
10	0.31926554900-11
11	-0.423050400-12
12	-0.98747000-13
13	-0.35691500-13
14	0.25814000-14
15	-0.25632500-14
16	-0.441500-16
17	-0.1971400-15
18	0.68500-18
19	0.1696200-16
20	-0.45000-18
21	-0.157000-17
22	0.13600-18
23	0.15000-18
24	-0.28200-19
25	-0.13500-19
26	0.4900-20
27	0.9100-21
28	-0.7200-21
29	0.1000-22
30	-0.9000-22
31	-0.2000-22
32	-0.1000-22

COEFFICIENTS FOR $J_0(z)$ $\phi = 82.5$

r	a_r	a_r
0	0.14786330077903400+03	-0.18685812523542100+03
1	0.10415768205398210+03	-0.14493435376763320+03
2	0.37773716029004830+02	-0.681000726343676870+02
3	0.71510684578255640+01	-0.199421710416300690+02
4	0.58718947463746490+00	-0.37945991286273710+01
5	-0.27336816292524010-01	-0.489446001776023350+00
6	-0.12633298385439300-01	-0.443441168850055760-01
7	-0.1611114168128550-02	-0.2894448387518540-02
8	-0.126923683133150-03	-0.137652845256220-03
9	-0.71010473576560-05	-0.470522276012430-05
10	-0.3002448370030-06	-0.1064111996390-06
11	-0.99086415010-08	-0.92761222840-09
12	-0.26027484120-09	0.40940007460-10
13	-0.549455020-11	0.236894200-11
14	-0.9309790-13	0.7217570-13
15	-0.124060-14	0.160580-14
16	-0.1210-16	0.2840-16

r	$a_r^{(2)}$
0	0.19852327715007720+01
1	-0.7268436398177990-02
2	0.112050489188790-03
3	-0.3009551455050-05
4	0.112254461530-06
5	-0.52468293640-08
6	0.2993899720-09
7	-0.132235290-10
8	0.12696520-11
9	-0.959030-13
10	0.771370-14
11	-0.64300-15
12	0.5600-16
13	-0.430-17
14	0.390-19
15	-0.30-19
16	0.00+00

r	$a_r^{(2)}$
0	0.201638648886053580+01
1	0.83516995539959070-02
2	0.164450312376310-03
3	0.6346545763420-05
4	0.372352063620-06
5	0.2592870566990-07
6	0.6390502880-09
7	-0.6006170410-09
8	-0.2371688590-09
9	-0.472304210-10
10	0.1819920-13
11	0.326286490-11
12	0.81044440-12
13	-0.92782650-13
14	-0.90348060-13
15	-0.6024600-14
16	0.8326520-14
17	0.1592200-14
18	-0.792060-15
19	-0.2398650-15
20	0.861900-16
21	0.325030-16
22	-0.112530-16
23	-0.423810-17
24	0.173100-17
25	0.51960-18
26	-0.29440-18
27	-0.5320-19
28	0.51630-19
29	0.2290-20
30	-0.9760-20
31	0.940-21
32	0.1340-20

0	0.2297755376037110-02
1	0.1196374380162480-02
2	0.50504308852510-04
3	0.3288389205410-05
4	0.317460531360-06
5	0.426519732250-07
6	0.69279940430-08
7	0.10562960400-08
8	0.729577700-10
9	-0.311816670-10
10	-0.1401327030-10
11	-0.174439590-11
12	0.63464050-12
13	0.29452940-12
14	-0.219170-14
15	-0.27563970-13
16	-0.3604230-14
17	0.2538400-14
18	0.632770-15
19	-0.2557930-15
20	-0.888500-16
21	0.304290-16
22	-0.117930-16
23	-0.433860-17
24	-0.150170-17
25	0.70820-18
26	0.17260-18
27	-0.12330-18
28	-0.14060-19
29	0.21440-19
30	-0.620-21
31	-0.3490-20
32	0.680-21

COEFFICIENTS FOR $J_1(z)$ $\phi = 7.5$

r	a_{2r}	a_{2r}
0	0.33165086130654000+00	0.48593997430256700-02
1	-0.28020674473706000	0.81346848297617300-01
2	-0.34611125276606000	-0.14881387653454340-01
3	-0.16691424951145510+00	-0.57255339954541560-01
4	-0.36569355422691320-01	0.29268693615916090-01
5	-0.4150292404715910-02	-0.6392889746787470-02
6	-0.2255569652975700-03	0.8102855026397040-03
7	-0.15209710883040-05	-0.672655518128580-04
8	-0.12464984222400-05	0.38945144437760-05
9	-0.1090898095490-06	-0.1622271914540-06
10	-0.577525134550-08	0.486614740470-08
11	-0.21945746200-09	-0.99130676380-10
12	-0.6376231530-11	0.966909760-12
13	0.146080980-12	0.17858350-13
14	-0.2676510-14	-0.1107820-14
15	0.391930-16	0.305520-16
16	-0.4480-18	-0.6010-18
17	0.40-20	0.90-20

r	a_{2r}
0	0.20073833669056160+01
1	0.3896804586714230-02
2	-0.201499491862720-03
3	-0.3747373028520-05
4	-0.113052228010-06
5	-0.124315284760-07
6	-0.2557555640-09
7	-0.468658910-10
8	0.55839870-11
9	-0.1299570-12
10	-0.4493150-13
11	0.726910-14
12	-0.35190-15
13	-0.67750-16
14	0.17860-16
15	-0.1780-17
16	-0.70-19
17	0.60-19
18	-0.10-19

r	a_{2r}
0	0.46014907031529380-C1
1	0.22922962485222800-C1
2	-0.89004053563680-04
3	-0.4312553098740-05
4	0.202799081200-06
5	-0.14879394330-03
6	-0.7976922490-09
7	0.458601270-10
8	0.17262580-11
9	-0.5656160-12
10	0.4434280-13
11	0.159220-14
12	-0.87130-15
13	0.10650-15
14	-0.1310-17
15	-0.1990-17
16	0.410-18
17	-0.30-19
18	0.00+00

r	a_{2r}	a_{2r}
0	0.19952377864814500+01	-0.46707946651731300-01
1	-0.2153532661257880-02	-0.23382687042118000-01
2	0.228122745261270-03	-0.22519344502930-04
3	0.281659722750-06	0.6192748306950-05
4	-0.264631369350-06	-0.169378999960-07
5	0.33889231690-08	-0.148373458340-07
6	0.9811384530-09	0.4540024060-09
7	-0.573574450-10	0.700690870-10
8	-0.48350220-11	-0.71851410-11
9	0.8926070-12	-0.24067580-12
10	-0.1059270-13	0.10777110-12
11	-0.1211570-13	-0.648840-14
12	0.147940-14	-0.114420-14
13	0.56860-16	0.26460-15
14	-0.40010-16	-0.10600-16
15	0.4540-17	-0.4860-17
16	0.340-18	0.1070-17
17	-0.190-18	-0.40-19
18	0.20-19	-0.30-19

COEFFICIENTS FOR $J_0(z)$ $\phi = 22.5$

r	a_{2r}
0	0.40374947400739900+00
1	-0.88138108175613040-01
2	0.63942710283276390+00
3	-0.119581425663184720+00
4	-0.377336922879738070-01
5	0.10559853770136720-01
6	-0.74422535185312140-03
7	-0.119310638369710-04
8	0.426034124101040-05
9	-0.23005440407360-06
10	0.416780509440-08
11	-0.95756893910-10
12	-0.6890784020-11
13	0.155917060-12
14	-0.1155750-14
15	-0.244640-16
16	-0.8010-18
17	-0.990-20

r	a_{2r}
0	0.201879670479666750+01
1	0.9532265270458230-02
2	-0.128529678279980-03
3	-0.5295345734020-05
4	0.93607437440-07
5	0.505222401300-08
6	-0.6797943030-09
7	0.369163230-10
8	-0.2155570-12
9	-0.3119640-12
10	0.4027500-13
11	-0.255090-14
12	-0.10470-15
13	0.5630-16
14	-0.8990-17
15	0.750-18
16	-0.10-19
17	-0.20-19

r	a_{2r}
0	0.423305048484136590-01
1	-0.20983776369815850-01
2	-0.169526262391710-03
3	-0.583594140580-06
4	0.185278326760-06
5	-0.94533334110-08
6	0.10330043490-09
7	0.342047380-10
8	-0.42355360-11
9	0.2464420-12
10	0.657190-14
11	-0.374500-14
12	-0.52890-15
13	-0.3840-16
14	-0.1480-17
15	0.1010-17
16	-0.190-18
17	0.20-19

r	a_{2r}	a_{2r}
0	0.29030589498529460+00	-0.44210282533773290-01
1	0.47435068271810570+00	-0.22249103858203930-01
2	0.10276879026948940-01	-0.139618700469370-03
3	-0.26100879276189320+00	0.4605779264900-05
4	-0.58215916946595900-01	0.2536139631070-06
5	-0.6129362732768200-03	-0.96056432490-08
6	-0.8283579228032100-03	-0.12744696790-08
7	0.852169335278430-04	0.689132390-10
8	-0.278466646651020-05	0.112302410-10
9	-0.6145026115430-07	-0.11128790-11
10	0.801903143030-08	-0.12499040-12
11	-0.267330218100-09	0.2521710-13
12	0.2988029750-11	0.83530-15
13	0.66674770-13	-0.68280-15
14	-0.3098460-14	0.42750-16
15	0.506790-16	0.15300-16
16	-0.2630-18	-0.3250-17
17	-0.520-20	-0.90-19
		0.140-19
		-0.20-19
		-0.30-20
		0.10-20

COEFFICIENTS FOR $J_1(z)$ $\phi = 37.5$

r	a_{2r}	a_{2r}	a_r
0	-0.43816218312422750+00	0.23777382028651190+01	0.20287910981550130+01
1	-0.268935225231081270+00	0.25747309246619630+01	0.14430518672985720-01
2	0.14718095080942920+01	0.10389342359726320+01	0.30692228172340-04
3	-0.44144324656702810+00	-0.45257631821928400+00	-0.4093612502260-05
4	-0.93443310751054220-01	-0.93005014696467870-01	0.187724520520-06
5	-0.11893011139311780-01	0.13755046260572590-01	-0.59799627920-08
6	0.1475622233424320-02	0.9960303086135680-03	-0.130390550-10
7	-0.556796744779950-04	-0.1175855980268730-03	0.271682040-10
8	-0.71034923884040-05	-0.19530242521530-05	-0.22500090-11
9	-0.2599056514500-07	-0.33172962453400-06	0.2500940-12
10	0.1217319690450-07	-0.151840424130-08	-0.1214900-13
11	-0.12389573470-09	-0.35519485970-09	-0.60180-15
12	-0.8286130380-11	0.5130534230-11	0.25370-15
13	0.151852620-12	0.153817430-12	-0.4170-16
14	0.2213510-14	-0.3510610-14	0.4870-17
15	-0.659180-16	-0.225370-16	-0.390-18
16	-0.960-19	0.10270-17	0.00+00
17	0.130-19	-0.20-20	0.00+00
0	0.35916123548842250-C1	0.19719513753107280+01	0.38548496277263230-C1
1	-0.17750744319525280-C1	-0.13931423457017560-01	-0.19502840670597850-01
2	-0.204411000976150-03	0.100187213159350-03	-0.229343329721660-03
3	0.2940992945130-05	0.7475463352770-05	-0.441571489290-06
4	0.27131660710-07	0.161687744000-06	0.327397600030-06
5	-0.73025559230-08	-0.172838519820-07	0.173328798570-07
6	0.5739479310-09	-0.17893273240-08	-0.10375224550-08
7	-0.304872870-10	0.6818999380-10	-0.1973409460-09
8	0.5433160-12	0.237871570-10	0.48450010-11
9	-0.1401550-12	-0.39524180-12	0.314223570-11
10	-0.2631610-13	-0.45248830-12	-0.4556070-13
11	-0.301500-14	0.862290-14	-0.7029920-13
12	-0.24010-15	0.11623350-13	0.218830-14
13	0.6470-17	-0.597720-15	0.201210-14
14	0.2280-17	-0.357720-15	-0.162690-15
15	-0.610-18	0.43400-16	-0.63800-16
16	-0.100-18	0.11040-16	0.11320-16
17	-0.10-19	-0.2880-17	0.1750-17
18		-0.2130-18	-0.7120-18
19		0.1690-18	-0.20-20
20		-0.110-19	0.380-19
21		-0.80-20	-0.50-20
22		0.20-20	-0.10-20

COEFFICIENTS FOR $J_1(z)$ $\phi = 52.5$

r	Q_r	$Q_{r(2)}$
0	-0.10940927246808720+02	0.395731591179575660+01
1	-0.334912789916133340+01	0.57507937113324470+01
2	-0.11814570335990530+01	0.48815414875917930+01
3	0.943563761405352880+00	0.10841395452803470+01
4	0.25587248917750550+00	-0.55395808501553490-01
5	0.7257413889462510-02	-0.31960179317434170-01
6	-0.2432126123650240-02	-0.1727141710520250-02
7	-0.1713718668135130-03	0.114175034779520-03
8	0.26737517545940-05	0.108141508144420-04
9	0.4827760412270-06	0.4262404048660-07
10	0.685933426390-08	-0.1590186732180-07
11	-0.38959009820-09	-0.34016665430-09
12	-0.1117105400-10	0.683946530-11
13	0.7981440-13	0.27589590-12
14	0.5370900-14	0.227800-15
15	0.29990-16	-0.84010-16
16	-0.10600-17	-0.750-18
0	0.27315204652010100-01	0.19625255239727340+01
1	-0.134625491022226010-01	-0.18777882455333770-01
2	-0.190221643609530-03	-0.35879742859880-04
3	-0.4701824708420-05	0.5181658032150-05
4	-0.1275582993630-06	0.436262518410-06
5	0.23409625690-08	0.182827680400-07
6	0.1176696030-09	-0.13532528080-08
7	-0.233152030-10	-0.3527791160-09
8	-0.25593880-11	-0.152736910-10
9	-0.2357530-12	0.57491890-11
10	-0.1394790-13	0.86190190-12
11	-0.120960-14	-0.11284240-12
12	0.2930-16	-0.3566040-13
13	-0.9300-17	0.315680-14
14	-0.2470-17	0.1609410-14
15	0.430-18	-0.150480-15
16	-0.60-19	-0.81300-16
17		0.11210-16
18		0.43250-17
19		-0.10070-17
20		-0.2100-18
21		0.930-19
22		0.50-20
23		-0.80-20
24		0.70-21
25		0.60-21
26		-0.20-21
		0.10-21

COEFFICIENTS FOR $J_1(z)$ $\phi = 67.5$

r	a_{2r}	a_{2r}
0	-0.21062112152594560+02	-0.2754319170190717D+02
1	-0.19618062895952710+02	-0.17287972760804510+02
2	-0.10451550397097540+02	-0.30917831418917030+01
3	-0.26466325973531990+01	-0.73004478697562500+00
4	-0.28523013984809640+00	0.35508311729530770+00
5	-0.534326718244450-03	0.5311808646305370-01
6	0.2891222119708890-02	0.354586987750110-02
7	0.29764093920420-03	0.52218049920170-04
8	0.13417240674840-04	-0.8282790903710-05
9	0.184524261950-06	-0.639648070560-06
10	-0.10375428460-07	-0.20744303350-07
11	-0.6352672680-09	-0.2364774360-09
12	-0.155251730-10	0.65476190-11
13	-0.1444680-12	0.3302660-12
14	0.230060-14	0.629950-14
15	0.9830-16	0.4820-16
16	0.150-17	-0.480-18

r	a_{2r}	a_{2r}
0	0.20423826622466760+01	0.17066779775250200-C1
1	-0.2104112998144451510-01	-0.939572226168891160-02
2	-0.1479384141222670-03	-0.1333786196664030-03
3	-0.223354211341140-05	-0.4113361123700-05
4	-0.23857238970-07	-0.166875473370-06
5	-0.11571646580-08	-0.79696703010-08
6	-0.2029299560-09	-0.4203694940-09
7	-0.2142447020-10	-0.231600730-10
8	-0.20773550-11	-0.12325050-11
9	-0.1986540-12	0.515400-13
10	-0.1905710-13	0.40450-15
11	-0.133290-14	-0.54750-15
12	-0.17480-15	0.10810-15
13	-0.1510-16	-0.1700-16
14	-0.1370-17	-0.2480-17
15	-0.90-19	-0.350-18
16	0.00+00	0.50-19

r	a_{2r}	a_{2r}
0	0.19556772162547030+01	-0.19048970227279180-01
1	-0.223344150554248140-01	-0.97222608982105830-02
2	-0.174643454392940-03	-0.2066532519784060-03
3	-0.1721000111960-05	-0.8867888141350-05
4	0.197419676050-06	-0.481234958440-06
5	0.379708036950-07	-0.212496656200-07
6	0.46292825530-08	0.125556136910-08
7	0.2673534090-09	0.5923515610-09
8	-0.548571930-10	0.8733373300-10
9	-0.184099350-10	-0.1243473360-11
10	-0.122274100-11	-0.341875940-11
11	0.59921160-12	-0.48623400-12
12	0.14279090-12	0.10087750-12
13	-0.16201440-13	0.38547460-13
14	-0.10199410-13	-0.2403900-14
15	0.302100-15	-0.2719030-14
16	0.740260-15	0.23020-15
17	0.2270-17	0.207110-15
18	-0.59640-16	0.1110-17
19	0.930-19	-0.176450-16
20	0.53400-17	0.3120-19
21	-0.2290-18	-0.16430-17
22	-0.5100-18	-0.1270-18
23	0.6130-19	-0.1530-18
24	0.4830-19	0.2790-19
25	-0.1200-19	0.1430-19
26	-0.400-20	-0.490-20
27	0.190-20	-0.990-21
28	0.130-21	0.740-21
29	-0.270-21	-0.00+00
30	0.30-22	-0.90-22
31	0.30-22	0.20-22
32	-0.10-22	0.90-23

COEFFICIENTS FOR $J_0(z)$ $\phi = 82.5$

r	a_r	b_r	c_r
0	0.437056187250519550+02	0.5803192092471250-02	0.19520627426271020+01
1	0.28604219418342190+02	0.2852078483063201-02	-0.24225147172738180-01
2	0.91755271882928420+01	-0.47825105695510-04	-0.264658076853590-03
3	0.15480330031503700+01	0.1613558945220-05	-0.85728657339980-05
4	0.12391565686220680+00	-0.74379989530-07	-0.463106235490-06
5	-0.1262998345261220-02	0.42167215270-08	-0.311621159520-07
6	-0.1425105567103680-02	-0.2791145760-09	-0.90663363300-09
7	-0.174407541264000-03	0.208996930-10	0.6315578270-09
8	-0.127185649761980-04	-0.17327240-11	0.2562668670-09
9	-0.6569688799650-06	-0.1566020-12	0.5214665660-10
10	-0.256793023330-07	-0.1524830-13	0.501933300-12
11	-0.78638698370-09	-0.158480-14	-0.341039450-11
12	-0.1924753230-10	-0.17450-15	-0.88271320-12
13	-0.38019760-12	-0.2020-16	0.87439250-13
14	-0.6052780-14	-0.2460-17	0.95122340-13
15	-0.76120-16	-0.310-18	0.7306580-14
16	-0.700-18	-0.40-19	-0.8583340-14
			-0.1749550-14
			0.803520-15
			0.257900-15
			-0.865610-16
			-0.346680-16
			0.112760-16
			0.451520-17
			-0.174190-17
			-0.55710-18
			0.29840-18
			0.5940-19
			-0.5270-19
			-0.2950-20
			0.9020-20
			-0.870-21
			-0.1390-20
			-0.670-21
			-0.6535360034816340-02
			-0.3343163263671980-02
			-0.79494560674590-04
			-0.43542583232690-05
			-0.385517393220-06
			-0.493082535220-07
			-0.78157967890-08
			-0.878275460-10
			0.319589170-10
			0.1501915320-10
			0.198030780-11
			-0.642248360-12
			-0.30378240-12
			-0.984770-15
			0.28693530-13
			-0.4063590-14
			-0.2594250-14
			-0.685960-15
			0.257960-15
			-0.950400-16
			-0.304960-16
			-0.125600-16
			0.435430-17
			0.160320-17
			-0.71500-18
			-0.18650-18
			0.12540-18
			0.1590-19
			-0.2200-19
			0.400-21
			-0.3610-20
			-0.670-21

APPENDIX B

Tables of Coefficients for the Chebyshev expansions

For $|z| \leq 8$

C:
$$J_n(z) = \frac{1}{n!} \left(\frac{z}{2}\right)^n \sum_{r=0}^{\infty} a_{nr} T_{nr}(t) \quad n=0,1,\dots,10$$

D:
$$Y_0(z) = \frac{2}{\pi} \log_e \left(\frac{z}{2}\right) J_0(z) + \sum_{r=0}^{\infty} b_{0r} T_{2r}(t)$$

and

E:
$$Y_1(z) = \frac{2}{\pi} \left(\log_e \left(\frac{z}{2}\right) J_1(z) - \frac{1}{z} \right) + \frac{z}{2} \sum_{r=0}^{\infty} b_{1r} T_{2r}(t),$$

with

$$t = \left(\frac{z}{8}\right) \text{Exp}(-i\phi) \quad \phi = 7.5^\circ(15^\circ)82.5^\circ$$

COEFFICIENTS FOR $J_N(Z)$, $N = 0$

$\text{ARG}(\phi) = 7.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.40795869055667137640715028153693Q+00	-0.56442646929690935200795128274590Q-01
0.76307829250131400361146778095084Q-01	-0.61302046672716605147332674516919Q-01
0.29061006595818788085586515391498Q+00	-0.20911435561449686363680202686931Q+00
-0.41389519311709364289690627937354Q+00	-0.49959512745050736116971017573850Q-01
0.14405373906065264429620605762981Q+00	0.91255766225367239948363268407435Q-01
-0.22191868433940324530320446112275Q-01	-0.29516142867063306110340315867003Q-01
0.15821011678673097559220170114149Q-02	0.47897681004629969864723074241209Q-02
-0.59234157455037023995012675293566Q-05	-0.47863710713778006789287843649731Q-03
-0.93656954962602142941518489788133Q-05	0.32225706258653841087281828118680Q-04
0.98007182750822194752002275702969Q-06	-0.15271062036972918272686508255232Q-05
-0.58849005869019316334881884219035Q-07	0.51370384201749153937317288042167Q-07
0.24841843116601105419586447412207Q-08	-0.11659304865213573041154924553111Q-08
-0.79166685333981694744509040991190Q-10	0.13002722707254774191498280825128Q-10
0.19713528931142066795752587275933Q-11	0.21989564315413682332706849016329Q-12
-0.38986101629433760169040062140733Q-13	-0.15719635840571344149732102786680Q-13
0.61288305780992463777428485624698Q-15	0.46989454558630322083751642792260Q-15
-0.74872296145083913456919480921153Q-17	-0.99111434114507891007149844922754Q-17
0.65125722022767106932468459658214Q-19	0.16392989156702043206417304659115Q-18
-0.24484306038263621355486069536909Q-21	-0.22147117906606840778028263028522Q-20
-0.39482281695962560448896899872804Q-23	0.24892188592384991764317178926069Q-22
0.10567641477407692569908964539183Q-24	-0.23401714881381196783886748810376Q-24
-0.15087853069717455406447080694267Q-26	0.18260278516002535548651669668619Q-26
0.16246331754032631900049937714552Q-28	-0.11458225051944390195654405947447Q-28
-0.14329199354165747403873789530818Q-30	0.52084078228743871177741921848437Q-31
0.10689014912284315842898385395235Q-32	-0.89267514828895868537298747317694Q-34
-0.68331426076520037960310045221448Q-35	-0.12555686975918394598649293578940Q-35
0.37492633224211049946541724110212Q-37	0.17950167586171787447862828877555Q-37
-0.17459403685946996851518989893939Q-39	-0.15040125647625372237119683989617Q-39
0.66565776420248262321144030300722Q-42	0.98283442944539753342027380529403Q-42
-0.18518818117174614520531911760205Q-44	-0.53871044191808775224137666081112Q-44
0.17106286956069944153166782398420Q-47	0.25472928825176313443730862834340Q-46

COEFFICIENTS FOR $J_N(Z)$, $N = 0$

$\text{ARG}(\phi) = 22.5$ DEGREES

REAL PART

IMAGINARY PART

0.18623881365254941206391184280746Q+01
 0.14586386625180950972545797220381Q+01
 0.99548777603655731948538246885643Q+00
 -0.6836306791980398818387983576158EQ+00
 -0.92103894031436975124900736670351Q-01
 0.5432515528670649700970670591326EQ-01
 -0.53126997414577367734725624978496Q-02
 -0.31338728717415638055222588885682Q-04
 0.34287393291279206654024854743375Q-04
 -0.22152426452254745284258733781922Q-05
 0.47791149661373039954614575852049Q-07
 0.9877246842914601155495190784251EQ-09
 -0.8555941841366383530592303102659EQ-10
 0.21497692022316954524086941727095Q-11
 -0.18044845865653460488125005161975Q-13
 -0.36611959638375363485072844099656Q-15
 0.13282565182822653728060423561167Q-16
 -0.1769798509974373184622959393144EQ-18
 0.73565583861815498696056484311028Q-21
 0.13439277020255245432805184079622Q-22
 -0.27343095881597148765216409975695Q-24
 0.22524555302524473310456212866351Q-26
 -0.53291005516586209617603948448949Q-29
 -0.86754651739648244417089788877725Q-31
 0.11362604863590713575985169424467Q-32
 -0.63512437869492818751052544852765Q-35
 0.9375573799748919097407679760512EQ-38
 0.13661645456438597285667311109735Q-39
 -0.12512496648563425010552281402351Q-41
 0.50528791030627037833593076928280Q-44
 -0.49692211992199977997653744395681Q-47

0.16486122859825242144718307919130Q+00
 -0.12544466638704219695403027896923Q+00
 -0.10844229108502024582959366154897Q+01
 -0.59323596240726279387806001876372Q+00
 0.27615145780052247518564222799368Q+00
 -0.12476381839349755533666766352444Q-01
 -0.46544745856727888381187414478990Q-02
 0.63622927278262264692972383255147Q-03
 -0.26013530174549737200216869491038Q-04
 -0.45992988672321494631582857389451Q-06
 0.82401441947674595322343130211135Q-07
 -0.31346016042881186165079671391767Q-08
 0.40492685731528486918403002058275Q-10
 0.84820179582633413317487408538878Q-12
 -0.44965043319601150272937439932755Q-13
 0.80228194969170118787213386821454Q-15
 -0.46368682453174704019240608040720Q-17
 -0.89566498342508217565158876682037Q-19
 0.23784602955667591021187110468363Q-20
 -0.24563383016281616649391002845311Q-22
 0.75979310571844082254091676766722Q-25
 0.13101506878535092988935606352495Q-26
 -0.21115269624392155974828639121683Q-28
 0.14195300876382134298811178252330Q-30
 -0.26277104068594085415779274347173Q-33
 -0.40436324862400408223202596401835Q-35
 0.43922566481873963909743193811676Q-37
 -0.20731565753726878313488826244435Q-39
 0.24804524270781308073626942673081Q-42
 0.34291931359358081498946228055910Q-44
 -0.26850205932367154274118374713516Q-46

COEFFICIENTS FOR $J_N(Z)$, $N = 0$ ARG(ϕ) = 37.5 DEGREES

REAL PART

IMAGINARY PART

0.67119273258906999397417562680239C+01
0.71500895090149274249562302933488Q+01
0.62159845055422611610087734646016C+01
0.91766948812533030008971126548282Q+00
-0.56777356588961925012223189451426Q+00
-0.5298496757710102865861144078445C-01
0.1099498809164804914674566909285C-01
0.33155441080985683550871115114717Q-03
-0.64158538295280157832171771619448C-04
-0.1926227848611983582288755909195C-07
0.13098460082289649995221225867198C-06
-0.1681607779631104118911277891405C-08
-0.10371152882321160186361693197984C-09
0.21859346943753071283032864827525C-11
0.31222454779154038507531746800889C-13
-0.10573078967372922849271977707796C-14
-0.11756726790979670550122091559254C-17
0.23918281026749472894061443142955C-18
-0.11947502045502751672080653424766C-20
-0.26729840697905951875410589489261C-22
0.26247196204098981414737826408936C-24
0.12897963883569176515585530211121Q-26
-0.25035651838797726597574836978744Q-28
0.11734694565831946831155271368519Q-31
0.12832433620734723751359383046365C-32
-0.45903276163115953174237946205508C-35
-0.35382723873725811024606758356275C-37
0.24657413518293437964736301762449C-39
0.36686604181049271304777475605924Q-42
-0.68854303543706907044843314712174Q-44
0.75736308133371409863328094870638Q-47

0.83712702818187066160128862554041Q+01
0.59935320789535874704309890908769Q+01
-0.19486400634296973199162619237438Q+00
-0.24065140778222976905395753482710Q+01
-0.31719376085575826201011612499029Q+00
0.92811127584290710768303757704639Q-01
0.53006984777873094100192858823990Q-02
-0.96600009434642741994234559722918Q-03
-0.11644140366370472243416450105202Q-04
0.32792409067351769904708187461371Q-05
-0.24389700433340074222958066472352Q-07
-0.41353051818623120210689188092807Q-08
0.70597257863178096934493061241108Q-10
0.20545036410632514002637938671880Q-11
-0.53302500363141758467434181670466Q-13
-0.32781289949432533628380676339244Q-15
0.17379430376588814964417894251052Q-16
-0.40755209566834685198756324933495Q-19
-0.27655685125209732758600102490881Q-20
0.20282438263448503305629880293553Q-22
0.21156346934994648642359041179737Q-24
-0.27869637139173354839687197440611Q-26
-0.47370861045896446416853047963321Q-29
0.19300680330736498144511520864952Q-30
-0.41046820113458423420150726959960Q-33
-0.73307874882075810998447859481484Q-35
0.37181894194269406039102168670998Q-37
0.13762270701873969701694544929432Q-39
-0.13969378383478819261520215595425Q-41
-0.27092262682857062761565080270755Q-46
0.29719079859615113795104140798596Q-46

COEFFICIENTS FOR $J_N(Z)$, $N = 0$ ARG(ϕ) = 52.5 DEGREES

REAL PART

IMAGINARY PART

-0.2461560768007209794640941478399EQ+02
-0.13674680433263381816930047837609Q+02
0.42050323986583392854024267002657Q+01
0.58622759766447914134576827477783Q+01
0.13204034634908637782731463139533Q+01
0.12011362049486119603651628069050Q-01
-0.19372490663037713599494764213266Q-01
-0.13193765886752324090956713249527Q-02
0.32882094123650831394692136588888Q-04
0.49561200024879495862816530660429Q-05
0.66621081843728297217183148014726Q-07
-0.49411940775348074211939481683830Q-08
-0.14323130562185905266344553318864Q-09
0.11426229129018497661619105639929Q-11
0.81399052533547393688503984541264Q-13
0.42963581930446140337461610709026Q-15
-0.18413047890098872367751566818597Q-16
-0.23556437667867229352963180294575Q-18
0.14780223743818327032278009370160Q-20
0.40956734504856370170797264766348Q-22
0.63981925565223277240096031917410Q-25
-0.33618720134599604516805907738920Q-26
-0.19927812351128876188982915416726Q-28
0.12653928079114571953761923102439Q-30
0.15148831459979218067536099487641Q-32
-0.35457362648932105717005203942104Q-36
-0.58150487026493486555445415501452Q-37
-0.15975891547896140605983712801556Q-39
0.11831408057472102297251922652125Q-41
0.68633517028189059100077279083896Q-44
-0.86383280198037442732074842918968Q-47

0.40708184519741684906894024764506Q+02
0.36750868601784119090822057617668Q+02
0.20367739691527432297883143363871Q+02
0.35548361381913597009511675099555Q+01
-0.61553461742536762917630456224676Q+00
-0.21071485764786350942543741438291Q+00
-0.10318073190657241386059601834323Q-01
0.10868685743074080331045584156917Q-02
0.97769240370192918375983719587556Q-04
0.10089071632680748236592179984534Q-06
-0.18223288264254629134116750106942Q-06
-0.39106755594795768418695956250335Q-08
0.96090174415940859625911547790801Q-10
0.38663761072019751013073789910377Q-11
0.87295719916708646689766836918206Q-15
-0.13679352041698177218734837566741Q-14
-0.12441337372666640599612697442972Q-16
0.19433016360925339500623608205876Q-18
0.34422016606709071729170561679391Q-20
-0.50882962561403372040902485809500Q-23
-0.40558120586652640015269127140345Q-24
-0.15617312296875415002587281808420Q-26
0.23097458034639807215285458498273Q-28
0.19179314075437628141322658676683Q-30
-0.48291887511308268323169572633154Q-33
-0.10136460884656706702109567689352Q-34
-0.14094518049251148437476781919616Q-37
0.28576217592464116638248112392485Q-39
0.11729079678680638855149063611130Q-41
-0.39214101947560129170821841454352Q-44
-0.33970818794065285978908534066290Q-46

COEFFICIENTS FOR $J_N(Z)$, $N=0$

$\text{ARG}(\phi) = 67.5$ DEGREES

REAL PART

IMAGINARY PART

-0.12023234004003584999818886107656Q+03
 -0.99170227887441285329135194491064Q+02
 -0.49482551698438323677964812571407Q+02
 -0.12834635332439317353249812196134Q+02
 -0.13924145634822307560273601121626Q+01
 0.35873442631228915638088468993558Q-01
 0.24623296955251348484432877021334Q-01
 0.25989029404040905193856549537677Q-02
 0.12385474204657920414516265382266Q-03
 0.16233818774304976462790471703260Q-05
 -0.12848188590205171272287441105036Q-06
 -0.79995001109688431240715126441317Q-08
 -0.20551599036084747083254899649849Q-09
 -0.19551294992334729981372286345167Q-11
 0.37507349404640345799651797934362Q-13
 0.16214600690405924996084918798866Q-14
 0.25894198253276515890314860610469Q-16
 0.16963885835163051058784661923728Q-18
 -0.13151889996350858678670178161917Q-20
 -0.43155758831589959275961716074911Q-22
 -0.46903719427128928309181255754079Q-24
 -0.22018698090879407279084234728340Q-26
 0.86281155469266477971682953927907Q-29
 0.22713070854464166764224304073967Q-30
 0.17873486006162661044327487865169Q-32
 0.62652933706147687905142320609119Q-35
 -0.14150553808635658780091448512327Q-37
 -0.31024687126385932720181561031139Q-39
 -0.18487614673649109979228641554924Q-41
 -0.50084778511551406989439808273949Q-44
 0.71176572747691937093496026753804Q-47

-0.66691018233263761381657206590178Q+02
 -0.39147826531356588336958685234818Q+02
 -0.14800978678558559364104741152456Q+01
 0.63351619838680025091559421924811Q+01
 0.23497334840795793775590047916078Q+01
 0.36184555175082579591418906704027Q+00
 0.25407683584997795664451288720100Q-01
 0.27328614896885497885059913797181Q-03
 -0.86478706066528572035583131906616Q-04
 -0.68164033686269323775951271638189Q-05
 -0.23322393287194919485736536781241Q-06
 -0.26718216601645247528390463650696Q-08
 0.93883769053122440787827556762722Q-10
 0.48015363459005741396881230966287Q-11
 0.95920761478909471667874560717069Q-13
 0.75440387851236339973278181017069Q-15
 -0.88611990878570833706219473950495Q-17
 -0.33091881056889251909482520124499Q-18
 -0.43198897528320881209066285578019Q-20
 -0.23825721607151014874315892698848Q-22
 0.12866821474808433187918726739619Q-24
 0.37606144136947100549130543087069Q-26
 0.34553644783555857374984659001160Q-28
 0.13950524503754715029219540715481Q-30
 -0.40963510166528041796508686348469Q-33
 -0.98027781360612562544596100114940Q-35
 -0.66805156657504333366626559124035Q-37
 -0.20507272069832033063755063045718Q-39
 0.36405841380286782369569079512925Q-42
 0.73617520664380580417787332142104Q-44
 0.38683765499870789188474037233299Q-46

COEFFICIENTS FOR $J_N(Z)$, $N = 0$

$\text{ARG}(\phi) = 82.5$ DEGREES

REAL PART

IMAGINARY PART

0.14786330077903399910834413043463E+03
 0.10415768205398205162399654475226E+03
 0.37773716029004829631634647285402E+02
 0.7151068457825563564755242774090E+01
 0.58718947463746490892023107893163E+00
 -0.27336816292524014223244975541634E-01
 -0.12633298385438303996328684848139E-01
 -0.16111141681285472669446158410552E-02
 -0.12682388318315677499352037464151E-03
 -0.71010473576556223110269068237272E-05
 -0.30024483700336217893916277191359E-06
 -0.99086415014185669218051899910643E-08
 -0.26027484117930187788192176623065E-09
 -0.54945502031913302315248827628845E-11
 -0.93097909164864721426484987076695E-13
 -0.12405514383185900794313961451409E-14
 -0.12100809941441867001603700742549E-16
 -0.61402516926354081139078013313073E-19
 0.54121172480611844676219493583295E-21
 0.19452840346739931358323830445185E-22
 0.31204514306599118157275391121056E-24
 0.36820345317120975520032057649888E-26
 0.35315464449440018810041928637612E-28
 0.28580566388241294610135183367357E-30
 0.19836067158649677864126489289905E-32
 0.11863994266786195581653997529741E-34
 0.6081555768467540007134523714420E-37
 0.2614092338201152310028094680528E-39
 0.88434651555956693190035115231847E-42
 0.18295586081000214890887670576021E-44
 -0.2794431546987037207418742391281E-47

-0.18685812523542100076518581169684E+03
 -0.14493435376763316555403379954426E+03
 -0.68100726343676867881843240961907E+02
 -0.19942171041630688504315865053873E+02
 -0.37945991286273713731645640517225E+01
 -0.48944600177602346303981707099740E+00
 -0.44344116850055764209395580208655E-01
 -0.28944488875185423952758626824804E-02
 -0.13765284525622497163004271726241E-03
 -0.47052276012432276254645764427057E-05
 -0.10641119963921452180967194363728E-06
 -0.92761222837942349840197804571025E-09
 0.40940074636336826992996865653823E-10
 0.23689419516543360236253155310313E-11
 0.72175746483862135156193005583481E-13
 0.16057793958388156658720457530673E-14
 0.28359465561914272852397225563504E-16
 0.41173060997307404969131501659123E-18
 0.49969014268740635137936784721376E-20
 0.51000584521037047207876926762789E-22
 0.43581643056407606329070172519971E-24
 0.30512590903047769797613238757884E-26
 0.16376144713659745783339920962069E-28
 0.50849901113658300951720520451353E-31
 -0.16369074613715444505432167096866E-33
 -0.42101148754094705704577267314263E-35
 -0.41597029420894998273193569008155E-37
 -0.30302663845261326002874852819372E-39
 -0.18190421665068652999707007238636E-41
 -0.93583279237706162808161717143004E-44
 -0.41938207208324406336812682175833E-46

COEFFICIENTS FOR $J_N(Z)$, $N = 1$

$\text{ARG}(\phi) = 7.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.33165086130653997604600350344184Q+00	0.48593997430256699465375462423286Q-
-0.28020674473706018265923536641445Q+00	0.81346848297617297782801638913665Q-
0.34611125276606388482375235700901Q+00	-0.14881387653454337076133925474944Q-
-0.16691424961145506535924281886925Q+00	-0.57255339954541558145714616435313Q-
0.36569355422691324747952979491212Q-01	0.29268693615916085505841393253322Q-
-0.41502924047159137283095984284926Q-02	-0.63928897467874703676258454980431Q-
0.22555696529756961672302548945317Q-03	0.81028550263970382178848993942696Q-
0.15209710883043568767233087953378Q-05	-0.67265551812858283025707285895073Q-
-0.12464984222397633027336984161773Q-05	0.38945144437760507464997175063693Q-
0.10908980954867918536642675121908Q-06	-0.16222719145416547549337232684003Q-
-0.57752513455056210899292078824517Q-08	0.48661474047348637475283503149842Q-
0.21945746198824051123807577467122Q-09	-0.99130676381298998137907296734863Q-
-0.63762315331630816503414103493347Q-11	0.96680975881599084980880224383144Q-
0.14608097533534666919616956854719Q-12	0.17858349957729693388065263762958Q-
-0.26765123039394560770827053789744Q-14	-0.11078169548851953812394692101090Q-
0.39192847989154024959016691973908Q-16	0.30551709794305953430172781216620Q-
-0.44768536013542468161935114956219Q-18	-0.60144451407608299630220580340237Q-
0.36434866106913209315345619239843Q-20	0.93427147672250877723963199358518Q-
-0.12481994664354918917576134007972Q-22	-0.11907426190414492712693446772511Q-
-0.20760742168313878200135884612938Q-24	0.12670553847602525469228558332698Q-
0.51707289101285615493893310481711Q-26	-0.11310629158223537173740539538092Q-
-0.70005351582556064872079927516915Q-28	0.84001782864102962740933761465353Q-
0.71838552811691688766668563004057Q-30	-0.50253077895558263046292282992253Q-
-0.60565181415817746475042673214826Q-32	0.21772175366030197188671463146932Q-
0.43284363698986344220638342081829Q-34	-0.34728319521611394144474329079696Q-
-0.26560244327395905958704200000379Q-36	-0.49631138636480988176999802523068Q-
0.14011313698688281314218301537436Q-38	0.67562957481977781156810715312794Q-
-0.62815278995322935897115439202192Q-41	-0.54396029802292208501872083462563Q-
0.23077280119695161982292241620196Q-43	0.34251575582187551874989629396549Q-
-0.61838480605700457963625457230486Q-46	-0.18121992924502119400041716420578Q-
0.53861301998113532422615953596505Q-49	0.82829411533041324672473983544281Q-

COEFFICIENTS FOR $J_N(Z)$, $N = 1$

$\text{ARG}(\phi) = 22.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.40374947400739902338453870603654Q+00	0.29030589498529461840121335816053Q+
-0.88138108175613036557239756955584Q-01	0.47435068271810566021069674229339Q+
0.63942710283276385088367676709279Q+00	0.10276879026948940920512851933717Q-
-0.11958142663184717566342473740303Q+00	-0.26100879276189324358970757626359Q+
-0.377336928797380680206622377779EQ-01	0.58215916946595901765042686098983Q-
0.10559853770136716991430919076343Q-01	-0.61293627327681952647610903719922Q-
-0.7442535185312136330812492093884EQ-03	-0.82885792280320990733827413627010Q-
-0.11931063836970950641432573874115Q-04	0.85216933527842977719908945573381Q-
0.42603412410103943460304496234092Q-05	-0.27846664665102053756186946492202Q-
-0.23005440407360849147791481113895Q-06	-0.61450261154334234166598634270232Q-
0.41678050943597051554066841494641Q-08	0.80190314302725333364238487329655Q-
0.9575689391248231295915311295187EQ-10	-0.26733021810351527618112387123096Q-
-0.6890784022242164059490476906431EQ-11	0.29880297475032918861014830487766Q-
0.15591705610438350125370621535845Q-12	0.66674768179946522159070694899207Q-
-0.11557490280015465119282951651162EQ-14	-0.30984550287166174445521694821064Q-
-0.2446363208903636959641780315991EQ-16	0.50678582961172157676640602187596Q-
0.80106345720037499759306573339584Q-18	-0.26256729611572430347944706552821Q-
-0.9908358661642744534962372952967EQ-20	-0.52286070629780640755188633381542Q-
0.37349945500819304709776999755489Q-22	0.12764520561942215835658543281793Q-
0.69819983790738622455537020424396Q-24	-0.12352904547766827781678429921744Q-
-0.1322572521455303003421346871056EQ-25	0.34971983515892016690402800742190Q-
0.10284075796948478676365609390929Q-27	0.61399725030866895267611728643793Q-
-0.22438307259806460919297069164112Q-30	-0.92990443815096385805089313296586Q-
-0.37061468256130320865899612850662Q-32	0.59354292870996098797177797276729Q-
0.45939634291541491455250653105002Q-34	-0.10196655907971881345962530320819Q-
-0.2449636309381936211518387788789EQ-36	-0.15879210418349082026705107546195Q-
0.33740174576013422980584343265537Q-39	0.16415862419538616285330051206266Q-
0.49657772946503430674320756815624Q-41	-0.74210151285781260581944685691338Q-
-0.43483704480270018161843698004638Q-43	0.83229933411456819933551135062198Q-
0.1687459274437614948171355073791EQ-45	0.11604438138044624210987543731646Q-
-0.15618962244555390879283290021941Q-48	-0.87200343259351583198351514446403Q-

COEFFICIENTS FOR $J_N(Z)$, $N = 1$

$\text{ARG}(\phi) = 37.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

-0.43816218312422748521447402532488Q+00	0.23777382028651191455818971645272Q+
-0.26885225231081266002910940730051Q+00	0.25747309246619626231005266475263Q+
0.14718095080942915840056756433482Q+01	0.10389342359726319562215619881985Q+
0.44144324656702805445101763453712Q+00	-0.45257631821928395505836840434952Q+
-0.93443310751054222530709126050471Q-01	-0.93005014696467869431995201960880Q-
-0.11893011139311777202518705869361Q-01	0.13755046260572593595389031777369Q-
0.14756222334243170819510250677735Q-02	0.99603030861356795719005623160517Q-
0.55679674477995456263297077881371Q-04	-0.11758559802687314506146035693468Q-
-0.71034923884038081093562428977654Q-05	-0.19530242521531124581780416637941Q-
-0.25980565144970626517837247558285Q-07	0.33172962453404536121438826535092Q-
0.12173196904521103267030295313217Q-07	-0.15184042413262830500117600999741Q-
-0.12389573472045318662097118819965Q-09	-0.35519485968886486517597865633557Q-
-0.82861303784049343269097000935301Q-11	0.51305342309898421916665459550042Q-
0.15185261680643643815882356277484Q-12	0.15381743493724142021368082990552Q-
0.22135080235417658132480659310637Q-14	-0.35106113611528249431002746014444Q-
-0.65918405516796886161574599591960Q-16	-0.22536931882504213189018277529083Q-
-0.95849052629609285799464543488336Q-19	0.10265720166102173184045188547896Q-
0.13411150333618504664879603246672Q-19	-0.19711685136277977037606222148366Q-
-0.60185031358122344706491862827886Q-22	-0.14756516523607960324735969513814Q-
-0.13613760026691481060727421809714Q-23	0.99373270561614902967744879127925Q-
0.12379336496632654310361887688322Q-25	0.10327404303977696087227055587926Q-
0.60804961167357798911003480672852Q-28	-0.12625286598338979621045179025075Q-
-0.10893859755312475183785103196152Q-29	-0.22207505610334869986166922610714Q-
0.38595017523338083180659550172736Q-33	0.80742446581822942585441907037466Q-
0.51680304267710195333088740105443Q-34	-0.15852186429369414162776563947638Q-
-0.17391320845712136974401680745915Q-36	-0.28469803800588745967701604002047Q-
-0.13280589021085557221900347302659Q-38	0.13671842300489378287411569485097Q-
0.87804749239330421106467824338385Q-41	0.50114336895901910028456004593304Q-
0.13110047074026216155753137812912Q-43	-0.48166609312509968021457111384547Q-
-0.22999617792488802730649675157149Q-45	-0.28241436574998738215630110995361Q-
0.23728801940044736980131953242610Q-48	0.96251610974220821000762454885176Q-

COEFFICIENTS FOR $J_N(Z)$, $N = 1$

$\text{ARG}(\phi) = 52.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

-0.10940927246808716129479366946385Q+02
 -0.83491278991613342749254616283719Q+01
 -0.11814570335990525524815512811297Q+01
 0.94356376140535276469469889793886Q+00
 0.25587248917750552546858028068995Q+00
 0.72574138884625078316483206086081Q-02
 -0.24321261236502427573487062738691Q-02
 -0.17137186813513031482635816187172Q-03
 0.26737517545941825886431190617516Q-05
 0.48277604122690481382780149588591Q-06
 0.68598342639081278651616621298922Q-08
 -0.38959009817263018784011872457180Q-09
 -0.11171053985580555859007085484820Q-10
 0.70814439815526978123722295836021Q-13
 0.53709002269942060570226490575051Q-14
 0.28993807809274387842116612256904Q-16
 -0.10568038502553286487665414801138Q-17
 -0.13238628692475815324420615857334Q-19
 0.73701024206502918041851855839560Q-22
 0.20414544418567161256565784054564Q-23
 0.33507232714118094960842434369708Q-26
 -0.15106955604823808629962447425567Q-27
 -0.87685562131398461427949642145405Q-30
 0.51512571955283736321765020770391Q-32
 0.60618476194210024288242227247953Q-34
 -0.93410512849783081364866831373596Q-38
 -0.21422322421356623626381144939564Q-38
 -0.57868297442938983906341167927955Q-41
 0.40339872426686616599738885834214Q-43
 0.22932691680283722938417314685591Q-45
 -0.26956804297955956014686251711005Q-48

0.39573159179575658160719671468391Q+0
 0.57507937113324469463743150446817Q+0
 0.48815414875917929001902841644347Q+0
 0.10841395452803469481805616803058Q+0
 -0.55395808501553487531591549673767Q-0
 -0.31960179317434170523227349873777Q-0
 -0.17271417105202489800027426611846Q-0
 0.11417503477951557347583972971305Q-0
 0.10814150814441503845979551199974Q-0
 0.42624040486572034975229623493027Q-0
 -0.15901867321818508449467437112872Q-0
 -0.34016665430631215551560972027989Q-0
 0.68394652527480149719826791395192Q-1
 0.27589583525122415740244499883531Q-1
 0.22780434505152393894241069031530Q-1
 -0.84012340888282492526072016348504Q-1
 -0.75309417230874672378125317700421Q-1
 0.10433337060800019353898155252635Q-1
 0.18164087377541996842071949078884Q-2
 -0.21592471144066054847355047044603Q-2
 -0.19169516918978938656134100033092Q-2
 -0.72980966914057343680822126384247Q-2
 0.98821830414667873828919719587461Q-3
 0.80292681141736342373043821086972Q-3
 -0.18538048290341634732473396329818Q-3
 -0.38880029041187950907296876653479Q-3
 -0.54235263594353271099611910110661Q-3
 0.10124651742292061270613564859295Q-4
 0.40704413799829145177274129758802Q-4
 -0.12852221900804684674861491953276Q-4
 -0.10956139903635251256837250133292Q-4

COEFFICIENTS FOR $J_N(Z)$, $N = 1$

$\text{ARG}(\phi) = 67.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

-0.21062112152594564669053666585503Q+02
 -0.19618062895952712895442082340300Q+02
 -0.10451550397097535860286217239061Q+02
 -0.26466325873531993719281743290462Q+01
 -0.28523013984809636983658492259600Q+00
 -0.53432671824445562063099204406652Q-03
 0.28912211970888601754883088473253Q-02
 0.29764093920420612277151002271515Q-03
 0.13417240674837476152630302603391Q-04
 0.18452426195142764065136040272766Q-06
 -0.10375428464636338612409160655577Q-07
 -0.63526726800831448623021311948901Q-09
 -0.15525173137111074406946019433247Q-10
 -0.14446750659750254534279293598156Q-12
 0.23006341328401762201708208685077Q-14
 0.98312667356936428391499501918831Q-16
 0.15042128717555419636123325716617Q-17
 0.95587859425419998948350170615573Q-20
 -0.64947036146494214783333532620223Q-22
 -0.21133085566725459837621655740407Q-23
 -0.2213423704567106934876800192415EQ-25
 -0.10081732515544430198880468303404Q-27
 0.35634422785272704991005992961784Q-30
 0.93212544440583627997579077655326Q-32
 0.71011012461943232331326329582230Q-34
 0.24199982071127197329484645418415Q-36
 -0.50205656278667425486737197452662Q-39
 -0.10953067005061329671945753453793Q-40
 -0.63416061288043903311542572212320Q-43
 -0.16740043430295218532805644670895Q-45
 0.22129381581571266166768866199305Q-48

-0.27543191701907173044698521355362Q+0
 -0.17287972760804514799105385734199Q+0
 -0.30917831418917028023374396511769Q+0
 0.73004478697562498072295132290189Q+0
 0.35508311729530773623504403642207Q+0
 0.53118086463053725117398885614330Q-0
 0.35458698775011299090688929268442Q-0
 0.52218048920171887019739849036804Q-0
 -0.82827909037130778959623603451413Q-0
 -0.63964807056049232429084520456664Q-0
 -0.2074430335006659981034416704508Q-0
 -0.23647743617114103131477376977710Q-0
 0.65476186214586529227646650447866Q-1
 0.33026588568828155434989900698979Q-1
 0.62994757527479159324355721580748Q-1
 0.48158664941240662972014038270232Q-1
 -0.48490409724200829658805698386195Q-1
 -0.17931914010878113095634591759522Q-1
 -0.22496874236790560308799535387124Q-2
 -0.12032181241517026311333856416615Q-2
 0.57880347965764956038227824728839Q-2
 0.16794348031416070719978905784691Q-2
 0.14906762856961256734623989082861Q-2
 0.58447712503372606844222137921304Q-3
 -0.15636081255731552868375521427330Q-3
 -0.37209550566376416671830074124986Q-3
 -0.24595932858353434566865671949768Q-3
 -0.73486622789130655942959318926882Q-3
 0.12065359164542154779250805294458Q-4
 0.24290164159668846627281101342191Q-4
 0.12420363116390780829630611834630Q-4

COEFFICIENTS FOR $J_N(Z)$, $N = 1$

$\text{ARG}(\phi) = 82.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.43705618725051947484347585682349Q+02
 0.28604219418342189849641811316528Q+02
 0.91755271882928422930782748337748Q+01
 0.15480330031503695938222900049909Q+01
 0.12391565686220676018195055839356Q+00
 -0.12629983452612194415691843738527Q-02
 -0.14251055671036787155002368900967Q-02
 -0.17440754126400095348040179606917Q-03
 -0.12718564976197552015411585408691Q-04
 -0.65696887996539806797364989244697Q-06
 -0.25679302333142161372360569704899Q-07
 -0.78638698373798369213913151686153Q-09
 -0.19247532307587339841244547499472Q-10
 -0.38019758145626080951814716454888Q-12
 -0.60527810491395520649490363170897Q-14
 -0.76117284960917883772717583941357Q-16
 -0.70493793349747147108740675573692Q-18
 -0.34664096911561194986692426062488Q-20
 0.26567399391173611188597568451623Q-22
 0.94323866817110571225096137822622Q-24
 0.14527465276411891143781524708393Q-25
 0.16431688648146850006693432846889Q-27
 0.1511788793826215796219457620753Q-29
 0.11750751252483752313320523937243Q-31
 0.78435961171214034575939576118808Q-34
 0.45182051324294693468773035376136Q-36
 0.22338145913959951880665237919400Q-38
 0.92751724293621062517802036056810Q-41
 0.30372566015432926706052442270415Q-43
 0.61162232282572711990769979224487Q-46
 -0.86687201040442531962537129095383Q-49

-0.41923771467787835211152012152582Q+0
 -0.31584658429505431833073499337817Q+0
 -0.13664310564945434006043559906726Q+0
 -0.35828118036049386796983480939276Q+0
 -0.60544156619452080411930287547991Q+0
 -0.69486323969685972382058150831310Q-0
 -0.56367882215703729076145171601647Q-0
 -0.33202506859076857680301623416888Q-0
 -0.14370784790881258460241441695177Q-0
 -0.45131931713130223280064569538351Q-0
 -0.95136922545545306276052828034417Q-0
 -0.83297261073526140761185475363179Q-1
 0.28195299878684876126591629996834Q-1
 0.15977023686601266706686926058032Q-1
 0.46140683881766576563809905634054Q-1
 0.97067233240961046043292955667481Q-1
 0.16229465614349770884640824549436Q-1
 0.22352712971661768676031800804496Q-1
 0.25793382684047941229111315518391Q-2
 0.25087851324398685030361998266601Q-2
 0.20477128509242162308538342335555Q-2
 0.13727363898429389158176061385632Q-2
 0.70775656841198186567859285647384Q-3
 0.21315335941138441718438519413034Q-3
 -0.62267129972523794555480203928082Q-3
 -0.15886693043481200396514310072876Q-3
 -0.15191061873385387636325017389106Q-3
 -0.10697527857763015195664366514750Q-4
 -0.62104306249702564821317224328693Q-4
 -0.30924847649001085256018023632763Q-4
 -0.13426007538609068163431704239594Q-4

COEFFICIENTS FOR $J_N(Z)$, $N = 2$

$\text{ARG}(\phi) = 7.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.403084940209225469544242962048470+00	-0.444113699042344162166189045417990-
-0.417545331668749378321991815615640+00	0.641521573007144232392903762590720-
0.278282531287512485697282901888140+00	0.311375619767751727492609470638270-
-0.870791203950120424231412557960580-01	-0.398023433049262488578824063345700-
0.142141959533837159506573411296840-01	0.129818246918579784931500061477900-
-0.128196002182927291713963031289840-02	-0.218926047524691973732185312800330-
0.555053531944859324780410885993710-04	0.229618206968522445606075255270660-
0.805832572528611973655130504142240-06	-0.163687093686110710440415183042440-
-0.286259021944877900663693352375160-06	0.833041967371828641993276322551470-
0.216305365531277186338118841369760-07	-0.309909147511219965637976513565710-
-0.102572253700441150740235091295980-08	0.839173068067660464996601376532510-
0.354725365628016540548492934745900-10	-0.155022949532888987363001438835290-
-0.947558827361431693620644512856970-12	0.133486211252570559646967252848130-
0.201089605511988246921693949303520-13	0.264507819823040707126658366029910-
-0.343286082450843319669698008579010-15	-0.145319929018860710865774079863190-
0.470515502740560851013261502827830-17	0.372165653405913111639240731998640-
-0.504691976654603850701692309849940-19	-0.687037473029397232657821714089780-
0.385857632403661744084982317167970-21	0.100619958441992337565832257901110-
-0.120986280988883979439924438791660-23	-0.121383778575049211538512621285800-
-0.206775463274567284214507391386500-25	0.122642504252239042900623262049250-
0.481787316114256097164953753143710-27	-0.104225341444820737332754207879490-
-0.620275953112625837400845356767890-29	0.738490714133062843384100707228970-
0.607997497278989541350602881352710-31	-0.422118482243681617774897036054460-
-0.490947664408455933955577167733920-33	0.174684019069742014103473050296710-
0.336757271135801661089172119178780-35	-0.259988165144938001074489191595940-
-0.198677482951228401188688183833820-37	-0.376997101221429672191944475652530-
0.100918945027537823855434834842860-39	0.489867044660199797787162114084170-
-0.436186609373528717163679210891540-42	-0.379574777941814758698607859493750-
0.154620432801366516765479969905890-44	0.230613264191581531289353746205180-
-0.399602369660111663452803025598210-47	-0.117921568373819721787418792609360-
0.328752782990973850974255597413720-50	0.521576870763543945255160359490440-

COEFFICIENTS FOR $J_N(Z)$, $N = 2$ ARG(ϕ) = 22.5 DEGREES

REAL PART

IMAGINARY PART

0.249365845180219764128139621642670+00	-0.293535198357588020460887540129750-
-0.485043474005584591627277682698920+00	0.309382535794104479526789260239780+
0.385912610399475517378749118259840+00	0.169516299355426414071680302290950+
-0.256333826686800164207934640477290-01	-0.135493853844021287266280352769180-
-0.177642770804090665913748535059140-01	0.195099249015640728109506195603780-
0.333285639423494317508494510906660-02	0.199385676787001905777957636698530-
-0.180445045577185244142569798000660-03	-0.240964109614058144680343314686970-
-0.421690895420068234781497740467320-05	0.199126328500718774652760707617580-
0.928228405307241859234791607834810-06	-0.535082610323421832738226068436720-
-0.430703947006304167615981727502520-07	-0.136580439253605650745616207903970-
0.66624042137477388867113709494830-09	0.141248725121908510176660783599670-
0.165790382663510020544108398984900-10	-0.419026353196582217765943137297400-
-0.102321415931140726700537389637550-11	0.409776345351974696127536992713980-
0.210541885151460560533516641525110-13	0.962192033809499837266425543264050-
-0.138934788535062105750495103376770-15	-0.398529887841558031893175227248380-
-0.304724425107896866583331306803040-17	0.601622939302686967947565133738920-
0.909829411334984379388988835637590-19	-0.281142262057871850604267306691980-
-0.104995391192909324294157943000360-20	-0.574753760088209848011683488969220-
0.360669452285274603410682903893000-23	0.129885565300634152794430467794220-
0.688000511040538993121575484639540-25	-0.118243438404154312903199056954580-
-0.121944149673629868756339843522210-26	0.307605568372033960745202863833370-
0.897794648176534128634294557621350-29	0.548814468062749532373019404295150-
-0.181243498218573420094809525522330-31	-0.784034358198337097234609717183140-
-0.303307420015568856438980926595930-33	0.476320228565814166754356945355750-
0.356875366273064794927724038889980-35	-0.761537642281393033247232632108830-
-0.181912776538033273218811453753070-37	-0.119890829678505298679526111253550-
0.234346368659302886459033545843030-40	0.118244136711455943481347467476240-
0.348084187396787655661151423483640-42	-0.512852550235476331813196437729990-
-0.291997831713907088405971238043510-44	0.540296653471928546424974188077370-
0.109055073609098966099890514451650-46	0.759253019898574480523149983276950-
-0.951764484973919492156836576450750-50	-0.548452336599490385163782105001060-

COEFFICIENTS FOR $J_N(Z)$, $N = 2$ ARG(ϕ) = 37.5 DEGREES

REAL PART

IMAGINARY PART

-0.74953051761511623986362630157393Q+00
 -0.11604411736034028293565233670991Q+01
 0.45917698871457204064854958229220Q+00
 0.22402456819623889651511768845330Q+00
 -0.25174863172514964306067460545658Q-01
 -0.40171400627983682733253590294246Q-02
 0.34567842352909733608451283292577Q-03
 0.14891545931972912539870663275633Q-04
 -0.14095966091596140189215546537368Q-05
 -0.84367935602192537801806804351055Q-08
 0.20621506329285415015315498803936Q-08
 -0.16846924073265012341785062432258Q-10
 -0.12209195731651344144917164592972Q-11
 0.19718912171740024795180283413710Q-13
 0.29121124914074773028722854015873Q-15
 -0.77328560103721758285519677867936Q-17
 -0.13325768000540332811915551532217Q-19
 0.14228945388846939790527756073265Q-20
 -0.57703443034957239269091450127002Q-23
 -0.13178968476423027427384359057786Q-24
 0.11151500954041243880281813786475Q-26
 0.54611037915713940148304255292429Q-29
 -0.90834646018198474116790259382628Q-31
 0.23221572561980018206304521388456Q-34
 0.40000194491745826267909312969293Q-35
 -0.12698727226283256126030430189510Q-37
 -0.96027666682404693367997097540993Q-40
 0.60383652169415542810222822993594Q-42
 0.90271633028939900067076630418121Q-45
 -0.14866792769620802000137777646987Q-46
 0.14415856059568308988391601741863Q-49

0.31493950776626674477435873677681Q+0
 0.10238644591262204445859764395518Q+0
 0.77797987440623588402954566895157Q+0
 -0.12889838924111171355832189071017Q+0
 -0.37325330420036801730568475613223Q-0
 0.34772868687132947762141287342793Q-0
 0.29425595573124696072476248789301Q-0
 -0.25363166513818696259013345212884Q-0
 -0.49944148751196647924075393763458Q-0
 0.60701017392192736231811359722265Q-0
 -0.15189172042210125264061325539141Q-0
 -0.55968679801069037553800424731115Q-1
 0.69420495741880734641151169233545Q-1
 0.21324017112899881194960596136447Q-1
 -0.43368034139247730090389441939854Q-1
 -0.28664150970003573354001490349138Q-1
 0.11443206076980097763978148755639Q-1
 -0.17938157391515768663385480041702Q-2
 -0.14934799524715009686455239467089Q-2
 0.92834977112275833825323616643052Q-2
 0.95978240369723093603597343769647Q-2
 -0.10942028495626780311389519629934Q-2
 -0.19746322757847392961064528039793Q-3
 0.64824498406073537262484949206081Q-3
 -0.11786778315934986457864893696309Q-3
 -0.21276224404177765902190095862218Q-3
 0.96983935038620181150758097580711Q-4
 0.35181646549320813021583855363001Q-4
 -0.32106500576013013278924197630937Q-4
 -0.29883739503320216661125385496643Q-4
 0.60379989698039396493128507399441Q-4

COEFFICIENTS FOR $J_N(Z)$, $N = 2$ ARG(ϕ) = 52.5 DEGREES

REAL PART

IMAGINARY PART

-0.49810229695014757620352087670975Q+01	-0.15037270521279964482410042710935Q+
-0.47784472437081878149626068981614Q+01	0.57950148249376936412268725349803Q+
-0.11231462306332141330370531407769Q+01	0.17843846117929431831908532510290Q+
0.24254333252404589779571220497067Q+00	0.45726543745111917125603196414154Q+
0.81332940917732494736838104822222Q-01	-0.36282398458974924279266803743093Q-
0.30775016905636889509450970764500Q-02	-0.83672730976178931497227724513757Q-
-0.54085795243008036965540644254909Q-03	-0.47379088262564775959745181638865Q-
-0.38928978317916385560288936720420Q-04	0.21538714081130593964489949862818Q-
0.37826084696641253543297810280562Q-06	0.21455568666621322632184680434644Q-
0.85620422438632885042979157667928Q-07	0.12497840184077284260566102616823Q-
0.12509708876289545565636166456040Q-08	-0.25530533010086300021041259466575Q-
-0.56977991931770952528870893146707Q-10	-0.54084635880092506961571491699432Q-
-0.16124193261076881086901667266775Q-11	0.90880273230822495252445932538134Q-
0.81939740650425315818835101975490Q-14	0.36718234788948390347094338729934Q-
0.66478901678563086709555338281691Q-15	0.46878331509950976715841278415013Q-
0.36298899845976511409392896481347Q-17	-0.97246101753540588089611570240564Q-
-0.11477902993575974387154403887537Q-18	-0.85722889352035132302989781003093Q-
-0.14071936188020184797632685837603Q-20	0.10637993065025930122342262361335Q-
0.70012077421071644110729790686567Q-23	0.18200346906245721156535814822151Q-
0.19382262865583033005476314676931Q-24	-0.17255361834892070137071407320611Q-
0.32956863323176117616456899350902Q-27	-0.17304328061460970420185103940613Q-
-0.12996014535718458465935896565582Q-28	-0.65026404692435631841751128455732Q-
-0.73853326616182932708216733002268Q-31	0.81118017809695419890247414397093Q-
0.40314675092211472256455928534769Q-33	0.64503069677714908275465152388596Q-
0.46640302552481432573890711768311Q-35	-0.13706861136716056723898675952414Q-
-0.39791936790084779706185806214373Q-39	-0.28724219321641868028882506534380Q-
-0.15224229801510004279279040037241Q-39	-0.40081411147563164338234101628915Q-
-0.40425921439742921187318175599319Q-42	0.69299747040476227748358054711630Q-
0.26606916955491203790215217901422Q-44	0.27295390479368539941825227961411Q-
0.14827413084961013706065617424111Q-46	-0.81592370530087324974146121321038Q-
-0.16315569362780249978361478621252Q-49	-0.68459335431997708071764747987381Q-

COEFFICIENTS FOR $J_N(Z)$, $N = 2$

$\text{ARG}(\phi) = 67.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

-0.44995534229269107853302059456158Q+01
 -0.6111008332570118023437243400826CQ+01
 -0.36757421552573258552333586200573Q+01
 -0.90686699845936955578265133980126CQ+00
 -0.94234951356679112634961056947006C-01
 -0.1327306039876384207380506473936CQ-02
 0.61001161211701446485623588369253C-03
 0.61413523766638477316742115936043C-04
 0.26209471931651662311800661332026C-05
 0.36565842017204958879603295614452C-07
 -0.15531101696408236435121406240641Q-08
 -0.93666706396240503356911953712056C-10
 -0.2181700659128028000667880185481CQ-11
 -0.1978200302134106489601185434517CQ-13
 0.2661469882858775261875251231188CQ-15
 0.11253162953264551119864674098257Q-16
 0.16522419749031455843391993028427Q-18
 0.10181744533558583749624242722231Q-20
 -0.61155410240200565655849989927468Q-23
 -0.19744909995542389926133848455525C-24
 -0.19954188994359969077048315873891C-26
 -0.88222975494377960027420820887340Q-29
 0.28275243523762499281920105678299Q-31
 0.73524296414463768831054797264234C-33
 0.54278527080407694131574343564776C-35
 0.17993690147996346694333608159714Q-37
 -0.34411643657669401982490333123053C-40
 -0.7472387659393037952449245880108CQ-42
 -0.42068336656446721217541811864557Q-44
 -0.10826360671916244437170219909273C-46
 0.13347306852420770458162373056944Q-49

-0.14987282147406928911182450982170Q+01
 -0.94641264126085413311786307220146Q+01
 -0.19390281076041520307860197731536Q+01
 0.11245657268195255702329714445702Q+01
 0.93820237975435851929664425337311Q-01
 0.13669544603964303552434583197207Q-01
 0.86440523583876638740447574862971Q-01
 0.14412354090356988972500033315384Q-01
 -0.14515845275738284868493135124299Q-01
 -0.11006291293812661069284216526437Q-01
 -0.33894365126973263747947197270293Q-01
 -0.38045963922569044655849905012182Q-01
 0.85474214081982120509718955146212Q-01
 0.42574136369385987769898032284051Q-01
 0.77673119418978835720469878355651Q-01
 0.57642360443643710501489002080788Q-01
 -0.50348191230640379666222359225397Q-01
 -0.1845128287382235328561545431848Q-01
 -0.22278317297371241301641030674482Q-01
 -0.11557073765193969887666605703030Q-01
 0.49850959967641238530946385375490Q-01
 0.14366917875978806128304318287737Q-01
 0.12332484547083110508269986296523Q-01
 0.46984464722049064624555608678343Q-01
 -0.11500689693764400708056155668525Q-01
 -0.27225143319484154885584201478130Q-01
 -0.17470646374106102551895450834472Q-01
 -0.50833049452608270615265170064758Q-01
 0.77419481158396077986668933903921Q-01
 0.15521136976744786294641051514745Q-01
 0.77284438964657187350832411421362Q-01

COEFFICIENTS FOR $J_N(Z)$, $N = 2$

ARG(ϕ) = 82.5 DEGREES

REAL PART

IMAGINARY PART

0.21577630050059540153560286526739Q+02	-0.16312562326469069320421825958462Q+
0.12952461486699565037709024321836Q+02	-0.11946898576373331884686626287394Q+
0.3675497344244604687405712299375EQ+01	-0.47600020585592129837295361263633Q+
0.55299899330652664888912046006489Q+00	-0.11229892884441387177722791201445Q+
0.41619863021846211417538296435140Q-01	-0.16989701630007108114834741808635Q+
0.21271076128623024726990230799813Q-03	-0.17509462216414392194801646926374Q-
-0.29119022132967329575680278087467Q-03	-0.12832089953326133963189347144179Q-
-0.34374856208393831597049588211566Q-04	-0.68770868659660298214304827329696Q-
-0.23340411166770530927556879028100Q-05	-0.27284582771386587928010231633593Q-
-0.11182586081339670961699885776993Q-06	-0.79192596587617789553051480856364Q-
-0.40618357896435009269489012518734Q-08	-0.15615859112124649078143169675196Q-
-0.11597791367626047078400681706624Q-09	-0.13528783601381119990710116213576Q-
-0.26566025069406584403685316862965Q-11	0.36406046951653031315422340507607Q-
-0.49294032281176354390305187426934Q-13	0.20242993524865955184132709740121Q-
-0.73992638304976658765153708829989Q-15	0.55561950689022425491549116100239Q-
-0.88081076585386892252341579614512Q-17	0.11083861509831164771348972264451Q-
-0.77633132834507698162833501897993Q-19	0.17592340656588119976608699495821Q-
-0.36977710609947574976479947316634Q-21	0.23044252493846029191023599353588Q-
0.24881771747158141134693525474858Q-23	0.25342409458445191533533532925555Q-
0.87339879511101687253303717475236Q-25	0.23540554395148570032254424725912Q-
0.12936075586806783274989800030974Q-26	0.18388658266696097045496542337568Q-
0.14048973099259056971137931142480Q-28	0.11824292304438070201093173723608Q-
0.12419070461631291026286627096278Q-30	0.58652091484410240351052317836089Q-
0.92853211218829721445160431070910Q-33	0.17142257189953688515439909518917Q-
0.59694444508020998272300433872962Q-35	-0.45653105755586483005352074258649Q-
0.33161982395112164109676672190944Q-37	-0.11560201616636927033661533778731Q-
0.15832802646139366825239628264032Q-39	-0.10709186857291252058999108141491Q-
0.63576388056973810249565050959556Q-42	-0.72979995680003681415697860697570Q-
0.20171996219592952990168287903911Q-44	-0.41018598394445250436045760980107Q-
0.39562558341425562080918840980665Q-47	-0.19789643077383578865807601644957Q-
-0.5217567542604764477977654746265EQ-50	-0.83314859764605237415660029059988Q-

COEFFICIENTS FOR $J_N(Z)$, $N = 3$

$\text{ARG}(\phi) = 7.5$ DEGREES

REAL PART

IMAGINARY PART

0.50925250556809636866863983468064Q+00	-0.84962239014809072050180230863410
-0.467066664946481771879639241447521Q+00	0.35401932285209651108593574906193
0.21921699100951471687225449461052Q+00	0.42563943169020852964286012039038
-0.51777732414787461646877862259929Q-01	-0.27520433318044407863523037672753
0.67864454435374947598713827823504Q-02	0.67700959179137651280409885894388
-0.50832508956174835785438257435275Q-03	-0.93870796204083060243567114934585
0.18068197140903437994015463442855Q-04	0.84342549893439423567194681648357
0.37119689943525489175467582499808Q-06	-0.52805500073885655386004535276244
-0.87138842186415858918057336085730Q-07	0.24004921318739126558003027917165
0.58097647594262650343524979011044Q-08	-0.80737831154086676181851915273460
-0.24977140610268428408389566620628Q-09	0.19930484676214861883422001840589
0.79303823291570260898803801723312Q-11	-0.33671148556354942057694597535576
-0.19611849307878977921283427225497Q-12	0.25807165030278999214567015630639
0.38774032443955133254072360462529Q-14	0.54159237088354695085967731039627
-0.61973819034473560999863447568482Q-16	-0.26753050217177078850705795077563
0.79843426260864603818222815352738Q-18	0.63975808610423454295825456408641
-0.80726193648016924362887681310088Q-20	-0.11120361983846865713385667643794
0.58185859831429703337425327334591Q-22	0.15407407364367918785758594772261
-0.16764739919615434069079080161075Q-24	-0.17645057360865911143464954022131
-0.29350254728067527141733820317434Q-26	0.16972460784183300688796704661704
0.64268204178565067998078001351337Q-28	-0.13763907364534496068677001019777
-0.78880384282497207438374505258741Q-30	0.93242939301778280619783205261932
0.74007571177293579357547207713388Q-32	-0.51025507243051348364462597462449
-0.57341788315585909341021168722879Q-34	0.20208568865732976818848184444281
0.37813261170437100795922476274845Q-36	-0.28131847828406713162389385067379
-0.21481425935726755394226619861129Q-38	-0.41339504687226625787253834995792
0.10521343460164403289642432890823Q-40	0.51386157557610919711561258704458
-0.43898559751401570076458819693292Q-43	-0.38375799382761814636799684965641
0.15033405749679821199050236108014Q-45	0.22524998926650868277945432407254
-0.37518181807483533990730790537165Q-48	-0.11144335000837427287017700888653
0.29201870746858362668646429719765Q-51	0.47751270112884295797947019642587

COEFFICIENTS FOR $J_N(Z)$, $N = 3$ $\text{ARG}(\phi) = 22.5$ DEGREES

REAL PART

IMAGINARY PART

0.34008587629008985463235816209296Q+00	-0.23830192209057947871152746977233
-0.59148516434357175168458871337294Q+00	0.15065120030892570429202581672065
0.24692759584089332027972554938454Q+00	0.18300609191966862080277639303607
-0.23638415114586139635395766226027Q-02	-0.78871748135575055848184695415196
-0.94242676960611657070183718898293Q-02	0.82191525766942548574152555025045
0.13392367242480653248746036919323Q-02	0.19877481873571435680559862675343
-0.57996491516045780438094938085516Q-04	-0.89716396894178234356628719956988
-0.16789965561354153603525047545804Q-05	0.62043361638035106932000808166786
0.27091069656607619647514810165606Q-06	-0.14004305136984180759158353524035
-0.11020210367203555539576629410251Q-07	-0.39666890209873199139290758825148
0.14751700577529921764624193482457Q-09	0.34123924601555549116631781585658
0.39078842807973538796204320087073Q-11	-0.91175673232489268165656446195156
-0.21151362824174737227082297081182Q-12	0.78745281161598445972608056206396
0.39899380146550614886167880593223Q-14	0.19284057617662403675497368748059
-0.23603483948875394596557380135883Q-16	-0.72103773677288787920318001566167
-0.53361714342230859270915905134987Q-18	0.10104834008462241606211159056287
0.14648231151088719803709774578091Q-19	-0.42823351949280228901424690109038
-0.15840762388593448035459485731768Q-21	-0.89574902390320766038445023822362
0.49804389085677601098922461506602Q-24	0.18847398634397020597013912800880
0.96726067033612608506435971882357Q-26	-0.16196076676939654793154960794858
-0.16110998068100851264684102609735Q-27	0.38856175005492050445668119508011
0.11261792227224280182288587691136Q-29	0.70334699710567977855840337149938
-0.21099395040356080229121511805235Q-32	-0.95096621681225044758271511566120
-0.25731980231755944316106768191017Q-34	0.55114302893202011251639646214537
0.40014119003600185971435581009223Q-36	-0.82217933284076818086584172113640
-0.19535140826680173859201031665833Q-38	-0.13073399668433373545485189037316
0.23590192476832162687714119416485Q-41	0.12327728605542977958984935912675
0.35337493866134632348152632276648Q-43	-0.51381740653004223795047905235750
-0.28449275369035163235782242828833Q-45	0.50947156931277310902916852731496
0.10239958038830100416688838960588Q-47	0.72117326727814487956944730989856
-0.84410025842273772860834226273101Q-51	-0.50154909245760968510622143312971

COEFFICIENTS FOR $J_N(Z)$, $N = 3$ ARG(ϕ) = 37.5 DEGREES

REAL PART

IMAGINARY PART

-0.37538318311886753109629621978658Q+00	-0.49529483263522665977461556150871
-0.11794407586607311808837899279676Q+01	0.32044517808709056005550321189946
0.14109145231099988648005913630334Q+00	0.54412695818840855855272053579704
0.12603114550066699911558786855985Q+00	-0.42817177964281707960143342109229
-0.85885788977404122299717643622327Q-02	-0.17816108677534487722400192635458
-0.16765585224834512442119770065357Q-02	0.11524544382455622682267677028485
0.10800424029507104182206682174489Q-03	0.10981907766054843355131903879049
0.50806175339334735588312783116020Q-05	-0.73785830524347327526359624491876
-0.38054954318743118238722586558252Q-06	-0.16106891091457236259162986677450
-0.29057452894068815274395664162351Q-08	0.15221750891303057015038773741491
0.48173012763417035487687229261802Q-09	-0.14611908256024532444195137846824
-0.31873735298493839724168719445803Q-11	-0.12226789316847215435737712496527
-0.25053156471202868533820341052886Q-12	0.13182179559972015552135174233651
0.36058014669806962212360929800223Q-14	0.41315886921920062829377153549387
0.53661754191517423172879721845284Q-16	-0.75666064745002276309997637407607
-0.12847921502030396325344789611068Q-17	-0.50988985178267778629358126346460
-0.24870787277071617538104158355655Q-20	0.18113527598286554935743936534791
0.21489344344007092496377996122265Q-21	-0.23049809735438414277769077899916
-0.79161461578995935576966449569653Q-24	-0.21562752308373163524544749900997
-0.18234807787833405421434366046254Q-25	0.12429771381361677543892846306150
0.14419216196657195799651394988559Q-27	0.12768316092817074718943005180945
0.70262497813910822890177844368037Q-30	-0.13633002649641131286001288687894
-0.10904429608937896147422909602770Q-31	-0.25073768141096667832774976177244
0.18568725583020640291524178606343Q-35	0.75035856350864229043465392377546
0.44695108678869723717037647133802Q-36	-0.12672759264206387229486940594703
-0.13420370041921706749672520350095Q-38	-0.22981736469199130768829027607431
-0.10046198322685886908880099810484Q-40	0.99668302655829522912484965831307
0.60218406066423915384371549635718Q-43	0.35761584061470942309182340435499
0.89969363955225030941036072949396Q-46	-0.31064700766799898577109171542768
-0.13961889782660211984386929210291Q-47	-0.38323958109069933599321133004565
0.12748676841321267651728046942091Q-50	0.55079446406115994764862655203203

COEFFICIENTS FOR $J_N(Z)$, $N = 3$

$\text{ARG}(\phi) = 52.5$ DEGREES

REAL PART

IMAGINARY PART

-0.21667951886465007911186552822031Q+01	-0.25529581818371703207042923946683
-0.29463291942798192660690801199005Q+01	-0.70459447082310676251090130511511
-0.81941373789435601849965920744855Q+00	0.79627150460509440716089277213250
0.75083028950052268153792265128555Q-01	0.22997037349883032450393589535272
0.33133001118626600253868709673999Q-01	0.28587908980680440282605614312491
0.141765492556010779029902775966631Q-02	-0.28972182656575530162178268937726
-0.16205349183318478686357050676928Q-03	-0.1675668399054373468778225159341
-0.11826373959457217628304489140617Q-04	0.55281922571502653035537413965657
0.68404419984756399175829387299522Q-07	0.57950981438054641365540629848077
0.20912108958503093372645595853051Q-07	0.40948535413590644440394232090073
0.30900673585352442678563897576486Q-09	-0.56936600586443293435177040252134
-0.11655115926705127616980508586394Q-10	-0.11893489893498042356831016512074
-0.32501053812422295809056211673064Q-12	0.16983807464115525037181978828928
0.13318171345152450542281040334637Q-14	0.68682372940500930954798135615005
0.11622554477516193424674599170696Q-15	0.11115697046711689405315942776765
0.63692660211223047243003504066151Q-18	-0.15965020615153011096636434073174
-0.17743984186147373023702282375169Q-19	-0.13817288517742388925301569034353
-0.21284352957664919464278183904284Q-21	0.15488684724944383107555642393066
0.95240746004046402217413949133425Q-24	0.26039955866776834481295190213896
0.26349917469016111087357402977634Q-25	-0.19437822493445456688275475480474
0.45940926437721336113698107026674Q-28	-0.22421044709752438652662616015310
-0.16081869313973237068634328549836Q-29	-0.83056743263886368836689778476353
-0.89472249353188228926658409543477Q-32	0.95969959038396162426034765481421
0.45560164082771501215107349792475Q-34	0.74703187138220317869568745803093
0.51828075469241458708202609086157Q-36	-0.14660556767934282512681983032629
-0.13447856684615257241152208314563Q-40	-0.30697888958015970299463566959604
-0.15673531412522437393693093587215Q-40	-0.42749847407912067449926004939565
-0.40904393289881513568830909166651Q-43	0.68805484543383989901348658193630
0.25488321670041641233269766598236Q-45	0.26557031075826293710937308606093
0.13928126799834351397208027961929Q-47	-0.75326071294472578272128627189997
-0.14377702406867630716204729621130Q-50	-0.62218643888510859610141278351753

COEFFICIENTS FOR $J_N(Z)$, $N = 3$

$\text{ARG}(\phi) = 67.5$ DEGREES

REAL PART

IMAGINARY PART

0.11693430530701552406002433680026Q+00	-0.94908533553291926889810574263470
-0.22417809065042875710705196776149Q+01	-0.59515464307962076634658557492874
-0.16613250940787737796704243681536Q+01	-0.12308908081716627526855901505664
-0.39971748638804349064008993011603Q+00	0.38167607146797907873458780785440
-0.39591222979810385602531267866611Q-01	0.33008439831471370556822888887862
-0.79218754586774303251138817646973Q-03	0.46965006470575256169646521026381
0.17518247032058274779362611295741Q-03	0.28113764263880218645770184154288
0.17288008041470808118961571663201Q-04	0.49349653740021292941526430142832
0.69952986114237041341408655530172Q-06	-0.35156757724576811269567793187783
0.97192634796348782244060674075982Q-08	-0.26220249033942731469492798473424
-0.32490074215445587478950002509007Q-09	-0.76820583842501076003267927584637
-0.19327592715727881025273210610633Q-10	-0.84369683498915137489723600060673
-0.42985768790523241276747467009680Q-12	0.15727096759051054475241737711218
-0.37890649741574615803484941955300Q-14	0.77439499493647480695613900851562
0.43689959785323456270706247601321Q-16	0.13536512143205719319070537627115
0.18293466372783473834145948199295Q-17	0.97448417427497285180485445298536
0.25813599716342829448427365014329Q-19	-0.74586484741905417724624962941465
0.15426246431159576119661305456464Q-21	-0.27106198715386705166508460413237
-0.82528687281866195654888517636265Q-24	-0.31540111451501227574565636588994
-0.26453215187624806965274551309934Q-25	-0.15875351937832931356578563466641
-0.25825347295399728381534294617777Q-27	0.61764757498597415223637271407514
-0.11088848732077670751878196562458Q-29	0.17688123851977881888789646682599
0.32378199000912788650350087716519Q-32	0.14699044567210975608931749267285
0.83725208828580657747672691283690Q-34	0.54444414527872654303459447660549
0.59951927460445977393722102293495Q-36	-0.12241091214703585432669200587771
0.19343926980866046856515738959107Q-38	-0.28834931909499854972459062465945
-0.34213118027340395345988524777050Q-41	-0.17978313940421202491755043683031
-0.73965516495964502432062656319990Q-43	-0.50970644474581507396220963512833
-0.40521207768169185630526123493160Q-45	0.72210648977808843562949417005322
-0.10172117994546187110744083369056Q-47	0.14419579042143780871043436695526
0.11723572682149307961451827403435Q-50	0.69964695733440197974688846032176

COEFFICIENTS FOR $J_N(Z)$, $N = 3$ ARG(ϕ) = 82.5 DEGREES

REAL PART

IMAGINARY PART

0.13575900469681263094854577400065C+02	-0.83228132860143043128094704522056
0.74260978594819319685049727927416Q+01	-0.59357243038778503156114061717056
0.18734990105628468231099551035865Q+01	-0.21822076620690445595743542037313
0.25274236436432026469574654451976Q+00	-0.465943990921999030803917189666231
0.17683204668119724240267223770665Q-01	-0.63612870900208725048291968200711
0.21951205242093691446730799338583Q-03	-0.59362829747544957612967680374721
-0.81464490506594895552782893489411C-04	-0.39616528840312343502466411203065
-0.93164202011210831260533841461468Q-05	-0.19456696097687094772978163247199
-0.59164865278790085046957189295997C-06	-0.71205403413879410694595600540360
-0.26418431154009965708462718760082C-07	-0.19200444354397972223066498011515
-0.89584303187714822408532063331022C-09	-0.35538568234861532270193078976442
-0.23951776687349306923189383923534Q-10	-0.30196890376565210432150358725738
-0.51545109834101556736260902337467Q-12	0.66346034944260410379720373341865
-0.90158948863212747495544145988602Q-14	0.36261903811086918214466485879833
-0.12799995574903340137184703946706C-15	0.94825086926193058591388939056546
-0.14462982033429893297924266729357Q-17	0.17983715329914395315678835210443
-0.12158147456825895765407174293631C-19	0.27163593534610703578980956129186
-0.56090393252331978319271121106524Q-22	0.33919048513474446716540990100221
0.3341343482709916500410629992637C-24	0.35625878055722559598854124836971
0.11605660745207039236613969093339Q-25	0.3166673915398389637745649232792
0.16554585074753903364158190370335Q-27	0.23716311687073832646711458010719
0.17289549750477798792856905937503Q-29	0.14651619482518552390026576463440
0.14706725409553425250980019186523C-31	0.70017832008660076927674055294686
0.10591911367691039353717601465591Q-33	0.19871335794095903925430899299327
0.65671655299119580507363909103302C-36	-0.48448655962631052754549237486530
0.35227488390307387442202130647481Q-38	-0.12181298557566038493976461039438
0.16260678830070716787254865737622Q-40	-0.10943186217701554612197268761751
0.63212569600875150804616250045993Q-43	-0.72241897291348648907563097209059
0.19451819268327589805654164142973C-45	-0.39349488565269600452916705445038
0.37177404591943802590076286759555Q-48	-0.18411205333787789927741077359729
-0.45738016172347818943801868962146C-51	-0.75232438791108561631250850576772

COEFFICIENTS FOR $J_N(Z)$, $N = 4$

$\text{ARG}(\phi) = 7.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.61707465320865149158462059664872Q+00	-0.111624231939634054192100716000160
-0.47899266909901679450721553667617Q+00	0.116532524805128919555641196925940
0.1750856082011228850692549764810EQ+00	0.434571529689046336002951734971760
-0.33465244490471403660999568595588Q-01	-0.195945881348332417094651578641100
0.36768244861877712050593385470246Q-02	0.391225087471626586380014493106190
-0.23551375710519716957164694938427Q-03	-0.462791477911743988494520493356920
0.70253335680836713756151966380339Q-05	0.364606267343078962196105581177430
0.17822176167867288429706136369577Q-06	-0.203795591325566859097501349116830
-0.31774102994679681144185589079522Q-07	0.837795408771318210008032939169410
0.18989616305575355841175466013849Q-08	-0.257250106287642384335552884145730
-0.74726089188694219963755258126763Q-10	0.583603281183741692573714593227680
0.21940135591934055424956326086021Q-11	-0.908106740871702491891788268707330
-0.50525012369727457632703918835801Q-13	0.623912880155556980508961923210460
0.93517918032201688558851868532484Q-15	0.137358868076965797043938396364710
-0.14053781175433950937641967523629Q-16	-0.617190729055224309054203163046990
0.17082254268020806594297386702620Q-18	0.138463456074102005009124616532650
-0.16333985858468608702029822386828Q-20	-0.227432275463437170723389893329610
0.11134678338825295811913149252484Q-22	0.299008355942503492080741769120410
-0.29586870916270095550832982668110Q-25	-0.325944997156729820130567332396200
-0.52922231569654998587627064117057Q-27	0.299180999766389218208752568114410
0.10933659567461159154274038408959Q-28	-0.232019071846835993799494986691160
-0.12822338432397996545470212937089Q-30	0.150574134056200655700976415131320
0.11536782404829670605307856020688Q-32	-0.790309802036923799294284934485230
-0.85915067567035891272346259758402Q-35	0.300093054087092255741981967689330
0.54549936091789251970666370258561Q-37	-0.391608746346874611464179793620090
-0.29881873230115815162231753816472Q-39	-0.582553404291855432427164121917010
0.14130676637375739180904968528082Q-41	0.694100997012334302909002283897690
-0.56983405419129610364929887904321Q-44	-0.500276912150495409123331718637880
0.18874231821179957395223511912640Q-46	0.284018866058451479131781507868370
-0.4553844073065789302054821049848EQ-49	-0.136107212827001247563057941326730
0.33584499326722136235133375949795Q-52	0.565519804266020632871265402455060

COEFFICIENTS FOR $J_N(Z)$, $N = 4$ ARG(ϕ) = 22.5 DEGREES

REAL PART

IMAGINARY PART

0.47858339845587016575111073716333Q+00	-0.35847880080382417788568569890391
-0.60398352290372192075444818440342Q+00	0.40632428794241340157156783452084
0.16686671310962534204250806450641Q+00	0.16629085520708918341439311126107
0.40345292483443152819878830098438Q-02	-0.49766228978439606117485686238686
-0.54599958296902657607385931551026Q-02	0.39930764421894391102585095867109
0.62594895628520195833684291377430Q-03	0.13689949431864912037055080931727
-0.22303871755186601890395851948738Q-04	-0.39065293174135859343023948506874
-0.74375409592514713567040461212137Q-06	0.23233331698105377436414273246902
0.95416511092663066265959827097328Q-07	-0.44787001805626294222260790979126
-0.34549010757675574265238001805361Q-08	-0.13722855455804840086976691363690
0.40467415510471252652986161110957Q-10	0.10133317075579841544802066321847
0.11275241187728454178113936728650Q-11	-0.24624481751371634772507775934581
-0.54408768179534532450735646862331Q-13	0.18934026176918988576081649872867
0.94733620521586490822347936808130Q-15	0.48034252984740057810298659561047
-0.50549494253955906945767499549168Q-17	-0.16378486563169320587055261988978
-0.11730705133561013754648955905740Q-18	0.21415725677342929483984053767426
0.29810397569169469443188598648929Q-20	-0.82707787754565565932805977103269
-0.30326156803101673086132273555546Q-22	-0.17652586095207331527300040590205
0.87615839855007810460101209678226Q-25	0.34755278459372069147965657032196
0.17291106535658113621398305726830Q-26	-0.28278333730192236382839722367796
-0.27167393490286492253396894648523Q-28	0.62772607288472691330528617791105
0.18075318540894420124976071168793Q-30	0.11512085536535432206883422631772
-0.31516590654169590205189505675429Q-33	-0.14774859465916066435518648510637
-0.53956065858586768965740091431130Q-35	0.81857224904708833701748903342447
0.57644606550824575210687542980848Q-37	-0.11421196838474935790855873900194
-0.27000794992421892848321668707222Q-39	-0.18327913293503432305610207781976
0.30627909725211442080843686338935Q-42	0.16556396963851835725884044634448
0.46240912045606052795970146003952Q-44	-0.66412829031121063709651579001610
-0.35786702586617823244401408021218Q-46	0.62091148638872115048586062942334
0.12429939179327096891519231765562Q-48	0.88490360464235837516568810837560
-0.96935100469648331206828051897891Q-52	-0.59334086572687432172048695077262

COEFFICIENTS FOR $J_N(Z)$, $N = 4$ ARG(ϕ) = 37.5 DEGREES

REAL PART

IMAGINARY PART

0.18596588160846084309094512679226Q-01	-0.83327456781790560048981191072186
-0.10472813165080234206378655940024Q+01	-0.23379409460784507546257516418254
0.22861130673403583133034143535860Q-01	0.38891503655067472812879085674954
0.7692555679908994934031979024119CQ-01	-0.14286325306712697278710371127870
-0.33557411451626374023868589900716Q-02	-0.95408818610942142823813271639459
-0.80215233972674472394427750221481C-03	0.45280242563351429654277595834360
0.40560003213306685681161103405422Q-04	0.47659805819213334908405515569687
0.20314863038278054137770465596451C-05	-0.26024717800668946227450502460502
-0.12542025840556014666837386183948Q-06	-0.60716746336200177495444405774914
-0.11064181450750886837317709182936C-08	0.46876617621800903037316614470047
0.13892236963532422654121118373402Q-09	0.97471498252537357004611771529370
-0.74902431978003088282855914655542C-12	-0.33123550731520711449250499164713
-0.64000382981119738235785867234274Q-13	0.31360205936665601104936473951791
0.82851810909758728164620685698416Q-15	0.99972584927366436385250274418122
0.12375099649773140426733646397718C-16	-0.16632040990425994850169468169542
-0.26960075718497058723171199394135Q-18	-0.11345106774640789016052704771030
-0.56715828224072714334552468326877Q-21	0.36297308048243790110594255415072
0.41175150786585550139516190632497Q-22	-0.37158818768928323924375676445762
-0.13842387820378683226010717207477C-24	-0.39575847168963653286210650736910
-0.32129274350128994447041821970772Q-26	0.21244245531527735315247877664500
0.23833200381048606904541909263522Q-28	0.21661759951264747608971649313377
0.11537287109204902126134490841740Q-30	-0.21743494457732360842346240047560
-0.16780081282800572964898498501302C-32	-0.40544100927353537115808182204488
0.16133733140509804883410636119110Q-36	0.11148187329989242148751333303105
0.64178896905790992470364690251291Q-37	-0.17536106784218317662247602493017
-0.18269610431381703756363969257346C-39	-0.31936600106966392042919728288923
-0.13534635626020483130880298257246C-41	0.13205510126486170795507352037558
0.77495636109654824166525796198628C-44	0.46845368098875221406609068741662
0.11553119581033604062170251472072Q-46	-0.38821534581559215861432687310234
-0.16950459436444485726195934498425Q-48	-0.58251291865902313649433300485941
0.14600885484873033958444159950309Q-51	0.65006354456736548999616528861831

COEFFICIENTS FOR $J_n(Z)$, $N = 4$

$\text{ARG}(\phi) = 52.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

-0.67431604934308611183906304556193Q+00	-0.26291341676897096640229140081480
-0.19384667399685394490526505110124Q+01	-0.10410272755675281411060305581265
-0.59233593715042212749278445262716Q+00	0.39916376193758198657851175917905
0.23971444475100381657099174545464Q-01	0.12977804720043558884324304795512
0.15768278412395521476452111461175Q-01	0.30277141884742357301873331260589
0.71515747310179804266183596790747Q-03	-0.11976768528914976635851117083853
-0.58791794171506037653065322718866Q-04	-0.69757013492491852547086596624727
-0.43301765932833001368036144420000Q-05	0.17300020915226922485073314291293
0.13207252087101348729621187583082Q-07	0.19117669108231520650317366797943
0.62964438119594076914561829577855Q-08	0.15079100737933837614766970667741
0.93106555854329299084908176321447Q-10	-0.15765879506999989760538123699161
-0.29777678342417099397480758707497Q-11	-0.32381332719630002149171751649436
-0.81738005087246192417330977175839Q-13	0.39835900861026599368285284087458
0.27096226233180734423719463966855Q-15	0.16116595124438932458661503901991
0.25600789931960762789312329266193Q-16	0.30274676467340444573433841510132
0.14002308335500771724017271579351Q-18	-0.33142010984589437929659727108294
-0.34796865459696038123610256608413Q-20	-0.28130837431868063598349335463571
-0.40833208222370697616729917513797Q-22	0.28690261828368936245397622616026
0.16525889450185823510745383235916Q-24	0.47399344680835209688268176432528
0.45689660829940314655429361016723Q-26	-0.27322360504227708170750262702342
0.81053913406772060929555714980226Q-29	-0.37134018313482015620695320627512
-0.25487874704578276168750331917181Q-30	-0.13547524329257155625584437724038
-0.13883416501916940941936153789928Q-32	0.14568537857486493310057992090527
0.66180267040878354479056124869563Q-35	0.11103812491615846988676409157209
0.74040615097239470932487882932761Q-37	-0.20188322501461295774039235548459
0.18851177948392022575357220109055Q-41	-0.42238500444041098639977615069091
-0.20802598699412793117441998738995Q-41	-0.58593252142294032781581871951920
-0.53353034213845290509154128351856Q-44	0.88180322326020389086419555206732
0.31554168191361693905140440009092Q-46	0.33360738326654237044021637058908
0.16912935832630815229653489814950Q-48	-0.89973622724197353612595862148582
-0.16411630275697748522917547359959Q-51	-0.73176324326869683943942789831000

COEFFICIENTS FOR $J_N(Z)$, $N = 4$ ARG(ϕ) = 67.5 DEGREES

REAL PART

IMAGINARY PART

0.17478308810061849594833468364952Q+01	-0.66194758598597459543856475851872
-0.81451244107349084752959623722662Q+00	-0.41068919137690145718272612108368
-0.87536947783269953602379997414406Q+00	-0.82591853917344660800762182964988
-0.20578643623327606002146209271395Q+00	-0.16680959495309474153986863319611
-0.1934407013066888400815142686989EQ-01	0.13913724227485082576341646704694
-0.44358069041949821486810380139798Q-03	0.19379629177747191486997735033417
0.61373074511090406296438527436609Q-04	0.10989287885121069186491975715268
0.59490346685145652447576302702390Q-05	0.19606401823644366683330741958572
0.22863284191338328246400186474230Q-06	-0.10522755025430343106763309244622
0.31296575380685502381399149337765Q-08	-0.77317419591320704186286848525878
-0.84815184622302783172770734723515Q-10	-0.21592121889232789927792895111167
-0.4982859006096144376873853112648EQ-11	-0.23112022582694850348317639711192
-0.10600687748105359961439338405128Q-12	0.36384575514272293447168754258605
-0.90716133638246787980117319828966Q-15	0.17727624818029435085248755522649
0.90722982730813810143142598398643Q-17	0.29737712108072386055985444703670
0.37646215242868884468272127345504Q-18	0.20761191689326657707855296423502
0.51126045085775139077471353345781Q-20	-0.14045941646176158486032102789351
0.29635295121127430306177011345921Q-22	-0.50650973522644762394264706258036
-0.14215662900131027264781995002076Q-24	-0.56867571852340752506809503393893
-0.45259194996881733388269066502637Q-26	-0.27784910222795835267899756716588
-0.42731147865953245801123781185082Q-28	0.98016706940191268100398434974134
-0.17828256119295104315586809671403Q-30	0.27904022336114498223982253053442
0.47629207735237014918243098782943Q-33	0.22470988052046831713649118116499
0.12251829152061523705586244100378Q-34	0.80964039689815454223922197685256
0.85168555512182592205664516640740Q-37	-0.16780270121148288672268754223810
0.26761777544667120227006334359403Q-39	-0.39343392534244726163139429798784
-0.43906583618681659487027139423517Q-42	-0.23852578685130268957350077272058
-0.94526112759914035895224981949802Q-44	-0.65929186258990028428286548651990
-0.50427838224905459493005678993593Q-46	0.87107045263259653563603792420786
-0.12354561327896986402262314335323Q-48	0.17328877807787881999392630386195
0.13340812748734404862883277202692Q-51	0.81984853823596959338196585973842

COEFFICIENTS FOR $J_N(Z)$, $N = 4$

$\text{ARG}(\phi) = 82.5$ DEGREES

REAL PART

IMAGINARY PART

0.98122292247818075407110420560794Q+01	-0.49939468763089119938816864418543
0.48832354861099685524819165983418Q+01	-0.34758105255632773289114962151392
0.11028910174367510904147561175444Q+01	-0.11824930302905818102046366018994
0.13431951982640030883170246899949Q+00	-0.22990349715530874614607155512091
0.87143496211064584183815281719839Q-02	-0.28534639188487899037317498514712
0.13997349394348195614743773324404Q-03	-0.24292381939137248494408125352582
-0.27927636890605182623288025847636Q-04	-0.14866350009098283006055144222095
-0.31053041052811411580125929655642Q-05	-0.67331227706390201337431243081123
-0.18518525881004409104918159788508Q-06	-0.22855444429099529823861539448674
-0.77395561630068892908515044357180Q-08	-0.57522055291281920056346591454779
-0.24602039490695152607333914045855Q-09	-0.10026046825515525470128333263163
-0.61827845802677576602538615895829Q-11	-0.83105966738023440241376553208248
-0.12544170650210006493739696961146Q-12	0.15219340037548777728432249696751
-0.20748344205751257170963245911014Q-14	0.81888196131404784201987820749980
-0.27939457251182820209275218884916Q-16	0.20446783439118173708310906820989
-0.3003999692246258546604101134564EQ-18	0.36952360624357722633526357856376
-0.24134006897302081300238199079291Q-20	0.53236032666123342988001889124282
-0.1078572766846865752886265627044CQ-22	0.63503605050550036250646666474350
0.57297902530709159204788946863691Q-25	0.63827737536700596846556843661887
0.19707262951691293808292859786752Q-26	0.54388005065971298526212239907721
0.27109674309251565800998203063078Q-28	0.39118281044915349604334950218645
0.27267056870161387207223405438325Q-30	0.23253380070455849461228732917042
0.22349322784755007962772743707569Q-32	0.10719801438885866629098741109729
0.1552557595489677508222310777831EQ-34	0.29559852202312589321669281779444
0.92951885673193185419279612253250Q-37	-0.66231573004925627882633328620308
0.4820198932976701703736366086691EQ-39	-0.16541428636695416901109739968881
0.21534425077380531316532338471303Q-41	-0.14423806142699217190895795342110
0.8112600423381417226260110278181EQ-44	-0.92329936323118382813668100798005
0.24233021164315233840642291443170Q-46	-0.48783625879867703740192006025696
0.45159891653534288880407522684542Q-49	-0.22156129286009529837710376141338
-0.51950778537904488178642440347368Q-52	-0.87948052367414230767332603921507

COEFFICIENTS FOR $J_N(Z)$, $N=5$

$\text{ARG}(\phi) = 7.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.71697914072940655358743725506113Q+00	-0.128109638944144250475546817044420
-0.47386022697282716563049859782970Q+00	-0.604019315499663040985247668958990
0.14241000822124513469978699293594Q+00	0.409261145039929752548468029617890
-0.22942102954627750376381887415771Q-01	-0.143879519953692748824370180369530
0.21735212462738713192394363868938Q-02	0.243057908479333880683036968023270
-0.12157912694591354508852595915341Q-03	-0.251392213940505909951278284447200
0.30958160927312461489488614225558Q-05	0.176616161747996759421275870987010
0.90791005990642409076600361788274Q-07	-0.892432342489305655237800020575630
-0.13170916173648693904323192527625Q-07	0.335036130692219343248387738270400
0.71414678281173427867145611364911Q-09	-0.946690049960980500527486489881440
-0.25923272126626176198775258705894Q-10	0.198707488959473888640618708880410
0.70809469332575257506240448526595Q-12	-0.286509857163579035768637560541220
-0.15260319342339457193827018152257Q-13	0.177548089299632340197307556607070
0.26555888369042496294964012284251Q-15	0.407093108069402758357649444371420
-0.37661221044883500514132163247608Q-17	-0.167930196525941826348520665473900
0.43330139470785595892640321717940Q-19	0.354883606797221528463145205028580
-0.39300755322157125335656254858107Q-21	-0.552574495523743031317984793401290
0.25410247270471242011113817105137Q-23	0.691220457129111462231792586552270
-0.62475343329925297792681888879026Q-26	-0.718921239514712730746652168705210
-0.11392523885996735011537255092424Q-27	0.631058222276479939573264116804690
0.22283904212773785077478549522770Q-29	-0.468918735950930949473086028312060
-0.25021610813047943440791944704873Q-31	0.292048371271085743761957285487100
0.21626872452540393437799926246575Q-33	-0.147267412566766839984437761025170
-0.15503792215307211064377192417985Q-35	0.537026467429503906738654270493010
0.94913020464894254857724757823214Q-38	-0.658312966008045842632221034578650
-0.50199261387064423797917935184703Q-40	-0.990423004822367103304988086329700
0.22946865654543267788085145016494Q-42	0.113318776939083012541397786710790
-0.89537870504448266688389759635979Q-45	-0.789238838289868353734244664286120
0.28715083805539291671157205126166Q-47	0.433859933573004885927619875339240
-0.67052186969410863999365125272740Q-50	-0.201587077387861063503548008804570
0.46923818429544725569882622053428Q-53	0.812959984956509972966878515839730

COEFFICIENTS FOR $J_N(Z)$, $N = 5$

$\text{ARG}(\phi) = 22.5$ DEGREES

REAL PART

IMAGINARY PART

0.61582012433420544087143086184347Q+00	-0.42496226419995419286442997941643
-0.58343349628173330704266007465410Q+00	-0.32313793252360843526984371325542
0.11788340079211215724294026554150Q+00	0.14480312418335805170809180369722
0.55114431191899834489904756538639Q-02	-0.33331775822068792851375680809110
-0.33810804366530376217085755938205Q-02	0.21423205265948833277155326541076
0.32509988741072670952114504187016Q-03	0.89881710394517952935349562779944
-0.97347339053242940752565900864295Q-05	-0.18993166481254630785621838678715
-0.35925766887001947290237286617942Q-06	0.99084978703447660425667315402069
0.38412997117060888382217399353174Q-07	-0.16532195444299687253879089967228
-0.12529754199096076266020138004850Q-08	-0.53876013108327919184685990995360
0.12962684468872675308852299710919Q-10	0.34909650696899313025517948282665
0.37653165423365109051489025444587Q-12	-0.77777763313530012606867003005537
-0.16406004710931099561085353213880Q-13	0.53615890849128839949177520330440
0.26519477025958821795692168548691Q-15	0.14017547397980150953644147806676
-0.12833434962539365149233053981520Q-17	-0.43948277315086309117485908004930
-0.30468544147510900160849941440818Q-19	0.53850779895023791374771257610717
0.72094795341659825117788492771806Q-21	-0.19035678596373409087110590798898
-0.69231210956933489758852324864241Q-23	-0.41361626678188108925478820369917
0.18446222343867152808710836554396Q-25	0.76529561000659887858070693167937
0.36932372130467720638983248404330Q-27	-0.59120094153413179613737357634784
-0.54919384021175787593228186921171Q-29	0.12179618719205713474560226369690
0.34857917153892919364824406367005Q-31	0.22603285587205648032473239983699
-0.56711142575671630963031283454505Q-34	-0.27610110044807089896316374895563
-0.98059167978067165415336582134552Q-36	0.14650759270848131886234730451142
0.10016372200594681966882082484507Q-37	-0.19161853593672500857112748670740
-0.45086065675090317186329685555353Q-40	-0.31010655302462742668594398336326
0.48134530731157193798330808692913Q-43	0.26884742460879747211331136541746
0.73202625326800317699746247565513Q-45	-0.10393302923292382143547373280210
-0.54543725582602672918138773426936Q-47	0.91779411546325856528312888675006
0.18303522105556231803122942271802Q-49	0.13163196117806070098622256426087
-0.13524684808351802967372348147330Q-52	-0.85206454223712966468409181811453

COEFFICIENTS FOR $J_n(Z)$, $N = 5$

$\text{ARG}(\phi) = 37.5$ DEGREES

REAL PART	IMAGINARY PART
0.34271673950163347231035737673710Q+00	-0.96997600846342206562275998669492
-0.90395145645904504079575341243072Q+00	-0.20010106263287621584900699048914
-0.24332167615711741973340595630920Q-01	0.28695528475334629442706664096804
0.49998478659224818611522032055104Q-01	-0.36599080067565330240198263053891
-0.14186604474643848213569897154760Q-02	-0.55520468259598492105133906234942
-0.42309091080541084254334064191816Q-03	0.20026752188886249285692876782968
0.17347393353851377306213391080437Q-04	0.23037880182879879602564535572771
0.91062882751462548428649117203691Q-06	-0.10526470076387639862285311108995
-0.47671039669559336403597319006625Q-07	-0.25670358996101463835218623527372
-0.46142139953847916911132687994639Q-09	0.16731566143544227154904424365605
0.46639403997328763238733161822253Q-10	0.18485270051813267883765069199952
-0.20568169740169357679978548848568Q-12	-0.10487591332221130417986276068500
-0.19173217938208356309614082976415Q-13	0.87956765871607055183013939264241
0.22502493730692870952488001784666Q-15	0.28448730690346123678701991110462
0.33629084127059927188361924102595Q-17	-0.43313414379833854058885931254798
-0.67175780662330658806086383663502Q-19	-0.29746229171361208816143622504195
-0.15021647882146584728224300184942Q-21	0.86550824282389367863091671135096
0.94065741761849437918248891109544Q-23	-0.70330548295066949916128271759833
-0.28981427086313735482005493702998Q-25	-0.86761261623173101256083530682410
-0.67728077561812485649204244424999Q-27	0.43532400850248549441176642983467
0.47290291910170865176932086864137Q-29	0.44025174631928035797790846433849
0.22712985897226654629537788448812Q-31	-0.41685131288108337968847344709817
-0.31077866036750750463580637244007Q-33	-0.78476726437269298322978283090087
0.98056591883695818529040791379658Q-38	0.19958683771471104547915481495776
0.11117563959062473154599691517676Q-37	-0.29314408553573895839707189569523
-0.30069488114383697187470214663688Q-40	-0.53595955518948538697058814951029
-0.22040763450304907755485113466757Q-42	0.21170976937587968419163609837721
0.12077744786114739398624158169293Q-44	0.74225129733534441407511066885997
0.17943104936990429284669869658933Q-47	-0.58804257558962812083410259189938
-0.24963558961590501760396491216548Q-49	-0.10207215535770114618693572554814
0.20319543904364873830470737754352Q-52	0.93142766633216217154426800012190

COEFFICIENTS FOR $J_N(Z)$, $N = 5$

$\text{ARG}(\phi) = 52.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.19134774173865152867894503974156Q+00
 -0.13410036861085022606683030321361Q+01
 -0.43845603418680898540349596696354Q+00
 0.61723344099748239624731835591553Q-02
 0.83628449662742907965501530518177Q-02
 0.38941724828587909271943702193489Q-03
 -0.24426146492270525619863766217234Q-04
 -0.18102382380689528814035564973803Q-05
 0.19798766584757163921666068826589Q-08
 0.22039112038813480231647834129789Q-08
 0.32385885237303927509881225123713Q-10
 -0.89444335278397792082285264911973Q-12
 -0.24150785889810286212583154564851Q-13
 0.64815422360608350759096065615807Q-16
 0.66810843898091770639647078808109Q-17
 0.36323083665441114503133301659175Q-19
 -0.81343852763051679999975690106615Q-21
 -0.93381340412760644772153161873733Q-23
 0.34354871054664556888469183956357Q-25
 0.94911484550258641248308930651720Q-27
 0.17031444136464931770506196305733Q-29
 -0.48578743909211572065632292397634Q-31
 -0.25909913755614495647699509562557Q-33
 0.11599071965002320372923680273074Q-35
 0.12764681207709623193648329237417Q-37
 0.89474368220430176750940843347451Q-42
 -0.33406772105841167518488778953415Q-42
 -0.84197635403236285158640135286311Q-45
 0.47371904850503272461772572020089Q-47
 0.24912839938172068348751578684156Q-49
 -0.22767390100986997949332285831625Q-52

-0.24621797733264907490726025545557Q
 -0.10925911015053865326946488806677Q
 0.21513619283107714741827276750709Q
 0.79512388065904021625782908215002Q
 0.23474394674254074409846915746912Q
 -0.56069918286336443459608592104707Q
 -0.32607368361384709229266348331966Q
 0.62298702757953877198009569117274Q
 0.72760890415806648753213503892222Q
 0.61308240928066997917732599110159Q
 -0.51061877122492282614434471318734Q
 -0.10291920254667079559672065587164Q
 0.11032363335313317721143242043792Q
 0.44637875532339116908256921772867Q
 0.92695868497277372864323456939681Q
 -0.81777623158300666102154970805874Q
 -0.68024248613859478551929741503987Q
 0.63518486528845012919658225809825Q
 0.10312858033414556302512365206851Q
 -0.44571115598291411135547343877146Q
 -0.73827439523320975453933519285429Q
 -0.26507283915298855460223939861484Q
 0.26640567095113023715126789125897Q
 0.19887083669250565000767186380717Q
 -0.33594843850021357839936790641538Q
 -0.70232016002124310119030206419005Q
 -0.96899025295758418518831775562328Q
 0.13689552677635972937037121469478Q
 0.50777783887743460639513354670058Q
 -0.13046961777552005092621075246251Q
 -0.10450387871500442866662451515285Q

COEFFICIENTS FOR $J_N(Z)$, $N = 5$

$\text{ARG}(\phi) = 67.5$ DEGREES

REAL PART

IMAGINARY PART

0.23981379237132517651678309376418Q+01	-0.49332005550322876240403378150219
-0.20525674795803005230639017010004Q+00	-0.30257707611769453018524393008394
-0.51194189051474125210311990172914Q+00	-0.58325256469461066400547963303387
-0.11783901280303569235507495247674Q+00	-0.18290027235794489385091299792875
-0.10500271911249658774046290593610Q-01	0.66532007276168685709121517785292
-0.25495321324383426263268840799403Q-03	0.90914482585477594825575055970468
0.24763307785399520504172435767361Q-04	0.48901936069782801616763133898460
0.23619067867422398562538461954382Q-05	0.86994828831022909926947095052887
0.86375897366343845473995894075233Q-07	-0.36680265543016231410038321609717
0.11573140285259174918407815161511Q-08	-0.26588973541614487286309642604648
-0.26006557181715955841502576722893Q-10	-0.70908148587686079184055972174759
-0.15105450592156427261576262018224Q-11	-0.73797572583348057577439218023133
-0.30791385089310038253611344832790Q-13	0.99539206309860815708016081160496
-0.25562927563562604519602401344136Q-15	0.48030180444792037369900694485769
0.22398653515025010501105501440846Q-17	0.77434853790876539735917255964337
0.92173892471594089605724009443909Q-19	0.52427875323017189629167236493557
0.12063536516977147054900456112827Q-20	-0.31590465190207526796035066376035
0.67849484691602976594861482687328Q-23	-0.11309855460939566989780327034812
-0.29354428529449662659315802493393Q-25	-0.12266683691817696698583548881740
-0.92869223791201824147869710746236Q-27	-0.58206345625496193087921775258394
-0.84885366148441779760122510864205Q-29	0.18706323500715382067432324155565
-0.34431947043564024345830517583079Q-31	0.52958975093840873151205902842675
0.84491575361621556843577567692772Q-34	0.41366296116698784770203645319536
0.21627059020978384782016328300234Q-35	0.14506697038797912087563179517765
0.14607248631959377750377825159495Q-37	-0.27805533221413310449715808418083
0.44723917842882512800766843018665Q-40	-0.64906731846530337450547760833622
-0.68253755818347128492248362790094Q-43	-0.38292295113147857913636315688168
-0.14636224629711351429111469867286Q-44	-0.10324211072637959651653408350369
-0.76086249459202642169895219398768Q-47	0.12751643615142106246013260423691
-0.18201660115155374344210634802803Q-49	0.25277283802978419582854387204241
0.18453475319867493876068106760685Q-52	0.11667993587791944742690100267513

COEFFICIENTS FOR $J_N(Z)$, $N = 5$ ARG(ϕ) = 82.5 DEGREES

REAL PART

IMAGINARY PART

0.77349979465329318658546296981541Q+01	-0.33337737945430332020785331822040
0.35075052598263660970318803005207Q+01	-0.22695468047467481713854286943611
0.71443640519736975728515979971239Q+00	-0.71689778403900381159275803332267
0.7909867416491719654042517763099CQ-01	-0.12766841129833068042824049968300
0.47635422928683202567967169104111Q-02	-0.14503000552541207882153714433030
0.85057037674741627071044035892395Q-04	-0.11338848564491021574635009275998
-0.11064722576980576856811563465542Q-04	-0.64024904622688522266296189188318
-0.11998577899169309507249831446954Q-05	-0.26889678760726523523917542463344
-0.67433584450771308705753034016955Q-07	-0.85075706289431982940497870704962
-0.26479788437897489042152562565948Q-08	-0.20067530116799382346078393854608
-0.7919500548864083297544777679414CQ-10	-0.33036628169209694031425533556169
-0.18771615506343912389516237387905Q-11	-0.26627006678828183716080535920222
-0.36018508382575450479477611315621Q-13	0.41322902176562134088424054770373
-0.56496473798025474379797909202836Q-15	0.21916958512725268521762054207747
-0.72343925527921217857359116190832C-17	0.5235928564900008489629717777410
-0.74182790674065784409172309471776Q-19	0.90365643127296627958175243504617
-0.57063901756023131531684348247675Q-21	0.12442865762629000926192798331197
-0.24712867953199784017560938568249Q-23	0.14206683857334947311902894521244
0.1178320323405318448437221258660EQ-25	0.13689180881125465761860082471472
0.4015861996054932842312439237948EQ-27	0.11200908316562191312083639072849
0.53342875845033502394389804656701Q-29	0.77487445341208910722474285286176
0.51739318135375228665958337607733Q-31	0.44382679251979921401725076996149
0.40917121152564182734512483890653Q-33	0.19761425011009313541383961161105
0.27450445228161537587158702453834C-35	0.52978002051457576193702224869911
0.15888089856084433682882839977000Q-37	-0.10946677236507294452981794496274
0.7973615365144997204637594449593CQ-40	-0.27167554237844459130698663625829
0.34512592410292499136373996343494Q-42	-0.23013817276606964471805676181622
0.12611767901219927208604462626159Q-44	-0.14297612472677462627865456524270
0.36599881383366475896982404340221Q-47	-0.73343530257379338679838909221485
0.66540749249636374486452199874224Q-50	-0.32361378570259356900254620438174
-0.71734115361082564402661579165059Q-53	-0.12488761173150582953980327891663

COEFFICIENTS FOR $J_N(Z)$, $N = 6$ ARG(ϕ) = 7.5 DEGREES

REAL PART

IMAGINARY PART

0.80676508810935050866389000355579Q+00	-0.13780071664201512085207947763377
-0.46088913551145985266485501920205Q+00	-0.18877525023441000943662164734721
0.11784936564605370833193210254487Q+00	0.37413792697462656845640131191305
-0.16439247295602547933852153917099Q-01	-0.10858019138824082019527292154003
0.13702597724933993845292787810515Q-02	0.15945071157794719799293484585418
-0.68004514384715340674986265768709Q-04	-0.14675663460834850139640320266140
0.14984368957087634251930639081130Q-05	0.93128275351076748719106021578018
0.48907771938461336916799467250149Q-07	-0.42963931755820887873490317796317
-0.6014751107125381016425874178274CQ-08	0.14847697452774943088954273552573
0.29871258699758837733144888272972Q-09	-0.38864854125057143105408927816290
-0.10067118833415732687586251703927Q-10	0.75908314042510323655122519566534
0.25714008298803007167080962317312Q-12	-0.10194998987014208984259612299192
-0.52085557979931708110624289056366Q-14	0.57296657549389538341171236980920
0.85533572528254522637681521551969Q-16	0.13604447749454550773291581890830
-0.11484804084195065137036093104727Q-17	-0.51910382580478751212989367180947
0.12543856070590710026338481175622Q-19	0.10370506717878638289860709415343
-0.10820953383619076771267990889336Q-21	-0.15350300239760288212798659446430
0.66529788222605675753767399130950Q-24	0.18314411070053210158080693285258
-0.15181435765504198352168669757113Q-26	-0.18213774138905665935837338477823
-0.2816990212194604074708389267698EQ-28	0.15319125317303572112809519567618
0.52326981151304620988392011896808Q-30	-0.10926336792826809853150020984078
-0.56361937864428499900027807519209Q-32	0.65414987687464802365145659058527
0.46872188032304092440963531114329Q-34	-0.31739969881432443050767599975827
-0.32392014252392090825629299740567Q-36	0.11132485146649071026084162054107
0.19144995405801748828637010710979Q-38	-0.12844500861993789314933591807996
-0.97883352708582646572984119947608Q-41	-0.19527451137261176376289147312296
0.4330065674735327607290260141882EQ-43	0.21490278726656124647434238403550
-0.16365735297342243078434089688647Q-45	-0.14480117089327158795730669824456
0.50869852603275634441680436450723Q-48	0.77154607838054486499981199242130
-0.11507952003693125685828801035859Q-50	-0.34790610489354615754037572472705
0.76523084257061105999162035958139Q-54	0.13629762043775460550940469259246

COEFFICIENTS FOR $J_n(z)$, $N = 6$ ARG(ϕ) = 22.5 DEGREES

REAL PART

IMAGINARY PART

0.73941693799319228961106522299145Q+00	-0.45971495114286007763486274352170
-0.55180401914729574996363085620735Q+00	-0.80479776234320073956900562516981
0.8631831210300063399204651193055EQ-01	0.12483576774351821342803827812942
0.54382502621573430575577853728728Q-02	-0.23374419361471786644166227334193
-0.22054444878210194530871195651259Q-02	0.12374019193679886385256173857852
0.18263706617239145650712816288541Q-03	0.59386296477694136575075309886541
-0.4672223753141893615729222943562CQ-05	-0.10030840830753316298240537266467
-0.18617238102292422537462799875633Q-06	0.46599235930515123961537620118947
0.17108478687766381081311195318362Q-07	-0.68043880471541829682885757677792
-0.50762213563545302848008517837359Q-09	-0.23305973971329560538443986491257
0.46758520652284039340202686071032Q-11	0.13469351254590762883082706067710
0.14065676883593647314003180503707Q-12	-0.27701001864890544396071744140653
-0.55898527572378171541145066119484Q-14	0.17224972850190811584449685634060
0.84307534733797442829638516981381Q-16	0.46218647893576294325489884079498
-0.37180967588731086650773466698002Q-18	-0.13415624637610936868959230383668
-0.90062438769262620466710445185483Q-20	0.15464245199993946575365379856939
0.19941688098865125079249766782967Q-21	-0.50232911394772528881879588348899
-0.18131758796889015011207028205135Q-23	-0.11090626445905494972724738214666
0.44701103851198128010420778283332Q-26	0.19357287722151146235931792731209
0.90672499423026645301242810932216Q-28	-0.14233335988179936439343269871507
-0.12798948096355301168670326352053Q-29	0.27289832919070749477427671199167
0.77658089929968644277993282954271Q-32	0.51196639361199554940067913418846
-0.11817064506487098491077379610602Q-34	-0.59662884332280296847808989684503
-0.20620583492394106488828686482966Q-36	0.30374723840670816634872818597694
0.20178278281496036132684253933191Q-38	-0.37317995163634399211174852462389
-0.87413287826039312443309883788475Q-41	-0.60868250157054892054540565110174
0.87995512896511146746270779997995Q-44	0.50727956222046226241474681553836
0.13473264754759400666350387190308Q-45	-0.18924141422471322672348539284928
-0.9678915505918888254531661279449EQ-48	0.15809776124216565249523738998825
0.31415755621425619002455061015929Q-50	0.22809406016992515769116748844928
-0.22026638667568696687844098357002Q-53	-0.14271067803962783390192804577425

COEFFICIENTS FOR $J_n(Z)$, $N = 6$

$\text{ARG}(\phi) = 37.5$ DEGREES

REAL PART

IMAGINARY PART

0.59791847516846201333321681382456Q+00	-0.10138925181793876448333670749771
-0.77849966495065979972747277041191Q+00	-0.29282108681861015407153689452553
-0.42945480398654701289458123021156Q-01	0.21814761914676599578892088176609
0.34140011486793917731170260648967Q-01	0.49659887117033386828275016270880
-0.62136562949498050837089239138918Q-03	-0.34417215373434461522083172221258
-0.24023907499596121082702947254469Q-03	0.96670713657808698138744126685588
0.81812657536546559208930049415029Q-05	0.12084610622312775048643040663767
0.44539811416463988084534015236778Q-06	-0.47216432464143905095808786318155
-0.20188140517604763424445517259966Q-07	-0.11861835758633735952366351436439
-0.20791412455281848822623704018136Q-09	0.66820231937899467629150810885066
0.17586761769357432323298838027133Q-10	0.12034333526320469599434862708072
-0.63534828267235255916744913938097Q-13	-0.37424336233842247560899463436901
-0.64931327063186620593941493513262Q-14	0.28013834829232669085557966024143
0.69561637234886707742987194042774Q-16	0.91747569846380190725986464065819
0.10376332378111067224382832257124Q-17	-0.12864921290694863295364739837467
-0.19128844001354909212443527856585Q-19	-0.88587412758242879672463714356421
-0.44766107260206429151075898163259Q-22	0.23631423071994323435265648552157
0.24650947035124268072056739844988Q-23	-0.14943509069864245254315609666142
-0.69870467631892834907611679606149Q-26	-0.21854853491350164911550408190271
-0.1642888827929692113943665367192Q-27	0.10284384633464902878753953023408
0.10830984376836693162501731557028Q-29	0.10309134192521732359597010925520
0.51558567295837970585015559677152Q-32	-0.92355555780688418976424318522244
-0.66596522074846735906277637549837Q-34	-0.17495422635480049175033659393981
-0.16575332039199958930475129285537Q-38	0.41389793570252495496020444667584
0.22331685716401407271224555916573Q-38	-0.56895707546759042182309679956386
-0.57503403233200369974051239833068Q-41	-0.10439849210680716681529308689065
-0.4169664281813708699666688458061Q-43	0.39466690057785341995293153229063
0.21905315668683213928852398440782Q-45	0.13671555343500451504690585333735
0.3239471836903993219273406510222Q-48	-0.10374056730774920385083177935764
-0.42852143265120158332731040041578Q-50	-0.20162267049982052139714734053368
0.33012664563064316332028508212176Q-53	0.15566981093014673948227191071558

COEFFICIENTS FOR $J_N(Z)$, $N = 6$ ARG(ϕ) = 52.5 DEGREES

REAL PART

IMAGINARY PART

0.72880865190574268679251864008511Q+00	-0.22459881214715900397338701099624
-0.96425133112969332306567531815111Q+00	-0.10516793395805829880270997232893
-0.33361835159442340246772120900557Q+00	0.12115209723867529112195795650587
-0.3922003915614545153182563027829CQ-03	0.51824649772787858073574832098151
0.48066299068718551832987551154652CQ-02	0.17160781821095187766920945207494
0.2257327607880816100032549753557CQ-03	-0.28805008063744348656371976693551
-0.11238257038418484659226522482956CQ-04	-0.16635944565181479102434657222697
-0.83636506418762052004298852364586CQ-06	0.24920244839226122173722253352699
-0.28142696275241310774390909064381Q-09	0.30862496941498428388700093916047
0.86525673536666457414188936236397Q-09	0.27063028605929173413858419402053
0.12575220341073990450268172917758Q-10	-0.18647143117600610332050033946506
-0.30431078333128292401326000942775CQ-12	-0.36836414010873652760060908320147
-0.80786807888660165726994678427954CQ-14	0.34740081727322211800426016440181
0.17520903794704418362416769645794Q-16	0.14053719582279686737102962343645
0.19886069656763866606086903924863Q-17	0.31300739967123650924913170787750
0.10714280370520276066182180575875CQ-19	-0.23081127182031535167810768481553
-0.21807546224960095710274407319404Q-21	-0.18806132103101893757879022323771
-0.24492901858384869886254453639789CQ-23	0.16166412666855368968914077731454
0.82284000250688778135442619613680Q-26	0.25797608461232145428484565928193
0.22714897333221194852725237124653CQ-27	-0.79764900852587435441835232538654
0.41041951631005733249448858259595CQ-30	-0.16941412674490923525954193103548
-0.10704565248484143343946107311342Q-31	-0.59831371216260648078577241908481
-0.55912932553793273024180400363555CQ-34	0.56410297900239726744206664777723
0.23575547002003021125449578013627Q-36	0.41255495873639618055061305583028
0.25525811763476865673300313593224Q-38	-0.64926086084096782241692868129469
0.27807861757454375705148390663103Q-42	-0.13562546777422851091580668819366
-0.62378436261728790621511188375537Q-43	-0.18587588043146318203375723014668
-0.15450037385497971959287147954777Q-45	0.24737780950238512537191477208309
0.82868500499325788951305294026105Q-48	0.89987375773766129842624500649176
0.4277204387050557544662337872637CQ-50	-0.22067678160476405316778681394844
-0.36878990471325826240149884323139Q-53	-0.17411409639774390625054979359825

COEFFICIENTS FOR $J_N(Z)$, $N = 6$

$\text{ARG}(\phi) = 67.5$ DEGREES

REAL PART

IMAGINARY PART

0.26686814968878098551887841638511Q+01	-0.38566887899593543481131043492392
0.78344190259833645257475667331147Q-01	-0.23391107953248144311102470020680
-0.32292538088035792049451217961411Q+00	-0.42966659709492329767249275191072
-0.72928593590876668489275913799486Q-01	-0.15852517335496444552537338613264
-0.61624484835869074586556264483015Q-02	0.34933558805000733815923309545206
-0.15257264783412751807465136932435Q-03	0.46922339442817807998490223225794
0.11113647007517654960210735373676Q-04	0.23980939455874832722294775482232
0.10446419017318243649827129474330Q-05	0.42065321833744243742467190080273
0.36420306870829659951916745774345Q-07	-0.14360257633639851221839538802067
0.47573461176410952203207817121446Q-09	-0.10282007540569263275635336221926
-0.90239155539605360968670027433462Q-11	-0.26230980647170245807111349212806
-0.51868477432666022843953184657758Q-12	-0.26506936327833174718157851245421
-0.10146813534128707325341291686457Q-13	0.31002161574732930026119969395554
-0.81698752532820199192667794266615Q-16	0.14826030352995616222852315930778
0.63268152510156339798508869414395Q-18	0.23005184179786545013189110899713
0.25835361350463755925242547218396Q-19	0.15108518112747853044125373357102
0.32626445596393749713102914094517Q-21	-0.81620051818932454386731755523738
0.17812898331546827364286979354464Q-23	-0.29025126027545644832112180755510
-0.69873980349349170822723406591566Q-26	-0.30445156731113085242411128191022
-0.219758839291373579422035111506363Q-27	-0.14037532498599880860542690802761
-0.19465161070503496760879472617773Q-29	0.41275664426924290142191362094455
-0.76806148996394754474596404542289Q-32	0.11624499924391237638826995448328
0.17373161044497604085357584971928Q-34	0.88148427342493473293395077000794
0.44262985360055269424960176250173Q-36	0.30104606993124039674227384936926
0.29070092432914705260599398129742Q-38	-0.53524915524312592878913533326825
0.86774203134134549296923952438545Q-41	-0.12442362340790862181691499357944
-0.12350006180597162245485213538793Q-43	-0.71481040393821279801595298208891
-0.26383820729270324954745055137253Q-45	-0.18808988300036882743511207596817
-0.13373721587310016863216633381405Q-47	0.21765951183228411904752456701227
-0.31254920859389443884919275507354Q-50	0.42999593030318774997823761987975
0.29808943754834403623852840982306Q-53	0.19376983186970346674239593811871

COEFFICIENTS FOR $J_N(Z)$, $N=6$

$\text{ARG}(\phi) = 82.5$ DEGREES

REAL PART

IMAGINARY PART

0.64584092221188028552340685610685Q+01	-0.23974937669811204548064557301272
0.26770998424946845036935829961222Q+01	-0.15999201326218025089360214907410
0.49532144242220337184368949200128Q+00	-0.47088670102653785094903161723906
0.50181717727637203427036067840617Q-01	-0.77239123922754673535049245358143
0.28123341582207865707769231210398Q-02	-0.80798254150276132694612483617170
0.52430051046393929415193963286143Q-04	-0.58355997372349834647120258458809
-0.48892307795184468956121478553307Q-05	-0.30568272937352116868443013372650
-0.51841936816051268917370766111635Q-06	-0.119634523146981706221104491801058
-0.27546723355146446428590534779156Q-07	-0.35432504795850204635849456200963
-0.10198355566130362696530598846601Q-08	-0.78621750820313444448260036784148
-0.28792420152776095660738152225744Q-10	-0.12258919504338644054869398451337
-0.64566823672422396943329654403846Q-12	-0.95887309022033657535598792080733
-0.11749704866514541744472586032596Q-13	0.12783518898239791145488128828380
-0.17522489575848900193874925069082Q-15	0.66912825477448611410011102668206
-0.21386316441855869046529149162578Q-17	0.15322819128240611777168088878110
-0.20958409888510879433570836945748Q-19	0.25304363795000196854733494355366
-0.15463022803048504581452200750894Q-21	0.33364996111982332439324005662975
-0.64921526534518305191274241116932Q-24	0.36527463452070668357254896124227
0.27942593012517230458565867021654Q-26	0.33798862614968225334958424454515
0.94421459741907238483671624993016Q-28	0.26596962448870945271942907201894
0.12124984698756172756456453678313Q-29	0.17722793224895087736408622903220
0.11355292096120059856756880317380Q-31	0.97939162990825860274534107570517
0.86749684240994056202719252732028Q-34	0.42165620999321678539083366676771
0.56269666288943928484204326294901Q-36	0.10996662096849169512242760144136
0.31519652203709272099261003842970Q-38	-0.21021548093890056980432627261695
0.15324505445793998275566982427042Q-40	-0.51861362268575169934261335527270
0.64324877223416789374838597833587Q-43	-0.42713667894365505983675052477353
0.22821012988003485059247487424566Q-45	-0.25776649967215234542950560511858
0.64393299367210773058260377200490Q-48	-0.12848580123192079244402545507344
0.11427215486278640734366537409242Q-50	-0.55120636588251145883983950126820
-0.11568409573653474904151691798186Q-53	-0.20696570512003992931044896319682

COEFFICIENTS FOR $J_N(Z)$, $N = 7$

$\text{ARG}(\phi) = 7.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.88665053525695709821664032960456Q+00	-0.14302587433459123412575197209746
-0.44450430309116214029745547534549Q+00	-0.28119796502019966586890435731479
0.99027016674659560919573209892910Q-01	0.33816170406033174365998517126072
-0.12194285129914436396082726549395Q-01	-0.83892872999271506905463273206294
0.90789939250908908928092113101901Q-03	0.10916035943177903635777370271745
-0.40478345273411547169861443266056Q-04	-0.90603357609524684962577164751264
0.78051568203016266291798854507995Q-06	0.52459436898970143272706567192580
0.27673004529168129288851670538218Q-07	-0.22275304237591171555795735552998
-0.29644020892467845113459975360786Q-08	0.71334881858772650444008168095279
0.13589273992049095694435080707637Q-09	-0.17395041499018701297685616388557
-0.42752993036088469921223240621076Q-11	0.31770104197449878974959704440649
0.10256893663101488782001994153247Q-12	-0.39926844204143267486890247896549
-0.19600775712210566774509166066264Q-14	0.20450011102538363639392292115801
0.30474709925016589277773755930300Q-16	0.50054641214661228307350022377313
-0.38854684908765449939702893841613Q-18	-0.17777448607244123096424511659308
0.40392474262579901534966456500912Q-20	0.33679849828253496638583474851277
-0.33220460743053090952008523377684Q-22	-0.47511286691556044551419178240082
0.19467750613033206046729574940427Q-24	0.54184893735316026760784529720936
-0.41345150088356450953290706418443Q-27	-0.51627571332963318528127694544875
-0.77940165758503185804837480523773Q-29	0.41680882321448726796766720641214
0.13786364906882110721559690693281Q-30	-0.28582556857216014785102709610974
-0.14268919829818911216897277961268Q-32	0.16474360777379809369367817690803
0.11434246811124855347077887353634Q-34	-0.77026004178718318181703328965969
-0.76275018676544476294997779576619Q-37	0.26021811749411649983785176671060
0.43576857910745334563744105653734Q-39	-0.28310420974997790330506171632966
-0.21561567429900934507597983893771Q-41	-0.43459968436394046312542793439380
0.92402253917718363533866466741362Q-44	0.46074817092988675831184796404767
-0.33861961309185506797537201825382Q-46	-0.30066856200842829758450621158326
0.10211066743432958166480730029203Q-48	0.15543295367101343005597846305886
-0.22400507134798906229749081570421Q-51	-0.68079102177762572421527088091320
0.14171956714886892215509990144347Q-54	0.25930916147768182637768563985671

COEFFICIENTS FOR $J_n(z)$, $N = 7$ ARG(ϕ) = 22.5 DEGREES

REAL PART

IMAGINARY PART

0.84749605082875813998779687463946Q+00	-0.47551549254326782349968011606494
-0.51767905736368488285138240531923Q+00	-0.11232703724051578979200425757025
0.65084771968519124975736466882291Q-01	0.10771850198414833867170756479256
0.49006287076922720600517790092045Q-02	-0.17001197969398738115015863778062
-0.14998587243086056579988481579817Q-02	0.75692309063034860956142508640716
0.10906245527960706458620217913989Q-03	0.40022318350886595760940709593911
-0.24155661234486758255755511241262Q-05	-0.56521409484930209819224149441226
-0.10222977110163495338499703619618Q-06	0.23661638059528713597539380327024
0.82507774480551021983841482236135Q-08	-0.30520968678819455698598070825520
-0.22447797850684710361162993852820Q-09	-0.10892256692851399621592254089402
0.18539140682612102957802958621308Q-11	0.56858113603397871940342191474537
0.57456978576674299474156989559835Q-13	-0.10856605327467600231944023572481
-0.20998526455834716951603261067802Q-14	0.61226948133927318315747465141227
0.29679703714110004906196687719143Q-16	0.16805052405390594807388729742036
-0.11980905920097460727285612061969Q-18	-0.45422508663574000872484928795276
-0.29542226065281848121833530271092Q-20	0.49424820778658436153520149302739
0.61474670912852531754330143528492Q-22	-0.14806548147502867557056927667680
-0.53070329172271642942647066040934Q-24	-0.33163338587079018799404653353843
0.12142791660950098893710510370081Q-26	0.54783395190963642998434095942989
0.24923837291854654659130766383687Q-28	-0.38428366931540691150721820667231
-0.33484691996813793927930930799352Q-30	0.68748672548266669332193027582388
0.19458934814311065341091122371678Q-32	0.13026069984937644994769785588573
-0.27756718365014847348201943622432Q-35	-0.14513880977317029958213155134707
-0.48845465149940292096125794709562Q-37	0.71007959211552105567342113322885
0.45872442550435784546392998388602Q-39	-0.82108696361858507904932833860944
-0.19151717838689109884741571433845Q-41	-0.13490166181586541828413015056002
0.18209866605302693754589679150818Q-44	0.10824261978115043634364727958340
0.28058534861287764279953884878736Q-46	-0.39013091194993182832442699921837
-0.19459102840393890932061045999209Q-48	0.30881748956212841485113603219867
0.61155050271100521579475733624074Q-51	0.44802413708251611133199513777072
-0.40741371873723387874915324635961Q-54	-0.27124890629599776451630928806276

COEFFICIENTS FOR $J_N(Z)$, $N = 7$ ARG(ϕ) = 37.5 DEGREES

REAL PART

IMAGINARY PART

0.79840790820707010652547660055424Q+00	-0.101311383687535613585996843720
-0.67434527056406032429108181429594Q+00	-0.340716139767008419281161299545
-0.49176574529689381052510396526442Q-01	0.170210364724313138954782807535
0.24252318107733975917951279238219Q-01	0.208403514741215428714458664850
-0.26881957889417616575842630010118Q-03	-0.224263046182497242399720173627
-0.14456474317140137026782443460077Q-03	0.499053384670436442301051713312
0.41654671904946240215641271941497Q-05	0.676134035605226913055445382220
0.23345818231333431609472350366597Q-06	-0.229730798565224124438673601209
-0.93107696109679154376293639293117Q-08	-0.588624226649946289862016527857
-0.10002029342244149454725129940963Q-09	0.291683725701414539348731180772
0.72728301679441563030898810315655Q-11	0.684196987301925247084488566616
-0.21509167919496024504724237279371Q-13	-0.146902944834860787892234824063
-0.24253692750975431832838162157455Q-14	0.987957255614796887474763266894
0.23859303663079852072092046930289Q-16	0.327105891428443604646948162397
0.35456495970459510028591958766407Q-18	-0.424764146962781415033679778278
-0.60661976386582105561530082453538Q-20	-0.292332854215307190966474279097
-0.14688578686920805297886923463194Q-22	0.719821535064404001590162170371
0.72188960531465301434282346191580Q-24	-0.343869973636066020975500526073
-0.18889341474872463047924562512329Q-26	-0.616124761131705118909441424746
-0.44662619765729030628392738981514Q-28	0.272757432610485120828935119283
0.27878514539158947295410967122617Q-30	0.270861970808974977371345681059
0.13142243646916331843385767885171Q-32	-0.230217057361641743035205126897
-0.16073963317014840041687656108286Q-34	-0.437633511825632596095678790365
-0.11947248793343852343814448798480Q-38	0.967796519472146223618626450821
0.50628435006532945236785686788101Q-39	-0.124782880617109010687444008012
-0.12434704600548708091409941570493Q-41	-0.229729359266603359334489931931
-0.89184758819701831761782455530350Q-44	0.832545579702951696655606957249
0.44991614062430543109712164751377Q-46	0.284890114239151232686717868850
0.66170876159700668379185940512206Q-49	-0.207413078882863882711502675008
-0.83426724868836495214598330648895Q-51	-0.441178158504377729932226527877
0.60920878273401838744569777168485Q-54	0.295277432460493305819103387494

COEFFICIENTS FOR $J_N(Z)$, $N = 7$

$\text{ARG}(\phi) = 52.5$ DEGREES

REAL PART

IMAGINARY PART

0.10805845587705430596482545684742Q+01	-0.203660546373119561552940337289
-0.71455466164237510857826262138897Q+00	-0.982679462146886657793071817262
-0.26040128727842228488808444141860Q+00	0.696768721119429383397658440937
-0.27571317714417320814848259619819Q-02	0.354499291865686034298026814975
0.29401647915386156918236067693070Q-02	0.124643024364588640661588397777
0.13776634216917879849558674080515Q-03	-0.159082579561992685007988129116
-0.56010187553550806903295952088593Q-05	-0.909200223807816786686178140719
-0.41800005250114989730369758423369Q-06	0.108151465538867591016195429393
-0.58472555797818629494612261490584Q-09	0.142591426111932792129210657659
0.37204874141962532549562502626530Q-09	0.127963710087384194057290782469
0.53302474883762926371159686262540Q-11	-0.749245238663356454073764124738
-0.11437235789001530275035903900267Q-12	-0.144940567418129859651518209410
-0.29844551553285805154922178093929Q-14	0.121263511121124645255319509271
0.52060171029971428198243485624971Q-17	0.490382875587145983650495637384
0.65796224903619507373851193188537Q-18	0.114795398226778390210309618768
0.35051959226210179111008642225901Q-20	-0.726045934868373543723808245220
-0.65313274252583875691615594832535Q-22	-0.579283728846037769317815399779
-0.71778966369081498916717136621209Q-24	0.460684026195554810120400241655
0.22110752304244249944714006825488Q-26	0.722627080698343794098696825285
0.60989716795503016111170838927503Q-28	-0.147295961367661003941334676476
0.11055692524883226714333588711638Q-30	-0.436883982236592101259208593898
-0.26548864863343487143590087707706Q-32	-0.151709942173409146530707627739
-0.13582972943379099626749519684819Q-34	0.134634581246952857589746518454
0.54087449456297907980737365869257Q-37	0.964946858975016834579661583642
0.57627871863767255223760384476148Q-39	-0.141825867317086183986756432174
0.82300352385280313618885572487650Q-43	-0.296035064528572753822148277333
-0.13179573656582221162809087081985Q-43	-0.402598297445537845018962777562
-0.32081027395168897036417223321291Q-46	0.506338594243282777512126044187
0.16435806742531699340447788131713Q-48	0.180682370731943093961482823692
0.83282928897075248281856230090333Q-51	-0.423604397365714361201759401291
-0.67862609840254306364229587102086Q-54	-0.329289302997156452636320323446

COEFFICIENTS FOR $J_n(Z)$, $N = 7$

$\text{ARG}(\phi) = 67.5$ DEGREES

REAL PART

IMAGINARY PART

0.27757220557402990688168113244589Q+01	-0.312516714139649831300005364539
0.21628220729771000203308663259539Q+00	-0.187552067122228479533006234792
-0.21572586791993218879984437357075Q+00	-0.327611196433808085715243806461
-0.47888825063255233790486375358588Q-01	-0.129505523970711998993765912668
-0.38417449605080035635072473758651Q-02	0.197186284708787551504725810215
-0.95031857659289807256447790836800Q-04	0.260781957731681777133919033920
0.54217719800385241666612428448998Q-05	0.126863079740585947687762096207
0.50291467102024244808698404842641Q-06	0.217946984056109575472355474989
0.16745769743568661997796733640857Q-07	-0.616344498795803608739657262034
0.21272982996446152112083430453548Q-09	-0.436361668854966474491723476958
-0.34558627349646436769108849543077Q-11	-0.106672373962763268091267598393
-0.19673724075954714287681301353715Q-12	-0.104585812636460666088369408545
-0.36990783751744152033399281138329Q-14	0.107151428113134166026315761401
-0.28885723809553503263622180417731Q-16	0.508199393513110562807202992531
0.19920539794538959112477537560883Q-18	0.759958393945918353717678923084
0.80762236755645100998300175866586Q-20	0.484287827463263873153032893970
0.98528599897722977835582181740570Q-22	-0.23594987722837337869445427821
0.52243652678512105438129210349385Q-24	-0.833805575071935222491927814844
-0.18669113195983391431107773739691Q-26	-0.846712003979222236139827956848
-0.58390948295527161848558704951923Q-28	-0.379557373403371668069099807384
-0.50166204665818462762337275869122Q-30	0.102507648991536648593672896926
-0.19266560917133184751677760427848Q-32	0.287274911731943817517415439044
0.40302762123403962588495282377733Q-35	0.211656796264626835905919777811
0.10223176115049176444497272315968Q-36	0.704349837823628843376416504607
0.65335741094562545385947435475121Q-39	-0.116487050911874790106389464915
0.19023900851116904593348292680310Q-41	-0.269716511379038738633519027674
-0.25310782324068450078617565913628Q-44	-0.150992348150994710791178716992
-0.53879707185266385606318669848610Q-46	-0.387952338731915604115768427065
-0.26646674470231544824130673276188Q-48	0.421501953859653864045451559492
-0.60866372429084114029382852101808Q-51	0.830001327089871212505221703225
0.54709618562114124337551794850091Q-54	0.365341146430375745952064864291

COEFFICIENTS FOR $J_N(Z)$, $N = 7$

$\text{ARG}(\phi) = 82.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.56111388441060315511084309626058Q+01	-0.182024002738611291828505509953
0.21348010418955653994959362972693Q+01	-0.119311079186459413448372433683
0.36119190693143105460364716986167Q+00	-0.328330462749980093992173116036
0.33687272380229078129062639907608Q-01	-0.498602812558346510079203602902
0.17623462721252877656041286796807Q-02	-0.482924100141902940669587464247
0.33240121578708466056655686105865Q-04	-0.323896463240259506362150313653
-0.23535138368850414957745447199615Q-05	-0.158156533800904691264867653344
-0.24455620156274546189214104316973Q-06	-0.579304702994036741892933479827
-0.12321691759127232725845279984125Q-07	-0.161232448012536158211293497553
-0.43141910141269011244156224977066Q-09	-0.337669497362627011313872250798
-0.11532110872926472960418710236535Q-10	-0.499938769651280732405359098681
-0.24534464315326537800397893771188Q-12	-0.379086316246370681063877097492
-0.42452292490378379500293654454256Q-14	0.439148572092753872789108148213
-0.60334010595574691196485668374444Q-16	0.227084301105649898297596075224
-0.70337669837572590128014041901186Q-18	0.499314677296007861552558002049
-0.66002467736610312065574694919341Q-20	0.790432749840772167885121389295
-0.46780955868897130003319527235830Q-22	0.999768746880819752950082717824
-0.19051043713450492619157292602870Q-24	0.105123855964283711174117181259
0.74398351846134580795974905347807Q-27	0.935517697423632644959952849212
0.24939744086255454384723266937771Q-28	0.709022147873708279737296145379
0.30995251242919495795207053801736Q-30	0.455679360915236946426730008680
0.28060292048726175583082061520497Q-32	0.243250704439898284086799446971
0.20731943114677931416623679676091Q-34	0.101371668940887265397631053774
0.13016017006701920894085972980736Q-36	0.257337949937023909122823488127
0.70634009840601383523695551365319Q-39	-0.456467450033857366380570814711
0.33301085494168660671173827689262Q-41	-0.111979543774130743869619154693
0.13567915718650329241513589411602Q-43	-0.897388973790644072731999998469
0.46772546964820834988720506626757Q-46	-0.526474205970465733079414305407
0.12841816193480049603565085742259Q-48	-0.255199586630879526926134075725
0.22255623392462292347226445715331Q-51	-0.106527766099528302492274725702
-0.21198670390518689182774290393753Q-54	-0.389451401572109874141519833934

COEFFICIENTS FOR $J_N(Z)$, $N = 8$

$\text{ARG}(\phi) = 7.5$ DEGREES

REAL PART

IMAGINARY PART

0.95759866477106535702280604751455Q+00	-0.14531167521678207037686034262090
-0.42692134123091872170433115135486Q+00	-0.34749254153383774671998635719911
0.84326414508050852399541519952233Q-01	0.30457407980661859262520650230201
-0.93012538712166909059664584032293Q-02	-0.66139163682128015654304506543204
0.62592089900115097931953475195652Q-03	0.77351177590467780331438939622679
-0.25323704748731113257281190267446Q-04	-0.58504768187462249516613593561643
0.43136295501460315878999050614097Q-06	0.31162119791821753014029277314088
0.16339950196487954026772089639625Q-07	-0.12260614963295412298792435858107
-0.15542843648742610537033159857896Q-08	0.36589394641877844357136370315434
0.66191204649973112930269027394254Q-10	-0.83522620064934512396808717386922
-0.19530826283722441613942082972665Q-11	0.14325689122153518754655604703160
0.4418046292716662185233244055871CQ-13	-0.16914102761094812839067643332279
-0.79917049956956480363737637568504Q-15	0.79299484863058739002964192912322
0.11798348444541987783600922649404Q-16	0.19934955940330656892438986231548
-0.14321052375796553970884944795892Q-18	-0.66249769064311163421315375276199
0.14203929377059938843547403666838Q-20	0.11935929961018500348729347786743
-0.1116180890368917307122438074704CQ-22	-0.16083594275913141849920825320677
0.62478632785863289953176101361239Q-25	0.17568639896019620509817993310109
-0.12381585968572490147732822833505Q-27	-0.16066418113252331769233794216507
-0.23679566029878262917138720314006Q-29	0.12471179046710723921052323807968
0.39982206776516031223315431111546Q-31	-0.82347595085204192622563409712312
-0.3982662560019144272722302532166CQ-33	0.45758655281018177908188068570459
0.30793989793850410428737053884014Q-35	-0.20643290177463888096373621155616
-0.19852982876002179948538705757294Q-37	0.67261796947811029258612587135758
0.10976069442825882997613208818001Q-39	-0.69120734450630317983473362064145
-0.52613734710206156062470484392381CQ-42	-0.10707176959139890540771711666763
0.21864837396171779838766422820036Q-44	0.10950593294582907103496409865389
-0.77762274913619247543350831804136Q-47	-0.69278251360105709932816552232368
0.22769319235433768405591915453844Q-49	0.34778326212961962840256792110569
-0.48481544295004608784315594117814Q-52	-0.14808505353820076227117354582382
0.29218708821230693750438977901098Q-55	0.54882153519126655459269531760336

COEFFICIENTS FOR $J_N(Z)$, $N = 8$

$\text{ARG}(\phi) = 22.5$ DEGREES

REAL PART

IMAGINARY PART

0.94124601619466089828982017641775Q+00	-0.47984791179505345772240945292058
-0.4844903908546081423421246897610EQ+00	-0.13332509313405877030255267934388
0.50282852827693130256304794960422Q-01	0.93406412598464015831120077467929
0.42710764380632958475140893239045Q-02	-0.12738694728747291196517330777144
-0.10555146593276918554375086936339Q-02	0.48483161600312265873191809179329
0.68397841591133171810099457788826Q-04	0.27591071267258598126380695671168
-0.13261152698573448814644454286998Q-05	-0.33558739603375791499779762895444
-0.58922959226501362977803298357221Q-07	0.12782318798035684349080192552149
0.42442903734740556385238879227333Q-08	-0.14681973235506724366989019105125
-0.10660256308929580041598146659748Q-09	-0.54237230816693601968645765063910
0.79421595858104790620152851728382Q-12	0.25829487864804830936619628773543
0.25256487672475219607199223153001Q-13	-0.46019615910996705028685575649969
-0.85465399974175079556167864352377Q-15	0.23653006520379788106106358490767
0.11364240090872462140278102896245Q-16	0.66228254790206171158831350416834
-0.42156620305644806347301683839270Q-19	-0.16751918720371348410620060590691
-0.10561494330550914349635851823814Q-20	0.17259383961214045164004152696774
0.20732342592350531582624228708816Q-22	-0.47842628356915017421220423811317
-0.17035873109008583476984172226465Q-24	-0.10855667071784190461933688066320
0.36277101416543547193093650536540Q-27	0.17022964826955197132302471518197
0.75269329203563279537284568631424Q-29	-0.11415041987297510548488602507705
-0.96476262919876163615644147616873Q-31	0.19100735360093520730013179127122
0.53791185846001651774529010976682Q-33	0.36522571658189678625841358683132
-0.72076208568677599013955221413816Q-36	-0.38984817933110404294614504408939
-0.12783226680605055840398807741581Q-37	0.18356153806837091573302349504960
0.11540690594997071997782765258982Q-39	-0.20014046789882885649779087486653
-0.46495365254405809830892674346773Q-42	-0.33105533372626405787067407469619
0.41824276213262118875853577974451Q-45	0.25610539929200054403841442503386
0.6482689096985919560099775279523EQ-47	-0.89281686736784439728178550219012
-0.43455563848677954464808603675557Q-49	0.67060013575186828601208783044513
0.13236510948604973532704731873225Q-51	0.97797603922381905995297414119909
-0.83896342995482259668130781728075Q-55	-0.57356072565099136403356714662957

COEFFICIENTS FOR $J_N(Z)$, $N = 8$

$\text{ARG}(\phi) = 37.5$ DEGREES

REAL PART

IMAGINARY PART

0.95743429319235820528978554674261Q+00	-0.99056962527153816169244473609479
-0.58893586923288254345916899212294Q+00	-0.36362506075793193727272868392628
-0.49845542470162557537324931158402Q-01	0.13579774818486417173883245021845
0.17795143279552256078088818906653Q-01	0.25933690466867680094437470180248
-0.10563135448759995267515908334495Q-03	-0.15215663400856413963132724699841
-0.91179189356333689392364692045371Q-04	0.27163440436143930019728540344995
0.22561106314149961071837488847562Q-05	0.39863703909700447300326422062958
0.12947677435962785681725486271748Q-06	-0.11938376335931981948802849916332
-0.4602195131363061628563142777532CQ-08	-0.30975190133467920743713920949988
-0.50872438554805487646616598704019Q-10	0.13689627594165368619110184522889
0.32432333543040415948763672193704Q-11	0.37976935239412378222108437773337
-0.78285635211766956725892659694178Q-14	-0.62349628666646600055243891254126
-0.98192056320765819808182877251611Q-15	0.37893198562018070298930879725965
0.89165268136359357578618697831107Q-17	0.12666723874401231258874332009461
0.13180775677456747655829221594688Q-18	-0.15306250732812023308883400908197
-0.21030557563511425065497119365738Q-20	-0.10501980209116595768245834424500
-0.52237604617550445246995042409809Q-23	0.24008614461508973546918389030169
0.23183386047082245499445848696216Q-24	-0.82692273092567819034666673603079
-0.56181960734392896805348287176859Q-27	-0.19075315286896859680978422871206
-0.13351130075453183992904857457111Q-28	0.79666281188215964659488536336778
0.79106062355917249804696843339311Q-31	0.78340926144432688629856822415052
0.36904786534613258666593400004355Q-33	-0.63329238333724024787762838438808
-0.42858193549043441621407460185373Q-35	-0.12053422289515461912523358074884
-0.5045393302480258269211167419135CQ-39	0.25023247127870738068057988985235
0.12704033765569269628501281359286Q-39	-0.30323896333689320587309113819984
-0.29813181053480562967218462747827Q-42	-0.56000079514620929170007219429959
-0.21147847630344015853574769548405Q-44	0.19485473400223190894855167123559
0.10260103942565481465818669593928Q-46	0.65855117939981949364007402520666
0.14995375018744411350519734972307Q-49	-0.46075838377646278147979617093709
-0.18058825750888295085976516445178Q-51	-0.10547949563883151467114450971567
0.12517582090351897926722957412904Q-54	0.62315511196977072145114512027895

COEFFICIENTS FOR $J_N(Z)$, $N = 8$ ARG(ϕ) = 52.5 DEGREES

REAL PART

IMAGINARY PART

0.13206668380068808166699544419665Q+01	-0.18491006007898918869576294036574
-0.54225415132118554463607171188192Q+00	-0.90877097052066620482434039773912
-0.2078433944965078853672033278623EQ+00	0.39985323230773487607272951865945
-0.3456140181846886243847270994166CQ-02	0.25207537780948039358604843577857
0.18902076258817583760368702286212Q-02	0.91373101176167418978972479963292
0.87767971875201788606094171813273Q-04	-0.93110148174314151838093086521040
-0.29776265093238908040532272697525Q-05	-0.52536538373839183830996715499138
-0.22262150770302491253800077953002Q-06	0.50075905561426299628412993934757
-0.48915513771333258878198895372631Q-09	0.70615815098310966877126052087205
0.17228780823082845697784213025254Q-09	0.64127051549613147508516528314370
0.24276057629542471223781234095798Q-11	-0.32552882756407194302767958047503
-0.46648321015328227958533350128250Q-13	-0.61635197396842567902219939422730
-0.1196280646564361666700229277358EQ-14	0.46078257631998532740190325230839
0.16647103669224135016496567058523Q-17	0.18624612564109657093419837133616
0.23758685438614973900731703699975Q-18	0.45178906266626806149575525229941
0.12493754876140269226162467582384Q-20	-0.24984396088703900157542778994153
-0.2144533001672561572922961251279EQ-22	-0.19517100296760365012751951403564
-0.23066181219218979089740971371499EQ-24	0.14421687065045626342570132542012
0.65393934348294591421007077176285Q-27	0.22240276847164391792773051720652
0.18023743497707459765454425739667Q-28	-0.25178816822747754535287970544402
0.3268371032767142374587744600708EQ-31	-0.12419370911485414839211277666168
-0.7268765610359183094622554693233CQ-33	-0.42393902330368641323495341871828
-0.36434169347232816751903855935952Q-35	0.35520454954722839489458363503558
0.13734993267684665191300760156535Q-37	0.24956176414250275717226332426141
0.14403591269107890610465820896090Q-39	-0.34335920200524381149561677426251
0.24831774574892943695116868411895Q-43	-0.71615805633079145895917162168792
-0.3089397368698159093627743390863EQ-44	-0.96563553189217402876009159862184
-0.73910788294215233072675547340841Q-47	0.11509127015592797595599542778326
0.3623376644998017578953830360395EQ-49	0.40298544151138070601248775965227
0.18030061388017137856645820892791Q-51	-0.90466114595241669689489047409697
-0.13906331877730908642349800627004Q-54	-0.69299066645457249509431046791763

COEFFICIENTS FOR $J_N(Z)$, $N=8$

$\text{ARG}(\phi) = 67.5$ DEGREES

REAL PART

IMAGINARY PART

0.28075123091421335177989379213860Q+01	-0.26034579935727398642125958322395
0.28351138365662222973167226231144Q+00	-0.15472131695983158246201194683108
-0.15074369147192290984639961475436Q+00	-0.25694045467950479077708781384663
-0.32948114373162691677826432369269Q-01	-0.10423878900230564248788083382465
-0.25131543207029416600644330500442Q-02	0.11791635126840991446804694361337
-0.61382341881863290255599973434823Q-04	0.15377167911987994428630689591647
0.28292963252019390310363074004528Q-05	0.71333152487269084004235262565451
0.25929008068089268591423323455229Q-06	0.11952556215683605949853442539196
0.82601510025675873614119348948188Q-08	-0.28510556989432406626476412896721
0.10190326775220603081623637346782Q-09	-0.19977536973270630087653251235061
-0.14350050960466027470284165173125Q-11	-0.46870831067570332798605134977298
-0.80970741600640941920359404846442Q-13	-0.44572249164110035605960162326635
-0.14653021657009812252389978260337Q-14	0.40352969433865363104563873167053
-0.11099579379853819166098189329533Q-16	0.18992169493434442189344776930910
0.68622219489843275014388262247474Q-19	0.27404804946630290663010784857056
0.27635093402926568404315849618554Q-20	0.16952910159459677181165835155460
0.32605722325398705753628639583741Q-22	-0.74884084397347242679041247119398
0.16799679532599068472441856520348Q-24	-0.26307278045122289843576962796494
-0.54923783120119699152410831443685Q-27	-0.25888128976571540149995583084159
-0.17089032653761279133721197888938Q-28	-0.11289007320825196976585516650206
-0.14253402828310676308419287870987Q-30	0.28103145637557560314976706040271
-0.53310250806008514063378902393671Q-33	0.78393239554815342528016701218234
0.10344135955141504976340804274827Q-35	0.56162879033824713959754248611766
0.26129849606314735080935419495121Q-37	0.18221335997894918051462880667197
0.16261825753827027624424913066847Q-39	-0.28103079284024160835311329882108
0.46211327408595690500847191026206Q-42	-0.64826760564129853120421416972084
-0.57604100465552391878026898162084Q-45	-0.35386692021917662394173133101669
-0.12220739130779775763913578789743Q-46	-0.88823039626412636558792793123375
-0.59002697361020476382182364288516Q-49	0.90783071912010785961836516085643
-0.13178775368855747854943426382545Q-51	0.17821398036977581303371505046666
0.11183284840583881961070226404276Q-54	0.76663869649247991446143497240014

COEFFICIENTS FOR $J_N(Z)$, $N = 8$

$\text{ARG}(\phi) = 82.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.5015541859856764814650060113967CQ+01	-0.14395645706499038085286698136265
0.17590903515957699006143605127206EQ+01	-0.92849752586101145683124462963580
0.27382334744147572439848859920403Q+00	-0.23970584997203179792479865005296
0.23639279601028286437641950481563CQ-01	-0.33859409553430887074792890912112
0.1158166686620206976598341368443CQ-02	-0.30512957631036962532314787127745
0.21717011689949774266770824239101Q-04	-0.19093095060473194536931664091109
-0.12139656777004974757105657491452Q-05	-0.87279394900814959579039726456902
-0.12385840522859638888869107369899CQ-06	-0.30036344436948433390496220270489
-0.59328234723695296394503092591336CQ-08	-0.78830194311094566250736134920968
-0.19700570380386667602479372872625CQ-09	-0.15629153337660447734600416286759
-0.49994798261878117138274639633531Q-11	-0.22024049633187585182392548708469
-0.10116432226834993680783301674225CQ-12	-0.16181059789236239757012847085203
-0.16682970337928802706923150070411Q-14	0.16447360541249488745738785983475
-0.22644246336100131363347615117555CQ-16	0.84098055215226629054062257664803
-0.25264968658149395522405102911822Q-18	0.17783458569598912202094251114627
-0.2274072574807420808505850794085EQ-20	0.27031304384343477898455800052714
-0.15507165220799252099828931701512Q-22	0.32850613220889807750163106022012
-0.61288683879883357903252626903986CQ-25	0.33226352512863549580247280334974
0.21817317733537715850542719422452Q-27	0.28478933132918887270758189107540
0.72588712982651552235544765895356CQ-29	0.20815630000230733548645394641401
0.87400628120907031516558606714194Q-31	0.12918945045124353374978507638046
0.76571372378321851434289745577263Q-33	0.66693987168344336099338269498976
0.54771503107271499420452543987456Q-35	0.26930529442353889869823399986371
0.33316949004853293340328507193159CQ-37	0.66584310465682387557589513369958
0.17532628740334170592490446010785CQ-39	-0.10989283373074264822765268313073
0.80227776152006845657425433067443CQ-42	-0.26814874039876230978814902206310
0.31754966346370464475866957700193Q-44	-0.20924417148917644856697026416403
0.10645252411815159697436582060167Q-46	-0.11943170113429511543504137759941
0.28459800477884326559923702441234CQ-49	-0.56340705664632311273877506102927
0.48191828722439463817878849477694Q-52	-0.22900323011003549286192419255468
-0.43268077365239924944537737125335CQ-55	-0.81571189841737152543006113292942

COEFFICIENTS FOR $J_N(Z)$, $N = 9$

$\text{ARG}(\phi) = 7.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.10207229191335687423520311711623Q+01	-0.14565576227903458559701710686234
-0.40927172272696725342199428117045Q+00	-0.39480008867590826128674824956516
0.72644653483007661816415890008482Q-01	0.27437124361500349160560760569406
-0.72596378500539349614995025176434Q-02	-0.53055918605613511142319236162031
0.44578374626267148591598163699029Q-03	0.56395019406422018958409541209740
-0.16503183083552379070895549785475Q-04	-0.39199093498599502721446637707421
0.25034872132604691736764332185943Q-06	0.19338375064689284206319212445248
0.10011888263833458822842039689594Q-07	-0.70900281151119255019425071018344
-0.85789883092459455880601559521499Q-09	0.19813034563690865019408242200220
0.34124821867357158724726974049495Q-10	-0.42515783526842934144175381421762
-0.94820810732064322011641166730131Q-12	0.68740268882403614468836487655002
0.2029310869876206649647751914907EQ-13	-0.76519764343332771686398158076950
-0.34849104487188334194067685731805Q-15	0.32969491014899723386949475724316
0.48980775456472644913545796672434Q-17	0.84864802708421717438591459797476
-0.56735057908047780002560760617657Q-19	-0.26509665595277232500639033913114
0.53801360400918962000996654592274Q-21	0.45534169127653773483383158462649
-0.40476823904861947722163393240497Q-23	-0.58730396052910065008939421544970
0.21683991734942756851308933203965Q-25	0.61558007301890003397188322303091
-0.4019414428561604912793299966856EQ-28	-0.54120254504968265010739635285272
-0.77892320330003502163382156816331Q-30	0.40451802993936445299815817895618
0.12581854956536336301119704178505Q-31	-0.25755327896244892400936831691496
-0.12079470390768682309853907448325Q-33	0.13815618741437583436337545593515
0.90232479991611370155666208106017Q-36	-0.60212969076319383673855751331497
-0.5628652646438267346880547531240EQ-38	0.18945454777485518447337531633738
0.3014617014337373956959869269527CQ-40	-0.18419582612110275029097853280541
-0.14013304706783637287160760803062Q-42	-0.28774528705049277527103498871138
0.5652425136075311857308998346560EQ-45	0.28426562846827475640694957242393
-0.19526817937073512723721070170105Q-47	-0.17451447244947190087184032766036
0.55564883204431105647250920835373Q-50	0.85146800946361951981742010316552
-0.1149304194180668827211638296108EQ-52	-0.35273105151887495484687363185155
0.66060359153142398965835552338762Q-56	0.12729143386650328505826813157397

COEFFICIENTS FOR $J_N(Z)$, $N = 9$

$\text{ARG}(\phi) = 22.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.10225897585221670738274843692417Q+01	-0.47719089152783541260252735271934
-0.45353997959298971265501805905428Q+00	-0.14700158197519581583050440153546
0.39651342458687760976526395992957Q-01	0.81509675977249559556858685135954
0.36764632564828101771123341989795Q-02	-0.97832550325261013538818562850273
-0.76435215974503344024327435401911Q-03	0.32253534248810169013219288002115
0.44658607895997630516476690138975Q-04	0.19448637430636356567167718793972
-0.76503373498260766446715567069393Q-06	-0.20805276731522011822933281769357
-0.35387614530761650313777943662975Q-07	0.72679618789203103409540640778628
0.23034139075923625264975779119987Q-08	-0.74856710791798879485719442994335
-0.53718556259606901830296415096271Q-10	-0.28479394993261915443775374971153
0.36300124203259804568851569997671Q-12	0.12474934501477124271871980093728
0.11805819778895655633582673809862Q-13	-0.20829965641608964170463050521962
-0.37204017598679301560178667617512Q-15	0.97999682124919073847085775119391
0.46698843072610020799773704743027Q-17	0.27929582249237061794556928216558
-0.15976871827816647974701842780769Q-19	-0.66395790460512109624244249623040
-0.40603096892075513215008399238413Q-21	0.64950836554408729236809179010561
0.75438961815922103837330167724896Q-23	-0.16709774920329790787667362867722
-0.59136578604962854597348179064365Q-25	-0.38364772992307696044492791863989
0.11750275453030970612208074925681Q-27	0.57259922168480868485649312539398
0.24622358887350230498692206586313Q-29	-0.36775263344479477116692596690442
-0.30174748644982576385569940440338Q-31	0.57684418749181185247026865227558
0.16167814162242449841861110250331Q-33	0.11123103491493152311791505855823
-0.20389721241579299534354748235659Q-36	-0.11394816977750194881981862536103
-0.36425818689968252173149074653450Q-38	0.51707933546218889164999105950787
0.31661168593544527379154689566608Q-40	-0.53251110171783464951987807000121
-0.12323919777102505167972135615617Q-42	-0.88641221157759999100997350450853
0.10503920347854597831521153426575Q-45	0.66200175830490586084623204765822
0.16371408650893434307511300490958Q-47	-0.22345603515861205780812622107100
-0.10619431352893993471548779929204Q-49	0.15947674619303636749292886162642
0.31379971680086032304199921596907Q-52	0.23371884547162630737327460560371
-0.18946256279155679959903574108904Q-55	-0.13291099808970913259355570929650

COEFFICIENTS FOR $J_N(Z)$, $N = 9$ ARG(ϕ) = 37.5 DEGREES

REAL PART

IMAGINARY PART

0.10852151871498635971643124282546Q+01	-0.95777318779619010847200024121095
-0.51879934718479929553272237443727Q+00	-0.37218220119290686955881971267275
-0.48025790140501868861518170813133Q-01	0.11042618122229946935940758293929
0.13413746700318794177003978606839Q-01	0.26414302002829215648308472547390
-0.28402260506634830565611725583076Q-04	-0.10674512155113251747686016670426
-0.59789975101214106951950727720934Q-04	0.15425555762210166251296855352153
0.12860161575802240562085934260199Q-05	0.24547167547924533036309638939485
0.75268825102523294915227595974734Q-07	-0.65518033538740695610788969951896
-0.24095237785455473687205138593795Q-08	-0.17123375060938830497095556154147
-0.27143407859938025157351196277812Q-10	0.68246453372627069332916620135332
0.15402942968986896588848396901165Q-11	0.21239859604881368742768522445519
-0.3017190324069451935672872077512EQ-14	-0.28251247690263310319532421880506
-0.42532671344409951903649774061934Q-15	0.15596931162172652218321238254608
0.35817930336106738023091930782788Q-17	0.52579529139395699316125384459883
0.52604954463498176185990076467997Q-19	-0.59377334221275013561470328970804
-0.78610664703174678236573463546900Q-21	-0.40534944592530828237787740453320
-0.19897763203663402067829485341777Q-23	0.86467119784116891036487464964831
0.80507388640712794041006100369380Q-25	-0.19807826160193289777451417896038
-0.18122043944835437817232012205784Q-27	-0.63943488984458999966063768099792
-0.43264430310713845892558833241417Q-29	0.25258614527361909763097120155913
0.24389057246987028616249488355763Q-31	0.24587641589193332654457714982521
0.11253918770968542425358514468985Q-33	-0.18946994472310725663863977484572
-0.12440360754792577042216635489772Q-35	-0.36038679087543200139932245027724
-0.19392859356259971091814984741151Q-39	0.70500880302599771068179850236628
0.34766532191314936035041941174052Q-40	-0.80451833652947280245225243808211
-0.78083569834085745263692172541547Q-43	-0.14900054951012355320237937900364
-0.54775632864857140093591107636631Q-45	0.49850593930095951616938158918901
0.25593166615164128526237251617316Q-47	0.16638099178502769452048959621878
0.37145943857774103031180478032250Q-50	-0.11203597433021005075672411385189
-0.42816139374226625147613374424708Q-52	-0.27254415482183584366651266326620
0.28209345358739383715080090748057Q-55	0.14413508331530099505640224831164

COEFFICIENTS FOR $J_n(Z)$, $N = 9$

$\text{ARG}(\phi) = 52.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.14901587876237078639644643242678Q+01	-0.168553146625802616344785190457670
-0.41935747691740330950925705827827Q+00	-0.838352065487400541478883260990750
-0.16912542253599059452467930380097Q+00	0.221837126984216881565267909340120
-0.34823931773245912768779716661251Q-02	0.185034617723682559139636209468890
0.12656824283325412512843222023167Q-02	0.679456069148356889645476471729450
0.57978670566006744954801029711202Q-04	-0.571603851238138773147917107257030
-0.16696658455455617747040585967991Q-05	-0.317898090034072710404773847323400
-0.12497133429781679846192026999238Q-06	0.244270906985977919183418980632620
-0.35017567726455584775520469151823Q-09	0.370453471320784422716687047786210
0.84864870124175846854673866197114Q-10	0.337750498392568076846830567140110
0.11741599757026453150464700386209Q-11	-0.150983867933773587515510378010370
-0.20376391312041416426509948212023Q-13	-0.279713983889550667121054829886690
-0.5135123593488226987663744725232EQ-15	0.188049340212785058504358566008020
0.5628334859299880692383791021856EQ-18	0.759638233359870104620521260114080
0.92354984473327268107343354317424Q-19	0.188983162831699137835354500754850
0.47877786671570704890166047881329Q-21	-0.927526938905541922224520595321260
-0.7611634642885189790080593536868EQ-23	-0.709372048995153189789132062788990
-0.80143471083886086621873346253805Q-25	0.488947177613632452105176420484690
0.20983159555373565156988246868467Q-27	0.741436099579338082706387663589730
0.57787319916283497614720459722321Q-29	-0.259432680128321668544450986337210
0.10458392802209250378149765494597Q-31	-0.383580570594827182474945088450240
-0.2165091054751818723571571713005EQ-33	-0.128689674912944052713304390078880
-0.10634717427153137976054318884921Q-35	0.102081755353025436492603128835040
0.38040480182879044523722399634758Q-38	0.703283657745841045360858732167210
0.39271973981955606056882579796150Q-40	-0.907729790746224200611649076016220
0.77866777500699107214184724686957Q-44	-0.189189271968020063348647322231550
-0.79155858882688467232509163004560Q-45	-0.252737663377141795276165584183870
-0.18614444376485889725597485316454Q-47	0.286201600167007763692084299075840
0.87466522371406139282689368442175Q-50	0.983579909748109451759465020777860
0.4275298753486621763979726595861EQ-52	-0.211737107600343356278836287037030
-0.31258644494194354112416381893310Q-55	-0.159862213227577185236892463584700

COEFFICIENTS FOR $J_N(Z)$, $N = 9$

$\text{ARG}(\phi) = 67.5$ DEGREES

REAL PART

IMAGINARY PART

0.28034474322191192608193328728190Q+01
 0.31437732007530896934607561546251Q+00
 -0.10921749648625950896312724842387Q+00
 -0.23538711243335439179540181866114Q-01
 -0.17099788895775039989248377040687Q-02
 -0.40944438779461664435848115852173Q-04
 0.15608669837079204926176351956846Q-05
 0.14147427746469514158293022108193Q-06
 0.43190723301963783514918403936564Q-08
 0.51704560228225790745288045135541Q-10
 -0.63762847379053134580170257357905Q-12
 -0.35684093996780337870082557096429Q-13
 -0.62235832628429628481084041837072Q-15
 -0.45746062299811684071119861380747Q-17
 0.25504533098548597290099342323586Q-19
 0.10206883465563043155542036880182Q-20
 0.11658892050574234114388341144133Q-22
 0.58403563287322845053682570438405Q-25
 -0.17537189507411510580697441525761Q-27
 -0.54298019925813554725174836976316Q-29
 -0.44002894275054373203954687124595Q-31
 -0.16036649990360654811215671334478Q-33
 0.28943863136944793706872777428400Q-36
 0.72825814798311295605531477917760Q-38
 0.44164985626135820301320262883108Q-40
 0.12254829469855025684982835508145Q-42
 -0.14342258442334544405261526918478Q-45
 -0.30328852725337080725369002229257Q-47
 -0.14303057345016156655482094935250Q-49
 -0.31253235019473693992942880898498Q-52
 0.25078662253834067163084730357517Q-55

-0.22168113161764802686448039675227Q
 -0.13056373037273132576838685540433Q
 -0.20627408614340475317877152146504Q
 -0.83942942606640410352981936563336Q
 0.73917011765269562352773029827696Q
 0.95171256851767575578257753911374Q
 0.42174403614562450408860759399778Q
 0.68749121356512272483106309644459Q
 -0.14036189978173417209376458232879Q
 -0.97422090475217296443622542996218Q
 -0.21969510796320723848422157440046Q
 -0.20263602663466943144440441270413Q
 0.16334784776688658525220529450233Q
 0.76332301255886023525836020269285Q
 0.10640604655280035751855542972636Q
 0.63929255646737121352441696479633Q
 -0.25723622493695918750883418272077Q
 -0.89870587413964431000831259447947Q
 -0.85782613544627708503532014023758Q
 -0.36409355977478066253111339476316Q
 0.83820489794358796915736661339038Q
 0.23279156453701257930271111413308Q
 0.16229255917354773751069893767741Q
 0.51361259043849766427847960419543Q
 -0.74051497139181772286496912550755Q
 -0.17021028413411951197026437462458Q
 -0.90651572893514574671664532119642Q
 -0.22239761851124248205733353231595Q
 0.21422229239532068611002609598837Q
 0.41929535560653559381988676654213Q
 0.17636779784355912179584888515659Q

COEFFICIENTS FOR $J_N(Z)$, $N = 9$

$\text{ARG}(\phi) = 82.5$ DEGREES

REAL PART

IMAGINARY PART

0.45778659072789435803243880792026Q+01	-0.11750221819701168519828243218294
0.14865911988951922495747745377008Q+01	-0.74694952932887756282107403006862
0.21407270736618212997212830790839Q+00	-0.18148145861256552829324664063437
0.17190528410662748007631924748333Q-01	-0.23946346828468768929276869118400
0.79126493965605218843812299238222Q-03	-0.20166275657361019514539439253517
0.14599285289090907177866000783479Q-04	-0.11821876051966164953545309570053
-0.66293408719324839740283297516924Q-06	-0.50785157478030162645431084729645
-0.66522927825808089822811274261290Q-07	-0.16477590281856967607776655379971
-0.30366094068877039218135082219804Q-08	-0.40906206517409217252675578801802
-0.95874470163768310920473470576199Q-10	-0.76987075071327321737086991058371
-0.23155528512801822301739124945351Q-11	-0.10348044356796781823183329200605
-0.44667680475846809936170561665811Q-13	-0.73654269259075211459263663529964
-0.70353919802577003695169410034367Q-15	0.66245988770603706098404184861597
-0.91379821424626194937826050237015Q-17	0.33521087090180482253367061302181
-0.97752542298021561978953254907562Q-19	0.68269364808600618612002886641996
-0.84534220094714737066362367388177Q-21	0.99795112002979000063433125050842
-0.55536947925795873896040840423211Q-23	0.11670220886594546233543154434652
-0.21314877223079242153626326680179Q-25	0.11370313712336180633335476270873
0.69454920565742270940222977862866Q-28	0.93990234785394433459433661377437
0.22945899058966714150834056417505Q-29	0.66335757568424907697473073106158
0.26792606017403100545991261488987Q-31	0.39803977856878961921443762540167
0.22738792245203343218290001209774Q-33	0.19893644721257648143801079203635
0.15762682839553318554361114948239Q-35	0.77907325196748499363687952535833
0.92988180632415942732054774593386Q-38	0.18771291219537093782831528280670
0.47495095554569864470476393241994Q-40	-0.28899288697921946550294033575800
0.21112084222432560557817968807386Q-42	-0.70159936933568687551485176184254
0.81245578481247655775878350756323Q-45	-0.53346073020585476304811085461725
0.26505539772445594136616626066297Q-47	-0.29645192111165589139948124066226
0.69047537490178655826353358349260Q-50	-0.13619635246725904539952402427983
0.11429456439797865372249321677330Q-52	-0.53940889092100702545067367899825
-0.96892149560262123839875621498527Q-56	-0.18732754308268549123272393345062

COEFFICIENTS FOR $J_N(Z)$, $N=10$

$\text{ARG}(\phi) = 7.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.10770779426106073890192043832212Q+01
 -0.39212966583325400750535674345730Q+00
 0.63217179741744383728147191719876Q-01
 -0.57766794899547661487696606603912Q-02
 0.32623043229235075669154216020844Q-03
 -0.11128648273865611237436324967911Q-04
 0.15139848520002283027176413868088Q-06
 0.63358108594102329667377990652636Q-08
 -0.49454316880328669096898436930727Q-09
 0.18459489460356737865707045114105Q-10
 -0.48474284098336005334805655417731Q-12
 0.98446594731482008322392190423090Q-14
 -0.16092574988635954642081991227612Q-15
 0.21584244415346523125745667406023Q-17
 -0.23909029935970118910907688810855Q-19
 0.21720066854263152721545196496415Q-21
 -0.15673082615050054274117119393181Q-23
 0.80501395598236888611468356761614Q-26
 -0.13988665369483780824487652497635Q-28
 -0.27439783084603088637083456334797Q-30
 0.42486511969182125404789613673033Q-32
 -0.39367209373285906562400771746547Q-34
 0.28443362465281128612221320828633Q-36
 -0.17185740972647282211843368652655Q-38
 0.89255031371157453272610224573363Q-41
 -0.40271650501305446811450701094978Q-43
 0.15780448320322486421184122945988Q-45
 -0.52996713551233774079793198970871Q-48
 0.14667329821418100086201404598724Q-50
 -0.29494577652425156628027794825673Q-53
 0.16186533619042864589367328454462Q-56

-0.14471617726677994164829168585454Q+01
 -0.42822693172595757977721559943020Q-01
 0.24765213424044801892177336985751Q-02
 -0.43206573045827685773998282895837Q-03
 0.42114593111328692199373708640293Q-04
 -0.27089968008436003028620257681448Q-06
 0.12449474810360664978704310744351Q-08
 -0.42739393207804263528855321489781Q-09
 0.11230807711054676115592435286881Q-10
 -0.22738519060877352783612498051743Q-12
 0.34771037680945009470704147923679Q-14
 -0.36608836053076755707826611586407Q-15
 0.14548228989709033084115748630499Q-17
 0.38246403485211142502971379747897Q-19
 -0.11273968700904676704470255784396Q-21
 0.18503519123490771051149676962184Q-23
 -0.22887453424008915733606621226772Q-26
 0.23057539499173694923701488800925Q-28
 -0.19518350103661280837287638319327Q-30
 0.14067528371550764518621536654098Q-32
 -0.86476004598322880177270084352663Q-34
 0.44833905306905288982561962152829Q-36
 -0.18899112188947303819005768775349Q-38
 0.57488754259333355761025717908944Q-41
 -0.52961039159847705513267306666401Q-43
 -0.83389340474374919543956016835359Q-45
 0.79671770321043473046348405508266Q-48
 -0.47505880175707503010957889954265Q-50
 0.22545343954568930493691392625562Q-53
 -0.90934226042688915493898962491234Q-56
 0.31975785447339136437482027426421Q-59

COEFFICIENTS FOR $J_N(Z)$, $N=10$

$\text{ARG}(\phi) = 22.5$ DEGREES

REAL PART

IMAGINARY PART

0.10934365869883215738394308304922Q+01	-0.47028602446269440118915689945615Q+
-0.42521461236315023627007955311524Q+00	-0.15566301656672756046348216474698Q+
0.3181906056526526716E071183275582Q-01	0.71602424661306323596915881812180Q-
0.31542763944476711461019401953149Q-02	-0.76714591660115322777735474948080Q-
-0.56709350574311854570514383181545Q-03	0.22148909866535180484739570299698Q-
0.30159719201330213545099359232924Q-04	0.13996066148054386442657645937180Q-
-0.46015707246229465823701740070630Q-06	-0.13376647042674452136864451628413Q-
-0.22017318292751463419500380330363Q-07	0.43145883711639812681855190560882Q-
0.13079367033826251356243750667839Q-08	-0.40091793666705322622005627536410Q-
-0.28463977632441842039992068840130Q-10	-0.15645730759689277826822406831856Q-
0.17530694536689453709873179018724Q-12	0.63466019804804903761488810487431Q-
0.58153151991331585746944579237765Q-14	-0.99698030677668387211514985389115Q-
-0.17150981648592178836950576946626Q-15	0.43105592882390350540053173161582Q-
0.20384532499130557704954872545272Q-17	0.12480651957886549317103378075187Q-
-0.64532392780238604688709208229600Q-20	-0.27989837324220134634599715035091Q-
-0.16613318327725543452309762656183Q-21	0.26062150414425241285394934214502Q-
0.29301551513187396232117620799296Q-23	-0.62402887418460114669138161693870Q-
-0.21958047473642545743166366089863Q-25	-0.14482351658154859035216129810815Q-
0.40808524177136503848660786259430Q-28	0.20622157878201833216783187477040Q-
0.86294549510274699551258563531163Q-30	-0.12707473004385527797409603267730Q-
-0.10131298417960660731707743332042Q-31	0.18723949489372310548225920747591Q-
0.52244061160892101031525866621885Q-34	0.36387266537721670106359309018600Q-
-0.62125636604351486479613785149378Q-37	-0.35834137852552893633384428818349Q-
-0.11173795483626256857449710871045Q-38	0.15691693477317070412427369090201Q-
0.93639085759250130459588803809473Q-41	-0.15288441968328379328017314198386Q-
-0.35254244552994616950521581637437Q-43	-0.25599684525958349714597681406665Q-
0.28511850610976042993105025370479Q-46	0.1847955779932439953018901551077Q-
0.44670612584875300546018404878063Q-48	-0.60456595263285699080420351167459Q-
-0.28068717767492755444456297018307Q-50	0.41050316306662456631194103872693Q-
0.80533826700881921265531663978337Q-53	0.60439777526461537929395834428985Q-
-0.46372384893806546987618902678121Q-56	-0.33358845033829961271679578818470Q-

COEFFICIENTS FOR $J_N(Z)$, $N=10$

$\text{ARG}(\phi) = 37.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.11892731397061021040654756324953Q+01	-0.92070712182592098061603414817541Q+
-0.46082377180951198736839907799622Q+00	-0.37235126135848637966983735515245Q+
-0.45165739932707199292543477714636Q-01	0.91277493651089151601757756180436Q-
0.10343784173674319839242823089156Q-01	0.24974876561291916107608167599555Q-
0.79507749186446390767994724675178Q-05	-0.77024577137716274268397855639896Q-
-0.40513259416953563357639481366671Q-04	0.90653826910150025246570868723397Q-
0.76525533340996492925024774236890Q-06	0.15680875727619770982993471257640Q-
0.45537336594143751690101859150321Q-07	-0.37648906373435473209324875593666Q-
-0.13243833551223220538747463928333Q-08	-0.98722401096745947386437086375846Q-
-0.15096685345703534113808439983121Q-10	0.35807397250830660866046807847830Q-
0.77171652528247393555037980816695Q-12	0.12100160889017580884524260391945Q-
-0.12156334218876028024705413991347Q-14	-0.13533740412746199432675615031962Q-
-0.19516712330693265724118699355359Q-15	0.68189324409909834515786520341586Q-
0.15305493251388436393481737342540Q-17	0.23161500199510543895649978286785Q-
0.22311578319588139056008532566540Q-19	-0.24536994316450377063715577503711Q-
-0.31345371296322153212933411246127Q-21	-0.16639683173760065235013111472695Q-
-0.80431383887768639269400963609724Q-24	0.33265357338027656563360596461776Q-
0.29903245821330885972100599008225Q-25	-0.42878122783181978180491684135084Q-
-0.62693296790304631866496038461793Q-28	-0.22954801378603195901403579619828Q-
-0.15030726715889159520837011563137Q-29	0.85965255951477178039204037964469Q-
0.80786592894846431706005408122002Q-32	0.82813885541594805183371996391602Q-
0.36854718930376072903891613156706Q-34	-0.60958298692728767195676509194550Q-
-0.38867074916762003688099378913567Q-36	-0.11569332064387194220272824361761Q-
-0.73653499469989608896364535141621Q-40	0.21398055036140281096845725195902Q-
0.10258158926465044574865126701273Q-40	-0.23035345007292336547012618599439Q-
-0.22083059221419849490531625155202Q-43	-0.42777008870140030223903118672166Q-
-0.15319139404231290368473338873328Q-45	0.13779724754099566569509672484470Q-
0.69021955425802322147376347750959Q-48	0.45414013854402513048734340347319Q-
0.99428687552751270426791507940520Q-51	-0.29472207386639069570012329321266Q-
-0.10989264133111235764429538994789Q-52	-0.75430765786330559529532446048065Q-
0.68906997742274749733737821100824Q-56	0.36111617066616194965118257450540Q-

COEFFICIENTS FOR $J_N(Z)$, $N=10$

$\text{ARG}(\phi) = 52.5$ DEGREES

REAL PART

IMAGINARY PART

0.16131901692933982020058639015388E+01	-0.15439212736918901332324849988329Q+
-0.32925121694190330163095275667469E+00	-0.77415763657918279029279595027441Q+
-0.13993035531835114601183784733108E+00	0.11202972435901450391668133278642Q-
-0.32477100333648170948629780892145E-02	0.13947594281469011753483699924640Q-
0.87675427047145050806737058743826E-03	0.51311143781636042535110904478778Q-
0.39504441786992409929595299616255E-04	-0.36520882658532342937431039504513Q-
-0.97918432765922958778034720904758E-06	-0.19997994108134104885064675107526Q-
-0.73334710023407880688057358897999E-07	0.12429779126227047852441180310511Q-
-0.23924209620577277766222412866763E-09	0.20403426459801682898748947123482Q-
0.44047742183406948178751261136048E-10	0.18570993957155022926230610786376Q-
0.59773999635131568020684186639394E-12	-0.74026278898078383207261331317165Q-
-0.94362568794462894070374595497220E-14	-0.13417029872131442977716707717604Q-
-0.23369752927692986482454342541443E-15	0.81575011540892412976223261456772Q-
0.19790524831005259460894762097039E-18	0.32931082974101630348739273399272Q-
0.38241075858849583826140549090876E-19	0.83373331803903805291106730813360Q-
0.19524909074511975140875204101316E-21	-0.36751282172291512821128944880172Q-
-0.28887091736785663835650503045987E-23	-0.27519401805042829973427544817706Q-
-0.29781886052901580270841998082916E-25	0.17756087997486642841890546664815Q-
0.72237079243055572287707626729999E-28	0.26480508571246431838347298734745Q-
0.19878240906129993364767420760478E-29	0.86698360564704439086045727663089Q-
0.35837360484424353113441861988638E-32	-0.12727735517113735941903850811010Q-
-0.69369306948066985023305391540348E-34	-0.41964393844151244608774565376343Q-
-0.33398516121575415608752136415887E-36	0.31594037532879267414227608286185Q-
0.11359424283166134017417206560414E-38	0.21350122312764593977994122083172Q-
0.11547164900803300405485156245012E-40	-0.25903484617452265166831964390329Q-
0.25510943162098450077533937589163E-44	-0.53949243706967857363317030145751Q-
-0.21912110489853883337082128591615E-45	-0.71363331409637988600969460769551Q-
-0.50656867911546071622073659753358E-48	0.76959629169699179435507740927286Q-
0.22850126897083686270599679914753E-50	0.25966240520705018274788479116694Q-
0.10974225638919435684874321396094E-52	-0.53676787509358374721516988745198Q-
-0.76169062126935885500855521294288E-56	-0.39950709078496150627555314745569Q-

COEFFICIENTS FOR $J_N(Z)$, $N=10$

$\text{ARG}(\phi) = 67.5$ DEGREES

REAL PART

IMAGINARY PART

0.27822907138463913982077228239064Q+01
 0.32580066855842345192718396283317Q+00
 -0.81517601434128351486625630962365Q-01
 -0.17346194968614217143376700852984Q-01
 -0.12022018424009118429572782447224Q-02
 -0.28096257126468491302399908789547Q-04
 0.90230018366711993716471965884418Q-06
 0.80958625413888494762821598493514Q-07
 0.23722903455583893267916837534972Q-08
 0.27548520369482274784784077741642Q-10
 -0.30013872256094967813333178412419Q-12
 -0.16669346718516516649564482342204Q-13
 -0.28053780805678310593151879714947Q-15
 -0.20018077646409445146235950127044Q-17
 0.10117738986138448080457622054619Q-19
 0.40254237052003516289369140896282Q-21
 0.44558803427369570093113741115989Q-23
 0.21713708815461493035836976482057Q-25
 -0.60098274186338796587864559864839Q-28
 -0.18521392886501332093735154871307Q-29
 -0.14595161220914792130334097827895Q-31
 -0.51858284005809331672942661423354Q-34
 0.87284902527443447414007744443399Q-37
 0.21879706167675198333460340564104Q-38
 0.12938184908698778721367959098851Q-40
 0.35072427907614927879582394621915Q-43
 -0.38612746049153302910881766293428Q-46
 -0.81400786697637970641412370912304Q-48
 -0.37517263200483345273113356861754Q-50
 -0.80232072700061882663560246762198Q-53
 0.60973356483030893412653042011155Q-56

-0.19211850738764318831286117319236Q+0
 -0.11222345126969609690751959093825Q+0
 -0.16886843297843570184702308467810Q+0
 -0.68033252734030429613633911500232Q-0
 0.48190253815463468175585909930113Q-0
 0.61329385231121013017460389312020Q-0
 0.26006533909254724720472649915863Q-0
 0.41180617936743956242181187348924Q-0
 -0.72844930932444296167968487288868Q-0
 -0.50118494418729065294094909080112Q-1
 -0.10878570475951727749222044764682Q-1
 -0.97338177825989185336634979773048Q-1
 0.70334161119495955003495937124837Q-1
 0.32648772613562729731021026691739Q-1
 0.44015944790935392325526145255116Q-1
 0.25697299617608944753652829857738Q-2
 -0.94595287247135573635906636649412Q-2
 -0.32877419328717899362768592678456Q-2
 -0.30466136372478204540092384484482Q-2
 -0.12593126879328347638024896938318Q-2
 0.26891482779783100245124983466398Q-3
 0.74375073603579126121327254409973Q-3
 0.50492564515150912624136340449118Q-3
 0.15595394971255342244936048926047Q-3
 -0.21066331538343720110360088473163Q-4
 -0.48257804498128586104664533050125Q-4
 -0.25090791358394678576366476746410Q-4
 -0.60191991048523126286989588381375Q-4
 0.54736708832091399121065888457811Q-5
 0.10683396196357965851457116385584Q-5
 0.43961298506966719755036932639659Q-5

COEFFICIENTS FOR $J_N(Z)$, $N=10$

$\text{ARG}(\phi) = 82.5$ DEGREES

REAL PART

IMAGINARY PART

0.42447314438858021139768539021454Q+01	-0.98334124575471919723004940647087Q+
0.12816185950022786480302670673817Q+01	-0.61696510345942212007588790523339Q+
0.17157745371687049153830943040939Q+00	-0.14149192886131537194456163290311Q+
0.12872522149607439830678476187445Q-01	-0.17507432642116695670720268525378Q-
0.5583588724716266011470288128811EQ-03	-0.13832386200317289901890398165733Q-
0.10073534186631920691166692869692Q-04	-0.76250525655960568692329827102811Q-
-0.37979524084068634629698613527629Q-06	-0.30888592802055494189618835666205Q-
-0.37537419925526154983853718447047Q-07	-0.94783344481441606245572624760200Q-
-0.16364772723703878577677766033556Q-08	-0.22320172581098813248723308640775Q-
-0.4924153542762533088963134635877CQ-10	-0.39974879281919812397246662178058Q-
-0.11343973206188397919203464791644Q-11	-0.51354242457369746175507841595217Q-
-0.2090529566655836397048974540408EQ-13	-0.35415960637573766224972180313220Q-
-0.3151018902423778059520110844240CQ-15	0.28394096061830276096936639591218Q-
-0.3923524500254147878290365290521CQ-17	0.14228957518635015326895721023317Q-
-0.4030812395014452933952900148908EQ-19	0.27947787308485203846718520533009Q-
-0.33540523345508838684005531973484Q-21	0.39344829641267875848821222453201Q-
-0.2125704194744693623135484170560EQ-23	0.44335968524975969670958912395662Q-
-0.79271557432924675698555316914276Q-26	0.41665618539189018229260641529583Q-
0.23735454055847514179232386015774Q-28	0.33258089962189097469586745634458Q-
0.77895478163617758143210152994354Q-30	0.22691701731190163068449207765829Q-
0.88284331343533283843199434252525Q-32	0.13178276940464517975593295985025Q-
0.72653356264619964407755637392408Q-34	0.63827807520755921114421630049285Q-
0.48854145646350888618452057204284Q-36	0.24264220522136749138904946517078Q-
0.2797542468355250286492451405754EQ-38	0.57005087473485524126006460289620Q-
0.13880587224784616649936733648024Q-40	-0.82059700108623087145068438062453Q-
0.59985432051894836742212767092778Q-43	-0.19826086110294003030394191514909Q-
0.2246089111757502641726373384794EQ-45	-0.14698426083363886378587189216027Q-
0.7136180014888513668396626061348EQ-48	-0.79580705508134970881013554920667Q-
0.18125655482174948734864600532207Q-50	-0.35630400382439371429769289610840Q-
0.29343207113616831655464794788975Q-53	-0.13759059108864575839476319980849Q-
-0.23525597783363804989803641103955Q-56	-0.46615512223085639938498020530894Q-

COEFFICIENTS FOR $Y_N(Z)$, $N = 0$

$\text{ARG}(\phi) = 7.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.287172129467966274308063479659Q-02
 -0.315637161452491146160345369752Q+00
 0.334865235657286400512166314221Q+00
 0.150427907100657108718423350824Q+00
 -0.1100648355540342734566479718999Q+00
 0.220937941966796911065705972117Q-01
 -0.187828075221057523527565588207Q-02
 0.194725134575721085397173420383Q-04
 0.115185795081090988930442722098Q-04
 -0.133013875318738913949915608748Q-05
 0.849478807388054025279247742283Q-07
 -0.376231258339377421380465073652Q-08
 0.124799462600334325671914409671Q-09
 -0.321705405619908963628198565981Q-11
 0.655954171899764999475652745399Q-13
 -0.105994013957885675803999163332Q-14
 0.132800880967539146677884713710Q-16
 -0.118361570034516777062556024029Q-18
 0.459841671094723146491002870230Q-21
 0.733196156204235866974381363552Q-23
 -0.201287237807042408169004640430Q-24
 0.292633882398674567444154219373Q-26
 -0.320206995614655283783593976279Q-28
 0.286635600570986695388261083303Q-30
 -0.216799319050532778524369768063Q-32
 0.140411650374743546920384696439Q-34
 -0.779998289737202829156889774814Q-37
 0.367530449595450355084004793748Q-39
 -0.141725574208817004356116135729Q-41
 0.398788657903525494699667175534Q-44
 -0.374868953426172642536880068437Q-47
 -0.457698512889497176350890418816Q-49

0.189175526283888869155234658493Q+00
 0.265039064095993662940135840354Q+00
 0.178975025484857816145548661325Q+00
 -0.735216715810214891905031214297Q-01
 -0.503268718590368360890100342784Q-01
 0.259976183740834830309255006196Q-01
 -0.510347514456898656862768602823Q-02
 0.573254561294709730469045559176Q-03
 -0.419751446205880724341723403541Q-04
 0.212378424463630405521893377551Q-05
 -0.754266473595541168191883129145Q-07
 0.179651578649456500059832044095Q-08
 -0.211890662425691986644577124813Q-10
 -0.344384585787282581762623571635Q-12
 0.261654024937918764227269753719Q-13
 -0.807344449703155930726675356635Q-15
 0.174804817760828251064669954306Q-16
 -0.295897288647215584854992320266Q-18
 0.408233540206645980327204865393Q-20
 -0.467755695293795169751230039028Q-22
 0.447672849624917534936472412091Q-24
 -0.355208400102381739810373494293Q-26
 0.226459266602804832923678981262Q-28
 -0.104567859728879799723724404852Q-30
 0.1833810720120C6592025462940872Q-33
 0.256593960935643836165518111388Q-35
 -0.372585492050160538960368183016Q-37
 0.316082054715164010265491324788Q-39
 -0.208918115116909194553018765962Q-41
 0.115746253684701085208465872357Q-43
 -0.552908065428936609478906132227Q-46
 0.230208071881553537531466077206Q-48

COEFFICIENTS FOR $Y_N(Z)$, $N = 0$

$$\text{ARG}(\phi) = 22.5 \text{ DEGREES}$$

REAL PART

IMAGINARY PART

-0.149982401841825042207821066617Q+01	0.107859591040152735387221625774Q+01
-0.141357093233175097919703960627Q+01	0.148986082032130339624541018790Q+01
0.282951921154269806808589056208Q+00	0.133966569289799879053736615991Q+01
0.657552965009547521396763736648Q+00	0.126672405937401965979281578336Q+00
0.198291706529368157392915460618Q-01	-0.248569829627057726857390461395Q+00
-0.522518701407516862859822726742Q-01	0.194210168500914138659457696360Q-01
0.637179398047106469894454574630Q-02	0.474178746600371388992805121038Q-02
-0.328634348978159485614814420281Q-05	-0.783284338912632757302061535032Q-03
-0.439092520757651666348547108820Q-04	0.360316803150178065168213046779Q-04
0.311469670381302876533120119137Q-05	0.554579750608498406761396553401Q-06
-0.728051503366116481284280539682Q-07	-0.119380750534273396772761336039Q-06
-0.142712677403921807369950040297Q-08	0.482286911005137130473876802019Q-08
0.134851828657556199671937219998Q-09	-0.661943242988190217983811484321Q-10
-0.353932973478850986153770653065Q-11	-0.134831224477900589647254954717Q-11
0.311877766002957957951895944222Q-13	0.755336107340111135024372308735Q-13
0.621967416488506945329328598647Q-15	-0.139406218976751538861636227753Q-14
-0.234810461498766217579620115046Q-16	0.838971365080555841659818271420Q-17
0.321559240901043744735354451325Q-18	0.160182372762271410187426277948Q-18
-0.138364867392323959748222645644Q-20	-0.438692734349015130211087780655Q-20
-0.250668665976145974374741957813Q-22	0.463549011924315275878111807280Q-22
0.522919590406060196877774286877Q-24	-0.147770094088740172304545740072Q-24
-0.439280016942636565029332641684Q-26	-0.253215861539776800269773816881Q-26
0.106736554307622893072702514977Q-28	0.416734871991164514710446161458Q-28
0.172917868679241206801702001513Q-30	-0.284970575964110297693018241003Q-30
-0.230577487360277615155178584082Q-32	0.540249891588032775311371188516Q-33
0.130833560937310589395468340261Q-34	0.828143464106095904552579893535Q-35
-0.197344861622563937926334032317Q-37	-0.913731401882880974859355400171Q-37
-0.286655588122102846401879178211Q-39	0.437101783150790746689468155574Q-39
0.266204149094617817073086313546Q-41	-0.533358782721014378644233062524Q-42
-0.108806062887745602150902243316Q-43	-0.735435554601903342305089007853Q-44
0.108952809466947606650166863959Q-46	0.583017946925049396861830701816Q-46
0.142879013281696972870821207824Q-48	-0.205948927316980664622675106645Q-48

COEFFICIENTS FOR $Y_N(Z)$, $N=0$

$\text{ARG}(\phi) = 37.5\text{DEGREES}$

REAL PART

IMAGINARY PART

-0.107922180850610026963964637326Q+02
 -0.988163426452300218561664667187Q+01
 -0.467892581986132737538413725781Q+01
 -0.717989813690795418116315725300Q-02
 0.610323799934335457685267403055Q+00
 0.435422950018807878197528969854Q-01
 -0.135251065108790500047797078630Q-01
 -0.343814423298775911184250099449Q-03
 0.874496121331639152789130244914Q-04
 -0.139011163823896903424103396102Q-06
 -0.193960340054676960306171341278Q-06
 0.274689911969373331712706233831Q-08
 0.163994041735505369725107525278Q-09
 -0.366194998115547044306280417119Q-11
 -0.518113847845265705306031247786Q-13
 0.184647250698689144543333006394Q-14
 0.179382676240224621132507214827Q-17
 -0.434321720497827011904449601915Q-18
 0.225337779680624431328445006622Q-20
 0.502144294788007211580592248533Q-22
 -0.506355441218174987134646899896Q-24
 -0.248740707507716633635470461924Q-26
 0.496101382859511596194619205491Q-28
 -0.252187963407082842421892591126Q-31
 -0.260732923841807634184749949096Q-32
 0.950960010783038225449765792403Q-35
 0.735061871407931412803492699548Q-37
 -0.520716535570809045597617942256Q-39
 -0.773629478709789265413313016973Q-42
 0.148257045723851696789934974680Q-43
 -0.166215108542638588883192344113Q-46
 -0.245642346248618848497290110525Q-48

-0.333324268974311560250880686962Q+01
 -0.485318846118956569918482178475Q+00
 0.320586782202616462059376153913Q+01
 0.231138375488762331691777820764Q+01
 0.185697764512417610039951894297Q+00
 -0.107448555939270962562695163217Q+00
 -0.509014616775542648794833978378Q-02
 0.125414408649802251144933583053Q-02
 0.116965558658103692901872786620Q-04
 -0.466953823081837792992535942891Q-05
 0.417758606606484211651215020718Q-07
 0.634061918897420083057736240819Q-08
 -0.116140461991463290099257168600Q-09
 -0.333658016058466586306759872511Q-11
 0.911664077428161326572993154683Q-13
 0.549902645466658894696224029104Q-15
 -0.309663349107683685342262013836Q-16
 0.777853918633221669322134190596Q-19
 0.511145658470170493508707527559Q-20
 -0.386252838062239940082631366364Q-22
 -0.403272212885982151280620588842Q-24
 0.544963649857128012317334760885Q-26
 0.914401742421107605060446518772Q-29
 -0.387393253620527405375990961652Q-30
 0.845203199486687922189804574837Q-33
 0.150676217767493445941643393574Q-34
 -0.777490027601082226986859149853Q-37
 -0.288593094932095386491103624381Q-39
 0.297921031509196806188617543013Q-41
 0.205349518158695416249384551169Q-46
 -0.645848032264863330609385145345Q-46
 0.136022895874588912330551993222Q-48

COEFFICIENTS FOR $Y_N(Z)$, $N = 0$

$\text{ARG}(\phi) = 52.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.278167761515106343592401499865Q+01
 -0.409726348594874336920167506482Q+01
 -0.102162634946850591205525466191Q+02
 -0.636260294002297895818821996975Q+01
 -0.127830646407103653024327374375Q+01
 0.117452687478855487713985948879Q-01
 0.247716662571573063449689550834Q-01
 0.165073130221949145269995525072Q-02
 -0.511238201622499100284924022574Q-04
 -0.718909800583566714744321985007Q-05
 -0.934441077611202995842996650576Q-07
 0.786606023754497392436913229562Q-08
 0.228871028861010840863539644649Q-09
 -0.199705735215906325706067937223Q-11
 -0.139131450618618233879301652410Q-12
 -0.726254356984184455553459166247Q-15
 0.331629497666836080230637476783Q-16
 0.427366822607021257364487400306Q-18
 -0.279770508258151397678353176072Q-20
 -0.775343963389745240582509297751Q-22
 -0.118751444420606791486381113404Q-24
 0.659191322605295429093358741752Q-26
 0.393393103713127189801590604052Q-28
 -0.256254622485587094890894499766Q-30
 -0.308442887134976732268466203945Q-32
 0.805119661976124636751663478848Q-36
 0.121524654628079249889565687968Q-36
 0.335570584137414017039144024983Q-39
 -0.253196110860095366156518581976Q-41
 -0.147767773639152669793806374173Q-43
 0.189807859279370788892000558473Q-46
 0.319901220306211542797851430255Q-48

-0.460574085750361160686512577276Q+02
 -0.366477599417306281669988700194Q+02
 -0.162700427624759510508719089756Q+02
 -0.202121828801310636329876939073Q+01
 0.859322104720186302722953615135Q+00
 0.241848173579418698286686769216Q+00
 0.109089755531801989258334310407Q-01
 -0.151908082960960518635981905487Q-02
 -0.133351361347304610932596557722Q-03
 0.876061949950436645836818914507Q-07
 0.277806030396004536427358012773Q-06
 0.595873198432559270192103135936Q-08
 -0.159444737505312073289944077079Q-09
 -0.640789647023392195194565102688Q-11
 0.255908347943311093728653133124Q-15
 0.240285567091300863973249332574Q-14
 0.219573952165709969956838965491Q-16
 -0.358494752273880446573529289753Q-18
 -0.638769964045017023239631346997Q-20
 0.101388900189541263637654742692Q-22
 0.781890036866025040733620740770Q-24
 0.302102278316077601416083681586Q-26
 -0.460274491716130597121283604214Q-28
 -0.384861622312278153980282473148Q-30
 0.996314037908744213698725214234Q-33
 0.209177749527261752132646125695Q-34
 0.290419869824897123032148909942Q-37
 -0.604426163970873417495094298452Q-39
 -0.249640598821033550720506711270Q-41
 0.849232414423968866333933982658Q-44
 0.739182158470702333800686622149Q-46
 0.588010358096686655650150194553Q-50

COEFFICIENTS FOR $Y_N(Z)$, $N=0$

$\text{ARG}(\phi) = 67.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.119463175477615061881038647996Q+03
 0.943571239815540364613386116460Q+02
 0.456863789220643102434320101218Q+02
 0.120198192259166127525402702926Q+02
 0.128515372976718187438719627123Q+01
 -0.706436640425005879550919390904Q-01
 -0.326781304025297306062939912286Q-01
 -0.349267581753677725126593557575Q-02
 -0.170840564845140039913808228329Q-03
 -0.216294970425069764444628362362Q-05
 0.202557058906811012276416635323Q-06
 0.127013552348433686973373683530Q-07
 0.332976751225946054883460055403Q-09
 0.319194415329908309113057036956Q-11
 -0.659235142791450764445921079264Q-13
 -0.286239641849782611479754182275Q-14
 -0.464311637610152527604325735084Q-16
 -0.307443084874502695143683440008Q-18
 0.249804357724251719863478750620Q-20
 0.822088729953723267399120576233Q-22
 0.904726635503040321526930430046Q-24
 0.428978834966685291699752282326Q-26
 -0.173917029030267102661157824101Q-28
 -0.458776106353609015849801953984Q-30
 -0.364770255070797020479926771364Q-32
 -0.129000541803953428802548581058Q-34
 0.299178302254708202761294617563Q-37
 0.656947439799182076315591620219Q-39
 0.394913390640329574696687033594Q-41
 0.107822019677176398369899746121Q-43
 -0.156567537930113973258179567056Q-46
 -0.294034058004412321907840543556Q-48

0.227458388304280638157926956174Q+02
 0.932418344759904044384453254760Q+01
 -0.785297841966779507421109550104Q+01
 -0.812619329258559138769301669269Q+01
 -0.271703765848739280849064637025Q+01
 -0.425359591635636377102062836763Q+00
 -0.306295314718138396750830873201Q-01
 -0.239930337752566347307078100477Q-03
 0.126547530178929849462276578642Q-03
 0.100684637769034704566671838372Q-04
 0.352570780584632729954073215989Q-06
 0.40349751472803003149430161364Q-08
 -0.157092858973373342212290781976Q-09
 -0.807863825301744684545950088158Q-11
 -0.164269141684299785210509476574Q-12
 -0.130512472501919799339926358299Q-14
 0.162367309626134991957429608903Q-16
 0.608507996404088915397343624849Q-18
 0.805490532577108897376382007580Q-20
 0.448927266770169139266047243361Q-22
 -0.252208634657242428125013288914Q-24
 -0.738938339627675641294098736404Q-26
 -0.686690752096620888253111922734Q-28
 -0.279863728242825305874987181170Q-30
 0.846656919769721782389072354633Q-33
 0.202968976354208046890453975297Q-34
 0.139638891022022131808723403693Q-36
 0.432221555345661139996438470234Q-39
 -0.785782547920678826311709359557Q-42
 -0.159108883528420000343267188776Q-43
 -0.842875242246647094466285945894Q-46
 -0.204418718677636608891814720513Q-48

COEFFICIENTS FOR $Y_N(Z)$, $N = 0$ ARG(ϕ) = 82.5 DEGREES

REAL PART

IMAGINARY PART

-0.105542264171543564071520438503Q+03
 -0.786454959742101137798108226126Q+02
 -0.314062393504746738823798662881Q+02
 -0.645617231729705975540108910410Q+01
 -0.525897620533831067063786460931Q+00
 0.460025433876543768533964347604Q-01
 0.171279779866541465392643891199Q-01
 0.222954768191032223626178909764Q-02
 0.181899173757394841124506372776Q-03
 0.105624491738584652036684642917Q-04
 0.462151697221401511824997085012Q-06
 0.157401237840098723418452753118Q-07
 0.425548512696002523953044723605Q-09
 0.922344697059932542603491259717Q-11
 0.160083155775149914360304636191Q-12
 0.218015096161939951418285191851Q-14
 0.216753385426864853409838346719Q-16
 0.111198996315245818550660117690Q-18
 -0.103012855256134166084231121283Q-20
 -0.371845960655719469367960226927Q-22
 -0.604728803138919725350217158798Q-24
 -0.723687321526021144836591979543Q-26
 -0.703654881490028781381384171013Q-28
 -0.576954434515810638833185004931Q-30
 -0.405448982636371774641061299170Q-32
 -0.245391867206137431968235213445Q-34
 -0.127214356857863649200323622058Q-36
 -0.552676349759311455264891428418Q-39
 -0.188832905672543734451483873709Q-41
 -0.393840808489456309703426174283Q-44
 0.615103383532513096500352830331Q-47
 0.122679143902130988823571991299Q-48

0.169032416773011957509624836121Q+03
 0.133791944879590687355835253701Q+03
 0.664185803052271919867833761270Q+02
 0.208756985872904668262115650612Q+02
 0.426842613334153813691775459994Q+01
 0.588437584721592236767841336077Q+00
 0.565691928530921856999691876466Q-01
 0.388983345611003337422526856101Q-02
 0.193588612957359139021487214153Q-03
 0.688052032694014646891954911923Q-05
 0.160493583181161166957368438916Q-06
 0.139019743917576904551641636542Q-08
 -0.688561152868893916693681704816Q-10
 -0.401748264894704598017362687557Q-11
 -0.124928157493799097298229956202Q-12
 -0.283818509696517572876772260834Q-14
 -0.511396210303886631918587047333Q-16
 -0.756628091779054453756654081973Q-18
 -0.934713600659930377647827746528Q-20
 -0.970019178698256293113047412831Q-22
 -0.841927940663335139546503884607Q-24
 -0.598069506016744417938690829514Q-26
 -0.325251680579870427887919029497Q-28
 -0.101992478080088821193751760908Q-30
 0.338698681221392395622956437559Q-33
 0.873358609498524480738860608389Q-35
 0.871711232667847601568972581220Q-37
 0.641675668165550663357829763045Q-39
 0.389129270848897280184078410393Q-41
 0.202168566422885980982426010536Q-43
 0.914594100174680165314291277348Q-46
 0.361881072814514425761921993754Q-48

COEFFICIENTS FOR $Y_N(Z)$, $N = 1$

$$\text{ARG}(\phi) = 7.5 \text{ DEGREES}$$

REAL PART

IMAGINARY PART

0.123748557806745811056135432336Q+00	-0.515168267682041884177266904100Q-01
-0.193838690455536387474256807103Q+00	-0.610494832678543787921293439351Q-02
-0.634618056986256598023429631538Q-01	0.676435364314541567768894428441Q-01
0.105760828190668091803609110273Q+00	0.180390199187001934207711382491Q-01
-0.325932423164314776926268260171Q-01	-0.226031413729368570106504904468Q-01
0.446152884151434668274525830978Q-02	0.633494552256200851427689143782Q-02
-0.280844397053881951668433700629Q-03	-0.923147544582503937655419202145Q-03
-0.525893018132882161021067230126Q-06	0.843283216584284551488938784305Q-04
0.161145216994395102324930678646Q-05	-0.524941639444993719960530163623Q-05
-0.152856766553300750093316609584Q-06	0.231814714579735416035535860517Q-06
0.854648301634342315800685772603Q-08	-0.730542134747220603219322849795Q-08
-0.339294477936013940697600811272Q-09	0.155597457527462822510830937430Q-09
0.102305992687996096812899052888Q-10	-0.160023801562510040897161976874Q-11
-0.242096440468855133590423964792Q-12	-0.286373390502693068011096071816Q-13
0.456536263195830152536756718730Q-14	0.187197230703066271313515432616Q-14
-0.686175965714311671155256056347Q-16	-0.531774044196844647042232373163Q-16
0.802889587240727866675653913956Q-18	0.107314548223636836161420762354Q-17
-0.668818015404528756828163611320Q-20	-0.170421574995371970661920717459Q-19
0.236529480318469956183961748183Q-22	0.221613526672372778454580002068Q-21
0.389486429654360978674936635945Q-24	-0.240223356498914567026588351206Q-23
-0.993423180737253136350800644806Q-26	0.218164410729927690245549543924Q-25
0.136855689871462335409146923331Q-27	-0.164665261447144543031896582440Q-27
-0.142636137684616794461307525376Q-29	0.100034598549146430880147648094Q-29
0.121990654606387512022111784894Q-31	-0.440043642612906247892523027864Q-32
-0.883630370712384978956953601878Q-34	0.717817926907847945625931199325Q-35
0.549132769671228875602534424085Q-36	0.102095389042727538146098749008Q-36
-0.293189070193334715526711269833Q-38	-0.141075259543392363205464118030Q-38
0.132960034454766960974985721680Q-40	0.114960678659966164743537759830Q-40
-0.493915918014890342007552521647Q-43	-0.731956429972298377856188054876Q-43
0.133825834568631859940836213739Q-45	0.391341827209621467559478584875Q-45
-0.118580144177632238121203874371Q-48	-0.180658248337403526701616084426Q-47
-0.146225986660949306876362630425Q-50	0.727786943570877354246446781467Q-50

COEFFICIENTS FOR $Y_N(Z)$, $N = 1$

$\text{ARG}(\phi) = 22.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

-0.242825766386393991868368328900Q+00
 -0.581053479084877644991684664899Q+00
 -0.302824695808903405217159243338Q+00
 0.146136562472512918417778962528Q+00
 0.264101624271980280833400171705Q-01
 -0.111493645011921025726052782356Q-01
 0.934862756292923378408244483221Q-03
 0.106910845189902056295293234594Q-04
 -0.567884046548716871600282628189Q-05
 0.3320347603874306011410380954423Q-06
 -0.647672277585484770185924835843Q-08
 -0.143024622738837045956724406358Q-09
 0.110584635898943389753296089672Q-10
 -0.260486138171834383328392373605Q-12
 0.202235356097057642050857568805Q-14
 0.421860782044955867304831146969Q-16
 -0.143245241798367195593789422576Q-17
 0.181840855337160521490376791331Q-19
 -0.708725092449964290927007161787Q-22
 -0.131504137006064352405160459858Q-23
 0.255046012192962724451300479835Q-25
 -0.202081035601418595191975241764Q-27
 0.452490717117115691216808987405Q-30
 0.744066888706386755962745451205Q-32
 -0.938297183776031130941439561033Q-34
 0.507654818142582216725263539694Q-36
 -0.714117920777066643467587809077Q-39
 -0.104793987861206912852605089104Q-40
 0.930014481419320156059457677985Q-43
 -0.365174727148313633165366709741Q-45
 0.344035554089943639643612423944Q-48
 0.454527528859774729511519974567Q-50

-0.352681634483825046064836820031Q+00
 -0.132128841713672614502911245256Q+00
 0.266733531216838924164990291598Q+00
 0.163219890999840252221514388141Q+00
 -0.586749824735128397301684700748Q-01
 0.166456507941861115990395850473Q-02
 0.924621782378134271100847454228Q-03
 -0.109386934708065166115350915234Q-03
 0.396681130838238994138425448285Q-05
 0.803281786685600381491013662806Q-07
 -0.119120581029948827225290722686Q-07
 0.419138865544663377935110765910Q-09
 -0.495981649808279736339699029161Q-11
 -0.108191145827876199261075124440Q-12
 0.527851379530953799627376460392Q-14
 -0.891166888824095853416444173829Q-16
 0.480003130650915589092232792156Q-18
 0.946196911254922161050281497942Q-20
 -0.237717404125617122067695745887Q-21
 0.235136098152520977794428568353Q-23
 -0.685420984793591509631444531484Q-26
 -0.119658866822919756513887992049Q-27
 0.184870620215186997188991879046Q-29
 -0.119953542146356296592946977991Q-31
 0.210923381889845618463269644157Q-34
 0.327298062802918533055491878558Q-36
 -0.343496887549063650147959808378Q-38
 0.157316290188515794341736043277Q-40
 -0.179866230490163910154683413053Q-43
 -0.250168276529252043283346642995Q-45
 0.190259703528151649037340695141Q-47
 -0.647541742051235062702875939918Q-50

COEFFICIENTS FOR $Y_w(Z)$, $N=1$

$\text{ARG}(\phi) = 37.5\text{DEGREES}$

REAL PART

IMAGINARY PART

-0.856690795730996964622143251062Q+00
 -0.158395689653744208271062718310Q+01
 -0.148072480474776520520945972773Q+01
 -0.278773637457742476347525251961Q+00
 0.109110390859522699544327218384Q+00
 0.117969543056272346268510564538Q-01
 -0.189179123073902363033630909403Q-02
 -0.651225377888054657478483293291Q-04
 0.996168059700034457814816201690Q-05
 0.232924883638330421839650963411Q-07
 -0.184256653057881356158945279622Q-07
 0.206079303091028827618876135780Q-09
 0.133411928059774969989003442939Q-10
 -0.257801779518344023523940913133Q-12
 -0.373425313841958595329823154950Q-14
 0.116455415541597987796176771810Q-15
 0.156266801023774355021132460642Q-18
 -0.246005332334029941175017747033Q-19
 0.114496510579485505174852409306Q-21
 0.258076763101821274513861114278Q-23
 -0.240677786679539122276810259293Q-25
 -0.118289624353552934488455639391Q-27
 0.217394951526542957459471226235Q-29
 -0.845299598867817883493569624069Q-33
 -0.105681658001927465685440936058Q-33
 0.362370894137148761892142913284Q-36
 0.277544929742359696462326543017Q-38
 -0.186422370816902536606736927460Q-40
 -0.278150637354411518193857386626Q-43
 0.497692764157270246863468350102Q-45
 -0.523146659939577861397895373262Q-48
 -0.775773549826535926397108632395Q-50

-0.278521911276997064547443170033Q+01
 -0.199301986229430556588618110171Q+01
 -0.194006212734653497490120511358Q+00
 0.500870075297875246972322039137Q+00
 0.788869299610031856478013507377Q-01
 -0.168263866252793129140045857612Q-01
 -0.109228218706498112565142791126Q-02
 0.157868350451040202433056446355Q-03
 0.232967406018006204860861412351Q-05
 -0.484214087318193438114403203314Q-06
 0.275007106582743148476161613356Q-08
 0.555472079344939811130593911371Q-09
 -0.856456054962319288464619476687Q-11
 -0.254061070224798288279458815613Q-12
 0.607918017761030913443903643140Q-14
 0.385065005554616530334529432159Q-16
 -0.184896881771605088157487625292Q-17
 0.380963339683060050780577851689Q-20
 0.275353241761211777209428171597Q-21
 -0.190788661442690492856633233906Q-23
 -0.198582597017435217887055423372Q-25
 0.248705096404292280786825199477Q-27
 0.433200331826501757523411214495Q-30
 -0.163153707476025012996918440162Q-31
 0.328384050705110949833496664536Q-34
 0.588780683549317071955218749578Q-36
 -0.287478417766917419524042004271Q-38
 -0.105703821492393790499878671545Q-40
 0.103254597638372845884951162411Q-42
 0.490920009348699956583740648877Q-47
 -0.210175574216298434075402961533Q-47
 0.422055298299143508617979316013Q-50

COEFFICIENTS FOR $Y_N(Z)$, $N = 1$ $\text{ARG}(\phi) = 52.5\text{DEGREES}$

REAL PART

IMAGINARY PART

0.770393645646066783404247235355Q+01	-0.883876692102317344576985332864Q+01
0.435526489072981676517272360793Q+01	-0.804477524357592879047166816238Q+01
-0.309989275303315516280187733583Q+00	-0.456485826850671406812689487910Q+01
-0.112363880067804649428500856689Q+01	-0.881500124496982236874037287733Q+00
-0.269546734610729570760603415776Q+00	0.942449121633046390023124480124Q-01
-0.536367402102601421163722766929Q-02	0.384272906956217380644235668758Q-01
0.321427404769108647428093176257Q-02	0.199978572243102088738999909088Q-02
0.223439508477124237993655977612Q-03	-0.164055725100444361259959150138Q-03
-0.432292913962386321706918879508Q-05	-0.151692083813606295296663403449Q-04
-0.715721141219107711217623987407Q-06	-0.430969464371546354192444012064Q-07
-0.999236883943139736480692348000Q-08	0.246822504352548514143681055217Q-07
0.629832579148817211379370761677Q-09	0.529598679217760468927704107950Q-09
0.181502528893741062035646441528Q-10	-0.115047227600270500949957493032Q-10
-0.125566957476041612146967890753Q-12	-0.463724515747442735294587949572Q-12
-0.929383250105790144587902390375Q-14	-0.289001358455420699305784733152Q-15
-0.498635978614817225836706679258Q-16	0.149201337146421882672469506003Q-15
0.192232348371608671969045352899Q-17	0.134527484663981801655166432367Q-17
0.242663029877591569421115402676Q-19	-0.194215043391509531098286423275Q-19
-0.140675290182577092964801323664Q-21	-0.340187692658571847083460525801Q-21
-0.389722901496325701141581159904Q-23	0.435600734202198522388351726872Q-24
-0.630595568328556777384662011793Q-26	0.372428642957051022286200641092Q-25
0.298352821299457760130805860633Q-27	0.142374002923177846014210810622Q-27
0.174379373676672516596900548186Q-29	-0.198250995474329927211184110875Q-29
-0.104972473965767807562406209961Q-31	-0.162210204865002042139672596027Q-31
-0.124202466423854127876077448235Q-33	0.384721331397776349257909601283Q-34
0.220514990676905852117050377493Q-37	0.807084146948174842152819456711Q-36
0.450189947433100475344567391150Q-38	0.112525940298204265123512175306Q-38
0.122249974254186826838135068022Q-40	-0.215281442881600616864984725722Q-40
-0.867606284531915526496855611942Q-43	-0.870940015035947102563046998149Q-43
-0.496203188232293911591371210061Q-45	0.279650144034476013819366778872Q-45
0.594988488487687161138714935586Q-48	0.239526743097598768193593918859Q-47
0.100201005283935493955340088978Q-49	0.258330249741415183038953988274Q-51

COEFFICIENTS FOR $Y_N(Z)$, $N = 1$

$\text{ARG}(\phi) = 67.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

0.255657066505864807831990574804Q+02
 0.206534107462896958721819510861Q+02
 0.102817151171232204939369132391Q+02
 0.265289842223579826569294118313Q+01
 0.291576517168570625997743861642Q+00
 -0.286592503970144173908569153106Q-02
 -0.393591952694306755189503100748Q-02
 -0.409930022181211863619379612387Q-03
 -0.189808983387945240534258556638Q-04
 -0.257704318378345332145111942231Q-06
 0.166108302613723668817989908115Q-07
 0.102374103987820485905140083379Q-08
 0.255232764327928265758158910956Q-10
 0.239837008751282475333286922319Q-12
 -0.408713113603419411569526467771Q-14
 -0.175369393939459123462910029110Q-15
 -0.272466932228476627594355289269Q-17
 -0.175065855774373180103531843949Q-19
 0.124357799589234382802440282247Q-21
 0.405761715162839121074366000771Q-23
 0.430234098406504553974669910035Q-25
 0.197934791159546137484609195580Q-27
 -0.722893988282935863924546978208Q-30
 -0.189466322424750148422300874839Q-31
 -0.145811909451500130181619138841Q-33
 -0.501301524968391497377195260303Q-36
 0.106707574411616198175428353904Q-38
 0.233138766300758317249055789067Q-40
 0.136149770487006501255403505716Q-42
 0.362179798215036764352470281216Q-45
 -0.488950749690909731499231282142Q-48
 -0.914568109347026479111258336952Q-50

0.166859929567612648083519648368Q+02
 0.102454232535935227529487766093Q+02
 0.120510286191943786612618190410Q+01
 -0.106278404989436287379885867120Q+01
 -0.425791146973324326538394535318Q+00
 -0.647101966452086034740234645956Q-01
 -0.444964138235927667729619935589Q-02
 -0.590292289112889153609848144664Q-04
 0.123568497029215075541802771725Q-04
 0.962538375130008238234386728922Q-06
 0.319458258575827014636891976923Q-07
 0.365956436900496077811730006701Q-09
 -0.110955992339734238981860167486Q-10
 -0.562539626699077596447103683372Q-12
 -0.109181575861547456785460964090Q-13
 -0.843935141198521856817315297993Q-16
 0.896736067440713025909149187046Q-18
 0.332719801382866332121387487319Q-19
 0.423161430720030794332539230538Q-21
 0.228734410929847354184495454081Q-23
 -0.114267506023676747155290367842Q-25
 -0.332320961923323840505992312596Q-27
 -0.298270151710681939601256955858Q-29
 -0.118051426512619025072279893815Q-31
 0.325049076948093373776655173625Q-34
 0.774833638786775990151395583988Q-36
 0.516971604564267834037289730458Q-38
 0.155734592824313629674065597965Q-40
 -0.261677643625918002690284341990Q-43
 -0.527489220043431773186706910066Q-45
 -0.271885754372020616127083760794Q-47
 -0.643152077302248599343706649486Q-50

COEFFICIENTS FOR $Y_N(Z)$, $N = 1$

$\text{ARG}(\phi) = 82.5 \text{ DEGREES}$

REAL PART

IMAGINARY PART

-0.317036928918682085992406038860Q+02
 -0.226495515099745260057510243012Q+02
 -0.812449841623329387309440129067Q+01
 -0.150223929520897908239580504945Q+01
 -0.124876048778886492300820243089Q+00
 0.315861912030447738538255687957Q-02
 0.197425006956782445704570416374Q-02
 0.246131560670728769407995702152Q-03
 0.185804603466345099023199902505Q-04
 0.993988343051537199855654790448Q-06
 0.401510844140920395830920325712Q-07
 0.126733059340307836630542766487Q-08
 0.318905656317122270003056354372Q-10
 0.646114306867542719177153614921Q-12
 0.105275005778788683355192547433Q-13
 0.135209093420141199282338079678Q-15
 0.127564159690697043604483150898Q-17
 0.634485749104378485876772732689Q-20
 -0.509682841674276525747943523512Q-22
 -0.181683227766324454373250426764Q-23
 -0.283599824452188238232586652841Q-25
 -0.325211535165193134052689736184Q-27
 -0.303221074247875992463722116746Q-29
 -0.238710716399359102416490575615Q-31
 -0.161288638385395165341987797479Q-33
 -0.939908456050182047637568476645Q-36
 -0.469841589314670734325458759093Q-38
 -0.197132416309947801898176177561Q-40
 -0.651839343342953979697641874425Q-43
 -0.132320444080776837568217349846Q-45
 0.191657486334752997247170256546Q-48
 0.380065662084014583718319897007Q-50

0.385315137878782881372825795854Q+02
 0.297010381412445792878546024327Q+02
 0.136755121810163659100427007169Q+02
 0.386269294908597356620506361705Q+01
 0.701409013000716783003902364363Q+00
 0.858858339421387801680505609886Q-01
 0.737546533450758405912429107646Q-02
 0.456610101814328051495753864609Q-03
 0.206395039371693566412449516615Q-04
 0.672907583322916905936505473735Q-06
 0.146202231888547022457609844047Q-07
 0.128271862162680111857230185172Q-09
 -0.479972413893064537113049993312Q-11
 -0.274070554621815090076104492841Q-12
 -0.807342069820989629921922660080Q-14
 -0.173319146228936405867394050683Q-15
 -0.295466642011050685225814044233Q-17
 -0.414467901538641026628589697656Q-19
 -0.486571743396785789478472741728Q-21
 -0.480973778584023330476379343267Q-23
 -0.398572236778421630496420801560Q-25
 -0.270996262861443689831158402894Q-27
 -0.141535141171829861058012490761Q-29
 -0.430456976851794654305334971981Q-32
 0.129577149163367351693842693917Q-34
 0.331403233704093056778331458457Q-36
 0.320065053758083791099433920801Q-38
 0.227701720251752263403122925116Q-40
 0.133515526383480086476741479016Q-42
 0.671270161866919749659156791780Q-45
 0.294144346635971982472643216904Q-47
 0.112839888164485364642740769871Q-49

APPENDIX C

Fortran Subroutine for $J_0(z)$, $|z| \leq 8$

This subroutine can be used to calculate $J_0(z)$ for $|z| \leq 8$ to a user supplied accuracy. It can be adapted for different orders by changing, NU, DENOM and the coefficients in A.

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1.000      SUBROUTINE BESAC (NACC, Z, BESS, IFAIL)
2.000      C
3.000      C-----EXPLANATION OF VARIABLES-----
4.000      C
5.000      C   NACC.....INTEGER .
6.000      C           ON ENTRY CONTAINS THE NUMBER OF
7.000      C           DECIMAL PLACE ACCURACY REQUIRED .
8.000      C           MAXIMUM ACCURACY OF 28 TO 31 PLACES
9.000      C           DEPENDS ON ARG(Z) .
10.000     C           NACC SHOULD BE NO GREATER THAN 28 ,
11.000     C           FOR Z NEAR THE IMAGINARY AXIS .
12.000     C           UNCHANGED ON EXIT .
13.000     C
14.000     C   Z.....QUADRUPLE PRECISION COMPLEX .
15.000     C           ON ENTRY CONTAINS THE POINT AT WHICH
16.000     C           THE FUNCTION IS TO BE CALCULATED .
17.000     C           THE MODULUS OF Z MUST NOT BE GREATER
18.000     C           THAN 8.000 .
19.000     C           UNCHANGED ON EXIT .
20.000     C
21.000     C   BESS.....QUADRUPLE PRECISION COMPLEX .
22.000     C           UNSPECIFIED ON ENTRY .
23.000     C           ON SUCCESSFUL EXIT ( SEE IFAIL )
24.000     C           CONTAINS THE VALUE OF THE BESSEL
25.000     C           FUNCTION AT Z .
26.000     C
27.000     C   IFAIL.....INTEGER .
28.000     C           UNSPECIFIED ON ENTRY .
29.000     C           ON EXIT HAS ONE OF THE FOLLOWING VALUES
30.000     C
31.000     C   0.....CALCULATION COMPLETED TO REQUIRED ACCURACY
32.000     C
33.000     C   1.....REQUIRED ACCURACY TOO HIGH ,
34.000     C           CALCULATION COMPLETED TO MAXIMUM ACCURACY .
35.000     C
36.000     C   2.....ABSOLUTE VALUE OF Z IS GREATER THAN EIGHT .
37.000     C           NO CALCULATION PERFORMED .
38.000     C
39.000     C   IMPLICIT REAL*16(Q)
40.000     C   COMPLEX*32 A, BESS, C, C1, T, Z, ZT

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41.000 REAL*16 ABSZ, ARGZT, COSIN, DENOM, PIFAC, X, Y
42.000 REAL*4 FNACC, FNU, R0, R1, R2
43.000 DIMENSION A(6, 28), COSIN(6)
44.000 C
45.000 C NU.....INTEGER .
46.000 C THE ORDER OF THE BESSEL FUNCTION OF THE FIRST KIND .
47.000 C
48.000 C DENOM.....QUADRUPLE PRECISION REAL .
49.000 C SET TO NU FACTORIAL .
50.000 C
51.000 DATA NU, DENOM / 0, 1.000 /
52.000 C
53.000 C PIFAC.....QUADRUPLE PRECISION REAL . SET TO PI/24 .
54.000 C
55.000 DATA PIFAC / 1.30899693899574718269276E0763664Q-01 /
56.000 C
57.000 C COSIN.....QUADRUPLE PRECISION REAL ARRAY OF DIMENSION 6
58.000 C COSIN( K ) SET TO COS( PI * ( 2K-1 ) / 24 ) .
59.000 C
60.000 DATA COSIN(1), COSIN(2), COSIN(3), COSIN(4), COSIN(5), COSIN(6) /
61.000 * 0.991444861373810411144557526928563Q+00 ,
62.000 * 0.923879532511286756128183189396789Q+00 ,
63.000 * 0.793353340291235164579776961501302Q+00 ,
64.000 * 0.608761429008720639416097542898169Q+00 ,
65.000 * 0.382683432365089771728459984030406Q+00 ,
66.000 * 0.130526192220051591548406227895500Q+00 /
67.000 C
68.000 C A.....QUADRUPLE PRECISION COMPLEX , TWO DIMENSIONAL ARRAY
69.000 C OF DIMENSIONS 6 BY 28 ,
70.000 C ROW K CONTAINS THE COEFFICIENTS IN THE EXPANSION OF THE
71.000 C FUNCTION ALONG THE RAY WITH ARGUMENT PI * ( 2K-1 ) / 24 .
72.000 C
73.000 DATA A(1,1), A(1,2), A(1,3), A(1,4), A(1,5), A(1,6), A(1,7) /
74.000 * ( 0.40795869055667137640715028153693Q+00 ,
75.000 * -0.56442646929590935200795128274590Q-01 ) ,
76.000 * ( 0.763078292501314003611467780950E4Q-01 ,
77.000 * -0.61302046672716605147332674516919Q-01 ) ,
78.000 * ( 0.29061006595818788085586515391498Q+00 ,
79.000 * -0.20911435561449686363680202686931Q+00 ) ,
80.000 * ( -0.41389519311709364289690627937354Q+00 ,

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81.000 * -0.49959512745050736116971017573850Q-01) 0
82.000 * (0.14405373906065264429620605762981Q+00) 0
83.000 * 0.91255766225367239943263268407435Q-01) 0
84.000 * (-0.22191868433940324530320446112275Q-01) 0
85.000 * -0.29516142867063306110240315867003Q-01) 0
86.000 * (0.15821011678673097559220170114149Q-02) 0
87.000 * 0.47897681004629969864723074241209Q-02) /
88.000 DATA A(1,8), A(1,9), A(1,10), A(1,11), A(1,12), A(1,13), A(1,14) /
89.000 * (-0.59234157455037023995012675293566Q-05) 0
90.000 * -0.47863710713778006789287843649731Q-03) 0
91.000 * (-0.93656954962602142941518489788133Q-05) 0
92.000 * 0.32225706258653841087281828118680Q-04) 0
93.000 * (0.98007182750822194752002275702969Q-06) 0
94.000 * -0.15271062036972918272686508255232Q-05) 0
95.000 * (-0.58849005869019316334881884219035Q-07) 0
96.000 * 0.51370384201749153937317288042167Q-07) 0
97.000 * (0.24841843116601105419586447412207Q-08) 0
98.000 * -0.11659304865213573041154924553111Q-08) 0
99.000 * (-0.79166685333981694744509040991150Q-10) 0
100.000 * 0.13002722707254774191498280825128Q-10) 0
101.000 * (0.19713528931142066795752587275933Q-11) 0
102.000 * 0.21989564315413682332706849016329Q-12) /
103.000 DATA A(1,15), A(1,16), A(1,17), A(1,18), A(1,19), A(1,20) /
104.000 * (-0.38986101629433760169040062140733Q-13) 0
105.000 * -0.15719635840571344149732102786680Q-13) 0
106.000 * (0.61288305780992463777428485624658Q-15) 0
107.000 * 0.46989454558630322083751642792260Q-15) 0
108.000 * (-0.74872296145083913456919480921153Q-17) 0
109.000 * -0.99111434114507891007149844922754Q-17) 0
110.000 * (0.65125722022767106932468459658214Q-19) 0
111.000 * 0.16392989156702043206417304659115Q-18) 0
112.000 * (-0.24484306038263621355486069536909Q-21) 0
113.000 * -0.22147117906606840779028263028522Q-20) 0
114.000 * (-0.39482281695962560448896899872804Q-23) 0
115.000 * 0.24892188592384991764317178926069Q-22) /
116.000 DATA A(1,21), A(1,22), A(1,23), A(1,24) /
117.000 * (0.10567641477407692569908964539183Q-24) 0
118.000 * -0.23401714881381196783886748810376Q-24) 0
119.000 * (-0.15087853069717455406447080694267Q-26) 0
120.000 * 0.18260278516002535548651669668619Q-26) 0

121.000 * (0.16246331754032631900049937714552Q-28) D
122.000 * -0.11458225051944390195654405947447Q-28) D
123.000 * (-0.143291993541657474C3873789530818Q-30) D
124.000 * 0.52094078228743871177741921848437Q-31) /
125.000 DATA A(1,25), A(1,26), A(1,27), A(1,28) /
126.000 * (0.10689014912284315842898385395235Q-32) D
127.000 * -0.89267514828995868537298747317694Q-34) D
128.000 * (-0.68331426076520037960310045221448Q-35) D
129.000 * -0.12555686975919394598649293578940Q-35) D
130.000 * (0.37492633224211049946541724110212Q-37) D
131.000 * 0.17950167586171787447E62828877555Q-37) D
132.000 * (-0.17459403685946996851518989893939Q-39) D
133.000 * -0.15040125647625372237119683989617Q-39) /
134.000 DATA A(2,1), A(2,2), A(2,3), A(2,4), A(2,5), A(2,6), A(2,7) /
135.000 * (0.186238813652549412C6391184280746Q+01) D
136.000 * 0.16486122859825242144719307919130Q+00) D
137.000 * (0.14586386625180950972545797220381Q+01) D
138.000 * -0.12544466638704219695403027896923Q+00) D
139.000 * (0.99548777603655731948538246885643Q+00) D
140.000 * -0.10844229108502024582959366154897Q+01) D
141.000 * (-0.683630679198039881838798357615E8Q+00) D
142.000 * -0.59323596240726279387806001876372Q+00) D
143.000 * (-0.92103894031436975124900736670351Q-01) D
144.000 * 0.27615145780052247518564222799368Q+00) D
145.000 * (0.54325155286706497009706705913268Q-01) D
146.000 * -0.12476381839349755533666766352444Q-01) D
147.000 * (-0.53126997414577367734725624978496Q-02) D
148.000 * -0.46544745856727888381187414478990Q-02) /
149.000 DATA A(2,8), A(2,9), A(2,10), A(2,11), A(2,12), A(2,13) /
150.000 * (-0.31338728717415638055222588885682Q-04) D
151.000 * 0.63622927278262264692972383255147Q-03) D
152.000 * (0.34287393291279206654024854743379Q-04) D
153.000 * -0.26013530174549737200216869491038Q-04) D
154.000 * (-0.221524264522547452E4258733781922Q-05) D
155.000 * -0.45992988672321494631582857389451Q-06) D
156.000 * (0.47791149661373039954614575852049Q-07) D
157.000 * 0.82401441947674595322343130211135Q-07) D
158.000 * (0.98772468429146011554951907842518Q-09) D
159.000 * -0.31346016042881186165079671391767Q-08) D
160.000 * (-0.85559418413663835305923031026596Q-10) D

161.000 * 0.40492685731528486919403002058275Q-10) /
162.000 DATA A(2,14), A(2,15), A(2,16), A(2,17), A(2,18), A(2,19) /
163.000 * (0.21497692022316954524086941727099Q-11)
164.000 * 0.84820179582633413317487408538878Q-12)
165.000 * (-0.18044845865653460488125005161979Q-13)
166.000 * -0.44965043319601150272937439932755Q-13)
167.000 * (-0.36611959638375363485072844099656Q-15)
168.000 * 0.80228194969170118787213386821454Q-15)
169.000 * (0.13282565182822653729060423561167Q-16)
170.000 * -0.46368682453174704019240608040720Q-17)
171.000 * (-0.17697985099743731846229593931446Q-18)
172.000 * -0.89566498342508217565158876682037Q-19)
173.000 * (0.73565583861815498696056484311028Q-21)
174.000 * 0.23784602955667591021187110468363Q-20) /
175.000 DATA A(2,20), A(2,21), A(2,22), A(2,23), A(2,24) /
176.000 * (0.13439277020255245432805184079622Q-22)
177.000 * -0.24563383016281616649391002845311Q-22)
178.000 * (-0.27343095881597148765216409975695Q-24)
179.000 * 0.75979310571844082254091676766722Q-25)
180.000 * (0.22524555302524473310456212866351Q-26)
181.000 * 0.13101506878535092988935606352495Q-26)
182.000 * (-0.53291005516586209617603948448949Q-29)
183.000 * -0.21115269624392155974828639121683Q-28)
184.000 * (-0.86754651739648244417089788877729Q-31)
185.000 * 0.14195300876382134298811178252330Q-30) /
186.000 DATA A(2,25), A(2,26), A(2,27), A(2,28) /
187.000 * (0.11362604863590713575985169424467Q-32)
188.000 * -0.26277104068594085415779274347173Q-33)
189.000 * (-0.63512437869492818751052544852765Q-35)
190.000 * -0.40436324862400408223202596401835Q-35)
191.000 * (0.93755737997489190974076797605123Q-38)
192.000 * 0.43922566481873963909743193811676Q-37)
193.000 * (0.13661645456438597289667311109739Q-39)
194.000 * -0.20731565753726878313488826244435Q-39) /
195.000 DATA A(3,1), A(3,2), A(3,3), A(3,4), A(3,5), A(3,6) /
196.000 * (0.67119273258906999397417562680239Q+01)
197.000 * 0.83712702818187066160128862554041Q+01)
198.000 * (0.71500895090149274249562302933468Q+01)
199.000 * 0.59935320789535874704309890908769Q+01)
200.000 * (0.62159845055422611610087734646016Q+01)

201.000 * -0.19486400634296973199162619237439Q+00) 0
202.000 * (0.917669488125330300089711265482E2Q+00) 0
203.000 * -0.24065140778222976905395753482710Q+01) 0
204.000 * (-0.56777356588961925012223189451426Q+00) 0
205.000 * -0.31719376085575826201011612499029Q+00) 0
206.000 * (-0.52984967577101028658611440784450Q-01) 0
207.000 * 0.92811127584290710768203757704639Q-01) /
208.000 DATA A(3,7), A(3,8), A(3,9), A(3,10), A(3,11), A(3,12) /
209.000 * (0.10994988091648049146745669092850Q-01) 0
210.000 * 0.53006984777873094100192858823990Q-02) 0
211.000 * (0.33155441080985683550871115114717Q-03) 0
212.000 * -0.96600009434642741994234559722918Q-03) 0
213.000 * (-0.64158538295280157832171771619448Q-04) 0
214.000 * -0.11644140366370472243416450105202Q-04) 0
215.000 * (-0.19262278486119835822887559091950Q-07) 0
216.000 * 0.32792409067351769904708187461371Q-05) 0
217.000 * (0.13098460082289649955221225867198Q-06) 0
218.000 * -0.24389700433340074222558066472352Q-07) 0
219.000 * (-0.16816077796311041189112778914050Q-08) 0
220.000 * -0.41353051818623120210689188092807Q-08) /
221.000 DATA A(3,13), A(3,14), A(3,15), A(3,16), A(3,17), A(3,18) /
222.000 * (-0.103711528823211601863616931979E4Q-09) 0
223.000 * 0.70597257863178096934493061241108Q-10) 0
224.000 * (0.21859346943753071283032864827525Q-11) 0
225.000 * 0.20545036410632514002637938671890Q-11) 0
226.000 * (0.312224547791540385075317468008E9Q-13) 0
227.000 * -0.53302500363141758467434181670466Q-13) 0
228.000 * (-0.10573078967372922849271977707796Q-14) 0
229.000 * -0.32781289949432533628380676339244Q-15) 0
230.000 * (-0.11756726790979670550122091559254Q-17) 0
231.000 * 0.17379430376588814964417894251052Q-16) 0
232.000 * (0.23918281026749472894061443142955Q-18) 0
233.000 * -0.40755209566834685198756324933495Q-19) /
234.000 DATA A(3,19), A(3,20), A(3,21), A(3,22), A(3,23) /
235.000 * (-0.11947502045502751672080653424766Q-20) 0
236.000 * -0.27655685125209732758600102490881Q-20) 0
237.000 * (-0.26729840697905951975410589489261Q-22) 0
238.000 * 0.20282438263448503305629880293553Q-22) 0
239.000 * (0.26247196204098981414737826408926Q-24) 0
240.000 * 0.21156346934994648642359041179737Q-24) 0

241.000 * (0.12897963883569176515585530211121Q-26)
242.000 * (-0.27869637139173354839687197440611Q-26)
243.000 * (-0.25035651838797726597574836978744Q-28)
244.000 * (-0.47370861045896446416853047963321Q-29) /
245.000 DATA A(3,24), A(3,25), A(3,26), A(3,27), A(3,28) /
246.000 * (0.11734694565831946821155271368519Q-31)
247.000 * (0.19300680330736498144511520864952Q-30)
248.000 * (0.12832433620734723751359383046365Q-32)
249.000 * (-0.41046820113458423420150726959960Q-33)
250.000 * (-0.45903276163115953174237946205508Q-35)
251.000 * (-0.73307874882075810998447859481484Q-35)
252.000 * (-0.35382723873725811024606758356275Q-37)
253.000 * (0.37181894194269406039102168670998Q-37)
254.000 * (0.24657413518293437964736301762449Q-39)
255.000 * (0.13762270701873969701694544929432Q-39) /
256.000 DATA A(4,1), A(4,2), A(4,3), A(4,4), A(4,5), A(4,6) /
257.000 * (-0.24615607680072097946409414783958Q+02)
258.000 * (0.40708184519741684906894024764506Q+02)
259.000 * (-0.13674680432263381816930047837609Q+02)
260.000 * (0.36750868601784119090822057617668Q+02)
261.000 * (0.42050323986583392854024267002657Q+01)
262.000 * (0.20367739691527432297883143363871Q+02)
263.000 * (0.58622759766447914134576827477783Q+01)
264.000 * (0.35548361381913597009511675099555Q+01)
265.000 * (0.13204034634908637782731463139523Q+01)
266.000 * (-0.61553461742526762917630456224676Q+00)
267.000 * (0.12011362049486119603651628069050Q-01)
268.000 * (-0.21071485764786350942543741438291Q+00) /
269.000 DATA A(4,7), A(4,8), A(4,9), A(4,10), A(4,11), A(4,12) /
270.000 * (-0.19372490663037713599494764213266Q-01)
271.000 * (-0.10318073190657241386059601834323Q-01)
272.000 * (-0.13193765886752324090956713249527Q-02)
273.000 * (0.10868685743074080331045584156917Q-02)
274.000 * (0.32882094123650831394692136589888Q-04)
275.000 * (0.97769240370192918375983719587556Q-04)
276.000 * (0.49561200024879495862816530660429Q-05)
277.000 * (0.10089071632680748236592179984534Q-06)
278.000 * (0.66621081843728297217183148014726Q-07)
279.000 * (-0.18223288264254629134116750106942Q-06)
280.000 * (-0.49411940775348074211939481683820Q-08)

281.000 * -0.39106755594795768418695956250335Q-08) /
282.000 DATA A(4,13), A(4,14), A(4,15), A(4,16), A(4,17), A(4,18) /
283.000 * (-0.14323130562185905266344553318864Q-09)
284.000 * (0.96090174415940859625911547790801Q-10)
285.000 * (0.11426229129018497661619105639929Q-11)
286.000 * (0.38663761072019751013073789910377Q-11)
287.000 * (0.81399052533547393688503984541264Q-13)
288.000 * (0.87295719916708646689766836918206Q-15)
289.000 * (0.42963581930446140327461610709026Q-15)
290.000 * -0.13679352041698177218734937566741Q-14)
291.000 * (-0.18413047890098872367751566818597Q-16)
292.000 * -0.12441337372666640599612697442972Q-16)
293.000 * (-0.23556437667867229352963180294575Q-18)
294.000 * (0.19433016360925339500623608205876Q-18) /
295.000 DATA A(4,19), A(4,20), A(4,21), A(4,22), A(4,23) /
296.000 * (0.14780223743818327032278009370160Q-20)
297.000 * (0.34422016606709071729170561679391Q-20)
298.000 * (0.40956734504856370170797264766348Q-22)
299.000 * -0.50882962561403372040902485809500Q-23)
300.000 * (0.63981925565223277240096031917410Q-25)
301.000 * -0.40558120586652640015269127140345Q-24)
302.000 * (-0.33618720134599604516805907738920Q-26)
303.000 * -0.15617312296875415002587281308420Q-26)
304.000 * (-0.19927812351128876188982915416726Q-28)
305.000 * (0.23097458034639807215285459498273Q-28) /
306.000 DATA A(4,24), A(4,25), A(4,26), A(4,27), A(4,28) /
307.000 * (0.12653928079114571953761923102439Q-30)
308.000 * (0.19179314075437628141322658676683Q-30)
309.000 * (0.15148831459979218067536099487641Q-32)
310.000 * -0.48291887511308268323169572633154Q-33)
311.000 * (-0.35457362648932105717005203942104Q-36)
312.000 * -0.10136460884656706702109567689352Q-34)
313.000 * (-0.58150487026493486555445415501452Q-37)
314.000 * -0.14094518049251148437476781919616Q-37)
315.000 * (-0.15975891547896140605983712801556Q-39)
316.000 * (0.28576217592464116638248112392485Q-39) /
317.000 DATA A(5,1), A(5,2), A(5,3), A(5,4), A(5,5), A(5,6) /
318.000 * (-0.12023234004003584999818886107656Q+03)
319.000 * -0.66691018233263761381657206590178Q+02)
320.000 * (-0.99170227887441285329135194491064Q+02)

321.000 * -0.39147826531356588336958685234818Q+02) 0
322.000 * (-0.49482551698438323677964812571407Q+02) 0
323.000 * -0.14800978678558559364104741152456Q+01) 0
324.000 * (-0.12834635332439317353249812196134Q+02) 0
325.000 * 0.63351619838680025091559421924811Q+01) 0
326.000 * (-0.13924145634822307560273601121626Q+01) 0
327.000 * 0.23497334840795793775590047916078Q+01) 0
328.000 * (0.35873442631228915638088468993558Q-01) 0
329.000 * 0.36184555175082579591418906704027Q+00) /
330.000 DATA A(5,7), A(5,8), A(5,9), A(5,10), A(5,11), A(5,12) /
331.000 * (0.24623296955251348484432877021334Q-01) 0
332.000 * 0.25407683584997795664451288720100Q-01) 0
333.000 * (0.25989029404040905193856549537677Q-02) 0
334.000 * 0.27328614896885497885059913797181Q-03) 0
335.000 * (0.12385474204657920414516265382266Q-03) 0
336.000 * -0.86478706066528572035583131906616Q-04) 0
337.000 * (0.16233818774304976462790471703260Q-05) 0
338.000 * -0.68164033686269323775951271638189Q-05) 0
339.000 * (-0.12848188590205171272287441105036Q-06) 0
340.000 * -0.23322393287194919485736536781241Q-06) 0
341.000 * (-0.79995001109688431240715126441317Q-08) 0
342.000 * -0.26718216601645247528390463650696Q-08) /
343.000 DATA A(5,13), A(5,14), A(5,15), A(5,16), A(5,17), A(5,18) /
344.000 * (-0.20551599036084747083254899649849Q-09) 0
345.000 * 0.93883769053122440787827556762722Q-10) 0
346.000 * (-0.19551294992334729981372286345167Q-11) 0
347.000 * 0.48015363459005741396881230966287Q-11) 0
348.000 * (0.37507349404640345799651797934362Q-13) 0
349.000 * 0.95920761478909471667874560717069Q-13) 0
350.000 * (0.16214600690405924996084918798866Q-14) 0
351.000 * 0.75440387851336339973278181017069Q-15) 0
352.000 * (0.25894198253276515890314860610469Q-16) 0
353.000 * -0.88611990878570833706219473950495Q-17) 0
354.000 * (0.16963885835163051058784661923728Q-18) 0
355.000 * -0.33091881056889251909482520124499Q-18) /
356.000 DATA A(5,19), A(5,20), A(5,21), A(5,22), A(5,23) /
357.000 * (-0.13151889996350858678670178161917Q-20) 0
358.000 * -0.43198897528320881209066285578019Q-20) 0
359.000 * (-0.4315575831589959275961716074911Q-22) 0
360.000 * -0.23325721607151014874315892698848Q-22) 0

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361.000 * ( -0.46903719427128928309181255754079Q-24 )
362.000 * 0.12866821474808433187918726739619Q-24 )
363.000 * ( -0.22018698090879407279084234728340Q-26 )
364.000 * 0.37606144136947100549130543087069Q-26 )
365.000 * ( 0.86281155469266477971682953927907Q-29 )
366.000 * 0.34553644783555857374984659001160Q-28 ) /
367.000 DATA A(5,24), A(5,25), A(5,26), A(5,27), A(5,28) /
368.000 * ( 0.22713070854464166764224304073967Q-30 )
369.000 * 0.13950524503754715029219540715481Q-30 )
370.000 * ( 0.17873486006162661044327487865169Q-32 )
371.000 * -0.40963510166528041796508686348469Q-33 )
372.000 * ( 0.62652933706147687905142320609119Q-35 )
373.000 * -0.98027781360612562544596100114940Q-35 )
374.000 * ( -0.14150553808635658780091448512327Q-37 )
375.000 * -0.66805156657504333366626559124035Q-37 )
376.000 * ( -0.31024687126385932720181561031139Q-39 )
377.000 * -0.20507272069832033063755063045718Q-39 ) /
378.000 DATA A(6,1), A(6,2), A(6,3), A(6,4), A(6,5), A(6,6) /
379.000 * ( 0.14786330077903399910834413043463Q+03 )
380.000 * -0.18685812523542100076518581169684Q+03 )
381.000 * ( 0.10415768205398205162399654475226Q+03 )
382.000 * -0.14493435376763316555403379954426Q+03 )
383.000 * ( 0.37773716029004829631634647285402Q+02 )
384.000 * -0.68100726343676867881843240961907Q+02 )
385.000 * ( 0.71510684578255635647552427740806Q+01 )
386.000 * -0.19942171041630688504315865053873Q+02 )
387.000 * ( 0.58718947462746490892023107893163Q+00 )
388.000 * -0.37945991286273713731645640517225Q+01 )
389.000 * ( -0.27336816292524014223244975541634Q-01 )
390.000 * -0.48944600177602346303981707099740Q+00 ) /
391.000 DATA A(6,7), A(6,8), A(6,9), A(6,10), A(6,11), A(6,12) /
392.000 * ( -0.12633298385438303996328684848139Q-01 )
393.000 * -0.44344116850055764209395590208655Q-01 )
394.000 * ( -0.16111141681285472669446158410552Q-02 )
395.000 * -0.28944488875185423952758626824804Q-02 )
396.000 * ( -0.12682388318315677499352037464151Q-03 )
397.000 * -0.13765284525622497163004271726241Q-03 )
398.000 * ( -0.71010473576556223110269068237272Q-05 )
399.000 * -0.47052276012432276254645764427057Q-05 )
400.000 * ( -0.30024483700336217893916277191359Q-06 )

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401.000 * -0.10641119963921452180967194363728Q-06)
402.000 * (-0.99086415014185669218051899910643Q-08)
403.000 * -0.92761222837942349840197804571025Q-09) /
404.000 DATA A(6,13), A(6,14), A(6,15), A(6,16), A(6,17), A(6,18) /
405.000 * (-0.26027484117930187788192176623065Q-09)
406.000 * 0.40940074636336826992996865653823Q-10)
407.000 * (-0.54945502031913302315248827628845Q-11)
408.000 * 0.23689419516543360236253155310313Q-11)
409.000 * (-0.93097909164864721426484987076695Q-13)
410.000 * 0.72175746483862135156193005583481Q-13)
411.000 * (-0.12405514383185900794313961451409Q-14)
412.000 * 0.16057793958388156658720457530673Q-14)
413.000 * (-0.12100809941441867001603700742549Q-16)
414.000 * 0.28359465561914272852397225563504Q-16)
415.000 * (-0.61402516926354081139078013313073Q-19)
416.000 * 0.41173060997307404969131501659123Q-18) /
417.000 DATA A(6,19), A(6,20), A(6,21), A(6,22), A(6,23) /
418.000 * (0.54121172480611844676219493583295Q-21)
419.000 * 0.49969014268740635137936784721376Q-20)
420.000 * (0.19452840346739931358323830445185Q-22)
421.000 * 0.51000584521037047207876926762789Q-22)
422.000 * (0.31204514306599118157275391121056Q-24)
423.000 * 0.43581643056407606329070172519971Q-24)
424.000 * (0.36820345317120975520032057649888Q-26)
425.000 * 0.30512590903047769797613238757884Q-26)
426.000 * (0.35315464449440018810041928637612Q-28)
427.000 * 0.16376144713659745783339920962069Q-28) /
428.000 DATA A(6,24), A(6,25), A(6,26), A(6,27), A(6,28) /
429.000 * (0.28580566388241294610135183367357Q-30)
430.000 * 0.50849901113658300951720520451353Q-31)
431.000 * (0.19836067158649677964126489289905Q-32)
432.000 * -0.16369074613715444505432167096866Q-33)
433.000 * (0.11863994266786195581653997529741Q-34)
434.000 * -0.42101148754094705704577267314263Q-35)
435.000 * (0.60815557684675400071345237144200Q-37)
436.000 * -0.41597029420894998273193569008155Q-37)
437.000 * (0.26140923382011523100280946805280Q-39)
438.000 * -0.30302663845261326002874852819372Q-39) /

IFAIL = 0

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441.000 C CHECK THE MODULUS OF Z IS LESS THAN EIGHT .
442.000 ABSZ = CQABS( Z )
443.000 IF ( ABSZ .GT. 8.000 ) GO TO 70
444.000 C
445.000 C IF THE MODULUS IS LESS THAN 1.0Q-16 .
446.000 C THE TAYLOR SERIES IS USED FOR THE CALCULATION ,
447.000 IF ( ABSZ .LT. 1.0Q-16 ) GO TO 60
448.000 C
449.000 C IF THE REAL AND IMAGINARY PARTS OF Z HAVE OPPOSITE SIGNS ,
450.000 C J1 IS SET TO 1 .
451.000 X = QREAL( Z )
452.000 Y = QIMAG( Z )
453.000 J1 = 0
454.000 IF ( ( X*Y ) .LT. 0.000 ) J1 = 1
455.000 C
456.000 C THE NEXT FEW LINES ARE USED TO TRANSPOSE THE POINT Z INTO ZT
457.000 C IN THE UPPER RIGHT QUADRANT BY TAKING THE ABSOLUTE VALUES OF
458.000 C ITS REAL AND IMAGINARY PARTS .
459.000 X = QABS( X )
460.000 Y = QABS( Y )
461.000 ZT = QCPLX( X, Y )
462.000 C
463.000 C NOW DECIDE WHICH SET OF COEFFICIENTS ARE REQUIRED .
464.000 C I.E. THE VALUE OF K SUCH THAT (2K-1)*PI/24 IS CLOSEST TO ARG(ZT) .
465.000 C K IS POSITIVE AND NO GREATER THAN 6 .
466.000 C
467.000 C FLOATING POINT DIVISION BY ZERO CHECK .
468.000 C IF X IS ZERO THEN ARG(ZT) IS PI/2 , SO K IS 6 .
469.000 IF ( X .NE. 0.000 ) GO TO 10
470.000 K = 6
471.000 GO TO 20
472.000 10 ARGZT = QATAN( Y /X )
473.000 K = 1 + ( IQINT( ARGZT /PIFAC ) ) /2
474.000 C
475.000 C CHECK K IS LESS THAN 7
476.000 C IF NOT K IS SET EQUAL TO 6
477.000 IF ( K .GT. 6 ) K = 6
478.000 C
479.000 20 CONTINUE
480.000 C

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481.000 C      CHECK NACC , IF NACC IS GREATER THAN 28 IFAIL
482.000 C      IS SET TO 1 AND NACC TO 32 .
483.000 C
484.000      IF( NACC .GT. 28 ) IFAIL = 1
485.000      IF( IFAIL .EQ. 1 ) NACC = 32
486.000 C
487.000 C      THE NEXT FEW LINES CALCULATE THE NUMBER OF COEFFICIENTS
488.000 C      REQUIRED FOR THE GIVEN ACCURACY .
489.000      FNACC = FLOAT( NACC )
490.000      FNU = FLOAT( NU )
491.000      RO = 8.0 + 0.5 * FNACC
492.000      R1 = ALOG( RO * ( RO + FNU ) )
493.000      R2 = ALOG( ( RO + FNU ) / 4.0 )
494.000      R1 = ( RO * 2.0 - ( 0.5 * R1 ) - ( FNU * R2 ) ) +
495.000 1 ( 0.5 * RO / ( RO + FNU ) ) + ( FNACC * 2.3025851 ) + 0.4909109 )
496.000 2 / ( R1 + ( 0.5 / RO ) + ( 0.5 / ( RO + FNU ) ) ) - 2.0940008 )
497.000      NUM = INT( R1 ) + 1
498.000 C
499.000 C      CALCULATION OF THE TRUNCATED CHEBYSHEV SERIES AT ZT
500.000      T = ZT * QCPLX( COSIN( K ) , -COSIN( 7-K ) )
501.000      T = 0.625Q-01 * T * T - 2.0Q0
502.000      NUMM1 = NUM - 1
503.000      BESS = T * A( K , NUM ) + A( K , NUMM1 )
504.000      C = A( K , NUM )
505.000      DO 30 J = 2 , NUMM1
506.000          C1 = C
507.000          C = BESS
508.000 30  BESS = T * C - C1 + A( K , NUM-J )
509.000      BESS = 0.5Q0 * ( BESS - C1 ) / DENOM
510.000 C
511.000 C      SYMMETRY RELATIONS ARE USED TO GET THE VALUE AT Z
512.000 C
513.000      IF ( J1 .EQ. 1 ) BESS = QCONJG( BESS )
514.000      IF ( NU .EQ. 0 ) GO TO 50
515.000      Z2 = 0.5Q0 * Z
516.000      DO 40 J = 1 , NU
517.000 40  BESS = Z2 * BESS
518.000 50 CONTINUE
519.000      RETURN
520.000 C

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521.000 C FOR SMALL MODULUS THE TERMS IN THE TAYLOR
522.000 C SERIES GREATER THAN 1.0Q-32 ARE USED .
523.000 60 BESS = 0.0Q0
524.000 IF ( NU .EQ. 1 ) BESS = Z * 0.5Q0
525.000 IF ( NU .EQ. 0 ) BESS = 1.0Q0
526.000 RETURN
527.000 C
528.000 C MODULUS OF Z TOO LARGE . NO CALCULATION
529.000 C HAS BEEN PERFORMED .
530.000 70 IFAIL = 2
531.000 BESS = 0.0Q0
532.000 RETURN
533.000 END
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